1 *problem goes here...*

solution goes here... ■

2 *problem goes here...*

solution goes here... ∎

Example Find the curve $x(t)$ that minimizes the functional

$$J(x) = \int_0^1 \left\{ \frac{1}{2}\dot{x}^2(t) + 3x(t)\dot{x}(t) + 2x^2(t) + 4x(t) \right\} dt$$

and passes through the points $x(0) = 1$ and $x(1) = 4$.

The function, $x^*(t)$, which minimizes the cost above, $J(x)$, must satisfy the following first-order necessary condition:

$$\frac{\partial g(x(t), \dot{x}(t), t)}{\partial x} - \frac{d}{dt}\left(\frac{\partial g(x(t), \dot{x}(t), t)}{\partial \dot{x}}\right) = 0. \tag{1}$$

Where

$$g(x(t), \dot{x}(t), t) = \frac{1}{2}\dot{x}^2(t) + 3x(t)\dot{x}(t) + 2x^2(t) + 4x(t).$$

First, compute the partials of g with respect to x and $\dot{x}$.

$$\frac{\partial g}{\partial x} = g_x = 3\dot{x}(t) + 4x(t) + 4 \tag{2}$$

$$\frac{\partial g}{\partial \dot{x}} = g_{\dot{x}} = \dot{x}(t) + 3x(t) \tag{3}$$

Substituting equations (2) and (3) into (1) results in the following:

$$\begin{aligned} 0 &= 3\dot{x}(t) + 4x(t) + 4 - \frac{d}{dt}\left\{\dot{x}(t) + 3x(t)\right\} \\ &= 3\dot{x}(t) + 4x(t) + 4 - \ddot{x}(t) - 3\dot{x}(t) \end{aligned}$$

Rearranging this equation results in the following second-order, nonhomogeneous ODE:

$$= \ddot{x}(t) - 4x(t) - 4,$$

the solution to which has the form

$$x(t) = c_1 e^{2t} + c_2 e^{-2t} - 1. \tag{4}$$

c_1 and c_2 are unknown constants. We can solve for them using the terminal conditions on x.

$$\begin{aligned} x(0) = 1 &= c_1 e^{2(0)} + c_2 e^{-2(0)} - 1 \\ c_2 &= 2 - c_1 \end{aligned}$$

Substituting this into equation (4) and using the final constraint, we have a solution for both unknown parameters.

$$\begin{aligned} x(1) = 4 &= c_1 e^{2(1)} + (2 - c_1)e^{-2(1)} - 1 \\ 5 &= c_1 e^2 + 2e^{-2} - c_1 e^{-2} \\ 5 - 2e^{-2} &= c_1(e^2 - e^{-2}) \\ c_1 &= \frac{5 - 2e^{-2}}{e^2 - e^{-2}} \approx 0.652 \\ c_2 &= 2 - c1 \approx 1.348 \end{aligned}$$

Thus, the optimal function which minimizes the functional is

$$\boxed{x^*(t) = 0.652e^{2t} + 1.348e^{-2t} - 1}$$

■