9.3.11

EE24BTECH11022 - ESHAN SHARMA

Question: Solve the differential equation A) $\frac{d^2y}{dx^2} + y = 0$ and verify if the general solution is $y = C_1e^x + C_2e^{-x}$.

Solution:

Solution Using Laplace Transform:
Given:

$$\frac{d^2y}{dx^2} + y = 0 \tag{0.1}$$

Taking the Laplace Transform of both sides:

$$\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} + \mathcal{L}\{y\} = \mathcal{L}\{0\} \tag{0.2}$$

Using properties of the Laplace Transform:

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 0 \tag{0.3}$$

Substituting the initial conditions $y(0) = C_1$ and $y'(0) = C_2$:

$$s^2Y(s) - sC_1 - C_2 + Y(s) = 0 \tag{0.4}$$

$$\left(s^2 + 1\right)Y(s) = sC_1 + C_2 \tag{0.5}$$

$$Y(s) = \frac{sC_1 + C_2}{s^2 + 1} \tag{0.6}$$

The Region of Convergence (ROC) is the entire s-plane since $s^2 + 1 \neq 0$ for all real s.

Taking the inverse Laplace Transform:

$$y(x) = \mathcal{L}^{-1}\left\{\frac{sC_1 + C_2}{s^2 + 1}\right\} \tag{0.7}$$

$$= C_1\cos(x) + C_2\sin(x) \tag{0.8}$$

Thus, the general solution is:

$$y(x) = C_1\cos(x) + C_2\sin(x) \tag{0.9}$$

Solving the Differential Equation using Z-Transform and Bilinear Transform:
Let the differential equation be:

$$\frac{d^2y}{dx^2} + y = 0$$

Using the Z-Transform:
Taking the Z-transform of both sides:

$$Z\left\{\frac{d^2y}{dx^2}\right\} + Z\{y\} = Z\{0\} \tag{0.10}$$

$$z^2Y(z) - zy(0) - y'(0) + Y(z) = 0 \tag{0.11}$$

Rearranging terms:

$$(z^2 + 1)Y(z) = zy(0) + y'(0) \tag{0.12}$$

$$Y(z) = \frac{zy(0) + y'(0)}{z^2 + 1} \tag{0.13}$$

Substituting the initial conditions $y(0) = y_0$ and $y'(0) = y_1$, we get:

$$Y(z) = \frac{zy_0 + y_1}{z^2 + 1} \tag{0.14}$$

Taking the inverse Z-transform:

$$y(x) = \mathcal{Z}^{-1}\left(\frac{1}{z^2 + 1}\right) \tag{0.15}$$

$$y(x) = y_0 \cos(x) + y_1 \sin(x) \tag{0.16}$$

Thus, the solution using Z-transform is:

$$y(x) = y_0 \cos(x) + y_1 \sin(x) \tag{0.17}$$

Using the Bilinear Transform:
The Bilinear Transform is a method used to convert a continuous-time system (in the Laplace domain) into a discrete-time system (in the Z-domain). The mapping is given by the relation:

$$s = \frac{2}{T} \cdot \frac{1 - z^{-1}}{1 + z^{-1}}$$

where T is the sampling period, and z^{-1} is the inverse Z-transform variable.

The continuous-time transfer function for the system is:

$$H(s) = \frac{1}{s^2 + 1}$$

Step 1: Apply the Bilinear Transform
Substitute the Bilinear Transform relationship for s into the continuous-time transfer function $H(s)$:

$$H(z) = \frac{1}{\left(\frac{2}{T} \cdot \frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 1}$$

This equation expresses the continuous-time transfer function $H(s)$ in terms of z, mapping the system to the discrete-time domain.

Step 2: Simplifying the Expression

Simplify the denominator by expanding the square and combining terms:

$$H(z) = \frac{1}{\frac{4}{T^2} \cdot \frac{(1-z^{-1})^2}{(1+z^{-1})^2} + 1}$$

The result is a transfer function $H(z)$ that describes the discrete-time system.

Step 3: Inverse Z-Transform to Find the Discrete-Time Response

Once we have $H(z)$, we can apply the inverse Z-transform to find the response in the discrete-time domain. The inverse Z-transform of a system with a denominator like $z^2 + 1$ corresponds to sinusoidal functions, specifically:

$$\mathcal{Z}^{-1}\left(\frac{1}{z^2+1}\right) = \cos(x)$$

Thus, the discrete-time response becomes:

$$y[n] = y_0 \cos\left(\frac{\pi n}{T}\right) + y_1 \sin\left(\frac{\pi n}{T}\right)$$

where: - y_0 and y_1 are the initial conditions for the discrete-time system, - n is the discrete time index, and - T is the sampling period.

Using Difference Equation to Approximate Solution:

This method approximates the solution by discretizing the function.
From the definition of the second-order differentiation:

$$\frac{d^2y}{dx^2} \approx \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} \tag{0.18}$$

Substituting into the differential equation:

$$\frac{d^2y}{dx^2} + y = 0 \tag{0.19}$$

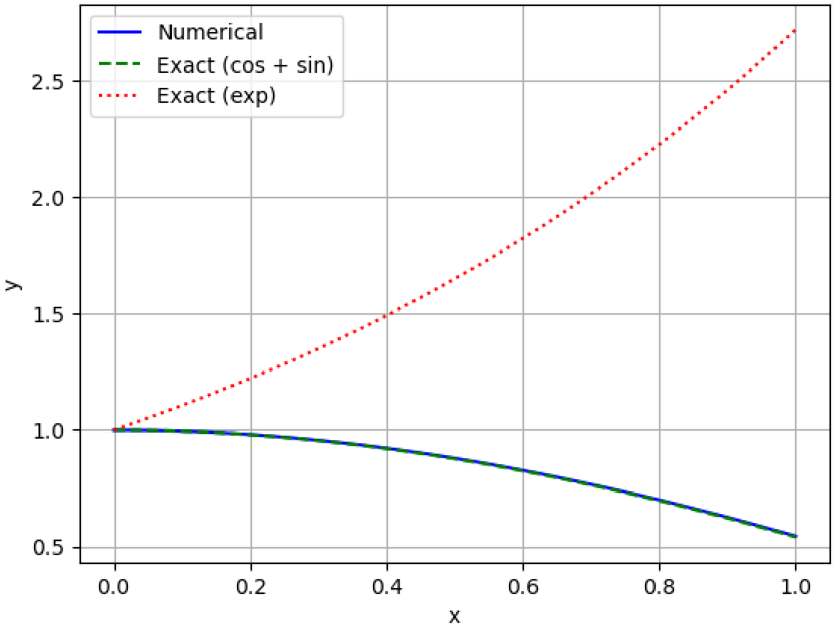
$$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} + y_n = 0 \tag{0.20}$$

Simplifying:

$$y_{n+1} = 2y_n - y_{n-1} - h^2 y_n \tag{0.21}$$

Let $x_0 = 0$, $y_0 = C_1$, $y_1 = C_1 + hC_2$. For small step size h, iteratively calculate y_{n+1}.

By comparing the plots, the numerical solution matches the exact solution, verifying the correctness. Additionally, the plot of the given question does not match our solution, so Option A is not correct.


Numerical
Exact (cos + sin)
Exact (exp)
2.5
2.0
1.5
1.0
0.5
y
0.0
0.2
0.4
0.6
0.8
1.0
x