LEMMA 7 (EXTENSIONS):

The Continuum Limit

Extending the Action Quota to Phase Space via Group Contraction

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Abstract

We extend the reconstruction from finite-dimensional spins (qubits) to continuous variables. We demonstrate that the canonical Heisenberg relation $[X, P] = i\hbar_{\text{phys}}$ arises as the tangent-space (flat) limit of a global $SU(2)$ geometry. By applying the **Wigner–Inönü group contraction** in the limit of large system size ($J \to \infty$), we show that the Action Quota established in Lemma 1 naturally manifests as the canonical commutator and its associated uncertainty bounds. This extension reveals that the uncertainty principle represents the physical persistence of state-space curvature in the tangent plane, providing a unified structural bridge between discrete and continuous quantum systems.

Keywords: Group contraction, phase space, Heisenberg algebra, canonical quantization, Action Quota, $SU(2)$, Wigner–Inönü, Large-J limit.

1 Introduction: Role in the Reconstruction

Lemmas 1–6 established the geometry and dynamics of finite-dimensional quantum systems (qubits) based on the **Action Quota**: a fundamental budget on certainty. However, macroscopic physics operates in phase spaces where coordinates and momenta can be arbitrarily large.

Lemma 7 bridges this gap. We show that the Heisenberg algebra $\mathfrak{h}_1$ of continuous variables is not a new postulate, but the result of zooming into a local patch of a high-dimensional spin manifold. As the system size J increases, the "sphere" of information becomes locally indistinguishable from a "plane." We derive the specific rescaling required to preserve the Action Quota in this limit, ensuring that the fundamental non-commutativity of quantum theory remains physically relevant even as coordinates become unbounded.

2 The Contraction Map

Consider a generalized spin system with total angular momentum J, characterized by the compact Lie algebra $\mathfrak{su}(2)$. The generators $\{J_x, J_y, J_z\}$ satisfy:

$$[J_x, J_y] = i\hbar_{\text{spin}} J_z, \quad [J_y, J_z] = i\hbar_{\text{spin}} J_x, \quad [J_z, J_x] = i\hbar_{\text{spin}} J_y. \tag{1}$$

Remark 2.1 (Normalization). *The constant $\hbar_{spin}$ fixes the $\mathfrak{su}(2)$ commutator normalization. In the usual angular-momentum convention, $\hbar_{spin} = \hbar$, but we keep $\hbar_{spin}$ distinct from the emergent phase-space scale $\hbar_{phys} = L_0 P_0$ until Lemma 8.*

Definition 2.1 (Contraction Rescaling). *To obtain the flat limit, we define a J-dependent linear map $\Phi_J : \mathfrak{su}(2) \to \mathfrak{h}_1$. For a polarized system near the north pole ($J_z \approx J\hbar_{spin}$), we define dimensionless contracted coordinates:*

$$x_J := \frac{J_x}{\hbar_{spin}\sqrt{J}}, \qquad p_J := \frac{J_y}{\hbar_{spin}\sqrt{J}}. \tag{2}$$

We may view this as the linear map Φ_J sending $(J_x, J_y, J_z) \mapsto (x_J, p_J, \mathbb{I})$ on the north-pole sector, with the central element realized by $J_z/(J\hbar_{spin}) \to \mathbb{I}$. Physical position X and momentum P are obtained by introducing a length scale L_0 and momentum scale P_0 such that $X = L_0 x_J$ and $P = P_0 p_J$.

Remark 2.2 (The Action Scale). *The physical action scale $\hbar_{phys}$ is defined by the product of the coordinate scales: $\hbar_{phys} := L_0 P_0$. This constant identifies the "area" in phase space corresponding to a single quantum of information.*

3 Derivation of the Heisenberg Algebra

Definition 3.1 (North-pole sector). *Let $\mathcal{H}_J$ be the spin-J Hilbert space and define the excitation operator*

$$N_J := J\mathbb{I} - \frac{J_z}{\hbar_{spin}},$$

so that on the standard eigenbasis $|J, m\rangle$ we have $N_J |J, m\rangle = (J - m) |J, m\rangle$. For a fixed integer $n_{\max} \geq 0$, define the low-excitation subspace $\mathcal{K}_J(n_{\max}) := \mathrm{span}\{|J, J - n\rangle : 0 \leq n \leq n_{\max}\}$. We call any family of states $\psi_J \in \mathcal{K}_J(n_{\max})$ a north-pole family.

Theorem 3.1 (Continuum limit (strong on the north-pole sector)). *Let x_J, p_J be defined by (2), and set $X = L_0 x_J$, $P = P_0 p_J$ with $\hbar_{phys} = L_0 P_0$. Then the exact finite-J identity holds:*

$$[X, P] = i\hbar_{phys}\Big(\mathbb{I} - \frac{N_J}{J}\Big), \qquad N_J := J\mathbb{I} - \frac{J_z}{\hbar_{spin}}.$$

Consequently, for every fixed $n_{\max}$ and every north-pole family $\psi_J \in \mathcal{K}_J(n_{\max})$,

$$\Big\|([X, P] - i\hbar_{phys}\mathbb{I})\psi_J\Big\| = \frac{\hbar_{phys}}{J} \|N_J \psi_J\| \xrightarrow[J\to\infty]{} 0,$$

so $[X, P] \to i\hbar_{phys}\mathbb{I}$ strongly on $\mathcal{K}_J(n_{\max})$.

Proof. Using $[J_x, J_y] = i\hbar_{\rm spin} J_z$ and (2),

$$[X, P] = L_0 P_0 [x_J, p_J] = \hbar_{\rm phys} \frac{1}{J\hbar_{\rm spin}^2} [J_x, J_y] = i\hbar_{\rm phys} \frac{J_z}{J\hbar_{\rm spin}}.$$

Since $J_z = \hbar_{\rm spin}(J\mathbb{I} - N_J)$ by the definition of N_J, we obtain $[X, P] = i\hbar_{\rm phys}(\mathbb{I} - N_J/J)$. The stated bound follows by taking norms and restricting to the sector $\mathcal{K}_J(n_{\max})$. □

Remark 3.1 (Uniqueness of $\sqrt{J}$ Scaling). *The $1/\sqrt{J}$ scaling is uniquely determined by requiring: (1) finite variances in the limit; (2) a non-vanishing commutator; and (3) correspondence with the classical limit where non-commutativity persists as a finite constant.*

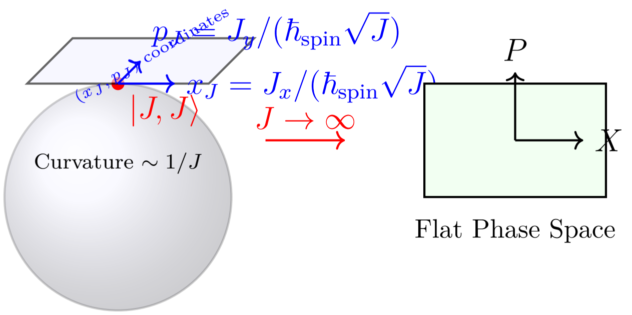

Figure 1: The Wigner–Inönü contraction. As $J \to \infty$, the $SU(2)$ sphere's radius becomes infinite, and the local tangent space near the pole becomes the Heisenberg phase space.

4 From Bounded to Unbounded Spectra

For finite J, J_x has eigenvalues $m\hbar_{\text{spin}}$ with $m \in \{-J, \dots, J\}$. The contracted operator x_J has eigenvalues in $\text{spec}(x_J) \subseteq [-\sqrt{J}, \sqrt{J}]$. As $J \to \infty$, these intervals expand to cover the entire real line $\mathbb{R}$, producing the unbounded operators of continuous-variable quantum mechanics.

5 Survival of the Action Quota

The variance bound establishes that non-classicality is preserved in the flat limit.

Proposition 5.1 (Action Quota in the continuum (Robertson bound))**.** *On the north-pole sector where $[X, P] \approx i\hbar_{phys}\mathbb{I}$, the standard Robertson inequality yields*

$$\Delta X \, \Delta P \geq \frac{1}{2} \, | \, \langle [X, P] \rangle \, | = \frac{\hbar_{phys}}{2} \left| 1 - \frac{\langle N_J \rangle}{J} \right| .$$

In particular, if $\psi_J \in \mathcal{K}_J(n_{\max})$ then $\langle N_J \rangle \leq n_{\max}$, so $\Delta X \, \Delta P \geq \frac{\hbar_{phys}}{2} \left(1 - \frac{n_{\max}}{J}\right)$ and hence $\Delta X \, \Delta P \to \hbar_{phys}/2$ as $J \to \infty$.

Remark 5.1 (Optional sum-form bound)**.** *By the AM–GM inequality, $\frac{\text{Var}(X)}{L_0^2} + \frac{\text{Var}(P)}{P_0^2} \geq 2\sqrt{\frac{\text{Var}(X)}{L_0^2} \frac{\text{Var}(P)}{P_0^2}} = 2\frac{\Delta X \, \Delta P}{\hbar_{phys}}$, so the Robertson bound implies a corresponding sum-form lower bound consistent with Lemma 1.*

6 Limits of Validity and Geometric Corrections

The exact Heisenberg relation $[X, P] = i\hbar_{\text{phys}}$ is a tangent-space approximation. The departure from exact CCR at finite J is controlled by the excitation ratio N_J/J: the curvature of the compact $SU(2)$ manifold manifests as the multiplicative correction $[X, P] = i\hbar_{\text{phys}}(\mathbb{I} - N_J/J)$, so flat phase space is valid precisely when $N_J \ll J$.

Proposition 6.1 (Curvature corrections (Holstein–Primakoff interpretation))**.** *On the low-excitation sector where the Holstein–Primakoff mapping is valid, $N_J \approx b^\dagger b$, so the exact finite-J identity $[X, P] = i\hbar_{phys}(\mathbb{I} - N_J/J)$ reads*

$$[X, P] \approx i\hbar_{phys} \left(1 - \frac{b^\dagger b}{J}\right) .$$

Thus the Heisenberg algebra is recovered whenever $b^\dagger b \ll J$, while highly excited states probe the underlying compact curvature through the factor $1 - b^\dagger b/J$.

7 Conclusion

Lemma 7 proves that continuous-variable quantum mechanics is a direct consequence of the Action Quota. The Heisenberg algebra is the unique "flat" limit of the $\mathfrak{su}(2)$ algebra that preserves the non-classical uncertainty relations.

Feature	**Discrete (Spin)**	**Contraction map**	**Continuous**
Algebra	$[J_x, J_y] = i\hbar_{\text{spin}} J_z$	$\frac{1}{J\hbar^2_{\text{spin}}}[J_x, J_y]$	$[X, P] = i\hbar_{\text{phys}}$
Certainty	$\text{Var}(J_x) + \text{Var}(J_y) \geq \ldots$	Rescale variances	$\Delta X \Delta P \geq \hbar_{\text{phys}}/2$
Manifold	Compact (Sphere)	$J \to \infty$	Flat (Plane)

Table 1: Summary of the transition from discrete spins to the continuous phase space.

Outlook This derivation provides the geometric unification of spin and continuous systems. Lemma 8 will perform the final calibration, fixing $\hbar_{\text{phys}} = \hbar$ by matching macroscopic thermodynamic constants to the predicted spectrum of field modes.

References

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A Formal limit of the commutator

A. Uniqueness of the $1/\sqrt{J}$ scaling

Let $X = c(J)J_x$ and $P = c(J)J_y$. Then

$$[X, P] = c(J)^2[J_x, J_y] = i\hbar_{\text{spin}}\, c(J)^2\, J_z.$$

On the north-pole sector (where $J_z \sim J\hbar_{\text{spin}}$) the commutator scale is $\hbar^2_{\text{spin}}\, J\, c(J)^2$. For a nonzero finite limit we require $Jc(J)^2 \to \text{const}$, hence $c(J) \propto 1/\sqrt{J}$.

B. Strong CCR limit on low excitations

Define $x_J = \frac{J_x}{\hbar_{\text{spin}}\sqrt{J}}$, $p_J = \frac{J_y}{\hbar_{\text{spin}}\sqrt{J}}$, $X = L_0 x_J$, $P = P_0 p_J$, $\hbar_{\text{phys}} = L_0 P_0$, and the excitation operator $N_J := J\mathbb{I} - \frac{J_z}{\hbar_{\text{spin}}}$. Using $[J_x, J_y] = i\hbar_{\text{spin}} J_z$ we compute

$$[X, P] = L_0 P_0 [x_J, p_J] = \hbar_{\text{phys}} \frac{1}{J\hbar_{\text{spin}}^2} [J_x, J_y] = i\hbar_{\text{phys}} \frac{J_z}{J\hbar_{\text{spin}}} = i\hbar_{\text{phys}} \left(\mathbb{I} - \frac{N_J}{J}\right).$$

Therefore, for any $\psi \in \mathcal{H}_J$, $\|([X, P] - i\hbar_{\text{phys}}\mathbb{I})\psi\| = \hbar_{\text{phys}} \left\| \frac{N_J}{J}\psi \right\|$. If $\psi_J \in \mathcal{K}_J(n_{\max})$ (Definition 3.1), then $\|N_J\psi_J\| \le n_{\max}\|\psi_J\|$ and

$$\left\| ([X, P] - i\hbar_{\text{phys}}\mathbb{I})\psi_J \right\| \le \frac{\hbar_{\text{phys}}\, n_{\max}}{J} \|\psi_J\| \xrightarrow[J\to\infty]{} 0,$$

which is the strong convergence of $[X, P]$ to $i\hbar_{\text{phys}}\mathbb{I}$ on the north-pole sector. $\square$