Dimensional Analysis/2-D Motion

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1 Dimensional Analysis

Before we use any numbers today, let's talk about units. A unit is a label that distinguishes physical quantities; a dimension is a physical quantity. From freshmen physics or other classes, you should already know several examples of units: seconds, meters, hours, kilograms, etc. Some teachers may have required you to always write the units of your answer, while some don't really care. While you may have thought it was dumb, dimensions can actually be very useful in solving problems.

Via unit conversion, we can add certain units. We can add minutes to seconds, for example. But we can't add seconds to meters. What's going on? This is where dimensions come in. Seconds and minutes both have dimension time, while meters have dimension length. We identify dimensions with brackets: time is denoted [T], for example, and length as [L]. The following are the rules for dimensional analysis: (Note: if you leave today only having learned one thing, make it this list.)

1. The dimensions on both sides of an equation must match.
2. We can add quantities only if they have the same dimension.
3. We may multiply quantities of different dimensions.
4. We may raise dimensions to real powers.

Let's do an example. From the Big 5 equations, we have a lot of variables with different dimensions: position, velocity, and acceleration. Position is measured in some unit of length, so it has dimension [L]. Velocity is the change in position over the change in time, so that has dimension $\frac{[L]}{[T]}$. Going one step further, acceleration is the change in velocity over the change in time, so its dimension is $\frac{\frac{[L]}{[T]}}{[T]} = \frac{[L]}{[T]^2}$. Newton's Second Law states that $F = ma$, so we introduce a new dimension for mass: [M]. Thus, from number 1 on our list above, we can determine the dimensions of force: $\frac{[M][L]}{[T]^2}$, or $[M][L][T]^{-2}$.

Now it's your turn. Newton's Law of Universal Gravitation states that $F = \frac{GMm}{r^2}$. If you haven't seen this formula before, fear not. You'll learn it soon. Regardless, you can still do what we're asking: find the units of the gravitational constant G. You can have the rest of this page to work.

While many important constants in physics have dimension, some don't. They are termed dimensionless. An example would be radians: by definition, $s = r\theta$, where s is the arc's length, r is the circle's radius, and θ is the measure of the central angle in radians. This means that θ has dimension [L][L]$^{-1}$, which is clearly dimensionless. This is pretty important, so let's add another point to our earlier list

5. We may only exponentiate, apply trigonometric functions to, or apply logarithmic functions to dimensionless quantities.

Think about why this rule is necessary. Imagine if we raised a number to the power of a length, like $2^{[L]}$. What does this even mean? Answer: it's meaningless. Thus, it's important that if you ever use one of those three functions, the value you're plugging in better be dimensionless. If it's not, you should go back and check your work, because something went wrong.

In all, dimensional analysis is a pretty basic but fundamental tool for physics problems. Not only do they help you solve problems sometimes, but more importantly, they help you check your work. Always understand what units you're dealing with. If you just memorize words like "Newtons", "Joules", or other fancy words without understanding what those units actually represent, there's no point. Also, dimensional analysis sounds pretty complicated, so you can sound like you're doing some complicated math when really all you're doing is dividing [L] by [T].

Alright, let's talk about the big test now: $F = ma$. Most of the time, all the answers have the same dimensions, so you're out of luck there. However, if you ever have no idea how to do a problem, a good idea is to try to use dimensional analysis to eliminate some choices. For example, if a question asks for how long a process takes, find the dimensions of each answer, and eliminate all of them that aren't [T].

In fact, one year (2013), a very intimidating problem (16) was on the test, but I am pretty sure all of you guys can solve it now. It was actually inspired by a problem from the previous year's International Physics Olympiad. Here it is:

A very large number of small particles forms a spherical cloud. Initially they are at rest, have uniform mass density per unit volume ρ_0, and occupy a region of radius r_0. The cloud collapses due to gravitation; the particles do not interact with each other in any other way.

How much time passes until the cloud collapses fully? (The constant 0.5427 is actually $\sqrt{\frac{3\pi}{32}}$.)

(a) $\frac{0.5427}{{r_0}^2\sqrt{G\rho_0}}$

(b) $\frac{0.5427}{r_0\sqrt{G\rho_0}}$

(c) $\frac{0.5427}{\sqrt{r_0}\sqrt{G\rho_0}}$

(d) $\frac{0.5427}{\sqrt{G\rho_0}}$

(e) $\frac{0.5427}{\sqrt{G\rho_0}}\rho_0$

Note: Some of these notes were taken from Alex Bourzutschky's notes with permission.