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Problem 1 (Frequency response of CT LTI systems, 5 Points): You want to design a system whose unit impulse response has the form:

$$h(t) = u(t) - u(t - N).$$

Find all possible values of $N > 0$ such that when the input signal $x(t) = \cos(\pi t)$, the output signal $y(t) = 0$ for all t.

given $x(t) = \cos(\pi t) = \frac{1}{2}e^{j\pi t} + \frac{1}{2}e^{-j\pi t}$

so $y(t) = \frac{1}{2}h(j\pi)e^{j\pi t} + \frac{1}{2}h(-j\pi)e^{-j\pi t}$

$= h(j\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t} dt$

given that $h(t) = u(t) - u(t-N) = \begin{cases} 1, & t \in (0, N) \\ 0, & \text{otherwise} \end{cases}$

so $h(j\omega) = \int_0^N e^{-j\omega t} dt$

$= -\frac{1}{j\omega} e^{-j\omega t}\Big|_0^N = \frac{1}{j\omega}(1 - e^{-jN\omega})$

$= \frac{e^{-j\frac{N\omega}{2}}}{j\omega}\, j\,2\sin\left(\frac{N\omega}{2}\right)$

$= \frac{2\sin\left(\frac{N\omega}{2}\right)}{\omega} e^{-j\frac{N\omega}{2}}$

$\rightarrow$ so $h(j\omega) = \frac{2\sin\left(\frac{N\pi}{2}\right)}{\pi} e^{-j\frac{N\pi}{2}}$

$h(-j\omega) = \frac{2\sin\left(\frac{N\pi}{2}\right)}{\pi} e^{j\frac{N\pi}{2}}$

output signal: $y(t) = \frac{1}{2}\frac{2\sin\left(\frac{N\pi}{2}\right)}{\pi} e^{-j\frac{N\pi}{2}} e^{j\pi t}$

$+ \frac{1}{2}\frac{2\sin\left(\frac{N\pi}{2}\right)}{\pi} e^{j\frac{N\pi}{2}} e^{-j\pi t}$

$= \frac{2\sin\left(\frac{N\pi}{2}\right)}{\pi} \cos\left(\pi\left(t - \frac{N}{2}\right)\right)$

for $y(t) = 0$

$\sin\left(\frac{N\pi}{2}\right) = 0 \rightarrow \frac{N\pi}{2} = k\pi$ for + int k

so $\boxed{N = 2k}$ for pos. int. k

Problem 2 (CTFS, 5 Points): A continuous-time periodic signal $x(t)$ is real-valued and has a fundamental period of $T_0 = 8$. The non-zero FS coefficients for $x(t)$ are specified as follows:

$$a_1 = a_{-1}^* = j$$
$$a_2 = a_{-2}^* = 1 + j\sqrt{3}$$
$$a_5 = a_{-5}^* = -3.$$

Express $x(t)$ in the form:

$$x(t) = \sum_{k=0}^{\infty} C_k \cos(k\omega_0 t + \phi_k).$$

FS coefficients in polar form: $a_1 = a_{-1}^* = j = e^{j\pi/2}$

$a_2 = a_{-2}^* = 1 + j\sqrt{3} = 2e^{j\pi/3}$

$a_5 = a_{-5}^* = -3 = 3e^{j\pi}$

when $T_0 = 8$

$= \omega_0 = \frac{2\pi}{8} = \frac{\pi}{4}$

so $x(t) = \left(a_1 e^{j\pi/4 t} + a_{-1} e^{-j\pi/4 t}\right) + \left(a_2 e^{j\pi/2 t} + a_{-2} e^{-j\pi/2 t}\right) + \left(a_5 e^{j\frac{5\pi}{4} t} + a_{-5} e^{-j\frac{5\pi}{4}}\right)$

$= \left(e^{j\pi/2} \cdot e^{j\pi/4 t} + e^{-j\pi/2} \cdot e^{-j\pi/4 t}\right) + \left(2e^{j\pi/3} \cdot e^{j\pi/2 t} + 2e^{-j\pi/3} e^{-j\pi/2 t}\right) + \left(e^{j\pi} e^{j5\pi/4 t} + e^{-j\pi} e^{-j\frac{5\pi}{4} t}\right)$

$e^{j(\pi/4 t + \pi/2)} + e^{-j(\pi/2 + \pi/4 t)}$ $\quad 2\left(e^{j(\pi/3 + \pi/2 t)}\right)$ .....

$= 2\cos\left(\pi/2 + \pi/4 t\right) + 4\cos\left(\pi/3 + \pi/2 t\right) + 2\cos\left(\pi + 5\pi/4 t\right)$

$= \boxed{2\cos\left(\omega_0 t + \frac{\pi}{2}\right) + 4\cos\left(2\omega_0 t + \frac{\pi}{3}\right) + 2\cos\left(5\omega_0 t + \pi\right)}$

Problem 3 (CTFS, 15 Points): Determine the fundamental period T_0 and the FS coefficients $\{a_k\}$ of the following periodic CT signals:

i) (5 Points) $x(t) = 2 - \cos(\pi t) + 4\sin(3\pi t + \pi/3) + 5e^{j2\pi(t-1/2)}$

ii) (5 Points) $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-k)$

iii) (5 Points) $x(t) = |\sin(t)|$

i) Fundamental Frequency $\omega_0 = \pi$

$$x(t) = 2 - \cos(\omega_0 t) + 4\sin(3\cos t + \tfrac{\pi}{3}) + 5e^{j(2\omega_0 t - \pi)}$$

$$= 2 - \left(\tfrac{1}{2}e^{j\omega_0 t} + \tfrac{1}{2}e^{-j\omega_0 t}\right) + 4\left(\tfrac{1}{j2}e^{j(3\omega_0 t + \pi/3)} - \tfrac{1}{j2}e^{-j(3\omega_0 t + \pi/3)}\right) + 5e^{-j\pi}e^{j2\omega_0 t}$$

$$= 2 - \left(\tfrac{1}{2}e^{j\omega_0 t} + \tfrac{1}{2}e^{-j\omega_0 t}\right) + \left(\tfrac{2}{j}e^{j\pi/3}e^{j3\omega_0 t} - \tfrac{2}{j}e^{-j\pi/3}e^{-j3\omega_0 t}\right) + 5e^{-j\pi}e^{j2\omega_0 t}$$

$$= 2 - \left(\tfrac{1}{2}e^{j\omega_0 t} + \tfrac{1}{2}e^{-j\omega_0 t}\right) + \left(\tfrac{2}{j}\left(\tfrac{1}{2} + j\tfrac{\sqrt{3}}{2}\right)e^{j3\omega_0 t} - \tfrac{2}{j}\left(\tfrac{1}{2} - j\tfrac{\sqrt{3}}{2}\right)e^{-j3\omega_0 t}\right) + 5(-)e^{j2\omega_0 t}$$

$$= 2 - \left(\tfrac{1}{2}e^{j\omega_0 t} + \tfrac{1}{2}e^{-j\omega_0 t}\right) + \left((\sqrt{3} - j)e^{j3\omega_0 t} + (\sqrt{3} + j)e^{-j3\omega_0 t}\right) - 5e^{j2\omega_0 t}$$

so FS coeff.

$$\boxed{a_k = \begin{cases} 2, & \text{if } k = 0 \\ -1/2, & \text{if } k = \pm 1 \\ \sqrt{3} - j, & \text{if } k = 2 \\ \sqrt{3} + j, & \text{if } k = 2 \\ -5, & \text{if } k = 3 \\ 0, & \text{otherwise} \end{cases}}$$

ii) $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-k)$

fund. pd. is $T_0 = 1$

$$\omega_0 = \frac{2\pi}{T_0} = 2\pi$$

$[T_0 = 1/2 - (-1/2) = 1]$

$$a_k = \frac{1}{T_0}\int_{-T/2}^{T/2} x(t)e^{-jn\omega_0 t}\,dt$$

$$a_k = \frac{1}{T_0}\int_{-T/2}^{T/2} \delta(t)e^{jn\omega_0 t}\,dt$$

$$= \frac{1}{T}\left(e^{-jn\omega_0 t}\right)\Big|_{t=0} \qquad \left[x(t) = \int \delta + x'(t) = x'(0)\right]$$

$$= \frac{1}{T}\cdot 1 = \frac{1}{1} = 1 \qquad \boxed{a_k = 1}$$

iii) fund. pd. $T_0 = \pi$

$$\omega_0 = \frac{2\pi}{T_0} = 2$$

$$a_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{\pi}\int_0^{\pi} |\sin(t)|\, e^{-j2kt}\, dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin(t)\, e^{-j2kt}\, dt$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\frac{1}{j2}e^{jt} - \frac{1}{j2}e^{-jt}\right)e^{-j2kt}\, dt$$

$$= \frac{1}{j2\pi}\int_0^{\pi}\left(e^{-j(2k-1)t} - e^{-j(2k+1)t}\right)dt$$

$$= \frac{1}{j2\pi}\left(\int_0^{\pi} e^{-j(2k-1)t}\, dt - \int_0^{\pi} e^{-j(2k+1)t}\, dt\right)$$

$$= \frac{1}{j2\pi}\left(-\frac{1}{j(2k-1)} e^{-j(2k-1)t}\Big|_0^{\pi} + \frac{1}{j2k+1} e^{-j(2k+1)t}\Big|_0^{\pi}\right)$$

$$= \frac{1}{2\pi j}\left(\frac{1}{j2k-1}\left(1 - e^{j(2k-1)\pi}\right) - \frac{1}{j2k+1}\left(1 - e^{j(2k+1)\pi}\right)\right)$$

$$= \frac{1}{2\pi j}\left(2\cdot\frac{1}{j(2k-1)} - 2\cdot\frac{1}{j(2k+1)}\right)$$

$$= -\frac{1}{\pi}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$$

$$= -\frac{2}{\pi(4k^2-1)}$$

$$\boxed{-\frac{2}{\pi(4k^2-1)}}$$

Problem 4 (Properties of CTFS, 10 Points): Let $x(t)$ be a periodic CT signal with fundamental frequency ω_0 and FS coefficients $\{a_k\}$. Derive the FS coefficients of the following signals in terms of $\{a_k\}$.

a) (1 Point) $-2x(t) + jx(t)$

b) (1 Point) $x(t-1)$

c) (1 Point) $dx(t)/dt$

d) (1 Point) $1 + x(-t)$

e) (1 Point) $x(1-t)$

f) (2 Points) $x(t)\cos(2\pi t/T_0)$

g) (3 Points) $|x(t)|^2$

a) $x(t) \longleftrightarrow a_k$

$-2x(t) + jx(t) \longleftrightarrow -2a_k + ja_k = \boxed{(-2+j)a_k}$

by linearity

b) $x(t-1) \longrightarrow a_k - e^{-jk\omega_0} = \boxed{a_k - e^{-jk\omega_0}}$

by time shift prop.

c) $dx(t)/dt \longleftrightarrow \quad = \boxed{a_k \cdot jk\omega_0}$

by differentiation

d) FS const. sign. is 1 so $\delta[k] = \begin{cases} 1, & k=0 \\ 0, & k \neq 0 \end{cases}$

DC compt.

by time flip $\rightarrow x(-t) \longleftrightarrow a_{-k}$

by linearity $\rightarrow 1 + x(-t) \longleftrightarrow \delta[k] + a_{-k} = \boxed{\begin{cases} 1 + a_0, & k=0 \\ a_k, & k \neq 0 \end{cases}}$

e) on time ver. the affine transf. is:

$x(t) \xrightarrow[\text{flip}]{t \to -t} x(-t) \xrightarrow[\text{time shift}]{t \to t-1} x(-(t-1)) = x(1-t)$

$a_k \nearrow \quad a_{-k} \qquad a_{-k} \cdot e^{-jk\omega_0}$

$= \boxed{a_{-k} \cdot e^{-jk\omega_0}}$

f) $\cos(2\pi t/T_0) = \cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$

FS $\rightarrow \frac{1}{2}\delta(k-1) + \frac{1}{2}\delta(k+1) = \begin{cases} \frac{1}{2}, & k = \pm 1 \\ 0, & k \neq \pm 1 \end{cases}$

$x(t)\cos(2\pi t/T_0) \longleftrightarrow a_k * \left(\frac{1}{2}\delta(k-1) + \frac{1}{2}\delta[k+1]\right)$

mult. prop

$= \boxed{\frac{1}{2}a_{k-1} + \frac{1}{2}a_{k+1}}$

g) $|x(t)|^2 = x(t) \cdot x^*(t)$, let $x^*(t) \longleftrightarrow b_k$

$|x(t)|^2 \longleftrightarrow a_k * b_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$

mult. prop

conjugation prop: $b_k = a_{-k}^*$

so $b_{k-l} = a_{-(k-l)}^* = a_{l-k}^*$

$|x(t)|^2 \longleftrightarrow \boxed{\sum_{l=-\infty}^{\infty} a_l a_{l-k}^*}$