Solve the differential equation and Verify the solution

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Problem Statement

Solve the differential equation,

$$\frac{d^2y}{dx^2} + y = 0 \tag{2.1}$$

with inital conditions $x = 0$ and $y = 0$

Laplace Transform Solution I

Given:

$$\frac{d^2y}{dx^2} + y = 0 \tag{3.1}$$

Taking the Laplace Transform of both sides:

$$\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} + \mathcal{L}\{y\} = \mathcal{L}\{0\} \tag{3.2}$$

Using properties of the Laplace Transform:

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 0 \tag{3.3}$$

Laplace Transform Solution II

Substituting the initial conditions $y(0) = C_1$ and $y'(0) = C_2$:

$$s^2 Y(s) - sC_1 - C_2 + Y(s) = 0 \tag{3.4}$$

$$\left(s^2 + 1\right) Y(s) = sC_1 + C_2 \tag{3.5}$$

$$Y(s) = \frac{sC_1 + C_2}{s^2 + 1} \tag{3.6}$$

The Region of Convergence (ROC) is the entire s-plane since $s^2 + 1 \neq 0$ for all real s.

Taking the inverse Laplace Transform:

$$y(x) = \mathcal{L}^{-1}\left\{\frac{sC_1 + C_2}{s^2 + 1}\right\} \tag{3.7}$$

$$= C_1 \cos(x) + C_2 \sin(x) \tag{3.8}$$

Thus, the general solution is:

$$y(x) = C_1 \cos(x) + C_2 \sin(x) \tag{3.9}$$

Z-Transform I

Let the differential equation be:

$$\frac{d^2y}{dx^2} + y = 0$$

Using the Z-Transform:
Taking the Z-transform of both sides:

$$Z\left\{\frac{d^2y}{dx^2}\right\} + Z\{y\} = Z\{0\} \tag{3.10}$$

$$z^2Y(z) - zy(0) - y'(0) + Y(z) = 0 \tag{3.11}$$

Rearranging terms:

$$(z^2+1)Y(z) = zy(0) + y'(0) \tag{3.12}$$

$$Y(z) = \frac{zy(0) + y'(0)}{z^2+1} \tag{3.13}$$

Z-Transform II

Substituting the initial conditions $y(0) = y_0$ and $y'(0) = y_1$, we get:

$$Y(z) = \frac{zy_0 + y_1}{z^2 + 1} \tag{3.14}$$

Taking the inverse Z-transform:

$$y(x) = \mathcal{Z}^{-1}\left(\frac{1}{z^2 + 1}\right) \tag{3.15}$$

$$y(x) = y_0 \cos(x) + y_1 \sin(x) \tag{3.16}$$

Thus, the solution using Z-transform is:

$$y(x) = y_0 \cos(x) + y_1 \sin(x) \tag{3.17}$$

Bilinear Transform I

Bilinear Transform Mapping: The Bilinear Transform is used to convert a continuous-time system into a discrete-time system using:

$$s = \frac{2}{T} \cdot \frac{1 - z^{-1}}{1 + z^{-1}}$$

where T is the sampling period.

Continuous-Time Transfer Function: The given system has the transfer function:

$$H(s) = \frac{1}{s^2 + 1}$$

Discrete-Time Transfer Function: Substituting s from the Bilinear Transform, we obtain:

$$H(z) = \frac{(1 + z^{-1})^2}{(1 + z^{-1})^2 + \frac{4}{T^2}(1 - z^{-1})^2}$$

This expression represents the discrete equivalent of the continuous-time system.

Bilinear Transform II

Difference Equation Derivation: Expanding and simplifying the denominator, the discrete-time representation can be rewritten as a second-order difference equation:

$$y[n+2] - 2y[n+1] + y[n] = 0$$

General Solution of the Discrete System: The difference equation solution follows the sinusoidal form:

$$y[n] = C_1 \cos\left(\frac{\pi n}{T}\right) + C_2 \sin\left(\frac{\pi n}{T}\right)$$

where C_1 and C_2 are constants determined by initial conditions.

Difference Equation

From the definition of the second-order differentiation:

$$\frac{d^2y}{dx^2} \approx \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} \tag{3.18}$$

Substituting into the differential equation:

$$\frac{d^2y}{dx^2} + y = 0 \tag{3.19}$$

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} + y_n = 0 \tag{3.20}$$

Simplifying:

$$y_{n+1} = 2y_n - y_{n-1} - h^2 y_n \tag{3.21}$$

Let $x_0 = 0$, $y_0 = C_1$, $y_1 = C_1 + hC_2$. For small step size h, iteratively calculate y_{n+1}.

Plotting the curve