0.1 Esercizi finali

1) Sia T_1 la teoria dei tipi definita del tipo singoletto con le regole strutturali, definite in questo capitolo, incluse quelle di sostituzione e indebolimento. Allora stabilire se i seguenti termini sono tipabili come termini del tipo singoletto, secondo T_1 e quali sono uguali definizionalmente.

- $\mathrm{El}_{N1}(*, *)$
- $\mathrm{El}_{N1}(\mathrm{x}, *)$
- $\mathrm{El}_{N1}(*, \mathrm{y})$
- $\mathrm{El}_{N1}(\mathrm{x}, \mathrm{y})$
- $\mathrm{El}_{N1}(\mathrm{El}_{N1}(*, \mathrm{y}), \mathrm{El}_{N1}(\mathrm{x}, *))$

Soluzione

Per una maggiore comprensione delle derivazioni, ho ritenuto opportuno, ove necessario, spezzare l'albero in più parti.
1

$$\text{E-S}\ \dfrac{\text{I-S}\,\dfrac{[\,]\ \text{cont}}{* \in \mathrm{N1}[\,]} \qquad \text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\ \text{cont}}{\mathrm{N}_1\ \text{type}[\,]}\quad (\mathrm{z} \in \mathrm{N}_1) \notin [\,]}{\mathrm{z} \in \mathrm{N}_1\ \text{cont}}}{\mathrm{N}_1\ \text{type}[\mathrm{z} \in \mathrm{N}_1]} \qquad \text{I-S}\,\dfrac{[\,]\ \text{cont}}{* \in \mathrm{N}_1[\,]}}{\mathrm{El}_{N1}(*, *) \in N_1[\,]}$$

Applicando la β N_1-*red* allora $\mathrm{El}_{N1}(*, *) \to_1 *$
$\mathrm{El}_{N1}(*, *)$ è uguale definizionalmente.

2

$$\text{E-S}\ \dfrac{\text{var}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\ \text{cont}}{\mathrm{N}_1\ \text{type}[\,]}\quad (\mathrm{x} \in \mathrm{N}_1) \notin [\,]}{\mathrm{x} \in \mathrm{N}_1\ \text{cont}}}{\mathrm{x} \in \mathrm{N}_1[\mathrm{x} \in \mathrm{N}_1]} \qquad \text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\ \text{cont}}{\mathrm{N}_1\ \text{type}\ [\,]}\quad (\mathrm{x} \in \mathrm{N}_1) \notin [\,]}{\mathrm{x} \in \mathrm{N}_1\ \text{cont}}}{\mathrm{N}_1\ \text{type}[\mathrm{x} \in \mathrm{N}_1]}\quad (\mathrm{z} \in \mathrm{N}_1) \notin \mathrm{x} \in \mathrm{N}_1}{\mathrm{x} \in \mathrm{N}_1,\ \mathrm{z} \in \mathrm{N}_1\ \text{cont}}}{\mathrm{N}_1\ \text{type}[\mathrm{x} \in \mathrm{N}_1,\ \mathrm{z} \in \mathrm{N}_1]} \qquad \text{I-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\ \text{cont}}{\mathrm{N}_1\ \text{type}[\,]}\quad (\mathrm{x} \in \mathrm{N}_1) \notin [\,]}{\mathrm{x} \in \mathrm{N}_1\ \text{cont}}}{* \in \mathrm{N}_1[\mathrm{x} \in \mathrm{N}_1]}}{\mathrm{El}_{N1}(\mathrm{x}, *) \in \mathrm{N}_1[\mathrm{x} \in \mathrm{N}_1]}$$

Applicando la β N_1-*red* allora $\mathrm{El}_{N1}(\mathrm{x}, *) \not\twoheadrightarrow_1$
$\mathrm{El}_{N1}(\mathrm{x}, *)$ non è uguale definizionalmente.

3

$$\text{E-S}\ \dfrac{\text{I-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\ \text{cont}}{\mathrm{N}_1\ \text{type}[\,]}\quad (\mathrm{y} \in \mathrm{N}_1) \notin [\,]}{\mathrm{y} \in \mathrm{N}_1\ \text{cont}}}{* \in \mathrm{N}_1[\mathrm{y} \in \mathrm{N}_1]} \qquad \text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\ \text{cont}}{\mathrm{N}_1\ \text{type}[\,]}\quad (\mathrm{y} \in \mathrm{N}_1) \notin [\,]}{\mathrm{y} \in \mathrm{N}_1\ \text{cont}}}{\mathrm{N}_1\ \text{type}[\mathrm{y} \in \mathrm{N}_1]}\quad (\mathrm{z} \in \mathrm{N}_1) \notin \mathrm{y} \in \mathrm{N}_1}{\mathrm{y} \in \mathrm{N}_1,\ \mathrm{z} \in \mathrm{N}_1\ \text{cont}}}{\mathrm{N}_1\ \text{type}[\mathrm{y} \in \mathrm{N}_1,\ \mathrm{z} \in \mathrm{N}_1]} \qquad \text{var}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\ \text{cont}}{\mathrm{N}_1\ \text{type}[\,]}\quad (\mathrm{y} \in \mathrm{N}_1) \notin [\,]}{\mathrm{y} \in \mathrm{N}_1\ \text{cont}}}{\mathrm{y} \in \mathrm{N}_1[\mathrm{y} \in \mathrm{N}_1]}}{\mathrm{El}_{N1}(*, \mathrm{y}) \in \mathrm{N}_1[\mathrm{y} \in \mathrm{N}_1]}$$

Applicando la β N_1-*red* allora $\mathrm{El}_{N1}(*, \mathrm{y}) \to_1 \mathrm{y}$
$\mathrm{El}_{N1}(*, \mathrm{y})$ è uguale definizionalmente.

4

$$\text{E-S}\,\dfrac{\text{var}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\ ]\ \mathrm{cont}}{\mathrm{N}_1\ \mathrm{type}\,[\ ]}}{\mathrm{x} \in \mathrm{N}_1\ \mathrm{cont}}\ (\mathrm{x} \in \mathrm{N}_1) \notin [\ ]}{\mathrm{N}_1\ \mathrm{type}[\mathrm{x} \in \mathrm{N}_1]}}{\mathrm{x} \in \mathrm{N}_1,\ \mathrm{y} \in \mathrm{N}_1\ \mathrm{cont}}\ \begin{matrix}(\mathrm{y} \in \mathrm{N}_1)\\ \notin \mathrm{x} \in \mathrm{N}_1\end{matrix}}{\mathrm{x} \in \mathrm{N}_1[\mathrm{x} \in \mathrm{N}_1,\ \mathrm{y} \in \mathrm{N}_1]} \quad \text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\ ]\ \mathrm{cont}}{\mathrm{N}_1\ \mathrm{type}[\ ]}}{\mathrm{x} \in \mathrm{N}_1\ \mathrm{cont}}\ (\mathrm{x} \in \mathrm{N}_1) \notin [\ ]}{\mathrm{N}_1\ \mathrm{type}[\mathrm{x} \in \mathrm{N}_1]}}{\mathrm{x} \in \mathrm{N}_1,\ \mathrm{y} \in \mathrm{N}_1\ \mathrm{cont}}\ \begin{matrix}(\mathrm{y} \in \mathrm{N}_1)\\ \notin (\mathrm{x} \in \mathrm{N}_1)\end{matrix}}{\mathrm{N}_1\ \mathrm{type}[\mathrm{x} \in \mathrm{N}_1,\ \mathrm{y} \in \mathrm{N}_1]}}{\mathrm{x} \in \mathrm{N}_1,\ \mathrm{y} \in \mathrm{N}_1,\ \mathrm{z} \in \mathrm{N}_1\ \mathrm{cont}}\ \begin{matrix}(\mathrm{z} \in \mathrm{N}_1) \notin\\ (\mathrm{x} \in \mathrm{N}_1,\ \mathrm{y} \in \mathrm{N}_1)\end{matrix}}{\mathrm{N}_1\ \mathrm{type}[\mathrm{x} \in \mathrm{N}_1,\ \mathrm{y} \in \mathrm{N}_1,\ \mathrm{z} \in \mathrm{N}_1]} \quad \text{var}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\ ]\ \mathrm{cont}}{\mathrm{N}_1\ \mathrm{type}\,[\ ]}}{\mathrm{x} \in \mathrm{N}_1\ \mathrm{cont}}\ (\mathrm{x} \in \mathrm{N}_1) \notin [\ ]}{\mathrm{N}_1\ \mathrm{type}[\mathrm{x} \in \mathrm{N}_1]}}{\mathrm{x} \in \mathrm{N}_1,\ \mathrm{y} \in \mathrm{N}_1\ \mathrm{cont}}\ \begin{matrix}(\mathrm{y} \in \mathrm{N}_1)\\ \notin \mathrm{x} \in \mathrm{N}_1\end{matrix}}{\mathrm{y} \in \mathrm{N}_1[\mathrm{x} \in \mathrm{N}_1,\ \mathrm{y} \in \mathrm{N}_1]}}{\mathrm{El}_{N1}(\mathrm{x},\ \mathrm{y}) \in \mathrm{N}_1[\mathrm{x} \in \mathrm{N}_1,\ \mathrm{y} \in \mathrm{N}_1]}$$

Applicando la β N_1-*red* allora $\mathrm{El}_{N1}(\mathrm{x}, \mathrm{y}) \not\to_1$
$\mathrm{El}_{N1}(\mathrm{x}, \mathrm{y})$ non è uguale definizionalmente.

5

$$\text{E-S}\,\dfrac{\dfrac{\mathbf{5}_A}{\mathrm{El}_{N1}(*,\ \mathrm{y}) \in \mathrm{N}_1[\mathrm{y} \in \mathrm{N}_1,\ \mathrm{x} \in \mathrm{N}_1]} \quad \dfrac{\mathbf{5}_B}{\mathrm{N}_1\ \mathrm{type}[\mathrm{y} \in \mathrm{N}_1,\ \mathrm{x} \in \mathrm{N}_1,\ \mathrm{z} \in \mathrm{N}_1]} \quad \dfrac{\mathbf{5}_C}{\mathrm{El}_{N1}(\mathrm{x},\ *) \in \mathrm{N}_1[\mathrm{y} \in \mathrm{N}_1,\ \mathrm{x} \in \mathrm{N}_1]}}{\mathrm{El}_{N1}(\mathrm{El}_{N1}(*,\ \mathrm{y}),\ \mathrm{El}_{N1}(\mathrm{x},\ *)) \in \mathrm{N}_1[\mathrm{y} \in \mathrm{N}_1,\ \mathrm{x} \in \mathrm{N}_1]}$$

$\mathbf{5}_A$

$$\text{ind-te}\,\dfrac{\dfrac{}{\mathrm{El}_{N1}(*,\ \mathrm{y}) \in \mathrm{N}_1[\mathrm{y} \in \mathrm{N}_1]} \quad \text{F-c}\,\dfrac{\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\ ]\ \mathrm{cont}}{\mathrm{N}_1\ \mathrm{type}[\ ]}}{\mathrm{y} \in \mathrm{N}_1\ \mathrm{cont}}\ (\mathrm{y} \in \mathrm{N}_1) \notin [\ ]}{\mathrm{N}_1\ \mathrm{type}[\mathrm{y} \in \mathrm{N}_1]}}{\mathrm{y} \in \mathrm{N}_1,\ \mathrm{x} \in \mathrm{N}_1\ \mathrm{cont}}\ (\mathrm{x} \in \mathrm{N}_1) \notin \mathrm{y} \in \mathrm{N}_1}{\mathrm{El}_{N1}(*,\ \mathrm{y}) \in \mathrm{N}_1[\mathrm{y} \in \mathrm{N}_1,\ \mathrm{x} \in \mathrm{N}_1]}$$

$\mathbf{5}_B$

$$\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\ ]\ \mathrm{cont}}{\mathrm{N}_1\ \mathrm{type}[\ ]}}{\mathrm{y} \in \mathrm{N}_1\ \mathrm{cont}}\ (\mathrm{y} \in \mathrm{N}_1) \notin [\ ]}{\mathrm{N}_1\ \mathrm{type}[\mathrm{y} \in \mathrm{N}_1]}}{\mathrm{y} \in \mathrm{N}_1,\ \mathrm{x} \in \mathrm{N}_1\ \mathrm{cont}}\ (\mathrm{x} \in \mathrm{N}_1) \notin \mathrm{y} \in \mathrm{N}_1}{\mathrm{N}_1\ \mathrm{type}[\mathrm{y} \in \mathrm{N}_1,\ \mathrm{x} \in \mathrm{N}_1]}}{\mathrm{y} \in \mathrm{N}_1,\ \mathrm{x} \in \mathrm{N}_1,\ \mathrm{z} \in \mathrm{N}_1\ \mathrm{cont}}\ (\mathrm{z} \in \mathrm{N}_1) \notin (\mathrm{y} \in \mathrm{N}_1,\ \mathrm{x} \in \mathrm{N}_1)}{\mathrm{N}_1\ \mathrm{type}[\mathrm{y} \in \mathrm{N}_1,\ \mathrm{x} \in \mathrm{N}_1,\ \mathrm{z} \in \mathrm{N}_1]}$$

$\mathbf{5}_C$

$$\text{ex-te}\ \dfrac{\text{ind-te}\ \dfrac{\dfrac{}{\mathrm{El}_{N1}(x,\ *) \in \mathrm{N}_1[x \in \mathrm{N}_1]} \qquad \text{F-c}\ \dfrac{\text{F-S}\ \dfrac{\text{F-c}\ \dfrac{\text{F-S}\ \dfrac{[\ ]\ \text{cont}}{\mathrm{N}_1\ \text{type}\ [\ ]}\ (x \in \mathrm{N}_1) \notin [\ ]}{x \in \mathrm{N}_1\ \text{cont}}}{\mathrm{N}_1\ \text{type}[x \in \mathrm{N}_1]}\ (y \in \mathrm{N}_1) \notin x \in \mathrm{N}_1}{x \in \mathrm{N}_1,\ y \in \mathrm{N}_1\ \text{cont}}}{\mathrm{El}_{N1}(x,\ *) \in \mathrm{N}_1[x \in \mathrm{N}_1,\ y \in \mathrm{N}_1]} \qquad \text{F-c}\ \dfrac{\text{F-S}\ \dfrac{\text{F-c}\ \dfrac{\text{F-S}\ \dfrac{[\ ]\ \text{cont}}{\mathrm{N}_1\ \text{type}\ [\ ]}\ (y \in \mathrm{N}_1) \notin [\ ]}{y \in \mathrm{N}_1\ \text{count}}}{\mathrm{N}_1\ \text{type}[y \in \mathrm{N}_1]}\ (x \in \mathrm{N}_1) \notin y \in \mathrm{N}_1}{y \in \mathrm{N}_1,\ x \in \mathrm{N}_1\ \text{cont}}}{\mathrm{El}_{N1}(x,\ *) \in \mathrm{N}_1[y \in \mathrm{N}_1,\ x \in \mathrm{N}_1]}$$

Per le premesse *$El_{N1}(*,\ y) \in N_1[y \in N_1]$* e *$El_{N1}(x,\ *) \in N_1[x \in N_1]$* ho già dimostrato sopra (in 3 e 2) la loro tipabilità per il tipo singoletto.
Applicando la *red_I* e β *N_1-red* allora $\mathrm{El}_{N1}(\mathrm{El}_{N1}(*,\ y),\ \mathrm{El}_{N1}(x,\ *)) \to_1 \mathrm{El}_{N1}(y,\ \mathrm{El}_{N1}(x,\ *))$.
Più nel dettaglio la riduzione è la seguente

$$\text{red}_I\ \dfrac{\beta\ \text{N}_1\text{-red}\ \dfrac{}{\mathrm{El}_{N1}(*,\ y) \to_1 y}}{\mathrm{El}_{N1}(\mathrm{El}_{N1}(*,\ y),\ \mathrm{El}_{N1}(x,\ *)) \to_1 \mathrm{El}_{N1}(y,\ \mathrm{El}_{N1}(x,\ *))}$$

$\mathrm{El}_{N1}(\mathrm{El}_{N1}(*,\ y),\ \mathrm{El}_{N1}(x,\ *))$ è uguale definizionalmente.

2) Definire w + 2 ∈ Nat[w ∈ Nat], ove 2 è l'abbreviazione del termine ottenuto applicando 2 ≡ succ(succ(0)).

Soluzione

La ricorsione la faccio su w, usando lo schema di ricorsione primitiva, vale che $w + 2 \equiv \mathrm{El}_{Nat}(w,\ 2,\ (x,y).\mathrm{succ}(y)) \in \mathrm{Nat}[w \in \mathrm{Nat}]$.

$$\text{E-Nat}_{dip}\ \dfrac{\text{F-Nat}\ \dfrac{\text{F-c}\ \dfrac{\text{F-Nat}\ \dfrac{[\ ]\ \text{cont}}{\text{Nat type}[\ ]}\ (w \in \text{Nat}) \notin [\ ]}{w \in \text{Nat cont}}}{\text{Nat type}[w \in \text{Nat}]} \qquad \text{I}_2\text{-Nat}\ \dfrac{\text{I}_2\text{-Nat}\ \dfrac{\text{I}_1\text{-Nat}\ \dfrac{[\ ]\ \text{cont}}{0 \in \text{Nat}[\ ]}}{1 \in \text{Nat}[\ ]}}{2 \in \text{Nat}[\ ]} \qquad \text{I}_2\text{-Nat}_{prog}\ \dfrac{\text{F-c}\ \dfrac{\text{F-Nat}\ \dfrac{[\ ]\ \text{cont}}{\text{Nat type}\ [\ ]}\ (x \in \text{Nat}) \notin [\ ]}{x \in \text{Nat cont}}}{\mathrm{succ}(y) \in \text{Nat}[x \in \text{Nat},\ y \in \text{Nat}]}}{\mathrm{El}_{Nat}(w,\ 2,\ (x,y).\mathrm{succ}(y)) \in \text{Nat}[w \in \text{Nat}]}$$

Dimostrazione di correttezza di $El_{Nat}(w,\ 2,\ (x,y).succ(y)) \in Nat[w \in Nat]$

- $\mathrm{El}_{Nat}(0,\ 2,\ (x,y).\mathrm{succ}(y)) \to_1 2$ per *β_{1Nat}-red*
- $\mathrm{El}_{Nat}(\mathrm{succ}(m),\ 2,\ (x,y).\mathrm{succ}(y)) \to_1 \mathrm{succ}(\mathrm{El}_{Nat}(m,\ 2,\ (x,y).\mathrm{succ}(y)))$ per *β_{2Nat}-red* $\Rightarrow$ per $m = 0 \equiv \mathrm{succ}(\mathrm{El}_{Nat}(0,\ 2,\ (x,y).\mathrm{succ}(y))) \to_1 \mathrm{succ}(2) \in \mathrm{Nat} = 3$ (dal punto precedente).

4) Si dimostri che esiste un *proof-term* pf del tipo

$$\mathbf{pf \in Id(N_1,x,*)[x \in N_1]}$$

Soluzione

$z \equiv x$
$\mathrm{El}_{N1}(z,c) \equiv \mathrm{pf} \equiv \mathrm{El}_{N1}(x,x.\mathrm{id}(*))$

$M(z) \equiv Id(N_1,x,*)$
$c \in M(*)[\Gamma] \equiv id(*) \in Id(N_1,*,*)[x \in N_1]$

$$\dfrac{\text{F-Id}\ \dfrac{\dfrac{}{N_1\ type[x \in N_1]} \quad \dfrac{}{x \in N_1[x \in N_1]} \quad \dfrac{}{* \in N_1[x \in N_1]}}{Id(N_1,x,*)type[x \in N_1]} \quad \text{I-Id}\ \dfrac{\text{I-S}\ \dfrac{[\ ]\ cont}{* \in N_1[\ ]}}{id(*) \in Id(N_1,*,*)[\ ]}}{El_{N1}(x,x.id(*)) \in Id(N_1,x,*)[x \in N_1]}\ \text{E-S}_{dip}$$