Lemma 7: The Continuum Limit

From Bloch Spheres to Phase Space via Group Contraction

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Abstract

We extend the reconstruction from finite-dimensional spins (qubits) to continuous variables (position and momentum). We demonstrate that the Heisenberg algebra $[x, p] = i\hbar_{\text{phys}}$ is not a fundamental postulate but the first-order flat approximation to the geometry of the unitary group $SU(2)$. By applying the Wigner-Inönü group contraction in the limit of large system size ($J \to \infty$), we derive the canonical commutation relation as the local linearization of angular momentum. Crucially, we prove that the Action Quota derived in Lemma 1 survives this limit unchanged: the uncertainty principle is simply the persistence of the state space's intrinsic curvature in the tangent plane approximation. Canonical quantization emerges as the linear approximation of a curved group manifold.

1 Introduction: The Emergence of Phase Space

Lemmas 1-6 established the physics of finite-dimensional systems (qubits). Standard quantum mechanics, however, relies on continuous degrees of freedom like position x and momentum p, governed by the unbounded Heisenberg algebra $\mathfrak{h}_1$:

$$[x, p] = i\hbar_{\text{phys}} \tag{1.1}$$

In this reconstruction, we assert that the Heisenberg algebra is not fundamental. It is the geometric limit of the compact spin algebra $\mathfrak{su}(2)$ when the radius of the state space goes to infinity. In this approach, $SU(2)$ is the fundamental kinematic structure; the Heisenberg algebra is an emergent limit.

The "flat" phase space of standard quantum mechanics arises as the **tangent plane approximation** to the curved geometry of a macroscopic spin system.

2 The Geometric Setup

Consider a generalized spin system with total angular momentum J. The dimension of the state space is $d = 2J + 1$. From Lemma 4, the generators of rotation satisfy the Lie algebra $\mathfrak{su}(2)$:

$$[J_x, J_y] = i\hbar_{\text{spin}} J_z \tag{2.1}$$

(and cyclic permutations).

Distinction of Scales:

- $\hbar_{\text{spin}}$: The intrinsic parameter of the angular momentum algebra (dimensionless in natural units, typically 1 or 1/2). Throughout this reconstruction, $\hbar_{\text{spin}}$ is the only fundamental quantity.
- $\hbar_{\text{phys}}$: The physical Planck's constant (with units of Action). This emergent constant appears only after attaching dimensional units to the tangent coordinates.

Remark 1 (Global Topology)**.** *The continuum phase space $\mathbb{R}^2$ does not exist globally on the Bloch sphere; it is a local chart valid only near a pole. Attempts to extend flat coordinates globally fail due to the compact topology of the sphere.*

Definition 1 (The Tangent Patch)**.** *We restrict our attention to states localized near the "North Pole" of the Bloch sphere (eigenstates of J_z with $m \approx J$). These states correspond to **North Polar Coherent States**: they possess minimum uncertainty and maximum localization, effectively defining points on the manifold.*

Remark 2 (Coherent State Properties)**.** *Formally, North Polar Coherent States are defined as $|\theta, \phi\rangle = e^{-i\phi J_z} e^{-i\theta J_y} |J, J\rangle$ with $\theta \ll 1$. In the contraction limit, these become harmonic oscillator coherent states $|\alpha\rangle$ with $\alpha = (x + ip)/\sqrt{2}$ (see Perelomov [**?**]).*

3 The Contraction Mechanism

To map discrete spin operators to continuous variables, we define a scaling transformation dependent on the system size J.

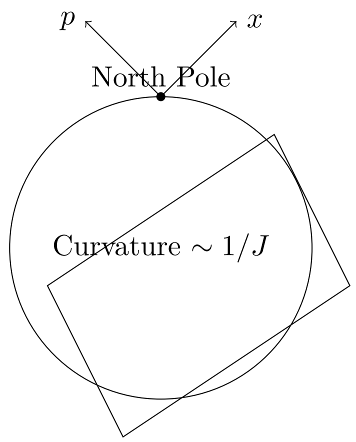

Figure 1: Phase space as the tangent plane to the Bloch sphere at the North Pole. The local coordinates (x, p) become the canonical variables in the limit $J \to \infty$.

3.1 Scaling Relations

Let $\epsilon = 1/J$ be a dimensionless scaling parameter. We define the continuous operators x and p via the transformation:

$$x = \frac{J_x}{\sqrt{J}}, \qquad p = \frac{J_y}{\sqrt{J}} \tag{3.1}$$

Formalized as a limit $\epsilon \to 0$:

$$x = \lim_{\epsilon \to 0} \sqrt{\epsilon} J_x, \quad p = \lim_{\epsilon \to 0} \sqrt{\epsilon} J_y \tag{3.2}$$

Motivation: As $J \to \infty$, the transverse fluctuations of a spin polarized along z scale as $\sqrt{J}$. The factor $\sqrt{\epsilon} = 1/\sqrt{J}$ rescales transverse fluctuations to order unity, ensuring the commutator remains finite and non-zero in the limit.

Remark 3 (Historical Context)**.** *The group contraction procedure was formally introduced by Wigner and Inönü [**?**] and applied to quantum mechanical oscillators by Holstein and Primakoff [**?**]. Our geometric interpretation reveals the underlying state-space curvature as the physical origin of this mathematical procedure.*

Remark 4 (Ladder Operators)**.** *Readers familiar with the Holstein-Primakoff transformation will recognize this as the coordinate space equivalent of the mapping to bosonic ladder operators: $J_- \approx \sqrt{2J} b$ and $J_z \approx J - b^\dagger b$. The limit $J \to \infty$ transforms the $\mathfrak{su}(2)$ algebra into the oscillator algebra.*

4 Derivation of the Canonical Commutation Relation

We compute the commutator of the rescaled variables using the underlying $\mathfrak{su}(2)$ structure from (**??**).

Substituting the definitions into the commutator:

$$[x,p] = \lim_{\epsilon\to 0}\left[\sqrt{\epsilon}J_x, \sqrt{\epsilon}J_y\right] = \lim_{\epsilon\to 0}\epsilon[J_x, J_y] \tag{4.1}$$

Using the structure constant $[J_x, J_y] = i\hbar_{\text{spin}}J_z$:

$$[x,p] = \lim_{\epsilon\to 0}\epsilon(i\hbar_{\text{spin}}J_z) = i\hbar_{\text{spin}}\lim_{J\to\infty}\frac{J_z}{J} \tag{4.2}$$

4.1 The Flat Limit

For coherent states near the North Pole, the system is macroscopically polarized. The operator J_z acts as:

$$J_z\left|\psi_{NP}\right\rangle \approx J\left|\psi_{NP}\right\rangle \implies \frac{J_z}{J}\to\hat{I} \tag{4.3}$$

Therefore, in the limit:

$$[x,p] = i\hbar_{\text{spin}}\hat{I} \tag{4.4}$$

To recover physical units, we attach dimensional constants for length (L_0) and momentum (P_0) such that $X = L_0 x$ and $P = P_0 p$:

$$[X,P] = L_0P_0[x,p] = i(L_0P_0\hbar_{\text{spin}})\hat{I} \tag{4.5}$$

Identifying the physical action scale as $\hbar_{\text{phys}} \equiv L_0P_0\hbar_{\text{spin}}$, we recover the standard CCR:

$$[X,P] = i\hbar_{\text{phys}} \tag{4.6}$$

Remark 5 (Physical Interpretation of Scales)**.** *The dimensional constants L_0 and P_0 represent the characteristic length and momentum scales at which the tangent plane approximation becomes valid. The choice of L_0 and P_0 defines which experimental observable realizes the tangent plane (e.g., lattice spacing vs. oscillator length).*

Remark 6 (Scale Invariance)**.** *The product L_0P_0 is fixed by $\hbar_{phys}$, but the individual scales L_0 and P_0 represent a choice of units. Different experimental realizations (optical cavities, trapped ions, condensed matter systems) correspond to different choices of these scales while preserving the fundamental relation $\hbar_{phys} = L_0P_0\hbar_{spin}$.*

5 Survival of the Action Quota

5.1 Statement

Does the variance bound derived in Lemma 1 ($Var(A) + Var(B) \geq 1$) survive this limit? We show that because the curvature of the state space survives in the commutator structure, the variance product remains bounded below even as $J\to\infty$. This result completes the promise of Lemma 1: the fundamental uncertainty relation persists through all scales of the reconstruction, from single qubits to continuum fields.

5.2 Derivation

For a spin-J system, the uncertainty relation is:

$$\Delta J_x\Delta J_y \geq \frac{1}{2}\hbar_{\text{spin}}|\langle J_z\rangle| \tag{5.1}$$

Using the scaling relations $J_x = \sqrt{J}x$:

$$(\sqrt{J}\Delta x)(\sqrt{J}\Delta p) \geq \frac{1}{2}\hbar_{\text{spin}}|\langle J_z\rangle| \tag{5.2}$$

$$J\Delta x\Delta p \geq \frac{1}{2}\hbar_{\text{spin}}|\langle J_z\rangle| \tag{5.3}$$

In the North Pole limit where $\langle J_z\rangle \approx J$:

$$J\Delta x\Delta p \geq \frac{1}{2}\hbar_{\text{spin}}J \implies \Delta x\Delta p \geq \frac{\hbar_{\text{spin}}}{2} \tag{5.4}$$

5.3 Interpretation

This confirms that the **Action Quota** is a scale-invariant feature of the geometry. The uncertainty principle is not an artifact of small systems; it is the local persistence of the non-commutative geometry in the tangent plane.

Remark 7 (Coherent State Saturation)**.** *Spin coherent states saturate the uncertainty relation* $\Delta J_x\Delta J_y = \frac{1}{2}\hbar_{spin}|\langle J_z\rangle|$*. This saturation property is preserved under contraction, explaining why harmonic oscillator coherent states similarly saturate* $\Delta x\Delta p = \hbar_{spin}/2$*.*

6 Limits of Validity and Corrections

Standard quantum mechanics is an effective theory valid only for the flat tangent patch. The exact relation retains curvature corrections. From the Holstein-Primakoff expansion (see Remark in §3), we know that $J_z = J - b^\dagger b$, where $b^\dagger b \propto x^2 + p^2$ measures the excitation number (distance from the pole). Thus:

$$[x,p] = i\hbar_{\text{spin}}\frac{J_z}{J} = i\hbar_{\text{spin}}\left(1 - \frac{x^2+p^2}{2J} + \dots\right) \tag{6.1}$$

6.1 State-Dependent Effective Action

The term $(x^2+p^2)/2J$ represents a geometric correction. For highly excited states, the effective commutator diminishes. This is not because $\hbar$ changes, but because the state explores a region of the sphere where the tangent plane approximation breaks down. In semiclassical language, these $1/J$ corrections correspond to the truncation of the **Moyal expansion**.

6.2 The Equatorial Limit

The contraction is strictly local. For states near the equator ($m \approx 0$), the topology is cylindrical, not planar. The equator corresponds to a momentum-like variable with compact domain; its conjugate is an angle, not a Cartesian coordinate. This explains why phase and photon number require different uncertainty relations than position and momentum—they live on topologically distinct patches of the same underlying manifold.

7 Physical Interpretation

- **Geometry:** Phase space is the tangent plane to the coherent ground state of a macroscopic spin.
- **Canonical Quantization:** The historical recipe of replacing $\{x,p\}$ with commutators works because it assumes the world is locally flat, implicitly reversing this contraction limit to reconstruct the quantum geometry from its shadow.

- **Classicality:** Macroscopic systems appear classical not because $\hbar_{\text{phys}} \to 0$, but because the radius of curvature J is so large that the curvature corrections $\sim 1/J$ are undetectable.

8 Conclusion

We have successfully reconstructed the physics of continuous variables. Canonical quantization is not a postulate but a consequence of taking the macroscopic limit of a compact state space. This completes the reconstruction of continuous-variable quantum mechanics from finite-dimensional principles.

References

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[3] A. M. Perelomov, *Generalized Coherent States and Their Applications*, Springer-Verlag (1986).