MATH 182 DISCUSSION 1 WORKSHEET

1. Arithmetic with extra effort

In some circumstances (for example, in ordinal arithmetic), addition of natural numbers is defined as repeated incrementation. That is, we define

$$m + n = m \overbrace{+1 + \cdots + 1}^{n \text{ times}}.$$

The following pseudocode implements this definition:

```
InductiveAdd(m, n)
sum = m
for j = 1 to n
    sum = sum + 1
return sum
```

Similarly, we can define multiplication as repeated addition:

```
InductiveMult(m, n)
1  prod = 0
2  for j = 1 to n
3      prod = prod + m
4  return prod
```

Prove carefully that InductiveAdd and InductiveMult perform addition and multiplication by identifying a loop invariant and proving that it is a loop invariant. Then analyze their running times.

If we replace line 3 of InductiveMult by

$$prod = \textsc{InductiveAdd}(prod, m),$$

what is the running time of the new algorithm?

This is an extremely inefficient way to do arithmetic. The methods that we use to do arithmetic by hand are much faster. The standard algorithm for adding two numbers written in decimal form is to start with the least significant digits and add the digits in each place, keeping track of carries.

Write pseudocode to implement this algorithm, taking as input two lists of digits, and producing a list of the digits of the sum of the corresponding numbers. What is the running time of this algorithm? How would you prove that it gives the correct answer? If you have time, you can try writing and analyzing the standard multiplication algorithm as well.

2. Fun with floor functions

Show that for any $x, y \in \mathbb{R}$,

$$\lfloor x \rfloor + \lfloor y \rfloor \;\le\; \lfloor x+y \rfloor \;\le\; \lceil x \rceil + \lfloor y \rfloor \;\le\; \lceil x+y \rceil \;\le\; \lceil x \rceil + \lceil y \rceil.$$

For each of the inequalities in the above chain, find an example where it is strict and an example where the quantities are equal. What similar inequalities hold for a sum of three numbers x, y, z?

Show that $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$ for any $x \in \mathbb{R}$. Derive similar formulas for $\lfloor nx \rfloor$ and $\lceil nx \rceil$ for arbitrary $n \in \mathbb{N}$.

It is frequently necessary to round real numbers to integers. We have already been introduced to floor and ceiling. Truncation, also known as rounding toward zero, is the same as floor for nonnegative numbers, and the same as ceiling for negative numbers. Round-half-up rounds a number to the nearest integer; if the number a half-integer (so that there is not a unique nearest integer), it is rounded up. Round-half-down is the same, but rounds half-integers down. Round-half-to-even rounds a half-integer up or down, to whichever of the possibilities is even. Round-half-to-even tends to produce the smallest rounding errors when used many times, since its rounding errors are not consistently biased in one direction; it is therefore often the default method of rounding.

Show that

$$\begin{aligned}
\text{Round-half-up}(x) &= \left\lfloor x + \frac{1}{2} \right\rfloor, \\
\text{Round-half-down}(x) &= \left\lceil x - \frac{1}{2} \right\rceil, \\
\text{Round-half-even}(x) &= \left\lfloor \frac{x}{2} + \frac{1}{4} \right\rfloor + \left\lceil \frac{x}{2} - \frac{1}{4} \right\rceil.
\end{aligned}$$

Is it possible to write truncation in a similar way, using only floors, ceilings, addition, and multiplication by constants?