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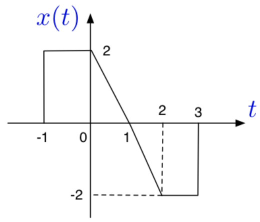
Problem 1 (Properties of CTFT, 10 Points): Let $X(j\omega)$ be the Fourier transform of the signal $x(t)$ as shown below:



a) Find $X(j0)$.

b) Find $\int_{-\infty}^{\infty} X(j\omega)d\omega$.

c) Evaluate $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$.

d) Sketch the inverse Fourier transform of $Re\{X(j\omega)\}$.

e) Evaluate $\int_{-\infty}^{\infty} X(j\omega)\frac{2\sin\omega}{\omega}e^{j2\omega}d\omega$.

a) $x(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

$\Rightarrow x(j0) = \int_{-\infty}^{\infty} x(t)\,dt = \boxed{0}$

b) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(j\omega)e^{j\omega t}d\omega$

$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(j\omega)d\omega$

$= \int_{-\infty}^{\infty} x(j\omega)d\omega = 2\pi \cdot x(0) = \boxed{4\pi}$

c) Parseval Id: $\int_{-\infty}^{\infty} |x(t)|^2 dt$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |x(j\omega)|^2 d\omega$

$\rightarrow \int_{-\infty}^{\infty} |x(j\omega)|^2 d\omega = 2\pi\int_{-\infty}^{\infty} |x(t)|^2 dt$

$= 2\pi \cdot 2\left(\int_{-1}^{0} 2^2 dt + \int_0^1 (2-2t)^2 dt\right)$

$= 4\pi\left(4 + 4\int_0^1 (1-t)^2 dt\right)$

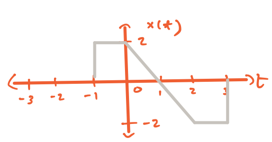
$= \boxed{\frac{64\pi}{3}}$

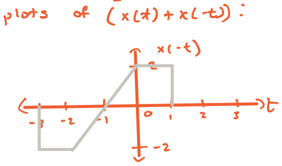
d) $Re\{x(j\omega)\} = \frac{1}{2}x(j\omega) + \frac{1}{2}x^*(j\omega)$

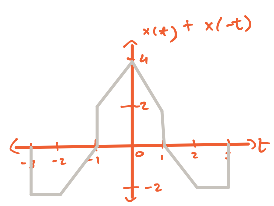
so inverse is given by

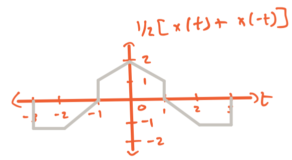
$\frac{1}{2}x(t) + \frac{1}{2}x^*(-t) = \frac{1}{2}x(t) + \frac{1}{2}x(-t)$

$x(t)$ is real so plots of $(x(t)+x(-t))$:









e) if $H(j\omega) = \frac{2\sin\omega}{\omega}$ and $Y(j\omega) = X(j\omega)H(j\omega)$

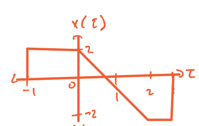
from FT table : $h(t) = \begin{cases} 1, & |t|<1 \\ 0, & |t|>1 \end{cases}$; convolution prop : $y(t) = x(t) * h(t)$

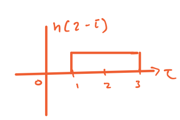
syn. equation $\rightarrow \int_{-\infty}^{\infty} x(j\omega)\frac{2\sin\omega}{\omega}e^{j2\omega}d\omega = \int_{-\infty}^{\infty} x(j\omega)H(j\omega)e^{j2\omega}d\omega$

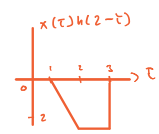
$= \int_{-\infty}^{\infty} y(j\omega)e^{j2\omega}d\omega = 2\pi \cdot y(2)$

$\downarrow$

$y(2)$ graphically determined:





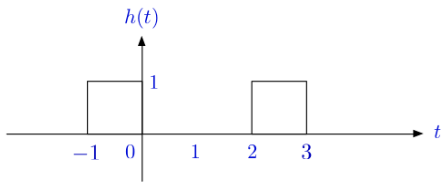


so $y(2) = \int_{-\infty}^{\infty} x(\tau)h(2-\tau)d\tau$

$= -3$

$\hookrightarrow \int_{-\infty}^{\infty} x(j\omega)\frac{2\sin\omega}{\omega}e^{j2\omega}d\omega = 2\pi \cdot y(2) = \boxed{-6\pi}$

Problem 2 (Frequency-domain characterization of CT LTI systems, 10 Points):
Consider a CT LTI system whose unit impulse response $h(t)$ is shown as below:



a) Find a closed-form expression for the frequency response $H(j\omega)$ of the system.

b) Determine and sketch the magnitude response $|H(j\omega)|$ of the system.

c) Find a closed-form expression for the output signal $y(t)$ when the input signal $x(t) = \cos^2\left(\frac{\pi}{2}t\right)$.

a) $h(t) = \begin{cases} 1, & t \in (-1, 0) \cup (2, 3) \\ 0, & \text{otherwise} \end{cases}$

when $\omega = 0 \rightarrow H(j0) = \int_{-\infty}^{\infty} h(t)\,dt = 2$

$\omega \neq 0 \rightarrow H(j\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\,dt$

$$= \int_{-1}^{0} e^{-j\omega t}\,dt + \int_{2}^{3} e^{-j\omega t}\,dt$$

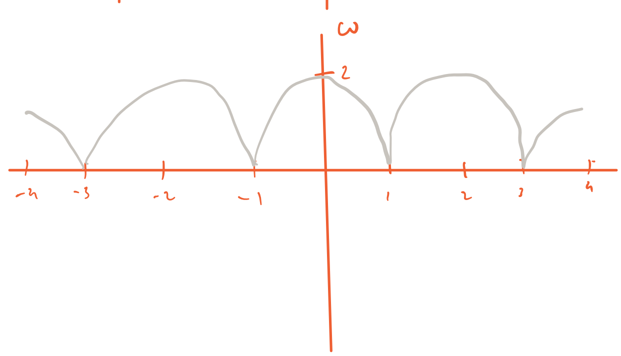
$$= -\frac{1}{j\omega} e^{-j\omega t}\Big|_{-1}^{0} - \frac{1}{j\omega} e^{-j\omega t}\Big|_{2}^{3}$$

$$= \frac{1}{j\omega}\left(e^{j\omega} - 1\right) + \frac{1}{j\omega}\left(e^{-j2\omega} - e^{-j3\omega}\right)$$

$$= \frac{1}{j\omega}\left(e^{j\omega} - 1\right) + \frac{1}{j\omega} e^{-j2\omega}\left(e^{j\omega} - 1\right)$$

$$= \underline{\frac{1}{j\omega}\left(e^{j\omega} - 1\right)\left(1 + e^{-j3\omega}\right)} \rightarrow H(j\omega) = \underline{\frac{2\left(\sin(2\omega) - \sin(\omega)\right)}{\omega}} \cdot e^{-j\omega t}$$

b) $|H(j\omega)| = 2\left|\frac{\sin(2\omega) - \sin(\omega)}{\omega}\right|$



c) $x(t) = \left(\cos\left(\frac{\pi}{2}t\right)\right)^2 = \frac{1}{4}\left(e^{j\pi/2\,t} + e^{-j\frac{\pi}{2}t}\right)^2 - \frac{1}{4}\left(e^{j\pi t} + e^{-j\pi t} + 2\right)$

$= \frac{1}{4}e^{j\pi t} + \frac{1}{4}e^{-j\pi t} + \frac{1}{2}$

so $y(t) = \frac{1}{4}H(j\pi)e^{j\pi t} + \frac{1}{4}H(-j\pi)e^{-j\pi t} + \frac{1}{2}H(j0)$

we know $H(j0) = 2$

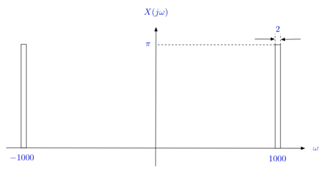
$H(j\pi) = H(-j\pi) = 0$

so $\boxed{y(t) = 1}$ for all t

Problem 3 (Frequency-domain characterization of CT LTI systems, 10 Points):
Consider an input signal:

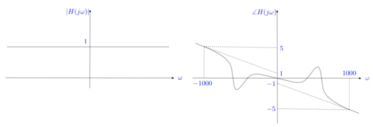
$$x(t) = s(t)\cos(1000t)$$

where $s(t)$ is a slowly-varying (relative to $\omega_0 = 1000$) signal. The Fourier transform $X(j\omega)$ of $x(t)$ (which is real in this case) is shown below:



a) Find an expression for the envelop signal $s(t)$.

b) Given an all-pass system whose frequency response $H(j\omega)$ is shown below:



find an approximate expression for the output $y(t)$ that incorporates the group and the phase delays.

a) $x(t) = s(t)\cos(1000t)$

$= \frac{1}{2} s(t) e^{j1000t} + \frac{1}{2} s(t) e^{-j1000t}$

$X(j\omega) = \frac{1}{2} S(j(\omega - 1000)) + \frac{1}{2} S(j(\omega + 1000))$

$$S(j\omega) = \begin{cases} 2\pi, & |\omega| < 1 \\ 0, & |\omega| > 1 \end{cases}$$

from FT table

$$s(t) = 2\pi \cdot \frac{\sin(t)}{\pi t} = \boxed{\frac{2\sin(t)}{t}}$$

b) $y(t) \approx |H(j\omega_0)|\, s(t - \tau_g(\omega_0)) \cos(\omega_0(t - \tau_p(\omega_0)))$

$\omega_0 = 1000$

$|H(j\omega_0)| = 1$

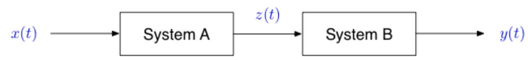
$\tau_g(\omega_0) = \frac{5-1}{1000} = 0.004$

$\tau_p(\omega_0) = \frac{5-0}{1000} = 0.005$

$$y(t) \approx \frac{2\sin(t - 0.004)}{t - 0.004} \cos(1000(t - 0.005))$$

$$\boxed{y(t) \approx \frac{2\sin(t - 0.004)}{t - 0.004} \cos(1000(t - 0.005))}$$

Problem 4 (LCCDE, 10 Points): Consider the serial interconnection of two causal CT LTI systems A and B as shown below:



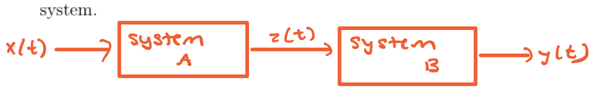
The input-output relation for system A is characterized by the following LCCDE:

$$\frac{d^2z(t)}{dt^2} + 5\frac{dz(t)}{dt} + 6z(t) = \frac{dx(t)}{dt} + 5x(t)$$

The unit impulse response $h_b(t)$ for system B is given by

$$h_b(t) = e^{-2t}u(t).$$

Determine the frequency response $H(j\omega)$ and the unit impulse response $h(t)$ of the overall system.



(1) $\frac{d^2 z(t)}{dt^2} + \frac{5d(z)dt}{dt} + 6z(t) = \frac{dx(t)}{dt} + 5x(t)$

$s^2 Z(s) + 5s Z(s) + 6Z(s) = sX(s) + 5X(s)$

$H_A(s) = \frac{Z(s)}{X(s)} = \frac{s+5}{s^2+5s+6}$

$h_b(t) = e^{-2t}u(t)$

$H_B(s) = \frac{1}{s+2}$

$H(s) = H_A(s)\, H_B(s)$

$H(s) = \frac{s+5}{(s^2+5s+6)(s+2)} = \frac{s+5}{(s+2)(s+3)(s+2)} = \frac{s+5}{(s+2)^2(s+3)}$

$H(s) = \frac{A}{(s+2)^2} + \frac{B}{(s+2)} + \frac{C}{(s+3)}$

$A = \frac{5+s}{s+3}\Big)_{s=-2} = 3$

$B = \frac{d}{ds}\left(\frac{5+s}{3+s}\right)_{s=-2} = -2$

$C = \frac{s+5}{(s+2)^2}\Big|_{s=-3} = 2$

$\Rightarrow H(s) = \frac{3}{(s+2)^2} - \frac{2}{s+2} + \frac{2}{s+3}$

$h(t) = 3te^{-2t} - 2e^{-2t} + 2e^{-3t})u(t)$

$\frac{1}{(s+a)^2} \; (-)\; te^{-at}u(t)$

$\underline{h(t) = (3t-2)e^{-2t}u(t) + 2e^{-3t}u(t)}$

$H(s) = \frac{5+s}{(s+2)^2(s+3)}$ $\quad s = j\omega$

$$H(j\omega) = \frac{5+j\omega}{(2+j\omega)^2(3+j\omega)}$$