n-Body Problem: Binary Star Systems

Project Report

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UG 24

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1 Introduction

The gravitational interactions between two bodies can easily be modelled analytically. Introducing a third body, however, makes the system much more complex since every body interacts with every other body leading to a very large number of interactions. In order to model these interactions analytically, one would need to solve a large number of equations simultaneously. This problem cannot be solved analytically. Using forward integration, we can model the three-body problem and solve it computationally. This approach can also be extended to any arbitrary number of bodies for sufficient computing power.

The principle of such a solution is that knowing the state of a body at an initial time allows us to compute its position at any later time by using the laws of motion and the laws of force. Thus, in order to simulate the motion of planets around the sun, we need only know the masses of the bodies, the distances between them at an initial time and Newton's Law of Gravitation. The module used to visualise this computation is VisualPython (VPython), which is a 3-D visualisation module built upon the programming language Python.

For adequate computing power, we can model the n-body problem to any arbitrary accuracy by discretising an appropriate time step for our calculations. Using an adequately small time step (Δt), we can model the relationships between position (x(t)), velocity (v(t)) and acceleration (a(t)) for any body as shown below:

$$x(t + \Delta t) = x(t) + v(t)\Delta t \tag{1}$$

$$v(t + \Delta t) = v(t) + a(t)\Delta t \tag{2}$$

In order to find a(t), we can use the laws governing the relevant forces (in this case, Newton's Law of Gravitation) and the relation F = ma where m is the mass of the body and a is the acceleration produced as a result of the external force F. To make the results obtained through forward integration even more accurate, we can use the Leap Frog Method. In this method, we do our first velocity calculation at half the time step. This simple step increases the precision of the forward integration process.

Extending upon the ideas discussed here, I have attempted to simulate the orbits of hypothetical exoplanets in orbit around the binary star system Alpha Centauri AB in order to discover the conditions that would allow such an orbit to form.

2 Planetary Motion

Planetary motion in classical mechanics can be described using Newton's law of gravitation. The force experienced by a planet of mass M_a due to a planet of mass M_b at a distance R is given by:

$$F_{ab} = \frac{GM_aM_b}{R^2}$$

where G ($= 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$)is the universal gravitational constant.

This can be written in vector form as:

$$\vec{F}_{ab} = -\frac{GM_aM_b}{R^3}\hat{R}_{ba}$$

From the laws of motion, we know that:

$$F = M_a a \implies a = \frac{F}{M_a}$$

Combining the two, we get an equation for the acceleration produced:

$$a = \frac{GM_b}{R^2} \tag{3}$$

Thus, we can use gravitation to find the value of acceleration. Through forward integration, we can use this to find the value of velocity at any given instant. This in turn allows us to find the position of the body. Further, gravitational forces obey the principle of superposition according to which the net gravitational force experienced by a body in the presence of other bodies is given by the sum of the gravitational force experienced by it due to each of the bodies individually. Thus, for a many body system, the forces due to separate bodies can be simply added up to find the total acceleration produced.

For instance, in order to simulate the Earth's orbit around the Sun, the following information is needed:

Body	Mass (in kg)	Initial Distance from Origin (in m)
Sun	1.99e30	0
Earth	5.97e24	1.49e11

Here, the Sun is taken as the origin (0,0,0) of the system.

Using these, the average velocities for each body may be calculated by velocity $= \frac{\text{circumference}}{\text{time period}}$ $\implies$ velocity $= \frac{2\pi R}{T}$ where R is the radius of the orbit and T is the period of the orbit. Further, the orbit of the Earth around the sun can be traced out through forward integration as per equations (1) and (2).

Here, the Sun is shown in yellow and the Earth's orbit is shown in green.

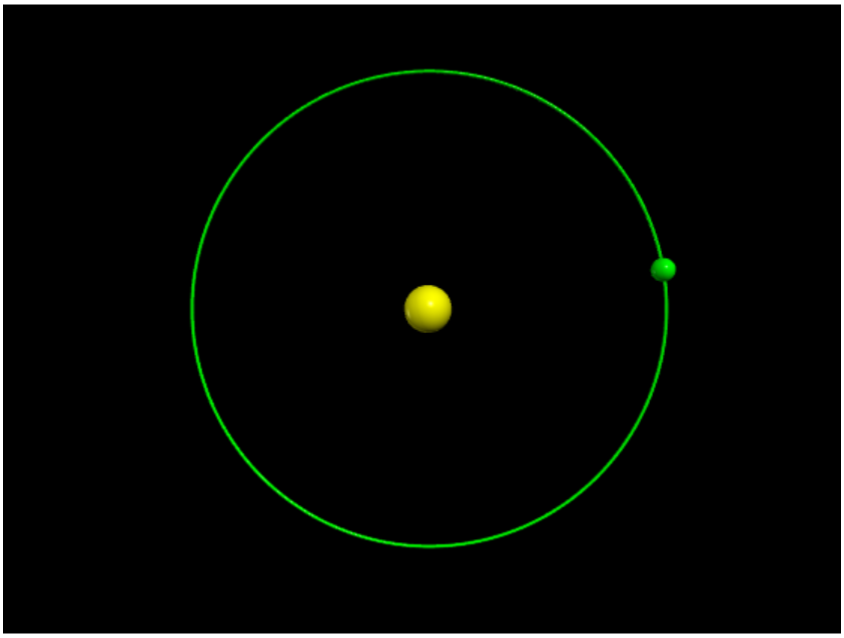

Figure 1: Simulation of the Earth's orbit around the Sun

2.1 Simulating the Sun-Earth-Moon System

This is a particularly interesting system to simulate since the Moon orbits the Earth which in turn orbits the Sun. Even though the Sun is a more massive body than the Earth, the Moon orbits the Earth because it falls within the Hill sphere of the Earth. The Hill sphere of a planet is the region within which it dominates the gravitational attraction of smaller bodies such as satellites. Thus, despite the Sun being much more massive than the Earth, the Moon orbits the Earth rather than the Sun.

In its revolution around the Earth, the Moon also follows the Earth in its orbit around the Sun. Thus, the Moon has two velocities- one associated with its orbit around the Earth and another associated with its orbit around the Sun. The code for the animation must thus factor in both these velocities in order to predict the orbit of the moon as shown below.

Here, the Sun is taken as the origin (0,0,0) of the system. The Sun is shown in yellow, the Earth's orbit is shown in green and the Moon's orbit is shown in white.

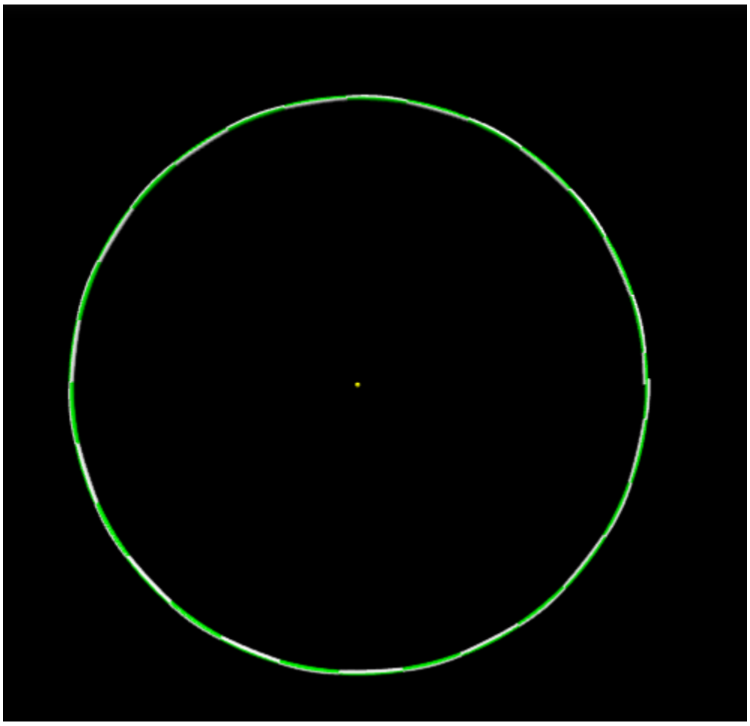

Figure 2: Orbits of the Earth and Moon around the Sun

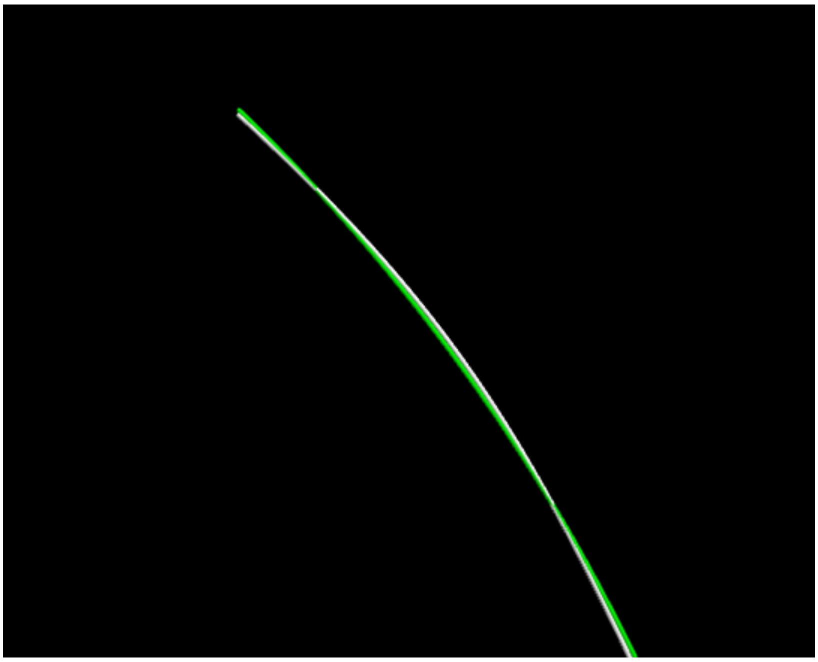

Figure 3: Orbit of the Moon around the Earth while the Earth moves in its orbit

3 Simulating a Binary Star System

Alpha Centauri is a triple star system located 4.47 light years from the Sun. It is a hierarchical system with Alpha Centauri A and B forming the inner binary system and Proxima Centauri (or Alpha Centauri C) orbiting almost 13,000 AU from the binary system. Proxima Centauri's orbit around the AB system has a period of about 550,000 years.

I first attempted to simulate the orbit of the inner binary system (Alpha Centauri AB). The two stars orbit each other at a distance of $\sim$ 11.2 AU with an orbital period of $\sim$ 79 years. Alpha Centauri A is the primary star in the system. The two stars orbit a common centre. Using their initial positions, orbital period and masses as listed below, I obtained their orbit.

Here, Alpha Centauri A is taken as the origin (0,0,0) of the system.

Body	Mass (in kg)	Initial Distance from Origin (in m)
Alpha Centauri A	2.188e30	0
Alpha Centauri B	1.804e30	1.68e12

Alpha Centauri A's orbit is shown in yellow, Alpha Centauri B's orbit is shown in red.

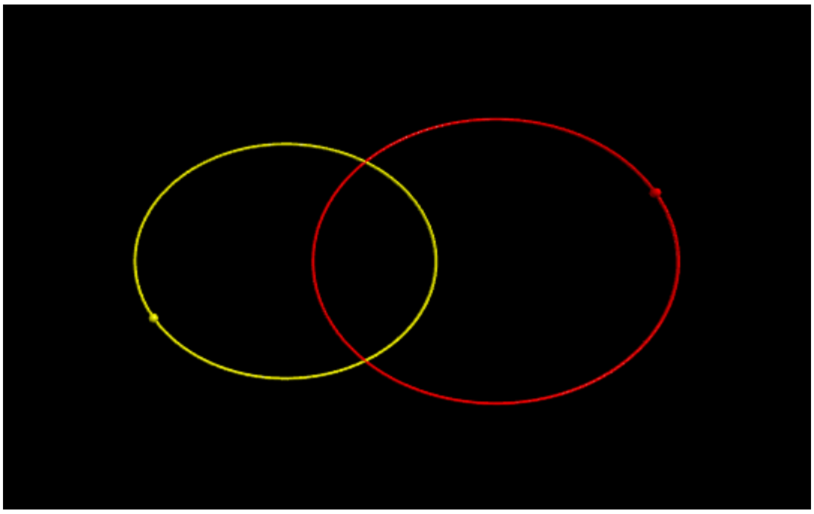

Figure 4: Alpha Centauri A and B in orbit around each other

Since Alpha Centauri C is quite distant from the main binary system, it is not possible to view its complete orbit in a simulation with a smaller time step due to computational restraints. Upon increasing the time step, the orbits of the binary system start showing significant errors. Its orbit can be simulated by approximating the inner binary system to a point mass.

The focus of this report, however, is the possibility of finding planets in orbit around the Alpha Centauri AB system.

Candidate A: The Alpha Centauri stellar system forms one of the best-suited zones for imaging habitable-zone exoplanets. Further, the primary stars (A and B) are similar in mass and temperature to the Sun. In 2021, a possible exoplanet in orbit around Alpha Centauri A was imaged by a team of astronomers observing the habitable zone around the star. This planet candidate, called 'Candidate 1 (C1)' or 'Alpha Centauri Ab' was observed using the Very Large Telescope (VLT) and is yet to be confirmed. It presents a fascinating case study since planets around a binary star are rare to find. There are no planets currently known (and confirmed) in orbit around either star.

Planets that orbit only one of the stars in a binary system are called non-circumbinary planets (or S-type planets). There is some contention as to whether planets can form in binary systems as the gravitational forces may interfere with planetary formation. Alpha Centauri AB is an exception in the sense that it has a stable habitable zone. It is believed that there could be planets in orbit around either or both of the stars. At least 2 planets are believed to be in orbit around Proxima Centauri, the tertiary star in the system.

Due to the recent nature of C1's discovery, its exact physical and orbital properties are not known. It is posited to have a mass similar to that of Neptune ($\sim$ 1.04e26) and is not further than 1.1 AU from Alpha Centauri A.

I simulated the motion of C1 around the Alpha Centauri AB system with several variations of these parameters as shown below.

Symbols used: M_e = 5.97e24 kg, AU = 1.49e11 m,

Alpha Centauri A's orbit is shown in yellow, Alpha Centauri B's orbit is shown in red. C1's orbit is shown in green.

1. Mass: $20M_e$, Radius of orbit: 0.9AU

These parameters give an orbital period of 57.3 days.

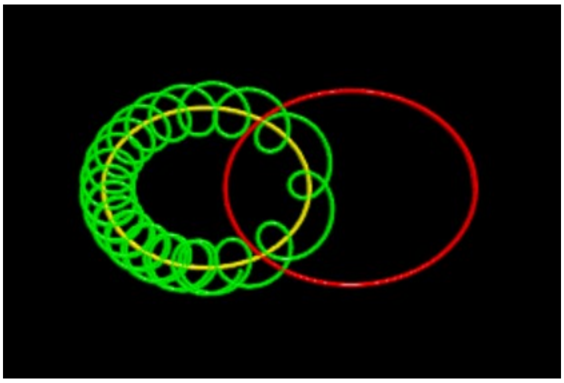

Figure 5: Possible orbit of C1: Case 1

2. Mass: $20M_e$, Radius of orbit: 1 AU

These parameters give an orbital period of 57.3 days.

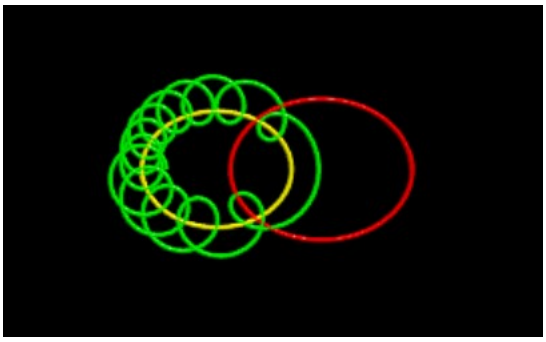

Figure 6: Possible orbit of C1: Case 2

3. Mass: $20M_e$, Radius of orbit: 1.05 AU

In this case, the planet's orbit is highly unstable. It starts out in orbit around Alpha

Centauri A but then enters into orbit around Alpha Centauri B before finally leaving the system. It is thus unlikely that these parameters represent C1.

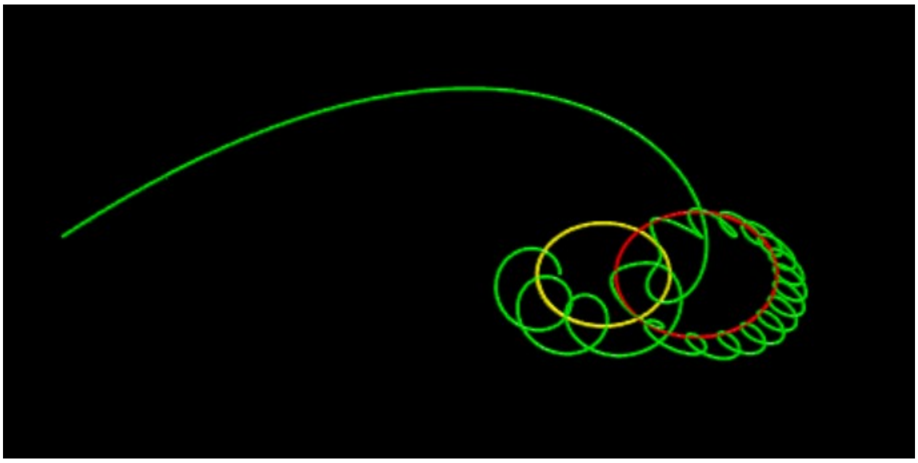

Figure 7: Possible orbit of C1: Case 3

4. Mass: $20M_e$, Radius of orbit: 1.1 AU

These parameters also lead to an unstable orbit where the planet leaves the system before completing an orbit. It is thus unlikely that these parameters represent C1.

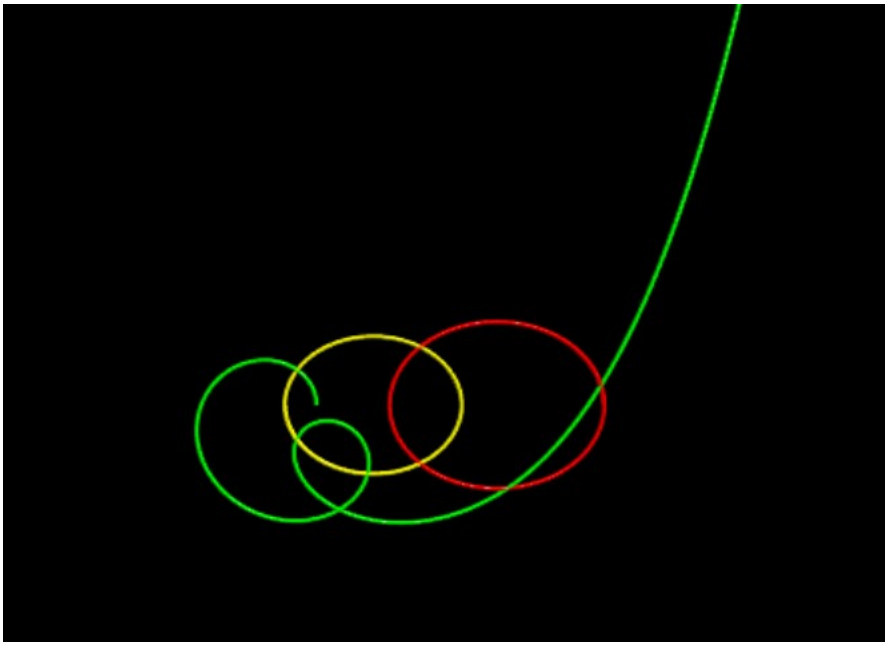

Figure 8: Possible orbit of C1: Case 4

5. Mass: $50M_e$, Radius of orbit: 0.9 AU

These parameters lead to an orbit very similar to the one observed in case (1).

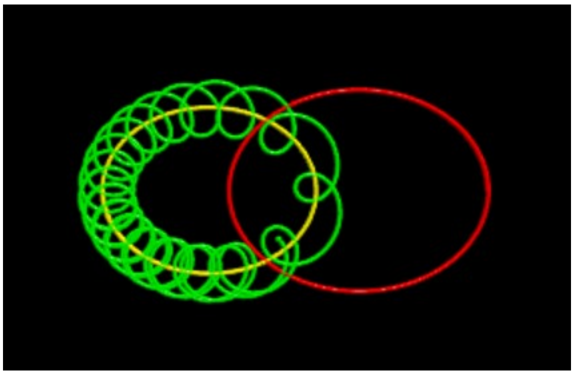

Figure 9: Possible orbit of C1: Case 5

It is found that changing the mass of the exoplanet does not have a significant impact on its orbit. The mass of C1 varies greatly between cases (1) and (5) for the same distance from Alpha Centauri A but the orbits are found to be highly similar. I believe that this is because the change in the mass of C1 is not comparable to the masses of the stars around it. Thus, this change does not produce a noticeable difference.

An interesting feature in all of the above cases is that C1's orbit around Alpha Centauri A changes its radius depending upon the distance between C1 and Alpha Centauri B. This is the most fascinating feature of the three-body problem: not only is C1's orbit affected by Alpha Centauri A, but it is also greatly affected by Alpha Centauri B.

From these results, the orbital separation observed in (1) and (5) seems to represent the most realistic orbit for C1. Using these parameters of an orbital separation of 0.9 AU, I also simulated the motion of a similar exoplanet around Alpha Centauri B.

Mass: $10M_e$, Radius of orbit: 0.9 AU

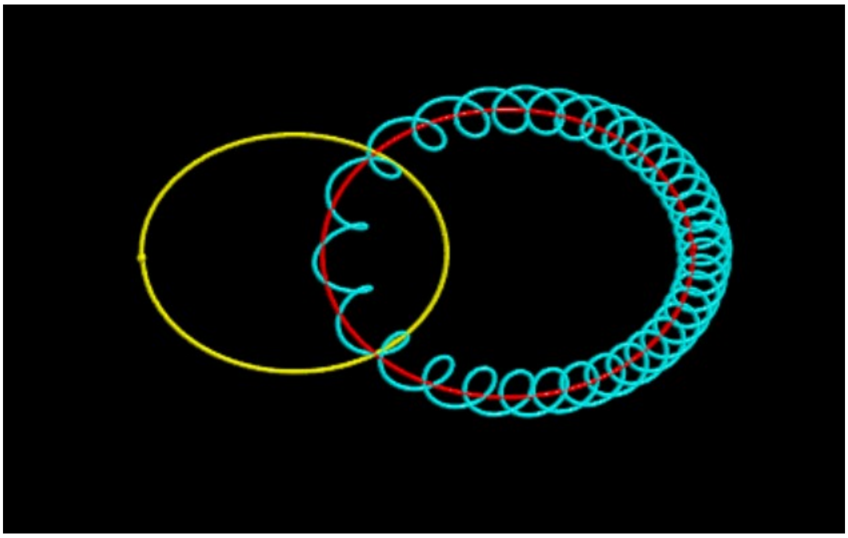

Figure 10: Possible orbit of an exoplanet around Alpha Centauri B

Combining both C1 and the exoplanet around Alpha Centauri B, we get an interesting system with a non-circumbinary planet in orbit around each star of a binary system. It is evident from the obtained orbit that the orbit of each planet is affected not just by its own star but also by the other star. This impact is largest when the two are closest together.

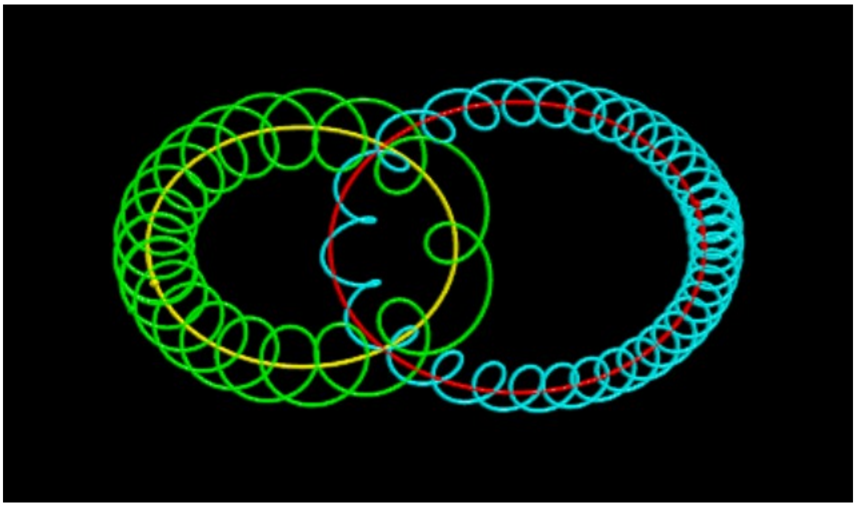

Figure 11: Possible orbit of exoplanets around Alpha Centauri A and B

At the completion of one orbit, it is found that the relative change in the total energy of the system is -0.14%. This is indicative of the error accumulation in the process of forward integration.

Another interesting feature in their orbits is that they do not repeat their orbits exactly. The orbits shift by a little each time but the planets remain within the gravitational influence of their respective stars.

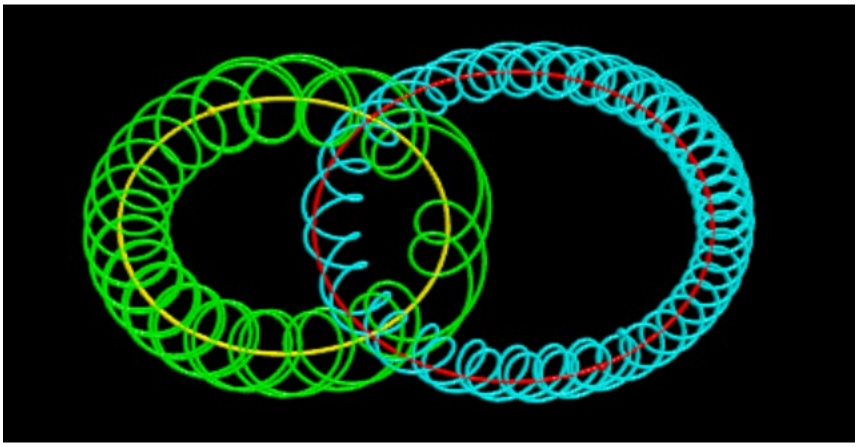

Figure 12: Changes observed in the orbits of S-type planets in orbit around Alpha Centauri A and B

4 Discussion: Circumbinary Planets

A circumbinary planet is a planet that orbits a binary system of two stars instead of one star. The two stars in the binary system orbit each other and the planet orbits them at a greater distance. It is believed that the planets and the stars in such systems are formed from the same disk.

Circumbinary planets usually orbit the stellar system with a semi-major axis equal to about two to four times the stellar separation with an orbital period about three to eight times larger than the binary period. Due to these constraints on the orbital distance, circumbinary planets are often found in the habitable zone of the binary systems.

After completing the above simulations on S-type planetary orbits around Alpha Centauri AB, I thought it might be interesting to see the characterstics necessary for a circumbinary planetary orbit around the system.

The motion of a circumbinary planet can be modelled as equivalent to a planet orbiting the centre of mass of the binary stellar system. Let m be the mass of the planet and R be the distance from the planet to the centre of mass of the stellar system. Let R_a be the distance between the planet and the primary star (of mass M_a) and R_b be the distance between the planet and the secondary star (of mass M_b). v is the orbital velocity of the planet.

Now, for a stable planetary orbit, the centripetal force of the centre of mass must balance out the gravitational attraction of the binary stars. Thus,

$$\frac{mv^2}{R} = Gm\left[\frac{M_a}{R_a^2} + \frac{M_b}{R_b^2}\right]$$
$$\implies v^2 = GR\left[\frac{M_a}{R_a^2} + \frac{M_b}{R_b^2}\right] \tag{4}$$

Using this equation, we can find the velocity for any planet to be in a stable circumbinary orbit of a given semi-major axis around the binary system. Using this framework, I simulated the motion of a planet 22.4 AU away from the binary system with mass approximately equal to the Earth's. I chose this separation since it falls within the critical radius requirements discussed above. The following orbit was observed:

(Alpha Centauri A's orbit is shown in yellow, Alpha Centauri B's orbit is shown in red. The orbit of the circumbinary planet is shown in cyan.)

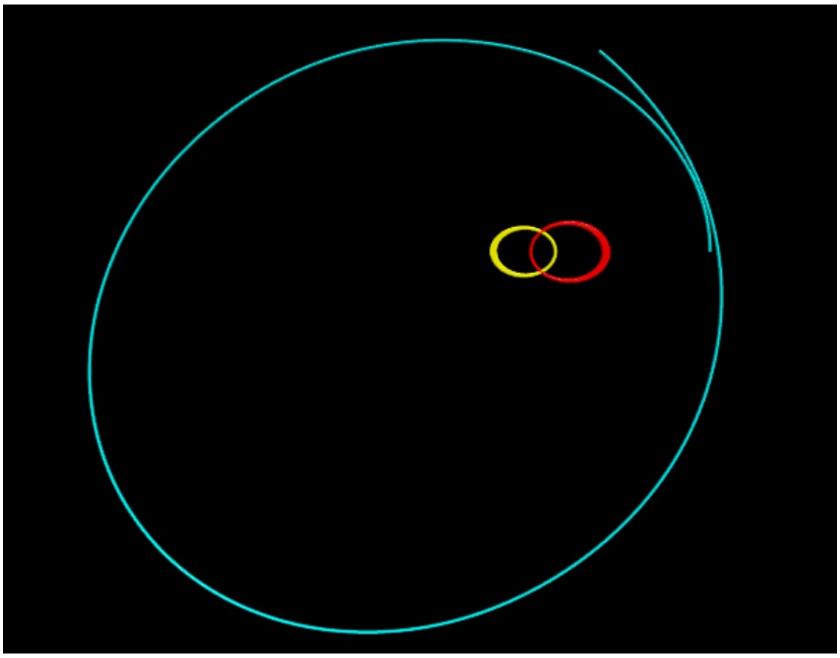

Figure 13: Hypothetical Planet in circumbinary orbit around Alpha Centauri AB

The plantery orbit is not found to be perfectly closed. This could be a consequence of the time step used for the simulation.

5 References

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