<u>Ex 1</u>: Analyze the conv. rate of

$F(h) = \sin(h) - h\cos(h)$ as $h \to 0$ $(L=0)$

<u>Sol</u>: Note by Taylor's Thm, that

$$\sin(h) = h - \frac{h^3}{6}\cos(\xi) \quad \text{where } 0 \le \xi \le h$$

$$\cos(h) = 1 - \frac{h^2}{2}\cos(\eta) \quad \text{where } 0 \le \eta \le h$$

$$|\sin(h) - h\cos(h)| = \left| \cancel{h} - \frac{h^3}{6}\cos(\xi) - \cancel{h} + \frac{h^3}{2}\cos(\eta) \right|$$

$$\le \left| \frac{h^3}{6}\underbrace{\cos(\xi)}_{\le 1} \right| + \left| \frac{h^3}{2}\underbrace{\cos(\eta)}_{\le 1} \right|$$

$$\le \left(\frac{1}{6} + \frac{1}{2}\right)|h^3|$$

$$\Rightarrow \underbrace{\sin(h) - h\cos(h)}_{F(h)} = 0 + O(h^3)$$

---

<u>Note</u>: $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

If $\sin h = h \cdot \cos(\xi)$ $\quad 0 \le \xi \le h$

$h \cdot \cosh = h \cdot \cos(\eta)$ $\quad 0 \le \eta \le h$

$\Rightarrow |\sin h - h\cosh| \le h \cdot (\cos(\xi) - \cos(\eta))$

$\Rightarrow F(h) = 0 + O(h)$

If $\sin h = h - \frac{h^3}{3!} + \frac{h^5}{5!}\cos(\xi)$

$h \cdot \cosh = h - \frac{h^3}{2!} + \frac{h^5}{4!}\cos(\eta)$ $\Rightarrow$

If $\sin h = h - \frac{h^3}{3!} + \frac{h^5}{5!}\cos(\xi)$

$$h\cdot\cos h = h - \frac{h^3}{2!} + \frac{h^5}{4!}\cos(\eta)$$

$$\Rightarrow |\sin h - h\cosh| = \left| -\frac{h^3}{3!} + \frac{h^3}{2!} + \frac{h^5}{5!}\underbrace{\cos(\xi)}_{\le 1} - \frac{h^5}{4!}\underbrace{\cos(\eta)}_{\le 1} \right|$$

$$\le \left(\frac{1}{3!} + \frac{1}{2!}\right)|h|^3 + \left(\frac{1}{5!} + \frac{1}{4!}\right)\underbrace{h^5}_{\le h^3}$$

$$\le K h^3$$

$$\Rightarrow F(h) = 0 + O(h^3)$$

---

Taylor's Thm:

$$f(x) = \underbrace{f(x^*) + f'(x^*)(x-x^*) + \frac{f''(x^*)}{2!}(x-x^*)^2 + \cdots + \frac{f^{(n)}(x^*)}{n!}(x-x^*)^n}_{P_n(x)}$$

$$+ \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x^*)^{n+1}}_{R_n(x)} \quad \text{where } \xi \text{ between } x \text{ and } x^*$$

Truncation error $R_n(x) = f(x) - P_n(x)$

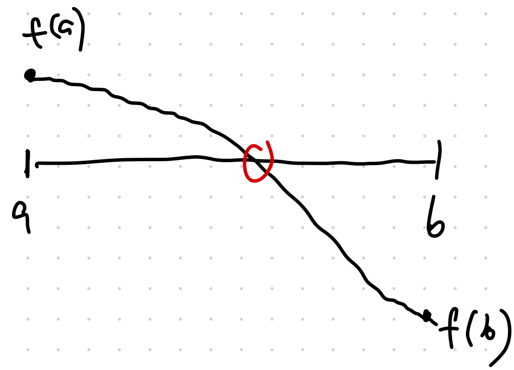
# 2.1 Bisection Method

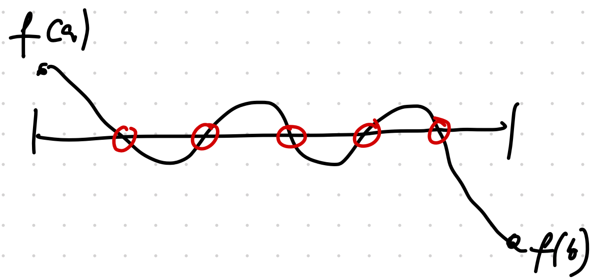
**Goal:** Given $f(x) \in C([a,b])$, want to find root $p \in [a,b]$ s.t. $f(p) = 0$

**Q1:** Is there a root? (Existence)

**Intermediate Value Thm (IVT):**
If $f \in C([a,b])$, and $K$ between $f(a)$ and $f(b)$, then there exists $p \in [a,b]$ s.t. $f(p) = K$

**Corollary:** If $f \in C([a,b])$ and $f(a) \cdot f(b) < 0$, then there exists $p \in [a,b]$ s.t. $f(p) = 0$



## Bisection:

Find interval $[a_1, b_1]$ s.t. $f(a_1) \cdot f(b_1) < 0$

Let $p_1 = \frac{a_1 + b_1}{2}$ be the midpoint

### Three possibilities:

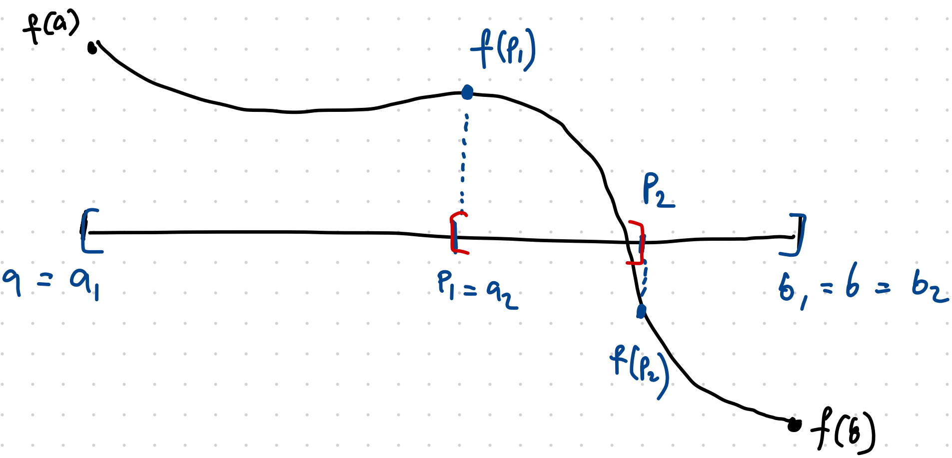
1) $f(p_1) = 0$, then $p = p_1$. Done!

2) If $f(p_1)$ has the same sign as $f(a_1)$, then set $a_2 = p_1$ and $b_2 = b_1$

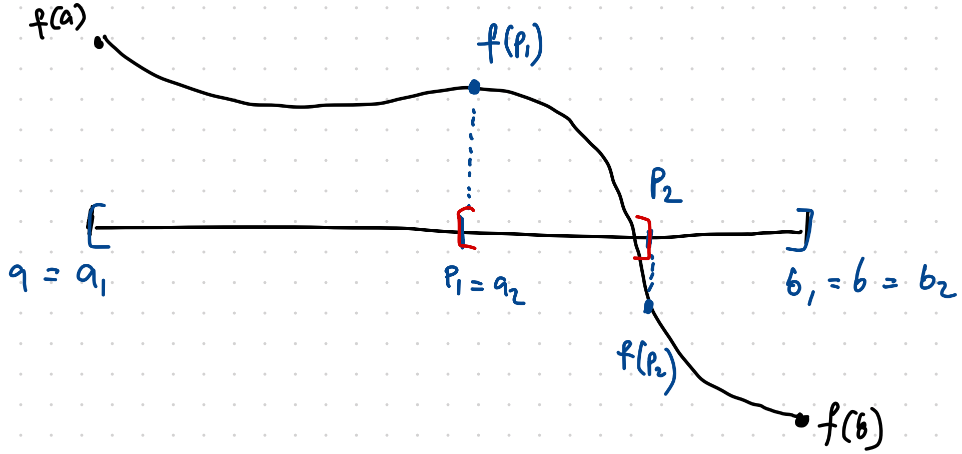
   Consider new interval $[a_2, b_2] = [p_1, b_1]$

3) If $f(p_1)$ has the same sign as $f(b_1)$, then set $b_2 = p_1$ and $a_2 = a_1$

   Consider new interval $[a_2, b_2] = [a_1, p_1]$




Bisection generates $p_1, p_2, \dots, p_n, \dots \longrightarrow p$




Remarks: 1) Each halved interval $[a_{n+1}, b_{n+1}]$ contains root since it satisfies $f(a_{n+1}) \cdot f(b_{n+1}) < 0$

2) For stopping criterion, choose

- $|p_n - p_{n-1}| < \varepsilon$
- $|f(p_n)| < \varepsilon$

can combine these 2.

$\varepsilon$ chosen by user

- max number of iters reached

3) To avoid overflow/underflow when computing $f(a) \cdot f(b)$, compute $\text{sgn}(f(a_n)) \cdot \text{sgn}(f(b_n))$, where

$$\text{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(see Alg. 2.1 in book)

## Thm (Convergence of Bisection)

Suppose that $f \in C[a,b]$ and $f(a)\cdot f(b) < 0$

Then the sequence $\{p_n\}_{n=1}^{\infty}$ generated by Bisection Method approximates a zero $p$ of $f(x)$ with rate

$$|p_n - p| \le \frac{b-a}{2^n}, \qquad n \ge 1$$

That is, $p_n = p + O\left(\frac{1}{2^n}\right)$

Pf: Note: since $a_1 = a$ and $b_1 = b$, $b_2 - a_2 = \frac{1}{2}(b_1 - a_1)$

By induction, we have

$$|b_2 - a_2| = \frac{1}{2}|b-a|$$

$$|b_3 - a_3| = \frac{1}{2}|b_2 - a_2| = \frac{1}{4}|b-a|$$

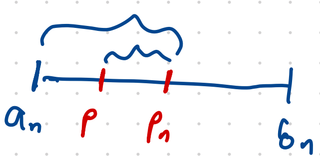
$$\vdots$$

$$|b_n - a_n| = \frac{1}{2^{n-1}}|b-a|.$$

By construction, $p_n = \frac{1}{2}(a_n + b_n)$ and $p \in (a_n, b_n)$

$$\Rightarrow |p_n - p| \le \frac{1}{2}(b_n - a_n) = \frac{1}{2^n}(b-a)$$

Thus, $p_n = p + O\left(\frac{1}{2^n}\right)$ ∎



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---

**Remark:** Conv. of Bisection Thm can be used to estimate error bound

**Ex1:** Determine the number of iterations needed in Bisection Method to solve

$$f(x) = x^3 + 4x^2 - 10 = 0$$

with accuracy $10^{-3}$ in $[\underset{a}{1}, \underset{b}{2}]$

**Sol:** By Thm, we have

$$|p_N - p| \le \frac{1}{2^N}(\underbrace{b-a}_{=1}) = \frac{1}{2^N} < 10^{-3}$$

$$\Rightarrow N > \frac{3}{\log(2)} \approx 9.96$$

$\Rightarrow$ at least 10 iterations needed to achieve accuracy of $10^{-3}$

## 2.2 Fixed Point Iteration

Two related/equivalent problems:

1. **Root finding:** Given a function $f(x)$, find $p$ s.t. $f(p) = 0$

2. **Fixed Point:** Given a function $g(x)$, find $p$ s.t.

$$g(p) = p$$

**Ex:** if $g(x) = x - f(x)$ or $g(x) = x + 3f(x)$

then $g(p) = p \iff f(p) = 0$

**Ex:** find fixed pt. of $g(x) = x^2 - 2$

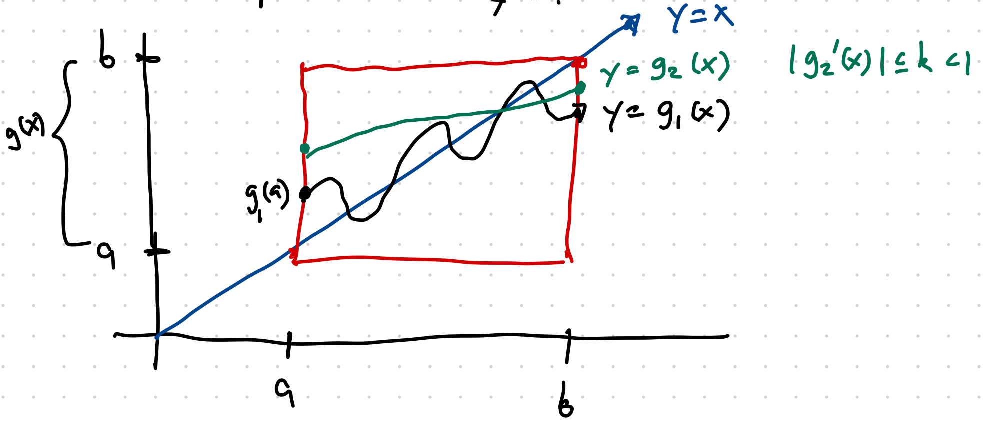
**Sol:** $x = g(x) \Rightarrow x = x^2 - 2 \Rightarrow x^2 - x - 2 = 0$

$\Rightarrow$ fixed pts of $g$ are $p = -1, 2$

# Thm (Existence and Uniqueness)

**Existence:** If $g(x) \in C[a,b]$ and $g(x) \in [a,b]$ ($a \le g(x) \le b$) for any $x \in [a,b]$, then there exists $p \in [a,b]$ s.t. $g(p) = p$.

**Uniqueness:** If in addition, $g \in C^1(a,b)$ (cont. diff'ble funcs.) and there exist $k \in (0,1)$ s.t. $|g'(x)| \le k < 1$ for any $x \in (a,b)$, then the fixed point is unique.




**Pf:** (Existence): Case 1: $g(a)=a$ or $g(b)=b$. $\Rightarrow$ Done

Case 2: we have $g(a) > a$ and $g(b) < b$.

Let $h(x) = g(x) - x$. $\Rightarrow$ $h(a) > 0$ and $h(b) < 0$

Since $h(a) \cdot h(b) < 0$ and $h \in C[a,b]$, by IVT,

$\exists\, p$ s.t. $h(p) = 0$

$\Rightarrow$ $p$ is fixed pt. ∎

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## Thm (Existence and Uniqueness)

**Existence:** If $g(x) \in C[a,b]$ and $g(x) \in [a,b]$ ($a \le g(x) \le b$) for any $x \in [a,b]$, then there exists $p \in [a,b]$ s.t. $g(p) = p$.

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---

**Recall Mean Value Thm (MVT):** $f \in C'(a,b)$, $C[a,b]$, then $\exists$ $\xi \in (a,b)$ s.t. $f'(\xi) = \dfrac{f(b) - f(a)}{b - a}$

**Pf (Uniqueness):** (By Contradiction)

Assume $p, q \in (a,b)$, $p \ne q$, and $g(p) = p$, $g(q) = q$

By MVT, we can find $\xi \in (a,b)$ s.t.

$$\frac{g(p) - g(q)}{p - q} = g'(\xi)$$

$$\Rightarrow |p-q| = |g(p) - g(q)| = |(p-q)\cdot g'(\xi)| \le \underset{<1}{k} \cdot |p-q| < |p-q| \quad \lightning$$

$\Rightarrow p = q.$ ∎

## Thm (Existence and Uniqueness)

<u>Existence</u>: If $g(x) \in C[a,b]$ and $g(x) \in [a,b]$ ($a \le g(x) \le b$) for any $x \in [a,b]$, then there exists $p \in [a,b]$ s.t. $g(p) = p$.

<u>Uniqueness</u>: If in addition, $g \in C^1(a,b)$ (cont. diff'ble funcs.) and there exist $k \in (0,1)$ s.t. $|g'(x)| \le k < 1$ for any $x \in (a,b)$, then the fixed point is unique.

---

<u>Remark</u>: 1) can show uniqueness with $|g'(\xi)| < 1$ (rather than $|g'(\xi)| \le k < 1$)

So condition is sufficient but not necessary.

We will use $|g'(x)| \le k$ later for algorithms.

2) This Thm gives <u>sufficient</u> but <u>NOT neccesary</u> conditions

<u>Ex</u>: Let $g(x) = 3^{-x}$ on $[0,1]$. Discuss existence and uniqueness of fixed pt. of $g$.

<u>Sol</u>: <u>Existence</u>: $g(1) = \frac{1}{3}$, $g(0) = 1 \Rightarrow g(x) \in [0,1]$ $\forall x \in [0,1]$.

$\Rightarrow \exists$ a sol. by Thm.

<u>Uniqueness</u>: $g'(x) = -3^{-x}\ln(3)$

$\Rightarrow g'(0) = -\ln(3) \approx -1.0986 \Rightarrow |g'(0)| > 1$

$\Rightarrow$ Cannot use uniqueness from Thm. However, fixed pt. is actually unique because $g'(x) < 0$ $\forall x \in [0,1]$. $g$ monotonically decreasing

# Fixed Point Iteration

Algorithm goes as follows:

1) Choose initial guess $p_0$

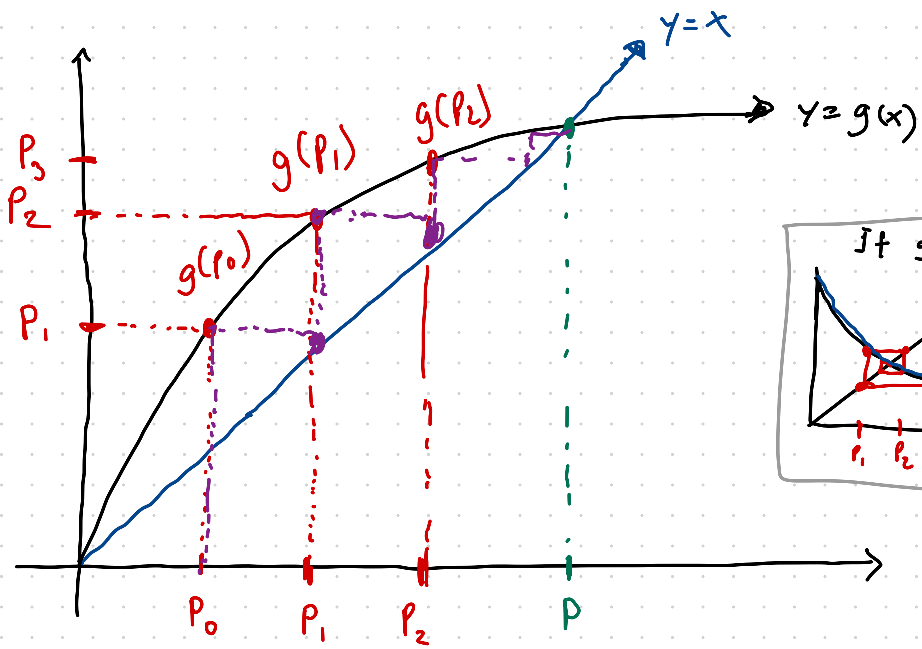
2) Generate $\{p_n\}_{n=1}^{\infty}$ by setting

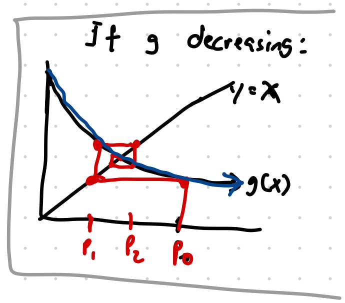
$$p_n = g(p_{n-1}), \quad n \geq 1$$

**Note:** If $p_n \to p$ and $g$ continuous, then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g\left(\lim_{n\to\infty} p_{n-1}\right) = g(p)$$

## Geometrically:



**Thm (Fixed Point Thm)**

Let • $g \in C[a,b]$ with $g(x) \in [a,b]$ $\forall\, x \in [a,b]$

• $g \in C'(a,b)$ s.t. $|g'(x)| \le k < 1$ $\forall\, x \in (a,b)$

Then for any $p_0 \in [a,b]$, the sequence $\{p_n\}_{n=1}^{\infty}$ defined

$$p_n = g(p_{n-1}) \quad , \; n \ge 1$$

converges to the unique fixed point of $g$ in $[a,b]$ with rate $O(k^n)$

---

Pf: By Existence/Uniqueness of fixed points thm,

$\exists\; p \in [a,b]$ s.t. $g(p) = p$.

Since $g(x)$ maps $[a,b]$ to itself, it is well-defined $\left(g(p_n) \in [a,b] \;\; \forall\; n \ge 1\right)$, and $p_n \in [a,b]$ $\forall\; n \ge 1$.

By MVT, $\exists\; \xi_n \in (a,b)$ s.t.

$$\begin{aligned} 0 \le |p_n - p| &= |g(p_{n-1}) - g(p)| \\ &= \underbrace{|g'(\xi_n)|}_{\le k} \cdot |p_{n-1} - p| \quad \text{By MVT} \\ &\le k \cdot |p_{n-1} - p| \\ &\le k \cdot \left(k\, |p_{n-2} - p|\right) \\ &\le \cdots \le k^n |p_0 - p| \end{aligned}$$

$$\begin{aligned}0 \le |p_n - p| &= |g(p_{n-1}) - g(p)| \\ &= \underbrace{|g'(\xi_n)|}_{\le k} \cdot |p_{n-1} - p| \quad \text{By MVT} \\ &\le k \cdot |p_{n-1} - p| \\ &\le k \cdot (k\,|p_{n-2} - p|) \\ &\le \dots \le k^n |p_0 - p|\end{aligned}$$

Since $k \in (0,1)$, we have $\lim_{n\to\infty} k^n = 0$

$$\Rightarrow \lim_{n\to\infty} k^n |p_0 - p| = 0$$

$$\Rightarrow \lim_{n\to\infty} |p_n - p| \le \lim_{n\to\infty} k^n |p_0 - p| = 0$$

$$\Rightarrow \lim_{n\to\infty} |p_n - p| = 0 \quad \text{by squeeze thm.}$$

Thus, $|p_n - p| \le k^n |p_0 - p|$, and $p_n$ converges to $p$ with rate $k^n$ $(0 < k < 1)$, i.e.,

$$p_n = p + O(k^n) \quad \blacksquare$$

<u>Note:</u> Error bound $|p_0 - p|$ not useful b/c we do not know $p$

**Note:** Error bound $|p_0 - p|$ not useful b/c we do not know $p$

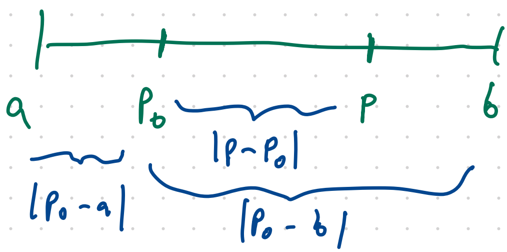
**Corollary:** Error bounds for $p_n$ in fixed point iteration can be given by

$$|p_n - p| \le k^n \cdot \max\{|p_0 - a|, |p_0 - b|\} \quad (1)$$

and

$$|p_n - p| \le \frac{k^n}{1-k} |p_0 - p_1| \quad \forall\ n \ge 1 \quad \text{(see book)} \ (2)$$

note $|p - p_0| \le \max\{|p_0 - a|, |p_0 - b|\}$




**Remark 1:** If $n_1$ and $n_2$ are min. number of iters required to achieve $\varepsilon$ for (1) and (2), respectively, the take $n = \min\{n_1, n_2\}$

**Remark 2:** Convergence rate depends on $k$ (upper bound for $|g'(x)|$ in $(a,b)$)

So $k \approx 0 \implies$ fast convergence

$k \approx 1 \implies$ slow convergence

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**Ex:** a) Show that $g(x) = 2^{-x}$ has unique sol. in $[\frac{1}{3}, 1]$

b) Estimate the # of iters required to achieve accuracy $\varepsilon = 10^{-4}$

**Sol.:** a) $g$ continuous on $[\frac{1}{3}, 1]$,

$$g(x) \in \left[\frac{1}{2}, \frac{1}{\sqrt[3]{2}}\right] \subset \left[\frac{1}{3}, 1\right]$$

$\Rightarrow$ sol. exists.

**Uniqueness:** $g'(x) = -\ln(2) \cdot 2^{-x}$,

$$|g'(x)| \in \left[\frac{\ln(2)}{2}, \frac{\ln(2)}{\sqrt[3]{2}}\right] \approx [0.347, 0.552]$$

$$\Rightarrow |g'(x)| \le k = \frac{\ln(2)}{\sqrt[3]{2}} < 1 \Rightarrow g \text{ has unique sol. in } [\tfrac{1}{3}, 1].$$

b) **First bound:**

Since $p_0 \in [\frac{1}{3}, 1]$ and $\max\{|p_0 - a|, |b - p_0|\} \le \frac{2}{3}$

$$\Rightarrow |p_n - p| \le \underbrace{\left(\frac{\ln(2)}{\sqrt[3]{2}}\right)^n}_{k^n} \cdot \frac{2}{3} \le 10^{-4}$$

$$\Rightarrow n_1 \ge 14.7347 \Rightarrow n_1 \ge 15$$

b) First bound:

Since $p_0 \in \left[\frac{1}{3}, 1\right]$ and $\max\{|p_0 - a|, |b - p_0|\} \le \frac{2}{3}$

$$\Rightarrow |p_n - p| \le \underbrace{\left(\frac{\ln(2)}{\sqrt[3]{2}}\right)^n}_{k^n} \cdot \frac{2}{3} \le 10^{-4}$$

$\Rightarrow n_1 \ge 14.7347 \Rightarrow n_1 \ge 15$

Second Bound:

$$|p_n - p| \le \frac{k^n}{1-k}|p_1 - p_0| \le \frac{k^n}{1-k}|b-a| \le 10^{-4}$$

$$\Rightarrow n_2 \ge 16.07$$

$\Rightarrow$ need at least 15 iterations.