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İSTANBUL TEKNİK ÜNİVERSİTESİ 1773

HOMEWORK-2

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TESLIM TARIHI : 24.03.2024

DERS ADI : COMPUTATIONAL FLUID DYNAMICS

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Preparation of the equation

The term on the left side of the equation represents the inertia of the fluid element, while the terms on the right side represent the viscous forces and the pressure forces acting on the fluid element. The unknown in the equation is the velocity of the fluid, u(r, t), which depends on both the radial distance from the center of the pipe (r) and time (t).

The given Navier-Stokes equation is:

$$\rho * \frac{\delta u}{\delta t} = -\frac{\delta P}{\delta x} + \mu * \frac{1}{r} * \frac{\delta}{\delta r}(r * \frac{\delta u}{\delta r})$$

Here, $\frac{\delta}{\delta r}(r * \frac{\delta u}{\delta r})$ is not suitable for the calculation. To make suitable this term, chain rule was used.

$$\frac{\delta}{\delta r}\left(r * \frac{\delta u}{\delta r}\right) = \frac{\delta u}{\delta r} + r * \frac{\delta^2 u}{\delta r^2}$$

If this expression is substituted into the equation, the new equation is:

$$\rho * \frac{\delta u}{\delta t} = -\frac{\delta P}{\delta x} + \mu * \frac{1}{r}(\frac{\delta u}{\delta r} + r * \frac{\delta^2 u}{\delta r^2})$$

If $\frac{\delta u}{\delta t}$ leave alone on the lefts side:

$$\frac{\delta u}{\delta t} = -\frac{1}{\rho} * \frac{\delta P}{\delta x} + \mu * \frac{1}{\rho * r}(\frac{\delta u}{\delta r} + r * \frac{\delta^2 u}{\delta r^2})$$

And,

$$\frac{\delta u}{\delta t} = -\frac{1}{\rho} * \frac{\delta P}{\delta x} + \frac{\mu}{\rho * r} * \frac{\delta u}{\delta r} + \frac{\mu}{\rho} * \frac{\delta^2 u}{\delta r^2})$$

Also, $\upsilon = \frac{\mu}{\rho}$. So,

$$\frac{\delta u}{\delta t} = -\frac{1}{\rho} * \frac{\delta P}{\delta x} + \frac{\upsilon}{r} * \frac{\delta u}{\delta r} + \upsilon * \frac{\delta^2 u}{\delta r^2}$$

Calculating the u_{avg} and u_{max}

$$Re = \frac{\rho * u_{avg} * D}{\mu} \rightarrow 2000 = \frac{1000 * u_{avg} * 0.05}{0.001} \rightarrow u_{avg} = 0.04\ [m/s]$$

$$u_{max} = 2 * u_{avg} \rightarrow u_{max} = 2 * 0.04 = 0.08\ [m/s]$$

Hydrodynamically Fully Developed Region

To calculate the number of seconds after which it passes into the fully developed flow region, this equation is used (for Laminar Flow):

$$\frac{L}{D} = 0.05Re$$

This equation calculates the entry lengths.

$$L_{h_{laminar}} = 0.05 * 2000 * 0.05 = 5[m]$$

After this length, the flow is fully developed. In the fully developed flow region, the velocity profile depends only on the diameter. Also, the wall shear stress also remains constant in that region.

$$t_{required} = \frac{L_{h_{laminar}}}{u_{avg}} = \frac{5}{0.04} = 125\ [s]$$

But after this time, the flow still is not in the full developed area. The time for the flow to pass into the fully developed region was found to be approximately 650 seconds by trial and error.

*An attempt was made to find this time using u_{avg}, but as a result of the calculated time, the flow could not reach its maximum speed.

FTCS Scheme

If the equation is discrete,

$$\frac{\delta u}{\delta t} = \frac{u_i^{j+1} - u_i^j}{\Delta t}$$

$$\frac{\delta u}{\delta r} = \frac{u_{i+1}^j - u_{i-1}^j}{2\Delta r}$$

$$\frac{\delta^2 u}{\delta r^2} = \frac{u_{i+1}^j - 2 * u_i^j + u_{i-1}^j}{(\Delta r)^2}$$

If they are substituted into the equation,

$$\frac{u_i^{j+1} - u_i^j}{\Delta t} = -\frac{1}{\rho} * \frac{\delta P}{\delta x} + \frac{\upsilon}{r} * \frac{u_{i+1}^j - u_{i-1}^j}{2\Delta r} + \upsilon * \frac{u_{i+1}^j - 2 * u_i^j + u_{i-1}^j}{(\Delta r)^2}$$

If u_i^{j+1} left alone,

$$u_i^{j+1} = u_i^j - \frac{\Delta t}{\rho} * \frac{\delta P}{\delta x} + \frac{\upsilon * \Delta t}{r} * \frac{u_{i+1}^j - u_{i-1}^j}{2\Delta r} + \upsilon * \Delta t * \frac{u_{i+1}^j - 2 * u_i^j + u_{i-1}^j}{(\Delta r)^2}$$

If the equation is made similar to the equation in the source (Eq. 19).

$$u_i^{j+1} = u_i^j - \frac{\Delta t}{\rho} * \frac{\delta P}{\delta x} + \frac{1}{2r} * \frac{\upsilon * \Delta t}{\Delta r}\left(u_{i+1}^j - u_{i-1}^j\right) + \frac{\upsilon * \Delta t}{(\Delta r)^2}\left(u_{i+1}^j - 2 * u_i^j + u_{i-1}^j\right)$$

$$u_i^{j+1} = u_i^j + \frac{\upsilon \Delta t}{\Delta x^2}\left(u_{i+1}^j - 2u_i^j + u_{i-1}^j\right) - \frac{1}{2}\frac{c\Delta t}{\Delta x}\left(u_{i+1}^j - u_{i-1}^j\right) \qquad (19)$$

*Figure 1. Canbolat, G., Köse, H., Yildizeli, A., Cadirci, S. (2019). Analytical and Numerical Solutions of the 1D Advection-Diffusion Equation. *This equation is used as an example.*

u(r), velocity profile graphics

5 nodes:

Since the number of 5 node is an odd number, the program did not create a graph. Therefore, if the graph is obtained with the number of 6 node.

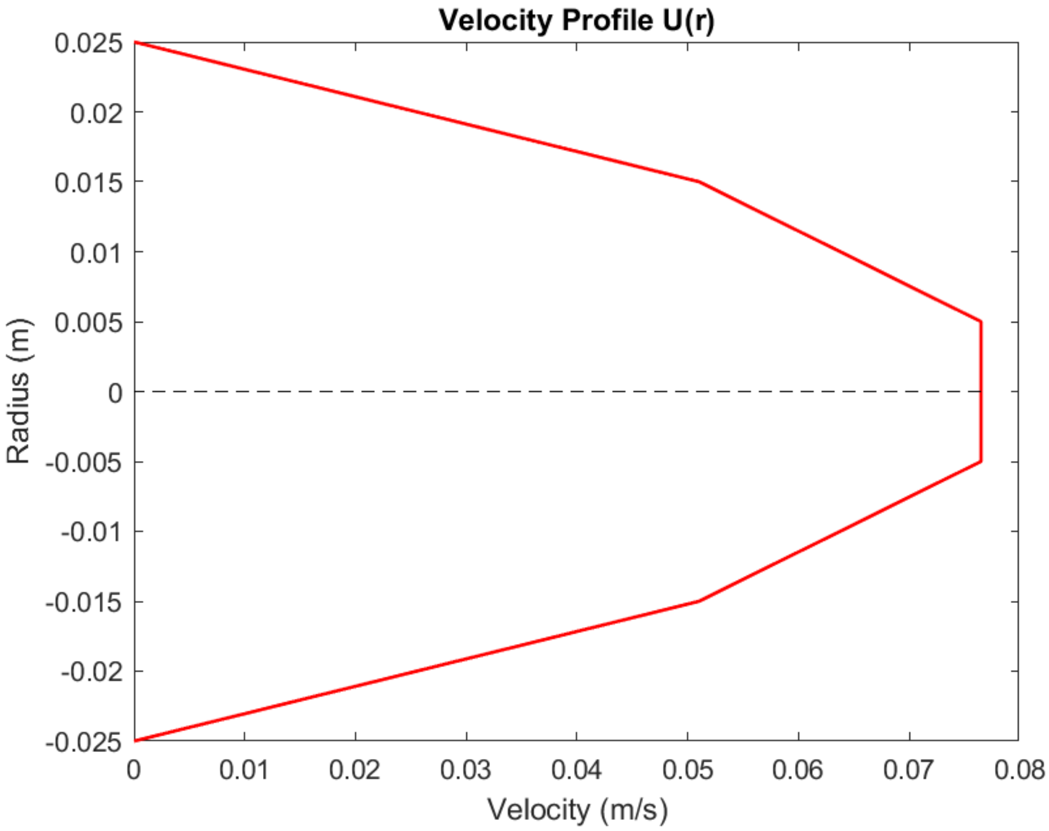

Figure 2. 6 nodes velocity profile [FTCS method]

As can be seen from the graph, a smooth graph could not be obtained because the number of nodes was low. Additionally, it was observed that the velocity profile could not reach maximum velocity even though it was in the fully developed flow region.

25 nodes:

Again, since the number of 25 node is an odd number, the program did not create a graph. Therefore, if the graph is obtained with the number of 26 node.

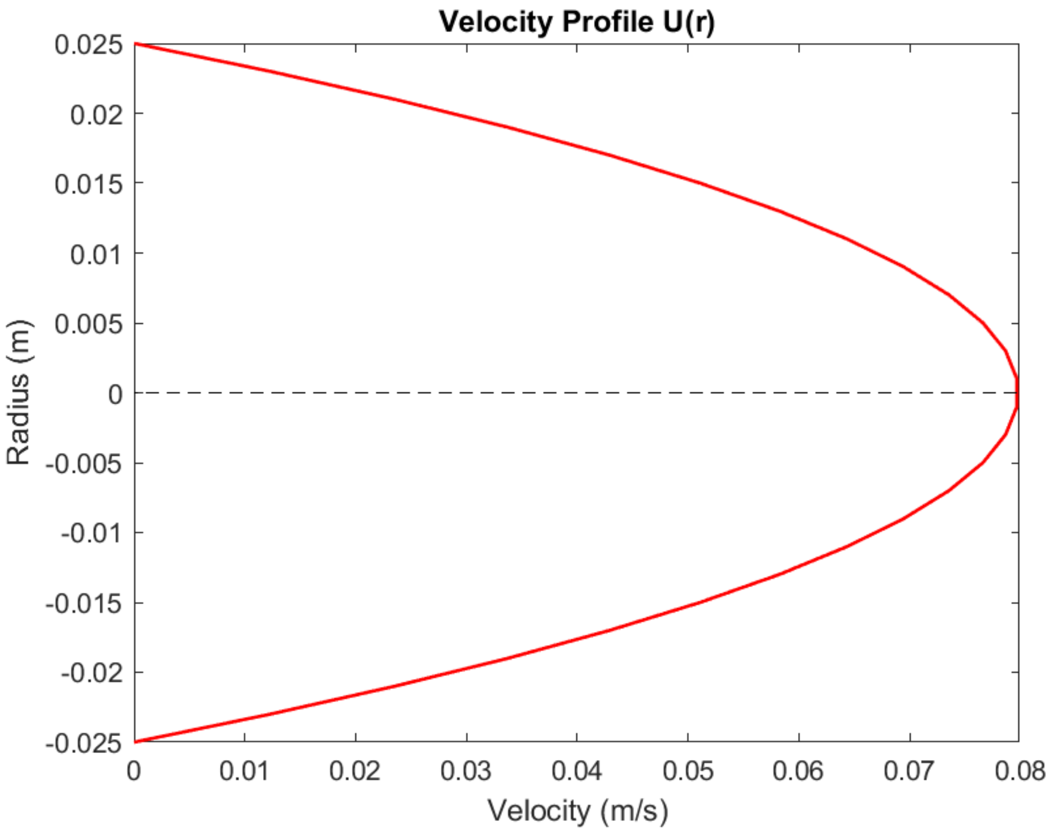

Figure 3. 26 nodes velocity profile [FTCS method]

Although a smoother graph is obtained compared to the previous graph, it is seen that there are still angular breaks in the graph.

50 nodes:

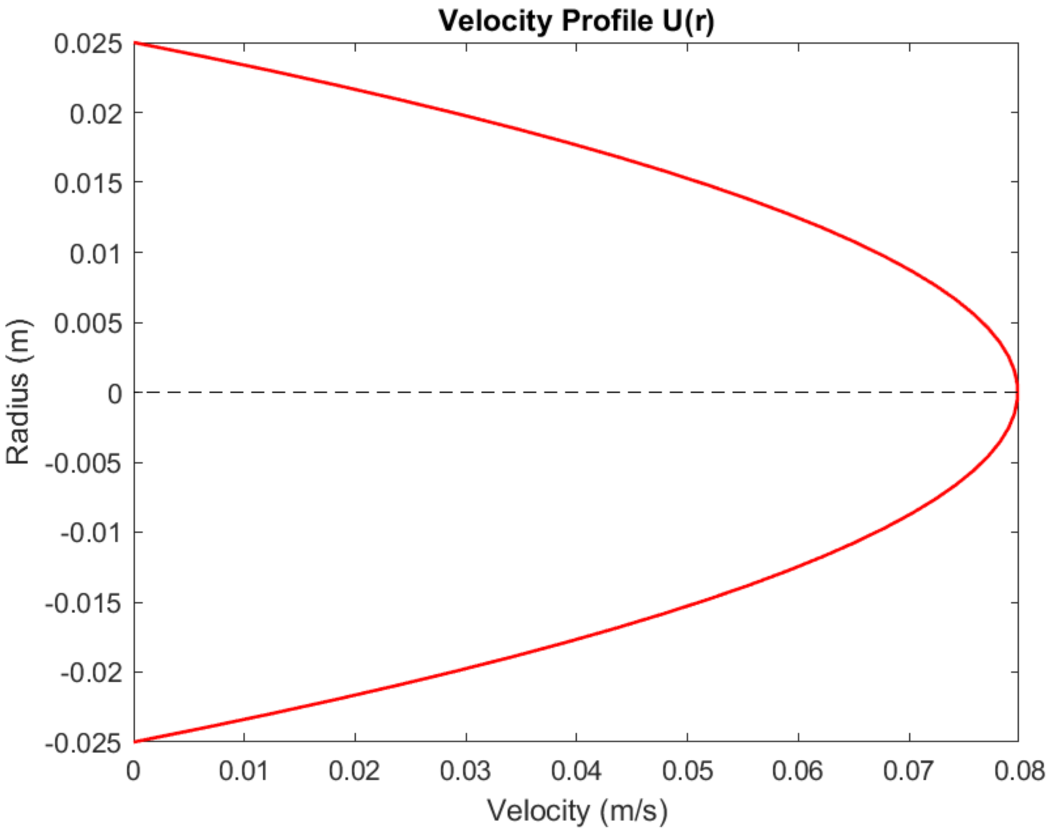

Figure 4. 50 nodes velocity profile [FTCS method]

It is understood from the graph that 50 node is sufficient. The graph is both smooth enough and reaches the maximum value of the speed.

Flow Rate

The flow rate was found using the given equation in homework file: $7.85*10^{(-5)}$ m^3/s

In FTCS method with using 50 nodes the flow rate's value is $7.82*10^{(-5)}$ m^3/s

Wall Shear Stress

Wall shear stress was calculated to using this formula:

Shear stress: $$\tau = \mu \frac{du}{dy} \quad (\mathrm{N/m^2}) \qquad \textbf{(2–33)}$$

Figure 5. Çengel, Y. and Cimbala, J., Fluid Mechanics Fundamentals and Applications, Equation 2-33.

It was found as 0.0063 N/m^2.

*It should be noted that wall shear stress is constant in the fully developed flow region.

BTCS Scheme

If the equation is discrete,

$$\frac{\delta u}{\delta t} = \frac{u_i^{j+1} - u_i^j}{\Delta t}$$

$$\frac{\delta u}{\delta r} = \frac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2\Delta r}$$

$$\frac{\delta^2 u}{\delta r^2} = \frac{u_{i+1}^{j+1} - 2 * u_i^{j+1} + u_{i-1}^{j+1}}{(\Delta r)^2}$$

If they are substituted into the equation,

$$\frac{u_i^{j+1} - u_i^j}{\Delta t} = -\frac{1}{\rho} * \frac{\delta P}{\delta x} + \frac{\upsilon}{r} * \frac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2\Delta r} + \upsilon * \frac{u_{i+1}^{j+1} - 2 * u_i^{j+1} + u_{i-1}^{j+1}}{(\Delta r)^2}$$

If u_i^j left alone,

$$-u_i^j = -u_i^{j+1} - \frac{\Delta t}{\rho} * \frac{\delta P}{\delta x} + \frac{\upsilon * \Delta t}{r} * \frac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2\Delta r} + \upsilon * \Delta t * \frac{u_{i+1}^{j+1} - 2 * u_i^{j+1} + u_{i-1}^{j+1}}{(\Delta r)^2}$$

If the equation is made similar to the equation in the source (Eq. 20).

$$u_i^j = u_i^{j+1} + \frac{\Delta t}{\rho} * \frac{\delta P}{\delta x} - \frac{1}{2r} * \frac{\upsilon * \Delta t}{\Delta r}\left(u_{i+1}^{j+1} - u_{i-1}^{j+1}\right) - \frac{\upsilon * \Delta t}{(\Delta r)^2}\left(u_{i+1}^{j+1} - 2 * u_i^{j+1} + u_{i-1}^{j+1}\right)$$

$$u_i^{j+1} - \frac{\upsilon \Delta t}{\Delta x^2}(u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}) + \frac{1}{2}\frac{c\Delta t}{\Delta x}(u_{i+1}^{j+1} - u_{i-1}^{j+1}) = u_i^j \qquad (20)$$

*Figure 6. Canbolat, G., Köse, H., Yildizeli, A., Cadirci, S. (2019). Analytical and Numerical Solutions of the 1D Advection-Diffusion Equation. *This equation is used as an example*

u(r), velocity profile graphics

5 nodes:

Since the number of 5 node is an odd number, the program did not create a graph. Therefore, if the graph is obtained with the number of 6 node.

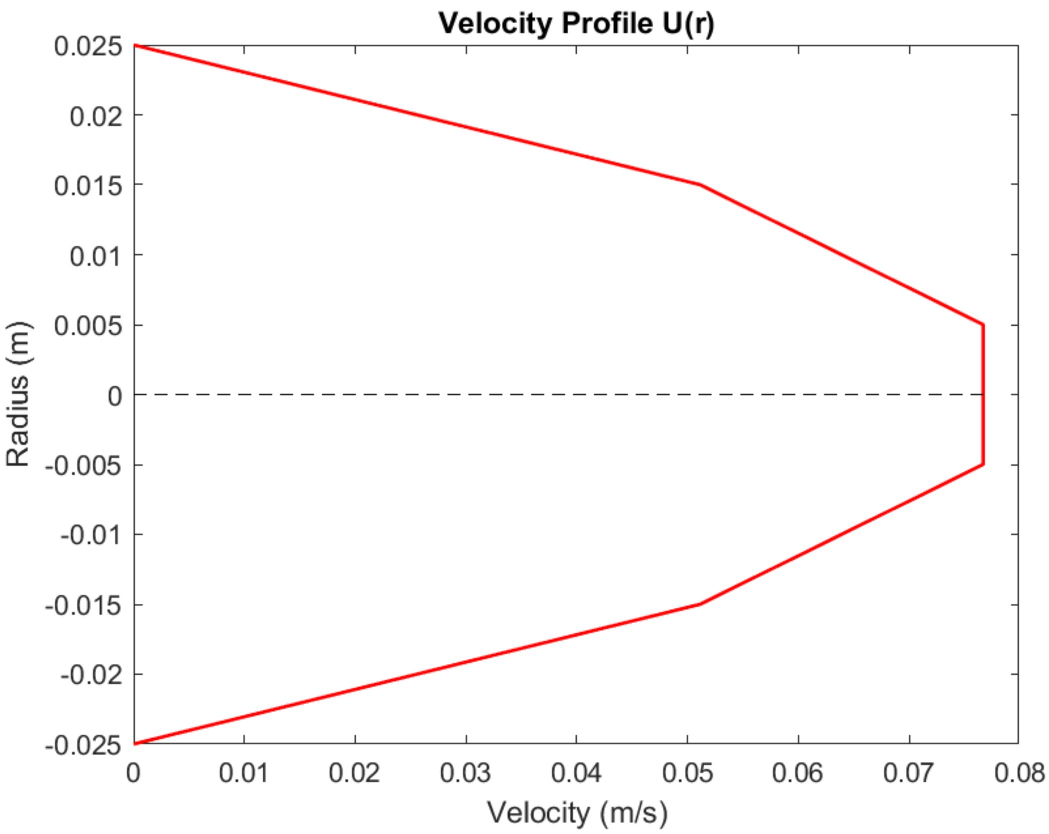

Figure 7. 6 nodes velocity profile [BTCS method]

As can be seen from the graph, a smooth graph could not be obtained because the number of nodes was low. Additionally, it was observed that the velocity profile could not reach maximum velocity even though it was in the fully developed flow region.

25 nodes:

Again, since the number of 25 node is an odd number, the program did not create a graph. Therefore, if the graph is obtained with the number of 26 node.

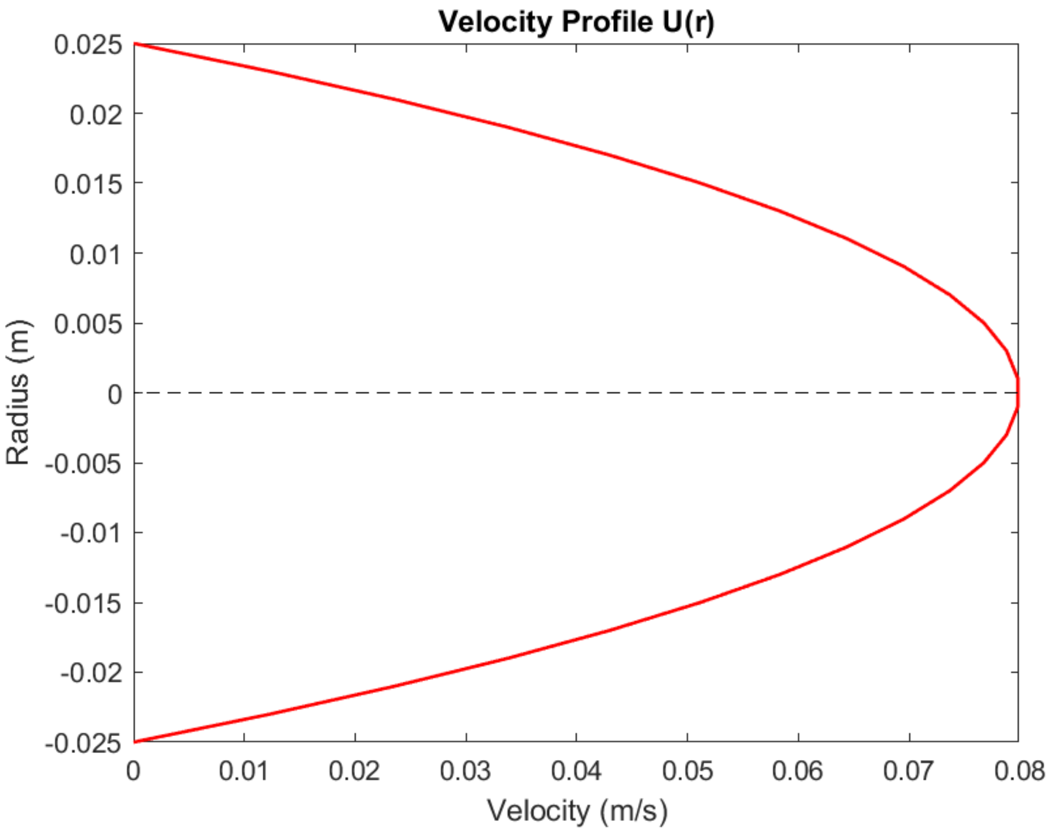

Figure 8. 26 nodes velocity profile [BTCS method]

Although a smoother graph is obtained compared to the previous graph, it is seen that there are still angular breaks in the graph.

50 nodes:

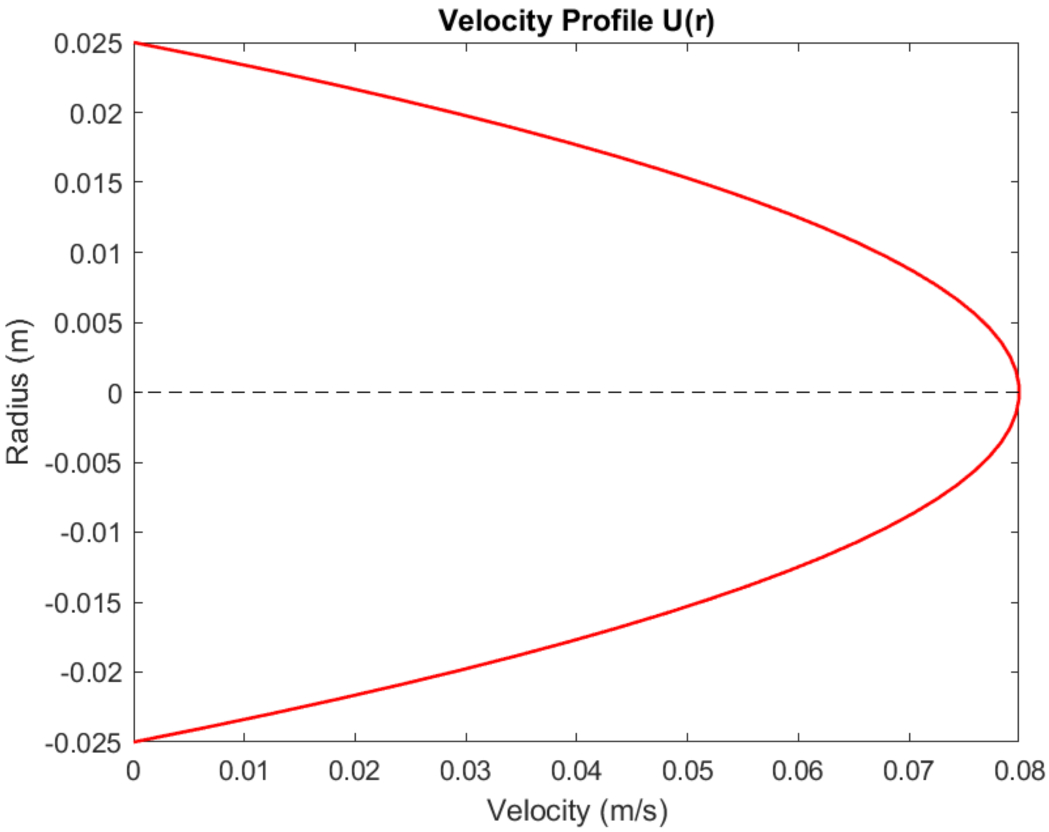

Figure 9. 50 nodes velocity profile [BTCS method]

It is understood from the graph that 50 node is sufficient. The graph is both smooth enough and reaches the maximum value of the speed.

Flow Rate

The flow rate calculated the early section as $7.85*10^{(-5)}$ m^3/s

In BTCS method with using 50 nodes the flow rate's value is $7.82*10^{(-5)}$ m^3/s and it is similar to FTCS method. The discussions above are also valid here.

Wall Shear Stress

The wall shear stress found same as in FTCS method: 0.0063 N/m^2

Crank Nicholson Scheme

The Crank-Nicholson method is a finite difference scheme that combines elements of both the explicit FTCS (Forward-Time Central-Space) and implicit BTCS (Backward-Time Central-Space) methods.

Correction of the equation will not be done because it has been done before. If the final form of the equation is written:

$$u_i^{j+1} = u_i^j + \frac{1}{2}\left\{\left[-\frac{\Delta t}{\rho} * \frac{\delta P}{\delta x} + \frac{1}{2r} * \frac{\upsilon * \Delta t}{\Delta r}\left(u_{i+1}^j - u_{i-1}^j\right) + \frac{\upsilon * \Delta t}{(\Delta r)^2}\left(u_{i+1}^j - 2 * u_i^j + u_{i-1}^j\right)\right] + \left[-\frac{\Delta t}{\rho} * \frac{\delta P}{\delta x} + \frac{1}{2r} * \frac{\upsilon * \Delta t}{\Delta r}\left(u_{i+1}^{j+1} - u_{i-1}^{j+1}\right) + \frac{\upsilon * \Delta t}{(\Delta r)^2}\left(u_{i+1}^{j+1} - 2 * u_i^{j+1} + u_{i-1}^{j+1}\right)\right]\right\}$$

Explanation

Below loop was planned to use for Crank-Nicolson Method, but velocity profile graph is not correct. Because of the velocity profile is not correct, other questions are not answered.

```
%% Crank-Nicolson Method
for j = 1:time_number-1
    for i = 2:nodes_number-1
        U(j+1, i) = U(j, i)+(1/2)*((- (dt/rho) * (delta_P / L) + ...
            (nu * dt / dr^2) * (U(j, i+1) - 2 * U(j, i) + U(j, i-1)) + ...
            (1 / (2 * r(i))) * (nu * dt / dr) * (U(j, i+1) - U(j, i-1)))+...
            (- (dt/rho) * (delta_P / L) + ...
        (nu * dt / dr^2) * (U(j+1, i+1) - 2 * U(j+1, i) + U(j+1, i-1)) + ...
        (1 / (2 * r(i))) * (nu * dt / dr) * (U(j+1, i+1) - U(j+1, i-1))));

        U(j, i) = U(j, nodes_number -(i - 1));
    end
    U(j+1, 1)      = 0; % u(r = -R, t) = 0
    U(j+1, end)    = 0; % u(r = R, t) = 0
end
```

Figure 10. A loop for Crank-Nicolson method.

Results and Discussion

FTCS, BTCS and Crank-Nicholson are three common methods used to numerically solve the linear parabolic differential equations. Each has its own advantages and disadvantages:

FTCS method:

Advantages	Disadvantages
Simple and easy to apply	Can propagate high frequency waves
unconditionally determined	Requires time step constraints
Computationally efficient	

BTCS method:

Advantages	Disadvantages
Provides higher accuracy without spillover	conditionally stable
Allows larger time steps	A tridiagonal system needs to be inverted to solve
	Computationally more expensive than FTCS

Crank-Nicholson:

Advantages	Disadvantages
Offers higher accuracy than FTCS and greater stability than BTCS	More complex than FTCS and BTCS
In terms of calculation, it falls somewhere between FTCS and BTCS	A tridiagonal system needs to be inverted to solve

In short,

Method	Accuracy	Stability	Computational Complexity	Conditional Stability
FTCS	*Low*	*Unconditional*	*Low*	*No*
BTCS	*High*	*Conditional*	*High*	*Yes*
CN	*Medium*	*Conditional*	*Medium*	*Yes*

Tabulating the flow rates and wall shear stress

	FTCS	BTCS	Crank-Nicholson
Flow Rate	$7.82*10^{(-5)}$ m^3/s	$7.82*10^{(-5)}$ m^3/s	Null
Wall Shear Stress	0.0063 N/m^2	0.0063 N/m^2	Null

Flow rates were found to be close to the value calculated with the formula given in the homework file, and it is surprising that the solution obtained by both methods is the same.

The fact that the wall shear stress calculated by both methods is the same shows us the real value of the wall shear stress.

Resources

A. I. (2021, December 26). *The FTCS Method with MATLAB code (Lecture # 02)*. YouTube. www.youtube.com/watch?v=sA2HVEeXkKs

A. I. (2022, January 1). *The BTCS / Laasonen Method with MATLAB code (Lecture # 03)*. YouTube. www.youtube.com/watch?v=niC7MMrtpw8

Canbolat, Gökkan & Köse, Haluk & Yildizeli, Alperen & Cadirci, Sertac. (2019). Analytical and Numerical Solutions of the 1D Advection-Diffusion Equation.

Cengel, Y. A., & Cimbala, J. M. (2017). *Fluid mechanics: Fundamentals and applications* (4th ed.). McGraw-Hill Education.

CFD Lecture Notes, Çadırcı, S.