Remark 1: If $f(p)=0$ and $f'(p)\neq 0$,
then for any p_0 sufficiently close to
(s.t. $g\in C^1(a,b)$, $g(x)\in[a,b)$, $|g'(x)|\le k$)
Newton's method will converge at least quadratically.

Remark 2: If $f(p)=0$, then for p_0 close to p,
secant method converges to p with order

$$\frac{\sqrt{5}+1}{2}\approx 1.618$$

Method	Order α
Bisection	1
Fixed Pt. Iteration	1 if $g'(p)\neq 0$ ≥ 2 if $g'(p)=0$
Newton	≥ 2 if $f'(p)\neq 0$ ≤ 1 if $f'(p)=0$
Secant	≈ 1.618 if $f'(p)\neq 0$ ≤ 1 if $f'(p)=0$

2.4 Multiple Roots

Def: Let $f(x) = (x-p)^m q(x)$ and $\lim_{x\to p} q(x) \neq 0$

Then p is a zero of multiplicity m (multiple root) of f.

$m=1 \Rightarrow p$ is a simple root.

Thm 1: A function $f \in C^1(a,b)$ has a simple root at $p \in (a,b)$ if and only if $f(p)=0$ and $f'(p) \neq 0$

Pf: ($\Rightarrow$) If f has a simple root at p in (a,b),

then 1) $f(p)=0$

2) $f(x) = (x-p)\cdot q(x)$ where $\lim_{x\to p} q(x) \neq 0$

Since $f \in C^1(a,b)$,

$$f'(p) = \lim_{x\to p} f'(x) = \lim_{x\to p}\left(q(x) + \overset{0}{\cancel{(x-p)q'(x)}}\right)$$

$$= \lim_{x\to p} q(x) \neq 0$$

($\Leftarrow$) If $f(p)=0$ and $f'(p)\neq 0$,

$$f(x) = \overset{=0}{\widetilde{f(p)}} + f'(\xi(x))(x-p) = f'(\xi(x))\cdot(x-p)$$

ξ between x and p

Taylor

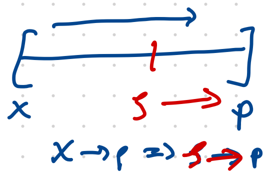
($\Leftarrow$) If $f(p)=0$ and $f'(p)\neq 0$,

$$f(x) = \overbrace{f(p)}^{=0} + f'(\xi(x))(x-p) = f'(\xi(x))\cdot(x-p)$$

$\uparrow$ Taylor

ξ between x and p

Then $\lim\limits_{x\to p} f'(\xi(x)) = f'(p)$



Let $q(x) = f'(\xi(x))$, then p is a simple root of f. ■

<u>Remark</u>: When f has simple root, Newton's method converges quadratically.

Generalization

$\boxed{\begin{array}{l}\underline{\text{Thm 2}}:\ f\in C^m(a,b) \text{ has a zero of multiplicity } m \text{ at } p\in(a,b) \text{ if and only if} \\ \qquad 0=f(p)=f'(p)=f''(p)=\cdots=f^{(m-1)}(p), \\ \qquad f^{(m)}(p)\neq 0\end{array}}$

<u>Ex</u>: Let $f(x) = e^x - x - 1$.

a) Show that $x=0$ is a zero of mult. 2 of $f(x)$

b) Show that Newton's Method does not converge quadratically

Ex: Let $f(x) = e^x - x - 1$.

a) Show that $x=0$ is a zero of mult. 2 of $f(x)$

b) Show that Newton's Method does not converge quadratically

Sol: a) Compute $f(0)$, $f'(0)$, $f''(0)$

$$f(x) = e^x - x - 1 \implies f(0) = 0$$

$$f'(x) = e^x - 1 \implies f'(0) = 0$$

$$f''(x) = e^x \implies f''(0) = 1 \neq 0$$

By Thm 2, $p=0$ is a zero of mult. 2

b) Since $f'(0) = 0$, Newton's method does not converge quadratically.

2.4 Modified Newton's Method

Recall: we lose quadratic convergence when $f'(p)=0$, i.e., when multiplicity of p is $m>1$

Let $$\mu(x) = \frac{f(x)}{f'(x)} = \frac{(x-p)^m q(x)}{(x-p)^m q'(x) + m(x-p)^{m-1} q(x)}$$

$$= \underbrace{\frac{(x-p)^{m-1}}{(x-p)^{m-1}}}_{=1} \cdot \frac{(x-p)q(x)}{(x-p)q'(x) + m\,q(x)}$$

Note that p is a simple root of $\mu(x)$ since

$$\lim_{x\to p} \frac{q(x)}{(x-p)q'(x)+m\,q(x)} = \frac{1}{m} \neq 0$$

Idea: Apply Newton's Method to $\mu(x)$ rather than $f(x)$ if $f'(p)=0$.

Since p is simple root $\Rightarrow$ quadratic convergence to p.

$$\boxed{\text{Modified Newton:}\quad p_{n+1} = p_n - \frac{\mu(p_n)}{\mu'(p_n)} = p_n - \frac{f(p_n)\cdot f'(p_n)}{[f'(p_n)]^2 - f(p_n)f''(p_n)}}$$

Remarks: 1) Modified Newton converges quadratically regardless of multiplicity of p.

2) Require second derivative information (expensive!)

2.5 Accelarating Convergence

Last time: Modified Newton improved linearly convergent to quadratically convergent scheme, but required higher derivatives (expensive!)

Q: Given a linearly convergent sequence, how to modify it to achieve faster convergence?

Def: Assume $\{p_n\}_{n=1}^{\infty}$ converges to p and

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = \lambda$$

a) If $\lambda = 0$, $\{p_n\}_{n=1}^{\infty}$ converges superlinearly to p

b) If $0<\lambda<1$, $\{p_n\}_{n=1}^{\infty}$ converges linearly to p

c) If $\lambda = 1$, $\{p_n\}_{n=1}^{\infty}$ converges sublinearly to p

Ex: Show that the seq. $p_n = \frac{1}{n+1}$ converges sublinearly to 0.

Pf:

$$\lim_{n\to\infty} \frac{|p_{n+1}-0|}{|p_n-0|} = \lim_{n\to\infty} \frac{\left|\frac{1}{n+2}\right|}{\left|\frac{1}{n+1}\right|} = \lim_{n\to\infty} \frac{|n+1|}{|n+2|} = 1$$

Aitken's Δ^2 Method

Idea: speed up convergence of linearly convergent seq.

Let $\{p_n\}_{n=1}^{\infty}$ converge linearly to p,

i.e., $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = \lambda \in (0,1)$

Assume that $p_n - p$, $p_{n+1}-p$, $p_{n+2}-p$ have the same sign.

Then

$$\underbrace{\frac{p_{n+2}-p}{p_{n+1}-p}}_{\approx\lambda} \approx \underbrace{\frac{p_{n+1}-p}{p_n-p}}_{\approx\lambda} \quad \text{for sufficiently large } n.$$

Isolating p:

$$p \approx \frac{p_{n+2} - 2p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p} = p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}$$

Define new sequence

$$\boxed{\hat{p}_n = p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}}$$

Remark: need to compute two sequences:

$$\begin{array}{lcl} p_0 \ p_1 \ p_2 & \longrightarrow & \hat{p}_0 \\ \quad\quad\;\; p_3 & \longrightarrow & \hat{p}_1 \\ \quad\quad\;\; \vdots & & \vdots \end{array}$$

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Def: Given $\{p_n\}_{n=1}^{\infty}$, the first order forward difference is defined as

$$\Delta p_n = p_{n+1} - p_n$$

2nd order fwd diff: $\Delta^2(p_n) = \Delta(\Delta p_n)$

$$= \Delta p_{n+1} - \Delta p_n$$

$$= p_{n+2} - 2p_{n+1} + p_n$$

3rd order fwd diff: $\Delta^3(p_n) = p_{n+3} - 3p_{n+2} + 3p_{n+1} - p_n$

k^{th} order fwd diff: $\Delta^k(p_n) = \Delta(\Delta^{k-1} p_n)$, $k \geq 2$.

Aitken's Δ^2 Method:

$$\boxed{\hat{p}_n = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n}}$$

Thm (Aitken's Method)

Assume that $\{p_n\}_{n=1}^{\infty}$ converges to p linearly, and

$$\lim_{n\to\infty} \frac{p_{n+1} - p}{p_n - p} < 1$$

Then Aitken's Δ^2 sequence $\{\hat{p}_n\}_{n=1}^{\infty}$ converges to p faster than $\{p_n\}$ in the sense that $\displaystyle\lim_{n\to\infty} \frac{\hat{p}_n - p}{p_n - p} = 0$

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Then Aitken's Δ^2 sequence $\{\hat{p}_n\}_{n=1}^{\infty}$ converges to p faster than $\{p_n\}$ in the sense that

$$\lim_{n\to\infty} \frac{\hat{p}_n-p}{p_n-p} = 0$$

Pf: Let $\displaystyle\lim_{n\to\infty} \frac{p_{n+1}-p}{\underbrace{p_n-p}_{\delta_n}} = \lambda$, $\quad \delta_n = \dfrac{p_{n+1}-p}{p_n-p}$

Then we have

$$p_{n+1}-p_n = (p_{n+1}-p) - (p_n-p) = (\delta_n - 1)(p_n-p) \quad (*)$$

$$p_{n+2}-2p_{n+1}+p_n = (p_{n+2}-p) - 2(p_{n+1}-p) + p_n - p$$

$$= (\delta_{n+1}\cdot\delta_n - 2\delta_n + 1)(p_n-p) \quad (**)$$

And

$$\lim_{n\to\infty}\frac{\hat{p}_n-p}{p_n-p} = \lim_{n\to\infty} \frac{1}{p_n-p}\left(p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n} - p\right)$$

$$= \lim_{n\to\infty} \underbrace{\frac{1}{p_n-p}(p_n-p)}_{=1} - \left[\frac{(p_{n+1}-p_n)^2}{(p_n-p)(p_{n+2}-2p_{n+1}+p_n)}\right]$$

$$= \underbrace{\lim_{n\to\infty} 1 - \frac{(\delta_n-1)^2}{\delta_{n+1}\delta_n - 2\delta_n+1}}_{\text{from } (*),\,(**)} = 1 - \frac{(\lambda-1)^2}{\lambda^2-2\lambda+1} = 1-1 = 0$$

∎

Remarks: 1) If $\{p_n\}$ converges quadratically, Aitken's may not accelerate convergence

2) Require $p_n - p$, $p_{n+1} - p$, $p_{n+2} - p$ to have same sign.

Aitken's Method for Fixed Pt. Iteration

$$p_0, \quad p_1 = g(p_0), \quad p_2 = g(p_1) \longrightarrow \hat{p}_0 = p_0 - \frac{(p_1 - p_0)^2}{p_2 - 2p_1 + p_0}$$

$$p_3 = g(p_2) \longrightarrow \hat{p}_1 = p_1 - \frac{(p_2 - p_1)^2}{p_3 - 2p_2 + p_1}$$

$$\vdots \qquad\qquad \vdots$$

Steffensen's Method. Let $\{\Delta^2\}(p_n) = p_n - \dfrac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$

0) $p_0^{(0)}$, $\quad p_1^{(0)} = g(p_0^{(0)})$, $\quad p_2^{(0)} = g(p_1^{(0)})$

1) $p_0^{(1)} = \{\Delta^2\}(p_0^{(0)}) = p_0^{(0)} - \dfrac{(p_1^{(0)} - p_0^{(0)})^2}{p_2^{(0)} - 2p_1^{(0)} + p_0^{(0)}}$ $\quad p_1^{(1)} = g(p_0^{(1)})$ $\quad p_2^{(1)} = g(p_1^{(1)})$

2) $p_0^{(2)} = \{\Delta^2\}(p_0^{(1)})$ $\quad p_1^{(2)} = g(p_0^{(2)})$ $\quad p_2^{(2)} = g(p_1^{(2)})$

$\vdots$

n) $p_0^{(n)} = \{\Delta^2\}(p_0^{(n-1)})$ $\quad p_1^{(n)} = g(p_0^{(n)})$ $\quad p_2^{(n)} = g(p_1^{(n)})$

Steffensen's Method. Let $\{\Delta^2\}(p_n) = p_n - \dfrac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}$

0) $p_0^{(0)}$, $\quad p_1^{(0)} = g(p_0^{(0)})$, $\quad p_2^{(0)} = g(p_1^{(0)})$

1) $p_0^{(1)} = \{\Delta^2\}(p_0^{(0)})$ $\quad p_1^{(1)} = g(p_0^{(1)})$ $\quad p_2^{(1)} = g(p_1^{(1)})$

$= p_0^{(0)} - \dfrac{(p_1^{(0)} - p_0^{(0)})}{p_2^{(0)} - 2p_1^{(0)} + p_0^{(0)}}$

2) $p_0^{(2)} = \{\Delta^2\}(p_0^{(1)})$ $\quad p_1^{(2)} = g(p_0^{(2)})$ $\quad p_2^{(2)} = g(p_1^{(2)})$

$\vdots$

n) $p_0^{(n)} = \{\Delta^2\}(p_0^{(n-1)})$ $\quad p_1^{(n)} = g(p_0^{(n)})$ $\quad p_2^{(n)} = g(p_1^{(n)})$

Thm (Steffensen's): Suppose $g(x) = x$ has sol. p with $g'(p) \neq 1$. If there exists $\delta > 0$ s.t. $g \in C^3[p-\delta, p+\delta]$, then Steffensen's gives quadratic convergence for $p_0 \in [p-\delta, p+\delta]$

2.6 Zeros of Polynomials

Def: A polynomial of degree n has the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, \quad a_n \neq 0.$$

Remarks:
- a_i's are coefficients of p
- $p(x) = 0$ is a polynomial of degree zero.

Q: Does $\underbrace{P_n}_{\text{deg. } n}$ have any zeros? How many?

Thm: (Fundamental Thm of Algebra): If $p(x)$ is a polynomial of degree $n \geq 1$ with $a_i \in \mathbb{C}$, $i = 0, 1, \dots, n$. Then $p(x) = 0$ has at least one complex root

Cor 1: If $p(x)$ is a polynomial of degree $n \geq 1$ with complex coefficients, then there exist unique $x_1, x_2, \dots, x_k$ and unique $\underbrace{m_1, m_2, \dots, m_k}_{\text{integers}}$ satisfying $\sum_{i=1}^{k} m_i = n$ s.t.

$$p(x) = a_n (x - x_1)^{m_1} (x - x_2)^{m_2} \cdots (x - x_k)^{m_k}$$

Remark: Cor 1 $\Rightarrow$ collection of zeros of P_n are unique, and if each zero x_i counted as many times as its multiplicity, m_i, then P_n has exactly n zeros.

$\boxed{\begin{array}{l}\textbf{Cor 2: } \text{Let } P(x) \text{ and } Q(x) \text{ be polynomials of degree at most } n. \\ \text{If } x_1, x_2, \dots, x_k \text{ with } k > n \text{ are distinct numbers s.t.} \\ \qquad P(x_i) = Q(x_i), \quad i = 1, \dots, k, \\ \text{Then } P(x) = Q(x) \text{ for all values of } X.\end{array}}$

<u>Remark</u>: 1) To show two polynomials of deg. $\leq n$ are the same, we only need to show that they agree on $n+1$ values (will use in Chpt 3)

2) <u>Proof sketch of Cor 2:</u>

$R(x) = P(x) - Q(x)$, $\deg(R) \leq n$, R has $n+1$ roots.

$$\Rightarrow R(x) \equiv 0 \Rightarrow P(x) = Q(x)$$

<u>Ex</u>: If $p(x)$ with $\deg(p(x)) = n$, and

$p(x_i) = x_i^n$ for $x_1 = 1, x_2 = 2, \dots, x_n = n, x_{n+1} = n+1$

then $p(x) = x^n$ (By Cor 2).

Horner's Method:

Newton's Method to find zeros of $p(x)$
$\Rightarrow$ need to evaluate $p(x)$ and $p'(x)$ repeatedly

Note: p and p' are both polynomials

Q: How can we evaluate polynomials efficiently?

Idea: nested evaluation / synthetic division

Recall from Chpt 1:

Ex: Evaluate $f(x) = x^3 - 6.1x^2 + 3.2x + 1.5$ at $x = 4.71$

mult. 2 2 1 = 5
add/sub. 1 1 1 = 3
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Now consider nested formulation:

$$f(x) = (x^3 - 6.1x^2 + 3.2x) + 1.5$$
$$= (x^2 - 6.1x + 3.2)x + 1.5$$
$$= ((x - 6.1)x + 3.2)x + 1.5$$

2 multiplications
3 additions

Thm (Horner's Method)

Let

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Define $b_n = a_n$

$$b_k = a_k + b_{k+1} x_0, \quad k = n-1, n-2, \dots, 1, 0$$

Then $\boxed{b_0 = p(x_0)}$

Moreover, if $Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \cdots + b_2 x + b_1$,

then $\boxed{p(x) = (x - x_0) Q(x) + b_0}$