MATH182 MIDTERM
DUE July 10, 2020

Question 1. *Consider the following pseudo-code:*

```
1  sum = A[1]
2  max = sum
3  for j = 2 to A.length
4        sum = sum + A[j]
5        if sum > max
6              max = sum
7  return max
```

This algorithm takes as input an array $A[1\,..\,n]$ and outputs the value of the maximum subarray of the form $A[1\,..\,j]$, i.e., it outputs the number

$$\max\left\{\sum_{i=1}^{j} A[i] : 1 \leq j \leq A.length\right\}$$

*(1) Give a proof of the correctness of this algorithm. Your proof should include: a precise statement of a loop invariant for the **for** loop, and a proof of this loop invariant. (5pts)*
(2) Analyze the running-time of this algorithm. This includes deducing a tight asymptotic bound. (5pts)
(3) Is this algorithm asymptotically optimal (i.e., is there another algorithm with asymptotically smaller running time which can do the same thing this algorithm does)? Justify your answer. (2pts)

Solution.
(1) We first define T_J as

$$T_J := \left\{\sum_{i=1}^{k} A[i] : 1 \leq k \leq J\right\}$$

The loop invariant for the **for** loop is then as follows:

Loop Invariant: *After line 3 is run, `sum` is the sum of all elements in $A[1\,..\,j-1]$, and `max` is the sum of the maximal subarray of the form $A[1\,..\,k]$ in $A[j\,..\,j-1]$, i.e.,*

$$\texttt{sum} = \sum_{i=1}^{j-1} A[i] \ ; \ \texttt{max} = \max\left\{\sum_{i=1}^{k} A[i] : 1 \leq k \leq j-1\right\} \equiv \max T_{j-1}$$

We show this loop invariant holds.

Initialisation: Before the first iteration, we have $\texttt{sum} = A[1]$, $\texttt{max} = \texttt{sum} = A[1]$, and $j = 2$. Since the subarray $A[j\,..\,j-1] = A[1\,..\,1]$ is a one-element array, $\texttt{max} = \texttt{sum} = A[1]$ is (trivially) the sum of its only (and therefore maximal) subarray, $A[1\,..\,1]$. The loop invariant is therefore true in the initialisation step.

Maintenance: Assume the loop invariant is true before some iteration with $j = j_0 \in [2\,..\,A.length]$. By assumption and definition of the loop invariant, we have

$$\texttt{sum} = \sum_{i=1}^{j_0-1} A[i] \ ; \ \texttt{max} = \max T_{j_0-1}$$

In line 4, we add $A[j_0]$ sum, and therefore have $\texttt{sum} = \sum_{i=1}^{j_0} A[i]$. The first requirement of the loop invariant for $j = j_0 + 1$ is therefore satisfied. Comparing `sum` to `max` in line 5, we have two cases:

Case 1: ($\texttt{sum} > \texttt{max}$) By definition of `sum` and `max`, we have

$$\sum_{i=1}^{j_0} A[i] \ > \ \max T_{j_0-1}$$

Updating `max` to `sum` in line 6 therefore gives us

$$\texttt{max} = \sum_{i=1}^{j_0} A[i] \ = \ \max \left\{ T_{j_0-1} \cup \left\{ \sum_{i=1}^{j_0} A[i] \right\} \right\} = \max T_{j_0}$$

We see that this state of `max` is precisely the second requirement of the loop invariant for $j = j_0+1$.

Case 2: ($\texttt{sum} \leq \texttt{max}$) In this case, condition of the **if** statement in line 5 fails and `max` is left unmodified. By definition of `sum` and `max`, we therefore have

$$\sum_{i=1}^{j_0} A[i] \ \leq \ \max T_{j_0-1} \implies \texttt{max} = \ \max \left\{ T_{j_0-1} \cup \left\{ \sum_{i=1}^{j_0} A[i] \right\} \right\} = \max T_{j_0}$$

We see that this state of `max` is also precisely the second requirement of the loop invariant for $j = j_0 + 1$.

The loop invariant is therefore true for $j = j_0 + 1$.

Termination: In the final iteration, we have $j = A.length + 1$ and the loop terminates. From the loop invariant, we know

$$\texttt{max} = \max T_{A.length+1-1} = \max T_{A.length} = \max \left\{ \sum_{i=1}^{k} A[i] : 1 \leq k \leq A.length \right\}$$

which is precisely the sum of the maximal subarray of A of the form $A[1\,..\,j]$, and is then returned in line 7. The algorithm is therefore correct. ■

(2) We consider the cost and number of times each line is run for $n := A.length$:

1	$sum = A[1]$	$cost : c_1$ $times : 1$
2	$max = sum$	$cost : c_2$ $times : 1$
3	**for** $j = 2$ **to** n	$cost : c_3$ $times : n$
4	$\quad sum = sum + A[j]$	$cost : c_4$ $times : n$
5	$\quad$ **if** $sum > max$	$cost : c_5$ $times : n$
6	$\quad\quad max = sum$	$cost : c_6$ $times : k \in [0, n]$
7	**return** max	$cost : c_7$ $times : 1$

Since $0 \leq k \leq n$, we may define $c_{6'} := k/n \in [0, 1]$. We therefore have $k = c_{6'}n$ for some $c_8 \in [0, 1]$. Let $T(n)$ be the running time of the algorithm for an array of length n. Adding up the total cost of each line, we have

$$\begin{aligned} T(n) &= c_1 \cdot 1 + c_2 \cdot 1 + c_3 \cdot n + c_4 \cdot n + c_5 \cdot n + c_6 c_{6'} \cdot n + c_7 \cdot 1 \\ &= (c_3 + c_4 + c_5 + c_6 c_{6'}) \cdot n + (c_1 + c_2 + c_7) = \Theta(n) \end{aligned}$$

The running time of this algorithm is therefore $\Theta(n)$.

(3) Yes, this algorithm is asymptotically optimal, since any algorithm searching for the maximal subarray of the form $A[1\,..\,j]$ must examine each element of the array (of length n) at least once and will therefore have running time $\Omega(n)$.

Question 2. *Recall that for $0 \leq k \leq n$, the **binomial coefficent** $\binom{n}{k}$ is defined by*

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

In particular, we have $\binom{n}{0} = \binom{n}{n} = 1$ for every n.

(1) Prove for every $0 < k < n$:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

You can use any valid method you know to prove this (from the definition, combinatorial, generating function, etc.) (5pts)

(2) Write pseudocode for a recursive algorithm Binomial(n, k) *which returns $\binom{n}{k}$. Your algorithm should use the above fact you proved in (1). (5pts)*

(3) Give a proof of correctness of your algorithm in (2). You should prove the statement: "For every $n \geq 0$ and for every $0 \leq k \leq n$, Binomial(n, k) *returns $\binom{n}{k}$." (5pts)*

Solution.

(1) Let k and n be such that $0 < k < n$. Therefore, since $k \leq n-1$ and $k-1 \leq n-1$, the binomial coefficients $\binom{n-1}{k}$ and $\binom{n-1}{k-1}$ are well-defined. We therefore have

$$\begin{aligned}
\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \frac{(n-1)!}{k!(n-k-1)!} \cdot \frac{n-k}{n-k} + \frac{(n-1)!}{(k-1)!(n-k)!} \cdot \frac{k}{k} \\
&= \frac{(n-k)\cdot(n-1)!}{k!(n-k)!} + \frac{k\cdot(n-1)!}{k!(n-k)!} \\
&= \frac{(n-k)\cdot(n-1)! + k\cdot(n-1)!}{k!(n-k)!} \\
&= \frac{(n\cdot(n-1)!}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k} \\
\therefore\ \binom{n}{k} &= \binom{n-1}{k} + \binom{n-1}{k-1} \quad \blacksquare
\end{aligned}$$

(2) Following is pseudocode for a recursive Binomial(n, k) algorithm based on the fact from (1):

Binomial(n, k):

```
1  // base cases: k = 0 or n = k imply (n choose k) = 1
2  if k == 0 or n == k
3        return 1
4  // recursive step
5  else return Binomial(n − 1, k) + Binomial(n − 1, k − 1)
```

(3) For $n \geq 0$, we define the property

C(n): "BINOMIAL(n, k) returns $\binom{n}{k}$ for every $k \in \mathbb{Z}$ such that $0 \leq k \leq n$."

We show this is true for all $n \geq 0$ by strong induction.

Base case: $(n = 0)$ Since $0 \leq k \leq n$, we only have one possible value of k, i.e., $k = 0$. We therefore have

$$\binom{n}{k} = \binom{0}{0} = \frac{0!}{0! \cdot 0!} = 1$$

which, since $n = k = 0$, is what BINOMIAL$(0, 0)$ returns in line 3. $P(0)$ is therefore true.

Inductive step: Let $n \geq 0$ such that $P(0), \ldots, P(n)$ holds. We show $P(n + 1)$ holds.

Let k be arbitrary such that $0 \leq k \leq n + 1$. We have the following two cases for k:

Case 1: $(k = 0)$ In this case, we have

$$\binom{n+1}{k} = \binom{n+1}{0} = \frac{(n+1)!}{0! \cdot (n+1)!} = 1$$

which, since $k = 0$, is what BINOMIAL$(n + 1, k)$ returns in line 3. BINOMIAL$(n + 1, k)$ therefore returns $\binom{n+1}{k}$ for $k = 0$.

Case 2: $(k = n + 1)$ In this case, we have

$$\binom{n+1}{k} = \binom{k}{k} = \frac{k!}{k! \cdot (k-k)!} = \frac{k!}{k! \cdot 0!} = 1$$

which, since $n + 1 = k$, is what BINOMIAL$(n + 1, k)$ returns in line 3. BINOMIAL$(n + 1, k)$ therefore returns $\binom{n+1}{k}$ for $k = n + 1$.

Case 3: $(0 < k < n+1)$[1] In this case, the condition in line 1 fails and we go to line 5. In line 5, we call BINOMIAL(n, k) and BINOMIAL$(n, k-1)$. Since $k < n+1$, we have $0 \leq k \leq n$ and $0 \leq k-1 \leq n$. By the inductive hypothesis, we know BINOMIAL(n, k) returns $\binom{n}{k}$ and BINOMIAL$(n, k-1)$ returns $\binom{n}{k-1}$. Furthermore, from (1), we have

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Therefore, in line 5, we return

$$\text{BINOMIAL}(n, k-1) + \text{BINOMIAL}(n, k-1) = \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

which is the desired value. BINOMIAL$(n + 1, k)$ therefore returns $\binom{n+1}{k}$ for $0 < k < n + 1$.

From cases 1, 2, and 3, we therefore know BINOMIAL$(n + 1, k)$ returns $\binom{n+1}{k}$ for $0 \leq k \leq n + 1$.

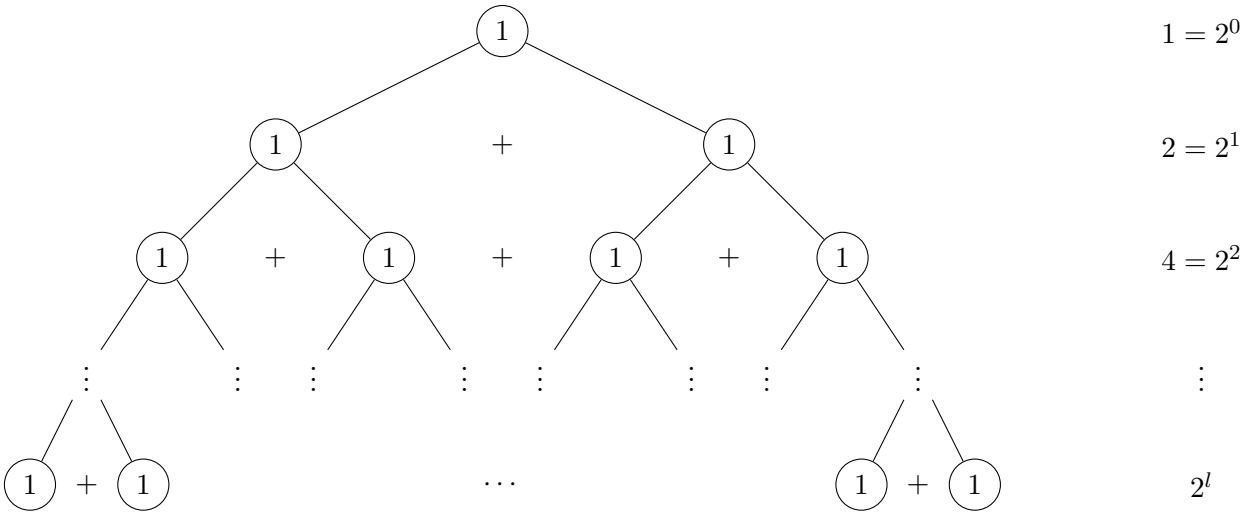
$\therefore P(n + 1)$ is true. By the Principle of Induction we conclude that $P(n)$ is true for $n \in N$. ■

Question 3. *Use a recursion tree and the substitution method to guess and verify an asymptotically tight bound for the following recurrence (5pts):*

$$T(n) \;=\; 2T(n-1) + 1$$

Following is the recursion tree for the given recurrence relation, with the cost per level in the right column:

[1]Note that this case is only possible for $n + 1 \geq 2$, and, thus far, we have only shown $P(0)$ is true. However, cases 1 and 2 are sufficient to show $P(1)$ is true, since the only values k can take for $n + 1 = 1$ are 0 (case 1) and 1 (case 2). Case 3 is therefore only to be considered for $n + 1 \geq 2$.



On each level k, we call $T(n-k)$ 2^k times. Assuming running $T(0)$ costs 1, the leaves of the recursion tree each represent a $T(0)$ call. In the last layer l, we therefore have $T(n-l) = T(0) \implies l = n$. We guess the total cost of $T(n)$ is therefore given by

$$\sum_{k=0}^{l} 2^k = \sum_{k=0}^{n} 2^k = \frac{2^{n+1}-1}{2-1} = 2^{n+1} = \Theta(2^n)$$

We now want to verify our guess by showing $T(n) = \Theta(2^n)$, i.e., we need to find some $c_1, c_2 > 0$, $n_0 \geq 0$ such that for all $n \geq n_0$, $c_1 2^n \leq T(n) \leq c_2 2^n$.

We assume that there is some $n_0 \geq 0$ such that for some $c_1, c_2 > 0$, $c_1 2^m \leq T(m) \leq c_2 2^m - k$ for all $m < n$.[2] We therefore have

$$\begin{aligned} T(n) = 2T(n-1) + 1 &\implies 2c_1 \cdot 2^{n-1} + 1 \leq T(n) \leq 2 \cdot (c_2 2^{n-1} - k) + 1 \quad \text{(by assumption)} \\ &\implies c_1 2^n \leq T(n) \leq c_2 2^n - 2k + 1 \\ &\implies c_1 2^n \leq T(n) \leq c_2 2^n - k \end{aligned}$$

where the last inequality holds for $k \geq 1$. Let $k := 1$. We now consider the base case of $n = 0$ with $T(n) = 1$:

$$c_1 2^n \leq T(n) \leq c_2 2^n - k \implies c_1 \leq 1 \leq c_2 - 1$$

which holds true for $c_1 \leq 1$ and $c_2 \geq 2$. We also consider the base case $n = 1$ with $T(n) = 2T(n-1) + 1 = 2T(0) + 1 = 3$:

$$c_1 2^n \leq T(n) \leq c_2 2^n - k \implies 2c_1 \leq 3 \leq 2c_2 - 1$$

which holds true for $c_1 \leq 1.5$ and $c_2 \geq 2$. Choosing $c_1 = 1$ and $c_2 = 2$ should therefore satisfy our inductive hypothesis. We now show directly by induction (tracing our steps backwards) that $c_1 2^n \leq T(n) \leq c_2 2^n$ for all $n \geq 0$.

We have already shown for chosen c_1, c_2 that for $n = 0$ and $n = 1$, $c_1 2^n \leq T(n) \leq c_2 2^n - 1$.

Inductive step: Let $n_0 \geq 0$ be such that for $n = n_0$, $2^n \leq T(n) \leq 2 \cdot 2^n - 1$. We show the inequality also holds for $n = n_0 + 1$:

$$\begin{aligned} T(n_0+1) = 2T(n_0) + 1 &\implies 2 \cdot 2^{n_0} + 1 \leq T(n_0+1) \leq 2 \cdot (2^{n_0+1} - 1) + 1 \quad \text{(by assumption on } T(n_0)) \\ &\implies 2^{n_0+1} \leq T(n_0+1) \leq 2 \cdot 2^{n_0+1} - 1 \end{aligned}$$

[2] We subtract k to account for $+1$ in the recurrence relation.

By the Principle of Induction, we therefore have for all $n \geq 0$

$$2^n \leq T(n) \leq 2 \cdot 2^n - 1$$

We therefore have $2^n \leq T(n) \leq 2 \cdot 2^n$ for all $n \geq 0$. By definition of Θ-notation, $T(n) = \Theta(2^n)$. ■

Question 4. *For the following recurrence determine an asymptotically tight bound using any method (recursion tree and substitution, master method, etc.). (5pts)*

$$T(n) \;=\; 25T(n/5) + \frac{n^2}{\lg n}$$

Solution. We find an asymptotically tight bound for $T(n)$ using the Master Theorem. From the recursion relation, we have $a = 25$, $b = 5$, and $f(n) = n^2 \lg^{-1} n$. We therefore have

$$f(n) = n^2 \lg^{-1} n = n^{\log_5 25} \lg^{-1} n = n^{\log_b a} \lg^{-1} n$$

The recursion relation is therefore characterised by the middle case of the Master Theorem for $k = -1$. We therefore have, from the Master Theorem,

$$\begin{aligned} T(n) &= \Theta(n^{\log_b a} \lg \lg n) = \Theta(n^{\log_5 25} \lg \lg n) = \Theta(n^2 \lg \lg n) \\ \therefore\; T(n) &= \Theta(n^2 \lg \lg n) \end{aligned}$$

Question 5. *For the following functions $f(n)$ and $g(n)$, determine whether they satisfy:*

(1) $f(n) = o(g(n))$,
(2) $f(n) = \Theta(g(n))$, or
(3) $f(n) = \omega(g_n)$.

The functions are:

$$f(n) \;=\; (\lg n)^{\sqrt{\lg n}} \quad \textit{and} \quad g(n) \;=\; \sqrt{n}$$

Justify your answer. (5pts)

Solution. We show $f(n) = o(g(n))$.

$$\begin{aligned} \lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{(\lg n)^{\sqrt{\lg n}}}{\sqrt{n}} = \lim_{n\to\infty} \operatorname{two}\left(\lg\left(\frac{(\lg n)^{\sqrt{\lg n}}}{\sqrt{n}}\right)\right) && (\text{for } \operatorname{two}(n) := 2^n) \\ &= \operatorname{two}\left(\lim_{n\to\infty}\left(\sqrt{\lg n}\lg\lg n - \frac{1}{2}\lg n\right)\right) \\ &= \operatorname{two}\left(\lim_{n\to\infty} \frac{1}{2}\lg n\left(\frac{2\lg\lg n}{\sqrt{\lg n}} - 1\right)\right) \\ &= \operatorname{two}\left(\lim_{k\to\infty} \frac{k}{2}\left(\frac{2\lg k}{\sqrt{k}} - 1\right)\right) && (\text{for } k := \lg n) \end{aligned}$$

Since $\lim_{k\to\infty} 2\lg k = \infty$ and $\lim_{k\to\infty} \sqrt{k} = \infty$, we can apply L'Hôpital's Rule to show $\lim_{k\to\infty} 2\lg k/\sqrt{k}$ exists and equals 0:

$$\lim_{k\to\infty} \frac{2\lg k}{\sqrt{k}} = \lim_{k\to\infty} 2 \cdot \frac{1}{k} \cdot \frac{2\sqrt{k}}{1} = 4 \lim_{n\to\infty} \frac{1}{\sqrt{k}} = 0 \implies \lim_{n\to\infty}\left(\frac{2\lg k}{\sqrt{k}} - 1\right) = -1$$

Applying this limit, we get

$$\begin{aligned}
\lim_{n\to\infty}\frac{f(n)}{g(n)} &= \text{two}\left(\lim_{k\to\infty}\frac{k}{2}\left(\frac{2\lg k}{\sqrt{k}}-1\right)\right)\\
&= \text{two}\left(\lim_{n\to\infty}k\cdot\lim_{n\to\infty}\left(\frac{2\lg k}{\sqrt{k}}-1\right)\right) && \text{(the limit exists)}\\
&= \text{two}\left(\lim_{k\to\infty}k\cdot(-1)\right) = \text{two}\left(\lim_{k\to\infty}-k\right)\\
&= \text{two}(-\infty) = 0\\
\implies f(n) &= o(g(n)) \ \blacksquare && \text{(by definition of } o\text{−notation)}
\end{aligned}$$

Question 6. *(True/False) For each of the following statements indicate whether they are **true** or **false**. Each question is worth 2pts, a blank answer will receive 1pt. Recall that "true" means "always true" and "false" means "there exists a counterexample".*

(1) For every $n \geq 1$ and $a, b \in \mathbb{Z}$, if $ab \bmod n = 0$, then either $a \bmod n = 0$ or $b \bmod n = 0$.

(2) Let $(F_n)_{n\geq 0}$ be the sequence of Fibonacci numbers, so $F_0 = 0, F_1 = 1$ and for every $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Then for every $n \geq 2$, $F_{2n} = F_{2(n-1)} + F_{2(n-2)}$.

(3) Suppose $f(n)$ and $g(n)$ are asymptotically positive, polynomially bounded functions. If $f(n) = \Theta(g(n))$, then $2^{2^{f(n)}} = \Theta(2^{2^{g(n)}})$.

(4) $\Omega(n) = O(n^2)$.

(5) The best-case running time of Insertion-Sort *is $O(n \lg n)$.*

(6) Merge-Sort *is an asymptotically optimal comparison-based sorting algorithm.*

Solution.

(1) False. We present a counter-example: let $a := 2 \in \mathbb{Z}$, $b := 2 \in \mathbb{Z}$ and $n := 4 \geq 1$. We have $ab \bmod n = 4 \bmod 4 = 0$, but $a \bmod n = 2 \bmod 4 = 2 \neq 0$ and $b \bmod n = 2 \bmod 4 = 2 \neq 0$.

(2) False. We present a counter-example: let $n := 2 \geq 2$. We have $F_{2n} = F_4 = 3$, but $F_{2(n-1)} + F_{2(n-2)} = F_2 + F_0 = 1 + 0 = 1 \neq 3$.

(3) False. We present a counter-example: let $g(n) := 2n$, $f(n) := n = \Theta(g(n))$. Both $f(n)$ and $g(n)$ are asymptotically positive and polynomially bounded functions. We show $f(n) = o(g(n))$:

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{2^{2^n}}{2^{2^{2n}}} = \lim_{n\to\infty}\frac{2^{2^n}}{2^{2^n\cdot 2^n}} = \lim_{n\to\infty}\frac{1}{2^{2^n\cdot 2^n - 2^n}} = \lim_{n\to\infty}\frac{1}{2^{2^n(2^n-1)}} = 0$$

We therefore have $f(n) = o(g(n))$, and therefore, $f(n) \neq \Theta(g(n))$.

(4) False. We present a counter-example: let $f(n) = n^3$. We therefore clearly have $f(n) = \Omega(n)$, but $f(n) \neq O(n^2)$.

(5) True. The best-case running time of Insertion-Sort is $\Theta(n)$, which in turn is $O(n \lg n)$.

(6) True. From Theorem 3.3.1 (proven in class), we know any comparison-based sorting algorithm performs at $\Omega(n \lg n)$. Since the running time of Merge-Sort is $\Theta(n \lg n)$, there does not exist any comparison-based sorting algorithm that is asymptotically better than Merge-Sort.