Ewan Short 29/8/2025

Non-dimensional governing equations are

$$u_t = \frac{f}{\omega} v - p_x - \frac{\alpha}{\omega} u, \qquad (1)$$

$$v_t = \frac{-f}{\omega} u - \frac{\alpha}{\omega} v, \qquad (2)$$

$$\frac{\omega^2}{N^2} w_t = b - p_z - \frac{\alpha}{\omega} \frac{\omega^2}{N^2} w, \qquad (3)$$

$$b_t + w = Q - \frac{\alpha}{\omega} b, \qquad (4)$$

$$u_x + w_z = 0, \qquad (5)$$

$$w(z=0) = 0, \qquad (6)$$

where ω is a representative frequency, not necessarily the diurnal frequency as in Rotunno (1983) etc. Define ψ such that $(u, w) = (\psi_z, -\psi_x)$. Then (1) and (2) imply

$$\left(\frac{\partial}{\partial t} + \frac{\alpha}{\omega}\right) u = \frac{f}{\omega} v - p_x, \qquad (7)$$

$$\left(\frac{\partial}{\partial t} + \frac{\alpha}{\omega}\right) v = -\frac{f}{\omega} u. \qquad (8)$$

Taking $\left(\frac{\partial}{\partial t} + \frac{\alpha}{\omega}\right)$ of (7) and subbing in (8) gives

$$\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 u = -\frac{f^2}{\omega^2}u - \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)p_x . \tag{9}$$

Cross-differentiating (3) and (9) gives

$$\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 u_z = \frac{-f^2}{\omega^2}u_z - \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)p_{xz}, \tag{10}$$

$$\frac{\omega^2}{N^2}\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)w_x = b_x - p_{zx} . \tag{11}$$

Using (11) to substitute p_{xz} in (10) gives

$$\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 u_z = \frac{-f^2}{\omega^2}u_z + \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)\left(\frac{\omega^2}{N^2}\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)w_x - b_x\right)$$

$$\Rightarrow \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 u_z = -\frac{f^2}{\omega^2}u_z + \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2\frac{\omega^2}{N^2}w_x - \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)b_x . \tag{12}$$

Note (4) implies

$$\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)b + w = Q$$

$$\Rightarrow \quad -\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)b_x = w_x - Q_x . \tag{13}$$

Subbing (13) into (12) then subbing ψ gives

$$\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 u_z = -\frac{f^2}{\omega^2}u_z + \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2\frac{\omega^2}{N^2}w_x + w_x - Q_x$$

$$\Rightarrow \left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2\psi_{zz} + \frac{f^2}{\omega^2}\psi_{zz} + \frac{\omega^2}{N^2}\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2\psi_{xx} + \psi_{xx} = -Q_x$$

$$\Rightarrow \left[\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 + \frac{f^2}{\omega^2}\right]\psi_{zz} + \left[\frac{\omega^2}{N^2}\left(\frac{\partial}{\partial t}+\frac{\alpha}{\omega}\right)^2 + 1\right]\psi_{xx} = -Q_x . \tag{14}$$

Now, for a general forcing Q for which the time dependence is unknown, we can pursue analytic solutions by first taking a Fourier transform from t to σ. Applying this transform to both sides of (14) gives

$$\left[\left(i\sigma + \frac{\alpha}{\omega}\right)^2 + \frac{f^2}{\omega^2}\right]\tilde{\Psi}_{zz} + \left[\frac{\omega^2}{N^2}\left(i\sigma + \frac{\alpha}{\omega}\right)^2 + 1\right]\tilde{\Psi}_{xx} = -\tilde{Q}_x, \quad (15)$$

where $\tilde{\Psi}$, $\tilde{Q}$ are the Fourier transforms from t to σ of Ψ, Q, respectively. Note because Ψ is real, we have

$$\Psi = \mathrm{Re}\left\{\frac{1}{\pi}\int_0^\infty \tilde{\Psi} e^{i\sigma t}\, d\sigma\right\},$$

so we can assume $\sigma > 0$. Now we can proceed much the same as Rotunno (1983), Qian (2009) etc, taking a Fourier transform of (15) from x to k, to get

$$\left[\left(i\sigma + \frac{\alpha}{\omega}\right)^2 + \frac{f^2}{\omega^2}\right]\hat{\Psi}_{zz} - k^2\left[\frac{\omega^2}{N^2}\left(i\sigma + \frac{\alpha}{\omega}\right)^2 + 1\right]\hat{\Psi} = -ik\hat{Q}, \quad (16)$$

where $\hat{\Psi}$, $\hat{Q}$ are the transforms of Ψ, Q, respectively. Writing

$$B = \left(-\left(i\sigma + \frac{\alpha}{\omega}\right)^2 - \frac{f^2}{\omega^2}\right)^{1/2}, \quad C = \left[\frac{\omega^2}{N^2}\left(i\sigma + \frac{\alpha}{\omega}\right)^2 + 1\right]^{1/2},$$

and $A = \frac{B}{C}$, (16) can be rewritten

$$\hat{\psi}_{zz} + \frac{k^2}{A^2}\hat{\psi} = \frac{ik}{B^2}\hat{Q}. \quad (17)$$

We solve (17) using Green's method. We first solve for the Green's function G satisfying

$$G_{zz} + \frac{k^2}{A^2}G = \delta(z - z'). \quad (18)$$

For $z < z'$ we have

$G = a_1 e^{imz} + a_2 e^{-imz}$, defining $m = \frac{k}{A}$.

The boundary condition (6) implies

$$a_1 + a_2 = 0 \Rightarrow a_2 = -a_1$$

$$\Rightarrow G = a_1\left(e^{imz} - e^{-imz}\right) = 2ia_1 \sin(mz)$$

$$= a\sin(mz),$$

defining $a = 2ia_1$ for simplicity.

For $z > z'$ we again have

$G = b_1 e^{imz} + b_2 e^{-imz}$, but we also require $G \to 0$ as $z \to \infty$, noting $\alpha > 0$.

Thus if $\mathrm{Im}(m) > 0$, $b_2 = 0$, but if $\mathrm{Im}(m) < 0$, $b_1 = 0$. Thus in either case we can write

$$G = b e^{\mathrm{sgn}(\mathrm{Im}(m)) i m z} \quad \text{for some constant } b.$$

To solve for a and b, note continuity of w at z' implies G must be continuous at z'. Furthermore, (18) implies

$$\lim_{\varepsilon \to 0} \int_{z'-\varepsilon}^{z'+\varepsilon} G_{zz} + \frac{k^2}{\lambda^2} G \, dz = 1$$

$$\Rightarrow \lim_{z \to z'^+} G_z - \lim_{z \to z'^-} G_z = 1.$$

Thus

$$a \sin(m z') = b e^{\mathrm{sgn}(\mathrm{Im}(m)) i m z'}, \quad (19)$$

$$b \,\mathrm{sgn}(\mathrm{Im}(m)) i m e^{\mathrm{sgn}(\mathrm{Im}(m)) i m z'} - a m \cos(m z') = 1 \quad (20)$$

Taking $\mathrm{sgn}(\mathrm{Im}(m)) i m \times (19) - (20)$ gives

$$a \,\mathrm{sgn}(\mathrm{Im}(m)) i m \sin(m z') - 1 - a m \cos(m z') = 0$$

$$\Rightarrow a\left(i m \sin(\mathrm{sgn}(\mathrm{Im}(m)) m z') - a m \cos(\mathrm{sgn}(\mathrm{Im}(m)) m z')\right) = 1$$

$$= -a m e^{-\mathrm{sgn}(\mathrm{Im}(m)) i m z'} = 1$$

$$\Rightarrow \quad a = -\frac{1}{m} e^{\operatorname{sgn}(\operatorname{Im}(m)) i m z'} .$$

Subbing a back into (19),

$$-\frac{1}{m} e^{\operatorname{sgn}(\operatorname{Im}(m)) i m z'} \sin(m z') = b e^{\operatorname{sgn}(\operatorname{Im}(m)) i m z'}$$

$$\Rightarrow \quad b = -\frac{1}{m} \sin(m z'). \quad \text{So}$$

$$G = \begin{cases} -\frac{1}{m} e^{\operatorname{sgn}(\operatorname{Im}(m)) i m z'} \sin(m z), & z \leq z', \\ -\frac{1}{m} \sin(m z') e^{\operatorname{sgn}(\operatorname{Im}(m)) i m z}, & z > z'. \end{cases}$$

We then have

$$\hat{\psi} = \int_0^\infty \frac{ik}{B^2} \hat{Q}(z') G(z, z') \, dz',$$

$$\tilde{\psi} = \frac{1}{2\pi} \int_{-\infty}^\infty \hat{\psi} e^{ikx} \, dk,$$

$$\psi = \operatorname{Re}\left\{ \frac{1}{\pi} \int_0^\infty \tilde{\psi} e^{i\sigma t} \, d\sigma \right\} = \operatorname{Re}\left\{ \frac{1}{2\pi^2} \int_0^\infty \int_{-\infty}^\infty \hat{\psi} e^{i(kx + \sigma t)} \, dk \, d\sigma \right\}.$$

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25/9/25

Let's try $Q = \frac{1}{1+(\frac{x}{L})^2} \delta(z - z_f) e^{it}$.

Then $\hat{Q} = 2\pi \delta(\sigma - 1) L\pi e^{-L|k|} \delta(z - z_f)$.

So $\hat{\psi} = \frac{2\pi\delta(\sigma-1)L\pi ik}{B^2} G(z, z_f) e^{-L|k|}$.

Now for $z \leq z_f$ we find

$$\tilde{\psi} = \frac{-\delta(\sigma-1)L\pi ik}{B^2 m} \left\{ \int_{-\infty}^{\infty} e^{\chi(m) i m z_f} \sin(mz) e^{ikx - L|k|} dk \right\}$$

$$= \frac{-\delta(\sigma-1)L\pi iA}{B^2} \left\{ \int_{-\infty}^{\infty} e^{\chi(m) i m z_f} \left[\frac{e^{imz} - e^{-imz}}{2i} \right] e^{ikx - L|k|} dk \right\}$$

$$= \frac{-\delta(\sigma-1)L\pi iA}{B^2 2i} \left\{ \int_0^{\infty} e^{ik(L_1 + iL)} - e^{ik(L_2 + iL)} dk \right.$$

$$\left. + \int_{-\infty}^{0} e^{ik(L_3 - iL)} - e^{ik(L_4 - iL)} dk \right\}$$

where $\chi(m) = \mathrm{sgn}(k) \cdot \chi(\frac{1}{A})$ and by convention $\chi(1/A) > 0$, as we are free to choose the positive or negative root of $1/A^2$; also

$L_1 = \frac{1}{A}(z_f + z) + x$, $L_2 = \frac{1}{A}(z_f - z) + x$,

$L_3 = \frac{1}{A}(-z_f + z) + x$, $L_4 = 1/A(-z_f - z) + x$.

Now, note that because $\chi(1/A) > 0$ and $z \leq z_f$, $L > 0$, $e^{ik(L_1 + iL)} \to 0$ and

$e^{ik(L_2+iL)} \to 0$ as $k \to \infty$, noting

$$x(L_1+iL) = x(1/A)(z_f+z)+L > 0$$ and

$$x(L_2+iL) = x(1/A)(z_f-z)+L > 0.$$ Similarly,

$e^{ik(L_3-iL)} \to 0$ and $e^{ik(L_4-iL)} \to 0$ as

$k \to -\infty$. So

$$\tilde{\psi} = \frac{-\delta(\sigma-1)L\pi iA}{B^2 2i}\left\{-\frac{1}{i(L_1+iL)}+\frac{1}{i(L_2+iL)} + \frac{1}{i(L_3-iL)}-\frac{1}{i(L_4-iL)}\right\}$$

$$= \frac{\delta(\sigma-1)L\pi iA}{2B^2}\left\{-\frac{1}{(L_1+iL)}+\frac{1}{(L_2+iL)}+\frac{1}{(L_3-iL)}-\frac{1}{(L_4-iL)}\right\}.$$

Now, if $z > z_f$,

$$\tilde{\psi} = -\frac{\delta(\sigma-1)L\pi ik}{B^2 m}\left\{\int_{-\infty}^{\infty} \sin(mz_f)e^{x(m)imz}e^{ikx-L|k|}\,dk\right\}$$

$$= \frac{-\delta(\sigma-1)L\pi ik}{B^2 m}\left\{\int_{-\infty}^{\infty}\left(\frac{e^{imz_f}-e^{-imz_f}}{2i}\right)e^{x(m)imz}e^{ikx-L|k|}\,dk\right\}$$

$$= \frac{-\delta(\sigma-1)L\pi ik}{B^2 m 2i}\left\{\int_0^{\infty} e^{ik(\frac{1}{A}(z_f+z)+x+iL)}-e^{ik(\frac{1}{A}(-z_f+z)+x+iL)}\,dk\right.$$
$$\left.+\int_{-\infty}^{0} e^{ik(\frac{1}{A}(z_f-z)+x-iL)}-e^{ik(1/A(-z_f-z)+x-iL)}\,dk\right\}$$

$$= \frac{-\delta(\sigma-1)L\pi\cdot A}{2B^2}\left\{\int_0^{\infty} e^{ik(L_1+iL)}-e^{ik(L_3+iL)}\,dk\right.$$
$$\left.+\int_{-\infty}^{0} e^{ik(L_2-iL)}-e^{ik(L_4-iL)}\,dk\right\}.$$

As before, the respective anti-derivatives $\to 0$

as $k \to \pm\infty$, and we find

$$\tilde{\psi} = \frac{-\delta(\sigma-1)L\pi A}{2B^2}\left\{\frac{-1}{i(L_1+iL)} + \frac{1}{i(L_3+iL)} + \frac{1}{i(L_2-iL)} - \frac{1}{i(L_4-iL)}\right\}$$

$$= \frac{\delta(\sigma-1)L\pi A i}{2B^2}\left\{\frac{-1}{(L_1+iL)} + \frac{1}{(L_3+iL)} + \frac{1}{(L_2-iL)} - \frac{1}{(L_4-iL)}\right\}.$$ So almost identical to the $z \leq z_f$ case, but with L_2 and L_3 switched. Note we have continuity at $z = z_f$, as $L_2 = L_3 = x$ when $z = z_f$. We then have for $z \leq z_f$, with $D = \frac{\delta(\sigma-1)L\pi A i}{2B^2}$

$$\tilde{u} = \frac{\partial\tilde{\psi}}{\partial z} = D\cdot\left\{\frac{1}{(L_1+iL)^2}\cdot\frac{1}{A} + \frac{1}{(L_2+iL)^2}\cdot\frac{1}{A} - \frac{1}{(L_3-iL)^2}\cdot\frac{1}{A} - \frac{1}{(L_4-iL)^2}\frac{1}{A}\right\}$$ and for $z > z_f$,

$$\tilde{u} = D/A\cdot\left\{\frac{1}{(L_1+iL)^2} - \frac{1}{(L_3+iL)^2} + \frac{1}{(L_2-iL)^2} - \frac{1}{(L_4-iL)^2}\right\}.$$

Similarly for $z \leq z_f$,

$$\tilde{w} = -\frac{\partial\tilde{\psi}}{\partial x} = -D\cdot\left\{\frac{1}{(L_1+iL)^2} - \frac{1}{(L_2+iL)^2} - \frac{1}{(L_3-iL)^2} + \frac{1}{(L_4-iL)^2}\right\}$$

and for $z > z_f$,

$$\tilde{w} = -\frac{\partial \psi}{\partial x} = -D \cdot \left\{ \frac{1}{(L_1 + iL)^2} - \frac{1}{(L_3 + iL)^2} - \frac{1}{(L_2 - iL)^2} + \frac{1}{(L_4 - iL)^2} \right\}$$

Note we thus have

$$\psi = \mathrm{Re}\left\{ \frac{1}{2\pi} \tilde{\psi} e^{it} \right\} \quad \text{etc.}$$

For completeness, note

$$\tilde{v} = -\frac{\delta}{\omega} \cdot \frac{1}{\underbrace{(i + \frac{\alpha}{\omega})}_{= \hat{\sigma}}} \tilde{u}, \quad \text{so}$$

$$-\tilde{p}_x = \hat{\sigma} \tilde{u} + \left(\frac{\delta}{\omega}\right)^2 \frac{1}{\hat{\sigma}} \tilde{u} = \left(\hat{\sigma} + \frac{\delta^2}{\omega^2} \frac{1}{\hat{\sigma}}\right) \tilde{u}$$

$$\Rightarrow p_x = -\left(\hat{\sigma} + \frac{\delta^2}{\omega^2} \frac{1}{\hat{\sigma}}\right) \tilde{u}$$

$$\Rightarrow p = -\left(\hat{\sigma} + \frac{\delta^2}{\omega^2} \frac{1}{\hat{\sigma}}\right) \cdot D/A \cdot \left\{ -\frac{1}{L_1 + iL} - \frac{1}{L_2 + iL} + \frac{1}{L_3 - iL} + \frac{1}{L_4 - iL} \right\} \quad \text{if } z \le z_f,$$

and $$p = -\left(\hat{\sigma} + \frac{\delta^2}{\omega^2} \frac{1}{\hat{\sigma}}\right) \cdot D/A \cdot \left\{ -\frac{1}{L_1 + iL} + \frac{1}{L_3 + iL} - \frac{1}{L_2 - iL} + \frac{1}{L_4 - iL} \right\} \quad \text{if } z > z_f.$$