0.1 Esercizi finali

1) Sia T_1 la teoria dei tipi definita del tipo singoletto con le regole strutturali, definite in questo capitolo, incluse quelle di sostituzione e indebolimento. Allora stabilire se i seguenti termini sono tipabili come termini del tipo singoletto, secondo T_1 e quali sono uguali definizionalmente.

- $\mathrm{El}_{N1}(*, *)$
- $\mathrm{El}_{N1}(\mathrm{x}, *)$
- $\mathrm{El}_{N1}(*, \mathrm{y})$
- $\mathrm{El}_{N1}(\mathrm{x}, \mathrm{y})$
- $\mathrm{El}_{N1}(\mathrm{El}_{N1}(*, \mathrm{y}), \mathrm{El}_{N1}(\mathrm{x}, *))$

Soluzione

1

$$\text{E-S}\;\dfrac{\text{I-S}\;\dfrac{[\,]\ \text{cont}}{* \in \mathrm{N1}[\,]} \qquad \text{F-S}\;\dfrac{\text{F-c}\;\dfrac{\text{F-S}\;\dfrac{[\,]\ \text{cont}}{\mathrm{N}_1\ \text{type}[\,]}}{\mathrm{z} \in \mathrm{N}_1\ \text{cont}}\;(\mathrm{z}\in \mathrm{N}_1)\notin[\,]}{\mathrm{N}_1\ \text{type}[\mathrm{z}\in\mathrm{N}_1]} \qquad \text{I-S}\;\dfrac{[\,]\ \text{cont}}{*\in\mathrm{N}_1[\,]}}{\mathrm{El}_{N1}(*,*)\in N_1[\,]}$$

Applicando la β N_1-*red* allora $\mathrm{El}_{N1}(*, *) \to_1 *$
$\mathrm{El}_{N1}(*, *)$ è uguale definizionalmente.

2

$$\text{E-S}\;\dfrac{\text{var}\;\dfrac{\text{F-c}\;\dfrac{\text{F-S}\;\dfrac{[\,]\ \text{cont}}{\mathrm{N}_1\ \text{type}[\,]}}{\mathrm{x}\in\mathrm{N}_1\ \text{cont}}\;(\mathrm{x}\in\mathrm{N}_1)\notin[\,]}{\mathrm{x}\in\mathrm{N}_1[\mathrm{x}\in\mathrm{N}_1]} \qquad \text{F-S}\;\dfrac{\text{F-c}\;\dfrac{\text{F-S}\;\dfrac{\text{F-c}\;\dfrac{\text{F-S}\;\dfrac{[\,]\ \text{cont}}{\mathrm{N}_1\ \text{type}\ [\,]}}{\mathrm{x}\in\mathrm{N}_1\ \text{cont}}\;(\mathrm{x}\in\mathrm{N}_1)\notin[\,]}{\mathrm{N}_1\ \text{type}[\mathrm{x}\in\mathrm{N}_1]}}{\mathrm{x}\in\mathrm{N}_1,\ \mathrm{z}\in\mathrm{N}_1\ \text{cont}}\;(\mathrm{z}\in\mathrm{N}_1)\notin \mathrm{x}\in\mathrm{N}_1}{\mathrm{N}_1\ \text{type}[\mathrm{x}\in\mathrm{N}_1,\ \mathrm{z}\in\mathrm{N}_1]} \qquad \text{I-S}\;\dfrac{\text{F-c}\;\dfrac{\text{F-S}\;\dfrac{[\,]\ \text{cont}}{\mathrm{N}_1\ \text{type}[\,]}}{\mathrm{x}\in\mathrm{N}_1\ \text{cont}}\;(\mathrm{x}\in\mathrm{N}_1)\notin[\,]}{*\in\mathrm{N}_1[\mathrm{x}\in\mathrm{N}_1]}}{\mathrm{El}_{N1}(\mathrm{x},*)\in\mathrm{N}_1[\mathrm{x}\in\mathrm{N}_1]}$$

Applicando la β N_1-*red* allora $\mathrm{El}_{N1}(\mathrm{x}, *) \not\twoheadrightarrow_1$
$\mathrm{El}_{N1}(\mathrm{x}, *)$ non è uguale definizionalmente.

3

$$\text{E-S}\;\dfrac{\text{I-S}\;\dfrac{\text{F-c}\;\dfrac{\text{F-S}\;\dfrac{[\,]\ \text{cont}}{\mathrm{N}_1\ \text{type}[\,]}}{\mathrm{y}\in\mathrm{N}_1\ \text{cont}}\;(\mathrm{y}\in\mathrm{N}_1)\notin[\,]}{*\in\mathrm{N}_1[\mathrm{y}\in\mathrm{N}_1]} \qquad \text{F-S}\;\dfrac{\text{F-c}\;\dfrac{\text{F-S}\;\dfrac{\text{F-c}\;\dfrac{\text{F-S}\;\dfrac{[\,]\ \text{cont}}{\mathrm{N}_1\ \text{type}[\,]}}{\mathrm{y}\in\mathrm{N}_1\ \text{cont}}\;(\mathrm{y}\in\mathrm{N}_1)\notin[\,]}{\mathrm{N}_1\ \text{type}[\mathrm{y}\in\mathrm{N}_1]}}{\mathrm{y}\in\mathrm{N}_1,\ \mathrm{z}\in\mathrm{N}_1\ \text{cont}}\;(\mathrm{z}\in\mathrm{N}_1)\notin \mathrm{y}\in\mathrm{N}_1}{\mathrm{N}_1\ \text{type}[\mathrm{y}\in\mathrm{N}_1,\ \mathrm{z}\in\mathrm{N}_1]} \qquad \text{var}\;\dfrac{\text{F-c}\;\dfrac{\text{F-S}\;\dfrac{[\,]\ \text{cont}}{\mathrm{N}_1\ \text{type}[\,]}}{\mathrm{y}\in\mathrm{N}_1\ \text{cont}}\;(\mathrm{y}\in\mathrm{N}_1)\notin[\,]}{\mathrm{y}\in\mathrm{N}_1[\mathrm{y}\in\mathrm{N}_1]}}{\mathrm{El}_{N1}(*,\mathrm{y})\in\mathrm{N}_1[\mathrm{y}\in\mathrm{N}_1]}$$

Applicando la β N_1-*red* allora $\mathrm{El}_{N1}(*, y) \to_1$ y
$\mathrm{El}_{N1}(*, y)$ è uguale definizionalmente.

4

$$\text{E-S}\ \dfrac{\text{var}\ \dfrac{\text{F-c}\ \dfrac{\text{F-S}\ \dfrac{\text{F-c}\ \dfrac{\text{F-S}\ \dfrac{[\,]\ \mathrm{cont}}{N_1\ \mathrm{type}\,[\,]} \quad (x \in N_1) \notin [\,]}{x \in N_1\ \mathrm{cont}}}{N_1\ \mathrm{type}[x \in N_1]} \quad (y \in N_1) \notin x \in N_1}{x \in N_1,\ y \in N_1\ \mathrm{cont}}}{x \in N_1[x \in N_1,\ y \in N_1]} \quad \text{F-S}\ \dfrac{\text{F-c}\ \dfrac{\text{F-S}\ \dfrac{\text{F-c}\ \dfrac{\text{F-S}\ \dfrac{\text{F-c}\ \dfrac{\text{F-S}\ \dfrac{[\,]\ \mathrm{cont}}{N_1\ \mathrm{type}[\,]} \quad (x \in N_1) \notin [\,]}{x \in N_1\ \mathrm{cont}}}{N_1\ \mathrm{type}[x \in N_1]} \quad (y \in N_1) \notin (x \in N_1)}{x \in N_1,\ y \in N_1\ \mathrm{cont}}}{N_1\ \mathrm{type}[x \in N_1,\ y \in N_1]} \quad (z \in N_1) \notin (x \in N_1,\ y \in N_1)}{x \in N_1,\ y \in N_1,\ z \in N_1\ \mathrm{cont}}}{N_1\ \mathrm{type}[x \in N_1,\ y \in N_1,\ z \in N_1]} \quad \text{var}\ \dfrac{\text{F-c}\ \dfrac{\text{F-S}\ \dfrac{\text{F-c}\ \dfrac{\text{F-S}\ \dfrac{[\,]\ \mathrm{cont}}{N_1\ \mathrm{type}\,[\,]} \quad (x \in N_1) \notin [\,]}{x \in N_1\ \mathrm{cont}}}{N_1\ \mathrm{type}[x \in N_1]} \quad (y \in N_1) \notin x \in N_1}{x \in N_1,\ y \in N_1\ \mathrm{cont}}}{y \in N_1[x \in N_1,\ y \in N_1]}}{\mathrm{El}_{N1}(x, y) \in N_1[x \in N_1,\ y \in N_1]}$$

Applicando la β N_1-*red* allora $\mathrm{El}_{N1}(x, y) \not\to_1$
$\mathrm{El}_{N1}(x, y)$ non è uguale definizionalmente.

5

$$\text{E-S}\ \dfrac{\dfrac{\mathbf{5}_A}{\mathrm{El}_{N1}(*, y) \in N_1[y \in N_1,\ x \in N_1]} \quad \dfrac{\mathbf{5}_B}{N_1\ \mathrm{type}[y \in N_1,\ x \in N_1,\ z \in N_1]} \quad \dfrac{\mathbf{5}_C}{\mathrm{El}_{N1}(x, *) \in N_1[y \in N_1,\ x \in N_1]}}{\mathrm{El}_{N1}(\mathrm{El}_{N1}(*, y),\ \mathrm{El}_{N1}(x, *)) \in N_1[y \in N_1,\ x \in N_1]}$$

$\mathbf{5}_A$

$$\text{ind-te}\ \dfrac{\dfrac{}{\mathrm{El}_{N1}(*, y) \in N_1[y \in N_1]} \quad \text{F-c}\ \dfrac{\text{F-S}\ \dfrac{\text{F-c}\ \dfrac{\text{F-S}\ \dfrac{[\,]\ \mathrm{cont}}{N_1\ \mathrm{type}[\,]} \quad (y \in N_1) \notin [\,]}{y \in N_1\ \mathrm{cont}}}{N_1\ \mathrm{type}[y \in N_1]} \quad (x \in N_1) \notin y \in N_1}{y \in N_1,\ x \in N_1\ \mathrm{cont}}}{\mathrm{El}_{N1}(*, y) \in N_1[y \in N_1,\ x \in N_1]}$$

$\mathbf{5}_B$

$$\text{F-S}\ \dfrac{\text{F-c}\ \dfrac{\text{F-S}\ \dfrac{\text{F-c}\ \dfrac{\text{F-S}\ \dfrac{\text{F-c}\ \dfrac{\text{F-S}\ \dfrac{[\,]\ \mathrm{cont}}{N_1\ \mathrm{type}[\,]} \quad (y \in N_1) \notin [\,]}{y \in N_1\ \mathrm{cont}}}{N_1\ \mathrm{type}[y \in N_1]} \quad (x \in N_1) \notin y \in N_1}{y \in N_1,\ x \in N_1\ \mathrm{cont}}}{N_1\ \mathrm{type}[y \in N_1,\ x \in N_1]} \quad (z \in N_1) \notin (y \in N_1,\ x \in N_1)}{y \in N_1,\ x \in N_1,\ z \in N_1\ \mathrm{cont}}}{N_1\ \mathrm{type}[y \in N_1,\ x \in N_1,\ z \in N_1]}$$

5$_C$

$$
\text{ex-te}\,\dfrac{\text{ind-te}\,\dfrac{\dfrac{}{\mathrm{El}_{N1}(x,\ *) \in N_1[x \in N_1]} \quad \text{F-c}\,\dfrac{\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\ \text{cont}}{N_1\ \text{type}\ [\,]}}{x \in N_1\ \text{cont}}\,(x \in N_1) \notin [\,]}{N_1\ \text{type}[x \in N_1]}}{x \in N_1,\ y \in N_1\ \text{cont}}\,(y \in N_1) \notin x \in N_1}{\mathrm{El}_{N1}(x,\ *) \in N_1[x \in N_1,\ y \in N_1]} \quad \text{F-c}\,\dfrac{\text{F-S}\,\dfrac{\text{F-c}\,\dfrac{\text{F-S}\,\dfrac{[\,]\ \text{cont}}{N_1\ \text{type}\ [\,]}}{y \in N_1\ \text{count}}\,(y \in N_1) \notin [\,]}{N_1\ \text{type}[y \in N_1]}}{y \in N_1,\ x \in N_1\ \text{cont}}\,(x \in N_1) \notin y \in N_1}{\mathrm{El}_{N1}(x,\ *) \in N_1[y \in N_1,\ x \in N_1]}
$$

Per le premesse $El_{N1}(*,\ y) \in N_1[y \in N_1]$ e $El_{N1}(x,\ *) \in N_1[x \in N_1]$ ho già dimostrato sopra (in 3 e 2) la loro tipabilità per il tipo singoletto.
Applicando la red_I e β N_1*-red* allora $\mathrm{El}_{N1}(\mathrm{El}_{N1}(*,\ y),\ \mathrm{El}_{N1}(x,\ *)) \to_1 \mathrm{El}_{N1}(y,\ \mathrm{El}_{N1}(x,\ *))$.
Più nel dettaglio la riduzione è la seguente

$$
\text{red}_I\,\dfrac{\beta\ \text{N}_1\text{-red}\,\dfrac{}{\mathrm{El}_{N1}(*,\ y) \to_1 y}}{\mathrm{El}_{N1}(\mathrm{El}_{N1}(*,\ y),\ \mathrm{El}_{N1}(x,\ *)) \to_1 \mathrm{El}_{N1}(y,\ \mathrm{El}_{N1}(x,\ *))}
$$

$\mathrm{El}_{N1}(\mathrm{El}_{N1}(*,\ y),\ \mathrm{El}_{N1}(x,\ *))$ è uguale definizionalmente.

2) Definire w + 2 ∈ Nat[w ∈ Nat], ove 2 è l'abbreviazione del termine ottenuto applicando 2 ≡ succ(succ(0)).

Soluzione

La ricorsione la faccio su w, usando lo schema di ricorsione primitiva, vale che $w + 2 \equiv \mathrm{El}_{Nat}(w,\ 2,\ (x,y).succ(y))$.

$$
\text{E-Nat}_{dip}\,\dfrac{\text{F-Nat}\,\dfrac{\text{F-c}\,\dfrac{\text{F-Nat}\,\dfrac{[\,]\ \text{cont}}{\text{Nat type}[\,]}}{w \in \text{Nat cont}}\,(w \in \text{Nat}) \notin [\,]}{\text{Nat type}[w \in \text{Nat}]} \quad \text{I}_2\text{-Nat}\,\dfrac{\text{I}_2\text{-Nat}\,\dfrac{\text{I}_1\text{-Nat}\,\dfrac{[\,]\ \text{cont}}{0 \in \text{Nat}[\,]}}{1 \in \text{Nat}[\,]}}{2 \in \text{Nat}[\,]} \quad \text{I}_2\text{-Nat}_{prog}\,\dfrac{\text{F-c}\,\dfrac{\text{F-Nat}\,\dfrac{[\,]\ \text{cont}}{\text{Nat type}\ [\,]}}{x \in \text{Nat cont}}\,(x \in \text{Nat}) \notin [\,]}{succ(y) \in \text{Nat}[x \in \text{Nat},\ y \in \text{Nat}]}}{\mathrm{El}_{Nat}(w,\ 2,\ (x,y).succ(y)) \in \text{Nat}[w \in \text{Nat}]}
$$

Dimostrazione di correttezza di $El_{Nat}(w,\ 2,\ (x,y).succ(y)) \in Nat[w \in Nat]$

- $\mathrm{El}_{Nat}(0,\ 2,\ (x,y).succ(y)) \to_1 2$ per β_{1Nat}*-red*
- $\mathrm{El}_{Nat}(succ(m),\ 2,\ (x,y).succ(y)) \to_1 succ(\mathrm{El}_{Nat}(m,\ 2,\ (x,y).succ(y)))$ per β_{2Nat}*-red* $\Rightarrow$ per $m = 0 \equiv succ(\mathrm{El}_{Nat}(0,\ 2,\ (x,y).succ(y))) \to_1 succ(2) \in$ Nat = 3 (dal punto precedente).

4) Si dimostri che esiste un *proof-term* pf del tipo

$$\mathbf{pf \in Id(N_1,x,*)[x \in N_1]}$$

Soluzione

$\mathrm{El}_{N1}(z,c) \equiv pf \equiv \mathrm{El}_{N1}(x,x.id(*))$
$M(z) \equiv Id(N_1,x,*)$
$c \in M(*)[\Gamma] \equiv id(*) \in Id(N_1,*,*)[\,]$

$$
\mathrm{E\text{-}S}_{dip}\ \dfrac{\mathrm{F\text{-}Id}\ \dfrac{\dfrac{}{\mathrm{N}_1\ \mathrm{type}[\mathrm{x} \in \mathrm{N}_1]} \qquad \dfrac{}{\mathrm{x} \in \mathrm{N}_1[\mathrm{x} \in \mathrm{N}_1]} \qquad \dfrac{}{* \in \mathrm{N}_1[\mathrm{x} \in \mathrm{N}_1]}}{\mathrm{Id}(\mathrm{N}_1,\mathrm{x},*)\mathrm{type}[\mathrm{x} \in \mathrm{N}_1]} \qquad \mathrm{I\text{-}Id}\ \dfrac{\mathrm{I\text{-}S}\ \dfrac{[\,]\ \mathrm{cont}}{* \in \mathrm{N}_1[\,]}}{\mathrm{id}(*) \in \mathrm{Id}(\mathrm{N}_1,*,*)[\,]}}{\mathrm{El}_{N1}(\mathrm{x},\mathrm{x.id}(*)) \in \mathrm{Id}(\mathrm{N}_1,\mathrm{x},*)[\mathrm{x} \in \mathrm{N}_1]}
$$