Lemma 1 (The Static Core):

From the Action Quota to the Unit Disk

Deriving the 2D Information Geometry of Complementarity

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Abstract

We derive the Euclidean unit disk $a^2 + b^2 \leq 1$ as the necessary geometry for the expectation values of two **maximally complementary** measurements. The derivation relies on the **Action Quota**—a variance-based complementarity condition $\mathrm{Var}(A) + \mathrm{Var}(B) \geq 1$—within a framework of a convex state space and affine response, plus the natural requirement that the complementary pair admits no preferred direction in their measurement plane. The main result is that the allowed expectation values $(a, b) = (\langle A \rangle, \langle B \rangle)$ form exactly the unit disk $a^2 + b^2 \leq 1$. We show that the "Bloch disk" is not a postulate of quantum mechanics, but a geometric consequence of finite information constraints, providing the foundation for the 3-dimensional "inflation" developed in Lemma 2.

1 Operational Framework

To reconstruct the theory from the ground up, we define our system using only experimental primitives: preparations (states) and outcomes (statistics).

Assumption 1.1 (Convex State Space)**.** *The set of physical preparations (states) forms a convex set Ω. For any two states $\omega_1, \omega_2 \in \Omega$, their mixture $\omega = p\omega_1 + (1-p)\omega_2$ for $p \in [0, 1]$ is also a valid state.*

Definition 1.1 (Pure State)**.** *A state $\omega \in \Omega$ is* ***pure*** *if it cannot be written as a non-trivial convex combination of other states. Geometrically, pure states represent the extremal points (the frontier) of the set Ω.*

Definition 1.2 (Dichotomic Measurement)**.** *A measurement A is dichotomic if it yields outcomes in $\{+1, -1\}$. For any state $\omega \in \Omega$, the expectation value is $a := \langle A \rangle_\omega \in [-1, 1]$.*

Two dichotomic measurements are **maximally complementary** (for Lemma 3.1) if they satisfy the Action Quota (Assumption 1.4) and admit Planar Rotational Symmetry (Assumption 1.5).

Assumption 1.2 (Affine Response)**.** *The map $\omega \mapsto \langle A \rangle_\omega$ is affine for every dichotomic measurement. This ensures operational consistency: the statistics of a mixture of preparations are the weighted average of the statistics of each individual preparation.*

Assumption 1.3 (Sharpness)**.** *For every dichotomic measurement A, there exist states $\omega_{A\pm}$ yielding outcomes ± 1 with certainty (i.e., $\langle A\rangle_{\omega_{A\pm}} = \pm 1$).*

Definition 1.3 (Operational Variance)**.** *For a dichotomic measurement with outcomes ± 1, the variance is $\mathrm{Var}(A)_\omega = 1 - a^2$. This definition follows from $\mathrm{Var}(A) = \langle A^2\rangle - \langle A\rangle^2$ with $A^2 = 1$ always.*

Assumption 1.4 (The Action Quota)**.** *There exists at least one pair of dichotomic measurements (A, B) such that:*

(i) For all states $\omega \in \Omega$, $\mathrm{Var}(A)_\omega + \mathrm{Var}(B)_\omega \geq 1$.

(ii) There exists at least one state saturating this bound.

Assumption 1.5 (Planar Rotational Symmetry)**.** *This assumption formalizes the requirement that there is no preferred direction in the (A, B)-plane. Formally, there exists a continuous one-parameter family of reversible transformations $\{T_\theta\}_{\theta\in[0,2\pi)}$ acting on states such that for every $\omega \in \Omega$:*

$$(\langle A\rangle_{T_\theta\omega}, \langle B\rangle_{T_\theta\omega}) = (a\cos\theta + b\sin\theta, -a\sin\theta + b\cos\theta),$$

where $(a, b) = (\langle A\rangle_\omega, \langle B\rangle_\omega)$.

Remark 1.1. *Planar Rotational Symmetry ensures that rotating the preparation by θ rotates the pair of expectation values $(\langle A\rangle, \langle B\rangle)$ by θ. Reversible means each T_θ is an invertible affine map on Ω (preserving convex mixtures) with inverse $T_{-\theta}$.*

2 Motivation and Interpretation

Physical Content. The Action Quota captures the irreducible operational limitation observed in quantum systems: the impossibility of preparing a state that has simultaneous definite outcomes for two incompatible measurements.

Accessibility Trade-offs. Classically, if A and B were jointly sharp observables, the accessible region would be the entire square $[-1, 1] \times [-1, 1]$. The diamond $|a| + |b| \leq 1$ is the region guaranteed reachable by convex combinations of the cardinal sharp preparations. The Action Quota plus symmetry extends this to the full unit disk—a larger set that remains constrained by the irreducible uncertainty trade-off.

Normalization. The value "1" is a normalization convention. For dichotomic variables, the maximum variance is 1. A sum of 1 dictates that if one variable is certain ($\mathrm{Var} = 0$), the other must be random ($\mathrm{Var} = 1$).

3 Lemma 1: The Unit Disk

Lemma 3.1 (Emergence of the Bloch Disk)**.** *Let $a = \langle A\rangle_\omega$ and $b = \langle B\rangle_\omega$. If the Action Quota* (1.4) *and Planar Rotational Symmetry* (1.5) *hold, the set of allowed expectation-value pairs (a, b) form exactly the unit disk in $\mathbb{R}^2$:*

$$\mathcal{D} = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 1\}.$$

Proof. The proof establishes the geometric bound and demonstrates reachability.

I. Algebraic Bound. Substituting the definition of dichotomic variance ($\mathrm{Var}(M) = 1 - m^2$) into part (i) of the Action Quota (1.4):

$$(1 - a^2) + (1 - b^2) \geq 1 \implies 2 - (a^2 + b^2) \geq 1 \implies a^2 + b^2 \leq 1.$$

This proves that the set of expectation values is a subset of the unit disk.

II. Reachability of the Origin and Cardinal Points. By Assumption 1.3 (Sharpness), there exist states $\omega_{A\pm}$ and $\omega_{B\pm}$. The Action Quota implies that if $a = \pm 1$, then $\mathrm{Var}(B) \geq 1$, i.e., $1 - b^2 \geq 1$, so $b^2 \leq 0 \implies b = 0$. Thus, the cardinal points $(\pm 1, 0)$ and $(0, \pm 1)$ are achievable. By Assumptions 1.1 and 1.2, the state $\rho_{\mathrm{mix}} := \frac{1}{2}\omega_{A+} + \frac{1}{2}\omega_{A-}$ is achievable and yields $(a, b) = (0, 0)$, since $b = 0$ on both endpoints.

III. Reachability of the Entire Boundary and Disk. By part (ii) of the Action Quota, there exists a state ω_* saturating the bound, so (a_*, b_*) lies on the circle $a^2 + b^2 = 1$. By Assumption 1.5, the orbit $\{T_\theta \omega_* : \theta \in [0, 2\pi)\}$ is physically realizable, and as θ ranges over $[0, 2\pi)$ it traces the entire unit circle.

Finally, for any boundary point $\omega_\theta := T_\theta \omega_*$ and any $\lambda \in [0, 1]$, convexity and affine response imply that the mixture $\omega_{\lambda,\theta} = \lambda \omega_\theta + (1 - \lambda)\rho_{\mathrm{mix}}$ achieves $(\langle A \rangle, \langle B \rangle) = \lambda(a_\theta, b_\theta)$, where (a_θ, b_θ) lies on the unit circle. Since θ ranges over $[0, 2\pi)$, this fills every point in the closed disk $\mathcal{D}$. □

Corollary 3.1.1 (Certainty Trade-off)**.** *By Lemma 3.1, we have $a^2 + b^2 \in [0, 1]$, hence for maximally complementary A, B, the total variance budget is bounded:*

$$\mathrm{Var}(A)_\omega + \mathrm{Var}(B)_\omega = 2 - (a^2 + b^2) \in [1, 2].$$

The lower bound 1 (maximal certainty) is achieved on the circle $a^2 + b^2 = 1$; the upper bound 2 (maximal uncertainty) at the origin.

Remark 3.1 (The Information Frontier)**.** *We call the boundary $a^2 + b^2 = 1$ the* Information Frontier*: it is the locus of states that saturate the Action Quota for the pair (A, B). Equivalently, ω lies on the Information Frontier iff $\mathrm{Var}(A)_\omega + \mathrm{Var}(B)_\omega = 1$. The diamond $|a| + |b| \leq 1$ represents the maximal set of states achievable if one only had access to mixtures of the four cardinal axis preparations.*

4 Interpretation and Consequences

Remark 4.1 (Connection to Standard Qubit Representation)**.** *In standard quantum mechanics, a qubit state is represented by a point $\vec{r}$ in the Bloch ball with $|\vec{r}| \leq 1$. Our disk corresponds to one equatorial cross-section of the Bloch ball. Lemma 1 thus derives the fundamental 2D slice of the qubit from first principles.*

4.1 Roadmap to Lemma 2 (The 3D Inflation)

Lemma 1 establishes that for a fixed maximally complementary pair (A, B), the set of achievable expectation-value pairs $(\langle A \rangle, \langle B \rangle)$ is exactly the unit disk. However, a complete physical theory requires more than one privileged pair of measurements; it must accommodate measurements in arbitrary spatial directions.

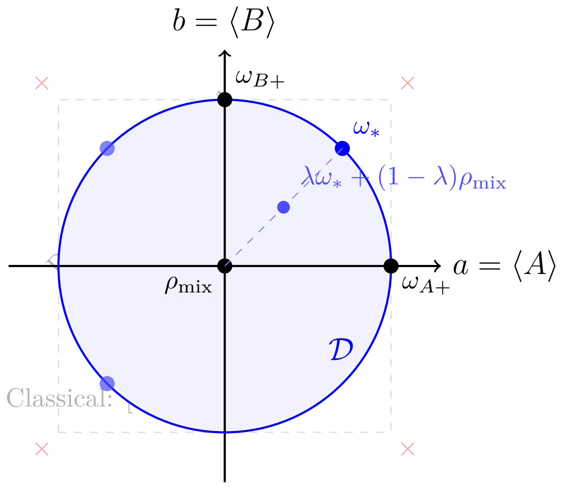

Figure 1: The Bloch Disk $\mathcal{D}$ (blue). The state ω_* saturates the bound; the reversible transformations T_θ generate the full boundary circle. The blue dots on the circle represent states obtained by applying T_θ to ω_*. Convex combinations with the central state ρ_{mix} fill the interior radial segments.

If we demand **isotropy**—that the physics is invariant under spatial rotations—then the state space must contain disks for *every* orthogonal pair of directions. These disks, all sharing the same origin and obeying the same variance complementarity, naturally fill a 3D ball. Lemma 2 will show that the minimal isotropic extension consistent with our operational constraints is a 3-ball, thereby deriving the three-dimensionality of the qubit state space from informational constraints.

5 Conclusion

The unit disk, bounded by the **Information Frontier** $a^2 + b^2 = 1$, is the unique geometric solution to a budget constraint on information. By deriving this from the Action Quota and planar symmetry, we demonstrate that the non-classicality of the qubit is rooted in the statistical trade-off between incompatible measurement outcomes.

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