Lemma 3 (The Measurement Rule):

The Unique Measurement Rule on the Bloch Ball

From Variance Complementarity to the Born Rule

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Abstract

Lemma 1 established that complementary binary observables obey a fundamental variance bound—the **Action Quota**. Lemma 2 showed that isotropy and operational efficiency inflate the resulting 2D geometry into a 3D Euclidean ball (B^3). In this paper, we prove Lemma 3: given this geometry and the operational requirement of **Affine Probability**, the unique expectation map is the **Born Rule**: $p(+1 \mid \vec{s}, \vec{m}) = (1 + \vec{s} \cdot \vec{m})/2$. We show that the Born rule is the unique solution consistent with the geometric structure of the state space and the existence of sharp eigenstates.

1 Framework and Operational Principles

Following the "Dimensional Inflation" established in Lemma 2, our system is defined by a state space Ω affinely isomorphic to the closed unit 3-ball $B^3 = \{\vec{s} \in \mathbb{R}^3 : \|\vec{s}\| \leq 1\}$. Pure states lie on the **Information Frontier** S^2. Measurements M are dichotomic properties (outcomes ± 1) represented by unit vectors $\vec{m} \in S^2$.

Assumption (O1): Affine Probability Postulate

For any dichotomic measurement M, the probability of an outcome is an affine functional of the state. For any statistical mixture $\vec{s} = \lambda \vec{s}_1 + (1-\lambda)\vec{s}_0$ with $\lambda \in [0, 1]$:

$$p(M = +1 \mid \vec{s}) = \lambda p(M = +1 \mid \vec{s}_1) + (1 - \lambda)p(M = +1 \mid \vec{s}_0).$$

This is equivalent to the expectation value $\langle M \rangle_{\vec{s}}$ being affine in $\vec{s}$.

Why Affineness is Operational The requirement of affineness arises from **randomized preparation procedures**. Suppose an experimenter chooses between two preparations, P_1 and P_0, by flipping a biased coin that yields P_1 with probability λ. Operationally, the "state" is an equivalence class of preparation procedures; thus, the randomized procedure itself corresponds to a state $\vec{s}$, which convexity identifies as the vector sum $\lambda \vec{s}_1 + (1 - \lambda)\vec{s}_0$. Consistency with classical data processing requires that the probability calculus respects these weighted averages.

Assumption 1.1 (Isotropy - O2). *Measurement statistics satisfy rotation-covariance: for each $R \in SO(3)$ and all $\vec{s} \in B^3, \vec{m} \in S^2$:*

$$\langle M_{R\vec{m}} \rangle_{R\vec{s}} = \langle M_{\vec{m}} \rangle_{\vec{s}} .$$

Equivalently, for each fixed $\vec{m}$ there exists a function $f_{\vec{m}}$ such that $\langle M_{\vec{m}} \rangle_{\vec{s}} = f_{\vec{m}}(\vec{s})$ and $f_{R\vec{m}}(R\vec{s}) = f_{\vec{m}}(\vec{s})$.

Assumption 1.2 (Existence of a complementary partner - O3). *For every sharp measurement $M_{\vec{s}}$ (with $+1$ eigenstate $\vec{s} \in S^2$), there exists at least one sharp measurement $M_{\vec{m}}$ that is maximally complementary to $M_{\vec{s}}$.*

Proposition 1.1 (Affine-Isotropic Form). *For fixed $\vec{m}$, (O1) implies $\langle M_{\vec{m}} \rangle_{\vec{s}} = \vec{a}(\vec{m}) \cdot \vec{s} + b(\vec{m})$. Covariance (O2) requires $\vec{a}(R\vec{m}) = R\vec{a}(\vec{m})$ and $b(R\vec{m}) = b(\vec{m})$. Consequently, $b(\vec{m})$ is a constant β.*

To identify $\vec{a}(\vec{m})$, let R range over the stabilizer of $\vec{m}$ in $SO(3)$, which is isomorphic to $SO(2)$ (rotations about the axis $\vec{m}$). This group acts trivially on $\vec{m}$ but irreducibly on the plane $\vec{m}^\perp$. Any vector fixed by all such R must have a zero component in $\vec{m}^\perp$; hence $\vec{a}(\vec{m}) = \alpha\vec{m}$. Thus $\langle M_{\vec{m}} \rangle_{\vec{s}} = \alpha(\vec{s} \cdot \vec{m}) + \beta$.

Definition 1.1 (Binary Expectation). *For outcomes $\{+1, -1\}$, the expectation value relates to the probability of the positive outcome via the relation $\langle M \rangle_{\vec{s}} = 2p(+1 \mid \vec{s}, \vec{m}) - 1$.*

Remark 1.1 (Automatic Continuity). *Continuity of the measurement rule is not an independent postulate. Because the expectation map $\vec{s} \mapsto \langle M \rangle_{\vec{s}}$ is affine on a finite-dimensional compact convex set (B^3), it is necessarily continuous.*

2 Boundary Conditions from the Action Quota

The variance complementarity established in Lemma 1 provides the specific constraints required to fix the measurement function at the Information Frontier.

Lemma 2.1 (Maximal complementarity implies unbiasedness). *Let $\vec{s} \in S^2$ be a $+1$ eigenstate of a sharp measurement $M_{\vec{s}}$. If $M_{\vec{m}}$ is maximally complementary to $M_{\vec{s}}$, then $\mathrm{Var}(M_{\vec{m}}) = 1$ and hence $\langle M_{\vec{m}} \rangle_{\vec{s}} = 0$.*

Proof. By the Action Quota for the maximally complementary pair $(M_{\vec{m}}, M_{\vec{s}})$, $\mathrm{Var}(M_{\vec{m}}) + \mathrm{Var}(M_{\vec{s}}) \geq 1$. Since the system is in an eigenstate of $M_{\vec{s}}$, we have $\mathrm{Var}(M_{\vec{s}}) = 0$, forcing $\mathrm{Var}(M_{\vec{m}}) \geq 1$. For ± 1 outcomes, $\mathrm{Var}(M) = 1 - \langle M \rangle^2 \leq 1$. Thus $\mathrm{Var}(M_{\vec{m}}) = 1$, which implies $\langle M_{\vec{m}} \rangle_{\vec{s}} = 0$. □

Proposition 2.1 (Boundary conditions at sharp points). *For pure states $\vec{s} \in S^2$, the expectation value function $\langle M_{\vec{m}} \rangle_{\vec{s}}$ must satisfy:*

(i) ***Aligned Case ($\vec{s} \cdot \vec{m} = 1$):*** $\langle M \rangle = +1$.
(ii) ***Anti-aligned Case ($\vec{s} \cdot \vec{m} = -1$):*** $\langle M \rangle = -1$.
(iii) ***Maximally complementary case:*** *If $M_{\vec{m}}$ is maximally complementary to $M_{\vec{s}}$ and $\vec{s}$ is a $+1$ eigenstate of $M_{\vec{s}}$, then $\langle M_{\vec{m}} \rangle_{\vec{s}} = 0$.*

Proof. Cases (i) and (ii) follow from the Sharpness postulate: each dichotomic measurement has sharp $+1$ and -1 eigenstates located at $\pm\vec{m}$ on S^2.

For (iii), let $M_{\vec{s}}$ be the sharp measurement with $+1$ eigenstate $\vec{s}$. By (O3), there exists a measurement $M_{\vec{m}}$ maximally complementary to $M_{\vec{s}}$. Applying Lemma 2.1 yields $\langle M_{\vec{m}} \rangle_{\vec{s}} = 0$. □

3 The Born Rule Derivation

Theorem 3.1 (Uniqueness of the Born Rule)**.** *Given the Euclidean 3-ball geometry (B^3) and the Affine Probability Postulate, the unique measurement rule satisfying the sharpness boundary conditions (aligned and anti-aligned) is:*

$$p(+1 \mid \vec{s}, \vec{m}) = \frac{1 + \vec{s} \cdot \vec{m}}{2}.$$

Equivalently, $\langle M \rangle_{\vec{s}} = \vec{s} \cdot \vec{m}$.

Proof. Fix $\vec{m} \in S^2$. By the Affine Probability Postulate (O1), the map $\vec{s} \mapsto \langle M_{\vec{m}} \rangle_{\vec{s}}$ is affine on B^3. As shown in Proposition 1.1, any functional that is both affine and rotation-covariant must take the form:

$$\langle M_{\vec{m}} \rangle_{\vec{s}} = \alpha(\vec{s} \cdot \vec{m}) + \beta.$$

We now determine α and β using the aligned and anti-aligned cases from Proposition 2.1. For $\vec{s} = \vec{m}$ we have $\langle M_{\vec{m}} \rangle_{\vec{m}} = +1$, hence

$$1 = \alpha(\vec{m} \cdot \vec{m}) + \beta = \alpha + \beta.$$

For $\vec{s} = -\vec{m}$ we have $\langle M_{\vec{m}} \rangle_{-\vec{m}} = -1$, hence

$$-1 = \alpha(-\vec{m} \cdot \vec{m}) + \beta = -\alpha + \beta.$$

Solving this system of linear equations gives $\beta = 0$ and $\alpha = 1$.

Therefore, $\langle M_{\vec{m}} \rangle_{\vec{s}} = \vec{s} \cdot \vec{m}$. Applying the probability relation from Definition 1.1:

$$2p(+1 \mid \vec{s}, \vec{m}) - 1 = \vec{s} \cdot \vec{m} \implies p(+1 \mid \vec{s}, \vec{m}) = \frac{1 + \vec{s} \cdot \vec{m}}{2}.$$

□

Remark 3.1 (Consistency Check)**.** *While the algebra above uses sharpness, the derived rule is consistent with the Action Quota: if $M_{\vec{m}}$ is maximally complementary to $M_{\vec{s}}$ and the system is in the $+1$ eigenstate $\vec{s}$ of $M_{\vec{s}}$, then the derived rule yields $\langle M_{\vec{m}} \rangle_{\vec{s}} = \vec{s} \cdot \vec{m} = 0$, consistent with Lemma 2.1. Thus, (O3) ensures the probability rule is consistent with the fundamental information limit of Lemma 1.*

Corollary 3.1 (Pure State Probabilities)**.** *For pure states, let θ be the angle between the state $\vec{s}$ and the measurement $\vec{m}$. Using $\vec{s} \cdot \vec{m} = \cos\theta$ and the half-angle identity $\cos\theta = 2\cos^2(\theta/2) - 1$, the Born rule recovers its standard form:*

$$p(+1) = \frac{1 + \cos\theta}{2} = \cos^2\left(\frac{\theta}{2}\right).$$

Remark 3.2 (Relation to Gleason's Theorem)**.** *Our derivation provides a constructive alternative to Gleason's theorem for qubits. We derive the rule from geometric constraints (convexity, isotropy) and boundary conditions from the Action Quota, without invoking the full machinery of projection lattices.*

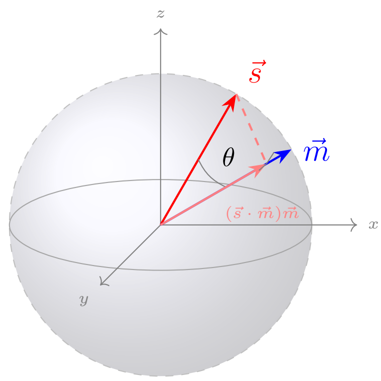

Figure 1: The geometry of the Born rule: $p(+1) = (1 + \vec{s}\cdot\vec{m})/2$. The inner product $\vec{s}\cdot\vec{m} = \cos\theta$ measures the projection of the state onto the measurement axis; the probability interpolates linearly across the ball.

4 Conclusion

We have shown that the Born rule is not an independent postulate. Instead, it is the unique mathematical bridge between the geometric state space (B^3) and the operational outcomes of dichotomic measurements. By deriving the rule from the requirement that statistics be affine in state, we demonstrate that the "weirdness" of quantum probability is actually the unique consistent way to assign probabilities to states on a curved Information Frontier.

References

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