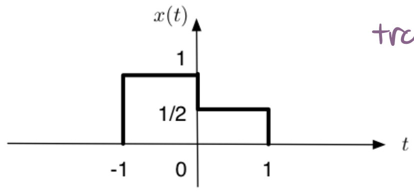
[redacted]

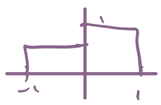
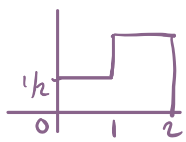
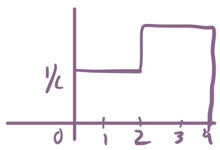
Problem 1 (Affine transformation on the time variable, 6 Points): A CT signal $x(t)$ is given as below:



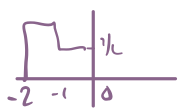
transform $x(t)$ into $x(1-0.5t)$

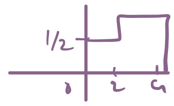
Transform $x(t)$ into $x(1-0.5t)$ in all six possible orders between time shift, time scaling, and time reversal. Carefully determine the time shift parameter in each order. Plot all intermediate and final signals. Carefully label your plots.

a) $x(t) \xrightarrow[\text{flip}]{t\to -t} x(-t) \xrightarrow[\text{shift}]{t\to t-1} x(1-t) \quad x(-(t-1)) \xrightarrow{t\to 0.5t} x(1-0.5t)$

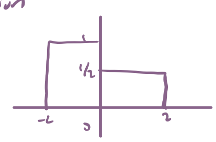
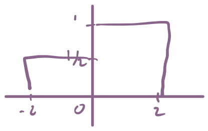
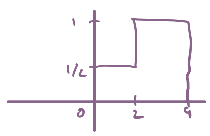




b) $x(t) \xrightarrow[\text{shift}]{t\to t+1} x(1+t) \xrightarrow[\text{flip}]{t\to -t} x(1-t) \xrightarrow{t\to 0.5t} x(1-0.5t)$

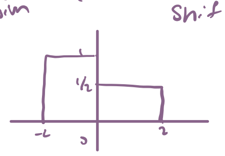
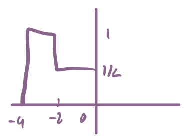
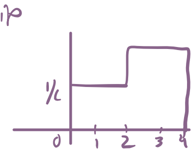




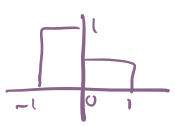
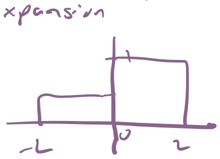
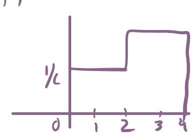
c) $x(t) \xrightarrow[\text{expansion}]{t\to 0.5t} x(0.5t) \xrightarrow[\text{flip}]{t\to -t} x(-0.5t) \xrightarrow[\text{shift}]{t\to t-2} x(1-0.5t)$

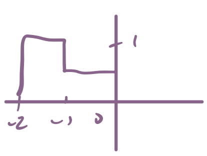
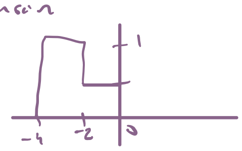
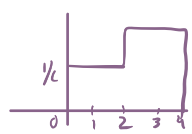
d) $x(t) \xrightarrow[\text{expansion}]{t\to 0.5t} x(0.5t) \xrightarrow[\text{shift}]{\cancel{t\to t+2}} x(0.5(t+2)) \xrightarrow[\text{flip}]{t\to -t} x(-0.5t+1)$

e) $x(t) \xrightarrow[\text{flip}]{t\to -t} x(-t) \xrightarrow[\text{expansion}]{t\to 0.5t} x(-0.5t) \xrightarrow[\text{shift}]{t\to t-2} x(1-0.5t)$

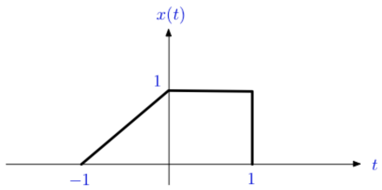
f) $x(t) \xrightarrow[\text{shift}]{\cancel{t\to t+1}} x(t+1) \xrightarrow[\text{expansion}]{t\to 0.5t} x(0.5t+1) \xrightarrow[\text{flip}]{t\to -t} x(-0.5t+1)$

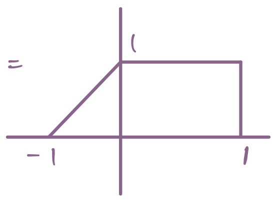
Problem 2 (Even and odd parts of a CT signal, 4 Points): For any CT signal $x(t)$, the even and odd parts of $x(t)$ are defined as:

$$x_e(t) := \frac{x(t) + x(-t)}{2}$$
$$x_o(t) := \frac{x(t) - x(-t)}{2}$$

Plot the even and odd parts $x_e(t)$ and $x_o(t)$ for a CT signal $x(t)$ as shown below:



$x(t) =$



$\rightarrow$ $x(-1) = 0$

$x(0) = 1$

$x(1) = 1$

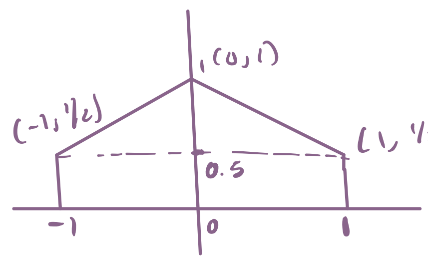
even

$$x_e(-1) = \frac{x(-1) + x(1)}{2} = 1/2$$

$$x_e(0) = \frac{1 + 1}{2} = 1$$

$$x_e(1) = 1/2$$

$x_e(t) \rightarrow$



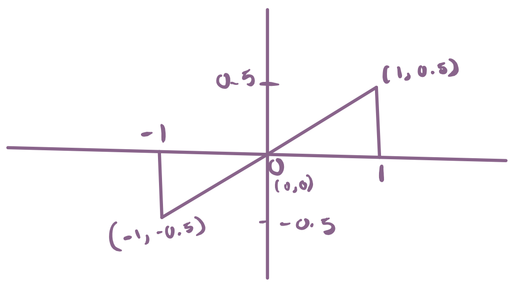
odd

$$x_o(-1) = \frac{x(-1) - x(1)}{2} = -1/2$$

$$x_o(0) = \frac{1 - 1}{2} = 0$$

$$x_o(1) = \frac{x(1) - x(-1)}{2} = 1/2$$

$x_o(t) \rightarrow$



Problem 3 (DT unit impulse response, 3 Points): The unit impulse response of a DT system is defined as the output corresponding to the input $x[n] = \delta[n]$, where

$$\delta[n] = \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases}$$

is the *DT unit impulse function*. Calculate and sketch the unit impulse response of the following DT systems:

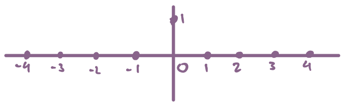
a) $y[n] = \max\{x[n-1], x[n], x[n+1]\}$;

b) $y[n] = \min\{x[n-1], x[n], x[n+1]\}$;

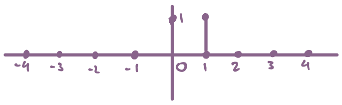
c) $y[n] = \frac{1}{3}\{x[n-1] + x[n] + x[n+1]\}$.

$\delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$ → general plots

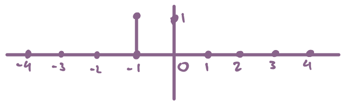
$\delta[n] =$



$\delta[n-1] =$

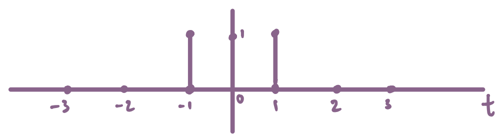


$\delta[n+1] =$



a) $y[-2] = \max\{\delta[-3], \delta[-2], \delta[-1]\} = 0$
$y[-1] = \max\{\delta[-2], \delta[-1], \delta[0]\} = 1$
$y[0] = \max\{\delta[-1], \delta[0], \delta[1]\} = 1$
$y[1] = \max\{\delta[0], \delta[1], \delta[2]\} = 1$
$y[2] = \max\{\delta[1], \delta[2], \delta[3]\} = 0$

$y[n] =$

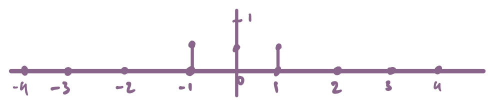


b) $y[n] = \min\{x[n-1], x[n], x[n+1]\}$
so, $y[-2] = \min\{x[-3], x[-2], x[-1]\} = 0$
$y[-1] = \min\{x[-2], x[-1], x[0]\} = 0$
$y[0] = \min\{x[-1], x[0], x[1]\} = 0$
$y[1] = \min\{x[0], x[1], x[2]\} = 0$
$y[2] = \min\{x[1], x[2], x[3]\} = 0$

$y[n] =$

c) $y[n] = \frac{1}{3}\{x[n-1] + x[n] + x[n+1]\}$
$y[-2] = \frac{1}{3}[x[-3] + x[-2] + x[-1]] = 0$
$y[-1] = \frac{1}{3}[x[-2] + x[-1] + x[0]] = 1/3$
$y[0] = \frac{1}{3}[x[-1] + x[0] + x[1]] = 1/3$
$y[1] = \frac{1}{3}[x[0] + x[1] + x[2]] = 1/3$
$y[2] = \frac{1}{3}[x[1] + x[2] + x[3]] = 0$

$y[n] =$



Problem 4 (CT systems described by LCCDE, 6 Points): Consider the CT system specified by the LCCDE

$$\frac{dy(t)}{dt} + y(t) = x(t)$$

and the condition of initial rest. Determine the output $y(t)$ that corresponds to the input $x(t) = 2\sin(t)u\left(t - \frac{\pi}{2}\right)$, where $u(t)$ is the unit step function.

Notes:
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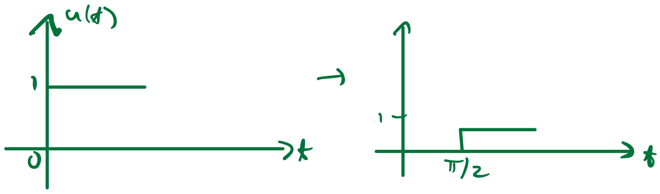
1st order LCCDE → $x(t) = 2\sin(t)\, u(t - \pi/2)$



$u(t-\pi/2) = \begin{cases} 1, & t > \pi/2 \\ 0, & t < \pi/2 \end{cases}$

$y(t) = y_h(t) + y_p(t)$

$x(t) = 2\sin(t)u(t-\pi/2) = \begin{cases} 2\sin(t), & t > \pi/2 \\ 0, & t < \pi/2 \end{cases}$

→ $y(t) = 0, \ t < \pi/2$

$x(t) = 2\sin(t), \ t > \pi/2$

guess! $y(t) = k\sin(t), \ t > \pi/2$

LCCDE: $\frac{dy(t)}{dt} + y(t) = x(t) = 2\sin(t), \ t > \pi/2$

$k\cos(t) + k\sin(t) = 2\sin(t), \ t > \pi/2$

Q: is there a soln for k? → no soln for k

guess2: $y(t) = k_1\sin(t) + k_2\cos(t), \ t \to \pi/2$

$x(t) = 2\sin(t)$

$y(t) = k\sin(t+\theta)$

→ $\frac{dy(t)}{dt} + y(t) = 2\sin t$

$k_1\cos(t) - k_2\sin(t)] + [k_1\sin(t) + k_2\cos(t)] = 2\sin(t)$

$\underbrace{(k_1+k_2)}_{=0}(\cos(t)) + \underbrace{(k_1-k_2)}_{=2}\sin(t) = 2\sin(t), \ t > \pi/2$

$\begin{cases} k_1 - k_2 = 2 \\ k_1 + k_2 = 0 \end{cases}$ ←

↳ $\begin{cases} k_1 = ? \\ k_2 = ? \end{cases}$

$\frac{dy(t)}{dt} + y(t) = x(t)$ → $x(t) = 2\sin(t)u(t-\pi/2)$

$u(t) = \begin{cases} 1 & t > \pi/2 \\ 0 & t \le \pi/2 \end{cases}$

suppose, $y(t) = y_H(t) + y_p(t)$

$y_H(t) = Ae^{st}$ → homogenous soln: $\frac{dy(t)}{dt} + Ae^{st} = 0$

$\frac{d}{dt}Ae^{st} + Ae^{st} = 0$

$Ase^{st} + Ae^{st} = 0$

$Ae^{st}(s+1) = 0$ → $s = -1$

$\therefore y_h(t) = Ae^{-t}$

let: $y_p(t) = k_1\sin(t) + k_2\cos(t)$

$k_1\cos(t) - k_2\sin(t) + k_1\sin(t) + k_2\cos(t) = 2\sin(t)$

$(k_1 + k_2)\cos(t) + (k_1 - k_2)\sin(t) = 2\sin(t)$ → $\begin{cases} k_1 + k_2 = 0 \\ k_1 - k_2 = 2 \end{cases}$ so $\begin{cases} k_1 = 1 \\ k_2 = -1 \end{cases}$

$\therefore y_p(t) = \sin(t) - \cos(t)$

$y(t) = Ae^{-t} + \sin(t) - \cos(t)$

Integrating the LCCDE from $\pi/2^-$ to $\pi/2^+$

$\int_{\pi/2^-}^{\pi/2^+} \frac{dy(t)}{dt}dt + \underbrace{\int_{\pi/2^-}^{\pi/2^+} \cancel{y(t)}\,dt}_{0} = \underbrace{\int_{\pi/2^-}^{\pi/2^+} \cancel{x(t)}\,dt}_{0}$

$\Rightarrow y(\pi/2^-) = y(\pi/2^+)$ → $y(t)$ is continuous now: $y(t) = Ae^{-t} + \sin t - \cos t$

since, $y(\pi/2^+) = 0$

$0 = Ae^{-\pi/2} + \sin \pi/2 - \cancel{\cos \pi/2}^{0}$

$0 = Ae^{-\pi/2} + 1$

$Ae^{-\pi/2} = -1$

$A = -e^{\pi/2}$

$\therefore\ y(t) = \left[-e^{\pi/2} e^{-t} + \sin(t) - \cos(t)\right] u(t - \pi/2)$

$$\boxed{y(t) = \left[-e^{-(t-\pi/2)} + \sin t - \cos t\right] u(t-\pi/2)}$$

Problem 5 (Arithmetic-geometric sequence, 6 Points): Alpha writes the infinite arithmetic sequence

$$10, 8, 6, 4, 2, 0, \ldots.$$

Beta writes the infinite geometric sequence

$$9, 6, 4, \frac{8}{3}, \frac{16}{9}, \ldots.$$

Gamma makes a sequence whose n^{th} term is the product of the n^{th} term of Alpha's sequence and the n^{th} term of Beta's sequence:

$$10 \cdot 9 \quad , \quad 8 \cdot 6 \quad , \quad 6 \cdot 4 \quad , \quad 4 \cdot \frac{8}{3} \quad , \quad 2 \cdot \frac{16}{9} \quad , \quad \ldots.$$

What is the sum of Gamma's entire sequence?

geometric ratio $\rightarrow q = 6/9 = 2/3$

$S = 10 \times 9 + 8 \times 6 + 6 \times 4 + 4 \times 8/3 + \ldots$

$2/3\, S = 10 \times 6 + 8 \times 4 + 6 \times 8/3 + 4 \times 16/9 + \ldots$

$S - 2/3\, S = (10 \times 9) - (2 \times 6) - (2 \times 4) - (2 \times 8/3) \ldots$

$= 90 - 2(6 + 4 + 8/3 + \ldots)$

$= 90 - 2\left(\frac{6}{1 - 2/3}\right) = 90 - 2\left(6/1/3\right)$

$= 90 - 12 \cdot 3$

$= 90 - 36$

$= 54$

$S = a_0 + a_0 q + a_0 q^2 + \ldots$

$|q| < 1: S = \frac{a}{1-q}$ $\rightarrow S = 3 \times 54 = 162$

$\boxed{S = 162}$

2) $\frac{dy(t)}{dt} + 3y(t) = x(t)$

$y(t) = ?$

$x(t) = \cos(t)\,u(t) \rightarrow$ unit ~~step~~ func

$y(t) = y_h(t) + y_p(t)$

$y_h(t) = Ae^{st} \rightarrow \frac{dy(t)}{dt} + y(t) = 0 \rightarrow 3 \rightarrow Ae^{-3t}$

$sAe^{st} + Ae^{st} = 0$

$Ae^{st}(s+1) = 0 \quad s = -1$

$Ae^{-t} = y_h(t), \ \forall \in \mathbb{R}$

$y_p(t) = k_1\cos(t) + k_2\sin(t), \ t>0$

$(-k_1\sin t + k_2\cos t) + 3(k_1\cos t + k_2\sin t) = \cos t$

$(3k_1 + k_2)\cos t + (3k_2 - k_1)\sin t = \cos t$

$\begin{cases} 3k_1 + k_2 = 1 \\ 3k_2 - k_1 = 0 \end{cases} = \begin{cases} k_1 = 3/10 \\ k_2 = 1/10 \end{cases}$

$y_p t = \frac{1}{10}(3\cos t + \sin t), \ t>0$

$= Ae^{-3t} + \frac{1}{10}(3\cos t + \sin t)$

$0 = y(0^+) = y(0^-) = A + \frac{3}{10} \Rightarrow A = -3/10$

$y(t) = \begin{cases} -3/10\, e^{-3t} + \frac{1}{10}(3\cos t + \sin t), & t>0 \\ 0, & t<0 \end{cases}$

$\boxed{\left\{ -\frac{3}{10}e^{-3t} + \frac{1}{10}(3\cos t + \sin t) \right\} u(t)}$

3) $y[n] = 2^{n+1}x[n-1]$

i) causal: $y[n]$ depends on $x[n-1]$, but n to $n-1$ represents past time $\rightarrow$ system is causal

ii) time invarient: time shift test

① $x[n] \rightarrow y[n] = 2^{n+1}x[n-1]$

② $x'[n] = x[n-n_0] \Rightarrow y'[n] = 2^{n+1}x[n-1]$ $\Big\}$ $y'[n] = 2^{n+1}x[n-n_0-1]$

$x[n-n_0-1]$

③ $y[n-n_0] = 2^{n-n_0+1}x[n-n_0-1]$

$y'[n]$ & $y[n-n_0]$ not always the same so time varint

iii) linear: linear combo test

① $x_1[n] \rightarrow y_1[n] = 2^{n+1}x_1[n-1]$

$x_2[n] \rightarrow y_2[n] = 2^{n+1}x_2[n-1]$

② $x'[n] = \alpha x_1[n] + \beta x_2[n] \rightarrow y'[n] = 2^{n+1}x'[n-1]$

$x'[n-1] = \alpha x_1[n-1] + \beta x_2[n-1]$

$y'[n] = 2^{n+1}(\alpha x_1[n-1] + \beta x_2[n-1])$

$= \underline{\alpha 2^{n+1}x_1[n-1] + \beta 2^{n+1}x_2[n-1]}$

③ $\alpha y_1[n] + \beta y_2[n] = \underline{\alpha 2^{n+1}x_1[n-1] + \beta 2^{n+1}x_2[n-1]}$

always the same : linear

4) $x(t) = e^{-2t}u(t) \rightarrow e^{-2t}u(t)$

$h(t) = e^{-t}u(t) \rightarrow e^{(-t-\tau)}u(t-\tau)$

$y(t) = x(t) * h(t)$

$$y = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2\tau} u(t) \cdot e^{-t-\tau} u(t-\tau)\, d\tau$$

$$= e^{-t}\int_{-\infty}^{\infty} e^{-\tau} u(\tau)\, u(t-\tau)\, d\tau$$

when $t<0$: $y(t) = e^{-t}\int_{-\infty}^{\infty} e^{-t}\cdot 0\, d\tau = 0$

$u(t)\, u(t-\tau) = 0$

when $t>0$: $u(\tau)\, u(t-\tau) = \begin{cases} 1, & \tau \in (0,t) \\ 0, & \text{otherwise} \end{cases}$

$$y(t) = e^{-t}\int_0^t e^{-\tau}\cdot 1\, d\tau = e^{-t}\int_0^t e^{-\tau}\, d\tau$$

$$= e^{-t}\left(-e^{-\tau}\right)\Big|_0^t$$

$$= e^{-t}\left(1-e^{-t}\right) = e^{-t} - e^{-2t}$$

$$y(t) = \begin{cases} e^{-t} - e^{-2t}, & t>0 \\ 0, & t<0 \end{cases}$$

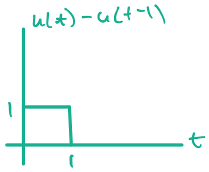
$(e^{-t} - e^{-2t})\, u(t)$

$$\boxed{y(t) = \begin{cases} e^{-t} - e^{-2t}, & t>0 \\ 0, & t<0 \end{cases} \quad (e^{-t}-e^{-2t})\, u(t)}$$

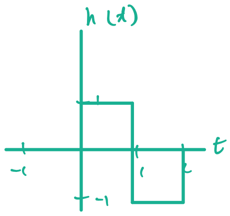
5) $x(t) = u(t) - u(t-1)$

a) $h(t) = u(t) + u(t-2) - 2u(t-1)$

$x(t) \rightarrow$ $u(t)-u(t-1)$



$h(t) \rightarrow$ $h(t)$



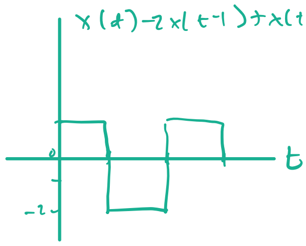
b) $y = x * h = ((u_{-1} * h) * x) * u$

$u_{-1} * h = \delta_0 - 2\delta_1 + \delta_2$

$[((u_{-1} * h) * x)](t) = (\delta_0 * x - 2\delta_1 * x + \delta_2 * x)(t)$

$= x(t) - 2x(t-1) + x(t-2)$

$x(t) - 2x(t-1) + x(t-2)$



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