Esercizi finali

1) Sia T_1 la teoria dei tipi definita del tipo singoletto con le regole strutturali, definite all'interno di *"Esame di Teoria dei Tipi"*, incluse quelle di sostituzione e indebolimento. Allora stabilire se i seguenti termini sono tipabili come termini del tipo singoletto, secondo T_1 e quali sono uguali definizionalmente.

- $\text{El}_{N1}(*, *)$
- $\text{El}_{N1}(\text{x}, *)$
- $\text{El}_{N1}(*, \text{y})$
- $\text{El}_{N1}(\text{x}, \text{y})$
- $\text{El}_{N1}(\text{El}_{N1}(*, \text{y}), \text{El}_{N1}(\text{x}, *))$

Soluzione

1

$$\text{E-S}\,\dfrac{\text{I-S}\,\dfrac{[\,]\text{ cont}}{* \in \text{N1}[\,]} \qquad \text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\text{ cont}}{\text{N}_1\text{ type}[\,]}}{\text{z} \in \text{N}_1\text{ cont}}\ (\text{z} \in \text{N}_1) \notin [\,]}{\text{N}_1\text{ type}[\text{z} \in \text{N}_1]} \qquad \text{I-S}\,\dfrac{[\,]\text{ cont}}{* \in \text{N}_1[\,]}}{\text{El}_{N1}(*, *) \in N_1[\,]}$$

Applicando la β N_1-*red* allora $\text{El}_{N1}(*, *) \rightarrow_1 *$
$\text{El}_{N1}(*, *)$ è uguale definizionalmente.

2

$$\text{E-S}\,\dfrac{\text{var}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\text{ cont}}{\text{N}_1\text{ type}[\,]}}{\text{x} \in \text{N}_1\text{ cont}}\ (\text{x} \in \text{N}_1) \notin [\,]}{\text{x} \in \text{N}_1[\text{x} \in \text{N}_1]} \quad \text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\text{ cont}}{\text{N}_1\text{ type }[\,]}}{\text{x} \in \text{N}_1\text{ cont}}\ (\text{x} \in \text{N}_1) \notin [\,]}{\text{N}_1\text{ type}[\text{x} \in \text{N}_1]}}{\text{x} \in \text{N}_1,\ \text{z} \in \text{N}_1\text{ cont}}\ (\text{z} \in \text{N}_1) \notin \text{x} \in \text{N}_1}{\text{N}_1\text{ type}[\text{x} \in \text{N}_1,\ \text{z} \in \text{N}_1]} \quad \text{I-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\text{ cont}}{\text{N}_1\text{ type}[\,]}}{\text{x} \in \text{N}_1\text{ cont}}\ (\text{x} \in \text{N}_1) \notin [\,]}{* \in \text{N}_1[\text{x} \in \text{N}_1]}}{\text{El}_{N1}(\text{x}, *) \in \text{N}_1[\text{x} \in \text{N}_1]}$$

Applicando la β N_1-*red* allora $\text{El}_{N1}(\text{x}, *) \nrightarrow_1$
$\text{El}_{N1}(\text{x}, *)$ non è uguale definizionalmente.

3

$$\text{E-S}\,\dfrac{\text{I-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\text{ cont}}{\text{N}_1\text{ type}[\,]}}{\text{y} \in \text{N}_1\text{ cont}}\ (\text{y} \in \text{N}_1) \notin [\,]}{* \in \text{N}_1[\text{y} \in \text{N}_1]} \quad \text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\text{ cont}}{\text{N}_1\text{ type}[\,]}}{\text{y} \in \text{N}_1\text{ cont}}\ (\text{y} \in \text{N}_1) \notin [\,]}{\text{N}_1\text{ type}[\text{y} \in \text{N}_1]}}{\text{y} \in \text{N}_1,\ \text{z} \in \text{N}_1\text{ cont}}\ (\text{z} \in \text{N}_1) \notin \text{y} \in \text{N}_1}{\text{N}_1\text{ type}[\text{y} \in \text{N}_1,\ \text{z} \in \text{N}_1]} \quad \text{var}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\text{ cont}}{\text{N}_1\text{ type}[\,]}}{\text{y} \in \text{N}_1\text{ cont}}\ (\text{y} \in \text{N}_1) \notin [\,]}{\text{y} \in \text{N}_1[\text{y} \in \text{N}_1]}}{\text{El}_{N1}(*, \text{y}) \in \text{N}_1[\text{y} \in \text{N}_1]}$$

Applicando la β N_1-*red* allora $\mathrm{El}_{N1}(*, y) \to_1 y$
$\mathrm{El}_{N1}(*, y)$ è uguale definizionalmente.

4

$$\text{E-S}\,\dfrac{\text{var}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\ \text{cont}}{N_1\ \text{type}\,[\,]}}{x \in N_1\ \text{cont}}\,(x \in N_1) \notin [\,]}{N_1\ \text{type}[x \in N_1]}}{x \in N_1,\, y \in N_1\ \text{cont}}\,\begin{matrix}(y \in N_1)\\ \notin x \in N_1\end{matrix}}{x \in N_1[x \in N_1,\, y \in N_1]} \quad \text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\ \text{cont}}{N_1\ \text{type}[\,]}}{x \in N_1\ \text{cont}}\,(x \in N_1) \notin [\,]}{N_1\ \text{type}[x \in N_1]}}{x \in N_1,\, y \in N_1\ \text{cont}}\,\begin{matrix}(y \in N_1)\\ \notin (x \in N_1)\end{matrix}}{N_1\ \text{type}[x \in N_1,\, y \in N_1]}}{x \in N_1,\, y \in N_1,\, z \in N_1\ \text{cont}}\,\begin{matrix}(z \in N_1) \notin\\ (x \in N_1,\, y \in N_1)\end{matrix}}{N_1\ \text{type}[x \in N_1,\, y \in N_1,\, z \in N_1]} \quad \text{var}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\ \text{cont}}{N_1\ \text{type}\,[\,]}}{x \in N_1\ \text{cont}}\,(x \in N_1) \notin [\,]}{N_1\ \text{type}[x \in N_1]}}{x \in N_1,\, y \in N_1\ \text{cont}}\,\begin{matrix}(y \in N_1)\\ \notin x \in N_1\end{matrix}}{y \in N_1[x \in N_1,\, y \in N_1]}}{\mathrm{El}_{N1}(x, y) \in N_1[x \in N_1,\, y \in N_1]}$$

Applicando la β N_1-*red* allora $\mathrm{El}_{N1}(x, y) \not\twoheadrightarrow_1$
$\mathrm{El}_{N1}(x, y)$ non è uguale definizionalmente.

5

$$\text{E-S}\,\dfrac{\dfrac{\mathbf{5}_A}{\mathrm{El}_{N1}(*, y) \in N_1[y \in N_1,\, x \in N_1]} \quad \dfrac{\mathbf{5}_B}{N_1\ \text{type}[y \in N_1,\, x \in N_1,\, z \in N_1]} \quad \dfrac{\mathbf{5}_C}{\mathrm{El}_{N1}(x, *) \in N_1[y \in N_1,\, x \in N_1]}}{\mathrm{El}_{N1}(\mathrm{El}_{N1}(*, y),\, \mathrm{El}_{N1}(x, *)) \in N_1[y \in N_1,\, x \in N_1]}$$

$\mathbf{5}_A$

$$\text{ind-te}\,\dfrac{\dfrac{\mathbf{3}}{\mathrm{El}_{N1}(*, y) \in N_1[y \in N_1]} \quad \text{F-c}\,\dfrac{\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\ \text{cont}}{N_1\ \text{type}[\,]}}{y \in N_1\ \text{cont}}\,(y \in N_1) \notin [\,]}{N_1\ \text{type}[y \in N_1]}}{y \in N_1,\, x \in N_1\ \text{cont}}\,(x \in N_1) \notin y \in N_1}{\mathrm{El}_{N1}(*, y) \in N_1[y \in N_1,\, x \in N_1]}$$

$\mathbf{5}_B$

$$\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\ \text{cont}}{N_1\ \text{type}[\,]}}{y \in N_1\ \text{cont}}\,(y \in N_1) \notin [\,]}{N_1\ \text{type}[y \in N_1]}}{y \in N_1,\, x \in N_1\ \text{cont}}\,(x \in N_1) \notin y \in N_1}{N_1\ \text{type}[y \in N_1,\, x \in N_1]}}{y \in N_1,\, x \in N_1,\, z \in N_1\ \text{cont}}\,(z \in N_1) \notin (y \in N_1,\, x \in N_1)}{N_1\ \text{type}[y \in N_1,\, x \in N_1,\, z \in N_1]}$$

$\mathbf{5}_C$

$$\text{ex-te}\ \dfrac{\text{ind-te}\ \dfrac{\dfrac{\mathbf{2}}{\mathrm{El}_{N1}(x,\ *) \in \mathrm{N}_1[x \in \mathrm{N}_1]} \qquad \text{F-c}\ \dfrac{\text{F-S}\ \dfrac{\text{F-c}\ \dfrac{\text{F-S}\ \dfrac{[\ ]\ \text{cont}}{\mathrm{N}_1\ \text{type}\ [\ ]}}{x \in \mathrm{N}_1\ \text{cont}}\ (x \in \mathrm{N}_1) \notin [\ ]}{\mathrm{N}_1\ \text{type}[x \in \mathrm{N}_1]}}{x \in \mathrm{N}_1,\ y \in \mathrm{N}_1\ \text{cont}}\ (y \in \mathrm{N}_1) \notin x \in \mathrm{N}_1}{\mathrm{El}_{N1}(x,\ *) \in \mathrm{N}_1[x \in \mathrm{N}_1,\ y \in \mathrm{N}_1]} \qquad \text{F-c}\ \dfrac{\text{F-S}\ \dfrac{\text{F-c}\ \dfrac{\text{F-S}\ \dfrac{[\ ]\ \text{cont}}{\mathrm{N}_1\ \text{type}\ [\ ]}}{y \in \mathrm{N}_1\ \text{count}}\ (y \in \mathrm{N}_1) \notin [\ ]}{\mathrm{N}_1\ \text{type}[y \in \mathrm{N}_1]}}{y \in \mathrm{N}_1,\ x \in \mathrm{N}_1\ \text{cont}}\ (x \in \mathrm{N}_1) \notin y \in \mathrm{N}_1}{\mathrm{El}_{N1}(x,\ *) \in \mathrm{N}_1[y \in \mathrm{N}_1,\ x \in \mathrm{N}_1]}$$

Per le premesse $El_{N1}(*,\ y) \in N_1[y \in N_1]$ e $El_{N1}(x,\ *) \in N_1[x \in N_1]$ ho già dimostrato sopra (in 3 e 2) la loro tipabilità per il tipo singoletto.
Applicando la red_I e β N_1*-red* allora $\mathrm{El}_{N1}(\mathrm{El}_{N1}(*,\ y),\ \mathrm{El}_{N1}(x,\ *)) \to_1 \mathrm{El}_{N1}(y,\ \mathrm{El}_{N1}(x,\ *))$.
Più nel dettaglio la riduzione è la seguente

$$\text{red}_I\ \dfrac{\beta\ \text{N}_1\text{-red}\ \dfrac{}{\mathrm{El}_{N1}(*,\ y) \to_1 y}}{\mathrm{El}_{N1}(\mathrm{El}_{N1}(*,\ y),\ \mathrm{El}_{N1}(x,\ *)) \to_1 \mathrm{El}_{N1}(y,\ \mathrm{El}_{N1}(x,\ *))}$$

$\mathrm{El}_{N1}(\mathrm{El}_{N1}(*,\ y),\ \mathrm{El}_{N1}(x,\ *))$ è uguale definizionalmente.

2) Definire w + 2 $\in$ Nat[w $\in$ Nat], ove 2 è l'abbreviazione del termine ottenuto applicando 2 $\equiv$ succ(succ(0)).

Soluzione

La ricorsione la faccio su w, usando lo schema di ricorsione primitiva, vale che w + 2 $\equiv \mathrm{El}_{Nat}$(w, 2, (x,y).succ(y)).

$$\text{E-Nat}_{dip}\ \dfrac{\text{F-Nat}\ \dfrac{\text{F-c}\ \dfrac{\text{F-Nat}\ \dfrac{[\ ]\ \text{cont}}{\text{Nat type}[\ ]}}{w \in \text{Nat cont}}\ (w \in \text{Nat}) \notin [\ ]}{\text{Nat type}[w \in \text{Nat}]} \qquad \text{I}_2\text{-Nat}\ \dfrac{\text{I}_2\text{-Nat}\ \dfrac{\text{I}_1\text{-Nat}\ \dfrac{[\ ]\ \text{cont}}{0 \in \text{Nat}[\ ]}}{1 \in \text{Nat}[\ ]}}{2 \in \text{Nat}[\ ]} \qquad \text{I}_2\text{-Nat}_{prog}\ \dfrac{\text{F-c}\ \dfrac{\text{F-Nat}\ \dfrac{[\ ]\ \text{cont}}{\text{Nat type}\ [\ ]}}{x \in \text{Nat cont}}\ (x \in \text{Nat}) \notin [\ ]}{\text{succ}(y) \in \text{Nat}[x \in \text{Nat},\ y \in \text{Nat}]}}{\mathrm{El}_{Nat}(w,\ 2,\ (x,y).\text{succ}(y)) \in \text{Nat}[w \in \text{Nat}]}$$

Dimostrazione di correttezza di $El_{Nat}(w,\ 2,\ (x,y).succ(y)) \in Nat[w \in Nat]$

- $\mathrm{El}_{Nat}(0,\ 2,\ (x,y).\text{succ}(y)) \to_1 2$ per β_{1Nat}*-red*
- $\mathrm{El}_{Nat}(\text{succ}(m),\ 2,\ (x,y).\text{succ}(y)) \to_1 \text{succ}(\mathrm{El}_{Nat}(m,\ 2,\ (x,y).\text{succ}(y)))$ per β_{2Nat}*-red* $\Rightarrow$ per m = 0 $\equiv \text{succ}(\mathrm{El}_{Nat}(0,\ 2,\ (x,y).\text{succ}(y))) \to_1 \text{succ}(2) \in$ Nat = 3 (dal punto precedente).

3) Si dimostri che esiste un *proof-term* pf del tipo

$$\mathbf{pf} \in \mathbf{Id(N_1,x,*)[x \in N_1]}$$

Soluzione

$\mathrm{El}_{N1}(z,c) \equiv pf \equiv \mathrm{El}_{N1}(x,x.\mathrm{id}(*))$
$M(z) \equiv \mathrm{Id}(N_1,x,*)$
$c \in M(*)[\Gamma] \equiv \mathrm{id}(*) \in \mathrm{Id}(N_1,*,*)[\ ]$

$$
\text{E-S}_{dip}\,\dfrac{
\text{F-Id}\,\dfrac{
\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\ \mathrm{cont}}{\mathrm{N}_1\ \mathrm{type}[\,]}}{\mathrm{x}\in\mathrm{N}_1\ \mathrm{cont}}\ (\mathrm{x}\in\mathrm{N}_1)\notin[\,]}{\mathrm{N}_1\ \mathrm{type}[\mathrm{x}\in\mathrm{N}_1]}
\qquad
\text{var}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\ \mathrm{cont}}{\mathrm{N}_1\ \mathrm{type}[\,]}}{\mathrm{x}\in\mathrm{N}_1\ \mathrm{cont}}\ (\mathrm{x}\in\mathrm{N}_1)\notin[\,]}{\mathrm{x}\in\mathrm{N}_1[\mathrm{x}\in\mathrm{N}_1]}
\qquad
\text{I-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\ \mathrm{cont}}{\mathrm{N}_1\ \mathrm{type}[\,]}}{\mathrm{x}\in\mathrm{N}_1\ \mathrm{cont}}\ (\mathrm{x}\in\mathrm{N}_1)\notin[\,]}{*\in\mathrm{N}_1[\mathrm{x}\in\mathrm{N}_1]}
}{\mathrm{Id}(\mathrm{N}_1,\mathrm{x},*)\mathrm{type}[\mathrm{x}\in\mathrm{N}_1]}
\qquad
\text{I-Id}\,\dfrac{\text{I-S}\,\dfrac{[\,]\ \mathrm{cont}}{*\in\mathrm{N}_1[\,]}}{\mathrm{id}(*)\in\mathrm{Id}(\mathrm{N}_1,*,*)[\,]}
}{\mathrm{El}_{N1}(\mathrm{x},\mathrm{x}.\mathrm{id}(*))\in\mathrm{Id}(\mathrm{N}_1,\mathrm{x},*)[\mathrm{x}\in\mathrm{N}_1]}
$$