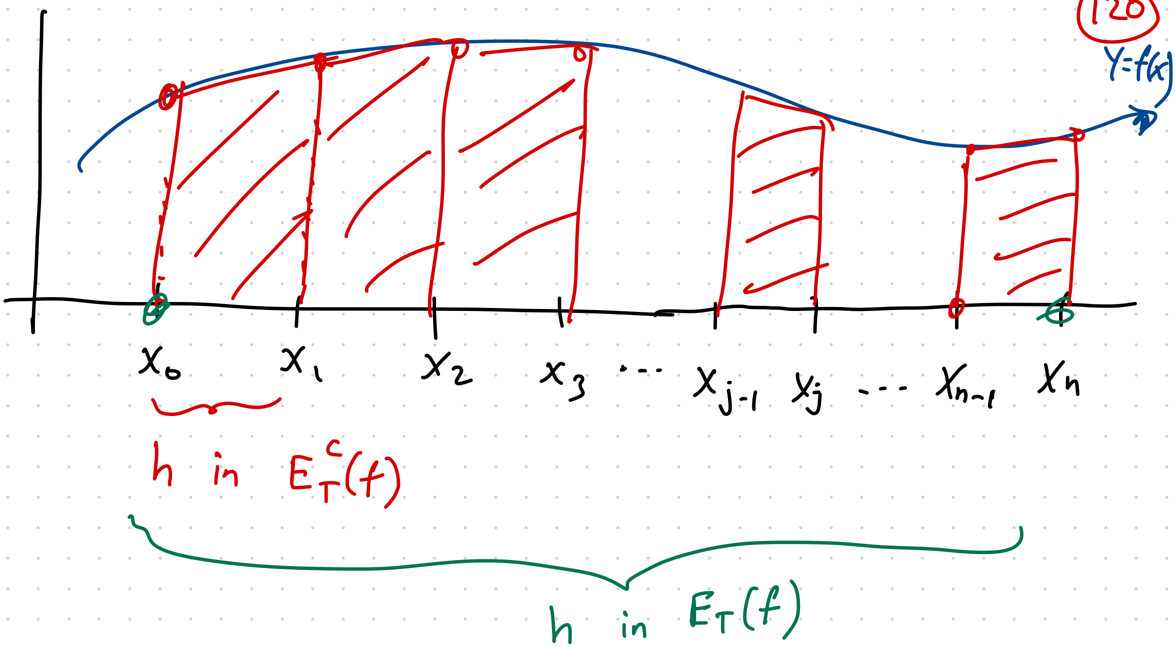


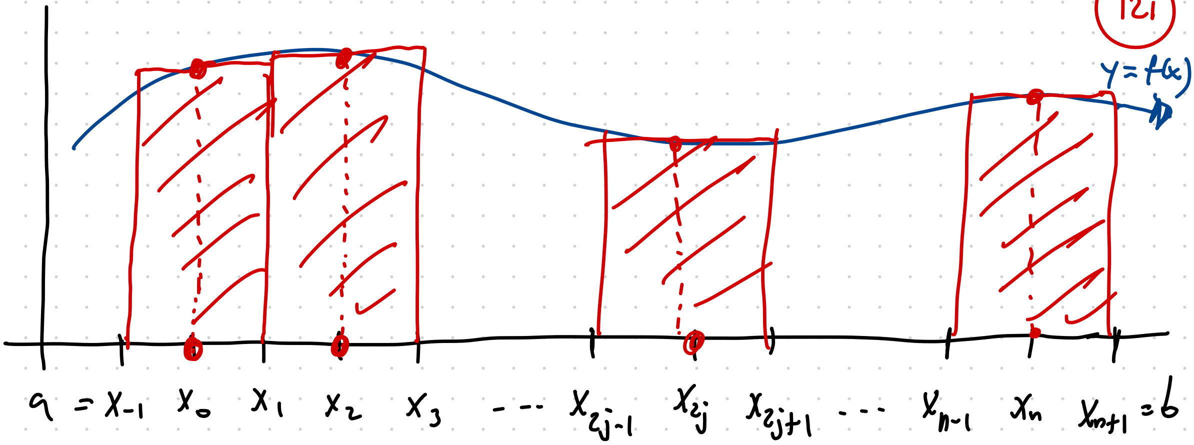
Remark: h in $R_T^c(f) = \dfrac{b-a}{n}$

h in $R_T(f) = b-a$

Finally, for the Composite Midpoint Rule:

Thm: Let $f \in C^2(a,b)$, n be even, $h = \dfrac{b-a}{n+2}$, and $x_j = a + (j+1)\cdot h$ for $j = -1, 0, \dots, n, n+1$. There exists $\mu \in (a,b)$ s.t. the Composite Midpoint Rule for $n+2$ subintervals can be written as

$$\int_a^b f(x)\,dx = 2h \sum_{j=0}^{n/2} f(x_{2j}) + \frac{b-a}{6} h^2 f''(\mu)$$



every even node = midpt.

Remark: h in $R_M^c(f) = \dfrac{b-a}{n+2}$

but h in $R_M(f) = \dfrac{b-a}{2}$

Composite Integration Formulae

Rule	Formula	Error
C- Midpt. n-even $h=\frac{(b-a)}{n+2}$	$R_M^C(f) = 2h\sum_{j=0}^{n/2} f(x_{2j})$	$E_M^C(f) = \frac{b-a}{6}h^2 f''(\mu)$
C- Trap. $h=\frac{(b-a)}{n}$	$R_T^C(f) = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right]$	$E_T^C = -\frac{(b-a)}{12}h^2 f''(\mu)$
C- Simpson's n-even $h=\frac{b-a}{n}$	$R_S^C(f) = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right]$	$E_S^C(f) = \frac{-(b-a)}{180}h^4 f^{(4)}(\mu)$

Remarks: 1) Composite Rules $\Rightarrow$ n can be very large whereas standard Trapezoid ($n=1$), Midpt. ($n=0$), Simpson's ($n=2$) have fixed n.

2) Can use errors to control accuracy of integration.

Ex: Determine the value of h that will ensure approx. error less than 2×10^{-5} when approximating $\int_0^{\pi} \sin x \, dx$ employing Composite Trap. Rule.

Sol: Error for $R_T^C(f)$ by

$$\left|E_T^C(f)\right| = \left|\frac{b-a}{12} h^2 f''(\mu)\right|$$

$$= \left|\frac{\pi}{12} h^2 (-\sin(\mu))\right|$$

$$= \frac{\pi}{12} h^2 \underbrace{\left|\sin(\mu)\right|}_{\leq 1}$$

$$\leq \frac{\pi}{12} h^2$$

Need $\frac{\pi}{12} h^2 < 2 \cdot 10^{-5}$

$$\Rightarrow h^2 = \frac{24}{\pi} \times 10^{-5} \Rightarrow \boxed{h = \sqrt{\frac{24}{\pi} \times 10^{-5}}}$$

↳ use to determine n

Gaussian Quadrature Rules

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Basic Idea: choose weights w_i and $x_i \in [a,b]$ (optimally)

s.t.

$$R(f) = \sum_{i=0}^{n} w_i f(x_i)$$

have DOP $= 2n+1$ (here, $n =$ # of intervals)

Basic Properties of Gauss Rules

- all $w_i > 0$
- all pts. $x_i \in (-1,1)$
- weights satisfy symmetry condition:

$$w_0 = w_n, \quad w_1 = w_{n-1}, \ldots$$

- x_i satisfy symmetry condition:

n odd $\Rightarrow$ $x_0 = -x_n$, $x_1 = -x_{n-1}$, $x_2 = -x_{n-2}, \ldots$

n even $\Rightarrow$ $x_0 = -x_n$, $x_1 = -x_{n-1}$, $\ldots$

and $x_{\frac{n}{2}} = 0$

Ex: Find a "1-pt." Gauss rule for

$\int_{-1}^{1} f(x)\,dx$. $\quad (\text{DOP} = 2n+1 = 2\cdot 0+1 = 1)$

Sol: $R(f) = w_0 f(x_0)$

- Symmetry $\Rightarrow \boxed{x_0 = 0}$

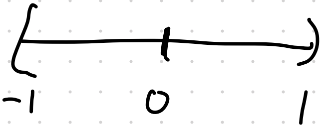
- DOP $= 2 \cdot 0 + 1 = 1$

$\Rightarrow R(f) = \int_{-1}^{1} f(x)\,dx$ when $f(x) = 1$, $f(x) = x$

$$f(x) = 1 \Rightarrow \int_{-1}^{1} x\,dx = x\Big|_{-1}^{1} = \boxed{2 = w_0}$$

$\Rightarrow$ "1-pt" Gauss-Rule is:

$$R(f) = 2 f(x_0)$$



<u>Remark:</u> For interval $[-1, 1]$, this is simply the midpoint rule!

<u>Ex</u>: Find a "2-pt" Gauss Rule for $\int_{-1}^{1} f(x)\,dx$

That is,

$$R(f) = w_0 f(x_0) + w_1 f(x_1)$$

<u>Sol</u>:
- symmetry $\Rightarrow \boxed{x_0 = -x_1, \quad w_0 = w_1}$
- DOP $= 2 \cdot 1 + 1 = 3$

$\Rightarrow R(f) = \int_{-1}^{1} f(x)\,dx$ for $f(x) = 1, x, x^2, x^3$

$$\int_{-1}^{1} 1\,dx = \boxed{2 = w_0 + w_1}$$

$$\int_{-1}^{1} x\,dx = \frac{1}{2}x^2\Big|_{-1}^{1} = \boxed{0 = w_0 x_0 + w_1 x_1}$$

$$\int_{-1}^{1} x^2\,dx = \frac{1}{3}x^3\Big|_{-1}^{1} = \boxed{\frac{2}{3} = w_0 x_0^2 + w_1 x_1^2}$$

$$\int_{-1}^{1} x^3\,dx = \frac{1}{4}x^4\Big|_{-1}^{1} = \boxed{0 = w_0 x_0^3 + w_1 x_1^3}$$

Symmetry $\Rightarrow$ $2w_0 = 2 \Rightarrow w_0 = 1 = w_1$

Symmetry $\Rightarrow$ $x_0 = -x_1 \Rightarrow x_0^2 = x_1^2$

$$\Rightarrow \frac{2}{3} = 2x_0^2 \Rightarrow x_0 = \pm \frac{1}{\sqrt{3}}$$

$$\Rightarrow \boxed{x_0 = \frac{-\sqrt{3}}{3}, \quad x_1 = \frac{\sqrt{3}}{3}}$$

$$\Rightarrow \boxed{R(f) = f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)}$$

Changing Intervals

Suppose we have a quadrature formula for interval $[c,d]$ (e.g., $c=-1$, $d=1$)

Q: What is the rule for the interval $[a,b]$?

- Use change of variables:

$$\int_{g(c)=a}^{g(d)=b} f(x)\,dx = \int_c^d f(g(t))\,g'(t)\,dt$$

where $x = g(t)$

- we want $g(t)$ to satisfy

$g(c) = a$ and $g(d) = b$

From this, we can use interpolation:

$$g(t) = \frac{t-d}{c-d} \cdot a + \frac{t-c}{d-c} b$$

$$g'(t) = \frac{a}{c-d} + \frac{b}{d-c} = \frac{b-a}{d-c}$$

$$\therefore \int_{a=g(c)}^{b=g(d)} f(x)\,dx = \int_c^d f(g(t)) \cdot \frac{b-a}{d-c}\,dt$$

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$$= \frac{b-a}{d-c} \int_c^d f(g(t))\,dt$$

$$\approx \frac{b-a}{d-c} \sum_{i=0}^{n} w_i f(g(t_i))$$

$$\Rightarrow \boxed{\int_a^b f(x)\,dx \approx \frac{b-a}{d-c} \sum_{i=0}^{n} w_i f(g(t_i))}$$

Ex: Suppose we have

$$\int_{-1}^{1} f(x)\,dx \approx \frac{4}{3} f\left(-\frac{1}{2}\right) - \frac{2}{3} f(0) + \frac{4}{3} f\left(\frac{1}{2}\right)$$

Then

$$\int_a^b f(x)\,dx = \frac{b-a}{c-d} \int_c^d f(g(t))\,dt$$

where $c=-1$, $d=1$, $g(t) = \frac{t-1}{-2} a + \frac{t+1}{2} b$

$$= \frac{1}{2}(b-a)t + \frac{1}{2}(b+a)$$

$$\Rightarrow \int_a^b f(x)\,dx = \frac{b-a}{2} \int_{-1}^{1} f\left(\frac{b-a}{2} t + \frac{a+b}{2}\right) dt$$

$\downarrow$

$$\approx \frac{b-a}{2}\Big[\frac{4}{3} f\Big(\frac{b-a}{2}\Big(-\frac{1}{2}\Big) + \frac{a+b}{2}\Big)$$

$$- \frac{2}{3} f\Big(\frac{b-a}{2}(0) + \frac{a+b}{2}\Big)$$

$$+ \frac{4}{3} f\Big(\frac{b-a}{2}\Big(\frac{1}{2}\Big) + \frac{a+b}{2}\Big)\Big]$$

$$= \frac{b-a}{2}\Big[\frac{4}{3} f\Big(\frac{b+3a}{4}\Big) - \frac{2}{3} f\Big(\frac{a+b}{2}\Big) + \frac{4}{3} f\Big(\frac{a+3b}{4}\Big)\Big]$$

A more systematic way to determine w_i, x_i:

Orthogonal polynomials on $[-1,1]$

Def: The functions $f, g \in L^2([-1,1])$

$\left(\text{i.e., } \int_{-1}^{1} (f(x))^2 dx < \infty \text{ and } \int_{-1}^{1} (g(x))^2 dx < \infty\right)$

are orthogonal if

$$\langle f, g \rangle := \int_{-1}^{1} f(x)\, g(x)\, dx = 0$$

Ex:

$$\left\langle \sin(\pi x), \cos(\pi x) \right\rangle = \int_{-1}^{1} \sin(\pi x)\cos(\pi x)\,dx$$

$$= \frac{1}{2\pi} \sin^2(\pi x) \Big|_{-1}^{1} = 0$$

$$\langle 1, x \rangle = \int_{-1}^{1} 1 \cdot x \, dx = 0$$

$$\langle 1, x^2 \rangle = \int_{-1}^{1} x^2 \, dx = \frac{2}{3} \neq 0$$

$$\langle 1, x^3 \rangle = \int_{-1}^{1} x^3 \, dx = 0$$

Remark: If $f(x)$ is even and $g(x)$ is odd in $[-1, 1]$, then f and g are orthogonal.

Def: Starting from $1, x, x^2, \dots, x^n, \dots,$ the Gram - Schmidt process without normalization generates a set of orthogonal polynomials

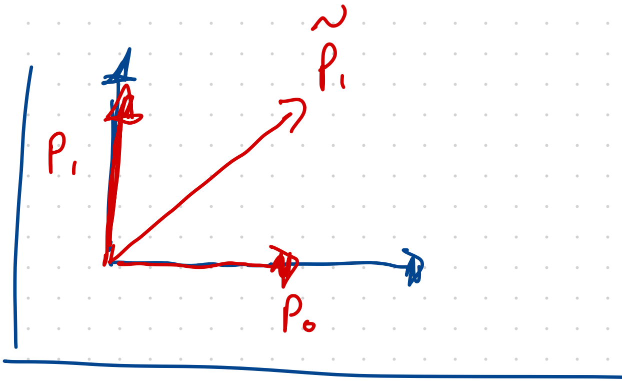
$$\{ P_0(x), P_1(x), \dots, P_n(x) \}$$

called <u>Legendre Polynomials</u>.

Here,

$P_0(x) = 1$

$P_1(x) = x - \frac{\langle x, P_0\rangle}{\langle P_0, P_0\rangle} P_0 = x$



$$P_2(x) = x^2 - \frac{\langle x^2, P_0\rangle}{\langle P_0, P_0\rangle} P_0 - \frac{\langle x^2, P_1\rangle}{\langle P_1, P_1\rangle} P_1 = x^2 - \frac{1}{3}$$

$$P_3(x) = x^3 - \frac{3}{5}x \quad , \quad P_4(x) = x^4 - \frac{6x^2}{7} + \frac{3}{35}$$

<u>Remarks</u>: 1) The Legendre polynomials $P_0, P_1, \dots, P_n$ satisfy:

a) For each n, P_n is a monic polynomial of degree n (leading coeff. $=1$)

b) $\int_{-1}^{1} p(x) \cdot P_n(x)\, dx = 0$ whenever the degree of $p(x)$ is less than n.

2) Root of $P_n(x)$ are distinct on $[-1, 1]$ and symmetric w.r.t. origin, and are exactly the Gauss-Quadrature nodes!!