9.3.11

EE24BTECH11022 - ESHAN SHARMA

Question: Solve the differential equation $\frac{d^2y}{dx^2} + y = 0$ and verify if the general solution is $y = C_1e^x + C_2e^{-x}$.

Solution:

Solution Using Laplace Transform:
Given:

$$\frac{d^2y}{dx^2} + y = 0 \tag{0.1}$$

Taking the Laplace Transform of both sides:

$$\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} + \mathcal{L}\{y\} = \mathcal{L}\{0\} \tag{0.2}$$

Using properties of the Laplace Transform:

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 0 \tag{0.3}$$

Substituting the initial conditions $y(0) = C_1$ and $y'(0) = C_2$:

$$s^2Y(s) - sC_1 - C_2 + Y(s) = 0 \tag{0.4}$$

$$\left(s^2 + 1\right)Y(s) = sC_1 + C_2 \tag{0.5}$$

$$Y(s) = \frac{sC_1 + C_2}{s^2 + 1} \tag{0.6}$$

The Region of Convergence (ROC) is the entire s-plane since $s^2 + 1 \neq 0$ for all real s.

Taking the inverse Laplace Transform:

$$y(x) = \mathcal{L}^{-1}\left\{\frac{sC_1 + C_2}{s^2 + 1}\right\} \tag{0.7}$$

$$= C_1\cos(x) + C_2\sin(x) \tag{0.8}$$

Thus, the general solution is:

$$y(x) = C_1\cos(x) + C_2\sin(x) \tag{0.9}$$

Using Difference Equation to Approximate Solution:
This method approximates the solution by discretizing the function.
From the definition of the second-order differentiation:

$$\frac{d^2y}{dx^2} \approx \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} \tag{0.10}$$

Substituting into the differential equation:

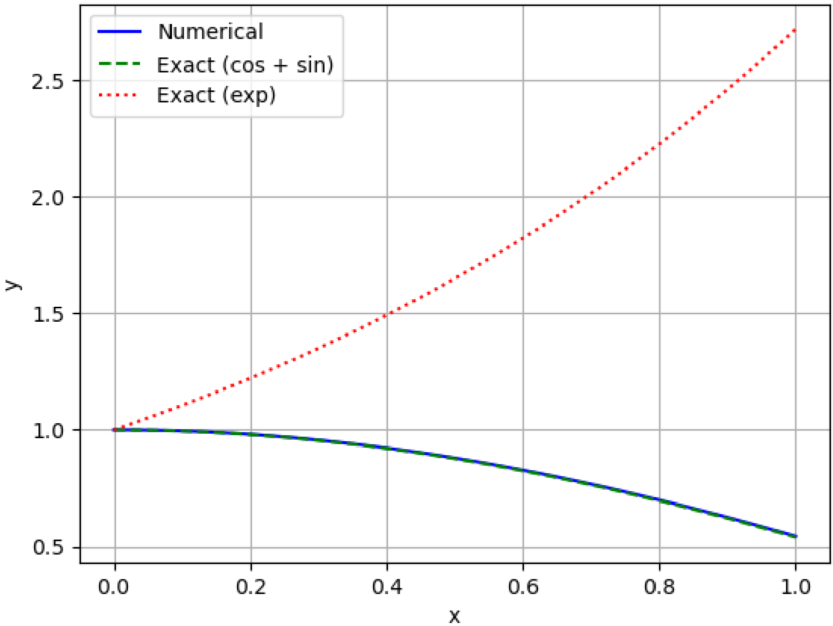
$$\frac{d^2y}{dx^2} + y = 0 \tag{0.11}$$

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} + y_n = 0 \tag{0.12}$$

Simplifying:

$$y_{n+1} = 2y_n - y_{n-1} - h^2 y_n \tag{0.13}$$

Let $x_0 = 0$, $y_0 = C_1$, $y_1 = C_1 + hC_2$. For small step size h, iteratively calculate y_{n+1}.



By comparing the plots, the numerical solution matches the exact solution, verifying the correctness. Additionally, the plot of the given question does not match our solution, so Option A is not correct.