Lemma 2 (Dimensional Inflation):

From Disk Geometry to the Bloch Ball

An operational derivation of the three-dimensional state space

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Abstract

We build upon the 2D disk constraint established in Lemma 1. By introducing structural assumptions—(R0) **Sharp Eigenstates**, (R1) **Continuity**, (R2) **Symmetry**, and (R3) **Threefold Complementarity**—we prove that the expectation-value image of the binary state space Ω is necessarily three-dimensional. We replace fixed spatial rotation assumptions with an abstract symmetry group acting on internal measurement frames. By defining a global expectation map F, we show that the Euclidean 3-ball (B^3) is the unique convex set consistent with the Action Quota across all orientations. This identifies the global Information Frontier as the 2-sphere S^2, completing the geometric reconstruction of the qubit state space.

1 Recalling the Foundational Constraint

Building on Lemma 1, we consider a binary system with a convex state space Ω. Throughout this derivation, we retain the operational primitives of convexity, affine response, and sharpness for dichotomic measurements. We assume the universal validity of the geometric result derived from the Action Quota:

Lemma 1.1 (Lemma 1: The Disk Constraint). *For any pair of maximally complementary measurements (A, B), the set of achievable expectation-value pairs $(\langle A\rangle_\omega, \langle B\rangle_\omega)$ is exactly the unit disk $\mathcal{D} = \{(a,b) \in \mathbb{R}^2 : a^2 + b^2 \leq 1\}$.*

2 Structural Assumptions

To determine the full dimensionality, we need principles governing how the 2D slices fit together without assuming $SO(3)$ at the outset.

Assumption 2.1 (Pure states as eigenstates - R0). *Every pure state is an eigenstate of some sharp dichotomic measurement.*

Definition 2.1 (Sharp dichotomic measurement). *A dichotomic measurement A (outcomes ± 1) is* sharp *if there exist states $\omega_A^{\pm} \in \Omega$ such that $\langle A\rangle_{\omega_A^{\pm}} = \pm 1$.*

Assumption 2.2 (Continuity - R1). *The state space Ω can be embedded as a convex body in some finite-dimensional Euclidean space $\mathbb{R}^N$.*

Assumption 2.3 (Symmetry - R2). *There exists a connected Lie group G of reversible affine automorphisms of Ω that acts transitively on the set of sharp measurements (equivalently, on their $\pm$ eigenfaces). This represents an internal symmetry of the measurement set.*

Remark 2.1. *For $g \in G$, measurement statistics are covariant: $\langle g \cdot A \rangle_{g \cdot \omega} = \langle A \rangle_\omega$ for all dichotomic A and all states ω.*

Definition 2.2 (Induced action on expectation coordinates). *For each $g \in G$ define the rotated triple $(X_g, Y_g, Z_g) := (g \cdot X,\ g \cdot Y,\ g \cdot Z)$.*

Assumption 2.4 (Linear action - R2"). *For each $g \in G$, there exists an invertible linear map $R_g \in GL(3)$ (hence $R_g(0) = 0$) such that $F_g(\omega) = R_g F(\omega)$ for all $\omega \in \Omega$, where $F(\omega) = (\langle X \rangle_\omega, \langle Y \rangle_\omega, \langle Z \rangle_\omega)$.*

Assumption 2.5 (Richness of Orientations - R2'). *The induced representation $g \mapsto R_g$ acts transitively on the unit sphere S^2; i.e., $\{R_g e_z : g \in G\} = S^2$.*

Assumption 2.6 (Nontrivial Stabilizer - R2"'). *For any sharp measurement Z, the stabilizer subgroup $G_Z := \{g \in G : g \cdot Z = Z\}$ contains a nontrivial one-parameter subgroup.*

Remark 2.2 (Physical motivation for R2"'). *If the symmetry group G represents rotations of an internal measurement frame, and if Z corresponds to a preferred axis, then the stabilizer G_Z represents rotations* about *that axis. For a 3D rotation group, this is $SO(2) \cong U(1)$, which is a one-parameter group. The assumption R2"' thus encodes the expectation that fixing one measurement direction leaves a continuous family of internal transformations available.*

Assumption 2.7 (Threefold Complementarity - R3). *The structure of measurement incompatibility satisfies two conditions:*

(i) ***Existence:*** *There exist at least three dichotomic measurements X, Y, Z that are pairwise maximally complementary.*

(ii) ***Covariance:*** *The symmetry group G maps any maximally complementary pair to another maximally complementary pair.*

3 Formalizing the Inflation

Lemma 3.1 (Covariance of Complementarity). *If the Action Quota variance inequality $\mathrm{Var}(A)_\omega + \mathrm{Var}(B)_\omega \geq 1$ holds for one pair (A, B), then symmetry (R2) and covariance (R3ii) imply that every orbit of the pair under G satisfies the identical bound.*

Lemma 3.2 (Eigenstates move with measurements). *Let A be a sharp dichotomic measurement with eigenstate ω_A^+ such that $\langle A \rangle_{\omega_A^+} = 1$. For any $g \in G$, define $A_g := g \cdot A$ and $\omega_g := g \cdot \omega_A^+$. Then $\langle A_g \rangle_{\omega_g} = 1$, so ω_g is a $+1$ eigenstate of A_g.*

Lemma 3.3 (3D Identification). *If the image $F(\Omega)$ is convex, invariant under the induced action of G, and its orthogonal projection onto each plane* $\mathrm{span}\{g \cdot X,\ g \cdot Y\}$ *is exactly the unit disk, then $F(\Omega)$ is the Euclidean 3-ball B^3.*

Proof. By the disk constraint (Lemma 1.1) and covariance, for each plane spanned by $g \cdot X$ and $g \cdot Y$, the projection of $F(\Omega)$ onto that plane is the unit disk. By R2', for every direction $\vec{n} \in S^2$ there exists such a plane containing $\vec{n}$. Therefore, the support function $h(\vec{n}) = \sup_{\vec{r} \in F(\Omega)}(\vec{r} \cdot \vec{n})$ is exactly 1. This is a standard result in convex geometry: a compact convex body with constant support function 1 is the unit Euclidean ball B^3 [3]. $\square$

4 The Dimensional Sandwich

Lemma 4.1 (Lemma 2: The Bloch Ball)**.** *Given Assumptions R0–R3 and the Disk Constraint, the expectation-value image $F(\Omega)$ is exactly the unit 3-ball B^3.*

Corollary 4.1 (Faithful Representation)**.** *If the map F is informationally complete (faithful/injective), then the state space Ω is affinely isomorphic to the unit 3-ball B^3.*

5 Proof Execution

5.1 Stage I: Dimensional Forcing ($\dim > 2$)

Assume $\dim(F(\Omega)) = 2$. Let Z be a sharp measurement. By R2"', the stabilizer G_Z fixes Z but must act nontrivially on measurements complementary to Z (otherwise G_Z would be trivial, violating R2"'). Let X be complementary to Z, and let ω_X^+ be a +1 eigenstate of X.

The orbit $\{\omega_t = g_t \cdot \omega_X^+ : g_t \in G_Z\}$ is a continuous family of pure states. By Remark 2.1, for every state in this orbit:

$$\langle Z \rangle_{\omega_t} = \langle g_t \cdot Z \rangle_{\omega_t} = \langle Z \rangle_{\omega_X^+} = 0.$$

The equality $\langle Z \rangle_{\omega_X^+} = 0$ follows from Lemma 1, as X and Z are maximally complementary. Thus, the orbit forms a continuous circle of states all satisfying $\langle Z \rangle = 0$. In a 2D image, the set of points satisfying $\langle Z \rangle = 0$ is a diameter, which intersects the pure-state boundary at exactly **two points**. A continuous orbit of pure states (which must map to the boundary) cannot be embedded into a set of only two points. Thus, $\dim(F(\Omega)) > 2$.

5.2 Stage II: Dimensional Identification

Stage I concludes that the dimensionality of the expectation image exceeds two. By R3, there exist exactly three pairwise maximally complementary measurements X, Y, Z. The map $F(\omega) = (\langle X \rangle_\omega, \langle Y \rangle_\omega, \langle Z \rangle_\omega)$ thus maps to $\mathbb{R}^3$. We now show $F(\Omega) = B^3$.

1. **Bounding:** For each $g \in G$, the projection of $F(\Omega)$ onto $\mathrm{span}\{g \cdot X, g \cdot Y\}$ is the unit disk. In particular, for any unit direction $\vec{n}$ there exists a disk-plane containing $\vec{n}$ (R2'), so the maximal support in direction $\vec{n}$ is at most 1; hence $F(\Omega) \subseteq B^3$.

2. **Filling:** Pick a reference sharp measurement Z and a reference eigenstate ω_Z^+ such that $F(\omega_Z^+) = (0, 0, 1)$. For each $g \in G$, $\omega_g = g \cdot \omega_Z^+$ is a +1 eigenstate of $g \cdot Z$ (Lemma 3.2). By R2' (transitivity of R_g on S^2), the set of expectation values $\{F(\omega_g)\}$ covers the entire unit sphere S^2. Thus $S^2 \subseteq F(\Omega)$.

3. **Conclusion:** Since $F(\Omega)$ is convex (R1), it must contain the convex hull of S^2, which is B^3. Combined with the bound, we have $F(\Omega) = B^3$. $\square$

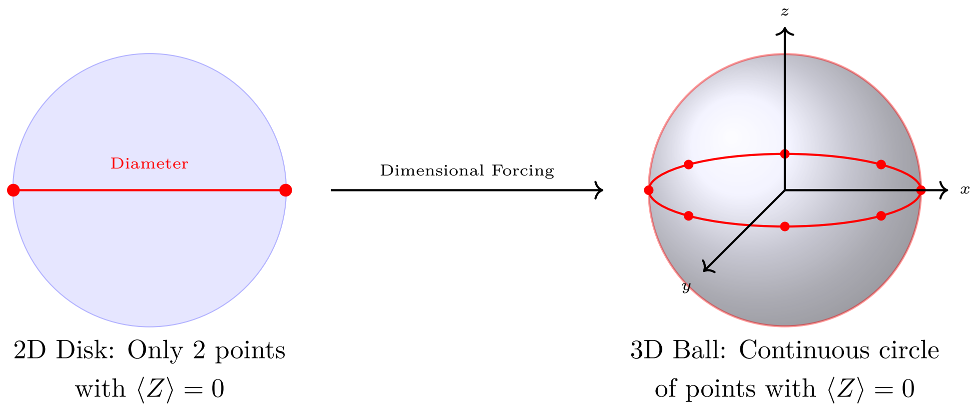

Figure 1: The dimensional forcing argument. In 2D, the set of pure states with $\langle Z \rangle = 0$ consists of only 2 points (the diameter endpoints). In 3D, it forms a continuous circle (the equator). Symmetry demands continuous orbits, forcing the upgrade to 3D.

6 Conclusion

Building on the Disk Constraint from Lemma 1, we have derived the Bloch ball as the unique minimal geometry satisfying complementarity and internal symmetry. Every point on the global Information Frontier S^2 represents a pure state—one that saturates the Action Quota for any complementary pair lying in its tangent plane.

References

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