Polar coordinates

$\vec{r} = \langle x, y \rangle = \langle r\cos\theta, r\sin\theta \rangle = r\langle \cos\theta, \sin\theta \rangle \quad \vec{e}_r = \langle \cos\theta, \sin\theta \rangle$

Describing 2D space, need a second basis vecter (need 2 vecters to sweep out [redacted] 2D space).

[redacted]

$\vec{e}_r$ is our radial vecter; we want a tangential vecter, which we call $\vec{e}_\theta$

$$\vec{e}_r = \langle \cos\theta, \sin\theta \rangle$$

$$\vec{e}_\theta = \langle -\sin\theta, \cos\theta \rangle$$

$$\vec{r} = r\,\vec{e}_r$$

$$\frac{d\vec{r}}{dt} = \dot{r}\,\vec{e}_r + r\,\frac{d\vec{e}_r}{dt} = \dot{r}\,\vec{e}_r + r\,\frac{d\vec{e}_r}{d\theta}\frac{d\theta}{dt} = \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta$$

$$\frac{d^2\vec{r}}{dt^2} = \ddot{r}\,\vec{e}_r + \dot{r}\,\frac{d\vec{e}_r}{dt} + \vec{e}_\theta\,\frac{d}{dt}(r\dot{\theta}) + r\dot{\theta}\,\frac{d\vec{e}_\theta}{dt}$$

$$= (\ddot{r} - r\dot{\theta}^2)\,\vec{e}_r + (\dot{r}\dot{\theta} + \dot{r}\dot{\theta} + r\ddot{\theta})\,\vec{e}_\theta$$

$$= (\ddot{r} - r\dot{\theta}^2)\,\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\vec{e}_\theta$$

Circular Orbit

$$\vec{a}_c = -r\dot{\theta}^2 = -\omega^2 r$$

$$\vec{a}_T = r\ddot{\theta} = \alpha r$$

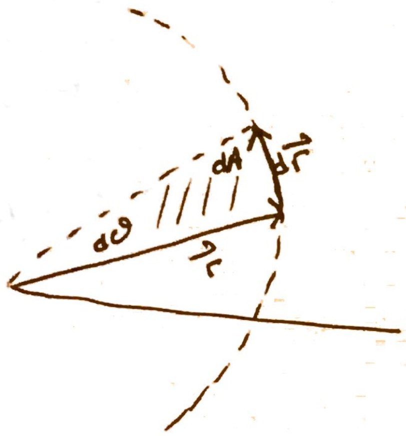
Conservation of angular momentum

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\frac{d\vec{L}}{dt} = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = (\vec{v} \times m\vec{v}) + \vec{r} \times \vec{F} = \vec{0}$$

(central force)

Kepler's second law



$$L = |\vec{r} \times m\vec{v}| = |r\vec{e}_r \times (m(\dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta))|$$

$$= |(r\vec{e}_r \times m\dot{r}\vec{e}_r) + (r\vec{e}_r \times mr\dot{\theta}\vec{e}_\theta)|$$

$$= mr^2\dot{\theta}$$

$$dA = \frac{1}{2}|\vec{r} \times d\vec{r}| = \frac{1}{2}|\vec{r} \times \vec{v}\,dt| = \frac{L}{2m}\,dt$$

$$\dot{A} = \frac{L}{2m} = \frac{1}{2}r^2\dot{\theta} = \text{constant}$$

~~central fo~~

$$\frac{d}{dt}(r^2\dot{\theta}) = 0 \rightarrow r^2\dot{\theta} = \ell \text{ (constant)}$$

$$|\ell| = \frac{L}{m} \quad \text{(angular momentum per unit mass)}$$

Central Force Equation

$$m\ddot{\vec{r}} = f(r)\,\vec{e}_r \qquad \dot{\theta} = \ell u^2$$

$$m(\ddot{r} - r\dot{\theta}^2) = f(r) \qquad m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0$$

Let $r = \frac{1}{u} \rightarrow \dot{r} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$

$$= -r^2\dot{\theta}\frac{du}{d\theta}$$

$$= -\ell\frac{du}{d\theta}$$

$$\ddot{r} = -\ell\frac{d}{dt}\frac{du}{d\theta} = -\ell\frac{d\theta}{dt}\frac{d}{d\theta}\frac{du}{d\theta}$$

$$= -\ell\dot{\theta}\frac{d^2u}{d\theta^2}$$

$$= -\ell^2u^2\frac{d^2u}{d\theta^2}$$

$$m\left[-\ell^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}(\ell^2u^4)\right] = f(u^{-1})$$

$$\boxed{\frac{d^2u}{d\theta^2} + u = -\frac{1}{m\ell^2u^2}f\left(\frac{1}{u}\right)}$$

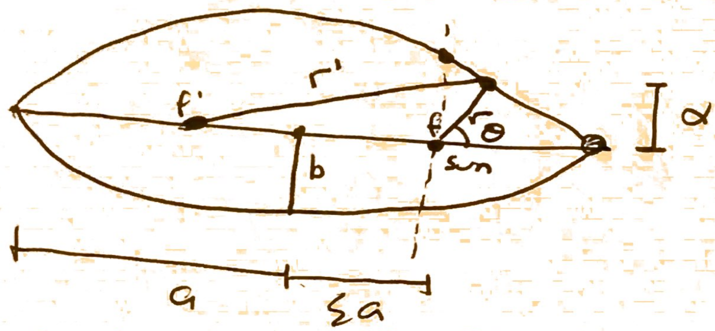
Central Force under Inverse-square

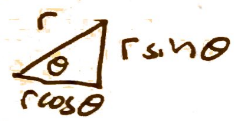
$$f(r) = -\frac{Gmm}{r^2} = -\frac{k}{r^2}$$

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{m\ell^2} \qquad \left(\frac{d^2u}{d\theta^2} = -u\right)$$

$$u = A\cos\theta + \frac{k}{m\ell^2}$$

$$r = \frac{1}{\frac{k}{m\ell^2} + A\cos\theta} = \frac{m\ell^2/k}{1 + \frac{Am\ell^2}{k}\cos\theta}$$





$$r + r' = 2a$$

$$r'^2 = r^2\sin^2\theta + (2\varepsilon a + r\cos\theta)^2$$

$$= r^2 + 4\varepsilon^2 a^2 + 4\varepsilon a r\cos\theta$$

$$= r^2 + 4\varepsilon a(\varepsilon a + r\cos\theta)$$

$$4a^2 - 4ar + r^2 = r^2 + 4\varepsilon a(\varepsilon a + r\cos\theta)$$

$$a - r = \varepsilon(\varepsilon a + r\cos\theta)$$

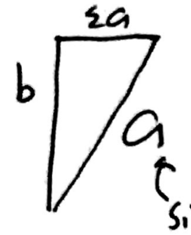
$$\boxed{r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta} \qquad \alpha = r\left(\frac{\pi}{2}\right) = a(1-\varepsilon^2) = m\ell^2/k \qquad \varepsilon = \frac{Am\ell^2}{k}}$$

Kepler's Third Law

$$\dot{A} = \frac{L}{2m}$$

$$\int_0^\tau \dot{A}\,dt = A = \frac{L}{2m}\tau = \frac{\ell}{2}\tau$$

$$\tau = \frac{2A}{\ell}$$



since r+r' = 2a

$$= \frac{2\pi ab}{\ell}$$

$$= \frac{2\pi a^2\sqrt{1-\varepsilon^2}}{\ell}$$

$$\tau^2 = \frac{4\pi^2 a^4}{\ell^2}(1-\varepsilon^2)$$

$$= \frac{4\pi^2 a^4}{\ell^2}\frac{\alpha}{a} = 4\pi^2\frac{\alpha}{\ell^2}a^3$$

$$\alpha = m\ell^2/k$$

$$\boxed{\tau^2 = \frac{4\pi^2}{Gm}a^3}$$