2.3 Newton's Method

Idea: approximate f by linear function at each iteration.

Suppose $f \in C^2[a,b]$ (twice cont. diff'ble), and $p \in [a,b]$ is a root of f.

Let $p_0 \in [a,b]$ be an approx. to p s.t. $f'(p_0) \neq 0$ and $|p-p_0|$ is "small"

Then,

$$\underbrace{f(p)}_{=0} = f(p_0) + f'(p_0)(p-p_0) + \underbrace{f''(\xi(p)) \cdot \frac{(p-p_0)^2}{2}}_{\approx 0 \text{ since } p_0 \text{ close to } p}$$

where $\xi(p)$ between p and p_0.

$$f(p) = 0 \Rightarrow 0 \approx f(p_0) + f'(p_0)(p-p_0)$$

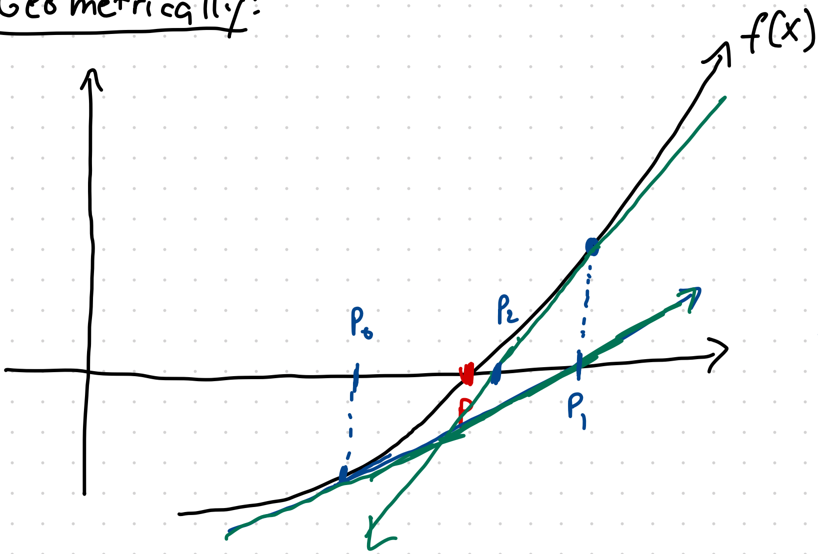
$$\Rightarrow p \approx p_0 - \frac{f(p_0)}{f'(p_0)} = p_1$$

$$\boxed{\begin{array}{l}\text{Newton Iteration:} \\ \qquad p_n = p_{n-1} - \dfrac{f(p_{n-1})}{f'(p_{n-1})}, \quad \forall\, n \geq 1\end{array}}$$

Newton Iteration:

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \forall\ n \geq 1$$

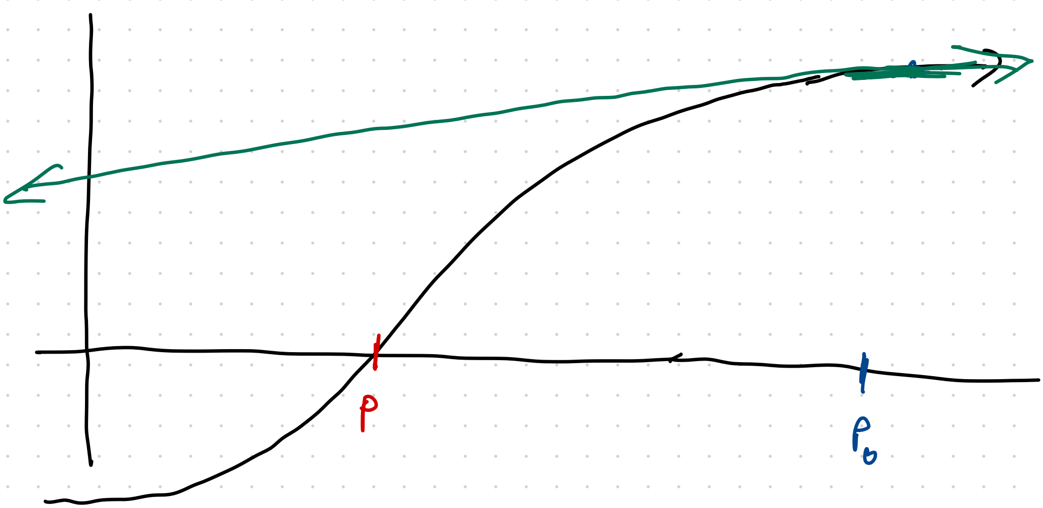
Geometrically:



Stopping Criteria

1) $|p_n - p_{n-1}| < \varepsilon$
2) $|f(p_n)| < \varepsilon$
3) max # iters

If p_0 is not close to p, Newton's method might diverge



Ex: Let $f(x) = x^2 - 3$. Use Newton's method to find a root of f with accuracy $\varepsilon = 10^{-8}$, with $p_0 = 1.5$

Sol: Newton's

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} = p_{n-1} - \frac{p_{n-1}^2 - 3}{2p_{n-1}}$$

By direct computation, we get $p_3 = 1.73205081$

$f(p_3) < 10^{-8}$ and $|p_3 - p| < 10^{-8}$

Remark 1: Equivalent fixed point iteration to Newton:

$$p_n = g(p_{n-1}) \quad \text{where} \quad g(x) = x - \frac{f(x)}{f'(x)}, \quad f'(p) \neq 0.$$

Also,

$$g'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2}$$

If p is a zero of f, then $g'(p) = 0$!

Remark 2: two caveats:

1) need $f'(p_n)$ at each iteration
   (can be expensive in high dimensions!)

We could approximate f' by

$$f'(p_{n-1}) \approx \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}$$ (secant method)

2) need $p_0 \approx p$ (NOT obvious in many cases!)

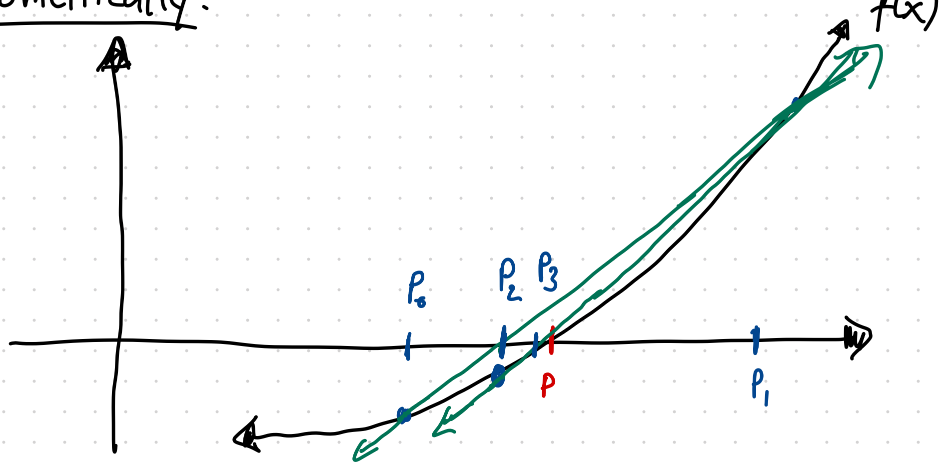
Secant Method

$$\boxed{p_n = p_{n-1} - \frac{f(p_{n-1})\,(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}}$$

← approx. of first order approx. of f.

Remark: need 2 initial conditions p_0, p_1 (close to p)

Geometrically:



Note: Newton's and Secant method are NOT root-bracketing (p may not be between p_n and p_{n-1})

Method of False Position (MFP)

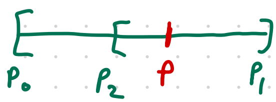
Idea: MFP based on Secant method, but with root-bracketing.

MFP: 1) Initialize p_0 and p_1 s.t. $f(p_0)\cdot f(p_1) < 0$

Let $$p_2 = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)}$$ (p_2 is root of line passing through $(p_0, f(p_0))$, $(p_1, f(p_1))$)

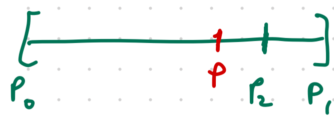
While $n \leq maxIter$

- If $sgn(f(p_1))\cdot sgn(f(p_2)) < 0$,



set $$p_3 = p_2 - \frac{f(p_2)(p_2 - p_1)}{f(p_2) - f(p_1)}$$

- If $sgn(f(p_1))\cdot sgn(f(p_2)) > 0$

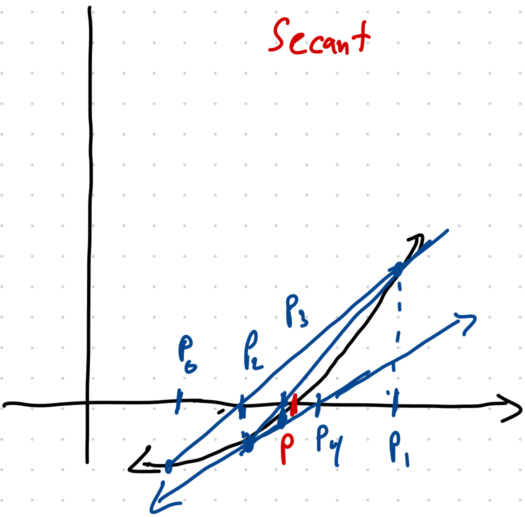


set $$p_3 = p_2 - \frac{f(p_2)(p_2 - p_0)}{f(p_2) - f(p_0)}$$

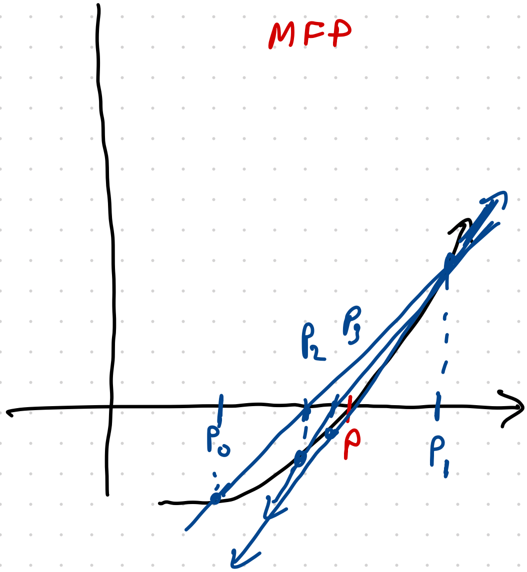
In general: $p_n = p_3$, $p_{n-1} = p_2$, $p_{n-2} = p_1$, $p_{n-3} = p_0$

Note: may require 3 previous points

Geometrically:



here, use $p_2, p_3 \to p_4$



here, use p_1, p_3

$\to p_4$

2.4 Convergence Order

Def: Suppose $p_n \to p$ as $n \to \infty$, $p_n \neq p$ $\forall n$

If $\lambda, \alpha > 0$ exist with

$$\boxed{\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda}$$

then $\{p_n\}$ converges to p with order α, and asymptotic error constant λ

Remarks:

1) Similar def: $|p_{n+1} - p| \leq \lambda \cdot |p_n - p|^{\alpha}$ for large n

2) α reflects convergence speed more than λ.

3) If
   - $\alpha = 1$ $(\lambda < 1) \Rightarrow \{p_n\}$ converges linearly
   - $\alpha = 2$ $\Rightarrow \{p_n\}$ converges quadratically

4) Different from $O(n^{-p})$ where

$$|p_n - p| \leq K n^{-p}$$

Ex: Assume $p_n \to 0$ as $n \to \infty$ with

$$\lim_{n\to\infty} \frac{|p_{n+1}|}{|p_n|} = \frac{1}{2},$$ and

$q_n \to 0$ as $n \to \infty$ with

$$\lim_{n\to\infty} \frac{|q_{n+1}|}{|q_n|^2} = \frac{1}{2}$$

Then $\forall\, n$:

$$|p_{n+1}| \approx \frac{1}{2}|p_n| \approx \cdots \approx \left(\frac{1}{2}\right)^{n+1} |p_0|$$

$$|q_{n+1}| \approx \frac{1}{2}|q_n|^2 \approx \cdots \approx \left(\frac{1}{2}\right)^{2^{n+1}-1} |q_0|^{2^{n+1}}$$

If $p_0 = q_0 = 1$, we can see that $|q_{n+1}| \ll |p_{n+1}|$

$\Rightarrow$ $\{q_n\}$ converges much faster than $\{p_n\}$.

Thm (Convergence Order of Bisection)

Let $f \in C[a,b]$, $f(a)\cdot f(b) < 0$, $a_1 = a$, $b_1 = b$.

The the seq. $\{p_n\}$ generated by Bisection method converges linearly to root of $f(x)$

Pf: Let p be root of $f(x)$ with $p \in [a,b]$.

Recall that $|p_n - p| \le |b-a| \cdot \left(\frac{1}{2}\right)^n$ for Bisection

$$\Rightarrow \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{\left(\frac{1}{2}\right)^{n+1}}{\left(\frac{1}{2}\right)^n} = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} = \frac{1}{2}$$

$$\Rightarrow \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \frac{1}{2}$$

$\Rightarrow$ linear convergence ∎

Convergence Order of Fixed Point Iteration

Thm 1: Let 1) $g \in C[a,b]$, $g(x) \in [a,b]$ $\forall x \in [a,b]$

2) $g \in C^1(a,b)$ and $0<k<1$ exists with

$$|g'(x)| \le k \quad \forall x \in (a,b)$$

If $g'(p) \neq 0$, then $\forall p_0 \neq p$ and $p_0 \in [a,b]$, the sequence $\{p_n\}$ generated by

$$p_n = g(p_{n-1})$$

converges linearly to unique fixed pt. p in $[a,b]$

Pf: By fixed pt. thm, we know $p_n \to p$ as $n \to \infty$.

By MVT,
$$p_{n+1} - p = g(p_n) - g(p)$$
$$= g'(\xi_n)(p_n - p) \quad \text{By MVT}$$

ξ_n is between p_n and p.

Since $p_n \to p$ as $n \to \infty$, $\xi_n \to p$ as $n \to \infty$

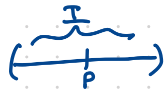
$$\Rightarrow \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} |g'(\xi_n)| = |g'(p)| < 1$$

Thus, if $g'(p) \neq 0$, then $p_n \to p$ linearly with asymptotic error constant $g'(p)$ ∎

Note: higher order of convergence for fixed pt. methods can be achieved under additional assumptions.

Thm 2: Let p be a sol. of $x = g(x)$.

Assume
- $g'(p) = 0$, and
- g'' continuous with $|g''(x)| < M$ on an open interval I containing p



Then $\exists\ \delta > 0$ s.t.

For $p_0 \in [p-\delta, p+\delta]$, the seq. $p_n = g(p_{n-1})$

converges at least quadratically to p, i.e.,

$$|p_{n+1} - p| \leq \frac{M}{2} |p_n - p|^2 \qquad \forall \quad n \geq n_0$$

Pf: 1) Show that fixed pt. iteration converges under given assumptions

By continuity of $g'(x)$, we can choose $k \in (0,1)$ and $\delta > 0$ s.t.

$$\left.\begin{array}{l} 1)\ [p-\delta, p+\delta] \subseteq I, \\ \text{and } \ g'(p) = \lim\limits_{x \to p} g'(x) = 0 < k \end{array}\right\} \Rightarrow |g'(x)| \leq k \text{ for } x \in [p-\delta, p+\delta]$$

Thm 2: Let p be a sol. of $x = g(x)$.

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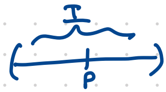

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$$|p_{n+1} - p| \le \frac{M}{2} |p_n - p|^2 \qquad \forall \quad n \ge n_0$$

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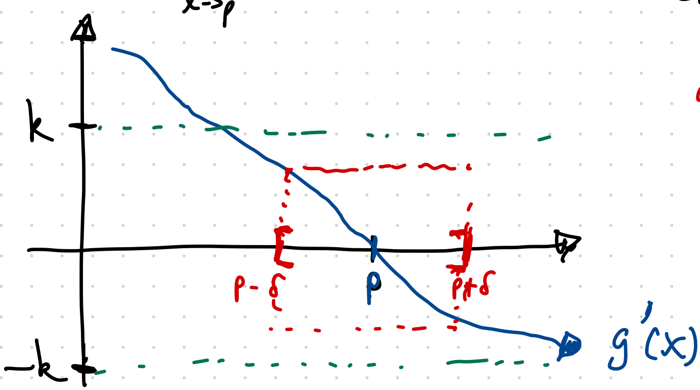


given k, can find δ s.t. $x \in [p-\delta, p+\delta] \Rightarrow |g'(x)| \le k$

4.1

know: $|g'(x)| \le k < 1$ on $[p-\delta, p+\delta]$.

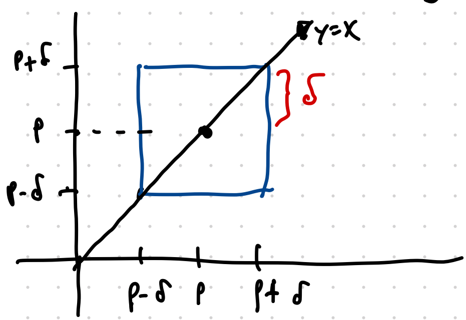
Next, need to show g maps to itself on $[p-\delta, p+\delta]$

Let $x \in [p-\delta, p+\delta]$

$$
\begin{aligned}
|g(x)-p| &= |g(x)-g(p)| \\
&= |g'(\xi)| \cdot |x-p| \quad \text{By MVT} \\
&\le k|x-p| \\
&< |x-p| \quad (*)
\end{aligned}
$$

$x \in [p-\delta, p+\delta] \Rightarrow |x-p| < \delta$

$\Rightarrow |g(x)-p| < \delta$ by $(*)$



Thus, g maps $[p-\delta, p+\delta]$ into $[p-\delta, p+\delta]$

Thus, $g(x) \in [p-\delta, p+\delta] \quad \forall x \in [p-\delta, p+\delta]$

$\Rightarrow$ By fixed pt. thm, g converges to unique sol.

2) Show quadratic convergence

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Expand $g(x)$ at p for $x \in [p-\delta, p+\delta]$

$$g(x) = g(p) + \underbrace{g'(p)}_{\substack{=0 \\ \text{by assumption}}}(x-p) + \frac{g''(\xi_n)}{2}(x-p)^2$$

ξ_n between x and p

$$\Rightarrow g(x) = \underbrace{g(p)}_{=p} + \frac{g''(\xi_n)}{2}(x-p)^2$$

Letting $x = p_n$, we have

$$\underbrace{g(p_n)}_{=p_{n+1}} = p + \frac{g''(\xi_n)}{2}(p_n - p)^2$$

$$\Rightarrow \frac{|p_{n+1} - p|}{|p_n - p|^2} = \frac{|g''(\xi_n)|}{2}$$

$$\Rightarrow \lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n - p|^2} = \lim_{n\to\infty} \frac{|g''(\xi_n)|}{2} = \frac{1}{2}\Big|g''\big(\underbrace{\lim_{n\to\infty}\xi_n}_{=p}\big)\Big| \underset{\text{by assumption}}{<} \frac{M}{2}$$

$|g''(p)| \neq 0 \Rightarrow \{p_n\}$ converges quadratically to p

$|g''(p)| = 0 \Rightarrow \{p_n\}$ converges at higher order (≥ 3) to p

∎

<u>Remark:</u> Need p_0 to be sufficiently close to p

$p_0 \in [p-\delta, p+\delta]$

Convergence Order of Newton's Method

Ex: Let $p \in (a,b)$ be a zero of $f \in C^2[a,b]$.

Construct a fixed pt. problem $g(x) = x$ associated with root-finding problem, $f(x) = 0$ s.t.

$$p_n = g(p_{n-1}) \text{ converges quadratically.}$$

Sol: Set $g(x) = x - \phi(x) \cdot f(x)$

Goal: find ϕ to get quadratic conv.

$$g'(x) = 1 - \phi'(x) f(x) - \phi(x) \cdot f'(x)$$

$\Rightarrow g'(p) = 1 - \phi(p) \cdot f'(p)$ (want $g'(p) = 0$)

To obtain quadratic convergence, want $g'(p) = 0$.

$$\Rightarrow \phi(p) = \frac{1}{f'(p)}.$$

Choose $\phi(x) = \dfrac{1}{f'(x)} \Rightarrow g(x) = x - \dfrac{f(x)}{f'(x)}$

$\Rightarrow$ fixed pt. iter is $\underbrace{p_n = g(p_{n-1}) = p_{n-1} - \dfrac{f(p_{n-1})}{f'(p_{n-1})}}_{\text{Newton's method!}}$

Remark 1: If $f(p) = 0$ and $f'(p) \neq 0$,

then for any p_0 sufficiently close to

($s.t.$ $g \in C^1(a,b)$, $g(x) \in [a,b)$, $|g'(x)| \leq k$)

Newton's method will converge at least quadratically.

Remark 2: If $f(p) = 0$, then for p_0 close to p,

secant method converges to p with order

$$\frac{\sqrt{5}+1}{2} \approx 1.618$$