NUM-PDE, Exercise 2

⚠ *The following exercises need to be uploaded in TUWEL not later than 23:59 18-03-2024.*

1. **Symmetries of the plate capacitor** (*1 point*): Reconsider the plate capacitor (2d case) from Exercise 1.

   Plotting the solution of the capacitor one may observe that the solution is symmetric with respect to one axis and antisymmetric with respect to the other axis.

   Therefore, it would be clever to recast our problem only on a quarter of the domain and apply proper boundary conditions on the symmetry axes.

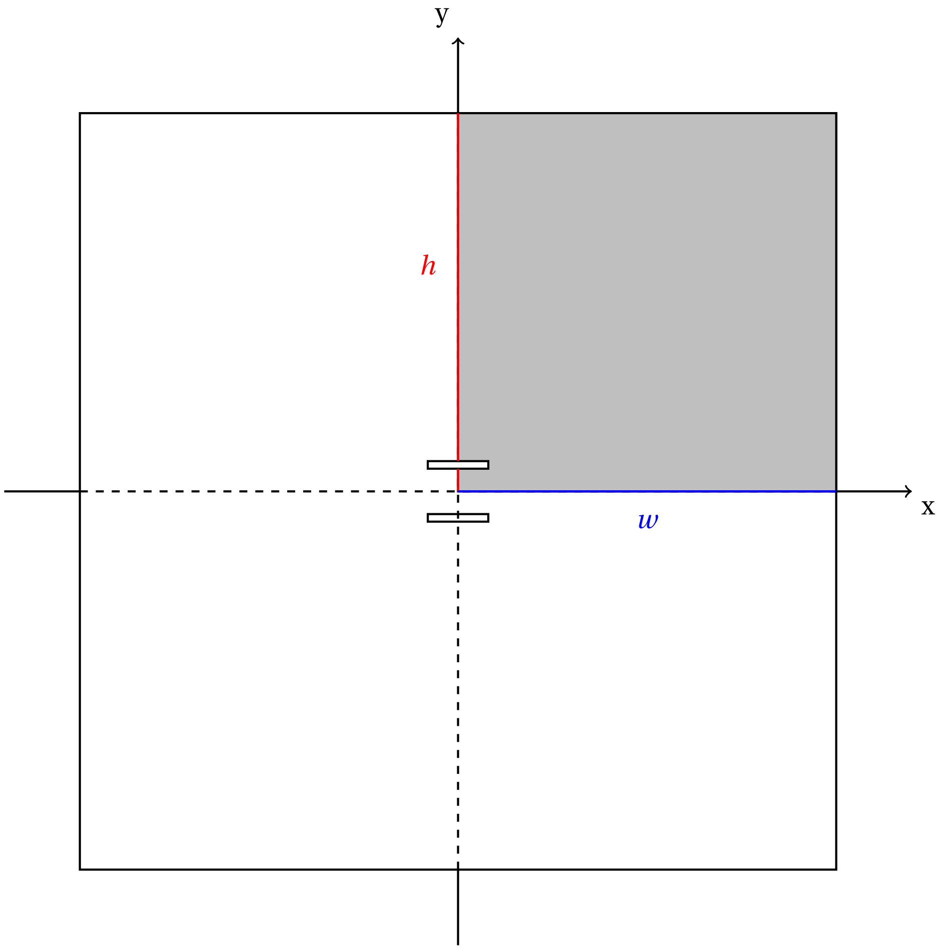

Figure 1: The domain Ω is divided into four subdomains. Our new domain is still called Ω, but it considers ONLY the shaded area.

- *0.2 points*: What are the correct boundary conditions to apply on the edge w? Neumann or Dirichlet? Why? (*Hint:* In order to deduce the correct w boundary condition look at the solution of Exercise 1 `Draw(gfu)` along w.)
- *0.2 points*: What are the correct boundary conditions to apply on the edge h? (*Hint:* In order to deduce the correct h boundary condition look at the solution of Exercise 1 `Draw(gfu)` along h.)

- *0.2 points*: Write down the new variational formulation of the problem considering only the quarter of the domain.
- *0.2 points*: Solve the derived variational formulation.
- *0.2 points*: Plot the solution, print the energy and compare it with the solution of the full domain. Does it make sense?

2. **Minimization problem** (*1 point*): Solve Exercise 12.1 from the lecture notes. Prove the following proposition:

   Proposition 1. *Let V be a Hilbert space, $A(.,.)$ a symmetric, coercive and continuous bilinear-form on V, and $f(.)$ a continuous linear from. Let $u \in V$ solve the variational problem*

   $$A(u,v) = f(v) \qquad \forall v \in V.$$

   Show that u is the unique solution of the minimization problem

   $$\min_{v\in V} J(v) \quad with \quad J(v) := \tfrac{1}{2}A(v,v) - f(v)$$

   Here you have (0.3 points) for the uniqueness and (0.7 points) for showing that is minimizer.

   (*Hint:* Show that $J(v) - J(u) = \frac{1}{2}A(u-v, u-v)$)

3. **Convection Diffusion Problem** (*1 point*):

   Consider the variational problem on $\Omega = (0,1)^2$: find $u \in H_0^1(\Omega)$ such that

   $$\int \nabla u \nabla v - \vec{b} \cdot u \nabla v \, dx = \int 1 \, v \, dx \quad \forall v \in H_0^1(\Omega)$$

   with $\vec{b} = (10, 2)$.

   - *0.25 points*: Solve this equation using finite elements, plot u_h.
   - *0.25 points*: What is the strong form of the PDE of that variational formulation?
   - *0.25 points*: Try also with larger $\vec{b}$, what happens?.
   - *0.25 points*: Prove that the bilinear-form is coercive on $H_0^1(\Omega)$ w.r.t. the norm $\|u\|_V = \|\nabla u\|_{L_2}$ for arbitrary (bounded and smooth) $\vec{b}$ such that $\operatorname{div} \vec{b} = 0$.

4. **Complex-valued problem as a real system** (*1 point*): Solve Exercise 12.3.3 from the lecture notes.

   Let $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ be two symmetric, continuous, and coercive bilinear-forms on the Hilbert-space V.

   Define the form $B(\cdot,\cdot) : V^2 \times V^2 \mapsto \mathbb{R}$ as

   $$B((u_1,u_2),(v_1,v_2)) := a(u_1,v_1) + b(u_2,v_1) + b(u_1,v_2) - a(u_2,v_2)$$

   Define the norms:

   $$\|u\|_V := \sqrt{a(u,u) + b(u,u)} \qquad \text{for } u \in V$$

   and

   $$\|(u_1,u_2)\|_{V^2} := \sqrt{\|u_1\|_V^2 + \|u_2\|_V^2} \quad \text{for } (u_1,u_2) \in V^2.$$

   - *0.1 points*: Show that the above are in fact norms.
   - *0.4 points*: Show that $B(\cdot,\cdot)$ is continuous and give explicit constant.

- *0.5 points*: Show that $B(\cdot,\cdot)$ is inf-sup stable and give explicit constant.

  Hint: Try first with the simplified forms $\tilde{B}((u_1,u_2),(v_1,v_2)) := a(u_1,v_1) - a(u_2,v_2)$, and $\hat{B}((u_1,u_2),(v_1,v_2)) := b(u_2,v_1) + b(u_1,v_2)$ and prove continuity and inf-sup. For given $v = (v_1,v_2)$, come up with some candidates $u = (u_1,u_2)$ for the supremum. Try a combination of both for the original bilinear-from $B(.,.)$

Remark 1 (Note for the kind students:)**.** *As a quick note before the submission. The exercises presented today were good. Let's keep up with this level of quality. To do so we want to specify some points: Please, before submitting the code check that:*

- *For those of you that NEVER coded in LaTeX or markdown before, must do at least ONE of the 2 exercises in a Jupyter notebook or in LaTeX. The other one can be sent as a picture.*
- *The code runs without errors and within a reasonable time.*
- *The code is clean and commented (where it is necessary: e.g. explaining how to create a geometry is not necessary anymore.)*
- *The LaTeX part is coincise and clear. You need to know that you have submitted!*

If these criterias are not met we take the freedom to subtract points.

Remark 2 (Note for the ambitious students:)**.** *If you are done with the exercises and you want to improve your skills try to integrate the conjugate gradient method and the adaptivity in your code. This will NOT result in extra points, but it will be a good exercise for you.*