

10.4.2.3

EE24BTECH11021 - Eshan Ray

Question:

Find two numbers whose sum is 27 and product is 182

Solution: Let one of the numbers be x

So, the other number is $27 - x$

Given,

$$x(27 - x) = 182 \quad (1)$$

$$27x - x^2 = 182 \quad (2)$$

$$x^2 - 27x + 182 = 0 \quad (3)$$

$$(x - 13)(x - 14) = 0 \quad (4)$$

$$\Rightarrow x = 13, 14 \quad (5)$$

So, the numbers are 13 and 14

Computational Solution:

Using Newton- Raphson Method we get,

We start by taking an initial guess and then iteratively we use the following equation to find the roots of the quadratic equation :-

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (6)$$

$$f(x) = x^2 - 27x + 182 \quad (7)$$

$$f'(x) = 2x - 27 \quad (8)$$

$$x_{n+1} = x_n - \frac{x_n^2 - 27x_n + 182}{2x_n - 27} \quad (9)$$

After running the code, we obtained the following results:-

$$\text{Root-1: } 14.00000000 \quad (10)$$

$$\text{Root-2: } 13.00000000 \quad (11)$$

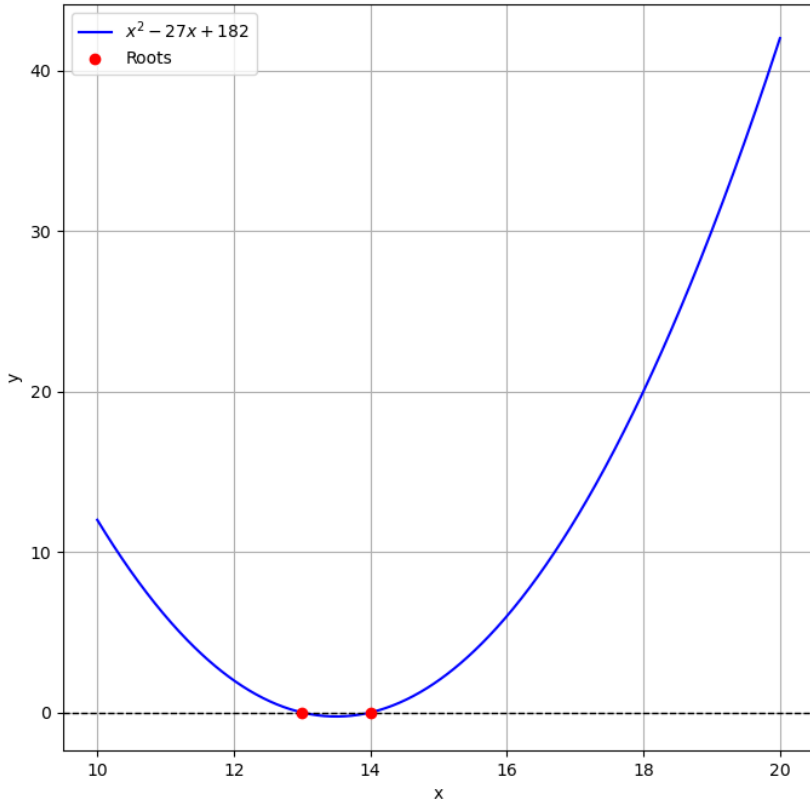


Fig. 0: Plot of the quadratic equation using newton-Raphson method

Alternate Method: Eigenvalues of Companion Matrix

In this method, we find the roots of any polynomial of the form $x^n + a_{n-1}x^{n-1} + \dots + ax + a_0 = 0$ by finding the eigenvalues of the Companion Matrix (C) given below:-

$$C = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix} \quad (12)$$

For the Quadratic Equation $x^2 - 27x + 182 = 0$, we get the following companion Matrix

$$C = \begin{pmatrix} 0 & 1 \\ -182 & 27 \end{pmatrix} \quad (13)$$

The roots of the equation is the eigenvalues of the matrix C which has been calculated using the QR Decomposition with shifts process.

The details of the process is given below:-

QR Decomposition : Gram-schmidt Process

- 1) In the QR Decomposition, the matrix A is decomposed into matrices Q and R as:

$$A = QR \quad (14)$$

where Q is an orthogonal matrix and R is an upper triangular matrix.

- 2) We start by producing an orthogonal set of column vectors of $Q \{q_1, q_2, \dots, q_n\}$ from a set of column vectors of $A \{a_1, a_2, \dots, a_n\}$.
- 3) For orthogonalization we subtract each vector a_i with the projections of all previously obtained orthogonal vectors q_1, q_2, \dots, q_{i-1} to make q_i orthogonal to them.

The projection of a_i onto a vector q_j is calculated as:

$$proj_{q_j}(a_i) = \frac{\langle a_i, q_j \rangle}{\langle q_j, q_j \rangle} q_j \quad (15)$$

Then q_i is computed as:

$$q_i = a_i - \sum_{j=1}^{i-1} proj_{q_j}(a_i) \quad (16)$$

Then all the q_i 's are normalized by :

$$q_i = \frac{q_i}{\|q_i\|} \quad (17)$$

The process is repeated for all the columns of A

- 3) As Q is an orthonormal matrix

$$Q^T Q = I \quad (18)$$

So, R can be represented as follows

$$R = Q^T A \quad (19)$$

$$r_{ij} = \langle a_j, q_i \rangle, \text{ for } i \leq j \quad (20)$$

QR With shift

Basic QR algorithm:

In the basic QR algorithm, the matrix A is decomposed into matrices Q and R as:

$$A = QR \quad (21)$$

Then, the new matrix A_{new} is computed as:

$$A_{new} = RQ \quad (22)$$

This process is repeated until the off-diagonal elements of the matrix become negligibly small, at which point the diagonal elements approximate the eigenvalues of the original matrix.

The shifted QR iteration is as follows:

- 1) Modify the matrix by subtracting it with μI where μ is a shift parameter, thus giving the following matrix:

$$A' = A - \mu I \quad (23)$$

- 2) Compute the QR decomposition of A' and revert the shift in the next matrix as

$$A'' = RQ + \mu I \quad (24)$$

- 3) Repeat the process until the matrix becomes nearly upper triangular, i.e., the off-diagonal elements are sufficiently small, indicating that the process is completed and eigenvalues of the matrix is computed.

$$\text{Roots computed : [14.0, 13.0]} \quad (25)$$

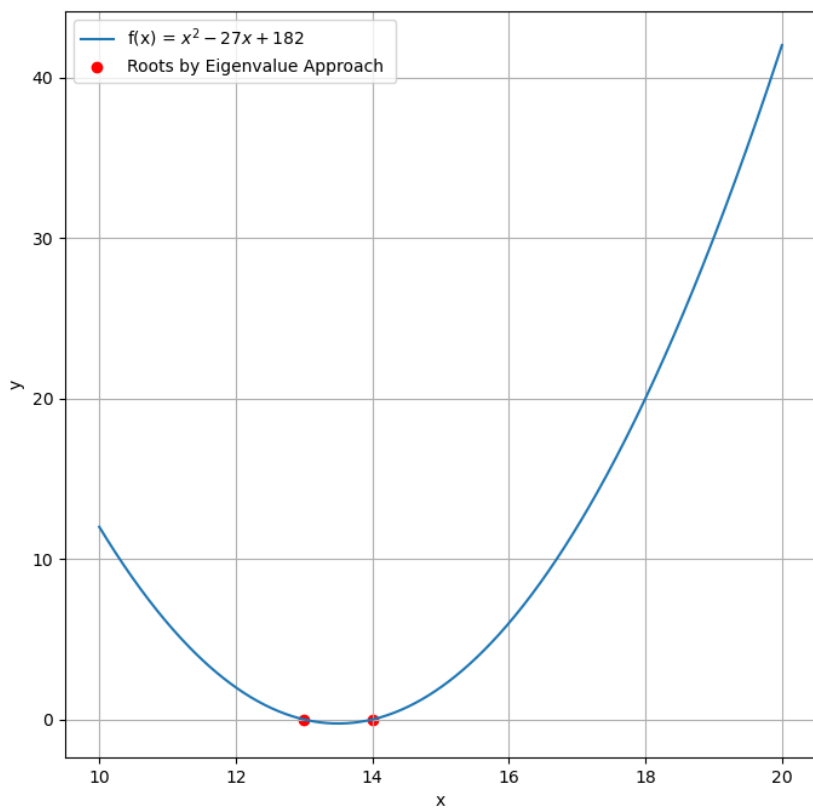


Fig. 3: Plot of the quadratic equation by eigenvalue approach