Problem Set 8

> **Due dates:** Electronic submission of this homework is due on **Friday 11/5/2021 before 11:59pm** on canvas.

Name: [redacted]

Resources. (All people, books, articles, web pages, etc. that have been consulted when producing your answers to this homework)

On my honor, as an Aggie, I have neither given nor received any unauthorized aid on any portion of the academic work included in this assignment. Furthermore, I have disclosed all resources (people, books, web sites, etc.) that have been used to prepare this homework.

Signature: [redacted]

This homework needs to be typeset in LaTeX to receive any credit. All answers need to be formulated in your own words.

Problem 1 (20 points)**.** Suppose that the sample space Ω is given by the set of positive integers. Let $\mathcal{F}$ denote the smallest family of subsets of Ω such that (a) $\mathcal{F}$ contains all finite sets, (b) $\mathcal{F}$ is closed under complements (meaning if A is in $\mathcal{F}$, then A^c is in $\mathcal{F}$), and (c) $\mathcal{F}$ is closed under countable unions (so if the sets $E_1, E_2, \ldots$ are contained in $\mathcal{F}$, then $\bigcup_{k=1}^{\infty} E_k$ is contained in $\mathcal{F}$).
(a) Show that $\mathcal{F}$ is a σ-algebra.
(b) Prove of disprove: $\mathcal{F}$ is equal to the power set $P(\Omega)$.
[Hint: You might have to qualify your answer depending on the cardinality of the sample space.]

Solution. a) There are a set of conditions to prove that $\mathcal{F}$ is a σ-algebra. The first one is that $\mathcal{F}$ is countable, which means it contain the empty set. Another one is that $\mathcal{F}$ is closed under compliments so as given that for set A there is a set A^c. And finally that $\mathcal{F}$ is a countable collection. If $E_1 \cup E_2 ... E_n \cup \sigma \in \mathcal{F}$, then $E_1 \cap E_2 ... A_n, \mathcal{F}$ because each $E_i^c \in \mathcal{F}$ and its closed under finite unions which belongs to $\mathcal{F}$. Considering these conditions, $\mathcal{F}$ meets all of them so its a σ-algebra.

b) For this part we must Prove of disprove: $\mathcal{F}$ is equal to the power set $P(\Omega)$. We know that the power set of a set Ω is the set of all subsets of Ω, which also includes an empty set. So $E_1 \cup E_2 ... E_n = \Omega$ and $E_i \cap E_j ... E_n = \sigma$. Since $\sigma \in \mathcal{F}$. then we can conclude that $\mathcal{F}$ is equal to set of $P(\Omega)$.

Problem 2 (20 points)**.** Let $B_1, B_2, \ldots, B_t$ denote a partition of the sample space Ω.
(a) Prove that $\Pr[A] = \sum_{k=1}^{t} \Pr[A \mid B_k] \Pr[B_k]$.
(b) Deduce that $\Pr[A] \leq \max_{1 \leq k \leq t} \Pr[A \mid B_k]$.

Solution. a) For this part we must prove $\Pr[A] = \sum_{k=1}^{t} \Pr[A \mid B_k] \Pr[B_k]$. Since we are given that $B_1, B_2, \ldots, B_t$ is a partition, A = $\cup_{k=1}^{t}(A \cap B_k)$. Then P(A) = P($\cup_{k=1}^{t}(A \cap B_k)$), which is equal to $\sum_{k=1}^{t} A \cap B_k$. This then simplifies to $\sum_{k=1}^{t} P(A|B_k)P(B_k)$, hence proving the statement.

b) For this problem we must deduce that $\Pr[A] \leq \max_{1 \leq k \leq t} \Pr[A \mid B_k]$. So using, $\sum_{k=1}^{t} P(A|B_k)P(B_k)$, we can say $\sum_{k=1}^{t} P(A|B_k)P(B_k) \max_{k=1}^{t} P(A|B_k)(\sum_{k=1}^{t} P(B_k))$. This is equal to $\max_{k=1}^{t} P(A|B_k)(1)$. Further simplifying to $P(A) \leq \max_{k=1}^{t} P(A|B_k)$. This shows that $\Pr[A] \leq \max_{1 \leq k \leq t} \Pr[A \mid B_k]$.

Problem 3 (20 points)**.** Consider an experiment, where you toss two fair coins. Give examples of events where (a) $\Pr[A_1 \mid B_1] < \Pr[A_1]$, (b) $\Pr[A_2 \mid B_2] = \Pr[A_2]$, and (c) $\Pr[A_3 \mid B_3] > \Pr[A_3]$. Make sure that your proofs are complete and self-contained.

Solution. a) For this part we must give an example of $\Pr[A_1 \mid B_1] < \Pr[A_1]$. We can consider A to be heads for both of the coins and B to be tails for both of the coins. The probability of getting A given B would be 0 since there can only be one tails out of the two coins. Both coins can't be heads or both be tails at the same time, also that probability would be 1/2 * 1/2 = 1/4. So we can say that $\Pr[A_1 \mid B_1] < \Pr[A_1]$.

b) For this part we must give an example of $\Pr[A_2 \mid B_2] = \Pr[A_2]$. Here we can consider that both A and B are heads in any given event, so then we would get the probability of 1/2, since they can be considered independent events. Because they can also be considered independently, A would also be 1/2 and B would be 1/2. Hence, we can say $\Pr[A_2 \mid B_2] = \Pr[A_2]$.

c) Now lets consider a situation where in A both coins are heads and in B only one coin is heads. The probability of A given B is 1/2 since one coin is heads and the other has 0.5 chance of being heads. This leaves A with the probability of 1/4 because each coin has the probability if 1/2 and 1/2. Hence we can prove $\Pr[A_3 \mid B_3] > \Pr[A_3]$.

Problem 4 (20 points)**.** There may be several different min-cut sets in a graph. Using the analysis of the randomized min-cut algorithm, argue that there can be at most $n(n-1)/2$ distinct min-cut sets.

Solution. For this problem we must use a randomized min-cut algorithm to argue that there can be at most $n(n-1)/2$ distinct min-cut sets. The meaning of a min cut is essentially the smallest cut to be made on a graph and every graph has 2^{n-1} cuts. Out of this $n(n-1)/2$ can be the most minimum cuts in a graph where n represents vertices. We can consider a graph where there is at least 2k/n edges and the degree of the vertices is k or more. The probability of finding an edge that passes a cut on the first try is 2/n and the probability of finding a different cut is 1-(2/n). Using this information, we can further infer that the remaining vertices are n-m+1, and theres (k/2)*(n-m+1) edges. The probability to pick an edge that crosses this cut is 2/(n-m+1). The conditional probability is $Pr[E_m|E_{m-1} \cap ... \cap E_1] \geq 1 - (2/(n-m+1)) = (n-m-1)/(n-m+1)$. So, the summation of the conditional probabilities gives us $Pr[\cap_{j=1}^{n-2} E_i] \geq \sum_{m=1}^{n-2} (n-m-1)/(n-m+1) = n(n-1)/2$

Problem 5 (20 points)**.** A popular choice for pivot selection in Quicksort is the median of three randomly selected elements. Approximate the probability of obtaining at worst an a-to-$(1-a)$ split in the partition (assuming that a is a real number in the range $0 < a < 1/2$).
[Hint: Suppose that the median-of-three is the m-th smallest element of the array. Then it gives at worst an a-to-$(1-a)$ split if and only if $an \leq m \leq (1-a)n$. Now count how many sets of three elements can lead to the the pivot (= median-of-three) being the m-th smallest element.]

Solution. In order to find the approximate probability we can take a specific number of possible solutions and divide it by all solutions. To implement this we can take some element m between a*n and (1-a)n, where we can find the probability of one element on the right then the left. Then we can take the summation of all possible m values so $(a*n \leq m \leq (1-a)n)$. Hence, utilizing the hint, and making m some median, we get $Pr(\frac{n}{3} \leq i \leq \frac{2n}{3}) = \sum_{i=n/3}^{2n/3} \frac{6(i-1)(n-i)}{n(n-1)(n-2)}$. This is the probability of obtaining the worst split partition.

Checklist:

☐ Did you add your name?
☐ Did you disclose all resources that you have used?
(This includes all people, books, websites, etc. that you have consulted)
☐ Did you sign that you followed the Aggie honor code?
☐ Did you solve all problems?
☐ Did you submit the pdf file of your homework?