Problem Sheet 4 - All Groups

discussion: Tuesday, 14.11.2023

4.1. **a)** Show that the weights of the Newton-Cotes formulas satisfy $\sum_{i=0}^n w_i = 1$(= length of the interval $[0,1]$).
Hint: apply the quadrature formula to a suitable function f.

b) Show the symmetry property $w_{n-i} = w_i$, $i = 0, \dots, n$.
Hint: use the symmetry of the points, i.e., $x_j = 1 - x_{n-j}$.

c) Let $n = 2m$ be even. Consider the function $f = (x-1/2)^{n+1}$, which is antisymmetric with respect to $1/2$. Show:

$$\int_0^1 f(x)\,dx = 0 = \widehat{Q}_n^{cNC}(f) = \widehat{Q}_n^{oNC}(f).$$

Conclude that the quadrature formulas $\widehat{Q}_n^{cNC}$ and $\widehat{Q}_n^{oNC}$ are exact for polynomials of degree $n+1$. In particular, the midpoint rule is exact for polynomials in $\mathcal{P}_1$, and the Simpson rule is exact for polynomials in $\mathcal{P}_3$.

4.2. We wish to show that the extrapolation of the composite trapezoidal rule is the composite Simpson rule. To that end, let $T(h)$ be the composite trapezoidal rule with step size $h = (b-a)/N$ and $S(h)$ be the composite Simpson rule with step size $h = (b-a)/N$. Use Romberg extrapolation with step sizes $h_i = (b-a)2^{-i}$, $i = 0, 1, \dots,$

a) Extrapolation of the composite trapezoidal rule (with step size h) has in column $m = 0$ the values $T(h_i)$. Show: in column $m = 1$ of the Neville scheme are the values

$$N_i := T(h_{i+1}) + \frac{1}{3}(T(h_{i+1}) - T(h_i))$$

b) Show: $N_i = S(h_i)$.

4.3. Consider the quadrature rule

$$\int_0^1 f(x)\,dx \simeq \frac{1}{2}f\left(\frac{1}{3}\right) + \frac{1}{2}f\left(\frac{2}{3}\right).$$

a) Take $f(x) = 3x^2 + 1$ and compute the value of the quadrature rule for this function. What is the error between quadrature and exact value of the integral?

b) What is the maximal degree of exactness of this quadrature formula?

c) Is it possible to choose different weights to achieve higher degree of exactness? If not, is it possible to choose different knots to achieve higher degree of exactness?

4.4. Write a program with signature $I = \texttt{adapt}(f, a, b, \tau, h_{min})$ that realizes an adaptive quadrature for $\int_a^b f(x)\,dx$. The quadrature should be based on the Simpson rule. τ is the desired (absolute) accuracy and h_{min} the minimal interval length. To estimate the accuracy, compare the value of the Simpson rule $S_{\{a,b\}}(f)$ for the integration on $[a,b]$ with the value $S_{\{a,m\}}(f) + S_{\{m,b\}}(f)$ with $m = (a+b)/2$. Use your algorithm for the integration of the function

$$\begin{cases} \frac{1}{2}e^x & x < 1/3 \\ e^x & x \geq 1/3 \end{cases}$$

over $[0,1]$. Use $\tau = h_{min} = 2^{-j}$, $j = 0, \dots, 10$. Plot the error versus τ. What convergence do you observe? Why was $h_{min} = \tau$ chosen?

4.5. Develop an adaptive algorithm for the integration of functions over the rectangle $[a_x, b_x] \times [a_y, b_y]$. Base your algorithm on the midpoint rule, i.e., $Q_{[a,b]\times[c,d]}(f) = (b-a)(d-c)f((a+b)/2, (c+d)/2)$.

Hint: adapt the ideas of the 1d-adaptive algorithm of Problem 4.4.

Test your adaptive algorithm for the integration over $[0,1]^2$ of the following functions:

$$f_1(x,y) = x^2 \quad \text{and} \quad f_2(x,y) = \begin{cases} 0 & x < y \\ 1 & x \geq y \end{cases}$$

Use the tolerances $\tau = 2^{-i}$, $i = 0, \ldots, 15$, and make a convergence plot (quadrature error versus tolerance) in `loglog` scale.