MATH182 HOMEWORK #3
DUE July 12, 2020

Exercise 1. *Obtain asymptotically tight bounds on* $\lg(n!)$ *without using Stirling's approximation. Instead, find a way to approximate the summation* $\sum_{k=1}^{n} \lg k$ *from above and below.*

Solution. We first find an upper bound on $\lg(n!)$:

$$\lg(n!) = \sum_{k=1}^{n} \lg k \leq \sum_{k=1}^{n} \lg n = n \lg n \implies \lg(n!) = \underline{O(n \lg n)}$$

We now prove the same $\Omega(n \lg n)$ is also a lower bound on $\lg(n!)$:

$$\lg(n!) = \sum_{k=1}^{n} \lg k \geq \sum_{k=n/2}^{n} \lg k \geq \sum_{k=n/2}^{n} \lg \frac{n}{2} \geq \frac{n}{2} \cdot \lg \frac{n}{2} = \frac{n}{2} \cdot (\lg n - 1) \tag{1}$$

For $n \geq 4$, we know $\lg n \geq 2$. We therefore have, for $n \geq 4$,

$$\lg n - 2 \geq 0 \implies 2 \lg n - 2 \geq \lg n \implies \frac{n}{4} \cdot (2 \lg n - 2) \geq \frac{n}{4} \lg n \implies \frac{n}{2} \cdot (\lg n - 1) \geq \frac{n}{4} \lg n \tag{2}$$

Therefore, from (1) and (2), we also have

$$\lg(n!) \geq \frac{n}{2} \cdot (\lg n - 1) \geq \frac{n}{4} \lg n = \underline{\Omega(n \lg n)}$$

$\Theta(n \lg n)$ is therefore an asymptotically tight bound on $\lg(n!)$.

Exercise 2. *Use the following ideas to develop a nonrecursive, linear-time algorithm for the maximum-subarray problem. Start at the left end of the array, and progress toward the right, keeping track of the maximum subarray seen so far. Knowing a maximum subarray of* $A[1 \,..\, j]$*, extend the answer to find a maximum subarray ending at an index* $j + 1$ *by using the following observation: a maximum subarray of* $A[1 \,..\, j+1]$ *is either a maximum subarray of* $A[1 \,..\, j]$ *or a subarray* $A[i \,..\, j+1]$*, for some* $1 \leq i \leq j+1$*. Determine a maximum subarray of the form* $A[i \,..\, j+1]$ *in constant time based on knowing a maximum subarray ending at index* j*.*

Your answer should consist of:

(1) Pseudocode for your algorithm.
(2) Proof of correctness.
(3) An analysis of the running time.

Solution. *(1)* We note that a maximum subarray ending at $j + 1$ is either an extension of the maximum subarray ending at j (if the sum of that subarray is positive) or simply the subarray $A[j+1]$. We can then extend the given observation as follows: a maximum subarray of $A[1 \,..\, j+1]$ is either a maximum subarray of $A[1 \,..\, j]$, a maximum subarray ending at j with the addition of $A[j + 1]$, or simply the subarray $A[j + 1]$. Following is pseudocode[1] for an algorithm that applies this observation to find the maximal subarray of $A[1 \,..\, n]$ (for $n := A.length$) in linear time:

MaxSubarray(A) :

[1] See Appendix I for C++ code

```
1   i2j_startIndex = 1          // starting index of maximal subarray of form A[i..j] for some i
2   i2j_sum = A[1]              // sum of maximal subarray of form A[i..j]
3   one2j_startIndex = 1        // starting index of maximal subarray of A[1..j]
4   one2j_endIndex = 1          // ending index of maximal subarray of A[1..j]
5   one2j_sum = A[1]            // sum of maximal subarray of A[1..j]
6   for j = 1 to A.length - 1
7       // updating i2j_sum, startIndex for j + 1 (finding max subarray ending at j + 1)
8       if i2j_sum > 0
9           // max subarray ending at j + 1 is max subarray ending at j, plus A[j + 1]
10          i2j_sum = i2j_sum + A[j + 1]
11      else
12          // max subarray ending at j + 1 is simply A[j + 1]
13          i2j_startIndex = j + 1
14          i2j_sum = A[j + 1]
15      // updating one2j_startIndex, endIndex and sum (finding max subarray of A[1..j + 1])
16      if i2j_sum > one2j_sum
17          // max subarray of A[1..j + 1] is max subarray ending at j + 1
18          one2j_startIndex = i2j_startIndex
19          one2j_endIndex = j + 1
20          one2j_sum = i2j_sum
21      // else, max subarray of A[1j + 1] is max subarray of A[1..j]
22      // no need to update variables
23  return one2j_startIndex, one2j_endIndex, one2j_sum
```

(2) We prove correctness of this algorithm by using the following loop invariant:

Loop Invariant: After line 6 is run, *one2j_startIndex* and *one2j_endIndex* are the starting and ending indices of the maximal subarray of $A[1\,..\,j]$, and *one2j_sum* is the sum of the same. Additionally, *i2j_startIndex* is the starting index of the maximal subarray of the form $A[i\,..\,j]$ for some $1 \le i \le j$, and *i2j_sum* is the sum of the same.

We show this loop invariant holds.

Initialisation: When line 6 is first run, we have $j = 1$. Both the maximal subarray of the form $A[i\,..\,j]$ and the maximal subarray of $A[1\,..\,j]$ are simply $A[1]$, with starting and ending indices 1 and sum $A[1]$. *i2j_startIndex*, *i2j_sum*, *one2j_startIndex*, *one2j_endIndex*, and *one2j_sum* are therefore all appropriately initialised from lines 1-5 and the loop invariant holds for $j = 1$.

Maintenance: Assume the loop invariant is true before some iteration with $j = j_0 \in [1\,..\,A.length - 1]$. By assumption and definition of the loop invariant, we have

$$\begin{aligned}
\textit{i2j_startIndex} &= \text{starting index of maximal subarray ending at } j_0 \\
\textit{i2j_sum} &= \text{sum of maximal subarray ending at } j_0 \\
\textit{one2j_startIndex} &= \text{starting index of maximal subarray of } A[1\,..\,j_0] \\
\textit{one2j_endIndex} &= \text{ending index of maximal subarray of } A[1\,..\,j_0] \\
\textit{one2j_sum} &= \text{sum of maximal subarray of } A[1\,..\,j_0]
\end{aligned}$$

We first consider how we find a maximal subarray ending at $j_0 + 1$. We have two cases:

Case 1: In this case, the sum of a maximal subarray ending at j_0, given by `i2j_sum`, is positive, and the **if** condition in line 8 holds true. A maximal subarray ending at $j_0 + 1$ is therefore given by the the same maximal subarray with $A[j_0 + 1]$ included. We therefore leave `i2j_startIndex` unchanged, and correctly update `i2j_sum` by adding $A[j_0 + 1]$ to it in line 10.

Case 2: In this case, the sum of a maximal subarray ending at j_0, given by `i2j_sum`, is non-positive, and we run the `else` block comprising lines 13 and 14. Since `i2j_sum`$+A[j_0+1] \leq A[j_0+1]$, $A[j_0+1]$ is a maximal subarray ending at j_0+1. In line 13, we therefore correctly set the starting index of the maximal subarray ending at j_0+1, `i2j_startIndex`, to j_0+1; similarly in line 14, we correctly set the sum of the same subarray to $A[j_0+1]$.

In either case, then, `i2j_startIndex` and `i2j_sum` reflect the starting index and sum of a maximal subarray ending at j_0+1 respectively.

We now consider how we find a maximal subarray of $A[1\mathinner{..}j_0+1]$. We have two cases:

Case 1: A maximal subarray of $A[1\mathinner{..}j_0+1]$ is also a maximal subarray of $A[1\mathinner{..}j_0]$. In this case, we may leave `one2j_startIndex`, `one2j_endIndex`, and `one2j_sum` may be left unchanged.

Case 2: A maximal subarray of $A[1\mathinner{..}j_0+1]$ is not a maximal subarray of $A[1 dots j_0]$. This implies a maximal subarray of $A[1\mathinner{..}j_0+1]$ includes j_0+1; in other words, it ends at j_0+1. We have already found such a maximal array in lines 7 through 14, and test for this case by comparing `i2j_sum` (the sum of a maximal subarray ending at j_0+1) with `one2j_sum` (the sum of a maximal subarray of $A[1\mathinner{..}j]$. This case applies when the former is larger.

We therefore correctly update, in lines 18 and 19, `one2j_startIndex` and `one2j_endIndex` to the starting index of the maximal subarray ending at j_0+1, `i2j_startIndex`, and the ending index of the same array, j_0+1, respectively. We also correctly update, in line 20, the variable `one2j_sum` to the sum of the maximal subarray ending at j_0+1, `i2j_sum`.

After completing the iteration and running line 6 again, we now have $j = j_0+1$. We have shown the following now holds:

$$\begin{aligned}
i2j_startIndex &= \text{starting index of maximal subarray ending at } j_0+1 && \text{(from lines 7-14)}\\
i2j_sum &= \text{sum of maximal subarray ending at } j_0+1 && \text{(from lines 7-14)}\\
one2j_startIndex &= \text{starting index of maximal subarray of } A[1\mathinner{..}j_0+1] && \text{(from lines 15-22)}\\
one2j_endIndex &= \text{ending index of maximal subarray of } A[1\mathinner{..}j_0+1] && \text{(from lines 15-22)}\\
one2j_sum &= \text{sum of maximal subarray of } A[1\mathinner{..}j_0+1] && \text{(from lines 15-22)}
\end{aligned}$$

Therefore, we have, by its definition, the loop invariant holds for $j = j_0+1$.

Termination: After the last iteration of the **for** loop, we have $j = A.length - 1 + 1 = A.length$, and therefore $A[1\mathinner{..}j] = A[1\mathinner{..}A.length] = A$. By the loop invariant, we have

$$\begin{aligned}
one2j_startIndex &= \text{starting index of maximal subarray of } A\\
one2j_endIndex &= \text{ending index of maximal subarray of } A\\
one2j_sum &= \text{sum of maximal subarray of } A
\end{aligned}$$

which are the desired outputs of MaxSubArray(A) and are returned in line 23.

The algorithm is therefore correct. ■

(3) Analysing the running time of the algorithm is rather simple — since lines 1-5 and 15 run in constant time and only once, and lines 6 through 15 all run in constant time and a maximum of $n := A.length$ times (by scope of the **for** loop), the algorithm is $\Theta(n)$.

Exercise 3. *How quickly can you multiply a $kn \times n$ matrix by an $n \times kn$ matrix, using Strassen's algorithm as a subroutine? Answer the same question with the order of the input matrices reversed.*

Exercise 4. *Show how to multiply the complex numbers $a+bi$ and $c+di$ using only three multiplications of real numbers. The algorithm should take a, b, c, d as input and produce the real component $ac - bd$ and the imaginary component $ad + bc$ separately.*

Solution. Given imaginary numbers $a + bi$, $c + di$, we first compute the following three products:

$$k_1 := a \cdot (c + d),\ \ k_2 := d \cdot (b + a),\ \ k_3 := b \cdot (d - c)$$

We now show these products are sufficient to find the real and imaginary components of the product of given imaginary numbers. We first show $k_1 - k_2$ gives the real component:

$$k_1 - k_2 = a \cdot (c + d) - d \cdot (b + a) = ac + ad - bd - ad = \underline{ac - bd}$$

We now show $k_2 - k_3$ gives the imaginary component:

$$k_2 - k_3 = d \cdot (b + a) - b \cdot (d - c) = bd + ad - bd + bc = \underline{ad + bc}$$

Hence shown, using the above algorithm, that three multiplications of real numbers are sufficient to generate the real and imaginary components of a product of any two imaginary numbers.

Exercise 5. *Show that the solution of $T(n) = T(n-1) + n$ is $O(n^2)$.*

Solution. We show, using the substitution method, that $T(n) = O(n^2)$. Assume inductively that there is some $d > 0$, $c > 0$ such that $T(m) \le cm^2 - dm$ for all $m < n$. We then have:

$$\begin{aligned} T(n) = T(n-1) + n &\le c(n-1)^2 - d(n-1) + n && \text{(by inductive assumption)} \\ &= c(n^2 - 2n + 1) - dn + d + n \\ &= cn^2 - ((2c + d - 1)n - (c + d)) \\ &\le cn^2 - dn \end{aligned}$$

where the last inequality applies if

$$dn \le (2c + d - 1)n - (c + d) \implies 0 \le (2c - 1)n - (c + d)$$

We see that for all $n \ge 2$, this is indeed true for $c = 1$ and $d = 1$, since then

$$(2c - 1)n - (c + d) = n - 2 \ge 0$$

Since this inductive proof will work for $c = d = 1$, we therefore have $\underline{T(n) = O(n^2)}$. ∎

Exercise 6. *Solve the recurrence $T(n) = 3T(\sqrt{n}) + \log n$ by making a change of variables. Your solution should be asymptotically tight. Do not worry about whether values are integers.*

Solution. We define $m := \log_k n$ where k is the base of the log in the given recurrence relation. We can therefore rewrite the recurrence as:

$$\begin{aligned} T(n) = 3T(\sqrt{n}) + \log n &\implies T(k^m) = 3T(\sqrt{k^m}) + m && (m = \log_k n \implies n = k^m) \\ &\implies T(k^m) = 3T(k^{m/2}) + m \\ &\implies S(m) = 3S(m/2) + m && (\text{for } S(m) := T(k^m)) \end{aligned}$$

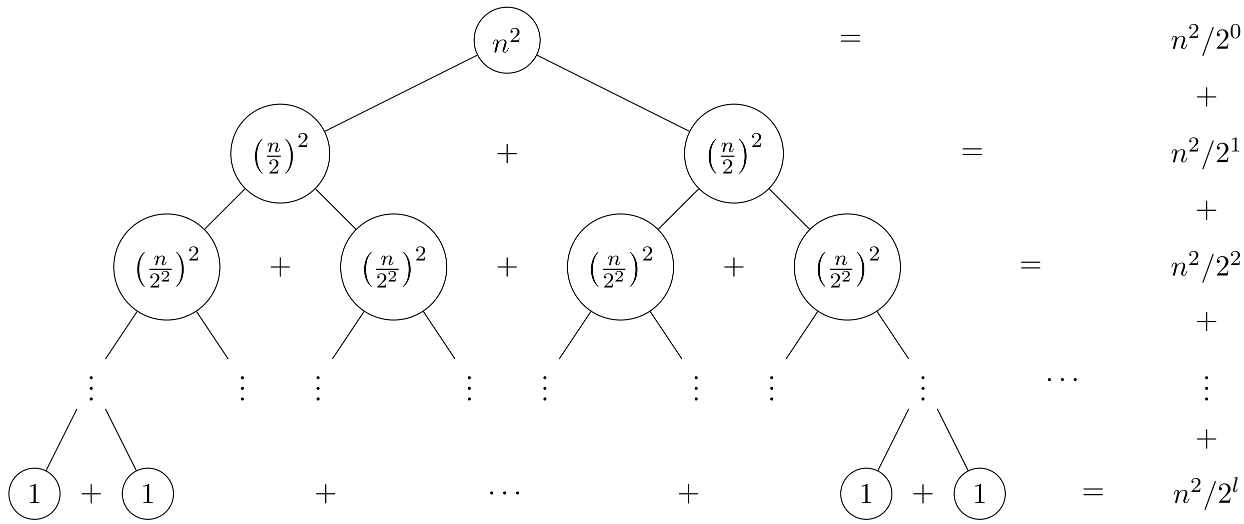
We now apply Master Theorem to find a tight asymptotic bound for $S(m)$. From the above recurrence relation, we have $a = 3$, $b = 2$, and $f(m) = m$. Since $\log_b a = \lg 3 > 1$, we have $f(m) = m^1 = O(m^{\log_b a - \epsilon})$ for any $\epsilon \in (0, \lg 3 - 1]$. We are therefore in the leaf-heavy case of the Master Theorem and have $S(m) = \Theta(m^{\log_b a}) = \Theta(m^{\lg 3})$. By definition of $S(m)$ and m, we have

$$T(n) = T(k^m) = S(m) = \Theta(m^{\lg 3}) = \underline{\Theta(\log_k^{\lg 3} n)}$$

which is a tight asymptotic bound on $T(n)$.

Exercise 7. *Use a recursion tree to determine a good asymptotic upper bound on the recurrence $T(n) = T(n/2) + n^2$. Use the substitution method to verify your answer.*

Solution. Below is a recursion tree for the given recurrence relation:



On each level k, we call $T(n/2^k)$ (with cost $(n/2^k)^2$ at that level) 2^k times, giving a total cost per level of

$$\left(\frac{n}{2^k}\right)^2 \cdot 2^k = \frac{n^2}{2^k}$$

Assuming running $T(1)$ costs 1, the leaves of the recursion tree each represent a $T(1)$ call. In the last layer l, we therefore have $T(n/2^l) = T(1) \implies n/2^l = 1 \implies l = \lg n$. Our guess of the total cost of $T(n)$ is therefore given by

$$T(n) = \sum_{k=0}^{l} \frac{n^2}{2^k} = n^2 \sum_{k=0}^{\lg n} \frac{1}{2^k} \leq= n^2 \sum_{k=0}^{\infty} \frac{1}{2^k} = n^2 \cdot \frac{1}{1 - 1/2} = 2n^2$$

We now use substitution method to verify that $T(n) = O(n^3)$. Assume inductively that there is some $c > 0$ such that $T(m) \leq cm^2$ for all $m < n$. We then have:

$$\begin{aligned} T(n) = T(n/2) + n^2 &\leq c\left(\frac{n}{2}\right)^2 + n^2 && \text{(by inductive assumption)} \\ &= \left(\frac{c}{4} + 1\right) n^2 \\ &\leq cn^2 && \text{(for any } c \geq 4/3\text{)} \end{aligned}$$

Since this inductive proof will work for any value of $c > 4/3$ and any $n \geq 0$ (assuming $T(0) = 0$), we therefore have $\underline{T(n) = O(n^2)}$.

Exercise 8. *Use a recursion tree to determine a good asymptotic upper bound on the recurrence $T(n) = T(n-1) + T(n/2) + n$. Use the substitution method to verify your answer.*

Exercise 9. *Use the master method to give tight asymptotic bounds for the following recurrences:*

(1) $T(n) = 2T(n/4) + 1$

(2) $T(n) = 2T(n/4) + \sqrt{n}$
(3) $T(n) = 2T(n/4) + n$
(4) $T(n) = 2T(n/4) + n^2$

Solution. For reach of the given recurrence relations, we have $a = 2$ and $b = 4$. We therefore have $\log_b a = 1/2$.

(1) For the recurrence $T(n) = 2T(n/4) + 1$, we have $f(n) = 1$. Since $f(n) = 1 = O(n^0) = O(n^{1/2-\epsilon})$ for $\epsilon \in (0, 1/2]$, we have $f(n) = O(n^{\log_b a - \epsilon})$ and are therefore in the leaf-heavy case of Master Theorem. $T(n)$ is therefore $\Theta(n^{\log_b a}) = \underline{\Theta(\sqrt{n})}$.

(2) For the recurrence $T(n) = 2T(n/4) + \sqrt{n}$, we have $f(n) = \sqrt{n}$. Since $f(n) = \sqrt{n} = \Theta(\sqrt{n}) = \Theta(n^{1/2}) = \Theta(n^{1/2} \lg^0 n)$, we have $f(n) = \Theta(n^{\log_b a} \lg^k n)$ for $k = 0$ and are therefore in the middle case of Master Theorem. Since $k = 0 > -1$ $T(n)$ is therefore $\Theta(n^{\log_b a} \lg^{k+1} n) = \underline{\Theta(\sqrt{n} \lg n)}$.

(3) For the recurrence $T(n) = 2T(n/4) + n$, we have $f(n) = n$. Since $f(n) = n = \Omega(n^1) = \Omega(n^{1/2+\epsilon})$ for $\epsilon \in (0, 1/2]$, we have $f(n) = \Omega(n^{\log_b a + \epsilon})$ and are therefore in the root-heavy case of Master Theorem. We now test the Regularity Condition by searching for $c < 1$ such that $2f(n/4) \leq cf(n)$ for all $n \geq n_0$ for some $n_0 \in \mathbb{Z}$. In other words, we have
$$2f(n/4) \leq cf(n) \implies 2 \cdot \frac{n}{4} \leq cn \implies \frac{1}{2} \leq c$$
for all $n \geq 0$. We can therefore choose any $c \in [1/2, 1)$ to satisfy the Regularity Condition. $T(n)$ is therefore $\Theta(f(n)) = \underline{\Theta(n)}$.

(4) For the recurrence $T(n) = 2T(n/4) + n^2$, we have $f(n) = n^2$. Since $f(n) = n^2 = \Omega(n^2) = \Omega(n^{1/2+\epsilon})$ for $\epsilon \in (0, 3/2]$, we have $f(n) = \Omega(n^{\log_b a + \epsilon})$ and are therefore in the root-heavy case of Master Theorem. We now test the Regularity Condition by searching for $c < 1$ such that $2f(n/4) \leq cf(n)$ for all $n \geq n_0$ for some $n_0 \in \mathbb{Z}$. In other words, we have
$$2f(n/4) \leq cf(n) \implies 2 \cdot \left(\frac{n}{4}\right)^2 \leq cn^2 \implies \frac{1}{8} \leq c$$
for all $n \geq 0$. We can therefore choose any $c \in [1/8, 1)$ to satisfy the Regularity Condition. $T(n)$ is therefore $\Theta(f(n)) = \underline{\Theta(n^2)}$.

Exercise 10. *Professor Diogenes has n supposedly identical integrated-circuit chips that in principle are capable of testing each other. The professor's test jig accommodates two chips at a time. When the jig is loaded, each chip tests the other and reports whether it is good or bad. A good chip always reports accurately whether the other chip is good or bad, but the professor cannot trust the answer of a bad chip. Thus, the four possible outcomes of a test are as follows:*

Chip A says	*Chip B says*	*Conclusion*
B is good	*A is good*	*both are good, or both are bad*
B is good	*A is bad*	*at least one is bad*
B is bad	*A is good*	*at least one is bad*
B is bad	*A is bad*	*at least one is bad*

(1) Show that if at least $n/2$ chips are bad, the professor cannot necessarily determine whip chips are good using any strategy based on this kind of pairwise test. Assume that the bad chips can conspire to fool the professor.

(2) Consider the problem of finding a single good chip from among n chips, assuming that more than $n/2$ of the chips are good. Show that $\lfloor n/2 \rfloor$ pairwise tests are sufficient to reduce the problem to one of nearly half the size.

(3) Show that the good chips can be identified with $\Theta(n)$ pairwise tests, assuming that more than $n/2$ of the chips are good. Give and solve the recurrence that describes the number of tests.

Exercise 11. *Suppose you're consulting for a bank that's concerned about fraud detection, and they come to you with the following problem. They have a collection of n bank cards that they've confiscated, suspecting them of being used in fraud. Each bank card is a small plastic object, containing a magnetic stripe with some encrypted data, and it corresponds to a unique account in the bank. Each account can have many bank cards corresponding to it, and we'll say that two bank cards are* equivalent *if they correspond to the same account.*

It's very difficult to read the account number off a bank card directly, but the bank has a high-tech "equivalence tester" that takes two bank cards and, after performing some computations, determines whether they are equivalent.

Their question is the following: among the collection of n cards, is there a set of more than $n/2$ of them that are all equivalent to one another? Assume that the only feasible operations you can do with the cards are to pick two of them and plug them in to the equivalence tester. Show how to decide the answer to their question with only $O(n \log n)$ invocations of the equivalence tester.

Exercise 12. *Consider an n-node complete binary tree T, where $n = 2^d - 1$ for some d. Each node v of T is labeled with a real number x_v. You may assume that the real numbers labeling the nodes are all distinct. A node v of T is a* local minimum *if the label x_v is less than the label x_w for all nodes w that are joined to v by an edge.*

You are given such a complete binary tree T, but the labeling is only specified in the following implicit *way: for each node v, you can determine the value x_v by* probing *the node v. Show how to find a local minimum of T using only $O(\log n)$* probes *to the nodes of T.*

Exercise 13 (Programming exercise)**.** *There are exactly ten ways of selecting three from five, 12345:*

$$123, 124, 125, 134, 135, 145, 234, 235, 245, \text{ and } 345$$

In combinatorics, we use the notation $\binom{5}{3} = 10$.

In general, $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, where $r \leq n$, $n! = n \times (n-1) \times \cdots \times 3 \times 2 \times 1$, and $0! = 1$.

It is not until $n = 23$, that a value exceeds one-million: $\binom{23}{10} = 1144066$.

How many, not necessarily distinct, values of $\binom{n}{r}$, for $1 \leq n \leq 100$, are greater than one-million?

Solution. The following function, `largeCombinations`, takes as input integers `minCombinations`, the lower-bound for the values of combinations we're searching for, and `nBound`, the upper bound for n in our search.

```
#include <vector>
using namespace std;

#include <boost/multiprecision/cpp_int.hpp>
    // for large ints, from the non-standard boost library
using namespace boost::multiprecision;

int largeCombinations(int minCombinations, int nBound) {
  // returns the number of values of (n r), with 0 <= r <= n <= nBound, that are greater than minCombinations

```

```
  // computing a vector of factorials
  vector<cpp_int> factorials = { 1 };   // initialised to 0!
  // iterating to build factorials
  for (int n = 1; n <= nBound; n++)
    factorials.push_back(factorials.back() * n);

  int count = 0;  // number of combinations found that are greater than min

  // iterating over all (n, r) for 0 <= n <= nBound
  for (int n = 1; n <= nBound; n++)
    for (int r = 0; r <= n; r++) {
      // computing nCr
      cpp_int nCr = factorials[n] / (factorials[r] * factorials[n - r]);
      // checking if nCr > min
      if (nCr > minCombinations)
        count++;
    }

  return count;
}
```

Calling `largeCombinations(1000000, 100)` returns <u>4075</u>.

Exercise 14 (Programming exercise)**.** *Euler's Totient function, $\varphi(n)$ is used to determine the number of numbers less than n which are relatively prime to n (d is* ***relatively prime*** *to n if $\gcd(d, n) = 1$). For example, as $1, 2, 4, 5, 7, 8$ are all less than nine and relatively prime to nine, $\varphi(9) = 6$.*

n	*Relatively Prime*	$\varphi(n)$	$n/\varphi(n)$
2	1	1	2
3	$1, 2$	2	1.5
4	$1, 3$	2	2
5	$1, 2, 3, 4$	4	1.25
6	$1, 5$	2	3
7	$1, 2, 3, 4, 5, 6$	6	1.1666...
8	$1, 3, 5, 7$	4	2
9	$1, 2, 4, 5, 7, 8$	6	1.5
10	$1, 3, 7, 9$	4	2.5

It can be seen that $n = 6$ produces a maximum $n/\varphi(n)$ for $n \leq 10$.

Find the value of $n \leq 1000000$ for which $n/\varphi(n)$ is a maximum.

Solution. Even with some optimisations, the brute force solution[2], using Euler's Product Formula, is quite inefficient and takes 10 seconds to run (in C++). We find a more efficient algorithm.

From Euler's Product Formula, we have

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) = n \prod_{p|n} \left(\frac{p-1}{p}\right) \implies \frac{n}{\varphi(n)} = \prod_{p|n} \left(\frac{p}{p-1}\right)$$

where p is prime. In other words, for $f(p) := p/(p-1)$, the totient ratio (i.e., $n/\varphi(n)$) is given by the product of $f(p)$ over the set of primes that divide n. We approach the problem by searching

[2] C++ code in Appendix II

for the set of primes that comprise a prime factorisation for some $n \leq 1000000$ such that the above product over that set is maximal.

Since $f(p)$ is strictly decreasing (easily verified by differentiating with respect to n), smaller prime factors yield a larger product. Furthermore, since $f(p) > 1$ for all primes p, each unique prime factor increases the totient ratio. We therefore want to maximise the number of unique factors (so more terms contribute to the product) and minimise the value of each unique prime factor (so each term contributes more to the product).

In fact, these goals go hand-in-hand — in minimising the value of each prime factor, we can only increase (non-strictly) the number of unique prime factors. Let $P_k := \{p_1, p_2 \dots, p_k\}$ where $p_1 = 2$ and each p_{i+1} is the prime immediately after p_i. We propose the value of n that gives a maximal totient ratio under a given bound B is given by the product over P_k where k is chosen such that

$$\prod_{p \in P_k} p \leq B < \prod_{p \in P_{k+1}} p$$

We show this proposition is true. Assume towards contradiction that $n := \prod_{p \in P_k} p$ does not give a maximal totient ratio. This means there exists $N \in [2, B]$ such that for the (sorted) set of prime factors of N, $PF(N)$, we have

$$\frac{N}{\varphi(N)} > \frac{n}{\varphi(n)} \implies \prod_{p \in PF(N)} f(p) > \prod_{p \in P_k} f(p)$$

From our discussion on the product of $f(p)$ over a set of primes, we know this implies one of the following two cases must be true:[3]

Case 1: $PF(N)$ has more terms than P_k, i.e., $|PF(N)| \geq k+1$. Since $PF(N)$ is (by assumption) sorted in increasing order, it is clear that, by definition of p_i, $PF(N)[i] \geq p_i$ for all $i \in [1, k+1]$. We therefore have

$$N \geq \prod_{p \in PF(N)} p \geq \prod_{1 \leq i \leq k+1} PF(N)[i] \geq \prod_{1 \leq i \leq k+1} p_i = \prod_{p_{k+1}} p > B \quad \text{(by definition of } P_k\text{)}$$

which contradicts our assumption $N \leq B$. Case 1 is therefore not possible.

Case 2: $PF(N)$ has at most as many terms as P_k (i.e, $|PF(N)| \leq k$), and $f(PF[i]) > f(p_i)$ for some $i \in [1, |PF(N)|]$. From our discussion in Case 1, we already know $PF(N)[i] \geq p_i$ for all $i \in [1, |PF(N)|]$. Since $f(p)$ is strictly decreasing, this implies $f(PF(N)[i]) \leq f(p_i)$ for all $i \in [1, |PF(N)|]$, which contradicts our assumption there exists $i \in [1, |PF(N)|]$ such that $f(PF[i]) > f(p_i)$. Case 2 is therefore not possible.

Hence shown that value of $n \leq B$ such that the totient ratio is maximal is given by the maximal product over consecutive primes starting from 2, i.e., the product of P_k. ■

This results in the following algorithm, which runs in effectively constant time for even large n:

```
#include <vector>
using namespace std;

int maxEulerTotientRatio(int bound) {
  // returns that maximum value of n / totient(n) for n <= bound

  double totientRatio = 1;  // to keep track of the optimal totient ratio
  int n = 1;  // to keep track of n that gives the optimal totient ratio

```

[3] Note that for any $N \in [2, B]$, if either case is true, that itself is not sufficient to say N has a higher totient ratio than n. However, if we assume N has a higher totient ratio, then one of the cases be true.

```
  vector<int> primes;

  // iterating over natural numbers to identify primes until n exceeds bound
  for (int p = 2; n*p <= bound; p++) {
    // checking if p is prime by iterating over primes
    bool primeFound = true;
    for (int prime : primes)
      if (p % prime == 0) {
        primeFound = false;
        break;
      }
    // prime found
    if (primeFound) {
      primes.push_back(p);  // update primes
      totientRatio *= p / double(p - 1);  // update totientRatio
      n *= p; // update n
    }
  }
  return n;
}
```

Calling `maxEulerTotientRatio(1000000)` returns <u>510510</u> (i.e., the product over $\{2, 3, 5, 7, 11, 13, 17\}$), which has totient ratio ≈ 5.53939.

Exercise 15 (Programming exercise)**.** *It is possible to write five as a sum in exactly six different ways:*

$$4 + 1$$
$$3 + 2$$
$$3 + 1 + 1$$
$$2 + 2 + 1$$
$$2 + 1 + 1 + 1$$
$$1 + 1 + 1 + 1 + 1$$

How many different ways can one hundred be written as a sum of at least two positive integers?

Solution. The solution to this programming exercise can be obtained with a few simple modifications to the solution of Exercise 14 from Homework 2 — instead of searching for the number of partitions of $200p$ using $\{1p, 2p, 5p, 10p, 20p, 50p, 100p, 200p\}$, we are now searching for the number of partitions of 100 using $\{1, 2, \ldots, 99\}$ (we exclude 100 since we don't want to include any partitions with only one integer). The following function, **`nPartitions`**, takes input integer n and returns the number of partitions of n with at least two positive integers:

```
#include <vector>
using namespace std;

int nPartitions(int n) {
  // returns the number of partitions of n with at least two integers

  vector<int> partitionSizes(n + 1, 0);
  // a list containing, for each number from 1 through n, the number of partitions
     it has
  // initialised to 0

```

```
  // base case: partitionSize[0] represents the termination of a partition
  partitionSizes[0] = 1;

  // iterating over 1 through n - 1
  for (int part = 1; part < n; part++)
    // iterating from 1 through n
    for (int i = 1; i <= n; i++)
      // partitions(i) = partitions[i] + partitions[i - part]
        // (for only parts that have already been processed for smaller i)
      if (i - part >= 0)
        partitionSizes[i] += partitionSizes[i - part];

  // return partitions of n
  return partitionSizes[n];
}
```

Calling `nPartitions(100)` returns 190569291.

Exercise 16 (Programming exercise)**.** *The most naive way of computing n^{15} requires fourteen multiplications:*

$$n \times n \times \cdots \times n \;=\; n^{15}$$

But using a "binary" method you can compute it in six multiplications:

$$\begin{aligned} n^2 &= n \times n \\ n^4 &= n^2 \times n^2 \\ n^8 &= n^4 \times n^4 \\ n^{12} &= n^8 \times n^4 \\ n^{14} &= n^{12} \times n^2 \\ n^{15} &= n^{14} \times n \end{aligned}$$

However it is yet possible to compute it in only five multiplications:

$$\begin{aligned} n^2 &= n \times n \\ n^3 &= n^2 \times n \\ n^6 &= n^3 \times n^3 \\ n^{12} &= n^6 \times n^6 \\ n^{15} &= n^{12} \times n^3 \end{aligned}$$

We shall define $m(k)$ to be the minimum number of multiplications to compute n^k, for example $m(15) = 5$.

For $1 \le k \le 200$, find $\sum m(k)$.

Exercise 17 (Programming exercise)**.** *Looking at the table below, it is easy to verify that the maximum possible sum of adjacent numbers in any direction (horizontal, vertical, diagonal, or anti-diagonal) is $16 (= 8 + 7 + 1)$.*

-2	5	3	2
9	-6	5	1
3	2	7	3
-1	8	-4	8

Now, let us repeat the search, but on a much larger scale:

First, generate four million pseudo-random numbers using a specific form of what is known as a "Lagged Fibonacci Generator":

- *For* $1 \le k \le 55$, $s_k = [100003 - 200003k + 300007k^3] \pmod{1000000} - 500000$.
- *For* $56 \le k \le 4000000$, $s_k = [s_{k-24} + s_{k-55} + 1000000] \pmod{1000000} - 500000$

Thus, $s_{10} = -393027$ *and* $s_{100} = 86613$.

The terms of s *are then arranged in a* 2000×2000 *table, using the first* 2000 *numbers to fill the first row (sequentially), the next* 2000 *numbers to fill the second row, and so on.*

Finally, find the greatest sum of (any number of) adjacent entries in any direction (horizontal, vertical, diagonal, or anti-diagonal).

Solution. We first generate the matrix of specified size `sz`× `sz` using the given Lagged Fibonacci Generator in the `generateMatrix` function. Then iterating over the matrix, we generate all the matrix's rows, columns, diagonals and anti-diagonals and call on each `checkArrayAndUpdateMax`, which uses the algorithm designed in Exercise 2 (see Appendix I for code) to find the sum of a maximal subarray of given vector.

```cpp
#include <vector>
#include<iostream>
using namespace std;

#include <boost/multiprecision/cpp_int.hpp> // for large ints, from the non-standard boost library
using namespace boost::multiprecision;

vector<vector<int>> generateMatrix(int sz);
vector<int> maxSubArray(const vector<int>& A);
  // from exercise 2, for finding maximal subarray

void checkArrayAndUpdateMax(const vector<int>& arr, int& max) {
  // checks input array for sum of maximal subarray (arrMax) and updates max is arrMax > max

  // computing sum of maximal subarray
  int arrMax = maxSubArray(arr)[2];
  // updating max if necessary
  if (arrMax > max)
    max = arrMax;
}

int greatestAdjacentEntrySum(int sz) {
  // returns the greatest sum of any number of adjacent entries (horizontal, vertical, diagonal, or anti-diagonal)
  // in a matrix of size sz*sz (assumed sz <= 2000)

  vector<vector<int>> matrix = generateMatrix(sz);

  int max = matrix[0][0]; // tracks the greatest adjacent entry sum

  // checking rows
  for (int row = 0; row < sz; row++)
    checkArrayAndUpdateMax(matrix[row], max);
```

```cpp

// checking columns
for (int col = 0; col < sz; col++) {
  // creating column vector
  vector<int> column(sz, 0);

  // iterating over rows to insert elements
  for (int row = 0; row < sz; row++)
    column[row] = matrix[row][col];

  checkArrayAndUpdateMax(column, max);
}

// checking diagonals starting in top row
for (int startCol = 0; startCol < sz; startCol++) {

  // creating diagonal vector
  vector<int> diagonal; diagonal.reserve(sz);

  // iterating over rows to insert elements
  for (int rowNum = 0; startCol + rowNum < sz; rowNum++)
    diagonal.push_back(matrix[rowNum][startCol + rowNum]);

  checkArrayAndUpdateMax(diagonal, max);
}

// checking diagonals starting in leftmost column
for (int startRow = 1; startRow < sz; startRow++) {
    // starting row at 1 since diagonal starting at [0][0] already checked
  // creating diagonal vector
  vector<int> diagonal; diagonal.reserve(sz);

  // iterating over columns to insert elements
  for (int colNum = 0; startRow + colNum < sz; colNum++)
    diagonal.push_back(matrix[startRow + colNum][colNum]);

  checkArrayAndUpdateMax(diagonal, max);
}


// checking anti-diagonals starting in top row
for (int startCol = sz - 1; startCol >= 0; startCol--) {

  // creating anti-diagonal vector
  vector<int> antiDiagonal; antiDiagonal.reserve(sz);

  // iterating over rows to insert elements
  for (int rowNum = 0; startCol - rowNum >= 0; rowNum++)
    antiDiagonal.push_back(matrix[rowNum][startCol - rowNum]);

  checkArrayAndUpdateMax(antiDiagonal, max);
}

```

```
  // checking anti-diagonals starting in rightmost column
  for (int startRow = 1; startRow < sz; startRow++) {

    // creating anti-diagonal vector
    vector<int> antiDiagonal; antiDiagonal.reserve(sz);

    // iterating over columns to insert elements
    for (int colNum = 0; startRow + colNum < sz; colNum++)
      antiDiagonal.push_back(matrix[startRow + colNum][sz - 1 - colNum]);

    checkArrayAndUpdateMax(antiDiagonal, max);
  }

  return max;
}

vector<vector<int>> generateMatrix(int sz) {
  // generates a matrix of size sz*sz (assumed sz <= 2000) using lagged fibonacci numbers

  // creating a vector of generated numbers
  vector<int> generatedNums(sz*sz, 0);

  // itearting to sz^2 times to generate all numbers
  for (int k = 1; k <= sz * sz; k++) {
    cpp_int Sk;   // cpp_int is variable size
    // generating Sk
    if (k <= 55)
      Sk = (100003 - 200003 * k + 300007 * cpp_int(pow(k, 3))) % 1000000 - 500000;
    else
      Sk = (generatedNums[k - 25] + generatedNums[k - 56] + 1000000) % 1000000 - 500000;
    // inserting Sk into generatedNums
    generatedNums[k - 1] = int(Sk);
  }

  // creating a matrix from generatedNums
  vector<vector<int>> matrix(sz, vector<int>(sz, 0));
    // initialising to a matrix of zeros

  int k = 0;
  for (int rowNum = 0; rowNum < sz; rowNum++) {
    // iterating over columns
    for (int colNum = 0; colNum < sz; colNum++) {
      // inserting generatedNums[k]
      matrix[rowNum][colNum] = generatedNums[k];
      k++;  // incrementing k
    }
  }
  return matrix;
}
```

Calling `greatestAdjacentEntrySum(2000)` returns 52852124.

APPENDIX

I. C++ code for Exercise 2:

```
#include <vector>
using namespace std;

vector<int> maxSubArray(const vector<int>& A) {
  // returns a vector { startingIndex, endingIndex, sum } to represent maximal subarray
  // finding maximal subarray starting at index 1

  int i2j_startIndex = 0;   // starting index of maximum subarray ending at j
  int i2j_sum = A[0];      // sum of maximum subarray ending at j
  int one2j_startIndex = 0; // starting index of maximum subarray of A[1 .. j]
  int one2j_endIndex = 0;   // ending index of maximum subarray of A[1 .. j]
  int one2j_sum = A[0];    // sum of maximum subarray of A[1 .. j]

  // iterating from j + 1 to n (= A.length)
  for (int j = 0; j < A.size() - 1; j++) {

    // updating i2j_sum, startIndex for j+1
    // (finding max subarray ending at j + 1)
    if (i2j_sum > 0) {
      // max subarray ending at j + 1 is max subarray ending at j, plus A[j+1]
      i2j_sum += A[j + 1];
    }
    else {
      // max subarray ending at j+1 is simply A[j+1]
      i2j_startIndex = j + 1;
      i2j_sum = A[j + 1];
    }

    // updating one2j_startIndex, endIndex and sum
    // (finding max subarray of A[1 .. j + 1])
    if (i2j_sum > one2j_sum) {
      // max subarray of A[1 .. j + 1] is max subarray ending at j + 1
      one2j_startIndex = i2j_startIndex;
      one2j_endIndex = j + 1;
      one2j_sum = i2j_sum;
    }
    // else, max subarray of A[1 .. j + 1] is max subarray of A[1 .. j]
      // no need to update anything
  }

  // at end of loop, j = A.length
  return vector<int> {one2j_startIndex + 1, one2j_endIndex + 1, one2j_sum};
    // returning 1-based indices
}
```

II. C++ code (brute force algorithm) for Exercise 14:

```
#include <vector>
using namespace std;

vector<int> generatePrimes(int primeBound); // used by brute force algorithm

int maxEulerTotientRatioBruteForce(int bound) {
  // returns that maximum value of n / totient(n) for n <= bound
  // brute force algorithm

  vector<int> primes = generatePrimes(bound);

  // vector of prime factors
  vector<double> totientRatios; // totientRatios[k] is k/totient(k)
  totientRatios.push_back(0); totientRatios.push_back(0);
    // setting base cases 0 and 1

  int bestNum = 0;
  double bestTotientRatio = 0;

  // iterating up to bound to find prime factorisations for each
  for (int n = 2; n <= bound; n++) {
    int k = n;
    // iterating until k == 1 (i.e. all primes found)
    for (int prime : primes) {
      // checking if k is divisible by prime factor
      if (n % prime == 0) {
        // reducing n (represented by k) to a number that doesn't have the same prime factor
        while (k % prime == 0)
          k /= prime;
        // finding nTotientRatio (using Euler's Product Formula)
        double nTotientRatio;
        if (k == 1)
          // n only has one prime factor
          nTotientRatio = double(prime) / (prime - 1);
        else
          // n has more than one prime factor
              // -> nTotientRatio = kTotentRatio * p / (p-1)
          nTotientRatio = totientRatios[k] * (double(prime) / (prime - 1));
        // updating bestTotientRatio and bestNum, if necessary
        if (nTotientRatio > bestTotientRatio) {
          bestTotientRatio = nTotientRatio;
          bestNum = n;
        }
        // updating totientRatios
        totientRatios.push_back(nTotientRatio);
        // don't need to check further primes
        break;
      }
    }
  }

```

```
  return bestNum;
}

vector<int> generatePrimes(int primeBound) {
  vector<int> primes;   // vector of all primes upto and including largest factor

  // searching for primes (primitive algorithm) by iterating over the range (2, primeBound) (inclusive)
  for (int n = 2; n <= primeBound; n++) {
    bool primeFound = true;
        // we assume we have a prime until and unless we find a prime factor
    int maxPrimeFactor = ceil(sqrt(n));

    // searching for prime factor
    for (int prime : primes) {
      if (n % prime == 0)
        primeFound = false;
      if (prime > maxPrimeFactor || !primeFound)
        break;
    }
    // no factor of i found -> i is a prime
    if (primeFound)
      primes.push_back(n);
  }
  return primes;
}
```