LEMMA 2 (DIMENSIONAL INFLATION):

From Disk Geometry to the Bloch Ball

An operational derivation of the three-dimensional state space

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Abstract

We build upon the 2D disk constraint established in Lemma 1. By introducing structural assumptions—(R1) **Continuity**, (R2) **Isotropy**, (R3) **Complementarity as Orthogonality**, (R4) **Local Charts**, and (R5) **Boundary Regularity**—we prove that the expectation-value image of the state space Ω is necessarily three-dimensional. Assumptions (R4) and (R5) serve as necessary regularity conditions to exclude pathological geometries. The proof utilizes a **dimensional sandwich**: isotropy forces $\dim(\Omega) > 2$ to accommodate a continuous family of complementary states, while the local chart requirement caps $\dim(\Omega) \leq 3$ to ensure operational efficiency. We conclude that the global Information Frontier is the 2-sphere S^2, characterizing the state space as the Euclidean 3-ball B^3 (the Bloch ball).

1 Recalling the Foundational Constraint

Building on Lemma 1, we consider a binary system with a convex state space Ω. Throughout this derivation, we retain the operational primitives of convexity, affine response, and sharpness for dichotomic measurements as established in the preceding core. We assume the universal validity of the geometric result:

Lemma 1.1 (Lemma 1: The Disk Constraint)**.** *For any pair of maximally complementary measurements (A, B), the set of achievable expectation-value pairs $(\langle A\rangle, \langle B\rangle)$ is exactly the unit disk $\mathcal{D} = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 1\}$.*

2 Structural Assumptions

To determine the full dimensionality, we need additional principles that govern how the 2D slices provided by Lemma 1 fit together.

Assumption 2.1 (Continuity - R1)**.** *The state space Ω can be embedded as a convex body in some finite-dimensional Euclidean space $\mathbb{R}^N$. The expectation-value functions $\omega \mapsto \langle M\rangle_\omega$ are continuous for all measurements M.*

Assumption 2.2 (Isotropy - R2)**.** *The state space Ω is isotropic: for every rotation $R \in SO(3)$, there exists a reversible affine transformation $T_R : \Omega \to \Omega$ that preserves measurement statistics (i.e. $\langle M_{R\vec{v}}\rangle_{T_R\omega} = \langle M_{\vec{v}}\rangle_\omega$ for all ω and all directions $\vec{v}$). Moreover, the action of $SO(3)$ is transitive on measurement directions.*

Assumption 2.3 (Complementarity as Orthogonality - R3). *There exist three dichotomic measurements X, Y, Z that are pairwise maximally complementary. For every rotation $R \in SO(3)$, the rotated triple (X_R, Y_R, Z_R) obtained by rotating the axes is again pairwise maximally complementary.*

Remark 2.1. *Operationally, Assumption R3 asserts that there exists a triple of pairwise maximally complementary binary tests whose complementarity relations are preserved under the physical rotation action. Physically, this encodes the requirement that a binary system's internal degrees of freedom transform coherently under spatial rotations—the hallmark of a spin-1/2 particle.*

Assumption 2.4 (Two-Observable Local Chart - R4). *The Information Frontier is locally exhausted by complementary pairs: for every pure state ω, there exists a pair of maximally complementary measurements (A, B) and an open neighborhood $U \subset \partial\Omega$ of ω such that the map $\omega' \mapsto (\langle A \rangle_{\omega'}, \langle B \rangle_{\omega'})$ is injective on U.*

Assumption 2.5 (Regular Pure-State Boundary - R5). *The set of pure states $\partial\Omega$ is a topological manifold of dimension* $\dim(\Omega) - 1$.

Remark 2.2 (On Regularity). *Assumptions R4 and R5 are mild regularity conditions that exclude pathological state spaces. They ensure the pure state space is locally well-behaved and can be parameterized by complementary measurements, which is necessary for a smooth information geometry [1, 3].*

3 The Dimensional Sandwich

Lemma 3.1 (Lemma 2). *Given Assumptions R1–R5 and the Disk Constraint, the state space Ω is affinely isomorphic to the unit 3-ball:*

$$B^3 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}.$$

Proof Overview. We demonstrate that the dimension n of the state space must be exactly 3. We use **Stage I** (Dimensional Forcing) to show $n > 2$ by contradiction with isotropy, and **Stage II** (Dimensional Capping) to show $n \leq 3$ by contradiction with local distinguishability. Finally, **Stage III** uses the geometry of Lemma 1 to select the 3-ball as the unique solution.

4 Proof of the Lemma

4.1 Stage I: Dimensional Forcing ($\dim > 2$)

Assume for contradiction that $\dim(\Omega) = 2$. Since Ω is 2-dimensional and the expectation map to a maximally complementary pair is affine and surjective onto the unit disk (Lemma 1.1), Ω is affinely isomorphic to a 2D disk. Let Z be a measurement and ω_Z its sharp eigenstate ($\langle Z \rangle = 1$), guaranteed by the Sharpness postulate from the operational framework.

By Assumption 2.2 (Isotropy), rotating the measurement frame about Z generates a continuous family of measurements $\{X_\phi\}_{\phi \in [0, 2\pi)}$ all complementary to Z. Since rotations fix the Z-axis, they correspond to transformations on Ω that preserve the value of $\langle Z \rangle$.

Let ω_0 be the $+1$ eigenstate of X_0 ($\langle Z \rangle_{\omega_0} = 0$). The states $\omega_\phi = T_{R_\phi}\omega_0$ form a continuous orbit of pure states, all satisfying $\langle Z \rangle = 0$.

However, because the expectation map $\omega \mapsto \langle Z \rangle$ is affine, the set of states with $\langle Z \rangle = 0$ in a 2D disk defines a **diameter** of the disk (a line segment through the center). The boundary of Ω (where pure states lie) intersects this segment at exactly **two points**. A continuous 1-parameter circle of distinct states $\{\omega_\phi\}$ cannot be embedded into two discrete points. Thus, $\dim(\Omega) > 2$.

4.2 Stage II: Dimensional Capping ($\dim \leq 3$)

Assume for contradiction that $\dim(\Omega) = n \geq 4$. By Assumption 2.5, $\partial\Omega$ is an $(n-1)$-manifold with $n - 1 \geq 3$. By Assumption 2.4, there exist a pure state ω_0, an open neighborhood $U \subset \partial\Omega$ of ω_0, and complementary measurements (A, B) such that the map $E : U \to \mathbb{R}^2$ is continuous and injective.

Since $\partial\Omega$ is an $(n-1)$-manifold, every point has a neighborhood $V \subset U$ homeomorphic to an open subset of $\mathbb{R}^{n-1}$. The restriction $E|_V : V \to \mathbb{R}^2$ is a continuous injection, hence a topological embedding of V into $\mathbb{R}^2$. But topological dimension is monotone under embeddings: if V embeds into $\mathbb{R}^2$, then $\dim_{\text{top}}(V) \leq 2$ (see, e.g., the discussion of dimension theory in [5]). On the other hand, V is an open subset of an $(n-1)$-manifold, so $\dim_{\text{top}}(V) = n - 1$. Thus $n - 1 \leq 2$, contradicting $n - 1 \geq 3$. Therefore $\dim(\Omega) \leq 3$.

4.3 Stage III: Selection of the Ball

Combined, Stages I and II fix $\dim(\Omega) = 3$. We map Ω to the space of expectation values $\vec{r}(\omega) = (\langle X \rangle, \langle Y \rangle, \langle Z \rangle)$.

Proposition 4.1 (Ball from Disk Sections). *Let $K \subset \mathbb{R}^3$ be a compact convex body containing the origin. If for every 2D linear subspace P (through the origin), the central section $K \cap P$ is the unit disk in P, then K is the unit Euclidean ball.*

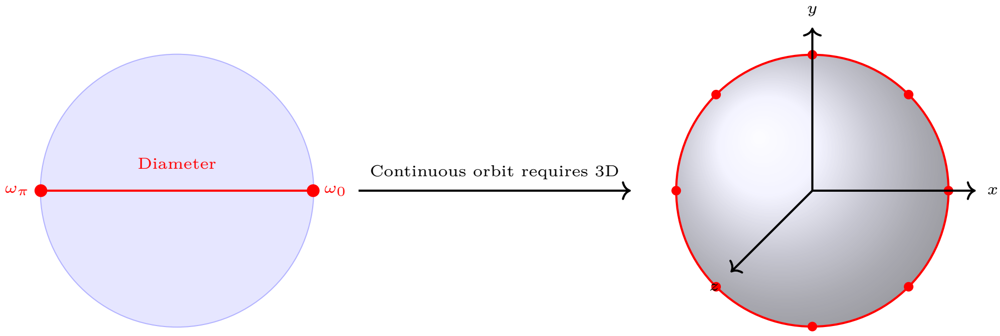
Proof. For any unit vector $u \in S^2$, choose a plane P through the origin containing u. The support function of the section $K \cap P$ in the plane P is $h_{K\cap P}(u) = \sup_{x \in K\cap P} u \cdot x = 1$ (since $K \cap P$ is a unit disk). Because $K \supseteq K \cap P$, the support function of the whole body satisfies $h_K(u) \geq h_{K\cap P}(u) = 1$. Conversely, if K contained a point x with $\|x\| > 1$, then for $u = x/\|x\|$ we would have $h_K(u) \geq u \cdot x = \|x\| > 1$. Take any plane P containing u (e.g., $P = \text{span}\{u, v\}$ for some unit vector $v \perp u$). Then the section $K \cap P$ would satisfy $h_{K\cap P}(u) > 1$, contradicting the hypothesis that $K \cap P$ is a unit disk (which has support function 1 in every direction). Hence K is contained in the unit ball, so $h_K(u) \leq 1$ for all directions u. Thus $h_K(u) = 1$ for all u, uniquely characterizing the unit Euclidean ball. $\square$

Any plane through the origin is the orthogonal complement of some unit vector $\vec{n}$. By Assumption 2.2, we can rotate the canonical (X, Y)-plane into that plane, and by Assumption 2.3, the complementarity of the measurement pair is preserved under this rotation. Consequently, Lemma 1.1 implies every central section of the expectation-value body K is a unit disk. Applying Proposition 4.1, we conclude K is the unit ball, i.e., $\Omega \cong B^3$.

Finally, the map $\omega \mapsto \vec{r}(\omega)$ is affine, and its image K has nonempty interior in $\mathbb{R}^3$ (indeed $K = B^3$), so the linear part of this affine map has rank 3 and is therefore invertible.

Hence $\omega \mapsto \vec{r}(\omega)$ is an affine isomorphism between Ω and B^3, and in particular every point of B^3 corresponds to a physical state. □

Corollary 4.1. *The pure states of a binary system satisfying R1–R5 form a 2-sphere S^2. The mixed states fill the interior of the Bloch ball B^3, with the maximally mixed state at the origin.*



2D Disk: Only 2 points with $\langle Z \rangle = 0$

3D Ball: Continuous circle of points with $\langle Z \rangle = 0$

Figure 1: Dimensional Forcing: In 2D, the constraint $\langle Z \rangle = 0$ identifies only two boundary points (left). Isotropy requires a continuous family of such states related by rotations (right), forcing a third dimension to accommodate the orbit.

5 Discussion: The Dimensional Sandwich Principle

The dimensional sandwich argument demonstrates an important methodological principle in our reconstruction: constraints from different physical principles (symmetry, information efficiency) can combine to uniquely determine mathematical structure. This approach contrasts with traditional axiomatizations that postulate state space dimensionality directly, instead deriving it from the necessary preconditions of measurement.

Remark 5.1 (Methodological Innovation)**.** *The dimensional sandwich demonstrates how operational principles can uniquely determine mathematical structure. Unlike traditional reconstructions that postulate the Bloch sphere, we derive its dimensionality and geometry from the interplay of complementarity, symmetry, and information efficiency.*

6 Consequences and Physical Interpretation

Remark 6.1 (Topological Foundation)**.** *The dimensional capping argument relies on fundamental topological facts: the Lebesgue covering dimension of $\mathbb{R}^2$ is 2, and dimension is monotone under embeddings. This provides a rigorous basis for ruling out dimensions ≥ 4 without additional physical postulates.*

Remark 6.2 (Connection to the Pauli Matrices)**.** *The three coordinate axes (x, y, z) correspond to three mutually complementary measurements. In standard quantum mechanics, these three directions correspond to the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$. Our derivation shows that such a triple must exist on geometric grounds; the algebraic representation will be derived in Lemma 4.*

7 Conclusion

Lemma 2 completes the static reconstruction of the qubit state space. Building on the Disk Constraint from Lemma 1 (which itself follows from the Action Quota), we have derived the Bloch ball as the unique geometry satisfying complementarity, isotropy, and operational efficiency. This provides a foundation for the standard quantum mechanical representation of qubits via the Pauli matrices and the Bloch sphere. Every point on the global Information Frontier S^2 represents a pure state—one that saturates the Certainty Budget for at least one complementary pair of measurements.

References

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A Alternative Proof via Convex Geometry

The following provides an alternative, more geometric proof of Stage III. Let $K \subset \mathbb{R}^3$ be the convex body representing the state space (the image of $\omega \mapsto \vec{r}(\omega)$). By Lemma 1.1, every central section of K (intersection with a plane through the origin) is a unit disk. For any $u \in S^2$, choose a plane P containing u. Since $K \cap P$ is a unit disk in P, we have $\sup_{x \in K \cap P} u \cdot x = 1$. The support function $h_K(u) = \sup_{x \in K} u \cdot x$ then satisfies $h_K(u) = 1$ for all directions $u \in S^2$. A convex body with constant support function is necessarily the Euclidean ball of radius 1 centered at the origin. This provides an independent verification that the state space must be the unit 3-ball.