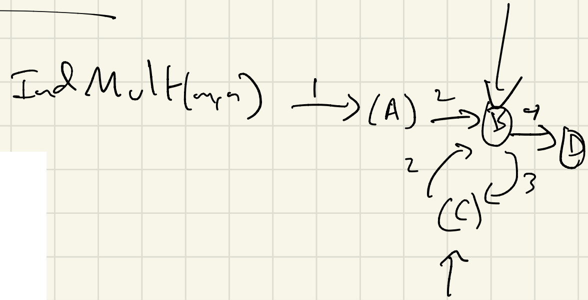
Math 182 Lecture 3



INDUCTIVEMULT(m, n)

1 $prod = 0$ (A)
2 **for** $j = 1$ **to** n (B)
3 $\quad prod = prod + m$ (C)
4 **return** $prod$ (D)

InductiveMult(5,3)

line	prod	m	n	j
1	0	5	3	
→ 2	0	5	3	1
3	5	5	3	1
→ 2	5	5	3	2
3	10	5	3	2
→ 2	10	5	3	3
3	15	5	3	3
→ 2	15	5	3	4
4	15	5	3	4

return 15

$\boxed{m = 5 \quad n = 3}$

Prod	j
0	1 ✓
5	2 ✓
10	3 ✓
15 ←	4 ✓

* $prod = 5(j-1)$ *

Thm: For every $m, n \geq 0$, IndMult(m,n) returns $m \cdot n$.

Proof: Let $m, n \in \mathbb{N}$ be arbitrary.

After line 1, $prod = 0$.

Each time we end line 2, we claim the following is true:

$*$ (Loop Invariant) Each time we end line 2, $prod = m(j-1)$.

To prove $(*)$

(Init.) The first time we finish line 2, ✓
$j=1$, $prod = 0$ so $0 = prod \overset{?}{=} m(j-1) = m \cdot 0 = 0$

(Maintenance) Sps we have just finished line 2 and current value of j is $j=j_0$ for some $1 \leq j_0 \leq n$. Furthermore assume $(*)$ is true for this j. In particular $sum = m(j_0-1)$. Then in line 3, we now get $sum = sum + m$
$= m(j_0)$
then we go to line 2, $j = j_0+1$
so $sum = m j_0 = m(j-1)$. Thus $(*)$ is still true ✓

Now we know $(*)$ is true.

(Termination) The last time line 2 is run, $j = n+1$, so $sum = m(j-1) = m \cdot n$.
The go to line 4 and return $sum = m \cdot n$.

✓

Exponentials and logarithms

$b>0$ $\quad k\in\mathbb{Z}$

$$b^k := \begin{cases} \underbrace{b\times\cdots\times b}_{k \text{ times}} & \text{if } k>0 \\ 1 & \text{if } k=0 \\ \underbrace{b^{-1}\times\cdots\times b^{-1}}_{-k \text{ times}} & \text{if } k<0 \end{cases}$$

initially b^k only makes sense if $k\in\mathbb{Z}$.
Using analysis (131a) can extend b^k to a function $x\mapsto b^x:\mathbb{R}\to\mathbb{R}$

s.t.

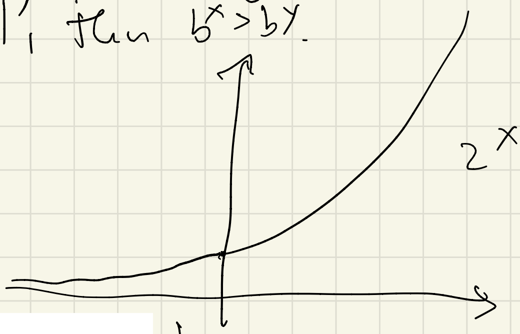
(Exponent Rule) For every $x,y\in\mathbb{R}$, $b^{x+y}=b^x b^y$

(Monotonicity) For every $x,y\in\mathbb{R}$ s.t. $x<y$

(a) if $b>1$, then $b^x<b^y$

(b) if $b=1$, then $b^x=b^y=1$

(c) if $b<1$, then $b^x>b^y$.



Fact 1.4.12. Suppose $b>0$, and $x,y\in\mathbb{R}$. Then

(1) $b^0=1$
(2) $b^1=b$
(3) $b^{-1}=1/b$
(4) $(b^x)^y=b^{xy}$
(5) $(b^x)^y=(b^y)^x$
(6) $b^xb^y=b^{x+y}$
(7) if $b>1$, then
 (a) $x\mapsto b^x$ is strictly increasing: if $x<y$, then $b^x<b^y$
 (b) $\lim_{x\to+\infty} b^x=+\infty$
 (c) $\lim_{x\to-\infty} b^x=0$
(8) $b^x>0$

$b > 0,\ b \neq 0,\ x > 0$

Define $\log_b x := y$

s.t. $b^y = x$.

Definition 1.4.13. Suppose $b > 0$ and $b \neq 1$. If $x > 0$, then we define the **logarithm of x to the base b** (notation: $\log_b x$) to be the unique $y \in \mathbb{R}$ such that $b^y = x$. In other words:

$$y = \log_b x \;:\Longleftrightarrow\; b^y = x.$$

$\lg x := \log_2 x$ (the **binary logarithm**) $\leftarrow$

$\ln x := \log_e x$ (the **natural logarithm**) $\leftarrow$

important for us

Facts about logarithms

Sps $a, b, c, x \in \mathbb{R}$, $a, b, c > 0$. Then

(1) $a = b^{\log_b a}$ $(b \neq 1)$

$\rightarrow$ (2) $\log_c(ab) = \log_c(a) + \log_c(b)$

hard

easy

(3) $\log_b a^x = x \log_b a$ $(b \neq 1)$

$\rightarrow$ (4) $\log_b a = \log_c a / \log_c a$ (if $b, c \neq 1$) (Base change)

(5) $\log_b (1/a) = -\log_b(a)$ $(b \neq 1)$

(6) $\log_b a = 1/\log_a b$ (if $a, b \neq 1$)

$2^{10} = 1024$

$2^5 = 32$

$1024 \cdot 32 = 2^{10} \cdot 2^5 = 2^{10+5} = 2^{15}$

(7) $a^{\log_b c} = c^{\log_b a}$ $(b \neq 1)$

Fact 1.4.15. Suppose $b > 0$ is such that $b \neq 1$. Then

(1) $\frac{d}{dx} \log_b x = 1/x \ln b$

(2) in particular, $\frac{d}{dx} \ln x = \frac{1}{x}$

(3) $\frac{d}{dx} \ln f(x) = f'(x)/f(x)$

(4) $\int \ln x dx = x \ln(x) - x + C$

(5) $\ln(t) = \int_1^t dx/x$

Chapter 2 Asymptotics

E.g. $\lim_{n\to\infty} \frac{2n+1}{4n^3+5n+2} = 0$

$$= \underbrace{\frac{1}{n^2}}_{\to 0} \underbrace{\left(\frac{2+\frac{1}{n}}{4+5/n+2/n^2} \right)}_{\to 1/2}$$

this limit is 0 bcs
numerator ~~has~~ is linear (deg 1)
denom. is cubic (deg 3)
and $3 > 1$.

§2.1 Asymptotic notation

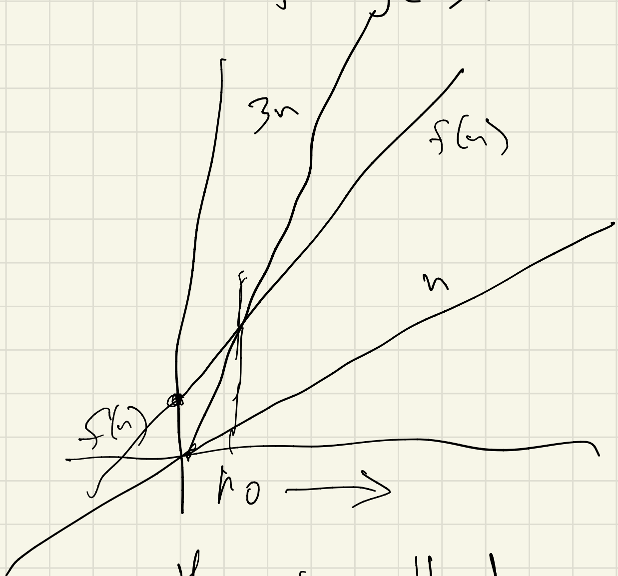
Θ-notation

Sps $g(n)$ is a function. Define the set

$$\Theta(g(n)) := \{f(n) : \text{ there exists positive constants } c_1, c_2, \text{ and } n_0 \text{ such that } 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0\}$$

"big theta of g of n"

i.e., $\Theta(g(n))$ is all functions which can be eventually "sandwiched" between two constant multiples of $g(n)$.



$g(n) = n$

$f(n) = 2n+1$

WTS $f(n) \in \Theta(n)$

$2n+1 \in \Theta(n)$

this is called a tight asymptotic bound

Example 2.1.1. $2n + 1 \in \Theta(n)$.

Proof: We need to find $c_1, c_2, n_0 > 0$
s.t. $0 \le c_1 n \le 2n+1 \le c_2 n$
for every $n \ge n_0$.
Notice for $n \ge 1$, $n \le 2n+1$
and for $n \ge 1$, $2n+1 \le 2n + n = 3n$.
Thus $c_1 = 1$, $c_2 = 3$, $n_0 = 1$ works ✓

Remark: Instead of writing $f(n) \in \Theta(g(n))$
we write $f(n) = \Theta(g(n))$.

Definition 2.1.3. We say that a function $f(n)$ is **asymptotically nonnegative** if there exists an n_0 such that for all $n \ge n_0$, $f(n) \ge 0$. Likewise, we say that $f(n)$ is **asymptotically positive** if there exists an n_0 such that for all $n \ge n_0$, $f(n) > 0$.

Fact 2.1.4. Suppose $g(n)$ is asymptotically positive. Given a function $f(n)$, if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)}$$

exists and is a positive real number (not $+\infty$), then $f(n)$ is also asymptotically positive and $f(n) = \Theta(g(n))$.

O-notation

"Big oh of g of n" $\quad$ $g(n)$ function

$$O(g(n)) := \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0\}$$

E.g. $\log n = O(n)$ but $\log n \neq \Theta(n)$

Example 2.1.5. $n = O(n^2)$, but $n \neq \Theta(n^2)$.

Proof: $n = O(n^2)$ need to find $c > 0, n_0 > 0$ s.t. for all $n \geq n_0$, $0 \leq n \leq cn^2$.
$c = 1, n_0 = 1$ works ✓

$n \neq \Theta(n^2)$ Need to show for all $c, n_0 > 0$ there exists $n \geq n_0$ s.t. $n < cn^2$. $\Leftarrow$ $(\frac{1}{c} < n)$

Let $c, n_0 > 0$ be arbitrary. Take $n = \max(\lceil 1/c \rceil + 1, n_0)$. This works.

Fact 2.1.6. Suppose $g(n)$ is asymptotically positive. Given an asymptotically nonnegative function $f(n)$, if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)}$$

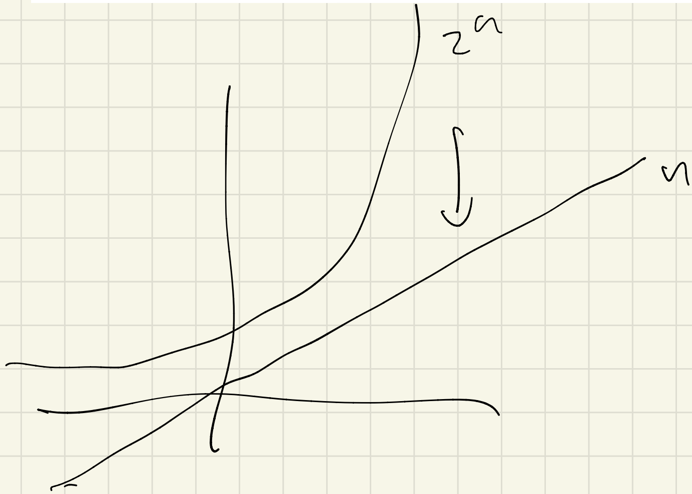
exists and is a nonnegative real number (not $+\infty$), then $f(n) = O(g(n))$.

including 0

Ω-notation

"big omega of g of n"

$$\Omega(g(n)) := \{f(n) : \text{there exists positive constants } c \text{ and } n_0 \text{ such that } 0 \le cg(n) \le f(n) \text{ for all } n \ge n_0\}$$



Example 2.1.7. $2^n = \Omega(n)$, but $2^n \neq \Theta(n)$.

Proof: ($2^n = \Omega(n)$) Need to find $c, n_0 > 0$ s.t. for all $n \geq n_0$, $0 \le cn < 2^n$.
Use $n_0 = 1 = c$. Need to show for all $n \ge 1$, $n \le 2^n$. (Induction)

($2^n \neq \Theta(n)$) Let $c_2, n_0 > 0$ be given. need to find $n \ge n_0$ s.t. $c_2 n < 2^n$
$\iff \frac{c_2 n}{2^n} < 1$. Note That $\lim_{n\to\infty} \frac{c_2 n}{2^n} = 0$.

In particular, can find $n \geq n_0$ s.t.

$$\frac{C_2 n}{2^n} < 1 \Rightarrow C_2 n < 2^n$$

Fact 2.1.8. Suppose $g(n)$ is asymptotically positive. Given a function $f(n)$, if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)}$$

exists and is a positive real number or $+\infty$, then $f(n) = \Omega(g(n))$.

Theorem 2.1.9. *For any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.*

o-notation

"little oh of g of n"

$$o(g(n)) := \{f(n) : \text{for every positive constant } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \leq f(n) < cg(n) \text{ for all } n \geq n_0\}$$

E.g. $n = o(n^2)$

$n = o(n \lg n)$

$n^2 = o(2^n)$

$n^3 \neq o(n^3)$

Fact 2.1.10. Suppose $g(n)$ is asymptotically positive. Given an asymptotically nonnegative function $f(n)$, if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0,$$

then $f(n) = o(g(n))$.

ω-notation

"little omega"

$$\omega(g(n)) := \{f(n) : \text{for every positive constant } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \leq cg(n) < f(n) \text{ for all } n \geq n_0\}$$

Fact 2.1.12. Suppose $g(n)$ is asymptotically positive. Given a function $f(n)$, if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty,$$

then $f(n) = \omega(g(n))$.

§2.2 Properties of asymptotic notation

Asymptotic notation in equations and inequalities

E.g. $2n^2+3n+1 = 2n^2 + \Theta(n)$

Interpretation:

The function $2n^2+3n+1$ is equal to something of the form $2n^2 + f(n)$ where $f(n) \in \Theta(n)$.

Note: "=" is not symmetric

rule: The right hand side can never contain more info. than the left hand side.

E.g. $2n^2 + O(n) = \Theta(n^2)$

Interpretation: regardless of which anonymous function which $O(n)$ refers to, can always find a function in $\Theta(n)$ to make equality true.

E.g. Can string together eqs:

$$2n^2+3n+1 = 2n^2 + \Theta(n)$$
$$= \Theta(n^2)$$

Interpretation: same as

$$2n^2+3n+1 = 2n^2 + \Theta(n)$$

and

$$2n^2 + \Theta(n) = \Theta(n^2)$$