Lemma 6: The Inflation of the State Space

From Interaction Generators to the Tensor Product

Emiliano Shea

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Abstract

We extend the reconstruction to composite systems ($N = 2$). We address the **Interaction Gap**: the observation that the direct sum algebra of two independent qubits (dimension 6) is insufficient to support reversible interactions. We invoke the principle of **Global Action Consistency**: any generator of reversible evolution must exist as an observable within the algebra. We prove that adjoining a single generic nonlocal interaction generator to the local algebra forces the observable space to expand from $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ to the full tensor product $\mathfrak{su}(4)$ via commutator closure. This algebraic inflation ($6 \to 15$ dimensions) physically corresponds to the emergence of entanglement.

1 Introduction

To bridge the gap between single-system and composite-system physics, we invoke the central consistency principle of this reconstruction immediately.

Axiom 1 (Global Action Consistency)**.** *The dual role of the Hamiltonian established in Lemma 5 must hold globally: any generator of reversible time evolution must be a measurable observable in the system's algebra.*

This axiom generalizes Lemma 5's identification of Hamiltonians as observables from single systems to composite systems. The logical chain is: Interactivity $\Rightarrow$ Observability $\Rightarrow$ Closure $\Rightarrow$ Tensor Product.

For a single qubit, the state space is the Bloch ball B^3 and the algebra of observables is the 3-dimensional space of traceless Hermitian matrices. For a composite system of two qubits, A and B, a classical combination implies the Cartesian product $B^3 \times B^3$. This "Naive Composition" corresponds to the direct sum algebra $\mathcal{L}_0 = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, which has dimension $3 + 3 = 6$. The observables are limited to local operations:

$$\mathcal{O}_{naive} = \{\vec{a} \cdot \vec{\sigma}_A \oplus \vec{b} \cdot \vec{\sigma}_B\}$$

We show that this structure is dynamically trivial: it cannot support interaction.

1.1 Roadmap

Our strategy is constructive:

1. Start with the non-interacting local algebra $\mathcal{L}_0$.
2. Postulate the existence of at least one nonlocal reversible transformation (Axiom 2).
3. Apply the commutator closure required by Axiom 1.
4. Show that for generic interactions, this closure spans the full algebra $\mathfrak{su}(4)$.

Theorem Summary: We prove that a single generic nonlocal generator plus local generators implies that the Lie algebra closure is $\mathfrak{su}(4)$.

2 The Interaction Gap

Definition 1 (Local Algebra)**.** *The local algebra is the direct sum space $\mathcal{L}_0 = span\{\sigma_i \otimes \mathbb{I}, \mathbb{I} \otimes \sigma_j\}$ for $i, j \in \{x, y, z\}$.*

Hamiltonians in $\mathcal{L}_0$ generate factorizable dynamics $U(t) = U_A(t) \otimes U_B(t)$, mapping separable states to separable states. To allow system A to influence system B, we require a Hamiltonian outside this set.

Axiom 2 (Interaction)**.** *It is possible to perform a reversible operation on system B that depends on the state of system A (e.g., a Conditional Rotation).*

Operational Justification: This axiom is empirically necessary. We observe conditional operations throughout physics: collision processes, beam splitters, and exchange couplings. More fundamentally, measurement requires coupling a system to a probe whose evolution depends on the system's state. Without interaction, quantum theory would be empirically vacuous.

3 Derivation of the Tensor Product

3.1 Conventions

We work with the real vector space of Hermitian observables. By Lemma 5, this space forms a Lie algebra under the scaled commutator.

Conventions

- **Units:** We set $\hbar = 1$.
- **Lie Bracket:** We use the bracket $\frac{1}{2i}[A, B]$.
- **Pauli Algebra:** With this bracket, $\frac{1}{2i}[\sigma_x, \sigma_y] = \sigma_z$.
- **Identity:** $\mathbb{I}$ denotes the identity matrix.

This matches the standard angular momentum convention $\vec{J} = \frac{1}{2}\vec{\sigma}$. Equivalently, one can work with the Lie algebra $\mathfrak{su}(4)$ of traceless skew-Hermitian matrices.

3.2 Algebraic Inflation

Let $\mathcal{L}_{gen}$ be the algebra generated by the local terms $\mathcal{L}_0$ and a single interaction generator H_{int}.

Theorem 1 (Algebraic Closure)**.** *Let $H_{int} = \sigma_u \otimes \sigma_v$ be any generic nonlocal interaction (with $u, v \in \{x, y, z\}$). Then the Lie algebra generated by $\mathcal{L}_0 \cup \{H_{int}\}$ is exactly $\mathfrak{su}(4)$ (dimension 15).*

By "generic," we mean any H_{int} with at least one nonzero coefficient on a nonlocal Pauli product that does not belong to the normalizer of $\mathcal{L}_0$.

Proof. The set

$$\{\sigma_i \otimes \sigma_j\}_{i,j \in \{0,x,y,z\},(i,j)\neq(0,0)} \tag{3.1}$$

forms a basis for traceless Hermitian operators on $\mathbb{C}^4$. Starting with the 6 local terms and one seed H_{int}, repeated commutation with local terms rotates the interaction into orthogonal directions. Iterating this process generates the complete basis. (See Appendix A for the explicit constructive closure). □

Lemma 1 (Exceptional Interactions). *There exist measure-zero interactions that fail to generate $\mathfrak{su}(4)$—for instance, interactions proportional to a local operator or those lying in the normalizer of $\mathcal{L}_0$ (such as the Ising interaction $\sigma_z \otimes \sigma_z$ in a restricted symmetry subspace). These interactions are non-generic and do not provide the requisite interactivity postulated in Axiom 2.*

Remark 1 (Universality). *This Lie-algebraic closure is the infinitesimal analogue of the universal two-qubit gate result: almost any entangling two-qubit gate together with local unitaries generates $SU(4)$ [?].*

3.3 Example: The CNOT Gate

Example: CNOT Generation. The controlled-NOT operation flips qubit B if qubit A is $|1\rangle$. The generator is:

$$H_{\text{CNOT}} = \frac{\pi}{4}(\mathbb{I} - \sigma_z) \otimes \sigma_x$$

The term $\sigma_z \otimes \sigma_x$ is the nonlocal seed. Time evolution yields:

$$U_{\text{CNOT}} = e^{-iH_{\text{CNOT}}} = |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes \sigma_x$$

The existence of this gate requires the full $\mathfrak{su}(4)$ algebra.
Remark: The prefactor in H_{CNOT} is chosen so that $e^{-iH_{\text{CNOT}}}$ yields the standard CNOT up to an overall phase; additive multiples of the identity commute with all operators and do not affect the gate action on states.

3.4 Dimension Counting

The closure forces a dramatic inflation of the observable space:

- **Local Dimensions (6):** Parameters describing independent subsystems.
- **Correlation Dimensions (9):** Coefficients of $\sigma_i \otimes \sigma_j$. These new dimensions correspond physically to entanglement.

3.5 Entanglement as Algebraic Necessity

A state ρ is *separable* if it can be written as a convex combination of product states: $\rho = \sum_k p_k(\rho_A^{(k)} \otimes \rho_B^{(k)})$. Separable states lie in a 6-dimensional submanifold of the full 15-dimensional state space. The 9 correlation dimensions correspond to **entangled** states that cannot be factored. These arise necessarily when generic nonlocal Hamiltonians evolve separable states, making entanglement an algebraic necessity.

4 Alternatives: Sketch and Elimination

4.1 Real Quantum Mechanics

In a real vector space theory, the observable algebra for two qubits would be $\mathfrak{so}(4)$ (dimension 6). Real QM does not naturally provide the 9-dimensional correlation space required for entanglement and fails to support Bell non-locality or quantum computational advantage [?].

4.2 Generalized Probabilistic Theories (Polytopes)

State spaces like hypercubes ("Boxworld") allow correlations but are incompatible with continuous reversible dynamics. A continuous Hamiltonian flow on a polytope generally rotates vertices (pure states) into the interior (mixed states), violating the conservation of purity. Only smooth, constant-curvature manifolds support the transitive reversible dynamics required by Lemma 4's continuous unitary evolution [?].

5 State Space Geometry: The Convex Structure

Having established the observable algebra $\mathfrak{su}(4)$ and ruled out alternatives, we identify the state space geometry.

Proposition 1 (Convex Hull of Unitary Orbits)**.** *Transitive action of $SU(4)$ on pure states plus convexity (operational mixing) implies the state space is the convex hull of pure states—i.e., the set of unit-trace positive semi-definite 4×4 matrices.*

The "Interaction Gap" between the factorizable states is filled by the entangled interior of this body.

6 Conclusion

The tensor product structure of quantum mechanics need not be taken as a primitive postulate; it follows from the following operational requirements:

1. **Interactivity:** Systems must be able to influence each other.
2. **Global Action Consistency:** Interaction generators must be observables.
3. **Closure:** The algebra of observables must be closed under commutation.

Lemma 6 bridges single-system structure (Lemmas 1–5) and many-body consequences (Lemma 7 onward). See Theorem **??** and Proposition **??** for the formal derivation.

Remark 2 (Computational Consequences)**.** *This algebraic inflation underlies the exponential growth of the state space (2^N dimensions for N qubits), providing the algebraic origin of quantum computational capacity [?].*

References

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A Explicit Closure Construction

Starting with $\mathcal{L}_0$ and $H_{int} = \sigma_z \otimes \sigma_x$:

1. **Commute with Local A ($\sigma_x \otimes \mathbb{I}$):**

$$\frac{1}{2i}[\sigma_x \otimes \mathbb{I}, \sigma_z \otimes \sigma_x] = -\sigma_y \otimes \sigma_x \tag{A.1}$$

   Normalization constants like $-2i$ are scalars; rescaling yields the basis element $\sigma_y \otimes \sigma_x$.

2. **Commute with Local B ($\mathbb{I} \otimes \sigma_z$):**

$$\frac{1}{2i}[\sigma_y \otimes \sigma_x, \mathbb{I} \otimes \sigma_z] = -\sigma_y \otimes \sigma_y \tag{A.2}$$

3. **Rotate A:** Commuting with $\sigma_z \otimes \mathbb{I}$ rotates $\sigma_y \otimes \sigma_y \to \sigma_x \otimes \sigma_y$.

4. **Rotate B:** Commuting with $\mathbb{I} \otimes \sigma_x$ rotates $\sigma_x \otimes \sigma_y \to \sigma_x \otimes \sigma_z$.

5. **Generate Cross-Terms ($\sigma_x \otimes \sigma_x$):** From step 3, we have $\sigma_x \otimes \sigma_y$. Commuting with $\mathbb{I} \otimes \sigma_z$:

$$\frac{1}{2i}[\sigma_x \otimes \sigma_y, \mathbb{I} \otimes \sigma_z] = \sigma_x \otimes \sigma_x \tag{A.3}$$

   Finally, commuting distinct product terms generates the remaining basis elements.

By systematically applying local rotations, we span the full algebra $\mathfrak{su}(4)$.