Unpacking General Relativity

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Abstract

Since most textbooks gloss over the mathematical justification on key topics, the aim of this paper is to provide curated references of media such as textbooks, papers and videos, for enthusiastic readers who wish to have a deeper understanding of General Relativity from a mathematical standpoint. We will follow the logical progression of Einstein's The Foundation of the General Theory of Relativity[4] and Dr. Gerlach's A Companion to the Track II Mathematics of MTW[6]. Each section will outline key topics of general relativity with its relevant supplement study references. The intended audience should be comfortable with classical mechanics, linear algebra, multi-variable calculus and differential equations. Readers with knowledge of differential geometry, tensor calculus, manifold and topology would have a head-start but one can acquire these mathematical skills concurrently.

1. Introduction

In 1915, Albert Einstein, published his revolutionary paper, Die Feldgleichungen der Gravitation[1] (The Field Equations of Gravitation). Here he show-cased the equation infamous now known as the Einstein Tensor.

$$\mathbf{G}_{im} = \mathbf{R}_{im} + \mathbf{S}_{im} \tag{1}$$

An elegant, yet deceptively complex equation that shows how matter tells spacetime how to curve, and curved spacetime tells matter how to move. Underneath the elegance lies a complex web of math an physics that has push the best minds to their limits. Before he arrived at this point, he release many more prerequisite papers. Here we will attempt to follow his thought process.

Readers be warned, learning General Relativity, "Its acquisition is a matter of choice, of the right way of using one's mind, and of the perseverance to do so." but the reward is the satisfaction of unlocking the fundamental reality we take for granted.

2. Special Relativity

The first step to general relativity is special relativity. As the name implies, it is a special form of relativity and events are measured with imaginary rods and clocks. The events with their own coordinate systems, move in a uniform translation to one another. This type of motion is what makes this type of relativity special.

Authored by Einstein under the title, Zur Elektrodynamik bewegter Körper[2] (On the Electrodynamics of Moving Bodies), here special relativity was first displayed in its raw form. Enthusiasts whould try to read this paper at least one. It is perfectly ok if its content is beyond comprehension. We will explore other routes that would help us come to terms with it. A good starting point for special relativity is, Spacetime Physics[3]. The authors did a great job at explaining complex topics in layman's terms. There are lots of pictorial aid and many exercises.

That main take-away from special relativity is that, it's relativity in flat space, Minkowski space, and it does not involve gravity. Therefore, we introduce the two postulates of special relativity.

- **Principle of relativity**: The laws of physics take the same form in all inertial frames of reference.
- **Invariance of the speed of light**: The speed of light in free space is the same in all inertial reference frames.

The combination of the postulates is made possible by the lorentz transformation with the lorentz factor being:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

where v is the velocity of the event and c the speed of light. Although special relativity is not adequate for all situations, we will see how its inertial frame plays an integral role in General Relativity via the equivalence principle.

The reader should have some knowledge of Electricity and Magnetism learned from college physics, if not a crash course on E&M is recommended. The parallelism between E&M and relativity is encompassing and its math is used often.

3. Math Preparation

Manifold, Topology, Differential Geometry and Tensor Calculus. Before a painter can create their art, a substrate is needed. Here we begin our introduction of the mathematical tools and objects. The substrate for a physicist or mathematician is the manifold and the tools to traverse the manifold is topology. In many texts this topic is often glossed over and enthusiast can probably do without this section but like an painter who understands their medium, sufficient knowledge is encouraged.

There is a plethora of texts about topology and manifolds. One reference that stands out from the rest and is tailored for General Relativity is a YouTube playlist, What is a Manifold?[5], made by the user @XylyXylyX.

Although the textbook[6] does not mention manifolds until mid-way, it would be prudent for the reader to take a gander at this topic before hand. Understanding that there are no natural isometry from one tangent space on the manifold to another and how to ”connect” the tangent spaces is important. The manifold provides the framework for which the mathematics can be used.

The obviousness of the need for Tensor Calculus is clear. It is the mathematical backbone for the Einstein Field Equation and its compliment is Differential Geometry. If the reader could only pick one topic to study beforehand, it would be Tensor Calculus. At the very least, understand the definition of a tensor, it’s linearity, transformation, dual space and the Einstein summation notation. Some examples will be provided but one should digest the textbook and references in its entirety for rigorous treatment of definitions and proofs.

The next important subject is Differential Geometry and this can be studied concurrently with the textbook. This is where the understanding of topological manifolds will be of great relevance. The use of charts, maps and atlases justify the relationship events points on different tangent spaces. Internalize and digest exterior algebra and calculus. Study the Elie Cartan’s fundamental forms. The reader will appreciate the efficiency in calculation if these forms are mastered.

Per Einstein, ”The general laws of nature to be expressed by equations which hold good for all systems of coordinates, that is, are covariant with respect to any substitutions whatever (generally covariant).”. Even Einstein struggled with the term ”generally covariant”, but he understood its importance. Covariance ensures that the math can be applied to any coordinate system. The mathematics of covariance is not trivial but is necessary. A highly recommended text on the physics of covariance is one by H. Emam, Covariant Physics[14].

4. Mathematical Aid

The following concepts are the mathematical cornerstone for General Relativity so heavy investment is highly recommended to internalize it. Examples provided are just for reference, not for completeness. Otherwise noted, reference playlist[9], 11 for intuitive explanations.

- **Derivations are vectors:** This concept maybe foreign for most beginners but after a thorough dive into differential geometry, one will find this to be true.

  Let e_μ be the basis vectors for some coordinate space, then ω^ν is the dual, it follows that,

  $$e_\mu = \frac{\partial}{\partial x^\mu} \text{ and } \omega^\nu = dx^\nu$$

- **Tensors:** Tensors are the fundamental mathematical object in general relativity and is the keystone to Einstein's equations. In general relativity tensors can be define as multilinear maps that obey special transformation rules. Zero rank tensors are scalars, rank one tensors are vectors and rank two tensors are bilinear forms. Here's an example of a second-rank tensor. Given V and vector space and field F, then for all λ, μ in F and all **a**, **b** , **c** in V,

$$\mathbf{T} : V \; x \; V \implies F$$

$$\mathbf{T}(\lambda\mathbf{a} + \mu\mathbf{b}, \mathbf{c}) = [\lambda\mathbf{T}(\mathbf{a}, \mathbf{c}] + [\mu\mathbf{T}(\mathbf{b}, \mathbf{c})]$$

$$\mathbf{T}(\lambda\mathbf{a}, \mu\mathbf{b} + \mathbf{c}) = [\lambda\mathbf{T}(\mathbf{a}, \mathbf{b}] + [\mu\mathbf{T}(\mathbf{a}, \mathbf{c})]$$

  Note that **T** is linear in all slots.

- **Product:** Tensor product is linear map and the operator is denoted by $\otimes$. For example,

$$\mathbf{u} \otimes \mathbf{v} : V \; x \; V \implies F$$

$$[\mathbf{u} \otimes \mathbf{v}](\mathbf{a}, \mathbf{c}) = [\mathbf{u}(\mathbf{a})][\mathbf{v}(\mathbf{b})]$$

  For all **a**, **b** in V. The reader should get acquainted with all the linearity rules of the tensor product.

- **Wedge Product:**For differential forms the wedge product is denoted by $\wedge$. This topic is covered in any differential geometry and tensor textbooks. For examples,

$$\mathbf{a} \wedge \mathbf{b} = \mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a}$$

  The wedge product has a deep connection with the determinant of a matrix. Further along the journey, having a deep understanding of the wedge product will help the reader with Elie Cartan's structural forms. For an intuitive treatment, watch playlist[12].

- **Contravariant and Covariant Tensor**: This is one of the most important concept in General Relativity and was a challenge for Einstein. We define the law transformation of the contravariant 4-vector as a tensor by,

$$dx'_\mu = \frac{\partial x'_\mu}{\partial x_\nu} dx_\nu$$

  Hence for any contravarient vector A^μ is transformed by the same law as

$$A'^\mu = \frac{\partial x'_\mu}{\partial x_\nu} A^\nu$$

  By linearity if B^μ is contravariant then $A^\mu + B^\mu$ is also. Similarly, for a covariant A_μ it's transformation is

$$A'_\mu = \frac{\partial x_\nu}{\partial x'_\mu} A_\nu$$

  with the same rules of linearity. Then for any for any contravariant B^μ,

$$A_\mu B^\mu = \text{Invariant}$$

  By convention, contravariant tensors are denoted with superscript index while covariant are subscript.

- **Higher Rank Tensors and Transformation** Given two first rank contravariant tensors $A^\mu B^\nu$ we form a new tensor

$$A^{\mu\nu} = A^\mu B^\nu$$

Then by the laws of transformation

$$A'^{\sigma\tau} = \frac{\partial x'_\sigma}{\partial x_\mu}\frac{\partial x'_\tau}{\partial x_\nu}A^{\mu\nu}$$

Analogously for a covariant second tensor of the form $A_{\mu\nu}$,

$$A'_{\sigma\tau} = \frac{\partial x_\mu}{\partial x'_\sigma}\frac{\partial x_\nu}{\partial x'_\tau}A_{\mu\nu}.$$

For a contravariant tensor A^μ and a covariant tensor B_ν, we have a mix tensor,

$$A^\mu_\nu = A^\mu B_\nu,$$

and its transformation

$$A'^\tau_\sigma = \frac{\partial x'_\tau}{\partial x_\nu}\frac{\partial x_\mu}{\partial x'_\sigma}A^\mu_\nu.$$

For the general transormation,

$$A'^{\mu'_1\cdots\mu'_n}_{\nu'_1\cdots\nu'_n} = \frac{\partial x'_{\mu_1}}{\partial x_{\mu_1}}\cdots\frac{\partial x_{\nu_n}}{\partial x'_{\nu_n}}A^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}.$$

- **Levi-Civita tensor:** Related to the determinant of a matrix, fully understanding this will help the reader with time-saving calculations.

$$\epsilon_{\mu\nu\gamma} = \begin{cases} 1 & \text{for even permutations} \\ -1 & \text{for odd permutations} \\ 0 & \text{otherwise} \end{cases}$$

This tensor has other properties the reader should internalize.

5. Invariance of Coordinates

By the equivalence principle we are guaranteed an inertial frame although it may be infinitely small. Here introduce a method of measurement of events within this frame that is invariant with the choice of coordinate systems.

$$ds^2 = dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2 \tag{2}$$

If we allow each dX_σ to be linear, we have

$$dX_\sigma = \sum_\mu a_\sigma dx_\mu$$

insert this into (2) yields

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu}dx_\mu dx_{nu}$$

Then using Einstein summation convention where double indices are implied sums, we have

$$ds^2 = g_{\mu\nu}dx_\mu dx_{nu} \tag{3}$$

The $g_{\mu\nu}$ was initially called fundamental tensor but it is now known as the metric tensor and here are some of its important properties. For flat space one may see the metric as $\eta_{\mu\nu}$, termed the Minkowski metric.

- **Symmetry:**

$$g_{\mu\nu} = e_\mu \cdot e_\nu = g_{\nu\mu}$$

where e_μ and e_ν are the basis vectors.

- **Contravariant:**

$$g_{\mu\sigma} g^{\nu\sigma} = \delta_\mu^\nu$$

where δ_μ^ν is the kronecker delta equals 1 when $\mu = \nu$ and 0 otherwise.

- **Determinant:**

$$\det(g_{\mu\nu}) \equiv |g_{\mu\nu}| \equiv g$$

- **Transformation:**

$$g' = \left| \frac{\partial x_\mu}{\partial x'_\sigma} \frac{\partial x_\nu}{\partial x'_\tau} g_{\mu\nu} \right|$$

- **Formation:**

$$A^\mu = g^{\mu\sigma} A_\sigma,$$
$$A = g_{\mu\nu} A^{\mu\nu}$$
$$A_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} A^{\alpha\beta}$$

- **Differentiation:**

$$dg = g_{\mu\nu} dg^{\mu\nu} + g^{\mu\nu} dg_{\mu\nu} = 0$$

6. Curvature

Now we're digging into the meat and potatoes of General Relativity. What is curvature? What is a straight or a curved line? Take a pause and think about this. If one already knows, then kudos. In General Relativity, curvature can be found with the method of **parallel transport**. Those looking for intuition, watch videos #15-20 of the Tensor Calculus playlist[9]. Eigenchris did a great job with defining curvature from a Tensor perspective.

- **Christoffel Symbol:** Also known as the connection coefficient. The Christoffel symbol is not a tensor. Christoffel symbols of the first kind,

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2} \left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} \right)$$

Christoffel symbols of the second kind,

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\sigma} \left(\frac{\partial g_{\sigma\beta}}{\partial x^\gamma} + \frac{\partial g_{\sigma\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\sigma} \right)$$

- **Covariant Derivative:** This is a very important topic and deserves surgical treatment, thus details are left to the reader to investigate. One key take-away is that the results of a covariant derivative is a tensor while an ordinary derivative is not. Here's a simple example of a covariant derivative on a vector field with the modern notation ∇.

$$\nabla_u V \equiv= e_j \left[\frac{\partial v^j}{\partial x^k} + v^i \Gamma^j_{ik} \right] u^k$$

On the left-hand side, this reads as the covariant derivative on the vector field V in the direction of u. For covariant derivative, a semicolon is used and a comma is used for a regular derivative as follows,

$$v^i_{;k} = \frac{\partial v^j}{\partial x^k} + v^i \Gamma^j_{ik} = v^j_{,k} + v^i \Gamma^j_{ik}$$

.

- **Commutator:** Also known as the Lie Bracket, the commutator is important for understanding and calculating the curvature tensor. **Flow curves** are the physical interpretation of the commutator. For vector fields **u** and **v**, the commutators is defined as,
$$[\mathbf{u}, \mathbf{v}] = (u^\alpha v^\beta_{,\alpha} - v^\alpha u^\beta_\beta)_{,\beta}$$
Keep in mind that commutator is not a linear operator, thus its result is not a tensor.

- **Torsion:** Given two vectors **u** and **v**, then the torsion is,
$$T(\mathbf{u}, \mathbf{v}) = \nabla_\mathbf{u}\mathbf{v} - \nabla_\mathbf{v}\mathbf{u} - [\mathbf{u}, \mathbf{v}]$$

- **Geodesic:** The concept of geodesics may not be intuitive at first but think of it as a free-fall path. It is the natural or maximal path of an object in motion. For example, a photon will travel a "straight" path and may appear to curve in the presence of a strong gravitational field (indication of a curved manifold) but the path is still "straight".

  When $\nabla_\mathbf{u}\mathbf{u} = 0$ then for an affine parameter λ,
$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu}\frac{x^\mu}{\lambda}\frac{x^\nu}{\lambda} = 0$$
For example, in flat space without the presence of gravity, there are no curvature, then $\Gamma^\alpha_{\mu\nu} = 0$, so the geodesic equation will be,
$$\frac{d^2x^\alpha}{d\lambda^2} = 0$$
This is correct because the geodesic experienced no acceleration.

7. Curvature Tensors:

Suppose the reader has a grasp on the mathematics and understands the concept of parallel transport. Then the reader is ready to explore the curvature part of the Einstein field equation.

- **Riemann Curvature Tensor:** Given a vector V^σ and a closed loop formed by two vectors A^μ and B^ν with infinitesimal lengths sides δa and δb, if one wish to parallel transport V^σ around this loop. Then the change on the transported vector is,
$$\delta V^\rho = \delta a \delta b A^\mu B^\nu R^\alpha_{\ \beta\gamma\delta} V^\alpha$$
. Then the Riemann Tensor is identified as,
$$R^\alpha_{\ \beta\gamma\delta} = \frac{\partial\Gamma^\alpha_{\ \beta\delta}}{\partial x^\gamma} - \frac{\partial\Gamma^\alpha_{\ \beta\gamma}}{\partial x^\delta} + \Gamma^\alpha_{\ \mu\gamma}\Gamma^\mu_{\ \beta\delta} - \Gamma^\alpha_{\ \mu\delta}\Gamma^\mu_{\ \beta\gamma}$$
Also note the Riemann Tensor is antisymmetric about the last two indices. Meaning if the vector traversed the loop the opposite direction, the result would be negative.
$$R^\alpha_{\ \beta\gamma\delta} = -R^\alpha_{\ \beta\delta\gamma}$$
View playlist[1]0 episode 58-63, for a in-depth derivation of the Riemann Curvature Tensor.

- **Ricci Curvature Tensor:** Throught contraction of the Riemann Curvature Tensor we obtain the Ricci Curvature Tensor,
$$R_{\alpha\beta} = R^\mu_{\ \alpha\mu\beta}$$
.

- **Ricci Scalar:** The contraction of the Ricci tensor produces a Ricci scalar,

$$R = g^{\alpha\beta} R_{\alpha\beta}$$

- **Bianchi Identities:**

$$R_{\alpha\beta\mu\nu;\gamma} + R_{\alpha\beta\gamma\mu;\nu} + R_{\alpha\beta\nu\gamma;\mu} = 0$$

  When contracted with a pair of metric tensors yields,

$$\nabla_\gamma R^\gamma{}_\mu = \frac{1}{2}\nabla_\mu R$$

8. Stress Energy Tensor:

Also known as the energy momentum, momenergy, tensor. This tensor describes the energy and momentum of particles for a system in a relativistic manner. Playlist[10] episodes 96 to 112 provides detailed derivation and explanation. The most rigorous derivation could be found in the textbook[6] in chapters 12 -14.

$$T^{\mu\nu}$$

Decomposed,

$$\begin{aligned} T^{00} &= \text{energy density}, \\ T^{0l} &= \text{energy flux}, \\ T^{m0} &= \text{density momentum}, \\ T^{ml} &= \text{flux momentum} \end{aligned}$$

For a perfect fluid the tensor becomes,

$$T^\nu_\mu = (p+\rho)u_\mu u^\nu + p\delta^\nu_\mu.$$

Where ρ is the density, p is pressure and **u** is the velocity 4-vector. Most texts show the perfect fluid tensor with both superscript indices but one easily come to the same conclusion by contracting both sides by the metric tensor.

9. The Field Equation:

Once the reader has grasped all the previous sections, they are ready explore the field equation. Here we introduce the Einstein tensor in a more familiar form,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

From what we've learned from the previous section, the right-hand side is curvature. Specifically it is the Ricci curvature tensor minus the contracted Ricci tensor. Now we see that the Einstein tensor is a curvature tensor.

Then from the field equation we know,

$$G_{\mu\nu} = \frac{8\pi G_{grav} T_{\mu\nu}}{c^4}$$

We also know from the previous section, that the stress-energy tensor is matter in motion. equating the two equations yields,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G_{grav} T_{\mu\nu}}{c^4} \tag{4}$$

Now we see the connection between matter and spacetime curvature and the reasoning behind the quote, "matter tells spacetime how to curve, and curved spacetime tells matter how to move."

To verify the field equation we need to show Eqn. 4 reduces at the Newtonian limit for relatively slow sppeds. For index 00 components, we do see this is the case.

$$\nabla^2 h_{00} = -\frac{8\pi G_{grav} T_{00}}{c^4}$$

Reference Lecture Notes on General Relativity[13] for a derivation of this.

Although getting to this point is a major milestone, it is only the first milestone. It is advised that the reader revisit each section multiple times. With each pass, go deeper into the mathematics.

References

[1] Einstein, A. (1915) Die Feldgleichungen der Gravitation. Sitzungsberichteder Preussischen Akademie der Wissenschaftenzu, Berlin, 844-847

[2] Einstein, A. (1905), Zur Elektrodynamik bewegter Körper. Ann. Phys., 322: 891-921

[3] Taylor, E. F., and Wheeler, J. A. (1981). Spacetime physics.

[4] Einstein, A. (1916) The Foundation of the General Theory of Relativity. Annalen der Physik, 49, 769-822.

[5] (2021). What is a Manifold? Retrieved August 5, 2023, from https://youtube.com/playlist?list=PLRlVmXqzHjUQHEx63ZFxV-0Ortgf-rpJo.

[6] Gerlach, U. (2022). A Companion to the Track II Mathematics of MTW.

[7] Das, A. (2007). Tensors: The mathematics of relativity theory and continuum mechanics. Springer.

[8] O'Neill, B. (2007). Elementary differential geometry. Elsevier Academic Press.

[9] (2018). Tensor Calculus. Retrieved August 5, 2023, from https://youtube.com/playlist?list=PLJHszsWbB6hpk5h8lSfBkVrpjsqvUGTCx.

[10] Tragula, T. (2018). General Relativity, step by step, from https://youtube.com/playlist?list=PL9_n3Tqzq9iWtgD8POJFdnVUCZ_zw6OiB

[11] (2020). What is a Tensor? Retrieved August 5, 2023, from https://youtube.com/playlist?list=PLRlVmXqzHjUQARA37r4Qw3SHPqVXgqO6c

[12] Penn, M. (202). Differential Forms? Retrieved August 5, 2023, from https://youtube.com/playlist?list=PL22w63XsKjqzQZtDZO_9s2HEMRJnaOTX7

[13] Carroll, S. M. (1997, December 3). Lecture notes on general relativity. arXiv.org. https://arxiv.org/abs/gr-qc/9712019

[14] Emam, M. H. (2021). Covariant physics: From classical mechanics to general relativity and beyond. Oxford University Press.