Reconstructing Quantum Mechanics from the Action Quota:

An Eight-Lemma Derivation
from Variance Complementarity

A Preface to the Reconstruction Programme

Emiliano Shea

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Abstract

We reconstruct quantum mechanics from a single axiom: variance complementarity between incompatible measurements. From this constraint plus generic structural assumptions, we derive the Bloch sphere, Born rule, tensor product, canonical commutation relations, and Planck's constant in eight sequential lemmas. This preface outlines the logical architecture; companion papers provide full proofs.

Scope of the Derivation:

1. **The Discrete Core (Lemmas 1–5):** The emergence of qubit geometry, complex dynamics, and the unification of scale.
2. **The Extensions (Lemmas 6–7):** The emergence of entanglement via interaction and phase space as a limit.
3. **The Calibration (Lemma 8):** The unique fixing of the numerical value of $\hbar$ via thermodynamic consistency.

1 Introduction: The Reconstruction Programme

Quantum mechanics is normally introduced as a list of formal axioms. This reconstruction asks whether these features are inevitable consequences of a single operational constraint.

Novelty of this Approach While previous programmes by Hardy [1], Chiribella [2], and Masanes [3] have successfully derived quantum theory from information-theoretic principles, they typically require separate postulates for state-space structure, composition, and measurement. Our approach replaces theory-specific postulates with **generic structural assumptions** (e.g., isotropy, reversibility) that define the preconditions for any experimental science. We show that the **Action Quota** alone, interpreted as a fundamental certainty budget, contains the seeds of the entire geometry.

The Deep Questions Addressed Our programme answers core puzzles that often feel taken for granted:

- **Geometry:** Why is the state space a 3D ball (the Bloch ball)?
- **Probability:** Why are probabilities given by the Born rule?
- **Complexity:** Why do composite systems combine via the tensor product ($\otimes$)?
- **Calibration:** Why is Planck's constant $h \approx 6.626 \times 10^{-34}$ J s?

A Roadmap for the Reader This preface is intended as a roadmap. It does not contain proofs, but it does contain the complete logical structure of the reconstruction. Readers seeking technical details should consult the individual lemma papers; readers seeking a conceptual overview should find this document self-contained.

2 The Axiom and Its Context

Our derivation rests on one foundational axiom and a set of minimal operational requirements.

Foundational Axiom: The Action Quota

Variance Complementarity. There exist pairs of dichotomic observables A and B (outcomes ± 1) such that for all states:

$$\mathrm{Var}(A) + \mathrm{Var}(B) \geq 1$$

This quantifies the irreducible trade-off between incompatible measurements.

Structural Assumptions (Operational Requirements): To build a theory around this axiom, we assume standard properties that define an operational physical theory:

- **Isotropy (L2):** The state space admits no preferred measurement direction (Symmetry).
- **Reversibility (L4):** Dynamics are continuous and reversible (Evolution).
- **Interaction (L6):** Systems can influence one another (Composition).
- **Thermodynamic Consistency (L8):** Thermal limits match observation (Calibration).

The structural assumptions constrain how the geometry can be realized; the Action Quota constrains what geometry is possible. Neither alone determines the structure—but together, they leave no room for alternatives.

3 The Action Quota in Detail

Why this Specific Bound? The form $\mathrm{Var}(A) + \mathrm{Var}(B) \geq 1$ is naturally motivated by the dichotomic nature of the variables. For outcomes ± 1, the variance is $\mathrm{Var}(A) = 1 - \langle A \rangle^2$. A variance of 0 implies absolute certainty, while a variance of 1 implies a random 50/50 distribution. The bound "1" dictates a simple "Certainty Budget": *If one variable is perfectly certain, the other must be perfectly random.*

The Action Link: While expressed here as a variance sum, in the continuous limit (Lemma 7), this constraint directly maps to the phase-space area $\Delta x \Delta p \geq \hbar/2$. This identifies the "quota" as the fundamental unit of physical action, justifying the nomenclature of this programme.

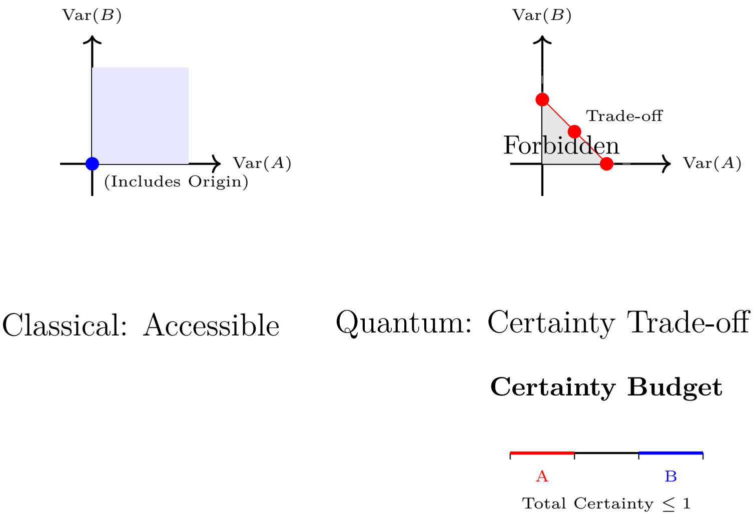

Figure 1: In classical physics, any variance is accessible. The Action Quota forbids this region. The budget bar (inset) illustrates that total certainty about A and B cannot exceed 1; the central gap represents the unavoidable uncertainty.

What We Do NOT Assume We maintain strict consistency by using $\mathrm{Var}(A) = (\Delta A)^2$ as our primary object. We assume **no** complex numbers, Hilbert spaces, operators, or Planck's constant. These are emergent properties.

4 Reconstruction Architecture: Eight Lemmas

Table 1: Logical dependency and insights

Lemma	Derives	Critical Inference	Paper[1]
L1	Unit Disk	Uncertainty creates a bounded information geometry.	§1
L2	Bloch Ball (B^3)	Isotropy + L1 geometry → requires a 3D manifold.[2]	§2
L3	Born Rule	Probabilities are affine maps on the convex state space.	§3
L4	$SU(2)$	Topology of S^2 forces a complex lift for reversibility.	§4
L5	$\hbar = \kappa$	Dual role unifies the noise scale and change rate.	§5
L6	Tensor Product	Interaction must preserve subsystem uncertainty.	§6
L7	$[x, p] = i\hbar$	Phase space as the local tangent of the sphere.	§7
L8	$\rho_0 = h$	Heat capacity of vacuum fixes the energy scale $\kappa \to h$.	§8

5 Guide for Readers

5.1 Conceptual Hurdles: Short Answers

- **Why 3D (L2)?** Isotropy requires a continuous family of measurements. A 2D disk's boundary is a 1D circle, which cannot support a continuous, non-commutative set of projective measurements. Only a 3D ball (B^3) provides the minimal degrees of freedom for rotational symmetry under the Action Quota.

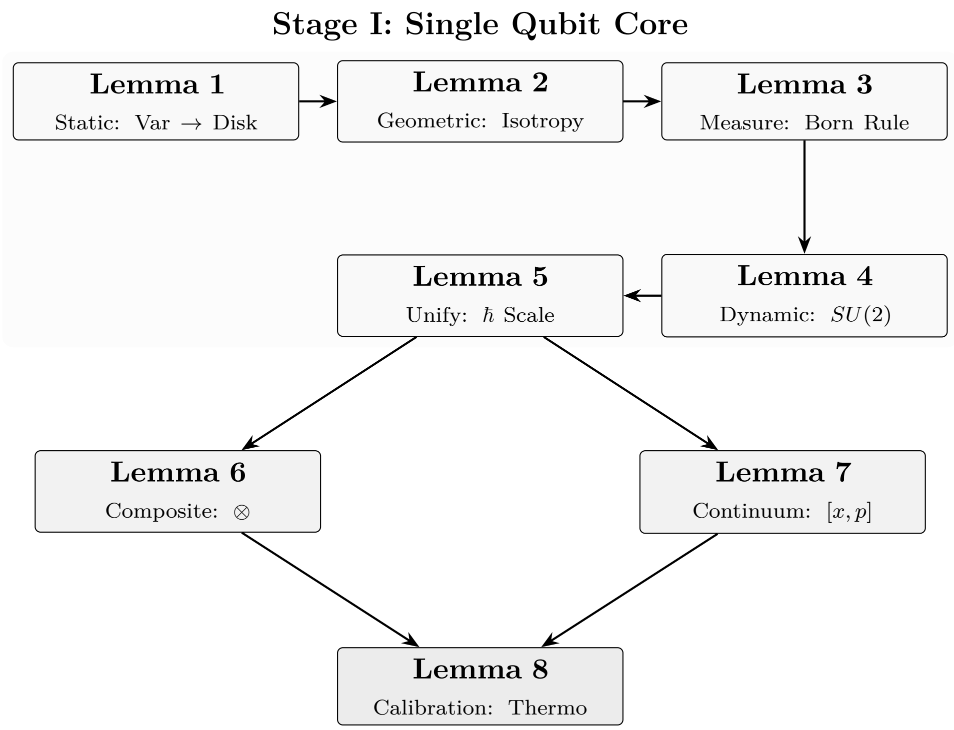

Figure 2: Logical flow from a single uncertainty bound to the full physical scale.

- **Why Complex Numbers (L4)?** Continuous, reversible dynamics on the Bloch ball are described by $SO(3)$. Consistent representations of these rotations on the state space require the universal cover $SU(2)$, which acts on $\mathbb{C}^2$.
- **Why the Tensor Product (L6)?** A Cartesian product of state spaces would allow composite states with higher joint certainty than the Action Quota permits for subsystems. The tensor product structure enforces uncertainty relationships across composite systems, leading directly to entanglement.
- **Why h has a specific value (L8)?** We apply the uncertainty bound to each mode of a radiation field. Demanding that the Stefan-Boltzmann law emerges from the statistical mechanics of these quantized modes uniquely fixes the scale κ to the physical value h.

6 Conclusion and Philosophical Implications

By deriving quantum mechanics from the Action Quota, we show that quantum "weirdness" is the inevitable geometric consequence of a single constraint on information.

Reframing the Measurement Problem This reconstruction suggests that if quantum states represent knowledge constrained by a fundamental information budget, then "collapse" corresponds to updating that knowledge when the budget is reallocated. Exploring whether fully unitary dynamics can model this budget reallocation is a key test for this interpretation. Whether this reframing dissolves the problem or relocates it remains an open question for future investigation.

References

[1] L. Hardy, *Quantum Theory From Five Reasonable Axioms*, arXiv:quant-ph/0101012 (2001).

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