Problem Sheet 7 (CSE/Inf/Geo)

discussion: Tuesday, 5.12.2023

7.1. **a)** Prove that the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}$$

does not have a factorization $\mathbf{A} = \mathbf{LU}$ with normalized lower triangular matrix $\mathbf{L}$ and upper triangular matrix $\mathbf{U}$.

b) Let $\mathbf{P}$ be given by

$$\mathbf{P} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & 1 & \\ & & & & & & \ddots \end{pmatrix}$$

where the off-diagonal 1 are in the positions (i_1, i_2) and (i_2, i_1) (with $i_1 \neq i_2$).

Show: For $\mathbf{A} \in \mathbb{R}^{n\times n}$, the matrix $\mathbf{PA}$ is the matrix $\mathbf{A}$ with rows i_1 and i_2 interchanged. Furthermore, $\mathbf{P}^{-1} = \mathbf{P}^\top = \mathbf{P}$.

c) Given the factorization $\mathbf{PA} = \mathbf{LU}$, how is the solution $\mathbf{x}$ to $\mathbf{Ax} = \mathbf{b}$ obtained?

7.2. ("arrowhead matrix") Let $n = 10$, $\mathbf{e} = (1, 1, \ldots, 1)^\top \in \mathbb{R}^n$, $\mathbf{b} = (1, 0, \ldots, 0)^\top \in \mathbb{R}^n$. Consider the matrix $\mathbf{A} = 10\mathbf{I} + \mathbf{be}^\top + \mathbf{eb}^\top$ and the matrix $\widetilde{\mathbf{A}} := \mathbf{A}(n : -1 : 1, n : -1 : 1)$ that is obtained from $\mathbf{A}$ by reversing the numbering of the rows and columns.

Use the commands `spy` (`matplotlib.pyplot.spy`) and `lu` (`scipy.linalg.lu`) to visualize the sparsity patterns of $\mathbf{A}$, $\widetilde{\mathbf{A}}$ and the corresponding factors $\mathbf{L}$, $\mathbf{U}$ of the LU-factorization. What do you observe? Which variant of the numbering is to be preferred from a cost (i.e., number of floating point operations or storage requirement) point of view?

Take $n = 10, 100, 1000$ and test the reverse Cuthill-McKee and the approximate minimum degree ordering for this example. Which one produces the least non-zero entries in the LU-factorization?

7.3. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 10^{-6} & 1 \\ 1 & 0 \end{pmatrix}$$

a) Compute by hand the LU-factorization of $\mathbf{A}$. Calculate $\mathbf{A}^{-1}$, $\mathbf{L}^{-1}$, $\mathbf{U}^{-1}$ and compute the three condition numbers $\kappa_\infty(\mathbf{A}) = \|\mathbf{A}\|_\infty \|\mathbf{A}^{-1}\|_\infty$, $\kappa_\infty(\mathbf{L}) = \|\mathbf{L}\|_\infty \|\mathbf{L}^{-1}\|_\infty$, $\kappa_\infty(\mathbf{U}) = \|\mathbf{U}\|_\infty \|\mathbf{U}^{-1}\|_\infty$. Here, $\|\cdot\|_\infty$ is the row-sum norm.

b) Repeat the calculation of a) for the matrix $\widetilde{\mathbf{A}}$ that is obtained from $\mathbf{A}$ by interchanging the two rows. What do you observe?

7.4. Consider the linear system

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1.001 & 3 \\ 2 & -2 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$

a) Perform Gaussian elimination without pivoting to transform the matrix to an upper triangular matrix and solve the triangular system using backward substitution. Do you expect this to be numerically stable? What is the size of the (possible) error amplification?

b) Perform row pivoting in the elimination process and answer the same questions as in (a).

7.5. Let $\mathbf{Q}$ be an orthogonal matrix. Show:

a) $\|\mathbf{Qx}\|_2 = \|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{R}^n$.

b) $\mathbf{x}^\top \mathbf{y} = ((\mathbf{Q})\mathbf{x})^\top (\mathbf{Qy})$ for all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$.

c) Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ with $m > n$ and its QR-factorization $\mathbf{A} = \mathbf{QR}$. Show: If $\mathbf{A}$ has full rank (i.e., $\operatorname{rank}(\mathbf{A}) = n$), then the diagonal entries of $\mathbf{R}$ are non-zero.