Lemma 4: Dynamics on the Bloch Ball
From Rotations to the Unitary Lift

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Abstract

Building on Lemma 2 (Bloch-ball geometry) and Lemma 3 (Born-rule measurement $\langle M \rangle_{\vec{s}} = \vec{s} \cdot \vec{m}$), we characterize all admissible dynamical transformations. The operational axioms of **Reversibility**, **Continuity**, and **Probability Preservation** imply that every physical evolution on the Bloch ball is a proper rotation, $R(t) \in SO(3)$. We then show that the requirement of a consistent projective representation forces a lift to the universal double cover, $SU(2)$. This topological necessity, grounded in empirical observations of spin-$1/2$ systems (the $2\pi/4\pi$ phenomenon), implies that the underlying state space is a two-dimensional **complex Hilbert space** $\mathbb{C}^2$ with **linear and unitary** dynamics. The dynamical action scale κ emerges from Stone's theorem but remains undetermined until unified with the static scale in Lemma 5.

Contents

1 Operational Axioms for Dynamics

The state space established in Lemma 2 is the Bloch ball $B^3 \subset \mathbb{R}^3$, with measurement statistics governed by the Born rule (Lemma 3). We impose three operational requirements on any admissible time evolution $T(t) : B^3 \to B^3$:

D1. Reversibility: $T(t)$ is invertible for all $t \in \mathbb{R}$.

D2. Continuity: $T(t)$ depends continuously on t with $T(0) = I$ (no discontinuous jumps).

D3. Probability Preservation: The evolution preserves all measurement statistics derived in Lemma 3 (equivalently, preserves the inner product $\vec{s} \cdot \vec{m}$ that determines probabilities).

Proposition 1 (Physical justification of dynamical axioms)**.**
(D1) Reversibility: Fundamental physical transformations are reversible. Time evolution must admit an inverse $T(-t)$ that recovers the initial state. This is the principle of deterministic, information-preserving dynamics. (Measurement is a distinct process and is not assumed reversible.)
(D2) Continuity: Since measurement statistics depend continuously on state parameters, physical time evolution should also be continuous. Infinitesimal time steps produce infinitesimal state changes, excluding unphysical discontinuous "jumps."
(D3) Probability Preservation: Evolution cannot change the fundamental probabilistic structure. The evolution must be a symmetry of the theory, preserving the inner product $\vec{s} \cdot \vec{m}$.

1.1 Probability Preservation Implies Isometry

Lemma 1 (Isometry from probability preservation)**.** *Axiom D3 (Probability Preservation) implies that T is a linear isometry of $\mathbb{R}^3$ that fixes the origin.*

Proof. Lemma 3 established the Born rule:

$$p(+1 \mid \vec{s}, \vec{m}) = \frac{1 + \vec{s} \cdot \vec{m}}{2}.$$

Probability preservation therefore requires that, for all states and measurement directions,

$$(T\vec{s}) \cdot (T\vec{m}) = \vec{s} \cdot \vec{m}.$$

Hence T preserves inner products and norms, i.e. it is an isometry.

Linearity from convexity and continuity. The map T preserves expectation values, which are affine in the state. Hence T maps convex combinations to convex combinations:

$$T(t\vec{s}_1 + (1-t)\vec{s}_0) = tT\vec{s}_1 + (1-t)T\vec{s}_0, \qquad t \in [0,1].$$

An affine map that fixes the origin is linear. Preservation of the maximally mixed state (the origin) is required because probabilities for the maximally mixed state are uniform and unaffected by any reversible symmetry; thus $T(0) = 0$. Together with continuity (D2) this yields a bona fide linear operator on $\mathbb{R}^3$ which is orthogonal because it preserves the inner product. $\square$

1.2 From Isometries to Rotations

Lemma 2 (Dynamics restricted to $SO(3)$). *Under axioms D1–D3, the allowed time evolutions are precisely the proper rotations: $T(t) \in SO(3)$ for all t.*

Proof. By Lemma 1, each $T(t)$ is orthogonal, so $T(t) \in O(3)$. Continuity (D2) implies $T(t)$ forms a continuous one-parameter path starting at $T(0) = I$. Determinant is continuous and $\det I = +1$, hence $\det T(t) = +1$ for all t. Therefore $T(t)$ lies in the identity component of $O(3)$, i.e. $SO(3)$. $\qquad\square$

Remark 1. Continuity naturally selects proper rotations ($\det = +1$) and excludes reflections ($\det = -1$), which are physically distinct and not generated by continuous time evolution starting at the identity.

2 The Topological Necessity of the Unitary Lift

Rotations of Bloch vectors are elements of $SO(3)$, but empirical spinor phenomena and the requirement of a consistent projective representation force us to lift to $SU(2)$.

2.1 Empirical Evidence: The 2π vs 4π Phenomenon

Proposition 2 (Experimental observation requires double cover). *Neutron interferometry and related spin-1/2 experiments show that a 2π spatial rotation produces an observable sign change in interference (a relative phase of π), while a 4π rotation restores the original interference pattern.*

Operational argument. If physical states were identified with Bloch vectors, $R(2\pi) = I$ would have no observable effect. The observed sign change therefore implies the existence of an underlying state $|\psi\rangle$ on which rotations act projectively: a 2π rotation yields $|\psi\rangle \mapsto -|\psi\rangle$, so interference observables can detect the minus sign. This is empirical evidence for a 2-to-1 map from a state vector representation to Bloch vectors. $\qquad\square$

2.2 The Universal Cover $SU(2)$ and Projective Representations

Theorem 1 (Projective representation requires $SU(2)$). *To represent rotations $R(t) \in SO(3)$ consistently with the empirical $2\pi/4\pi$ phenomenology, one must lift to the universal cover $SU(2)$. Consequently:*

1. *The minimal faithful representation space is $\mathbb{C}^2$ (a complex 2D Hilbert space).*

2. *Time evolution is given by unitary operators $U(t) \in SU(2)$ (projectively representing $SO(3)$).*

3. *The mapping from state vectors to Bloch vectors is 2-to-1: $|\psi\rangle$ and $-|\psi\rangle$ map to the same Bloch vector.*

Topological and representation-theoretic argument. Topologically, $SO(3)$ has fundamental group $\pi_1(SO(3)) \cong \mathbb{Z}_2$, so paths corresponding to a 2π rotation are non-contractible. To represent such topological features continuously on state space we require a simply connected cover; the universal cover of $SO(3)$ is $SU(2)$ with covering map $\phi : SU(2) \to SO(3)$ and kernel $\{\pm I\}$. Representations of $SU(2)$ act naturally on a 2-dimensional complex vector space; the nontrivial kernel element $-I$ is what produces the observed minus sign after 2π rotations. Wigner's theorem (projective symmetry representations) justifies passing from projective representations on rays to true unitary representations on vectors (up to a phase), which is implemented here by choosing the double cover $SU(2)$. $\square$

Remark 2. This is not a mere mathematical convenience: the double-cover structure is empirically motivated (spin-$\frac{1}{2}$ interference experiments) and mathematically minimal (smallest faithful compact group covering $SO(3)$).

2.3 Why Complex Numbers Are Forced

Proposition 3 (Complex numbers are minimal and necessary). *A faithful, minimal-dimensional representation of $\mathfrak{su}(2)$ requires complex scalars; equivalently, a real 2×2 representation cannot realize the $\mathfrak{su}(2)$ commutation relations.*

Proof. The Lie algebra $\mathfrak{su}(2)$ is non-abelian and its structure constants appear multiplied by i in the fundamental (Pauli) representation: $[\sigma_a, \sigma_b] = 2i\varepsilon_{abc}\sigma_c$. Any 2D real matrix algebra lacks the required imaginary structure to satisfy these commutation relations. Thus $\mathbb{C}$ is the minimal division-algebra extension of $\mathbb{R}$ that supports a faithful 2D representation of $\mathfrak{su}(2)$. (This is a standard fact in Lie algebra/representation theory; see e.g. Fulton–Harris.) $\square$

3 Stone's Theorem and the Schrödinger Equation

With states in $\mathbb{C}^2$ and evolution $U(t) \in SU(2)$, continuous time evolution is generated by a Hermitian operator via Stone's theorem.

Theorem 2 (Stone's theorem for $SU(2)$). *Every strongly continuous one-parameter unitary group $U(t)$ on $\mathbb{C}^2$ has the form*

$$U(t) = \exp\left(-\frac{i}{\kappa}Ht\right),$$

where H is a Hermitian (here traceless) generator in $\mathfrak{su}(2)$ and $\kappa > 0$ is a real constant (the dynamical action scale).

Proof. Direct application of Stone's theorem (see [4]) for strongly continuous unitary groups on Hilbert space. The κ factor is included to make the exponent dimensionless; it represents the physical action scale converting energy×time to a pure number. $\square$

Remark 3 (Emergent Schrödinger equation). Differentiation yields the Schrödinger equation for state vectors:

$$i\kappa \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle.$$

This is the Schrödinger equation for a two-level system, with κ playing the role of the action quantum. The fact that it emerges from purely geometric and operational considerations is remarkable.

3.1 The generator correspondence

The Hamiltonian $H \in \mathfrak{su}(2)$ corresponds under the covering map to an antisymmetric generator $\Omega \in \mathfrak{so}(3)$ so that the one-parameter rotation $R(t) = \exp(t\Omega)$ on Bloch vectors is realized projectively by $U(t) = \exp(-iHt/\kappa)$ on state vectors. In other words, $H \in \mathfrak{su}(2)$ corresponds to the geometric rotation generator $\Omega \in \mathfrak{so}(3)$ via the double-cover map, completing the fundamental bridge between the real geometry of the Bloch ball and the complex, unitary dynamics of quantum states.

4 Preview: Unification of Action Scales in Lemma 5

Definition 1 (Two action scales). The reconstruction has introduced two action scales:

- **Static scale $\hbar_{\mathbf{static}}$ (Lemma 1):** Governs measurement uncertainty $(\mathrm{Var}(A)+\mathrm{Var}(B) \geq 1)$.

- **Dynamic scale κ (Lemma 4):** Governs temporal evolution (Schrödinger dynamics).

Remark 4 (The unification problem). The Hamiltonian H serves both as generator of dynamics (scaled by κ) and as an observable participating in the static algebra (scaled by $\hbar_{\mathrm{static}}$). Lemma 5 will show that consistency of expectation-value evolution forces $\kappa = \hbar_{\mathrm{static}}$, producing a single fundamental action scale.

A The Lie Algebra $\mathfrak{su}(2)$ and Pauli Matrices

The Pauli matrices form a basis for $\mathfrak{su}(2)$:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with $[\sigma_a, \sigma_b] = 2i\varepsilon_{abc}\sigma_c$. Any $H \in \mathfrak{su}(2)$ can be written $H = \frac{1}{2}\vec{h} \cdot \vec{\sigma}$, linking the Hermitian generator to the rotation axis $\vec{h} \in \mathbb{R}^3$.

B Geometric Visualization of the Double Cover

Remark 5 (Dirac belt trick and topology). The Dirac belt trick gives an intuitive picture of why a 2π rotation is not homotopic to the identity in $SO(3)$ but a 4π rotation is. This topological fact underlies the need for the double cover $SU(2)$.

C   Why linearity was not assumed earlier

Remark 6. Linearity is not assumed in Lemma 1: it is derived here from the combination of convexity (affine mixing), preservation of convex structure by dynamics (D3), and continuity (D2). This avoids circularity in the reconstruction.

References

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