Identify and Solve a Quadratic Equation

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Problem Statement

A pole has to be erected at a point on the boundary of a circular park of diameter 13 metres in such a way that the difference of its distances from two diametrically opposite fixed gates A and B on the boundary is 7 metres. Is it possible to do so? If yes, at what distances from the two gates should the pole be erected?

Quadratic Equation

Let the distance of the pole from gate B be x metres. Then the distance from gate A will be $(x + 7)$ metres.
From the Pythagoras theorem, since $\triangle APB$ is a right triangle:

$$AB^2 = AP^2 + BP^2 \tag{3.1}$$

Given $AB = 13$ metres, substituting values:

$$13^2 = (x + 7)^2 + x^2 \tag{3.2}$$

$$169 = x^2 + 14x + 49 + x^2 \tag{3.3}$$

$$2x^2 + 14x - 120 = 0 \tag{3.4}$$

Simplify the equation:

$$x^2 + 7x - 60 = 0 \tag{3.5}$$

Theoretical Solution

Using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $a = 1, b = 7, c = -60$:

$$x = \frac{-7 \pm \sqrt{7^2 - 4(1)(-60)}}{2(1)} \tag{3.6}$$

$$x = \frac{-7 \pm \sqrt{289}}{2} \tag{3.7}$$

$$x = \frac{-7 \pm 17}{2} \tag{3.8}$$

Thus, $x = 5$ or $x = -12$. Since distance cannot be negative, $x = 5$. The distances are: $BP = 5$ metres, $AP = 12$ metres.

Computational Newton's Method I

Newton's method formula is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{3.9}$$

Define $f(x) = x^2 + 7x - 60$ and $f'(x) = 2x + 7$. The iterative formula becomes:

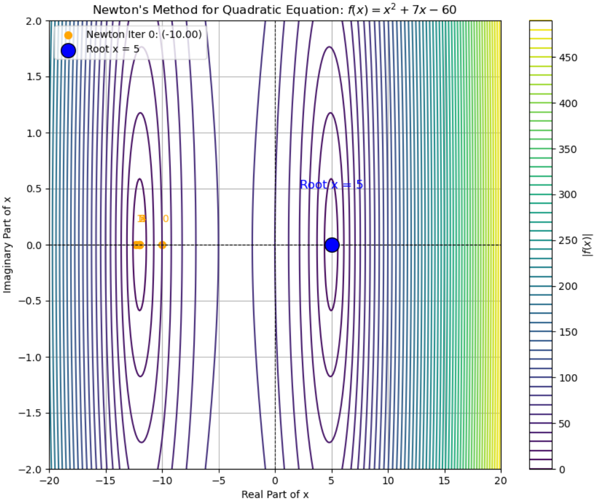
$$x_{n+1} = x_n - \frac{x_n^2 + 7x_n - 60}{2x_n + 7} \tag{3.10}$$

Starting with an initial guess $x_0 = 0$:

$$x_1 = 5 \quad \text{(converges to 5)} \tag{3.11}$$

Thus, $x = 5$ is the root.

Computational Newton's Method II



Companion Matrix I

We are given the quadratic equation $x^2 + 7x - 60 = 0$ and need to solve it using the QR decomposition method. The companion matrix is:

$$A_0 = \begin{bmatrix} 0 & -60 \\ 1 & -7 \end{bmatrix}$$

Step 1: QR Decomposition of A_0 The QR decomposition of A_0 is:

$$Q_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad R_0 = \begin{bmatrix} 1 & -7 \\ 0 & 60 \end{bmatrix}$$

Step 2: Update A_1 Now, we update $A_1 = R_0 Q_0$:

$$A_1 = \begin{bmatrix} -7 & 60 \\ 60 & 0 \end{bmatrix}$$

Companion Matrix II

Step 3: QR Decomposition of A_1 Next, we compute the QR decomposition of A_1:

$$Q_1 = \begin{bmatrix} -0.116 & 0.993 \\ 0.993 & 0.116 \end{bmatrix}, \quad R_1 = \begin{bmatrix} -60.57 & 6.93 \\ 0 & 60.28 \end{bmatrix}$$

Step 4: Update A_2 We update $A_2 = R_1 Q_1$:

$$A_2 = \begin{bmatrix} -6.93 & 60.28 \\ 60.28 & 0.57 \end{bmatrix}$$

Step 5: Convergence Continuing the iterations, we eventually reach the matrix:

$$T = \begin{bmatrix} 5 & 0 \\ 0 & -12 \end{bmatrix}$$

The eigenvalues of T are $\lambda_1 = 5$ and $\lambda_2 = -12$, which are the roots of the quadratic equation.

Companion Matrix III

Thus, the solutions to the equation $x^2 + 7x - 60 = 0$ are:

$$x = 5 \quad \text{and} \quad x = -12$$

Plotting the curve