9.3.23

EE24BTECH11021 - Eshan Ray

Question:
Find the area bounded by the ellipse $x^2 + 4y^2 = 16$ and the ordinates $x = 0$ and $x = 2$, using integration.
Solution:

Variable	Description	value
V	Quadratic Form Matrix	$\begin{pmatrix} 4 & 0 \\ 0 & 16 \end{pmatrix}$
u	Linear Coefficient Vector	0
f	Constant Term	-64
h	Point on the line $x = 2$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
m	slope of the line $x = 2$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
k_i	constant for point of intersection of the curve and the line	$-$

TABLE 0: Input parameters

The point of intersection of the line with the ellipse is $x_i = h + k_i m$,
where,k_i is a constant and is calculated as follows:-

$$k_i = \frac{1}{m^\top V m}\left(-m^\top (Vh + u) \pm \sqrt{\left[m^\top (Vh + u)\right]^2 - g(h)\left(m^\top V m\right)}\right)$$

$$k_i = \frac{1}{\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 16 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}\left(-\begin{pmatrix} 0 & 1 \end{pmatrix}\left(\begin{pmatrix} 4 & 0 \\ 0 & 16 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} + 0\right) \pm \right.$$
$$\left.\sqrt{\left[\begin{pmatrix} 0 & 1 \end{pmatrix}\left(\begin{pmatrix} 4 & 0 \\ 0 & 16 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} + 0\right)\right]^2 - g(h)\left(\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 16 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)}\right) \tag{1}$$

We get,

$$k_i = \sqrt{3}, -\sqrt{3}$$

$$x_1 = \begin{pmatrix}2\\0\end{pmatrix} + \left(\sqrt{3}\right)\begin{pmatrix}0\\1\end{pmatrix} \tag{2}$$

$$\implies x_1 = \begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}0\\\sqrt{3}\end{pmatrix} \tag{3}$$

$$\implies x_1 = \begin{pmatrix}2\\\sqrt{3}\end{pmatrix} \tag{4}$$

$$x_2 = \begin{pmatrix}2\\0\end{pmatrix} + \left(-\sqrt{3}\right)\begin{pmatrix}0\\1\end{pmatrix} \tag{5}$$

$$\implies x_2 = \begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}0\\-\sqrt{3}\end{pmatrix} \tag{6}$$

$$\implies x_2 = \begin{pmatrix}2\\-\sqrt{3}\end{pmatrix} \tag{7}$$

$\therefore$ The bounded area is given by:-

$$Area = 2\int_0^2 \frac{\sqrt{16-x^2}}{2}\,dx \tag{8}$$

$$\implies Area = \int_0^2 \sqrt{16-x^2}\,dx \tag{9}$$

$$(Substituting, x = 4\sin\theta) \tag{10}$$

$$\implies dx = 4\cos\theta d\theta \tag{11}$$

$$\implies Area = \int_0^{\frac{\pi}{6}} \sqrt{16-(4\sin\theta)^2}\,4\cos\theta d\theta \tag{12}$$

$$\implies = 16\int_0^{\frac{\pi}{6}} \sqrt{1-\sin^2\theta}\,\cos\theta d\theta \tag{13}$$

$$\implies = 16\int_0^{\frac{\pi}{6}} \cos^2\theta\,d\theta \tag{14}$$

$$\implies = 16\int_0^{\frac{\pi}{6}} \frac{(1+\cos 2\theta)}{2}\,d\theta \tag{15}$$

$$\implies = 8\left([\theta]_0^{\frac{\pi}{6}} + \left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{6}} \right) \tag{16}$$

$$\implies Area = \frac{4\pi}{3} + 2\sqrt{3} \tag{17}$$

So, the required area is $\frac{4\pi}{3} + 2\sqrt{3}units$.

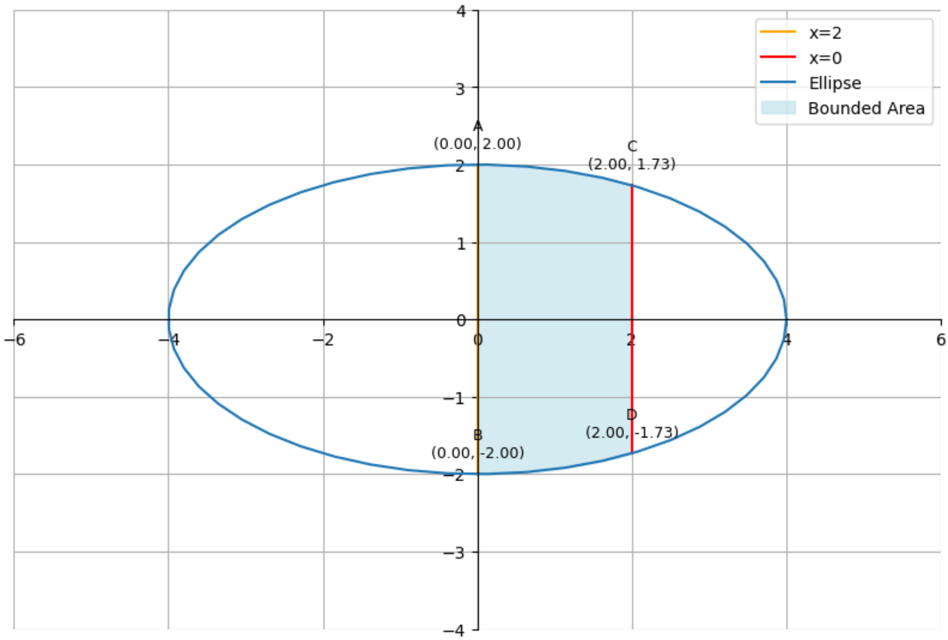

Fig. 0: Intersection of line and ellipse