Final Exam

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Problem 1 *Let $f(x) = x^3 + 4x - 2$ and $I = [0, 1]$.*

(a) *Prove that I contains a root of f.*

Since f is a polynomial, we know f is continuous.

Let $a := 0$, $b := 1$. Since $f(a) = -2 < 0$ and $f(b) = 3 > 0$, we have $f(a) \cdot f(b) < 0$.

$\therefore$ By the Intermediate Value Theorem, there exists at least one $x \in [a, b] = [0, 1]$ such that $f(x) = 0$. i.e. there exists at least one root of f in $[0, 1]$. $\square$

(b) *Show that finding the root x of f (i.e., $f(x) = 0$), is equivalent to finding the fixed-point of the function $g(x) = \frac{2-x^3}{4}$. Is the fixed point unique?*

By definition, at a fixed point p of g, $g(p) = p$. We therefore have:

$$\begin{aligned} g(p) = p \quad &\iff \quad \frac{2-p^3}{4} = p \\ &\iff \quad 2 - p^3 = 4p \\ &\iff \quad p^3 + 4p - 2 = 0 \\ &\iff \quad f(p) = 0 \end{aligned}$$

Therefore, $g(x)$ has a fixed point at p precisely when $f(p) = 0$. Since we have already shown that f contains a root in $[0, 1]$, g must also contain a fixed point in $[0, 1]$.

We now test for uniqueness. To satisfy the condition of the Fixed Point Uniqueness Theorem, we need an upper bound k on $|g'(x)|$ on the interval I such that $k \in (0, 1)$. Since g is a polynomial function, it is also continuously differentiable on I and we have

$$|g'(x)| = \left| \frac{d}{dx} \left(\frac{2-x^3}{4} \right) \right| = \left| \frac{-3x^2}{4} \right| = \frac{3}{4}x^2 \leq \frac{3}{4} \quad \forall x \in I = [0, 1]$$

Therefore, $g(x)$ is continuously differentiable on I and there exists $k = \frac{3}{4} \in (0, 1)$ such that $|g'(x)| \leq k$ for all $x \in I$. Therefore, by the Fixed Point Uniqueness Theorem, there exists a unique fixed point $p \in I$ such that $g(p) = p$.

(c) *State the fixed point iteration. Using any of the two error bounds for fixed point iterations we saw in class, estimate the number of iterations required for the fixed point iteration to give an approximation accurate to within* 10^{-5} *(you can consider any initial guess).*

To approximate the fixed point of a function g, we choose an initial approximation p_0 and generate the sequence $\{p_n\}_{n=0}^{\infty}$ by letting $p_n = g(p_{n-1})$ for each $n \geq 1$. If the sequence converges to p and and g is continuous, then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g\left(\lim_{n\to\infty} p_{n-1}\right) = g(p)$$

and a solution to $g(x) = x$ is obtained. This technique is called fixed-point iteration.[1]

The error bound for p_n in fixed point iteration can be given by:

$$|p_n - p| \leq k^n \cdot \max\{|p_0 - a|, |p_0 - b|\}$$

For $[a, b] = [0, 1]$, any arbitrary $p_0 \in [a, b]$ gives $\max\{|p_0 - a|\,|p_0 - b|\} = \dfrac{a+b}{2} = 0.5$. From (b), we also have $k = 0.75$.

$$\therefore |p_n - p| \leq k^n \cdot \max\{|p_0 - a|, |p_0 - b|\} \leq 10^{-5}$$

$$\begin{aligned}
\implies && 0.75^n \cdot 0.5 &\leq 10^{-5} \\
\implies && 0.75^n &\leq 2 \cdot 10^{-5} \\
\implies && n \cdot \log_{10}(0.75) &\leq \log_{10}(2 \cdot 10^{-5}) \\
\implies && n \cdot \log_{10}(0.75) &\leq \log_{10}(10^{-5}) + \log_{10}(2) \\
\implies && n &\geq \frac{\log_{10}(10^{-5}) + \log_{10}(2)}{\log_{10}(0.75)} \quad (\because \log_{10}(0.75) < 0) \\
\implies && n &\geq 37.61 \\
\implies && n &\geq 38
\end{aligned}$$

We therefore find that for any arbitrary $p_0 \in [0, 1]$, 38 fixed point iterations give a result with at least 10^{-5} accuracy.

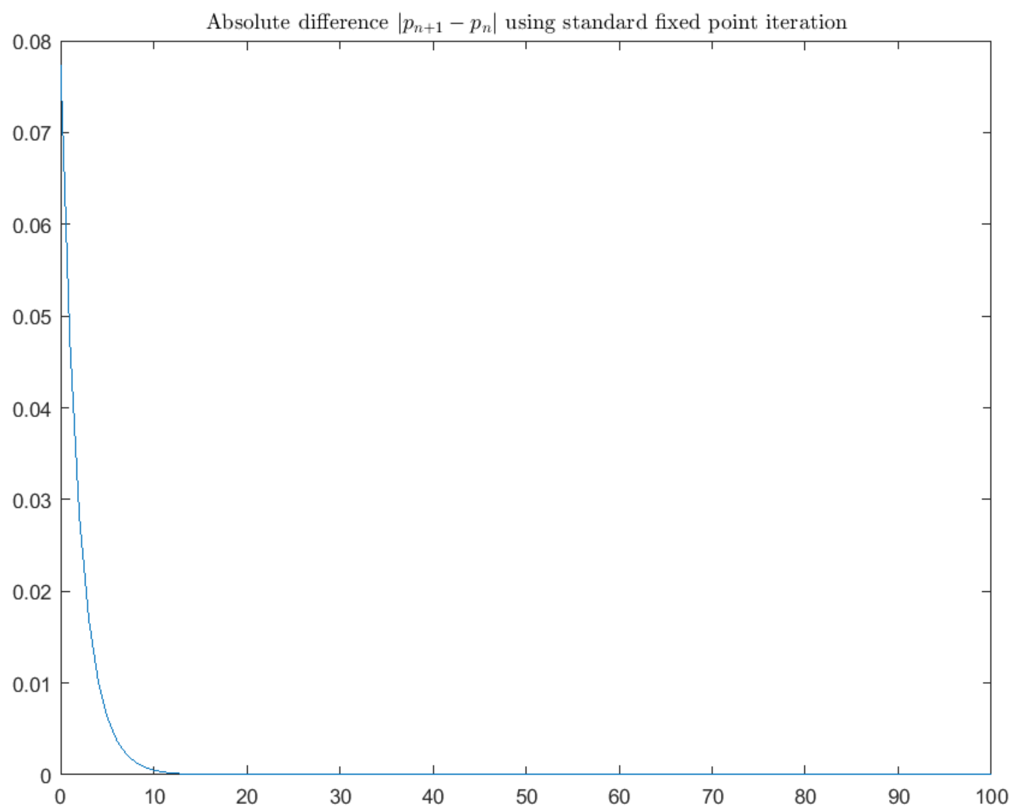
1. Burden, *Fixed Point Iteration*, p. 59

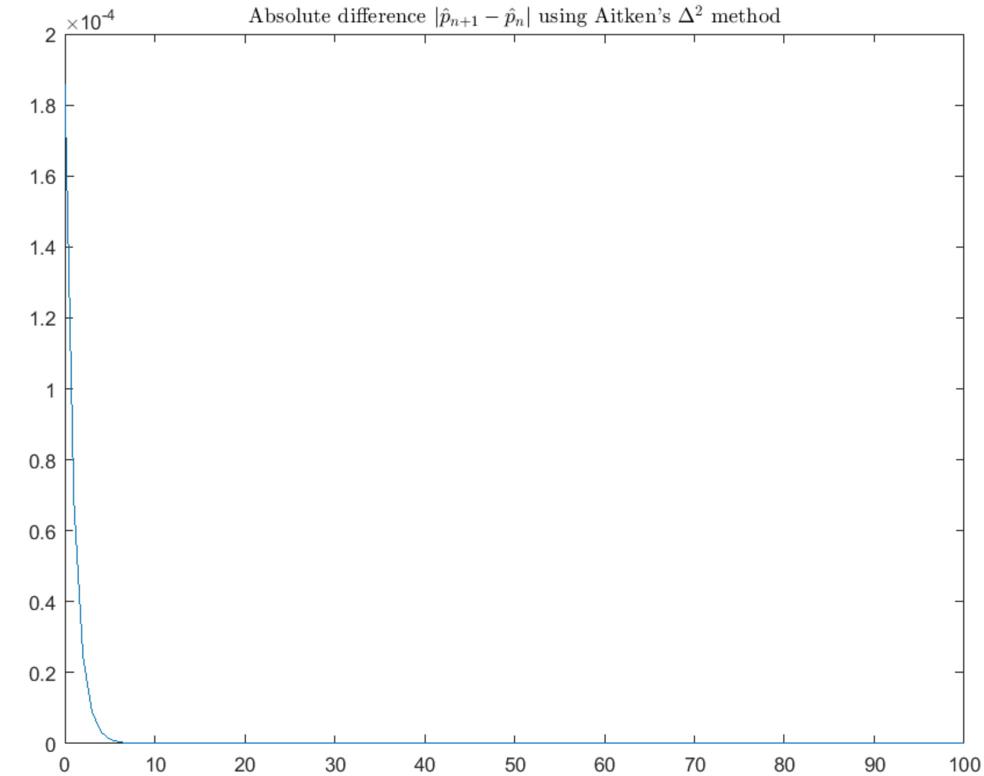
Problem 2 *(Programming) Consider the following linearly convergent sequence:*

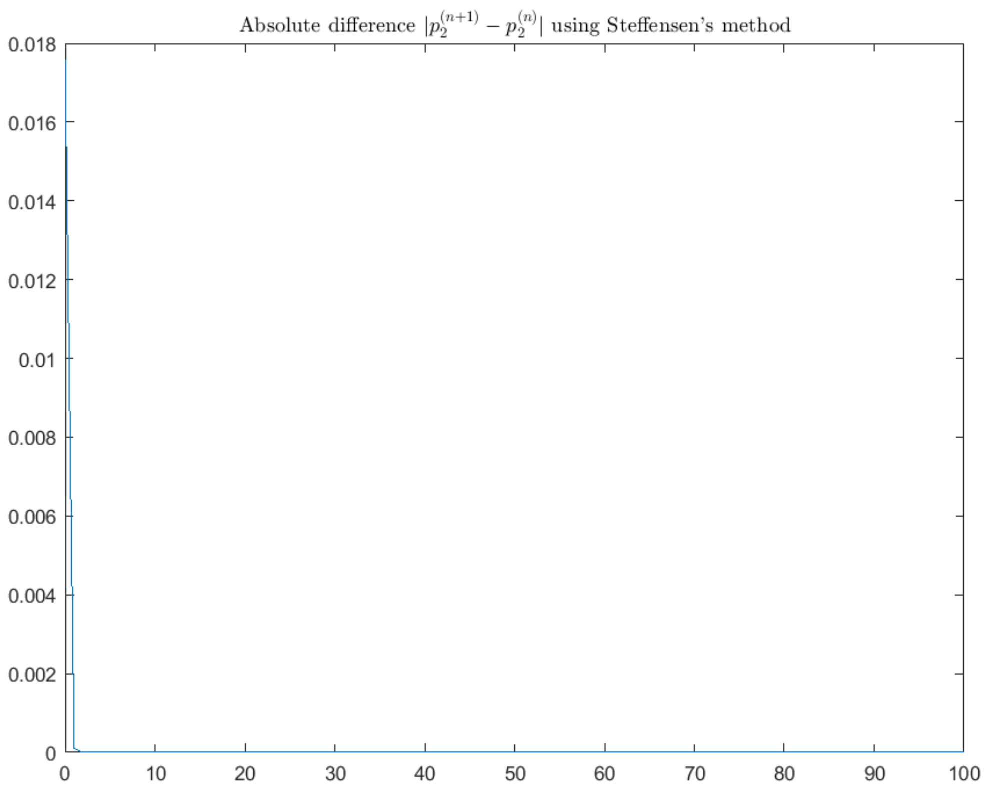
$$p_0 = 0.5, \;\; p_n = 3^{-p_{n-1}}, \;\; n \geq 1$$

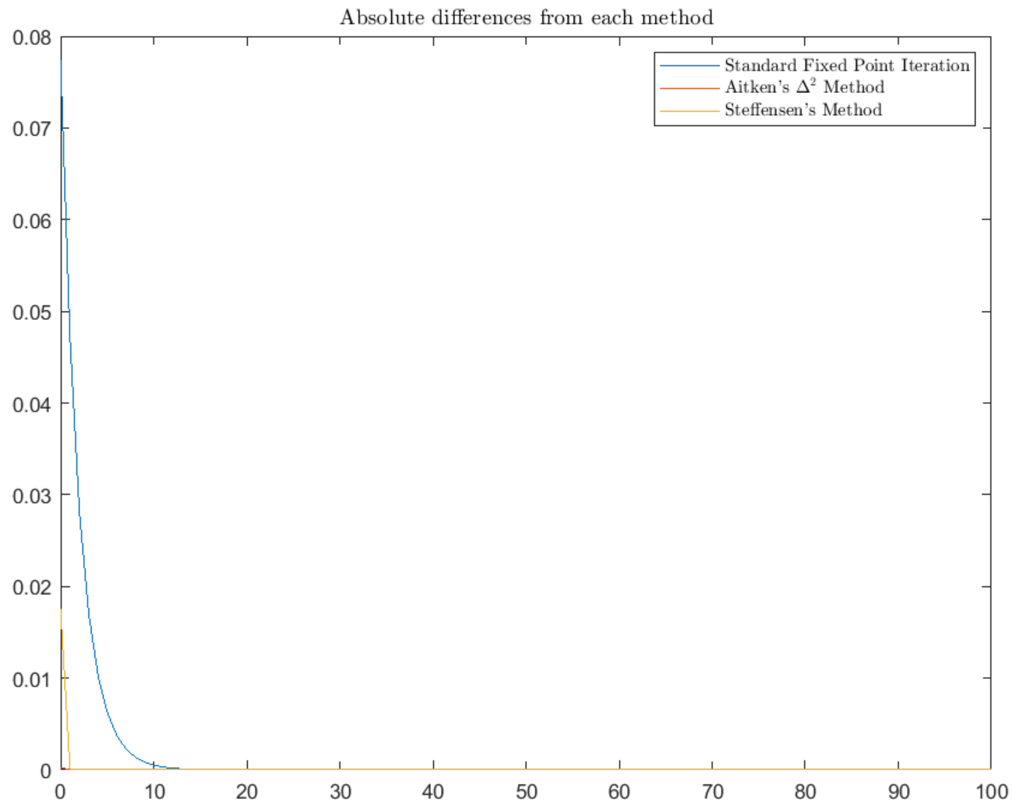
Generate the first 100 *terms of the sequences* p_n, $\{\hat{p}_n\}$ *using Aitken's* Δ^2 *method, and* $p_2^{(n)}$ *using Steffensen's method. Plot the values of* $|p_{n+1} - p_n|$, $|\hat{p}_{n+1} - \hat{p}_n|$ *for Aitken's, and* $|p_2^{(n+1)} - p_2^{(n)}|$ *for Steffensen's. Explain your findings.*

Since we are generating $p_n = 3^{-p_{n-1}}$, we effectively have $g(x) = 3^{-x}$. Following are the plots generated from the first 100 terms of the sequences generated by the standard fixed point iteration, Aitken's Δ^2 method, and Steffensen's method:




Absolute difference $|\hat{p}_{n+1} - \hat{p}_n|$ using Aitken's Δ^2 method
$\times 10^{-4}$
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
10
20
30
40
50
60
70
80
90
100


Absolute difference $|p_2^{(n+1)} - p_2^{(n)}|$ using Steffensen's method
0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
0
10
20
30
40
50
60
70
80
90
100



It's clear from the graphs that Steffensen's Method converges to the fixed point (approximately 5.4781) much quicker than Aitken's Δ^2 method — in fact, the plot for the sequence generated by Steffensen's method is barely visible on the graph above. In turn, we see that the sequence given by Aitken's Δ^2 method converges much more quickly than that given by standard fixed point iteration.

This is consistent with our theoretical understanding of each method. The standard fixed point iteration converges to the fixed point linearly, and since Aitken's Δ^2 method accelerates this convergence, it converges more rapidly. Since $g(x) = 3^{-x}$ is thrice-differentiable and has a continuous third derivative over all of $\mathbb{R}$, Steffensen's method in fact gives quadratic convergence — this is also in line with the rapid convergence we observe in the above plots.

Problem 3 *Consider the function $f(x) = x^4 + \sin\left(\frac{\pi}{2}x\right)$ on the interval $[-1, 1]$.*

(a) *Build the power series form of the polynomial interpolating f at the three nodes $x_0 = -1$, $x_1 = 0$, $x_2 = 1$.*

We take $(x_0, f_0) = (-1, f(-1)) = (-1, 0)$, $(x_1, f_1) = (0, f(0)) = (0, 0)$, and $(x_2, f_2) = (1, f(1)) = (1, 2)$ for $f_i := f(x_i)$. We now compute the Power Series polynomial by taking $p(x) = a_2x^2 + a_1x + a_0$:

$$\begin{aligned}
p(x_0) = f_0 &\implies a_2(-1)^2 + a_1(-1) + a_0 = 0 \implies a_2 - a_1 + a_0 = 0 \\
p(x_1) = f_1 &\implies a_2(0)^2 + a_1(0) + a_0 = 0 \implies a_0 = 0 \\
p(x_2) = f_2 &\implies a_2(1)^2 + a_1(1) + a_0 = 2 \implies a_2 + a_1 + a_0 = 2
\end{aligned}$$

Solving the above system of equations for a_2, a_1, and a_0, we get $a_2 = 1$, $a_1 = 1$, $a_0 = 0$. We therefore get:

$$p(x) = x^2 + x$$

(b) *Derive the Lagrange form of the polynomial interpolating f at the three nodes $x_0 = -1$, $x_1 = 0$, $x_2 = 1$.*

We first compute the Lagrange polynomials:

$$\begin{aligned}
L_{n,0}(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 0)(x - 1)}{(-1 - 0)(-1 - 1)} = \frac{x(x - 1)}{(-1)(-2)} = 0.5x^2 - 0.5x \\
L_{n,1}(x) &= \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x - (-1))(x - 1)}{(0 - (-1))(0 - 1)} = \frac{(x + 1)(x - 1)}{(1)(-1)} = 1 - x^2 \\
L_{n,1}(x) &= \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x - (-1))(x - 0)}{(1 - (-1))(1 - 0)} = \frac{(x + 1) \cdot x}{(2)(1)} = 0.5x^2 + 0.5x
\end{aligned}$$

We can now compute the Lagrange interpolating polynomial $p_L(x)$:

$$\begin{aligned}
p(x) &= f(x_0)L_{n,0}(x) + f(x_1)L_{n,1}(x) + f(x_2)L_{n,2}(x) \\
&= 0 \cdot (0.5x^2 - 0.5x) + 0 \cdot (1 - x^2) + 2(0.5x^2 + 0.5x) \\
&= x^2 + x \\
\therefore\ \ p(x) &= x^2 + x
\end{aligned}$$

(c) *State the formula for the error when interpolating $f(x)$ for $x \in [-1, 1]$, and compute an upper bound for the error.*

The (absolute) error form of an interpolating polynomial $p_n(x)$ of a function $f(x)$ on an interval $[a, b]$ is given by:

$$R(x) = \left| \frac{f^{(n+1)}(\xi(x))}{(n + 1)!}(x - x_0)(x - x_1) \cdots (x - x_n) \right|$$

for some $\xi(x) \in [a, b]$ lying between the interpolating points $(x_0, x_1, \ldots, x_n)$.

In our case, we have $n = 2$ and $[a, b] = [-1, 1]$. We first compute $f^{(n+1)}(x) = f^{(3)}(x)$:

$$\begin{aligned} f(x) = x^4 + \sin\left(\frac{\pi}{2}x\right) &\implies f'(x) = 4x^3 + \frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right) \\ &\implies f''(x) = 12x^2 - \frac{\pi^2}{4}\sin\left(\frac{\pi}{2}x\right) \\ &\implies f^{(3)}(x) = 24x - \frac{\pi^3}{8}\cos\left(\frac{\pi}{2}x\right) \end{aligned}$$

Differentiating once more, we get

$$f^{(4)}(x) = 24 + \frac{\pi^4}{16}\sin\left(\frac{\pi}{2}x\right) \approx 24 + 6.1\sin\left(\frac{\pi}{2}x\right) > 0 \ \ \forall x \in \mathbb{R}$$

$f^{(3)}(x)$ is therefore strictly increasing on $[-1, 1]$.

$\therefore$ The most extreme values of $f^{(3)}(x)$ lie at the endpoints -1 and 1.

$$\therefore \max_{x\in[-1,1]} |f^{(3)}(x)| \in \{|f^{(3)}(-1)|, |f^{(3)}(-1)|\} = \{24, 24\} \implies \max_{x\in[-1,1]} |f^{(3)}(x)| = 24$$

We now find an upper bound for the expression

$$|(x - x_0)(x - x_1)\cdots(x - x_n)| = |(x + 1)(x - 0)(x - 1)| = |x^3 - x|$$

on the interval $[-1, 1]$. Differentiating $g(x) := x^3 - x$, we get $g'(x) = 3x^2 - 1$, giving us critical points $x = \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$. At each critical point, we have $|g(x)| = \frac{2\sqrt{3}}{9}$.

$\therefore max\{|(x + 1)(x - 0)(x - 1)|\}_{-1}^{1} = \frac{2\sqrt{3}}{9}$.

We can now find an upper bound on $R(x)$:

$$\begin{aligned} R(x) &= \left|\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)\right| \\ &= \left|f^{(n+1)}(\xi(x))\right| \frac{|(x - x_0)(x - x_1)\cdots(x - x_n)|}{(n+1)!} \\ &\leq \max_{\xi(x)\in[-1,2]} \left|f^{(n+1)}(\xi(x))\right| \frac{\max_{x\in[-1,1]}\{|(x - x_0)(x - x_1)\cdots(x - x_n)|\}}{(n+1)!} \\ &= \max_{\xi(x)\in[-1,1]} |f^{(3)}(\xi(x))| \cdot \frac{\max_{x\in[-1,1]}\{|(x - (-1))(x - 0)(x - 1)|\}}{3!} \\ &= 24\frac{2\sqrt{3}}{6 \cdot 9} \approx 1.54 \end{aligned}$$

We therefore have the upper bound 1.54 on the error for an interpolating polynomial for $f(x)$.

Problem 4 *Derive the error for linear splines when using equidistant nodes $x_0, x_1, \ldots, x_n$.*

For linear interpolation, we use two points (x_{i-1}, f_{i-1}) and (x_i, f_i) and have $n = 1$. For a single spline, we therefore have the error form:

$$R_i(x) = \left| \frac{f''(\xi_i)}{2}(x - x_{i-1})(x - x_i) \right|$$

where ξ_i lies between x_{i-1} and x_i. We define M_i as

$$M_i := \max_{\xi \in [x_{i-1}, x_i]} |f''(\xi)|$$

We now search for an upper bound on $|W_i(x)|$ for $W_i(x) := (x - x_{i-1})(x - x_i)$. We have:

$$\begin{aligned} W_i(x) &= (x - x_{i-1})(x - x_i) \\ &= x^2 - (x_{i-1} + x_i)x + x_{i-1}x_i \\ \implies W_i'(x) &= 2x - (x_{i-1} + x_i) \\ \therefore W_i'(x) &= 0 \implies x = \frac{x_{i-1} + x_i}{2} \end{aligned}$$

$\therefore W_i(x)$ is largest either at critical point $\frac{x_{i-1}+x_i}{2}$ or at one of the endpoints. However, by definition, $W_i(x_i) = W_i(x_{i-1}) = 0$.

$$\therefore \max_{x \in [x_{i-1}, x_i]} |W_i(x)| = \left| W_i\left(\frac{x_{i-1} + x_i}{2}\right) \right| = \frac{(x_i - x_{i-1})^2}{4}$$

$\therefore$ For linear interpolation, the error for $x \in [x_{i-1}, x_i]$ is

$$\begin{aligned} R_i(x) &= \left| \frac{f''(\xi_i)}{2}(x - x_{i-1})(x - x_i) \right| \\ &\leq \frac{1}{2} \max_{\xi \in [x_{i-1}, x_i]} f''(\xi) \max_{x \in [x_{i-1}, x_i]} |W_i(x)| \\ &= \frac{1}{2} M_i \cdot \frac{(x_i - x_{i-1})^2}{4} \\ &= \frac{1}{8} M_i h^2 \end{aligned}$$

for $h = x_i - x_{i-1}$, the distance between each pair of consecutive nodes.

Now, for any $x \in [x_0, x_n]$, we have $R(x) \leq \frac{1}{8} M h^2$, where

$$M := \max\{M_i\}_{i=1}^n = \max\left\{ \max_{\xi \in [x_{i-1}, x_i]} |f''(\xi)| \right\}_{i=1}^n = \max_{\xi \in [x_0, x_n]} |f''(\xi)|$$

Problem 5 *Consider the integral $\int_{-2}^{2} x^3 e^x dx$. (This problem can optionally be done with programming).*

(a) *For $n = 4$, approximate the integral using the Composite Trapezoidal rule.*

For $n = 4$ and $[a, b] = [-2, 2]$, we have $h = 1$ and the points $x_0 = -2$, $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, and $x_4 = 2$. Therefore, for $f(x) := x^3 e^x$, we have

$$\begin{aligned}\int_a^b f(x)dx \approx R_T(f) &= \frac{h}{2}\left(f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right)\\ &= \frac{1}{2}\left(f(-2) + 2(f(-1) + f(0) + f(1)) + f(2)\right)\\ &\approx 31.365\end{aligned}$$

We therefore have $\int_{-2}^{2} x^3 e^x dx \approx 31.365$ from the Composite Trapezoidal Rule.

(b) *For $n = 4$, approximate the integral using the Composite Simpson's rule.*

For $n = 4$ and $[a, b] = [-2, 2]$, we have $h = 1$ and the points $x_0 = -2$, $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, and $x_4 = 2$. Therefore, for $f(x) := x^3 e^x$, we have

$$\begin{aligned}\int_a^b f(x)dx \approx R_S(f) &= \frac{h}{3}\left(f(a) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{\frac{n}{2}} f(x_{2j-1}) + f(b)\right)\\ &= \frac{h}{3}\left(f(a) + 2f(x_2) + 4(f(x_1) + f(x_3)) + f(b)\right)\\ &= \frac{1}{3}\left(f(-2) + 2f(0) + 4(f(-1) + f(1)) + f(2)\right)\\ &\approx 22.477\end{aligned}$$

We therefore have $\int_{-2}^{2} x^3 e^x dx \approx 22.477$ from the Composite Simpson's Rule.

(c) *For Composite Trapezoidal and Composite Simpson's Rules, determine the value of n required to approximate the integral to within 10^{-4}.*

We first find an upper-bound for the error term for Composite Trapezoidal Rule. The error term $E_T(f)$ is given by $E_T(f) = \frac{b-a}{12}h^2|f''(\mu)|$, for some $\mu \in (a, b)$. We first compute $f''(x)$:

$$\begin{aligned}f(x) = x^3 e^x \implies f'(x) &= (3x^2 + x^3)e^x\\ \implies f''(x) &= (6x + 3x^2 + 3x^2 + x^3)e^x\\ &= (x^3 + 6x^2 + 6x)e^x\end{aligned}$$

Differentiating once more, we get $f^{(3)}(x) = e^x(x^3 + 9x^2 + 18x + 6)$. This polynomial only has one root in the interval $[-2, 2]$, at approximately -0.41577. $f''(x)$ therefore attains its maximum absolute value at either the endpoints or this critical point.

Comparing $f''(-2), f''(-0.41577)$, and $f''(2)$, we find

$$\max_{\mu\in[-2,2]} |f''(\mu)| = |f''(2)| \approx 325.12$$

We can now find the upper bound for $E_T(f)$:

$$\begin{aligned} E_T(f) &= \frac{b-a}{12}h^2|f''(\mu)| \\ &\leq \frac{b-a}{12}\left(\frac{b-a}{n}\right)^2 \max_{\mu\in[-2,2]} |f''(\mu)| \\ &= \frac{4}{12}\left(\frac{4}{n}\right)^2 \cdot 325.12 \approx \frac{1734}{n^2} \end{aligned}$$

For $E_T(f) < 10^{-4}$, we have

$$\frac{1734}{n^2} \leq 10^{-4} \implies n^2 \geq 1734 \cdot 10^4 \implies n \geq \sqrt{1734 \cdot 10^4} \approx 4164.13$$

We therefore require $n \geq 4165$ for an error of less than 10^{-4} with the Composite Trapezoidal Rule.

We now find an upper-bound for the error term for Composite Simpson's Rule. The error term $E_S(f)$ is given by $E_S(f) = \frac{b-a}{180}h^4|f^{(4)}(\mu)|$, for some $\mu \in (a, b)$. We first compute $f^{(4)}(x)$:

$$f^{(3)}(x) = e^x(x^3 + 9x^2 + 18x + 6) \implies f^{(4)}(x) = e^x(x^3 + 9x^2 + 30x + 24)$$

Differentiating once more, we get $f^{(5)}(x) = e^x(x^3 + 12x^2 + 48x + 54)$. This polynomial only has one root in the interval $[-2, 2]$, at approximately -1.846. $f^{(4)}(x)$ therefore attains its maximum absolute value at either the endpoints or this critical point.

Comparing $f^{(4)}(-2), f^{(4)}(-1.846)$, and $f^{(4)}(2)$, we find

$$\max_{\mu\in[-2,2]} |f^{(4)}(\mu)| = |f^{(4)}(2)| \approx 945.8$$

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We can now find the upper bound for $E_T(f)$:

$$\begin{aligned} E_S(f) &= \frac{b-a}{180}h^4|f^{(4)}(\mu)| \\ &\leq \frac{b-a}{180}\left(\frac{b-a}{n}\right)^4 \max_{\mu\in[-2,2]} |f^{(4)}(\mu)| \\ &= \frac{4}{180}\left(\frac{4}{n}\right)^4 \cdot 945.8 \approx \frac{5381}{n^4} \end{aligned}$$

For $E_S(f) < 10^{-4}$, we have

$$\frac{5381}{n^4} \leq 10^{-4} \implies n^4 \geq 5381 \cdot 10^4 \implies n \geq \sqrt{1734 \cdot 10^4} \approx 85.65$$

We therefore require $n \geq 86$ for an error of less than 10^{-4} with the Composite Simpson's Rule.