3.Limiti notevoli

Tenendo conto che:

$\lim\limits_{x\to 0} \frac{sinx}{x} = 1$ si ha:

$\lim\limits_{x\to 0} \frac{1-cosx}{x^2} = \frac{1}{2}$ poichè

$$\lim_{x\to 0} \frac{(1-cosx)(1+cosx)}{x^2(1+cosx)} = \lim_{x\to 0} \frac{sin^2x}{x^2(1+cosx)} = \lim_{x\to 0} \frac{sin^2x}{x^2} \cdot \frac{1}{1+cosx} = \frac{1}{2}$$

$\lim\limits_{x\to 0} \frac{1-cosx}{x} = 0$ poichè $\lim\limits_{x\to 0} \frac{(1-cosx)x}{x^2} = \frac{1}{2} \cdot 0 = 0$

$\lim\limits_{x\to 0} \frac{tgx}{x} = 1$ poichè $\lim\limits_{x\to 0} \frac{sinx}{x \cdot cosx} = 1$

$\lim\limits_{x\to 0} \frac{arcsinx}{x} = 1$ in quanto posto $y = arcsinx\ si\ ha\ x = siny$ e

$$\lim_{x\to 0} \frac{arcsinx}{x} = \lim_{y\to 0} \frac{y}{siny} = 1$$

$\lim\limits_{x\to 0} \frac{arctgx}{x} = 1$ in quanto posto $y = arctgx$ si ha $x = tgy$, quindi

$$\lim_{x\to 0} \frac{arctgx}{x} = \lim_{y\to 0} \frac{y}{tgy} = 1$$

$\lim\limits_{x\to 0} \frac{sin(mx)}{nx} = \lim\limits_{x\to 0} \frac{sin(mx)}{mx} \cdot \frac{m}{n} = \frac{m}{n}$

Esercizi

(gli esercizi con asterisco sono avviati)

Calcolare:

1) $\lim\limits_{x\to 0} \frac{sin5x}{x}$

2) $\lim\limits_{x\to 0} \frac{sin(3x)}{4x}$

3) $\lim\limits_{x\to 0} \frac{tg(3x)}{x}$

*4) $\lim\limits_{x\to 0} \frac{1-cos(2x)}{x^2} = \lim\limits_{x\to 0} \frac{2sin^2x}{x^2} =$..

5) $\lim\limits_{x\to 0} \frac{\sqrt{cosx}-1}{x}$

6) $\lim\limits_{x\to 0} \frac{1-cos(5x)}{x}$

*7) $\lim\limits_{x\to \pi} \frac{sin5x}{x-\pi}$

*8) $\lim\limits_{x\to 0} \frac{sinx^2}{sin^2x}$

9) $\lim\limits_{x\to 1} \frac{sin(\pi x-\pi)}{2(x-1)}$

*10) $\lim\limits_{x\to 0} \frac{sin^2(2x)}{1-cosx}$;

$* \ 11) \lim_{x\to 0} \frac{x\cdot sin(2x)}{1-\cos(5x)}$

12)$\lim_{x\to 0}(1-cosx)\left(\frac{1}{x}-\frac{1}{x^2}\right)$

*13) $\lim_{x\to \frac{\pi}{6}} \frac{2sinx-1}{6x-\pi}$

*14)$\lim_{x\to 0} \frac{sin2x+3x}{2x-sin3x}$

15) $\lim_{x\to 0} \frac{sin(2x)-3x}{\sin(3x)-7x}$

*16)$\lim_{x\to \frac{\pi}{4}} \frac{sinx-cosx}{x-\frac{\pi}{4}}$

17) $\lim_{x\to 3} \frac{x-3}{\arcsin(x-3)}$

18) $\lim_{x\to 0} \frac{arcsin(4x)}{x}$

19) $\lim_{x\to 2} \frac{3arcsin(x-2)}{x-2}$

20) $\lim_{x\to 0} \frac{arcsin(6x)}{3x}$

21)$\lim_{x\to 0} \frac{1-cos^2x+sin^2(4x)}{x^2}$

22) $\lim_{x\to 0^+}(log(sinx)-logx)$

$* \ 23) \lim_{x\to \pi} \frac{cos^2(\frac{x}{2})}{x-\pi} =$

*24)$\lim_{x\to 0} \frac{(2sin\pi x-sin4\pi x)x}{1-\cos(2\pi x)}$

Ricordando che

$\lim_{x\to\infty}\left(\mathbf{1}+\frac{\mathbf{1}}{x}\right)^{x} = e$ si ha

$\lim_{x\to 0}(\mathbf{1}+x)^{\frac{1}{x}}$ **=e**

in quanto, posto $\frac{1}{x} = t$, risulta $\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{t} = e$. Inoltre

$\lim_{x\to 0}\frac{\log_a(1+x)}{x} = \log_a e$ poiché $\lim_{x\to 0}\log_a(1+x)^{\frac{1}{x}}$ $=\log_a e$.

In particolare, se $a = e$,

$\lim_{x\to 0}\frac{\log(1+x)}{x} = \mathbf{1}$ **.**

Ricordiamo anche che:

$\lim_{x\to 0}\frac{a^x-1}{x} = \log a$

in quanto, posto $a^x - 1 = t$, risulta $x = \log_a(1+t)$, e si ha:

$\lim_{t\to 0}\frac{t}{\log_a(1+t)} = \log a.$

In particolare, se $a = e$

$\lim_{x\to 0}\frac{e^x-1}{x} = \mathbf{1}$

Esempi

1) $\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^{x} = \lim_{x\to\infty}\left[\left(1+\frac{1}{\frac{x}{4}}\right)^{\frac{x}{4}}\right]^{4} = e^{4}$;

2) $\lim_{x\to\infty}\left(\frac{3x+7}{3x+5}\right)^{x} = \lim_{x\to\infty}\left(1+\frac{2}{3x+5}\right)^{x} =$ (posto $\frac{3x+5}{2} = t \Rightarrow x = \frac{2t-5}{3}$)

$= \lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{\frac{2t-5}{3}} = \lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{\frac{2t}{3}} \cdot \lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{\frac{-5}{3}} = \lim_{t\to\infty}\left[\left(1+\frac{1}{t}\right)^{t}\right]^{\frac{2}{3}} = \sqrt[3]{e^{2}}$;

3) $\lim_{x\to\infty}\left(\frac{3+x^2}{x^2}\right)^{4x^2+1} = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x^2}{3}}\right)^{12\cdot\frac{x^2}{3}} \cdot \lim_{x\to\infty}\left(1+\frac{1}{\frac{x^2}{3}}\right) = e^{12}$.

Esercizi

Calcolare:

25) $\lim_{x\to\infty}\left(1-\frac{1}{2x}\right)^{x}$

26) $\lim_{x\to\infty}\left(1+\frac{1}{x-1}\right)^{2x}$

27) $\lim_{x\to\infty}\left(1-\frac{\sqrt{2}}{x}\right)^{x+3}$

* 28) $\lim_{x\to+\infty}\left(1+\frac{1}{3x+2}\right)^{-x}$

29) $\lim_{x\to\infty}\left(\frac{3+x}{x+2}\right)^{x+1}$

30) $\lim_{x\to\infty}\left(\frac{x-4}{x+1}\right)^{x-2}$

31) $\lim_{x\to\infty}\left(\frac{x+2}{x-3}\right)^{3x}$

32) $\lim_{x\to\infty}\left(\frac{4+x^2}{x^2-4}\right)^{x^2}$

33) $\lim_{x\to\infty}\left(\frac{4+x^3}{x^3}\right)^{2x^3}$

*34) $\lim_{x\to 0}(1+2x)^{\frac{x+3}{x}}$

35) $\lim_{x\to+\infty}\left(\frac{e^x+1}{e^x}\right)^{2e^x}$

*36) $\lim_{x\to 2}(3-x)^{\frac{1}{2-x}}$

$*$ 37) $\lim_{x\to 1}(x)^{\frac{1}{x-1}}$

38) $\lim_{x\to 0} log\left[(1+sinx)^{\frac{1}{sinx}}\right]$

39) $\lim_{x\to 0} log[(1+3tgx)^{ctgx}]$

*40) $\lim_{x\to 0}\frac{\log(1+sinx)}{sinx}$

41) $\lim_{x\to 0}\frac{\log(1+x^2)}{x^2}$

42) $\lim_{x\to 0}\frac{\log(1+7x)}{4x}$

*43) $\lim_{x\to 2}\frac{\log(x-1)}{x-2}$

*44) $\lim_{x\to 0}\frac{\log(1+x)}{sinx}$

45) $\lim_{x\to 0}\frac{e^{3x}-1}{x}$

46) $\lim_{x\to 0}\frac{e^{-2x}-1}{x}$

47) $\lim_{x\to 0}\frac{e^{\sin(2x)}-1}{x}$

48) $\lim_{x\to 0}\frac{e^{x}-e^{-x}}{3x}$

Esercizi di ricapitolazione

Calcolare i limiti delle seguenti forme indeterminate:

*1) $\lim_{x\to\pi}\frac{sin^2(2x)}{xtg(4x)}$

2) $\lim_{x\to1}\frac{x^2-1}{tg(x-1)}$

3) $\lim_{x\to0}\frac{sin(3x)+x}{tgx}$

4) $\lim_{x\to0}\frac{arcsin(2x)+sinx}{3x}$

*5) $\lim_{x\to1}\frac{cos(\pi x)+1}{1-sin\left(\frac{\pi}{2}x\right)}$

6) $\lim_{x\to0}\frac{e^{2x}-1}{tgx}$

7) $\lim_{x\to0}\frac{\log(1+3x)}{2x}$

8) $\lim_{x\to0}\frac{\log(1+x)}{\sin(2x)}$

*9) $\lim_{x\to0}\frac{\log(1+tgx)+e^{sinx}-1}{\sqrt{1+x}-1}$

10) $\lim_{x\to0}\frac{tgx-1+e^x}{x+sinx}$

11) $\lim_{x\to0}\frac{e^{x^2}-cosx}{tg^2(2x)}$

12) $\lim_{x\to0}\frac{e^{sinx}-cos\sqrt{x}}{sinx}$

*13) $\lim_{x\to0}\frac{(e^{x^2}-1)x}{sinx(1-cos2x)}$

*14) $\lim_{x\to0^+}\frac{\sqrt{e^{x^3}-1}}{\log(1-x\sqrt{x})}$

15) $\lim_{x\to0^+}\frac{1}{x^2}\cdot(e^x-1)$

16) $\lim_{x\to0^+}\frac{1}{\sqrt{x}}\cdot(e^x-1)$

*17) $\lim_{x\to-1}\frac{2}{x+1}\cdot\left(e^{x^3+1}-1\right)$

*18) $\lim_{x\to0}\frac{x+1}{x^2-x}\cdot sinx$

19) $\lim_{x\to-1}\frac{5x+2}{x^2+x}\cdot tg(x+1)$

20) $\lim_{x\to0}\frac{x+2}{x}\cdot\log(1+x^2)$

21) $\lim_{x\to0}\frac{x^2+1}{sinx}\cdot\log(x+1)$

22) $\lim_{x\to0}\frac{x+3}{x}\cdot\log(1+3x)$

Soluzioni

1. S. 5 ; **2. S.** $\frac{3}{4}$; **3.S.**3 ;

***4.S.** 2 $\left(= \lim_{x\to 0}\frac{2sin^2x}{x^2}\right.$ =..);

5.S. 0 ; **6.S.** 0;

*** 7.S.** -5 ; (posto $x-\pi=t, si\ ha \lim_{x\to\pi}\frac{sin5x}{x-\pi} = \lim_{t\to 0}\frac{\sin(5t+5\pi)}{t} = -\lim_{t\to 0}\frac{\sin(5t)}{t} = \cdots$);

***8.S.** 1; $\left(\lim_{x\to 0}\frac{sinx^2}{sin^2x} = \lim_{x\to 0}\frac{sinx^2}{x^2}\cdot\left(\frac{x}{sinx}\right)^2 = \cdots\right)$;

9.S. $\frac{\pi}{2}$;

***10.S** 8 ; $\left(\lim_{x\to 0}\frac{sin^2(2x)}{1-cosx} = \lim_{x\to 0}\frac{4sin^2x\cdot cos^2x}{1-cosx} = \lim_{x\to 0}\frac{4(1-cos^2x)\cdot cos^2x}{1-cosx} = \lim_{x\to 0}4(1+cosx)cos^2x = \cdots\right)$;

***11.S.** . $\frac{4}{25}\left(\lim_{x\to 0}\frac{x\cdot sin(2x)}{1-\cos(5x)} = \lim_{x\to 0}\frac{(5x)^2}{1-\cos(5x)}\cdot\frac{sin2x}{2x}\cdot\frac{2}{25} = \cdots\right)$;

12. S. $-\frac{1}{2}$;

$*$ **13. S.** $\frac{\sqrt{3}}{6}$;

$$\left(posto\ 6x-\pi=6y, si\ ha \lim_{x\to\frac{\pi}{6}}\frac{2sinx-1}{6x-\pi} = \lim_{y\to 0}\frac{2\sin\left(x+\frac{\pi}{6}\right)-1}{6y} = \lim_{y\to 0}\frac{\sqrt{3}}{6}\cdot\frac{siny}{y} = \cdots\right);$$

***14.S.** -5 $\left(\lim_{x\to 0}\frac{sin2x+3x}{2x-sin3x} = \lim_{x\to 0}\frac{2\frac{sin2x}{2x}+3}{2-3\frac{sin3x}{3x}}\right.$ =...) ;

15.S. $\frac{1}{4}$;

***16.S.** $\sqrt{2}$; ($sinx-cosx=\sqrt{2}sin\left(x-\frac{\pi}{4}\right)$, porre $x-\frac{\pi}{4}=t$ ...);

17.S. 1 ; **18. S.** 4; **19. S.** 3 ; **20.** 2;

21. S. 17; **22. S.** 0;

***23. S.** 0 $\left(porre\ \ x-\pi=y \ldots \lim_{y\to o}\frac{sin^2\left(\frac{y}{2}\right)}{y} = \cdots\right)$;

$*$ **24S.** $-\frac{1}{\pi}$;

$\left(posto\ \pi x=y\ , \lim_{x\to 0}\frac{(2sin\pi x-sin4\pi x)x}{1-\cos(2\pi x)} = \lim_{y\to o}\frac{(2siny-sin4y)}{1-cos2y}\cdot\frac{y}{\pi} = \frac{1}{\pi}\lim_{y\to o}\frac{2siny-\sin(4y)}{y}\cdot\frac{y^2}{1-cos2y} = \cdots\right)$;

25. S. $\frac{1}{\sqrt{e}}$; **26. S.** e^2 ; **27.S.** $e^{-\sqrt{2}}$;

***28. S.** $\frac{1}{\sqrt[3]{e}}$;

$$\left(\lim_{x\to+\infty}\left(1+\frac{1}{3x+2}\right)^{-x}=\lim_{x\to+\infty}\left[\left(1+\frac{1}{3x+2}\right)^{3x+2}\right]^{-\frac{x}{3x+2}}, ma \lim_{x\to+\infty}\left(1+\frac{1}{3x+2}\right)^{3x+2}=e\ldots\right);$$

29. S. e ; **30. S.** e^{-5}; **31.S.** e^{15}; **32. S.** e^8; **33. S.** e^8 ;

***34. S.** e^6 $\left(posto\ 2x=\frac{1}{y}\to\frac{1}{x}=2y, si\ ha\ \lim_{x\to 0}(1+2x)^{1+\frac{3}{x}}=\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{1+6y}=\cdots\right)$;

35.S. e^2 ;

***36. S.** e; $\left(\lim_{x\to 2}(3-x)^{\frac{1}{2-x}}=\lim_{x\to 2}[1+(2-x)]^{\frac{1}{2-x}}\ldots.\right)$;

***37. S.** e; $\left(\lim_{x\to 1}(x)^{\frac{1}{x-1}}=\lim_{x\to 1}[1+(x-1)]^{\frac{1}{x-1}}\ldots.\right)$;

38. S.1; **39.S.**3 ;

***40.S.** 1 ($posto\ sinx=t, si\ ha\ \lim_{t\to 0}\frac{\log(1+t)}{t}=\cdots$);

41. S. 1 ; **42.S.** $\frac{7}{4}$;

* **43. S.** 1 $\left(\lim_{x\to 2}\frac{\log(x-1)}{x-2}=\lim_{x\to 2}\frac{\log[(x-2)+1]}{x-2}=\cdots\right)$;

***44. S.** 1 $\left(\lim_{x\to 0}\frac{\log(1+x)}{sinx}=\lim_{x\to 0}\left(\frac{\log(1+x)}{x}\cdot\frac{x}{sinx}\right)=\cdots\right)$;

45. S. 3 ; **46. S.** -2 ; **47.S.** 2 ; **48. S.** $\frac{2}{3}$;

Esercizi di ricapitolazione

***1.S.** 0;(ponendo $x-\pi=y, si\ ha\ \lim_{x\to\pi}\frac{sin^2(2x)}{xtg(4x)}=\lim_{y\to 0}\frac{sin^2(2y+2\pi)}{(y+\pi)tg(4y+4\pi)}=\frac{1}{\pi}\lim_{y\to 0}\frac{sin^2(2y)}{tg(4y)}=\cdots$);

2.S. 2; **3. S.** 4 ; **4.S.** 1;

5 *S. 4 ;$\left(\lim_{x\to 1}\frac{cos(\pi x)+1}{1-sin\left(\frac{\pi}{2}x\right)}=\lim_{x\to 1}\frac{2cos^2\frac{\pi}{2}x}{1-sin\left(\frac{\pi}{2}x\right)}=2\lim_{x\to 1}\frac{1-sin^2\frac{\pi}{2}x}{1-sin\left(\frac{\pi}{2}x\right)}=\cdots\right)$;

6.S. 2; **7. S.** $\frac{3}{2}$; **8. S.** $\frac{1}{2}$;

***9.S.** 4;

$$(\lim_{x\to 0}\frac{\log(1+tgx)+e^{sinx}-1}{\sqrt{1+x}-1}=\lim_{x\to 0}\left(\sqrt{1+x}+1\right)\frac{\log(1+tgx)+e^{sinx}-1}{x}=$$

$$=2(\lim_{x\to 0}\frac{\log(1+tgx)}{x}+\lim_{x\to 0}\frac{e^{sinx}-1}{x})=2(\lim_{x\to 0}(\frac{\log(1+tgx)}{tgx}\cdot\frac{tgx}{x}+\lim_{x\to 0}\left(\frac{e^{sinx}-1}{sinx}\cdot\frac{sinx}{x}\right)=\cdots);$$

10.S. 1 ; **11. S.** $\frac{3}{8}$; **12.S.** $\frac{3}{2}$;

$*$ **13. S.** $\frac{1}{2}$ $\left(\lim_{x\to 0} \frac{(e^{x^2}-1)x}{sinx(1-cos2x)} = \lim_{x\to 0}\left(\frac{x}{sinx} \cdot \frac{e^{x^2}-1}{x^2} \cdot \frac{x^2}{1-\cos(2x)} = \cdots \right)\right.$;

***14 S.** -1; $\left(\lim_{x\to 0^+} \frac{\sqrt{e^{x^3}-1}}{\log(1-x\sqrt{x})} = \lim_{x\to 0^+} \left(\frac{x\sqrt{x}}{\log(1-x\sqrt{x})} \cdot \sqrt{\frac{e^{x^3}-1}{x^3}}\right) = \cdots \ldots \right)$;

15.S. $+\infty$; **16. S.** 0 ;

***17. S.** 6 ; $\left(\lim_{x\to -1} \frac{2}{x+1} \cdot \left(e^{x^3+1} - 1\right) = \lim_{x\to -1} \frac{2(x^2-x+1)}{x^3+1} \cdot \left(e^{x^3+1} - 1\right) = \cdots\right)$;

*** 18.S.** -1 ; $\left(\lim_{x\to 0} \frac{x+1}{x^2-x} \cdot sinx = \lim_{x\to 0} \frac{x+1}{x-1} \cdot \frac{sinx}{x} = \cdots.\right)$;

19. S. 3; **20. S.** 0; **21. S.** 1 ; **22. S.** 9;