Problem Sheet 3 (CSE/Inf/Geo)

discussion: Tuesday, 07.11.2023

3.1. Let C, $\alpha > 0$ and consider the function $h \mapsto f(h) = Ch^\alpha$. Why is this function a straight line in a `loglog`-plot? What is its slope? Other popular plotting schemes are, `semilogx` and `semilogy`. Which one would you use to plot functions of the form $N \mapsto Ce^{-bN}$? How would you proceed if you suspect that a function $h \mapsto f(h)$ has the form $f(h) = Ce^{-b/h}$? What if you suspect $f(h) = Ce^{-b/h^2}$?

3.2. For $n \in \{10, 20, 40\}$ consider interpolation on the interval $[-5, 5]$ in $n+1$ points. Compare the uniformly distributed points $x_i^{unif} := 5(-1 + 2\frac{i}{n})$, $i = 0, \ldots, n$ and the Chebyshev points $x_i^{Cheb} = 5\cos(\frac{i+0.5}{n+1}\pi)$, $i = 0, \ldots, n$.

a) For the function $f(x) = (1 + x^2)^{-1}$, plot the two interpolating polynomials on $[-5, 5]$.

b) Investigate numerically the Lebesgue constant

$$\Lambda_n := \max_{x \in [-5,5]} \sum_{i=0}^{n} |\ell_i(x)|,$$

for both the uniform interpolation point distribution and the Chebyshev points. To that end, plot Λ_n versus n in semilogarithmic scale (`semilogy`).

c) For the uniform point distribution, one can in fact show that $\Lambda_n \approx Ce^{bn}$ for some C, $b > 0$. Determine C and b from your data as follows by taking the logarithm $\log \Lambda_n \approx \log C + bn$ and fit your data for the values for $n \in \{20, 40\}$. *Hint:* you can let `polyfit` do the work for you to compute $\log C$ and b or you solve a 2×2 system.

3.3. Write a program that realizes the composite trapezoidal rule for integration on $[a, b]$. The rule is based on a subdivision of $[a, b]$ into N subintervals of length $h = (b - a)/N$. Input are a function handle for f, N, and a, b.

Consider, for $[a, b] = [-1, 1]$ the 5 integrands

$$f_1(x) = x^2, \qquad f_2(x) = |x|, \qquad f_3(x) = \begin{cases} \frac{1}{2}e^x & x < 1/3 \\ e^x & x \geq 1/3 \end{cases}, \qquad f_4(x) = \sin(\pi x), \qquad f_5(x) = \sin(4\pi x).$$

Plot in `loglog`-scale the quadrature error versus h for $h = (b - a)2^{-i}$, $i = 1, 2, \ldots, 20$. What do you observe? Explain your observations for the functions f_1, f_2, f_3.

3.4. Explain the convergence behavior in Exercise 3.3 for the integrand f_4. You may use Euler's formula $e^{\mathbf{i}z} = \cos z + \mathbf{i} \sin z$ in the form $\sin z = \frac{1}{2\mathbf{i}}(e^{\mathbf{i}z} - e^{-\mathbf{i}z})$ with the imaginary unit $\mathbf{i}$ and the geometric series

$$\sum_{j=0}^{N-1} q^j = \frac{q^N - 1}{q - 1}, \qquad q \neq 1.$$

*H*int: what is is the exact value of the integral?

3.5. Let $\mathbf{A} \in \mathbb{R}^{N \times N}$ be given by

$$\mathbf{A} = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & & & & -1 \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ -1 & & & & -1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \qquad \text{with } h = 1/N.$$

The matrix $\mathbf{A}$ is a circulant matrix (compare Ex. 1.64 in the lecture notes). Read Example 1.63 in the lecture notes on how linear systems with circulant matrices can be solved using FFT. For $\mathbf{b} = (1, \dots, 1)^T$, write a code that implements this solution procedure for the linear system $\mathbf{Ax} = \mathbf{b}$.

Use different values of $N = 2^j$ and plot the computational times of computing the corresponding solution vectors $\mathbf{x} \in \mathbb{R}^N$ versus N.