Project Euler Problem 700

Solved 30 November, 2021

Leonhard Euler was born on 15 April 1707.

Consider the sequence $1504170715041707n \pmod{4503599627370517}$.

An element of this sequence is defined to be an Eulercoin if it is strictly smaller than all previously found Eulercoins.

For example, the first term is 1504170715041707 which is the first Eulercoin. The second term is 3008341430083414 which is greater than 1504170715041707 and so is not an Eulercoin. However, the third term is 8912517754604 which is small enough to be a new Eulercoin.

The sum of the first two Eulercoins is therefore 1513083232796311.

Find the sum of all Eulercoins.

Analysis

Let an $\text{coin}(k, m, i)$ be the ith element of the sequence $kn \pmod m$ that is strictly smaller than all previous elements in the sequence.

Given k, m, and i, define $c_i := \text{coin}(k, m, i)$ and n_i the smallest integer such that $kn_i \pmod m = e_i$. Let $\tilde{n}$ be the smallest positive integer such that $n_i + \tilde{n}$ yields a coin, i.e., $k(n_i + \tilde{n}) \pmod m < c_i$, or equivalently, $(c_i + k\tilde{n}) \pmod m < c_i$.

If $c_i + k\tilde{n} \pmod m < m$, the inequality cannot hold since then $(c_i + k\tilde{n}) \pmod m = c_i + k\tilde{n} > c_i$. We must therefore have $c_i + k\tilde{n} \pmod m \geq m$, and so

$$\begin{aligned} c_{i+1} &= (c_i + k\tilde{n}) \pmod m \\ &= (c_i \pmod m) + k\tilde{n} \pmod m) \pmod m \\ &= (c_i + k\tilde{n} \pmod m)) \pmod m \end{aligned}$$

Clearly, $k\tilde{n} \pmod m$ is itself an element of the sequence. We prove that it must be an element of the sequence strictly larger than element before it; let this class of elements be termed *anticoins*.

Assume towards contradiction the desired $\tilde{n}$ does not yield an anticoin. In other words, there exists some $n' < n$ such that $kn' \pmod m > k\tilde{n} \pmod m$. Also note that since $c_i < m$ and $kn \pmod m < m$ for all n, we have $c_i + kn \pmod m < 2m$ for all n. Furthermore, $c_i + k\tilde{n} \pmod m \geq m$ implies $c_i + kn' \pmod m > m$, and so $m < c_i + kn' \pmod m < 2m$. Thus

$$(c_i + kn') \pmod m = (c_i + kn' \pmod m)) \pmod m = c_i + kn' \pmod m - m < c_i$$

which implies $(c_i + kn') \pmod m$ is a coin. However, this contradicts our assumption $\tilde{n}$ (which is greater than n') is the smallest integer such that $(c_i + k\tilde{n}) \pmod m$ is a coin. Thus $\tilde{n}$ does yield an anticoin.

Thus, given c_i, we can generate c_{i+1} by testing successive anticoins a until we find one satisfying $(c_i + a) \pmod m < c_i$. It remains to show we can also generate every anticoin from a coin and a preceding anticoin.

Solution

```
(1504170715041707, 4503599627370517)
```

```
k, m = 1504170715041707, 4503599627370517
```

```
eulercoin_sum (generic function with 1 method)
```

```julia
function eulercoin_sum(k, m)
    last_coin, last_coin_n = k % m, 1
    anticoins, anticoins_ns = [k % m], [1]
    coins_sum = last_coin

    while last_coin_n < m
        new_mods = (last_coin .+ anticoins) .% m
        new_ns = last_coin_n .+ anticoins_ns

        for i in eachindex(new_mods)
            if new_mods[i] > anticoins[end]
                push!(anticoins, new_mods[i])
                push!(anticoins_ns, new_ns[i])
                @show new_mods[i], new_ns[i]
            end

            if new_mods[i] < last_coin
                last_coin, last_coin_n = new_mods[i], new_ns[i]
                @show last_coin, last_coin_n
                coins_sum += last_coin
            end
        end
    end

    return coins_sum
end
```

```
1517926517777556
```

```julia
eulercoin_sum(k, m)
```

Benchmark

```
BenchmarkTools.Trial: 9658 samples with 1 evaluation.
 Range (min … max):  401.600 μs …  13.216 ms  ┊ GC (min … max): 0.00% … 0.00%
 Time  (median):     465.600 μs              ┊ GC (median):    0.00%
 Time  (mean ± σ):   508.864 μs ± 251.262 μs  ┊ GC (mean ± σ):  4.24% ± 8.82%
```

```
  402 μs          Histogram: frequency by time        1.77 ms <

 Memory estimate: 680.55 KiB, allocs estimate: 615.
```

```julia
@benchmark eulercoin_sum(k, m)
```