Problem Sheet 6 - All Groups

discussion: Tuesday, 28.11.2023

Note that this week there are only 4 examples, the remaining time of the exercise is used to discuss the correct solutions to the first exam.

6.1. Consider the three recursions

- $x_{n+1} = \frac{x_n}{3}$ with $x_0 = 1$,
- $y_{n+1} = \frac{4}{3}y_n - \frac{1}{3}y_{n-1}$ with $y_0 = 1, y_1 = \frac{1}{3}$,
- $z_{n+1} = \frac{10}{3}z_n - z_{n-1}$ with $z_0 = 1, z_1 = \frac{1}{3}$.

These recursions in explicit form lead to the sequence $(\frac{1}{3})^n$. Write a program (in `matlab`/`python`) that realizes each recursion and computes the absolute and relative errors between $\left(\frac{1}{3}\right)^N$ and x_N, y_N, z_N for different values of N. Also try your program with slightly perturbed initial values, i.e., $x_0 = y_0 = z_0 = 1 + 10^{-14}$ and $y_1 = z_1 = \frac{1}{3} + 10^{-14}$. What do you observe?

Note: `matlab` has the possibility to compute results in higher precision (which you can use to compute $\left(\frac{1}{3}\right)^N$) using the `vpa` library (use `help vpa` for a documentation).

6.2. The sequence u_k, $k = 0, 1, \ldots$, given by[1]

$$u_1 := 2, \qquad u_{k+1} = 2^k \sqrt{2\left(1 - \sqrt{1 - (2^{-k}u_k)^2}\right)} \tag{1}$$

converges to the number $\pi = 3.1415\ldots$.

a) Compute (in `matlab`/`python`) the first 30 members of the sequence and the absolute error $|\pi - u_k|$. When is the error minimal?

b) Explain why you should expect that the error grows for $k \geq k_0$ for some k_0.

6.3. **a)** Compute the number of additions and multiplications in Algorithms 4.2 and 4.3 (forward and backward substitution).

b) Show that the product $\mathbf{L}_1\mathbf{L}_2$ of two lower triangular matrices $\mathbf{L}_1, \mathbf{L}_2$ is again a lower triangular matrix. Also show that the inverse of a (invertible) lower triangular matrix is lower triangular.

6.4. **a)** Explain Crout's algorithm from Chapter 4.3.1 in the lecture notes.

b) Modify the algorithm to compute a Cholesky factorization

$$\mathbf{C}^\top \mathbf{C} = \mathbf{A}$$

and realize your algorithm in Matlab/Python.

[1] The u_k correspond to the circumference of regular polygons with 2^k edges; this method of approximating π is due to Archimedes