From thm above, we can derive error bounds for common NC rules.

<u>Ex</u>: Compute/Derive error bound for Trap. Rule

$$R_T(f) = \frac{h}{2}\left(f(x_0) + f(x_1)\right)$$

<u>Sol</u>: $n=1$:

$$E_T(f) = \frac{h^3 \cdot f^{(2)}(\xi)}{2} \cdot \int_0^1 t(t-1)\,dt$$

$$= \frac{h^3 f''(\xi)}{2} \cdot \underbrace{\int_0^1 t^2 - t\,dt}_{= \frac{-1}{6}}$$

$$= \frac{-h^3 f''(\xi)}{12}, \quad x_0 < \xi < x_1$$

In the same manner, we can derive the following bounds:

n	rule	formula	error	DOP
1	Trapezoid	$\frac{h}{2}\left(f(x_0)+f(x_1)\right) - \frac{h^3 f''(\xi)}{12}$	$O(h^3)$	1
2	Simpson's 1/3	$\frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right] - \frac{h^5}{90} f^{(4)}(\xi)$	$O(h^5)$	3
3	Simpson's 3/8	$\frac{3h}{8}\left[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right] - \frac{3h^5}{80} f^{(4)}(\xi)$	$O(h^5)$	3

Remarks: 1) n even $\Rightarrow$ error term depends on $f^{(n+2)}(\xi)$

$\Rightarrow$ exactly integrate functions whose $n+2$-derivative is zero (poly. of degree $n+1$)

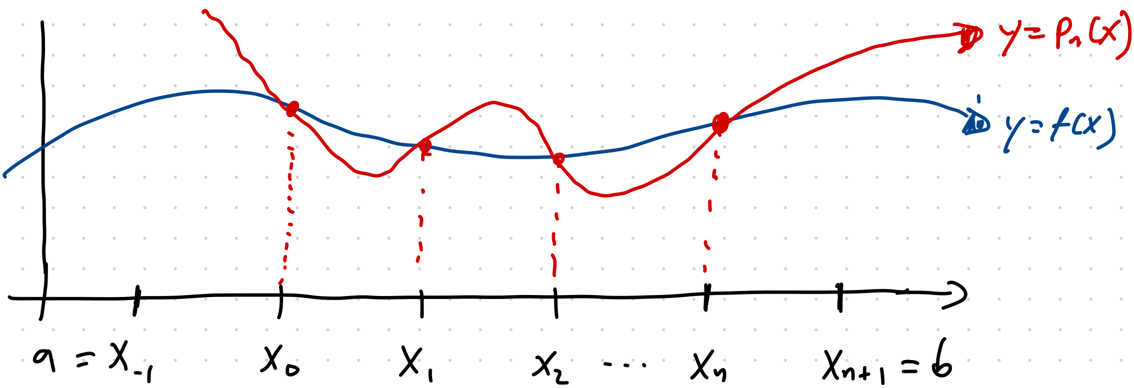
$\Rightarrow$ DOP $= n+1$

2) This is NOT the case for n odd! e.g., Trap. rule: $n=1$, DOP$=1$

Similar case holds for open NC Formulae.

Let $a = x_{-1} < x_0 < x_1 < \cdots < x_n < x_{n+1} = b$

and $h = \dfrac{b-a}{n+2}$. Then $x_i = x_0 + ih$ for $i = -1, 0, 1, \ldots, n+1$



Open - NC Error Thm:

Suppose $R(f) = \sum_{i=0}^{n} w_i f(x_i)$ denotes $(n+1)$ - open NC rule with $x_{-1} = a$, $x_{n+1} = b$, $h = \frac{b-a}{n+2}$.

Then there exists $\xi \in (a,b)$ s.t.

- When n is even and $f \in C^{n+2}(a,b)$

$$\int_a^b f(x)\,dx = R(f) + \frac{h^{n+3} f^{(n+2)}(\xi)}{(n+2)!} \int_{-1}^{n+1} t^2 (t-1)\cdots(t-n)\,dt$$

- When n is odd and $f \in C^{n+1}(a,b)$

$$\int_a^b f(x)\,dx = R(f) + \frac{h^{n+2} f^{(n+1)}(\xi)}{(n+1)!} \int_{-1}^{n+1} t\,(t-1)\cdots(t-n)\,dt$$

From this thm, we can obtain the error bounds for the following common open NC Rules.

n	rule	formula	error	DOP
0	Midpoint	$2h\, f(x_0) + \frac{h^3}{3} f''(\xi)$	$O(h^3)$	1
1	open Trap.	$\frac{3h}{2}[f(x_0) + f(x_1)] + \frac{3h^3}{4} f''(\xi)$	$O(h^3)$	1
2	Milne's	$\frac{4h}{3}[2f(x_0) - f(x_1) + 2f(x_2)] + \frac{14h^5}{45} f^{(4)}(\xi)$	$O(h^5)$	3

Remarks:

1) Both NC rules are not suitable when integrating over large intervals using high-deg. polynomials due to oscillations.

- One Remedy: Use piece-wise polynomial interpolation i.e., Composite numerical integration.

Composite Integration

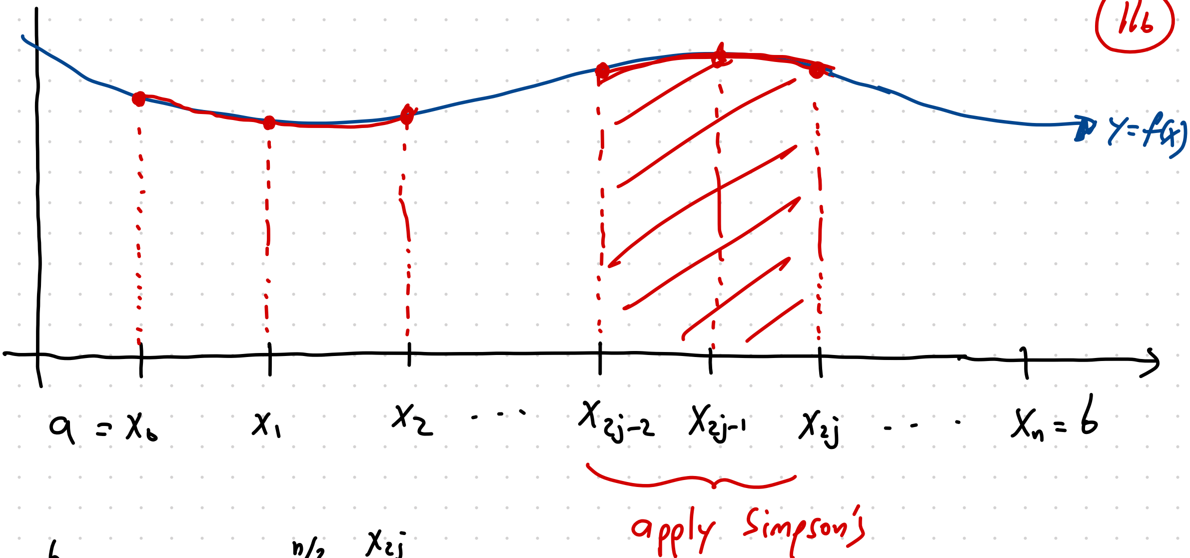
Idea:
- we consider $I(f) = \int_a^b f(x)\,dx$
- partition $[a,b]$ into n subintervals
- use low-order NC formula at each subinterval (e.g. Midpoint, Trapezoid, Simpson's)

Ex: Composite Simpson's Rule

Let $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. Let n be even and consider $x_{2j-2}, x_{2j-1}, x_{2j}$ for $j = 1, \dots, \frac{n}{2}$

(need 3 pts for Simpson's)

Goal: Partition $[a,b]$ in $\frac{n}{2}$ subintervals and apply Simpson's rule on $[x_{2j-2}, x_{2j}]$

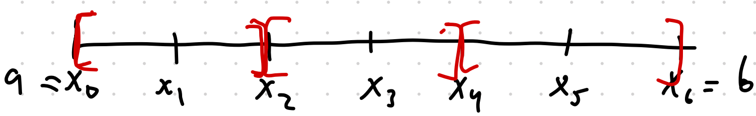


$$\int_a^b f(x)\,dx = \sum_{j=1}^{n/2} \int_{x_{2j-2}}^{x_{2j}} f(x)\,dx$$

$$= \sum_{j=1}^{n/2}\left[\frac{h}{3}\left(f(x_{2j-2})+4f(x_{2j-1})+f(x_{2j})\right) - \frac{h^5}{90} f^{(4)}(\xi_j)\right]$$

$$x_{2j-2} \le \xi_j \le x_{2j}$$

$$= \frac{h}{3}\left[f(x_0)+2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(x_n)\right] - \underbrace{\frac{h^5}{90}\sum_{j=1}^{n/2} f^{(4)}(\xi_j)}_{E_S^c(f)}$$



Note: (*) $\displaystyle\min_{x\in[a,b]} f^{(4)}(x) \le f^{(4)}(\xi_j) \le \max_{x\in[a,b]} f^{(4)}(x)$

$$\Rightarrow \min_{x\in[a,b]} f^{(4)}(x) \le \frac{2}{n}\sum_{j=1}^{n/2} f^{(4)}(\xi_j(x)) \le \max_{x\in[a,b]} f^{(4)}(x) \quad 117$$

the average of $\sum_{j=1}^{n/2} f^{(4)}(\xi_j(x))$ must also satisfy (*)

<u>Note:</u> $\frac{2}{n}\sum_{j=1}^{n/2} f^{(4)}(\xi_j(x))$ continuous on $[a,b]$

By IVT, there exists $\xi\in(a,b)$ s.t.

$$\frac{2}{n}\sum_{j=1}^{n/2} f^{(4)}(\xi_j) = f^{(4)}(\xi)$$

$$\Rightarrow E_s^c(f) = \frac{-h^5}{90}\sum_{j=1}^{n/2} f^{(4)}(\xi_j) = \frac{-h^5 n}{180} f^{(4)}(\xi), \quad a<\xi<b$$

$$h = \frac{b-a}{n} \Rightarrow \boxed{E_s^c(f) = \frac{-(b-a)}{180}\cdot h^4 f^{(4)}(\xi)}$$

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Thm (Composite Simpson's): Let $f \in C^4(a,b)$, n be even, $h = \frac{b-a}{n}$, $x_j = a_j + j \cdot h$, for each $j = 0, 1, \ldots, n$.

There exists a $\mu \in (a,b)$ s.t. Composite Simpson's Rule for n subintervals can be written with error term as

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right] - \frac{b-a}{180} h^4 f^{(4)}(\mu)$$

Remarks:

1) Composite Simpson's Quadrature rule is given by

$$\boxed{R_s^c(f) = f(a) + 2\sum_{j=1}^{n/2-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)}$$

2) Compare Standard Simpson's Rule error:

$$E_s(f) = \frac{-h^5}{90} f^{(4)}(\xi), \quad a < \xi < b$$

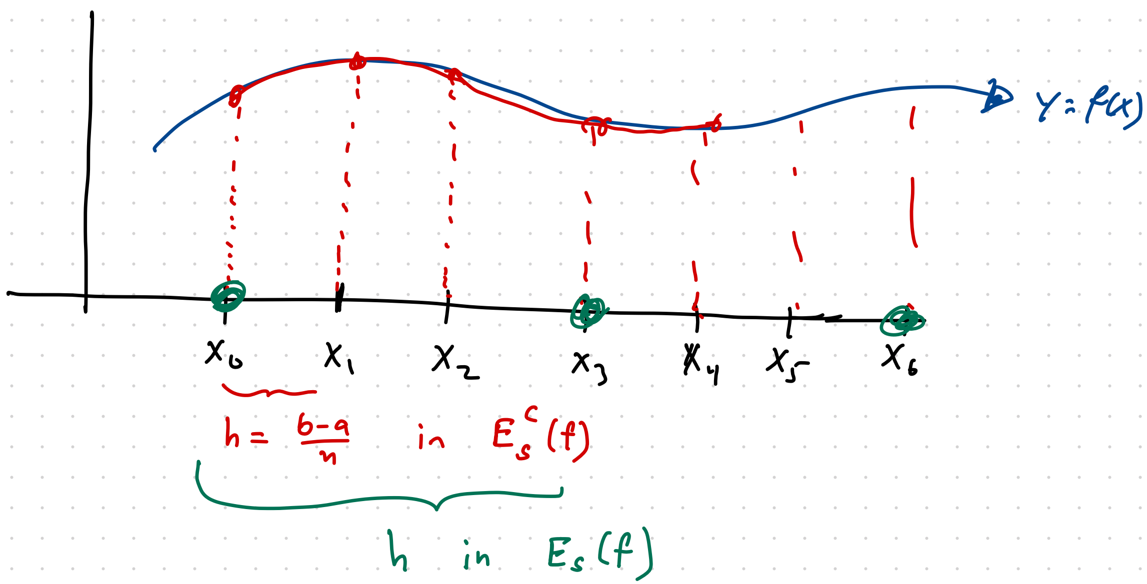
and composite Simpson's Rule error:

$$E_s^c(f) = -\frac{(b-a)}{180} h^4 f^{(4)}(\mu) \quad a < \mu < b$$

Is Composite Simpson $(O(h^4))$ worse than Standard Simpson $(O(h^5))$?

No! In standard Simpson's, $h = \frac{b-a}{2}$

whereas in composite Simpson's, $h = \frac{b-a}{n}$



Similarly, we can obtain Composite Trapezoidal rule:

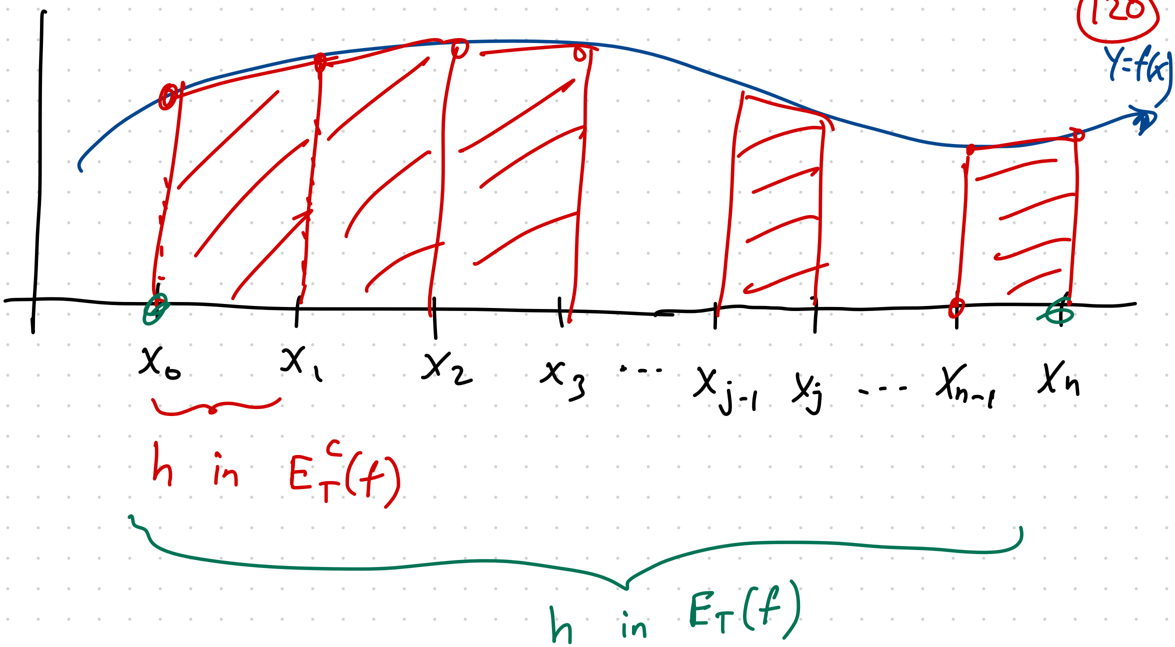
Thm (Composite Trap.): Let $f \in C^2(a,b)$, $h = \frac{(b-a)}{n}$,

and $x_j = a + jh$, $j = 0, 1, \dots, n$.

↳ no need to be even.

There exists a $\mu \in (a,b)$ s.t. Composite Trap. Rule for n subintervals can be written with error term as

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b) \right] - \frac{b-a}{12} h^2 f''(\mu)$$



Remark: h in $R_T^c(f) = \dfrac{b-a}{n}$

h in $R_T(f) = b-a$