Resource-Bounded Incompleteness

Operational Limits in Quota-Bounded Worlds

Part I of the Beyond the Ledger Series

Emiliano Shea

December 22, 2025

Abstract

We assume the *Action-Quota Principle*, which posits that stable physical distinctions require a finite expenditure of action. We model systems constrained by this principle as *quota-bounded worlds* $\mathsf{Q}(V, N)$, which are limited not only in memory but in total action expenditure. We prove an *Operational Bounded Diagonal Lemma*, demonstrating that for any finite resource budget N, there exists a physically specifiable unary predicate on codes that is undecidable by any procedure admissible within that budget. This yields a strict resource hierarchy: the set of decidable truths grows as action expenditure increases. We establish the *Principle of Action-Indexed Progress*, showing that extending the set of decidable truths requires a strictly non-zero investment of additional action. This implies that any finitely-resourced physical theory is necessarily operationally incomplete, necessitating the existence of an unmeasurable sector ($\mathcal{U}$) explored in subsequent parts of this series.

Contents

1 Introduction: The Pilot's Dilemma

To ensure this work is self-contained, we briefly summarize the core results of the *Action-Quota Reconstruction of Quantum Mechanics* (AQR) that serve as the foundation for this paper. The novelty here is not diagonalization per se, but the identification of physical action as the governing resource and the operational interpretation of "truth growth" as action-indexed.

In AQR, we establish that physical reality operates as a transaction between two distinct sectors, metaphorically termed the **Dreamer** and the **Pilot**.

- The **Dreamer** (Semantics) represents the unmeasurable potential of the quantum state space, where contradictions can coexist in superposition.
- The **Pilot** (Syntax) represents the finite, measurable record of facts—the actual history composed of definite outcomes.

The metaphor is motivational; the theorems below depend only on the operational model $\mathsf{Q}(V, N)$.

In this paper, we operationalize this distinction through the model of a *quota-bounded world* $\mathsf{Q}(V, N)$, where the Pilot's finite capacity is explicitly quantified.

Operationally, these correspond to the distinction between the continuous, reversible evolution of a quantum state and the discrete, irreversible recording of a measurement result. The central tenet of AQR, the **Action-Quota Principle**, states that converting potential into fact is not free; it requires paying a toll. Specifically, to stabilize one distinguishable bit of information against the entropy of the unmeasurable sector, the system must expend a minimum quantum of action. We identify this fundamental unit with the reduced Planck constant, $\hbar$.

This thermodynamic cost implies that "truth" is not an abstract property of the universe but a thermodynamically paid-for achievement [3]. A measurement is not a passive reading; it is an active inscription that consumes a finite resource.

1.1 Roadmap of the Series

This paper constitutes the logical foundation of the *Beyond the Ledger* series:

1. **Part I (Logic):** We prove that any system with a finite action budget N contains specifiable but undecidable facts, creating a "Pilot's Dilemma" [6].
2. **Part II (Ontology):** We resolve this dilemma by constructing the **Unmeasurable Sector** ($\mathcal{U}$) as an inverse limit of finite ledgers, physically realized by Infinite Derivative Field Theory (IDFT) [4].
3. **Part III (Experiment):** We propose **Quota-Critical Witnesses** to detect the saturation of these resource limits in laboratory systems like QRNGs and Fermi gases [5].

In this first paper, we treat the laboratory not as an abstract oracle, but as a *Quota-Bounded World* $\mathsf{Q}(V, N)$. We prove that for any such world, there exist operational questions that are well-posed but answering them requires "overdrawing" the ledger.

2 Formal Model: The Finite Ledger

We begin by defining the operational limits of a finite observer.

2.1 Persistence and Computation

The fundamental unit of our model is the ***h*-cell**.

Definition 2.1 (h-cell)**.** An h-cell is the physical resource required to stabilize one bit of classical information. In our model, this corresponds to the expenditure of at least one quantum of action $\hbar$.

This definition bridges the informational and physical domains. While Landauer's Principle sets a lower bound on the energy cost of erasure ($k_B T \ln 2$), the Action-Quota Principle sets a lower bound on the *action* cost of distinction.

Definition 2.2 (Persistence)**.** A record is *persistent* if maintaining recoverability at error $\leq \epsilon$ over the procedure duration requires at most a fixed maintenance overhead already charged to the budget.

This links persistence directly to the thermodynamic cost of maintenance.

2.2 Admissibility and the Description-Action Link

We define our system not just by memory, but by a total action budget, resolving potential halting paradoxes.

Definition 2.3 (Quota-bounded world $\mathsf{Q}(V, N)$)**.** A physical setup localized to a spacetime region V with a total action budget of N units (where N is dimensionless and units are scaled to $\hbar$). An *admissible procedure* is any preparation-measurement-computation protocol whose total action expenditure (including the cost of persistent memory maintenance and computation steps) does not exceed $N\hbar$.

In this framework, a procedure that runs indefinitely or requires infinite tape is physically inadmissible because it consumes infinite action. To rigorously link the informational complexity of a procedure to the physical resources of the world, we introduce the following crucial assumption.

Assumption 2.4 (Description-Action Link)**.** For a procedure to be admissible in $\mathsf{Q}(V, N)$, its complete description (code) must be physically instantiated and maintained within V. The action cost of storing and maintaining a description of length $|s|$ scales at least linearly: $\mathrm{Cost}(s) \geq \alpha |s|$ for some $\alpha > 0$. We normalize units so $\alpha = 1$. Consequently, any code s describing an admissible procedure in $\mathsf{Q}(V, N)$ satisfies $|s| \leq N$.

The linear lower bound is encoding-independent up to constants: any physical realization of a length-$|s|$ description requires $\Omega(|s|)$ persistent degrees of freedom, hence $\Omega(|s|)$ action to stabilize them.

We define two distinct sets of codes based on this link:

- $\mathrm{Code}_{\leq N} := \{s \in \{0,1\}^* : |s| \leq N\}$, the set of codes with description length within the budget.
- $\mathrm{Adm}(N)$, the set of admissible procedures in $\mathsf{Q}(V, N)$.

By the Description-Action Link, we have the inclusion:

$$\mathrm{Adm}(N) \subseteq \mathrm{Code}_{\leq N}.$$

This distinction ensures we do not assume every short code corresponds to an admissible physical procedure, only that every admissible procedure has a short code.

Assumption 2.5 (Physical Church-Turing Thesis)**.** We assume the physical dynamics within the region V are sufficiently complex to support universal computation. This means physical states can map to computational states, and dynamics can map to state transitions [7].

2.3 Specifiability vs. Decidability

Fix a *prefix-free* coding scheme for tasks and a universal interpreter U whose description costs a constant overhead c_0. We define a universal bounded evaluator $\mathrm{Eval}_N(s, x)$ here for use in subsequent sections.

Definition 2.6 (Bounded Evaluation)**.** The universal bounded evaluator $\mathrm{Eval}_N : \mathrm{Code} \times \mathrm{Code} \to \{0, 1\}$ is defined as:

$$\mathrm{Eval}_N(s, x) = \begin{cases} \text{output of } \mathsf{U}(s) \text{ on } x & \text{if } \mathsf{U}(s) \text{ halts within action budget } N \\ 0 & \text{otherwise} \end{cases}$$

To ensure this mathematical definition is physically grounded, we state:

Assumption 2.7 (Universal bounded simulation and shared accounting)**.** There exists a fixed universal physical procedure U_{phys} in V that, given (s, x, N), executes the procedure encoded by s on input x while enforcing the action cap N under the same action-accounting convention used to define admissibility. The additional overhead of the wrapper is bounded by $c_{\mathrm{eval}} + O(|s| + |x|)$. In particular, the wrapper overhead is independent of N except through the explicit storage of (s, x).

Definition 2.8 (Specifiability)**.** A yes/no property P is *specifiable* if there exists a description code s such that $\mathsf{U}(s)$ uniquely defines the property.

Definition 2.9 (Uniform specifiability)**.** A family of predicates $\{P_N\}$ is uniformly specifiable if there exists a single code s_0 such that, for each N, the interpreter $\mathsf{U}(s_0)$ with access to the parameter N uniquely defines P_N.

Definition 2.10 (Decidability in $\mathsf{Q}(V, N)$)**.** A predicate P is *decidable* in $\mathsf{Q}(V, N)$ if there exists an admissible physical procedure $E \in \mathrm{Adm}(N)$ that, strictly staying within the action budget N, halts and outputs the correct bit for $P(\cdot)$ on every valid input.

We denote the set of all predicates decidable in $\mathsf{Q}(V, N)$ as $\mathcal{D}(N)$.

3 Operational Incompleteness

We now present the core theorem of this paper: the physical analogue of Gödel's incompleteness theorem [2]. To avoid ambiguity regarding halting within budget, we employ a bounded diagonal argument.

Lemma 3.1 (Operational Bounded Diagonal Lemma)**.** *There exists a constant c_0 such that for every resource budget N, there exists a unary predicate $G_N : \mathrm{Code}_{\le N - c_0} \to \{0, 1\}$ such that the family $\{G_N\}$ is uniformly specifiable, but no admissible procedure in $\mathsf{Q}(V, N - c_0)$ can correctly compute $G_N(s)$ for every code $s \in \mathrm{Code}_{\le N - c_0}$. In particular, any such procedure fails on its own description t (the code of the decider).*

Proof. Assume $N > c_0$ so that $\mathrm{Code}_{\le N - c_0}$ is nonempty. We construct the unary predicate $G_N(s)$ defined by the following logic:

$$G_N(s) := 1 - \mathrm{Eval}_N(s, s) \tag{3.1}$$

This predicate is well-defined for all s, as Eval_N is total (it always returns a bit within budget N or returns 0). This is physically specifiable because it is a finite wrapper around a bounded execution of a physically instantiable code.

Uniformity: The predicate G_N is produced by the same fixed interpreter and routine for all N. We treat N as a fixed structural parameter of the world $\mathsf{Q}(V, N)$, not as freely writable memory available to the agent. The code for this logic, C_{diag}, combined with the interpreter U, has a length $c_0 = |C_{\mathrm{diag}}| + |\mathsf{U}|$ which is constant.

Now, consider the task: "Decide $G_N(s)$ for any given $s \in \mathrm{Code}_{\le N - c_0}$." Suppose there is a decider E^* *inside* the world $\mathsf{Q}(V, N - c_0)$ that performs this task. Since E^* is admissible, it must have a code $t \in \mathrm{Adm}(N - c_0) \subseteq \mathrm{Code}_{\le N - c_0}$. What happens when E^* is run on its own code t?

$$E^*(t) = G_N(t) = 1 - \mathrm{Eval}_N(t, t)$$

Because t encodes E^* and E^* is admissible under the shared accounting, the capped execution agrees with $\mathrm{Eval}_N(t,t)$ (it halts within budget N by definition of admissibility). Therefore:

$$\mathrm{Eval}_N(t,t) = E^*(t)$$

Substituting this back:

$$E^*(t) = 1 - E^*(t)$$

This is a contradiction. Therefore, no such decider E^* can exist within the resource bounds of $\mathsf{Q}(V, N - c_0)$. The truth of $G_N(t)$ is well-defined by the external logic, but physically undecidable to the observer restricted to budget $N - c_0$. $\square$

Remark 3.2. By using the bounded evaluator Eval_N, we remove the meta-claim that the simulation "must halt" and replace it with a definition that is always budget-bounded. The contradiction arises purely from the inability of a system to simulate its own negation within its own resource limits [1].

4 Quota Forcing: The Principle of Action-Indexed Progress

If the ledger is incomplete, how does physics proceed? It proceeds by expansion. We introduce the concept of *Quota Forcing.*[1]

4.1 The Resource Hierarchy

Since the diagonal argument fails due to a lack of resources, adding resources should resolve it.

Proposition 4.1 (Strict Resource Hierarchy)**.** *For every N, the set of decidable truths in $\mathsf{Q}(V,N)$ is a strict subset of those decidable in $\mathsf{Q}(V, 2N + C)$, for some constant C independent of N.*

$$\mathcal{D}(N) \subsetneq \mathcal{D}(2N + C)$$

Proof. Let $G_N(s) = 1 - \mathrm{Eval}_N(s,s)$ be the undecidable diagonal predicate from Lemma 3.1. We construct a concrete decider $E^\star$ admissible in a larger world $\mathsf{Q}(V, N')$ that decides $G_N(s)$ for inputs in $\mathrm{Code}_{\leq N}$.

The procedure for $E^\star$ is: 1. Load input s (where $s \in \mathrm{Code}_{\leq N}$). 2. Compute $\mathrm{Eval}_N(s,s)$. Since the evaluator is strictly bounded by N, this step consumes at most N action units plus a small constant overhead c_{eval} for the universal interpreter logic. 3. Compute $1 - \mathrm{result}$. 4. Output the bit.

Resource Analysis: The action cost of $E^\star$ is the sum of storage and computation costs.

- The simulation cost: $\leq N + c_{\mathrm{eval}}$ (by definition of Eval_N and interpreter overhead).
- The storage of the input code s: $|s| \leq N$ (since the domain is $\mathrm{Code}_{\leq N}$).
- The overhead of the monitor logic, negation, and storage: $c_{\mathrm{storage}} + c_{\mathrm{negation}}$.

Thus $E^\star$ is admissible whenever

$$N' \geq (N + c_{\mathrm{eval}}) + |s| + c_{\mathrm{misc}}$$

and since $|s| \leq N$ on $\mathrm{Code}_{\leq N}$, it suffices to take $N' = 2N + C$, where $C = c_{\mathrm{eval}} + c_{\mathrm{storage}} + c_{\mathrm{negation}}$. This is strictly greater than N. Therefore, $E^\star$ is admissible in $\mathsf{Q}(V, N')$ with $N' = 2N + C$. Since $G_N(s)$ is decidable in this expanded world but not in $\mathsf{Q}(V,N)$, we have $G_N \in \mathcal{D}(2N + C)$ but $G_N \notin \mathcal{D}(N)$, so the hierarchy is strict. $\square$

[1] We borrow the term 'forcing' from set theory but use it in a physical sense: expansion of decidable truths through resource extension.

4.2 The Forcing Principle

This leads to a rigorous principle of experimental progress.

Principle 4.2 (Action-Indexed Progress)**.** Extending the set of decidable physical truths requires a strictly non-zero investment of additional action. Truth grows with action expenditure.

Remark 4.3. We can view this as a form of "stewardship." We cannot resolve all possible questions at once. We must choose which specific truths to fund with our limited action budget, leaving others (like the diagonal questions for our current budget) formally undecidable until we expand our capacity.

5 Conclusion: The Syntax is Never Finished

In this paper, we have proven that any physical description of reality bounded by a finite action ledger is operationally incomplete.

1. **The Ledger is Finite:** Thermodynamics (and stability requirements) impose an effective cap N on the total action budget for distinct records and computations. 2. **Incompleteness is Inevitable:** For any N, there are physical facts (G_N) that are specifiable but undecidable. 3. **Truth is Dynamic:** The only way to decide G_N is to expand the ledger—to perform a **Forcing Step**.

This leads to a necessary question: *Where do the fresh bits come from?* If the measurable ledger is always incomplete, there must be a source of novelty that lies outside the ledger but can project into it.

In **Part II: The Unmeasurable Sector** [4], we identify this source as the **Inverse Limit** ($\mathcal{U}$) of finite ledgers, physically realized by Infinite Derivative Field Theory (IDFT) and the topological mechanism of the Fold. Finally, in **Part III: Quota-Critical Witnesses** [5], we define the experimental protocols to detect the signature of these limits in the laboratory.

References

[1] Thomas Breuer. The impossibility of accurate state self-measurements. *Philosophy of Science*, 62 (2):197–214, 1995.

[2] Samuel R Buss. *Bounded arithmetic*. Bibliopolis Naples, 1986.

[3] Rolf Landauer. Irreversibility and heat generation in the computing process. *IBM Journal of Research and Development*, 5(3):183–191, 1961.

[4] Emiliano Shea. The unmeasurable sector and the architecture of action. *Preprint*, 2025. Part II of the Beyond the Ledger Series.

[5] Emiliano Shea. Quota-critical witnesses: Experimental signatures of the action limit. *Preprint*, 2025. Part III of the Beyond the Ledger Series.

[6] Emiliano Shea. Resource-bounded incompleteness: Operational limits in quota-bounded worlds. *Preprint*, 2025. Part I of the Beyond the Ledger Series.

[7] David H Wolpert. Physical limits of inference. *Physica D: Nonlinear Phenomena*, 237(9):1257–1281, 2008.