Problem Sheet 8 (CSE/Inf/Geo)

discussion: Tuesday, 12.12.2023

8.1. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0.01 & 0 & 0.01 \\ 0 & 0.01 & 0.01 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0.02 \end{pmatrix}$$

Then, the solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x} = (-1, 1, 1)^T$.

a) Compute the QR-factorization using the Gram-Schmidt process (applied to the columns of $\mathbf{A}$) by hand with 3 digits of accuracy. Then, solve the linear system by solving $\mathbf{R}\mathbf{x} = \mathbf{Q}^T\mathbf{b}$. Is the computed solution accurate?

b) Compute the QR-factorization using either Householder transformations or Givens rotations (your choice) and then solve $\mathbf{R}\mathbf{x} = \mathbf{Q}^T\mathbf{b}$. What do you observe?

8.2. The function $f(x) = \sin x$ is to be approximated by a polynomial of the form $\pi(x) = a_1 x + a_3 x^3$. To this end, the coefficients a_1, a_3 are determined using the least squares method by minimizing $\sum_{j=0}^{m}(\pi(x_j) - f(x_j))^2$, where $x_0, \ldots, x_m$ are given points.

Set up the least squares problem for a_1 and a_3. Write a program that computes the coefficients a_1, a_3 for the following 9 choices of knots x_j: the x_j are N randomly chosen points in the interval $[-1/N, 1/N]$ for $N = 2^n$, $n = 2, \ldots, 10$. Do the values a_1, a_3 converge to a limit as $N \to \infty$? Which limit do you expect?

8.3. In 1801 italian astronomer Giuseppe Piazzi made 22 observations of the dwarf planet Ceres, which then is obscured by the sun. The question when and where Ceres does reappear, was very accurately answered by Carl Friedrich Gauß, who used the method of least squares to do so, as the orbit of Ceres is essentially determined by 6 parameters.

Simplified, planets move on ellipsoidal orbits around the sun, i.e., on curves described by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

where $a, b \in \mathbb{R}$ are to be computed. Let measurements (x_i, y_i) of the orbit be given as

$$(0.2, 1, 8), (-0.7, 1.7), (-0.3, -1.8), (1.2, -1.6), (2.1, 1.1), (-2.4, 0.5)$$

Set up the overdetermined linear system for the unknowns $\alpha = 1/a^2, \beta = 1/b^2$ and compute the least squares solution for this system. Draw the corresponding ellipsoidal orbit.

8.4. Program Newton's method in 1D. To that end, realize a `Matlab/python` function `newton(x,f,df)` that realizes one step of the method. f and df are *function handles* for the function f and its derivative f'. Plot (use `semilogy`) the error versus the number of Newton steps for the following 3 functions:

$$f_1(x) = x^2, \qquad f_2(x) = e^x - 2, \qquad f_3(x) = |x|^{3/2}.$$

Use $x_0 = 0.5$ as the starting value. What do you observe? Which assumptions that underlie the proof of quadratic convergence are not satisfied? Consider Newton's method for

$$f_4(x) = \frac{1}{x} - 1$$

and initial value $x_0 = 2.1$. What do you observe?

8.5. Take $f(x) = x^2 + 3x - 4$. We want to find approximations to the zero $x^* = 1$ of f.

a) Write the equation $f(x) = 0$ in fixed point form $x = x^2 + 4x - 4 =: \Phi(x)$. Is the function Φ a contraction near $x^* = 1$? Do you expect the fixed point iteration

$$x_{n+1} = \Phi(x_n)$$

with x_0 close to x^* converge to x^*?

b) Write down Newton's method for the problem and use $x_0 = 0$ as starting value. Compute the iterates x_1, x_2, x_3 and graphically draw the iteration process.