Lemma 4 (Dynamics):

The Unitary Lift and the Emergence of $\mathbb{C}^2$

From Bloch Rotations to the Schrödinger Equation

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Abstract

Building on the Bloch-ball geometry (Lemma 2) and the Born rule (Lemma 3), we characterize all admissible dynamical transformations. The operational axioms of *Reversibility*, *Continuity*, and *Probability Preservation* force every physical evolution on the Bloch ball to be a proper rotation, $R(t) \in SO(3)$. We then show that the requirement of a consistent representation of the rotation path—grounded in the topological requirement to distinguish between contractible and non-contractible loops—forces a lift to the universal double cover, $SU(2)$. This necessity implies that the underlying state space is a two-dimensional *complex Hilbert space* ($\mathbb{C}^2$) with *unitary* dynamics. The dynamic action scale κ emerges via an application of Stone's theorem, setting the stage for scale unification in Lemma 5.

1 Operational Axioms for Dynamics

A physical theory must describe how states change over time while maintaining the consistency of the probabilistic framework. Let $T(t) : B^3 \to B^3$ be the map representing time evolution. We impose four operational requirements:

Assumption 1.1 (Reversibility - D1)**.** *Fundamental evolution forms a one-parameter group of reversible transformations:* $T(0) = I$, $T(t+s) = T(t) \circ T(s)$, *and* $T(-t) = T(t)^{-1}$. *This ensures evolution is information-preserving and deterministic in both time directions.*

Remark 1.1 (On Time Independence)**.** *The group property in D1 implies that the physical laws (the generators of evolution) are time-independent. While time-dependent Hamiltonians* $H(t)$ *are physically relevant, this reconstruction focuses on the stationary symmetries that define the theory's structural core.*

Assumption 1.2 (Continuity - D2)**.** *The evolution is strongly continuous: the map* $t \mapsto T(t)\vec{s}$ *is continuous for every state* $\vec{s} \in B^3$. *This ensures that infinitesimal time evolution is well-defined.*

Assumption 1.3 (Probability Preservation (Born covariance) - D3)**.** *Evolution is compatible with the Born rule and acts consistently on preparations and measurement frames: for all* $\vec{s} \in B^3$, $\vec{m} \in S^2$, *and all* t,

$$p(+1 \mid \vec{s}, \vec{m}) \;=\; p(+1 \mid T(t)\vec{s},\, T(t)\vec{m}).$$

In particular, $T(t)$ *maps the information frontier to itself,* $T(t)(S^2) = S^2$, *and the maximally mixed state is fixed:* $T(t)\,0 = 0$ *for all* t.

Assumption 1.4 (Spinorial Path Sensitivity - D4)**.** *The operational implementation of state transformations depends on the homotopy class of the path in the group of reversible transformations. A closed* 2π *rotation path is not operationally equivalent to the identity transformation. This is an operational postulate supported by spin-*$1/2$ *interferometry (see* §*4.1), where a* 2π *loop acts trivially on Bloch vectors but nontrivially on the lifted state (ray) representation.*

2 Lemma: Dynamics as Proper Rotations

Lemma 2.1 (Dynamics as Proper Rotations)**.** *Under Assumptions D1–D3, the allowed time evolutions on the Bloch ball are linear isometries belonging to the rotation group* $SO(3)$*.*

Proof. By Lemma 3 (Born rule), $p(+1 \mid \vec{s}, \vec{m}) = (1+\vec{s}\cdot\vec{m})/2$. Born covariance (D3) implies that for all $\vec{s} \in B^3$ and $\vec{m} \in S^2$, $p(+1 \mid \vec{s}, \vec{m}) = p(+1 \mid T(t)\vec{s},\, T(t)\vec{m})$. Using invertibility (D1) and the fact that $T(t)$ maps the information frontier onto itself (D3), for any $\vec{m} \in S^2$ we may write $\vec{m} = T(t)\vec{m}'$ with $\vec{m}' = T(-t)\vec{m} \in S^2$, hence

$$p(+1 \mid T(t)\vec{s},\, \vec{m}) = p(+1 \mid T(t)\vec{s},\, T(t)\vec{m}') = p(+1 \mid \vec{s},\, \vec{m}').$$

Define the operational distinguishability between two states by

$$d(\vec{s}_1, \vec{s}_2) := \sup_{\vec{m}\in S^2} \Big| p(+1 \mid \vec{s}_1, \vec{m}) - p(+1 \mid \vec{s}_2, \vec{m}) \Big|.$$

Using the Born rule,

$$d(\vec{s}_1, \vec{s}_2) = \frac{1}{2} \sup_{\vec{m}\in S^2} \Big| (\vec{s}_1 - \vec{s}_2)\cdot\vec{m} \Big| = \frac{1}{2}\, \|\vec{s}_1 - \vec{s}_2\|.$$

Substituting the covariance relation into the distinguishability metric for evolved states:

$$d(T(t)\vec{s}_1, T(t)\vec{s}_2) = \sup_{\vec{m}\in S^2} \Big| p(+1 \mid T(t)\vec{s}_1, \vec{m}) - p(+1 \mid T(t)\vec{s}_2, \vec{m}) \Big| = \sup_{\vec{m}'\in S^2} \Big| p(+1 \mid \vec{s}_1, \vec{m}') - p(+1 \mid \vec{s}_2, \vec{m}') \Big|,$$

which yields $d(T(t)\vec{s}_1, T(t)\vec{s}_2) = d(\vec{s}_1, \vec{s}_2)$. Therefore, $\|T(t)\vec{s}_1 - T(t)\vec{s}_2\| = \|\vec{s}_1 - \vec{s}_2\|$ for all $\vec{s}_1, \vec{s}_2 \in B^3$: $T(t)$ is a surjective isometry of the ball B^3.

By the Mankiewicz extension theorem [7], $T(t)$ extends uniquely to an affine isometry of $\mathbb{R}^3$. Since $T(t)0 = 0$ (from D3), this affine map has zero translation part and is therefore linear. Thus $T(t) \in O(3)$. By Continuity (D2), $t \mapsto T(t)$ is a continuous path in $O(3)$ starting at I, so $\det T(t) = 1$ for all t. Thus, $T(t) \in SO(3)$ for all $t \in \mathbb{R}$. □

3 Beyond SO(3): The Path-Dependence Problem

While Bloch vectors rotate via $SO(3)$, the state representation must identify the *path* history to satisfy D4. In 3D, the topology of rotations allows us to distinguish between contractible and non-contractible loops.

Proposition 3.1 (Topology of SO(3))**.** *The rotation group* $SO(3)$ *has fundamental group* $\pi_1(SO(3)) \cong \mathbb{Z}_2$*. A* 2π *rotation about any axis represents a non-contractible loop, while a* 4π *rotation is contractible to the identity.*

Proposition 3.2 (Projective nature of the representation). *Assumption D4 implies that the state-level representation of rotations cannot be an ordinary representation of $SO(3)$; it must be a projective representation of $SO(3)$, equivalently a linear representation of its universal cover $SU(2)$.*

Proof. Since $\pi_1(SO(3)) \cong \mathbb{Z}_2$ (Proposition 3.1), there exist loops (e.g., 2π rotations) that are non-contractible. D4 asserts that traversing such a loop is not operationally equivalent to the identity. Therefore, the state-level (ray) implementation cannot descend to a single-valued representation of $SO(3)$; it must lift to the simply connected cover $SU(2)$. Equivalently, a projective representation of $SO(3)$ is a genuine linear representation of a central extension; for $SO(3)$ the universal such extension is its simply connected double cover $SU(2)$. □

4 The Unitary Lift and Hilbert Space

Theorem 4.1 (The Unitary Lift). *To consistently represent $SO(3)$ rotations and satisfy topological path-sensitivity, the system admits a unitary lift to a complex Hilbert space representation of $SU(2)$. For a binary system (qubit), the minimal faithful lift is the fundamental representation on $\mathbb{C}^2$.*

Proof. The derivation follows from the properties of the universal cover:

Step 1: Topology. To represent the path history (D4), we lift the representation to the universal cover of $SO(3)$, which is $SU(2)$. This simply connected group provides a 2-to-1 mapping $\phi : SU(2) \to SO(3)$. Physical rotations $R(t)$ remain in $SO(3)$, while the operator $U(t)$ acts on the state space to track the path history.

Step 2: Representation theory and minimality. Lemma 2 identifies the state space as the 3-ball B^3 with pure states S^2. By D2 (continuity), the induced symmetry action on rays is continuous, hence the projective $SO(3)$ action lifts to a continuous linear representation of the universal cover $SU(2)$. Among irreducible $SU(2)$ representations, only the fundamental spin-1/2 representation acts on a 2-dimensional complex Hilbert space, whose ray space is $\mathbb{C}P^1 \cong S^2$ and hence matches the Bloch sphere of pure states. Higher-spin irreducible representations act on $\mathbb{C}^{2j+1}$ and have ray spaces $\mathbb{C}P^{2j}$, which are not S^2 and therefore describe different (non-qubit) state spaces.

Step 3: Unitarity via Wigner's Theorem. Operationally, pure states are identified with equivalence classes of preparations that yield the same transition probabilities, i.e., rays in the representing Hilbert space (global phase ignored). Any bijection on rays preserving transition probabilities is implemented by a unitary or anti-unitary operator [8, 9]. A continuous one-parameter group cannot include anti-unitaries because they are not connected to the identity in the strong operator topology; thus continuity (D2) and $U(0) = I$ force unitary operators. □

Corollary 4.1 (Global Phase Redundancy). *The map from $\mathbb{C}^2$ vectors to Bloch vectors is 2-to-1: $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ map to the same $\vec{s}$. The global $U(1)$ phase is physically unobservable; physical states are rays in $\mathbb{C}^2$.*

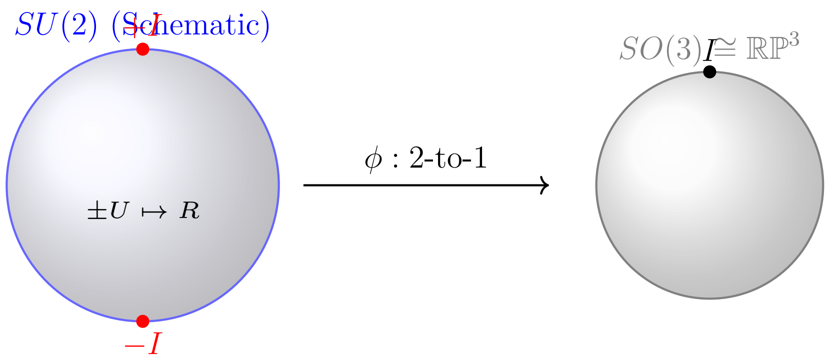

Figure 1: The Unitary Lift (Conceptual Schematic): Rotations in $SO(3)$ are lifted to the universal cover $SU(2)$. The spinor distinguishes whether the rotation path belongs to the trivial or non-trivial homotopy class, as required by D4.

4.1 Empirical Validation

While this derivation is based on logical consistency, it is validated by neutron interferometry experiments [3, 4] demonstrating that spin-1/2 systems exhibit a sign change under 2π rotations. This confirms that physical systems implement the $SU(2)$ lift.

5 The Schrödinger Equation and Dynamic Action

Lemma 5.1 (Generators of Unitary Evolution)**.** *Let $\{U(t)\}_{t\in\mathbb{R}}$ be a strongly continuous one-parameter unitary group on $\mathbb{C}^2$ with $U(0) = I$. By Stone's theorem, there exists a unique self-adjoint operator H on $\mathbb{C}^2$ such that:*

$$U(t) = \exp\left(-\frac{i}{\kappa}Ht\right),$$

where $\kappa > 0$ is a constant with dimensions of action. The operator H satisfies the Schrödinger equation:

$$i\kappa\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle.$$

Proof. We take $U(t)$ to be a strongly continuous unitary representation of the one-parameter group $t \mapsto T(t)$ on $\mathbb{C}^2$ (the standard regularity assumption for reversible continuous dynamics), so Stone's theorem applies. Stone's theorem guarantees a self-adjoint generator A such that $U(t) = e^{-iAt}$. Defining $H = \kappa A$, where κ has units of action (J s), converts the frequency units of A into energy units. The Schrödinger equation follows by differentiation. □

Remark 5.1 (Why Complex Numbers?)**.** *The emergence of complex numbers arises from three fundamental requirements:*

1. ***3D Isotropy:*** *The state space must represent non-abelian rotations in 3D ($SO(3)$).*
2. ***Topological Consistency:*** *The representation must be simply connected to handle closed transformation paths consistently (D4).*
3. ***Minimality:*** *The representation should be the smallest faithful structure satisfying the above.*

$SU(2)$ is the minimal simply-connected cover of $SO(3)$ and naturally acts on $\mathbb{C}^2$. Thus, complex scalars are the minimal extension required to support the non-abelian isotropy of the 3-ball.

6 Outlook: The Scale Identification Problem

In Lemma 1, we normalized the variance bound (the Action Quota) to 1. In Lemma 4, we introduced the dynamic scale κ. In standard quantum mechanics, these are assumed to be identical ($\kappa = \hbar$). In **Lemma 5**, we will prove that this identification is a necessary requirement for logical consistency: if $\kappa \neq \hbar_{\text{static}}$, the dual role of the Hamiltonian—as both an observable and a generator—leads to a violation of the Action Quota over time.

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A Representing States and Operators

The mapping from vector $|\psi\rangle \in \mathbb{C}^2$ to Bloch vector $\vec{s} \in B^3$ is:

$$s_i = \langle\psi|\sigma_i|\psi\rangle$$

The Pauli matrices σ_i are the generators of $SU(2)$, satisfying $[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c$. They provide an orthonormal basis for traceless Hermitian operators on $\mathbb{C}^2$. This map is 2-to-1, confirming the double-cover structure.