The Unmeasurable Sector

The Architecture of the Action Substrate

Part II of the Beyond the Ledger Series

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Abstract

In Part I, we proved that any finite physical ledger is operationally incomplete. Part II identifies the necessary source of the "fresh bits" that expand the ledger: an infinite structure $\mathcal{U}$ (the Inverse Limit) that cannot be fully measured but projects partial information into finite experiments. We identify this sector with the "Dreamer" (Semantics) and propose *Dynamic Field Theory* (DFT) as its candidate physical realization. We introduce the *Mechanism of the Fold*, showing how local, quota-saturating updates in the continuous substrate $\mathcal{U}$ implement the discrete bits required by the forcing calculus, thereby providing a physical mechanism for the ledger expansion proven necessary in Part I.

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1 Introduction: The Source of the Fresh Bit

In the first paper of this series, *Resource-Bounded Incompleteness*, we demonstrated a hard logical limit on physical knowledge. We proved the **Operational Bounded Diagonal Lemma**: for any finite action budget N, there are physical facts that are specifiable but undecidable. To decide them, the ledger must expand; the system must perform a **Quota Forcing** step, expending action to stabilize one fresh bit of truth.

This result creates a paradox of origins. If the fresh bit was not decidable in the previous ledger $\mathsf{Q}(V, N)$, it was not "there" to be read. Yet, when the budget expands to N', the bit appears as a stable, historical fact. *Where did it come from?*

It cannot come from the ledger itself (by the Diagonal Lemma). It cannot simply appear from nowhere without a mechanism. It must, therefore, come from a sector of reality that is **real but unmeasurable**.

In the *Action-Quota Reconstruction* (AQR), we termed this sector the **Dreamer** (Semantics)—the reservoir of infinite potential from which the **Pilot** (Syntax) carves finite records. In this paper, we formalize the Dreamer not as a metaphor, but as a rigorous mathematical object: the **Inverse Limit** of the projective system of finite ledgers.

We then proceed to the physics. We introduce **Dynamic Field Theory (DFT)** as the architecture of this unmeasurable substrate. We show that the transition from unmeasurable potential to measurable fact is governed by a specific topological event: the **Fold**.

2 The Existence Theorem

We first prove that the incompleteness of the ledger forces the existence of a larger structure.

2.1 Premises

We rely on two operational assumptions consistent with the Action-Quota framework:

Assumption 1 (Indefinite Extensibility)**.** For every quota-bounded world $\mathsf{Q}(V, N)$, there exists an operational extension (a larger budget, longer time, or larger volume) allowing the realization of a larger ledger $\mathsf{Q}(V', N+1)$. The universe does not run out of potential.

Remark 2. This assumption holds in asymptotically flat spacetimes. In cosmologies with finite causal horizons (like de Sitter space), Assumption 1 applies up to the Bekenstein bound, which sets the ultimate capacity of the ledger.

Assumption 3 (Coherent Extensibility)**.** Every physically realizable state $s_N \in \mathcal{S}_N$ admits at least one extension $s_{N+1} \in \mathcal{S}_{N+1}$ with $r_{N+1\to N}(s_{N+1}) = s_N$. The history is always extendable.

Assumption 4 (No Hidden Stash)**.** All persistent classical information is accounted for in the ledger N. There is no "off-book" memory available to the Pilot. If a bit is physically accessible, it is in the ledger.

2.2 The Inverse Limit Construction

Definition 5 (Stabilized Ledger States)**.** Let $\mathcal{S}_N$ be the set of all physically realizable stabilized ledger states (transcripts) admissible at budget N. We define $\mathcal{S}_N$ as a subset of finite binary strings with length $|s| \leq N$ (since each bit costs at least one action unit by the Description-Action Link). Since the alphabet is finite and length is bounded, $\mathcal{S}_N$ is a finite set.

Since $N+1$ extends N, there exists a natural restriction map:

$$r_{N+1\to N} : \mathcal{S}_{N+1} \to \mathcal{S}_N$$

This map simply "forgets" the fresh bits added by the extension, returning the history to its state at budget N. This forms a **Projective (Inverse) System** of measurable states $(\mathcal{S}_N, r_{N+1\to N})$.

Definition 6 (The Unmeasurable Sector $\mathcal{U}$)**.** We define the unmeasurable sector $\mathcal{U}$ as the inverse limit of the system $(\mathcal{S}_N, r_{N+1\to N})_{N\in\mathbb{N}}$:

$$\mathcal{U} \;:=\; \varprojlim_{N\in\mathbb{N}} (\mathcal{S}_N, r_{N+1\to N}) \;=\; \left\{ (s_N)_{N\in\mathbb{N}} \;\middle|\; \begin{array}{l} s_N \in \mathcal{S}_N \text{ for all } N, \\ r_{N+1\to N}(s_{N+1}) = s_N \text{ for all } N \end{array} \right\}.$$

Definition 7 (Unmeasurability)**.** A property P of $u \in \mathcal{U}$ is *unmeasurable* at budget N if there exist $u, u' \in \mathcal{U}$ with $\pi_N(u) = \pi_N(u')$ (same projection) but $P(u) \neq P(u')$ (different properties). Equivalently, P is not a function of any finite projection $\pi_N(u)$.

Theorem 8 (Existence of the Unmeasurable)**.** *Given Indefinite and Coherent Extensibility, the set $\mathcal{U}$ is non-empty, and its content is strictly unmeasurable.*

Proof. **(1) Non-emptiness.** The set of ledger states forms a tree where nodes at level N are states in $\mathcal{S}_N$.

- By Assumption 1, the tree has infinite depth.
- By Assumption 3, every node in the tree has at least one child.
- Since $\mathcal{S}_N$ is finite, the tree is finitely branching.
- By König's Lemma, every infinite, finitely branching tree contains at least one infinite path. Thus, $\mathcal{U} \neq \emptyset$.

(2) Unmeasurability. Consider the property $P(u) := G_N(\pi_N(u))$, where G_N is the diagonal predicate from Lemma 3.1 of Paper I. By that lemma, $P(u)$ is undecidable at budget N. Its truth value depends on the tail of u, which can differ for distinct u, u' with $\pi_N(u) = \pi_N(u')$. Thus, the tail information resides in $\mathcal{U}$ but is not measurable in any $\mathsf{Q}(V, N)$.

□

3 The Architecture of the Substrate: Dynamic Field Theory

The mathematical object $\mathcal{U}$ requires a physical realization. We connect the two by positing that an element $u \in \mathcal{U}$ corresponds to a complete specification of a field configuration $\Phi[x]$ for all spacetime, modulo gauge equivalence. Standard Quantum Field Theory (QFT) assumes infinite capacity in every finite volume. DFT replaces this with an action-constrained field. We use natural units $c = \hbar = 1$ for scaling estimates in this section; factors of $\hbar$ can be restored by dimensional analysis.

3.1 The Kernel of the Limit

We seek a physical theory that naturally suppresses information density beyond the Action Quota. We modify the kinetic operator of the field with a non-local kernel $K(\Box)$. To preserve unitarity and Lorentz invariance, $K(\Box)$ must be an **entire function**.

We introduce a fundamental length scale ℓ_0 (equivalently an energy scale $\Lambda = \ell_0^{-1}$) associated with the action-quota microphysics. In Planck units, ℓ_0 corresponds to the fundamental action-quota scale; its value relative to the Planck length is a free parameter of the theory, to be constrained by experiment.

Definition 9 (The DFT Kernel)**.** The kinetic operator for the field is modified by:

$$\mathcal{L} = \frac{1}{2}\phi\, K(\Box)(\Box + m^2)\,\phi$$

A canonical choice compatible with the Action Quota is the Gaussian kernel:

$$K(\Box) = \exp\left(-\ell_0^2\,\Box\right)$$

This kernel suppresses high-frequency modes (the "blur" of the Dreamer) not by a hard cutoff, but by an exponential cost.

Remark 10. While other entire functions (e.g., polynomials) could preserve unitarity, the exponential form is distinguished in providing maximal UV suppression (decaying faster than any polynomial) while maintaining the pole structure of the standard propagator. It is the canonical choice for analytical tractability and appears naturally in various non-local field theories (e.g., String Field Theory).

3.2 Properties of the Substrate

1. **Finiteness:** Formal power counting suggests all loop diagrams are rendered finite by the exponential suppression $\tilde{\Delta}(k) \sim e^{-\ell_0^2 k^2}$. A full non-perturbative proof of finiteness is beyond our present scope, but the exponential decay of the propagator in momentum space removes the standard UV divergences in perturbation theory. 2. **Ghost-Free:** Because $K(\Box)$ is an entire function (exponential), it does not introduce additional poles in the propagator. The only pole is at the physical mass $p^2 = m^2$, avoiding the Ostrogradsky instabilities and ghosts typical of higher-derivative theories. 3. **Measurable Projection:** In the low-energy limit ($\ell_0^2\Box \ll 1$, or wavelengths $\lambda \gg \ell_0$), $K(\Box) \to 1$, recovering standard QFT.

4 The Mechanism of the Fold

How does the continuous, unmeasurable substrate $\mathcal{U}$ produce the discrete, measurable bit in the ledger? This is the mechanism of the **Fold**. In DFT, we propose that measurement corresponds to a dynamical event: the localized saturation of the action density.

4.1 Fold Creation

We define a "Fold" as a localized topological event in the substrate where the action (magnitude) integrated over a region saturates the quota. We assume the field Φ has a compact target space (e.g., S^1) admitting topologically stable defects. We define the fold in Euclidean signature, where the action S_E is positive and the path integral weight $e^{-S_E/\hbar}$ is well-defined for topological configurations. Euclidean instantons in path integrals correspond to tunneling events in real time, providing the mechanism for bit stabilization.

Definition 11 (Fold)**.** A Fold is a localized topological defect (e.g., an instanton or vortex) in the field configuration, characterized by a topological charge $Q \in \mathbb{Z}$, such that the Euclidean action localized to the defect core region W satisfies:

$$(1-\delta)h \leq S_E(W) \leq (1+\delta)h$$

where $\delta \in (0,1)$ is a fixed tolerance parameter determined by finite resolution.

Remark 12. We model Fold creation as a stochastic process where local action density fluctuations reach the threshold h. In thermal equilibrium at temperature T, the rate might scale as $\Gamma \sim (kT/h) \cdot e^{-h/kT}$, making new bits rare at low energies.

Remark 13. In the minimal realization, N bits correspond to N persistent defects; implementations with erasure/rewriting would have different dynamics but must still pay h per net bit. Transient Folds (decaying after readout) would require energy input to maintain, matching the thermodynamic cost of memory.

4.2 The Synthesis: Projection to the Ledger

The Fold is the physical implementation of the Forcing Step defined in Paper I. We model this as the creation of a topological defect.

Definition 14 (Ledger Readout)**.** We define the **readout region** R as a designated spacetime volume of size $O(\ell_0)$ spatially coincident with the defect core, where the presence ($b = 1$) or absence ($b = 0$) of a minimal $|Q| = 1$ defect is recorded as the fresh bit b_{fresh}.

Proposition 15 (Fold-Bit Correspondence)**.** *Within this framework, a Fold creation event in the substrate $\mathcal{U}$ implies a normalized one-bit extension in the ledger $\mathsf{Q}(V, N)$.*

$$u \xrightarrow{\textit{Fold }(W)} u' \quad \Longrightarrow \quad \pi_N(u) \to \pi_{N+1}(u') = (\pi_N(u), b_{fresh})$$

Proof. **Topology as Bit:** We model the Fold as a localized topological defect (e.g., a kink or vortex) carrying a quantized topological charge $Q \in \mathbb{Z}$. The fresh bit is defined by the presence or absence of a minimal $|Q| = 1$ defect. The existence ($Q = 1$) or absence ($Q = 0$) of such a defect constitutes a binary distinction.

Dimensional Analysis: The energy of such a defect scales as $E \sim 1/\ell_0$ due to field gradients over the scale ℓ_0. The localization time scales as $\tau \sim \ell_0$. Thus, the action cost scales as $S \sim E\tau \sim (1/\ell_0) \cdot \ell_0 \sim 1$ (in natural units). Restoring units, the minimal action cost to establish this distinction is of order $\hbar$. (For a concrete example, the kink soliton in 1+1 dimensions has action $S = (8/3)m^3/\lambda$, which can be set to order $\hbar$ by appropriate scaling of the coupling λ). For defects with winding (e.g., 2π circulation), the cost is $h = 2\pi\hbar$, matching Definition 4.1. By the Action-Quota principle, this matches the cost of a persistent bit.

Persistence: The Fold is localized in W. Due to the macrocausality of the DFT kernel, the modification in W is exponentially suppressed outside the lightcone (at scales $\gtrsim \ell_0$), effectively preserving the historical record $\pi_N(u)$ stored in the past. $\square$

Remark 16 (The Cost of Being)*.* This formalizes the "metabolic" view of reality. The "Dreamer" ($\mathcal{U}$) flows continuously. To create a fact (a bit in the Pilot's ledger), the system must "knot" itself—it must gather enough action density to form a Fold (a topological defect). This requires work. Truth is not free; it is paid for in action.

5 Conclusion: The Necessity of the Dreamer

In this paper, we have moved from the logical incompleteness of the ledger to the physical necessity of the unmeasurable.

1. **The Inverse Limit:** The sequence of finite ledgers points to a limit object $\mathcal{U}$ that contains infinite information but is never fully measurable. 2. **The Substrate:** This object is physically realized as a field theory regularized by the Action Quota (DFT). 3. **The Fold:** The transaction between the unmeasurable and the measurable occurs through discrete events of action saturation, physically realized as the creation of topological defects.

This resolves the Pilot's Dilemma: the fresh bits required to expand the incomplete ledger do not arise from the finite ledger itself, nor from nothingness, but originate from the unmeasurable sector $\mathcal{U}$, entering measurable reality through the costly mechanism of the Fold.

We have now defined the **Logic** (Paper I) and the **Ontology** (Paper II) of Action Realism. In the final paper of this series, **"Quota-Critical Witnesses,"** we will design experiments to detect signatures of these Folds, including introspective interferometry protocols that push qubits to their action-quota limits.

A Consistency Checks for DFT

We briefly record the consistency of the chosen kernel $K(\Box) = e^{-\ell_0^2\Box}$.

Unitarity (Optical Theorem). At tree level, the modified propagator $\Delta(p) = \frac{i}{K(-p^2)(p^2-m^2)}$ has the same pole structure as standard QFT (at $p^2 = m^2$). The factor $1/K(-m^2)$ is real and positive. Thus, the imaginary part of the forward scattering amplitude remains positive. There are no negative-norm states (ghosts). Full loop-level unitarity check is deferred to future work; we verify only tree-level consistency here.

Macrocausality. While the kernel is non-local, acausal leakage is exponentially suppressed outside the lightcone at scales $\gtrsim \ell_0$. In the free theory, the commutator $[\phi(x), \phi(y)]$ vanishes for spacelike separation, as the kernel $K(\Box)$ is a function of the d'Alembertian and does not introduce non-local interactions in the equation of motion. Smeared observables obey Paley-Wiener bounds, ensuring that violations of causality (if present in interacting theories) are unobservable at macroscopic scales.