The Caustic Ladder

Paper II of Series III: The Geometry of the Limit

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Abstract

We derive the discrete energy spectrum of quantum systems from the **constraint imposed by the Action Quota** (ρ_0) on the geometry of phase-space flows. While Paper I established the local cost of phase, this paper addresses global stability. We show that when a continuous action flow saturates the Liouville capacity of a phase-space channel, the field must fold. Applying semiclassical methods (stationary phase approximation) to the canonical fold catastrophe, we find that constructive interference is only possible at discrete "stations" quantized in units set by ρ_0 (with Maslov shift). This mechanism generates the *Caustic Ladder*—a discrete spectrum $E_n \approx E_0 + n\rho_0\nu$—without postulating field discreteness. We conclude that quantum levels are the stable interference fringes of a capacity-limited continuum.

Contents

1 Introduction: The Steps of the Limit

In classical mechanics, energy is continuous. A planet can orbit a star at any radius; a pendulum can swing with any amplitude. In quantum mechanics, energy is discrete. The electron in an atom is confined to a "ladder" of allowed levels. Standard quantum mechanics attributes discrete spectra to boundary conditions: the wavefunction must vanish at the walls (particle in a box) or decay exponentially (bound states).

We propose a complementary geometric view: discreteness arises from the interference conditions imposed by fold caustics in a capacity-constrained phase space. We view discreteness not as a property of the field itself, but as a consequence of **saturation**.

Imagine a fluid flowing through a channel. If the flow remains laminar, the fluid level can vary continuously. But if the flow exceeds the channel's capacity, the fluid buckles—it folds over itself. These folds create distinct, quantized structures (like the hydraulic jump).

We apply this logic to the flow of Action. In Paper I, we defined the Action Quota ρ_0 as the capacity of a phase-space cycle. In this paper, we ask: what happens when a system tries to sustain a continuous flow of action that fills this quota? We will show that the field is forced to "fold," and the stable configurations of this fold form the discrete energy ladder we observe in nature.

2 The Geometry of Saturation

Consider a dynamical system described by a Lagrangian manifold $\mathcal{L}$ in phase space (q, p). The state evolves by flowing along this manifold.

2.1 Liouville Incompressibility

Liouville's Theorem guarantees that the phase-space volume (symplectic area) is conserved under Hamiltonian flow. This is usually interpreted as "information conservation." We interpret it as **incompressibility**.

Just as water resists compression, the action fluid cannot be squeezed into a volume smaller than ρ_0 per cycle (as derived in Paper I via the Heisenberg Floor).

Consequently, when a continuous flow is bounded (e.g., in a potential well), the Lagrangian manifold cannot simply terminate; to preserve the symplectic area and satisfy the quota, it must turn back on itself. This reversal creates a fold.

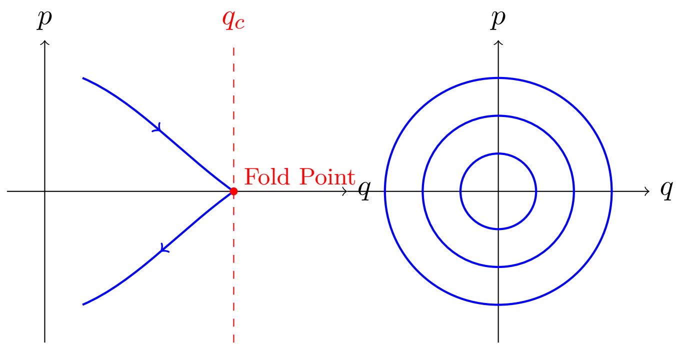
2.2 The Fold Catastrophe

What happens when we project this incompressible 2D sheet $\mathcal{L}$ onto the 1D configuration space q? For most points, the projection is simple. But if the sheet twists or turns to remain within a bounded region (a bound state), the projection must become singular. The map $q(p)$ loses invertibility.

When a smooth 2D surface is projected onto a 1D space, the generic singularities (by Thom's classification) are the fold and cusp. For a Lagrangian manifold constrained by $\omega = dp \wedge dq$, the codimension-1 singularity is the **Fold**. The fold occurs where the projection $\pi : \mathcal{L} \to q$ is singular (i.e., the Jacobian of the projection vanishes; equivalently $\partial q/\partial p = 0$ in local coordinates), so two classical branches coalesce at a turning point. The manifold $\mathcal{L}$ folds over itself, creating a multi-valued region (the "caustic zone") where the density of classical paths—and thus the semiclassical amplitude—formally diverges.

3 Derivation of the Caustic Ladder

We now apply the Action Quota to this geometry. The physical state is not the manifold $\mathcal{L}$ itself, but the wave field $\psi(q)$ supported by it.



(a) Single Fold (b) Quantized Orbits (Schematic)

Figure 1: (a) A Lagrangian manifold $\mathcal{L}$ folding in phase space (q: configuration, p: momentum). The projection onto q becomes singular (vertical) at the fold point (caustic). (b) Schematic of quantized orbits (harmonic oscillator example) showing nested stable manifolds. Each orbit satisfies the Bohr-Sommerfeld-Maslov condition with a different quantum number n.

3.1 The Phase Integral

The field at a point q is the superposition of contributions from all paths (branches of $\mathcal{L}$) reaching that point. In the semiclassical limit, we express this using a local coordinate u along the fold:

$$\psi(q) \sim \int du \, \exp\left(\frac{i}{\hbar} S(u;q)\right) \tag{1}$$

where $S(u;q)$ is the generating function of the manifold (the action) parametrized near the caustic by a local coordinate u along the Lagrangian manifold.

The cubic normal form $S(u;q) \approx \frac{u^3}{3} + qu$ is the universal unfolding of the fold catastrophe (Thom's theorem). This universal form is obtained by a smooth change of variables and Taylor expansion around the singular point, a standard result in catastrophe theory. Higher-order terms are subleading in the limit $\hbar \to 0$ provided the fold is non-degenerate (in normal form, the cubic term is present, i.e., $\partial_u^3 S \neq 0$ at the caustic). The integral is then exactly the Airy function, which describes the canonical wave pattern near any codimension-1 caustic:

$$\psi(q) \propto \mathrm{Ai}\left(\frac{q - q_c}{\ell_\hbar}\right) \tag{2}$$

where q_c is the caustic location and $\ell_\hbar := C\,\hbar^{2/3}$ ($C > 0$ set by local curvature of S) is the quantum resolution scale of the fold. This universal Airy structure fixes the phase matching across turning points and determines the Maslov contribution in the global quantization condition.

3.2 The Condition for Stability

A bound state is formed by two such folds (turning points) connected by the flow. For the state to be stationary (stable in time), the phase accumulated during a round trip between the folds must interfere constructively.

The total phase θ_{total} accumulated on a round trip consists of two parts: (i) the **Dynamical Phase** $\frac{1}{\hbar}\oint p\,dq = \frac{A(\mathcal{L})}{\hbar}$, and (ii) the **Topological Phase (Maslov Index)**.

The Maslov Phase Shift. The Maslov index accounts for phase shifts accumulated when trajectories pass through caustics (turning points). The Airy function has asymptotic behavior:

$$\text{For } z \to -\infty : \quad \text{Ai}(z) \sim \frac{1}{\sqrt{\pi |z|^{1/2}}} \cos\left(\frac{2}{3}|z|^{3/2} - \frac{\pi}{4}\right)$$
$$\text{For } z \to +\infty : \quad \text{Ai}(z) \sim \frac{1}{2\sqrt{\pi} z^{1/4}} \exp\left(-\frac{2}{3}z^{3/2}\right)$$

The oscillatory phase in the allowed region includes a constant shift of $-\pi/4$ relative to the WKB phase. When a closed orbit passes through a turning point, this contributes a topological phase shift of $-\pi/2$ (the connection formula). For a trajectory with two simple turning points (one at each fold), the total Maslov index is $m = 2$, contributing $-m\pi/2 = -\pi$ to the total phase.

The condition for constructive interference (single-valuedness of the wavefunction) requires:

$$\frac{1}{\hbar} \oint p\, dq - \frac{m\pi}{2} = 2\pi n \tag{3}$$

Rearranging:

$$\frac{1}{\hbar} \oint p\, dq = 2\pi n + \frac{m\pi}{2} = 2\pi \left(n + \frac{m}{4}\right) \tag{4}$$

where $m = 2$ is the Maslov index for a cycle with two simple turning points.

3.3 The Quantization Condition

For a harmonic oscillator with Hamiltonian $H = p^2/(2m) + m\omega^2 q^2/2$ and frequency ω, the classical action variable is $J = \oint p\, dq = E/\nu$, so $E = \nu J$, where $\nu = \omega/(2\pi)$ is the frequency in Hertz.

Result: The Caustic Ladder

$$E_n = \rho_0 \nu \left(n + \frac{1}{2}\right) = E_0 + n\rho_0\nu, \quad \text{where } E_0 = \frac{\rho_0 \nu}{2}. \tag{5}$$

3.4 Extension to General Potentials

This matches the exact harmonic oscillator spectrum and provides the leading-order (WKB) approximation for general smooth potentials with isolated turning points [**?**]. Corrections enter at order $O(\hbar^2)$ and depend on potential anharmonicity. For general potentials, Hamilton-Jacobi theory provides $dE/dJ = \omega(E)$, leading to the expansion $E \approx \omega_0 J + \dots$, recovering the ladder as the leading-order behavior. This WKB quantization breaks down when: (i) turning points merge (near the classical threshold), (ii) the potential is not smooth, or (iii) quantum tunneling between separate wells becomes significant.

4 Physical Interpretation

4.1 Why Discrete?

Why can't the energy be $E = 1.1\rho_0\nu$? Because such an orbit would accumulate a phase mismatch of $0.1 \times 2\pi$ per cycle. Such a phase mismatch fails the stationarity condition: after one complete cycle, the wavefunction would return to the same point with a different phase, making it multi-valued. Attempting to "glue" these branches together creates destructive interference—the state self-cancels. Only when the accumulated phase is an integer multiple of 2π (accounting for the

Maslov shift) can the wave close smoothly onto itself, forming a stable, self-consistent eigenmode. The system can only "rest" on the discrete rungs of the caustic ladder; intermediate energies correspond to unstable, destructive interference between the folds.

4.2 The Universality of Quantization

This result explains the universality of *quantization* across different physical systems—energy is always discrete when constrained by fold boundaries. The specific *functional form* $E(n)$ depends on the potential: equal spacing (harmonic potential), quadratic (anharmonic potential), or inverse square (Coulomb potential). But the discreteness itself—the existence of stable caustic stations—is universal.

5 Discussion

We have derived the discrete energy spectrum from the geometry of a folded action sheet. The "quantum levels" of an atom are revealed to be the stable interference fringes of a capacity-limited continuum. While the fold caustic itself represents the spatial boundary of the wavefunction, the resonance condition for these folds generates the discrete energy spectrum.

This derivation demystifies the "quantum jump." In the fold picture, a jump corresponds to moving between adjacent resonant stations; the intermediate states are physically unstable—they are "off the ladder" and cannot sustain a persistent fold.

Thus, quantization is not a property imposed on the field, but an emergent stability condition for folds in an incompressible action fluid. The quantum numbers label the stable resonance modes of a geometric structure, not arbitrary labels on a mysterious "quantum state."

In Paper III, we will ask: what if the fold is not simple? What if the sheet twists before it folds? We will find that the topology of the twisted fold forces the emergence of Spin.

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