Lemma 5 (The Unification):

Unifying the Algebraic and Dynamical Action Scales

The Dual Role of the Hamiltonian and the Identity $\hbar = \kappa$

Emiliano Shea

December 30, 2025

Abstract

We prove that consistency between algebraic uncertainty bounds and dynamical evolution forces the identity $\hbar = \kappa$, completing the reconstruction of single-qubit quantum mechanics from information-theoretic principles. Lemma 1 established a variance trade-off (*Action Quota*) which defines a static scale $\hbar$. Lemma 4, through topological consistency, defined a *dynamic* scale κ governing the rate of state evolution. By identifying the Hamiltonian's dual role as both a physical observable and a generator of evolution, we demonstrate that a unified action scale is necessary to maintain the integrity of the Information Frontier.

Keywords: Quantum foundations, action scales, uncertainty principle, Hamiltonian, information-theoretic reconstruction

1 Introduction: Two Scales of Action

Our reconstruction has produced two fundamental constants that appear in distinct operational contexts. In the initial derivations, these scales are logically independent.

Definition 1.1 (Static Action Scale, $\hbar$). *The Action Quota established that for complementary observables, certainty is finite:* $\mathrm{Var}(A) + \mathrm{Var}(B) \geq 1$ *in normalized units. When physical dimensions are restored, this bound implies* $\Delta A \cdot \Delta B \geq \hbar/2$ *for canonically conjugate variables. The scale* $\hbar$ *quantifies the physical "size" of a quantum of certainty within the measurement algebra.*

Definition 1.2 (Dynamic Action Scale, κ). *In the derivation of dynamics, the Schrödinger equation appears as the unique representation of reversible evolution on Hilbert space:*

$$i\kappa \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle. \tag{1}$$

The scale κ *serves as the conversion factor between energy eigenvalues and the frequency of temporal state change.*

Structure After briefly recapping prior results (section 2), we elucidate the Hamiltonian's dual role (section 3), present the unification proof (section 4), summarize the complete reconstruction (section 5), and conclude with outlook (section 6).

2 Preliminaries: Recapping Lemmas 1–4

The reconstruction to this point has established the following foundation across the preceding papers:

- **Lemma 1:** Complementarity between measurements A and B implies a variance-sum bound, resulting in a unit disk geometry for expectation values.
- **Lemma 2:** Isotropy and informational efficiency "inflate" this geometry into the 3D Bloch Ball (B^3).
- **Lemma 3:** The statistics of these states are uniquely governed by the linear Born Rule.
- **Lemma 4:** Reversibility and topological path-sensitivity require representing states in a complex 2D Hilbert space ($\mathbb{C}^2$) evolving via a unitary generator H.

3 The Dual Role of the Hamiltonian

The unification $\hbar = \kappa$ is necessitated by the fact that the Hamiltonian H performs two distinct functions within the theory.

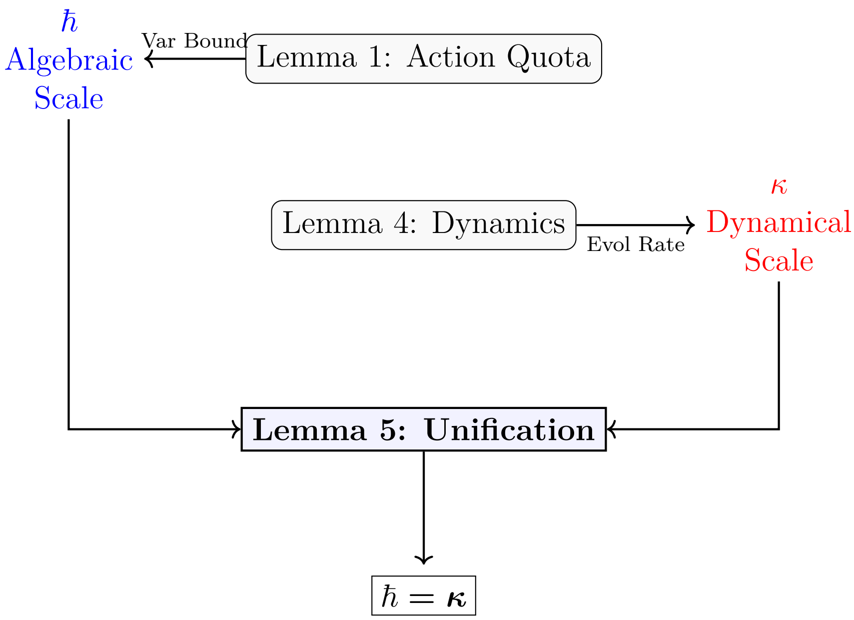

Figure 1: Conceptual flow showing the convergence of two independent action scales through Lemma 5.

As an **observable**, H represents energy; its variance and commutation relations are governed by $\hbar$. As a **generator**, H induces time translations at a rate governed by κ.

Assumption 3.1 (Single Action Calibration)**.** *All commutators in the measurement algebra and all generators of continuous symmetries are calibrated by the same fundamental physical action unit.*

With this conceptual framework established, we now prove the scale unification.

4 Main Result: Forced Unification

Theorem: Action Scale Unification

Internal consistency between the algebraic uncertainty budget and the dynamical implementation of time translations requires

$$\boxed{\hbar = \kappa}.$$

Proof. Fix a time-independent observable A and a Schrödinger-evolving state $|\psi(t)\rangle$ with

$$i\kappa \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle.$$

For reference, the corresponding unitary evolution operator is $U(t) = e^{-iHt/\kappa}$.

Step 1: Dynamical rate (Schrödinger/Heisenberg bridge). Using the Schrödinger equation (1) (Lemma 4) and the Hermiticity of H, for any time-independent A:

$$\frac{d}{dt}\langle A\rangle_t = \frac{d}{dt}\langle\psi(t)|A|\psi(t)\rangle = \frac{1}{i\kappa}\langle[A,H]\rangle_t. \tag{2}$$

Step 2: Mandelstam–Tamm operational time. Define the characteristic evolution time of the expectation value of A by

$$\tau_A := \frac{\Delta A}{\left|\frac{d}{dt}\langle A\rangle_t\right|}. \tag{3}$$

Combining (2) and (3) gives

$$\tau_A = \frac{\kappa\,\Delta A}{|\langle[A,H]\rangle_t|}. \tag{4}$$

Step 3: Algebraic uncertainty (Robertson). Because Lemma 4 has established a complex Hilbert-space representation, the Robertson inequality follows from the Cauchy–Schwarz inequality:

$$\Delta A\,\Delta H \geq \frac{1}{2}\,|\langle[A,H]\rangle_t|. \tag{5}$$

The static action scale $\hbar$ enters at the level of calibration: once physical units are restored, commutators in the observable algebra are measured in action units set by $\hbar$ (Lemma 1 with dimensional restoration).

Step 4: Energy–time complementarity from dynamics. Substituting (4) into (5) yields

$$\Delta H\,\tau_A = \Delta H\frac{\kappa\,\Delta A}{|\langle[A,H]\rangle_t|} \geq \Delta H\frac{\kappa\cdot\frac{1}{2}|\langle[A,H]\rangle_t|}{|\langle[A,H]\rangle_t|} = \frac{\kappa}{2}.$$

Thus the dynamical implementation predicts the universal bound

$$\Delta H\,\tau_A \geq \frac{\kappa}{2}. \tag{6}$$

Step 5: Matching the commutator scale to the generator scale. The algebraic action scale $\hbar$ is the constant that calibrates commutators in the measurement algebra. Because the Hamiltonian is itself an observable in that same algebra, for each observable A the quantity $|\langle [A, H] \rangle_t|$ is an algebraic object whose scale is fixed by $\hbar$.

Now compare (2) with the requirement of time-translation covariance: the rate of change of expectation values must be determined by the same commutator scale that governs all other algebraic relations. If $\kappa \neq \hbar$, then the same commutator $[A, H]$ would control algebraic uncertainty in units of $\hbar$ yet dynamical rates in units of κ. Equivalently: one could absorb a mismatch by redefining the generator as $H' = (\kappa/\hbar)H$, but then the operator that generates time translations would no longer coincide with the energy observable appearing in the measurement algebra, contradicting the Hamiltonian's dual role. This is precisely the physical content of Assumption 3.1: the Hamiltonian cannot have different calibrations in its two roles. Consequently, consistency requires $\hbar = \kappa$. □

Remark 4.1 (Historical Context)**.** *The identification $\hbar = \kappa$ is implicit in Dirac's original formulation of quantum mechanics but rarely justified axiomatically. Our derivation shows this identity is forced by consistency between information-theoretic and dynamical principles.*

Remark 4.2. *The quantity τ_A follows Mandelstam and Tamm's operational approach [3], defining time uncertainty via the evolution rate of an observable rather than through a problematic time operator. This captures the physical content of energy-time complementarity without mathematical inconsistencies.*

Corollary 4.1 (Unified Heisenberg Equation)**.** *The unification $\hbar = \kappa$ ensures that the rate derived from (2) recovers the standard Heisenberg equation of motion:*

$$\frac{d}{dt}\langle A \rangle = \frac{1}{i\hbar}\langle [A, H] \rangle \quad \text{or} \quad \frac{dA}{dt} = \frac{1}{i\hbar}[A, H].$$

Physical Significance The unification $\hbar = \kappa$ means that the same fundamental "grain size" of action governs both what we can know about a system and how quickly it evolves. This deepens the connection between information and dynamics in quantum theory.

5 Summary of the Qubit Core

Having unified the scales, we can now view the reconstruction as a coherent whole. Every postulate of the standard formalism has been derived from the Action Quota.

Lemma	**Input Principle**	**Quantum Result**	**Significance**	**Source**
L1	Variance Bound	Unit Disk Geometry ($\hbar$)	Non-classical uncertainty	Paper 1
L2	Isotropy/Efficiency	Bloch Ball (B^3)	3D state space	Paper 2
L3	Convexity/Sharpness	Born Rule	Probabilistic predictions	Paper 3
L4	Path Sensitivity	Hilbert Space ($\mathbb{C}^2, \kappa$)	Unitary dynamics	Paper 4
L5	Consistency	$\hbar = \kappa$	Unified action scale	Sec. 4

Table 1: Architecture of the single-qubit reconstruction from information principles.

6 Conclusion and Future Directions

Lemma 5 proves that the physical constant limiting knowledge and the constant governing the flow of time are identical. This reveals that quantum mechanics is a single geometric structure forced by a finite information budget.

This completes Stage I of the programme. Future directions include:

- **Lemma 6 (Composite Systems):** Showing that interaction consistency requires entangled subsystems to share the same $\hbar$.
- **Lemma 8 (Calibration):** Using thermodynamic consistency to fix the numerical value of $\hbar$ to the physical value of Planck's constant.

References

[1] L. Hardy, *Quantum Theory From Five Reasonable Axioms*, arXiv:quant-ph/0101012 (2001).

[2] J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*, Cambridge (2017).

[3] L. Mandelstam and I. Tamm, *The Uncertainty Relation Between Energy and Time in Non-relativistic Quantum Mechanics*, J. Phys. (USSR) **9**, 249–254 (1945).

[4] G. Chiribella et al., *Informational derivation of quantum theory*, Phys. Rev. A **84** (2011).

[5] B. Dakic and C. Brukner, *Quantum Theory and Beyond*, arXiv:0911.0695 (2009).