Lemma 8 (The Calibration):

The Thermodynamic Calibration of Action

From Algebraic Structure to Planck's Constant

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Abstract

In Lemmas 1–7, we reconstructed the structural framework of quantum mechanics (Hilbert space, unitary dynamics, and canonical commutation relations) from the Action Quota axiom. However, these derivations treated the action scale $\hbar$ as an algebraic free parameter. In this final lemma, we determine its numerical value. By applying the previously derived canonical algebra to electromagnetic field modes and requiring consistency with the macroscopic Stefan–Boltzmann radiation law, we derive the energy density as a function of the action scale. Matching this result to the measured Stefan–Boltzmann constant fixes the action scale to the reduced Planck constant. This completes the programme: structure is determined by informational and geometric constraints, while scale is fixed by thermodynamic consistency.

Keywords: Planck's constant, thermodynamic consistency, blackbody radiation, harmonic oscillator spectrum, Action Quota calibration

1 Introduction: Structure vs. Scale

Our reconstruction has established that if nature operates under a finite budget of certainty—the Action Quota—then physical states must reside in complex Hilbert spaces governed by non-commuting observables. Conceptually, Lemmas 1–7 fix the *dimensionless* structure of the theory; Lemma 8 fixes the remaining *dimensionful* conversion factor between time and phase (action) by matching a macroscopic radiometric constant. In **Lemma 5**, we proved the unification $\hbar = \kappa$, identifying the same action scale in both uncertainty relations and dynamics. In **Lemma 7**, we showed that continuous phase space variables X and P must satisfy:

$$[X, P] = i\hbar_0 \mathbb{I}, \tag{1}$$

where $\hbar_0$ is the same uncalibrated scale of physical action. We emphasize that $\hbar_0$ is *not* fixed by algebra alone: it is a dimensionful conversion factor whose numerical value must be calibrated empirically.

However, eq. (1) does not tell us the magnitude of $\hbar_0$. We answer this by coupling our reconstructed system to a thermal reservoir. We show that the algebraic structure of quantum mechanics forces energy to be processed in discrete packets, which allows us to calibrate $\hbar_0$ against the macroscopic Stefan–Boltzmann constant.

2 From the Action Quota to Energy Quantization

The connection between the Action Quota and energy quantization is a logical consequence of the results established in Lemmas 4 and 7.

2.1 The Harmonic Oscillator Spectrum

From Lemma 7, a continuous-variable degree of freedom with frequency ω may be represented as a harmonic oscillator with Hamiltonian

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2,$$

together with the canonical commutator $[X, P] = i\hbar_0\mathbb{I}$. (One may set $m = 1$ by a canonical rescaling; we keep m here to display the dimensionless normalization explicitly.) The algebra of eq. (1) allows us to define dimensionless ladder operators:

$$a = \sqrt{\frac{m\omega}{2\hbar_0}}X + \frac{i}{\sqrt{2m\omega\hbar_0}}P, \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar_0}}X - \frac{i}{\sqrt{2m\omega\hbar_0}}P,$$

which satisfy $\left[a, a^\dagger\right] = \mathbb{I}$ and yields the energy form $H = \hbar_0\omega(a^\dagger a + 1/2)$.

Commutation Check. Using $[X, P] = i\hbar_0\mathbb{I}$ and $[X, X] = [P, P] = 0$, one finds

$$\left[a, a^\dagger\right] = \frac{m\omega}{2\hbar_0}[X, X] + \frac{1}{2m\omega\hbar_0}[P, P] - \frac{i}{2\hbar_0}[X, P] + \frac{i}{2\hbar_0}[P, X] = 0 + 0 + \frac{1}{2} + \frac{1}{2} = 1.$$

Spectrum Derivation. Define $N = a^\dagger a$. The commutation relations imply $[N, a] = -a$ and $\left[N, a^\dagger\right] = a^\dagger$. Let $|n\rangle$ be an eigenstate of N with eigenvalue n. Then:

$$N(a^\dagger |n\rangle) = (n + 1)a^\dagger |n\rangle, \quad N(a |n\rangle) = (n - 1)a |n\rangle.$$

Since $\langle N\rangle \geq 0$, there exists a ground state $|0\rangle$ with $a |0\rangle = 0$ and $N |0\rangle = 0$. Repeated application of $a^\dagger$ generates states with $n = 0, 1, 2, \ldots$. From $H = \hbar_0\omega(N + 1/2)$, we obtain the quantized spectrum:

$$E_n = \left(n + \frac{1}{2}\right)\hbar_0\omega, \quad n \in \{0, 1, 2, \ldots\}. \tag{2}$$

2.2 Field Modes as Oscillators

In a cavity, the electromagnetic field decomposes into independent normal modes labeled by k (and polarization), each with a quadratic Hamiltonian of the oscillator form,

$$H_k = \frac{P_k^2}{2} + \frac{1}{2}\omega_k^2 X_k^2,$$

after a choice of canonical field coordinates (X_k, P_k).[1] By Lemma 7, each canonical pair satisfies $[X_k, P_k] = i\hbar_0\mathbb{I}$, and therefore each mode has the quantized spectrum $E_{n,k} = (n + 1/2)\hbar_0\omega_k$. By Lemma 6, the tensor product structure ensures that modes are independent quantum systems.

[1] Equivalently one may include mode-dependent effective masses; this is absorbed by a rescaling of (X_k, P_k) and does not affect the spectrum.

2.3 Zero-Point Energy and Thermal Increments

The factor 1/2 in eq. (2) represents the zero-point energy $E_0 = \hbar_0\omega/2$. In thermal equilibrium at temperature T, only *thermal* energy increments contribute to the temperature-dependent part of the internal energy (the T-independent vacuum offset may be removed by normal ordering for this purpose). Therefore, for thermodynamic averages we consider excitation energies relative to the ground state:

$$\Delta E = E_n - E_0 = n\hbar_0\omega = nh_0\nu,$$

where $h_0 = 2\pi\hbar_0$ and $\nu = \omega/2\pi$ is the cycle frequency. This establishes the quantum of energy exchange with a thermal reservoir.

3 Derivation of the Blackbody Spectrum

3.1 Statistical Mechanics of the Modes

For a single mode of frequency ν, we work with the thermal partition function (excluding the zero-point energy, which does not contribute to thermal exchanges):

$$Z_{\text{thermal}} = \sum_{n=0}^{\infty} e^{-\beta n h_0\nu} = \frac{1}{1 - e^{-\beta h_0\nu}} \tag{3}$$

where $\beta = 1/(k_B T)$. The mean energy per mode is:

$$\bar{E}_\nu = -\frac{\partial \ln Z_{\text{thermal}}}{\partial\beta} = \frac{h_0\nu}{e^{\beta h_0\nu} - 1}. \tag{4}$$

Density of States. In a cavity of volume V, the number of electromagnetic modes with frequencies between ν and $\nu + \mathrm{d}\nu$ is $g(\nu)\mathrm{d}\nu = \frac{8\pi V}{c^3}\nu^2\mathrm{d}\nu$. This follows from counting standing waves in a box and accounts for two polarization states per mode. We thus obtain the spectral energy density per unit volume:

$$u(\nu, T) = \frac{8\pi h_0}{c^3}\frac{\nu^3}{e^{h_0\nu/k_B T} - 1}. \tag{5}$$

4 Empirical Calibration

The total energy density U/V is found by integrating $u(\nu, T)$ over all frequencies:

$$\frac{U}{V} = \int_0^\infty \frac{8\pi h_0}{c^3}\frac{\nu^3}{e^{h_0\nu/k_B T} - 1}\mathrm{d}\nu = \frac{8\pi h_0}{c^3}\left(\frac{k_B T}{h_0}\right)^4 \int_0^\infty \frac{x^3}{e^x - 1}\mathrm{d}x \tag{6}$$

where $x = h_0\nu/(k_B T)$. The integral evaluates to $\int_0^\infty \frac{x^3}{e^x-1}\mathrm{d}x = \frac{\pi^4}{15}$ (a standard Bose integral obtained by series expansion of $(e^x - 1)^{-1}$; see e.g. [3]). Thus:

$$\frac{U}{V} = \frac{8\pi^5 k_B^4}{15c^3 h_0^3}T^4.$$

The radiated energy flux is $\Phi = \frac{c}{4}(U/V) = \sigma_{\text{pred}}T^4$, where:

$$\sigma_{\text{pred}}(h_0) = \frac{2\pi^5 k_B^4}{15c^2 h_0^3}. \tag{7}$$

Numerical verification. Using standard reference values for k_B and c together with the experimentally realized Stefan–Boltzmann constant σ_{exp} (see e.g. [6]):

$$k_B = 1.380\,649\times10^{-23}\,\mathrm{J\,K^{-1}}, \quad c = 299\,792\,458\,\mathrm{m\,s^{-1}}, \quad \sigma_{\text{exp}} = 5.670\,374\,419\times10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}},$$

we solve for h_0:

$$h_0 = \left(\frac{2\pi^5(1.380\,649 \times 10^{-23}\,\mathrm{J\,K^{-1}})^4}{15(299\,792\,458\,\mathrm{m\,s^{-1}})^2(5.670\,374\,419 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}})} \right)^{1/3} \approx 6.626\,070\times10^{-34}\,\mathrm{J\,s}.$$

This agrees with the measured value of Planck's constant h (within the displayed rounding). Thus:

$$\boxed{h_0 = h \implies \hbar_0 = \hbar = \frac{h}{2\pi} \approx 1.054\,571\,8 \times 10^{-34}\,\mathrm{J\,s}.} \tag{8}$$

Remark 4.1 (Historical Context)**.** *Our calibration via blackbody radiation mirrors Planck's original 1901 derivation [1], but with a crucial difference: Planck postulated energy quantization to explain the spectrum, whereas we derive quantization from the algebraic structure established in Lemmas 1–7, then calibrate the scale via the same thermodynamic data.*

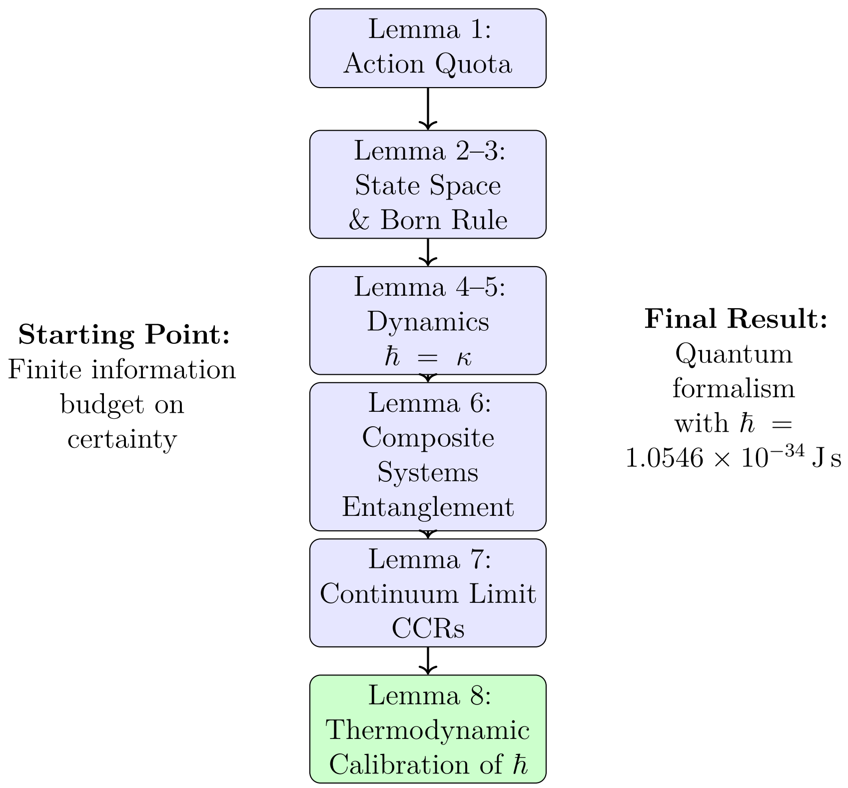

Figure 1: Architecture of the reconstruction programme. Each lemma builds upon the previous, ultimately calibrating the action scale via thermodynamics.

5 Discussion: Synthesis of the Reconstruction

Architecture of the Reconstruction. Starting from a single operational principle—the Action Quota—we have derived:

1. **Lemma 1–3:** The geometry of state space (Bloch ball) and measurement statistics (Born rule).
2. **Lemma 4–5:** Unitary dynamics (Schrödinger equation) and the unification $\hbar = \kappa$.
3. **Lemma 6:** Composite systems and entanglement via algebraic closure.
4. **Lemma 7:** Continuous variables via group contraction.
5. **Lemma 8:** Numerical calibration of $\hbar$ via thermodynamics.

Comparison to Other Programmes. Unlike Hardy's five axioms [4] or the operational approach of Chiribella et al. [5], our approach uses a single quantitative constraint (Action Quota) to drive the geometry, rather than multiple qualitative axioms. The thermodynamic calibration (Lemma 8) provides an explicit bridge from the reconstructed Hilbert-space structure to experimentally fixed constants.

Scope and Limitations. This reconstruction addresses the kinematic and dynamical framework of quantum mechanics. It does not derive specific Hamiltonians (which must be determined empirically), extend to relativistic quantum field theory, or resolve the measurement problem.

6 Conclusion

The reconstruction is now complete. We have shown that the distinctive features of quantum mechanics—superposition, entanglement, uncertainty, and quantization—are not arbitrary postulates but necessary geometric consequences of operating in a universe with finite information-processing capacity.

The Action Quota, initially a simple variance bound, cascades through mathematical necessity into complex Hilbert spaces, non-commuting observables, unitary dynamics, and finally to discrete energy levels. The numerical scale of this quota, $\hbar$, is not a free parameter but is fixed by the requirement that quantum statistics must reproduce macroscopic thermodynamics. This work demonstrates that quantum mechanics is the unique theory consistent with a fundamental limit on simultaneous knowledge. The reconstruction provides a coherent, principled foundation that explains both the structure and scale of quantum theory from a single operational starting point, completing a century-long quest to understand quantum theory's foundations from first principles.

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