# Chapter 2
# Laplace Transform

# Introduction to the Laplace Transform

- Defnition of Laplace Transform
- Properties of Laplace Transform
- The Inverse Laplace Transform
- Application to Integro-differential Equations
- Inverse Laplace Trabsform

# Definition of Laplace Transform

$$\boxed{\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt}$$

# Definition of Laplace Transform

a) Unit step, *u(t)*

$$\boxed{L\left[u(t)\right] = F(s) = \int_0^\infty 1 e^{-st}\,dt = \frac{1}{s}}$$

# Definition of Laplace Transform

b) Exponential function, $e^{-\alpha t}u(t), \alpha > 0$

$$\boxed{L[u(t)] = F(s) = \int_0^{\infty} e^{\alpha t} e^{-st}\, dt = \frac{1}{s+\alpha}}$$

# Definition of Laplace Transform

c) Impulse function, *δ(t)*

$$L[u(t)] = F(s) = \int_0^\infty \delta(t) e^{-st} dt = 1$$

| $f(t)$ | $F(s)$ |
|---|---|
| $\delta(t)$ | $1$ |
| $u(t)$ | $\frac{1}{s}$ |
| $e^{-at}$ | $\frac{1}{s+a}$ |
| $t$ | $\frac{1}{s^2}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |
| $te^{-at}$ | $\frac{1}{(s+a)^2}$ |
| $t^n e^{-at}$ | $\frac{n!}{(s+a)^{n+1}}$ |
| $\sin \omega t$ | $\frac{\omega}{s^2+\omega^2}$ |
| $\cos \omega t$ | $\frac{s}{s^2+\omega^2}$ |
| $\sin(\omega t+\theta)$ | $\frac{s \sin\theta + \omega \cos\theta}{s^2+\omega^2}$ |
| $\cos(\omega t+\theta)$ | $\frac{s \cos\theta - \omega \sin\theta}{s^2+\omega^2}$ |
| $e^{-at} \sin \omega t$ | $\frac{\omega}{(s+a)^2+\omega^2}$ |
| $e^{-at} \cos \omega t$ | $\frac{s+a}{(s+a)^2+\omega^2}$ |

# Properties of Laplace Transform

**Linearity:**

If $F_1(s)$ and $F_2(s)$ are, respectively, the Laplace Transforms of $f_1(t)$ and $f_2(t)$

$$L\left[a_1 f_1(t) + a_2 f_2(t)\right] = a_1 F_1(s) + a_2 F_2(s)$$

Example:

$$L\left[\cos(\omega t)u(t)\right] = L\left[\frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)u(t)\right] = \frac{s}{s^2 + \omega^2}$$

# Properties of Laplace Transform

**Scaling:**

If *F (s)* is the Laplace Transforms of *f (t)*, then

$$\boxed{L\left[f(at)\right] = \frac{1}{a}F\left(\frac{s}{a}\right)}$$

Example:

$$\boxed{L\left[\sin(2\omega t)u(t)\right] = \frac{2\omega}{s^2 + 4\omega^2}}$$

# Properties of Laplace Transform

**Time Shift:**

If $F(s)$ is the Laplace Transforms of $f(t)$, then

$$\boxed{L\left[f(t-a)u(t-a)\right] = e^{-as}F(s)}$$

Example:

$$\boxed{L\left[\cos(\omega(t-a))u(t-a)\right] = e^{-as}\frac{s}{s^2+\omega^2}}$$

# Properties of Laplace Transform

**Frequency Shift:**

If $F(s)$ is the Laplace Transforms of $f(t)$, then

$$\boxed{L\left[e^{-at} f(t)u(t)\right] = F(s+a)}$$

Example:

$$\boxed{L\left[e^{-at}\cos(\omega t)u(t)\right] = \frac{s+a}{(s+a)^2+\omega^2}}$$

# Properties of Laplace Transform

**Time Differentiation:**

If $F(s)$ is the Laplace Transforms of $f(t)$, then the Laplace Transform of its derivative is

$$\boxed{L\left[\frac{df}{dt}u(t)\right] = sF(s) - f(0^-)}$$

Example:

$$\boxed{L[\sin(\omega t)u(t)] = \frac{\omega}{s^2+\omega^2}}$$

# Properties of Laplace Transform

**Time Integration:**

If $F(s)$ is the Laplace Transforms of $f(t)$, then the Laplace Transform of its integral is

$$\boxed{L\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)}$$

Example:

$$\boxed{L\left[t^n\right] = \frac{n!}{s^{n+1}}}$$

# Properties of Laplace Transform

**Frequency Differentiation:**

If *F(s)* is the Laplace Transforms of *f (t)*, then the derivative with respect to s, is

$$\boxed{L[tf(t)] = -\frac{dF(s)}{ds}}$$

Example:

$$\boxed{L\left[te^{-at}u(t)\right] = \frac{1}{(s+a)^2}}$$

# Properties of Laplace Transform

**Initial and Final Values:**

The initial-value and final-value properties allow us to find the initial value *f(0)* and *f(∞)* of *f(t)* directly from its Laplace transform *F(s)*.

$$\boxed{f(0) = \lim_{s\to\infty} sF(s)}$$

Initial-value theorem

$$\boxed{f(\infty) = \lim_{s\to 0} sF(s)}$$

Final-value theorem

| Property | $f(t)$ | $F(s)$ |
| --- | --- | --- |
| Linearity | $a_1 f_1(t) + a_2 f_2(t)$ | $a_1 F_1(s) + a_2 F_2(s)$ |
| Scaling | $f(at)$ | $\frac{1}{a} F\left(\frac{s}{a}\right)$ |
| Time shift | $f(t-a)u(t-a)$ | $e^{-as} F(s)$ |
| Frequency shift | $e^{-at} f(t)$ | $F(s+a)$ |
| Time differentiation | $\frac{df}{dt}$ | $sF(s) - f(0^-)$ |
| | $\frac{d^2 f}{dt^2}$ | $s^2 F(s) - sf(0^-) - f'(0^-)$ |
| | $\frac{d^3 f}{dt^3}$ | $s^3 F(s) - s^2 f(0^-) - sf'(0^-) - f''(0^-)$ |
| | $\frac{d^n f}{dt^n}$ | $s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \cdots - f^{(n-1)}(0^-)$ |
| Time integration | $\int_0^t f(t)\, dt$ | $\frac{1}{s} F(s)$ |
| Frequency differentiation | $tf(t)$ | $-\frac{d}{ds} F(s)$ |
| Frequency integration | $\frac{f(t)}{t}$ | $\int_s^\infty F(s)\, ds$ |
| Time periodicity | $f(t) = f(t+nT)$ | $\frac{F_1(s)}{1-e^{-sT}}$ |
| Initial value | $f(0^+)$ | $\lim_{s\to\infty} sF(s)$ |
| Final value | $f(\infty)$ | $\lim_{s\to 0} sF(s)$ |
| Convolution | $f_1(t) * f_1(t)$ | $F_1(s)F_2(s)$ |

# The Inverse Laplace Transform

Suppose F(s) has the general form of

$$F(s) = \frac{N(s)\text{......numerator polynomial}}{D(s)\text{...denominator polynomial}}$$

The finding the inverse Laplace transform of *F(s)* involves two steps:

1. Decompose F(s) into simple terms using partial fraction expansion.
2. Find the inverse of each term by matching entries in Laplace Transform Table.

# The Inverse Laplace Transform

Example

Find the inverse Laplace transform of

$$F(s) = \frac{3}{s} - \frac{5}{s+1} + \frac{6}{s^2+4}$$

Solution:

$$\boxed{\begin{aligned} f(t) &= L^{-1}\left(\frac{3}{s}\right) - L^{-1}\left(\frac{5}{s+1}\right) + L^{-1}\left(\frac{6}{s^2+4}\right) \\ &= (3 - 5e^{-t} + 3\sin(2t)u(t), \quad \mathrm{t} \geq 0 \end{aligned}}$$

# Application to Integro-differential Equation

- The Laplace transform is useful in solving linear integro-differential equations.
- Each term in the integro-differential equation is transformed into s-domain.
- Initial conditions are automatically taken into account.
- The resulting algebraic equation in the s-domain can then be solved easily.
- The solution is then converted back to time domain.

# Application to Integro-differential Equation

Example 3:

Use the Laplace transform to solve the differential equation

$$\frac{d^2v(t)}{dt^2}+6\frac{dv(t)}{dt}+8v(t)=2u(t)$$

Given: $v(0) = 1$; $v'(0) = -2$

# Application to Integro-differential Equation

Solution:

Taking the Laplace transform of each term in the given differential equation and obtain

$$\left[s^2V(s) - sv(0) - v'(0)\right] + 6\left[sV(s) - v(0)\right] + 8V(s) = \frac{2}{s}$$

Substituting $v(0) = 1; v'(0) = -2$, we have

$$(s^2 + 6s + 8)V(s) = s + 4 + \frac{2}{s} = \frac{s^2 + 4s + 2}{s} \Rightarrow V(s) = \frac{\frac{1}{4}}{s} + \frac{\frac{1}{2}}{s+2} + \frac{\frac{1}{4}}{s+4}$$

By the inverse Laplace Transform,

$$v(t) = \frac{1}{4}(1 + 2e^{-2t} + e^{-4t})u(t)$$

# Applications of the Laplace Transform

# Application of the Laplace Transform

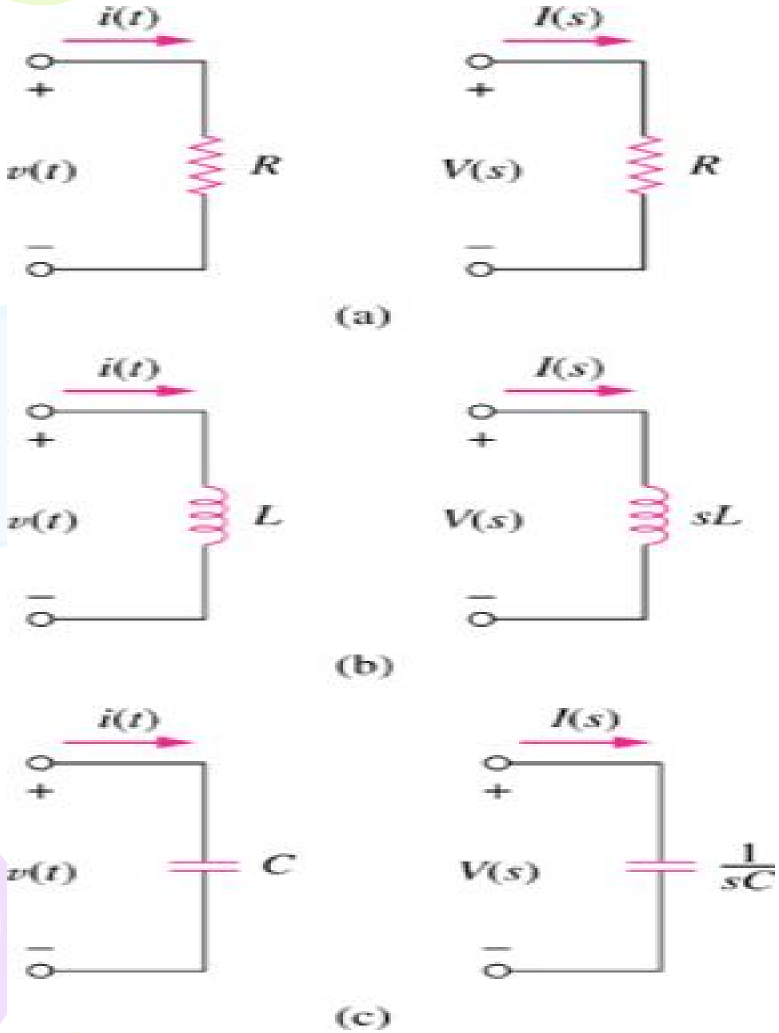
- Circuit Element Models
- Circuit Analysis
- Transfer Functions
- State Variables

# Circuit Element Models

Steps in Applying the Laplace Transform:

1. Transform the circuit from the time domain to the s-domain
2. Solve the circuit using nodal analysis, mesh analysis, source transformation, superposition, or any circuit analysis technique with which we are familiar
3. Take the inverse transform of the solution and thus obtain the solution in the time domain.

# Circuit Element Models




**Time-domain and s-domain representations of passive elements under zero initial conditions.**

Assume zero initial condition for the inductor and capacitor,

Resistor : $V(s)=RI(s)$

Inductor: $V(s)=sLI(s)$

Capacitor: $V(s) = I(s)/sC$

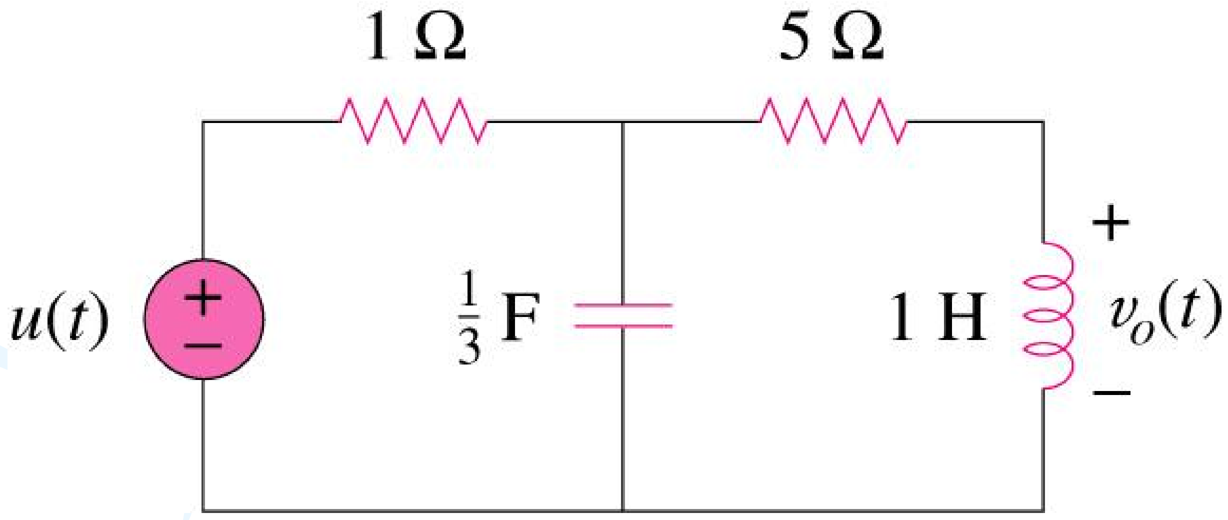
The impedance in the s-domain is defined as $Z(s) = V(s)/I(s)$

The admittance in the s-domain is defined as $Y(s) = I(s)/V(s)$

# Circuit Element Models

Example :

Find $v_0(t)$ in the circuit shown below, assuming zero initial conditions.



# Circuit Element Models

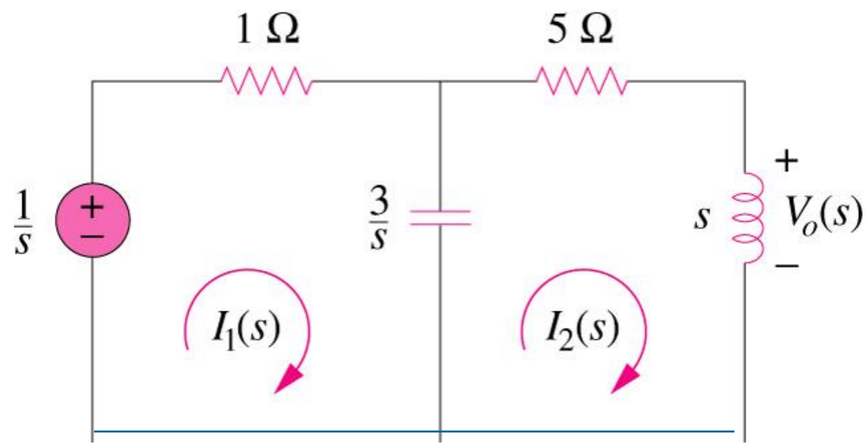
Solution:

Transform the circuit from the time domain to the s-domain, we have

$$u(t) \quad \Rightarrow \quad \frac{1}{s}$$

$$1\,\mathrm{H} \quad \Rightarrow \quad sL = s$$

$$\frac{1}{3}\mathrm{F} \quad \Rightarrow \quad \frac{1}{sC} = \frac{3}{s}$$

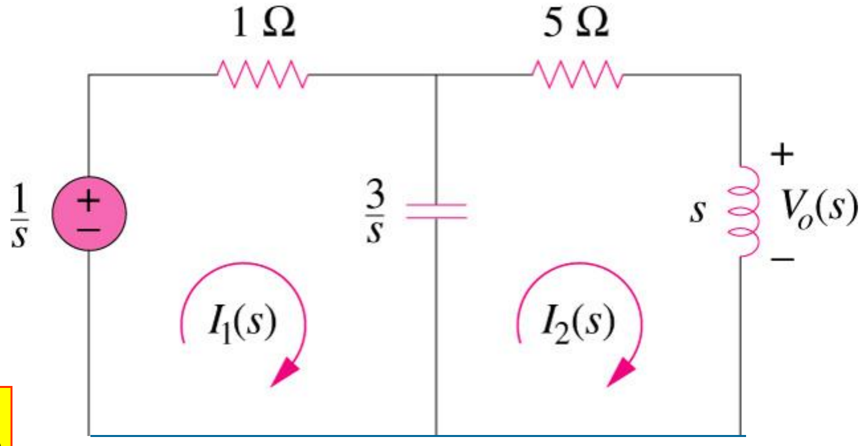


# Circuit Element Models

Solution:

Apply mesh analysis, on solving for $V_0(s)$

$$V_0(s) = \frac{3}{\sqrt{2}} \frac{\sqrt{2}}{(s+4)^2 + (\sqrt{2})^2}$$




Taking the inverse transform give

$$v_0(t) = \frac{3}{\sqrt{2}} e^{-4t} \sin(\sqrt{2}t) \text{ V}, \quad t \geq 0$$

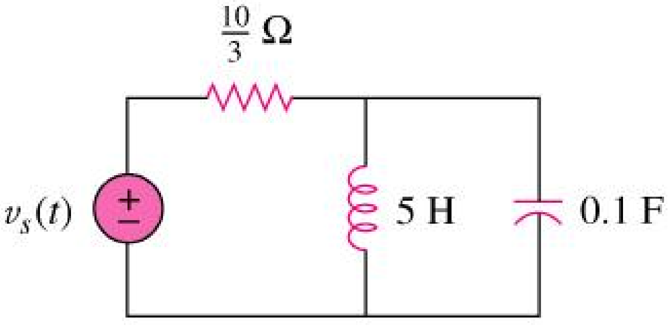
# Circuit Analysis

- Circuit analysis is relatively easy to do in the s-domain.
- By transforming a complicated set of mathematical relationships in the time domain into the s-domain where we convert operators (derivatives and integrals) into simple multipliers of s and 1/s.
- This allow us to use algebra to set up and solve the circuit equations.
- In this case, all the circuit theorems and relationships developed for dc circuits are perfectly valid in the s-domain.

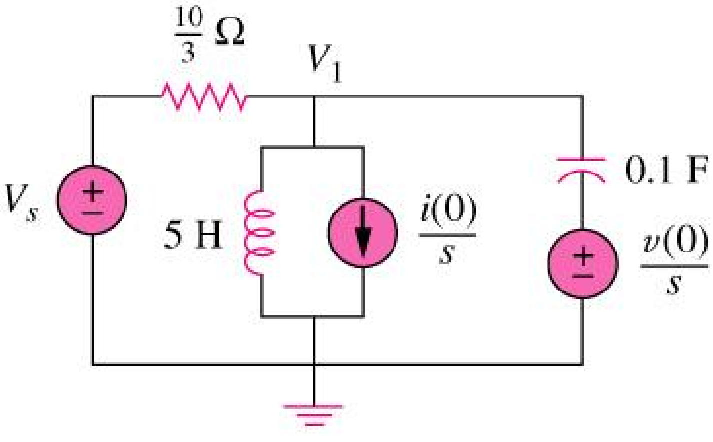
# Circuit Analysis

Example :

Consider the circuit below. Find the value of the voltage across the capacitor assuming that the value of $v_s(t)=10u(t)$ V and assume that at t=0, -1A flows through the inductor and +5 is across the capacitor.




(a)




(b)

# Circuit Analysis

Solution:

Transform the circuit from time-domain (a) into s-domain (b) using Laplace Transform. On rearranging the terms, we have

$$\boxed{V_1 = \frac{35}{s+1} - \frac{30}{s+2}}$$

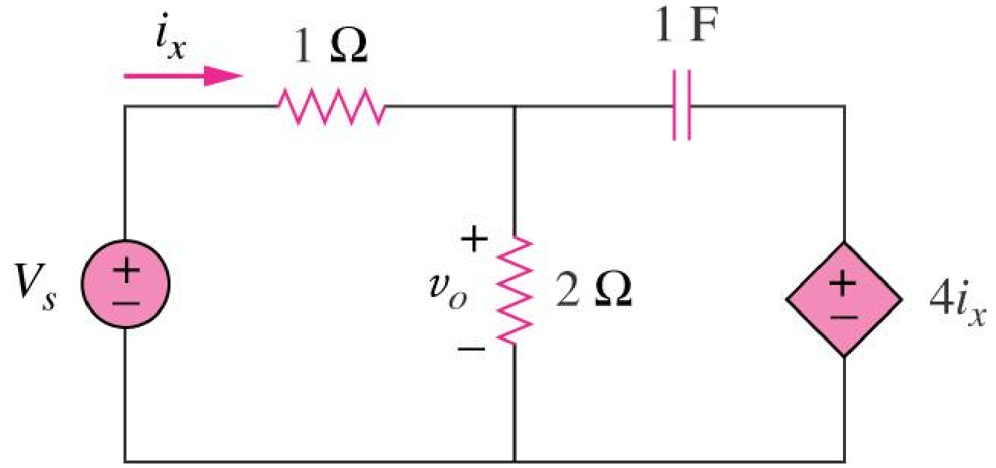
By taking the inverse transform, we get

$$\boxed{v_1(t) = (35e^{-t} - 30e^{-2t})u(t) \quad \text{V}}$$

# Circuit Analysis

## Example :

The initial energy in the circuit below is zero at $t=0$. Assume that $v_s=5u(t)$ V. (a) Find V0(s) using the thevenin theorem. (b) Apply the initial- and final-value theorem to find $v_0(0)$ and $v_0(\infty)$. (c) Obtain $v_0(t)$.




*Ans: (a) $V_0(s) = 4(s+0.25)/(s(s+0.3))$ (b) 4,3.333V, (c) (3.333+0.6667e-0.3t)u(t) V.*

*Refer to in-class illustration, textbook

# Transfer Functions

- The transfer function H(s) is the ratio of the output response Y(s) to the input response X(s), assuming all the initial conditions are zero.

$$\boxed{H(s) = \frac{Y(s)}{X(s)}}$$

**, h(t) is the impulse response function**.

- Four types of gain:

1. $H(s)$ = voltage gain = $V_0(s)/V_i(s)$
2. $H(s)$ = Current gain = $I_0(s)/I_i(s)$
3. $H(s)$ = Impedance = $V(s)/I(s)$
4. $H(s)$ = Admittance = $I(s)/V(s)$

# Transfer Function

## Example :

The output of a linear system is $y(t)=10e^{-t}\cos 4t$ when the input is $x(t)=e^{-t}u(t)$. Find the transfer function of the system and its impulse response.

## Solution:

Transform y(t) and x(t) into s-domain and apply H(s)=Y(s)/X(s), we get

$$\boxed{H(s)=\frac{Y(s)}{X(s)}=\frac{10(s+1)^2}{(s+1)^2+16}=10-40\frac{4}{(s+1)^2+16}}$$

Apply inverse transform for H(s), we get

$$\boxed{h(t)=10\delta(t)-40e^{-t}\sin(4t)u(t)}$$

# Transfer Function

Example 8:

The transfer function of a linear system is

$$H(s) = \frac{2s}{s+6}$$

Find the output y(t) due to the input e-3tu(t) and its impulse response.

$$\text{Ans}: \ -2e^{-3t} + 4e^{-6t}, \ \text{t} \geq 0; \ 2\delta(\text{t}) - 12\text{e}^{-6\text{t}}u(t)$$

*Refer to in-class illustration, textbook