Lemma 2: From Disk Geometry to the Bloch Ball

An operational derivation of the three-dimensional state space

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Abstract

We build upon the 2D disk constraint established in Lemma 1. By introducing three operational axioms—(R1) **Continuity**, (R2) **Isotropy**, and (R3) **Pairwise State Determination**—we show that the state space is necessarily three-dimensional. The core of the proof relies on a dimensional "sandwich":

1. A *dimensional-forcing* argument (using Isotropy) shows $\dim(\Omega) > 2$.
2. A *dimensional-capping* argument (using Pairwise Determination) shows $\dim(\Omega) \leq 3$.

With the dimension uniquely fixed at 3, we show that Isotropy and the universal disk constraint of Lemma 1 force the full state space to be the 3D unit ball, B^3. This derives the Bloch ball's geometry **from operational principles**, without presupposing quantum formalism.

1. Operational Premises

We adopt the operational framework from Lemma 1, adding three structural axioms.

- A binary system with expectation values in $[-1, 1]$.
- A convex state space Ω.

The three structural axioms are:

Axiom R1 (Continuity). The set of available measurement settings is continuous, and measurement probabilities vary continuously with state parameters.

Proposition 1 (Physical basis for R1)**.** *Continuity is operationally motivated: small changes in experimental apparatus (e.g., rotating a detector by a small angle) produce small changes in measured statistics. Discontinuous jumps in probabilities under continuous parameter changes would be unphysical and empirically unobserved.*

Axiom R2 (Isotropy). The state space Ω is isotropic: no measurement direction is physically privileged. All directional relationships are equivalent under physical rotations.

Operational form of R2 (Isotropy). We formalize isotropy as the requirement that the physical rotation group $SO(3)$ acts on measurement device settings and that this action is implemented on the state space by affine automorphisms: for each rotation $R \in SO(3)$ there is an affine bijection $T_R : \Omega \to \Omega$ such that measurement statistics are preserved under the joint action (T_R on states and R on measurement frames). In particular, the action is transitive on oriented measurement directions.

Proposition 2 (Physical basis for R2)**.** *Isotropy expresses the absence of an absolute reference frame for measurements. The physics of a binary system should not depend on laboratory orientation in space. This is a symmetry principle analogous to spatial homogeneity in mechanics. Operationally: for any spatial rotation of the measurement apparatus there exists an implementable reversible transformation on the set of preparations that reproduces the rotated statistics. This assumption captures the empirical fact that measurement orientation is physically implementable and composes continuously.*

Axiom R3 (Pairwise State Determination). For any pure state ω_p, there exists at least one pair of maximally complementary measurements (A, B) such that the expectation values $(\langle A\rangle_{\omega_p}, \langle B\rangle_{\omega_p})$ are sufficient to *uniquely* determine ω_p from all other pure states.

Proposition 3 (Physical basis for R3)**.** *This axiom embodies* operational efficiency *and is grounded in empirical observation. We justify it on information-theoretic grounds:*

Why exactly two measurements?

- ***One measurement is insufficient:*** *A single measurement yields one real number, which cannot uniquely specify a point on a manifold of dimension ≥ 2. Empirically, we observe that single measurements do not fully characterize quantum states.*
- ***Two measurements are necessary and sufficient:*** *If pure states form a 2-dimensional manifold (which we will derive), two real numbers (two measurement outcomes) provide the minimal information needed to specify a point.*
- ***More than two would indicate redundancy:*** *If three or more measurements were* required *to distinguish pure states, this would suggest the pure state manifold has*

dimension > 2, implying the state space has dimension > 3. We will show (Stage II) that this leads to operational inconsistencies.

Framing as an empirical constraint: *R3 asserts that nature is operationally efficient: the number of measurements needed to characterize a pure state equals the dimension of the pure state manifold. This is testable: if we empirically find that two complementary measurements suffice to fully characterize pure quantum states (as observed), then the pure state space is 2-dimensional, forcing the total state space dimension to be 3.*

Alternative formulations: *R3 is equivalent (in the presence of R1 and R2) to requiring that pure states form a 2-parameter homogeneous orbit under reversible transformations. Alternative formalizations in the literature (e.g., tomographic locality, spectrality) lead to equivalent conclusions. We choose R3 for its direct operational and information-theoretic interpretation.*

The final ingredient is the universal application of Lemma 1:

Lemma 1 (Disk Constraint). For *any* maximally complementary pair of dichotomic measurements A, B, the allowed expectation values (a, b) must lie within the unit disk:

$$a^2 + b^2 \leq 1.$$

2. Statement of Lemma 2

Lemma 1. *Let a binary system satisfy Axioms R1, R2, R3, and the universal Disk Constraint from Lemma 1. Then the state space Ω is affinely isomorphic to the unit 3-ball*

$$B^3 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\},$$

with pure states lying on the sphere S^2.

3. Proof

The argument proceeds in three stages: dimensional forcing (proving $\dim > 2$), dimensional capping (proving $\dim \leq 3$), and symmetry-based selection (proving the 3-ball shape).

3.1. Stage I: Dimensional Forcing (Proof that $\dim(\Omega) > 2$)

Strategy: We show that assuming $\dim(\Omega) = 2$ violates the isotropy requirement.

Assume, for contradiction, that the entire state space Ω is 2-dimensional. By Lemma 1, this space Ω must be (affinely isomorphic to) the unit disk $\mathcal{D} = \{(a, b) : a^2 + b^2 \leq 1\}$.

Step 1: Identifying a reference state. Let ω_Z be a pure state with $\langle Z \rangle = 1$ for some measurement Z. This state lies on the boundary of $\mathcal{D}$.

Step 2: Constructing complementary measurements. By the formalized isotropy (Axiom R2), rotating the measurement frame about the Z-axis yields a continuous one-parameter family of measurement settings X_ϕ for $\phi \in [0, 2\pi)$; each X_ϕ is by construction maximally complementary to Z (they are related by a rotation).

By **Axiom R1 (Continuity)**, this family varies continuously with the parameter ϕ.

Step 3: Pure states complementary to Z. For each measurement X_ϕ in this continuous family, there exists a *unique* pure state ω_ϕ with:

$$\langle Z \rangle_{\omega_\phi} = 0 \quad \text{(orthogonal to } Z\text{)}$$
$$\langle X_\phi \rangle_{\omega_\phi} = 1 \quad \text{(eigenstate of } X_\phi\text{)}$$

Uniqueness argument. By Lemma 1, the pair (Z, X_ϕ) constrains states to lie in the unit disk in the (z, x_ϕ)-coordinate plane, where $z = \langle Z \rangle$ and $x_\phi = \langle X_\phi \rangle$. The boundary of this disk satisfies $z^2 + x_\phi^2 = 1$.

The conditions $z = 0$ and $x_\phi = 1$ identify the unique boundary point $(z, x_\phi) = (0, 1)$ in this 2D slice. Since pure states correspond to boundary points, ω_ϕ is uniquely determined by these two measurement outcomes.

As ϕ varies continuously, the measurement direction X_ϕ rotates continuously around Z. By Axiom R1 (Continuity) and the operational implementability of measurement settings, the associated pure state ω_ϕ must vary continuously with ϕ. Thus the map $\phi \mapsto \omega_\phi$ is a continuous injection from $S^1 = [0, 2\pi)/\sim$ into the set of pure states, producing a continuous circle of distinct pure states.

Step 4: Geometric contradiction. The set of states with $\langle Z \rangle = 0$ forms a "slice" through Ω. In the 2D disk $\mathcal{D}$, if we coordinatize by (z, x) where $z = \langle Z \rangle$, the condition $z = 0$ defines a 1-dimensional line segment through the origin.

The boundary of $\mathcal{D}$ intersects this line at exactly **two points**: $(z, x) = (0, \pm 1)$.

However, isotropy demands a *continuous circle* of distinct pure states $\{\omega_\phi : \phi \in [0, 2\pi)\}$, all satisfying $\langle Z \rangle = 0$. This continuous circle cannot be embedded into the two discrete boundary points available in the 2D disk slice.

Conclusion: A 2D disk cannot accommodate a continuous 1-parameter family of distinct pure states on its boundary with a fixed measurement value. This geometric impossibility contradicts the requirements of isotropy.

Therefore, $\dim(\Omega) > 2$.

3.2. Stage II: Dimensional Capping (Proof that $\dim(\Omega) \leq 3$)

Strategy: We show that assuming $\dim(\Omega) \geq 4$ violates the pairwise determination axiom.

Suppose Ω is a full-dimensional compact convex body of dimension $n \geq 4$. Its boundary of extremal points (pure states) is then an $(n-1)$-dimensional manifold (at least locally, under mild regularity; see Appendix A). A point on the boundary requires $n-1$ real parameters to specify.

For $n = 4$: pure states on the boundary form a 3-dimensional manifold (topologically, S^3) and require 3 independent coordinates.

Step 1: Information from two measurements. Consider a maximally complementary pair (A, B). Measuring these on a pure state ω_p yields two expectation values:

$$(a, b) = (\langle A \rangle_{\omega_p}, \langle B \rangle_{\omega_p})$$

These provide only **two** pieces of information about ω_p.

Step 2: Geometric degeneracy in higher dimensions. By Lemma 1, for this measurement pair, the state projects onto a 2D disk where $a^2+b^2 \leq 1$. In the full n-dimensional space, fixing (a, b) determines the state's position in a 2D subspace but leaves it undetermined in the orthogonal $(n-2)$-dimensional subspace.

For $n = 4$: Fixing (a, b) constrains the state to lie in a 2-dimensional subspace (the disk slice spanned by the measurement directions A and B). The pure states satisfying this constraint form the intersection of the boundary S^3 (the 3-sphere of pure states in $\mathbb{R}^4$) with this 2-plane through the origin.

Geometric fact. The intersection of an $(n-1)$-sphere $S^{n-1} \subset \mathbb{R}^n$ with a k-dimensional plane through the origin is a $(k-1)$-sphere. In our case, $S^3 \cap \mathbb{R}^2 = S^1$, a circle.

Thus, there exists a **continuous circle** (S^1) of distinct pure states that all yield the same measurement outcomes (a, b).

Step 3: Contradiction with R3. This continuous degeneracy directly violates **Axiom R3 (Pairwise State Determination)**, which requires that the two measurements (A, B) uniquely determine the pure state ω_p.

Operational interpretation: If $\dim(\Omega) \geq 4$, then even after measuring two complementary observables, an infinite family of distinct pure states would remain indistinguishable. This violates the principle that pure states (maximal information states) should be fully characterized by minimal measurements. Such redundancy contradicts the information-theoretic efficiency embodied in R3.

Therefore, $\dim(\Omega) \leq 3$.

Combining Stages I and II: We have shown $\dim(\Omega) > 2$ and $\dim(\Omega) \leq 3$, which uniquely determines:

$$\boxed{\dim(\Omega) = 3}$$

3.3. Stage III: Selection by Isotropy (Uniqueness of the 3-Ball)

The state space Ω is a 3-dimensional, compact, convex body K.

We apply **Axiom R2 (Isotropy)** and the **Lemma 1 (Disk Constraint)**:

- **Isotropy** requires that K be invariant under all rotations: $R(K) = K$ for all $R \in SO(3)$. This means *every* 2D plane through the origin intersects K in the same shape.
- The **Disk Constraint** requires that every such cross-section corresponding to a complementary measurement pair must be a Euclidean disk of radius 1.

We now invoke a standard result from convex geometry:

Lemma 2 (Central section theorem)**.** *Let $K \subset \mathbb{R}^3$ be a compact, convex, origin-symmetric body such that every central planar section through the origin is a Euclidean disk of fixed radius r. Then K is the Euclidean ball of radius r.*

Sketch. For any unit vector $u \in S^2$, consider the support function $h_K(u) = \sup_{x \in K} u \cdot x$, which measures how far K extends in direction u.

For any 2D plane Π through the origin with normal vector u, the intersection $K \cap \Pi$ is by hypothesis a disk of radius r. The boundary points of this disk lie at distance r from the origin in all directions within Π.

Now consider the plane Π_u perpendicular to u (i.e., the plane with normal vector u). By hypothesis, $K \cap \Pi_u$ is a disk of radius r. The farthest point of this disk from the origin, measured in direction u, lies at distance r along the u-axis. Therefore, the support function in direction u satisfies $h_K(u) = r$.

Since this argument applies to any unit vector u, we have $h_K(u) = r$ for all $u \in S^2$.

A convex body is uniquely determined by its support function. Since $h_K(u) = r$ for all u, the body is

$$K = \{x \in \mathbb{R}^3 : u \cdot x \leq r \ \forall u \in S^2\} = \{x \in \mathbb{R}^3 : \|x\| \leq r\},$$

which is exactly the Euclidean ball of radius r.

For complete proofs using variational methods, see Schneider [1], Theorem 1.7.2, or Gardner [2], Section 3.2. □

Applying Lemma 2 with $r = 1$, we conclude that K is uniquely determined:

$$K = B^3 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}.$$

QED

4. Remarks

Remark 1 (The dimensional sandwich)**.** *The proof relies on a **dimensional sandwich**. Axiom R2 (Isotropy) forces* $\dim > 2$ *by requiring a continuous family of distinguishable pure states (Stage). Axiom R3 (Pairwise Determination) forces* $\dim \leq 3$ *by requiring operational efficiency (Stage). Together, they isolate 3D as the unique operationally consistent dimension.*

Remark 2 (Uniqueness)**.** *The spherical geometry is the* unique *structure compatible with the uncertainty principle (Lemma 1), rotational symmetry (Axiom R2), and operational efficiency (Axiom R3). No other convex body satisfies these constraints simultaneously.*

Remark 3 (The static scale)**.** *The derived radius $R = 1$ for the Bloch ball is a direct consequence of the unit bound in the Action Quota Axiom (Lemma 1). This "1" defines the natural unit of action for the statistical structure of the theory. The physical scale factor ($\hbar$) will be identified in Lemma 5 through unification with dynamics.*

Remark 4 (Pure states and the boundary)**.** *Pure states correspond to points on the boundary sphere S^2. Mixed states correspond to interior points. The center represents the maximally mixed state (equal probabilities for all measurement outcomes, maximum entropy).*

Remark 5 (Scope)**.** *This derivation addresses single binary systems (qubits). Extension to composite systems and multi-qubit entanglement structures requires additional principles (tensor product structure, local operations) and remains an open direction for this reconstruction program.*

Remark 6 (What was not assumed)**.** *No Hilbert space, complex numbers, linear operators, or specifically quantum structure was assumed. The Bloch ball emerges purely from:*

- *The variance constraint (Lemma 1)*
- *Convexity (operational mixing)*
- *Three physically motivated axioms (R1, R2, R3)*

The complex structure will emerge later (Lemma 4) from dynamical requirements.

Remark 7 (Comparison with classical systems)**.** *A classical 2-outcome system would have a state space that is a simplex (e.g., a line segment for a deterministic coin). The Bloch ball's spherical geometry, forced by variance complementarity and isotropy, is distinctively non-classical.*

Appendix A: Topological Regularity Assumption

For completeness, we note that Stage II's argument assumes the boundary of pure states forms a smooth (or at least locally manifold-like) structure. This is guaranteed under mild regularity conditions.

We assume the convex body has no nontrivial flat boundary facets (equivalently, the support function is strictly convex), which ensures the boundary of extreme points is locally homeomorphic to a sphere and admits a manifold structure.

Lemma 3 (Pure state boundary)**.** *For a full-dimensional compact convex body $K \subset \mathbb{R}^n$ whose extremal points (pure states) are continuously parameterizable and which has no flat boundary segments, the boundary is homeomorphic to an $(n-1)$-sphere S^{n-1}.*

The strict convexity condition (no flat faces) is guaranteed by the variance constraint: interior points strictly satisfy $\mathrm{Var}(A) + \mathrm{Var}(B) > 1$ for all complementary pairs, while boundary points saturate the inequality. This rules out flat boundary segments.

For details, see Schneider [1], Chapter 2, or Gardner [2], Section 1.7.

Appendix B: Support Function and the Central Section Theorem

We provide additional detail on Lemma 2.

Support function definition. For a compact convex body K, the support function is

$$h_K(u) = \sup_{x \in K} u \cdot x$$

This function uniquely determines K and satisfies $h_K(u) = h_K(-u)$ for origin-symmetric bodies.

Central section theorem (expanded). If every 2D plane through the origin intersects K in a disk of radius r, then for any unit vector u, consider the plane perpendicular to u. This plane intersects K in a disk of radius r (by hypothesis).

The boundary of this disk consists of points at distance r from the origin within the plane. The point on this boundary that extends furthest in direction u lies at distance r from the origin along the u-axis. Therefore, $h_K(u) = r$.

Since this applies to all directions $u \in S^2$, we have $h_K(u) = r$ for all u. The constancy of the support function is the defining property of a sphere: the body extends equally far in all directions from the origin. This characterizes the ball uniquely.

For complete proofs using Fourier-analytic and variational methods, see:

References

[1] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, 2nd ed., Cambridge University Press, 2014.

[2] R.J. Gardner, *Geometric Tomography*, 2nd ed., Cambridge University Press, 2006.