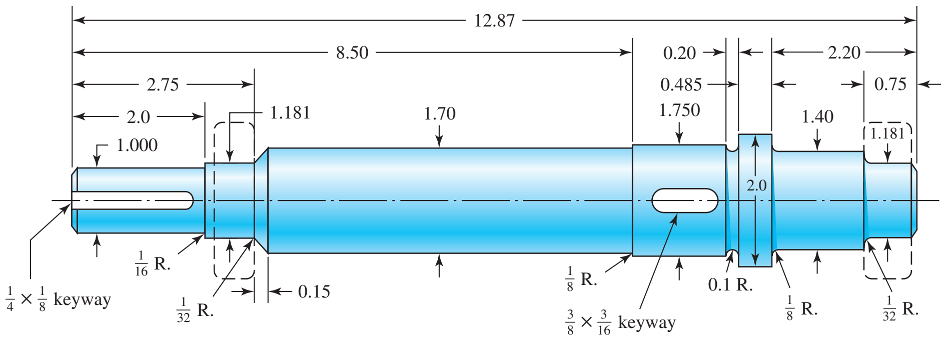
7 Shafts and Shaft Components



Chapter Outline

7–1 Introduction

A *shaft* is a rotating member, usually of circular cross section, used to transmit power or motion. It provides the axis of rotation, or oscillation, of elements such as gears, pulleys, flywheels, cranks, sprockets, and the like and controls the geometry of their motion. An *axle* is a nonrotating member that carries no torque and is used to support rotating wheels, pulleys, and the like. The automotive axle is not a true axle; the term is a carryover from the horse-and-buggy era, when the wheels rotated on nonrotating members. A nonrotating axle can readily be designed and analyzed as a static beam, and will not warrant the special attention given in this chapter to the rotating shafts, which are subject to fatigue loading.

There is really nothing unique about a shaft that requires any special treatment beyond the basic methods already developed in previous chapters. However, because of the ubiquity of the shaft in so many machine design applications, there is some advantage in giving the shaft and its design a closer inspection. A complete shaft design has much interdependence on the design of the components. The design of the machine itself will dictate that certain gears, pulleys, bearings, and other elements will have at least been partially analyzed and their size and spacing tentatively determined. Chapter 18 provides a complete case study of a power transmission, focusing on the overall design process. In this chapter, details of the shaft itself will be examined, including the following:

- Material selection
- Geometric layout
- Stress and strength
    - Static strength
    - Fatigue strength
- Deflection and rigidity
    - Bending deflection
    - Torsional deflection
    - Slope at bearings and shaft-supported elements
    - Shear deflection due to transverse loading of short shafts
- Vibration due to natural frequency

In deciding on an approach to shaft sizing, it is necessary to realize that a stress analysis at a specific point on a shaft can be made using only the shaft geometry in the vicinity of that point. Thus the geometry of the entire shaft is not needed. In design it is usually possible to locate the critical areas, size these to meet the strength requirements, and then size the rest of the shaft to meet the requirements of the shaft-supported elements.

The deflection and slope analyses cannot be made until the geometry of the entire shaft has been defined. Thus deflection is a function of the geometry *everywhere,* whereas the stress at a section of interest is a function of *local geometry*. For this reason, shaft design allows a consideration of stress first. Then, after tentative values for the shaft dimensions have been established, the determination of the deflections and slopes can be made.

7–2 Shaft Materials

Deflection is not affected by strength, but rather by stiffness as represented by the modulus of elasticity, which is essentially constant for all steels. For that reason, rigidity cannot be controlled by material decisions, but only by geometric decisions.

Necessary strength to resist loading stresses affects the choice of materials and their treatments. Many shafts are made from low-carbon, cold-drawn or hot-rolled steel, such as AISI 1020–1050 steels.

Significant strengthening from heat treatment and high alloy content are often not warranted. Fatigue failure is reduced moderately by increase in strength, and then only to a certain level before adverse effects in endurance limit and notch sensitivity begin to counteract the benefits of higher strength. A good practice is to start with an inexpensive, low or medium carbon steel for the first time through the design calculations. If strength considerations turn out to dominate over deflection, then a higher strength material should be tried, allowing the shaft sizes to be reduced until excess deflection becomes an issue. The cost of the material and its processing must be weighed against the need for smaller shaft diameters. When warranted, typical alloy steels for heat treatment include AISI 1340–50, 3140–50, 4140, 4340, 5140, and 8650.

Shafts usually don't need to be surface hardened unless they serve as the actual journal of a bearing surface. Typical material choices for surface hardening include carburizing grades of AISI 1020, 4320, 4820, and 8620.

Cold-drawn steel is usually used for diameters under about 3 inches. The nominal diameter of the bar can be left unmachined in areas that do not require fitting of components. Hot-rolled steel should be machined all over. For large shafts requiring much material removal, the residual stresses may tend to cause warping. If concentricity is important, it may be necessary to rough machine, then heat treat to remove residual stresses and increase the strength, then finish machine to the final dimensions.

In approaching material selection, the amount to be produced is a salient factor. For low production, turning is the usual primary shaping process. An economic viewpoint may require removing the least material. High production may permit a volume-conservative shaping method (hot or cold forming, casting), and minimum material in the shaft can become a design goal. Cast iron may be specified if the production quantity is high, and the gears are to be integrally cast with the shaft.

Properties of the shaft locally depend on its history—cold work, cold forming, rolling of fillet features, heat treatment, including quenching medium, agitation, and tempering regimen.[1]

Stainless steel may be appropriate for some environments.

7–3 Shaft Layout

The general layout of a shaft to accommodate shaft elements, e.g., gears, bearings, and pulleys, must be specified early in the design process in order to perform a free body force analysis and to obtain shear-moment diagrams. The geometry of a shaft is generally that of a stepped cylinder. The use of shaft shoulders is an excellent means of axially locating the shaft elements and to carry any thrust loads. Figure 7–1 shows an example of a stepped shaft supporting the gear of a worm-gear speed reducer. Each shoulder in the shaft serves a specific purpose, which you should attempt to determine by observation.

The geometric configuration of a shaft to be designed is often simply a revision of existing models in which a limited number of changes must be made. If there is no existing design to use as a starter, then the determination of the shaft layout may have

[1]See Joseph E. Shigley, Charles R. Mischke, and Thomas H. Brown, Jr. (eds-in-chief), *Standard Handbook of Machine Design,* 3rd ed., McGraw-Hill, New York, 2004. For cold-worked property prediction see Chapter 29, and for heat-treated property prediction see Chapters 29 and 33.

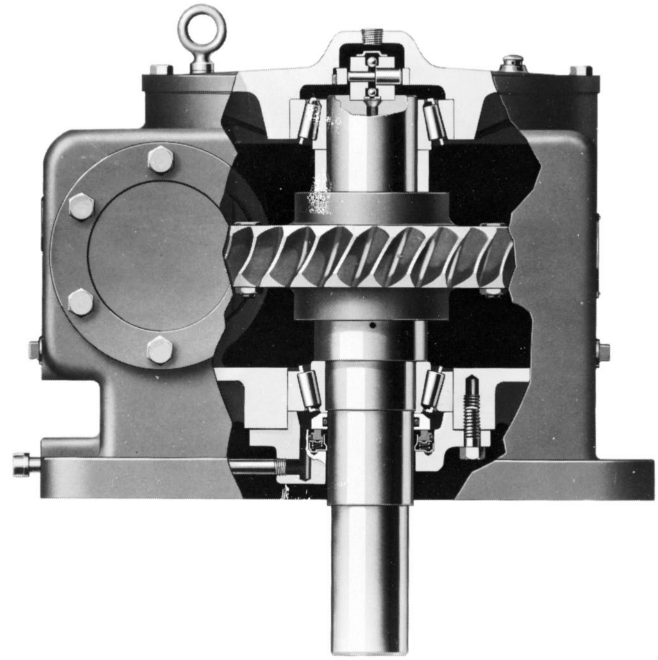


Figure 7–1

A vertical worm-gear speed reducer. *(Courtesy of the Cleveland Gear Company.)*

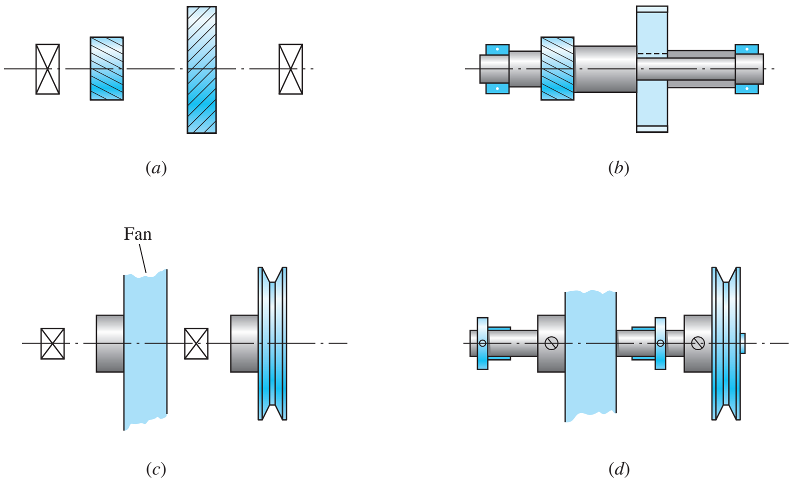


Figure 7–2

(*a*) Choose a shaft configuration to support and locate the two gears and two bearings. (*b*) Solution uses an integral pinion, three shaft shoulders, key and keyway, and sleeve. The housing locates the bearings on their outer rings and receives the thrust loads. (*c*) Choose fan-shaft configuration. (*d*) Solution uses sleeve bearings, a straight-through shaft, locating collars, and setscrews for collars, fan pulley, and fan itself. The fan housing supports the sleeve bearings.

many solutions. This problem is illustrated by the two examples of Figure 7–2. In Figure 7–2*a* a geared countershaft is to be supported by two bearings. In Figure 7–2*c* a fanshaft is to be configured. The solutions shown in Figure 7–2*b* and 7–2*d* are not necessarily the best ones, but they do illustrate how the shaft-mounted devices are fixed and located in the axial direction, and how provision is made for torque transfer from one element to another. There are no absolute rules for specifying the general layout, but the following guidelines may be helpful.

Axial Layout of Components

The axial positioning of components is often dictated by the layout of the housing and other meshing components. In general, it is best to support load-carrying components between bearings, such as in Figure 7–2*a,* rather than cantilevered outboard of the bearings, such as in Figure 7–2*c.* Pulleys and sprockets often need to be mounted outboard for ease of installation of the belt or chain. The length of the cantilever should be kept short to minimize the deflection.

Only two bearings should be used in most cases. For extremely long shafts carrying several load-bearing components, it may be necessary to provide more than two bearing supports. In this case, particular care must be given to the alignment of the bearings.

Shafts should be kept short to minimize bending moments and deflections. Some axial space between components is desirable to allow for lubricant flow and to provide access space for disassembly of components with a puller. Load-bearing components should be placed near the bearings, again to minimize the bending moment at the locations that will likely have stress concentrations, and to minimize the deflection at the load-carrying components.

The components must be accurately located on the shaft to line up with other mating components, and provision must be made to securely hold the components in position. The primary means of locating the components is to position them against a shoulder of the shaft. A shoulder also provides a solid support to minimize deflection and vibration of the component. Sometimes when the magnitudes of the forces are reasonably low, shoulders can be constructed with retaining rings in grooves, sleeves between components, or clamp-on collars. In cases where axial loads are very small, it may be feasible to do without the shoulders entirely, and rely on press fits, pins, or collars with setscrews to maintain an axial location. See Figures 7–2*b* and 7–2*d* for examples of some of these means of axial location.

Supporting Axial Loads

In cases where axial loads are not trivial, it is necessary to provide a means to transfer the axial loads into the shaft, then through a bearing to the ground. This will be particularly necessary with helical or bevel gears, or tapered roller bearings, as each of these produces axial force components. Often, the same means of providing axial location, e.g., shoulders, retaining rings, and pins, will be used to also transmit the axial load into the shaft.

It is generally best to have only one bearing carry the axial load, to allow greater tolerances on shaft length dimensions, and to prevent binding if the shaft expands due to temperature changes. This is particularly important for long shafts. Figures 7–3 and 7–4 show examples of shafts with only one bearing carrying the axial load against a shoulder, while the other bearing is simply press-fit onto the shaft with no shoulder.

Providing for Torque Transmission

Most shafts serve to transmit torque from an input gear or pulley, through the shaft, to an output gear or pulley. Of course, the shaft itself must be sized to support the torsional stress and torsional deflection. It is also necessary to provide a means of transmitting the torque between the shaft and the gears. Common torque-transfer elements are:

- Keys
- Splines
- Setscrews
- Pins
- Press or shrink fits
- Tapered fits

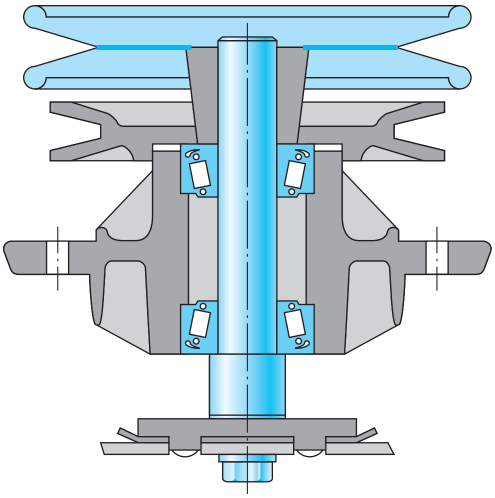


Figure 7–3

Tapered roller bearings used in a mowing machine spindle. This design represents good practice for the situation in which one or more torque-transfer elements must be mounted outboard. *(Source: Redrawn from material furnished by The Timken Company.)*

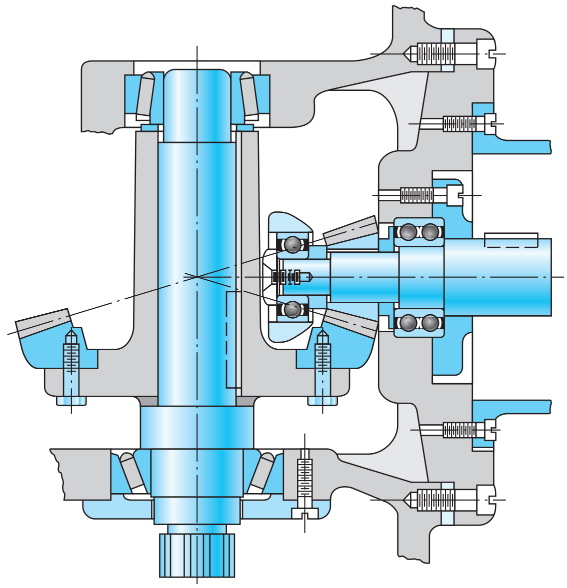


Figure 7–4

A bevel-gear drive in which both pinion and gear are straddle-mounted. *(Source: Redrawn from material furnished by Gleason Machine Division.)*

In addition to transmitting the torque, many of these devices are designed to fail if the torque exceeds acceptable operating limits, protecting more expensive components.

Details regarding hardware components such as *keys, pins,* and *setscrews* are addressed in detail in Section 7–7. One of the most effective and economical means of transmitting moderate to high levels of torque is through a key that fits in a groove in the shaft and gear. Keyed components generally have a slip fit onto the shaft, so assembly and disassembly is easy. The key provides for positive angular orientation of the component, which is useful in cases where phase angle timing is important.

Splines are essentially stubby gear teeth formed on the outside of the shaft and on the inside of the hub of the load-transmitting component. Splines are generally much more expensive to manufacture than keys, and are usually not necessary for simple torque transmission. They are typically used to transfer high torques. One feature of a spline is that it can be made with a reasonably loose slip fit to allow for large axial motion between the shaft and component while still transmitting torque. This is useful for connecting two shafts where relative motion between them is common, such as in connecting a power takeoff (PTO) shaft of a tractor to an implement. SAE and ANSI publish standards for splines. Stress-concentration factors are greatest where the spline ends and blends into the shaft, but are generally quite moderate.

For cases of low-torque transmission, various means of transmitting torque are available. These include pins, setscrews in hubs, tapered fits, and press fits.

Press and shrink fits for securing hubs to shafts are used both for torque transfer and for preserving axial location. The resulting stress-concentration factor is usually quite small. See Section 7–8 for guidelines regarding appropriate sizing and tolerancing to transmit torque with press and shrink fits. A similar method is to use a split hub with screws to clamp the hub to the shaft. This method allows for disassembly and lateral adjustments. Another similar method uses a two-part hub consisting of a split inner member that fits into a tapered hole. The assembly is then tightened to the shaft with screws, which forces the inner part into the wheel and clamps the whole assembly against the shaft.

Tapered fits between the shaft and the shaft-mounted device, such as a wheel, are often used on the overhanging end of a shaft. Screw threads at the shaft end then permit the use of a nut to lock the wheel tightly to the shaft. This approach is useful because it can be disassembled, but it does not provide good axial location of the wheel on the shaft.

At the early stages of the shaft layout, the important thing is to select an appropriate means of transmitting torque, and to determine how it affects the overall shaft layout. It is necessary to know where the shaft discontinuities, such as keyways, holes, and splines, will be in order to determine critical locations for analysis.

Assembly and Disassembly

Consideration should be given to the method of assembling the components onto the shaft, and the shaft assembly into the frame. This generally requires the largest diameter in the center of the shaft, with progressively smaller diameters toward the ends to allow components to be slid on from the ends. If a shoulder is needed on both sides of a component, one of them must be created by such means as a retaining ring or by a sleeve between two components. The gearbox itself will need means to physically position the shaft into its bearings, and the bearings into the frame. This is typically accomplished by providing access through the housing to the bearing at one end of the shaft. See Figures 7–5 through 7–8 for examples.

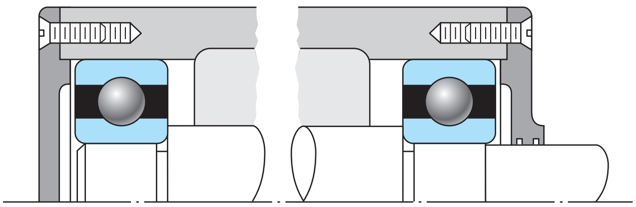


Figure 7–5

Arrangement showing bearing inner rings press-fitted to shaft while outer rings float in the housing. The axial clearance should be sufficient only to allow for machinery vibrations. Note the labyrinth seal on the right.

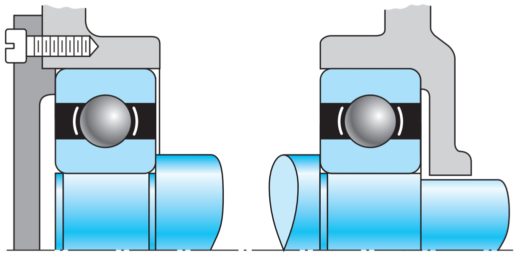


Figure 7–6

Similar to the arrangement of Figure 7–5 except that the outer bearing rings are preloaded.

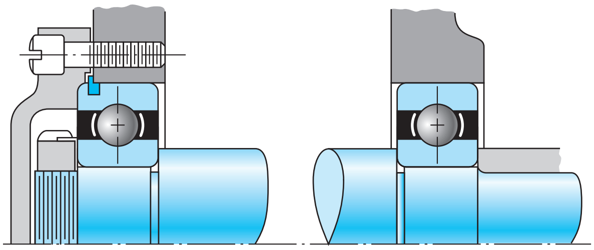


Figure 7–7

In this arrangement the inner ring of the left-hand bearing is locked to the shaft between a nut and a shaft shoulder. The locknut and washer are AFBMA standard. The snap ring in the outer race is used to positively locate the shaft assembly in the axial direction. Note the floating right-hand bearing and the grinding runout grooves in the shaft.

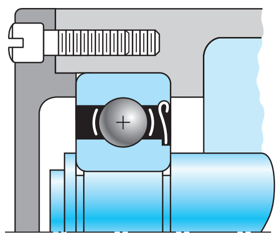


Figure 7–8

This arrangement is similar to Figure 7–7 in that the left-hand bearing positions the entire shaft assembly. In this case the inner ring is secured to the shaft using a snap ring. Note the use of a shield to prevent dirt generated from within the machine from entering the bearing.

When components are to be press-fit to the shaft, the shaft should be designed so that it is not necessary to press the component down a long length of shaft. This may require an extra change in diameter, but it will reduce manufacturing and assembly cost by only requiring the close tolerance for a short length.

Consideration should also be given to the necessity of disassembling the components from the shaft. This requires consideration of issues such as accessibility of retaining rings, space for pullers to access bearings, openings in the housing to allow pressing the shaft or bearings out, etc.

7–4 Shaft Design for Stress

Critical Locations

It is not necessary to evaluate the stresses in a shaft at every point; a few potentially critical locations will suffice. Critical locations will usually be on the outer surface, at axial locations where the bending moment is large, where the torque is present, and where stress concentrations exist. By direct comparison of various points along the shaft, a few critical locations can be identified upon which to base the design. An assessment of typical stress situations will help.

Most shafts will transmit torque through a portion of the shaft. Typically the torque comes into the shaft at one gear and leaves the shaft at another gear. A free body diagram of the shaft will allow the torque at any section to be determined. The torque is often relatively constant at steady state operation. The shear stress due to the torsion will be greatest on outer surfaces.

The bending moments on a shaft can be determined by shear and bending moment diagrams. Since most shaft problems incorporate gears or pulleys that introduce forces in two planes, the shear and bending moment diagrams will generally be needed in two planes. Resultant moments are obtained by summing moments as vectors at points of interest along the shaft. The phase angle of the moments is not important since the shaft rotates. A steady bending moment will produce a completely reversed moment on a rotating shaft, as a specific stress element will alternate from compression to tension in every revolution of the shaft. The normal stress due to bending moments will be greatest on the outer surfaces. In situations where a bearing is located at the end of the shaft, stresses near the bearing are often not critical since the bending moment is small.

Axial stresses on shafts due to the axial components transmitted through helical gears or tapered roller bearings will almost always be negligibly small compared to the bending moment stress. They are often also constant, so they contribute little to fatigue. Consequently, it is usually acceptable to neglect the axial stresses induced by the gears and bearings when bending is present in a shaft. If an axial load is applied to the shaft in some other way, it is not safe to assume it is negligible without checking magnitudes.

Shaft Stresses

Bending, torsion, and axial stresses may be present in both mean and alternating components. For analysis, it is simple enough to combine the different types of stresses into alternating and mean von Mises stresses, as shown in Section 6–16. It is sometimes convenient to customize the equations specifically for shaft applications. Axial loads are usually comparatively very small at critical locations where bending and torsion dominate, so they will be left out of the following equations. The fluctuating stresses due to bending and torsion are given by

$$\sigma_a = K_f \frac{M_a c}{I} \qquad \sigma_m = K_f \frac{M_m c}{I}$$
$$\tau_a = K_{fs} \frac{T_a r}{J} \qquad \tau_m = K_{fs} \frac{T_m r}{J} \tag{7–1}$$

where M_m and M_a are the mean and alternating bending moments, T_m and T_a are the mean and alternating torques, and K_f and K_{fs} are the fatigue stress-concentration factors for bending and torsion, respectively.

Assuming a solid shaft with round cross section, appropriate geometry terms can be introduced for c, I, r, and J resulting in

$$\sigma_a = K_f \frac{32M_a}{\pi d^3} \qquad \sigma_m = K_f \frac{32M_m}{\pi d^3} \tag{7–2}$$

$$\tau_a = K_{fs} \frac{16T_a}{\pi d^3} \qquad \tau_m = K_{fs} \frac{16T_m}{\pi d^3} \tag{7–3}$$

Using the distortion energy failure theory, the von Mises stress is given by Equation (5–15), with $\sigma_x = \sigma$, the bending stress, $\sigma_y = 0$, and $\tau_{xy} = \tau$, the torsional shear stress. Thus, for rotating round solid shafts, neglecting axial loads, the fluctuating von Mises stresses are given by

$$\sigma'_a = (\sigma_a^2 + 3\tau_a^2)^{1/2} = \left[\left(\frac{32K_f M_a}{\pi d^3} \right)^2 + 3 \left(\frac{16K_{fs} T_a}{\pi d^3} \right)^2 \right]^{1/2} \tag{7–4}$$

$$\sigma'_m = (\sigma_m^2 + 3\tau_m^2)^{1/2} = \left[\left(\frac{32K_f M_m}{\pi d^3} \right)^2 + 3 \left(\frac{16K_{fs} T_m}{\pi d^3} \right)^2 \right]^{1/2} \tag{7–5}$$

Note that the stress-concentration factors are sometimes considered optional for the mean components with ductile materials, because of the capacity of the ductile material to yield locally at the discontinuity.

Expressions can be obtained for any of the common failure criteria by substituting the von Mises stresses from Equations (7–4) and (7–5) into a failure criterion equation for any of the failure criteria from the fluctuating-stress diagram (Figure 6–36). The resulting equations for several of the commonly used failure curves are summarized below. For design purposes, it is also desirable to solve each equation for the diameter. The names given to each set of equations identifies the significant failure theory, followed by a fatigue failure locus name. For example, DE-Gerber indicates the stresses are combined using the distortion energy (DE) theory, and the Gerber criteria is used for the fatigue failure.

To keep the equations in simpler form, first establish a pair of terms to be used in each of the criteria equations.

$$A = \sqrt{4(K_f M_a)^2 + 3(K_{fs} T_a)^2}$$
$$B = \sqrt{4(K_f M_m)^2 + 3(K_{fs} T_m)^2} \tag{7–6}$$

DE-Goodman

$$n = \frac{\pi d^3}{16}\left(\frac{A}{S_e} + \frac{B}{S_{ut}}\right)^{-1} \tag{7–7}$$

$$d = \left[\frac{16n}{\pi}\left(\frac{A}{S_e} + \frac{B}{S_{ut}}\right)\right]^{1/3} \tag{7–8}$$

DE-Morrow

$$n = \frac{\pi d^3}{16}\left(\frac{A}{S_e} + \frac{B}{\tilde{\sigma}_f}\right)^{-1} \tag{7–9}$$

$$d = \left[\frac{16n}{\pi}\left(\frac{A}{S_e} + \frac{B}{\tilde{\sigma}_f}\right)\right]^{1/3} \tag{7–10}$$

DE-Gerber

$$\frac{1}{n} = \frac{8A}{\pi d^3 S_e}\left\{1 + \left[1 + \left(\frac{2BS_e}{AS_{ut}}\right)^2\right]^{1/2}\right\} \tag{7–11}$$

$$d = \left(\frac{8nA}{\pi S_e}\left\{1 + \left[1 + \left(\frac{2BS_e}{AS_{ut}}\right)^2\right]^{1/2}\right\}\right)^{1/3} \tag{7–12}$$

DE-SWT

$$n = \frac{\pi d^3}{16}\frac{S_e}{(A^2 + AB)^{1/2}} \tag{7–13}$$

$$d = \left[\frac{16n}{\pi S_e}(A^2 + AB)^{1/2}\right]^{1/3} \tag{7–14}$$

For a rotating shaft with constant bending and torsion, the bending stress is completely reversed and the torsion is steady. Equations (7–7) through (7–14) can be simplified by setting M_m and T_a equal to 0, which simply drops out some of the terms.

Note that in an analysis situation in which the diameter is known and the factor of safety is desired, as an alternative to using the specialized equations above, it is always still valid to calculate the alternating and mean stresses using Equations (7–4) and (7–5), and substitute them into one of the fatigue failure criterion equations from Section 6–13, and solve directly for n. In a design situation, however, having the equations pre-solved for diameter is quite helpful.

It is always necessary to consider the possibility of static failure in the first load cycle. A von Mises maximum stress is calculated for this purpose.

$$\begin{aligned}\sigma'_{\max} &= [(\sigma_m + \sigma_a)^2 + 3(\tau_m + \tau_a)^2]^{1/2}\\ &= \left[\left(\frac{32K_f(M_m + M_a)}{\pi d^3}\right)^2 + 3\left(\frac{16K_{fs}(T_m + T_a)}{\pi d^3}\right)^2\right]^{1/2}\end{aligned} \tag{7–15}$$

To check for yielding, this von Mises maximum stress is compared to the yield strength, as usual.

$$n_y = \frac{S_y}{\sigma'_{\max}} \tag{7–16}$$

For a quick, conservative check, an estimate for $\sigma'_{\max}$ can be obtained by simply adding σ'_a and σ'_m. $(\sigma'_a + \sigma'_m)$ will always be greater than or equal to $\sigma'_{\max}$, and will therefore be conservative.

EXAMPLE 7–1

At a machined shaft shoulder the small diameter d is 1.100 in, the large diameter D is 1.65 in, and the fillet radius is 0.11 in. The bending moment is 1260 lbf · in and the steady torsion moment is 1100 lbf · in. The heat-treated steel shaft has an ultimate strength of $S_{ut} = 105$ kpsi, a yield strength of $S_y = 82$ kpsi, and a true fracture strength of $\tilde{\sigma}_f = 155$ kpsi. The reliability goal for the endurance limit is 0.99.
(*a*) Determine the fatigue factor of safety of the design using each of the fatigue failure criteria described in this section.
(*b*) Determine the yielding factor of safety.

Solution

(*a*) $D/d = 1.65/1.100 = 1.50$, $r/d = 0.11/1.100 = 0.10$, $K_t = 1.68$ (Figure A–15–9), $K_{ts} = 1.42$ (Figure A–15–8), $q = 0.85$ (Figure 6–26), $q_{\text{shear}} = 0.88$ (Figure 6–27).

From Equation (6–32),

$$K_f = 1 + 0.85(1.68 - 1) = 1.58$$

$$K_{fs} = 1 + 0.88(1.42 - 1) = 1.37$$

Equation (6–10): $$S'_e = 0.5(105) = 52.5 \text{ kpsi}$$

Equation (6–18): $$k_a = 2.00(105)^{-0.217} = 0.729$$

Equation (6–19): $$k_b = \left(\frac{1.100}{0.30}\right)^{-0.107} = 0.870$$

$$k_c = k_d = k_f = 1$$

Table 6–4: $$k_e = 0.814$$

$$S_e = 0.729(0.870)0.814(52.5) = 27.1 \text{ kpsi}$$

For a rotating shaft, the constant bending moment will create a completely reversed bending stress.

$$M_a = 1260 \text{ lbf} \cdot \text{in} \qquad T_m = 1100 \text{ lbf} \cdot \text{in} \qquad M_m = T_a = 0$$

Applying Equations (7–6) and (7–7) for the DE-Goodman criteria gives

Equation (7–6): $$A = \sqrt{4(K_f M_a)^2} = \sqrt{4[(1.58)(1260)]^2} = 3981.6$$

$$B = \sqrt{3(K_{fs} T_m)^2} = \sqrt{3[(1.37)(1100)]^2} = 2610.2$$

Equation (7–7): $$n = \frac{\pi d^3}{16}\left(\frac{A}{S_e} + \frac{B}{S_{ut}}\right)^{-1} = \frac{\pi(1.1)^3}{16}\left(\frac{3981.6}{27100} + \frac{2610.2}{105000}\right)^{-1} = 1.52$$

Answer $n = 1.52$ DE-Goodman

Similarly, applying Equations (7–9), (7–11), and (7–13) for the other failure criteria,

Answer $n = 1.60$ DE-Morrow

Answer $n = 1.73$ DE-Gerber

Answer $n = 1.38$ DE-SWT

For comparison, consider an equivalent approach of calculating the stresses and applying the fatigue failure criteria directly. From Equations (7–4) and (7–5),

$$\sigma'_a = \left[\left(\frac{32 \cdot 1.58 \cdot 1260}{\pi(1.1)^3}\right)^2\right]^{1/2} = 15\,235 \text{ psi}$$

$$\sigma'_m = \left[3\left(\frac{16 \cdot 1.37 \cdot 1100}{\pi(1.1)^3}\right)^2\right]^{1/2} = 9988 \text{ psi}$$

Taking, for example, the Goodman failure critera, application of Equation (6–41) gives

$$\frac{1}{n} = \frac{\sigma'_a}{S_e} + \frac{\sigma'_m}{S_{ut}} = \frac{15\,235}{27\,100} + \frac{9988}{105\,000} = 0.657$$

$$n = 1.52$$

which is identical with the previous result. The same process could be used for the other failure criteria.

(*b*) For the yielding factor of safety, determine an equivalent von Mises maximum stress using Equation (7–15).

$$\sigma'_{\max} = \left[\left(\frac{32(1.58)(1260)}{\pi(1.1)^3}\right)^2 + 3\left(\frac{16(1.37)(1100)}{\pi(1.1)^3}\right)^2\right]^{1/2} = 18\,220 \text{ psi}$$

Answer

$$n_y = \frac{S_y}{\sigma'_{\max}} = \frac{82\,000}{18\,220} = 4.50$$

For comparison, a quick and very conservative check on yielding can be obtained by replacing $\sigma'_{\max}$ with $\sigma'_a + \sigma'_m$. This just saves the extra time of calculating $\sigma'_{\max}$ if σ'_a and σ'_m have already been determined. For this example,

$$n_y = \frac{S_y}{\sigma'_a + \sigma'_m} = \frac{82\,000}{15\,235 + 9988} = 3.25$$

which is quite conservative compared with $n_y = 4.50$.

Estimating Stress Concentrations

The stress analysis process for fatigue is highly dependent on stress concentrations. Stress concentrations for shoulders and keyways are dependent on size specifications that are not known the first time through the process. Fortunately, since these elements are usually of standard proportions, it is possible to estimate the stress-concentration factors for initial design of the shaft. These stress concentrations will be fine-tuned in successive iterations, once the details are known.

Shoulders for bearing and gear support should match the catalog recommendation for the specific bearing or gear. A look through bearing catalogs shows that a typical bearing calls for the ratio of D/d to be between 1.2 and 1.5. For a first approximation, the worst case of 1.5 can be assumed. Similarly, the fillet radius at the shoulder needs to be sized to avoid interference with the fillet radius of the mating component. There is a significant variation in typical bearings in the ratio of fillet radius versus bore diameter, with r/d typically ranging from around 0.02 to 0.06. A quick look at the stress concentration charts (Figures A–15–8 and A–15–9) shows that the stress concentrations for bending and torsion increase significantly in this range. For example, with $D/d = 1.5$ for bending, $K_t = 2.7$ at $r/d = 0.02$, and reduces to $K_t = 2.1$ at

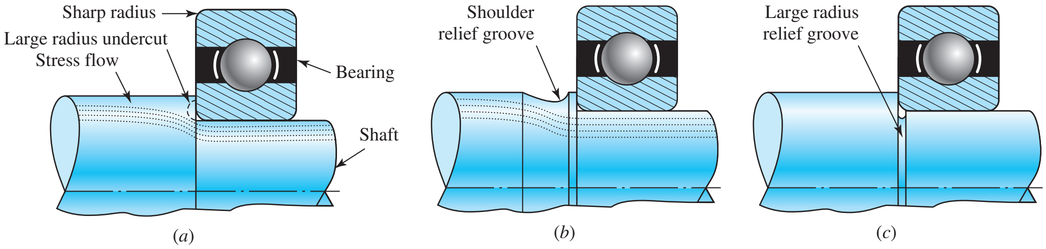


Figure 7–9

Techniques for reducing stress concentration at a shoulder supporting a bearing with a sharp radius. (*a*) Large radius undercut into the shoulder. (*b*) Large radius relief groove into the back of the shoulder. (*c*) Large radius relief groove into the small diameter.

$r/d = 0.05$, and further down to $K_t = 1.7$ at $r/d = 0.1$. This indicates that this is an area where some attention to detail could make a significant difference. Fortunately, in most cases the shear and bending moment diagrams show that bending moments are quite low near the bearings, since the bending moments from the ground reaction forces are small.

In cases where the shoulder at the bearing is found to be critical, the designer should plan to select a bearing with generous fillet radius, or consider providing for a larger fillet radius on the shaft by relieving it into the base of the shoulder as shown in Figure 7–9*a*. This effectively creates a dead zone in the shoulder area that does not carry the bending stresses, as shown by the stress flow lines. A shoulder relief groove as shown in Figure 7–9*b* can accomplish a similar purpose. Another option is to cut a large-radius relief groove into the small diameter of the shaft, as shown in Figure 7–9*c*. This has the disadvantage of reducing the cross-sectional area, but is often used in cases where it is useful to provide a relief groove before the shoulder to prevent the grinding or turning operation from having to go all the way to the shoulder.

For the standard shoulder fillet, for estimating K_t values for the first iteration, an r/d ratio should be selected so K_t values can be obtained. For the worst end of the spectrum, with $r/d = 0.02$ and $D/d = 1.5$, K_t values from the stress concentration charts for shoulders indicate 2.7 for bending, 2.2 for torsion, and 3.0 for axial.

A keyway will produce a stress concentration near a critical point where the load-transmitting component is located. The stress concentration in an end-milled keyseat is a function of the ratio of the radius r at the bottom of the groove and the shaft diameter d. For early stages of the design process, it is possible to estimate the stress concentration for keyways regardless of the actual shaft dimensions by assuming a typical ratio of $r/d = 0.02$. This gives $K_t = 2.14$ for bending and $K_{ts} = 3.0$ for torsion, assuming the key is in place.

Figures A–15–16 and A–15–17 give values for stress concentrations for flat-bottomed grooves such as used for retaining rings. By examining typical retaining ring specifications in vendor catalogs, it can be seen that the groove width is typically slightly greater than the groove depth, and the radius at the bottom of the groove is around 1/10 of the groove width. From Figures A–15–16 and A–15–17, stress-concentration factors for typical retaining ring dimensions are around 5 for bending and axial, and 3 for torsion. Fortunately, the small radius will often lead to a smaller notch sensitivity, reducing K_f.

Table 7–1 summarizes some typical stress-concentration factors for the first iteration in the design of a shaft. Similar estimates can be made for other features. The

Table 7–1 First Iteration Estimates for Stress-Concentration Factors K_t and K_{ts}

Warning: These factors are only estimates for use when actual dimensions are not yet determined. Do *not* use these once actual dimensions are available.

	Bending	Torsional	Axial
Shoulder fillet—sharp ($r/d = 0.02$)	2.7	2.2	3.0
Shoulder fillet—well rounded ($r/d = 0.1$)	1.7	1.5	1.9
End-mill keyseat ($r/d = 0.02$)	2.14	3.0	—
Sled runner keyseat	1.7	—	—
Retaining ring groove	5.0	3.0	5.0

Missing values in the table are not readily available.

point is to notice that stress concentrations are essentially normalized so that they are dependent on ratios of geometry features, not on the specific dimensions. Consequently, by estimating the appropriate ratios, the first iteration values for stress concentrations can be obtained. These values can be used for initial design, then actual values inserted once diameters have been determined.

EXAMPLE 7–2

This example problem is part of a larger case study. See Chapter 18 for the full context.

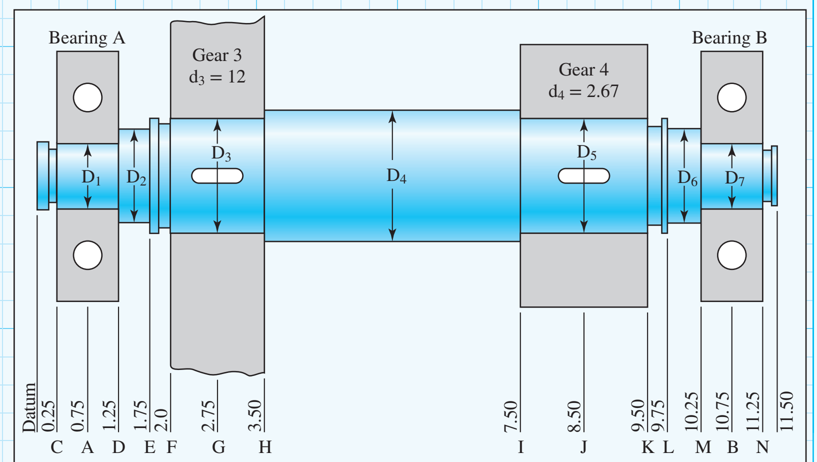
A double reduction gearbox design has developed to the point that the general layout and axial dimensions of the countershaft carrying two spur gears has been proposed, as shown in Figure 7–10. The gears and bearings are located and supported by shoulders, and held in place by retaining rings. The gears transmit torque through keys. Gears have been specified as shown, allowing the tangential and radial forces transmitted through the gears to the shaft to be determined as follows.

$$W_{23}^t = 540 \text{ lbf} \qquad W_{54}^t = 2431 \text{ lbf}$$

$$W_{23}^r = 197 \text{ lbf} \qquad W_{54}^r = 885 \text{ lbf}$$

Figure 7–10

Shaft layout for Example 7–2. Dimensions in inches.



where the superscripts t and r represent tangential and radial directions, respectively; and the subscripts 23 and 54 represent the forces exerted by gears 2 and 5 (not shown) on gears 3 and 4, respectively.

Proceed with the next phase of the design, in which a suitable material is selected, and appropriate diameters for each section of the shaft are estimated, based on providing sufficient fatigue and static stress capacity for infinite life of the shaft, with minimum safety factors of 1.5.

Solution

Perform free body diagram analysis to get reaction forces at the bearings.

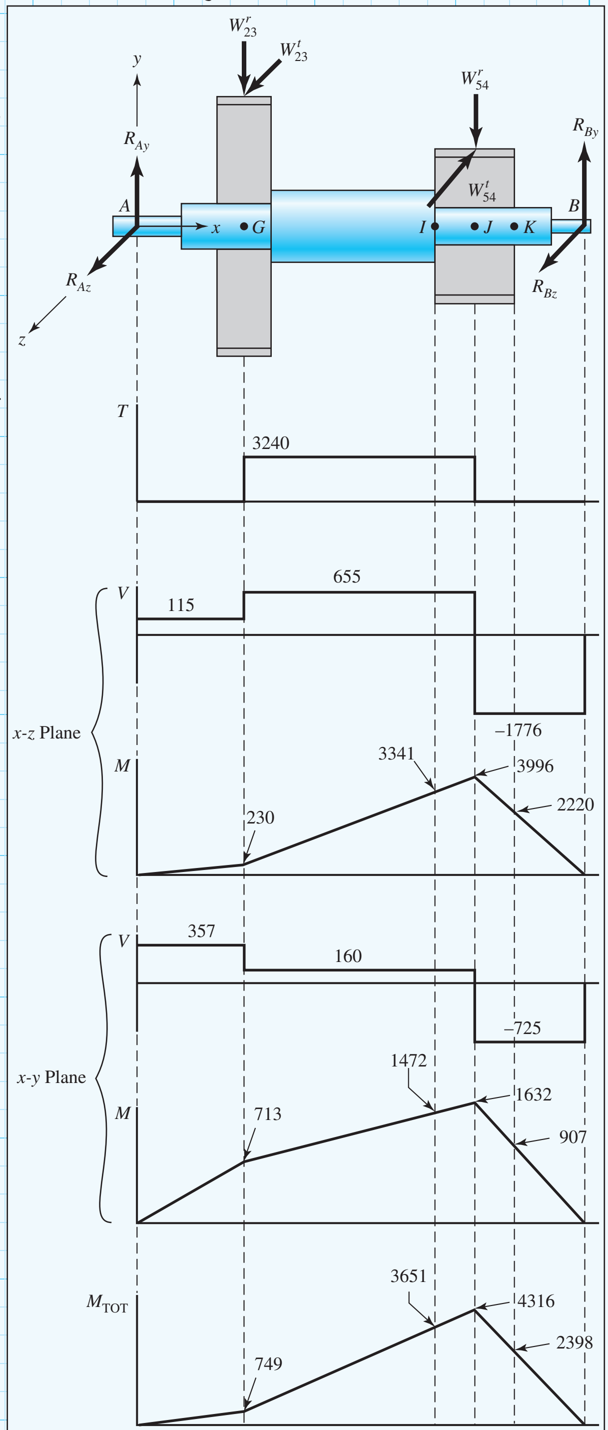
$$R_{Az} = 115.0 \text{ lbf}$$
$$R_{Ay} = 356.7 \text{ lbf}$$
$$R_{Bz} = 1776.0 \text{ lbf}$$
$$R_{By} = 725.3 \text{ lbf}$$

From ΣM_x, find the torque in the shaft between the gears,

$T = W^t_{23}(d_3/2) = 540(12/2) = 3240 \text{ lbf} \cdot \text{in.}$

Generate shear-moment diagrams for two planes.

Combine orthogonal planes as vectors to get total moments, e.g., at J, $\sqrt{3996^2 + 1632^2} = 4316 \text{ lbf} \cdot \text{in.}$



Start with point I, where the bending moment is high, there is a stress concentration at the shoulder, and the torque is present.

$$\text{At } I,\ M_a = 3651 \text{ lbf} \cdot \text{in},\ T_m = 3240 \text{ lbf} \cdot \text{in},\ M_m = T_a = 0$$

Assume generous fillet radius for gear at I.

From Table 7-1, estimate $K_t = 1.7$, $K_{ts} = 1.5$. For quick, conservative first pass, assume $K_f = K_t$, $K_{fs} = K_{ts}$.

Choose inexpensive steel, 1020 CD, with $S_{ut} = 68$ kpsi. For S_e,

Equation (6–18) $$k_a = aS_{ut}^b = 2.00(68)^{-0.217} = 0.801$$

Guess $k_b = 0.9$. Check later when d is known.

$$k_c = k_d = k_e = 1$$

Equation (6–17) $$S_e = (0.801)(0.9)(0.5)(68) = 24.5 \text{ kpsi}$$

For first estimate of the small diameter at the shoulder at point I, use the DE-Goodman criterion of Equation (7-8). This criterion is good for the initial design, since it is simple and conservative. With $M_m = T_a = 0$, Equations (7-6) and (7-8) reduce to

$$d = \left\{ \frac{16n}{\pi} \left(\frac{2(K_f M_a)}{S_e} + \frac{[3(K_{fs} T_m)^2]^{1/2}}{S_{ut}} \right) \right\}^{1/3}$$

$$d = \left\{ \frac{16(1.5)}{\pi} \left(\frac{2(1.7)(3651)}{24\,500} + \frac{\{3[(1.5)(3240)]^2\}^{1/2}}{68\,000} \right) \right\}^{1/3}$$

$$d = 1.69 \text{ in}$$

All estimates have probably been conservative, so select the next standard size below 1.69 in and check, $d = 1.625$ in.

A typical D/d ratio for support at a shoulder is $D/d = 1.2$, thus, $D = 1.2(1.625) = 1.95$ in. Increase to $D = 2.0$ in. A nominal 2-in. cold-drawn shaft diameter can be used. Check if estimates were acceptable.

$$D/d = 2/1.625 = 1.23$$

Assume fillet radius $r = d/10 = 0.16$ in, $\quad r/d = 0.1$

$$K_t = 1.6 \text{ (Fig. A–15–9)},\ q = 0.82 \text{ (Fig. 6–26)}$$

Equation (6–32) $$K_f = 1 + 0.82(1.6 - 1) = 1.49$$

$$K_{ts} = 1.35 \text{ (Fig. A–15–8)},\ q_s = 0.85 \text{ (Fig. 6–27)}$$

$$K_{fs} = 1 + 0.85(1.35 - 1) = 1.30$$

$$k_a = 0.801 \text{ (no change)}$$

Equation (6–19) $$k_b = \left(\frac{1.625}{0.3} \right)^{-0.107} = 0.835$$

$$S_e = (0.801)(0.835)(0.5)(68) = 22.7 \text{ kpsi}$$

Equation (7–4) $$\sigma'_a = \frac{32K_f M_a}{\pi d^3} = \frac{32(1.49)(3651)}{\pi(1.625)^3} = 12\,910 \text{ psi}$$

Equation (7–5) $$\sigma'_m = \left[3 \left(\frac{16K_{fs} T_m}{\pi d^3} \right)^2 \right]^{1/2} = \frac{\sqrt{3}(16)(1.30)(3240)}{\pi(1.625)^3} = 8659 \text{ psi}$$

Using Goodman criterion

$$\frac{1}{n_f} = \frac{\sigma'_a}{S_e} + \frac{\sigma'_m}{S_{ut}} = \frac{12\,910}{22\,700} + \frac{8659}{68\,000} = 0.696$$

$$n_f = 1.44$$

Note that we could have used Equation (7–7) directly.

Check yielding.

$$n_y = \frac{S_y}{\sigma'_{\max}} > \frac{S_y}{\sigma'_a + \sigma'_m} = \frac{57\,000}{12\,910 + 8659} = 2.64$$

Also check this diameter at the end of the keyway, just to the right of point I, and at the groove at point K. From moment diagram, estimate M at end of keyway to be $M = 3750$ lbf-in.

Assume the radius at the bottom of the keyway will be the standard $r/d = 0.02$, $r = 0.02d = 0.02(1.625) = 0.0325$ in.

$$K_t = 2.14 \text{ (Table 7–1)}, \; q = 0.65 \text{ (Figure 6–26)}$$

$$K_f = 1 + 0.65(2.14 - 1) = 1.74$$

$$K_{ts} = 3.0 \text{ (Table 7–1)}, \; q_s = 0.71 \text{ (Figure 6–27)}$$

$$K_{fs} = 1 + 0.71(3 - 1) = 2.42$$

$$\sigma'_a = \frac{32K_f M_a}{\pi d^3} = \frac{32(1.74)(3750)}{\pi(1.625)^3} = 15\,490 \text{ psi}$$

$$\sigma'_m = \sqrt{3}(16)\frac{K_{fs}T_m}{\pi d^3} = \frac{\sqrt{3}(16)(2.42)(3240)}{\pi(1.625)^3} = 16\,120 \text{ psi}$$

$$\frac{1}{n_f} = \frac{\sigma'_a}{S_e} + \frac{\sigma'_m}{S_{ut}} = \frac{15\,490}{22\,700} + \frac{16\,120}{68\,000} = 0.919$$

$$n_f = 1.09$$

The keyway turns out to be more critical than the shoulder. We can either increase the diameter or use a higher strength material. Unless the deflection analysis shows a need for larger diameters, let us choose to increase the strength. We started with a very low strength and can afford to increase it some to avoid larger sizes. Try 1050 CD with $S_{ut} = 100$ kpsi.

Recalculate factors affected by S_{ut}, i.e., $k_a \rightarrow S_e$; $q \rightarrow K_f \rightarrow \sigma'_a$

$$k_a = 2.00(100)^{-0.217} = 0.736, \quad S_e = 0.736(0.835)(0.5)(100) = 30.7 \text{ kpsi}$$

$$q = 0.72, \; K_f = 1 + 0.72(2.14 - 1) = 1.82$$

$$\sigma'_a = \frac{32(1.82)(3750)}{\pi(1.625)^3} = 16\,200 \text{ psi}$$

$$\frac{1}{n_f} = \frac{16\,200}{30\,700} + \frac{16\,120}{100\,000} = 0.689$$

$$n_f = 1.45$$

This is slightly under the goal for the design factor to be 1.5. If we round to 2 significant figures, which is actually more realistic for fatigue, then we get 1.5. If the situation calls for a more conservative decision, a higher strength material can be selected.

Check at the groove at K, since K_t for flat-bottomed grooves are often very high. From the torque diagram, note that no torque is present at the groove. From the moment diagram, $M_a = 2398$ lbf · in,

$M_m = T_a = T_m = 0$. To quickly check if this location is potentially critical, just use $K_f = K_t = 5.0$ as an estimate, from Table 7-1.

$$\sigma_a = \frac{32K_f M_a}{\pi d^3} = \frac{32(5)(2398)}{\pi(1.625)^3} = 28\,460 \text{ psi}$$

$$n_f = \frac{S_e}{\sigma_a} = \frac{30\,700}{28\,460} = 1.08$$

This is low. We will look up data for a specific retaining ring to obtain K_f more accurately. With a quick online search of a retaining ring specification using the website www.globalspec.com, appropriate groove specifications for a retaining ring for a shaft diameter of 1.625 in are obtained as follows: width, $a = 0.068$ in; depth, $t = 0.048$ in; and corner radius at bottom of groove, $r = 0.01$ in. From Figure A-15-16, with $r/t = 0.01/0.048 = 0.208$, and $a/t = 0.068/0.048 = 1.42$

$$K_t = 4.3,\ q = 0.65 \text{ (Figure 6–26)}$$

$$K_f = 1 + 0.65(4.3 - 1) = 3.15$$

$$\sigma_a = \frac{32K_f M_a}{\pi d^3} = \frac{32(3.15)(2398)}{\pi(1.625)^3} = 17\,930 \text{ psi}$$

$$n_f = \frac{S_e}{\sigma_a} = \frac{30\,700}{17\,930} = 1.71$$

Quickly check if point M might be critical. Only bending is present, and the moment is small, but the diameter is small and the stress concentration is high for a sharp fillet required for a bearing. From the moment diagram, $M_a = 959$ lbf · in, and $M_m = T_m = T_a = 0$.

Estimate $K_t = 2.7$ from Table 7-1, $d = 1.0$ in, and fillet radius r to fit a typical bearing.

$$r/d = 0.02,\ r = 0.02(1) = 0.02$$

$$q = 0.7 \text{ (Figure 6–26)}$$

$$K_f = 1 + (0.7)(2.7 - 1) = 2.19$$

$$\sigma_a = \frac{32K_f M_a}{\pi d^3} = \frac{32(2.19)(959)}{\pi(1)^3} = 21\,390 \text{ psi}$$

$$n_f = \frac{S_e}{\sigma_a} = \frac{30\,700}{21\,390} = 1.44$$

This location is more critical than perhaps anticipated. The estimate for stress concentration is likely on the high side, so we will choose to continue and recheck after a specific bearing is selected.

With the diameters specified for the critical locations, fill in trial values for the rest of the diameters, taking into account typical shoulder heights for bearing and gear support.

$$D_1 = D_7 = 1.0 \text{ in}$$
$$D_2 = D_6 = 1.4 \text{ in}$$
$$D_3 = D_5 = 1.625 \text{ in}$$
$$D_4 = 2.0 \text{ in}$$

The bending moments are much less on the left end of the shaft, so D_1, D_2, and D_3 could be smaller. However, unless weight is an issue, there is little advantage to requiring more material removal. Also, the extra rigidity may be needed to keep deflections small.

7–5 Deflection Considerations

Deflection analysis at even a single point of interest requires complete geometry information for the entire shaft. For this reason, it is desirable to design the dimensions at critical locations to handle the stresses, and fill in reasonable estimates for all other dimensions, before performing a deflection analysis. Deflection of the shaft, both linear and angular, should be checked at gears and bearings. Allowable deflections will depend on many factors, and bearing and gear catalogs should be used for guidance on allowable misalignment for specific bearings and gears. As a rough guideline, typical ranges for maximum slopes and transverse deflections of the shaft centerline are given in Table 7–2. The allowable transverse deflections for spur gears are dependent on the size of the teeth, as represented by the diametral pitch P, which equals the number of teeth divided by the pitch diameter.

In Section 4–4 several beam deflection methods are described. For shafts, where the deflections may be sought at a number of different points, integration using either singularity functions or numerical integration is practical. In a stepped shaft, the cross-sectional properties change along the shaft at each step, increasing the complexity of integration, since both M and I vary. Fortunately, only the gross geometric dimensions need to be included, as the local factors such as fillets, grooves, and keyways do not have much impact on deflection. Example 4–7 demonstrates the use of singularity functions for a stepped shaft. Many shafts will include forces in multiple planes, requiring either a three-dimensional analysis, or the use of superposition to obtain deflections in two planes which can then be summed as vectors.

A deflection analysis is straightforward, but it is lengthy and tedious to carry out manually, particularly for multiple points of interest. Consequently, practically all shaft deflection analysis will be evaluated with the assistance of software. Any general-purpose finite-element software can readily handle a shaft problem (see Chapter 19). This is practical if the designer is already familiar with using the software and with how to properly model the shaft. Special-purpose software solutions for 3-D shaft analysis are available, but somewhat expensive if only used occasionally. Software requiring very little training is readily available for planar beam analysis, and can be downloaded from the Internet. Example 7–3 demonstrates how to incorporate such a program for a shaft with forces in multiple planes.

Table 7–2 **Typical Maximum Ranges for Slopes and Transverse Deflections**

Slopes	
Tapered roller	0.0005–0.0012 rad
Cylindrical roller	0.0008–0.0012 rad
Deep-groove ball	0.001–0.003 rad
Spherical ball	0.026–0.052 rad
Self-align ball	0.026–0.052 rad
Uncrowned spur gear	$<$0.0005 rad
Transverse Deflections	
Spur gears with $P < 10$ teeth/in	0.010 in
Spur gears with $11 < P < 19$	0.005 in
Spur gears with $20 < P < 50$	0.003 in

EXAMPLE 7–3

This example problem is part of a larger case study. See Chapter 18 for the full context.

In Example 7-2, a preliminary shaft geometry was obtained on the basis of design for stress. The resulting shaft is shown in Figure 7-10, with proposed diameters of

$$D_1 = D_7 = 1 \text{ in}$$
$$D_2 = D_6 = 1.4 \text{ in}$$
$$D_3 = D_5 = 1.625 \text{ in}$$
$$D_4 = 2.0 \text{ in}$$

Check that the deflections and slopes at the gears and bearings are acceptable. If necessary, propose changes in the geometry to resolve any problems.

Solution

A simple planar beam analysis program will be used. By modeling the shaft twice, with loads in two orthogonal planes, and combining the results, the shaft deflections can readily be obtained. For both planes, the material is selected (steel with $E = 30$ Mpsi), the shaft lengths and diameters are entered, and the bearing locations are specified. Local details like grooves and keyways are ignored, as they will have insignificant effect on the deflections. Then the tangential gear forces are entered in the horizontal xz plane model, and the radial gear forces are entered in the vertical xy plane model. The software can calculate the bearing reaction forces, and numerically integrate to generate plots for shear, moment, slope, and deflection, as shown in Figure 7-11.

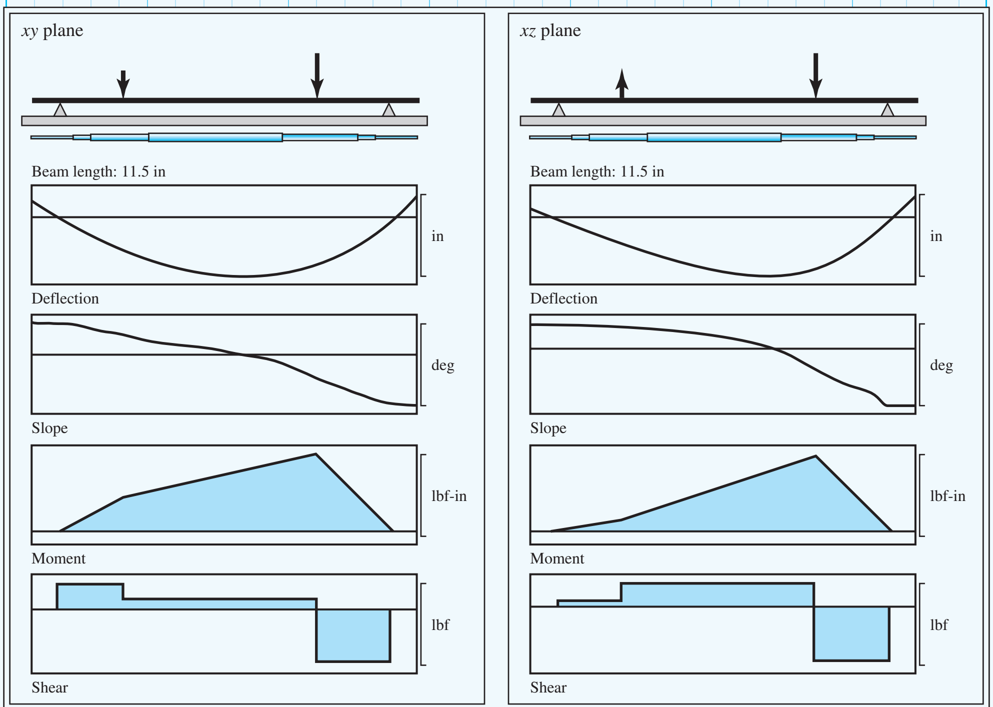


Figure 7–11

Shear, moment, slope, and deflection plots from two planes. *(Source: Beam 2D Stress Analysis, Orand Systems, Inc.)*

Table 7–3 Slope and Deflection Values at Key Locations

Point of Interest	*xz* Plane	*xy* Plane	Total
Left bearing slope	0.02263 deg	0.01770 deg	0.02872 deg 0.000501 rad
Right bearing slope	0.05711 deg	0.02599 deg	0.06274 deg 0.001095 rad
Left gear slope	0.02067 deg	0.01162 deg	0.02371 deg 0.000414 rad
Right gear slope	0.02155 deg	0.01149 deg	0.02442 deg 0.000426 rad
Left gear deflection	0.0007568 in	0.0005153 in	0.0009155 in
Right gear deflection	0.0015870 in	0.0007535 in	0.0017567 in

The deflections and slopes at points of interest are obtained from the plots, and combined with orthogonal vector addition, that is, $\delta = \sqrt{\delta_{xz}^2 + \delta_{xy}^2}$. Results are shown in Table 7-3.

Whether these values are acceptable will depend on the specific bearings and gears selected, as well as the level of performance expected. According to the guidelines in Table 7-2, all of the bearing slopes are well below typical limits for ball bearings. The right bearing slope is within the typical range for cylindrical bearings. Since the load on the right bearing is relatively high, a cylindrical bearing might be used. This constraint should be checked against the specific bearing specifications once the bearing is selected.

The gear slopes and deflections more than satisfy the limits recommended in Table 7-2. It is recommended to proceed with the design, with an awareness that changes that reduce rigidity should warrant another deflection check.

Once deflections at various points have been determined, if any value is larger than the allowable deflection at that point, a larger shaft diameter is warranted. Since I is proportional to d^4, a new diameter can be found from

$$d_{\text{new}} = d_{\text{old}} \left| \frac{n_d y_{\text{old}}}{y_{\text{all}}} \right|^{1/4} \tag{7–17}$$

where y_{all} is the allowable deflection at that station and n_d is the design factor. Similarly, if any slope is larger than the allowable slope θ_{all}, a new diameter can be found from

$$d_{\text{new}} = d_{\text{old}} \left| \frac{n_d (dy/dx)_{\text{old}}}{(\text{slope})_{\text{all}}} \right|^{1/4} \tag{7–18}$$

where $(\text{slope})_{\text{all}}$ is the allowable slope. As a result of these calculations, determine the largest $d_{\text{new}}/d_{\text{old}}$ ratio, then multiply *all* diameters by this ratio. The tight constraint will be just tight, and all others will be loose. Don't be too concerned about end journal sizes, as their influence is usually negligible. The beauty of the method is that the deflections need to be completed just once and constraints can be rendered loose but for one, with diameters all identified without reworking every deflection.

EXAMPLE 7–4

For the shaft in Example 7–3, it was noted that the slope at the right bearing is near the limit for a cylindrical roller bearing. Determine an appropriate increase in diameters to bring this slope down to 0.0005 rad.

Solution

Applying Equation (7–17) to the deflection at the right bearing gives

$$d_{\text{new}} = d_{\text{old}} \left| \frac{n_d \text{slope}_{\text{old}}}{\text{slope}_{\text{all}}} \right|^{1/4} = 1.0 \left| \frac{(1)(0.001095)}{(0.0005)} \right|^{1/4} = 1.216 \text{ in}$$

Multiplying all diameters by the ratio

$$\frac{d_{\text{new}}}{d_{\text{old}}} = \frac{1.216}{1.0} = 1.216$$

gives a new set of diameters,

$$D_1 = D_7 = 1.216 \text{ in}$$

$$D_2 = D_6 = 1.702 \text{ in}$$

$$D_3 = D_5 = 1.976 \text{ in}$$

$$D_4 = 2.432 \text{ in}$$

Repeating the beam deflection analysis of Example 7–3 with these new diameters produces a slope at the right bearing of 0.0005 in, with all other deflections less than their previous values.

The transverse shear V at a section of a beam in flexure imposes a shearing deflection, which is superposed on the bending deflection. Usually such shearing deflection is less than 1 percent of the transverse bending deflection, and it is seldom evaluated. However, when the shaft length-to-diameter ratio is less than 10, the shear component of transverse deflection merits attention. There are many short shafts. A tabular method is explained in detail elsewhere,[2] including examples.

For right-circular cylindrical shafts in torsion the angular deflection θ is given in Equation (4–5). For a stepped shaft with individual cylinder length l_i and torque T_i, the angular deflection can be estimated from

$$\theta = \sum \theta_i = \sum \frac{T_i l_i}{G_i J_i} \tag{7–19}$$

or, for a constant torque throughout homogeneous material, from

$$\theta = \frac{T}{G} \sum \frac{l_i}{J_i} \tag{7–20}$$

This should be treated only as an estimate, since experimental evidence shows that the actual θ is larger than given by Equations (7–19) and (7–20).[3]

[2]C. R. Mischke, "Tabular Method for Transverse Shear Deflection," Section 17.3 in Joseph E. Shigley, Charles R. Mischke, and Thomas H. Brown, Jr. (eds.), *Standard Handbook of Machine Design,* 3rd ed., McGraw-Hill, New York, 2004.

[3]R. Bruce Hopkins, *Design Analysis of Shafts and Beams,* McGraw-Hill, New York, 1970, pp. 93–99.

If torsional stiffness is defined as $k_i = T_i/\theta_i$ and, since $\theta_i = T_i/k_i$ and $\theta = \Sigma\theta_i = \Sigma(T_i/k_i)$, for constant torque $\theta = T\,\Sigma(1/k_i)$, it follows that the torsional stiffness of the shaft k in terms of segment stiffnesses is

$$\frac{1}{k} = \sum \frac{1}{k_i} \tag{7–21}$$

7–6 Critical Speeds for Shafts

When a shaft is turning, eccentricity causes a centrifugal force deflection, which is resisted by the shaft's flexural rigidity EI. As long as deflections are small, no harm is done. Another potential problem, however, is called *critical speeds:* at certain speeds the shaft is unstable, with deflections increasing without upper bound. It is fortunate that although the dynamic deflection shape is unknown, using a static deflection curve gives an excellent estimate of the lowest critical speed. Such a curve meets the boundary condition of the differential equation (zero moment and deflection at both bearings) and the shaft energy is not particularly sensitive to the exact shape of the deflection curve. Designers seek first critical speeds at least twice the operating speed.

The shaft, because of its own mass, has a critical speed. The ensemble of attachments to a shaft likewise has a critical speed that is much lower than the shaft's intrinsic critical speed. Estimating these critical speeds (and harmonics) is a task of the designer. When geometry is simple, as in a shaft of uniform diameter, simply supported, the task is easy. It can be expressed[4] as

$$\omega_1 = \left(\frac{\pi}{l}\right)^2 \sqrt{\frac{EI}{m}} = \left(\frac{\pi}{l}\right)^2 \sqrt{\frac{gEI}{A\gamma}} \tag{7–22}$$

where m is the mass per unit length, A the cross-sectional area, and γ the specific weight. For an ensemble of attachments, Rayleigh's method for lumped masses gives[5]

$$\omega_1 = \sqrt{\frac{g\,\Sigma w_i y_i}{\Sigma w_i y_i^2}} \tag{7–23}$$

where w_i is the weight of the ith location and y_i is the deflection at the ith body location. It is possible to use Equation (7–23) for the case of Equation (7–22) by partitioning the shaft into segments and placing its weight force at the segment centroid as seen in Figure 7–12. Computer assistance is often used to lessen the difficulty in

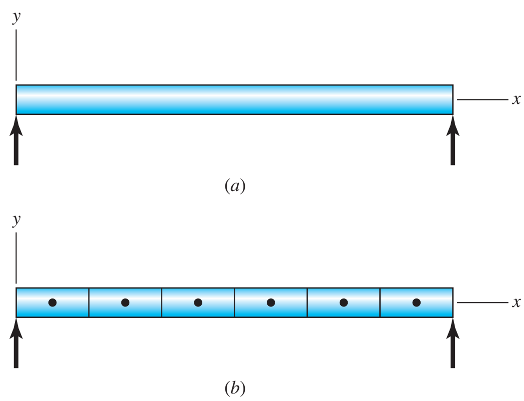


Figure 7–12

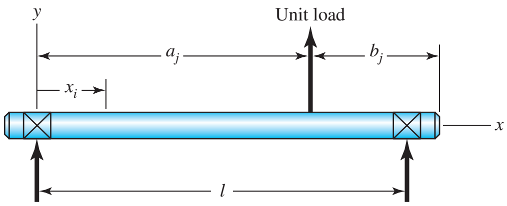
(*a*) A uniform-diameter shaft for Equation (7–22). (*b*) A segmented uniform-diameter shaft for Equation (7–23).

[4]William T. Thomson and Marie Dillon Dahleh, *Theory of Vibration with Applications,* Prentice Hall, 5th ed., 1998, p. 273.

[5]Thomson, op. cit., p. 357.

Figure 7–13

The influence coefficient δ_{ij} is the deflection at i due to a unit load at j.



finding transverse deflections of a stepped shaft. Rayleigh's equation overestimates the critical speed.

To counter the increasing complexity of detail, we adopt a useful viewpoint. Inasmuch as the shaft is an elastic body, we can use *influence coefficients*. An influence coefficient is the transverse deflection at location i on a shaft due to a unit load at location j on the shaft. From Table A–9–6 we obtain, for a simply supported beam with a single unit load as shown in Figure 7–13,

$$\delta_{ij} = \begin{cases} \dfrac{b_j x_i}{6EIl}(l^2 - b_j^2 - x_i^2) & x_i \le a_i \\ \dfrac{a_j(l - x_i)}{6EIl}(2lx_i - a_j^2 - x_i^2) & x_i > a_i \end{cases} \tag{7–24}$$

For three loads the influence coefficients may be displayed as

	j		
i	**1**	**2**	**3**
1	δ_{11}	δ_{12}	δ_{13}
2	δ_{21}	δ_{22}	δ_{23}
3	δ_{31}	δ_{32}	δ_{33}

Maxwell's reciprocity theorem[6] states that there is a symmetry about the main diagonal, composed of δ_{11}, δ_{22}, and δ_{33}, of the form $\delta_{ij} = \delta_{ji}$. This relation reduces the work of finding the influence coefficients. From the influence coefficients above, one can find the deflections y_1, y_2, and y_3 of Equation (7–23) as follows:

$$\begin{aligned} y_1 &= F_1\delta_{11} + F_2\delta_{12} + F_3\delta_{13} \\ y_2 &= F_1\delta_{21} + F_2\delta_{22} + F_3\delta_{23} \\ y_3 &= F_1\delta_{31} + F_2\delta_{32} + F_3\delta_{33} \end{aligned} \tag{7–25}$$

The forces F_i can arise from weight attached w_i or centrifugal forces $m_i\omega^2 y_i$. The Equation set (7–25) written with inertial forces can be displayed as

$$\begin{aligned} y_1 &= m_1\omega^2 y_1\delta_{11} + m_2\omega^2 y_2\delta_{12} + m_3\omega^2 y_3\delta_{13} \\ y_2 &= m_1\omega^2 y_1\delta_{21} + m_2\omega^2 y_2\delta_{22} + m_3\omega^2 y_3\delta_{23} \\ y_3 &= m_1\omega^2 y_1\delta_{31} + m_2\omega^2 y_2\delta_{32} + m_3\omega^2 y_3\delta_{33} \end{aligned}$$

[6]Thomson, op. cit., p. 167.

which can be rewritten as

$$
\begin{aligned}
(m_1\delta_{11} - 1/\omega^2)y_1 + (m_2\delta_{12})y_2 + (m_3\delta_{13})y_3 &= 0 \\
(m_1\delta_{21})y_1 + (m_2\delta_{22} - 1/\omega^2)y_2 + (m_3\delta_{23})y_3 &= 0 \\
(m_1\delta_{31})y_1 + (m_2\delta_{32})y_2 + (m_3\delta_{33} - 1/\omega^2)y_3 &= 0
\end{aligned}
\tag{a}
$$

Equation set (a) is three simultaneous equations in terms of y_1, y_2, and y_3. To avoid the trivial solution $y_1 = y_2 = y_3 = 0$, the determinant of the coefficients of y_1, y_2, and y_3 must be zero (eigenvalue problem). Thus,

$$
\begin{vmatrix}
(m_1\delta_{11} - 1/\omega^2) & m_2\delta_{12} & m_3\delta_{13} \\
m_1\delta_{21} & (m_2\delta_{22} - 1/\omega^2) & m_3\delta_{23} \\
m_1\delta_{31} & m_2\delta_{32} & (m_3\delta_{33} - 1/\omega^2)
\end{vmatrix} = 0
\tag{7–26}
$$

which says that a deflection other than zero exists only at three distinct values of ω, the critical speeds. Expanding the determinant, we obtain

$$
\left(\frac{1}{\omega^2}\right)^3 - (m_1\delta_{11} + m_2\delta_{22} + m_3\delta_{33})\left(\frac{1}{\omega^2}\right)^2 + \cdots = 0
\tag{7–27}
$$

The three roots of Equation (7–27) can be expressed as $1/\omega_1^2$, $1/\omega_2^2$, and $1/\omega_3^2$. Thus Equation (7–27) can be written in the form

$$
\left(\frac{1}{\omega^2} - \frac{1}{\omega_1^2}\right)\left(\frac{1}{\omega^2} - \frac{1}{\omega_2^2}\right)\left(\frac{1}{\omega^2} - \frac{1}{\omega_3^2}\right) = 0
$$

or

$$
\left(\frac{1}{\omega^2}\right)^3 - \left(\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2}\right)\left(\frac{1}{\omega^2}\right)^2 + \cdots = 0
\tag{7–28}
$$

Comparing Equations (7–27) and (7–28) we see that

$$
\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2} = m_1\delta_{11} + m_2\delta_{22} + m_3\delta_{33}
\tag{7–29}
$$

If we had only a single mass m_1 alone, the critical speed would be given by $1/\omega^2 = m_1\delta_{11}$. Denote this critical speed as ω_{11} (which considers only m_1 acting alone). Likewise for m_2 or m_3 acting alone, we similarly define the terms $1/\omega_{22}^2 = m_2\delta_{22}$ or $1/\omega_{33}^2 = m_3\delta_{33}$, respectively. Thus, Equation (7–29) can be rewritten as

$$
\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2} = \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} + \frac{1}{\omega_{33}^2}
\tag{7–30}
$$

If we order the critical speeds such that $\omega_1 < \omega_2 < \omega_3$, then $1/\omega_1^2$ is much greater than $1/\omega_2^2$ and $1/\omega_3^2$. So the first, or fundamental, critical speed ω_1 can be approximated by

$$
\frac{1}{\omega_1^2} \approx \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} + \frac{1}{\omega_{33}^2}
\tag{7–31}
$$

This idea can be extended to an n-body shaft:

$$
\frac{1}{\omega_1^2} \approx \sum_{i=1}^{n} \frac{1}{\omega_{ii}^2}
\tag{7–32}
$$

This is called *Dunkerley's equation*. By ignoring the higher mode term(s), the first critical speed estimate is *lower* than actually is the case.

Since Equation (7–32) has no loads appearing in the equation, it follows that if each load could be placed at some convenient location transformed into an equivalent load, then the critical speed of an array of loads could be found by summing the equivalent loads, all placed at a single convenient location. For the load at station 1, placed at the center of span, denoted with the subscript c, the equivalent load is found from

$$\omega_{11}^2 = \frac{1}{m_1\delta_{11}} = \frac{g}{w_1\delta_{11}} = \frac{g}{w_{1c}\delta_{cc}}$$

or

$$w_{1c} = w_1\frac{\delta_{11}}{\delta_{cc}} \tag{7–33}$$

EXAMPLE 7–5

Consider a simply supported steel shaft as depicted in Figure 7–14, with 1 in diameter and a 31-in span between bearings, carrying two gears weighing 35 and 55 lbf.

(*a*) Find the influence coefficients.

(*b*) Find $\Sigma\, wy$ and $\Sigma\, wy^2$ and the first critical speed using Rayleigh's equation, Equation (7–23).

(*c*) From the influence coefficients, find ω_{11} and ω_{22}.

(*d*) Using Dunkerley's equation, Equation (7–32), estimate the first critical speed.

(*e*) Use superposition to estimate the first critical speed.

(*f*) Estimate the shaft's intrinsic critical speed. Suggest a modification to Dunkerley's equation to include the effect of the shaft's mass on the first critical speed of the attachments.

Solution

(*a*)

$$I = \frac{\pi d^4}{64} = \frac{\pi(1)^4}{64} = 0.049\,09 \text{ in}^4$$

$$6EIl = 6(30)10^6(0.049\,09)31 = 0.2739(10^9)\text{ lbf}\cdot\text{in}^3$$

From Equation set (7–24),

$$\delta_{11} = \frac{24(7)(31^2 - 24^2 - 7^2)}{0.2739(10^9)} = 2.061(10^{-4})\text{ in/lbf}$$

$$\delta_{22} = \frac{11(20)(31^2 - 11^2 - 20^2)}{0.2739(10^9)} = 3.534(10^{-4})\text{ in/lbf}$$

$$\delta_{12} = \delta_{21} = \frac{11(7)(31^2 - 11^2 - 7^2)}{0.2739(10^9)} = 2.224(10^{-4})\text{ in/lbf}$$

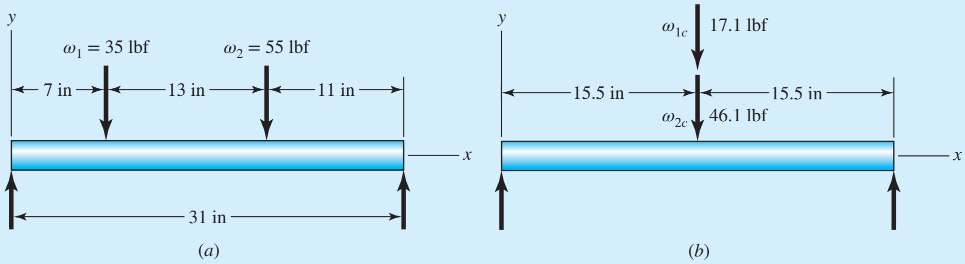


Figure 7–14

(*a*) A 1-in uniform-diameter shaft for Example 7–5. (*b*) Superposing of equivalent loads at the center of the shaft for the purpose of finding the first critical speed.

Answer

	j	
i	**1**	**2**
1	$2.061(10^{-4})$	$2.224(10^{-4})$
2	$2.224(10^{-4})$	$3.534(10^{-4})$

$$y_1 = w_1\delta_{11} + w_2\delta_{12} = 35(2.061)10^{-4} + 55(2.224)10^{-4} = 0.019\,45 \text{ in}$$

$$y_2 = w_1\delta_{21} + w_2\delta_{22} = 35(2.224)10^{-4} + 55(3.534)10^{-4} = 0.027\,22 \text{ in}$$

(*b*)

$$\sum w_i y_i = 35(0.019\,45) + 55(0.027\,22) = 2.178 \text{ lbf} \cdot \text{in}$$

Answer

$$\sum w_i y_i^2 = 35(0.019\,45)^2 + 55(0.027\,22)^2 = 0.053\,99 \text{ lbf} \cdot \text{in}^2$$

Answer

$$\omega = \sqrt{\frac{386.1(2.178)}{0.053\,99}} = 124.8 \text{ rad/s, or } 1192 \text{ rev/min}$$

(*c*)

Answer

$$\frac{1}{\omega_{11}^2} = \frac{w_1}{g}\delta_{11}$$

$$\omega_{11} = \sqrt{\frac{g}{w_1\delta_{11}}} = \sqrt{\frac{386.1}{35(2.061)10^{-4}}} = 231.4 \text{ rad/s, or } 2210 \text{ rev/min}$$

Answer

$$\omega_{22} = \sqrt{\frac{g}{w_2\delta_{22}}} = \sqrt{\frac{386.1}{55(3.534)10^{-4}}} = 140.9 \text{ rad/s, or } 1346 \text{ rev/min}$$

(*d*)

$$\frac{1}{\omega_1^2} \approx \sum \frac{1}{\omega_{ii}^2} = \frac{1}{231.4^2} + \frac{1}{140.9^2} = 6.905(10^{-5}) \qquad (1)$$

Answer

$$\omega_1 \approx \sqrt{\frac{1}{6.905(10^{-5})}} = 120.3 \text{ rad/s, or } 1149 \text{ rev/min}$$

which is less than part *b*, as expected.
(*e*) From Equation (7–24),

$$\delta_{cc} = \frac{b_{cc}x_{cc}(l^2 - b_{cc}^2 - x_{cc}^2)}{6EIl} = \frac{15.5(15.5)(31^2 - 15.5^2 - 15.5^2)}{0.2739(10^9)}$$
$$= 4.215(10^{-4}) \text{ in/lbf}$$

From Equation (7–33),

$$w_{1c} = w_1\frac{\delta_{11}}{\delta_{cc}} = 35\frac{2.061(10^{-4})}{4.215(10^{-4})} = 17.11 \text{ lbf}$$

$$w_{2c} = w_2\frac{\delta_{22}}{\delta_{cc}} = 55\frac{3.534(10^{-4})}{4.215(10^{-4})} = 46.11 \text{ lbf}$$

Answer

$$\omega = \sqrt{\frac{g}{\delta_{cc}\sum w_{ic}}} = \sqrt{\frac{386.1}{4.215(10^{-4})(17.11 + 46.11)}} = 120.4 \text{ rad/s, or } 1150 \text{ rev/min}$$

which, except for rounding, agrees with part *d*, as expected.

(f) For the shaft, $E = 30(10^6)$ psi, $\gamma = 0.282$ lbf/in^3, and $A = \pi(1^2)/4 = 0.7854$ in^2. Considering the shaft alone, the critical speed, from Equation (7–22), is

Answer
$$\omega_s = \left(\frac{\pi}{l}\right)^2 \sqrt{\frac{gEI}{A\gamma}} = \left(\frac{\pi}{31}\right)^2 \sqrt{\frac{386.1(30)10^6(0.049\ 09)}{0.7854(0.282)}}$$

$$= 520.4 \text{ rad/s, or } 4970 \text{ rev/min}$$

We can simply add $1/\omega_s^2$ to the right side of Dunkerley's equation, Equation (1), to include the shaft's contribution,

Answer
$$\frac{1}{\omega_1^2} \approx \frac{1}{520.4^2} + 6.905(10^{-5}) = 7.274(10^{-5})$$

$$\omega_1 \approx 117.3 \text{ rad/s, or } 1120 \text{ rev/min}$$

which is slightly less than part d, as expected.

The shaft's first critical speed ω_s is just one more single effect to add to Dunkerley's equation. Since it does not fit into the summation, it is usually written up front.

Answer
$$\frac{1}{\omega_1^2} \approx \frac{1}{\omega_s^2} + \sum_{i=1}^{n} \frac{1}{\omega_{ii}^2} \tag{7–34}$$

Common shafts are complicated by the stepped-cylinder geometry, which makes the influence-coefficient determination part of a numerical solution.

7–7 Miscellaneous Shaft Components

Setscrews

Unlike bolts and cap screws, which depend on tension to develop a clamping force, the setscrew depends on compression to develop the clamping force. The resistance to axial motion of the collar or hub relative to the shaft is called *holding power*. This holding power, which is really a force resistance, is due to frictional resistance of the contacting portions of the collar and shaft as well as any slight penetration of the setscrew into the shaft.

Figure 7–15 shows the point types available with socket setscrews. These are also manufactured with screwdriver slots and with square heads.

Table 7–4 lists values of the seating torque and the corresponding holding power for inch-series setscrews. The values listed apply to both axial holding power, for resisting thrust, and the tangential holding power, for resisting torsion. Typical factors of safety are 1.5 to 2.0 for static loads and 4 to 8 for various dynamic loads.

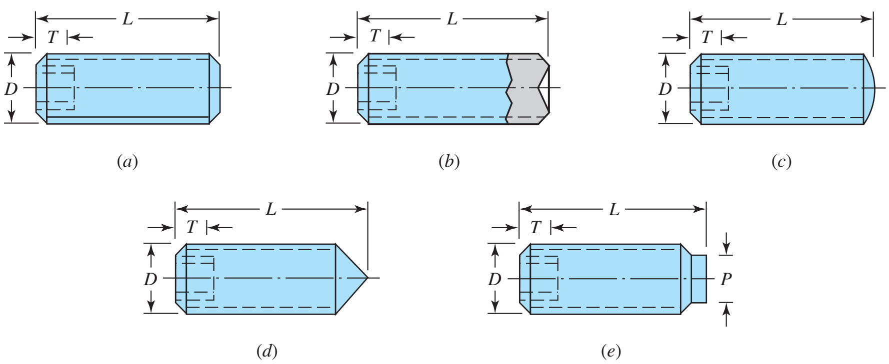


Figure 7–15

Socket setscrews: (a) flat point; (b) cup point; (c) oval point; (d) cone point; (e) half-dog point.

Table 7–4 Typical Holding Power (Force) for Socket Setscrews*

Size, in	Seating Torque, lbf · in	Holding Power, lbf
#0	1.0	50
#1	1.8	65
#2	1.8	85
#3	5	120
#4	5	160
#5	10	200
#6	10	250
#8	20	385
#10	36	540
$\frac{1}{4}$	87	1000
$\frac{5}{16}$	165	1500
$\frac{3}{8}$	290	2000
$\frac{7}{16}$	430	2500
$\frac{1}{2}$	620	3000
$\frac{9}{16}$	620	3500
$\frac{5}{8}$	1325	4000
$\frac{3}{4}$	2400	5000
$\frac{7}{8}$	5200	6000
1	7200	7000

Source: Unbrako Division, SPS Technologies, Jenkintown, Pa.
*Based on alloy-steel screw against steel shaft, class 3A coarse or fine threads in class 2B holes, and cup-point socket setscrews.

Setscrews should have a length of about half of the shaft diameter. Note that this practice also provides a rough rule for the radial thickness of a hub or collar.

Keys and Pins

Keys and pins are used on shafts to secure rotating elements, such as gears, pulleys, or other wheels. Keys are used to enable the transmission of torque from the shaft to the shaft-supported element. Pins are used for axial positioning and for the transfer of torque or thrust or both.

Figure 7–16 shows a variety of keys and pins. Pins are useful when the principal loading is shear and when both torsion and thrust are present. Taper pins are sized according to the diameter at the large end. Some of the most useful sizes of these are listed in Table 7–5. The diameter at the small end is

$$d = D - 0.0208L \tag{7–35}$$

where d = diameter at small end, in
D = diameter at large end, in
L = length, in

For less important applications, a dowel pin or a drive pin can be used. A large variety of these are listed in manufacturers' catalogs.[7]

[7]See also Joseph E. Shigley, "Unthreaded Fasteners," Chapter 24. In Joseph E. Shigley, Charles R. Mischke, and Thomas H. Brown, Jr. (eds.), *Standard Handbook of Machine Design,* 3rd ed., McGraw-Hill, New York, 2004.

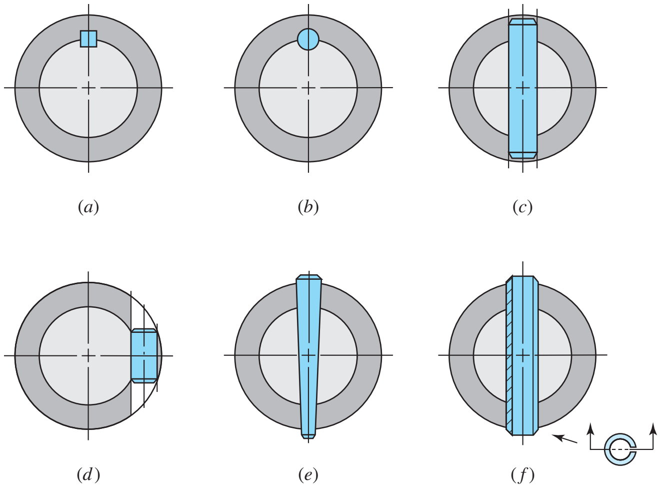


Figure 7–16

(*a*) Square key; (*b*) round key; (*c* and *d*) round pins; (*e*) taper pin; (*f*) split tubular spring pin. The pins in parts (*e*) and (*f*) are shown longer than necessary, to illustrate the chamfer on the ends, but their lengths should be kept smaller than the hub diameters to prevent injuries due to projections on rotating parts.

Table 7–5 Dimensions at Large End of Some Standard Taper Pins—Inch Series

	Commercial		Precision	
Size	**Maximum**	**Minimum**	**Maximum**	**Minimum**
4/0	0.1103	0.1083	0.1100	0.1090
2/0	0.1423	0.1403	0.1420	0.1410
0	0.1573	0.1553	0.1570	0.1560
2	0.1943	0.1923	0.1940	0.1930
4	0.2513	0.2493	0.2510	0.2500
6	0.3423	0.3403	0.3420	0.3410
8	0.4933	0.4913	0.4930	0.4920

The square key, shown in Figure 7–16*a*, is also available in rectangular sizes. Standard sizes of these, together with the range of applicable shaft diameters, are listed in Table 7–6. The shaft diameter determines standard sizes for width, height, and key depth. The designer chooses an appropriate key length to carry the torsional load. Failure of the key can be by direct shear, or by bearing stress. Example 7–6 demonstrates the process to size the length of a key. The maximum length of a key is limited by the hub length of the attached element, and should generally not exceed about 1.5 times the shaft diameter to avoid excessive twisting with the angular deflection of the shaft. Multiple keys may be used as necessary to carry greater loads, typically oriented at 90° from one another. Excessive safety factors should be avoided in key design, since it is desirable in an overload situation for the key to fail, rather than more costly components.

Stock key material is typically made from low carbon cold-rolled steel, and is manufactured such that its dimensions never exceed the nominal dimension. This allows standard cutter sizes to be used for the keyseats. A setscrew is sometimes used along with a key to hold the hub axially, and to minimize rotational backlash when the shaft rotates in both directions.

The gib-head key, in Figure 7–17*a*, is tapered so that, when firmly driven, it acts to prevent relative axial motion. This also gives the advantage that the hub position

Table 7–6 Inch Dimensions for Some Standard Square- and Rectangular-Key Applications

Shaft Diameter		Key Size		
Over	To (Incl.)	w	h	Keyway Depth
$\frac{5}{16}$	$\frac{7}{16}$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{3}{64}$
$\frac{7}{16}$	$\frac{9}{16}$	$\frac{1}{8}$	$\frac{3}{32}$	$\frac{3}{64}$
		$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{16}$
$\frac{9}{16}$	$\frac{7}{8}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{16}$
		$\frac{3}{16}$	$\frac{3}{16}$	$\frac{3}{32}$
$\frac{7}{8}$	$1\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{3}{32}$
		$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$
$1\frac{1}{4}$	$1\frac{3}{8}$	$\frac{5}{16}$	$\frac{1}{4}$	$\frac{1}{8}$
		$\frac{5}{16}$	$\frac{5}{16}$	$\frac{5}{32}$
$1\frac{3}{8}$	$1\frac{3}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
		$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{16}$
$1\frac{3}{4}$	$2\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{3}{16}$
		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$
$2\frac{1}{4}$	$2\frac{3}{4}$	$\frac{5}{8}$	$\frac{7}{16}$	$\frac{7}{32}$
		$\frac{5}{8}$	$\frac{5}{8}$	$\frac{5}{16}$
$2\frac{3}{4}$	$3\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
		$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{8}$

Source: Joseph E. Shigley, "Unthreaded Fasteners," Chapter 24 in Joseph E. Shigley, Charles R. Mischke, and Thomas H. Brown, Jr. (eds.), *Standard Handbook of Machine Design,* 3rd ed., McGraw-Hill, New York, 2004.

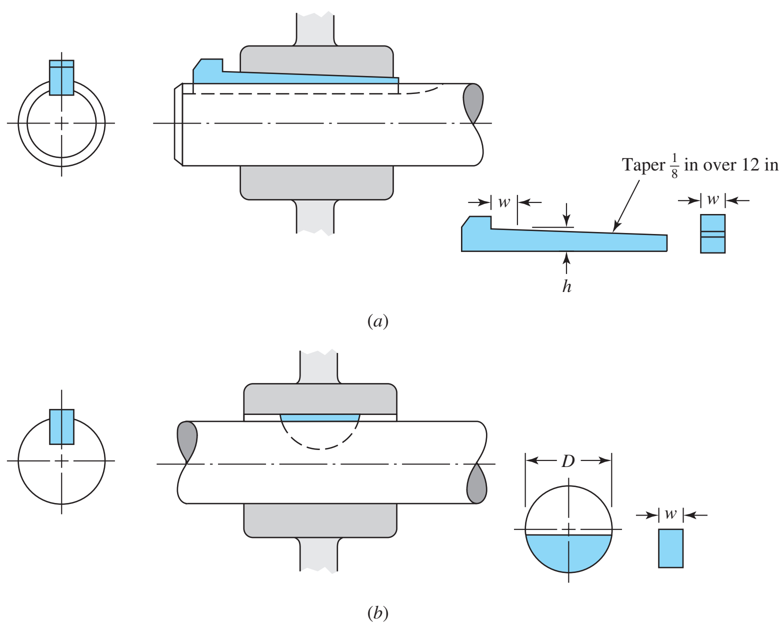


Figure 7–17
(*a*) Gib-head key;
(*b*) Woodruff key.

Table 7–7 Dimensions of Woodruff Keys—Inch Series

Key Size		Height	Offset	Keyseat Depth	
w	*D*	*b*	*e*	Shaft	Hub
$\frac{1}{16}$	$\frac{1}{4}$	0.109	$\frac{1}{64}$	0.0728	0.0372
$\frac{1}{16}$	$\frac{3}{8}$	0.172	$\frac{1}{64}$	0.1358	0.0372
$\frac{3}{32}$	$\frac{3}{8}$	0.172	$\frac{1}{64}$	0.1202	0.0529
$\frac{3}{32}$	$\frac{1}{2}$	0.203	$\frac{3}{64}$	0.1511	0.0529
$\frac{3}{32}$	$\frac{5}{8}$	0.250	$\frac{1}{16}$	0.1981	0.0529
$\frac{1}{8}$	$\frac{1}{2}$	0.203	$\frac{3}{64}$	0.1355	0.0685
$\frac{1}{8}$	$\frac{5}{8}$	0.250	$\frac{1}{16}$	0.1825	0.0685
$\frac{1}{8}$	$\frac{3}{4}$	0.313	$\frac{1}{16}$	0.2455	0.0685
$\frac{5}{32}$	$\frac{5}{8}$	0.250	$\frac{1}{16}$	0.1669	0.0841
$\frac{5}{32}$	$\frac{3}{4}$	0.313	$\frac{1}{16}$	0.2299	0.0841
$\frac{5}{32}$	$\frac{7}{8}$	0.375	$\frac{1}{16}$	0.2919	0.0841
$\frac{3}{16}$	$\frac{3}{4}$	0.313	$\frac{1}{16}$	0.2143	0.0997
$\frac{3}{16}$	$\frac{7}{8}$	0.375	$\frac{1}{16}$	0.2763	0.0997
$\frac{3}{16}$	1	0.438	$\frac{1}{16}$	0.3393	0.0997
$\frac{1}{4}$	$\frac{7}{8}$	0.375	$\frac{1}{16}$	0.2450	0.1310
$\frac{1}{4}$	1	0.438	$\frac{1}{16}$	0.3080	0.1310
$\frac{1}{4}$	$1\frac{1}{4}$	0.547	$\frac{5}{64}$	0.4170	0.1310
$\frac{5}{16}$	1	0.438	$\frac{1}{16}$	0.2768	0.1622
$\frac{5}{16}$	$1\frac{1}{4}$	0.547	$\frac{5}{64}$	0.3858	0.1622
$\frac{5}{16}$	$1\frac{1}{2}$	0.641	$\frac{7}{64}$	0.4798	0.1622
$\frac{3}{8}$	$1\frac{1}{4}$	0.547	$\frac{5}{64}$	0.3545	0.1935
$\frac{3}{8}$	$1\frac{1}{2}$	0.641	$\frac{7}{64}$	0.4485	0.1935

Table 7–8 Sizes of Woodruff Keys Suitable for Various Shaft Diameters

Keyseat	Shaft Diameter, in	
Width, in	From	To (inclusive)
$\frac{1}{16}$	$\frac{5}{16}$	$\frac{1}{2}$
$\frac{3}{32}$	$\frac{3}{8}$	$\frac{7}{8}$
$\frac{1}{8}$	$\frac{3}{8}$	$1\frac{1}{2}$
$\frac{5}{32}$	$\frac{1}{2}$	$1\frac{5}{8}$
$\frac{3}{16}$	$\frac{9}{16}$	2
$\frac{1}{4}$	$\frac{11}{16}$	$2\frac{1}{4}$
$\frac{5}{16}$	$\frac{3}{4}$	$2\frac{3}{8}$
$\frac{3}{8}$	1	$2\frac{5}{8}$

can be adjusted for the best axial location. The head makes removal possible without access to the other end, but the projection may be hazardous.

The Woodruff key, shown in Figure 7–17*b*, is of general usefulness, especially when a wheel is to be positioned against a shaft shoulder, since the keyslot need not be machined into the shoulder stress concentration region. The use of the Woodruff key also yields better concentricity after assembly of the wheel and shaft. This is especially important at high speeds, as, for example, with a turbine wheel and shaft. Woodruff keys are particularly useful in smaller shafts where their deeper penetration helps prevent key rolling. Dimensions for some standard Woodruff key sizes can be found in Table 7–7, and Table 7–8 gives the shaft diameters for which the different keyseat widths are suitable.

Pilkey[8] gives values for stress concentrations in an end-milled keyseat, as a function of the ratio of the radius r at the bottom of the groove and the shaft diameter d.

[8]W. D. Pilkey, *Peterson's Stress-Concentration Factors,* 2nd ed., John Wiley & Sons, New York, 1997, pp. 408–409.

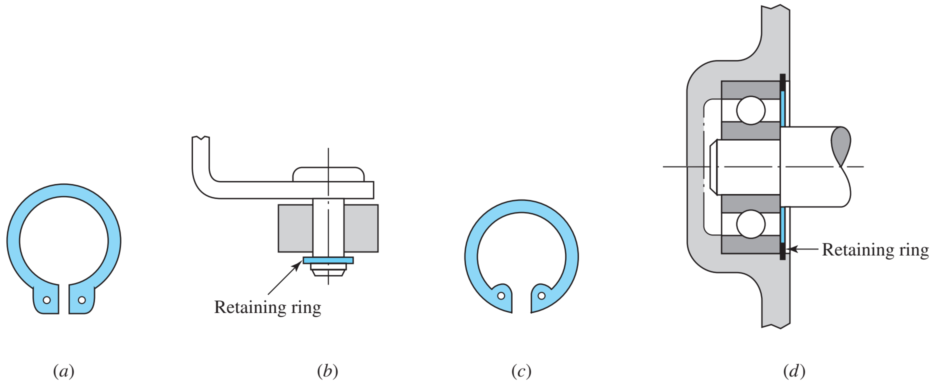


Figure 7–18

Typical uses for retaining rings. (*a*) External ring and (*b*) its application; (*c*) internal ring and (*d*) its application.

For fillets cut by standard milling-machine cutters, with a ratio of $r/d = 0.02$, Peterson's charts give $K_t = 2.14$ for bending and $K_{ts} = 2.62$ for torsion without the key in place, or $K_{ts} = 3.0$ for torsion with the key in place. The stress concentration at the end of the keyseat can be reduced somewhat by using a sled-runner keyseat, eliminating the abrupt end to the keyseat, as shown in Figure 7–17. It does, however, still have the sharp radius in the bottom of the groove on the sides. The sled-runner keyseat can only be used when definite longitudinal key positioning is not necessary. It is also not as suitable near a shoulder. Keeping the end of a keyseat at least a distance of $d/10$ from the start of the shoulder fillet will prevent the two stress concentrations from combining with each other.[9]

Retaining Rings

A retaining ring is frequently used instead of a shaft shoulder or a sleeve to axially position a component on a shaft or in a housing bore. As shown in Figure 7–18, a groove is cut in the shaft or bore to receive the spring retainer. For sizes, dimensions, and axial load ratings, the manufacturers' catalogs should be consulted.

Appendix Tables A–15–16 and A–15–17 give values for stress-concentration factors for flat-bottomed grooves in shafts, suitable for retaining rings. For the rings to seat nicely in the bottom of the groove, and support axial loads against the sides of the groove, the radius in the bottom of the groove must be reasonably sharp, typically about one-tenth of the groove width. This causes comparatively high values for stress-concentration factors, around 5 for bending and axial, and 3 for torsion. Care should be taken in using retaining rings, particularly in locations with high bending stresses.

EXAMPLE 7–6

A UNS G10350 steel shaft, heat-treated to a minimum yield strength of 75 kpsi, has a diameter of $1\frac{7}{16}$ in. The shaft rotates at 600 rev/min and transmits 40 hp through a gear. Select an appropriate key for the gear, with a design factor of 1.5.

Solution

From Table 7–6, a $\frac{3}{8}$-in square key is selected. Choose a cold-drawn low-carbon mild steel which is generally available for key stock, such as UNS G10180, with a yield strength of 54 kpsi.

[9]Ibid, p. 381.

The torque is obtained from the power and angular velocity using Equation (3–42):

$$T = \frac{63\,025H}{n} = \frac{(63\,025)(40)}{600} = 4200 \text{ lbf} \cdot \text{in}$$

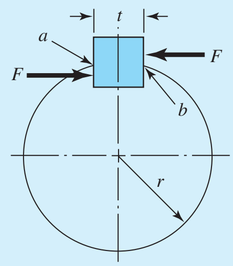


Figure 7–19

From Figure 7–19, the force F at the surface of the shaft is

$$F = \frac{T}{r} = \frac{4200}{1.4375/2} = 5850 \text{ lbf}$$

By the distortion-energy theory, the shear strength is

$$S_{sy} = 0.577S_y = (0.577)(54) = 31.2 \text{ kpsi}$$

Failure by shear across the area ab will create a stress of $\tau = F/tl$. Substituting the strength divided by the design factor for τ gives

$$\frac{S_{sy}}{n} = \frac{F}{tl} \quad \text{or} \quad \frac{31.2(10)^3}{1.5} = \frac{5850}{0.375l}$$

or $l = 0.75$ in. To resist crushing, the area of one-half the face of the key is used:

$$\frac{S_y}{n} = \frac{F}{tl/2} \quad \text{or} \quad \frac{54(10)^3}{1.5} = \frac{5850}{0.375l/2}$$

and $l = 0.87$ in. Failure by crushing the key is the dominant failure mode, so it defines the necessary length of the key to be $l = 0.87$ in.

7–8 Limits and Fits

The designer is free to adopt any geometry of fit for shafts and holes that will ensure the intended function. There is sufficient accumulated experience with commonly recurring situations to make standards useful. There are two standards for limits and fits in the United States, one based on inch units and the other based on metric units.[10] These differ in nomenclature, definitions, and organization. No point would be served by separately studying each of the two systems. The metric version is the newer of the two and is well organized, and so here we present only the metric version but include a set of inch conversions to enable the same system to be used with either system of units.

In using the standard, c*apital letters always refer to the hole; lowercase letters are used for the shaft*.

The definitions illustrated in Figure 7–20 are explained as follows:

- ***Basic size*** is the size to which limits or deviations are assigned and is the same for both members of the fit.
- ***Deviation*** is the algebraic difference between a size and the corresponding basic size.
- ***Upper deviation*** is the algebraic difference between the maximum limit and the corresponding basic size.
- ***Lower deviation*** is the algebraic difference between the minimum limit and the corresponding basic size.

[10] *Preferred Limits and Fits for Cylindrical Parts,* ANSI B4.1-1967. *Preferred Metric Limits and Fits,* ANSI B4.2-1978.

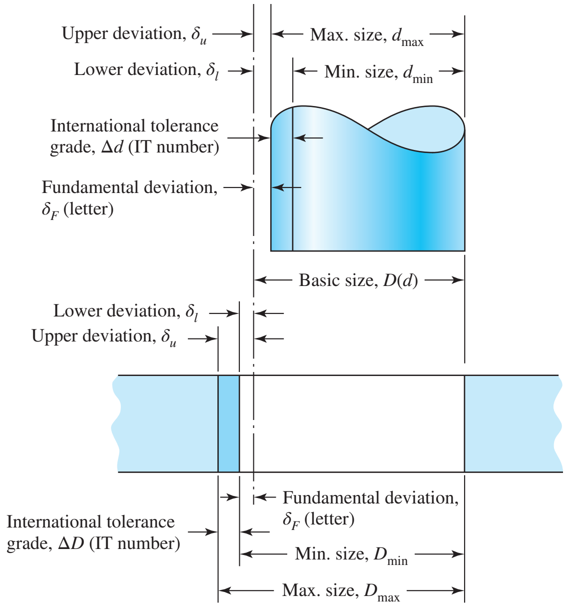


Figure 7–20

Definitions applied to a cylindrical fit.

- ***Fundamental deviation*** is either the upper or the lower deviation, depending on which is closer to the basic size.
- ***Tolerance*** is the difference between the maximum and minimum size limits of a part.
- ***International tolerance grade*** numbers (IT) designate groups of tolerances such that the tolerances for a particular IT number have the same relative level of accuracy but vary depending on the basic size.
- ***Hole basis*** represents a system of fits corresponding to a basic hole size. The fundamental deviation is H.
- ***Shaft basis*** represents a system of fits corresponding to a basic shaft size. The fundamental deviation is h. The shaft-basis system is not included here.

The magnitude of the tolerance zone is the variation in part size and is the same for both the internal and the external dimensions. The tolerance zones are specified in international tolerance grade numbers, called IT numbers. The smaller grade numbers specify a smaller tolerance zone. These range from IT0 to IT16, but only grades IT6 to IT11 are needed for the preferred fits. These are listed in Tables A–11 to A–14 for basic sizes up to 16 in or 400 mm.

The standard uses *tolerance position letters,* with capital letters for internal dimensions (holes) and lowercase letters for external dimensions (shafts). As shown in Figure 7–20, the fundamental deviation locates the tolerance zone relative to the basic size.

Table 7–9 shows how the letters are combined with the tolerance grades to establish a preferred fit. The ISO symbol for the hole for a sliding fit with a basic size of 32 mm is 32H7. Inch units are not a part of the standard. However, the designation ($1\frac{3}{8}$ in) H7 includes the same information and is recommended for use here. In both cases, the capital letter H establishes the fundamental deviation and the number 7 defines a tolerance grade of IT7.

For the sliding fit, the corresponding shaft dimensions are defined by the symbol 32g6 [($1\frac{3}{8}$ in)g6].

Table 7–9 Descriptions of Preferred Fits Using the Basic Hole System

Type of Fit	Description	Symbol
Clearance	*Loose running fit:* for wide commercial tolerances or allowances on external members	H11/c11
	Free running fit: not for use where accuracy is essential, but good for large temperature variations, high running speeds, or heavy journal pressures	H9/d9
	Close running fit: for running on accurate machines and for accurate location at moderate speeds and journal pressures	H8/f7
	Sliding fit: where parts are not intended to run freely, but must move and turn freely and locate accurately	H7/g6
	Locational clearance fit: provides snug fit for location of stationary parts, but can be freely assembled and disassembled	H7/h6
Transition	*Locational transition fit:* for accurate location, a compromise between clearance and interference	H7/k6
	Locational transition fit: for more accurate location where greater interference is permissible	H7/n6
Interference	*Locational interference fit:* for parts requiring rigidity and alignment with prime accuracy of location but without special bore pressure requirements	H7/p6
	Medium drive fit: for ordinary steel parts or shrink fits on light sections, the tightest fit usable with cast iron	H7/s6
	Force fit: suitable for parts that can be highly stressed or for shrink fits where the heavy pressing forces required are impractical	H7/u6

Source: Preferred Metric Limits and Fits, ANSI B4.2-1978. See also BS 4500.

The fundamental deviations for shafts are given in Tables A–11 and A–13. For letter codes c, d, f, g, and h,

$$\begin{aligned}\text{Upper deviation} &= \text{fundamental deviation}\\ \text{Lower deviation} &= \text{upper deviation} - \text{tolerance grade}\end{aligned}$$

For letter codes k, n, p, s, and u, the deviations for shafts are

$$\begin{aligned}\text{Lower deviation} &= \text{fundamental deviation}\\ \text{Upper deviation} &= \text{lower deviation} + \text{tolerance grade}\end{aligned}$$

The lower deviation H (for holes) is zero. For these, the upper deviation equals the tolerance grade.

As shown in Figure 7–20, we use the following notation:

$$\begin{aligned}D &= \text{basic size of hole}\\ d &= \text{basic size of shaft}\\ \delta_u &= \text{upper deviation}\\ \delta_l &= \text{lower deviation}\\ \delta_F &= \text{fundamental deviation}\\ \Delta D &= \text{tolerance grade for hole}\\ \Delta d &= \text{tolerance grade for shaft}\end{aligned}$$

Note that these quantities are all deterministic. Thus, for the hole,

$$D_{\max} = D + \Delta D \qquad D_{\min} = D \tag{7–36}$$

For shafts with clearance fits c, d, f, g, and h,

$$d_{\max} = d + \delta_F \qquad d_{\min} = d + \delta_F - \Delta d \tag{7–37}$$

For shafts with interference fits k, n, p, s, and u,

$$d_{\min} = d + \delta_F \qquad d_{\max} = d + \delta_F + \Delta d \tag{7–38}$$

EXAMPLE 7–7

Find the shaft and hole dimensions for a loose running fit with a 34-mm basic size.

Solution

From Table 7–9, the ISO symbol is 34H11/c11. From Table A–11, we find that tolerance grade IT11 is 0.160 mm. The symbol 34H11/c11 therefore says that $\Delta D = \Delta d = 0.160$ mm. Using Equation (7–36) for the hole, we get

Answer $$D_{\max} = D + \Delta D = 34 + 0.160 = 34.160 \text{ mm}$$

Answer $$D_{\min} = D = 34.000 \text{ mm}$$

The shaft is designated as a 34c11 shaft. From Table A–12, the fundamental deviation is $\delta_F = -0.120$ mm. Using Equation (7–37), we get for the shaft dimensions

Answer $$d_{\max} = d + \delta_F = 34 + (-0.120) = 33.880 \text{ mm}$$

Answer $$d_{\min} = d + \delta_F - \Delta d = 34 + (-0.120) - 0.160 = 33.720 \text{ mm}$$

EXAMPLE 7–8

Find the hole and shaft limits for a medium drive fit using a basic hole size of 2 in.

Solution

The symbol for the fit, from Table 7–8, in inch units is (2 in)H7/s6. For the hole, we use Table A–13 and find the IT7 grade to be $\Delta D = 0.0010$ in. Thus, from Equation (7–36),

Answer $$D_{\max} = D + \Delta D = 2 + 0.0010 = 2.0010 \text{ in}$$

Answer $$D_{\min} = D = 2.0000 \text{ in}$$

The IT6 tolerance for the shaft is $\Delta d = 0.0006$ in. Also, from Table A–14, the fundamental deviation is $\delta_F = 0.0017$ in. Using Equation (7–38), we get for the shaft that

Answer $$d_{\min} = d + \delta_F = 2 + 0.0017 = 2.0017 \text{ in}$$

Answer $$d_{\max} = d + \delta_F + \Delta d = 2 + 0.0017 + 0.0006 = 2.0023 \text{ in}$$

Stress and Torque Capacity in Interference Fits

Interference fits between a shaft and its components can sometimes be used effectively to minimize the need for shoulders and keyways. The stresses due to an interference fit can be obtained by treating the shaft as a cylinder with a uniform external pressure, and the hub as a hollow cylinder with a uniform internal pressure. Stress equations for these situations were developed in Section 3–16, and will be converted here from radius terms into diameter terms to match the terminology of this section.

The pressure p generated at the interface of the interference fit, from Equation (3–56) converted into terms of diameters, is given by

$$p = \frac{\delta}{\dfrac{d}{E_o}\left(\dfrac{d_o^2 + d^2}{d_o^2 - d^2} + \nu_o\right) + \dfrac{d}{E_i}\left(\dfrac{d^2 + d_i^2}{d^2 - d_i^2} - \nu_i\right)} \tag{7–39}$$

or, in the case where both members are of the same material,

$$p = \frac{E\delta}{2d^3}\left[\frac{(d_o^2 - d^2)(d^2 - d_i^2)}{d_o^2 - d_i^2}\right] \tag{7–40}$$

where d is the nominal shaft diameter, d_i is the inside diameter (if any) of the shaft, d_o is the outside diameter of the hub, E is Young's modulus, and ν is Poisson's ratio, with subscripts o and i for the outer member (hub) and inner member (shaft), respectively. The term δ is the *diametral* interference between the shaft and hub, that is, the difference between the shaft outside diameter and the hub inside diameter.

$$\delta = d_{\text{shaft}} - d_{\text{hub}} \tag{7–41}$$

Since there will be tolerances on both diameters, the maximum and minimum pressures can be found by applying the maximum and minimum interferences. Adopting the notation from Figure 7–20, we write

$$\delta_{\min} = d_{\min} - D_{\max} \tag{7–42}$$

$$\delta_{\max} = d_{\max} - D_{\min} \tag{7–43}$$

where the diameter terms are defined in Equations (7–36) and (7–38). The maximum interference should be used in Equation (7–39) or (7–40) to determine the maximum pressure to check for excessive stress.

From Equations (3–58) and (3–59), with radii converted to diameters, the tangential stresses at the interface of the shaft and hub are

$$\sigma_{t,\,\text{shaft}} = -p\frac{d^2 + d_i^2}{d^2 - d_i^2} \tag{7–44}$$

$$\sigma_{t,\,\text{hub}} = p\frac{d_o^2 + d^2}{d_o^2 - d^2} \tag{7–45}$$

The radial stresses at the interface are simply

$$\sigma_{r,\,\text{shaft}} = -p \tag{7–46}$$

$$\sigma_{r,\,\text{hub}} = -p \tag{7–47}$$

The tangential and radial stresses are orthogonal, and should be combined using a failure theory to compare with the yield strength. If either the shaft or hub yields during assembly, the full pressure will not be achieved, diminishing the torque that can be transmitted. The interaction of the stresses due to the interference fit with the other stresses in the shaft due to shaft loading is not trivial. Finite-element analysis of the interface would be appropriate when warranted. A stress element on the surface of a rotating shaft will experience a completely reversed bending stress in the longitudinal direction, as well as the steady compressive stresses in the tangential and radial directions. This is a three-dimensional stress element. Shear stress due to torsion in the shaft may also be present. Since the stresses due to the press fit are compressive, the fatigue situation is usually actually improved. For this reason, it may be acceptable to simplify the shaft analysis by ignoring the steady compressive stresses due to the press fit. There is, however, a stress concentration effect in the shaft bending stress near the ends of the hub, due to the sudden change from compressed to uncompressed material. The design of the hub geometry, and therefore its uniformity and rigidity, can have a significant effect on the specific value of the stress-concentration factor, making it difficult to report generalized values. For first estimates, values are typically not greater than 2.

The amount of torque that can be transmitted through an interference fit can be estimated with a simple friction analysis at the interface. The friction force is the product of the coefficient of friction f and the normal force acting at the interface. The normal force can be represented by the product of the pressure p and the surface area A of interface. Therefore, the friction force F_f is

$$F_f = fN = f(pA) = f[p2\pi(d/2)l] = \pi f pld \tag{7–48}$$

where l is the length of the hub. This friction force is acting with a moment arm of $d/2$ to provide the torque capacity of the joint, so

$$T = F_f d/2 = \pi f pld(d/2)$$

$$T = (\pi/2) f pld^2 \tag{7–49}$$

The minimum interference, from Equation (7–42), should be used to determine the minimum pressure to check for the maximum amount of torque that the joint should be designed to transmit without slipping.

PROBLEMS

Problems marked with an asterisk (*) are linked to problems in other chapters, as summarized in Table 1–2 of Section 1–17.

7–1 A shaft is loaded in bending and torsion such that M_a = 70 N · m, T_a = 45 N · m, M_m = 55 N · m, and T_m = 35 N · m. For the shaft, S_u = 700 MPa, S_y = 560 MPa, the true fracture strength is 1045 MPa, and a fully corrected endurance limit of S_e = 210 MPa is assumed. Let K_f = 2.2 and K_{fs} = 1.8. With a design factor of 2.0 determine the minimum acceptable diameter of the shaft using the
(*a*) DE-Goodman criterion.
(*b*) DE-Morrow criterion.
(*c*) DE-Gerber criterion.
(*d*) DE-SWT criterion.
Discuss and compare the results.

7–2 A shaft, made of AISI 1050 CD steel, is loaded in bending and torsion such that M_a = 650 lbf · in, T_a = 400 lbf · in, M_m = 500 lbf · in, and T_m = 300 lbf · in. The shaft has a fully corrected endurance limit of S_e = 30 kpsi and at the critical stress location, K_f = 2.3 and K_{fs} = 1.9. Estimate the true fracture strength as being 50 kpsi greater than the ultimate strength. With a design factor of 2.5 determine the minimum acceptable diameter of the shaft using the
(*a*) DE-Goodman criterion.
(*b*) DE-Morrow criterion.
(*c*) DE-Gerber criterion.
(*d*) DE-SWT criterion.
Discuss and compare the results.