Lecture 1

§1 Induction

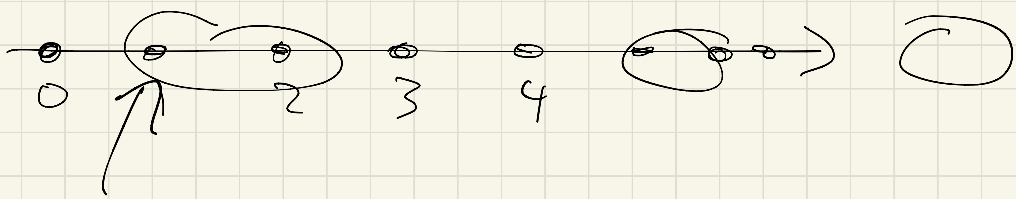
property of the natural numbers which we will take for granted.

Well-Ordering Principle 1.1.1. *Suppose* $S \subseteq \mathbb{N}$ *is such that* $S \neq \emptyset$. *Then* S *has a least element, i.e., there is some* $a \in S$ *such that for all* $b \in S$, $a \leq b$.

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$ $\quad +, \cdot, -, \leq$

Sps $S \subseteq \mathbb{N}$ and $S \neq \emptyset$

then $\min S$ exists.



Division Algorithm 1.1.2. *Given integers $a, b \in \mathbb{Z}$, with $b > 0$, there exist* $\boxed{\textit{unique}}$ *integers $q, r \in \mathbb{Z}$ satisfying*

(1) $a = bq + r$
(2) $0 \le r < b$.

The integer q is called the **quotient** and the integer r is called the **remainder** in the division of a by b.

$a = 100$
$b = 17$

$$\begin{array}{r} 5 \\ 17 \overline{)100} \\ 85 \\ \hline 15 \end{array}$$

remainder

$100 = 17 \cdot 5 + 15$

$0 \le 15 < 17$ ✓

quotient $= 5$
remainder $= 15$

Proof: Define the set

$$S = \{a - bq : q \in \mathbb{Z} \text{ and } a - bq \geq 0\}$$

(all possibles "r"s)

Can check that $S \neq \emptyset$

Let $r := \min S$

and let $q \in \mathbb{Z}$ be s.t.

$a - bq = r$.

Then $a = bq + r$ and $0 \le r < b$. ∎

Principle of Induction 1.1.3. *Suppose $P(n)$ is a property that a natural number n may or may not have. Suppose that*

(1) *$P(0)$ holds (this is called the "base case for the induction"), and* ← Base Case $P(0)$

(2) *for every $n \in \mathbb{N}$, if $P(0), \ldots, P(n)$ holds, then $P(n+1)$ holds (this is called the "inductive step").*

Then $P(n)$ holds for every natural number $n \in \mathbb{N}$.

Inductive Step: $n \geq 0$ show:

$$[P(0) \,\&\, P(1) \,\&\, \cdots \,\&\, P(n)] \longrightarrow P(n+1)$$

Proof: Define the set

$$S := \{n : P(n) \text{ is false}\} \subseteq \mathbb{N}.$$

"set of counterexamples".

Goal: show $S = \emptyset$.

Assume towards contr. that $S \neq \emptyset$.

By Well-ordering Principle $\min S$ exists, let $a := \min S$.

So $P(a)$ is false.

Since $P(0)$ is true (by (1)) then $a \geq 1$. Since $a = \min S$ then $0, \ldots, a-1 \notin S$ so $P(0), \ldots, P(a-1)$ True

so $P((a-1)+1) = P(a)$ is true by (2).

$\#$ Thus $S = \emptyset$. ∎

§2 Summations

$$a_1 + \cdots + a_n \qquad a_i \in \mathbb{R}$$

summation notation

$$\sum_{k=1}^{n} a_k \qquad \text{or} \qquad \sum_{1 \le k \le n} a_k$$

delimited notation

generalized notation

If $a_1, a_2, a_3, \ldots$

$$\sum_{P(k)} a_k$$ ← this means sum every a_k for which $P(k)$ is true.

E.g.

$$\sum_{k=0}^{49} (2k+1)^2 = \sum_{\substack{1 \le k < 100 \\ k \text{ odd}}} k^2$$

$$\sum_{\substack{0 \le p < 20 \\ p \text{ prime}}} p = 2+3+5+7+11+13+17+19$$

$$\sum_{k=1}^{n} a_k = \sum_{1 \le k \le n} a_k = \sum_{1 \le k+1 \le n} a_{k+1} = \sum_{0 \le k \le n-1} a_{k+1} = \sum_{k=0}^{n-1} a_{k+1}$$

Distributive Law 1.2.1. *Suppose $S(i)$ and $R(j)$ are relations which may or may not be true for integers i and j. Then*

$$\left(\sum_{R(i)} a_i\right)\left(\sum_{S(j)} b_j\right) = \sum_{R(i)}\left(\sum_{S(j)} a_i b_j\right).$$

E.g. $a \sum_{S(j)} b_j = \sum_{S(j)} a b_j$

if $R(i)$ only holds for one i.

Change of Variable 1.2.2. *Suppose $R(i)$ is a relation and $\pi : \mathbb{Z} \to \mathbb{Z}$ is a bijection. Then*

$$\sum_{R(i)} a_i = \sum_{R(\pi(i))} a_{\pi(i)}.$$

E.g. $\sum_{R(i)} a_i = \sum_{R(i+1)} a_{i+1}$

Interchanging Order of Summation 1.2.3. *Suppose $R(i)$ and $S(j)$ are relations on integers. Then*

$$\sum_{R(i)}\left(\sum_{S(j)} a_{ij}\right) = \sum_{S(j)}\left(\sum_{R(i)} a_{ij}\right).$$

E.g. $\sum_{R(i)} b_i + \sum_{R(i)} c_i = \sum_{R(i)} (b_i + c_i)$

Manipulating the Domain 1.2.4. *Suppose $R(i)$ and $S(i)$ are relations on integers. Then*

$$\sum_{R(i)} a_i + \sum_{S(i)} a_i = \sum_{R(i) \text{ or } S(i)} a_i + \sum_{R(i) \text{ and } S(i)} a_i.$$

E.g. $\sum_{k=0}^{n+1} a_k = \sum_{k=0}^{n} a_k + a_{n+1}$

Geometric Sum 1.2.5. *Suppose $x \neq 1$. Then*

$$\sum_{0 \le j \le n} x^j = \frac{1 - x^{n+1}}{1 - x}.$$

Proof:

$$\sum_{0 \le j \le n} x^j = 1 + \sum_{1 \le j \le n} x^j$$

$$= 1 + x \sum_{1 \le j \le n} x^{j-1}$$

$$= 1 + x \sum_{0 \le j \le n-1} x^j$$

$$= 1 + x \left(\sum_{0 \le j \le n} x^j \right) - x^{n+1}$$

now solve for $\sum_{0 \le j \le n} x^j$.

Triangular Numbers 1.2.6. *Suppose $n \geq 0$. Then*

$$\sum_{0\leq j\leq n} j = \frac{n(n+1)}{2}.$$

Proof:

$$\sum_{0\leq j\leq n} j = \sum_{0\leq n-j\leq n} (n-j)$$

$$= \sum_{0\leq j\leq n} (n-j)$$

$$= \sum_{0\leq j\leq n} n - \sum_{0\leq j\leq n} j$$

$$= n(n+1) - \sum_{0\leq j\leq n} j$$

Now solve for $\sum_{0\leq j\leq n} j$.

Infinite Summations

$a_1 + a_2 + a_3 + \cdots$

$$\sum_{k=1}^{\infty} a_k := \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} a_k}$$ (if this exists)

Geometric Series 1.2.7. *Suppose $|x| < 1$. Then*

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

<u>Proof!</u>
$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1-x}$$

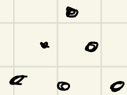
Since $|x|<1$ $\lim_{n\to\infty} x^{n+1} = 0$

thus
$$\lim_{n\to\infty} \frac{1-x^{n+1}}{1-x} = \frac{1-0}{1-x} = \frac{1}{1-x}.$$

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} \sum_{k=0}^{n} x^k$$

§1.3 Triangular Number Algorithms

$$\sum_{k=0}^{n} k \qquad \sum_{k=0}^{3} k = 0+1+2+3$$



$$\sum_{k=0}^{2} k = 0+1+2 = 3$$

```
TRIANGLE(n)
Sum = 0
// Initializes Sum to 0
for j = 0 to n
    Sum = Sum + j
    // Replaces the current value of Sum with Sum + j
    // This has the effect of adding j to Sum
return Sum
```

Triangle(2) $\quad n=2$

(1) $Sum = 0$

(2) $j=0 \quad Sum=0 \quad n=2$

(3) $Sum+j = 0+0 = 0$
$Sum = 0$

(4) $j=1 \quad Sum\ 0, \quad n=2$

(5) $j=1 \quad Sum+j = 0+1$
$Sum = 1, n=2$

(6) $j=2 \quad Sum=1, n=2$

(7) $Sum+j = 1+2 = 3$
$Sum = 3, \quad n=2$

(8) $j=3$ but $n=2$
$j>2$

(9) output $Sum=3$