LEMMA 6 (COMPOSITE SYSTEMS):

The Inflation of the Observable Algebra

From Interaction Generators to the Tensor Product

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Abstract

We extend the reconstruction to composite systems ($N = 2$). We address the **Interaction Gap**: the fact that the direct sum algebra of two independent qubits is insufficient to support reversible interactions. We invoke the principle of **Global Action Consistency**, which requires that any generator of reversible evolution exists as a measurable observable within a unified scale $\hbar$. We prove that adjoining a single entangling interaction generator and enforcing commutator closure forces algebraic inflation to the full $\mathfrak{su}(4)$ correlation algebra (generically). This algebraic inflation ($6 \to 15$ dimensions) provides the geometric necessity for **entanglement** and uniquely identifies the state space as the set of 4×4 density matrices.

Keywords: Composite systems, tensor product, entanglement, Lie algebra closure, interaction gap, Action Quota universality

1 Introduction

To bridge the gap between single-system and composite-system physics, we invoke the central consistency principle established in Lemma 5: the identity of generators and observables.

Axiom 1.1 (Global Action Consistency). *The dual role of the Hamiltonian must hold globally: any generator of reversible time evolution must be a measurable observable in the system's algebra.*

Remark 1.1 (Unified Scale $\hbar$). *By Lemma 5, the Hamiltonian's dual role requires $\hbar = \kappa$. For composite systems, this means any interaction generator H_{int} must be measurable with the same action scale $\hbar$ that governs local uncertainties. This forces H_{int} to be embedded in a unified global algebra where all observables share the same dimensional constant.*

The logical chain is: *Interactivity $\Rightarrow$ Observability $\Rightarrow$ Closure $\Rightarrow$ Tensor Product.* In this paper, we show that the requirement for systems to influence one another necessarily "inflates" the algebra of observables beyond the sum of its parts.

2 The Interaction Gap

Consider a composite system of two distinguishable qubits, A and B. A naive combination implies that we can only prepare and measure them independently.

Definition 2.1 (Local Algebra)**.** *The local algebra $\mathcal{L}_0$ is the direct sum space of observables acting on each subsystem separately:*

$$\mathcal{L}_0 = span\{\sigma_i \otimes \mathbb{I}, \mathbb{I} \otimes \sigma_j\} \quad for\ i, j \in \{x, y, z\}.$$

Remark 2.1 (Distinguishability)**.** *The assumption of distinguishable subsystems (A and B) is operationally justified by the existence of local measurement contexts. For indistinguishable particles, the algebra would be constrained by permutation symmetry, but the core argument—that interactions force algebraic extension—remains valid.*

This space has dimension $3+3=6$. Hamiltonians in $\mathcal{L}_0$ generate factorizable dynamics $U(t) = U_A(t) \otimes U_B(t)$, which map product states to product states. To allow for physical influence between A and B, we must extend this set.

Axiom 2.1 (Interaction)**.** *There exists at least one nontrivial reversible operation U on the composite system such that for some orthogonal initial states $|\psi_A\rangle, |\phi_A\rangle$ of system A and a fixed initial state $|\chi_B\rangle$ of system B, the final reduced state of B depends on the initial state of A:*

$$\mathrm{Tr}_A\Big(U(|\psi_A\rangle\langle\psi_A| \otimes |\chi_B\rangle\langle\chi_B|)U^\dagger\Big) \neq \mathrm{Tr}_A\Big(U(|\phi_A\rangle\langle\phi_A| \otimes |\chi_B\rangle\langle\chi_B|)U^\dagger\Big).$$

Remark 2.2 (Partial Trace Notation)**.** *We denote by Tr_A the partial trace over subsystem A, defined uniquely by $\mathrm{Tr}_A(X_A \otimes X_B) = \mathrm{Tr}(X_A)X_B$ for any operators X_A, X_B.*

Example 2.1 (CNOT Generation)**.** *The controlled-NOT gate $U_{CNOT} = |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes \sigma_x$ satisfies axiom 2.1. Here $\{|0\rangle, |1\rangle\}$ denotes the computational basis of subsystem A. With $|\psi_A\rangle = |0\rangle$, $|\phi_A\rangle = |1\rangle$, and $|\chi_B\rangle = |0\rangle$, we obtain:*

$$\rho_B(\psi) = |0\rangle\langle 0| \neq |1\rangle\langle 1| = \rho_B(\phi).$$

The generator H_{CNOT} (via $U = e^{-iHt}$) must therefore be an observable in the global algebra.

3 Derivation of the Tensor Product Algebra

3.1 Algebraic Conventions

We adopt the *physics convention* $\mathfrak{su}(n) \cong \{$traceless Hermitian $n \times n$ matrices$\}$. We define the scaled Lie bracket as an observable:

$$[A, B]_{\mathrm{L}} \equiv \frac{1}{2i}[A, B] = \frac{1}{2i}[A, B].$$

(Equivalently: we identify the usual anti-Hermitian $\mathfrak{su}(n)$ with traceless Hermitians by multiplication by i.) Following Lemma 5, this ensures that $\frac{1}{2i}[\sigma_x, \sigma_y] = \sigma_z$ for the local components.

3.2 Algebraic Inflation

Theorem 3.1 (Algebraic Inflation to $\mathfrak{su}(4)$)**.** *Assume (i) full local controllability on A and B (i.e., $\mathcal{L}_0$ is available), and (ii) there exists at least one entangling interaction generator H_{int} whose correlation component is* generic *in the sense that its projection onto $\mathcal{C}$ is not contained in any proper $\mathcal{L}_0$-invariant subspace of*

$$\mathcal{C} := \mathrm{span}\{\sigma_i \otimes \sigma_j : i, j \in \{x, y, z\}\}$$

(equivalently: the $\mathcal{L}_0$-adjoint orbit of the $\mathcal{C}$-projection of H_{int} spans $\mathcal{C}$). Then the Lie closure of $\langle \mathcal{L}_0, H_{int} \rangle$ equals the full 15-dimensional algebra $\mathfrak{su}(4)$.

Proof. Write the global observable space as the orthogonal decomposition $\mathfrak{g} = \mathcal{L}_A \oplus \mathcal{L}_B \oplus \mathcal{C}$, where $\mathcal{L}_A = \mathrm{span}\{\sigma_i \otimes \mathbb{I}\}$, $\mathcal{L}_B = \mathrm{span}\{\mathbb{I} \otimes \sigma_j\}$, and $\mathcal{C}$ is defined above. Thus $\dim \mathfrak{g} = 15$.

Because H_{int} is nonlocal, it has a nonzero projection onto $\mathcal{C}$; pick a nonzero component of the $\mathcal{C}$-projection and denote it by $X_{uv} = \sigma_u \otimes \sigma_v$. By linearity of the Lie closure, it suffices to show that the subalgebra generated by $\mathcal{L}_0$ together with any single nonzero $\mathcal{C}$-component X_{uv} already spans $\mathcal{C}$ under the stated hypothesis. Using the bracket $\frac{1}{2i}[\cdot, \cdot]$:

$$\frac{1}{2i}[\sigma_i \otimes \mathbb{I}, \sigma_u \otimes \sigma_v] = \frac{1}{2i}[\sigma_i, \sigma_u] \otimes \sigma_v, \qquad \frac{1}{2i}[\mathbb{I} \otimes \sigma_j, \sigma_u \otimes \sigma_v] = \sigma_u \otimes \frac{1}{2i}[\sigma_j, \sigma_v].$$

Hence commutators with $\mathcal{L}_A$ move the first index among $\{x, y, z\}$ and commutators with $\mathcal{L}_B$ move the second index. By assumption (ii), the $\mathcal{L}_0$-adjoint orbit generated by repeated brackets from X_{uv} spans all of $\mathcal{C}$. Therefore the Lie closure contains $\mathcal{L}_A \oplus \mathcal{L}_B$ and all of $\mathcal{C}$, spanning the full $\mathfrak{su}(4)$ [4, 6]. □

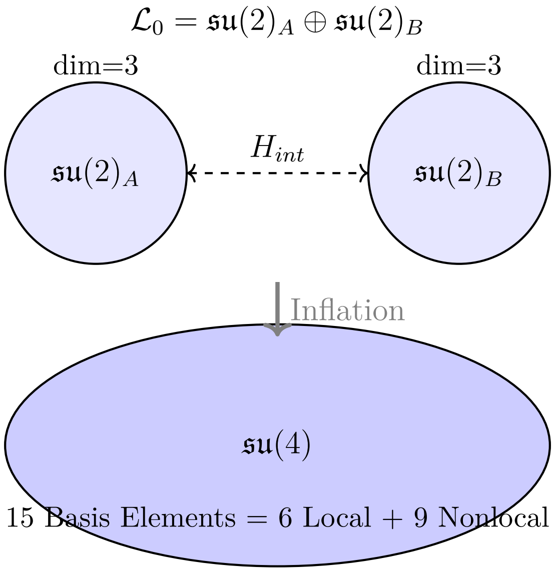

Figure 1: Algebraic Inflation: an entangling interaction H_{int} bridges the independent local algebras. For a generic such interaction, Lie closure with the local generators spans the 9-dimensional correlation sector and hence the full 15-dimensional algebra $\mathfrak{su}(4)$.

Remark 3.1 (Why closure is operational)**.** *If A and B are jointly part of the observable algebra, then the theory's universal variance constraints (Robertson-type bounds) involve the commutator $[A, B]$. Under our convention, the derived observable $\frac{1}{2i}[A, B] = \frac{1}{2i}[A, B]$ is therefore operationally meaningful. To keep the uncertainty calculus internally closed, the algebra must be closed under $\frac{1}{2i}[\cdot, \cdot]$ (otherwise the bound would refer to an object outside the measurable catalog).*

3.3 Entanglement and the Action Quota

The Action Quota (Lemma 1) requires $\mathrm{Var}(A)+\mathrm{Var}(B) \geq 1$ for complementary observables on a single system. For composite systems, this bound extends naturally to any pair of complementary observables in the global algebra $\mathfrak{su}(4)$, as the same algebraic structure governs uncertainty relations at all scales. Entanglement is the mechanism by which certainty can be concentrated in global correlation observables while local complementary observables remain maximally uncertain, consistent with the same global action budget.

Entanglement as redistribution of certainty. Consider $A = \sigma_x \otimes \mathbb{I}$ and $B = \sigma_z \otimes \sigma_z$. With our convention $\frac{1}{2i}[\cdot,\cdot] = \frac{1}{2i}[\cdot,\cdot]$, we have $\frac{1}{2i}[A,B] = \sigma_y \otimes \sigma_z$. The Robertson bound can be written as

$$\Delta A \cdot \Delta B \geq \left| \left\langle \frac{1}{2i}[A,B] \right\rangle \right| = \left| \langle \sigma_y \otimes \sigma_z \rangle \right|.$$

In the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, one has $\langle \sigma_y \otimes \sigma_z \rangle = 0$, so this lower bound does not obstruct having $\Delta B = 0$ (perfect correlation certainty) while $\Delta A = 1$. Operationally: entanglement permits certainty to concentrate in *global* correlation observables at the expense of local complementary ones. The state resides on the global Information Frontier while sitting at the center of the individual local Bloch balls.

4 From Algebra to State Space

In quantum theory, states are positive linear functionals $\omega : \mathcal{A} \to \mathbb{C}$ satisfying $\omega(\mathbb{I}) = 1$ and $\omega(A^\dagger A) \geq 0$. For finite-dimensional matrix algebras, every positive normalized linear functional ω has the form $\omega(A) = \mathrm{Tr}(\rho A)$ for a unique density matrix ρ (positive semi-definite with unit trace), by the finite-dimensional Riesz/Hilbert–Schmidt representation [5]. Since $\mathfrak{su}(4) \oplus \mathbb{C}\mathbb{I}$ is the full algebra of 4×4 Hermitian matrices, the state space is exactly the set of 4×4 density matrices. Because $\mathfrak{su}(4)$ acts irreducibly on $\mathbb{C}^4$, the pure states are the projectivization of $\mathbb{C}^4$.

Remark 4.1 (Why Tensor Product?)**.** *The algebra* $\mathfrak{su}(4)$ *acts on a 4-dimensional Hilbert space. The specific tensor product structure* $\mathbb{C}^2 \otimes \mathbb{C}^2$ *is distinguished by the requirement that local observables acting on different subsystems must be simultaneously definable and commute, consistent with the operational definition of distinguishable systems.*

5 Alternatives and Minimality

5.1 Comparison with Real Quantum Mechanics

Real-Hilbert formulations can exhibit entanglement, but they modify the correlation sector and typically fail some reconstruction desiderata used to single out complex quantum theory (for example, local tomography). A natural real-Hilbert analogue yields an $\mathfrak{so}(4)$-type restriction, effectively keeping only the 6 local generators rather than the full 15 correlation algebra available over $\mathbb{C}$. In a real vector space theory, the two-qubit algebra would be isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ (dimension 6), making it insufficient to provide the degrees of freedom required for complex QM correlations [2].

Remark 5.1 (Minimality of $\mathfrak{su}(4)$)**.** *Any proper subalgebra containing* $\mathcal{L}_0$ *and a nonlocal generator would correspond to a nontrivial invariant subspace under the adjoint action* *of $\mathcal{L}_0$. However, $\mathcal{L}_0$ acts irreducibly on the 9-dimensional correlation space $\mathcal{C}$. Thus no proper subspace is invariant, forcing the full 15-dimensional algebra.*

6 Summary of Dimensional Inflation

Algebra Stage	**Dimensions**	**Observable Types**
Local only ($\mathcal{L}_0$)	6	$\sigma_i \otimes \mathbb{I}$, $\mathbb{I} \otimes \sigma_j$
+ Interaction generator	+1	H_{int} (nonlocal)
Closure under commutation	+8	All $\sigma_i \otimes \sigma_j$ $(i, j \neq 0)$
Total ($\mathfrak{su}(4)$)	**15**	**All traceless Hermitian 4×4 matrices**

Table 1: Dimensional inflation from local to global algebra. For a generic entangling interaction, Lie closure with the local algebra spans the full 15-dimensional $\mathfrak{su}(4)$ correlation algebra.

7 Conclusion

The tensor product structure of quantum mechanics is not a primitive postulate; it is the geometric cost of interaction. It follows from interactivity, consistency, and algebraic closure. The tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$ is thus *derived*: it is the unique minimal algebraic structure that allows distinguishable systems to interact while maintaining a consistent accounting of uncertainty via the universal Action Quota $\hbar$.

Remark 7.1 (Generalization to N Systems)**.** *For N qubits, the local algebra has dimension $3N$. Adding pairwise interactions generates correlations that span the full $\mathfrak{su}(2^N)$ algebra. The exponential growth is unavoidable: each new qubit that can interact doubles the required correlation dimensions.*

Outlook This derivation completes the reconstruction of the two-qubit formalism. The next step (Lemma 7) will extend this approach to the continuous limit, showing how the canonical commutation relations $[x, p] = i\hbar$ emerge from the same principles of algebraic closure and interactivity, thereby connecting the qubit formalism to continuous-variable quantum mechanics and demonstrating the universality of the derived structure.

References

[1] A. Barenco et al., *Elementary gates for quantum computation*, Phys. Rev. A **52**, 3457 (1995).

[2] W. K. Wootters, *Local accessibility of quantum states*, in *Complexity, Entropy and the Physics of Information*, Addison-Wesley (1990).

[3] L. Hardy, *Quantum Theory From Five Reasonable Axioms*, arXiv:quant-ph/0101012 (2001).

[4] A. Kitaev, A. Shen, and M. Vyalyi, *Classical and Quantum Computation*, American Mathematical Society (2002).

[5] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Springer (1987).

[6] D. D'Alessandro, *Introduction to Quantum Control and Dynamics*, Chapman & Hall/CRC (2007).

[7] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press (2010).

A Explicit Closure Construction

We show that starting from $\mathcal{L}_0$ and $H_{int} = \sigma_z \otimes \sigma_x$, we can generate all 15 basis elements using the Lie bracket $\frac{1}{2i}[A, B] = \frac{1}{2i}[A, B]$.

Step 1: Generate all $\sigma_i \otimes \sigma_x$:

$$\frac{1}{2i}[\sigma_x \otimes \mathbb{I}, \sigma_z \otimes \sigma_x] = -\sigma_y \otimes \sigma_x$$
$$\frac{1}{2i}[\sigma_y \otimes \mathbb{I}, \sigma_z \otimes \sigma_x] = \sigma_x \otimes \sigma_x$$

Step 2: Generate all $\sigma_i \otimes \sigma_j$: For each $\sigma_i \otimes \sigma_x$ (where $i \in \{x, y, z\}$), commute with $\mathbb{I} \otimes \sigma_y$:

$$\frac{1}{2i}[\mathbb{I} \otimes \sigma_y, \sigma_i \otimes \sigma_x] = -\sigma_i \otimes \sigma_z$$

Then with $\mathbb{I} \otimes \sigma_x$:

$$\frac{1}{2i}[\mathbb{I} \otimes \sigma_x, \sigma_i \otimes \sigma_z] = \sigma_i \otimes \sigma_y$$

Thus we obtain all 9 combinations $\sigma_i \otimes \sigma_j$ $(i, j \in \{x, y, z\})$.

Genericity of the Example The choice $H_{int} = \sigma_z \otimes \sigma_x$ is not special: any H_{int} with a nonzero $\mathcal{C}$-projection that is not an eigenoperator of the $\mathcal{L}_0$ adjoint action will generate the full algebra. The calculation here demonstrates the pattern for a typical case.

Completeness Argument The 15 operators $\{\sigma_i \otimes \sigma_j\}$ (excluding $\sigma_0 \otimes \sigma_0$) are orthogonal under the Hilbert–Schmidt inner product

$$\langle A, B\rangle_{\mathrm{HS}} := \mathrm{Tr}(A^\dagger B),$$

and thus linearly independent. Therefore, the generated algebra cannot be a proper subspace of $\mathfrak{su}(4)$. These calculations demonstrate the general pattern: starting from any entangling nonlocal term, commutation with local generators rotates the indices independently, eventually spanning all 9 elements of the form $\sigma_i \otimes \sigma_j$.