Chapter 3
Convolution

The Convolution Integral

- It is used in finding the response y(t) of a system to an excitation x(t), knowing the system impulse response h(t).
- It is defined as

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda \text{ or } y(t) = x(t) * h(t)$$

- It consists of time-reversing one of the signals, shifting it, and multiplying it point by point with the second signal, and integrating the product.

Properties of convolution integral

1. $x(t) * h(t) = h(t) * x(t)$ (Commutative)
2. $f(t) * [x(t) + y(t)] = f(t) * x(t) + f(t) * y(t)$ (Distributive)
3. $f(t) * [x(t) * y(t)] = [f(t) * x(t)] * y(t)$ (Associative)
4. $f(t) * \delta(t) = \int_{-\infty}^{\infty} f(\lambda)\delta(t-\lambda)\, d\lambda = f(t)$
5. $f(t) * \delta(t - t_o) = f(t - t_o)$
6. $f(t) * \delta'(t) = \int_{-\infty}^{\infty} f(\lambda)\delta'(t-\lambda)\, d\lambda = f'(t)$
7. $f(t) * u(t) = \int_{-\infty}^{\infty} f(\lambda)u(t-\lambda)\, d\lambda = \int_{-\infty}^{t} f(\lambda)\, d\lambda$

Relation with Laplace Transform

- It is defined as $y(t) = \int_{-\infty}^{\infty} x(\lambda) h(t-\lambda) d\lambda$ or $y(t) = x(t) * h(t)$

- Given two functions, $f_1(t)$ and $f_2(t)$ with Laplace Transforms $F_1(s)$ and $F_2(s)$, respectively

$$\boxed{F_1(s)F_2(s) = L\left[f_1(t) * f_2(t)\right]}$$

- Example: $y(t) = 4e^{-t}$ and $h(t) = 5e^{-2t}$

$$\boxed{h(t) * x(t) = L^{-1}\left[H(s)X(s)\right] = L^{-1}\left[\left(\frac{5}{s+2}\right)\left(\frac{4}{s+1}\right)\right] = 20(e^{-t} - e^{-2t}),\quad \mathrm{t} \geq 0}$$

Graphical Method

1. Folding: Take the mirror image of $h(\lambda)$ about the ordinate axis to obtain $h(-\lambda)$.

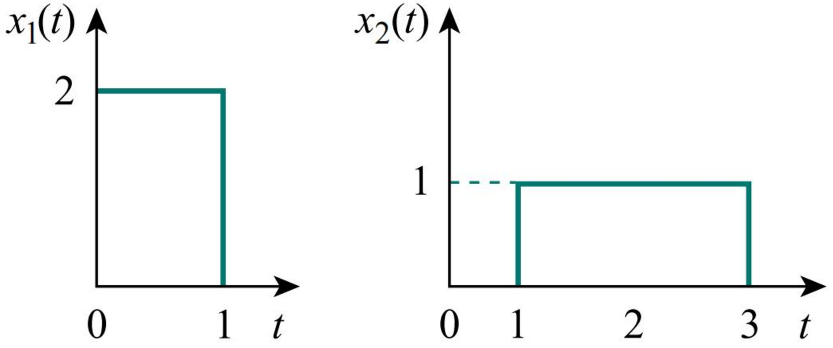
2. Displacement: Shift or delay $h(-\lambda)$ by t to obtain $h(t - \lambda)$.

3. Multiplication: Find the product of $h(t - \lambda)$ and $x(\lambda)$.

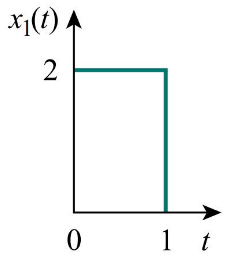
4. Integration: For a given time t, calculate the area under the product $h(t - \lambda)x(\lambda)$ for $0 < \lambda < t$ to get $y(t)$ at t.

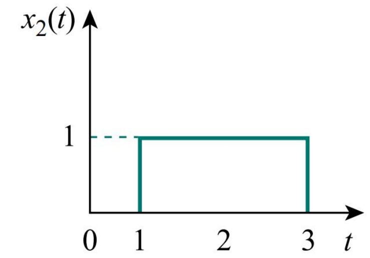
Graphical Method

Example: Find the convolution of the two signals

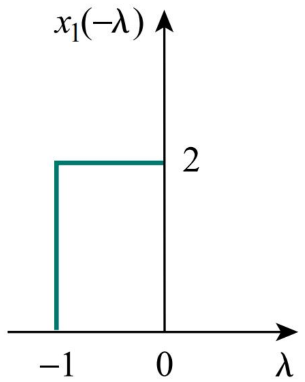


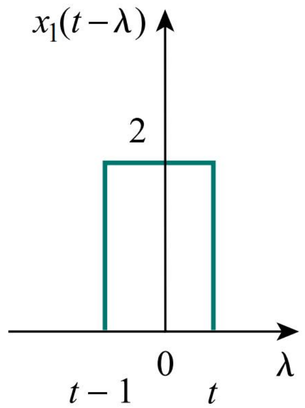
Solution



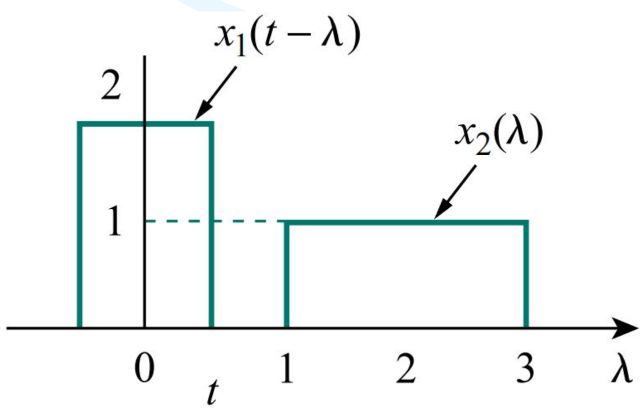


- First, fold x1(t) and shift it by t



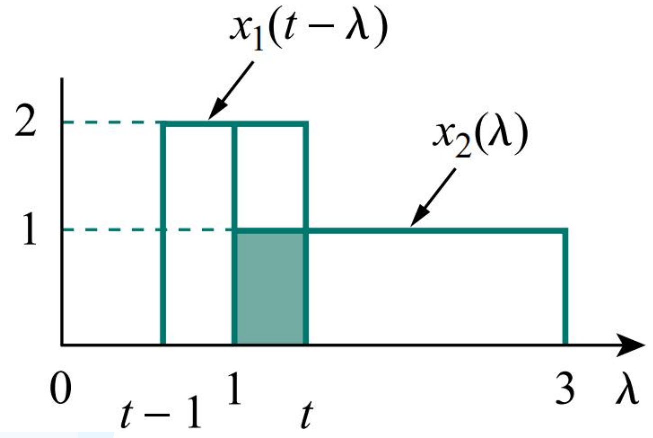


- For 0 <t< 1, there is no overlap



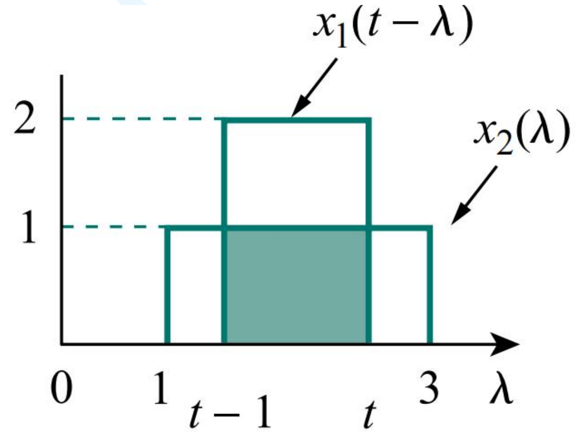
$$y(t) = x_1(t) * x_2(t) = 0, \qquad 0 < t < 1$$

- For 1 <t< 2



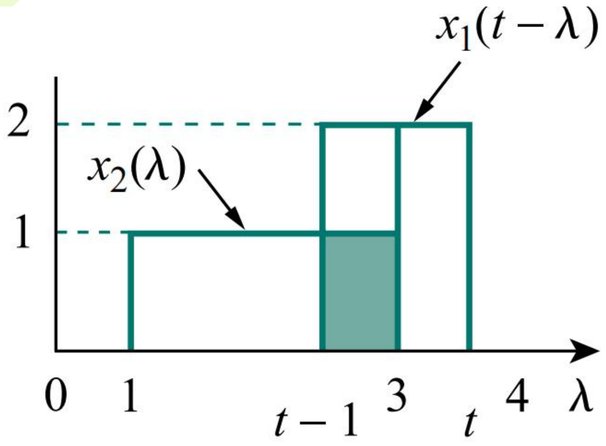
$$y(t) = \int_1^t (2)(1)\, d\lambda = 2\lambda \Big|_1^t = 2(t-1), \qquad 1 < t < 2$$

- For 2 <t< 3



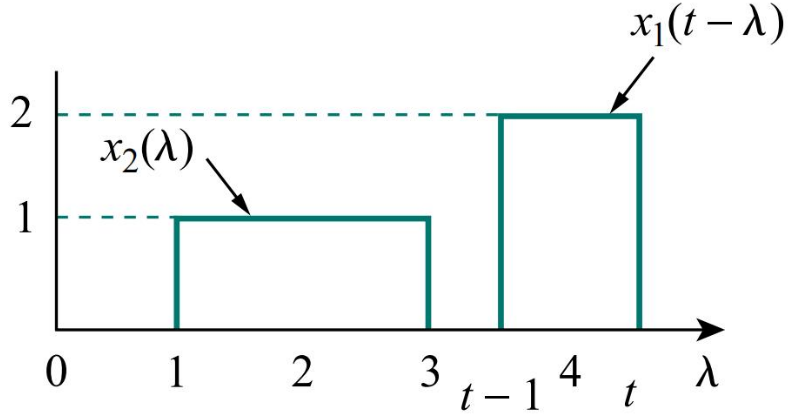
$$y(t) = \int_{t-1}^t (2)(1)\, d\lambda = 2\lambda \Big|_{t-1}^t = 2, \qquad 2 < t < 3$$

- For 3 <t< 4



$$y(t) = \int_{t-1}^{3} (2)(1)\, d\lambda = 2\lambda \Big|_{t-1}^{3}$$
$$= 2(3 - t + 1) = 8 - 2t, \qquad 3 < t < 4$$

- For t > 4



$$y(t) = 0, \qquad t > 4$$

- Combining the results

$$y(t) = \begin{cases} 0, & 0 \le t \le 1 \\ 2t - 2, & 1 \le t \le 2 \\ 2, & 2 \le t \le 3 \\ 8 - 2t, & 3 \le t \le 4 \\ 0, & t \ge 4 \end{cases}$$