LEMMA 7 (EXTENSIONS):

The Continuum Limit

Extending the Action Quota to Phase Space via Group Contraction

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Abstract

We extend the reconstruction from finite-dimensional spins (qubits) to continuous variables. We demonstrate that the canonical Heisenberg relation $[X, P] = i\hbar_{\text{phys}}$ arises as the tangent-space (flat) limit of a global $SU(2)$ geometry. By applying the Wigner–Inönü group contraction in the limit of large system size ($J \to \infty$), we show that the Action Quota established in Lemma 1 naturally manifests as the canonical commutator and its associated uncertainty bounds. This extension reveals that the uncertainty principle represents the physical persistence of state-space curvature in the tangent plane, providing a unified structural bridge between discrete and continuous quantum systems.

Keywords: Group contraction, phase space, Heisenberg algebra, canonical quantization, Action Quota, $SU(2)$, Wigner–Inönü

1 Introduction: Role in the Reconstruction

Lemmas 1–6 established the geometry and dynamics of finite-dimensional quantum systems (qubits) based on the **Action Quota**: a fundamental budget on certainty. However, many physical systems present continuous degrees of freedom, such as position x and momentum p.

Lemma 7 extends the reconstruction to these infinite-dimensional regimes. We demonstrate that the transition from discrete spins to continuous variables is governed by the mathematical procedure of **group contraction**. Rather than postulating a separate set of rules for the continuum, we show that the Heisenberg algebra $\mathfrak{h}_1$ inherits its structure from the same Action Quota that governs spins. This derivation ensures that the Action Quota remains the single governing principle across all physical scales, preparing the framework for the final calibration in Lemma 8.

2 The Contraction Setup

Consider a generalized spin system with total angular momentum J. The generators $\{J_x, J_y, J_z\}$ satisfy the Lie algebra $\mathfrak{su}(2)$:

$$[J_x, J_y] = i\hbar_{\text{spin}} J_z \quad \text{(and cyclic permutations).} \tag{1}$$

The Wigner–Inönü contraction procedure [1, 6] maps this compact algebra to a non-compact one. To obtain a nontrivial limit, we define the J-dependent *dimensionless* contracted generators:

$$x_J := \frac{J_x}{\sqrt{J}\,\hbar_{\text{spin}}}, \qquad p_J := \frac{J_y}{\sqrt{J}\,\hbar_{\text{spin}}}. \tag{2}$$

These are the natural tangent-plane coordinates near the north pole.

Dimensional considerations. Let $\hbar_{\text{spin}}$ carry dimensions of action, so that J_x, J_y, J_z also have dimensions of action. To obtain position and momentum with their standard dimensions, introduce characteristic scales L_0 (length) and P_0 (momentum) and define the physical position and momentum variables as

$$X := L_0\, x_J, \qquad P := P_0\, p_J.$$

Then the commutator of the contracted operators is

$$[x_J, p_J] = \frac{1}{J\hbar_{\text{spin}}^2}\,[J_x, J_y] = i\,\frac{J_z}{J\,\hbar_{\text{spin}}},$$

so that in the polarized (north-pole) sector where $J_z/(J\hbar_{\text{spin}}) \to \mathbb{I}$ as $J \to \infty$ (made precise below), we have $[x_J, p_J] \to i\,\mathbb{I}$. Consequently, the physical commutator becomes

$$[X, P] = L_0 P_0\,[x_J, p_J] \to i\,\hbar_{\text{phys}}\,\mathbb{I}, \qquad \hbar_{\text{phys}} := L_0 P_0,$$

which has the correct dimensions of action.

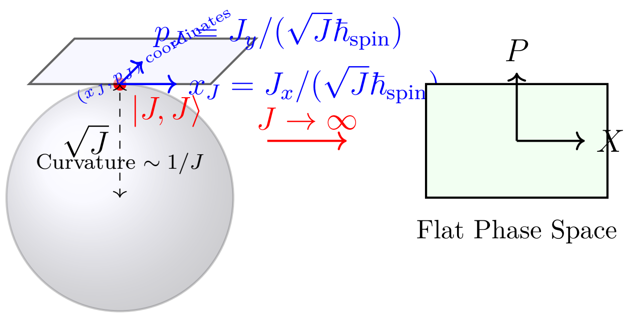

Figure 1: Group contraction from curved $SU(2)$ manifold to flat Heisenberg phase space. As $J \to \infty$, local curvature vanishes and the tangent plane becomes exact.

Remark 2.1 (Uniqueness of $\sqrt{J}$ Scaling)**.** *The $\sqrt{J}$ scaling is uniquely determined by requiring: (1) finite variance in the limit, since* $\mathrm{Var}(x_J) = \mathrm{Var}(J_x)/(J\hbar_{spin}^2)$ *remains finite; (2) a non-vanishing commutator, since* $[x_J, p_J] = i\,J_z/(J\hbar_{spin})$ *remains* $O(1)$ *on the polarized sector; (3) correspondence with classical limit, where (with appropriate state families)* $\frac{1}{i\hbar_{spin}}[\,\cdot\,,\,\cdot\,] \to \{\,\cdot\,,\,\cdot\,\}$.

3 Derivation of the Canonical Commutation Relation

3.1 The Polarization Limit

In the commutator we have

$$[X, P] = L_0 P_0 \, [x_J, p_J] = i \, L_0 P_0 \, \frac{J_z}{J \hbar_{\text{spin}}}.$$

For spin-coherent states $\{|\theta_J, \phi\rangle\}$ localized at the north pole (with $\theta_J \sim 1/\sqrt{J}$), J_z is sharply concentrated near $J\hbar_{\text{spin}}$: $\langle J_z \rangle / (J\hbar_{\text{spin}}) = \cos\theta_J \to 1$ and $\Delta J_z/(J\hbar_{\text{spin}}) \sim \theta_J/\sqrt{J} \to 0$ as $J \to \infty$. Hence $J_z/(J\hbar_{\text{spin}}) \to \mathbb{I}$ in the sense of expectation values with vanishing relative fluctuations on this state family, and we obtain the canonical form $[X, P] \to i \, \hbar_{\text{phys}} \, \mathbb{I}$.

4 From Bounded to Unbounded Operators

For finite J, J_x has eigenvalues $m\hbar_{\text{spin}}$ with $m \in \{-J, -J+1, \ldots, J\}$. Hence the contracted operator $x_J = J_x/(\sqrt{J}\hbar_{\text{spin}})$ has eigenvalues in

$$\text{spec}(x_J) \subseteq [-\sqrt{J}, \, \sqrt{J}],$$

and the physical coordinate $X = L_0 x_J$ has eigenvalues in $[-L_0\sqrt{J}, \, L_0\sqrt{J}]$. As $J \to \infty$, these intervals expand to cover the entire real line $\mathbb{R}$, producing the unbounded operators of continuous-variable quantum mechanics. This transition from compact to non-compact geometry is the origin of unbounded operators in the theory.

5 Survival of the Action Quota

5.1 Action quota in the continuum limit

Lemma 1 gives an additive uncertainty bound for spin-J, schematically of the form

$$\text{Var}(J_x) + \text{Var}(J_y) \;\geq\; \frac{1}{2} \, \hbar_{\text{spin}}^2 \, J,$$

where the constant $1/2$ is set by the normalization used in Lemma 1.[1] Under the contraction map $X = L_0 x_J$ and $P = P_0 p_J$, the variances are related by

$$\text{Var}(x_J) = \frac{1}{J\hbar_{\text{spin}}^2} \text{Var}(J_x), \qquad \text{Var}(p_J) = \frac{1}{J\hbar_{\text{spin}}^2} \text{Var}(J_y),$$

and therefore

$$\frac{\text{Var}(X)}{L_0^2} = \text{Var}(x_J), \qquad \frac{\text{Var}(P)}{P_0^2} = \text{Var}(p_J).$$

Dividing the spin bound by $J\hbar_{\text{spin}}^2$ yields the finite tangent-plane inequality

$$\text{Var}(x_J) + \text{Var}(p_J) \;\geq\; \frac{1}{2},$$

[1] The precise constant depends on the normalization convention; for spin-J coherent states, the bound is saturated at $\frac{1}{2}\hbar_{\text{spin}}^2 J$.

equivalently,

$$\frac{\mathrm{Var}(X)}{L_0^2}+\frac{\mathrm{Var}(P)}{P_0^2} \;\geq\; \frac{1}{2}.$$

This inequality constrains global uncertainties in the tangent plane using the scale $\hbar_{\text{phys}}$.

Remark 5.1 (Sum vs. product forms)**.** *The additive and product forms are related but distinct. In the contracted, dimensionless variables one may write*

$$\mathrm{Var}(x_J)+\mathrm{Var}(p_J) \;\geq\; \frac{1}{2} \qquad \textit{and} \qquad \Delta x_J\,\Delta p_J \;\geq\; \frac{1}{2},$$

where the $1/2$ *is the same constant appearing in the rescaled Lemma 1 bound. In physical units this becomes*

$$\frac{\mathrm{Var}(X)}{L_0^2}+\frac{\mathrm{Var}(P)}{P_0^2} \;\geq\; \frac{1}{2} \qquad \textit{and} \qquad \Delta X\,\Delta P \;\geq\; \hbar_{phys}/2,$$

with $\hbar_{phys} = L_0P_0$*. Both originate from non-commutativity; the additive form is the more restrictive "frontier"-type constraint inherited from the compact geometry prior to contraction.*

6 Limits of Validity and Geometric Corrections

Standard quantum mechanics with exact $[X,P]=i\hbar_{\text{phys}}$ is a tangent-space approximation. The underlying $SU(2)$ curvature introduces corrections of order $1/J$, which vanish in the strict contraction limit but become relevant away from the polarized, low-excitation regime.

Proposition 6.1 (Curvature corrections from Holstein–Primakoff)**.** *In the Holstein–Primakoff representation one has* $J_z=\hbar_{spin}(J-b^\dagger b)$*, and therefore*

$$[X,P]=i\,\hbar_{phys}\left(1-\frac{b^\dagger b}{J}\right)+\mathcal{O}(J^{-2}),$$

where $b^\dagger b$ *is the bosonic number operator. For states with* $\left\langle b^\dagger b\right\rangle \ll J$*, this reduces to* $[X,P]\approx i\hbar_{phys}$*. For highly excited states, the correction becomes significant, indicating departure from flat phase space.*

Remark 6.1 (Geometric Interpretation)**.** *The correction term* $-b^\dagger b/J$ *represents the curvature of the underlying* $SU(2)$ *manifold. As excitations increase, the state explores more of the sphere's curvature, modifying the effective commutation relations.*

Example 6.1 (Holstein–Primakoff Transformation)**.** *The Holstein–Primakoff representation maps spin operators to bosonic operators as*

$$J_+=\hbar_{spin}\sqrt{2J-b^\dagger b}\,b, \qquad J_-=\hbar_{spin}\,b^\dagger\sqrt{2J-b^\dagger b}, \qquad J_z=\hbar_{spin}\,(J-b^\dagger b),$$

where $[b,b^\dagger]=1$ *and* $b^\dagger b$ *is dimensionless. Define dimensionless quadratures*

$$q=\frac{b+b^\dagger}{\sqrt{2}}, \qquad \pi=\frac{i(b^\dagger-b)}{\sqrt{2}}, \qquad [q,\pi]=i.$$

In the polarized sector $b^\dagger b\ll J$*, one obtains*

$$\frac{J_z}{J\hbar_{spin}}=1-\frac{b^\dagger b}{J}, \qquad \textit{and hence} \qquad [X,P]=i\,\hbar_{phys}\left(1-\frac{b^\dagger b}{J}\right)+\mathcal{O}(J^{-2}),$$

which exhibits the curvature correction explicitly.

7 Conclusion

Lemma 7 demonstrates that continuous-variable quantum mechanics emerges as the macroscopic, flat limit of the same principles that govern qubits:

Quantity	**Spin representation**	**Contraction map**	**Continuum limit**
Generators	J_x, J_y, J_z	$x_J = \frac{J_x}{\sqrt{J}\hbar_{\text{spin}}}$, $p_J = \frac{J_y}{\sqrt{J}\hbar_{\text{spin}}}$	$X = L_0 x_J,\ P = P_0 p_J$
Algebra	$[J_x, J_y] = i\hbar_{\text{spin}} J_z$	$[x_J, p_J] = i\,\frac{J_z}{J\hbar_{\text{spin}}}$	$[X, P] = i\hbar_{\text{phys}}$ (polarized limit)
Casimir	$J_x^2 + J_y^2 + J_z^2 = J(J+1)\hbar_{\text{spin}}^2$	Expand for large J	$x_J^2 + p_J^2$ unbounded (hence $X^2 + P^2$ unbounded)
State space	$\mathbb{C}P^{2J}$ (real dim $4J$)	Local patch near pole	$\mathbb{R}^2$ (plane)

Table 1: Summary of the Wigner–Inönü contraction from spins to phase space. The $\sqrt{J}$ scaling ensures finite variances and non-vanishing commutators in the limit.

Outlook This derivation completes the extension to continuous variables and provides a geometric unification of spin and continuous systems. Lemma 8 will calibrate $\hbar_{\text{phys}}$ to its physical value $\hbar$ using thermodynamic consistency—specifically by matching the quantum of action to the scale of thermal fluctuations. This will relate the uncertainty bound to the equipartition theorem, fixing $\hbar$ in terms of Boltzmann's constant and measurable thermal energies, bridging the reconstructed formalism to experimental reality.

References

[1] E. P. Wigner and E. İnönü, Proc. Nat. Acad. Sci. **39**, 510 (1953).

[2] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).

[3] J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*, Cambridge (2017).

[4] A. M. Perelomov, *Generalized Coherent States and Their Applications*, Springer (1986).

[5] A. Messiah, *Quantum Mechanics*, North-Holland (1961).

[6] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications*, Wiley (1974).