Homework Assignment 3

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Exercise 1 Given data $(-1,0)$, $(0,1)$, $(1,3)$, construct the Power Series, Newton, and Lagrange interpolating polynomial of the data. Show that they are the same polynomial.

We take $(x_0, f_0) = (-1, 0)$, $(x_1, f_1) = (0, 1)$, $(x_2, f_2) = (1, 3)$ for $f_i := f(x_i)$. We first compute the Power Series polynomial by taking $p_P(x) = a_2x^2 + a_1x + a_0$:

$$\begin{aligned}
p_P(x_0) = f_0 &\implies a_2(-1)^2 + a_1(-1) + a_0 = 0 &&\implies a_2 - a_1 + a_0 = 0\\
p_P(x_1) = f_1 &\implies a_2(0)^2 + a_1(0) + a_0 = 1 &&\implies a_0 = 1\\
p_P(x_2) = f_2 &\implies a_2(1)^2 + a_1(1) + a_0 = 3 &&\implies a_2 + a_1 + a_0 = 3
\end{aligned}$$

Solving the above system of equations for a_2, a_1, and a_0, we get $a_2 = 0.5$, $a_1 = 1.5$, $a_0 = 3$. We therefore get:

$$p_P(x) = 0.5x^2 + 1.5x + 1$$

We now compute the Newton's form of the interpolating polynomial, taking

$$p_N(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

By definition of $f[x_i]$, we have $f[x_0] = 0$, $f[x_1] = 1$, $f[x_2] = 3$. We first find the necessary forward divided differences:

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{1 - 0}{0 - (-1)} = 1; \quad f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{3 - 1}{1 - 0} = 2$$

$$\implies \quad f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{2 - 1}{1 - (-1)} = 0.5$$

Plugging in these coefficients, we now compute $p_N(x)$:

$$\begin{aligned}
p_N(x) &= 0 + 1(x - (-1)) + 0.5(x - (-1))(x - 0)\\
&= (x + 1) + 0.5x(x + 1)\\
&= x + 1 + 0.5x^2 + 0.5x\\
&= 0.5x^2 + 1.5x + 1\\
\therefore \quad p_N(x) &= 0.5x^2 + 1.5x + 1
\end{aligned}$$

We now compute the Lagrange interpolating polynomial. We first compute the Lagrange polynomials:

$$
\begin{aligned}
L_{n,0}(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{x(x-1)}{(-1)(-2)} = 0.5x^2 - 0.5x \\
L_{n,1}(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-(-1))(x-1)}{(0-(-1))(0-1)} = \frac{(x+1)(x-1)}{(1)(-1)} = 1 - x^2 \\
L_{n,1}(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-(-1))(x-0)}{(1-(-1))(1-0)} = \frac{(x+1)\cdot x}{(2)(1)} = 0.5x^2 + 0.5x
\end{aligned}
$$

We can now compute the Lagrange interpolating polynomial $p_L(x)$:

$$
\begin{aligned}
p_L(x) &= f(x_0)L_{n,0}(x) + f(x_1)L_{n,1}(x) + f(x_2)L_{n,2}(x) \\
&= 0 \cdot (0.5x^2 - 0.5x) + 1 \cdot (1 - x^2) + 3(0.5x^2 + 0.5x) \\
&= 1 - x^2 + 1.5x^2 + 1.5x \\
&= 0.5x^2 + 1.5x + 1 \\
\therefore \quad p_L(x) &= 0.5x^2 + 1.5x + 1
\end{aligned}
$$

Since $p_P(x) \equiv p_N(x) \equiv p_L(x) = 0.5x^2 + 1.5x + 1$, the Power Series, Newton's Form, and Lagrange interpolating polynomial are all indeed the same.

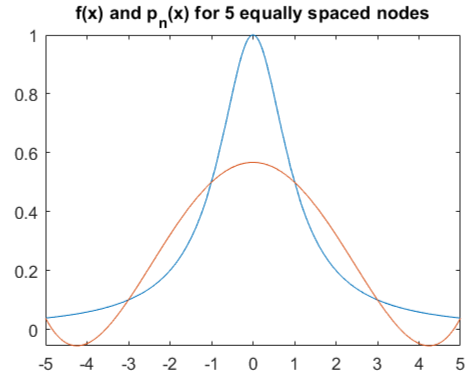
Exercise 2 (Programming) Consider the function $f(x) = \frac{1}{1+x^2}$ on the interval $[-5, 5]$. For $n = 5$, 10, 25, and 50, plot $f(x)$ and $p_n(x)$ in one figure, and $|e_n(x)| = |f(x) - p_n(x)|$ in another figure using

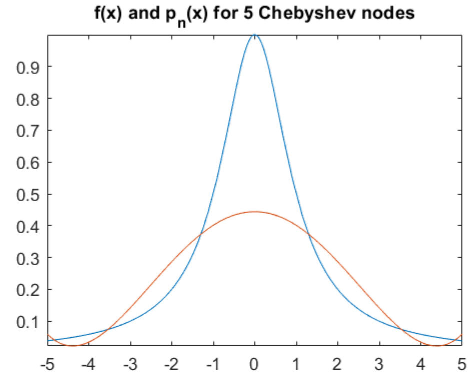
(a) $n + 1$ equally spaced nodes, and

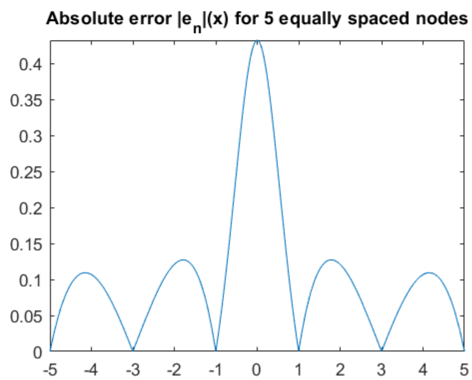
(b) $n + 1$ Chebyshev nodes

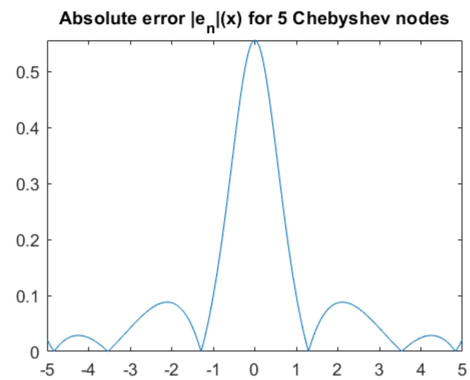
Explain your findings. Here, you may use any of the three interpolating polynomials mentioned in Exercise 1.

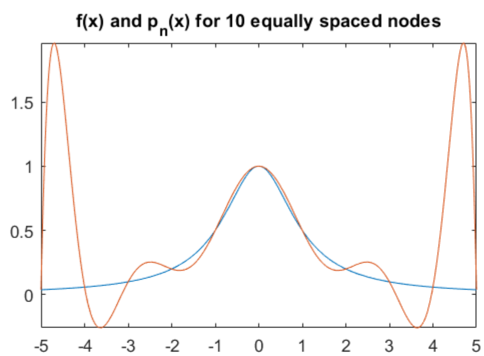
We use Lagrange polynomials to plot the required graphs. In each graph of $f(x)$ and $p_n(x)$, the blue line represents $f(x)$ and the orange line $p_n(x)$ respectively.

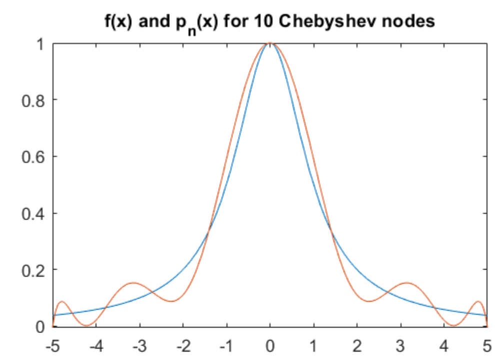

f(x) and p_n(x) for 5 equally spaced nodes
1 0.8 0.6 0.4 0.2 0
-5 -4 -3 -2 -1 0 1 2 3 4 5

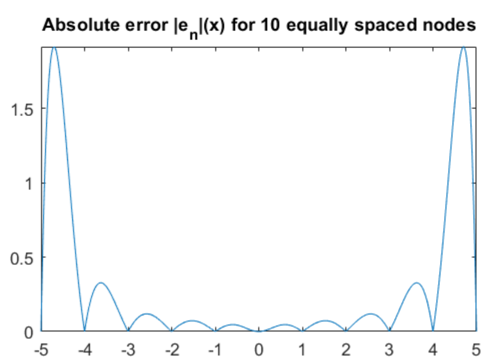

f(x) and p_n(x) for 5 Chebyshev nodes
0.9 0.8 0.7 0.6 0.5 0.4 0.3 0.2 0.1
-5 -4 -3 -2 -1 0 1 2 3 4 5

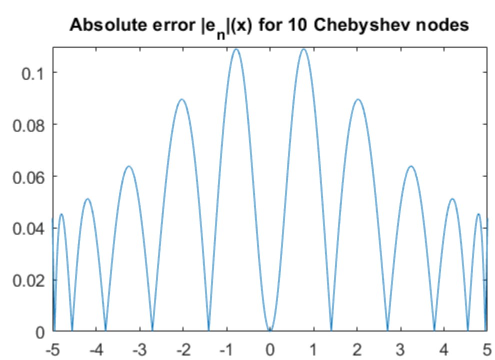

Absolute error |e_n|(x) for 5 equally spaced nodes
0.4 0.35 0.3 0.25 0.2 0.15 0.1 0.05 0
-5 -4 -3 -2 -1 0 1 2 3 4 5

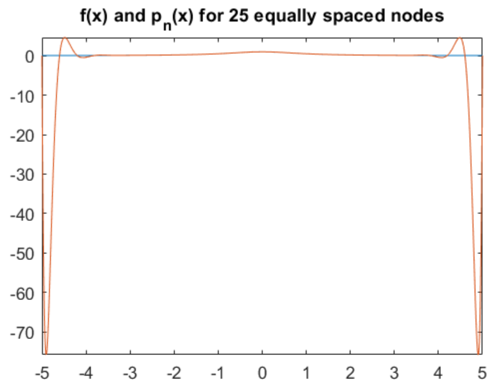

Absolute error |e_n|(x) for 5 Chebyshev nodes
0.5 0.4 0.3 0.2 0.1 0
-5 -4 -3 -2 -1 0 1 2 3 4 5

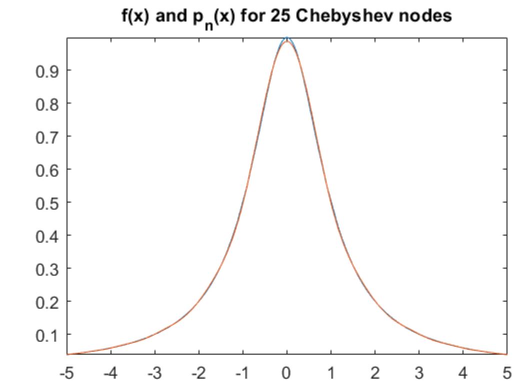

f(x) and p_n(x) for 10 equally spaced nodes
1.5 1 0.5 0
-5 -4 -3 -2 -1 0 1 2 3 4 5

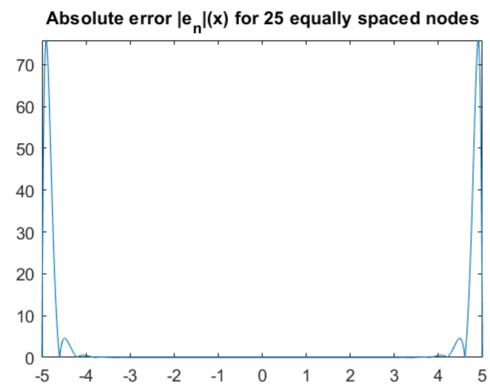

f(x) and p_n(x) for 10 Chebyshev nodes
1 0.8 0.6 0.4 0.2 0
-5 -4 -3 -2 -1 0 1 2 3 4 5

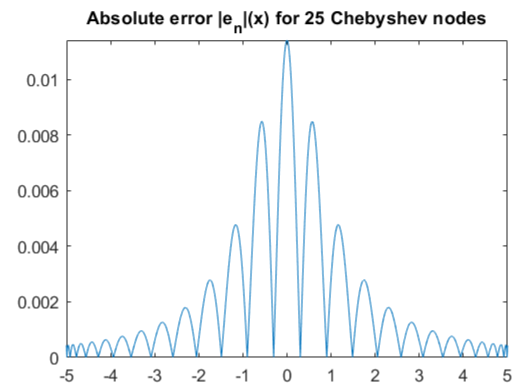

Absolute error |e_n|(x) for 10 equally spaced nodes
1.5 1 0.5 0
-5 -4 -3 -2 -1 0 1 2 3 4 5

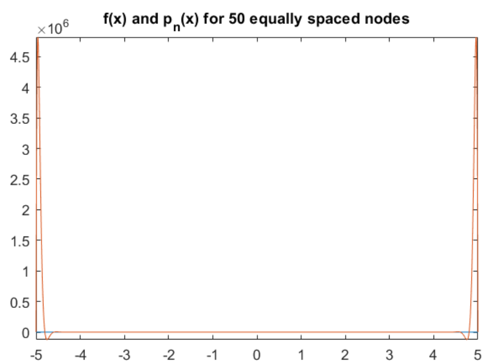

Absolute error |e_n|(x) for 10 Chebyshev nodes
0.1 0.08 0.06 0.04 0.02 0
-5 -4 -3 -2 -1 0 1 2 3 4 5

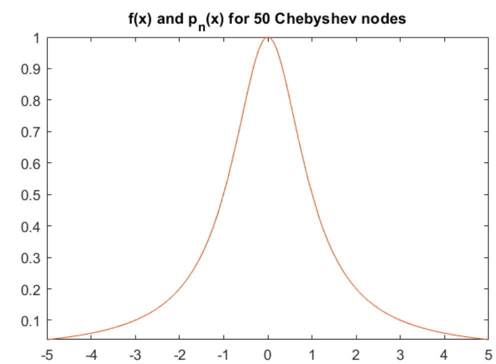

f(x) and p_n(x) for 25 equally spaced nodes

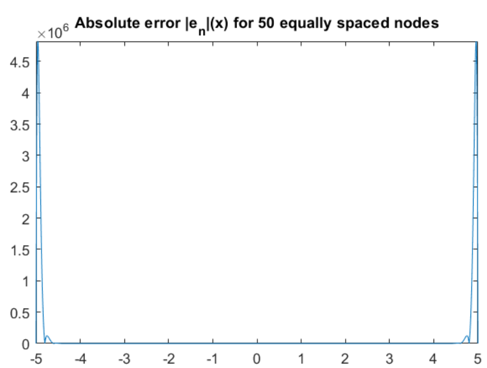

f(x) and p_n(x) for 25 Chebyshev nodes

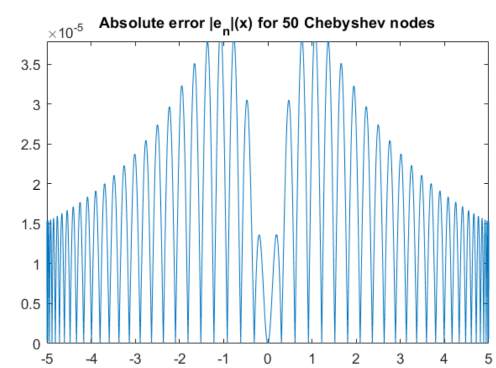

Absolute error $|e_n|$(x) for 25 equally spaced nodes


Absolute error $|e_n|$(x) for 25 Chebyshev nodes


$\times 10^6$
f(x) and p_n(x) for 50 equally spaced nodes


f(x) and p_n(x) for 50 Chebyshev nodes


$\times 10^6$
Absolute error $|e_n|$(x) for 50 equally spaced nodes


$\times 10^{-5}$
Absolute error $|e_n|$(x) for 50 Chebyshev nodes

Two observations emerge quickly from these graphs: first, that as the number of nodes increases, the interpolating polynomial (regardless of the kinds of nodes used) become closer approximations of the function; and second, that for any given number of nodes, the Lagrange polynomial with Chebyshev nodes generally approximates the function better than the polynomial with equally spaced nodes, and that for larger numbers of nodes, using Chebyshev nodes gives a vastly superior polynomial. These observations are consistent with our theoretical understanding of interpolating polynomials, since higher degree interpolating polynomials give better approximations, and since Chebyshev nodes tend to give interpolating polynomials that oscillate less (and our function is one that does not oscillate much on the interval $[-5, 5]$.

Exercise 3 Suppose we want to approximate the function $f(x) = e^{-2x} + 2x^2 + x + 1$ on the interval $[0, 1]$ using a piecewise linear polynomial $S_{1,n}$ that is constructed using the $n + 1$ equidistantly spaced nodes $x_i = ih$, where $h = \frac{1}{n}$ and $i = 0, ..., n$.

1. Using the error bound we learned in class, determine the smallest value of n that guarantees that $|f(x) - S_{1,n}(x)| \leq 10^{-5}$, $\forall x \in [0, 1]$.

For linear splines, we have the error function

$$e_n(x) = |f(x) - S_{1,n}(x)| = max \left\{ \frac{M_i}{8} \cdot h_i^2 \right\}_{i=1}^{n}$$

where $h_i := x_i - x_{i-1}$ and M_i is the maximum absolute value of $f''(x)$ on the interval $[x_{i-1}, x_i]$.

Since our nodes are equidistant, we have $h = \frac{1}{n}$. We therefore have the error bound

$$e_n(x) = max \left\{ \frac{M_i}{8} \cdot h_i^2 \right\}_{i=1}^{n} = \frac{h^2}{8} \cdot max \left\{ M_i \right\}_{i=1}^{n}$$

$max \left\{ M_i \right\}_{i=1}^{n}$ is simply the maximum absolute value of the second derivative of f on the entire interval $[0, 1]$. We first compute $f''(x)$:

$$f(x) = e^{-2x} + 2x^2 + x + 1 \implies f'(x) = -2e^{-2x} + 4x + 1 \implies f''(x) = 4e^{-2x} + 4$$

Since $e^{-2x} > 0$ for all $x \in \mathbb{R}$, $f''(x) > 0$ for all $x \in [0, 1]$. The maximum absolute value of $f''(x)$ is therefore simply its maximum value. Furthermore, since e^{-2x} is a decreasing function for $x \geq 0$, $f''(x)$ is decreasing on the interval $[0, 1]$. We therefore have the maximum value at $x = 0$, i.e.

$$max \left\{ M_i \right\}_{i=1}^{n} = f''(0) = 4e^{-2 \cdot 0} + 4 = 4e^0 + 4 = 8$$

Therefore, for $h = \frac{1}{n}$ and $max \left\{ M_i \right\}_{i=1}^{n} = 8$, the error term $e_n(x)$ is given by

$$e_n(x) = \frac{h^2}{8} \cdot max \left\{ M_i \right\}_{i=1}^{n} = \frac{8}{8n^2} \cdot 5 = \frac{1}{n^2}$$

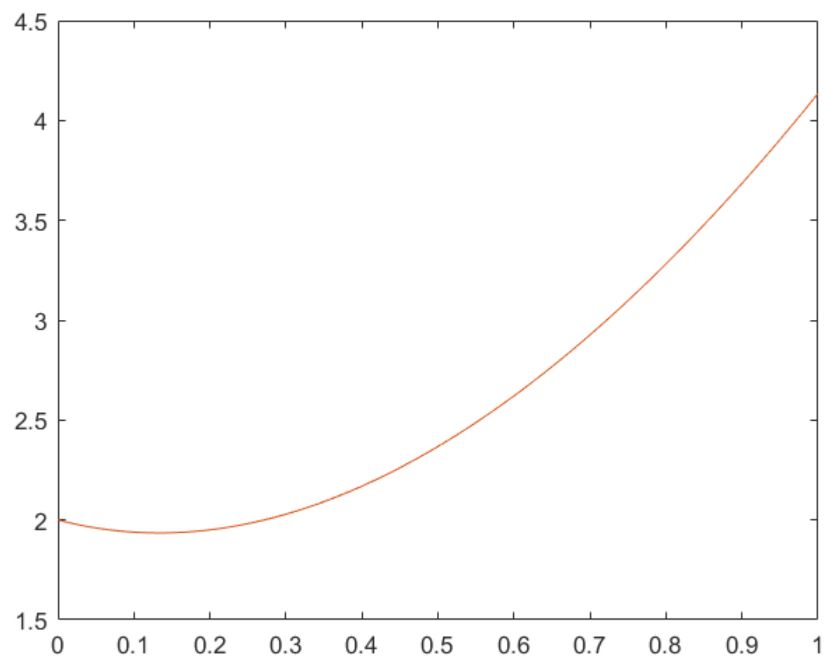
To guarantee $|f(x) - S_{1,n}(x)| \leq 10^{-5}$ for all $x \in [0, 1]$, we need n such that

$$\begin{aligned} e_n(x) \leq 10^{-5} &\implies \frac{1}{n^2} \leq 10^{-5} \\ &\implies n^2 \geq 10^5 \\ &\implies n \geq \sqrt{10^5} \approx 316.22 \\ &\implies n \geq 317 \end{aligned}$$

317 is therefore the smallest value of n that guarantees $|f(x) - S_{1,n}(x)| \leq 10^{-5}$.

2. (Programming) For the value of n determined in the first item, evaluate f and $S_{1,n}$ at 1000 equally spaced points between 0 and 1. Include a plot of f and $S_{1,n}$ that shows that the two curves are close to each other and report the maximum value of $|f(x) - S_{1,n}(x)|$ at those points. Is this value less than 10^{-5}?

When we evaluate f and $S_{1,317}$ at 1000 equally-spaced points between 0 and 1, we get the following graph:



The curves for f and $S_{1,317}$ are virtually indistinguishable. Computing the value of $|f(x) - S_{1,317}(x)|$ for each of the points, we get a maximum of $9.8392 \cdot 10^{-6}$, which is just slightly less than 10^{-5}.