Problem Sheet 9 (CSE/Inf/Geo)

discussion: Tuesday, 09.01.2024

9.1. Consider the nonlinear system of equations $\mathbf{f}(\mathbf{x}) = 0$ given by

$$\begin{aligned} 3x_1 - \cos(x_2 x_3) - 3/2 &= 0 \\ 4x_1^2 - 625x_2^2 + 2x_3 - 1 &= 0 \\ 20x_3 + e^{-x_1 x_2} + 9 &= 0 \end{aligned}$$

Compute the derivative $\mathbf{f}'(\mathbf{x})$ and formulate Newton's method. Program Newton's method in `matlab`/`python`. The program should additionally estimate the error (e.g., in the $\|\cdot\|_2$-norm) and also return it. Use the initial vector $(1, 1, 1)^\top$. Plot (using `semilogy`) the estimated error versus the iteration number.

9.2. Consider the system of equations

$$\mathbf{A}x = \mathbf{b} + \varepsilon \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$$

with

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \qquad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} (x_1 - x_2)^2 \\ 0 \end{pmatrix}, \qquad \varepsilon = 0.01.$$

a) Formulate the Newton method and write a program to compute the iterates $\mathbf{x}_n$, $n = 1, 2, \ldots,$. Initial value: $\mathbf{A}\mathbf{x}_0 = \mathbf{b}$.

b) Consider the following method with initial value $\mathbf{x}_0$ given by $\mathbf{A}\mathbf{x}_0 = \mathbf{b}$: For $n = 0, 1, \ldots$ one determines $\mathbf{x}_{n+1} \in \mathbb{R}^2$ such that $\mathbf{A}\mathbf{x}_{n+1} = \mathbf{b} + \varepsilon \mathbf{f}(\mathbf{x}_n)$. Write a program to compute the iterates. Taking the last value of the Newton method as the "exact solution" you can compute the errors. Plot the error for both methods in `loglog`-plot (error versus iteration index n). Can you relate this "simple" method to Newton's method for small ε?

9.3. Newton's method converges only for initial values close to the zero x^* of f. One possibility to address this difficulty is the so-called *continuation method*: One considers a function $H(x, s)$ with $H(x, 1) = f(x)$ and for which a zero x_0 of $H(x, 0)$ is known. One then selects points s_i, $i = 1, \ldots, N$ and employs Newton's method to compute the zero x_{i+1} of $H(x, s_{i+1})$, taking x_i as the initial value. Perform the method for

$$f(x) = \arctan x, \qquad H(x, s) = \arctan x - (1 - s) \arctan 4$$

and initial value $x_0 = 4$. Select $s_i = i/10$, $i = 0, \ldots, 10$.

9.4. Consider the for $N \in \mathbb{N}$ the system of equations

$$\frac{-u_{i+1} + 2u_i - u_{i-1}}{h^2} + u_i^3 = 1, \qquad i = 1, \ldots, N-1, \qquad u_0 = u_N = 0.$$

Formulate Newton's method for its solution and program it in `matlab`/`python`. Estimate the error (in the $\|\cdot\|_2$-norm) by considering the difference of two consecutive iterates. Plot the error versus the iteration number.

Remark: the above system of equations results from the numerical approximation of the "boundary value problem"

$$-u''(x) + (u(x))^3 = 1, \qquad x \in (0, 1), \qquad u(0) = u(1) = 0.$$

The values u_i are approximations to the true values $u(x_i)$ with $x_i = ih$, $i = 1, \ldots, N-1$.

9.5. Let $F(x) = 2 - x^2 - e^x$.

1. Compute, using Newton's method the positive zero x^* of F to machine precision.
2. Compute the zero x^* with the Broyden method. Set $x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$. Plot for $n \in \{1, \ldots, 8\}$ and $x_0 = 2.5$ the error $|x^* - x_n|$ versus the step number n. Also plot the *numerical convergence order* $p_n = \log(|x^* - x_{n+1}|)/\log(|x^* - x_n|)$ versus n. What convergence order do you observe?
3. Compare the *efficiency* of the secant method with that of the Newton method by comparing accuracy versus number of function evaluations. To that end, assume that a Newton step costs 3 function evaluations (this is realistic assuming that F' is approximated with a difference quotient) and plot achieved accuracy versus number of function evaluations. Which method is more efficient?