

Problem Sheet 11 (CSE/Inf/Geo)

discussion: Tuesday, 23.01.2023

11.1. Let $\mathbf{x}_\ell \in \mathbf{x}_0 + \mathcal{K}_\ell$ be such that

$$\|\mathbf{b} - \mathbf{A}\mathbf{x}_\ell\|_2 \leq \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2 \quad \forall \mathbf{x} \in \mathbf{x}_0 + \mathcal{K}_\ell.$$

Show that the residual $\mathbf{r}_\ell := \mathbf{b} - \mathbf{A}\mathbf{x}_\ell$ satisfies

$$(\mathbf{b} - \mathbf{A}\mathbf{x}_\ell, \mathbf{v})_2 = (\mathbf{r}_\ell, \mathbf{v})_2 = 0 \quad \forall \mathbf{v} \in \mathbf{A}\mathcal{K}_\ell. \quad (1)$$

Hint: Proceed as in the derivation of Lemma 8.3 or in the derivation of the normal equations in Least Squares.

11.2. For $n = 1000$, we look at the $n \times n$ matrix given by

$$\mathbf{A} := \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}, \quad \mathbf{b} := \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \quad (2)$$

- Apply the GMRES method to solve $\mathbf{A}\mathbf{x} = \mathbf{b}$ (you may use the implementations in MATLAB or scipy) and plot the convergence of the residual. What do you observe?
- How does the computational cost for reaching good accuracy (say $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 < 10^{-5}$) compare to a direct solver?

11.3. Implement the explicit and implicit Euler method to solve the general linear differential equation

$$y'(t) = f(t)y(t) + g(t), \quad y(0) = y_0.$$

Here, f, g are given functions and y_0 is a given value. Compare both methods for the ODE

$$y'(t) = \lambda y(t), \quad y(0) = 1,$$

which has the exact solution $y(t) = \exp(\lambda t)$. Take the different values $\lambda = 1, 10, -1, -10$ and plot the error $|y(1) - y_N|$ (in loglog-scale) over $N = 2^i$, $i = 1, \dots, 10$. What do you observe?

11.4. Formulate the Newton method to compute y_{i+1} for given y_i for the implicit Euler method to solve $y' = f(t, y(t))$.