Maximum Flight Trajectory

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1 Problem

Find the launch angle for which the length of the flight trajectory is maximized (In 2D, ignoring drag forces).

2 Solution

This may simultaneously be one of the coolest and most terrible problems of all time. Bear with me on this one. And persevere to the end. Let's begin.

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We want to maximize the length of the flight trajectory, or in other terms, the arc length

$$s = \int_0^{t_{max}} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$

We know that

$$\frac{dx}{dt} = v_0 cos(\theta)$$

And that as:

$$y(t) = vsin(\theta)t - \frac{gt^2}{2}$$

$$\frac{dy}{dt} = v_0 sin(\theta) - gt$$

In order to calculate t_{max}, we use the v_y formula that we just derived:

$$v_0 sin(\theta) - gt = 0 \rightarrow t = \frac{v_0 sin(\theta)}{g}$$

This represents half of the flight path, and hence the total flight time is

$$\frac{2v_0 sin(\theta)}{g}$$

(We could just roll with the half time, but I'm taking the full flight time because it leads to cancellation later on)

Plugging these back into the arc length integral, we are left with the following mess:

$$s = \int_0^{\frac{2v_0 sin(\theta)}{g}} \sqrt{(v_0^2 cos^2(\theta) +)(v_0^2 sin^2(\theta) - 2v_0 sin(\theta) gt + (gt)^2)} dt$$

Which can be simplified to:

$$s = \int_0^{\frac{2v_0 sin(\theta)}{g}} \sqrt{v_0^2 - 2v_0 sin(\theta) gt + (gt)^2} dt$$

And rearranged to:

$$s = \int_0^{\frac{2v_0 sin(\theta)}{g}} \sqrt{(gt - v_0 sin(\theta))^2 - v_o^2(sin^2(\theta) - 1)} dt$$

$$s = \int_0^{\frac{2v_0 sin(\theta)}{g}} \sqrt{(gt - v_0 sin(\theta))^2 + v_o^2 cos^2(\theta)} dt$$

$$s = v_0 cos(\theta) \int_0^{\frac{2v_0 sin(\theta)}{g}} \sqrt{1 + (\frac{gt - v_0 sin(\theta)}{v_0 cos(\theta)})^2} dt$$

Now we do the u-subsitution:

$$u = \frac{gt - v_0 sin(\theta)}{v_0 cos(\theta)} \rightarrow dt = \frac{v_0 cos(\theta)}{g} du$$

In terms of u,

$$t = \frac{2v_0 sin(\theta)}{g} \rightarrow u = tan(\theta)$$

$$t = 0 \rightarrow u = -tan(\theta)$$

With these things in mind, our integral becomes:

$$s = \frac{2v_0^2 cos^2(\theta)}{g} \int_0^{tan(\theta)} \sqrt{1+u^2} du$$

Doing a trig sub (or if you're fancy and want to do a sub for some inverse hyperbolic trig functions, but don't be that guy), we get

$$u = tan(\alpha)$$

$$du = sec^2(\alpha) d\alpha$$

$$\sqrt{1+u^2} = sec(\alpha)$$

And our integral becomes:

$$s = \frac{2v_0^2 cos^2(\theta)}{g} \int_0^{u=tan(\theta)} sec^3(\alpha) d\alpha$$

Now, you go home and cry for a bit, remember how to do integration by parts, and come back and do this:

$$s = \frac{2v_0^2 cos^2(\theta)}{g} \left(\frac{sec(\alpha)tan(\alpha) + ln|sec(\alpha) + tan(\alpha)|}{2} \right)$$

$$s = \frac{2v_0^2 cos^2(\theta)}{g} \left(\frac{u\sqrt{1+u^2} + ln|u + \sqrt{1+u^2}|}{2} \right)$$

Note that each of these integrals are bounded by $u = 0$ and $u = tan(\theta)$. Hence, with some simplification, we can get:

$$s = v_0^2 cos^2(\theta) g (tan(\theta)sec(\theta) + ln|tan(\theta) + sec(\theta)|) - \frac{1}{2}$$

Almost there. Pull through this.

We need to solve for the theta which maximizes s. So what do we do? Resort to Wolfram? Pretty much. In order to do this, we must take the derivative of s with respect to theta, and solve for when that is equivalent to 0. Cleveland Brown, it's yo time to shine.

Upon using Wolfram, realizing that Wolfram does not give a nice derivative, and then simplifying the derivative by hand, we get:

$$\frac{ds}{d\theta} = cos(\theta)(1 - sin(\theta) ln|tan(\theta) + sec(\theta)|) = 0$$

Now, split this up into two cases:

$$cos(\theta) = 0$$

$$(1 - sin(\theta)ln|tan(\theta) + sec(\theta)|) = 0$$

The former is pretty trivial. If all this work led to that and only that, I too would be completely done with physics. Fortunately, that represents a minimum. However, we have another equation. And I tip my hat to anyone who can do that by hand. But I'm not that smart, so we have no choice but to resort to Wolfram. And we get the final, coveted solution:

$$\boxed{\theta = 56.4658...^{\circ}}$$