Homework Assignment 4

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Exercise 1

(a) The following table was generated using the function $f(x) = e^{2x}$. Use the best possible finite-difference schemes from your notes to determine the missing entries in the following table.

x	$f(x)$	$f'(x)$
1.1	9.025013	
1.2	11.02318	
1.3	13.46374	
1.4	16.44465	

We have $h = 0.1$. We first compute $f'(1.1)$ using forward difference approximation:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$
$$\implies f'(1.1) \approx \frac{f(1.1+0.1) - f(1.1)}{0.1} = \frac{11.02318 - 9.025013}{0.1} = 19.98167$$

We now compute $f'(1.2)$ and $f'(1.3)$ using centered difference approximation:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$
$$\implies f'(1.2) \approx \frac{f(1.2+0.1) - f(1.2-0.1)}{2 \cdot 0.1} = \frac{13.46374 - 9.025013}{0.2} = 22.193635$$
$$\implies f'(1.3) \approx \frac{f(1.3+0.1) - f(1.3-0.1)}{2 \cdot 0.1} = \frac{16.44465 - 11.02318}{0.2} = 27.10735$$

We now compute $f'(1.4)$ using backward difference approximation:

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$
$$\implies f'(1.4) \approx \frac{f(1.4) - f(1.4-0.1)}{0.1} = \frac{16.44465 - 13.46374}{0.1} = 29.8091$$

We tabulate our results below:

x	$f(x)$	$f'(x)$
1.1	9.025013	19.98167
1.2	11.02318	22.193635
1.3	13.46374	27.10735
1.4	16.44465	29.8091

(b) Find the error bounds and compute the actual errors.

From Taylor's Theorem, we have truncation error $R_n(x)$ such that $f(x+h) = f(x) + f'(x) \cdot h + R_n(x)$ where $R_n(x) = \dfrac{|f''(\xi) \cdot h^2|}{2!}$ where ξ lies between x and $x+h$.

For $f(x) = e^{2x}$, we have

$$f'(x) = 2e^{2x} \implies f''(x) = 4e^{2x} \implies f^{(3)}(x) = 8e^{2x}$$

It is clear each of these derivatives is positive and increasing on $\mathbb{R}$. In each case, the derivative therefore reaches its maximum at the maximum possible value of ξ. We therefore, after simplifying have error bounds as follows:

For forward difference approximation, for ξ between x and $x+h$, we have error bound::

$$E(x) = max\left\{\frac{|f''(\xi)| \cdot h}{2}\right\} = \frac{h}{2}\{|f''(\xi)|\} \implies E(x) = \frac{h}{2} \cdot 4e^{2(x+h)}$$

For backward difference approximation, for ξ between $x-h$ and x, we have error bound:

$$E(x) = max\left\{\frac{|f''(\xi)| \cdot h}{2}\right\} = \frac{h}{2}\{|f''(\xi)|\} \implies E(x) = \frac{h}{2} \cdot 4e^{2x}$$

For centered difference approximation, for ξ between $x-h$ and $x+h$, we have error bound:

$$E(x) = max\left\{\frac{|f^{(3)}(\xi)| \cdot h^2}{3!}\right\} = \frac{h^2}{6}\{|f^{(3)}(\xi)|\} \implies E(x) = \frac{h^2}{6} \cdot 8e^{2(x+h)}$$

Applying the forward difference approximation error formula to $x = 1.1$, we get error bound:

$$E(1.1) = \frac{0.1}{2} \cdot 4e^{2(1.1+0.1)} = 2.2047$$

Applying the centered difference approximation error formula to $x = 1.2$ and $x = 1.3$, we get error bounds:

$$E(1.2) = \frac{0.1^2}{6} \cdot 8e^{2(1.2+0.1)} = 0.1796$$

$$E(1.3) = \frac{0.1^2}{6} \cdot 8e^{2(1.3+0.1)} = 0.2193$$

Applying the backward difference approximation error formula to $x = 1.4$, we get error bound:

$$E(1.4) = \frac{0.1}{2} \cdot 4e^{2 \cdot 1.4} = 3.2889$$

We tabulate the error bounds for each x and the actual errors using $f'(x) = 2e^{2x}$:

x	$f(x)$	$f'(x)$	$E(x)$	Actual Error
1.1	9.025013	19.98167	2.2047	1.9316
1.2	11.02318	22.193635	0.1796	0.1473
1.3	13.46374	27.10735	0.2193	0.1799
1.4	16.44465	29.8091	3.2889	3.0802

Exercise 2 Consider Poisson's Equation:

$$-\frac{\partial^2 u}{\partial x^2} = f(x), \quad 0 \leq x \leq 1$$

for $f(x) = 32\pi^2 \sin(2\pi(2x-1)) + 40$.

(a) Verify that the true solution to the differential equation is given by

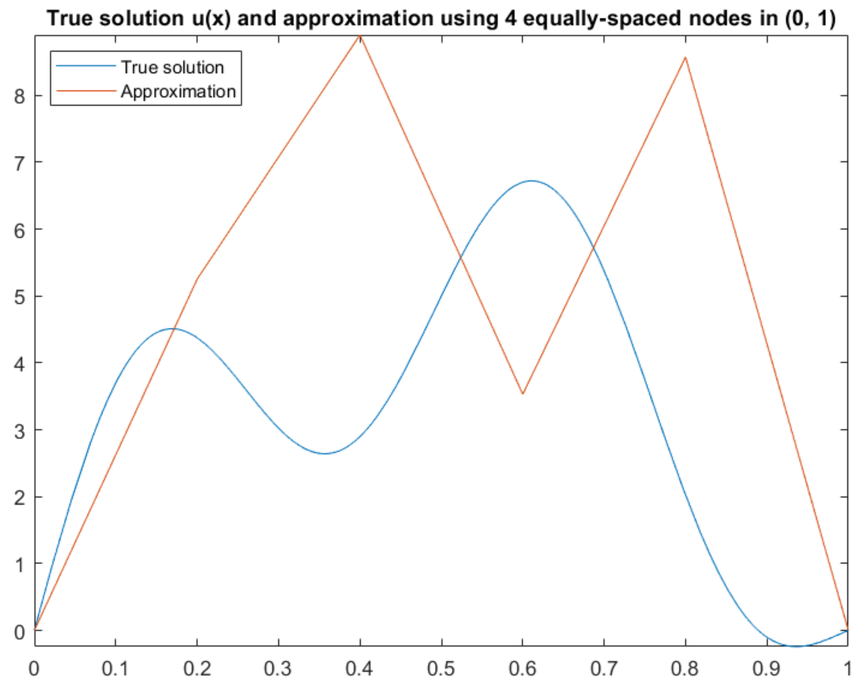
$$u(x) = 2\sin(2\pi(2x-1)) - 20x(x-1)$$

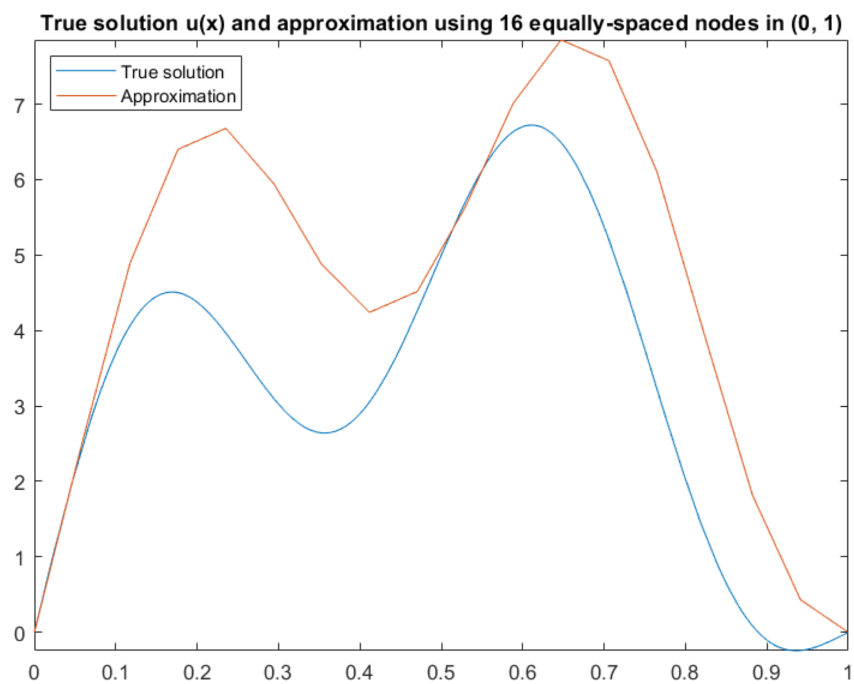
For given $u(x)$, we have

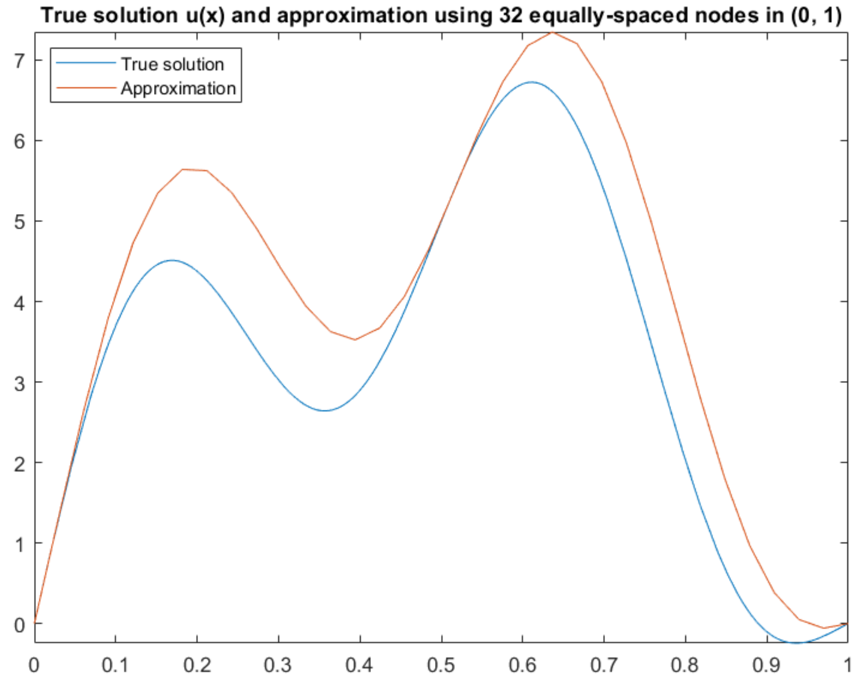
$$\begin{aligned}
-\frac{\partial^2 u}{\partial x^2} &= -\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(2\sin(2\pi(2x-1)) - 20x(x-1))\right) \\
&= -\frac{\partial}{\partial x}(4\pi \cdot 2\cos(2\pi(2x-1)) - 40x + 20) \\
&= -(-4\pi \cdot 4\pi \cdot 2\sin(2\pi(2x-1)) - 40) \\
&= 32\pi^2 \sin(2\pi(2x-1)) + 40 = f(x) \\
\implies -\frac{\partial^2 u}{\partial x^2} &= f(x)
\end{aligned}$$

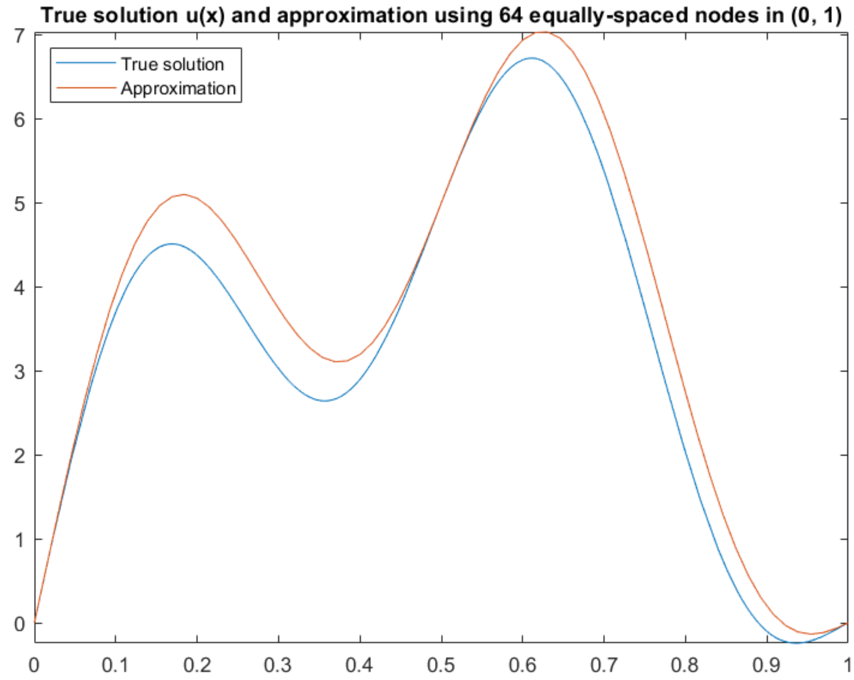
The true solution to the differential equation is therefore given by $u(x)$.

(b) (Programming) Use the centered difference scheme to approximate the solution using 4, 16, 32, and 64 nodes. Plot your solutions along with the true solution. Do your approximations approach the true solution as you increase the number of nodes?









Yes, our approximations clearly approach the true solutions as we increase the number of nodes.

Exercise 3 Approximate the integral

$$\int_1^{1.5} x^2 \ln x \mathrm{d}x$$

using the Midpoint, Trapezoidal, and Simpson's 1/3 rule. Compare each approximation with the true integral value (this you can compute using integration by parts).

We have $f(x) = x^2 \ln(x)$, $a = 1$, $b = 1.5$. We first approximate the integral using the Midpoint Rule:

$$R_M(f) = (b-a) \cdot f\left(\frac{a+b}{2}\right) = 0.5 \cdot f(1.25) \approx 0.17433$$

We now approximate the integral using the Trapezoidal Rule:

$$R_T(f) = \frac{b-a}{2} \cdot (f(a) + f(b)) = 0.25 \cdot (f(1) + f(1.5)) \approx 0.22807$$

We now approximate the integral using Simpson's 1/3 Rule:

$$\begin{aligned} R_S(f) &= \frac{b-a}{6} \cdot f(a) + \frac{2(b-a)}{3} \cdot f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} \cdot f(b) \\ &= \frac{1}{12} \cdot f(1) + \frac{1}{3} \cdot f(1.25) + \frac{1}{12} \cdot f(1.5) \\ &= 0.192245 \end{aligned}$$

From integral by parts, we have

$$\int_a^b f(x)\mathrm{d}x = \int_1^{1.5} x^2 \ln x \mathrm{d}x = \left(\frac{1}{3}x^3 \ln x - \frac{x^3}{9}\right)_1^{1.5} \approx 0.19226$$

We see that the approximation given by Simpson's 1/3 Rule, $R_S(f)$, is much closer to the true value of the integral than the approximations given by the Midpoint Rule and the Trapezoidal Rule. In this case, the approximation given by the Midpoint Rule differs from the true value by about 0.018, whereas the approximation given by the Trapezoidal Rule differs from the true value by about 0.036.

Exercise 4 The quadrature rule

$$\int_{-1}^{1} f(x)\mathrm{d}x = c_0 f(-1) + c_1 f(0) + c_2 f(1)$$

is exact for all polynomials of degree less than or equal to 2. Determine c_0, c_1, and c_2.

Since the quadrature rule is exact for all polynomials of degree less than equal or 2, we know $R(1) = \int_{-1}^{1} \mathrm{d}x$, $R(x) = \int_{-1}^{1} x\mathrm{d}x$, and $R(x^2) = \int_{1}^{1} x^2\mathrm{d}x$.

$$\begin{aligned} R(1) &= \int_{-1}^{1} \mathrm{d}x \implies 1c_0 + 1c_1 + 1c_2 = (x)_{-1}^{1} = 2 \\ R(x) &= \int_{-1}^{1} x\mathrm{d}x \implies -1c_0 + 0c_1 + 1c_2 = \left(\frac{x^2}{2}\right)_{-1}^{1} = 0 \\ R(x^2) &= \int_{1}^{1} x^2\mathrm{d}x \implies 1c_0 + 0c_1 + 1c_2 = \left(\frac{x^3}{3}\right)_{-1}^{1} = \frac{2}{3} \end{aligned}$$

From the above system of equations, we get the linear system:

$$\begin{aligned} c_0 + c_1 + c_2 &= 2 \\ -c_0 + c_2 &= 0 \\ c_0 + c_2 &= \frac{2}{3} \end{aligned}$$

Solving the above linear system, we get $c_0 = \frac{1}{3}$, $c_1 = \frac{4}{3}$, $c_2 = \frac{1}{3}$.