LEMMA 6 (COMPOSITE SYSTEMS):

The Inflation of the Observable Algebra

From Interaction Generators to the Tensor Product

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December 31, 2025

Abstract

We reconstruct the composition rule for quantum systems ($N = 2$). We address the **Interaction Gap**: the fact that the direct sum algebra of independent qubits is insufficient to support reversible interactions. Invoking the principle of **Global Action Consistency** and a new **Interaction Axiom**, we prove that adjoining a single entangling interaction generator forces algebraic inflation to the full $\mathfrak{su}(4)$ correlation algebra. This provides the geometric necessity for **entanglement** and identifies the composite Hilbert space as the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$.

Keywords: Composite systems, tensor product, entanglement, Lie algebra closure, interaction gap, Action Quota universality.

1 The Composition Starting Point

Following Lemma 4, a single qubit is described by the algebra $\mathfrak{su}(2)$ acting on $\mathbb{C}^2$. For two distinguishable systems A and B, we define their joint description through independent local contexts.

Definition 1.1 (Local Subalgebras). *On the composite Hilbert space $\mathcal{H}$ of dimension d, let $\mathcal{A}, \mathcal{B} \subset \mathfrak{su}(d)$ be two Lie subalgebras, each isomorphic to $\mathfrak{su}(2)$, such that $[\mathcal{A}, \mathcal{B}] = \{0\}$. These encode the independent controllability and measurement of two distinguishable subsystems. Operationally, this means there exist two independently tunable three-parameter families of reversible controls whose generators close as $\mathfrak{su}(2)$ and commute across the two control sets.*

Remark 1.1. *Once the minimal two-qubit sector $\mathcal{H}_{\frac{1}{2},\frac{1}{2}}$ from Lemma 1.1 is identified, we may choose a basis on that sector such that $\mathcal{H}_{\frac{1}{2},\frac{1}{2}} \cong \mathbb{C}^2 \otimes \mathbb{C}^2$ and*

$$\mathcal{A} = \mathrm{span}\{\sigma_i \otimes \mathbb{I}\}, \qquad \mathcal{B} = \mathrm{span}\{\mathbb{I} \otimes \sigma_j\},$$

with $i, j \in \{x, y, z\}$. Global identification $\mathcal{H} \cong \mathbb{C}^2 \otimes \mathbb{C}^2$ then follows under Axiom 1.1 for the full implementable algebra.

Axiom 1.1 (No superselection across the composite). *All states of the composite are operationally connected by reversible transformations generated from the global catalog. Equivalently, the representation of the physically implementable Lie algebra on $\mathcal{H}$ is irreducible: there is no nontrivial decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ preserved by all allowed dynamics.*

Lemma 1.1 (Local normal form and the minimal two-qubit sector). *Let $\mathcal{A}, \mathcal{B} \subset \mathfrak{su}(d)$ be commuting Lie subalgebras as in Definition 1.1. Assume (i) the restriction of the representation of $\mathcal{A}$ to $\mathcal{H}$ contains at least one spin-$\frac{1}{2}$ irreducible subrepresentation, and likewise for $\mathcal{B}$, and (ii) Axiom 1.1 holds for the* physically implementable *Lie algebra generated in this lemma. Then there exists an invariant subspace $\mathcal{H}_{\frac{1}{2},\frac{1}{2}} \subseteq \mathcal{H}$ on which $\mathcal{A} \oplus \mathcal{B}$ acts as the product irrep $(\frac{1}{2}) \otimes (\frac{1}{2})$, hence $\dim \mathcal{H}_{\frac{1}{2},\frac{1}{2}} = 4$, and there is a unitary W identifying $\mathcal{H}_{\frac{1}{2},\frac{1}{2}} \cong \mathbb{C}^2 \otimes \mathbb{C}^2$ such that on this sector*

$$W\mathcal{A}W^\dagger = \mathrm{span}\{\sigma_i \otimes \mathbb{I}\}, \qquad W\mathcal{B}W^\dagger = \mathrm{span}\{\mathbb{I} \otimes \sigma_j\}.$$

Moreover, if Axiom 1.1 holds for the full physically implementable algebra on $\mathcal{H}$, then $\mathcal{H} = \mathcal{H}_{\frac{1}{2},\frac{1}{2}}$ and $d = 4$.

Sketch. Since $[\mathcal{A}, \mathcal{B}] = 0$, $\mathcal{H}$ carries a representation of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and decomposes as a direct sum of tensor products of irreducibles: $\mathcal{H} \cong \bigoplus_{\alpha,\beta} (V_\alpha \otimes W_\beta)$. By assumption, $\alpha = \beta = \frac{1}{2}$ occurs at least once, yielding an invariant sector $\mathcal{H}_{\frac{1}{2},\frac{1}{2}} \cong \mathbb{C}^2 \otimes \mathbb{C}^2$. A standard change of basis gives the stated Pauli normal form on that sector. If the physically implementable algebra acts irreducibly (no superselection), there can be only one such sector, hence $\mathcal{H} = \mathcal{H}_{\frac{1}{2},\frac{1}{2}}$ and $d = 4$. □

2 The Interaction Gap

A naive "classical" composition would suggest that the global algebra of observables is simply the span of the two local ones: $\mathcal{L}_0 = \mathrm{span}\{\mathcal{A}, \mathcal{B}\}$.

Proposition 2.1 (Failure of the Direct Sum). *If the global observable algebra is restricted to $\mathcal{L}_0$, interactions are impossible. Every Hamiltonian $H \in \mathcal{L}_0$ takes the form $H = H_A + H_B$, generating factorizable dynamics. Equivalently, for all product inputs $\rho_A \otimes \rho_B$ and all t, the reduced state satisfies $\mathrm{Tr}_A(U(t)(\rho_A \otimes \rho_B)U(t)^\dagger) = U_B(t)\rho_B U_B(t)^\dagger$ and is independent of ρ_A.*
In particular, on a direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ every implementable generator is block-diagonal, so no commutator closure can ever produce a correlation sector and entanglement is impossible; such a structure cannot realize Axiom 3.1.

3 Derivation of the Tensor Product Algebra

Axiom 3.1 (Interaction). *There exists at least one nontrivial reversible generator G in the global algebra that couples the systems, such that $G \notin \mathrm{span}(\mathcal{L}_0)$.*

Remark 3.1 (Genericity of the interaction). *Let $\mathcal{C} := \mathrm{span}\{\sigma_i \otimes \sigma_j : \ i, j \in \{x, y, z\}\}$ denote the correlation sector on $\mathcal{H}_{\frac{1}{2},\frac{1}{2}} \cong \mathbb{C}^2 \otimes \mathbb{C}^2$. Under local conjugations $U_A \otimes U_B$, an element $C \in \mathcal{C}$ transforms by the adjoint action $C \mapsto \mathrm{Ad}_{U_A \otimes U_B}(C)$, which corresponds to the natural action of $(R_A, R_B) \in SO(3) \times SO(3)$ on $\mathbb{R}^3 \otimes \mathbb{R}^3 \cong \mathcal{C}$.*

The $SO(3) \times SO(3)$-module $\mathcal{C} \cong (\mathbf{3}) \boxtimes (\mathbf{3})$ is irreducible. Any nonzero orbit-span is an invariant subspace; since $\mathcal{C}$ is irreducible, it must be all of $\mathcal{C}$. Accordingly, we call G generic *if its correlation projection $G_{\text{corr}} := \Pi_{\mathcal{C}}(G)$ is nonzero.*

Axiom 3.2 (Lie Closure of Implementable Generators)**.** *If A and B are implementable generators in the global catalog, then the derived generator $\frac{1}{2i}[A, B] = \frac{1}{2i}[A, B]$ is also an implementable generator in the global catalog. Operationally, $\frac{1}{2i}[A, B]$ is the infinitesimal generator of the group commutator loop $e^{tA}e^{tB}e^{-tA}e^{-tB}$, so it must belong to the same implementable generator set if that loop is a physically admissible reversible protocol.*

Lie-algebraic Controllability. This is the standard Lie-algebraic controllability mechanism: local $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ plus a generic entangling Hamiltonian generates $\mathfrak{su}(4)$.

Definition 3.1 (Generated implementable algebra)**.** *On the two-qubit sector $\mathcal{H}_{\frac{1}{2},\frac{1}{2}}$ from Lemma 1.1, let $\mathfrak{g} := \langle \mathcal{A}, \mathcal{B}, G\rangle_{\text{Lie}}$ denote the smallest Lie subalgebra of $\mathfrak{su}(\mathcal{H}_{\frac{1}{2},\frac{1}{2}}) \cong \mathfrak{su}(4)$ containing $\mathcal{A}$, $\mathcal{B}$, and G.*

Theorem 3.1 (Algebraic Inflation to $\mathfrak{su}(4)$)**.** *Let $\mathcal{L}_0$ be the local algebra of two qubits. If the global algebra contains a generic entangling generator G and satisfies Axiom 3.2, then $\mathfrak{g} = \mathfrak{su}(4)$.*

Proof sketch (orbit/closure). Work on the sector $\mathcal{H}_{\frac{1}{2},\frac{1}{2}} \cong \mathbb{C}^2 \otimes \mathbb{C}^2$ from Lemma 1.1. In the Pauli basis we have the vector space decomposition

$$\mathfrak{su}(4) = \underbrace{\operatorname{span}\{\sigma_i \otimes \mathbb{I}\}}_{\mathcal{A}} \oplus \underbrace{\operatorname{span}\{\mathbb{I} \otimes \sigma_j\}}_{\mathcal{B}} \oplus \underbrace{\operatorname{span}\{\sigma_i \otimes \sigma_j\}}_{\mathcal{C}}.$$

Let $\mathfrak{g} := \langle \mathcal{A}, \mathcal{B}, G\rangle_{\text{Lie}}$. Since $K = \exp(\mathcal{A}) \times \exp(\mathcal{B}) \subseteq G_{\mathfrak{g}}$ (the connected Lie group generated by $\mathfrak{g}$), we have $\mathrm{Ad}_U(\mathfrak{g}) = \mathfrak{g}$ for all $U \in K$, as inner automorphisms of the generated group preserve the Lie algebra. In particular, $\mathrm{Ad}_U(G) \in \mathfrak{g}$ for all $U \in K$. Taking the $\mathcal{C}$-projection gives $\mathrm{Ad}_U(G_{\text{corr}}) \in \mathfrak{g} \cap \mathcal{C}$. By Remark 3.1, the linear span of this orbit set is all of $\mathcal{C}$, hence $\mathcal{C} \subseteq \mathfrak{g}$. Consequently $\mathfrak{g} \supseteq \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C} = \mathfrak{su}(4)$, so $\mathfrak{g} = \mathfrak{su}(4)$. □

Corollary 3.1 (Tensor product identification)**.** *If $\mathfrak{g} = \mathfrak{su}(4)$ on $\mathcal{H}_{\frac{1}{2},\frac{1}{2}}$ and Axiom 1.1 holds for the full implementable algebra on $\mathcal{H}$, then $\mathcal{H} = \mathcal{H}_{\frac{1}{2},\frac{1}{2}} \cong \mathbb{C}^4$ and, by Lemma 1.1, the commuting factors $\mathcal{A}, \mathcal{B}$ select a decomposition $\mathcal{H} \cong \mathbb{C}^2 \otimes \mathbb{C}^2$ unique up to local unitaries.*

4 Summary: Entanglement as Algebraic Necessity

The tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$ is not a primitive assumption; it is the unique structure that preserves the operational separability of local contexts while permitting physical interactivity.

The Mechanism of Entanglement. The Action Quota (Lemma 1), which bounds local certainty, extends naturally to the composite system. The 9 additional correlation dimensions provided by algebraic inflation are the geometric mechanism of entanglement: they allow certainty to concentrate in global correlation observables (like $\sigma_z \otimes \sigma_z$) while local observables remain uncertain, saturating the global Action Quota while sitting at the center of the local Bloch balls.

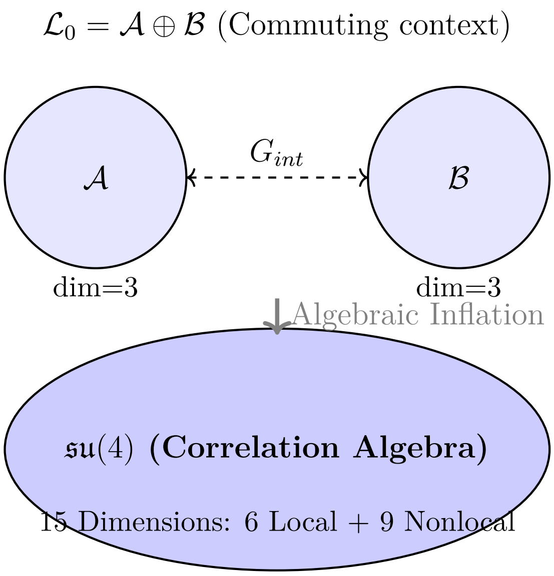

Figure 1: Algebraic Inflation: A single nonlocal interaction G_{int} bridges commuting local contexts. Commutator closure inflates the algebra to the full 15-dimensional correlation space, identifying the tensor product as the unique minimal structure for interacting systems.

5 Conclusion

The tensor product is the geometric cost of interaction. It follows from interactivity, consistency, and algebraic closure. The composite state space is thus uniquely identified as the set of 4×4 density matrices, whose pure states are the projectivization of $\mathbb{C}^4$.

References

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A Explicit Closure Construction

Starting from $\mathcal{L}_0$ and $G = \sigma_x \otimes \sigma_z$, we generate all 15 basis elements using $\frac{1}{2i}[A, B] = \frac{1}{2i}[A, B]$:

1. $\frac{1}{2i}[\sigma_y \otimes \mathbb{I}, \sigma_x \otimes \sigma_z] = \sigma_z \otimes \sigma_z$
2. $\frac{1}{2i}[\sigma_z \otimes \mathbb{I}, \sigma_x \otimes \sigma_z] = \sigma_y \otimes \sigma_z$
3. $\frac{1}{2i}[\mathbb{I} \otimes \sigma_x, \sigma_z \otimes \sigma_z] = -\sigma_z \otimes \sigma_y$

Orthogonality under the Hilbert–Schmidt product $\langle A, B \rangle_{\mathrm{HS}} = \mathrm{Tr}(A^\dagger B)$ ensures these are linearly independent, spanning all 9 correlation terms.