$$L_{2,1}(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{-16}{15}(x-2)(x-4)$$

$$L_{2,2}(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{2}{5}(x-2)(x-2.75)$$

$$\Rightarrow P_2(x) = \sum_{i=0}^{2} L_{2,i}(x) \cdot f(x_i)$$

A better error estimate using Lagrange Polynomials:

$\boxed{\begin{array}{l}\text{Thm: Suppose } x_0, x_1, \dots, x_n \text{ are } n+1 \text{ distinct numbers in the interval } [a,b] \text{ and } f \in C^{(n+1)}[a,b]. \text{ Then}\\ \text{for each } x \in [a,b], \text{ there exists } \xi(x) \in [a,b] \text{ between } x_0, x_1, \dots, x_n, \text{ s.t.}\\ \quad f(x) = P(x) + \underbrace{\dfrac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)}_{R_L}\\ \text{where } P(x) = \displaystyle\sum_{k=0}^{n} f(x_k) \cdot L_{n,k}(x)\end{array}}$

Remark: 1) Lagrange error R_L is similar to Taylor poly. error:

$$\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1} \quad \text{vs.} \quad \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)\cdots(x-x_n)$$

but error in R_L is "spread" across different nodes.

Ex: Given $x_0 = 2$, $x_1 = 2.75$, $x_2 = 4$ for $f(x) = \frac{1}{x}$ 67

a) Determine the error form for Lagrange polynomial $p(x)$

b) Determine the maximum error when $p(x)$ is used to approximate $f(x)$ for $x \in [2,4]$

Sol: a) $f'(x) = \frac{-1}{x^2}$, $f''(x) = \frac{2}{x^3}$, $f'''(x) = \frac{-6}{x^4}$

$\Rightarrow$ Error: $R_2(x) = \frac{f'''(\xi(x))}{3!} \underbrace{(x-x_0)(x-x_1)(x-x_2)}_{g(x)}$

$= \frac{-6}{3!} (\xi(x))^{-4} (x-2)(x-2.75)(x-4) \qquad \xi(x) \in (2,4).$

b) Want to find $\max\limits_{x \in (2,4)} |R_2(x)|$

Note: $|\xi(x)^{-4}| \leq 2^{-4} = \frac{1}{16}$

Let $g(x) = (x-2)(x-2.75)(x-4) = x^3 - \frac{35}{4}x^2 + \frac{49}{2}x - 22$

To find max. vals of g in $[2,4]$, need to find crit. pts.

$g'(x) = 3x^2 - \frac{35}{2}x + \frac{49}{2} = \frac{1}{2}(3x-7)(2x-7)$

$\Rightarrow$ critical pts occur at $x = \frac{7}{3}$ with $g\left(\frac{7}{3}\right) = \frac{25}{108}$

and $x = \frac{7}{2}$ with $g\left(\frac{7}{2}\right) = \frac{-9}{16}$

$\left(\left|\frac{9}{16}\right| > \left|\frac{25}{108}\right|\right)$

At boundary values $g(2) = g(4) = 0$.

$\Rightarrow \max\limits_{x \in (2,4)} |g(x)| \leq \frac{9}{16}$. Hence, max. error is

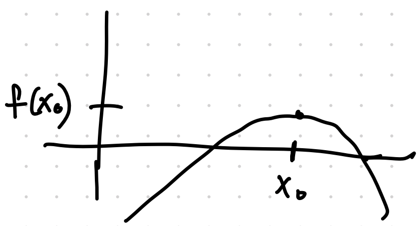
$$R_L(x) \leq |s(x)^{-4}| \cdot |g(x)| \leq \left|\frac{1}{16}\right| \cdot \left|\frac{9}{16}\right|$$

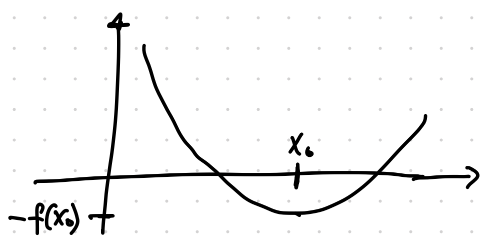
Remarks: 1) Power series form of $p(x) \equiv$ Lagrange form but with different basis:

$$\underbrace{\{x^n, x^{n-1}, \ldots, x^1, 1\}}_{\text{Power series}} \quad \text{vs.} \quad \underbrace{\{L_{n,n}(x), L_{n,n-1}(x), \ldots, L_{n,1}(x), L_{n,0}(x)\}}_{\text{Lagrange}}$$

2) Can use fminbound (fun, x_1, x_2) to find maximum of a function in interval (x_1, x_2).

Note: $\max\limits_{x} f(x) = -\min\limits_{x} (-f(x))$





3) Adding a new node x_j changes all bases $L_{n,j}$'s to $L_{n+1,j}$'s

3.3 Divided Differences and Newton's Form

Q: Given data (x_i, f_i) for $i=0,1,\dots,n$, how to find a_i's s.t. the polynomial

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

(Newton Form)

interpolates (x_i, f_i)?

Forward Divided Differences

- zeroth divided difference of f w.r.t. x_i:

$$f[x_i] = f(x_i)$$

- 1st div. diff. of f w.r.t. x_i, x_{i+1}:

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

- 2nd div. diff. of f w.r.t. x_i, x_{i+1}, x_{i+2}:

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

- k^{th} div. diff. w.r.t. $x_i, x_{i+1}, \dots, x_{i+k}$

$$f[x_i, x_{i+1}, \dots, x_{i+k}] = \frac{f[x_{i+1}, \dots, x_{i+k}] - f[x_i, \dots, x_{i+k-1}]}{x_{i+k} - x_i}$$

In particular,

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, \ldots x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0} \qquad (1)$$

Thm: Let p_n be the n^{th} degree polynomial s.t.

$p_n(x_i) = f(x_i)$ for $i = 0, 1, \ldots, n$. Then

$$p(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots + f[x_0, \ldots, x_n](x - x_0)\cdots(x - x_{n-1}) \qquad (*)$$

$$p(x) = f[x_0] + \sum_{j=1}^{n}\left[f[x_0, \ldots, x_j]\left(\prod_{i=0}^{j-1}(x - x_i)\right)\right]$$

Remarks: $(*)$ referred to as Newton form of polynomial

2) obtain $f[x_0, \ldots, x_j]$ recursively using (1)

Another equivalent way to obtain Newton Coefficients:

Recall: $p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$

$$\left.\begin{aligned} p(x_0) &= f(x_0) \\ p(x_1) &= f(x_1) \\ &\vdots \\ p(x_n) &= f(x_n) \end{aligned}\right\} \quad \begin{aligned} & a_0 \\ & a_0 + a_1(x_1 - x_0) \\ & \vdots \\ & a_0 + a_1(x_n - x_0) + a_2(x_n - x_0)(x_n - x_1) + \cdots \\ & \qquad + a_n(x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1}) \end{aligned}$$

$\Rightarrow$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & (x_1 - x_0) & 0 & & 0 \\ 1 & (x_2 - x_0) & (x_2 - x_0)(x_2 - x_1) & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \\ 1 & (x_n - x_0) & (x_n - x_0)(x_n - x_1) & & (x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1}) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}$$

$$L\underline{a} = \underline{f}$$

Remarks: 1) This can be easily solved by forward substitution

2) $\det(L) = 1 \cdot [(x_1 - x_0)] \cdot [(x_2 - x_0)(x_2 - x_1)] \cdots [(x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})]$

$\Rightarrow$ If all x_i are distinct, L is nonsingular

$\Rightarrow$ $L\underline{a} = \underline{f}$ has unique sol.

Ex: Given $(0,0), (\frac{\pi}{2}, 1), (\pi, 0)$. Find Newton form of poly.

$$\begin{aligned} p(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\ &= a_0 + a_1(x) + a_2(x)(x - \tfrac{\pi}{2}) \end{aligned}$$

$p(0) = 0 \Rightarrow a_0 = 0$

$p(\frac{\pi}{2}) = 1 \Rightarrow 0 + a_1 \frac{\pi}{2} = 1 \Rightarrow a_1 = \frac{2}{\pi}$

$p(\pi) = 0 \Rightarrow a_0 + \frac{2}{\pi}(\pi) + a_2 \cdot \pi \cdot (\frac{\pi}{2}) \Rightarrow a_2 = \frac{-4}{\pi^2}$

$$\Rightarrow p(x) = 0 + \frac{2}{\pi} x - \frac{4}{\pi^2} x \left(x - \frac{\pi}{2}\right)$$

← double check!

<u>Recap</u>: We have seen three forms of $p(x)$:

- power series (Vandermonde)
- Lagrange
- Newton

Although these may look different, they each give the <u>same</u> $p(x)$.

A general form for interpolation

- Given data $\{(x_i, f_i)\}_{i=0}^{n}$
- and a basis of polynomials of degree n:

$$\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots, \phi_n(x)\}$$

Find coefficients $a_0, a_1, \ldots, a_n$ s.t.

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$p(x) = a_0 \phi_0(x) + a_1 \phi_1(x) + \cdots + a_n \phi_n(x)$

and $p(x_i) = f_i$

To find $a_0, a_1, \ldots, a_n$, we need to solve a linear system:

$$\left.\begin{aligned} p(x_0) &= f_0 \\ p(x_1) &= f_1 \\ &\vdots \\ p(x_n) &= f_n \end{aligned}\right\} \Rightarrow \begin{bmatrix} \phi_0(x_0) & \phi_1(x_0) & \cdots & \phi_n(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \cdots & \phi_1(x_n) \\ \vdots & & & \\ \phi_0(x_n) & \phi_1(x_n) & \cdots & \phi_n(x_n) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}$$

$$M \underset{\sim}{a} = \underline{f}$$

<u>Ex</u>: a) Power series basis $\{1, x, x^2, \ldots, x^n\}$

$\Rightarrow M = V =$ Vandermonde matrix $O(n^3)$

b) Newton Basis: $\{1, x-x_0, (x-x_0)(x-x_1), \ldots, (x-x_0)\cdots(x-x_{n-1})\}$

$\Rightarrow M = L =$ lower triangular $O(n^2)$

c) Lagrange basis: $\{L_{n,0}(x), L_{n,1}(x), \ldots, L_{n,n}(x)\}$

where $L_{n,j} = \dfrac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_n)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_n)}$

In this case, $M = I$ = identity matrix.

d) Chebyshev Basis $\{T_0(x), T_1(x), \ldots, T_n(x)\}$

where

$$T_j(x) = \cos(j \cdot \arccos(x)), \quad -1 \le x \le 1$$

In this case, M does not have any special structure.

Solve $\quad M\underline{a} = \underline{f}$

Question Why different bases?

- In some cases, easier to solve $M\underline{a} = \underline{f}$
- In some cases, M is better conditioned
  $\Rightarrow$ solving $M\underline{a} = \underline{f}$ may produce less errors in $\underline{a}$

MATLAB: $\quad$ p = polyfit ($\underline{x}_{nodes}$, $\underline{f}_{nodes}$, n)

polyval (p, $\underline{x}$)

Error in Poly. interpolation

Given $\{(x_i, f_i)\}_{i=0}^{n}$, $\quad f_i = f(x_i)$.

Let $p_n(x)$ be poly. of deg. n with $p_n(x_i) = f_i$

Recall Thm:

Let $[a,b]$ be interval containing $x_0, x_1, \dots, x_n$ and $f \in C^{n+1}(a,b)$. Then

$$e_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \underbrace{(x-x_0)(x-x_1)\cdots(x-x_n)}_{W_{n+1}(x)}$$

where ξ between $x_0, x_1, \dots, x_n$.

Remarks: 1) At interpolating points $x_0, x_1, \dots, x_n$, $e_n(x_i) = 0$.

2) We may have little information, or control over the size of $\left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \right|$

$\Rightarrow$ We might investigate how to make small:

$$|W_{n+1}(x)| = |(x-x_0)(x-x_1)\cdots(x-x_n)|$$

- With equally spaced points, $x_0, x_1, \dots, x_n$, $W_{n+1}(x)$ tends to oscillate, with growing amplitudes near endpoints
- A different choice of points can reduce this. A good choice of points is often Chebyshev points, e.g.,

$$x_i = \frac{b+a}{2} - \frac{b-a}{2}\cos\left(\frac{2i+1}{2n+2}\pi\right), \quad i = 0, 1, \dots, n$$

on interval $[a,b]$

Problems with polynomial interpolation

- High degree poly. tend to oscillate
  $\Rightarrow$ large errors can occur (will see how)
- Low degree poly. do not oscillate as much, but be poor approx. to functions that do oscillate.

Splines (piece-polynomial) Interpolation

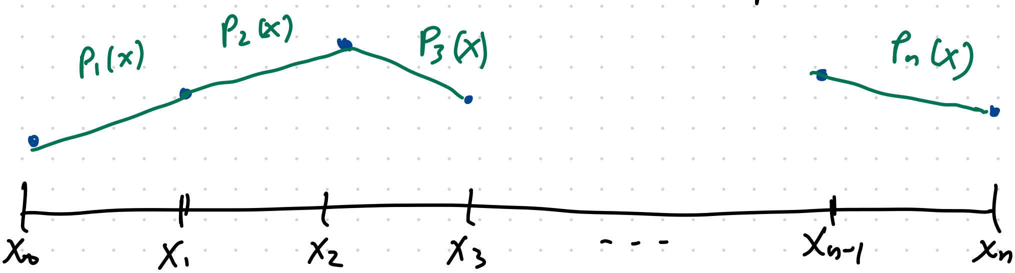
- partition interval into pieces
- construct low degree poly. approx. on each subinterval
- connect poly. pieces together

Linear Splines

Given $\{(x_i, f_i)\}_{i=0}^{n}$, define $s(x)$ as

$$S(x) = \begin{cases} p_1(x) & \text{if} \quad x \in [x_0, x_1] \\ p_2(x) & \text{if} \quad x \in [x_1, x_2] \\ \vdots & \\ p_n(x) & \text{if} \quad x \in [x_{n-1}, x_n] \end{cases}$$

where each $p_i(x)$ is a linear polynomial (line)



How to find $p_i(x)$?

Recall: Given two points (x_{i-1}, f_{i-1}), (x_i, f_i), point-slope form eqn. of line:

$$y - f_i = \frac{f_i - f_{i-1}}{x_i - x_{i-1}}(x - x_i)$$

$$\Rightarrow p_i(x) = f_i + \frac{f_i - f_{i-1}}{x_i - x_{i-1}}(x - x_i)$$

We will use notation:

$$a_i = f_i \,, \quad b_i = \frac{f_i - f_{i-1}}{x_i - x_{i-1}}$$

So $p_i(x) = a_i + b_i(x - x_i)$

A linear spline for the data $\{(x_i, f_i)\}_{i=0}^{n}$ can be represented as

$$S(x) = \begin{cases} a_1 + b_1(x - x_1) & \text{if } x_0 \le x < x_1 \\ a_2 + b_2(x - x_2) & \text{if } x_1 \le x < x_2 \\ \vdots & \\ a_n + b_n(x - x_n) & \text{if } x_{n-1} \le x \le x_n \end{cases}$$

where $\boxed{a_i = f_i \qquad b_i = \frac{f_i - f_{i-1}}{x_i - x_{i-1}}}$

Ex: Construct a linear spline for data 78

x_i	-1	0	1
f_i	0	1	3

$a_1 = 1$, $a_2 = 3$

$b_1 = \dfrac{f_1 - f_0}{x_1 - x_0} = 1$, $b_2 = 2$

$$s(x) = \begin{cases} 1 + (x-0) = 1 + x & -1 \le x < 0 \\ 3 + 2(x-1) = 1 + 2x & 0 \le x \le 1 \end{cases}$$

Error for Linear Splines

Recall:
- For linear interpolation, we use two points:

(x_{i-1}, f_{i-1}), (x_i, f_i)

- From them, the error is

$$|f(x) - p_i(x)| = \left|\frac{f''(\xi_i)}{2!}\right| \cdot |(x - x_{i-1})(x - x_i)|, \quad \xi_i \in [x_{i-1}, x_i]$$

- Let $M_i = \max\limits_{x_{i-1} \le x \le x_i} |f''(x)|$

Then $|f(x) - p_i(x)| \le \dfrac{M_i}{2} |(x - x_{i-1})(x - x_i)|$

- Can we find an upper bound on $|(x - x_{i-1})(x - x_i)|$?

Let $w_i(x) = (x - x_{i-1})(x - x_i) = x^2 - (x_{i-1} + x_i)x + x_{i-1} \cdot x_i$

$$w_i'(x) = 2x - (x_{i-1} + x_i) = 0 \Rightarrow \boxed{x = \frac{x_{i-1} + x_i}{2}}$$

- Notice that $|w_i(x)|$ is largest either at crit. pt. $x = \frac{x_{i-1}+x_i}{2}$, or at one of the endpts. x_{i-1}, x_i.

$$|w_i(x_{i-1})| = 0, \quad |w_i(x_i)| = 0$$

$$\left|w_i\left(\frac{x_{i-1}+x_i}{2}\right)\right| = \left|\left(\frac{x_{i-1}+x_i}{2} - x_{i-1}\right)\left(\frac{x_{i-1}+x_i}{2} - x_i\right)\right|$$

$$= \frac{(x_i - x_{i-1})^2}{4} \leftarrow \text{largest}$$

$\Rightarrow$ For linear interpolation, the error for $x \in [x_{i-1}, x_i]$ is

$$|f(x) - p_i(x)| \le \frac{1}{2} M_i \, |w_i(x)| \le \frac{1}{2} \cdot \frac{1}{4} M_i \, (x_i - x_{i-1})^2$$

$$= \frac{1}{8} M_i h_i^2$$

where $h_i = x_i - x_{i-1}$

Now consider linear splines

$$s(x) = \begin{cases} p_1(x) & x \in [x_0, x_1] \\ p_2(x) & x \in [x_1, x_2] \\ \vdots & \vdots \\ p_n(x) & x \in [x_{n-1}, x_n] \end{cases}$$

Let $|e_n(x)| = |f(x) - s(x)| \le \max\limits_i \frac{M_i}{8} \cdot h_i^2 = \frac{M}{8} \cdot h^2$

where $h = \max\limits_i h_i$, $M = \max\limits_i M_i$.