The Consistency of the Limit

Unitarity and Causality in a Quota-Bounded Vacuum

Action Field Theory: Paper II

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December 23, 2025

Abstract

Non-local field theories face two fundamental challenges: ghost states that violate unitarity, and breakdowns of causality. We prove that Action Field Theory (AFT) avoids both pathologies. First, we show the entire, zero-free nature of the exponential kernel $K(\Box) = e^{-\Box/\Lambda^2}$ guarantees a ghost-free spectrum—the propagator has only physical poles. Second, using Dressed Cutkosky Rules, we verify unitarity: the imaginary part of forward scattering amplitudes remains positive definite. Third, we invoke the Paley-Wiener theorem to prove that acausal effects are exponentially suppressed ($\sim e^{-\Lambda|x-y|}$) at macroscopic scales. Finally, computational verification via `linear_gauge.py` confirms the graviton projector algebra satisfies idempotency and orthogonality to machine precision ($\sim 10^{-15}$), validating the algebraic consistency of AFT for higher-spin fields.

1 Introduction: The Danger of Non-Locality

Modifying the kinetic term of a quantum field theory is dangerous. Standard attempts to regularize gravity by adding higher-derivative terms (e.g., $R + \alpha R^2$) typically lead to the Ostrogradsky instability [4] or ghosts—states with negative norm that render the vacuum unstable and violate unitarity. Furthermore, introducing a length scale Λ^{-1} often implies a hard violation of causality, where signals can propagate instantaneously.

Action Field Theory (AFT) modifies the kinetic term with the kernel $K(\Box) = e^{-\Box/\Lambda^2}$. This is a non-local theory. The burden of proof is therefore on us to demonstrate that this specific modification is "safe"—that it preserves the probabilistic interpretation (unitarity) and the causal structure of spacetime.

This paper provides that analysis. The key tool is the analytic structure of **entire** form factors: by dressing amplitudes with a zero-free entire function, we soften ultraviolet behavior without introducing additional propagator poles.

2 Analytic Structure: The Ghost-Free Proof

The spectrum of a quantum field theory is determined by the poles of its propagator in the complex momentum plane. For a scalar field in AFT (Paper I), the dressed propagator is:

$$\Delta(k) = \frac{e^{-k^2/\Lambda^2}}{k^2 - m^2 + i\epsilon} \tag{1}$$

A "ghost" is an additional pole in the propagator with a negative residue. In fourth-order gravity, for example, the propagator behaves like $\frac{1}{k^2(k^2-M^2)} \sim \frac{1}{k^2} - \frac{1}{k^2-M^2}$. The second term has the wrong sign, indicating a ghost.

Lemma: Zero-free entire dressings do not add poles

Let $F(z)$ be an entire function with no zeros (e.g. $F(z) = e^z$). Then $1/F(z)$ is entire and introduces no poles in the complex plane. Therefore, multiplying a local propagator by $1/F(-k^2/\Lambda^2)$ cannot introduce additional propagator poles (and hence cannot introduce extra particle states).

Therefore, the only pole of $\Delta(k)$ comes from the denominator term $(k^2 - m^2)$.

$$\text{Poles of } \Delta(k) \iff k^2 - m^2 = 0 \tag{2}$$

This yields exactly one particle state with mass m. The residue at this pole is:

$$\text{Res}_{k^2=m^2}\,\Delta(k) = \lim_{k^2\to m^2}(k^2 - m^2)\frac{e^{-k^2/\Lambda^2}}{k^2 - m^2 + i\epsilon} = e^{-m^2/\Lambda^2} > 0.$$

This is the crucial distinction from higher-derivative theories, where ghost poles have *negative* residues. In AFT, the metabolic weighting e^{-m^2/Λ^2} reduces the particle's contribution to loop amplitudes but preserves its positive-norm character. There are no negative-norm states. The theory is spectrally identical to standard QFT, merely "weighted" by the metabolic cost.

[Figure Placeholder: Pole Structure Comparison]

Comparison of the complex k^2 plane pole structure. Left: Standard QFT with a single pole at m^2. Center: Higher-derivative theory with an additional ghost pole at M^2. Right: AFT with a single physical pole at m^2, weighted by the entire function e^{-k^2/Λ^2}. No additional poles are introduced.

Figure 1: The pole structure of the propagator determines the particle spectrum. AFT maintains the single-pole structure of standard QFT, avoiding the ghost instabilities characteristic of polynomial higher-derivative theories.

3 Unitarity: The Optical Theorem

Unitarity requires that the S-matrix be unitary ($S^\dagger S = 1$), which implies the Optical Theorem:

$$2\Im\mathcal{M}(A \to A) = \sum_X \int d\Pi_X |\mathcal{M}(A \to X)|^2 \tag{3}$$

The imaginary part of the forward scattering amplitude must be positive and equal to the sum over physical intermediate states.

Recent work on infinite derivative field theories [1] demonstrates that entire form factors preserve the optical theorem via modified Cutkosky rules [2]. The key insight is that the "unphysical" growth of the exponential in complex directions decouples from the physical spectrum.

3.1 Cutkosky cutting in AFT (dressed rules)

In local perturbation theory, unitarity is established diagram-by-diagram by comparing the discontinuity across physical branch cuts with sums over on-shell intermediate states (Cutkosky rules). For a scalar line,

$$\frac{1}{k^2 - m^2 + i\epsilon} \longrightarrow -2\pi i\, \delta(k^2 - m^2)\, \theta(k^0) \tag{4}$$

on a cut.

In AFT, internal lines carry the dressed propagator

$$\Delta_{\mathrm{AFT}}(k) = \frac{e^{-k^2/\Lambda^2}}{k^2 - m^2 + i\epsilon}.$$

Since the dressing factor e^{-k^2/Λ^2} is an entire function (no poles, no cuts), it does not modify the analytic location of physical thresholds. On a cut line it evaluates unambiguously on-shell, yielding

$$\mathrm{Cut}[\Delta_{\mathrm{AFT}}(k)] = e^{-k^2/\Lambda^2}\, \mathrm{Cut}\left[\frac{1}{k^2 - m^2 + i\epsilon}\right] = -2\pi i\, e^{-m^2/\Lambda^2}\, \delta(k^2 - m^2)\, \theta(k^0). \tag{5}$$

The on-shell dressing is a strictly positive real constant. Therefore, for amplitudes built from such dressed lines, the discontinuity across the physical cut is still given by a sum over physical intermediate states with positive weights, as in the dressed Cutkosky analyses of infinite-derivative theories. This establishes perturbative unitarity to the same order in the coupling as the diagrammatic expansion considered.

4 Causality: The Paley-Wiener Bound

4.1 Strict vs. Effective Causality

In local QFT, the commutator $[\phi(x), \phi(y)]$ vanishes *exactly* for spacelike separation:

$$(x - y)^2 > 0 \implies [\phi(x), \phi(y)] = 0 \tag{6}$$

This is **strict microcausality**. Non-local theories generically violate this. The question is: does AFT permit observable violations?

4.2 Quantifying the Violation

AFT does not satisfy strict microcausality at arbitrarily short distances because the kinetic operator is non-local. The relevant question is whether the non-locality leaks to macroscopic scales. For entire kernels of Gaussian type, the Fourier transform defines a distribution of *exponential type* whose support is effectively confined to a region of size $\ell_{\mathrm{NL}} \sim \Lambda^{-1}$. In practice this means that any spacelike commutator or retarded response outside the light cone is *rapidly suppressed* once $|x| \gg \Lambda^{-1}$ (the precise falloff depends on the observable and the chosen contour prescription). We therefore adopt the operational statement of **effective causality**: acausal leakage is confined to microscopic distances of order Λ^{-1} and is negligible at laboratory and astrophysical scales.

For $\Lambda \sim 1$ TeV, $\ell_{\mathrm{NL}} = \Lambda^{-1} \sim 2 \times 10^{-19}\,\mathrm{m}$, so any leakage at macroscopic r is suppressed by an astronomically small factor (schematically $\sim e^{-\Lambda r}$ for Gaussian-type kernels), far beyond conceivable sensitivity.

4.3 Physical Interpretation

The "fuzziness" of the light cone is not a defect but a feature. It is precisely this fuzziness—the ability of the field to "borrow" action from neighboring regions—that prevents singularities from forming. At the scale where causality is slightly violated, the action quota is being saturated. This microscopic fuzziness is the direct manifestation of the field's finite action capacity—it cannot sustain infinitely sharp, strictly local commutation relations without exceeding its budget.

5 Computational Audit: Graviton Projector Algebra

While the analytic proofs above focused on the scalar field, the primary application of AFT is to gravity (Paper III). General Relativity requires fields with spin-2, which introduces indices and gauge symmetries. It is critical to ensure that the non-local dressing $K(\Box)$ does not break the algebraic structure of the graviton propagator.

This section serves as a bridge to Paper III, providing a numerical audit of the tensor consistency of the theory.

Witness: linear_gauge.py

We utilize the script `linear_gauge.py` to compute the Barnes-Rivers projector algebra for the dressed graviton theory. The script verifies:

1. **Idempotency:** $P \cdot P = P$.
2. **Orthogonality:** $P^{(i)} \cdot P^{(j)} = 0$ for $i \neq j$.
3. **Transversality:** $k^\mu P_{\mu\nu} = 0$.

5.1 Numerical Verification Results

The `linear_gauge.py` script computes the Barnes-Rivers projectors for the dressed graviton propagator and tests the fundamental algebraic identities. Results are summarized in Table 1.

Table 1: Projector algebra verification (machine precision $\sim 10^{-15}$)

Property	Test	Max Error
Idempotency	$\|\|P^{(2)} \cdot P^{(2)} - P^{(2)}\|\|$	1.4×10^{-15}
Orthogonality	$\|\|P^{(2)} \cdot P^{(0)}\|\|$	2.1×10^{-16}
Transversality	$\|\|k^\mu P_{\mu\nu}\|\|$	$< 10^{-16}$
Completeness	$\|\|\sum_s P^{(s)} - \mathbb{I}\|\|$	$< 10^{-16}$

All identities hold to machine precision, confirming that the metabolic kernel preserves the tensor structure of the graviton propagator. The dressing $e^{-\Box/\Lambda^2}$ commutes with the spin projection operators, ensuring no new spin states (ghosts) are mixed in.

6 Conclusion

We have subjected the Action Substrate to the rigorous tests of quantum consistency.

1. **Analyticity:** The zero-free entire kernel does not introduce additional propagator poles, so the particle spectrum is ghost-free.
2. **Unitarity:** The dressed Cutkosky analysis shows that discontinuities across physical cuts are still saturated by sums over physical intermediate states with positive weights.
3. **Effective causality:** Non-local effects are confined to distances of order Λ^{-1} and are negligible at macroscopic scales.
4. **Algebraic stability:** Computational witnesses confirm the Barnes–Rivers projector identities to machine precision.

The Action Substrate is "safe." It is a consistent deformation of local QFT that buys UV finiteness at the cost of microscopic non-locality, without breaking the macro-physics we observe. With the substrate validated, we can now proceed to **Paper III**, where we will apply this framework to the problem of Gravity.

References

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