Homework Assignment 2

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1. Exercise 1

Use algebraic manipulations to show that each of the following functions has a fixed point at p precisely when $f(p) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$.

(a) $g_1(x) = (3 + x - 2x^2)^{1/4}$

By definition, at a fixed point p of g_1, $g_1(p) = p$.

$$\begin{aligned} g_1(p) = p &\iff (3 + p - 2p^2)^{1/4} = p \\ &\iff 3 + p - 2p^2 = p^4 \\ &\iff p^4 + 2p^2 - p - 3 = 0 \\ &\iff f(p) = 0 \end{aligned}$$

Therefore, $g_1(x)$ has a fixed point at p precisely when $f(p) = 0$.

(b) $g_2(x) = \left(\dfrac{x + 3 - x^4}{2}\right)^{1/2}$

By definition, at a fixed point p of g_2, $g_2(p) = p$.

$$\begin{aligned} g_2(p) = p &\iff \left(\frac{p + 3 - p^4}{2}\right)^{1/2} = p \\ &\iff \frac{p + 3 - p^4}{2} = p^2 \\ &\iff p + 3 - p^4 = 2p^2 \\ &\iff p^4 + 2p^2 - p - 3 = 0 \\ &\iff f(p) = 0 \end{aligned}$$

Therefore, $g_2(x)$ has a fixed point at p precisely when $f(p) = 0$.

2. Exercise 2

Given the following sequence $p_{n+1} = \dfrac{p_n^2 + 3}{2p_n}$

(a) Calculate p_1 and p_2 with $p_0 = 3$.

We first find p_1, given $p_0 = 3$:

$$p_1 = \frac{p_0^2 + 3}{2p_0} = \frac{3^2 + 3}{2 \cdot 3} = \frac{12}{6} = 2$$

We now find p_2, given $p_1 = 2$:

$$p_2 = \frac{p_1^2 + 3}{2p_1} = \frac{2^2 + 3}{2 \cdot 2} = \frac{7}{4} = 1.75$$

(b) Show by definition that the given sequence is actually a sequence generated by Newton's method to find a solution of the equation $x^2 - 3 = 0$.

By Newton's method, for any equation $f(x)$ and given starting guess p_0, we find a root by computing the sequence:

$$p_{n+1} = p_n - \frac{f(p_n}{f'(p_n} \quad \forall n \geq 0$$

For $f(x) = x^2 - 3$, we have $f'(x) = 2x$. By Newton's method,

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} = p_n - \frac{p_n^2 - 3}{2p_n} = \frac{2p_n^2 - p_n^2 + 3}{2p_n} = \frac{p_n^2 + 3}{2p_n}$$

$$\implies p_{n+1} = \frac{p_n^2 + 3}{2p_n}$$

The given sequence is therefore the sequence generated by Newton's method to find a root of $f(x) = x^2 - 3$.

3. Exercise 3

Consider the following non-linear equation: $f(x) = x^2 - 3 = 0$ on $[0, 4]$.

(a) Write an expression for p_n using the secant method with the f provided above. Compute p_2 and p_3 using starting points $p_0 = 1$ and $p_1 = 3$.

By the secant method, we have:

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$$

For $f(x) = x^2 - 3$, we then have:

$$
\begin{aligned}
p_n &= p_{n-1} - \frac{(p_{n-1}^2 - 3) \cdot (p_{n-1} - p_{n-2})}{(p_{n-1}^2 - 3) - (p_{n-2}^2 - 3)} \\
&= p_{n-1} - \frac{(p_{n-1}^2 - 3) \cdot (p_{n-1} - p_{n-2})}{p_{n-1}^2 - p_{n-2}^2} \\
&= p_{n-1} - \frac{(p_{n-1}^2 - 3) \cdot (p_{n-1} - p_{n-2})}{(p_{n-1} + p_{n-2}) \cdot (p_{n-1} - p_{n-2})} \\
&= p_{n-1} - \frac{p_{n-1}^2 - 3}{p_{n-1} + p_{n-2}} \\
&= \frac{p_{n-1} \cdot (p_{n-1} + p_{n-2}) - (p_{n-1}^2 - 3)}{p_{n-1} + p_{n-2}} \\
&= \frac{p_{n-1}^2 + p_{n-1} \cdot p_{n-2} - p_{n-1}^2 + 3}{p_{n-1} + p_{n-2}} \\
&= \frac{p_{n-1} \cdot p_{n-2} + 3}{p_{n-1} + p_{n-2}} \\
\implies \quad p_n &= \frac{p_{n-1} \cdot p_{n-2} + 3}{p_{n-1} + p_{n-2}}
\end{aligned}
$$

With the above formula, we now compute p_2 given $p_0 = 1$ and $p_1 = 3$:

$$p_2 = \frac{p_1 \cdot p_0 + 3}{p_1 + p_0} = \frac{3 \cdot 1 + 3}{3 + 1} = \frac{6}{4} = 1.5$$

We now compute p_3 given $p_1 = 3$ and $p_2 = 1.5$:

$$p_3 = \frac{p_2 \cdot p_1 + 3}{p_2 + p_1} = \frac{1.5 \cdot 3 + 3}{1.5 + 3} = \frac{7.5}{4.5} = \frac{5}{3} = 1.\overline{6}$$

(b) Compute p_2 and p_3 using the method of false position with f given above, and with starting points $p_0 = 1$ and $p_1 = 3$.

We first compute $sgn(f(p_0))$ and $sgn(f(p_1))$ for $p_0 = 1$ and $p_1 = 3$:

$$sgn(f(p_0)) = sgn(f(1)) = sgn(1^2 - 3) = sgn(-2) = -1$$

$$sgn(f(p_1)) = sgn(f(3)) = sgn(3^2 - 3) = sgn(6) = 1$$

From (a), we know $p_2 = 1.5$ (since $sgn(f(p_0)) \cdot sgn(f(p_1)) < 0$). We compute $sgn(f(p_2))$:

$$sgn(f(p_2)) = sgn(f(1.5)) = sgn(1.5^2 - 3) = sgn(-0.75) = -1$$

Since $sgn(f(p_1)) = 1$ and $sgn(f(p_2) = -1$, it is clear that $sgn(f(p_1)) \cdot sgn(f(p_2)) = -1 < 0$. We therefore continue iterating with p_2 and p_1 as we did with the secant method in (a), and we get $p_3 = 5/3 = 1.\overline{6}$. While the secant method and the method of false position give consistent results up to p_3, as we continue iterating, we will see that the method of false position (which has root-bracketing) will get us close to the root over fewer iterations.

4. Exercise 4

(a) Use the Fixed Point Theorems from Section 2.2 to show that $g(x) = \pi + 0.5\sin(x/2)$ has a unique fixed point on $[0, 2\pi]$.

Since $\sin(x)$ is defined for all $x \in \mathbb{R}$, $g(x) = \pi + 0.5\sin(x/2)$ is continuous over $\mathbb{R}$. $g(x)$ is therefore necessarily continuous over $[0, 2\pi]$.

We now show that $g(x)$ is bounded from above and below as follows: .

$$\begin{array}{rrcccl} & -1 & \leq & \sin(x/2) & \leq & 1 \qquad \text{(by definition of sin(x)), for all } x \\ \implies & -0.5 & \leq & 0.5\sin(x/2) & \leq & 0.5 \\ \implies & \pi - 0.5 & \leq & \pi + 0.5\sin(x/2) & \leq & \pi + 0.5 \\ \implies & 2.64 & \leq & g(x) & \leq & 3.65 \text{ for all } x \end{array}$$

Therefore, $g(x) \in [2.64, 3.65] \subseteq [0, 2\pi]$ for all $x \in \mathbb{R}$. It then follows that $g(x) \in [0, 2\pi]$ for all $x \in [0, 2\pi]$. Therefore, by the Fixed Point Existence Theorem, there exists a fixed point $p \in [0, 2\pi]$ such that $g(p) = p$. Furthermore, since g is a trigonometric function defined on $\mathbb{R}$, it is continuously differentiable on $[0, 2\pi] \subseteq \mathbb{R}$. To satisfy the second condition of the Fixed Point Uniqueness Theorem, we need an upper bound k on $|g'(x)|$ on the interval $[0, 2\pi]$ such that $k \in (0, 1)$.

We find $g'(x) = 0.5\cos(x/2) \cdot \frac{1}{2} = 0.25\cos(x/2)$.

Since $|\cos(x)| \leq 1$ for all $x \in \mathbb{R}$, $|g(x)| = |0.25\cos(x/2)| = 0.25\,|\cos(x/2)| \leq 0.25 \cdot 1 = 0.25$ for all $x \in \mathbb{R}$. We can therefore take the upper-bound $k = 0.25 \in (0, 1)$ for $|g'(x)|$ on the interval $[0, 2\pi]$.

Therefore, $g(x)$ is continuously differentiable on $[0, 2\pi]$ and there exists $k = 0.25 \in (0, 1)$ such that $|g'(x)| \leq k$ for all $x \in [0, 2\pi]$. Therefore, by the Fixed Point Uniqueness Theorem, there exists a unique fixed point $p \in [0, 2\pi]$ such that $g(p) = p$. □

(b) Use the theoretical result to estimate the number of iterations required to achieve 10^{-4} accuracy.

The error bound for p_n in fixed point iteration can be given by:

$$|p_n - p| \;\leq\; k^n \cdot max\{|p_0 - a|\,, |p_0 - b|\}$$

For $[a, b] = [0, 2\pi]$, any arbitrary $p_0 \in [a, b]$ gives $max\{|p_0 - a|\,|p_0 - b|\} = \dfrac{a+b}{2} = \pi$. From (a), we also have $k = 0.25$.

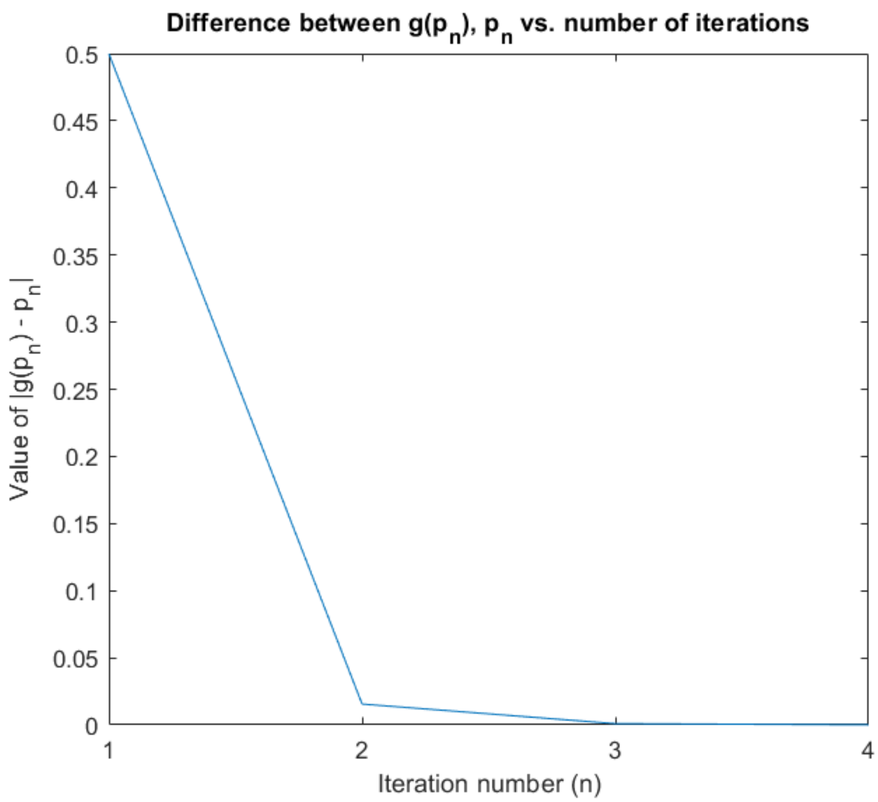
$$\therefore |p_n - p| \;\leq\; k^n \cdot max\{|p_0 - a|\,, |p_0 - b|\} \;\leq\; 10^{-4}$$

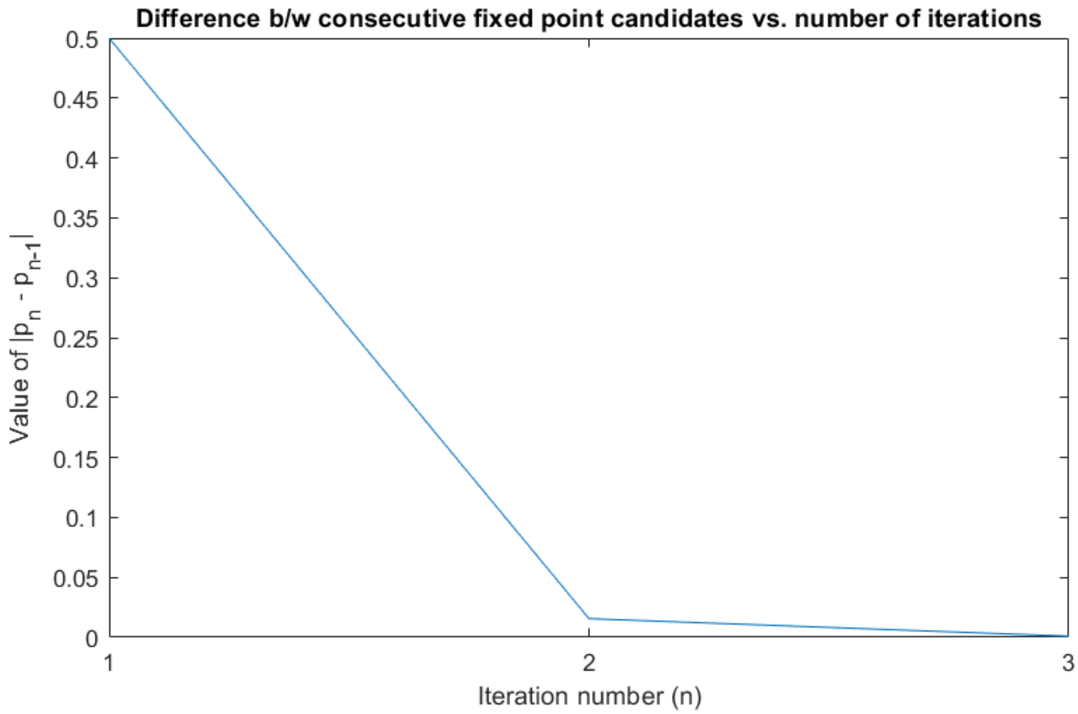
$$\begin{aligned}
\implies \quad 0.25^n \cdot \pi &\leq 10^{-4} \\
\implies \quad 0.25^n &\leq \frac{10^{-4}}{\pi} \\
\implies \quad n \cdot \log_{10}(0.25) &\leq \log_{10}\left(\frac{10^{-4}}{\pi}\right) \\
\implies \quad n \cdot \log_{10}(0.25) &\leq \log_{10}(10^{-4}) - \log_{10}(\pi) \\
\implies \quad n &\geq \frac{\log_{10}(10^{-4}) - \log_{10}(\pi)}{\log_{10}(0.25)} \quad (\because \log_{10}(0.25) < 0) \\
\implies \quad n &\geq 7.47 \\
\implies \quad n &\geq 8
\end{aligned}$$

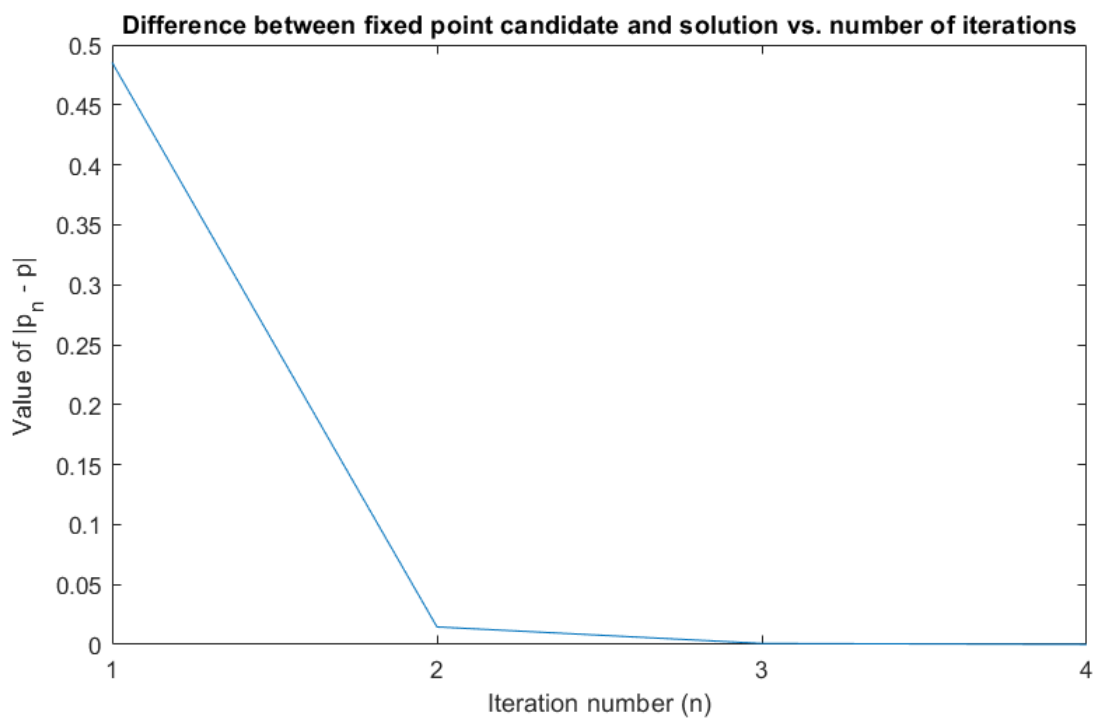
We therefore find that for any arbitrary $p_0 \in [0, 2\pi]$, eight fixed point iterations give a result with at least 10^{-4} accuracy.

(c) Use fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-4} using stopping criteria based on $|g(p_n) - p_n|$ and $|p_n - p_{n-1}|$. Create three figures for the following convergence histories: $|g(p_n) - p_n|$, $|p_n - p_{n-1}|$, and $|p_n - p|$.

Taking $p_0 = \frac{a+b}{2} = \frac{0+2\pi}{2} = \pi$, we get, in four iterations, $p_4 = 3.626995622438735$, which is within 10^{-4} of $p \approx 3.626942014871573$ (approximated to 3.626942015 in our code). Please find below graphs representing the convergence histories.




Difference b/w consecutive fixed point candidates vs. number of iterations
Value of $|p_n - p_{n-1}|$
Iteration number (n)
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
1
2
3


Difference between fixed point candidate and solution vs. number of iterations
Value of $|p_n - p|$
Iteration number (n)
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
1
2
3
4