Problem Sheet 5 - All Groups

discussion: Tuesday, 21.11.2023

5.1. (sinc quadrature) For certain integrals of the form $\int_{-\infty}^{\infty} f(x)\,dx$ the simple trapezoidal rule works astonishingly well: Define the quadrature rule

$$Q^N(f) := h \sum_{i=-N}^{N} f(x_i), \qquad h := \frac{1}{\sqrt{N}}, \qquad x_i := ih.$$

Apply this rule to the integrand $f(x) = e^{-x^2} \sin^2(x)$. Plot the error versus N in a suitable scale.

5.2. **a)** Write a program with signature $y = \texttt{composite\_gauss}(n, L, q)$ that realizes a composite Gauss rule for integration over $(0, 1)$. The composite Gauss rule uses n points for each of the L subintervals that are given by

$$(0, q^{L-1}), (q^{L-1}, q^{L-2}), (q^{L-2}, q^{L-3}) \cdots, (q, 1)$$

Check your program with $f(x) = x^m$, $m = 0, 1, 2$. *Hint:* Gauss points and weights can be obtained by `numpy.polynomial.legendre.leggauss` or `gauleg.m` (see homepage).

b) Use your routine `composite_gauss` for $n = L = 1, \ldots, 20$ and the three choices $q \in \{0.5, 0.15, 0.05\}$ and the integrand

$$f(x) = x^{0.1} \log x.$$

(The exact integral is $\int_0^1 f(x)\,x = -1/1.1^2 \approx -0.82644.$) Plot semilogarithmically (`semilogy`) the quadrature error versus n for these 3 values of q. Which choice of q is the best one?

c) Fit (using `polyfit`) the error curves to the law Ce^{-bn}.

5.3. Give an explicit error bound (in dependence on n) for the Gaussian quadrature error

$$\left| \int_{-1}^{1} f(x)\,dx - Q_n^{Gauss}(f) \right| \qquad \text{with } f(x) = (4 - x^2)^{-1}.$$

Hint: Use Theorem 2.18 and estimate the best-approximation error by choosing v as the Taylor polynomial of suitable degree. In order to further bound the occuring expression you can, e.g., use $|x| \leq 1$ and the formula for the geometric series.

5.4. Show that multiplication $f(x, y) = x \cdot y$ and divison $f(x, y) = \frac{x}{y}$ for $x, y \in \mathbb{R}$ with $y \neq 0$ are well conditioned (in terms of relative conditioning).

Is the computation of square roots $f(x) = \sqrt{x}$ for $x > 0$ well conditioned?

5.5. Consider the function

$$\varphi(x) = \sqrt{x + 1} - \sqrt{x}$$

a) Is the evaluation of φ well-conditioned for large x? Consider relative conditioning.

b) Formulate a stable numerical realization of φ (*Hint:* You may use that a stable realization of $\sqrt{\cdot}$ is available.)