LEMMA 1 (VARIANCE FORM):

# From Variance Complementarity to the Unit Disk

A minimal and operational derivation of the Bloch-disk geometry

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**Abstract**

We present a minimal, operational, and fully self-contained derivation of the unit disk $a^2 + b^2 \leq 1$ for the expectation values of two **maximally complementary** dichotomic measurements. The key postulate is a single, variance-based complementarity condition $\mathrm{Var}(A) + \mathrm{Var}(B) \geq 1$. This has a direct operational interpretation: no physical state can simultaneously minimize the uncertainties (statistical variance) of two maximally incompatible binary measurements. From this single assumption, we obtain the Bloch-disk geometry immediately, without **presupposing** Hilbert space, complex numbers, or any specifically quantum mathematical structure. This result forms the foundational base for Lemma 2 (disk-to-sphere inflation), which reconstructs the Bloch ball and the geometry of the qubit.

## 1 Operational Setup (Minimal Assumptions)

We work with a single binary system described entirely in operational terms.

**Assumption 1.1** (State Space)**.** *The set of preparations (states) forms a convex set $\Omega$.*

**Definition 1.1** (Dichotomic Measurement)**.** *A measurement $M$ has outcomes $\pm 1$. For any state $\omega \in \Omega$,*

$$m := \langle M \rangle_\omega \ \in [-1, 1].$$

**Assumption 1.2** (Affine Response)**.** *The map $\omega \mapsto \langle M \rangle_\omega$ is affine for every dichotomic measurement.*

**Definition 1.2** (Variance)**.** *For a dichotomic $\pm 1$ measurement, the variance is defined by the expectation values as*

$$\mathrm{Var}(M)_\omega = \langle M^2 \rangle_\omega - \langle M \rangle_\omega^2.$$

*Since $M^2$ is deterministically the outcome $(\pm 1)^2 = 1$, we have $\langle M^2 \rangle_\omega = 1$. Thus, the variance is simply*

$$\mathrm{Var}(M)_\omega = 1 - m^2.$$

## 2 The Variance Complementarity Axiom

**Assumption 2.1** (Variance Complementarity)**.** *There exist pairs of dichotomic measurements $A, B$ satisfying, for all states,*

$$\mathrm{Var}(A)_\omega + \mathrm{Var}(B)_\omega \ \geq \ 1. \tag{1}$$

*We call such pairs* ***maximally complementary****.*

This postulate is the operational core of our reconstruction.

### Physical Motivation and Justification

**Physical Content.** The axiom captures the essential operational limitation observed in quantum systems: the impossibility of preparing a state that has simultaneous definite outcomes (i.e., zero variance) for two incompatible measurements.

**Operational Testability.** This constraint can be verified experimentally by preparing states and measuring variances. It makes falsifiable predictions: observation of $\mathrm{Var}(A) + \mathrm{Var}(B) < 1$ for any complementary pair would falsify the axiom.

**Non-Classical Content.** Classical systems exhibit fundamentally different variance behavior:

- **Classical deterministic systems:** $\mathrm{Var}(A) = \mathrm{Var}(B) = 0$ (certainty about all observables simultaneously).
- **Classical probabilistic systems:** Independent observables have no variance trade-off; uncertainties are uncorrelated.
- **Quantum systems (our axiom):** $\mathrm{Var}(A) + \mathrm{Var}(B) \geq 1$ captures *irreducible complementarity*—incompatibility that cannot be reduced by better preparation.

The existence of a finite, non-trivial bound strictly between total certainty (sum $= 0$) and maximal randomness (sum $= 2$ for dichotomic observables) is the signature of quantum complementarity.

**Why the Bound is "1".** The numerical value "1" is a choice of normalization (a unit convention) that sets the natural scale for the state space. Any positive bound $C$ could be rescaled to 1 by redefining variance units. What matters physically is:

- The *existence* of a finite bound (distinguishing quantum from classical)
- The bound being *saturated* by pure states (states on the boundary circle)

The specific value "1" then defines the natural unit of action for the static structure.

**Definition of "Maximally Complementary".** We define this operationally: two measurements $A$ and $B$ are maximally complementary if there exists at least one state that saturates the variance bound with equality. This is not circular—it is a *classification* of measurement pairs based on their operational behavior. The physical content of the axiom is that such pairs exist.

**Relation to Standard Uncertainty Relations.** This variance-sum condition is operationally equivalent to the more familiar product-form uncertainty relations. For qubit observables satisfying the algebra $[A, B] = 2iC$, the relation $\Delta A \Delta B \geq |\langle C \rangle|$ becomes equivalent to our variance sum $\mathrm{Var}(A) + \mathrm{Var}(B) \geq 1$ when specialized to pure states where $|\langle C \rangle|$ is maximal.

## 3 Lemma 1: The Unit Disk

We now state and prove the core geometric lemma.

**Lemma 3.1** (Variance Complementarity Implies the Bloch Disk)**.** *Let $a = \langle A \rangle_\omega$ and $b = \langle B \rangle_\omega$ for any state $\omega$. If the variance complementarity condition* (1) *holds, then*

$$a^2 + b^2 \leq 1.$$

*Thus the allowed expectation-value pairs $(a, b)$ for two complementary dichotomic measurements form exactly the unit disk in $\mathbb{R}^2$.*

*Proof.* The proof is a direct substitution of the definition of variance (Section 1) into the complementarity axiom (Section 2).

Using the definition of variance for dichotomic observables, we have:

$$\mathrm{Var}(A) = 1 - a^2, \qquad \mathrm{Var}(B) = 1 - b^2.$$

Insert these expressions into the axiom (1):

$$(1 - a^2) + (1 - b^2) \;\geq\; 1.$$

We can now rearrange the terms algebraically:

$$2 - (a^2 + b^2) \geq 1$$

Subtracting 1 from both sides and multiplying by -1 yields:

$$1 \geq a^2 + b^2.$$

This is precisely the condition $a^2 + b^2 \leq 1$. No further assumptions are required. □

# 4 Consequences and Interpretation

## 4.1 Exact Geometry

The result shows that the physically achievable expectation-value pairs for two maximally complementary binary measurements form

$$\mathcal{D} := \{(a, b) \in \mathbb{R}^2 : \; a^2 + b^2 \leq 1\},$$

the Bloch disk. This geometric object—a Euclidean unit disk—is obtained without referencing quantum theory.

## 4.2 Boundary Conditions

The derived geometry $a^2 + b^2 \leq 1$ includes two physically significant boundaries:

**The Circle ($a^2 + b^2 = 1$).** States on the boundary of the disk are those that *saturate* the uncertainty relation. These are the minimal uncertainty states (pure states) for this pair of observables. They correspond to states with maximum knowledge: $\mathrm{Var}(A) + \mathrm{Var}(B) = 1$ exactly.

**The Origin ($a = b = 0$).** A state at the origin ($a = 0, b = 0$) corresponds to maximum variance ($\mathrm{Var}(A) = 1, \mathrm{Var}(B) = 1$) for both measurements. This state represents the **maximally mixed state** for this 2D cross-section, with equal probabilities for all outcomes.

## 4.3 Physical Meaning

The variance bound expresses the operational limitation:

*No state can make both complementary measurements simultaneously precise.*

This is a direct, experimentally meaningful uncertainty relation that follows from the Action Quota axiom.

## 4.4 Basis for the Full Reconstruction

Lemma 1 establishes the precise two-observable cross-section of the qubit state space. Lemma 2 (presented separately) uses rotational symmetry, operational homogeneity, and consistency across all measurement pairs to inflate the disk into the sphere, producing the full Bloch ball.

# 5 Roadmap to Lemma 2 (Disk-to-Sphere Inflation)

With Lemma 1 in place, the route to the Bloch ball follows these steps:

1. Show that every rotated pair of complementary dichotomic measurements yields the same disk (same radius, same geometry).
2. Use isotropy: there is no distinguished measurement direction.
3. Fit all 2D disks as slices of a single 3D convex body.
4. Symmetry plus convexity forces this body to be a Euclidean ball.

This produces the full geometry of the qubit without appealing to Hilbert space formalism.

# 6 Conclusion

**Remark 6.1** (Minimality)**.** *Variance complementarity is the correct operational axiom for deriving the Bloch disk from first principles. It is physically meaningful, algebraically clean, and mathematically decisive.*

**Remark 6.2** (Comparison with Other Approaches)**.** *Unlike reconstructions that begin from abstract informational principles or the Hilbert space formalism itself, this approach derives the core 2D geometry of quantum theory directly from a single, physically testable constraint on measurement statistics.*

**Remark 6.3** (What Was Not Assumed)**.** *No complex numbers, no Hilbert space, no linear operators, no Born rule. Only: convex state space, affine expectation maps, and the variance bound. The rest emerges from geometry and consistency.*