Bretherton (1966)

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1 Suspected Misprints

1.

2 Wave Energy Equation

2.1 Wave Kinetic Energy

Assume $w' = \Re w_0 e^{i\phi}$. Substituting into (30) from the paper, considering for now both real and imaginary parts, gives

$$-i\Omega u' + U_z w' + \frac{1}{\rho} ikp' = 0. \tag{1}$$

Substituting into (the corrected version of) (34) from the paper gives

$$\frac{g}{\gamma p}(-i\Omega)p' = \frac{g}{\rho}(-i\Omega)\rho' - N^2 w' \tag{2}$$

and into (33) gives

$$\frac{1}{\rho}(-i\Omega)\rho' + \frac{1}{\rho}\rho_z w' + iku' + ilv' + imw' = 0. \tag{3}$$

By cross multiplying (30) and (31) from the paper we can show $v' = \frac{l}{k}$. Using this result, and substituting equation (3) into (2) gives

$$\Rightarrow \frac{g}{\gamma p}(-i\Omega)p' = g\left[-\frac{1}{\rho}\rho_z w' - iku' - ilv' - imw'\right] - N^2 w' \tag{4}$$

$$\Rightarrow ip' = \frac{\gamma p}{\Omega}\left[\frac{1}{\rho}\rho_z w' + iku' + ilv' + imw'\right] + \frac{\gamma p}{\Omega g} N^2 w'. \tag{5}$$

Now substituting this into (1) gives

$$\Rightarrow -i\Omega u' + U_z w' + \frac{1}{\rho}k\left[\frac{\gamma p}{\Omega}\left[\frac{1}{\rho}\rho_z w' + iku' + ilv' + imw'\right] + \frac{\gamma p}{\Omega g}N^2 w'\right] = 0 \tag{6}$$

$$\Rightarrow -i\Omega u' + U_z w' + \frac{1}{\rho}k\frac{c^2}{\Omega}\rho_z w' + ik^2\frac{c^2}{\Omega}u' + il^2\frac{c^2}{\Omega}u' + imk\frac{c^2}{\Omega}w' + k\frac{c^2}{\Omega g}N^2 w' = 0. \tag{7}$$

Now taking just the real part, and then just the coefficients of the sin terms, gives

$$\Rightarrow \Omega u_0 - k^2\frac{c^2}{\Omega}u_0 - l^2\frac{c^2}{\Omega}u_0 - mk\frac{c^2}{\Omega}w_0 = 0 \tag{8}$$

$$\Rightarrow \left[\frac{\Omega^2}{c^2} - k^2 - l^2\right] u_0 = mkw_0 \tag{9}$$

$$\Rightarrow u_0 = -mk\frac{1}{k^2 + l^2 - \frac{\Omega^2}{c^2}}w_0 \tag{10}$$

Similarly we can show

$$v_0 = -ml\frac{1}{k^2 + l^2 - \frac{\Omega^2}{c^2}}w_0 \tag{11}$$

Thus $u'^2 + v'^2 + w'^2$

$$= \sin^2\phi \left\{ \frac{m^2}{\left(k^2 + l^2 - \frac{\Omega^2}{c^2}\right)^2}\left(k^2 + l^2\right) w_0^2 + w_0^2 \right\} \tag{12}$$

$$= \sin^2\phi \left\{ \frac{m^2}{\left(k^2 + l^2 - \frac{\Omega^2}{c^2}\right)^2}\left(k^2 + l^2\right) w_0^2 - \frac{m^2\frac{\Omega^2}{c^2}}{\left(k^2 + l^2 - \frac{\Omega^2}{c^2}\right)^2}w_0^2 + \frac{m^2\frac{\Omega^2}{c^2}}{\left(k^2 + l^2 - \frac{\Omega^2}{c^2}\right)^2}w_0^2 + w_0^2 \right\} \tag{13}$$

$$= \sin^2\phi \left\{ \frac{k^2 + l^2 + m^2 - \frac{\Omega^2}{c^2}}{\left(k^2 + l^2 - \frac{\Omega^2}{c^2}\right)}w_0^2 + \frac{m^2\frac{\Omega^2}{c^2}}{\left(k^2 + l^2 - \frac{\Omega^2}{c^2}\right)^2}w_0^2 \right\} \tag{14}$$

and so

$$\overline{u'^2} + \overline{v'^2} + \overline{w'^2} = \frac{1}{2}w_0^2 \left\{ \frac{k^2 + l^2 + m^2 - \frac{\Omega^2}{c^2}}{\left(k^2 + l^2 - \frac{\Omega^2}{c^2}\right)} + \frac{m^2\frac{\Omega^2}{c^2}}{\left(k^2 + l^2 - \frac{\Omega^2}{c^2}\right)^2} \right\} \tag{15}$$

2.2 Wave Potential Energy

By definition we have $w' = \frac{D_0\zeta'}{Dt} = \zeta'_t + U\zeta'_x + V\zeta'_y$. Following the mean flow, so that x, y are now functions of t and track the position of the particle at $(x_0, y_0, z, 0)$, we have

$$\begin{aligned}
\zeta' &= \int_0^t w'(x(s), y(s), z, s)\, ds \\
&= \int_0^t w_0 \cos(kx(s) + ly(s) + mz - \omega s)\, ds \\
&= \int_0^t w_0 \cos(k(Us + x_0) + l(Vs + v_0) + mz - \omega s)\, ds \\
&= \left[w_0 \frac{-1}{\Omega} \sin(k(Us + x_0) + l(Vs + v_0) + mz - \omega s) \right]_0^t \\
&= \frac{-w_0}{\Omega}\left[\sin(k(Ut + x_0) + l(Vt + y_0) + mz - \omega t) - \sin(kUx_0 + lVy_0 + mz)\right] \\
&= \frac{-w_0}{\Omega}\left[\sin(kUx_0 + lUy_0 + mz - \Omega t) - \sin(kUx_0 + lVy_0 + mz)\right]
\end{aligned}$$

Now

$$\begin{aligned}
\bar{\zeta'^2} &= \frac{w_0^2}{\Omega^2}\frac{-\Omega}{2\pi}\int_0^{\frac{2\pi}{-\Omega}} \frac{m}{2\pi}\int_0^{\frac{2\pi}{m}} \frac{w_0^2}{\Omega^2}\left[\sin(kUx_0 + lUy_0 + mz - \Omega t) - \sin(kUx_0 + lVy_0 + mz)\right]^2 \, dz\, dt \\
&= \frac{w_0^2}{\Omega^2}\frac{-\Omega}{2\pi}\int_0^{\frac{2\pi}{-\Omega}} 1 - \cos(-\Omega t) dt \\
&= \frac{w_0^2}{\Omega^2}
\end{aligned}$$

Thus using (15) from the paper, i.e. the dispersion relation,

$$
\begin{aligned}
N^2\bar{\zeta'^2} &= N^2\frac{w_0^2}{\Omega^2} \\
&= w_0^2\frac{k^2+l^2+m^2}{k^2+l^2}
\end{aligned}
$$

2.3 Internal Energy

From equation 5 we have

$$
ip' = \frac{\gamma p}{\Omega}\left[\frac{1}{\rho}\rho_z w' + iku' + i\frac{l^2}{k}u' + imw'\right] + \frac{\gamma p}{\Omega g}N^2 w'. \tag{16}
$$

Substituting equation 10 gives

$$
\begin{aligned}
ip' &= \frac{\gamma p}{\Omega}\left[\frac{1}{\rho}\rho_z w' - im\left(\frac{k^2+l^2}{k^2+l^2-\frac{\Omega^2}{c^2}}\right)w' + imw'\right] + \frac{\gamma p}{\Omega g}N^2 w' && (17)\\
&= \frac{\gamma p}{\Omega}\left[\frac{1}{\rho}\rho_z w' - im\left(\frac{-\frac{\Omega^2}{c^2}}{k^2+l^2-\frac{\Omega^2}{c^2}}\right)w'\right] + \frac{\gamma p}{\Omega g}N^2 w' && (18)
\end{aligned}
$$

Taking real parts and equating sin coefficients gives

$$
\begin{aligned}
&\Rightarrow -p_0 = \frac{\gamma p}{\Omega}\left[m\left(\frac{-\frac{\Omega^2}{c^2}}{k^2+l^2-\frac{\Omega^2}{c^2}}\right)w_0\right] && (19)\\
&\Rightarrow \frac{1}{c^2\rho^2}p'^2 = \frac{\gamma^2 p^2}{c^2\rho^2\Omega^2}m^2\frac{\frac{\Omega^4}{c^4}}{\left(k^2+l^2-\frac{\Omega^2}{c^2}\right)^2}w_0^2\sin^2\phi && (20)\\
&\Rightarrow \frac{1}{c^2\rho^2}p'^2 = \frac{c^4}{c^2\Omega^2}m^2\frac{\frac{\Omega^4}{c^4}}{\left(k^2+l^2-\frac{\Omega^2}{c^2}\right)^2}w_0^2\sin^2\phi && (21)\\
&\Rightarrow \frac{1}{c^2\rho^2}p'^2 = m^2\frac{\frac{\Omega^2}{c^2}}{\left(k^2+l^2-\frac{\Omega^2}{c^2}\right)^2}w_0^2\sin^2\phi && (22)
\end{aligned}
$$

References

Bretherton, F. P. (1966), 'The propagation of groups of internal gravity waves in a shear flow', *Quarterly Journal of the Royal Meteorological Society* **92**(394), 466–480.