3. Potenza n-ma di un numero complesso

Formula di De Moivre

Dato il numero complesso z in forma trigonometrica

$$z = \rho(cos\vartheta + isin\vartheta)$$

la sua potenza a esponente intero positivo n è data dalla formula

$$z^n = \rho^n\big(cos(n\vartheta) + isin(n\vartheta)\big)$$

Se z è dato in forma esponenziale

$$z = \rho e^{i\vartheta}$$

la sua potenza a esponente intero positivo n è

$$z^n = \rho^n e^{in\vartheta}$$

Esempio

Dato $z = 2\left(cos\left(\frac{5}{24}\pi\right) + isin\left(\frac{5}{24}\pi\right)\right)$ *calcolare* z^6 *e fornire il risultato in forma trigonometrica, esponenziale, algebrica.*

Applicando la formula , si ha :

$$z^6 = 2^6\left(cos\left(6\cdot\frac{5}{24}\pi\right) + isin\left(6\cdot\frac{5}{24}\pi\right)\right) = 64\left(cos\left(\frac{5}{4}\pi\right) + isin\left(\frac{5}{4}\pi\right)\right) =$$

$$= 64\, e^{\frac{5}{4}\pi i} = -32\sqrt{2}(1+i)\ .$$

Esercizi

(gli esercizi con asterisco sono avviati)

Risolvere i seguenti esercizi fornendo il risultato in forma trigonometrica, esponenziale, algebrica

***1.** $(1+i)^6$

***2.** $(1-i)^9$

***3.** $\frac{1}{(1-\sqrt{3}i)^4}$

4. $\left[2\sqrt{2}\left(cos\left(\frac{\pi}{12}\right) + isin\left(\frac{\pi}{12}\right)\right)\right]^3$

***5.** *Se* $z = \sqrt{3} - \frac{1}{i}$ *calcolare*:

a. $z \cdot |z|$

b. z^3

***6.** *Se* $z = \sqrt{3} - \frac{1}{i}$ *calcolare* $\left(\frac{1}{z-2i}\right)^6$

***7.** *Se* $z = \frac{1-i}{1+i}$ *calcolare*:

a. z^{11}

b. $z^3 - \bar{z}^3$

c. $\left(\sqrt{3} - z\right)^5$

Se $z_1 = \frac{i\sqrt{3}}{1+i}$ *e* $z_2 = -i(2+2i)$, *calcolare modulo e argomento dei seguenti numeri complessi:*

***8.** $\frac{z_1}{z_2}$ ***9.** ${z_1}^2 \cdot z_2$ **10.** ${z_2}^2 \cdot z_1$

11. *Se* $z = \sqrt{3} + i$ *calcolare argomento e modulo di :*

$$\frac{z^4 \cdot \bar{z}^5}{64}$$

Se $z_1 = (1+i)^2 - (1-i)^2$ *e* $z_2 = i^{24} - i^5$*calcolare modulo e argomento dei seguenti numeri complessi:*

12. $z_1 \cdot z_2$ **13.** $z_1 \cdot \bar{z_2}$ **14.** $\frac{\bar{z_1}}{z_2}$ **15.** $\frac{{z_1}^4}{{z_2}^3}$

Calcolare le seguenti potenze esprimendole in forma trigonometrica, esponenziale e algebrica:

16. $\left[2\left(cos\frac{\pi}{9} + isin\frac{\pi}{9}\right)\right]^3$

17. $\left[-3\left(cos\frac{\pi}{12} + isin\frac{\pi}{12}\right)\right]^8$

***18.** $\left(\sqrt{3} - i\right)^4$

19. $\left[2\left(cos\left(\frac{\pi}{4}\right) + isin\left(\frac{\pi}{4}\right)\right)\right]^4$

***20.** $(1-i)^6$

***21.** Scrivere in forma trigonometrica, esponenziale, algebrica il numero $z = \frac{1-i\sqrt{3}}{1-i}$.

***22.** Calcolare $z = \frac{(2+2i)^3}{(1+\sqrt{3}i)}(-1-i)^5$ e esprimere il risultato in forma trigonometrica, esponenziale e algebrica.

***23.** Calcolare $z = \left[1+\sqrt{3}i+\frac{2}{1-\sqrt{3}i}\right]^4$ e esprimere il risultato in forma trigonometrica, esponenziale e algebrica.

Soluzioni

***1. S.** $8\left(cos\left(\frac{3}{2}\pi\right)+isin\left(\frac{3}{2}\pi\right)\right) = 8\,e^{\frac{3}{2}\pi i} = -8i$; $\left((\sqrt{2})^6\left(cos\left(6\frac{\pi}{4}\right)+isin\left(6\frac{\pi}{4}\right)\right) = \cdots\right)$;

***2. S.** $16\sqrt{2}\left(cos\left(-\frac{\pi}{4}\right)+isin\left(-\frac{\pi}{4}\right)\right) = 16\sqrt{2}\,e^{-\frac{\pi}{4}} = 16(1-i)$;

$\left((\sqrt{2})^9\left(cos\left(-9\frac{\pi}{4}\right)+isin\left(-9\frac{\pi}{4}\right)\right) = 16\sqrt{2}\left(cos\left(-\frac{\pi}{4}\right)+isin\left(-\frac{\pi}{4}\right)\right)\right)$;

***3. S.** $\frac{1}{16}\left(cos\frac{4\pi}{3}+isin\frac{4\pi}{3}\right) = \frac{1}{16}\,e^{\frac{4\pi}{3}i} = -\frac{1}{32}(1+\sqrt{3}i)$;

$\left(\frac{1}{1-\sqrt{3}i} = \frac{1+\sqrt{3}i}{(1-\sqrt{3}i)(1+\sqrt{3}i)} = \frac{1+\sqrt{3}i}{4} = \frac{1}{2}\left(cos\frac{\pi}{3}+isin\frac{\pi}{3}\right), \ldots\right)$;

4. $16\sqrt{2}\left(cos\frac{\pi}{4}+isin\frac{\pi}{4}\right) = 16\sqrt{2}\,e^{\frac{\pi}{4}i} = 16(1+i)$;

***5. S. a.** $4\left(cos\left(\frac{\pi}{6}\right)+isin\left(\frac{\pi}{6}\right)\right) = 4\,e^{\frac{\pi}{6}i} = 2\sqrt{3}+2i$; **b.** $8\left(cos\left(\frac{\pi}{2}\right)+isin\left(\frac{\pi}{2}\right)\right) = 8\,e^{\frac{\pi}{2}i} = 8i$;

$\left(z = \sqrt{3}-\frac{1}{i} = \sqrt{3}+i = 2\left(cos\frac{\pi}{6}+isin\frac{\pi}{6}\right),\ |z| = 2, \ldots\right)$;

***6. S.** $\frac{\mathbf{1}}{64}\left(cos(\pi)+isin(\pi)\right) = \frac{1}{64}e^{\pi i} = -\frac{1}{64}$;

(dall'es. 5 si ha : $z-2i = \sqrt{3}-i$, $\frac{1}{\sqrt{3}-i} = \frac{\sqrt{3}+i}{4} = \frac{1}{2}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$ ...) ;

***7. S. a.** $cos\frac{\pi}{2}+isin\frac{\pi}{2}= e^{\frac{\pi}{2}i}=i$; **b.** $2\left(cos\frac{\pi}{2}+isin\frac{\pi}{2}\right)=2e^{\frac{\pi}{2}i}=2i$;

c. $32\left(cos\left(\frac{5}{6}\pi\right)+isin\left(\frac{5}{6}\pi\right)\right)=32\,e^{\frac{5}{6}\pi i}=16\left(-\sqrt{3}+i\right)$;

($z=\frac{1-i}{1+i}=\frac{(1-i)^2}{(1+i)(1-i)}=-i=cos\left(-\frac{\pi}{2}\right)+isin\left(-\frac{\pi}{2}\right)=e^{-\frac{\pi}{2}i}$;

a. $z^{11}=cos\left(11\left(-\frac{\pi}{2}\right)\right)+isin\left(11\left(-\frac{\pi}{2}\right)\right)=cos\frac{\pi}{2}+isin\frac{\pi}{2}$ …; b. $\bar{z}=\bar{\imath}=-i$ …;

c. $\sqrt{3}-z=\sqrt{3}+i=2\left(cos\frac{\pi}{6}+isin\frac{\pi}{6}\right)$=$2e^{\frac{\pi}{6}i}$ …);

***8. S.** $\rho=\frac{\sqrt{3}}{4}$; $\vartheta=\frac{\pi}{2}$; ($z_1=\frac{i\sqrt{3}}{1+i}=\frac{i\sqrt{3}(1-i)}{(1+i)(1-i)}=\frac{\sqrt{3}}{2}(1+i)=\frac{\sqrt{6}}{2}\left(cos\frac{\pi}{4}+isin\frac{\pi}{4}\right)$,

$z_2=-i(2+2i)=2(1-i)=2\sqrt{2}\left(cos\left(-\frac{\pi}{4}\right)+isin\left(-\frac{\pi}{4}\right)\right)$ …);

***9. S.** $\rho=3\sqrt{2}$; $\vartheta=\frac{\pi}{4}$; (vedi es. 8: $z_1^2=\frac{3}{2}\left(cos\left(2\cdot\frac{\pi}{4}\right)+isin\left(2\cdot\frac{\pi}{4}\right)\right)=\ldots$);

10. S. $\rho=4\sqrt{6}$; $\vartheta=-\frac{\pi}{4}$; **11. S.** $\rho=\mathbf{8}$; $\vartheta=-\frac{\pi}{6}$;

12. S. ρ=$4\sqrt{2}$; $\vartheta=\frac{\pi}{4}$; **13. S**. ρ=$4\sqrt{2}$; $\vartheta=\frac{3\pi}{4}$; **14. S**. ρ=$2\sqrt{2}$; $\vartheta=-\frac{\pi}{4}$; **15. S**. ρ=$64\sqrt{2}$; $\vartheta=\frac{3\pi}{4}$;

16. S. $8\left(cos\frac{\pi}{3}+isin\frac{\pi}{3}\right)=8e^{i\frac{\pi}{3}}=4\left(1+\sqrt{3}i\right)$;

17. S. $6561\left(cos\frac{2}{3}\pi+isin\frac{2}{3}\pi\right)=6561e^{i\frac{2}{3}\pi}=6561\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$;

***18. S.** $16\left(cos\left(-\frac{2}{3}\pi\right)+isin\left(-\frac{2}{3}\pi\right)\right)=16e^{-\frac{2}{3}\pi}=8\left(-1-i\sqrt{3}\right)$;

($\left[2\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)\right]^4=\left[2\left(cos\left(-\frac{\pi}{6}\right)+isin\left(-\frac{\pi}{6}\right)\right)\right]^4=\cdots$);

19. S. $16(cos\pi+isin\pi)=16\,e^{\pi i}=-16$;

***20. S.** $8\left(cos\frac{\pi}{2}+isin\frac{\pi}{2}\right)=8e^{\frac{\pi}{2}i}=8i$;

($(1-i)^6=\left[\sqrt{2}\left(cos\left(-\frac{\pi}{4}\right)+isin\left(-\frac{\pi}{4}\right)\right)\right]^6$ =$8\left(cos\left(-\frac{3}{2}\pi\right)+isin\left(-\frac{3}{2}\pi\right)\right)=$

$=8\left(cos\frac{\pi}{2}+isin\frac{\pi}{2}\right)=\cdots$) ;

***21. S.** $\sqrt{2}\left(cos\frac{-\pi}{12}+isin\frac{-\pi}{12}\right)=\sqrt{2}e^{-\frac{1}{12}\pi i}$; ($z=\frac{1-i\sqrt{3}}{1-i}=\frac{(1-i\sqrt{3})(1+i)}{(1-i)(1+i)}=\frac{1+\sqrt{3}}{2}+i\,\frac{1-\sqrt{3}}{2}$;

$z=\frac{2\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)}{\sqrt{2}\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)}=\sqrt{2}\,\frac{cos\frac{5}{3}\pi+isin\frac{5}{3}\pi}{cos\frac{7}{4}\pi+isin\frac{7}{4}\pi}=\sqrt{2}\,\frac{e^{\frac{5}{3}\pi i}}{e^{\frac{7}{4}\pi i}}=\sqrt{2}e^{-\frac{1}{12}\pi i}$);

***22. S.** $-64e^{\frac{5}{3}\pi i} = -64\left(cos\frac{5}{3}\pi + isin\frac{5}{3}\pi\right) = -32 + 32\sqrt{3}i;$

($z = \frac{(2+2i)^3}{(1+\sqrt{3}i)}(-1-i)^5 = \frac{-8(1+i)^8}{(1+\sqrt{3}i)}$

esprimiamo i numeri in forma esponenziale

$1 + i = \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \sqrt{2}\left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right) = \sqrt{2}e^{\frac{\pi}{4}i}$

$1 + \sqrt{3}i = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2\left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right) = 2e^{\frac{\pi}{3}i}$

e calcoliamo z applicando le proprietà delle potenze …);

***23. S.** $81\left(cos\frac{4}{3}\pi + isin\frac{4}{3}\pi\right) = 81e^{\frac{4}{3}\pi i} = -\frac{81}{2}\left(1 + \sqrt{3}i\right);$

$(1 + \sqrt{3}i + \frac{2}{1-\sqrt{3}i} = 1 + \sqrt{3}i + \frac{2(1+\sqrt{3}i)}{(1-\sqrt{3}i)(1+\sqrt{3}i)} = 1 + \sqrt{3}i + \frac{1}{2} + \frac{\sqrt{3}}{2}i = 3\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) =$

$= 3\left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right)$ …) ;