Problem Set 1

CSCE 411 (Dr. Klappenecker)

Due date: Electronic submission the .pdf file of this homework is due on **Friday, 9/10, 11:59pm** on e-campus (as a turnitin assignment).

Name: [redacted]

Resources. (All people, books, articles, web pages, etc. that have been consulted when producing your answers to this homework)

On my honor, as an Aggie, I have neither given nor received any unauthorized aid on any portion of the academic work included in this assignment. Furthermore, I have disclosed all resources (people, books, web sites, etc.) that have been used to prepare this homework.

Signature: [redacted]

Get familiar with LaTeX. All exercises in this homework are from the lecture notes on perusall, not from our textbook.

Reading assignment: Carefully read the lecture notes `dm_ch11.pdf` on Perusall. Skim Chapters 1-3 in the textbook.

Problem 1. Exercise 11.7

Solution. For Exercise 11.7, we must show that $\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}$.
We know that $\binom{n}{k}$ is the binomial coefficients and that the formula is $\binom{n}{k}=\frac{n!}{k!(n-k)!}$. To prove $\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}$ we can use Stirling approximation. Given in example 11.3, Stirling approximation is n! $\sim \sqrt{2\pi n}\ (\frac{n}{e})^n$. Given $\binom{2n}{n}$, we can now use $\binom{n}{k}=\frac{n!}{k!(n-k)!}$, where we get $\frac{2n!}{n!(2n-n)!}$, which simplifies to $\frac{2n!}{n!*n!}$ or $\frac{2n!}{n!^2}$. We can now use Stirling approximation and plug $\sqrt{2\pi n}\ (\frac{n}{e})^n$ for n. This gives us, $\binom{2n}{n}=\frac{\sqrt{2\pi 2n}(\frac{2n}{e})^{2n}}{(\sqrt{2\pi n}(\frac{n}{e})^n)^2}$ which simplifies to $\frac{\sqrt{4\pi n}(\frac{2n}{e})^{2n}}{2\pi n(\frac{n}{e})^{2n}}$, and this further simplifies to $\frac{4^n}{\sqrt{\pi n}}$. Thus, this shows that $\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}$.

Problem 2. Exercise 11.9

Solution. In exercise 11.9, we must find the accumulation points, along with their upper and lower accumulation points for the given functions.
a) f(n) = $(-1)^n$ lets say n = 0,1,2,3,...∞ then
f(n) = f(0)=$(-1)^0 = 1$
f(n) = f(1)=$(-1)^1 = -1$
f(n) = f(2)=$(-1)^2 = 1$
f(n) = f(5)=$(-1)^5 = -1$, etc
so its -1 when n is odd and 1 when n is even. Thus, the upper accumulation point is 1 and lower accumulation point is -1.
b) f(n) = 4 + $(-1)^n\frac{n}{n+10}$
So, for evens, we can take the limit of 4 + $(-1)^{2n}\frac{n}{n+10}$, which gives us 4 + $\lim_{n\to\infty}(-1)^{2n}\frac{2n}{2n+10}$, which gives us 4 + 1 = 5. For odds, we can take the limit of 4 + $(-1)^{2n+1}\frac{2n}{2n+10}$, which gives us 4 + $\lim_{n\to\infty}(-1)^{2n+1}\frac{2n}{2n+10}$, which gives us 4 - 1 = 3. Thus, the upper accumulation point is 5 and lower accumulation point is 3.
c) f(n) = $((-1)^n + (-1)^{\lfloor\frac{n}{2}\rfloor})(1 + \frac{1}{n})$
So, for evens, we can take the limit of $((-1)^{2n} + (-1)^{\lfloor\frac{2n}{2}\rfloor})(1 + \frac{1}{2n})$ as n approaches ∞, and for the odds, we can take the limit of $((-1)^{2n+1} + (-1)^{\lfloor\frac{2n+1}{2}\rfloor})(1+\frac{1}{2n+1})$ as n approaches ∞. This will give us the upper accumulation point of 2 and lower accumulation point of -2.

Problem 3. Exercise 11.17

Solution. In Exercise 11.17, we are given that b and d are positive real numbers that don't equal to 1. Using this information, we can answer parts a and b.
a) For this part, we show that $\Theta(\log_b n) = \Theta(\log_d n)$ so one can write $\Theta(\log n)$ using a baseless log.

To prove $\Theta(\log_b n) = \Theta(\log_d n)$, we can say that $\Theta(\log_b n) = C * (\log_b n)$, where C is the constant. Then we can do $\frac{C(\log_b n)}{(\log_b d)}(\log_b d)$ where we can multiply and divide by $(\log_b d)$. Then $(c)(\log_b d)(\log_d n)$, where $\frac{(\log_b n)}{(\log_b d)} = \log_d n$. We can set C1 = (C)$(\log_b d)$ where C1 will be a constant in (C1)$(\log_d n)$. Thus showing that $\Theta(\log_b n) = \Theta(\log_d n)$. Also, for Θ a base is not required and multiplicative constants don't matter.
b) For part b we must prove or disprove if $\Theta(n^{\log_b n}) = \Theta(n^{\log_d n})$.
Suppose $\Theta(n^{\log_b n}) = \Theta(n^{\log_d n})$ and if the $0 \leq \lim_{n\to\infty} \frac{n^{\log_b n}}{n^{\log_d n}} =$ C $\leq \infty$. So if b=d then the $0 \leq \lim_{n\to\infty} \frac{n^{\log_b n}}{n^{\log_d n}} =$ C $\leq \infty$ is true, since it will eventually reach 0 or ∞. However if $b > d$ or $b < d$ then $0 \leq \lim_{n\to\infty} \frac{n^{\log_b n}}{n^{\log_d n}} =$ C $\leq \infty$ does not hold true. Thus we can disprove $\Theta(n^{\log_b n}) = \Theta(n^{\log_d n})$ when b is not equal to d, but note that is does hold true when b=d.

Problem 4. Exercise 11.19

Solution. In Exercise 11.19, we must show that for all integers k, we have $1^k + 2^k + ... + n^k = \Theta(n^{k+1})$.
Θ means $c * n^{k+1} \leq 1^k + 2^k + ... + n^k \leq C * n^{k+1}$. In order to prove this, we need to show that $C > 0$ and N exists, such that $f(n) \leq C * g(n)$ for all $n \geq N$, where f(n) = $1^k + 2^k + ... + n^k$ and g(n) = n^{k+1}. We know that $1^k + 2^k + ... + n^k \leq n^k + n^k + ... + n^k$.
$n^k + n^k + ... + n^k$ can further be simplified to $n * n^k$ or n^{k+1} for all $n \geq 1$ and $k > 0$. Now, lets assume C=1 and N=1 then $1^k + 2^k + ... + n^k \leq C * n^{k+1}$, for all $n \geq N$.
Now, to find the lower bound, we want to delete some positive terms, so we can use $(\lfloor \frac{n}{2} \rfloor + 1)^k + \ . \ . \ . \ + n^k$. We know that $(\lfloor \frac{n}{2} \rfloor + 1)^k \geq (\frac{n}{2})^2$, and $(\lfloor \frac{n}{2} \rfloor + 2)^k \geq (\frac{n}{2})^2$, etc. Hence, $n^k \geq (\frac{n}{2})^k$ for all $n \geq 1$. Furthermore, $(\frac{n}{2})^k + (\frac{n}{2})^k + \ . \ . \ . \ + (\frac{n}{2})^k$ gives us $(\frac{n}{2})^{k+1}$. So $(\frac{n}{2})(n - \lfloor \frac{n}{2} \rfloor) \geq (\frac{n}{2})^{k+1}$ and $(\frac{1}{2^{k+1}})(n^{k+1}) = C(n^{k+1})$, where $n \geq 1$. Thus gives us our upper and lower bound for Θ and we can conclude, $f(n) \leq C * g(n)$ for all $n \geq N$, hence $1^k + 2^k + ... + n^k$ is equal to $\Theta(n^{k+1})$.

Problem 5. Exercise 11.34

Solution. In Exercise 11.34, we must Prove or Disprove: $n^{\ln n} \in O(e^{(\ln n)^2})$
$n^{\ln n} \in O(e^{(\ln n)(\ln n)})$
$n^{\ln n} \in e^{(\ln n)(\ln n)}$
Using log rules we can say $n^{\ln n} = e^{\ln(n^{\ln n})}$. The right hand side then simplifies to $e^{(\ln n)(\ln n)}$ or $e^{(\ln n)^2}$ which is the same from the question $O(e^{(\ln n)^2})$.
Hence, we proved that $n^{\ln n} \in O(e^{(\ln n)^2})$.

Homeworks must be typeset in LaTeX.

Checklist:

☐ Did you add your name?
☐ Did you disclose all resources that you have used?
(This includes all people, books, websites, etc. that you have consulted)
☐ Did you follow the Aggie honor code?
☐ Did you solve all problems?
☐ Did you submit the pdf file (resulting from your latex file) of your homework?