10.4.ex17

EE24BTECH11022 - ESHAN SHARMA

Question: A pole has to be erected at a point on the boundary of a circular park of diameter 13 metres in such a way that the difference of its distances from two diametrically opposite fixed gates A and B on the boundary is 7 metres. Is it possible to do so? If yes, at what distances from the two gates should the pole be erected?

Solution:

Let the distance of the pole from gate B be x metres. Then the distance from gate A will be $(x+7)$ metres.

From the Pythagoras theorem, since $\triangle APB$ is a right triangle:

$$AB^2 = AP^2 + BP^2 \tag{1}$$

Given $AB = 13$ metres, substituting values:

$$13^2 = (x+7)^2 + x^2 \tag{2}$$

$$169 = x^2 + 14x + 49 + x^2 \tag{3}$$

$$2x^2 + 14x - 120 = 0 \tag{4}$$

Simplify the equation:

$$x^2 + 7x - 60 = 0 \tag{5}$$

Theoretical Solution: Using the quadratic formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, where $a = 1, b = 7, c = -60$:

$$x = \frac{-7 \pm \sqrt{7^2 - 4(1)(-60)}}{2(1)} \tag{6}$$

$$x = \frac{-7 \pm \sqrt{289}}{2} \tag{7}$$

$$x = \frac{-7 \pm 17}{2} \tag{8}$$

Thus, $x = 5$ or $x = -12$. Since distance cannot be negative, $x = 5$. The distances are: $BP = 5$ metres, $AP = 12$ metres.

Computational Solution: Newton's Method

Newton's method formula is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{9}$$

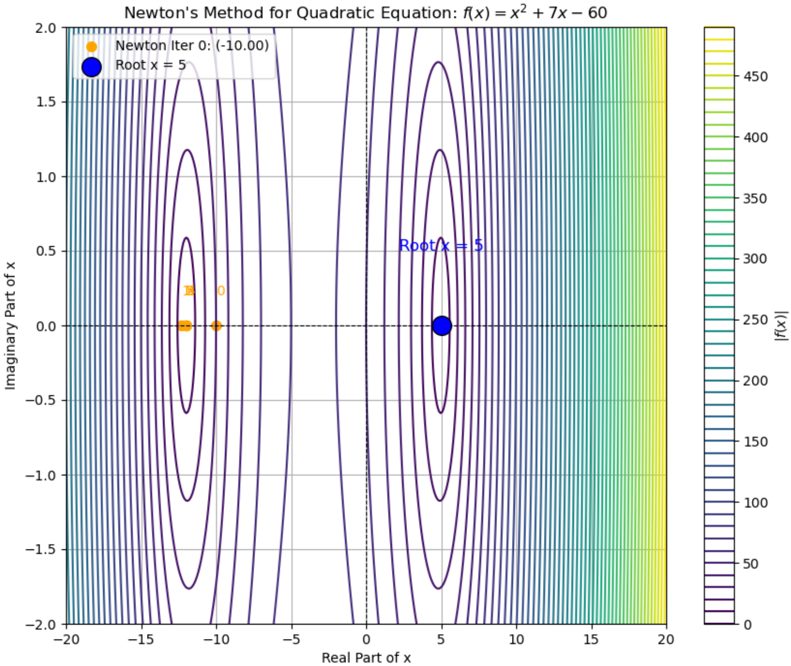
Define $f(x) = x^2 + 7x - 60$ and $f'(x) = 2x + 7$. The iterative formula becomes:

$$x_{n+1} = x_n - \frac{x_n^2 + 7x_n - 60}{2x_n + 7} \tag{10}$$

Starting with an initial guess $x_0 = 0$:

$$x_1 = 5 \quad \text{(converges to 5)} \tag{11}$$

Thus, $x = 5$ is the root.



Eigenvalue Solution Using Companion Matrix:

The quadratic equation $x^2 + 7x - 60 = 0$ can be solved by finding the eigenvalues of its companion matrix:

$$A = \begin{bmatrix} 0 & -60 \\ 1 & -7 \end{bmatrix} \tag{12}$$

QR Decomposition Process

Step 1: QR Decomposition

For each iteration, we decompose A_k into:

$$A_k = Q_k R_k,$$

where:

- Q_k is an orthogonal matrix, satisfying $Q_k^\top Q_k = I$,
- R_k is an upper triangular matrix.

Step 2: Matrix Updates

Once the QR decomposition is performed, we update the matrix as:

$$A_{k+1} = R_k Q_k.$$

The process is repeated until A_k converges to an upper triangular matrix T, whose diagonal elements are the eigenvalues of the original matrix.

Detailed Steps for QR Decomposition:

1) Start with $A_0 = A$:
$$A_0 = \begin{bmatrix} 0 & 60 \\ 1 & -7 \end{bmatrix}.$$
2) Compute the QR decomposition of A_0:
$$Q_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad R_0 = \begin{bmatrix} 1 & -7 \\ 0 & 60 \end{bmatrix}.$$
3) Update A_1 as:
$$A_1 = R_0 Q_0 = \begin{bmatrix} -7 & 60 \\ 60 & 0 \end{bmatrix}.$$
4) Repeat the QR decomposition for A_1:
$$Q_1 = \begin{bmatrix} -0.116 & 0.993 \\ 0.993 & 0.116 \end{bmatrix}, \quad R_1 = \begin{bmatrix} -60.57 & 6.93 \\ 0 & 60.28 \end{bmatrix}.$$
5) Update A_2:
$$A_2 = R_1 Q_1 = \begin{bmatrix} -6.93 & 60.28 \\ 60.28 & 0.57 \end{bmatrix}.$$

Final Step: Convergence

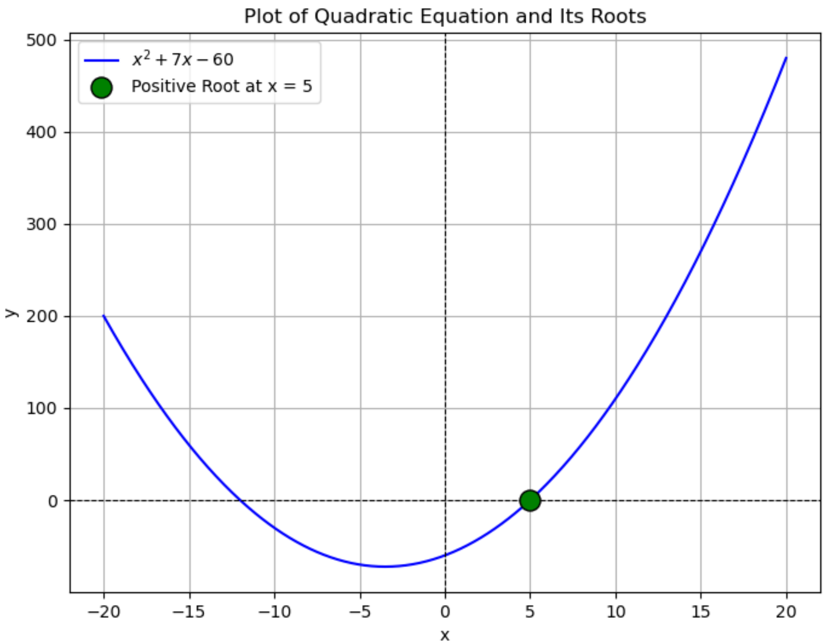
This iterative process is continued until A_k becomes an upper triangular matrix:

$$T = \begin{bmatrix} 5 & 0 \\ 0 & -12 \end{bmatrix}.$$

The eigenvalues of T are $\lambda_1 = 5$ and $\lambda_2 = -12$, which are the roots of the original quadratic equation.

Thus, the solution is $x = 5$.

Conclusion: The pole can be erected at distances of 5 metres from gate B and 12 metres from gate A.


Plot of Quadratic Equation and Its Roots
$x^2 + 7x - 60$
Positive Root at x = 5
500
400
300
200
100
0
y
−20
−15
−10
−5
0
5
10
15
20
x