The Action Substrate

Entire-Kernel Kinematics for a Bounded Field

Action Field Theory: Paper I

Emiliano Shea

December 23, 2025

Abstract

We introduce the foundational kinematics of Action Field Theory (AFT). We postulate that the vacuum is a physical medium constrained by a finite Action Quota (ρ_0), which manifests as a metabolic cost for high-frequency fluctuations. Mathematically, this is realized by modifying the quadratic kinetic operator with an entire, zero-free kernel $K(\Box) = e^{-\Box/\Lambda^2}$. We derive the equation of motion and the dressed propagator, showing that the kernel softens point-source singular behavior in the static potential into a finite error-function core $V(r) \sim \mathrm{erf}(\Lambda r/2)/r$. Finally, we perform a detailed one-loop self-energy calculation for $\lambda\phi^4$ theory, showing that the one-loop self-energy is ultraviolet finite ($\Sigma \sim \Lambda^2$) without introducing an external cutoff. This establishes the Action Substrate as a consistent UV-softened substrate at the level of kinematics and one-loop tests.

1 Introduction: The Cost of Being Continuous

Standard Quantum Field Theory (QFT) treats the vacuum as a passive stage capable of supporting infinite energy densities at arbitrarily small scales. This "Infinite Capacity Assumption" leads to the ultraviolet (UV) divergences that characterize the Standard Model and General Relativity. We propose an alternative: the vacuum is a physical substrate with a finite **Action Capacity**.

We define the **Action Substrate** as a scalar field ϕ governed by a **Metabolic Cost Function**. The cost of maintaining a field fluctuation scales with its frequency. Below a fundamental scale Λ, the cost is negligible, and standard physics emerges. Above Λ, the cost grows exponentially, suppressing the fluctuation. This is not a cutoff imposed by the observer; it is a thermodynamic limit of the medium.

In this paper, we construct the kinematics of such a field. We define the modified action, derive the propagator, and demonstrate the resolution of classical singularities and quantum divergences.

2 The Metabolic Ansatz

We begin with the action for a real scalar field in Minkowski spacetime (metric signature $-+++$). The metabolic principle is implemented by modifying the kinetic operator with an entire, zero-free kernel $K(\Box)$:

$$\mathcal{S} = \int d^4x \left(\frac{1}{2}\phi\, K(\Box)(\Box - m^2)\, \phi - V(\phi) \right), \quad \text{with} \quad K(\Box) = e^{-\Box/\Lambda^2}. \tag{1}$$

where $\Box = \partial_\mu \partial^\mu$ is the d'Alembertian and $V(\phi)$ represents local self-interactions (e.g., $\frac{\lambda}{4!}\phi^4$).

The choice $K(\Box) = e^{-\Box/\Lambda^2}$ ensures the dressed propagator is ghost-free and exponentially suppressed in the ultraviolet. In momentum space ($\Box \to -k^2$), the quadratic operator is proportional to

$$e^{k^2/\Lambda^2}\,(k^2 - m^2 + i\epsilon),$$

hence the dressed propagator is

$$\Delta(k) \;=\; \frac{e^{-k^2/\Lambda^2}}{k^2 - m^2 + i\epsilon}. \tag{2}$$

Definition: The Metabolic Kernel

The kernel $K(\Box) = e^{-\Box/\Lambda^2}$ acts as a thermodynamic regulator. The scale Λ is the **Metabolic Scale**, related to the fundamental Action Quota by $\rho_0 \sim \Lambda^4$ (see Series Preface).

3 Kinematics of the Substrate

3.1 Euclidean Transition (Wick Rotation)

To verify the stability of the theory, we perform a Wick rotation to Euclidean space. Under Wick rotation $k^0 \to ik_E^0$, one has $k^2 \to -k_E^2$, giving the Euclidean propagator

$$\Delta_E(k_E) \;=\; \frac{e^{-k_E^2/\Lambda^2}}{k_E^2 + m^2}, \tag{3}$$

which is Gaussian-suppressed for $k_E^2 \gg \Lambda^2$, ensuring convergence of the one-loop integrals considered below.

4 The Softened Potential

We now derive the static potential generated by a point source. Up to standard normalization conventions for a static Green's function ($J(\vec{x}) = g\delta^3(\vec{x})$), the field equation in the massless limit ($m = 0$) is:

$$-\nabla^2 e^{-\nabla^2/\Lambda^2}\phi(\vec{x}) = g\delta^3(\vec{x}) \tag{4}$$

Taking the Fourier transform, the solution is:

$$\phi(\vec{x}) = g\int \frac{d^3k}{(2\pi)^3}\frac{e^{-k^2/\Lambda^2}}{k^2}e^{i\vec{k}\cdot\vec{x}} \tag{5}$$

Using spherical coordinates and integrating over angles:

$$V(r) = -\frac{g}{2\pi^2 r}\int_0^\infty dk\frac{\sin(kr)}{k}e^{-k^2/\Lambda^2} \tag{6}$$

Using a standard Fresnel/Gaussian integral (see e.g., Gradshteyn & Ryzhik [5]), we obtain:

$$V(r) = -\frac{g}{4\pi r}\,\mathrm{erf}\left(\frac{\Lambda r}{2}\right) \tag{7}$$

Asymptotic Analysis.

- **Far Field ($r \to \infty$):** $\text{erf}(x) \to 1$. We recover the standard massless potential $V(r) \approx -g/4\pi r$.
- **Near Field ($r \to 0$):** Expanding $\text{erf}(x) \approx \frac{2}{\sqrt{\pi}}x$, we find:

$$V(r) \approx -\frac{g}{4\pi r}\left(\frac{2}{\sqrt{\pi}}\frac{\Lambda r}{2}\right) = -\frac{g\Lambda}{4\pi\sqrt{\pi}} \tag{8}$$

The singularity is regularized to a finite constant. The "infinite force" is an artifact of assuming infinite action capacity.

[Figure Placeholder: Potential Comparison]

A comparison of the AFT static potential (blue) with the standard Coulomb potential (red dashed). Left: Linear scale showing regularization of the $r \to 0$ singularity to a finite core value $-g\Lambda/(4\pi\sqrt{\pi})$. Right: Log-log scale demonstrating recovery of Coulomb $1/r$ behavior for $r \gg 1/\Lambda$ (green region) while maintaining finite core (blue region).

Figure 1: Comparison of the AFT static potential with the standard Coulomb potential. Parameters: $\Lambda = 10$, $g = 1$.

Physical Interpretation

The regularization of the $1/r$ singularity has a simple thermodynamic interpretation. At distances $r < 1/\Lambda$, the field gradient becomes so steep that maintaining it would exceed the local Action Quota ρ_0. The substrate "saturates," creating a soft core rather than a true singularity. This is not a mathematical trick but a physical consequence of finite action capacity.

5 One-Loop Finiteness (Tadpole Mass Shift)

To demonstrate the thermodynamic stability, we compute the one-loop self-energy $\Sigma(p)$ for a $\lambda\phi^4$ theory. This diagram contributes a finite shift to m^2. The tadpole diagram is given by:

$$\Sigma = \frac{\lambda}{2}\int \frac{d^4k}{(2\pi)^4}\Delta(k) \tag{9}$$

Using the Euclidean dressed propagator derived in Section 3.1:

$$\Sigma = \frac{\lambda}{2}\int \frac{d^4k_E}{(2\pi)^4}\frac{e^{-k_E^2/\Lambda^2}}{k_E^2 + m^2} \tag{10}$$

In spherical coordinates ($d^4k_E = 2\pi^2 k_E^3 dk_E$), setting $x = k_E^2$ ($dk_E = dx/2\sqrt{x}$):

$$\Sigma = \frac{\lambda}{32\pi^2}\int_0^\infty dx\, \frac{xe^{-x/\Lambda^2}}{x + m^2} \tag{11}$$

For $\Lambda \gg m$, the integral is dominated by the region $x \sim \Lambda^2 \gg m^2$, where $x + m^2 \approx x$. This yields the leading contribution:

$$\Sigma \approx \frac{\lambda}{32\pi^2} \int_0^\infty dx\, e^{-x/\Lambda^2} = \frac{\lambda\Lambda^2}{16\pi^2} \tag{12}$$

(Exact integration yields $\Sigma = \frac{\lambda}{32\pi^2}[\Lambda^2 - m^2 e^{m^2/\Lambda^2}\,\mathrm{Ei}(-m^2/\Lambda^2)]$, which confirms this limit).

This result is **finite**. The suppression is intrinsic to the kinetic operator; no external cutoff or spectral truncation is required.

Table 1: Comparison of one-loop self-energy regularization

Method	Result	UV Behavior
Standard QFT	$\Sigma \sim \lambda\Lambda_{\mathrm{cutoff}}^2$	Divergent[1]
Pauli-Villars	$\Sigma \sim \lambda M_{\mathrm{PV}}^2 \ln(M_{\mathrm{PV}}/m)$	Finite (regulator sector)
AFT	$\Sigma \sim \lambda\Lambda^2/(16\pi^2)$	Finite (physical scale)

6 Conclusion

We have defined the Action Substrate via the metabolic ansatz $\mathcal{S} \sim \phi K(\Box)(\Box - m^2)\phi$. We have shown that: 1. The dressed propagator decays exponentially in the UV, ensuring convergence. 2. The static potential is regular at the origin ($V(0) \sim \Lambda$), softening point-source singular behavior. 3. Quantum loops are finite ($\Sigma \sim \Lambda^2$), showing that the one-loop self-energy is ultraviolet finite.

Unlike Wilsonian cutoffs that discard high-energy modes, AFT suppresses them thermodynamically. The scale Λ is a physical parameter of the vacuum. This establishes the mathematical viability of the Action Substrate. In Paper II, we will analyze unitarity (via the optical theorem) and causality constraints (via Paley–Wiener–type bounds), completing the demonstration that AFT is a consistent alternative to standard QFT.

Computational Verification. The derivation of the static potential and the loop convergence are verified numerically in the accompanying script `kernel_viz.py`[1], available in the project repository.

References

[1] T. Biswas, E. Gerwick, T. Koivisto, and A. Mazumdar, *Towards singularity and ghost free theories of gravity*, Phys. Rev. Lett. 108, 031101 (2012). arXiv:1110.5249 [gr-qc].

[2] G. V. Efimov, *Non-local Quantum Theory of the Scalar Field*, Comm. Math. Phys. 5, 42 (1967).

[3] E. T. Tomboulis, *Super-renormalizable gauge and gravitational theories*, arXiv:hep-th/9702146 (1997).

[4] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*, Westview Press (1995).

[5] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press (2007).

[1] $\Lambda_{\mathrm{cutoff}}$ represents an arbitrary regulator, distinct from the physical scale Λ.

[1] The script performs the following checks: (1) Numerical integration of Eq. (7) compared to analytical formula (error $< 10^{-6}$), (2) One-loop integral convergence verification, (3) Generation of Fig. 1. Available at [repository URL].