Lemma 5 (The Unification):

Unifying the Algebraic and Dynamical Action Scales

The Dual Role of the Hamiltonian and the Identity $\hbar = \kappa$

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Abstract

We prove that consistency between algebraic uncertainty bounds and dynamical evolution forces the identity $\hbar = \kappa$, completing the reconstruction of single-qubit quantum mechanics. Lemma 1 established a variance trade-off (*Action Quota*) which defines a static scale $\hbar$. Lemma 4, through topological consistency, defined a *dynamic* scale κ governing the rate of state evolution. By identifying the Hamiltonian's dual role as both a physical observable and a generator of evolution, we show that a unified scale is the unique solution that makes the observable calibration of energy gaps and the dynamical calibration of phase agree in interferometric predictions.

Keywords: Action scales, uncertainty principle, Hamiltonian, scale unification, information-theoretic reconstruction, Ramsey interferometry.

1 Introduction: Two Scales of Action

Our reconstruction has produced two fundamental constants that appear in distinct operational contexts. In the initial derivations, these scales are logically independent.

Definition 1.1 (Static Action Scale, $\hbar$)**.** *The Action Quota established that for complementary observables, certainty is finite. The scale $\hbar$ is the commutator scale used to identify the generator of a symmetry as a measurable observable:*

$$[X, Y] = i\hbar Z.$$

This quantifies the physical "size" of a quantum of certainty within the measurement algebra.

Definition 1.2 (Dynamic Action Scale, κ)**.** *In the derivation of dynamics (Lemma 4), the Schrödinger equation appears as the unique representation of reversible evolution on Hilbert space:*

$$i\kappa \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle. \tag{1}$$

The scale κ serves as the conversion factor between energy and the frequency of temporal state change, governing how relative phases accumulate: $\Delta\phi = (\Delta E)t/\kappa$.

2 Preliminaries: Recapping Lemmas 1–4

The reconstruction to this point has established the following foundation:

- **Lemma 1:** Complementarity implies a variance-sum bound, establishing a unit disk geometry.
- **Lemma 2:** Isotropy and efficiency inflate this geometry into the 3D Bloch Ball (B^3).
- **Lemma 3:** State statistics are uniquely governed by the linear Born Rule.
- **Lemma 4:** Path-sensitivity requires a complex 2D Hilbert space ($\mathbb{C}^2$) with unitary generator H.

3 The Dual Role of the Hamiltonian

The unification $\hbar = \kappa$ is necessitated by the fact that the Hamiltonian H performs two distinct functions within the theory.

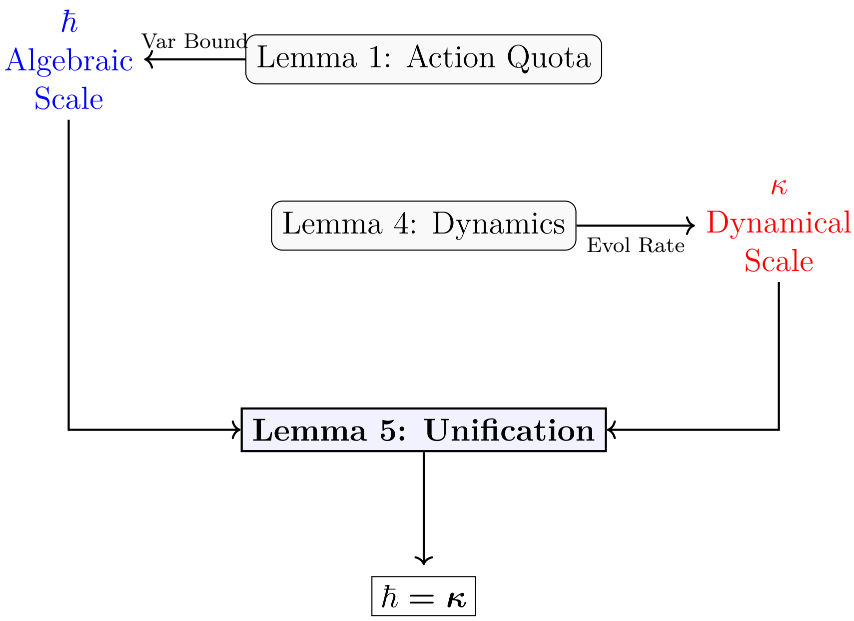

Figure 1: Conceptual flow showing the convergence of two independent action scales through Lemma 5.

As an **observable**, H represents energy; its properties are governed by $\hbar$. As a **generator**, H induces time translations at a rate governed by κ.

Assumption 3.1 (Same observable, same generator)**.** *The Hamiltonian observable calibrated by spectroscopy is the same operator that generates time translations in $U(t)$ (not merely proportional to it).*

4 Lemma 5: Scale Unification ($\hbar = \kappa$)

Lemma 4.1 (Lemma 5: Operational scale unification)**.** *Assuming Assumption 3.1, internal consistency between the algebraic uncertainty budget (Lemma 1) and the dynamical implementation of evolution (Lemma 4) forces $\hbar = \kappa$.*

Operational consistency argument. The existence of two routes reflects the logical independence of Lemmas 1 and 4: Lemma 1 establishes algebraic structure without dynamics, while Lemma 4 establishes dynamics without specifying the commutator scale. The theory thus has *two* places where an action unit enters: (i) the conversion between an *observable* and its associated *generator* (the commutator scale $\hbar$), and (ii) the conversion between a *generator* and *accumulated phase* (the dynamical scale κ).

Thought experiment (Ramsey-type phase accumulation). Prepare a two-level system with a Hamiltonian observable H having sharp eigenstates $|E_0\rangle, |E_1\rangle$ and eigenvalue gap $\Delta E := E_1 - E_0$. The gap ΔE is fixed by energy measurements (spectroscopy), not by fitting fringes. Prepare the superposition state $|\psi(0)\rangle = (|E_0\rangle + |E_1\rangle)/\sqrt{2}$. Let it evolve freely for time t, then apply a Ramsey readout.

Route A (direct time-translation postulate). By the definition of $U(t) = e^{-iHt/\kappa}$, the accumulated relative phase is $\Delta\phi_A(t) = \frac{\Delta E}{\kappa} t$. The Ramsey fringe probability is:

$$p_A(E_0 \mid t) = \frac{1}{2}\Big(1 + \cos(\Delta E\, t/\kappa)\Big).$$

Route B (generator normalization from the algebra). *Normalization bridge.* Independently, the generator of a continuous symmetry is operationally calibrated by how it moves other observables. The commutator relations $[X, Y] = i\hbar Z$ (which underlie the variance bounds of Lemma 1) fix $\hbar$ as the natural unit relating observables to their generators.

With this commutator/variance scale fixed to $\hbar$, the unique continuous unitary implementation of the one-parameter group generated by the observable H is $U(t) = \exp(-iHt/\hbar)$ (i.e. t is the physical time parameter and $\hbar$ is the action unit that converts energy eigenvalues into phase rates). Hence the Ramsey phase predicted from the algebraic normalization is

$$\Delta\phi_B(t) = \frac{\Delta E}{\hbar}\, t, \qquad p_B(E_0 \mid t) = \tfrac{1}{2}\Big(1 + \cos(\Delta E\, t/\hbar)\Big).$$

Unification. Both routes refer to the *same laboratory procedure*. For the operational prediction to be unique, we must have $p_A = p_B$ for all t and all calibrated ΔE. This forces:

$$\frac{\Delta E}{\kappa} = \frac{\Delta E}{\hbar} \quad \Longrightarrow \quad \hbar = \kappa.$$

□

Remark 4.1 (Consistency with Mandelstam–Tamm)**.** *The identity $\hbar = \kappa$ is also required to reconcile the two fundamental bounds of the theory. The algebraic Robertson relation requires $\Delta H \Delta M \geq \frac{1}{2}|\langle [H, M]\rangle|$, where the commutator is evaluated in units of $\hbar$. The dynamical Mandelstam–Tamm relation defines a characteristic time $\tau_M = \Delta M / |d\langle M\rangle/dt|$ and requires $\Delta H \tau_M \geq \kappa/2$. If $\hbar \neq \kappa$, the algebraic bound on information and the dynamical bound on evolution would disagree on the minimum uncertainty budget for the system.*

Corollary 4.1 (Unified Heisenberg Equation)**.** *The unification $\hbar = \kappa$ ensures that the rate derived from dynamics recovers the standard Heisenberg equation of motion:*

$$\frac{d}{dt}\langle A\rangle = \frac{1}{i\hbar}\langle [A, H]\rangle\,.$$

Remark 4.2 (Falsifiability Checklist)**.** *If $\hbar \neq \kappa$, the framework predicts two incompatible calibrations of the same Ramsey experiment unless one rejects Assumption 3.1. Specifically:*

- ***Ramsey mismatch:*** *Ramsey fringes would oscillate at frequency $\omega = \Delta E/\kappa$ by time-evolution, but at $\omega = \Delta E/\hbar$ by generator/commutator calibration.*
- ***Hamiltonian not an observable:*** *To avoid the mismatch, one must deny that the generator of time translations is the measured energy observable H.*
- ***Two-action-constant world:*** *there would exist two operationally distinct calibrations of action: one from commutator/variance protocols (Lemma 1) and one from Ramsey phase accumulation (Lemma 4). Measuring both would return different constants, $\hbar \neq \kappa$.*

5 Summary of the Qubit Core

Having unified the scales, we can now view the reconstruction as a coherent whole. Every postulate of the standard formalism has been derived from the Action Quota.

Lemma	Input Principle	Quantum Result	Significance
L1	Variance Bound	Unit Disk Geometry ($\hbar$)	Non-classical uncertainty
L2	Isotropy/Efficiency	Bloch Ball (B^3)	3D state space
L3	Convexity/Sharpness	Born Rule	Probabilistic predictions
L4	Path Sensitivity	Hilbert Space ($\mathbb{C}^2, \kappa$)	Unitary dynamics
L5	Consistency	$\hbar = \kappa$	Unified action scale

Table 1: Architecture of the single-qubit reconstruction from information principles.

6 Conclusion

Lemma 5 proves that the physical constant limiting knowledge and the constant governing the flow of time are identical. This reveals that quantum mechanics is a single geometric structure forced by a finite information budget. This completes the single-qubit core of the programme.

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