Homework Assignment 1

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1. Exercise 1

Consider the function $f(x) := \dfrac{1}{1+2x} - \dfrac{1-x}{1+x}$**, with** $x > 0$**.**

(a) For what values of x **do you expect cancellation of significant digits? Explain.**

We expect cancellation of significant digits (by subtraction) in one, or possibly both, of the following two cases:

Case 1: x is extremely close to 1. In this case, computing the numerator $1 - x$ in the second fraction of $f(x)$ can result in catastrophic cancellation. This may happen only if x is extremely close to 1.

Case 2: The two fractions subtracted in $f(x)$ are extremely close to each other. In this case, computing their difference can result in catastrophic cancellation. We may find estimate a range for the values of x give this result by finding values where the two fractions are equal.

$$\begin{aligned}
\frac{1}{1+2x} &= \frac{1-x}{1+x} \\
\implies 1 + x &= (1-x) \cdot (1+2x) \\
\implies 1 + x &= -2x^2 + x + 1 \\
\implies 2x^2 &= 0 \\
\implies x &= 0
\end{aligned}$$

We therefore see that values of x very close to 0 may result in catastrophic cancellation.

Values of x close to 0 or 1 may result in catastrophic cancellation.

(b) Rewrite the expression for computing f(x) so that it avoids cancellation for those values of x identified in part (a).

We rewrite $f(x)$ to remove subtractions as follows:

$$\begin{aligned}
f(x) &= \frac{1}{1+2x} - \frac{1-x}{1+x} && (1)\\
&= \frac{1}{1+2x} \cdot \frac{1+x}{1+x} - \frac{1-x}{1+x} \cdot \frac{1+2x}{1+2x} && (2)\\
&= \frac{1+x}{(1+2x)\cdot(1+x)} - \frac{(1-x)\cdot(1+2x)}{(1+x)\cdot(1+2x)} && (3)\\
&= \frac{(1+x)-(1-x)\cdot(1+2x)}{(1+2x)\cdot(1+x)} && (4)\\
&= \frac{(1+x)-(-2x^2+x+1)}{(1+2x)\cdot(1+x)} && (5)\\
&= \frac{2x^2}{(1+2x)\cdot(1+x)} && (6)\\
&= \frac{2x\cdot x}{(1+2x)\cdot(1+x)} && (7)\\
\therefore f(x) &= \frac{2x\cdot x}{(1+2x)\cdot(1+x)} && (8)
\end{aligned}$$

This final expression involves no subtraction, and we therefore avoid catastrophic cancellation of significant figures.

It is important to note, however, that if x is sufficiently close to 0, we may get rounding-off error to 0 when computing x^2 in the numerator of equation (6). Separating the x variables and multiplying the first with 2 before computing the product (per Horner's Method) as in equation (8) decreases the chance of such a round-off error by some nominal amount.

2. Exercise 2

Suppose $f(x)$ is continuous on $[a, b]$, and $f(x) \in [a, b]$ for any $x \in [a, b]$. Show that f has at least one fixed point on $[a, b]$.

To prove their exists a fixed point $x \in [a, b]$, we consider the following two cases:

Case 1: There exists a fixed point at the edges, i.e. $f(a) = a$ and/or $f(b) = b$. This is a trivial case, since if there exists a fixed point at at least one of the edges, there exists at least one fixed point in $[a, b]$.

Case 2: There does not exist a fixed point at either edge, i.e. $f(a) \neq a$ and $f(b) \neq b$.
Since $f(x) \in [a, b]$ for all $x \in [a, b]$, and $f(a) \neq a$ and $f(b) \neq b$, it must be that $f(a) > a$ and $f(b) < b$.
Let $h(x) := f(x) - x$. Since f is continuous over $[a, b]$, h is also continuous over $[a, b]$. This implies the following:

$$\begin{aligned}
h(a) &= f(a) - a > 0 \qquad \because f(a) > a\\
h(b) &= f(b) - b < 0 \qquad \because f(b) < b
\end{aligned}$$

Since $h(a) > 0$ and $h(b) < 0$, we have $h(a) \cdot h(b) < 0$. Furthermore, since h is continuous over $[a, b]$, by the Intermediate Value Theorem, there exists $x \in [a, b]$ such that $h(x) = 0$. For such a value of x, $h(x) = 0 \implies f(x) - x = 0$ (by definition of $h(x)$) $\implies f(x) = x$, i.e. x is a fixed point of f.

Therefore, there exists at least one fixed point of f on $[a, b]$. ■

3. Exercise 3

Consider the following non-linear equation: $f(x) = x^2 - 0.7x = 0$ **on** $[0.5, 1]$**.**

(a) Show that $f(x)$ **has exactly one root on** $[0.5, 1]$ **without solving the equation.**

Let $a := 0.5$, $b := 1$. Since $f(a) = -0.1 < 0$ and $f(b) = 0.3 > 0$, we have $f(a) \cdot f(b) < 0$.

$\therefore$ By the Intermediate Value Theorem, there exists at least one $x \in [a, b] = [0.5, 1]$ such that $f(x) = 0$.

i.e. there exists at least one root of f in $[0.5, 1]$.

Computing the derivative of $f(x)$, we have $f'(x) = 2x - 0.7 > 0$ for $x \geq 0.35$.

$\implies f'(x) > 0 \ \forall \ x \in [0.5, 1]$

$\implies f(x)$ is monotonically increasing.

Since we have shown f has at least one root in $[0.5, 1]$ and f is monotonically increasing over the interval, f must have exactly one root in $[0.5, 1]$. ■

(b) Consider the bisection algorithm starting with the interval $[0.5, 1]$**, i.e. consider** $[a_1, b_1] = [0.5, 1]$ **and** $p_1 = 0.75$**. Find the minimum number of iterations required to approximate the solution with an absolute error of less than** 10^{-5}**.**

Let $N :=$ the minimum number of iterations required to know we have accuracy of $\epsilon = 10^{-5}$. We find N by computing the following bound:

$$\begin{aligned}
&|p_N - p| \leq \frac{1}{2^N} \cdot (b - a) < \epsilon \\
\implies &|p_N - p| \leq \frac{1}{2^N} \cdot (1 - 0.5) < 10^{-5} \\
\implies &\frac{1}{2^N} \cdot 0.5 < 10^{-5} \\
\implies &\frac{1}{2^{N+1}} < 10^{-5} \\
\implies &2^{N+1} > 10^5
\end{aligned}$$

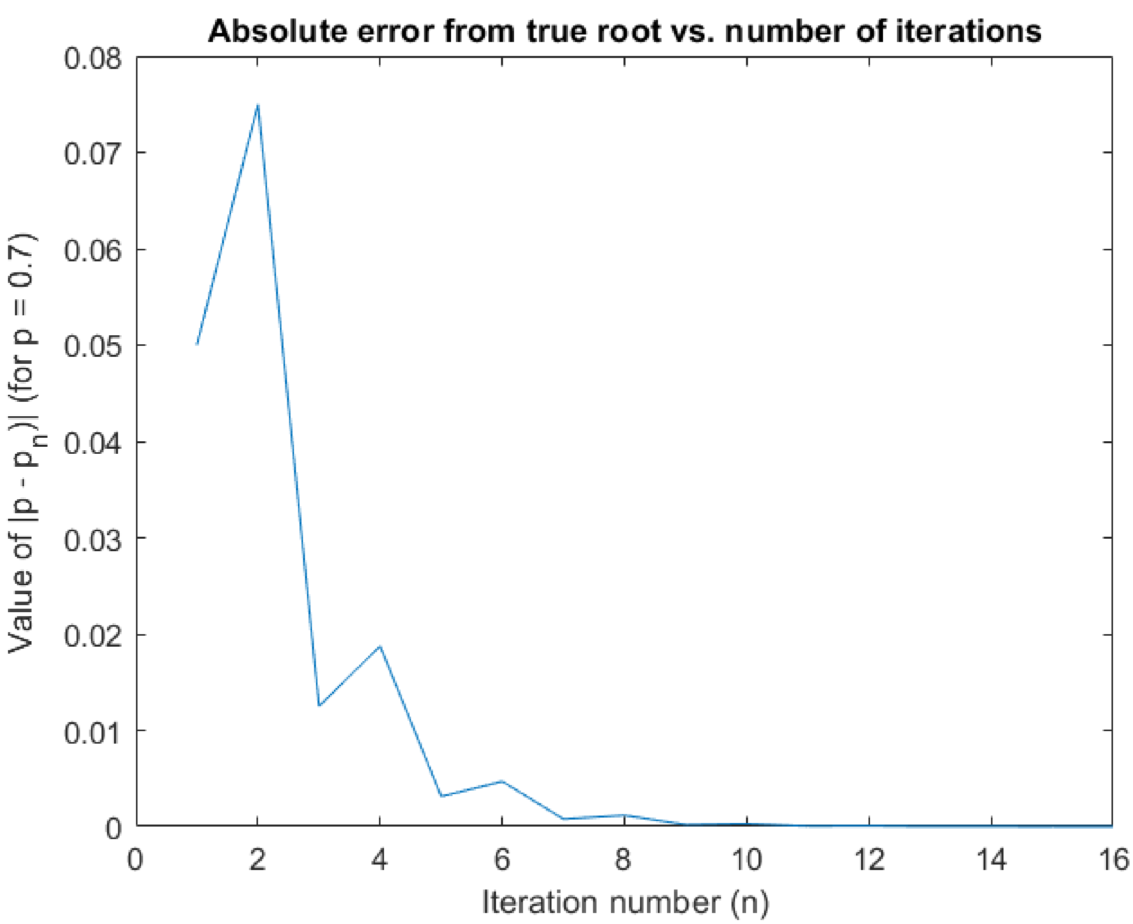
Taking $\log_{10}$ on both sides of the inequality, we get:

$$\implies \log_{10}(2^{N+1}) > log_{10}(10^5)$$
$$\implies (N+1) \cdot log_{10}(2) > 5$$
$$\implies N > \frac{5}{log_{10}(2)} - 1 \approx 15.61$$
$$\implies N > 15.61 \implies N = 16$$

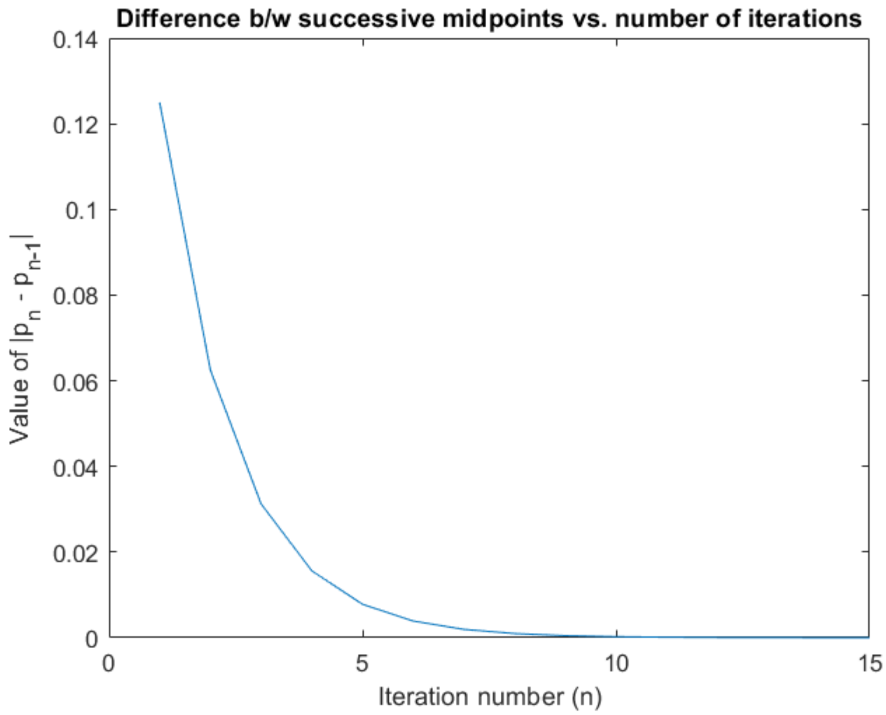
$\therefore$ We require 16 iterations to approximate the solution with an absolute error of less than 10^{-5}.

(c) (Programming) Now program a bisection algorithm to verify this. In particular, create three figures.

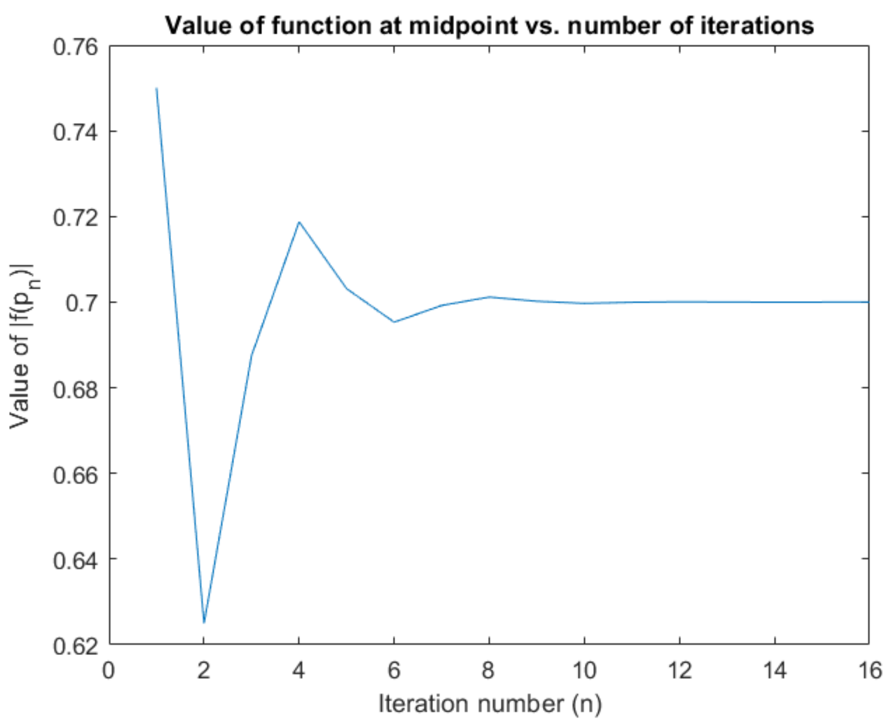
- In the first figure, plot the values $|p - p_n|$ on the y-axis, and the iteration number in the x-axis.



- In the second figure, plot $|p_n - p_{n1}|$ in the y-axis and the iteration number in the x-axis.



- In the third figure, plot the values for $|f(p_n)|$ on the y-axis and the iteration number in the x-axis.



Do your experiments coincide with part (b)?

Yes, the experiment confirms our expectation from part (b): the error $|p_{16} - p|$ (for $p = 0.7$) is $7.6294e - 07$, well below the error bound of 10^{-5}.

4. Exercise 4

Consider the following non-linear equation: $f(x) = \sqrt{x} - \cos x = 0$ on $[0, 1]$.

(a) Show that $f(x)$ has exactly one root on $[0, 1]$ without solving the equation.

Let $a := 0$, $b := 1$. Since $f(a) = -1 < 0$ and $f(b) \approx 0.46 > 0$, we have $f(a) \cdot f(b) < 0$.
$\therefore$ By the Intermediate Value Theorem, there exists at least one $x \in [a, b] = [0, 1]$ such that $f(x) = 0$. i.e. there exists at least one root of f in $[0, 1]$.

Computing the derivative of $f(x)$, we have $f'(x) = \dfrac{1}{2x^{1/2}} + \sin x$.

For $x \in (0, 1]$, $x > 0 \implies \dfrac{1}{2x^{1/2}} > 0$, and for $x \in [0, 1] \subset [0, \pi/2]$, $\sin x \geq 0$.

$\therefore f'(x) = \dfrac{1}{2x^{3/2}} + \sin x > 0 \;\forall\; x \in (0, 1]$.

$\implies f'(x) > 0 \;\forall\; x \in (0, 1]$

$\implies f(x)$ is monotonically increasing over the interval $(0, 1]$.

Since we have shown f has at least one root in $[0, 1]$ (and $x = 0$ is not a root of f) and f is monotonically increasing over the interval $(0, 1]$, f must have exactly one root in $[0, 1]$. $\blacksquare$

(b) Consider the bisection algorithm starting with the interval $[0, 1]$, i.e. consider $[a_1, b_1] = [0, 1]$ and $p_1 = 0.5$. Find the minimum number of iterations required to approximate the solution with an absolute error of less than 10^{-5}.

Let $N :=$ the minimum number of iterations required to know we have accuracy of $\epsilon = 10^{-5}$. We find N by computing the following bound:

$$
\begin{aligned}
|p_N - p| &\leq \frac{1}{2^N} \cdot (b - a) < \epsilon \\
\implies |p_N - p| &\leq \frac{1}{2^N} \cdot (1 - 0) < 10^{-5} \\
\implies \frac{1}{2^N} &\cdot 1 < 10^{-5} \\
\implies \frac{1}{2^N} &< 10^{-5} \\
\implies 2^N &> 10^5
\end{aligned}
$$

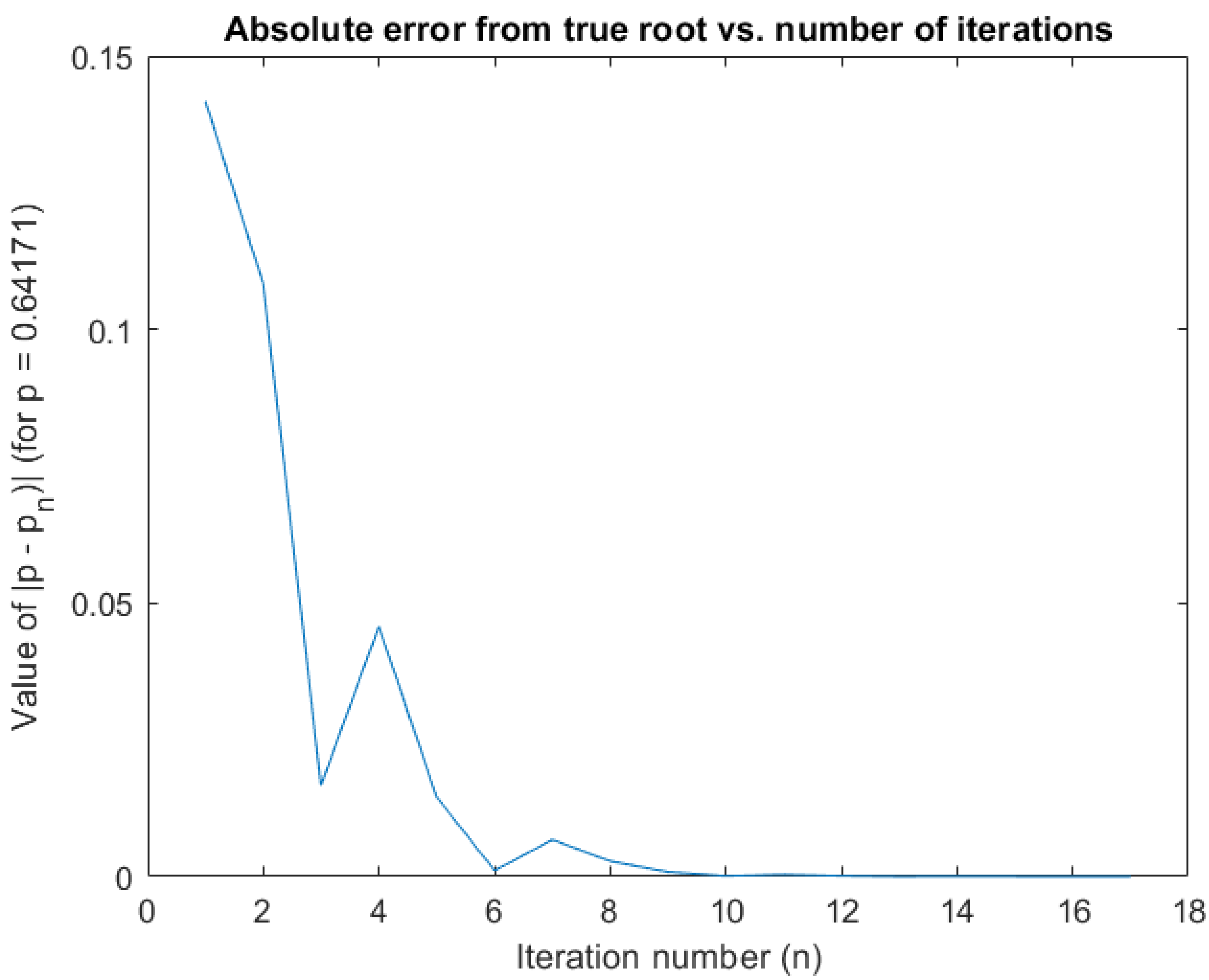
Taking $\log_{10}$ on both sides of the inequality, we get:

$$\implies \log_{10}(2^N) > log_{10}(10^5)$$
$$\implies N \cdot log_{10}(2) > 5$$
$$\implies N > \frac{5}{log_{10}(2)} \approx 16.61$$
$$\implies N > 16.61 \implies N = 17$$

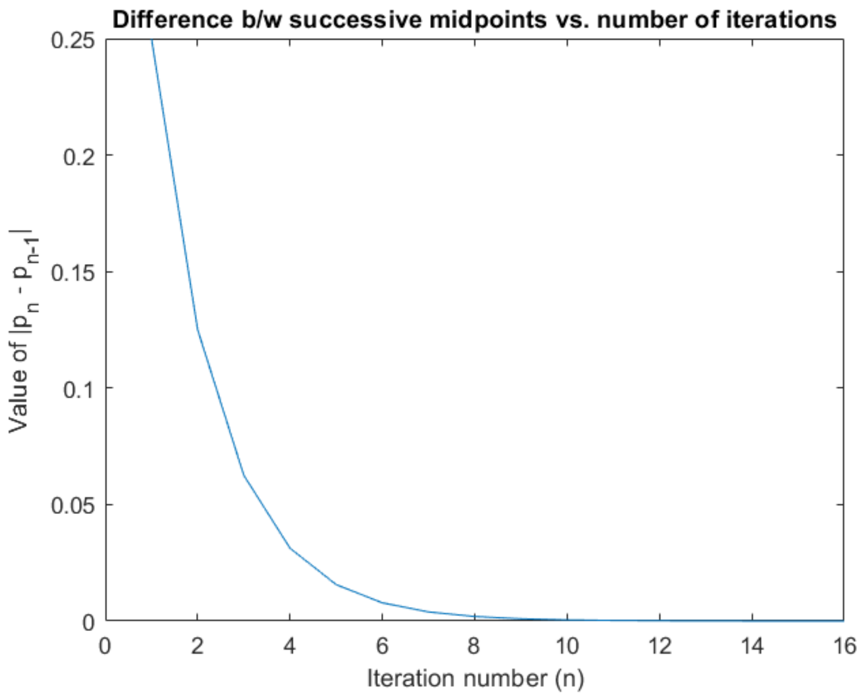
$\therefore$ We require 17 iterations to approximate the solution with an absolute error of less than 10^{-5}.

(c) (Programming) Now program a bisection algorithm to verify this. In particular, create three figures.

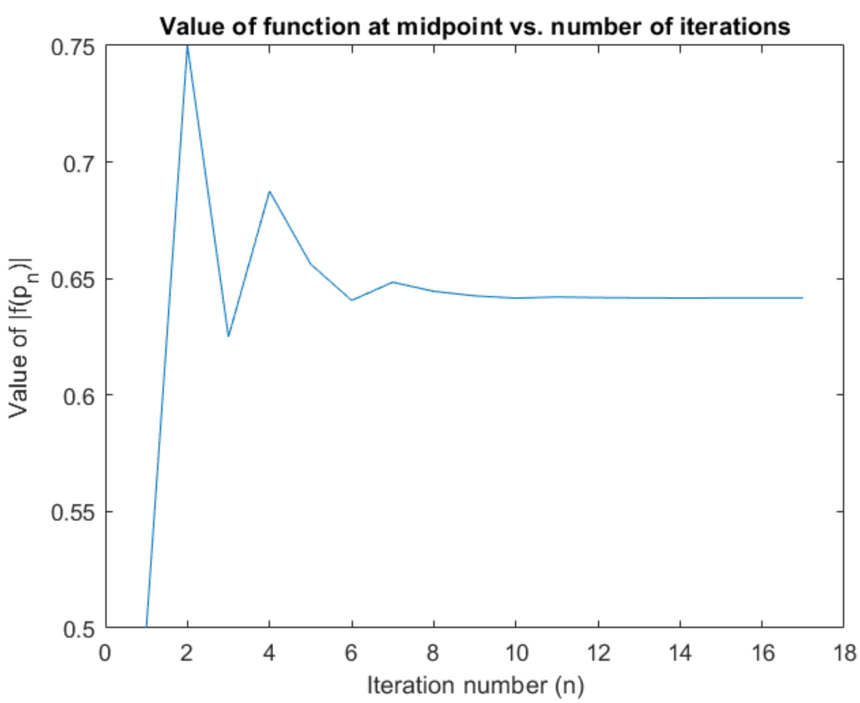
- In the first figure, plot the values $|p - p_n|$ on the y-axis, and the iteration number in the x-axis.



- In the second figure, plot $|p_n - p_{n1}|$ in the y-axis and the iteration number in the x-axis.



- In the third figure, plot the values for $|f(p_n)|$ on the y-axis and the iteration number in the x-axis.



Do your experiments coincide with part (b)?

Yes, the experiment confirms our expectation from part (b): the error $|p_{17} - p|$ (for $p = 0.641714371$) is $2.1823e - 06$, well below the error bound of 10^{-5}.