Lemma 3 (The Measurement Rule):

The Unique Measurement Rule on the Bloch Ball

From Variance Complementarity to the Born Rule

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December 30, 2025

Abstract

Lemma 1 established that complementary binary observables obey a fundamental variance bound—the **Action Quota**. Lemma 2 showed that isotropy and operational efficiency inflate the resulting 2D geometry into a 3D Euclidean ball (B^3). In this paper, we prove Lemma 3: given this geometry and the operational requirements of convexity and rotation-covariance, the unique **affine expectation map** is the **Born Rule**: $p(+1 \mid \vec{s}, \vec{m}) = (1 + \vec{s} \cdot \vec{m})/2$. We show that the Born rule is not an independent postulate of quantum mechanics, but the unique solution consistent with the Action Quota's boundary conditions.

1 Framework and Operational Principles

Following the "Dimensional Inflation" established in Lemma 2, our system is defined by a state space Ω affinely isomorphic to the closed unit 3-ball $B^3 = \{\vec{s} \in \mathbb{R}^3 : \|\vec{s}\| \leq 1\}$. Pure states lie on the **Information Frontier** S^2. Measurements M are dichotomic properties (outcomes ± 1) represented by unit vectors $\vec{m} \in S^2$.

We adopt the following operational requirements for the measurement process:

Assumption 1.1 (Convexity - O1)**.** *The expectation value map $\vec{s} \mapsto \langle M \rangle_{\vec{s}}$ is affine. For any statistical mixture $\vec{s}_t = t\vec{s}_1 + (1-t)\vec{s}_0$, the average outcome is the weighted average of individual outcomes:*

$$\langle M \rangle_{\vec{s}_t} = t\langle M \rangle_{\vec{s}_1} + (1-t)\langle M \rangle_{\vec{s}_0}.$$

Assumption 1.2 (Isotropy - O2)**.** *Measurement statistics satisfy rotation-covariance: for any rotation $R \in SO(3)$, we have $\langle M_{R\vec{m}} \rangle_{R\vec{s}} = \langle M_{\vec{m}} \rangle_{\vec{s}}$. Since $\|\vec{m}\| = 1$ and $\|\vec{s}\| \leq 1$, the inner product $x = \vec{s} \cdot \vec{m}$ always lies in the interval $[-1, 1]$. By the combination of O1 and O2, there exists a function $f : [-1, 1] \to \mathbb{R}$ such that $\langle M_{\vec{m}} \rangle_{\vec{s}} = f(\vec{s} \cdot \vec{m})$.*

Proposition 1.1 (Affine-Isotropic Form)**.** *For fixed $\vec{m}$, O1 implies $\langle M_{\vec{m}} \rangle_{\vec{s}} = \vec{a}(\vec{m}) \cdot \vec{s} + b(\vec{m})$. Covariance (O2) requires $\vec{a}(R\vec{m}) = R\vec{a}(\vec{m})$ and $b(R\vec{m}) = b(\vec{m})$. Consequently, $b(\vec{m})$ is a constant β, and $\vec{a}(\vec{m})$ must be proportional to $\vec{m}$ (as it is a vector function of $\vec{m}$ that rotates with it), so $\vec{a}(\vec{m}) = \alpha\vec{m}$. Thus $\langle M_{\vec{m}} \rangle_{\vec{s}} = \alpha(\vec{s} \cdot \vec{m}) + \beta$.*

Definition 1.1 (Binary Expectation)**.** *For outcomes $\{+1, -1\}$, the expectation value relates to the probability of the positive outcome via the relation $\langle M \rangle_{\vec{s}} = 2p(+1 \mid \vec{s}, \vec{m}) - 1$.*

Remark 1.1 (Automatic Continuity)**.** *Continuity of the measurement rule is not an independent postulate. Because the expectation map $\vec{s} \mapsto \langle M \rangle_{\vec{s}}$ is affine on a finite-dimensional compact convex set (B^3), it is necessarily continuous.*

2 Boundary Conditions from the Action Quota

The variance complementarity established in Lemma 1 provides the specific constraints required to fix the measurement function at the Information Frontier.

Proposition 2.1 (Geometric Boundary Conditions)**.** *For pure states $\vec{s} \in S^2$, the expectation value function $\langle M_{\vec{m}} \rangle_{\vec{s}}$ must satisfy:*

(i) ***Aligned Case ($\vec{s} \cdot \vec{m} = 1$):*** *$\langle M \rangle = +1$.*
(ii) ***Orthogonal Case ($\vec{s} \cdot \vec{m} = 0$):*** *$\langle M \rangle = 0$.*
(iii) ***Anti-aligned Case ($\vec{s} \cdot \vec{m} = -1$):*** *$\langle M \rangle = -1$.*

Proof. For (i), if $\vec{s} = \vec{m}$, the state is the $+1$ eigenstate of M by the Sharpness postulate; thus $\langle M \rangle = 1$. For (iii), the state $-\vec{s} = -\vec{m}$ is the -1 eigenstate of $M_{\vec{m}}$ by Sharpness (each dichotomic measurement has antipodal eigenstates on S^2), yielding $\langle M \rangle = -1$.

For (ii), if $\vec{s} \perp \vec{m}$, choose $\vec{m}' = \vec{s}$. By the complementarity–orthogonality identification from Lemma 2, $\vec{m} \perp \vec{m}'$ implies $(M_{\vec{m}}, M_{\vec{m}'})$ is maximally complementary. The Action Quota (Lemma 1) then requires $\mathrm{Var}(M) + \mathrm{Var}(M') \geq 1$. Since $\vec{s}$ is an eigenstate of M', $\mathrm{Var}(M') = 0$, implying $\mathrm{Var}(M) \geq 1$. Hence $\mathrm{Var}(M) = 1$ (since $\mathrm{Var} \leq 1$ for dichotomic outcomes where $M^2 = 1$), which gives $\langle M \rangle = 0$. ☐

3 The Born Rule Derivation

Lemma 3: The Born Rule

Given the unit ball geometry (Lemma 2) and Assumptions O1–O2, the unique **affine expectation map** is:

$$p(+1 \mid \vec{s}, \vec{m}) = \frac{1 + \vec{s} \cdot \vec{m}}{2}.$$

Equivalently, $\langle M \rangle_{\vec{s}} = \vec{s} \cdot \vec{m}$.

Proof. By Isotropy (O2), there exists a function $f : [-1, 1] \to \mathbb{R}$ such that $\langle M \rangle_{\vec{s}} = f(\vec{s} \cdot \vec{m})$. Fix $\vec{m}$ and consider $x_0, x_1 \in [-1, 1]$. Choose pure states $\vec{s}_0, \vec{s}_1 \in S^2$ in the plane spanned by $\vec{m}$ and some $\vec{m}_\perp$ as follows:

$$\vec{s}_i = x_i \vec{m} + \sqrt{1 - x_i^2}\, \vec{m}_\perp \quad (i = 0, 1).$$

Then $\vec{s}_i \cdot \vec{m} = x_i$. For any $t \in [0, 1]$, consider the convex mixture $\vec{s}_t = t\vec{s}_1 + (1 - t)\vec{s}_0$. Since B^3 is convex, $\vec{s}_t \in B^3$. Note $\vec{s}_t$ need not be pure; O1 applies to all states in the ball. By Assumption O1:

$$f(\vec{s}_t \cdot \vec{m}) = t f(\vec{s}_1 \cdot \vec{m}) + (1 - t) f(\vec{s}_0 \cdot \vec{m}).$$

Linearity of the inner product ensures $\vec{s}_t \cdot \vec{m} = t x_1 + (1 - t) x_0$, yielding the **Jensen-affine functional equation** on $[-1, 1]$:

$$f(t x_1 + (1 - t) x_0) = t f(x_1) + (1 - t) f(x_0).$$

Since f is continuous (Remark 1.1), the only solutions are affine: $f(x) = kx + c$ [3]. Applying the boundary conditions from Proposition 2.1:

- $f(0) = 0 \implies k(0) + c = 0 \implies c = 0$.
- $f(1) = 1 \implies k(1) + 0 = 1 \implies k = 1$.

Therefore $f(x) = x$, implying $\langle M \rangle_{\vec{s}} = \vec{s} \cdot \vec{m}$. Applying Definition 1.1 yields the result. □

Remark 3.1 (Extreme Points Determine the Rule)**.** *The derivation uses pure states to establish the functional equation. This is sufficient because B^3 is the convex hull of its extreme points S^2, and because O1 asserts affineness on all of B^3, the behavior on the boundary determines the function uniquely across the entire ball.*

Corollary 3.1 (Pure State Probabilities)**.** *For pure states, let θ be the angle between the state $\vec{s}$ and the measurement $\vec{m}$. Using $\vec{s} \cdot \vec{m} = \cos\theta$ and $\cos\theta = 2\cos^2(\theta/2) - 1$, the Born rule recovers its standard form: $p(+1) = \cos^2(\theta/2)$.*

Corollary 3.2 (Variance Form)**.** *For any state $\vec{s}$ and measurement $\vec{m}$, the variance is $\mathrm{Var}(M)_{\vec{s}} = 1 - (\vec{s} \cdot \vec{m})^2$. For pure states, this simplifies to $\mathrm{Var}(M)_{\vec{s}} = \sin^2\theta$.*

4 Physical Interpretation

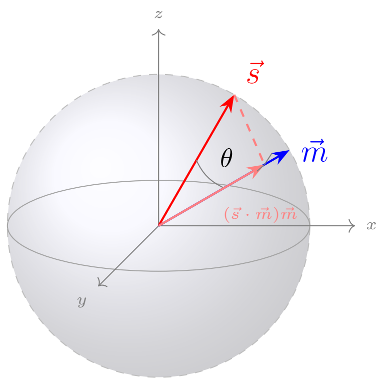

Figure 1: The geometry of the Born rule: $p(+1) = (1 + \vec{s} \cdot \vec{m})/2$. The inner product $\vec{s} \cdot \vec{m} = \cos\theta$ measures the projection of the state onto the measurement axis; the probability interpolates linearly from 0 (anti-aligned) through 1/2 (orthogonal) to 1 (aligned).

Remark 4.1 (Relation to Gleason's Theorem)**.** *Our derivation provides a constructive alternative to Gleason's theorem for qubits. We derive the rule from geometric constraints (convexity, isotropy) and boundary conditions from the Action Quota, without invoking the full machinery of projection lattices.*

5 Outlook: From Geometry to Dynamics

With $\langle M \rangle = \vec{s} \cdot \vec{m}$ established, we have a complete static description of qubit statistics. However, physical systems also evolve. In **Lemma 4**, we show that continuous, reversible dynamics must preserve this Born rule inner product structure. This requirement forces state transformations to be elements of $SO(3)$. To represent these rotations consistently, preserving the global phase structure, we must lift from $SO(3)$ to its universal double cover $SU(2)$, which naturally acts on $\mathbb{C}^2$, necessitating the introduction of complex amplitudes and unitary evolution.

References

[1] L. Hardy, *Quantum Theory From Five Reasonable Axioms*, arXiv:quant-ph/0101012 (2001).

[2] L. Masanes and M. P. Müller, *A derivation of quantum theory from physical requirements*, New J. Phys. **13**, 063001 (2011).

[3] J. Aczél, *Lectures on Functional Equations and Their Applications*, Academic Press (1966).

[4] A. M. Gleason, *Measures on the Closed Subspaces of a Hilbert Space*, J. Math. Mech. **6**, 885-893 (1957).