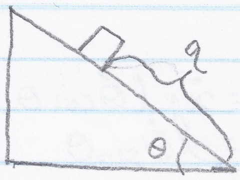
1) 

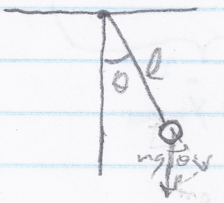
$T = \frac{1}{2} m(\dot{q})^2 \quad V = mgh = mgq\sin\theta$

$L = \frac{1}{2} m\dot{q}^2 - mgq\sin\theta$

$\frac{\partial L}{\partial \dot{q}} = m\dot{q} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = m\ddot{q} \quad \frac{\partial L}{\partial q} = -mg\sin\theta$

$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 = -mg\sin\theta - m\ddot{q} \Rightarrow \quad \cancel{m}\ddot{q} = -g\sin\theta$ ✓

2) <u>Newton's law</u>



$I = ml^2$

$\alpha = \ddot{\theta}$

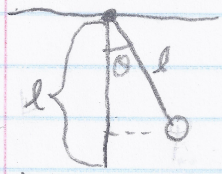
$\tau = -mgl\sin\theta$

$\tau = I\alpha$

$-mgl\sin\theta = ml^2\ddot{\theta}$

$\boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta}$

<u>Lagrangian</u>



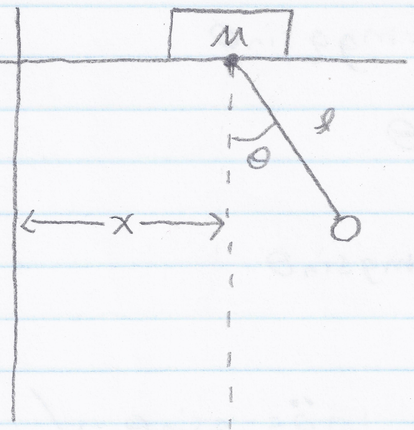
$T = \frac{1}{2} ml^2\dot{\theta}^2 \quad V = mgh = mgl(1-\cos\theta)$

$L = \frac{1}{2} ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$

$\frac{\partial L}{\partial \dot{q}} = ml^2\dot{\theta} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = ml^2\ddot{\theta} \quad \frac{\partial L}{\partial q} = -mgl\sin\theta$

$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 = -mgl\sin\theta - ml^2\ddot{\theta} \Rightarrow \boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta}$ ✓

3)



$$x_{mass} = x + l\sin\theta \qquad \dot{x}_{mass} = \dot{x} + l\dot{\theta}\cos\theta$$

$$y_{mass} = -l\cos\theta \qquad \dot{y}_{mass} = l\dot{\theta}\sin\theta$$

$$T = \frac{1}{2}M(\dot{x})^2 + \frac{1}{2}m\left(\dot{x}_{mass}^2 + \dot{y}_{mass}^2\right) = \frac{1}{2}M\dot{x}^2 + \overbrace{\frac{1}{2}m\dot{x}^2 + ml\dot{\theta}\dot{x}\cos\theta + \frac{1}{2}ml^2\dot{\theta}^2\cos^2\theta}^{\frac{1}{2}m\dot{x}_{mass}^2}$$

$$+\underbrace{\frac{1}{2}ml^2\dot{\theta}^2\sin^2\theta}_{\frac{1}{2}m\dot{y}_{mass}^2}$$

$$T = \frac{1}{2}(M+m)\dot{x}^2 + ml\dot{\theta}\dot{x}\cos\theta + \frac{1}{2}ml^2\dot{\theta}^2$$

$$V = mg\,y_{mass} = -mgl\cos\theta$$

$$L = \frac{1}{2}(M+m)\dot{x}^2 + ml\dot{\theta}\dot{x}\cos\theta + \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

$$\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + ml\dot{\theta}\cos\theta \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta$$

$$\frac{\partial L}{\partial x} = 0 \qquad \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \boxed{0 = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta}$$

$$\frac{\partial L}{\partial \theta} = -ml\dot{\theta}\dot{x}\sin\theta - mgl\sin\theta \qquad \frac{\partial L}{\partial \dot{\theta}} = ml\dot{x}\cos\theta + ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml\ddot{x}\cos\theta - ml\dot{x}\dot{\theta}\sin\theta + ml^2\ddot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right)-\frac{\partial L}{\partial \theta}=0=ml\ddot{x}\cos\theta-ml\dot{x}\dot\theta\sin\theta+ml^2\ddot\theta+ml\dot\theta\dot{x}\sin\theta+mgl\sin\theta$$

$$\ddot{x}\cos\theta\cancel{-\dot{x}\dot\theta\sin\theta}+l\ddot\theta\cancel{+\dot{x}\dot\theta\sin\theta}+g\sin\theta=0$$

$$\boxed{\ddot\theta+\frac{\ddot{x}}{l}\cos\theta+\frac{g}{l}\sin\theta=0}$$

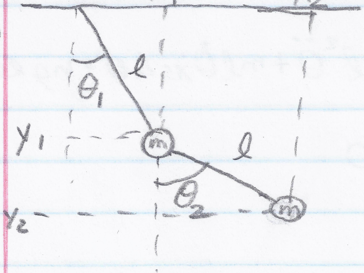
4) a) $H=\sum_j \dot{q}_j\frac{\partial L}{\partial \dot{q}_j}-L$

a) $$\frac{dH}{dt}=\left[\sum_j \cancel{\ddot{q}_j\frac{\partial L}{\partial \dot{q}_j}}+\dot{q}_j\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right]-\left[\sum_j\frac{\partial L}{\partial q_j}\dot{q}_j+\cancel{\frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j}\right]-\cancel{\frac{\partial L}{\partial t}}^{0}$$

$$=\sum_j \dot{q}_j\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}\right]=0 \text{ by E-L equation}$$

b) $p_j=\frac{\partial L}{\partial \dot{q}_j}$ $\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)=\frac{\partial L}{\partial q_j} \Rightarrow \frac{dp_j}{dt}=\cancel{\frac{\partial L}{\partial q_j}}^{0}=0$

5)



$x_1 = l\sin\theta_1 \qquad \dot{x}_1 = l\cos\theta_1\,\dot{\theta}_1$

$x_2 = l(\sin\theta_1 + \sin\theta_2) \qquad \dot{x}_2 = l(\cos\theta_1\dot{\theta}_1 + \cos\theta_2\dot{\theta}_2)$

$y_1 = -l\cos\theta_1 \qquad \dot{y}_1 = l\sin\theta_1\cdot\dot{\theta}_1$

$y_2 = -l(\cos\theta_1 + \cos\theta_2) \qquad \dot{y}_2 = l(\sin\theta_1\dot{\theta}_1 + \cos\theta_2\dot{\theta}_2)$

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2) =$$

$$\frac{1}{2}m\Big[l^2\cos^2\theta_1\,\dot{\theta}_1^2 + l^2\sin^2\theta_1\cdot\dot{\theta}_1^2 + l^2\cos^2\theta_1\,\dot{\theta}_1^2 + l^2\cos\theta_1\cos\theta_2\,\dot{\theta}_1\dot{\theta}_2 + l^2\cos^2\theta_2\,\dot{\theta}_2^2$$
$$+ l^2\sin^2\theta_1\,\dot{\theta}^2 + l^2\sin\theta_1\sin\theta_2\,\dot{\theta}_1\dot{\theta}_2 + l^2\sin^2\theta_2\,\dot{\theta}_2^2\Big]$$

$$= \frac{1}{2}m\left[l^2\dot{\theta}_1^2 + l^2\dot{\theta}_1^2 + l^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1) + l^2\dot{\theta}_2^2\right]$$

$$= ml^2\dot{\theta}_1^2 + \frac{1}{2}ml^2\dot{\theta}_2^2 + \frac{1}{2}ml^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1)$$

$$V = mgy_1 + mgy_2 = -mgl\cos\theta_1 - ml\cos\theta_1 - ml\cos\theta_2 = -2mgl\cos\theta_1 - mgl\cos\theta_2$$

$$L = T - V = ml^2\dot{\theta}_1^2 + \frac{1}{2}ml^2\dot{\theta}_2^2 + \frac{1}{2}ml^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1) + 2mgl\cos\theta_1 + mgl\cos\theta_2$$

This Lagrangian characterizes complete motion of the double pendulum. The equations of motion would be found by using

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) - \frac{\partial L}{\partial\theta_1} = 0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right) - \frac{\partial L}{\partial\theta_2} = 0$$