Problem Sheet 10 (CSE/Inf/Geo)

discussion: Tuesday, 16.01.2023

10.1. Show the following convergence result for the inverse iteration with shift: Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be diagonalizable and $\lambda \in \mathbb{R}$. Let the eigenvalues of $\mathbf{A}$ be numbered such that $|\lambda_1 - \lambda| \geq |\lambda_2 - \lambda| \geq \cdots |\lambda_{n-1} - \lambda| > |\lambda_n - \lambda| > 0$. Then there exists $C > 0$ such that there holds for the approximations $\widetilde{\lambda}_\ell$ of the inverse iteration with shift:

$$|\lambda_n - \widetilde{\lambda}_\ell| \leq C \left| \frac{\lambda_n - \lambda}{\lambda_{n-1} - \lambda} \right|^\ell, \qquad \ell = 0, 1, \ldots,$$

hint: use Theorem 7.1 applied to a suitable matrix.

10.2. Consider the power iteration method for the matrix $\mathbf{A}$ and the following three initial vectors $\mathbf{x}_0^{(j)}$, $j = 0, 1, 2$:

$$\mathbf{A} = \begin{pmatrix} 2 & \\ & -2 \end{pmatrix}, \qquad \mathbf{x}_0^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_0^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{x}_0^{(3)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Discuss the behavior of the vector iteration. Do the eigenvalue approximations $\widetilde{\lambda}_\ell$ and the iterates $\mathbf{x}_\ell$ converge? If so, what do they converge to?

10.3. Use power iteration and inverse iteration (take starting values of your choice) to compute approximations $\widetilde{\lambda}_\ell$ ($\ell = 0, 1, \ldots$ are the number of steps in the interations) to both eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 5 \\ 5 & 6 \end{pmatrix}.$$

Plot the error between the computed approximations and the true eigenvalues for $\ell = 0, 1, \ldots$.

10.4. **a)** Show that $\mathbf{A}_\ell$ and $\mathbf{A}_{\ell+1}$ in the QR-algorithm with shift are similar and hence have the same eigenvalues.

b) Make 5 steps of the QR-algorithm with shift applied to the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 0 & 1 \\ -1 & 3 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 1 & 1 & 0 & 3 \end{pmatrix}.$$

What is the maximal error of all four computed eigenvalues?

10.5. Implement the CG method from the lecture to iteratively solve a linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ with $\mathbf{A} \in \mathbb{R}^{n\times n}$. Stop the iteration, once the residual in the ℓ-th step is smaller than a prescribed tolerance.

Compare your CG implementation with a direct solver (e.g in MATLAB $\mathbf{A}\backslash\mathbf{b}$) by plotting computational times over matrix size n for stopping tolerances 10^{-1}, 10^{-4} and 10^{-8}. In MATLAB you can, e.g., use `A = gallery('poisson',n)` as test matrix.