LEMMA 4 (DYNAMICS):

The Unitary Lift and the Emergence of $\mathbb{C}^2$

From Bloch Rotations to the Schrödinger Equation

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Abstract

Building on the Bloch-ball geometry (Lemma 2) and the Born rule (Lemma 3), we characterize all admissible dynamical transformations. Reversibility, continuity, and probability preservation force physical evolution on the Bloch ball to be a proper rotation, $R \in SO(3)$. We show that the requirement of a consistent amplitude-level representation—grounded in the topological necessity to distinguish between contractible and non-contractible loops—forces a lift to the universal cover, $SU(2)$. This identifies the underlying state space as a two-dimensional *complex Hilbert space* ($\mathbb{C}^2$) with *unitary* dynamics. Complex numbers emerge as the unique minimal extension required to faithfully represent 3D isotropy at the amplitude level.

1 Operational Assumptions for Dynamics

A physical theory must describe how states change over time while maintaining the consistency of the probabilistic framework. Let $T(t) : B^3 \to B^3$ be the map representing time evolution. We define the group of reversible transformations $\mathcal{G}$ acting on Ω:

Assumption 1.1 (Reversibility - D1)**.** *Fundamental evolution forms a one-parameter group of continuous affine bijections:* $T(0) = I$*,* $T(t+s) = T(t) \circ T(s)$*, and* $T(-t) = T(t)^{-1}$*. This ensures evolution is information-preserving and deterministic.*

Assumption 1.2 (Isotropy/Transitivity - D2)**.** *The set of reversible transformations* $\mathcal{G}$ *acts transitively on the set of pure states* S^2*. No direction in the state space is privileged by the dynamics.*

Assumption 1.3 (Probability covariance (Born covariance) - D3)**.** *Evolution is compatible with the Born rule in the following covariant sense: for all* $\vec{s} \in B^3$*,* $\vec{m} \in S^2$*, and all* t*,*

$$p(+1 \mid T(t)\vec{s},\, \vec{m}) \;=\; p(+1 \mid \vec{s},\, T(-t)\vec{m}).$$

Equivalently, in the Schrödinger picture states are pushed forward by $T(t)$*, while in the Heisenberg picture measurement directions are pulled back by* $T(-t)$*. In particular,* $T(t)$ *maps* S^2 *to itself and fixes the maximally mixed state:* $T(t)0 = 0$*.*

Assumption 1.4 (Interferometric path sensitivity - D4)**.** *There exist operational scenarios (e.g. two-path interferometry) in which the implemented transformation depends on the homotopy class of the rotation path. In particular, in such scenarios a loop in physical rotations representing the nontrivial element of* $\pi_1(SO(3))$ *yields a detectable relative phase (a sign flip), whereas the corresponding* 4π *loop is operationally equivalent to identity.*

Remark 1.1 (Empirical Basis of D4). *Assumption D4 is an empirical input: interferometry experiments [1, 2] reveal that certain closed rotation paths produce detectable effects (such as sign flips in neutron interference patterns) that cannot be explained by Bloch-vector evolution alone. This forces the inclusion of amplitude-level structure to distinguish between different homotopy classes of transformation paths.*

2 Dynamics as Proper Rotations

Lemma 2.1 (Lemma 4.1: Reversible dynamics on the Bloch ball). *Under Assumptions D1–D3, each $T(t)$ is the restriction to B^3 of a linear orthogonal map $A(t) \in SO(3)$. Consequently, the connected component of the identity in the reversible transformation group $\mathcal{G}$ is isomorphic to $SO(3)$.*

Proof. Define the operational distinguishability between two states by

$$d(\vec{s}_1, \vec{s}_2) := \sup_{\vec{m} \in S^2} \Big| p(+1 \mid \vec{s}_1, \vec{m}) - p(+1 \mid \vec{s}_2, \vec{m}) \Big|.$$

Using the Born rule (Lemma 3), $d(\vec{s}_1, \vec{s}_2) = \frac{1}{2} \|\vec{s}_1 - \vec{s}_2\|$. Assumption D3 implies

$$d(T(t)\vec{s}_1, T(t)\vec{s}_2) = \sup_{\vec{m} \in S^2} \Big| p(+1 \mid \vec{s}_1, T(-t)\vec{m}) - p(+1 \mid \vec{s}_2, T(-t)\vec{m}) \Big| = d(\vec{s}_1, \vec{s}_2).$$

Thus $\|T(t)\vec{s}_1 - T(t)\vec{s}_2\| = \|\vec{s}_1 - \vec{s}_2\|$, showing $T(t)$ is a surjective isometry of B^3. Since $T(t)$ is affine by D1, there exist $A(t)$ and $\vec{b}(t)$ such that $T(t)\vec{s} = A(t)\vec{s} + \vec{b}(t)$. By D3, $T(t)0 = 0$, hence $\vec{b}(t) = 0$ and $T(t)\vec{s} = A(t)\vec{s}$ is linear. Distance preservation on the ball then forces $A(t) \in O(3)$.

Since $\det(A(t)) \in \{\pm 1\}$ and $t \mapsto \det(A(t))$ is continuous with $\det(A(0)) = 1$, we have $\det(A(t)) = 1$ for all t, hence $A(t) \in SO(3)$. Let $\mathcal{G}_0$ denote the connected component of the identity in $\mathcal{G}$. Now $\mathcal{G}_0$ is a connected Lie subgroup of $SO(3)$, hence (by the standard classification of connected subgroups of $SO(3)$ [5]) it is either trivial, conjugate to $SO(2)$, or all of $SO(3)$. Assumption D2 rules out the first two cases because $\mathcal{G}_0$ acts transitively on S^2. Therefore $\mathcal{G}_0 = SO(3)$, as claimed. □

Bridge. Lemma 2.1 shows that reversible state evolution acts as $SO(3)$ rotations on Bloch vectors. However, Bloch vectors capture only operational probabilities and are insensitive to global phase. Assumption D4 asserts that closed rotation loops can nevertheless be distinguished in interferometry through relative phase, so an amplitude-level model must remember the homotopy class of the rotation path. This is exactly the topological obstruction encoded by $\pi_1(SO(3)) \cong \mathbb{Z}_2$ and it forces a lift to the universal cover $SU(2)$ (a connected double cover of $SO(3)$).

3 The Double Cover and the Emergence of Spinors

Lemma 3.1 (Lemma 4.2: The spinorial lift). *Assume there exists an amplitude-level implementation assigning to each continuous rotation path $\gamma : [0, 1] \to SO(3)$ a projective unitary $[U_\gamma] \in PU(\mathcal{H})$ (i.e. unitaries defined up to overall phase) such that concatenation of paths corresponds to multiplication: $[U_{\gamma_2 * \gamma_1}] = [U_{\gamma_2}][U_{\gamma_1}]$. If, moreover, the nontrivial 2π loop yields a detectable relative phase (D4), then the implementation factors through the universal cover $SU(2) \to SO(3)$, and the minimal state space whose rays reproduce S^2 is $\mathbb{C}^2$.*

Proof. (1) The assignment $\gamma \mapsto [U_\gamma]$ defines a continuous representation of rotation *implementations* at the amplitude level. In particular, any closed loop γ in $SO(3)$ determines a well-defined relative phase in interferometric comparisons, and concatenation of loops corresponds to multiplication in the projective unitary group.

(2) By Assumption D4, the loop class of a 2π rotation is operationally nontrivial: it contributes a relative phase π (i.e. a factor -1) compared to the identity loop, while a 4π loop is operationally equivalent to identity. Thus the phase accumulated under loop concatenation defines an induced (and necessarily multiplicative) map on loop classes, i.e. a nontrivial homomorphism

$$\pi_1(SO(3)) \;\cong\; \mathbb{Z}_2 \;\longrightarrow\; U(1), \qquad [2\pi] \mapsto -1.$$

Hence the amplitude implementation cannot descend to a single-valued representation on $SO(3)$ itself.

(3) The obstruction is exactly removed by passing to the universal cover of $SO(3)$. Since $\pi_1(SO(3)) \cong \mathbb{Z}_2$, the universal cover is the unique connected double cover $SU(2) \to SO(3)$ with kernel $\{\pm I\}$. Under this lift, a 2π rotation corresponds to $-I \in SU(2)$, while a 4π rotation returns to $+I$. Therefore the path-sensitive implementation descends to a single-valued unitary representation of the universal cover $SU(2)$, whose projection to $SO(3)$ forgets the $\{\pm I\}$ sign.

(4) Minimality: the ray space of a two-dimensional complex Hilbert space is $\mathbb{C}P^1$, which is diffeomorphic to S^2 and matches the Information Frontier. Any higher-dimensional representation yields a ray space $\mathbb{C}P^{n-1}$ of real dimension $2(n-1) > 2$ and therefore does not reproduce the qubit state space. □

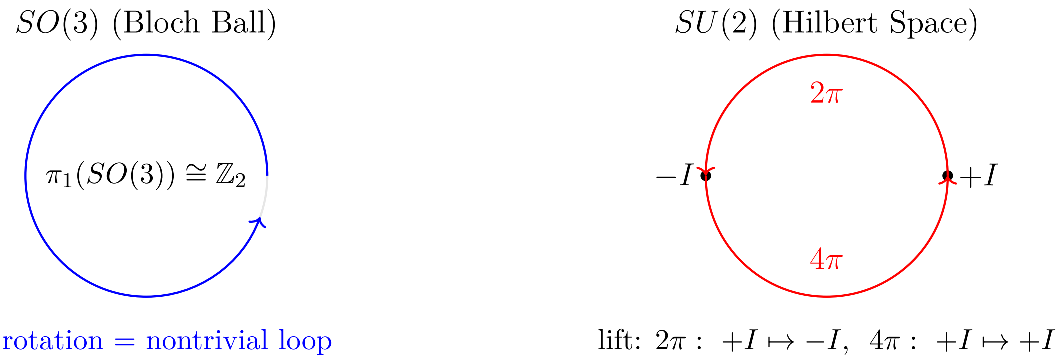

Figure 1: Schematic: the topological origin of complex spinors. The sign $\pm I$ acts trivially on Bloch vectors because U and $-U$ induce the same rotation via the adjoint action on $\vec{\sigma}$, yet the distinction matters at the amplitude level: a 2π loop produces a relative phase in interferometry (D4), so the lift is required for a faithful composition law for implemented transformations.

4 The Schrödinger Equation and Dynamic Action

Theorem 4.1 (Unitary Evolution and Schrödinger Equation)**.** *The lift to $\mathbb{C}^2$ ensures dynamics is implemented by a continuous one-parameter unitary group $\{U(t)\}_{t\in\mathbb{R}}$. By Stone's theorem [3], there exists a unique self-adjoint operator H such that:*

$$i\kappa \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle,$$

where κ is the dynamic action scale representing the rate of temporal evolution.

Calibration of Action Scales. The scale κ has dimensions of action and governs the rate at which phases accumulate. Its relationship to the algebraic action scale $\hbar$ (from the uncertainty relations of Lemma 1) will be established in Lemma 5, where we prove that consistency between observables and generators forces $\hbar = \kappa$.

Proof. Operationally, bijections on rays preserving transition probabilities are implemented by unitaries (Wigner's theorem [4]). Continuity and $U(0) = I$ rule out anti-unitaries. Stone's theorem then guarantees a self-adjoint generator A such that $U(t) = e^{-iAt}$. Defining $H = \kappa A$ restores physical units of energy. □

5 Summary: Why Complex Numbers?

The emergence of complex numbers is a geometric and topological necessity:

1. **3D Isotropy**: Requires representing $SO(3)$ rotations.
2. **Topological Faithfulness**: Requires distinguishing non-contractible rotation paths (D4), forcing the lift to $SU(2)$.
3. **Minimality**: The fundamental representation of $SU(2)$ acts on $\mathbb{C}^2$, the minimal structure capable of faithfully representing the dynamics of the 3-ball.

References

[1] S. A. Werner et al., Phys. Rev. Lett. **35**, 1053 (1975).

[2] H. Rauch et al., Phys. Lett. A **54**, 425 (1975).

[3] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I*, Academic Press (1972).

[4] E. P. Wigner, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra*, Academic Press (1959).

[5] B. C. Hall, *Lie Groups, Lie Algebras, and Representations*, Springer (2015).

A Representing States and Operators

The mapping from vector $|\psi\rangle \in \mathbb{C}^2$ to Bloch vector $\vec{s} \in B^3$ is $s_i = \langle\psi|\sigma_i|\psi\rangle$, where σ_i are the generators of $SU(2)$ satisfying $[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c$. For mixed states one may equivalently write $\rho(\vec{s}) = \frac{1}{2}(I + \vec{s} \cdot \vec{\sigma})$, so that $s_i = \mathrm{Tr}(\rho\,\sigma_i)$ and $\|\vec{s}\| \leq 1$. Moreover, $U \in SU(2)$ acts on Bloch vectors by $\rho(\vec{s}) \mapsto U\rho(\vec{s})U^\dagger$, which induces $\vec{s} \mapsto R_U\vec{s}$ with $R_U \in SO(3)$ and $R_{-U} = R_U$.