1 k-regular branching process

A branching process is called *k-regular* if in all generations the multiplicity probability vector is of the form $p_0 = q, \quad p_k = p = 1 - q$.

1.1 Write an iterative formula for the population size at generation n given $n_0 = 1$

We notice that in the first generation, we either die off or have k progeny. Thus, the PGF for the population size is:

$$H_1(x) = q + px^k$$

Therefore, the probability to have a population of size m at generation n is:

$$\begin{aligned} a_{m,n} &= \mathbb{P}\left[\text{population of size m at gen n} \mid \text{k births}\right] p + \mathbb{P}\left[\text{population of size m at gen n} \mid \text{death}\right] q \\ &= p \sum_{\sum k_i = m} \prod_{i=1}^{k} a_{k_i, n-1} + q\delta_{m0} \end{aligned}$$

We multiply this by x^m and sum over m to get:

$$\begin{aligned} H_n(x) &= p \sum_{m=0}^{\infty} \sum_{\sum k_i = m} \prod_{i=1}^{k} a_{k_i, n-1} x^{k_i} + q \\ &= p \prod_{i=1}^{k} \sum_{m=0}^{\infty} \sum_{\sum k_i = m} a_{k_i, n-1} x^{k_i} + q \end{aligned}$$

And if we change the order of summation:

$$H_n(x) = pH_{n-1}^k(x) + q = Q(H_{n-1}(x))$$

where $Q(x) = px^k + q$ is the PGF of the multiplicity vector, as we have seen for the general case in class.

1.2 Write the criticality condition in terms of k, p:

If we look at the first moment of $H_n(x)$, we get:

$$M_1(n) = H_n'(1) = pkH_{n-1}^{k-1}(1)\, H_{n-1}'(1) = pkM_1(n-1)$$

And therefore we have exponential growth of the expected value for $pk > 1$, exponential decay for $pk < 1$ and a critical condition at $pk = 1$. Thus, the criticality condition depends on ordering of $z = pk$ and 1,

2 A concrete case

Let a branching process with multiplicity vector $p_0 = 0.2, p_1 = 0.3, p_2 = 0.2, p_3 = 0.3$, and let the initial population be $n_0 = 1$.

In this equation, I use *SymPy*, a symbolic math package for Python. The code is fairly simple, and appears in Listing 1.

2.1 Write the PGF for the population size for generations $n = 1, 2, 3$

We know that the general case of the PGF is given by:

$$H_n(x) = H_{n-1}(Q(x)),\, H_0(x) = x$$

Since the initial population is 1 with probability 1 ($n_0 = 1$), and where $Q(x) = 0.2 + 0.3x + 0.2x^2 + 0.3x^3$. Thus:

$$\begin{aligned} &H_1(x) = H_0(Q(x)) = Q(x) = 0.2 + 0.3x + 0.2x^2 + 0.3x^3 \\ &H_2(x) = H_1(Q(x)) = Q(Q(x)) = 0.2 + 0.3Q(x) + 0.2Q^2(x) + 0.3Q^3(x) = \\ &\quad = 0.2704 + 0.1248x + 0.1174x^2 + 0.1785x^3 + 0.0998x^4 + 0.0807x^5 + 0.069x^6 + 0.0351x^7 + 0.0162x^8 + 0.0081x^9 \end{aligned}$$

Listing 1: SymPy Program for Question 2

```
import sympy
x = sympy.symbols("x")
def q(x): return 0.2 + 0.3*x+0.2*(x**2)+0.3*(x**3)
h0 = x
h1 = q(x)
h2 = q(h2)
h3 = q(h3)

print(h2.expand())

c3 = h3 ** 10
print(c3.as_poly().coeffs()[-3:])

def i(x): return 0.5 + 0.5 * (x**2)

hi1 = q(x)*i(x)
hi2 = q(q(x))*i(q(x))*i(x)
hi3 = q(q(q(x)))*i(q(q(x)))*i(q(x))*i(x)

print(h1.expand())
```

Where the last line was evaluated using SymPy as seen in Listing 1, because no one wants to do this kind of arithmetic by hand.

Getting H_3 is similar, but I won't write out the 28-term polynomial because that's just tedious.

$$H_3(x) = Q(H_2(x)) = 0.2 + 0.3H_2(x) + 0.2H_2^2(x) + 0.3H_2^3(x) = Q(Q(Q(x)))$$

2.2 Repeat the process for $n_0 = 100$

In this case, we mark the PGF for this initial population at generation n as C_n. $C_n(x) = H_n^{100}(x)$, because it is the sum of the independent population contributions of the 100 initial individuals, and the PGFs of two independent variables follow the rule of $H_{X+Y}(x) = H_X(x)H_Y(x)$.

Thus:

$$\begin{aligned}
C_0(x) &= H_0^{100}(x) = x^{100} \\
C_1(x) &= H_1^{100}(x) = Q^{100}(x) = \left(0.2 + 0.3x + 0.2x^2 + 0.3x^3\right)^{100} \\
C_2(x) &= H_2^{100}(x) = \left(0.2 + 0.3Q(x) + 0.2Q^2(x) + 0.3Q^3(x)\right)^{100} \\
C_3(x) &= H_3^{100}(x) = [Q(Q(Q(x)))]^{100}
\end{aligned}$$

And I am not going to expand these terms, obviously.

2.3 Find $\mathbb{P}[Z_3 = i]$ for $i \in \{0, 1, 2\}$, given $n_0 = 10$

Similarly to before, if we mark C_n as the PGF now, we get:

$$\begin{aligned}
C_0(x) &= H_0^{10}(x) = x^{10} \\
C_1(x) &= H_1^{10}(x) = Q^{10}(x) = \left(0.2 + 0.3x + 0.2x^2 + 0.3x^3\right)^{10} \\
C_2(x) &= H_2^{10}(x) = \left(0.2 + 0.3Q(x) + 0.2Q^2(x) + 0.3Q^3(x)\right)^{10} \\
C_3(x) &= H_3^{10}(x) = [Q(Q(Q(x)))]^{10}
\end{aligned}$$

Expanding this by hand is gruesome, but computers can do anything these days. This is done by printing the coefficients of the resulting polynomial, see Listing 1.

Plugging it in, we get:

$$\mathbb{P}\left[Z_3=0\right]\approx 0.301674415$$
$$\mathbb{P}\left[Z_3=1\right]\approx 0.059150775$$
$$\mathbb{P}\left[Z_3=2\right]\approx 0.062548787$$

2.4 Repeat subsection 1-2 with an immigration law of $p_{I,0}=0.5, P_{I,2}=0.5$

In this case, the PGFs follow the rule we've seen in class, that:

$$G_n\left(x\right)=G_{n-1}\left(Q\left(x\right)\right)I\left(x\right)$$

where I is the PGF for the immigration, and in our case $I\left(x\right)=0.5+0.5x^2$. We denote the PGF for this case with G to differentiate it from the case of no immigration.

$$\begin{aligned}
&G_0\left(x\right)=x\\
&G_1\left(x\right)=Q\left(x\right)I\left(x\right)=\left(0.2+0.3x+0.2x^2+0.3x^3\right)\left(0.5+0.5x^2\right)=0.1+0.15x+0.2x^2+0.3x^3+0.1x^4+0.15x^5\\
&G_2\left(x\right)=G_1\left(Q\left(x\right)\right)I\left(x\right)=Q\left(Q\left(x\right)\right)I\left(Q\left(x\right)\right)I\left(x\right)\\
&G_3\left(x\right)=G_2\left(Q\left(x\right)\right)I\left(x\right)=Q\left(Q\left(Q\left(x\right)\right)\right)I\left(Q\left(Q\left(x\right)\right)\right)I\left(Q\left(x\right)\right)I\left(x\right)
\end{aligned}$$

For $n_0=100$ things do differ slightly too. We can't just raise H^{100}, because we would be considering each individual's population as if it had immigration law I, but only the total population follows that rule.

In essence, we could say that we have a population of 99 individuals with no immigration, and one population with $n_0=1$ that does have immigration, and we count any immigration as part of that subpopulation. Then, the total PGF is just that of the contribution of both of these subpopulations, and thus its PGF is a multiplication of them.

Therefore, in general, if we denote the PGF of the population with $n_0=100$ as D_n:

$$D_n\left(x\right)=H_n^{99}\left(x\right)G_n\left(x\right)$$

Thus, we can say:

$$\begin{aligned}
&D_0\left(x\right)=H_0^{99}\left(x\right)G_0\left(x\right)=x^{99}x=x^{100}\\
&D_1\left(x\right)=H_1^{99}\left(x\right)G_1\left(x\right)=Q^{99}\left(x\right)Q\left(x\right)I\left(x\right)=Q^{100}\left(x\right)I\left(x\right)\\
&D_2\left(x\right)=H_2^{99}\left(x\right)G_2\left(x\right)=\left(Q\left(Q\left(x\right)\right)\right)^{99}\left(Q\left(Q\left(x\right)\right)I\left(Q\left(x\right)\right)I\left(x\right)\right)\\
&D_3\left(x\right)=H_3^{99}\left(x\right)G_3\left(x\right)=\left(Q\left(Q\left(Q\left(x\right)\right)\right)\right)^{99}\left(Q\left(Q\left(Q\left(x\right)\right)\right)I\left(Q\left(Q\left(x\right)\right)\right)I\left(Q\left(x\right)\right)I\left(x\right)\right)
\end{aligned}$$

Which is not quite friendly on the eyes even without expanding it with the actual coefficients.

3 Criticality of branching processes of a particular form

Given a branching process with multiplicity vector $p_0=0.5, p_1=0.3, p_k=0.2$.

For which values of k is the system subcritical, critical and supercritical?

Criticality is measured by finding the relation between the expected multiplication $\mathbb{E}\left[\nu\right]$ and 1. Thus:

$$\mathbb{E}\left[\nu\right]=0.3+0.2k$$

This is critical when $0.3+0.2k=1$, thus when $k=\frac{0.7}{0.2}=0.7\cdot 5=3.5$. Since that isn't possible, the system is subcritical for $k=2,3$ and supercritical for all $k>3$.

4 Extinction in a critical system

Prove that for any critical or subcritical branching process without immigration.

$$\lim_{n\to\infty}\mathbb{P}\left[Z_n=0\right]=1$$

We know that if Q is the PGF of the multiplicity vector, the PGF for Z_n, denoted H_n, follows:

$$H_{n+1}\left(x\right)=Q\left(H_n\left(x\right)\right)$$

and that:

$$\mathbb{P}\left[Z_n = 0\right] = H_n\left(0\right)$$

Thus:

$$\mathbb{P}\left[Z_n = 0\right] = H_n\left(0\right) = Q\left(H_{n-1}\left(0\right)\right)$$

And thus we get a recursive relation of the form:

$$p_n = Q\left(p_{n-1}\right)$$

The limit exists because p_n is monotonically increasing and bounded at 1, and therefore we just want to look at the roots of $f\left(x\right) = Q\left(x\right) - x$, as we already mentioned in class.

What we need to finish the proof is to show that if the system isn't supercritical, there is no root for f in $[0,1)$.

Notice that $f\left(0\right) = Q\left(0\right) > 0$. Otherwise death is **impossible** and the system is supercritical or deterministically staying at n_0, in contradiction to the case in point.

We look at $\frac{df}{dx}$, which exists because it is analytic.

$$\frac{df}{dx}\left(x\right) = \frac{dQ}{dx}\left(x\right) - 1 = \sum_{n=1}^{\infty} p_n n x^{n-1} - 1 \leq \sum_{n=1}^{\infty} p_n n - 1 = \overline{\nu} - 1 \leq 0$$

Where equality is only possible in the second to last step for $x \in [0,1)$ if $p_0 = 1$, in which case obviously, the system dies immediately in generation 1 with probability 1. The last inequality holds because the system isn't supercritical.

Thus, for all cases of interest, $\frac{df}{dx}\left(x\right) < 0$, so the function is strictly monotonically decreasing in this range, and can therefore only have 1 root at most in $[0,1]$. But we know from class, and from the fact that Q is a PGF, that $f\left(1\right) = Q\left(1\right) - 1 = 1 - 1 = 0$, so it must be **the only root** in $[0,1]$, and therefore there are no roots in $[0,1)$. Since as mentioned, $\lim_{n\to\infty} \mathbb{P}\left[Z_n = 0\right]$ must be a root of f, we have $\lim_{n\to\infty} \mathbb{P}\left[Z_n = 0\right] = 1$, completing the proof.

5 Subcritical branching process with immigration at long times

Let $k \in \mathbb{N}$. Let a subcritical branching process with immigration, where the PGF for immigration is of the form $p_{i,k} = \frac{1}{k}, p_{i,0} = 1 - \frac{1}{k}$, meaning its PGF is of the form $I\left(x\right) = 1 - \frac{1}{k} + \frac{1}{k}x^k$.

5.1 Prove that $\lim_{n\to\infty} \mathbb{E}\left[Z_n\right]$ does not depend on k, but that $\lim_{n\to\infty} V\left[Z_n\right]$ does.

We've seen in class the terms for the first moment, but I had to derive things to go for M_2, so I included it here as well.

We assume $Z_0 = 1$ for now, because it makes things a lot simpler.

We also notice that $\overline{\nu_i} = \frac{dI}{dx}\left(1\right) = 1$.

We know that the PGF follows a recursion relation of the form:

$$G_n\left(x\right) = G_{n-1}\left(Q\left(x\right)\right) I\left(x\right)$$

We derive to get:

$$\frac{dG_n}{dx}\left(x\right) = \frac{dG_{n-1}}{dx}\left(Q\left(x\right)\right)\frac{dQ}{dx}\left(x\right) I\left(x\right) + G_{n-1}\left(Q\left(x\right)\right)\frac{dI}{dx}\left(x\right)$$

We set $x = 1$ and get:

$$M_1\left(n\right) = M_1\left(n-1\right)\overline{\nu} + \overline{\nu_i}$$

The solution to which, when $\bar{\nu} < 1$ is clearly, if we solve the homogeneous part and guess a constant particular solution:

$$M_1\left(n\right) = \left(1 - \frac{\overline{\nu_i}}{1-\overline{\nu}}\right)\overline{\nu}^n + \frac{\overline{\nu_i}}{1-\overline{\nu}}$$

The limit of which is of course:

$$\lim_{n\to\infty} M_1\left(n\right) = \frac{\overline{\nu_i}}{1-\overline{\nu}} = \frac{1}{1-\overline{\nu}}$$

Which does not depend on k.

We derive G again for the second moment equation:

$$\begin{aligned}\frac{d^2G_n}{dx^2}(x) &= \frac{d^2G_{n-1}}{dx^2}(Q(x))\left(\frac{dQ}{dx}(x)\right)^2 I(x)\\ &+ \frac{dG_{n-1}}{dx}(Q(x))\frac{d^2Q}{dx^2}(x)\,I(x)\\ &+ 2\frac{dG_{n-1}}{dx}(Q(x))\frac{dQ}{dx}(x)\frac{dI}{dx}(x)\\ &+ G_{n-1}(Q(x))\frac{d^2I}{dx^2}\end{aligned}$$

And we set $x = 1$ to get:

$$F_2(n) = F_2(n-1)\overline{\nu}^2 + M_1(n-1)\frac{d^2Q}{dx^2}(1) + 2M_1(n-1)\overline{\nu}\overline{\nu_i} + (k-1)$$

The homogeneous solution is of the form $H\overline{\nu}^{2n}$, so it will die out for $n \to \infty$. We pick a particular solution of the form $F_p(n) = A\overline{\nu}^n + B$. We get a relation of the form:

$$A\overline{\nu}^n + B = A\overline{\nu}^n\overline{\nu} + B\overline{\nu}^2 + C_1\overline{\nu}^n + C_2 + (k-1)$$

For some C_1, C_2 that do not depend on neither n nor k, because they depend only on the properties of the PGF Q of the multiplicity (and $\overline{\nu_i} = 1$ for all k).

We can solve $A = \frac{C_1}{1-\overline{\nu}}, B = \frac{C_2+(k-1)}{1-\overline{\nu}^2}$,which means that this is the right form of the particular solution.

Thus, we can see the limit of $F_2(n)$ does depend on k:

$$\lim_{n\to\infty} F_2(n) = B = \frac{C_2 + (k-1)}{1-\overline{\nu}^2}$$

And since:

$$V(n) = \mathbb{E}\left[Z_n^2\right] - \mathbb{E}^2[Z_n] = \mathbb{E}\left[Z_n^2 - Z_n\right] + \mathbb{E}[Z_n] - \mathbb{E}^2[Z_n] = F_2(n) + M_1(n) - M_1^2(n)$$

And we can take the limit:

$$\lim_{n\to\infty} V(n) = \frac{C_2 + (k-1)}{1-\overline{\nu}^2} + \frac{1}{1-\overline{\nu}} - \left(\frac{1}{1-\overline{\nu}}\right)^2$$

Which clearly does depend on k, finishing the proof.

5.2 Suggest a statistical method to differentiate systems with different k values

We can run many (N) copies of each system until a sufficiently large time such that $H\overline{\nu}^{2n} \ll A\overline{\nu}^n \ll B$, and therefore we would expect $V(n) \approx \lim_{n\to\infty} V(n)$.

Then, we can just take the sample variance of the population sizes of both systems, and perform an *F-test*, which uses the statistic of the ratio of the two sample variances and checks it against an F distribution. This test is used to test that two samples come from a distribution with the same variance.

However, this test is sensitive to non-normality (though for large enough populations, normality is fair, as we'll see when we get to diffusion approximations). If we are concerned with non-normality, we can instead use a *Levene's Test*, which should be good enough for a distribution that is mostly symmetric and without too much of a tail (i.e., not too far from normal).