5. Equazioni

(gli esercizi con asterisco sono avviati)

Risolvere le seguenti equazioni ed esprimere le soluzioni in forma algebrica:

1. $z^2 - 5z + 8 = 0$

2. $z + \frac{1}{z} = 4$

3. $2iz + z = i(z - i)$

4. $\frac{iz}{1-i} = z + 1$

5. $\frac{z-1}{z+1} = 2i$

6. $z^3 - 8 = 0$

7. $z^3 + 22z + 52 = 0$

8. $z^4 - 4z^2 - 5 = 0$

9. $z^2 - (2i - 1)z - i - 1 = 0$

10. $z^3 + 2z - i = 0$

11. $(z - 1)^2 = (1 + i)^2$

***12.** $(z + 1)^4 = -81$

13. $(z + 1)^3 = 8i$

***14.** $(z - i)^2 + i - 1 = (z - i)(1 + 2i)$

***15.** $3z = i - |z|$

***16.** $z^4 = 64\left(1 + i\sqrt{3}\right)^2$

17. $iz + \frac{i}{z} - 2 = 0$

18. $i + \bar{z} = \bar{\imath}(1 + i)$

***19.** $z \cdot Re(z) = i + 2\bar{z}$

20. $z - 2\bar{z} + i(z + Re(z)) + 3 - 2iIm(z) = 0$

21. $3Re(z) - z + \bar{z} - iIm(z) + 3 - 6i = 0$

Risolvere le seguenti equazioni ed esprimere le soluzioni in forma esponenziale o algebrica:

22. $4z^2 + 1 = 0$

23. $z^5 = -32$

24. $z^3 = 3 - 3i$

25. $(z + i)\left(z - i\sqrt{3} - \sqrt{3}\right)(z + 1 - i) = 0$

26. $z^2 - 3iz + 4 = 0$

27. $z^4 - iz^2 + 2 = 0$

28. $(z^5 + 2i)(z^2 - i) = 0$

29. $(z^2 - 3)(z^2 + 1 + i) = 0$

***30.** $z^3 - (i + 1)z^2 + z - i - 1 = 0$

31. $z^3 - \left(i + \sqrt{3}\right)z^2 + 4z - 4\left(i + \sqrt{3}\right) = 0$

***32.** $z^2 = \bar{z}$

***33.** $z^3 = |z|$

***34.** $|z|^2 = \frac{1}{z^4}$

***35.** $|z| = 4iz^2$

***36..** $\frac{1}{\bar{z}} = -16i \cdot z^3$

***37.** $[iRe(z)]^3 - 2z^2 - 4Im(z) = 0$

***38.** $i|z|^2 + Re(z) = \bar{z} - 2z + 3e^{\pi i}$

Soluzioni

1.S. $\frac{1}{2}\left(5 \pm \sqrt{7}i\right)$; **2. S.** $2 \pm \sqrt{3}$; **3. S.** $\frac{1}{2} - \frac{1}{2}i$

4. S. $z = -\frac{3}{5} - \frac{1}{5}i$; **5.S.** $-\frac{3}{5} + \frac{4}{5}i$; **6.S.** 2 ; $-1 \pm \sqrt{3}i$;

7.S. -2; $1 \pm 5i$; **8. S.** $\pm\sqrt{5}$; $\pm i$; **9. S.** $-1 + i$; i ;

10. S. i ; $\left(\frac{-1\pm\sqrt{5}}{2}\right)i$; **11.S.** $2 + i$; $-i$;

***12. S.** $\frac{3\sqrt{2}}{2} - 1 + \frac{3\sqrt{2}}{2}i$; $-\frac{3\sqrt{2}}{2} - 1 + \frac{3\sqrt{2}}{2}i$; $-\frac{3\sqrt{2}}{2} - 1 - \frac{3\sqrt{2}}{2}i$; $\frac{3\sqrt{2}}{2} - 1 - \frac{3\sqrt{2}}{2}i$;

(posto $z + 1 = t$, si ha $t^4 = -81$ da cui $t = 3\left(cos\frac{\pi+2k\pi}{4} + isin\frac{\pi+2k\pi}{4}\right)$, $k = 0,1,2,3$.

Ne segue $z = -1 + 3\left(\cos(\frac{\pi}{4} + k\frac{\pi}{2}) + +isin(\frac{\pi}{4} + k\frac{\pi}{2})\right)$, $k = 0,1,2,3$.)

13. S. $-1 + \sqrt{3} + i; -1 - \sqrt{3} + i; -1 - 2i$; ***14. S**. $1 + 2i; 2i$ (porre $z - i = t$);

***15. S.** $-\frac{\sqrt{2}}{12} + \frac{1}{3}i$; (Si ha $3\rho(cos\vartheta + isin\vartheta) = i - \rho$, perciò deve essere

$\begin{cases} 3\rho cos\vartheta = -\rho \\ 3\rho sin\vartheta = 1 \end{cases} \Rightarrow cos\vartheta = -\frac{1}{3}, sin\vartheta = \frac{2\sqrt{2}}{3}, \rho = \frac{1}{2\sqrt{2}} \Rightarrow z = \frac{1}{2\sqrt{2}}\left(-\frac{1}{3} + \frac{2\sqrt{2}}{3}i\right)$);

***16. S.** $2\left(\sqrt{3} + i\right)$, $2\left(-1 + i\sqrt{3}\right)$, $2\left(-\sqrt{3} - i\right)$, $2\left(1 - i\sqrt{3}\right)$;

($z^4 = 64\left(-2 + 2\sqrt{3}i\right); z = \sqrt[4]{64\left(-2 + 2\sqrt{3}i\right)} = 4\left(cos\frac{\frac{2\pi}{3}+2k\pi}{4} + isin\frac{\frac{2\pi}{3}+2k\pi}{4}\right), k = 0,1,2,3$)

17. S. $-i\left(\sqrt{2} + 1\right)$, $i\left(\sqrt{2} - 1\right)$; **18. S.** $1 + 2i$;

***19.S.** $\frac{1}{2}i$; $2 + \frac{1}{4}i$; (sia $z = x + iy$, si ha $(x + iy)x = i + 2(x - iy)$, da cui $\begin{cases} x^2 - 2x = 0 \\ xy + 2y - 1 = 0 \end{cases}$

$\Rightarrow \begin{cases} x = 0\,, y = \frac{1}{2} \\ x = 2, y = \frac{1}{4} \end{cases}$);

20 S. $-3 + 6i$; **21.S.** $-1 - 2i$;

22. S. $\frac{1}{2}e^{\pm\frac{\pi}{2}i}$; **23.S.** $2e^{i\left(-\frac{\pi}{5}+k\frac{2\pi}{5}\right)}$ $k = 0,1,2,3,4$;

24.S. $\sqrt[6]{18}e^{i\left(-\frac{\pi}{12}+k\frac{2\pi}{3}\right)}$ $k = 0,1,2$; **25.S.** $e^{-\frac{\pi}{2}i}; \sqrt{6}e^{\frac{\pi}{4}i}; \sqrt{2}e^{\frac{3\pi}{4}i}$;

26. S. $e^{-\frac{\pi}{2}i}$; $4e^{\frac{\pi}{2}i}$; **27. S.** $e^{-\frac{\pi i}{4}}$; $e^{3\frac{\pi}{4}i}$; $\sqrt{2}e^{\frac{\pi}{4}i}$; $\sqrt{2}e^{\frac{5\pi}{4}i}$;

28.S. $\sqrt[5]{2}e^{i\left(-\frac{\pi}{10}+k\frac{2\pi}{5}\right)}$ $k = 0,1,2,3,4$; $e^{\frac{\pi}{4}i}$; $e^{\frac{5\pi}{4}i}$;

29.S. $\sqrt{3}$ $e^{k\pi i}k = 0,1$; $\sqrt[4]{2}\, e^{\frac{-3\pi}{8}i}$; $\sqrt[4]{2}\, e^{\frac{5\pi}{8}i}$

***30.S.** $e^{\frac{\pi}{2}i}$; $\sqrt{2}e^{\frac{\pi}{4}i}$; $e^{-\frac{\pi}{2}i}$

(si ha $z(z^2+1)-(1+i)(z^2+1)=0 \Rightarrow (z^2+1)\big(z-(1+i)\big)=0 \Rightarrow z=\pm i, z=1+i..)$

31 S. $2e^{\frac{\pi}{2}i}$; $2e^{\frac{\pi}{6}i}$; $2e^{-\frac{\pi}{2}i}$

***32.S** 0 ; 1 ; $e^{\frac{2\pi i}{3}}$; $e^{\frac{4\pi i}{3}}$ **;**

$((x+iy)^2 = x-iy \Rightarrow x^2-y^2-x+i(y+2xy)=0 \Rightarrow \begin{cases} x^2-y^2-x=0 \\ y+2xy=0 \end{cases} \rightarrow$

$x=y=0; x=1, y=0; x=-\frac{1}{2}, y=\pm\frac{\sqrt{3}}{2}$ cioè $z=0; z=1; z=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$

In questo caso può essere vantaggioso utilizzare la forma trigonometrica (o la forma esponenziale), Infatti si ha, tenendo conto che due numeri complessi sono uguali se hanno lo stesso modulo e stessa anomalia a meno di multipli di 2π

$\rho^2e^{2\vartheta i}=\rho e^{-\vartheta i}$ da cui $\begin{cases} \rho^2=\rho \rightarrow \rho=0 \vee \rho=1 \\ 2\vartheta=-\vartheta+2k\pi \rightarrow \vartheta=0 \vee \vartheta=\frac{2\pi}{3} \vee \vartheta=\frac{4\pi}{3} \end{cases}$

$\Rightarrow z=0,\ z=1, z=e^{\frac{2\pi i}{3}}, z=e^{\frac{4\pi i}{3}}$);

***33. S** 0; 1 ; $e^{\frac{2\pi i}{3}}$; $e^{\frac{4\pi i}{3}}$

(Si ha $\rho^3e^{3\vartheta i}=\rho$. Tenendo conto che il secondo membro ha anomalia 0 , risulta

:$\begin{cases} \rho^3=\rho \rightarrow \rho=0 \vee \rho=1 \\ 3\vartheta=2k\pi \rightarrow \vartheta=0 \vee \vartheta=\frac{2\pi}{3} \vee \vartheta=\frac{4\pi}{3} \end{cases}$ $\Rightarrow z=0,\ z=1, z=e^{\frac{2\pi i}{3}},\ z=e^{\frac{4\pi i}{3}}$);

***34 S.** $e^{\frac{3\pi i}{2}}$; (per $\rho \neq 0$ si ha: $\rho^6e^{4\vartheta i}=1 \rightarrow \rho=1 \wedge 4\vartheta=0+2k\pi$

$\begin{cases} \rho=1 \\ \vartheta=0 \end{cases} \Rightarrow z=1;$ $\begin{cases} \rho=1 \\ \vartheta=\frac{\pi}{2} \end{cases} \Rightarrow z=e^{\frac{\pi i}{2}};$ $\begin{cases} \rho=1 \\ \vartheta=\pi \end{cases} \Rightarrow z=e^{\pi i};$ $\begin{cases} \rho=1 \\ \vartheta=\frac{3\pi}{2} \end{cases} \Rightarrow z=e^{\frac{3\pi i}{2}}$).

***35. S.** $\frac{1}{2}e^{\frac{\pi i}{4}}$; $\frac{1}{2}e^{\frac{5\pi i}{4}}$;

($\rho=4e^{\frac{\pi i}{2}}\cdot\rho^2e^{2\vartheta i} \Rightarrow \begin{cases} \rho=4\rho^2 \\ \frac{\pi}{2}+2\vartheta=0+2k\pi \end{cases} \Rightarrow \begin{cases} \rho=0 \vee \rho=\frac{1}{4} \\ \vartheta=-\frac{\pi}{4}+k\pi \end{cases} \Rightarrow z=0;\quad z=\frac{1}{4}e^{-\frac{\pi i}{4}}\ ; z=\frac{1}{4}e^{\frac{3\pi i}{4}}$) ;

***36. S.** $\frac{1}{2}e^{\frac{\pi i}{4}}$; $\frac{1}{2}e^{\frac{5\pi i}{4}}$; $(\frac{1}{\rho e^{-\vartheta i}}=16e^{-\frac{\pi i}{2}}\cdot\rho^3e^{3\vartheta i} \Rightarrow \rho^4e^{\left(2\vartheta-\frac{\pi}{2}\right)i}=\frac{1}{16} \Rightarrow \begin{cases} \rho^4=\frac{1}{16} \\ 2\vartheta-\frac{\pi}{2}=0+2k\pi \end{cases} \Rightarrow$

$\begin{cases} \rho=\frac{1}{2} \\ \vartheta=\frac{\pi}{4}+k\pi \end{cases} \Rightarrow z=\frac{1}{2}e^{\frac{\pi i}{4}}\ ; z=\frac{1}{2}e^{\frac{5\pi i}{4}}$);

***37.S.** 0, $2e^{\frac{\pi i}{2}}$, $\pm 2\sqrt{2}-2i$

(se $z=x+iy \Rightarrow -ix^3-2(x+iy)^2-4y=0 \Rightarrow i(-x^3-4xy)-2x^2+2y^2-4y=0 \Rightarrow$

$\Rightarrow \begin{cases} -x^3 - 4xy = 0 \\ -2x^2 + 2y^2 - 4y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x = 0 \\ y = 2 \end{cases} \vee \begin{cases} x = \pm 2\sqrt{2} \\ y = -2 \end{cases}$);

***38. S.** $\frac{3\sqrt{2}}{2} e^{\frac{5\pi i}{4}}$ (se $z = x + iy \Rightarrow i(x^2 + y^2) + x = x - iy - 2(x + iy) - 3 \Rightarrow$
$\begin{cases} x^2 + y^2 + 3y = 0 \\ -2x - 3 = 0 \end{cases} \Rightarrow \begin{cases} x = -\frac{3}{2} \\ y = -\frac{3}{2} \end{cases}$).