NUM-PDE, Exercise 1

The following exercises need to be uploaded in TUWEL not later than 23:59 11-03-2024.

1. **The plate capacitor** (*2 point*): The electric potential for a plate capacitor is given by the following equations:

$$\begin{cases} -\varepsilon_0 \Delta u = 0 \quad \text{in } \Omega \\ u = 1 \quad \text{on } \Gamma_{\text{electorde 1}} \\ u = -1 \quad \text{on } \Gamma_{\text{electrode 2}} \\ \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_{\text{outer}} \end{cases} \tag{1}$$

with $\varepsilon_0 = 8.854 \cdot 10^{-12} \frac{As}{Vm}$, the dielectric constant in vacuum. The geometry of the capacitor Ω is given by:

$$\Omega = (-5, 5)^d - \Omega_{\text{electrode 1}} - \Omega_{\text{electrode 2}}$$

with

$$\begin{aligned} \Omega_{\text{electrode 1}} &= (-0.4, 0.4) \times (0.3, 0.4) \\ \Omega_{\text{electrode 2}} &= (-0.4, 0.4) \times (-0.4, -0.3) \\ \Gamma_1 &= \partial\Omega_{\text{electrode 1}} \\ \Gamma_2 &= \partial\Omega_{\text{electrode 2}} \end{aligned}$$

And Γ_{outer} is the outermost border (in 2d the edges of the square and in 3d the faces of the cube).

Sizes are given in m.

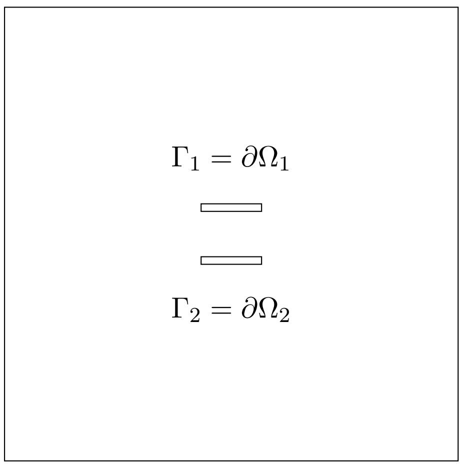

Figure 1: Not in scale, just to give an idea!!!

Derive the weak formulation of system (1) and find the discrete solution u

- plot the solution u and the electric field

$$E = -\nabla u.$$

- calculate the electric energy
$$\frac{1}{2}\int \varepsilon_0 |\nabla u|^2\,dx.$$
How does the energy vary with mesh-size and polynomial order?
Plot the energy vs mesh-size/polynomial order and try to estimate the error.

Consider now a 3d model of the plate capacitor:

$$\Omega_{\text{electrode 1}} = (-0.4, 0.4) \times (-0.4, 0.4) \times (0.3, 0.4)$$

and

$$\Omega_{\text{electrode 2}} = (-0.4, 0.4) \times (-0.4, 0.4) \times (-0.4, -0.3).$$

Recalculate the above bullet points for this case. What difference do we observe? Finally change the scale of the figures from meters m to millimeters mm. What do we observe?

Hint: Have a look at the documentation, especially poisson.py and dirichlet_boundary.py

2. **Unconventional method to calculate the flux** (*2 point*)

Of uttermost importance in FEA is the computation of the flux. In this example we show how one can use the "shape of the problem" to his/her advantage to derive, up to machine precision, the exact conservation of energy.

Consider the following PDE on $\Omega = (0,1)^2$ with boundary parts $\Gamma_b, \Gamma_t, \Gamma_l, \Gamma_r$ (bottom, top, left, right):

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_b \cup \Gamma_t \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_l \cup \Gamma_r \end{cases}$$

with

$$f = 25\chi_{[0.3,0.5]\times[0.5,0.7]}.$$

Here χ_A is the indicator function

$$\chi_A = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}.$$

Calculate

$$\int_{\Gamma_b} \frac{\partial u}{\partial n},$$
$$\int_{\Gamma_t} \frac{\partial u}{\partial n},$$

and the sum of both, using finite elements. Resolve the sub-domain for the non-zero source on the mesh.

- *method (A)* Interpolate the gradient ∇u into `VectorH1`, and integrate the normal derivative over the parts of the boundary.
- *method (B)* The relation
$$\int_\Omega \nabla u \nabla v - \int_{\partial\Omega} \frac{\partial u}{\partial n} v = \int_\Omega f v$$
is valid for all values of v. Therefore can be used for a particular case:
choose $v = 1$ on Γ_t and $v = 0$ on Γ_b (and vice versa). Compute inner products of the residual vector $r = f - Au$ and the coefficient vector of v.

Hint: In NGSolve similar to the special coordinate coefficientfunctions x, y, z, we have a builtin coefficientfunction for the normal vector `n = specialcf.normal(dim=mesh.dim)`