MA203: Module 3 - Partial Differential Equations

Vinay Kumar Gupta

These notes are taken largely from the textbook *Advanced Engineering Mathematics* by Erwin Kreyszig (10th Edition, John Wiley & Sons, 2011).

Contents

[Lecture 1]

Fourier series are the basic tool for representing *periodic functions*, which play an important role in applications.

1 Recap of some definitions

1.1 Periodic functions

Definition: A function $f(x)$ is said to be a *periodic function* if $f(x)$ is defined for all real x (perhaps expect at some points, such as $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$ for $\tan x$) and if there is some positive number p, called *a period* of $f(x)$ such that

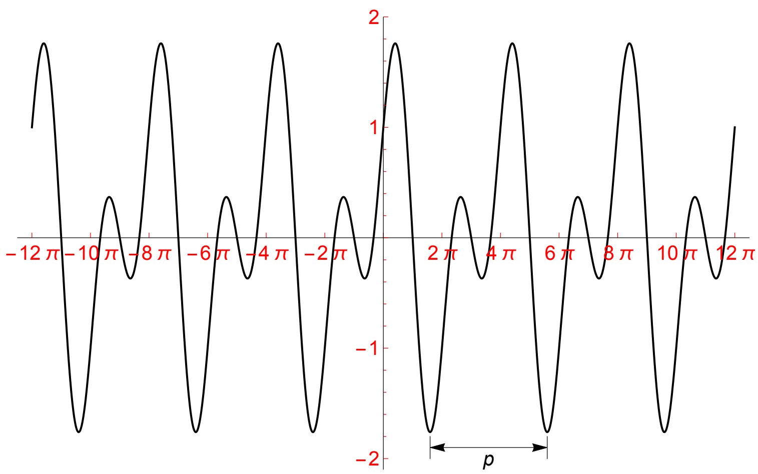
$$f(x+p) = f(x) \quad \text{for all } x.$$

Examples:

- The functions $\sin x$, $\cos x$ and $\tan x$ are periodic functions.
- On the other hand, the functions x, x^2, x^3, e^x and $\ln x$ are NOT periodic.

Remarks:

- The smallest positive period is often referred to as the *fundamental period.*
- The graph of a periodic function is obtained by periodic repetition of its graph in any interval of length p.



- If $f(x)$ has period p, then any positive integer multiple of p is also a period of $f(x)$. Reason: If $f(x)$ has period p, then $f(x+p) = f(x)$. Therefore,

  $$f(x+2p) = f\big((x+p)+p\big) = f(x+p) = f(x).$$

  Therefore $2p$ is also a period of $f(x)$. Similarly, it can be shown that any positive integer multiple of p is also a period of $f(x)$.
- If the functions $f(x)$ and $g(x)$ have period p, then $a\,f(x) + b\,g(x)$, for any constants a and b, also has the period p. [The statement is also true for more than two functions.]

1.2 Orthogonality of functions

Definition: Two distinct functions $f(x)$ and $g(x)$ defined on some interval $a \le x \le b$ are said to be *orthogonal* with respect to the *weight function* $r(x)$ on this interval if

$$\int_a^b r(x)\, f(x)\, g(x)\, \mathrm{d}x = 0.$$

Remark: If $r(x) = 1$, we more briefly call the functions *orthogonal* instead of orthogonal with respect to $r(x) = 1$.

Example: The functions $\sin x$ and $\cos x$ are orthogonal on the interval $[-\pi, \pi]$.

To show this, let $f(x) = \sin x$ and $g(x) = \cos x$. Note that $r(x) = 1$.

$$\int_{-\pi}^{\pi} \sin x \cos x\, \mathrm{d}x = \frac{1}{2}\int_{-\pi}^{\pi} \sin 2x\, \mathrm{d}x = \frac{1}{2}\left(-\frac{\cos 2x}{2}\right)_{-\pi}^{\pi} = \frac{1}{2}\left[-\frac{\cos 2\pi}{2} + \frac{\cos(-2\pi)}{2}\right] = 0$$

since $\cos(-\theta) = \cos\theta$.

Definition: A set of functions defined on some interval $[a, b]$ is said to be an *orthogonal set* with respect to the *weight function* $r(x)$ on this interval if every pair of distinct functions of the set is orthogonal with respect to $r(x)$.

Remark: If $r(x) = 1$, we more briefly call the set as the *orthogonal set* instead of orthogonal set with respect to $r(x) = 1$.

Example: See the following theorem.

Theorem 1 (Orthogonality of the trigonometric system)**:** The trigonometric system

$$\{1, \sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \cos 3x, \dots, \sin nx, \cos nx, \dots\}$$

is orthogonal on the interval $[-\pi, \pi]$ (hence also on $[0, 2\pi]$ or any other interval of length 2π because of periodicity). In other words, for any integers m and n

$$\int_{-\pi}^{\pi} \cos mx\, \cos nx\, \mathrm{d}x = 0 \qquad (m \ne n) \tag{1a}$$

$$\int_{-\pi}^{\pi} \sin mx\, \sin nx\, \mathrm{d}x = 0 \qquad (m \ne n) \tag{1b}$$

$$\int_{-\pi}^{\pi} \sin mx\, \cos nx\, \mathrm{d}x = 0 \qquad (m \ne n \text{ or } m = n). \tag{1c}$$

Proof. (1a) For $m \ne n$,

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos mx\, \cos nx\, \mathrm{d}x &= \int_{-\pi}^{\pi} \frac{\cos[(m+n)x] + \cos[(m-n)x]}{2}\, \mathrm{d}x \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \cos[(m+n)x]\, \mathrm{d}x + \frac{1}{2}\int_{-\pi}^{\pi} \cos[(m-n)x]\, \mathrm{d}x \\
&= \frac{1}{2}\, \frac{\sin[(m+n)x]}{m+n}\bigg|_{-\pi}^{\pi} + \frac{1}{2}\, \frac{\sin[(m-n)x]}{m-n}\bigg|_{-\pi}^{\pi} = 0.
\end{aligned}$$

(1b) For $m \neq n$,

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin mx \, \sin nx \, \mathrm{d}x &= \int_{-\pi}^{\pi} \frac{\cos[(m-n)x] - \cos[(m+n)x]}{2} \, \mathrm{d}x \\
&= \frac{1}{2} \int_{-\pi}^{\pi} \cos[(m-n)x] \, \mathrm{d}x - \frac{1}{2} \int_{-\pi}^{\pi} \cos[(m+n)x] \, \mathrm{d}x \\
&= \frac{1}{2} \left. \frac{\sin[(m-n)x]}{m-n} \right|_{-\pi}^{\pi} - \frac{1}{2} \left. \frac{\sin[(m+n)x]}{m+n} \right|_{-\pi}^{\pi} = 0.
\end{aligned}$$

(1c) Case I: $m \neq n$,

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin mx \, \cos nx \, \mathrm{d}x &= \int_{-\pi}^{\pi} \frac{\sin[(m+n)x] + \sin[(m-n)x]}{2} \, \mathrm{d}x \\
&= \frac{1}{2} \int_{-\pi}^{\pi} \sin[(m+n)x] \, \mathrm{d}x + \frac{1}{2} \int_{-\pi}^{\pi} \sin[(m-n)x] \, \mathrm{d}x \\
&= \frac{1}{2} \left(-\frac{\cos[(m+n)x]}{m+n} \right)_{-\pi}^{\pi} + \frac{1}{2} \left(-\frac{\cos[(m-n)x]}{m-n} \right)_{-\pi}^{\pi} \\
&= -\frac{\cos[(m+n)\pi] - \cos[-(m+n)\pi]}{2(m+n)} \\
&\quad - \frac{\cos[(m-n)\pi] - \cos[-(m-n)\pi]}{2(m-n)} \\
&= 0 \qquad \qquad \big[\because \cos(-\theta) = \cos\theta\big].
\end{aligned}$$

Case II: $m = n$,

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin mx \, \cos nx \, \mathrm{d}x &= \int_{-\pi}^{\pi} \sin nx \, \cos nx \, \mathrm{d}x = \frac{1}{2} \int_{-\pi}^{\pi} \sin(2nx) \, \mathrm{d}x \\
&= \frac{1}{2} \left(-\frac{\cos(2nx)}{2n} \right)_{-\pi}^{\pi} = -\frac{\cos(2n\pi) - \cos(-2n\pi)}{4n} = 0
\end{aligned}$$

since $\cos(-\theta) = \cos\theta$. $\blacksquare$

[Lecture 2]

2 Fourier series

2.1 Fourier series of periodic functions with period 2π

Definition: Let $f(x)$ be a periodic function with period 2π such that it can be represented by a series

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

i.e., this series converges and has the sum $f(x)$. Then we write

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{2}$$

and call (2) the *Fourier series* representation of $f(x)$. Moreover, the coefficients $a_0, a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots$ in (2) are referred to as the *Fourier coefficients* of $f(x)$ and are given by

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x, \tag{3a}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos nx\,\mathrm{d}x, \qquad n = 1, 2, 3, \ldots \tag{3b}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin nx\,\mathrm{d}x, \qquad n = 1, 2, 3, \ldots \tag{3c}$$

Derivation of the formulas for the Fourier coefficients:

(3a): Integrating both sides of (2) from $-\pi$ to π, we obtain

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x &= \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\mathrm{d}x \\
&= a_0\int_{-\pi}^{\pi}\mathrm{d}x + \sum_{n=1}^{\infty}\Bigg[a_n \underbrace{\int_{-\pi}^{\pi}\cos nx\,\mathrm{d}x}_{\substack{=0 \\ \text{(by theorem 1)}}} + b_n \underbrace{\int_{-\pi}^{\pi}\sin nx\,\mathrm{d}x}_{\substack{=0 \\ \text{(by theorem 1)}}}\Bigg] \\
&= a_0 \times 2\pi
\end{aligned}$$

$$\Longrightarrow \quad \boxed{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x}$$

(3b): Multiplying both sides of (2) with $\cos mx$—with m being any *fixed* positive integer—and integrating from $-\pi$ to π, we obtain

$$\int_{-\pi}^{\pi} \cos mx\, f(x)\, \mathrm{d}x = \int_{-\pi}^{\pi} \cos mx \left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right] \mathrm{d}x$$

$$= a_0 \underbrace{\int_{-\pi}^{\pi} \cos mx\, \mathrm{d}x}_{\substack{=0 \\ \text{(by theorem 1)}}} + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} \cos mx\, \cos nx\, \mathrm{d}x + b_n \underbrace{\int_{-\pi}^{\pi} \cos mx\, \sin nx\, \mathrm{d}x}_{\substack{=0 \\ \text{(by theorem 1)}}}\right]$$

$$= \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos mx\, \cos nx\, \mathrm{d}x.$$

The integrals obtained in the last equality also vanish for $m \neq n$ by theorem 1. Therefore, the only term that can be nonzero in the above series is for $n = m$. Consequently, we have

$$\int_{-\pi}^{\pi} \cos mx\, f(x)\, \mathrm{d}x = a_m \int_{-\pi}^{\pi} \cos mx\, \cos mx\, \mathrm{d}x = a_m \int_{-\pi}^{\pi} \frac{1 + \cos(2mx)}{2}\, \mathrm{d}x$$

$$= \frac{a_m}{2}\int_{-\pi}^{\pi} \mathrm{d}x + \frac{a_m}{2}\underbrace{\int_{-\pi}^{\pi} \cos(2mx)\, \mathrm{d}x}_{\substack{=0 \\ \text{(by theorem 1)}}} = \frac{a_m}{2} \times 2\pi = \pi a_m$$

$$\implies \qquad a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, \cos mx\, \mathrm{d}x,$$

which is true for all $m = 1, 2, 3, \ldots$. Therefore

$$\implies \quad \boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, \cos nx\, \mathrm{d}x} \qquad n = 1, 2, 3, \ldots$$

(3c): Multiplying both sides of (2) with $\sin mx$—with m being any *fixed* positive integer—and integrating from $-\pi$ to π, we obtain

$$\int_{-\pi}^{\pi} \sin mx\, f(x)\, \mathrm{d}x = \int_{-\pi}^{\pi} \sin mx \left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right] \mathrm{d}x$$

$$= a_0 \underbrace{\int_{-\pi}^{\pi} \sin mx\, \mathrm{d}x}_{\substack{=0 \\ \text{(by theorem 1)}}} + \sum_{n=1}^{\infty}\left[a_n \underbrace{\int_{-\pi}^{\pi} \sin mx\, \cos nx\, \mathrm{d}x}_{\substack{=0 \\ \text{(by theorem 1)}}} + b_n \int_{-\pi}^{\pi} \sin mx\, \sin nx\, \mathrm{d}x\right]$$

$$= \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin mx\, \sin nx\, \mathrm{d}x.$$

The integrals obtained in the last equality also vanish for $m \neq n$ by theorem 1. Therefore, the only term that can be nonzero in the above series is for $n = m$.

Consequently, we have

$$\int_{-\pi}^{\pi} \sin mx\, f(x)\,\mathrm{d}x = b_m \int_{-\pi}^{\pi} \sin mx\, \sin mx\,\mathrm{d}x = b_m \int_{-\pi}^{\pi} \frac{1-\cos(2mx)}{2}\,\mathrm{d}x$$

$$= \frac{b_m}{2}\int_{-\pi}^{\pi}\mathrm{d}x - \frac{b_m}{2}\underbrace{\int_{-\pi}^{\pi}\cos(2mx)\,\mathrm{d}x}_{\substack{=0 \\ \text{(by theorem 1)}}} = \frac{b_m}{2}\times 2\pi = \pi b_m$$

$$\Longrightarrow \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin mx\,\mathrm{d}x,$$

which is true for all $m = 1, 2, 3, \ldots$. Therefore

$$\Longrightarrow \qquad \boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin nx\,\mathrm{d}x} \qquad n = 1, 2, 3, \ldots \qquad \blacksquare$$

Remark: A Fourier series for $f(x)$ does NOT always converge to $f(x)$; the sum of the series at some specific point $x = x_0$ may differ from the value $f(x_0)$ of the function at $x = x_0$.

Theorem 2 (Representation by a Fourier series)**:** Let $f(x)$ be periodic with period 2π and piecewise continuous in the interval $[-\pi, \pi]$. Furthermore, let $f(x)$ has a left-hand derivative and a right-hand derivative at each point of that interval. Then the Fourier series (2) of $f(x)$ (with coefficients (3)) converges. If $f(x)$ is continuous at x_0, the sum of the Fourier series is $f(x_0)$. However, if $f(x)$ is discontinuous at x_0, the sum of the Fourier series is the average of the left- and right-hand limits of $f(x)$ at x_0.

Example: Find the Fourier series of the periodic function $f(x)$ defined by

$$f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0 \\ k & \text{if} \quad 0 < x < \pi \end{cases} \qquad \text{and} \qquad f(x + 2\pi) = f(x).$$

Hence show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

Solution: Let the Fourier series representation of $f(x)$ is given by

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where the *Fourier coefficients* are given by

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos nx\,\mathrm{d}x, \qquad n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin nx\,\mathrm{d}x, \qquad n = 1, 2, 3, \ldots$$

Let us compute the Fourier coefficients as follows.

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{2\pi}\left(\int_{-\pi}^{0} f(x)\,\mathrm{d}x + \int_{0}^{\pi} f(x)\,\mathrm{d}x\right)\\
&= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-k)\,\mathrm{d}x + \int_{0}^{\pi} k\,\mathrm{d}x\right] = \frac{1}{2\pi}\left(-k\int_{-\pi}^{0}\mathrm{d}x + k\int_{0}^{\pi}\mathrm{d}x\right)\\
&= \frac{1}{2\pi}\big(-k\times\pi + k\times\pi\big) = 0.
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos nx\,\mathrm{d}x = \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\,\cos nx\,\mathrm{d}x + \int_{0}^{\pi} f(x)\,\cos nx\,\mathrm{d}x\right)\\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\,\mathrm{d}x + \int_{0}^{\pi} k\,\cos nx\,\mathrm{d}x\right]\\
&= \frac{1}{\pi}\left(-k\int_{-\pi}^{0}\cos nx\,\mathrm{d}x + k\int_{0}^{\pi}\cos nx\,\mathrm{d}x\right)\\
&= \frac{1}{\pi}\left[-k\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + k\left(\frac{\sin nx}{n}\right)_{0}^{\pi}\right] = 0 \qquad \text{for all} \quad n = 1, 2, 3, \ldots
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin nx\,\mathrm{d}x = \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\,\sin nx\,\mathrm{d}x + \int_{0}^{\pi} f(x)\,\sin nx\,\mathrm{d}x\right)\\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,\mathrm{d}x + \int_{0}^{\pi} k\,\sin nx\,\mathrm{d}x\right]\\
&= \frac{1}{\pi}\left(-k\int_{-\pi}^{0}\sin nx\,\mathrm{d}x + k\int_{0}^{\pi}\sin nx\,\mathrm{d}x\right)\\
&= \frac{1}{\pi}\left[-k\left(-\frac{\cos nx}{n}\right)_{-\pi}^{0} + k\left(-\frac{\cos nx}{n}\right)_{0}^{\pi}\right]\\
&= \frac{k}{n\pi}[\cos 0 - \cos(-n\pi)] - \frac{k}{n\pi}[\cos n\pi - \cos 0] = \frac{2k}{n\pi}(1 - \cos n\pi) \quad \text{for} \quad n = 1, 2, 3, \ldots
\end{aligned}$$

Noting that $\cos n\pi = (-1)^n$,

$$b_n = \frac{2k}{n\pi}[1 - (-1)^n] \qquad \text{for} \quad n = 1, 2, 3, \ldots$$

Thus we have,

$$b_1 = \frac{4k}{\pi}, \quad b_2 = 0, \quad b_3 = \frac{4k}{3\pi}, \quad b_4 = 0, \quad b_5 = \frac{4k}{5\pi}, \quad \ldots$$

Therefore, the Fourier series of given $f(x)$ is

$$\boxed{\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \ldots\right)} \tag{\#}$$

Note that the function $f(x)$ is discontinuous at the points $x = n\pi$ for all integers n. Nevertheless, at all other points than these, the function $f(x)$ is continuous and its left-and right-hand derivatives exist. Hence, the Fourier series (#) converges to the given $f(x)$ for all $x \neq n\pi$, where n is an integer. Therefore, at $x = \pi/2$,

$$f\left(\frac{\pi}{2}\right) = \frac{4k}{\pi}\left[\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi}{2}\right) + \frac{1}{7}\sin\left(\frac{7\pi}{2}\right) + \dots\right]$$

$$\implies \quad k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$$

$$\implies \quad \boxed{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}} \qquad (*)$$

Remarks:

1. Note that the sum of the Fourier series (the right-hand side of (#)) in (#) at the point of discontinuity, i.e. at $x = 0$, is 0, which the average of the left-hand and right-hand limits of $f(x)$ at $x = 0$. The left-hand and right-hand limits of $f(x)$ at $x = 0$ are $(h > 0)$

$$\lim_{h\to 0} f(0 - h) = -k \qquad \text{and} \qquad \lim_{h\to 0} f(0 + h) = k,$$

   respectively.

2. (*) is a famous result obtained by Leibniz in 1673. It illustrates that the values of various series with constant terms can be obtained by evaluating Fourier series at specific points. ■

[Lecture 3]

2.2 Fourier series of periodic functions with any period $p = 2L$

Let $f(x)$ be a periodic function with period $2L$. The idea is to employ a change of scale that gives a periodic function $g(y)$ of period 2π.

We introduce a new variable y and the function $g(y) := f(ky)$, where the constant k is to be chosen in such a way that the function $g(y)$ has period 2π. In other words, the constant k is such that

$$g(y+2\pi) = g(y) \qquad \Longrightarrow \qquad f\big(k(y+2\pi)\big) = f(ky + 2k\pi) = f(ky).$$

From the fact that f is a periodic function with period $2L$, it follows that the above equation holds if $2k\pi = 2L$ or $k = L/\pi$. Thus, the function $g(y)$, defined as $g(y) = f\left(\frac{L}{\pi}y\right)$, is a periodic function with period 2π, and hence the Fourier series for the function $g(y) = f\left(\frac{L}{\pi}y\right)$ is given by

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos ny + b_n \sin ny), \tag{5}$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(y)\,\mathrm{d}y, \tag{6a}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(y)\,\cos ny\,\mathrm{d}y, \qquad n = 1, 2, 3, \ldots \tag{6b}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(y)\,\sin ny\,\mathrm{d}y, \qquad n = 1, 2, 3, \ldots \tag{6c}$$

Now, using the scale $y = \pi x/L$, we have $\mathrm{d}y = (\pi/L)\,\mathrm{d}x$, the Fourier series for the function $f(x) = g\left(\frac{\pi x}{L}\right)$ is given by [use the scaling in (5) and (6)]

$$a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right] \tag{7}$$

with the Fourier coefficients given by

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(y)\,\mathrm{d}y = \frac{1}{2\pi}\int_{-L}^{L} g\left(\frac{\pi}{L}x\right) \times \frac{\pi}{L}\mathrm{d}x = \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{d}x, \tag{8a}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(y)\,\cos ny\,\mathrm{d}y = \frac{1}{\pi}\int_{-L}^{L} g\left(\frac{\pi}{L}x\right)\,\cos\left(\frac{n\pi}{L}x\right) \times \frac{\pi}{L}\mathrm{d}x \\ &= \frac{1}{L}\int_{-L}^{L} f(x)\,\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x, \qquad n = 1, 2, 3, \ldots \end{aligned} \tag{8b}$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(y)\,\sin ny\,\mathrm{d}y = \frac{1}{\pi}\int_{-L}^{L} g\left(\frac{\pi}{L}x\right)\,\sin\left(\frac{n\pi}{L}x\right)\,\times\frac{\pi}{L}\mathrm{d}x \\
&= \frac{1}{L}\int_{-L}^{L} f(x)\,\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x, \qquad\qquad n=1,2,3,\ldots \qquad (8c)
\end{aligned}
$$

Here, the fact $g(\pi x/L) = f(x)$ has been used. This leads to the following definition.

Definition (Fourier series of a function with period $p = 2L$)**:** Let $f(x)$ be a periodic function with period $2L$. The series

$$
a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi}{L}x\right) + b_n\sin\left(\frac{n\pi}{L}x\right)\right], \tag{9}
$$

with the coefficients

$$
a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{d}x, \tag{10a}
$$

$$
a_n = \frac{1}{L}\int_{-L}^{L} f(x)\,\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x, \qquad n=1,2,3,\ldots \tag{10b}
$$

$$
b_n = \frac{1}{L}\int_{-L}^{L} f(x)\,\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x, \qquad n=1,2,3,\ldots, \tag{10c}
$$

is referred to as the *Fourier series* of $f(x)$. The coefficients $a_0, a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots$ in (9) are again referred to as the *Fourier coefficients* of $f(x)$.

Theorem 3 (Representation by a Fourier series)**:** Let $f(x)$ be periodic with period $2L$ and piecewise continuous in the interval $[-L, L]$. Furthermore, let $f(x)$ has a left-hand derivative and a right-hand derivative at each point of that interval. Then the Fourier series (9) of $f(x)$ (with coefficients (10)) converges. If $f(x)$ is continuous at x_0, the sum of the Fourier series is $f(x_0)$. However, if $f(x)$ is discontinuous at x_0, the sum of the Fourier series is the average of the left- and right-hand limits of $f(x)$ at x_0.

2.3 Fourier series of even and odd functions

Definition: A function $f(x)$ is said to be an *even* function if $f(-x) = f(x)$ for all x and a function $g(x)$ is said to be an *odd* function if $g(-x) = -g(x)$ for all x.

Remarks:

1. The graph of an even function is symmetric with respect to the vertical (or y-) axis.

2. The graph of an odd function is symmetric with respect to the origin. In other words, the graph of an odd function has 180° rotation symmetry with respect to the origin, meaning that its graph remains unchanged after rotation of 180° with respect to the origin.

3. For an even function $f(x)$ and an odd function $g(x)$, we have the following:

$$
\int_{-L}^{L} f(x)\,\mathrm{d}x = 2\int_{0}^{L} f(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_{-L}^{L} g(x)\,\mathrm{d}x = 0.
$$

Therefore, it is not difficult to obtain the following theorem.

Theorem 4: (i) The Fourier series of an *even* function $f(x)$ of period $2L$ is a **Fourier cosine series**

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right), \tag{11}$$

with the coefficients

$$a_0 = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x, \tag{12a}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\,\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x, \qquad n = 1, 2, 3, \ldots \tag{12b}$$

(ii) The Fourier series of an *odd* function $f(x)$ of period $2L$ is a **Fourier sine series**

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right), \tag{13}$$

with the coefficients

$$b_n = \frac{2}{L}\int_0^L f(x)\,\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x, \qquad n = 1, 2, 3, \ldots \tag{14}$$

Remarks:

1. **Even function of period** 2π: If $f(x)$ is even and $L = \pi$, the Fourier series of $f(x)$ is the Fourier cosine series, given by

$$a_0 + \sum_{n=1}^{\infty} a_n \cos nx \tag{15}$$

with coefficients

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x, \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\,\cos nx\,\mathrm{d}x, \quad n = 1, 2, 3, \ldots \tag{16}$$

2. **Odd function of period** 2π: If $f(x)$ is odd and $L = \pi$, the Fourier series of $f(x)$ is the Fourier sine series, given by

$$\sum_{n=1}^{\infty} b_n \sin nx \tag{17}$$

with coefficients

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\,\sin nx\,\mathrm{d}x, \qquad n = 1, 2, 3, \ldots \tag{18}$$

Theorem: (i) The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 and f_2.

(ii) The Fourier coefficients of cf, where c is a constant, are c times the corresponding Fourier coefficients of f.

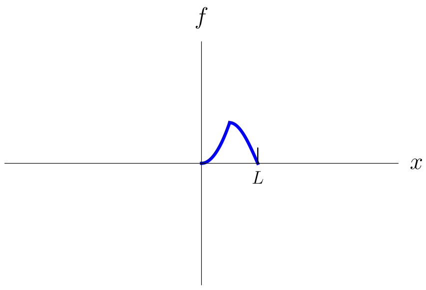
2.4 Half-range expansions

The goal is to represent a given function $f(x)$ defined on an interval $[0, L]$ by a Fourier series.

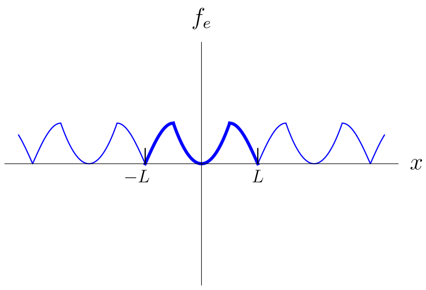
Definition (Even and odd periodic extensions of a function)**:** Let $f(x)$ be a function defined on an interval $[0, L]$. The function f can be extended to $[-L, L]$ as

(i) an even function, by letting $f(-x) = f(x)$, and then it can be extended periodically with period $p = 2L$. Let us call this function f_e. Note that f_e is an even and periodic function with period $2L$. Hence f_e is referred to as the *even periodic extension* of f.

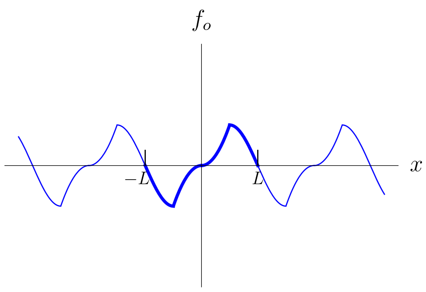
(ii) an odd function, by letting $f(-x) = -f(x)$, and then it can be extended periodically with period $p = 2L$. Let us call this function f_o. Note that f_o is an odd and periodic function with period $2L$. Hence f_o is referred to as the *odd periodic extension* of f.



(a) The given function $f(x)$ defined on the interval $[0, L]$



(b) Even extension of the given function $f(x)$ to interval $[-L, L]$ (depicted by heavy curve) and the periodic extension of period $2L$ to the x-axis.



(c) Odd extension of the given function $f(x)$ to interval $[-L, L]$ (depicted by heavy curve) and the periodic extension of period $2L$ to the x-axis.

Figure 1: Even, odd and periodic extensions of a given function.

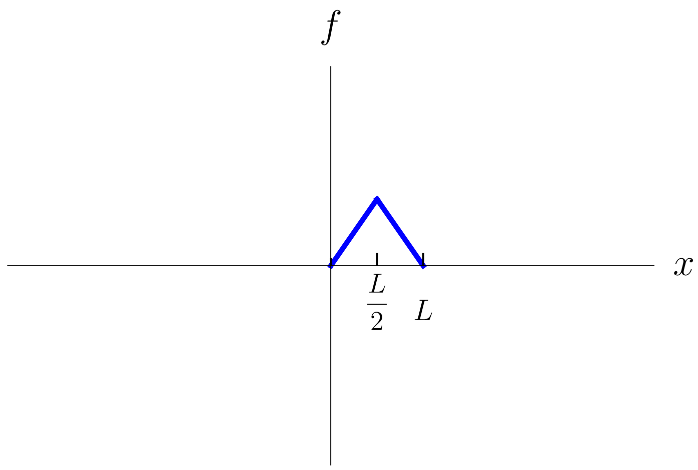
Figure 1 illustrates a function $f(x)$ defined on the interval $[0, L]$ and its even and odd periodic extensions.

Remark: The functions f_e and f_o are periodic with period $2L$ and can be represented by Fourier cosine series and Fourier sine series, respectively.

Example: Find the two half-range expansions of the function

$$f(x) = \begin{cases} \frac{2k}{L}x & \text{if} \quad 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x) & \text{if} \quad \frac{L}{2} < x < L \end{cases}$$

shown in the following figure.



Solution: (a) **Even periodic extension**: The even periodic extension f_e of given f is illustrated in figure 2. The function $f_e(x)$ is represented by a Fourier cosine series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right),$$

where

$$\begin{aligned} a_0 &= \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x = \frac{1}{L}\left[\int_0^{L/2} \frac{2k}{L}x\,\mathrm{d}x + \int_{L/2}^L \frac{2k}{L}(L-x)\,\mathrm{d}x\right] \\ &= \frac{2k}{L^2}\left[\frac{x^2}{2}\bigg|_0^{L/2} + \left(-\frac{(L-x)^2}{2}\right)\bigg|_{L/2}^L\right] = \frac{2k}{L^2}\left[\frac{L^2}{8} + \frac{L^2}{8}\right] = \frac{k}{2} \end{aligned}$$

and for $n = 1, 2, 3, \ldots$

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\,\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x \\ &= \frac{2}{L}\left[\int_0^{L/2} \frac{2k}{L}x\,\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x + \int_{L/2}^L \frac{2k}{L}(L-x)\,\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x\right] \\ &= \frac{4k}{L^2}\int_0^{L/2} x\,\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x + \frac{4k}{L^2}\int_{L/2}^L (L-x)\,\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x. \end{aligned}$$

Applying integration by parts, we get

$$
\begin{aligned}
a_n &= \frac{4k}{L^2}\left[\left\{x \times \frac{L}{n\pi}\sin\left(\frac{n\pi}{L}x\right)\right\}_0^{L/2} - \frac{L}{n\pi}\int_0^{L/2}\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x\right] \\
&\quad + \frac{4k}{L^2}\left[\left\{(L-x)\times \frac{L}{n\pi}\sin\left(\frac{n\pi}{L}x\right)\right\}_{L/2}^{L} + \frac{L}{n\pi}\int_{L/2}^{L}\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x\right] \\
&= \frac{4k}{L^2}\left[\cancel{\frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right)} + \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{L}x\right)\Big|_0^{L/2}\right] \\
&\quad + \frac{4k}{L^2}\left[-\cancel{\frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right)} - \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{L}x\right)\Big|_{L/2}^{L}\right] \\
&= \frac{4k}{n^2\pi^2}\left[\left\{\cos\left(\frac{n\pi}{2}\right) - 1\right\} - \left\{\cos n\pi - \cos\left(\frac{n\pi}{2}\right)\right\}\right] \\
&= \frac{4k}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}\right) - \cos n\pi - 1\right].
\end{aligned}
$$

This implies that

$$
a_1 = 0,\quad a_2 = -\frac{16k}{2^2\pi^2},\quad a_3 = 0,\quad a_4 = 0,\quad a_5 = 0,\quad a_6 = -\frac{16k}{6^2\pi^2},\quad a_7 = 0,\quad a_8 = 0,
$$
$$
a_9 = 0,\quad a_{10} = -\frac{16k}{10^2\pi^2},\quad a_{11} = 0,\quad \ldots
$$

Hence, the first half-range expansion of the given $f(x)$ is

$$
\frac{k}{2} - \frac{16k}{\pi^2}\left[\frac{1}{2^2}\cos\left(\frac{2\pi}{L}x\right) + \frac{1}{6^2}\cos\left(\frac{6\pi}{L}x\right) + \frac{1}{10^2}\cos\left(\frac{10\pi}{L}x\right) + \ldots\right]
$$

This Fourier cosine series represents the even periodic extension of the given function $f(x)$, of period $2L$.

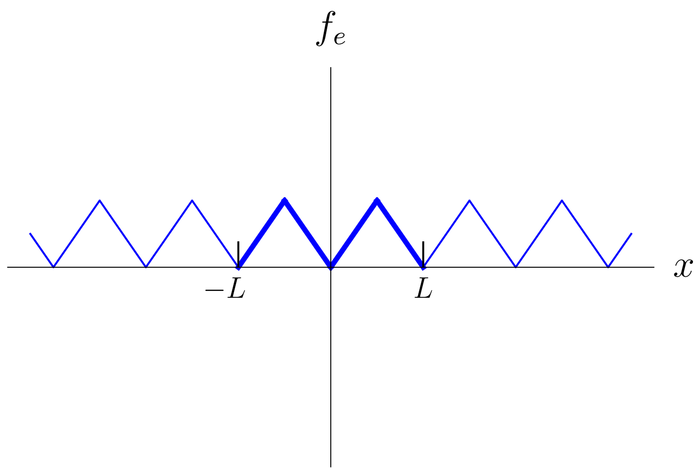


Figure 2: Even periodic extension of given f

(b) **Odd periodic extension**: The odd periodic extension f_o of given f is illustrated in figure 3. The function $f_o(x)$ is represented by a Fourier sine series

$$
\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right),
$$

where, for $n = 1, 2, 3, \ldots$,

$$\begin{aligned}
b_n &= \frac{2}{L}\int_0^L f(x)\,\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x \\
&= \frac{2}{L}\left[\int_0^{L/2}\frac{2k}{L}x\,\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x + \int_{L/2}^{L}\frac{2k}{L}(L-x)\,\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x\right] \\
&= \frac{4k}{L^2}\int_0^{L/2}x\,\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x + \frac{4k}{L^2}\int_{L/2}^{L}(L-x)\,\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x.
\end{aligned}$$

Applying integration by parts, we get

$$\begin{aligned}
b_n &= \frac{4k}{L^2}\left[\left\{-x\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\right\}_0^{L/2} + \frac{L}{n\pi}\int_0^{L/2}\cos\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x\right] \\
&\quad + \frac{4k}{L^2}\left[\left\{-(L-x)\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\right\}_{L/2}^{L} - \frac{L}{n\pi}\int_{L/2}^{L}\cos\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x\right] \\
&= \frac{4k}{L^2}\left[\cancel{-\frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)} + \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{L}x\right)\Big|_0^{L/2}\right] \\
&\quad + \frac{4k}{L^2}\left[\cancel{\frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)} - \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{L}x\right)\Big|_{L/2}^{L}\right] \\
&= \frac{4k}{n^2\pi^2}\left[\left\{\sin\left(\frac{n\pi}{2}\right) - 0\right\} - \left\{\sin n\pi - \sin\left(\frac{n\pi}{2}\right)\right\}\right] \\
&= \frac{8k}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right).
\end{aligned}$$

This implies that

$$b_1 = \frac{8k}{1^2\pi^2}, \quad b_2 = 0, \quad b_3 = -\frac{8k}{3^2\pi^2}, \quad b_4 = 0, \quad b_5 = \frac{8k}{5^2\pi^2}, \quad b_6 = 0, \quad \ldots$$

Hence, the second half-range expansion of the given $f(x)$ is

$$\frac{8k}{\pi^2}\left[\frac{1}{1^2}\sin\left(\frac{\pi}{L}x\right) - \frac{1}{3^2}\sin\left(\frac{3\pi}{L}x\right) + \frac{1}{5^2}\sin\left(\frac{5\pi}{L}x\right) - \frac{1}{7^2}\sin\left(\frac{7\pi}{L}x\right) + \ldots\right]$$

This Fourier sine series represents the odd periodic extension of the given function $f(x)$, of period $2L$.

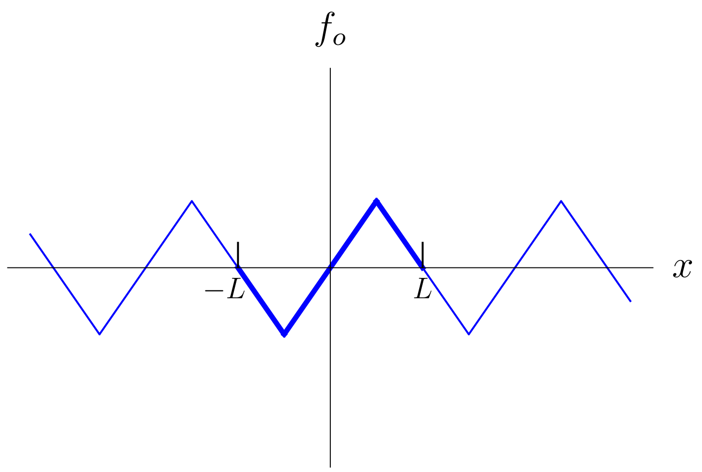


Figure 3: Odd periodic extension of given f

■

[Lectures 4 and 5]

3 Partial differential equations

3.1 Definitions

Definition: A *partial differential equation* (PDE) is an equation involving one or more partial derivatives of an (unknown) function, let us say u, that depends on two or more variables, often time t and one or several variables in space.

Examples: $\dfrac{\partial u}{\partial t} = u\dfrac{\partial u}{\partial x} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^3 u}{\partial z^3}, \quad u_{xx} + u\,u_y + u_{yz} = x^2 + y^2 + u$

Definition: The order of the highest derivative is called the *order* of the PDE.

Definition: A PDE is said to be *linear* if it is of the first degree in the unknown function u and its partial derivatives. Otherwise it is called *nonlinear*.

Definition: A linear PDE is said to be *homogeneous* if each of its terms contains either the unknown function u or one of its partial derivatives. Otherwise, the PDE is called *nonhomogeneous* or *inhomogeneous*.

Examples:

(i) $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}$ **One-dimensional wave equation**

(ii) $\dfrac{\partial u}{\partial t} = c^2\dfrac{\partial^2 u}{\partial x^2}$ **One-dimensional heat equation**

(iii) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ **Two-dimensional Laplace equation**

(iv) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x,y)$ **Two-dimensional Poisson equation**

(v) $\dfrac{\partial^2 u}{\partial t^2} = \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right)$ **Two-dimensional wave equation**

(vi) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ **Three-dimensional Laplace equation**

PDEs (i)–(iii), (v) and (vi) are homogeneous while (iv) is nonhomogeneous for $f(x,y) \neq 0$.

Remark: Second-order PDEs are the most important ones in applications. Our syllabus contains only linear second-order homogeneous PDEs in two variables. These are one-dimensional wave equation, one-dimensional heat equation and two-dimensional Laplace equation.

3.2 Classification of linear second-order PDEs in two variables

The general second-order linear PDE has the following form:

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D\,u_x + E\,u_y + F\,u = G, \tag{19}$$

where the coefficients A, B, C, D, F and the free term G are in general functions of the independent variables x and y, but do not depend on the unknown function u. The classification of second-order equations depends on the form of the leading part of the equations consisting of the second-order terms. So, for simplicity of notation, we combine the lower-order terms and rewrite the above equation in the following form

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + I(x, y, u, u_x, u_y) = 0. \tag{20}$$

The type of the above equation depends on the sign of the quantity

$$\Delta(x, y) = B^2(x, y) - 4A(x, y)\,C(x, y), \tag{21}$$

which is called the *discriminant* for (20). The classification of second-order linear PDEs is given by the following.

Definition: At the point (x_0, y_0), the second-order linear PDE (20) is called

(i) *elliptic*, if $\Delta(x_0, y_0) < 0$

(ii) *parabolic*, if $\Delta(x_0, y_0) = 0$

(iii) *hyperbolic*, if $\Delta(x_0, y_0) > 0$

Remarks:

1. For each of these categories, equation (20) and its solutions have distinct features.

2. In general a second order equation may be of one type at a specific point, and of another type at some other point.

3. The terminology is motivated from the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

which—for A, B, C, D, E, F being constants—represents a conic section in the xy-plane and the different types of conic sections arising are determined by $B^2 - 4AC$.

4. The canonical examples of the elliptic, parabolic and hyperbolic PDEs are the two-dimensional Laplace equation, one-dimensional heat equation and one-dimensional wave equations, i.e.,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (\textbf{Laplace equation})$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (\textbf{Heat equation})$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (\textbf{Wave equation})$$

3.3 Boundary conditions

Let u denotes the dependent variable in a boundary value problem (BVP).

Definition (Dirichlet condition)**:** A condition that prescribes the values of u itself along a portion of the boundary is known as a *Dirichlet condition*.

Definition (Neumann condition)**:** A condition that prescribes the values of the normal derivatives $\partial u/\partial \hat{n}$ on a portion of the boundary is known as a *Neumann condition*. Here, $\hat{n}$ denotes the unit outward normal to the boundary.

Definition (Robin condition)**:** A condition that prescribes the values of $hu + \partial u/\partial \hat{n}$ at boundary points is known as a *Robin condition*. Here, h is either a constant or a function of the independent variables.

Definition (Cauchy condition)**:** If a PDE in u is of second order with respect to one of the independent variables t (time) and if the values of both u and u_t are prescribed at $t = 0$, the boundary condition is known as a *Cauchy-type* condition with respect to t.

3.4 Laplace equation

Definition (Two-dimensional Laplace equation)**:** The two-dimensional Laplace equation is given by

$$\nabla^2 u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{22}$$

Note: the Laplace equation is also referred to as the *potential equation*.

3.4.1 Boundary value problems for the Laplace equation

The function u, in addition to satisfying the Laplace equation (22) in a bounded region D in the xy-plane, should also satisfy certain boundary conditions on the boundary ∂D of this region. Such problems are referred to as *boundary value problems* (*BVPs*) for the Laplace equation.

We shall consider the following two types of BVPs (or their combinations, in the sense that some boundary conditions in the problem are Dirichlet type while the others are of Neumann type; the problem in this case will neither be called as a Dirichlet problem nor a Neumann problem).

Definition (Dirichlet problem)**:** The *Dirichlet problem* for the Laplace equation is to determine the function $\varphi(x, y)$ continuous on D such that $\nabla^2\varphi(x, y) = 0$ within D and $\varphi(x, y) = f(x, y)$ on ∂D, where $f(x, y)$ is a given function.

Definition (Neumann problem)**:** The *Neumann problem* for the Laplace equation is to determine the function $\varphi(x, y)$ continuous on D such that $\nabla^2\varphi(x, y) = 0$ within D and $\partial\varphi(x, y)/\partial\hat{n} = f(x, y)$ on ∂D, where $f(x, y)$ is a given function, $\hat{n}$ denotes the unit outward normal to the boundary ∂D, and $\partial\varphi/\partial\hat{n}$ denotes the directional derivative or the derivative in the direction of $\hat{n}$.

3.4.2 Solution by the method of separation of variables

We first solve the Laplace equation (22) without considering the boundary conditions. Let $u(x, y) = X(x)\, Y(y)$ be a solution of the Laplace equation (22). Therefore, this solution must satisfy the Laplace equation (22). Substituting this solution in (22), we obtain

$$X''(x)\, Y(y) + X(x)\, Y''(y) = 0,$$

where prime denotes the derivative with respect to an independent variable, i.e. $X''(x) = \dfrac{\mathrm{d}^2X}{\mathrm{d}x^2}$ and $Y''(y) = \dfrac{\mathrm{d}^2Y}{\mathrm{d}y^2}$. The above equation can be rewritten as

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

The left-hand side of the above equation is a function of x alone whereas the right-hand side is a function of y alone. Therefore, the left- and right-hand sides of the above equation must be constant. Let this constant be k. In other words,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = k.$$

From this equation, we obtain two *ordinary* differential equations (ODEs), namely

$$\frac{\mathrm{d}^2X}{\mathrm{d}x^2} - kX = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2Y}{\mathrm{d}y^2} + kY = 0. \tag{23}$$

In order to solve these ODEs, there are three possibilities for k. It can be zero, positive or negative. In the following, we shall consider these three cases separately.

Case 1: $k = 0$

In this case, eqs. (23) reduce to

$$\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2Y}{\mathrm{d}y^2} = 0.$$

The above equations yield the solution $X(x) = c_1x + c_2$ and $Y(y) = c_3y + c_4$, where c_1, c_2, c_3, c_4 are the integration constants and will be determined from the boundary conditions. Consequently, the solution of the Laplace equation (22) in this case is

$$\boxed{u(x, y) = (c_1x + c_2)(c_3y + c_4)} \tag{24}$$

Case 2: $k > 0$

Let $k = \lambda^2$ and $\lambda \neq 0$. In this case, eqs. (23) reduce to

$$\frac{\mathrm{d}^2X}{\mathrm{d}x^2} - \lambda^2X = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2Y}{\mathrm{d}y^2} + \lambda^2Y = 0.$$

The above equations yield the solution $X(x) = c_5\mathrm{e}^{\lambda x} + c_6\mathrm{e}^{-\lambda x}$ and $Y(y) = c_7 \cos \lambda y + c_8 \sin \lambda y$, where c_5, c_6, c_7, c_8 are the integration constants and will be determined from the boundary conditions. Consequently, the solution of the Laplace equation (22) in this case is

$$\boxed{u(x, y) = \left(c_5\mathrm{e}^{\lambda x} + c_6\mathrm{e}^{-\lambda x}\right)(c_7 \cos \lambda y + c_8 \sin \lambda y)} \tag{25}$$

Case 3: $k < 0$

Let $k = -\lambda^2$ and $\lambda \neq 0$. In this case, eqs. (23) reduce to

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \lambda^2 X = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} - \lambda^2 Y = 0.$$

The above equations yield the solution $X(x) = c_9 \cos \lambda x + c_{10} \sin \lambda x$ and $Y(y) = c_{11} \mathrm{e}^{\lambda y} + c_{12} \mathrm{e}^{-\lambda y}$, where $c_9, c_{10}, c_{11}, c_{12}$ are the integration constants and will be determined from the boundary conditions. Consequently, the solution of the Laplace equation (22) in this case is

$$\boxed{u(x,y) = (c_9 \cos \lambda x + c_{10} \sin \lambda x)\big(c_{11} \mathrm{e}^{\lambda y} + c_{12} \mathrm{e}^{-\lambda y}\big)} \tag{26}$$

Example (Dirichlet problem for a rectangle)**:** Find the solution of the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{27}$$

satisfying the boundary conditions

$$\begin{aligned} &u(x,0) = 0, \qquad && u(x,b) = 0, \qquad && \text{for} \quad 0 < x < a \\ &u(0,y) = 0, && u(a,y) = f(y) && \text{for} \quad 0 < y < b. \end{aligned}$$

Solution: Let $u(x,y) = X(x)\,Y(y)$ be a solution of the Laplace equation (27). Then, as shown above, we obtain two ODEs from the Laplace equation (27), which read

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - kX = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} + kY = 0. \tag{28}$$

We shall consider the three cases, namely $k = 0$, $k < 0$ and $k > 0$.

Case 1: $k = 0$

In this case, we obtain the solution of the Laplace equation (27) as (see § 3.4.2)

$$u(x,y) = (c_1 x + c_2)(c_3 y + c_4). \tag{29}$$

We now compute the integration constants c_1, c_2, c_3, c_4 using the given boundary conditions.

$$u(x,0) = 0 \quad \Longrightarrow \quad (c_1 x + c_2)c_4 = 0 \quad \Longrightarrow \quad \boxed{c_4 = 0}$$

since $c_1 x + c_2$ cannot be zero for all $x \in (0,a)$.

$$u(0,y) = 0 \quad \Longrightarrow \quad c_2(c_3 y + c_4) = 0 \quad \Longrightarrow \quad \boxed{c_2 = 0}$$

since $c_3 y + c_4$ cannot be zero for all $y \in (0,b)$. Therefore, with these two boundary conditions the solution of the Laplace equation (27) in this case reduces to

$$u(x,y) = c_1 c_3 xy. \tag{30}$$

Now, let us apply the remaining boundary conditions.

$$u(x,b) = 0 \quad \Longrightarrow \quad c_1 c_3 x b = 0 \quad \Longrightarrow \quad c_1 c_3 x = 0$$

since $b \neq 0$. With this, the solution (30) further reduces to

$$u(x,y) = 0,$$

which cannot be the general solution of the Laplace equation with the given boundary conditions. Therefore, $k = 0$ is not possible.

Case 2: $k < 0$

Let $k = -\lambda^2$ and $\lambda \neq 0$. In this case, we obtain the solution of the Laplace equation (27) as (see § 3.4.2)

$$u(x, y) = (c_9 \cos \lambda x + c_{10} \sin \lambda x)\big(c_{11} \mathrm{e}^{\lambda y} + c_{12} \mathrm{e}^{-\lambda y}\big). \tag{31}$$

We now compute the integration constants $c_9, c_{10}, c_{11}, c_{12}$ using the given boundary conditions.

$$u(x, 0) = 0 \quad \Longrightarrow \quad (c_9 \cos \lambda x + c_{10} \sin \lambda x)(c_{11} + c_{12}) = 0 \quad \Longrightarrow \quad \boxed{c_{12} = -c_{11}}$$

since $c_9 \cos \lambda x + c_{10} \sin \lambda x$ cannot be zero for all $x \in (0, a)$.

$$u(0, y) = 0 \quad \Longrightarrow \quad c_9\big(c_{11} \mathrm{e}^{\lambda y} + c_{12} \mathrm{e}^{-\lambda y}\big) = 0 \quad \Longrightarrow \quad \boxed{c_9 = 0}$$

since $c_{11} \mathrm{e}^{\lambda y} + c_{12} \mathrm{e}^{-\lambda y}$ cannot be zero for all $y \in (0, b)$. Therefore, with these two boundary conditions the solution of the Laplace equation (27) in this case reduces to

$$u(x, y) = c_{10} c_{11} \sin \lambda x \big(\mathrm{e}^{\lambda y} - \mathrm{e}^{-\lambda y}\big). \tag{32}$$

Now, let us apply the remaining boundary conditions.

$$u(x, b) = 0 \quad \Longrightarrow \quad c_{10} c_{11} \sin \lambda x \big(\mathrm{e}^{\lambda b} - \mathrm{e}^{-\lambda b}\big) = 0 \quad \Longrightarrow \quad c_{10} c_{11} \sin \lambda x = 0$$

since $\mathrm{e}^{\lambda b} - \mathrm{e}^{-\lambda b} \neq 0$ (as $\lambda, b \neq 0$). With this, the solution (32) further reduces to

$$u(x, y) = 0,$$

which cannot be the general solution of the Laplace equation with the given boundary conditions. Therefore, $k < 0$ is also not possible.

Case 3: $k > 0$

Let $k = \lambda^2$ and $\lambda \neq 0$. In this case, we obtain the solution of the Laplace equation (27) as (see § 3.4.2)

$$u(x, y) = \big(c_5 \mathrm{e}^{\lambda x} + c_6 \mathrm{e}^{-\lambda x}\big)(c_7 \cos \lambda y + c_8 \sin \lambda y). \tag{33}$$

We now compute the integration constants c_5, c_6, c_7, c_8 using the given boundary conditions.

$$u(x, 0) = 0 \quad \Longrightarrow \quad \big(c_5 \mathrm{e}^{\lambda x} + c_6 \mathrm{e}^{-\lambda x}\big) c_7 = 0 \quad \Longrightarrow \quad \boxed{c_7 = 0}$$

since $c_5 \mathrm{e}^{\lambda x} + c_6 \mathrm{e}^{-\lambda x}$ cannot be zero for all $x \in (0, a)$.

$$u(0, y) = 0 \quad \Longrightarrow \quad (c_5 + c_6)(c_7 \cos \lambda y + c_8 \sin \lambda y) = 0 \quad \Longrightarrow \quad \boxed{c_6 = -c_5}$$

since $c_7 \cos \lambda y + c_8 \sin \lambda y$ cannot be zero for all $y \in (0, b)$. Therefore, with these two boundary conditions the solution of the Laplace equation (27) in this case reduces to

$$u(x, y) = c_5 c_8 \big(\mathrm{e}^{\lambda x} - \mathrm{e}^{-\lambda x}\big) \sin \lambda y = A\big(\mathrm{e}^{\lambda x} - \mathrm{e}^{-\lambda x}\big) \sin \lambda y, \tag{34}$$

where $A = c_5 c_8$ is another constant. Now, let us apply the remaining boundary conditions.

$$u(x,b) = 0 \quad \Longrightarrow \quad A\left(e^{\lambda x} - e^{-\lambda x}\right) \sin \lambda b = 0 \quad \Longrightarrow \quad \boxed{\sin \lambda b = 0}$$

since $e^{\lambda x} - e^{-\lambda x}$ cannot be zero for all $x \in (0,a)$ and $A \neq 0$ for a nontrivial solution. This yields

$$\sin \lambda b = \sin n\pi \quad \Longrightarrow \quad \lambda = \frac{n\pi}{b}, \qquad n = \pm 1, \pm 2, \pm 3, \ldots$$

Note that we have not taken $n = 0$ because $n = 0$ will yield $\lambda = 0$ whereas $\lambda \neq 0$ in this case. With this, the solution (34) becomes

$$u(x,y) = A\left(e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}}\right) \sin \frac{n\pi y}{b}, \qquad n = \pm 1, \pm 2, \pm 3, \ldots$$

Neglecting the constant of proportionality A, we conclude that functions

$$u_n(x,y) = \left(e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}}\right) \sin \frac{n\pi y}{b}, \qquad n = 1, 2, 3, \ldots \tag{35}$$

satisfy the Laplace equation (27) and the above three boundary conditions. Note that for negative integers n, we obtain essentially the same solutions. Therefore negative values of n have been omitted. The functions in (35) are referred to as the *eigenfunctions* for the given problem.

Therefore, the most general solution is given by

$$u(x,y) = \sum_{n=1}^{\infty} A_n \left(e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}}\right) \sin \frac{n\pi y}{b}.$$

Now we shall use the final given boundary condition.

$$u(a,y) = f(y) \quad \Longrightarrow \quad \sum_{n=1}^{\infty} A_n \left(e^{\frac{n\pi a}{b}} - e^{-\frac{n\pi a}{b}}\right) \sin \frac{n\pi y}{b} = f(y),$$

which is a Fourier sine series for $f(y)$, $0 < y < b$. Therefore, the Fourier coefficients in the series are given by

$$A_n \left(e^{\frac{n\pi a}{b}} - e^{-\frac{n\pi a}{b}}\right) = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi y}{b} \, dy,$$

which gives

$$A_n = \frac{2}{b} \frac{1}{e^{\frac{n\pi a}{b}} - e^{-\frac{n\pi a}{b}}} \int_0^b f(y) \sin \frac{n\pi y}{b} \, dy.$$

Therefore, the solution of the given problem is

$$\boxed{u(x,y) = \frac{2}{b} \sum_{n=1}^{\infty} \frac{e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}}}{e^{\frac{n\pi a}{b}} - e^{-\frac{n\pi a}{b}}} \left(\int_0^b f(y) \sin \frac{n\pi y}{b} \, dy \right) \sin \frac{n\pi y}{b}}$$

or

$$\boxed{u(x,y) = \frac{2}{b} \sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi x}{b}}{\sinh \frac{n\pi a}{b}} \sin \frac{n\pi y}{b} \int_0^b f(y) \sin \frac{n\pi y}{b} \, dy}$$

Example: Find the solution of the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{36}$$

satisfying the boundary conditions

$$\begin{aligned} &\frac{\partial u}{\partial x}(0, y) = 0, && 0 < y < 1 \\ &\frac{\partial u}{\partial x}(1, y) = 0, && 0 < y < 1 \\ &\frac{\partial u}{\partial y}(x, 0) = 0, && 0 < x < 1 \\ &u(x, 1) = x(1 - x), && 0 < x < 1. \end{aligned}$$

Solution: Let $u(x, y) = X(x)\, Y(y)$ be a solution of the Laplace equation (27). Then, as shown above, we obtain two ODEs from the Laplace equation (27), which read

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - kX = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} + kY = 0. \tag{37}$$

We shall consider the three cases, namely $k = 0$, $k > 0$ and $k < 0$.

Case 1: $k = 0$

In this case, we obtain the solution of the Laplace equation (27) as (see § 3.4.2)

$$u(x, y) = (c_1 x + c_2)(c_3 y + c_4). \tag{38}$$

This implies that

$$\begin{aligned} \frac{\partial u}{\partial x}(x, y) &= c_1(c_3 y + c_4), \\ \frac{\partial u}{\partial y}(x, y) &= c_3(c_1 x + c_2). \end{aligned}$$

We now compute the integration constants c_1, c_2, c_3, c_4 using the given boundary conditions. The boundary condition

$$\begin{aligned} \frac{\partial u}{\partial x}(0, y) = 0 &\implies c_1(c_3 y + c_4) = 0 \implies \boxed{c_1 = 0} \\ \frac{\partial u}{\partial x}(1, y) = 0 &\implies c_1(c_3 y + c_4) = 0 \implies \boxed{c_1 = 0} \\ \frac{\partial u}{\partial y}(x, 0) = 0 &\implies c_3(c_1 x + c_2) = 0 \implies \boxed{c_3 = 0} \end{aligned}$$

With these conditions, the solution becomes

$$u(x, y) = c_2 c_4.$$

Dropping the constant, $u(x, y) = 1$ is the eigensolution in this case that satisfies the Laplace equation and the above three boundary conditions. [The last boundary condition will be checked later.]

Case 2: $k > 0$

Let $k = \lambda^2$ and $\lambda \neq 0$. In this case, we obtain the solution of the Laplace equation (27) as (see § 3.4.2)

$$u(x,y) = \left(c_5 \mathrm{e}^{\lambda x} + c_6 \mathrm{e}^{-\lambda x}\right)\left(c_7 \cos \lambda y + c_8 \sin \lambda y\right). \tag{39}$$

Hence

$$\frac{\partial u}{\partial x}(x,y) = \lambda\left(c_5 \mathrm{e}^{\lambda x} - c_6 \mathrm{e}^{-\lambda x}\right)\left(c_7 \cos \lambda y + c_8 \sin \lambda y\right).$$

We now compute the integration constants c_5, c_6, c_7, c_8 using the given boundary conditions.

$$\frac{\partial u}{\partial x}(0,y) = 0 \quad \Longrightarrow \quad \lambda(c_5 - c_6)(c_7 \cos \lambda y + c_8 \sin \lambda y) = 0 \quad \Longrightarrow \quad \boxed{c_6 = c_5}$$

since $\lambda \neq 0$ and $c_7 \cos \lambda y + c_8 \sin \lambda y$ cannot be zero for all $y \in (0,1)$. Using this result, the next boundary condition gives

$$\frac{\partial u}{\partial x}(1,y) = 0 \quad \Longrightarrow \quad c_5 \lambda\left(\mathrm{e}^{\lambda} - \mathrm{e}^{-\lambda}\right)(c_7 \cos \lambda y + c_8 \sin \lambda y) = 0 \quad \Longrightarrow \quad \boxed{c_5 = 0}$$

since $\lambda \neq 0$, $\mathrm{e}^{\lambda} \neq \mathrm{e}^{-\lambda}$ and $c_7 \cos \lambda y + c_8 \sin \lambda y$ cannot be zero for all $y \in (0,1)$. Since c_5 and c_6 both vanish, solution (39) reduces to

$$u(x,y) = 0,$$

which cannot be the general solution of the Laplace equation with the given boundary conditions. Therefore, $k > 0$ is also not possible and we do not need to apply the remaining boundary conditions in this case as well.

Case 3: $k < 0$

Let $k = -\lambda^2$ and $\lambda \neq 0$. In this case, we obtain the solution of the Laplace equation (27) as (see § 3.4.2)

$$u(x,y) = (c_9 \cos \lambda x + c_{10} \sin \lambda x)\left(c_{11} \mathrm{e}^{\lambda y} + c_{12} \mathrm{e}^{-\lambda y}\right). \tag{40}$$

Hence

$$\begin{aligned}
\frac{\partial u}{\partial x}(x,y) &= \lambda(-c_9 \sin \lambda x + c_{10} \cos \lambda x)\left(c_{11} \mathrm{e}^{\lambda y} + c_{12} \mathrm{e}^{-\lambda y}\right),\\
\frac{\partial u}{\partial y}(x,y) &= \lambda(c_9 \cos \lambda x + c_{10} \sin \lambda x)\left(c_{11} \mathrm{e}^{\lambda y} - c_{12} \mathrm{e}^{-\lambda y}\right).
\end{aligned}$$

We now compute the integration constants $c_9, c_{10}, c_{11}, c_{12}$ using the given boundary conditions.

$$\frac{\partial u}{\partial x}(0,y) = 0 \quad \Longrightarrow \quad \lambda c_{10}\left(c_{11} \mathrm{e}^{\lambda y} + c_{12} \mathrm{e}^{-\lambda y}\right) = 0 \quad \Longrightarrow \quad \boxed{c_{10} = 0}$$

since $\lambda \neq 0$ and $c_{11} \mathrm{e}^{\lambda y} + c_{12} \mathrm{e}^{-\lambda y}$ cannot be zero for all $y \in (0,1)$.

$$\frac{\partial u}{\partial y}(x,0) = 0 \quad \Longrightarrow \quad \lambda(c_9 \cos \lambda x + c_{10} \sin \lambda x)(c_{11} - c_{12}) = 0 \quad \Longrightarrow \quad \boxed{c_{12} = c_{11}}$$

since $\lambda \neq 0$ and $c_9 \cos \lambda x + c_{10} \sin \lambda x$ cannot be zero for all $x \in (0, 1)$. Therefore, with these two boundary conditions the solution of the Laplace equation (36) in this case reduces to

$$u(x, y) = c_9 c_{11} \cos \lambda x \left(\mathrm{e}^{\lambda y} + \mathrm{e}^{-\lambda y}\right) = A \cos \lambda x \left(\mathrm{e}^{\lambda y} + \mathrm{e}^{-\lambda y}\right), \tag{41}$$

where $A = c_9 c_{11}$ is another constant. Hence

$$\frac{\partial u}{\partial x}(x, y) = -A\lambda \sin \lambda x \left(\mathrm{e}^{\lambda y} + \mathrm{e}^{-\lambda y}\right),$$
$$\frac{\partial u}{\partial y}(x, y) = A\lambda \cos \lambda x \left(\mathrm{e}^{\lambda y} - \mathrm{e}^{-\lambda y}\right).$$

Now, let us apply the remaining boundary conditions.

$$\frac{\partial u}{\partial x}(1, y) = 0 \qquad \Longrightarrow \qquad -A\lambda \sin \lambda \left(\mathrm{e}^{\lambda y} + \mathrm{e}^{-\lambda y}\right) = 0 \qquad \Longrightarrow \qquad \boxed{\sin \lambda = 0}$$

since $\lambda \neq 0$, $\mathrm{e}^{\lambda y} + \mathrm{e}^{-\lambda y}$ cannot be zero for all $y \in (0, 1)$ and $A \neq 0$ for a nontrivial solution. This yields

$$\sin \lambda = \sin n\pi \qquad \Longrightarrow \qquad \lambda = n\pi, \qquad n = \pm 1, \pm 2, \pm 3, \ldots$$

Note that we have not taken $n = 0$ because $n = 0$ will yield $\lambda = 0$ whereas $\lambda \neq 0$ in this case. With this, the solution (34) becomes

$$u(x, y) = A \cos(n\pi x) \left(\mathrm{e}^{n\pi y} + \mathrm{e}^{-n\pi y}\right), \qquad n = \pm 1, \pm 2, \pm 3, \ldots$$

Neglecting the constant of proportionality A, we conclude that functions

$$u_n(x, y) = \cos(n\pi x) \left(\mathrm{e}^{n\pi y} + \mathrm{e}^{-n\pi y}\right), \qquad n = 1, 2, 3, \ldots \tag{42}$$

satisfy the Laplace equation (36) and the above three boundary conditions. Note that for negative integers n, we obtain essentially the same solutions. Therefore negative values of n have been omitted. The functions in (42) are referred to as the *eigenfunctions* for the given problem.

Therefore, the most general solution is given by

$$u(x, y) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) \left(\mathrm{e}^{n\pi y} + \mathrm{e}^{-n\pi y}\right), \quad 0 < x < 1, \quad 0 < y < 1,$$

where $A_0, A_1, A_2, A_3, \ldots$ are constants to be determined. Note that A_0 in the above solution comes from the $k = 0$ case. Now we shall use the final given boundary condition.

$$u(x, 1) = x(1 - x) \qquad \Longrightarrow \qquad A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) \left(\mathrm{e}^{n\pi} + \mathrm{e}^{-n\pi}\right) = x(1 - x),$$

which is a Fourier cosine series for $f(x) = x(1 - x)$, $0 < x < 1$. Therefore, the Fourier coefficients in the series are given by

$$A_0 = \int_0^1 x(1 - x)\, \mathrm{d}x = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$
$$A_n \left(\mathrm{e}^{n\pi} + \mathrm{e}^{-n\pi}\right) = 2 \int_0^1 x(1 - x) \cos(n\pi x)\, \mathrm{d}x.$$

Let us compute the integral in the above equation separately.

$$
\begin{aligned}
&\int_0^1 x(1-x)\,\cos(n\pi x)\,\mathrm{d}x = \cancelto{0}{\left[x(1-x)\frac{\sin(n\pi x)}{n\pi}\right]_0^1} - \int_0^1 (1-2x)\frac{\sin(n\pi x)}{n\pi}\,\mathrm{d}x \\
&= -\frac{1}{n\pi}\left[\left\{(1-2x)\left(-\frac{\cos(n\pi x)}{n\pi}\right)\right\}_0^1 - \int_0^1 (-2)\left(-\frac{\cos(n\pi x)}{n\pi}\right)\,\mathrm{d}x\right] \\
&= -\frac{1}{n\pi}\left\{\frac{\cos(n\pi)}{n\pi} + \frac{1}{n\pi}\right\} + \frac{2}{n^2\pi^2}\cancelto{0}{\left.\frac{\sin(n\pi x)}{n\pi}\right|_0^1} = -\frac{1+\cos(n\pi)}{n^2\pi^2} \\
&= -\frac{1+(-1)^n}{n^2\pi^2} = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ -\dfrac{2}{n^2\pi^2} & \text{if } n \text{ is even.} \end{cases}
\end{aligned}
$$

$$
A_n\left(\mathrm{e}^{n\pi} + \mathrm{e}^{-n\pi}\right) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ -\dfrac{4}{n^2\pi^2} & \text{if } n \text{ is even.} \end{cases}
$$

Therefore, the solution of the given problem is given by (take $n = 2m$)

$$
\boxed{u(x,y) = \frac{1}{6} - \frac{1}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos(2m\pi x)\,\frac{\mathrm{e}^{2m\pi y} + \mathrm{e}^{-2m\pi y}}{\mathrm{e}^{2m\pi} + \mathrm{e}^{-2m\pi}}}
$$

or

$$
\boxed{u(x,y) = \frac{1}{6} - \frac{1}{\pi^2}\sum_{m=1}^{\infty}\frac{\cos(2m\pi x)\,\cosh(2m\pi y)}{m^2\cosh(2m\pi)}}
$$

[Lecture 6]

3.5 Heat equation

Definition (One-dimensional heat equation)**:** The one-dimensional heat equation is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \tag{43}$$

where α^2 is a constant known as the *thermal diffusivity* and the unknown $u \equiv u(x,t)$ is usually the temperature. The parameter α^2 depends only on the material and is defined and $\alpha^2 = \kappa/(\rho s)$, where κ is the thermal conductivity, ρ is the density and s is the specific heat of the material. The units of α^2 are $(\text{length})^2/\text{time}$.

Note: the heat equation is also referred to as the *heat conduction equation* or *diffusion equation*.

3.5.1 Solution by the method of separation of variables

We first solve the heat equation (43) without considering the boundary conditions. Let $u(x,t) = X(x)\,T(t)$ be a solution of the heat equation (43). Therefore, this solution must satisfy the heat equation (43). Substituting this solution in (43), we obtain

$$X(x)\,T'(t) = \alpha^2 X''(x)\,T(t),$$

where prime denotes the derivative with respect to an independent variable, i.e. $T'(t) = \dfrac{\mathrm{d}T}{\mathrm{d}t}$ and $X''(x) = \dfrac{\mathrm{d}^2X}{\mathrm{d}x^2}$. The above equation can be rewritten as

$$\frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)}.$$

The left-hand side of the above equation is a function of x alone whereas the right-hand side is a function of t alone. Therefore, the left- and right-hand sides of the above equation must be constant. Let this constant be k. In other words,

$$\frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = k.$$

From this equation, we obtain two *ordinary* differential equations (ODEs), namely

$$\frac{\mathrm{d}^2X}{\mathrm{d}x^2} - kX = 0 \qquad \text{and} \qquad \frac{\mathrm{d}T}{\mathrm{d}t} = k\alpha^2 T. \tag{44}$$

In order to solve these ODEs, there are three possibilities for k. It can be zero, positive or negative. In the following, we shall consider these three cases separately.

Case 1: $k = 0$

In this case, eqs. (44) reduce to

$$\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = 0 \qquad \text{and} \qquad \frac{\mathrm{d}T}{\mathrm{d}t} = 0.$$

The above equations yield the solution $X(x) = c_1 x + c_2$ and $T(t) = c_3$, where c_1, c_2, c_3 are the integration constants and will be determined from the boundary conditions. Consequently, the solution of the heat equation (43) in this case is

$$\boxed{u(x,t) = (c_1 x + c_2)c_3 = A_1 x + B_1} \tag{45}$$

where $A_1 = c_1 c_3$ and $B_1 = c_2 c_3$ are another constants.

Case 2: $k > 0$

Let $k = \lambda^2$ and $\lambda \neq 0$. In this case, eqs. (44) reduce to

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - \lambda^2 X = 0 \qquad \text{and} \qquad \frac{\mathrm{d}T}{\mathrm{d}t} = \lambda^2 \alpha^2 T.$$

The above equations yield the solution $X(x) = c_4 \mathrm{e}^{\lambda x} + c_5 \mathrm{e}^{-\lambda x}$ and $T(t) = c_6 \mathrm{e}^{\lambda^2 \alpha^2 t}$, where c_4, c_5, c_6 are the integration constants and will be determined from the boundary conditions. Consequently, the solution of the heat equation (43) in this case is

$$\boxed{u(x,t) = \left(c_4 \mathrm{e}^{\lambda x} + c_5 \mathrm{e}^{-\lambda x}\right) c_6 \mathrm{e}^{\lambda^2 \alpha^2 t} = \left(A_2 \mathrm{e}^{\lambda x} + B_2 \mathrm{e}^{-\lambda x}\right) \mathrm{e}^{\lambda^2 \alpha^2 t}} \tag{46}$$

where $A_2 = c_4 c_6$ and $B_2 = c_5 c_6$ are another constants.

Case 3: $k < 0$

Let $k = -\lambda^2$ and $\lambda \neq 0$. In this case, eqs. (44) reduce to

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \lambda^2 X = 0 \qquad \text{and} \qquad \frac{\mathrm{d}T}{\mathrm{d}t} = -\lambda^2 \alpha^2 T.$$

The above equations yield the solution $X(x) = c_7 \cos \lambda x + c_8 \sin \lambda x$ and $T(t) = c_9 \mathrm{e}^{-\lambda^2 \alpha^2 t}$, where c_7, c_8, c_9 are the integration constants and will be determined from the boundary conditions. Consequently, the solution of the heat equation (43) in this case is

$$\boxed{u(x,t) = (c_7 \cos \lambda x + c_8 \sin \lambda x) c_9 \mathrm{e}^{-\lambda^2 \alpha^2 t} = (A_3 \cos \lambda x + B_3 \sin \lambda x) \mathrm{e}^{-\lambda^2 \alpha^2 t}} \tag{47}$$

where $A_3 = c_7 c_9$ and $B_3 = c_8 c_9$ are another constants.

Example (Homogeneous boundary conditions)**:** Find the solution of the heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad t > 0, \quad 0 < x < L \tag{48}$$

satisfying the conditions

$$u(0,t) = u(L,t) = 0 \quad \text{for} \quad t > 0 \quad \text{and} \quad u(x,0) = f(x) \quad \text{for} \quad 0 \leq x \leq L.$$

Solution: Let $u(x,t) = X(x)T(t)$ be a solution of the heat equation (48). Then, as shown above, we obtain two ODEs from the heat equation (48), which read

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - \beta X = 0 \qquad \text{and} \qquad \frac{\mathrm{d}T}{\mathrm{d}t} = \beta k T. \tag{49}$$

We shall consider the three cases, namely $\beta = 0$, $\beta < 0$ and $\beta > 0$.

Case 1: $\beta = 0$

In this case, we obtain the solution of the heat equation (48) as (see above)

$$u(x,t) = A_1 x + B_1. \tag{50}$$

We now compute the constants A_1, B_1 using the given conditions.

$$u(0,t) = 0 \quad \Longrightarrow \quad \boxed{B_1 = 0}$$

With this

$$u(L,t) = 0 \quad \Longrightarrow \quad A_1 L + B_1 = 0 \quad \Longrightarrow \quad \boxed{A_1 = 0}$$

These two conditions yield

$$u(x,t) = 0,$$

which cannot satisfy the remaining condition for $f(x) \neq 0$. Hence it cannot be a solution of the given problem. Therefore $\beta = 0$ is not possible.

Case 2: $\beta > 0$

Let $\beta = \lambda^2$ and $\lambda \neq 0$. In this case, we obtain the solution of the heat equation (48) as (see above)

$$u(x,t) = \big(A_2 \mathrm{e}^{\lambda x} + B_2 \mathrm{e}^{-\lambda x}\big)\mathrm{e}^{\lambda^2 k t}. \tag{51}$$

We now compute the integration constants A_2, B_2 using the given conditions.

$$u(0,t) = 0 \quad \Longrightarrow \quad \big(A_2 + B_2\big)\mathrm{e}^{\lambda^2 k t} = 0 \quad \Longrightarrow \quad \boxed{B_2 = -A_2}$$

since $\mathrm{e}^{\lambda^2 k t} \neq 0$. With this

$$u(L,t) = 0 \quad \Longrightarrow \quad A_2\big(\mathrm{e}^{\lambda L} - \mathrm{e}^{-\lambda L}\big)\mathrm{e}^{\lambda^2 k t} = 0 \quad \Longrightarrow \quad \boxed{A_2 = 0}$$

since $\mathrm{e}^{\lambda^2 k t} \neq 0$ and $\mathrm{e}^{\lambda L} - \mathrm{e}^{-\lambda L}$ cannot be zero. These two conditions again yield

$$u(x,t) = 0,$$

which cannot satisfy the remaining condition for $f(x) \neq 0$. Hence it cannot be a solution of the given problem. Therefore $\beta > 0$ is not possible.

Case 3: $\beta < 0$

Let $\beta = -\lambda^2$ and $\lambda \neq 0$. In this case, we obtain the solution of the heat equation (48) as (see above)

$$u(x,t) = (A_3 \cos \lambda x + B_3 \sin \lambda x)\mathrm{e}^{-\lambda^2 k t}. \tag{52}$$

We now compute the integration constants A_3, B_3 using the given conditions.

$$u(0,t) = 0 \quad \Longrightarrow \quad A_3 \mathrm{e}^{-\lambda^2 k t} = 0 \quad \Longrightarrow \quad \boxed{A_3 = 0}$$

since $e^{\lambda^2 kt} \neq 0$. With this

$$u(L,t) = 0 \quad \Longrightarrow \quad B_3(\sin\lambda L)\,e^{-\lambda^2 kt} = 0 \quad \Longrightarrow \quad \boxed{\sin\lambda L = 0}$$

since $e^{\lambda^2 kt} \neq 0$ and B_3 cannot be zero for a nontrivial solution. Therefore

$$\lambda L = n\pi \quad \Longrightarrow \quad \lambda = \frac{n\pi}{L} \qquad n = \pm 1, \pm 2, \pm 3, \ldots$$

Note that we have not taken $n = 0$ because $n = 0$ will yield $\lambda = 0$ whereas $\lambda \neq 0$ in this case. With these, the solution (52) becomes

$$u(x,t) = B_3\,e^{-\frac{n^2\pi^2}{L^2}kt}\sin\frac{n\pi x}{L}, \qquad n = \pm 1, \pm 2, \pm 3, \ldots$$

Neglecting the constant of proportionality B_3, we conclude that functions

$$u_n(x,t) = e^{-\frac{n^2\pi^2}{L^2}kt}\sin\frac{n\pi x}{L}, \qquad n = 1, 2, 3, \ldots \tag{53}$$

satisfy the heat equation (48) and the first two boundary conditions. Note that for negative integers n, we obtain essentially the same solutions, except for a minus sign, therefore negative values of n have been omitted. The functions in (53) are referred to as the *eigenfunctions* for our problem.

Therefore, the most general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\frac{n^2\pi^2}{L^2}kt}\sin\frac{n\pi x}{L}$$

where $C_1, C_2, C_3, \ldots$ are constants to be determined. Now we shall use the final given condition.

$$u(x,0) = f(x) \quad \Longrightarrow \quad \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} = f(x),$$

which is a Fourier sine series for $f(x)$, $0 < x < L$. Therefore, the Fourier coefficients in the series are given by

$$C_n = \frac{2}{L}\int_0^L f(x)\,\sin\frac{n\pi x}{L}\,dx \qquad n = 1, 2, 3, \ldots$$

Therefore, the solution of the given problem is

$$\boxed{u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(x)\,\sin\frac{n\pi x}{L}\,dx\right)e^{-\frac{n^2\pi^2}{L^2}kt}\sin\frac{n\pi x}{L}}$$

Example (Innhomogeneous boundary conditions)**:** Find the solution of the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad t > 0, \quad 0 < x < L \tag{54}$$

satisfying the conditions

$$u(0,t) = T_1, \quad u(L,t) = T_2 \quad \text{for} \quad t > 0 \quad \text{and} \quad u(x,0) = f(x) \quad \text{for} \quad 0 \le x \le L. \tag{55}$$

Solution: This problem is slightly more difficult, because of the nonhomogeneous boundary conditions, than the previous one. We can solve it by reducing it to a problem having homogeneous boundary conditions, which can then be solved as the previous one. The idea for doing this is based on the following physical argument. After a long time—that is, as $t \to \infty$—we anticipate that a steady temperature distribution $v(x)$ will be reached, which is independent of the time t and the initial conditions. Since $v(x)$ must satisfy the heat equation (54), we have

$$v''(x) = 0, \qquad 0 < x < L. \tag{56}$$

This implies that the steady temperature distribution $v(x)$ is linear function of x, i.e.

$$v(x) = ax + b,$$

where a and b are the integration constants. Further, $v(x)$ must satisfy the boundary conditions

$$v(0) = T_1 \quad \text{and} \quad v(L) = T_2,$$

which holds even for $t \to \infty$. With these conditions, we obtain $b = T_1$ and $a = (T_2 - T_1)/L$ and the solution of ODE (56) as

$$v(x) = (T_2 - T_1)\frac{x}{L} + T_2. \tag{57}$$

Returning to the original problem, we shall try to express $u(x,t)$ as the sum of the steady state temperature distribution $v(x)$ and another (transient) temperature distribution $w(x,t)$; thus we write

$$u(x,t) = v(x) + w(x,t). \tag{58}$$

Since $v(x)$ is given by (57), the given problem will be solved if $w(x,t)$ can be determined.

The boundary value problem for $w(x,t)$ is found by substituting the expression in (58) for $u(x,t)$ in (54) and (55). Substituting the expression in (58) for $u(x,t)$ in (54), we obtain

$$\cancelto{0}{\frac{\partial v}{\partial t}} + \frac{\partial w}{\partial t} = k\left(\cancelto{0}{\frac{\partial^2 v}{\partial x^2}} + \frac{\partial^2 w}{\partial x^2}\right) \quad \Longrightarrow \quad \frac{\partial w}{\partial t} = k\frac{\partial^2 w}{\partial x^2}, \qquad t > 0, \quad 0 < x < L \tag{59}$$

Substituting the expression in (58) for $u(x,t)$ in (55), we obtain

$$\left.\begin{aligned} v(0) + w(0,t) = T_1 \quad &\Longrightarrow \quad w(0,t) = 0 \\ v(L) + w(L,t) = T_2 \quad &\Longrightarrow \quad w(L,t) = 0 \end{aligned}\right\} \quad \text{for} \quad t > 0 \tag{60}$$

$$v(x) + w(x,0) = f(x) \quad \Longrightarrow \quad w(x,0) = f(x) - v(x) \quad \text{for} \quad 0 \le x \le L, \tag{61}$$

where $v(x)$ is given by (57). Thus the transient part of the solution to the original problem is found by solving the problem consisting of PDE (59) and conditions (60) and (61). This latter problem is precisely the one solved in the previous example, provided that $f(x) - v(x)$ is now regarded as the initial temperature distribution. Hence (see the previous example)

$$w(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L \left[f(x) - v(x)\right] \sin\frac{n\pi x}{L}\,\mathrm{d}x\right) \mathrm{e}^{-\frac{n^2\pi^2}{L^2}kt} \sin\frac{n\pi x}{L}$$

Therefore, the solution of the given problem is

$$\boxed{\begin{aligned} u(x,t) = {} & (T_2 - T_1)\frac{x}{L} + T_2 \\ & + \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L \left[f(x) - (T_2 - T_1)\frac{x}{L} + T_2\right] \sin\frac{n\pi x}{L}\,\mathrm{d}x\right) \mathrm{e}^{-\frac{n^2\pi^2}{L^2}kt} \sin\frac{n\pi x}{L} \end{aligned}}$$

[Lecture 7]

Example (Bar with insulated ends)**:** Find the solution of the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad t > 0, \quad 0 < x < L \tag{62}$$

satisfying the conditions

$$u_x(0,t) = u_x(L,t) = 0 \quad \text{for} \quad t > 0 \quad \text{and} \quad u(x,0) = f(x) \quad \text{for} \quad 0 \le x \le L.$$

Solution: Let $u(x,t) = X(x)\,T(t)$ be a solution of the heat equation (62). Then, as shown above, we obtain two ODEs from the heat equation (62), which read

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - kX = 0 \qquad \text{and} \qquad \frac{\mathrm{d}T}{\mathrm{d}t} = k\alpha^2 T. \tag{63}$$

We shall consider the three cases, namely $k = 0$, $k < 0$ and $k > 0$.

Case 1: $k = 0$

In this case, we obtain the solution of the heat equation (62) as (see above)

$$u(x,t) = A_1 x + B_1. \tag{64}$$

Therefore, $u_x(x,t) = A_1$. We now apply the boundary conditions.

$$u_x(0,t) = 0 \qquad \Longrightarrow \qquad \boxed{A_1 = 0}$$

and

$$u_x(L,t) = 0 \qquad \Longrightarrow \qquad \boxed{A_1 = 0}$$

After applying the boundary conditions, the solution in this case is

$$u(x,t) = B_1.$$

Dropping the constant $u(x,t) = 1$ is the eigensolution in this case that satisfies the heat equation and the given boundary conditions. [Initial condition will be checked later.]

Case 2: $k > 0$

Let $k = \lambda^2$ and $\lambda \neq 0$. In this case, we obtain the solution of the heat equation (62) as (see above)

$$u(x,t) = \big(A_2 \mathrm{e}^{\lambda x} + B_2 \mathrm{e}^{-\lambda x}\big)\mathrm{e}^{\lambda^2 \alpha^2 t}. \tag{65}$$

This implies that

$$u_x(x,t) = \lambda\big(A_2 \mathrm{e}^{\lambda x} - B_2 \mathrm{e}^{-\lambda x}\big)\mathrm{e}^{\lambda^2 \alpha^2 t}.$$

We now apply the boundary conditions.

$$u_x(0,t) = 0 \qquad \Longrightarrow \qquad \lambda\big(A_2 - B_2\big)\mathrm{e}^{\lambda^2 \alpha^2 t} = 0 \qquad \Longrightarrow \qquad \boxed{B_2 = A_2}$$

since $\lambda \neq 0$ and $e^{\lambda^2\alpha^2 t} \neq 0$. With this

$$u_x(L,t) = 0 \quad \Longrightarrow \quad \lambda A_2\big(e^{\lambda L} - e^{-\lambda L}\big)e^{\lambda^2\alpha^2 t} = 0 \quad \Longrightarrow \quad \boxed{A_2 = 0}$$

since $\lambda \neq 0$ and $e^{\lambda^2\alpha^2 t} \neq 0$ and $e^{\lambda L} - e^{-\lambda L}$ cannot be zero. These two conditions again yield

$$u(x,t) = 0,$$

which cannot satisfy the remaining condition for $f(x) \neq 0$. Hence it cannot be a solution of the given problem. Therefore $k > 0$ is not possible.

Case 3: $k < 0$

Let $k = -\lambda^2$ and $\lambda \neq 0$. In this case, we obtain the solution of the heat equation (62) as (see above)

$$u(x,t) = (A_3 \cos \lambda x + B_3 \sin \lambda x)e^{-\lambda^2\alpha^2 t}. \tag{66}$$

This implies that

$$u_x(x,t) = \lambda(-A_3 \sin \lambda x + B_3 \cos \lambda x)e^{-\lambda^2\alpha^2 t}.$$

We now apply the given boundary conditions.

$$u_x(0,t) = 0 \quad \Longrightarrow \quad \lambda B_3 e^{-\lambda^2\alpha^2 t} = 0 \quad \Longrightarrow \quad \boxed{B_3 = 0}$$

since $\lambda \neq 0$ and $e^{\lambda^2\alpha^2 t} \neq 0$. With this

$$u_x(L,t) = 0 \quad \Longrightarrow \quad -A_3\lambda(\sin \lambda L)\, e^{-\lambda^2\alpha^2 t} = 0 \quad \Longrightarrow \quad \boxed{\sin \lambda L = 0}$$

since $e^{\lambda^2\alpha^2 t} \neq 0$ and A_3 cannot be zero for a nontrivial solution. Therefore

$$\lambda L = n\pi \quad \Longrightarrow \quad \lambda = \frac{n\pi}{L} \qquad n = \pm 1, \pm 2, \pm 3, \ldots$$

Note that we have not taken $n = 0$ because $n = 0$ will yield $\lambda = 0$ whereas $\lambda \neq 0$ in this case. With these, the solution (66) becomes

$$u(x,t) = A_3\, e^{-\frac{n^2\pi^2}{L^2}\alpha^2 t} \cos \frac{n\pi x}{L}, \qquad n = \pm 1, \pm 2, \pm 3, \ldots$$

Neglecting the constant of proportionality A_3, we conclude that functions

$$u_n(x,t) = e^{-\frac{n^2\pi^2}{L^2}\alpha^2 t} \cos \frac{n\pi x}{L}, \qquad n = 1, 2, 3, \ldots \tag{67}$$

satisfy the heat equation (62) and the first two boundary conditions. Note that for negative integers n, we obtain essentially the same solutions. Therefore negative values of n have been omitted. The functions in (67) are referred to as the *eigenfunctions* for our problem.

Therefore, the most general solution is given by

$$u(x,y) = C_0 + \sum_{n=1}^{\infty} C_n \mathrm{e}^{-\frac{n^2\pi^2}{L^2}\alpha^2 t} \cos\frac{n\pi x}{L}$$

where $C_0, C_1, C_2, C_3, \ldots$ are constants to be determined. Note that C_0 in the above solution comes from the $k = 0$ case. Now we shall use the final given condition.

$$u(x,0) = f(x) \qquad \Longrightarrow \qquad C_0 + \sum_{n=1}^{\infty} C_n \cos\frac{n\pi x}{L} = f(x),$$

which is a Fourier cosine series for $f(x)$, $0 < x < L$. Therefore, the Fourier coefficients in the series are given by

$$\begin{aligned} C_0 &= \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x, \\ C_n &= \frac{2}{L}\int_0^L f(x)\,\cos\frac{n\pi x}{L}\,\mathrm{d}x, \qquad n = 1, 2, 3, \ldots \end{aligned}$$

Therefore, the solution of the given problem is

$$\boxed{u(x,t) = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x + \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(x)\,\cos\frac{n\pi x}{L}\,\mathrm{d}x\right)\mathrm{e}^{-\frac{n^2\pi^2}{L^2}\alpha^2 t}\cos\frac{n\pi x}{L}}$$

3.6 Wave equation

Definition (One-dimensional wave equation)**:** The one-dimensional wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \tag{68}$$

where $c^2 = T/\rho$ with T being the tension in the string and ρ is the mass per unit length of the string material. The units of c are length/time, i.e. the units of the velocity. Indeed, c denotes the speed of propagation of waves along the string.

3.6.1 Solution by the method of separation of variables

We first solve the wave equation (68) without considering the boundary conditions. Let $u(x,t) = X(x)\,T(t)$ be a solution of the wave equation (68). Therefore, this solution must satisfy the wave equation (68). Substituting this solution in (68), we obtain

$$X(x)\,T''(t) = c^2 X''(x)\,T(t),$$

where prime denotes the derivative with respect to an independent variable, i.e. $T''(t) = \dfrac{\mathrm{d}^2 T}{\mathrm{d}t^2}$ and $X''(x) = \dfrac{\mathrm{d}^2 X}{\mathrm{d}x^2}$. The above equation can be rewritten as

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)}.$$

The left-hand side of the above equation is a function of x alone whereas the right-hand side is a function of t alone. Therefore, the left- and right-hand sides of the above equation must be constant. Let this constant be k. In other words,

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} = k.$$

From this equation, we obtain two *ordinary* differential equations (ODEs), namely

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - kX = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2 T}{\mathrm{d}t^2} - k\,c^2 T = 0. \tag{69}$$

In order to solve these ODEs, there are three possibilities for k. It can be zero, positive or negative. In the following, we shall consider these three cases separately.

Case 1: $k = 0$

In this case, eqs. (69) reduce to

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = 0.$$

The above equations yield the solution $X(x) = c_1 x + c_2$ and $T(t) = c_3 t + c_4$, where c_1, c_2, c_3, c_4 are the integration constants and will be determined from the boundary conditions. Consequently, the solution of the wave equation (68) in this case is

$$\boxed{u(x,t) = (c_1 x + c_2)(c_3 t + c_4)} \tag{70}$$

Case 2: $k > 0$

Let $k = \beta^2$ and $\beta \neq 0$. In this case, eqs. (69) reduce to

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - \beta^2 X = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2 T}{\mathrm{d}t^2} - \lambda^2 T = 0,$$

where $\lambda = \beta c$. The above equations yield the solution $X(x) = c_5 \mathrm{e}^{\beta x} + c_6 \mathrm{e}^{-\beta x}$ and $T(t) = c_7 \mathrm{e}^{\lambda t} + c_8 \mathrm{e}^{-\lambda t}$, where c_5, c_6, c_7, c_8 are the integration constants and will be determined from the boundary conditions. Consequently, the solution of the wave equation (68) in this case is

$$\boxed{u(x,t) = \left(c_5 \mathrm{e}^{\beta x} + c_6 \mathrm{e}^{-\beta x}\right)\left(c_7 \mathrm{e}^{\lambda t} + c_8 \mathrm{e}^{-\lambda t}\right)} \tag{71}$$

Case 3: $k < 0$

Let $k = -\beta^2$ and $\beta \neq 0$. In this case, eqs. (69) reduce to

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \beta^2 X = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2 T}{\mathrm{d}t^2} + \lambda^2 T = 0,$$

where $\lambda = \beta c$. The above equations yield the solution $X(x) = c_9 \cos\beta x + c_{10} \sin\beta x$ and $T(t) = c_{11} \cos\lambda t + c_{12} \sin\lambda t$, where $c_9, c_{10}, c_{11}, c_{12}$ are the integration constants and will be determined from the boundary conditions. Consequently, the solution of the wave equation (68) in this case is

$$\boxed{u(x,t) = (c_9 \cos\beta x + c_{10} \sin\beta x)(c_{11} \cos\lambda t + c_{12} \sin\lambda t)} \tag{72}$$

Example 1 (Elastic string with a nonzero initial displacement)**:** Find the solution of the wave equation

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad t > 0, \quad 0 < x < L \tag{73}$$

satisfying the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0 \quad \text{for} \quad t \geq 0 \tag{74}$$

and the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = 0 \quad \text{for} \quad 0 \leq x \leq L. \tag{75}$$

Solution: Let $u(x,t) = X(x)\,T(t)$ be a solution of the wave equation (73). Then, as shown above, we obtain two ODEs from the wave equation (73), which read

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - kX = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2 T}{\mathrm{d}t^2} - k\,c^2 T = 0. \tag{76}$$

We shall consider the three cases, namely $k = 0$, $k < 0$ and $k > 0$.

Case 1: $k = 0$

In this case, we obtain the solution of the wave equation (73) as (see above)

$$u(x,t) = (c_1 x + c_2)(c_3 t + c_4). \tag{77}$$

We now compute the constants c_1, c_2, c_3, c_4 using the given initial and boundary conditions.

$$u(0,t) = 0 \quad \Longrightarrow \quad c_2(c_3 t + c_4) = 0 \quad \Longrightarrow \quad \boxed{c_2 = 0}$$

since $c_3 t + c_4$ cannot be zero for all $t \geq 0$. With this

$$u(L,t) = 0 \quad \Longrightarrow \quad c_1 L(c_3 t + c_4) = 0 \quad \Longrightarrow \quad \boxed{c_1 = 0}$$

since $L > 0$ and $c_3 t + c_4$ cannot be zero for all $t \geq 0$. These two conditions yield

$$u(x,t) = 0,$$

which cannot satisfy the initial condition $u(x,0) = f(x)$ for $f(x) \neq 0$. Hence, $u(x,t) = 0$ cannot be a solution of the given problem. Therefore, $k = 0$ is not possible.

Case 2: $k > 0$

Let $k = \beta^2$ and $\beta \neq 0$. In this case, we obtain the solution of the wave equation (73) as (see above)

$$u(x,t) = \left(c_5 \mathrm{e}^{\beta x} + c_6 \mathrm{e}^{-\beta x}\right)\left(c_7 \mathrm{e}^{\lambda t} + c_8 \mathrm{e}^{-\lambda t}\right), \tag{78}$$

where $\lambda = \beta c$. We now compute the integration constants c_5, c_6, c_7, c_8 using the given initial and boundary conditions.

$$u(0,t) = 0 \quad \Longrightarrow \quad (c_5 + c_6)\left(c_7 \mathrm{e}^{\lambda t} + c_8 \mathrm{e}^{-\lambda t}\right) = 0 \quad \Longrightarrow \quad \boxed{c_6 = -c_5}$$

since $c_7 e^{\lambda t} + c_8 e^{-\lambda t} \neq 0$ for all $t \geq 0$. With this

$$u(L,t) = 0 \quad \Longrightarrow \quad c_5\left(e^{\beta L} - e^{-\beta L}\right)\left(c_7 e^{\lambda t} + c_8 e^{-\lambda t}\right) = 0 \quad \Longrightarrow \quad \boxed{c_5 = 0}$$

since $c_7 e^{\lambda t} + c_8 e^{-\lambda t} \neq 0$ for all $t \geq 0$ and $e^{\beta L} - e^{-\beta L} \neq 0$. These two conditions again yield

$$u(x,t) = 0,$$

which cannot satisfy the initial condition $u(x,0) = f(x)$ for $f(x) \neq 0$. Hence, $u(x,t) = 0$ cannot be a solution of the given problem. Therefore, $k > 0$ is also not possible.

Case 3: $k < 0$

Let $k = -\beta^2$ and $\beta \neq 0$. In this case, we obtain the solution of the wave equation (73) as (see above)

$$u(x,t) = (c_9 \cos \beta x + c_{10} \sin \beta x)(c_{11} \cos \lambda t + c_{12} \sin \lambda t), \tag{79}$$

where $\lambda = \beta c$. This implies that

$$u_t(x,t) = \lambda(c_9 \cos \beta x + c_{10} \sin \beta x)(-c_{11} \sin \lambda t + c_{12} \cos \lambda t).$$

We now compute the integration constants $c_9, c_{10}, c_{11}, c_{12}$ using the given initial and boundary conditions.

$$u(0,t) = 0 \quad \Longrightarrow \quad c_9(c_{11} \cos \lambda t + c_{12} \sin \lambda t) = 0 \quad \Longrightarrow \quad \boxed{c_9 = 0}$$

since $c_{11} \cos \lambda t + c_{12} \sin \lambda t$ cannot be zero for all $t \geq 0$. With this

$$u(L,t) = 0 \quad \Longrightarrow \quad c_{10}(\sin \beta L)(c_{11} \cos \lambda t + c_{12} \sin \lambda t) = 0 \quad \Longrightarrow \quad \boxed{\sin \beta L = 0}$$

since $c_{11} \cos \lambda t + c_{12} \sin \lambda t$ cannot be zero for all $t \geq 0$ and c_{10} cannot be zero for a nontrivial solution. Therefore

$$\beta L = n\pi \quad \Longrightarrow \quad \beta = \frac{n\pi}{L} \qquad n = \pm 1, \pm 2, \pm 3, \ldots$$

and, hence,

$$\lambda = \beta c = \frac{n\pi c}{L} \qquad n = \pm 1, \pm 2, \pm 3, \ldots$$

Note that we have not taken $n = 0$ because $n = 0$ will yield $\beta = 0$ whereas $\beta \neq 0$ in this case.

$$u_t(x,0) = 0 \quad \Longrightarrow \quad c_{12}\lambda(c_9 \cos \beta x + c_{10} \sin \beta x) = 0 \quad \Longrightarrow \quad \boxed{c_{12} = 0}$$

since $\beta \neq 0$, $c \neq 0$ and $c_9 \cos \beta x + c_{10} \sin \beta x$ cannot be zero for all $x \in [0, L]$. With these, solution (79) becomes

$$u(x,t) = c_{10} c_{11} \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L}, \qquad n = \pm 1, \pm 2, \pm 3, \ldots$$

Neglecting the constant of proportionality $c_{10}c_{11}$, we conclude that functions

$$u_n(x,t) = \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}, \qquad n = 1, 2, 3, \ldots \tag{80}$$

satisfy the wave equation (73) and the first two boundary conditions. Note that for negative integers n, we obtain essentially the same solutions, except for a minus sign, therefore negative values of n have been omitted. The functions in (80) are referred to as the *eigenfunctions* or *characteristic functions* and $\lambda_n = n\pi c/L$ are referred to as the *eigenvalues* or *characteristic values* of the vibrating string. The set $\{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ is referred to as the spectrum.

Discussion on eigenfunctions: We see that each u_n represents a harmonic motion having the frequency $\lambda_n/(2\pi) = nc/(2L)$ cycles per unit time. This motion is called the *nth normal mode* of the string. The first normal mode ($n = 1$) is known as the fundamental mode, and the others are known as *overtones*. Since, in (80),

$$\sin\frac{n\pi x}{L} = 0 \quad \text{at} \quad x = \frac{L}{n}, \frac{2L}{n}, \ldots \frac{(n-1)L}{n},$$

the nth normal mode has $n-1$ nodes, that is, points of the string that do not move (in addition to the fixed endpoints), see figure 4.

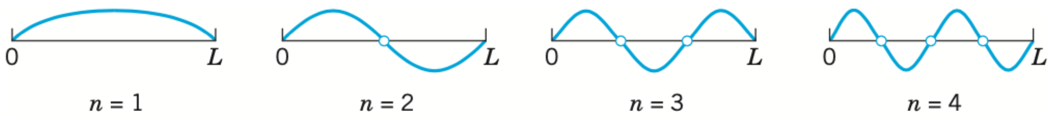


Figure 4: Normal modes of a vibrating string

Thus, the most general solution of the wave equation (73) that satisfies the boundary conditions (74) and the homogeneous initial condition $(75)_2$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$$

where $C_1, C_2, C_3, \ldots$ are constants to be determined from the remaining condition. Now we shall use the remaining initial condition.

$$u(x,0) = f(x) \quad \Longrightarrow \quad \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} = f(x),$$

which is a Fourier sine series for $f(x)$, $0 \le x \le L$. Therefore, the Fourier coefficients in the series are given by

$$C_n = \frac{2}{L}\int_0^L f(x)\,\sin\frac{n\pi x}{L}\,dx \qquad n = 1, 2, 3, \ldots$$

Therefore, the solution of the given problem is

$$\boxed{u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(x)\,\sin\frac{n\pi x}{L}\,dx\right)\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}} \tag{81}$$

[Lecture 8]

Example 2 (Elastic string with nonzero initial velocity)**:** Find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad t > 0, \quad 0 < x < L \tag{82}$$

satisfying the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0 \quad \text{for} \quad t \geq 0 \tag{83}$$

and the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = g(x) \quad \text{for} \quad 0 \leq x \leq L. \tag{84}$$

Solution: Let $u(x,t) = X(x)\,T(t)$ be a solution of the wave equation (82). Then, as shown above, we obtain two ODEs from the wave equation (82), which read

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - kX = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2 T}{\mathrm{d}t^2} - k\,c^2 T = 0. \tag{85}$$

We shall consider the three cases, namely $k = 0$, $k < 0$ and $k > 0$.

Case 1: $k = 0$

In this case, we obtain the solution of the wave equation (82) as (see above)

$$u(x,t) = (c_1 x + c_2)(c_3 t + c_4). \tag{86}$$

We now compute the constants c_1, c_2, c_3, c_4 using the given initial and boundary conditions.

$$u(0,t) = 0 \quad \Longrightarrow \quad c_2(c_3 t + c_4) = 0 \quad \Longrightarrow \quad \boxed{c_2 = 0}$$

since $c_3 t + c_4$ cannot be zero for all $t \geq 0$. With this

$$u(L,t) = 0 \quad \Longrightarrow \quad c_1 L(c_3 t + c_4) = 0 \quad \Longrightarrow \quad \boxed{c_1 = 0}$$

since $L > 0$ and $c_3 t + c_4$ cannot be zero for all $t \geq 0$. These two conditions yield

$$u(x,t) = 0,$$

which cannot satisfy the initial condition $u_t(x,0) = g(x)$ for $g(x) \neq 0$. Hence, $u(x,t) = 0$ cannot be a solution of the given problem. Therefore, $k = 0$ is not possible.

Case 2: $k > 0$

Let $k = \beta^2$ and $\beta \neq 0$. In this case, we obtain the solution of the wave equation (82) as (see above)

$$u(x,t) = \left(c_5 \mathrm{e}^{\beta x} + c_6 \mathrm{e}^{-\beta x}\right)\left(c_7 \mathrm{e}^{\lambda t} + c_8 \mathrm{e}^{-\lambda t}\right), \tag{87}$$

where $\lambda = \beta c$. We now compute the integration constants c_5, c_6, c_7, c_8 using the given initial and boundary conditions.

$$u(0,t) = 0 \quad \Longrightarrow \quad (c_5 + c_6)\left(c_7 e^{\lambda t} + c_8 e^{-\lambda t}\right) = 0 \quad \Longrightarrow \quad \boxed{c_6 = -c_5}$$

since $c_7 e^{\lambda t} + c_8 e^{-\lambda t} \neq 0$ for all $t \geq 0$. With this

$$u(L,t) = 0 \quad \Longrightarrow \quad c_5\left(e^{\beta L} - e^{-\beta L}\right)\left(c_7 e^{\lambda t} + c_8 e^{-\lambda t}\right) = 0 \quad \Longrightarrow \quad \boxed{c_5 = 0}$$

since $c_7 e^{\lambda t} + c_8 e^{-\lambda t} \neq 0$ for all $t \geq 0$ and $e^{\beta L} - e^{-\beta L} \neq 0$. These two conditions again yield

$$u(x,t) = 0,$$

which cannot satisfy the initial condition $u(x,0) = f(x)$ for $f(x) \neq 0$. Hence, $u(x,t) = 0$ cannot be a solution of the given problem. Therefore, $k > 0$ is also not possible.

Case 3: $k < 0$

Let $k = -\beta^2$ and $\beta \neq 0$. In this case, we obtain the solution of the wave equation (82) as (see above)

$$u(x,t) = (c_9 \cos\beta x + c_{10} \sin\beta x)(c_{11} \cos\lambda t + c_{12} \sin\lambda t), \tag{88}$$

where $\lambda = \beta c$. We now compute the integration constants $c_9, c_{10}, c_{11}, c_{12}$ using the given initial and boundary conditions.

$$u(0,t) = 0 \quad \Longrightarrow \quad c_9(c_{11} \cos\lambda t + c_{12} \sin\lambda t) = 0 \quad \Longrightarrow \quad \boxed{c_9 = 0}$$

since $c_{11} \cos\lambda t + c_{12} \sin\lambda t$ cannot be zero for all $t \geq 0$.

$$u(x,0) = 0 \quad \Longrightarrow \quad c_{11}(c_9 \cos\beta x + c_{10} \sin\beta x) = 0 \quad \Longrightarrow \quad \boxed{c_{11} = 0}$$

since $c_9 \cos\beta x + c_{10} \sin\beta x$ cannot be zero for all $x \in [0, L]$. With these boundary conditions solution (88) reduces to

$$u(x,t) = A \sin\beta x \sin\lambda t, \tag{89}$$

where $A = c_{10} c_{12}$ is another constant. We shall now apply the other boundary condition.

$$u(L,t) = 0 \quad \Longrightarrow \quad A \sin\beta L \sin\lambda t = 0 \quad \Longrightarrow \quad \boxed{\sin\beta L = 0}$$

since $\sin\lambda t$ cannot be zero for all $t \geq 0$ and A cannot be zero for a nontrivial solution. Therefore

$$\beta L = n\pi \quad \Longrightarrow \quad \beta = \frac{n\pi}{L} \qquad n = \pm 1, \pm 2, \pm 3, \ldots$$

and, hence,

$$\lambda = \beta c = \frac{n\pi c}{L} \qquad n = \pm 1, \pm 2, \pm 3, \ldots$$

Note that we have not taken $n = 0$ because $n = 0$ will yield $\beta = 0$ whereas $\beta \neq 0$ in this case. With these, the solution becomes

$$u(x,t) = A \sin \frac{n\pi x}{L} \sin \frac{n\pi ct}{L}, \qquad n = \pm 1, \pm 2, \pm 3, \ldots$$

Neglecting the constant of proportionality A, we conclude that functions

$$u_n(x,t) = \sin \frac{n\pi x}{L} \sin \frac{n\pi ct}{L}, \qquad n = 1, 2, 3, \ldots \tag{90}$$

satisfy the wave equation (82) and the first two boundary conditions. Note that for negative integers n, we obtain essentially the same solutions; therefore negative values of n have been omitted. The functions in (90) are referred to as the *eigenfunctions* or *characteristic functions* and $\lambda_n = n\pi c/L$ are referred to as the *eigenvalues* or *characteristic values* of the vibrating string. The set $\{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ is referred to as the spectrum.

Therefore, the most general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \sin \frac{n\pi ct}{L}$$

where $C_1, C_2, C_3, \ldots$ are constants to be determined from the remaining boundary condition. It follows that

$$u_t(x,t) = \sum_{n=1}^{\infty} C_n \frac{n\pi c}{L} \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L}$$

Now we shall use the remaining initial condition.

$$u_t(x,0) = g(x) \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} C_n \frac{n\pi c}{L} \sin \frac{n\pi x}{L} = g(x),$$

which is a Fourier sine series for $g(x)$, $0 \leq x \leq L$. Therefore, the Fourier coefficients in the series are given by

$$C_n \frac{n\pi c}{L} = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L}\, \mathrm{d}x \qquad n = 1, 2, 3, \ldots$$

$$\Longrightarrow \qquad C_n = \frac{2}{n\pi c} \int_0^L g(x) \sin \frac{n\pi x}{L}\, \mathrm{d}x \qquad n = 1, 2, 3, \ldots$$

Therefore, the solution of the given problem is

$$\boxed{u(x,t) = \frac{2}{\pi c} \sum_{n=1}^{\infty} \frac{1}{n} \left(\int_0^L g(x) \sin \frac{n\pi x}{L}\, \mathrm{d}x \right) \sin \frac{n\pi x}{L} \sin \frac{n\pi ct}{L}} \tag{91}$$

Example 3 (Elastic string with general initial conditions)**:** Find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad t > 0, \quad 0 < x < L \tag{92}$$

satisfying the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0 \quad \text{for} \quad t \geq 0 \tag{93}$$

and the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{for} \quad 0 \leq x \leq L. \tag{94}$$

Solution: Although the problem in this example can be solved by the method of separation of variables, similarly to the previous two examples, it is important to note that it can also be solved simply by adding together the two solutions that we obtained above. To show this, let $v(x,t)$ be the solution of the problem in example 1, and let $w(x,t)$ be the solution of the problem in example 2. Thus $v(x,t)$ is given by (81) and $w(x,t)$ is given by (91). Now let $u(x,t) = v(x,t) + w(x,t)$. Notice that

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 w}{\partial t^2} - c^2 \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} \right) \\
&= \left(\frac{\partial^2 v}{\partial t^2} - c^2 \frac{\partial^2 v}{\partial x^2} \right) + \left(\frac{\partial^2 w}{\partial t^2} - c^2 \frac{\partial^2 w}{\partial x^2} \right) \\
&= 0 + 0 = 0.
\end{aligned}$$

Therefore, $u(x,t)$ satisfies the wave equation (92). Next, we have

$$u(0,t) = v(0,t) + w(0,t) = 0 + 0 = 0, \qquad u(L,t) = v(L,t) + w(L,t) = 0 + 0 = 0,$$

so $u(x,t)$ also satisfies the boundary conditions (93). Finally, we have

$$\begin{aligned}
u(x,0) &= v(x,0) + w(x,0) = f(x) + 0 = f(x), \\
u_t(x,0) &= v_t(x,0) + w_t(x,0) = 0 + g(x) = g(x),
\end{aligned}$$

Thus $u(x,t)$ also satisfies the general initial conditions (94). Hence the solution of the given problem is

$$\boxed{\begin{aligned}
u(x,t) = \frac{2}{L} \sum_{n=1}^{\infty} &\left(\int_0^L f(x) \sin \frac{n\pi x}{L} \, \mathrm{d}x \right) \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L} \\
&+ \frac{2}{\pi c} \sum_{n=1}^{\infty} \frac{1}{n} \left(\int_0^L g(x) \sin \frac{n\pi x}{L} \, \mathrm{d}x \right) \sin \frac{n\pi x}{L} \sin \frac{n\pi ct}{L}
\end{aligned}}$$

[Lecture 9]

3.6.2 d'Alembert's solution of the one-dimensional wave equation

Consider the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx}. \tag{95}$$

Let us consider the the change of variables $\xi = x - ct$ and $\eta = x + ct$. This implies that

$$\frac{\partial \xi}{\partial x} = 1, \qquad \frac{\partial \xi}{\partial t} = -c, \qquad \frac{\partial \eta}{\partial x} = 1, \qquad \frac{\partial \eta}{\partial t} = c.$$

We assume all the partial derivatives of u involved are continuous, and apply the chain rule to obtain

$$\begin{aligned}
u_x &= u_\xi \frac{\partial \xi}{\partial x} + u_\eta \frac{\partial \eta}{\partial x} = u_\xi + u_\eta, \\
u_t &= u_\xi \frac{\partial \xi}{\partial t} + u_\eta \frac{\partial \eta}{\partial t} = u_\xi(-c) + u_\eta \times c = c(-u_\xi + u_\eta), \\
u_{xx} &= \frac{\partial(u_\xi + u_\eta)}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial(u_\xi + u_\eta)}{\partial \eta}\frac{\partial \eta}{\partial x} = u_{\xi\xi} + u_{\xi\eta} + u_{\eta\xi} + u_{\eta\eta} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \\
u_{tt} &= c\left[\frac{\partial(-u_\xi + u_\eta)}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial(-u_\xi + u_\eta)}{\partial \eta}\frac{\partial \eta}{\partial t}\right] = c\big[(-u_{\xi\xi} + u_{\xi\eta})(-c) + (-u_{\eta\xi} + u_{\eta\eta})c\big] \\
&= c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}).
\end{aligned}$$

Substituting these in the one-dimensional wave equation (95), we obtain

$$c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) = c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) \qquad \Longrightarrow \qquad \boxed{u_{\xi\eta} = 0}$$

We shall now integrate both sides of the above equation with respect to ξ and η successively. On integrating with respect to ξ, we obtain

$$u_\eta = \bar{\psi}(\eta),$$

where $\bar{\psi}$ is an arbitrary function of η only. Next, integrating this equation with respect to η, we obtain

$$u(\xi, \eta) = \int \bar{\psi}(\eta)\,\mathrm{d}\eta + \phi(\xi),$$

where ϕ is an arbitrary function of ξ only. Writing $\displaystyle\int \bar{\psi}(\eta)\,\mathrm{d}\eta$ as $\psi(\eta)$, we obtain the solution of the wave equation as

$$u(\xi, \eta) = \phi(\xi) + \psi(\eta), \tag{96}$$

where ϕ and ψ are two arbitrary functions of ξ and η, respectively. Rewriting ξ and η in terms of x and t, we obtain from (96)

$$\boxed{u(x,t) = \phi(x - ct) + \psi(x + ct)} \tag{97}$$

This is referred to as the *d'Alembert's solution* of the one-dimensional wave equation (95).

Example (d'Alembert's solution satisfying the initial conditions)**:** Determine the solution of the wave equation

$$u_{tt} = c^2 u_{xx}, \qquad t > 0, \quad -\infty < x < \infty \tag{98}$$

in an infinite one-dimensional medium subject to the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{for} \quad -\infty < x < \infty. \tag{99}$$

Solution: We know that the general solution of the wave equation (98) is given by the d'Alembert's solution

$$u(x,t) = \phi(x-ct) + \psi(x+ct). \tag{100}$$

Therefore,

$$u_t(x,t) = -c\,\phi'(x-ct) + c\,\psi'(x+ct), \tag{101}$$

where prime denotes the derivatives with respect to the entire arguments $x-ct$ and $x+ct$, respectively.

Let us now apply the given boundary conditions. The condition $u(x,0) = f(x)$ implies

$$\phi(x) + \psi(x) = f(x) \tag{102}$$

and $u_t(x,0) = g(x)$ implies the condition

$$-c\,\phi'(x) + c\,\psi'(x) = g(x). \tag{103}$$

Dividing the above equation by $(-c)$ and integrating with respect to x, we obtain

$$\phi(x) - \psi(x) = -\frac{1}{c}\int_{x_0}^{x} g(s)\,\mathrm{d}s + k, \tag{104}$$

where $k = \phi(x_0) - \psi(x_0)$. Adding (102) and (104), and dividing the resulting equation by 2, we obtain

$$\phi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)\,\mathrm{d}s + \frac{k}{2}. \tag{105}$$

Substituting this in (102), we obtain

$$\psi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)\,\mathrm{d}s - \frac{k}{2}. \tag{106}$$

From (105) and (106),

$$\phi(x-ct) = \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_{x_0}^{x-ct} g(s)\,\mathrm{d}s + \frac{k}{2}, \tag{107}$$

$$\psi(x+ct) = \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)\,\mathrm{d}s - \frac{k}{2}. \tag{108}$$

From the above equations, the solution of the given problem is

$$\begin{aligned} u(x,t) &= \phi(x-ct) + \psi(x+ct) \\ &= \frac{1}{2}\big[f(x-ct) + f(x+ct)\big] + \frac{1}{2c}\left(\int_{x-ct}^{x_0} g(s)\,\mathrm{d}s + \int_{x_0}^{x+ct} g(s)\,\mathrm{d}s\right) \end{aligned}$$

or

$$\boxed{u(x,t) = \frac{1}{2}\big[f(x-ct) + f(x+ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s} \tag{109}$$

Remark: Notice that if the initial velocity is zero, solution (109) reduces to

$$u(x,t) = \frac{1}{2}\big[f(x-ct) + f(x+ct)\big].$$

Example (d'Alembert's solution satisfying the initial conditions)**:** Determine the solution of the wave equation

$$u_{tt} = c^2 u_{xx}, \qquad t > 0, \quad -\infty < x < \infty \tag{110}$$

in an infinite one-dimensional medium subject to the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = \sin x \quad \text{for} \quad -\infty < x < \infty. \tag{111}$$

Solution: From the previous example, the general solution of the wave equation (110) with the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ is given by the d'Alembert's solution

$$u(x,t) = \frac{1}{2}\big[f(x-ct) + f(x+ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s \tag{112}$$

Here, $f(x) = 0$ and $g(x) = \sin x$. Therefore, the solution for the given problem is

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \sin s\,\mathrm{d}s = \frac{1}{2c}(-\cos s)\Big|_{x-ct}^{x+ct} = \frac{1}{2c}\Big[\cos(x-ct) - \cos(x+ct)\Big].$$

Example: Let $y(x,t)$ represents transverse displacement in a long stretched string one end of which is attached to a ring (of negligible diameter and weight) that can slide along the y-axis. The other end is so far out on the positive x-axis that it may be considered to be infinitely far from the origin. The ring is initially at the origin and is then moved along the y-axis (see figure 5) so that $y = f(t)$ when $x = 0$ and $t \geq 0$, where f is a given continuous function with $f(0) = 0$. Assume that the string is initially at rest on the x-axis.

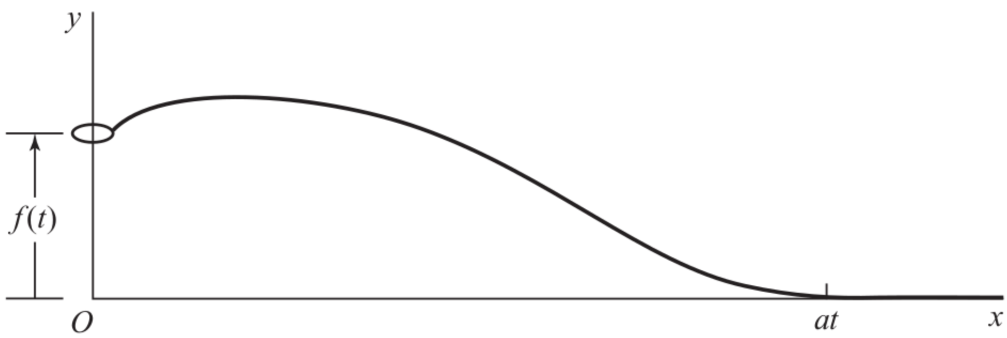


Figure 5: Schematic of the string in the given problem.

(a) Write down the boundary value problem describing the given problem.
Hint: Look at the figure carefully; the boundary value problem should include 'a' in it.

(b) The general solution of the partial differential equation identified in part (a) is given by $y(x,t) = \phi(x+at)+\psi(x-at)$, where ϕ and ψ are two arbitrary functions that need to be determined using the given conditions. Apply the *initial* condition(s) identified in part (a) to show that there is a constant c such that

$$\phi(x) = c \quad \text{and} \quad \psi(x) = -c \qquad (x \geq 0).$$

Then apply the *boundary* condition(s) identified in part (a) to show that

$$\psi(-x) = f\left(\frac{x}{a}\right) - c \qquad (x \geq 0),$$

where c is the same constant.

(c) With the aid of the results obtained in part (b), show that the solution of given problem is

$$y(x,t) = \begin{cases} 0 & x \geq at, \\ f\left(t - \dfrac{x}{a}\right) & x \leq at. \end{cases}$$

(d) What can you infer from this solution about the displacement in the string due to the movement of the ring?

Solution: (a) The boundary value problem, which describes the given problem is as follows.

$$y_{tt}(x,t) = a^2\, y_{xx}(x,t) \qquad (x > 0, \quad t > 0), \tag{113}$$

$$y(x,0) = 0 \quad \text{and} \quad y_t(x,0) = 0 \qquad (x \geq 0), \tag{114}$$

$$y(0,t) = f(t) \qquad (t \geq 0), \tag{115}$$

where $y(x,t)$ is the transverse displacement in the string and a is the wave speed in the string. The initial conditions are given by (114) and the boundary condition by (118).

(b) The general solution of (113) is given by (the d'Alembert's solution)

$$y(x,t) = \phi(x+at) + \psi(x-at). \tag{116}$$

This implies that

$$y_t(x,t) = a\,\phi'(x+at) - a\,\psi'(x-at) = a\big[\phi'(x+at) - \psi'(x-at)\big]. \tag{117}$$

Let us now apply the initial conditions (114), which hold for $x \geq 0$.

$$y(x,0) = 0 \quad \Longrightarrow \quad \phi(x) + \psi(x) = 0 \quad \Longrightarrow \quad \phi(x) = -\psi(x), \tag{118}$$

$$y_t(x,0) = 0 \quad \Longrightarrow \quad a\big[\phi'(x) - \psi'(x)\big] = 0 \quad \Longrightarrow \quad \phi'(x) = \psi'(x). \tag{119}$$

Differentiating (118) with respect to x and adding in (119), we obtain

$$\phi'(x) = 0 \quad \Longrightarrow \quad \phi(x) = c,$$

where c is a constant of integration. With this, eq. (118) implies $\psi(x) = -c$. Thus, applying after applying the initial conditions (114), we have

$$\boxed{\phi(x) = c} \quad \text{and} \quad \boxed{\psi(x) = -c} \qquad (x \geq 0). \tag{120}$$

Let us now apply the boundary condition (115), which holds for $t \geq 0$.

$$y(0,t) = f(t) \qquad \Longrightarrow \qquad \phi(at) + \psi(-at) = f(t).$$

Let us apply the change of variable $at = x$ to obtain

$$\phi(x) + \psi(-x) = f\left(\frac{x}{a}\right) \qquad (x \geq 0).$$

After applying the initial conditions, we obtained $\phi(x) = c$. Therefore, the above equation yields

$$\boxed{\psi(-x) = f\left(\frac{x}{a}\right) - c} \qquad (x \geq 0). \tag{121}$$

(c) The results obtained in part (b) above are (120) and (121) and they hold for $x \geq 0$. Note from the general solution (116) that we need to determine the values of $\phi(x+at)$ and $\psi(x-at)$. From (120) and (121), it is clear that $\phi(x) = c$ for all $x \geq 0$. Therefore, $\phi(x+at) = c$ for all $x \geq 0$ because $x + at \geq x \geq 0$.

Now, we need to determine $\psi(x - at)$ from (120) and (121). From (120), we have

$$\psi(x - at) = -c \qquad (x - at \geq 0 \quad \text{or} \quad x \geq at) \tag{122}$$

and from (121), we have

$$\psi(x - at) = f\left(\frac{at - x}{a}\right) - c = f\left(t - \frac{x}{a}\right) - c \qquad (at - x \geq 0 \quad \text{or} \quad x \leq at). \tag{123}$$

Consequently, the general solution $y(x,t) = \phi(x+at) + \psi(x-at)$ of the above problem is

$$y(x,t) = c + (-c) = 0 \quad \text{for} \quad x \geq at$$

and

$$y(x,t) = c + \left[f\left(t - \frac{x}{a}\right) - c\right] = f\left(t - \frac{x}{a}\right) \quad \text{for} \quad x \leq at.$$

Combining the above two results, the solution of the given problem is

$$y(x,t) = \begin{cases} 0 & x \geq at, \\ f\left(t - \frac{x}{a}\right) & x \leq at. \end{cases}$$

(d) The solution of the given problems reveals that the part of the string to the right of the point $x = at$ on the x-axis is unaffected by the movement of the ring <u>prior to time t</u>, as also shown in figure 5.

[Lecture 10]

3.7 Vibration of a circular membrane

Circular membranes are encountered in many engineering applications, such as in drums, pumps, microphones, etc. Whenever a circular membrane is plane and its material is elastic, but offers no resistance to bending (e.g., not a metallic membrane!), its vibrations are governed by the two-dimensional wave equation. Since the membrane is circular, it is convenient to use the polar coordinates defined by $x = r\cos\theta$ and $y = r\sin\theta$.

The two-dimensional wave equation in the polar coordinates reads

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right). \tag{124}$$

We shall consider a membrane of radius R with fixed end (figure 6) and determine solutions $u(r,t)$ that are radially symmetric (i.e., those solutions which do not depend on θ). In this case, $u_{\theta\theta} = 0$ and the two-dimensional wave equation (124) reduces to

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right). \tag{125}$$

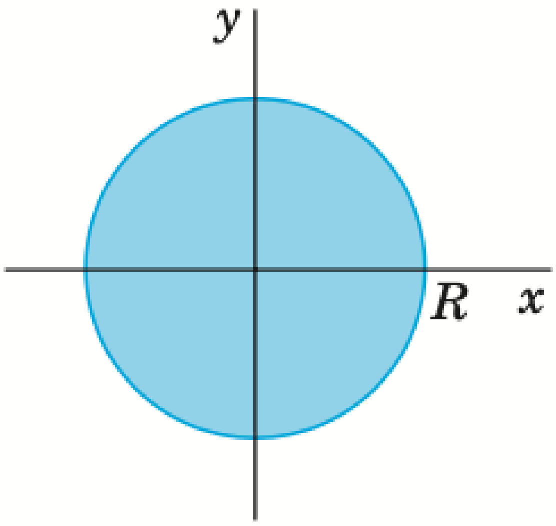


Figure 6: Circular membrane

Boundary Condition: Since the membrane is fixed along the boundary $r = R$, we have the boundary condition

$$u(R,t) = 0, \qquad \text{for all} \quad t \geq 0. \tag{126}$$

Initial Conditions: We can obtain radially symmetric solutions only if the initial conditions do not depend on θ. Let us assume that the initial deflection in the membrane is $f(r)$ and the initial velocity of the membrane is $g(r)$. Therefore, the initial conditions are

$$u(r,0) = f(r) \quad \text{and} \quad u_t(r,0) = g(r), \qquad 0 \leq r \leq R. \tag{127}$$

We would like to solve the reduced wave equation (125) along with the boundary condition (126) and the initial conditions (127) using the method of separation of variables.

Solution: First Step: To find the ordinary differential equations:
Let the solution of this problem be $u(r,t) = W(r)\,T(t)$. Therefore, it satisfies (125). Substituting this ansatz in (125), we obtain

$$W\,\ddot{T} = c^2\left[W''\,T + \frac{1}{r}W'\,T\right], \tag{128}$$

where the time derivative has been denoted with dots while the spacial derivative has been denoted by primes. The above equation can be written as

$$\frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{1}{W}\left(W'' + \frac{1}{r}W'\right). \tag{129}$$

Again, the left-hand side of the above equation is a function of t alone while the right-hand side is a function of r alone. Therefore, both of them must be equal to a constant, let us say, it is k. This gives two ordinary differential equations:

$$\frac{\mathrm{d}^2T}{\mathrm{d}t^2} - kc^2T = 0 \qquad \text{and} \qquad W'' + \frac{1}{r}W' - kW = 0. \tag{130}$$

The equation for $T(t)$ has solutions which grow or decay exponentially for $k > 0$, are linear or constant for $k = 0$, and and are periodic for $k < 0$. Physically, it is expected that a solution to the problem of a vibrating membrane will be oscillatory in time, and this leaves only the third case $k < 0$; let $k = -\beta^2$, $\beta \neq 0$. With this, the above ordinary differential equations become

$$\frac{\mathrm{d}^2T}{\mathrm{d}t^2} + \lambda^2T = 0 \qquad \text{and} \qquad rW'' + W' + \beta^2 rW = 0, \tag{131}$$

where $\lambda = \beta c$. The equation for W can be reduced to the Bessel equation, which is $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ by a scaling $s = \beta r$. With this scaling,

$$\begin{aligned}
W' &= \frac{\mathrm{d}W}{\mathrm{d}r} = \frac{\mathrm{d}W}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}r} = \beta\frac{\mathrm{d}W}{\mathrm{d}s},\\
W'' &= \frac{\mathrm{d}^2W}{\mathrm{d}r^2} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}W}{\mathrm{d}r}\right)\frac{\mathrm{d}s}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\beta\frac{\mathrm{d}W}{\mathrm{d}s}\right)\beta = \beta^2\frac{\mathrm{d}^2W}{\mathrm{d}s^2},
\end{aligned}$$

and the equation for W becomes

$$\begin{aligned}
\beta^2 r\frac{\mathrm{d}^2W}{\mathrm{d}s^2} + \beta\frac{\mathrm{d}W}{\mathrm{d}s} + \beta^2 rW = 0 \quad &\Longrightarrow \quad \beta^2r^2\frac{\mathrm{d}^2W}{\mathrm{d}s^2} + \beta r\frac{\mathrm{d}W}{\mathrm{d}s} + \beta^2r^2W = 0\\
&\Longrightarrow \quad s^2\frac{\mathrm{d}^2W}{\mathrm{d}s^2} + s\frac{\mathrm{d}W}{\mathrm{d}s} + s^2W = 0,
\end{aligned} \tag{132}$$

which is the Bessel equation with $\nu = 0$.

Second Step: Satisfying the boundary condition:
The boundary condition $u(R,t) = 0$ leads to $W(R)\,T(t) = 0$ and, hence, to

$$W(R) = 0 \tag{133}$$

because $T(t) = 0$ will result into the zero solution which is meaningless.

Solution of the Bessel equation (132) are the Bessel functions $J_0(s)$ and $Y_0(s)$ of the first and second kind, respectively. It turns out that $Y_0(s)$ becomes infinite at $s = 0$; therefore $Y_0(s)$ cannot be a part of the solution because the deflection of the membrane must always be finite. This leaves us with the solution $W(s) = J_0(s)$ or, in other words,

$$W(r) = J_0(\beta r). \tag{134}$$

The boundary condition (133) implies that

$$J_0(\beta R) = 0 \tag{135}$$

We can satisfy this condition because $J_0(s)$ has infinitely many positive zeros, $s = \alpha_1, \alpha_2, \alpha_3, \ldots$ (see figure 7), with numerical values

$$\alpha_1 = 2.4048, \quad \alpha_2 = 5.5201, \quad \alpha_3 = 8.6537, \quad \alpha_4 = 11.7915, \quad \alpha_5 = 14.9309$$

and so on. These zeros are slightly irregularly spaced, as we can see in the figure.

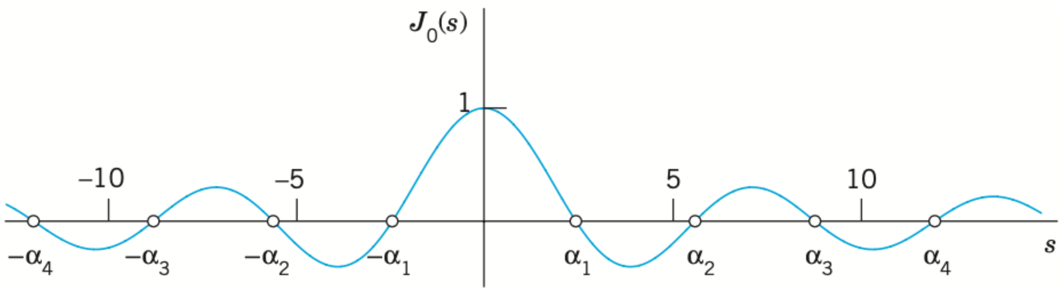


Figure 7: Bessel function $J_0(s)$

Equation (135) now implies that

$$\beta R = \alpha_n \quad \Longrightarrow \quad \beta = \beta_n = \frac{\alpha_n}{R}, \qquad n = 1, 2, 3, \ldots \tag{136}$$

Hence, the functions

$$W_n(r) = J_0(\beta_n r) = J_0\left(\frac{\alpha_n}{R} r\right), \qquad n = 1, 2, 3, \ldots \tag{137}$$

are solutions of (131) that vanish at $r = R$.

Eigenfunctions and eigenvalues: For W_n in (137), a corresponding general solution of $(131)_1$ with $\lambda = \lambda_n = \beta_n c = \alpha_n c/R$ is

$$T_n(t) = c_n \cos \lambda_n t + d_n \sin \lambda_n t. \tag{138}$$

Hence the functions

$$u_n(x,t) = W_n(r)\, T_n(t) = \Big(c_n \cos \lambda_n t + d_n \sin \lambda_n t\Big)\, J_0(\beta_n r) \tag{139}$$

with $n = 1, 2, 3, \ldots$ are solutions of the wave equation (125) satisfying the boundary condition (126). These are the *eigenfunctions* of our problem. The corresponding *eigenvalues* are λ_n.

The vibration of the membrane corresponding to u_n is referred to as the *nth normal mode*; it has the frequency $\lambda_n/(2\pi) = \alpha_n c/(2\pi R)$ cycles per unit time. Since the zeros of the Bessel function J_0 are not regularly spaced on the axis (in contrast to the zeros of the sine functions appearing in the case of the vibrating string), the sound of a drum is entirely different from that of a violin. The forms of the normal modes can be easily obtained from figure 7 and are shown in figure 8. For $n = 1$, all the points of the membrane move up (or down) at the same time. For $n = 2$, the situation is as follows. The function $W_2(r) = J_0(\alpha_2 r/R)$ is zero for $\alpha_2 r/R = \alpha_1$ or for $r = \alpha_1 R/\alpha_2$. The circle with the radius $r = \alpha_1 R/\alpha_2$ is referred to as a *nodal line*. When at some instant the central part $(r < \alpha_1 R/\alpha_2)$ of the membrane moves up, the outer part $(r > \alpha_1 R/\alpha_2)$ moves down, and vice versa. The solution $u_n(r,t)$ has $n-1$ nodal lines, which are circles (figure 8).

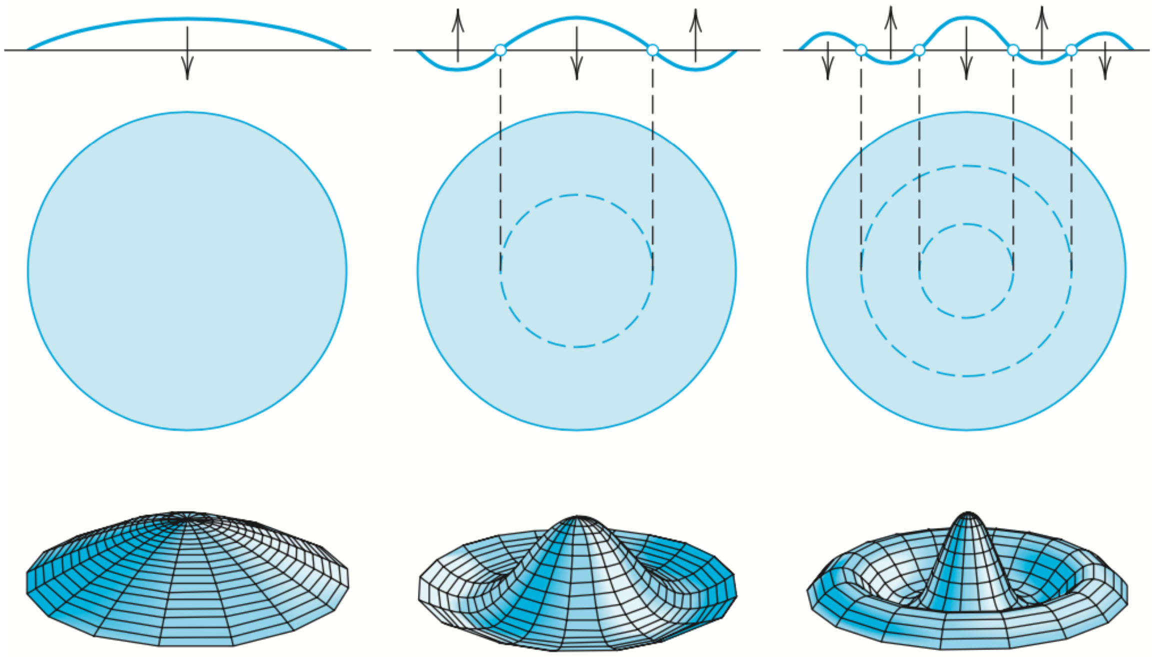


Figure 8: Normal modes of the circular membrane in the case of vibrations independent of the angle θ: (left) $n = 1$, (middle) $n = 2$ and (right) $n = 3$.

Third Step: Solution of the entire problem: From (139), the most general solution of the wave equation (125) that satisfies the given boundary condition (126) is

$$u(r,t) = \sum_{n=1}^{\infty} u_n(r,t) = \sum_{n=1}^{\infty} \Big(c_n \cos\lambda_n t + d_n \sin\lambda_n t\Big)\, J_0\left(\frac{\alpha_n}{R} r\right). \tag{140}$$

Let us now apply the initial conditions (127). The initial condition $u(r,0) = f(r)$ gives

$$f(r) = \sum_{n=1}^{\infty} c_n\, J_0\left(\frac{\alpha_n}{R} r\right). \tag{141}$$

To obtain the coefficients c_n, we shall use the orthogonality of the Bessel functions, which is given by

$$\int_0^1 x\, J_p(ax)\, J_p(bx)\, \mathrm{d}x = \begin{cases} 0 & \text{if} \quad a \neq b \\ \dfrac{1}{2} J_{p+1}^2(a) & \text{if} \quad a = b, \end{cases} \tag{142}$$

where p is a non-negative integer, and a and b are the zeros of $J_p(x)$.

To use the orthogonality of the Bessel functions for the problem under consideration, let us replace p with 0, x with r/R, a with α_m and b with α_n. With this, the above orthogonality relation changes to

$$\int_0^R r\, J_0\left(\frac{\alpha_m}{R}r\right) J_0\left(\frac{\alpha_n}{R}r\right) \mathrm{d}r = \begin{cases} 0 & \text{if} \quad m \neq n \\ \frac{1}{2}R^2 J_1^2(\alpha_m) & \text{if} \quad m = n. \end{cases} \tag{143}$$

Now, to find the coefficients c_n in (141), we shall use the orthogonality relation (143). For that, let us multiply both sides of (141) with $rJ_0\left(\frac{\alpha_m}{R}r\right)$ for some fixed m $(m = 1, 2, 3, \dots)$ and integrate both sides of the resulting equation with respect to r in $(0, R)$. Using the orthogonality relation (143), the only term that will be nonzero in the right-hand side will be for $n = m$ and we would have

$$\int_0^R r\, f(r)\, J_0\left(\frac{\alpha_m}{R}r\right) \mathrm{d}r = \frac{1}{2}R^2 J_1^2(\alpha_m) c_m, \qquad m = 1, 2, 3, \dots \tag{144}$$

or

$$\boxed{c_m = \frac{2}{R^2 J_1^2(\alpha_m)} \int_0^R r\, f(r)\, J_0\left(\frac{\alpha_m}{R}r\right) \mathrm{d}r} \qquad m = 1, 2, 3, \dots \tag{145}$$

Let us now apply the remaining initial condition $u_t(r, 0) = g(r)$. For that we first differentiate solution (140) partially with respect to t to obtain

$$u_t(r,t) = \sum_{n=1}^{\infty} \lambda_n \Big(-c_n \sin \lambda_n t + d_n \cos \lambda_n t \Big) J_0\left(\frac{\alpha_n}{R}r\right). \tag{146}$$

The initial condition $u_t(r, 0) = g(r)$ gives

$$g(r) = \sum_{n=1}^{\infty} \lambda_n d_n\, J_0\left(\frac{\alpha_n}{R}r\right). \tag{147}$$

To obtain the coefficients d_n, let us multiply both sides of (147) with $rJ_0\left(\frac{\alpha_m}{R}r\right)$ for some fixed m $(m = 1, 2, 3, \dots)$ and integrate both sides of the resulting equation with respect to r in $(0, R)$. Using the orthogonality relation (143), the only term that will be nonzero in the right-hand side will be for $n = m$ and we would have

$$\int_0^R r\, g(r)\, J_0\left(\frac{\alpha_m}{R}r\right) \mathrm{d}r = \frac{1}{2}R^2 J_1^2(\alpha_m) \lambda_m d_m, \qquad m = 1, 2, 3, \dots \tag{148}$$

or

$$\boxed{d_m = \frac{2}{\alpha_m c R J_1^2(\alpha_m)} \int_0^R r\, g(r)\, J_0\left(\frac{\alpha_m}{R}r\right) \mathrm{d}r} \qquad m = 1, 2, 3, \dots, \tag{149}$$

where the relation $\lambda_m = \alpha_m c/R$ has been used. Therefore, the deflection in a (radially symmetric) vibrating membrane fixed at the boundary and satisfying the initial conditions (127) is given by

$$\boxed{u(r,t) = \sum_{n=1}^{\infty} \left[c_n \cos\left(\frac{\alpha_n c}{R}t\right) + d_n \sin\left(\frac{\alpha_n c}{R}t\right) \right] J_0\left(\frac{\alpha_n}{R}r\right)} \tag{150}$$

where

$$\boxed{c_n = \frac{2}{R^2 J_1^2(\alpha_n)} \int_0^R r\, f(r)\, J_0\left(\frac{\alpha_n}{R} r\right)\, \mathrm{d}r} \qquad n = 1, 2, 3, \ldots \tag{151}$$

and

$$\boxed{d_n = \frac{2}{\alpha_n c R J_1^2(\alpha_n)} \int_0^R r\, g(r)\, J_0\left(\frac{\alpha_n}{R} r\right)\, \mathrm{d}r} \qquad n = 1, 2, 3, \ldots \tag{152}$$