

The Unmeasurable Sector and the Architecture of Action

A Unified Ontology of Inverse Limits, Non-Local Field Theory,
and the Origins of Information

Part II of the Beyond the Ledger Series

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Abstract

The central tension in foundational physics resides between the continuous, unitary evolution of the quantum state and the discrete, irreversible nature of the historical record. We develop a unified framework that links Resource-Bounded Incompleteness to inverse-limit constructions of state space, and we propose a concrete physical realization in terms of a ghost-free non-local (infinite-derivative) field theory. Formally, under explicit extensibility assumptions, the Unmeasurable Sector \mathcal{U} exists as an inverse limit of finite ledger state spaces (Theorem 2.5). Physically, we argue that non-local kernels of the form $e^{-\square/M^2}$ provide a natural UV-softened substrate for \mathcal{U} , and we introduce the Fold mechanism: robust topological defect nucleation as an operational route by which discrete ledger bits can emerge at finite action cost.

Contributions

- We provide a rigorous definition of the Unmeasurable Sector \mathcal{U} as the inverse limit of finite ledger states, grounded in König's Lemma.
- We propose Infinite Derivative Field Theory (IDFT) with ghost-free exponential kernel as a candidate physical architecture for the substrate.
- We introduce the “Mechanism of the Fold,” establishing a mapping between topological defect nucleation and information bit creation.
- We connect the Kibble-Zurek mechanism to information generation, providing a scaling law for bit density.

Key Results Summary

- **Existence Theorem:** Given indefinite extensibility and no dead ends, the set \mathcal{U} of unmeasurable states is non-empty (König's Lemma).
- **Physical Substrate:** We model \mathcal{U} using a non-local field ϕ with exponential kernel $e^{-\square/M^2}$.
- **Fold Mechanism:** A bit is recorded when action saturates $\int_{\Delta t} \int_{V_\xi} \mathcal{L} d^d x dt \geq \hbar$, nucleating a topological defect.
- **Information Scaling:** Defect density scales as $n \sim \tau_Q^{-dv/(1+z\nu)}$, linking forcing rate to information generation.

Key Equations

- **Inverse limit:** $\mathcal{U} = \varprojlim_N \mathcal{S}_N$
(mathematical definition of Dreamer)
- **IDFT action:** $S = \int d^D x \left(\frac{1}{2} \phi e^{-\square/M^2} (\square + m^2) \phi + V(\phi) \right)$
(ghost-free substrate)
- **Fold condition:** $\int_{\Delta t} \int_{V_\xi} \mathcal{L} d^d x dt \geq \hbar$
(action saturation triggers bit creation)
- **KZM scaling:** $n \sim \tau_Q^{-d\nu/(1+z\nu)}$
(information density vs. forcing rate)

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1 Introduction: The Pilot’s Dilemma and the Resource-Bounded Universe

Conventional quantum mechanics, formulated on infinite-dimensional Hilbert spaces, presumes a universe of unbounded potential—a continuous “Dreamer” where superpositions coexist without contradiction. Yet, the empirical reality inhabited by the observer—the “Pilot”—is one of distinct facts, finite resources, and a chronological ledger of outcomes.

Scope and Status of Claims. In this paper, we distinguish between formal mathematical results and physical hypotheses. Section 2 establishes the logical existence of the Unmeasurable Sector \mathcal{U} as a rigorous consequence of Bounded Arithmetic and inverse limit topology. Sections 3 and 4 then propose specific physical models—Infinite Derivative Field Theory and the Kibble-Zurek Mechanism—as the most consistent realizations of this structure. While the existence of \mathcal{U} is a theorem within our axiomatic framework, its physical architecture is a constructive hypothesis subject to experimental verification.

1.1 The Operational Limits of Physical Knowledge

The “Pilot’s Dilemma” arises from the thermodynamic constraints on knowledge [6, 15]. If information is physical, then the recording of a fact requires a finite expenditure of resources (energy and action). Consequently, any physical observer bounded by a finite spacetime volume V and a maximum action budget N cannot possess a complete description of the system they inhabit. This is not merely a practical limitation but a logical one, formalized by the Operational Bounded Diagonal Lemma (Lemma 9 in [13]) and subjective decoherence constraints [2].

This lemma demonstrates that for any finite resource budget N , there exist physical predicates akin to Gödel sentences that are well-defined but undecidable. To decide them, the system must expand; it must perform a “Forcing Step,” creating a “fresh bit” of information that was previously strictly unmeasurable.

1.2 The Necessity of the Unmeasurable Sector

If the fresh bit was not in the ledger, it must be sourced by degrees of freedom outside the Pilot’s operational description. We identify this source as the **Inverse Limit** of the system of finite state spaces. This implies a Strict Resource Hierarchy (Proposition 11 in [13]).

This report establishes that this unmeasurable sector is physically realized as a non-local field substrate governed by **Infinite Derivative Field Theory (IDFT)**. The interaction between the unmeasurable Dreamer and the measurable Pilot is mediated by the **Mechanism of the Fold**: a topological phase transition in the substrate that nucleates stable defects (solitons or instantons), thereby inscribing discrete bits into the historical ledger at the cost of a quantum of action, \hbar . This Action-Quota Reconstruction (AQR) unifies the logical and physical views.

Notation Throughout this paper:

- \mathcal{S}_N denotes the ledger state space at budget N
- S (or $S[\phi]$) denotes the action functional
- S_E denotes Euclidean action
- \mathcal{U} denotes the unmeasurable sector (inverse limit)

Roadmap The remainder of this paper is organized as follows: Section 2 establishes the logical necessity and mathematical construction of \mathcal{U} . Section 3 proposes IDFT as the physical realization. Section 4 introduces the Mechanism of the Fold. Section 5 addresses the thermodynamic costs. Section 6 synthesizes these results.

2 The Existence Theorem: Logical Foundations of the Substrate

We first establish the logical necessity of a “larger,” unmeasurable structure based on the incompleteness of finite physical descriptions.

2.1 Bounded Arithmetic and the Finite Observer

The logic of a resource-bounded observer is best captured not by standard Peano Arithmetic (PA), which assumes infinite resources, but by **Bounded Arithmetic** ($I\Delta_0$) [3]. Parikh’s Theorem [10] states that in $I\Delta_0$, one cannot prove the totality of functions with super-polynomial growth. This mirrors the physical constraint of the Pilot: a finite universe cannot simulate its own future if the complexity of that simulation grows exponentially [11].

The Operational Bounded Diagonal Lemma [13] extends this to physical decidability. It proves that there exists a predicate $G_N(s)$ that is specifiable with a short code but whose evaluation requires resources exceeding the budget N :

$$G_N(s) := 1 - \text{Eval}_N(s, s) \quad (2.1)$$

where $\text{Eval}_N(s, s)$ is the output of procedure s on input s when run with action budget N .

Definition 2.1 (Bounded Evaluation). The bounded evaluator $\text{Eval}_N : \text{Code} \times \text{Code} \rightarrow \{0, 1\}$ returns the output of procedure s on input x if it halts within budget N , and returns 0 otherwise.

The truth of G_N exists relative to a larger resource budget ($N' > N$). The limit of this hierarchy represents the “God’s eye view” or the complete state of the Dreamer, which is inaccessible to any finite N .

2.2 The Inverse Limit Construction

To rigorously define the object that lies at the limit of this hierarchy, we employ the category-theoretic notion of the Inverse Limit.

Definition 2.2 (Ledger State Space \mathcal{S}_N). We take N to be a dimensionless action budget, corresponding to available action $A_{\max} = N\hbar$. The ledger state space is:

$$\mathcal{S}_N = \{(s_1, \dots, s_k) : \sum_{i=1}^k A(s_i) \leq N\hbar\} \quad (2.2)$$

where s_i are distinct facts (bits) and $A(s_i) \geq \hbar$ is the operational cost of resolving each fact.

Physical examples of \mathcal{S}_N include:

- **Quantum Computer:** States reachable with N gate operations.
- **Neurology:** Percepts distinguishable with N neural firings.
- **Cosmology:** Facts decidable within the causal horizon given the universe’s total action.

Since $N + 1$ extends N , there exists a natural map that “forgets” the excess information.

Definition 2.3 (Restriction Map). The map $r_{N+1 \rightarrow N} : \mathcal{S}_{N+1} \rightarrow \mathcal{S}_N$ discards facts requiring more than $N\hbar$ action, preserving only those decidable within budget N .

Definition 2.4 (The Unmeasurable Sector \mathcal{U}). The unmeasurable sector is defined as the inverse limit of the projective system $(\mathcal{S}_N, r_{N+1 \rightarrow N})$:

$$\mathcal{U} := \varprojlim_N \mathcal{S}_N = \left\{ (s_N)_{N \in \mathbb{N}} \in \prod_N \mathcal{S}_N \mid r_{N+1 \rightarrow N}(s_{N+1}) = s_N \right\} \quad (2.3)$$

Theorem 2.5 (Existence of the Unmeasurable Sector). *Assume:*

1. **Indefinite extensibility:** for every N there exists $N' > N$ with $\mathcal{S}_{N'} \neq \emptyset$;
2. **No dead ends:** for every N and every $s_N \in \mathcal{S}_N$ there exists $s_{N+1} \in \mathcal{S}_{N+1}$ such that $r_{N+1 \rightarrow N}(s_{N+1}) = s_N$;
3. **Finite branching:** for every N and $s_N \in \mathcal{S}_N$, the fiber $\{s_{N+1} \in \mathcal{S}_{N+1} : r_{N+1 \rightarrow N}(s_{N+1}) = s_N\}$ is finite.

Then $\mathcal{U} = \varprojlim_N \mathcal{S}_N$ is non-empty (by König's Lemma). If, moreover, there exist infinitely many levels at which at least two incompatible extensions are available (i.e. the extension tree contains a full binary subtree), then \mathcal{U} has cardinality at least that of the continuum.

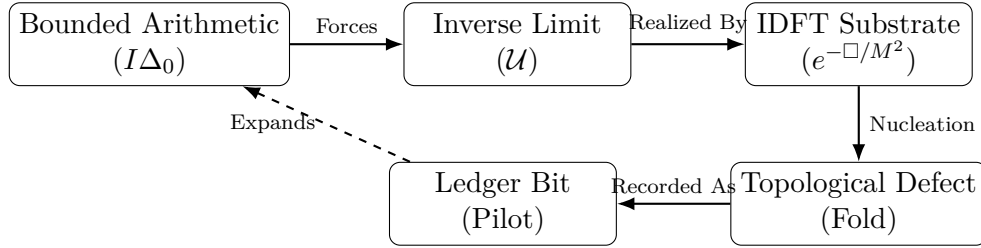


Figure 1: The complete architecture: Bounded Arithmetic forces the existence of the Inverse Limit (Theorem 2.5), physically realized as the IDFT substrate (Section 3). Topological defects (Folds) nucleate when action saturates (Section 4), becoming measurable bits in the Pilot's Ledger, which in turn requires expanding the arithmetic system (dashed feedback arrow).

Table 1: Correspondence Between Mathematical and Physical Structures

Mathematical Structure	Physical Interpretation
Inverse Limit \mathcal{U}	Dreamer's complete state space
Projection $\pi_N : \mathcal{U} \rightarrow \mathcal{S}_N$	Pilot's measurement at resolution N
Cylindrical measure on \mathcal{U}	Probability distribution over histories
Topological defect	Physical bit (h-cell)
Kibble-Zurek scaling	Information generation rate

Remark 2.6 (Loop Quantum Gravity Realization). In Loop Quantum Gravity (LQG), the labels λ correspond to finite graphs γ embedded in a spatial manifold. Each graph represents a finite network of quantum geometric excitations (spin networks). The corresponding continuum completion is obtained as a projective limit over graphs, yielding the space of generalized connections $\bar{\mathcal{A}}$ —a distributional extension of the classical configuration space that includes configurations too singular to be captured by any finite graph [9]. In this analogy, \mathcal{U} plays the role of the limit object, while each \mathcal{S}_N corresponds to a finite-resolution truncation.

Having established the mathematical existence of \mathcal{U} through inverse limits, we now seek its physical realization in field theory.

3 The Architecture of the Substrate: Infinite Derivative Field Theory

We propose that the Unmeasurable Sector is physically realized by **Infinite Derivative Field Theory (IDFT)**, also known as Non-local Gravity.

3.1 Physical Motivation and Selection Criteria

Why select IDFT as the substrate for \mathcal{U} ? The logical structure of the Pilot’s Dilemma requires a theory that is (1) continuous and unitary (preserving the Dreamer’s potential), but (2) naturally regularized to prevent the infinite information density of local QFT, which contradicts the Pilot’s finiteness. IDFT with an exponential kernel is a minimal candidate satisfying these criteria: it introduces a non-local correlation scale M^{-1} that softens UV behavior without introducing ghosts (negative norm states), unlike polynomial higher-derivative theories. We do not claim IDFT is the unique realization of \mathcal{U} ; rather, it is a minimal explicit candidate satisfying these selection criteria.

3.2 The Ghost-Free Condition

To resolve divergences without introducing pathologies, we modify the kinetic term with a non-local form factor $e^{-\square/M^2}$:

$$S = \int d^D x \left(\frac{1}{2} \phi e^{-\frac{\square}{M^2}} (\square + m^2) \phi + V(\phi) \right) \quad (3.1)$$

The choice of the kernel $K(\square) = e^{-\square/M^2}$ is critical. Polynomial kernels introduce ghosts and Ostrogradsky instabilities [14]. However, the exponential function is an *entire function* with no zeroes, introducing no new poles in the propagator at tree level [8].

Relation to Fundamental Scales The IDFT scale M sets the correlation length in the Dreamer. We distinguish two physical scenarios for M :

1. **Planckian Non-locality:** $M^{-1} \sim L_P$. Here, the Dreamer is the geometry of spacetime itself at the quantum gravity scale.
2. **TeV-Scale Non-locality:** $M > 10^3$ GeV. Here, non-locality is a mesoscale phenomenon potentially accessible at collider energies.

3.3 The Diffusion Equation Method

To bridge the non-local substrate and local physics, we employ the **Diffusion Equation Method** [4]. The non-local field $\phi(x)$ can be mapped to a local field $\Phi(x, r)$ in a space with an extra dimension r , satisfying:

$$(\square - \partial_r) \Phi(x, r) = 0 \quad (3.2)$$

The physical non-local field corresponds to the solution at “depth” $r = 1/M^2$. This establishes an isomorphism between \mathcal{U} and a bulk geometry where information diffuses to the slice of observable reality.

The IDFT provides the continuum substrate; we now show how discrete bits emerge via topological transitions.

4 The Mechanism of the Fold: Topology as Information

How does the continuous, non-local substrate generate a discrete bit? We propose the **Mechanism of the Fold**—the nucleation of a topological defect.

4.1 Topological Defects as Bits

In a field theory with Spontaneous Symmetry Breaking, the manifold of ground states \mathcal{M} may have non-trivial homotopy groups. The “Fold” is the event where the field locally saturates the action density and nucleates a stable defect.

The Interface Story. Why equate a defect to a bit? Unlike transient field fluctuations, topological defects are protected by global boundary conditions. They are robust against small perturbations and persist in time, making them suitable for storage in a macroscopic ledger. The Pilot’s measurement is a coarse-graining operation that projects the continuous field onto these discrete topological sectors.

As shown in Table 2, these defects map directly to informational primitives.

Table 2: Topological Defects and Information Encoding

Dim	Defect Type	Homotopy	AQR Interpretation	Action Cost
1+1	Kink / Soliton	$\pi_0(\mathcal{M})$	Binary decision (0/1)	$\sim \hbar$
2+1	Vortex	$\pi_1(\mathcal{M})$	Phase choice	$\sim 2\pi\hbar$
3+1	Monopole	$\pi_2(\mathcal{M})$	Charge Quantization	$\sim 4\pi\hbar/g^2$
Eucl.	Instanton	$\pi_3(\mathcal{M})$	Tunneling Event	$\geq 8\pi^2\hbar/g^2$

4.2 The Action Cost of the Fold

Information has a metabolic cost. A Fold occurs when the action integrated over a spacetime volume defined by the correlation scale $\xi \sim M^{-1}$ exceeds \hbar . Specifically:

$$\int_{\Delta t} \int_{V_\xi} \mathcal{L} d^d x dt \geq \hbar \quad (4.1)$$

This condition—local action saturation—triggers the topological phase transition.

Remark 4.1 (The Cost of Being). This formalizes the “metabolic” view of reality. The Dreamer flows continuously. To create a fact (a bit in the Pilot’s ledger), the system must “knot” itself—gathering enough action density to form a Fold. Truth is not free; it is paid for in action.

4.3 The Kibble-Zurek Mechanism

The dynamics of defect formation is governed by the **Kibble-Zurek Mechanism (KZM)** [5, 16]. When the system is forced through a phase transition at a finite rate τ_Q :

1. **Critical Slowing Down:** The relaxation time diverges.
2. **Freeze-Out:** At critical time \hat{t} , the system falls out of equilibrium, freezing the correlation length at $\hat{\xi}$.
3. **Defect Nucleation:** Independent domains form; topological defects are trapped at their boundaries.

The defect density scales as:

$$n \sim \tau_Q^{-\frac{d\nu}{1+z\nu}} \quad (4.2)$$

(up to non-universal prefactors), where d is spatial dimension, ν is the correlation length exponent, and z is the dynamical critical exponent.

Relation to Resource Bounds The forcing rate τ_Q^{-1} corresponds to the rate of action expenditure dN/dt . Thus, the defect density n scales with the budget expansion rate: faster budget growth produces denser information encoding.

5 Thermodynamics of Distinction: The Entropy of Becoming

The conversion of potential (Dreamer) into history (Pilot) is a thermodynamic process.

5.1 Landauer’s Principle vs. Entropic Purpose

Landauer’s Principle governs the cost of erasure ($\Delta E \geq k_B T \ln 2$). However, the Fold represents the *creation* of distinction. The Margolus-Levitin Theorem [7] limits the speed of evolution: $\tau_{\perp} \geq h/4E$. Rearranging, we find an action bound for distinguishability:

$$E\tau \geq \frac{h}{4} \quad (5.1)$$

This confirms the Action-Quota hypothesis: distinguishing a new state requires a minimum quantum of action. Recent work on “Entropic Purpose” [1] suggests that creating distinction is physically costly.

5.2 Quantum Darwinism and the Ledger

A single Fold is microscopic. To become part of the macroscopic “Ledger,” it must be amplified via **Quantum Darwinism** [17]. The environment scatters off the pointer state (the defect), carrying away redundant copies of the information. This redundancy renders the subjective quantum event into an objective classical fact, paid for by thermodynamic dissipation.

6 Conclusion

This report establishes a unified ontology where the “fresh bits” required to resolve the incompleteness of the finite physical ledger originate from a rigorously defined unmeasurable sector. The architecture, visualized in Figure 1, resolves several foundational issues:

1. **Measurement Problem:** Collapse is identified with Fold nucleation (action saturation).
2. **UV Divergences:** IDFT provides a natural physical cutoff at scale M .
3. **Arrow of Time:** Ledger growth requires irreversible action expenditure.
4. **Logic-Physics Bridge:** The inverse limit structure (\mathcal{U}) connects Bounded Arithmetic incompleteness to Non-local Field Theory.

6.1 Recommendations for Future Research

Experimental Signatures of Non-Locality IDFT predicts UV-softened propagators and vertices, often expressible via exponential form factors; in a low-momentum expansion this can appear as higher-order corrections such as $O(p^4/M^2)$ to the dispersion relation. We propose investigating high-energy scattering experiments for these characteristic deviations.

Information Scaling Laws We predict that defect density n follows the KZM scaling law related to the rate of action injection:

$$n \propto \left(\frac{dN}{dt} \right)^{\frac{d\nu}{1+z\nu}} \quad (6.1)$$

This can be tested in analogue systems like Bose-Einstein condensates or liquid crystals [12].

Quantum Simulation of Vacuum Decay Use large-scale quantum annealers to observe “quantized bubble formation”—validating the minimum action cost $\Delta S \geq \hbar$ for bit generation.

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A Consistency Checks for IDFT

Unitarity (linearized, tree level). For form factors $K(\Box)$ chosen as entire functions with no zeros (e.g. $K(\Box) = e^{-\Box/M^2}$), the modified propagator acquires no additional poles beyond the standard one, so no extra ghost-like degrees of freedom are introduced at tree level around the perturbative vacuum [8].

Macrocausality. Depending on the contour/prescription and the chosen entire form factor, non-local effects are confined to the scale M^{-1} and are strongly suppressed beyond it; in particular, acausal leakage is expected to be negligible for macroscopic observables at distances $\gg M^{-1}$ [4].

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