

9.4.5

EE24BTECH11021 - Eshan Ray

Question:

For the Differential Equation $(e^{-x} + e^x) dy - (e^x - e^{-x}) dx = 0$, find a general solution of the differential equation.

Solution: Solving the given D.E. , we get,

$$(e^{-x} + e^x) dy - (e^x - e^{-x}) dx = 0 \quad (1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x - e^{-x}}{e^{-x} + e^x} \quad (2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{2x} - 1}{e^{2x} + 1} \quad (3)$$

Substituting, $t = e^{2x}$, we get,

$$\Rightarrow dt = 2e^{2x} dx \quad (4)$$

$$\Rightarrow dx = \frac{dt}{2t} \quad (5)$$

$$\Rightarrow \frac{dy}{\left(\frac{dt}{2t}\right)} = \frac{t-1}{t+1} \quad (6)$$

$$\Rightarrow \frac{dy}{dt} = \frac{t-1}{2t(t+1)} \quad (7)$$

$$\Rightarrow \frac{dy}{dt} = \frac{1}{2} \left(\frac{1}{(t+1)} - \frac{1}{t(t+1)} \right) \quad (8)$$

$$\Rightarrow dy = \frac{1}{2} \left(\frac{2}{t+1} - \frac{1}{t} \right) dt \quad (9)$$

$$(10)$$

Integrating both sides, we get,

$$\Rightarrow \int dy = \int \frac{dt}{t+1} - \int \frac{dt}{2t} \quad (11)$$

$$\Rightarrow y = \ln|t+1| - \frac{1}{2} \ln|t| + C \quad (12)$$

$$(13)$$

substituting, $t = e^{2x}$

$$\Rightarrow y = \ln|e^{2x} + 1| - \frac{1}{2} \ln|e^{2x}| + C \quad (14)$$

$$\Rightarrow y = \ln(e^{2x} + 1) - x + C \quad (15)$$

Computational Solution:

Using Trapezoidal rule, we get,

$$f(x) = \frac{dy}{dx} \quad (16)$$

$$\int_{x_0}^{x_n} f(x) \approx \frac{h}{2} ((f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + \dots (f(x_{n-1}) + f(n))) \quad (17)$$

$$\int_{x_0}^{x_n} f(x) \approx \frac{h}{2} \left(f(x_0) + f(x_n) + 2 \sum_{i=1}^{n-1} f(x_i) \right) \quad (18)$$

$$(19)$$

Using Trapezoid rule for discretizing the steps , we get,

$$y_{n+1} - y_n = \frac{h}{2} (f(x_n) + f(x_{n+1})) \quad (20)$$

$$(21)$$

By the classical definition of derivative we know that $f(x_{n+1}) = f(x_n) + hf'(x_n)$

$$y_{n+1} - y_n = \frac{h}{2} (f(x_n) + f(x_n) + hf'(x_n)) \quad (22)$$

$$y_{n+1} = y_n + hf(x_n) + \frac{h^2}{2} f'(x_n) \quad (23)$$

The difference equation,

$$y_{n+1} = y_n + h \frac{e^{2x} - 1}{e^{2x} + 1} + \frac{h^2}{2} \frac{4e^{2x}}{(e^{2x} + 1)^2} \quad (24)$$

$$x_{n+1} = x + h \quad (25)$$

The initial conditions for the plotting of graph are as follows:-

$$x_0 = -5 \quad (26)$$

$$y_0 = 5 \quad (27)$$

$$h = 0.01 \quad (28)$$

$$C = 0 \quad (29)$$

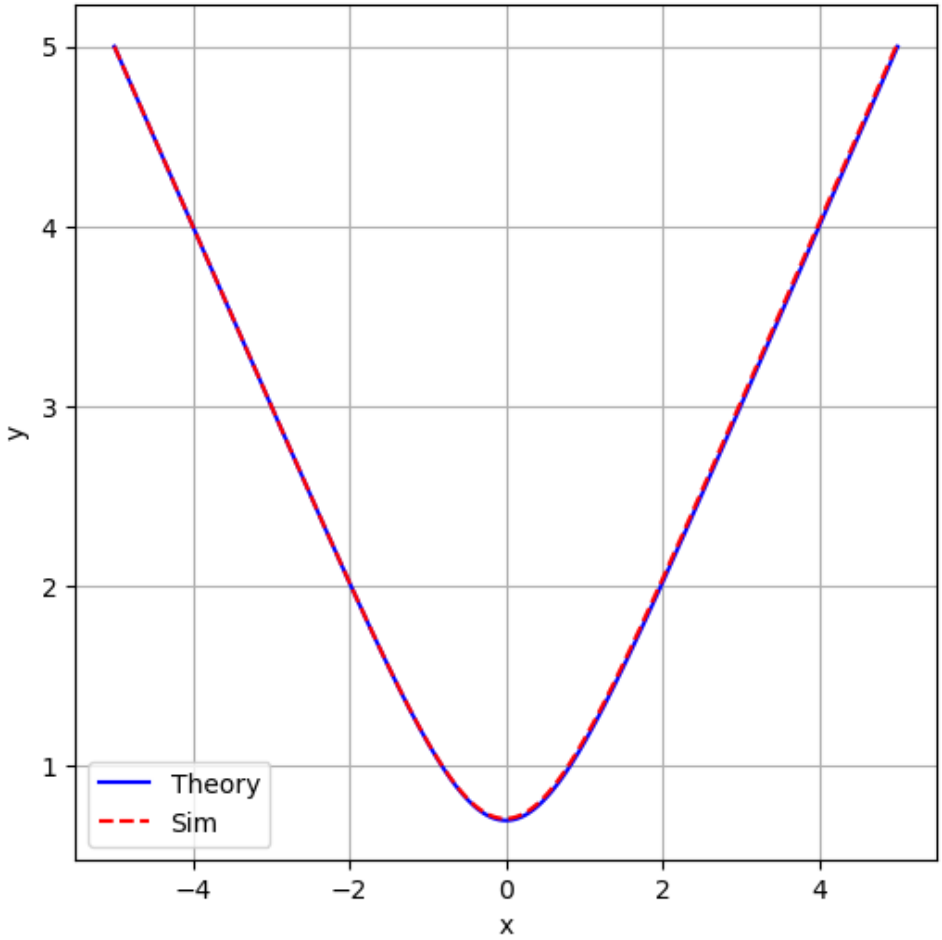


Fig. 0: Plot of the differential equation when $h = 0.01$