Reconstructing Quantum Mechanics from the Action Quota:

An Eight-Lemma Derivation from Variance Complementarity

A Preface to the Reconstruction

Emiliano Shea

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Abstract

We present an operational reconstruction of quantum mechanics from a single foundational axiom: a variance complementarity condition between incompatible measurements. Unlike traditional axiomatic approaches that postulate complex Hilbert spaces, we show that the full mathematical structure—including the Bloch sphere, Born rule, tensor product, Heisenberg commutator, and the specific value of Planck's constant—emerges necessarily from this one constraint through eight sequential derivations.

Scope: The reconstruction proceeds in three stages:

1. **The Discrete Core (Lemmas 1–5):** Deriving the qubit geometry, probability rules, and unitary dynamics.
2. **The Extensions (Lemmas 6–7):** Deriving entanglement (tensor products) and continuous variables (phase space).
3. **The Calibration (Lemma 8):** Fixing the numerical scale of action via thermodynamic consistency.

Main Theorem (Informal): From the Action Quota plus operational axioms (isotropy, reversibility, interaction), we derive:

(i) The single-qubit structure (Bloch ball, Born rule, $SU(2)$ dynamics).
(ii) The tensor product structure for composite systems.
(iii) The canonical commutation relation $[x, p] = i\hbar$.
(iv) The physical value of the quantum of action $\hbar = h/2\pi$.

1 Introduction: The Reconstruction Programme

Quantum mechanics is normally introduced as a list of formal axioms: complex state vectors, non-commuting observables, tensor products, and probabilistic outcomes. This reconstruction asks whether these features are inevitable consequences of a single operational constraint.

In this work we show that the full mathematical structure follows necessarily from one simple statistical constraint.

Why Reconstruction? Standard quantum mechanics is often taught as a collection of mysterious postulates. This leaves foundational questions unanswered: Why complex numbers? Why tensor products? Why does uncertainty exist? Reconstruction programmes replace arbitrary postulates with operational principles, revealing quantum mechanics as the unique mathematical consequence of well-posed physical constraints. This approach transforms quantum theory from a list of rules into an inevitable structure.

The Deep Questions Addressed Our programme answers core puzzles that often feel taken for granted:

- **Geometry:** Why is the state space a 3D ball (the Bloch ball)?
- **Probability:** Why are probabilities given by the Born rule?
- **Scale unification:** Why does one constant ($\hbar$) govern both uncertainty and dynamics?
- **Complexity:** Why do composite systems combine via the tensor product ($\otimes$)?
- **Continuum:** Why do position and momentum non-commute ($[x, p] = i\hbar$)?
- **Calibration:** Why is Planck's constant $h \approx 6.626 \times 10^{-34}\,\mathrm{J\,s}$?

2 Key Operational Axioms

We rely on one foundational constraint plus a set of standard operational definitions.

Foundational Axiom

Variance Complementarity (The Action Quota). There exist pairs of dichotomic observables A and B (outcomes ± 1) such that for all states:

$$\mathrm{Var}(A) + \mathrm{Var}(B) \geq 1$$

This quantifies the irreducible trade-off between incompatible measurements.

Operational Requirements:

- **Isotropy (L2):** No measurement direction is spatially privileged; the state space respects rotational symmetry.
- **Reversibility (L4):** Time evolution is a continuous, reversible transformation of the state space.
- **Interaction (L6):** Systems can influence one another via conditional operations.

- **Thermodynamic Consistency (L8):** The theory must recover the Stefan-Boltzmann law in the macroscopic thermal limit.

3 The Action Quota in Detail

3.1 Operational Meaning

The constraint has immediate physical content:

- **Dichotomic** means binary-valued (± 1), like measuring "spin up/down".
- **Variance** is the statistical spread: $\mathrm{Var}(A) = 1 - \langle A \rangle^2$.
- The inequality says: **if A is sharp (low variance), B must be fuzzy (high variance).**
- **Normalisation:** The bound "1" is a natural unit choice. For dichotomic variables, $\mathrm{Var}(A) + \mathrm{Var}(B) \geq 1$ is algebraically equivalent to $\Delta A \Delta B \geq 1/2$.

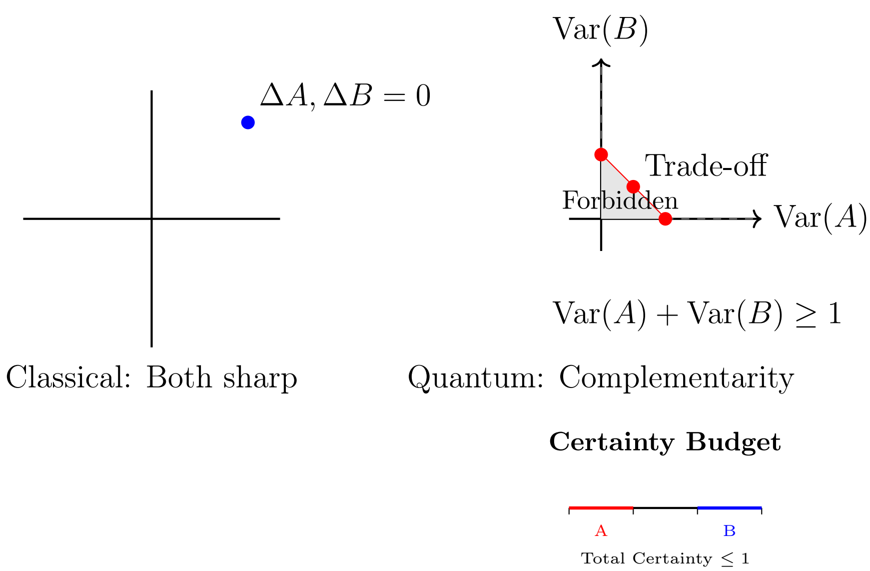

Figure 1: Classical vs. quantum certainty budgets. Classical physics allows infinite precision ($\mathrm{Var} = 0$). The Action Quota imposes a trade-off boundary.

3.2 What We Do NOT Assume

To clarify the minimality of our starting point, note what is **not** assumed:

- No complex numbers (derived in L4)
- No Hilbert space (constructed in L1–L3)

- No operators or matrices (emerge from geometry)
- No Born rule (derived in L3)
- No tensor products (derived in L6)
- No Planck's constant (derived in L8)

The **only** input is the Action Quota plus operational requirements.

4 Reconstruction Architecture: Eight Lemmas

The argument unfolds in eight sequential lemmas.

4.1 Logical Flow

4.2 The Eight Lemmas in Brief

Table 1: Logical dependency and key insights

Lemma	Requires	Derives	Key Insight
L1	Action Quota	Unit Disk	Uncertainty creates geometry
L2	L1 + Isotropy	Bloch Ball (B^3)	Dimensional sandwich
L3	L2 + Convexity	Born Rule	Probabilities are affine
L4	L3 + Reversibility	$\mathbb{C}^2, SU(2)$	Topology forces Complex field
L5	L4	$\hbar = \kappa$	Dual role unifies scales
L6	L1–L5 + Interaction	Tensor Product	Consistency forces entanglement
L7	L1–L6 + Limit	$[x, p] = i\hbar$	Phase space as tangent plane
L8	L7 + Thermodynamics	$\rho_0 = h$	Heat fixes the scale

5 Guide for Readers: Navigating the Reconstruction

5.1 Conceptual Hurdles: Short Answers

For readers encountering specific conceptual puzzles:

- **Why 3D?** Isotropy requires a continuous family of measurements. A 2D disk cannot accommodate this on its boundary; only a 3D ball can.

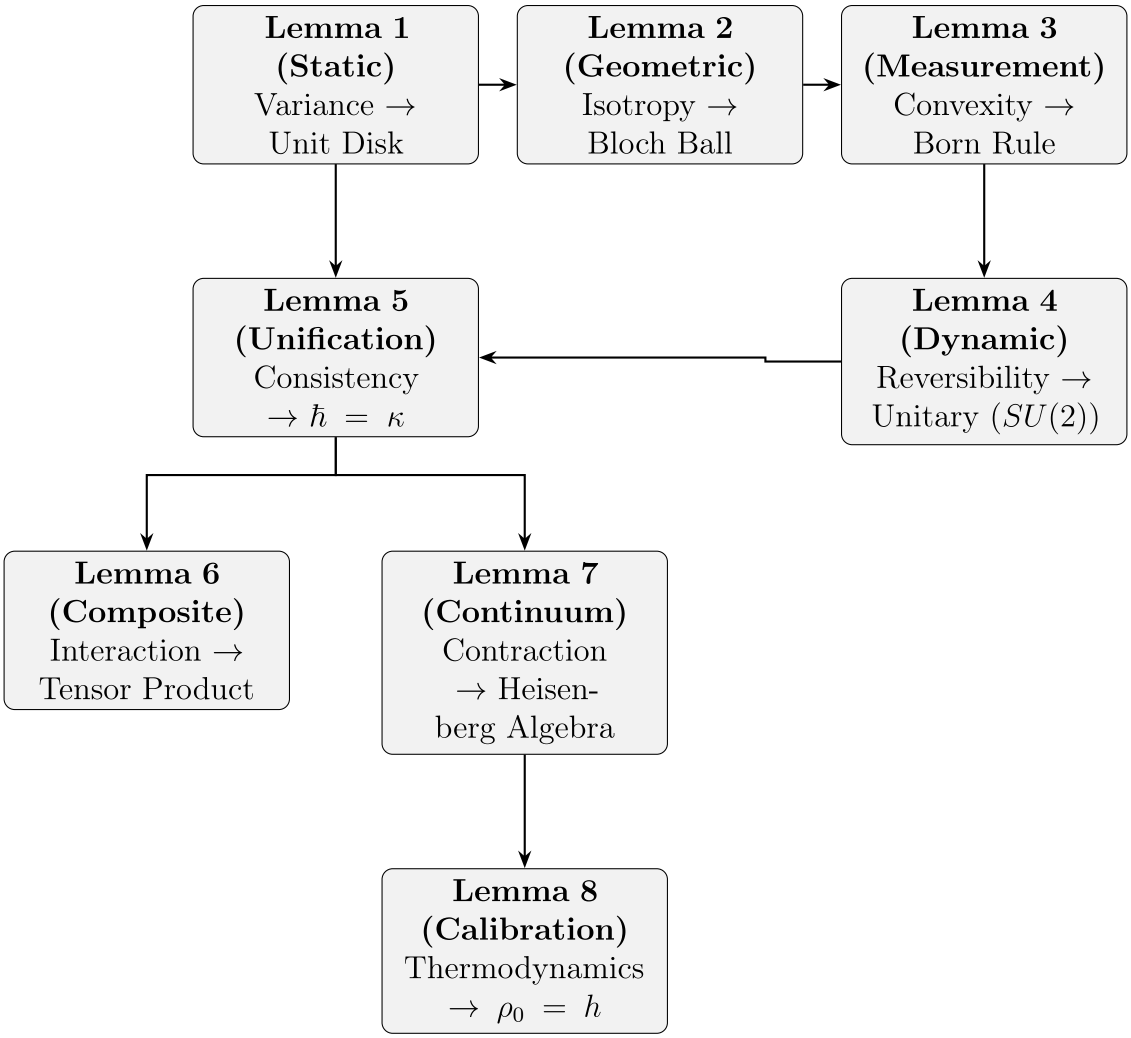

Figure 2: Logical dependency structure. Lemmas 1–5 reconstruct the single qubit. Lemmas 6–7 extend to composite systems and continuous variables. Lemma 8 uses the continuous modes (L7) to calibrate the scale against heat.

- **Why Complex Numbers?** Rotations on the Bloch ball must lift to a consistent representation. The universal cover of $SO(3)$ is $SU(2)$, which acts naturally on $\mathbb{C}^2$.
- **Why the Tensor Product (L6)?** If systems interact, the interaction generator must be an observable. The algebra generated by local terms plus one interaction term is the full $\mathfrak{su}(4)$. This inflation from $3+3$ to 15 dimensions is the origin of entanglement.
- **Why x and p don't commute (L7)?** Continuous variables are the limit of a large spin system. The non-commutativity of position and momentum is simply the local "flat earth" approximation of the non-commutativity of rotations on a giant sphere.
- **Why h has a specific value (L8)?** While structure is fixed by information constraints, the *size* of the quanta is fixed by thermodynamics. If h were any other value, the heat capacity of the vacuum would not match the Stefan-Boltzmann law.

5.2 Suggested Reading Paths

- **Physicists:** Read all eight lemmas sequentially.
- **Mathematicians:** Focus on L1–L2 (geometry), L4 (Lie groups), and L6–L7 (algebra).
- **Foundations researchers:** Read L1, L3 (operational principles), and L8 (thermodynamic calibration).

6 Conclusion and Scope

This reconstruction provides a clear, economical path from a single operational principle to the full mathematical structure of quantum mechanics. We have shown that:

1. **Qubits** are the unique geometry of finite uncertainty (L1–L5).
2. **Entanglement** is the algebraic price of interaction (L6).
3. **Phase Space** is the tangent geometry of macroscopic information (L7).
4. **Planck's Constant** is the thermodynamic exchange rate of action (L8).

6.1 Philosophical Implications

By deriving quantum mechanics from the Action Quota, we've shown that quantum "weirdness" (superposition, entanglement, uncertainty) is not a list of independent mysteries but the inevitable geometric consequence of a single constraint on information extraction. The question shifts from "Why is nature quantum?" to "Why does nature impose this particular certainty budget?"

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