Lemma 3: The Unique Measurement Rule on the Bloch Sphere

From Variance Complementarity to the Born Rule

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Abstract

Lemma 1 established that complementary binary observables obey a fundamental variance bound, a manifestation of the finite "Action Quota." Lemma 2 showed that isotropy inflates the resulting 2D disk into a 3D Euclidean ball. In this note we prove Lemma 3: given this geometry and the operational axioms of convexity and isotropy, the *only* admissible measurement rule is the Born Rule,

$$p(+1 \mid \vec{s}, \vec{m}) = \frac{1 + \vec{s} \cdot \vec{m}}{2}.$$

The proof explicitly uses the variance structure from Lemma 1 to fix the boundary conditions of the measurement function. The Born rule emerges as the unique affine function consistent with operational constraints, requiring no additional quantum postulates.

1 Operational Background

Following Lemmas 1 and 2, the state space Ω is the closed unit ball $B^3 = \{\vec{s} \in \mathbb{R}^3 : \|\vec{s}\| \le 1\}$, with pure states on the sphere S^2. Measurements correspond to unit vectors $\vec{m} \in S^2$.

We impose three operational principles:

(O1) Convexity: The expectation map is affine. If $\vec{s}_t = t\vec{s}_1 + (1-t)\vec{s}_0$, then

$$\langle M \rangle_{\vec{s}_t} = t\langle M \rangle_{\vec{s}_1} + (1-t)\langle M \rangle_{\vec{s}_0}.$$

(O2) Isotropy: Probabilities depend only on the relative orientation of $\vec{s}$ and $\vec{m}$, i.e., only on the invariant $\vec{s} \cdot \vec{m}$.

(O3) Binary Expectation: For ± 1 outcomes,

$$\langle M \rangle_{\vec{s}} = (+1)p(+1 \mid \vec{s}, \vec{m}) + (-1)p(-1 \mid \vec{s}, \vec{m}) = 2\,p(+1 \mid \vec{s}, \vec{m}) - 1.$$

Proposition 1 (Physical justification of axioms)**.** ***(O1) Convexity:*** *This expresses that statistical mixtures behave linearly. If we prepare state $\vec{s}_1$ with probability t and state $\vec{s}_0$ with probability $1-t$, the average measurement outcome is the weighted average of individual outcomes. This is the operational definition of a convex probabilistic theory.*

(O2) Isotropy: *Measurement statistics depend only on the relative angle between state and measurement direction, not on absolute orientation in space. This was established in Lemma 2 via the SO(3) symmetry of the state space.*

(O3) Binary expectation: *This is the definition of expectation value for dichotomic measurements with outcomes ± 1. Since $p(+1) + p(-1) = 1$, the relation follows directly.*

Boundary Conditions from Lemma 1

The variance complementarity axiom from Lemma 1 provides crucial boundary conditions that will uniquely determine the measurement function.

Proposition 2 (Boundary conditions). *For pure states (on the boundary sphere S^2), the measurement function must satisfy:*

- ***Aligned Case ($\vec{s}\cdot\vec{m} = 1$).*** *A pure state measured along its own direction yields zero variance:* $\mathrm{Var}(M) = 1 - \langle M\rangle^2 = 0$, *hence* $\langle M\rangle = \pm 1$. *Since* $\vec{s}$ *and* $\vec{m}$ *point in the same direction, we have* $\langle M\rangle = +1$.

- ***Orthogonal Case ($\vec{s}\cdot\vec{m} = 0$).*** *For any pure state* $\vec{s}$ *orthogonal to measurement direction* $\vec{m}_A$, *we can choose a complementary measurement* $\vec{m}_B$ *(orthogonal to* $\vec{m}_A$*) such that* $\vec{s}$ *lies in the plane spanned by* $\vec{m}_A$ *and* $\vec{m}_B$.

  By construction, the pair (A, B) *with measurement directions* $(\vec{m}_A, \vec{m}_B)$ *are maximally complementary (orthogonal measurement directions). Lemma 1 requires* $\mathrm{Var}(A) + \mathrm{Var}(B) = 1$ *for pure states saturating the bound.*

  Since $\vec{s}$ *lies in the* $(\vec{m}_A, \vec{m}_B)$ *plane and is perpendicular to* $\vec{m}_A$, *we can write* $\vec{s} = \alpha\vec{m}_B$ *for some* $|\alpha| = 1$ *(since* $\vec{s}$ *is pure). This gives* $\langle A\rangle = \vec{s}\cdot\vec{m}_A = 0$ *(hence* $\mathrm{Var}(A) = 1$*) and* $\langle B\rangle = \vec{s}\cdot\vec{m}_B = \pm 1$ *(hence* $\mathrm{Var}(B) = 0$*), confirming* $\mathrm{Var}(A) + \mathrm{Var}(B) = 1$.

  By isotropy (O2), $\langle M\rangle$ *depends only on* $\vec{s}\cdot\vec{m}$. *Therefore,* $\langle M\rangle_{\vec{s}} = 0$ *whenever* $\vec{s}\cdot\vec{m} = 0$ *for any pure state* $\vec{s}$ *and measurement direction* $\vec{m}$.

- ***Anti-aligned Case ($\vec{s}\cdot\vec{m} = -1$).*** *By the same deterministic argument as the aligned case,* $\langle M\rangle = -1$ *when* $\vec{s}$ *and* $\vec{m}$ *point in opposite directions.*

These three boundary conditions are sufficient to uniquely determine the measurement function.

2 Main Result: Isotropy and Convexity Force an Affine Function

Lemma 1 (Functional form). *Axioms O1 and O2 determine that the expectation value has the form*

$$\langle M\rangle_{\vec{s}} = f(\vec{s}\cdot\vec{m})$$

where $f : [-1, 1] \to [-1, 1]$ *is an affine function:* $f(x) = kx + c$ *for constants* k, c.

Proof. The argument proceeds in three steps.

Step 1: Isotropy implies dependence on inner product.
By axiom O2 (Isotropy), the expectation value can depend only on rotationally invariant quantities. For two vectors $\vec{s}$ and $\vec{m}$ on or in the unit ball, the only rotational invariant is their inner product $\vec{s}\cdot\vec{m}$.

Therefore, there exists a function $f : [-1, 1] \to [-1, 1]$ such that

$$\langle M\rangle_{\vec{s}} = f(\vec{s}\cdot\vec{m}). \tag{1}$$

Step 2: Convexity implies Jensen's functional equation.
Consider two states $\vec{s}_0$ and $\vec{s}_1$, and their convex combination $\vec{s}_t = t\vec{s}_1 + (1 - t)\vec{s}_0$ for $t \in [0, 1]$.

By axiom O1 (Convexity), the expectation value satisfies:

$$\langle M\rangle_{\vec{s}_t} = t\langle M\rangle_{\vec{s}_1} + (1-t)\langle M\rangle_{\vec{s}_0}.$$

Substituting the functional form from Step 1:

$$f(\vec{s}_t\cdot\vec{m}) = t\,f(\vec{s}_1\cdot\vec{m}) + (1-t)f(\vec{s}_0\cdot\vec{m}).$$

Since $\vec{s}_t\cdot\vec{m} = (t\vec{s}_1 + (1-t)\vec{s}_0)\cdot\vec{m} = t(\vec{s}_1\cdot\vec{m}) + (1-t)(\vec{s}_0\cdot\vec{m})$, we can write $x_t = tx_1 + (1-t)x_0$ where $x_i = \vec{s}_i\cdot\vec{m}$.

This gives Jensen's functional equation:

$$f(tx_1 + (1-t)x_0) = tf(x_1) + (1-t)f(x_0). \tag{2}$$

Note on functional equations. Jensen's equation naturally arises from convexity (affinity over convex combinations). While Cauchy's functional equation $f(x+y) = f(x) + f(y)$ is related, our operational framework provides convex combinations of states, not sums of inner products, making Jensen's equation the appropriate tool. Under continuity, both lead to affine solutions.

Step 3: Continuity implies affine solutions.

To solve equation (2), we invoke continuity of f. This is not an additional axiom but follows from operational physics:

> *Continuity of f follows from the physical requirement that small changes in the relative orientation of $\vec{s}$ and $\vec{m}$ should produce small changes in measurement probabilities. Discontinuous jumps in statistics under smooth parameter variations would be unphysical.*

Since measurement probabilities are empirically observed to vary smoothly with apparatus orientation, the map $x \mapsto f(x)$ must be continuous.

A standard result in functional analysis states that continuous solutions to Jensen's equation on a finite interval are affine functions:

$$f(x) = kx + c$$

for constants $k, c \in \mathbb{R}$.

(For a proof, see Aczél, J., *Lectures on Functional Equations and Their Applications*, Academic Press, 1966, Theorem 2.1.1.) □

3 Determining the Constants: The Born Rule Emerges

We now use the boundary conditions from Proposition 2 to uniquely fix the constants k and c.

Lemma 2 (The Born Rule)**.** *The measurement rule on the Bloch ball is uniquely determined to be*

$$\langle M\rangle_{\vec{s}} = \vec{s}\cdot\vec{m},$$

which yields the Born probability rule

$$p(+1 \mid \vec{s},\vec{m}) = \frac{1+\vec{s}\cdot\vec{m}}{2}.$$

Proof. From Lemma 1, we know $f(x) = kx + c$. We apply three boundary conditions:

Condition 1: Orthogonality ($x = 0$).
From Proposition 2, when $\vec{s} \cdot \vec{m} = 0$, we have $\langle M \rangle = 0$. Thus $f(0) = 0$, which gives:

$$k(0) + c = 0 \quad \Rightarrow \quad c = 0.$$

So $f(x) = kx$.

Condition 2: Alignment ($x = 1$).
From Proposition 2, when $\vec{s} \cdot \vec{m} = 1$, we have $\langle M \rangle = 1$. Thus $f(1) = 1$, which gives:

$$k(1) = 1 \quad \Rightarrow \quad k = 1.$$

Therefore, $f(x) = x$.

Verification with Condition 3: Anti-alignment ($x = -1$).
We can verify consistency: $f(-1) = -1$, which correctly gives $\langle M \rangle = -1$ when $\vec{s} \cdot \vec{m} = -1$.

Final form.
Substituting $f(x) = x$ into equation (1):

$$\langle M \rangle_{\vec{s}} = \vec{s} \cdot \vec{m}.$$

Using axiom O3 (Binary Expectation), since $\langle M \rangle = 2p(+1) - 1$:

$$p(+1 \mid \vec{s}, \vec{m}) = \frac{1 + \langle M \rangle_{\vec{s}}}{2} = \frac{1 + \vec{s} \cdot \vec{m}}{2}.$$

This is exactly the Born rule for qubit measurements. □

4 Consequences and Interpretation

Remark 1 (Uniqueness)**.** *The Born rule is the* unique *measurement rule consistent with:*

- *The Bloch ball geometry (Lemmas 1-2)*
- *Convexity (operational mixing)*
- *Isotropy (rotational symmetry)*
- *Variance complementarity boundary conditions (Lemma 1)*

No additional quantum postulates are required. The probabilistic structure of quantum mechanics emerges necessarily from operational geometry.

Remark 2 (Geometric interpretation)**.** *Writing $\vec{s} \cdot \vec{m} = \|\vec{s}\|\|\vec{m}\| \cos\theta = \|\vec{s}\| \cos\theta$ (since $\|\vec{m}\| = 1$), we have:*

$$p(+1 \mid \vec{s}, \vec{m}) = \frac{1 + \|\vec{s}\| \cos\theta}{2}.$$

For pure states ($\|\vec{s}\| = 1$), this reduces to:

$$p(+1) = \frac{1 + \cos\theta}{2}.$$

Using the half-angle identity $\cos\theta = 2\cos^2(\theta/2) - 1$*, we can verify:*

$$p(+1) = \frac{1 + 2\cos^2(\theta/2) - 1}{2} = \cos^2(\theta/2),$$

which is the standard textbook form of the Born rule for qubit measurements.

For mixed states ($\|\vec{s}\| < 1$*), the probability interpolates continuously between pure-state behavior and uniform randomness (at the origin, where* $\vec{s} = \vec{0}$ *gives* $p(+1) = 1/2$ *for all measurements).*

Remark 3 (What was derived, not assumed)**.** *The Born rule was not postulated. It was derived as the unique function satisfying:*

1. *Affinity (from convexity + isotropy, via Jensen's equation)*
2. *Three boundary conditions (from variance complementarity via Lemma 1)*

This shows that quantum probabilities are a consequence of the geometric structure forced by the Action Quota, not an independent postulate.

Remark 4 (Comparison with standard QM)**.** *In standard quantum mechanics, the Born rule* $p = |\langle\psi|m\rangle|^2$ *is postulated. Here, we have shown it emerges from operational constraints on a real geometric state space. The complex Hilbert space structure (which we derive in Lemma 4) will later be shown to provide an efficient computational tool for representing these geometric probabilities, but the probabilities themselves are fundamentally geometric, not algebraic.*

5 Outlook: From Geometry to Dynamics

The dependence on $\cos\theta = \vec{s} \cdot \vec{m}$ shows that measurement probabilities are purely geometric—determined by angles on the Bloch sphere. This raises the question: what transformations preserve this geometric structure?

In Lemma 4, we show that:

- Reversible transformations preserving the Born rule are precisely rotations of B^3 (elements of $SO(3)$)
- These rotations lift to a projective representation via the universal cover $SU(2)$
- This lift forces the introduction of complex numbers and unitary dynamics
- A dynamical action scale κ appears in the time evolution

Lemma 5 then shows that consistency between the static variance scale (from Lemma 1) and the dynamic evolution scale (from Lemma 4) requires their identification: $\hbar = \kappa$, completing the reconstruction.

Corollary 1 (Preview of scale identification)**.** *The measurement function* $\langle M\rangle = \vec{s} \cdot \vec{m}$ *will play a dual role:*

- *As a statistical prediction (this lemma)*
- *As the generator of rotations in dynamics (Lemma 4)*

This dual role will force the unification of action scales in Lemma 5.