LEMMA 1 (THE STATIC CORE):

From the Action Quota to the Unit Disk

Deriving the 2D Information Geometry of Complementarity

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December 31, 2025

Abstract

We derive the Euclidean unit disk $a^2 + b^2 \leq 1$ as the necessary geometry for the expectation values of two **maximally complementary** measurements. The derivation relies on the **Action Quota**—a variance-based complementarity condition $\mathrm{Var}(A)+\mathrm{Var}(B) \geq 1$—within a framework of a convex state space and affine response. The main result is that the allowed expectation values $(a, b) = (\langle A \rangle, \langle B \rangle)$ form exactly the unit disk $a^2 + b^2 \leq 1$. We show that the "Bloch disk" is not a postulate of quantum mechanics, but a geometric consequence of finite information constraints, providing the foundation for the 3-dimensional "inflation" developed in Lemma 2.

1 Operational Framework

To reconstruct the theory from the ground up, we define our system using only experimental primitives: preparations (states) and outcomes (statistics).

Assumption 1.1 (Convex State Space)**.** *The set of physical preparations (states) forms a convex set* Ω*. For any two states* $\omega_1, \omega_2 \in \Omega$*, their mixture* $\omega = p\omega_1 + (1 - p)\omega_2$ *for* $p \in [0, 1]$ *is also a valid state.*

Definition 1.1 (Dichotomic Measurement)**.** *A measurement* A *is dichotomic if it yields outcomes in* $\{+1, -1\}$*. For any state* $\omega \in \Omega$*, the expectation value is* $a := \langle A \rangle_\omega \in [-1, 1]$*.*

Assumption 1.2 (Affine Response)**.** *The map* $\omega \mapsto \langle A \rangle_\omega$ *is affine for every dichotomic measurement. This ensures operational consistency: the statistics of a mixture of preparations are the weighted average of the statistics of each individual preparation.*

Assumption 1.3 (Sharpness)**.** *For every dichotomic measurement* A*, there exist states* $\omega_{A\pm}$ *yielding outcomes* ± 1 *with certainty (i.e.,* $\langle A \rangle_{\omega_{A\pm}} = \pm 1$*).*

Definition 1.2 (Operational Variance)**.** *For a dichotomic measurement with outcomes* ± 1*, the variance is* $\mathrm{Var}(A)_\omega = 1 - a^2$*. This follows from the definition* $\mathrm{Var}(A) = \langle A^2 \rangle - \langle A \rangle^2$ *because* $A^2 = 1$ *identically for all outcomes.*

Foundational Axiom: The Action Quota

Assumption 1.4 (The Action Quota)**.** *For any physical system, there exists a specific class of measurements representing mutually exclusive information. Specifically:*

(i) ***Complementary Pair:*** *For each system, there exist at least two implementable dichotomic measurements A and B (outcomes ± 1) such that for* all *preparation procedures $\omega \in \Omega$, the operational variances satisfy:*

$$\mathrm{Var}(A)_\omega + \mathrm{Var}(B)_\omega \geq 1$$

(ii) ***Saturation:*** *There exists at least one state $\omega_* \in \Omega$ saturating this bound.*

At this stage, the axiom is ***existential****: it asserts that such a pair exists. The requirement that every orientation possesses such a partner (Isotropy) is developed in Lemma 2.*

Why Variance? For dichotomic outcomes, the variance $\mathrm{Var}(M) = 1 - \langle M \rangle^2$ is the unique operational measure of unpredictability. A variance of 0 implies absolute predictability (certainty), while a variance of 1 implies a perfectly random result ($p = 1/2$ for both outcomes). The Action Quota represents a *state-independent budget*: it asserts that nature cannot provide simultaneous certainty for A and B. If one property is known perfectly, the other must be perfectly random.

Assumption 1.5 (Planar Rotational Symmetry)**.** *There exists a continuous one-parameter family of reversible transformations $\{T_\theta\}_{\theta \in [0,2\pi)}$ acting on states such that for every $\omega \in \Omega$:*

$$(\langle A \rangle_{T_\theta \omega}, \langle B \rangle_{T_\theta \omega}) = (a \cos\theta + b \sin\theta, -a \sin\theta + b \cos\theta),$$

where $(a, b) = (\langle A \rangle_\omega, \langle B \rangle_\omega)$.

2 Motivation and Interpretation

Physical Content. The Action Quota captures the irreducible operational limitation observed in quantum systems: the impossibility of preparing a state that has simultaneous definite outcomes for two incompatible measurements.

Accessibility Trade-offs. Classically, if A and B were jointly sharp observables, the accessible region would be the entire square $[-1, 1] \times [-1, 1]$. The diamond $|a| + |b| \leq 1$ is the region guaranteed reachable by convex combinations of the cardinal sharp preparations. The Action Quota plus symmetry extends this to the full unit disk—a larger set that remains constrained by the irreducible uncertainty trade-off.

Normalization. The value "1" is a normalization convention. For dichotomic variables, the maximum variance is 1. A sum of 1 dictates that if one variable is certain ($\mathrm{Var} = 0$), the other must be random ($\mathrm{Var} = 1$).

3 Lemma 1: The Unit Disk

Lemma 3.1 (Emergence of the Bloch Disk)**.** *Let* $a = \langle A \rangle_\omega$ *and* $b = \langle B \rangle_\omega$*. If the Action Quota* (1.4) *and Planar Rotational Symmetry* (1.5) *hold, the set of allowed expectation-value pairs* (a, b) *form exactly the unit disk in* $\mathbb{R}^2$*:*

$$\mathcal{D} = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 1\}.$$

Proof.
I. Algebraic Bound. Substituting the definition of dichotomic variance ($\mathrm{Var}(M) = 1 - m^2$) into part (i) of the Action Quota (1.4):

$$(1 - a^2) + (1 - b^2) \geq 1 \implies 2 - (a^2 + b^2) \geq 1 \implies a^2 + b^2 \leq 1.$$

This proves that the set of expectation values is a subset of the unit disk.

II. Reachability of the Origin and Cardinal Points. By Assumption 1.3 (Sharpness), there exist states $\omega_{A\pm}$ and $\omega_{B\pm}$. The Action Quota implies that if $a = \pm 1$, then $\mathrm{Var}(B) \geq 1$, i.e., $1 - b^2 \geq 1$, so $b^2 \leq 0 \implies b = 0$. Thus, the cardinal points $(\pm 1, 0)$ and $(0, \pm 1)$ are achievable. By Assumptions 1.1 and 1.2, the mixed state $\rho_{\text{mix}} := \frac{1}{2}\omega_{A+} + \frac{1}{2}\omega_{A-}$ is achievable and yields $(a, b) = (0, 0)$.

III. Reachability of the Entire Boundary and Disk. By part (ii) of the Action Quota, there exists a state ω_* saturating the bound, so (a_*, b_*) lies on the circle $a^2 + b^2 = 1$. By Assumption 1.5, the orbit $\{T_\theta \omega_* : \theta \in [0, 2\pi)\}$ traces the entire unit circle.

Finally, for any boundary point $\omega_\theta := T_\theta \omega_*$ and any $\lambda \in [0, 1]$, convexity and affine response imply that the mixture $\omega_{\lambda,\theta} = \lambda \omega_\theta + (1 - \lambda)\rho_{\text{mix}}$ achieves $(\langle A \rangle, \langle B \rangle) = \lambda(a_\theta, b_\theta)$. As θ ranges over $[0, 2\pi)$, this fills every point in the closed disk $\mathcal{D}$. □

4 Non-circularity: Why this is not the Born Rule in disguise

A common criticism of informational reconstructions is the implicit assumption of quantum probability rules within the axioms. We emphasize that the Action Quota **is not** the Born Rule.

Expectation Geometry vs. Probability Calculus. The Action Quota constrains the *expectation geometry*—the range of possible statistics (a, b). It establishes that the state space is restricted by a variance bound, but it does not specify how to calculate the probability of an outcome for a given state. In standard QM, the Born Rule ($p = |\langle \psi | \phi \rangle|^2$) provides a specific map from vectors to probabilities. In this stage, we do not assume the existence of state vectors, inner products, or the $\cos^2(\theta/2)$ laws. The probability map is derived later in **Lemma 3**, where it emerges as the unique linear map consistent with the 3D geometry of the Bloch Ball. Lemma 1 merely establishes the "frontier" of what can be known, not the calculus of how we know it.

Corollary 4.0.1 (Certainty Trade-off)**.** *By Lemma 3.1, we have* $a^2 + b^2 \in [0, 1]$*, hence for maximally complementary* A, B*, the total variance budget is bounded:*

$$\mathrm{Var}(A)_\omega + \mathrm{Var}(B)_\omega = 2 - (a^2 + b^2) \in [1, 2].$$

The lower bound 1 (maximal certainty) is achieved on the circle $a^2 + b^2 = 1$*; the upper bound 2 (maximal uncertainty) at the origin.*

Remark 4.1 (The Information Frontier)**.** *We call the boundary* $a^2 + b^2 = 1$ *the* Information Frontier*: it is the locus of states that saturate the Action Quota for the pair* (A, B)*. Equivalently,* ω *lies on the Information Frontier iff* $\mathrm{Var}(A)_\omega + \mathrm{Var}(B)_\omega = 1$*. The diamond* $|a| + |b| \leq 1$ *represents the maximal set of states achievable if one only had access to mixtures of the four cardinal axis preparations.*

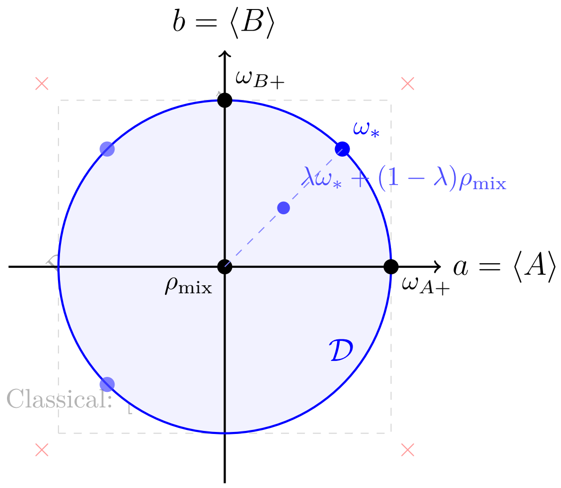

Figure 1: The Bloch Disk $\mathcal{D}$ (blue). The state ω_* saturates the bound; the reversible transformations T_θ generate the full boundary circle. Convex combinations with the central state ρ_{mix} fill the interior radial segments.

5 Roadmap to Lemma 2 (The 3D Inflation)

Lemma 1 establishes that for a fixed maximally complementary pair (A, B), the set of achievable expectation-value pairs $(\langle A \rangle, \langle B \rangle)$ is exactly the unit disk. However, a complete physical theory requires more than one privileged pair of measurements; it must accommodate measurements in arbitrary spatial directions.

If we demand **isotropy**—that the physics is invariant under spatial rotations—then the state space must contain disks for *every* orthogonal pair of directions. These disks naturally fill a 3D ball. Lemma 2 will show that the minimal isotropic extension consistent with our operational constraints is a 3-ball, thereby deriving the three-dimensionality of the qubit state space from informational constraints.

6 Conclusion

The unit disk, bounded by the **Information Frontier** $a^2 + b^2 = 1$, is the unique geometric solution to a budget constraint on information. By deriving this from the Action Quota and planar symmetry, we demonstrate that the non-classicality of the qubit is rooted in the statistical trade-off between incompatible measurement outcomes.

References

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