Reconstructing Quantum Mechanics from the Action Quota:

An Eight-Lemma Derivation from Variance Complementarity

A Preface to the Reconstruction Programme

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Abstract

We reconstruct quantum mechanics from a single operational axiom: the **Action Quota**, which imposes a fundamental limit on simultaneous knowledge of incompatible properties. Starting from this variance complementarity constraint and incorporating basic structural principles (isotropy, reversibility, interactivity), we derive in eight sequential lemmas the complete quantum formalism for qubits, entanglement, and continuous variables. The reconstruction culminates in the thermodynamic calibration of Planck's constant, showing that both the structure and scale of quantum theory emerge from information-theoretic principles.

Architecture of the Derivation:

1. **The Static Core (Lemmas 1–3):** Qubit geometry (Bloch ball) and measurement statistics (Born rule).
2. **The Dynamic Framework (Lemmas 4–5):** Complex amplitudes, Schrödinger dynamics, and scale unification ($\hbar = \kappa$).
3. **The Extensions (Lemmas 6–8):** Entanglement via interaction, continuous variables via group contraction, and thermodynamic calibration of $\hbar$.

1 Introduction: The Reconstruction Programme

Standard quantum mechanics is usually introduced as a collection of formal postulates (state vectors, operators, commutators). This programme asks: *Is this architecture arbitrary, or is it the only structure consistent with a fundamental limit on information?*

The Action Quota The central engine of this reconstruction is the **Action Quota**. Interpreted as a universal "certainty budget," it dictates that nature cannot process simultaneous definite outcomes for incompatible properties. We show that this single quantitative constraint, when embedded in a space that respects spatial isotropy and temporal reversibility, forces the entire mathematical apparatus of quantum theory.

Novelty and Scope Unlike earlier reconstructions (Hardy [1], Chiribella [2], Masanes–Müller [3]), this programme: (1) uses a single quantitative axiom (the Action Quota) rather than multiple qualitative postulates; (2) derives the numerical value of $\hbar$ from thermodynamic consistency, not just its algebraic role; and (3) extends to continuous variables via group contraction, bridging qubit and field theory.

2 Axioms and Assumptions

The Action Quota (Foundational Axiom)

Variance Complementarity. For any physical system, there exist *complementary* dichotomic observables A and B (outcomes ± 1) such that for all possible states:

$$\mathrm{Var}(A) + \mathrm{Var}(B) \geq 1$$

Equivalence: For dichotomic ± 1 observables, $\mathrm{Var}(A) = 1 - \langle A \rangle^2$, yielding:

$$\langle A \rangle^2 + \langle B \rangle^2 \leq 1$$

This defines a unit disk in the $(\langle A \rangle, \langle B \rangle)$ plane—the fundamental quantum uncertainty geometry.

Standing Operational Assumptions

1. **Operational states:** States are equivalence classes of preparation procedures.
2. **Convexity:** Probabilistic mixtures of preparations exist, and the state space is convex.
3. **Affineness:** Measurement outcome probabilities are affine in state (mixtures of states map linearly to mixture probabilities).
4. **Symmetry:** Isotropy of measurement directions; reversible transformations act continuously on the state space.
5. **Interactivity (for L6):** There exist reversible joint transformations that cannot be decomposed into independent local ones.
6. **Continuum idealization (for L7):** A limit procedure exists (group contraction) that defines canonical pairs (X, P).
7. **Thermal equilibrium (for L8):** Standard Gibbs equilibrium for modes, characterized by discrete energy exchange increments.

2.1 What we do NOT assume

A central goal of this reconstruction is to ensure that the "quantum" features are derived rather than postulated. In particular, **we do not assume**:

- The complex Hilbert space structure or state vectors.
- The Born Rule (inner-product based probabilities).
- The tensor product rule for composite systems.
- Canonical Commutation Relations (CCRs) like $[X, P] = i\hbar$.
- The existence or numerical value of $\hbar$.

3 Logical Architecture: The Eight Lemmas

Table 1: The Eight-Lemma Reconstruction: From Information to Quantum Mechanics

Lemma	**Title**	**Critical Result**	**Physical Meaning**
L1	The Action Quota	$\mathrm{Var}(A) + \mathrm{Var}(B) \geq 1$	Fundamental information limit
L2	The Bloch Ball	Isotropy $\rightarrow$ 3D ball B^3	Spatial degrees of freedom
L3	The Born Rule	$p = (1 + \vec{s} \cdot \vec{m})/2$	Quantum probability
L4	Complex Dynamics	Path sensitivity $\rightarrow \mathbb{C}^2$	Complex amplitudes, phases
L5	Scale Unification	$\hbar = \kappa$	Hamiltonian dual role
L6	Composite Systems	Interaction $\rightarrow \mathbb{C}^2 \otimes \mathbb{C}^2$	Entanglement
L7	Continuum Limit	Group contraction $\rightarrow [X, P] = i\hbar$	Canonical variables
L8	Calibration	$\sigma_{\mathrm{SB}} \Rightarrow \hbar = h/2\pi$	Fixes physical action scale

4 Guide for the Reader: Conceptual Hurdles

Why the Action Quota Matters The Action Quota echoes a broader theme—tradeoffs between incompatible descriptions (e.g. time–frequency in Fourier analysis). Quantum theory is distinctive because the tradeoff becomes a *structural constraint* on the state space itself, not merely a limitation of instruments or a property of signals.

Why the Bloch Ball is 3D (Lemma 2) A 2D disk would permit only one complementary pair (A, B). But physical space is isotropic: rotating a measurement setup should yield equally valid complementary observables. This rotational symmetry requires a continuous family of directions, which only a 3D ball can provide while respecting the Action Quota at every orientation.

Why Complex Amplitudes (Lemma 4) The Bloch ball's rotation group $SO(3)$ has a topological subtlety: a 2π rotation is not contractible to the identity. Representing reversible evolution requires the universal cover $SU(2) \cong \mathrm{Spin}(3)$, which acts on $\mathbb{C}^2$ vectors. Complex numbers enter as the price of representing rotations faithfully.

Why Scale Unification is Necessary (Lemma 5) The Action Quota establishes an algebraic scale $\hbar$ for uncertainties, while dynamics introduces a separate scale κ for time evolution. If $\hbar \neq \kappa$, energy would be measured in different "units" than other observables, breaking the consistency of the Hamiltonian as both an observable and a generator. The unification $\hbar = \kappa$ ensures a single action scale governs all aspects of the theory.

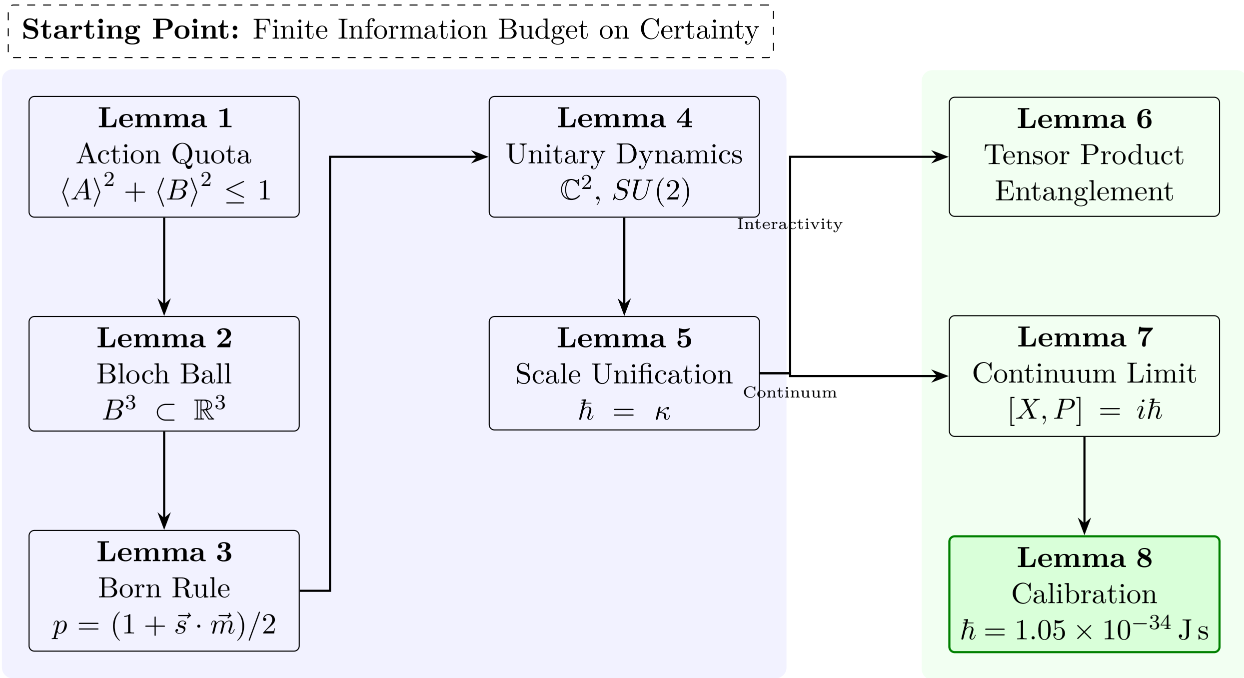

Figure 1: Logical architecture of the reconstruction programme. The derivation flows from the informational constraints of the qubit core (blue) through dynamical unification to the global interaction and physical scale calibration (green).

Why Entanglement Emerges (Lemma 6) If composite systems were described by a Cartesian product of local algebras (direct sum), interactions would be impossible—systems could not influence each other. The requirement of interactivity forces algebraic closure to the full tensor product, whose expanded state space necessarily includes entangled states that saturate global uncertainty bounds.

Why h has its Specific Value (Lemma 8) The algebraic structure forces energy quantization: $E_n = (n + 1/2)\hbar\omega$. Applying this to electromagnetic field modes yields Planck's blackbody spectrum. Matching the predicted Stefan–Boltzmann constant to its measured value uniquely determines $\hbar$, grounding the abstract formalism in experimental thermodynamics.

5 Conclusion

This reconstruction demonstrates that quantum mechanics is not a strange deviation from "classical intuition" but the **unique mathematical architecture** for any physical theory that respects a fundamental information budget and the basic symmetries of space and time.

Synthesis The eight lemmas form a tight logical chain: from information limits (L1) to geometry (L2–3), to dynamics (L4–5), to interaction (L6), to continuum (L7), and finally to empirical calibration (L8). Each step is forced by mathematical necessity once the initial principles are accepted.

Implications The reconstruction provides a coherent foundation that:

- Explains *why* quantum theory has its specific mathematical structure.
- Derives rather than postulates distinctive features (entanglement, uncertainty).
- Grounds the numerical value of $\hbar$ in thermodynamic consistency.
- Unifies discrete (qubit) and continuous (phase space) quantum mechanics.

By completing this programme, we bridge operational principles, geometric necessity, and empirical reality, offering a principled answer to the century-old question: "Why quantum mechanics?"

References

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