

## Esercizi finali

1) Sia  $T_1$  la teoria dei tipi definita del tipo singoletto con le regole strutturali, definite all'interno di "*Esame di Teoria dei Tipi*", incluse quelle di sostituzione e indebolimento. Allora stabilire se i seguenti termini sono tipabili come termini del tipo singoletto, secondo  $T_1$  e quali sono uguali definizionalmente.

- $\text{El}_{N_1}(*, *)$
- $\text{El}_{N_1}(x, *)$
- $\text{El}_{N_1}(*, y)$
- $\text{El}_{N_1}(x, y)$
- $\text{El}_{N_1}(\text{El}_{N_1}(*, y), \text{El}_{N_1}(x, *))$

### Soluzione

1

$$\frac{\text{I-S } \frac{[] \text{ cont}}{* \in N_1[]} \quad \text{F-S } \frac{\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type}[]} (z \in N_1) \notin [] \quad \text{F-c } \frac{z \in N_1 \text{ cont}}{N_1 \text{ type}[z \in N_1]}}{\text{E-S } \frac{}{\text{El}_{N_1}(*, *) \in N_1[]}}$$

Applicando la  $\beta$   $N_1$ -red allora  $\text{El}_{N_1}(*, *) \rightarrow_1 *$   
 $\text{El}_{N_1}(*, *)$  è uguale definizionalmente.

2

$$\frac{\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type}[]} \quad \text{F-c } \frac{x \in N_1 \text{ cont}}{x \in N_1[x \in N_1]} \quad \text{F-S } \frac{\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type}[]} (x \in N_1) \notin [] \quad \text{F-c } \frac{x \in N_1, z \in N_1 \text{ cont}}{N_1 \text{ type}[x \in N_1]} \quad \text{F-S } \frac{x \in N_1, z \in N_1 \text{ cont}}{N_1 \text{ type}[x \in N_1, z \in N_1]} \quad \text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type}[]} (x \in N_1) \notin [] \quad \text{F-c } \frac{x \in N_1 \text{ cont}}{x \in N_1[x \in N_1]} \quad \text{I-S } \frac{* \in N_1[x \in N_1]}{* \in N_1[x \in N_1]}}{\text{E-S } \frac{}{\text{El}_{N_1}(x, *) \in N_1[x \in N_1]}}$$

Applicando la  $\beta$   $N_1$ -red allora  $\text{El}_{N_1}(x, *) \not\rightarrow_1$   
 $\text{El}_{N_1}(x, *)$  non è uguale definizionalmente.

3

$$\frac{\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type}[]} \quad \text{F-c } \frac{y \in N_1 \text{ cont}}{y \in N_1[y \in N_1]} \quad \text{F-S } \frac{\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type}[]} (y \in N_1) \notin [] \quad \text{F-c } \frac{y \in N_1, z \in N_1 \text{ cont}}{N_1 \text{ type}[y \in N_1]} \quad \text{F-S } \frac{y \in N_1, z \in N_1 \text{ cont}}{N_1 \text{ type}[y \in N_1, z \in N_1]} \quad \text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type}[]} (y \in N_1) \notin [] \quad \text{F-c } \frac{y \in N_1 \text{ cont}}{y \in N_1[y \in N_1]} \quad \text{I-S } \frac{* \in N_1[y \in N_1]}{* \in N_1[y \in N_1]} \quad \text{var } \frac{y \in N_1[y \in N_1]}{y \in N_1[y \in N_1]}}{\text{E-S } \frac{}{\text{El}_{N_1}(*, y) \in N_1[y \in N_1]}}$$

2

Applicando la  $\beta$   $N_1$ -red allora  $\text{El}_{N_1}(*, y) \rightarrow_1 y$   
 $\text{El}_{N_1}(*, y)$  è uguale definizionalmente.

4

$$\begin{array}{c}
\begin{array}{c}
\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type } []} \\
\text{F-c } \frac{[] \text{ cont}}{x \in N_1 \text{ cont}} (x \in N_1) \notin [] \\
\text{F-S } \frac{N_1 \text{ type}[x \in N_1]}{x \in N_1, y \in N_1 \text{ cont}} (y \in N_1) \\
\text{F-c } \frac{x \in N_1, y \in N_1 \text{ cont}}{x \in N_1[x \in N_1, y \in N_1]} \notin x \in N_1 \\
\text{var } \frac{x \in N_1[x \in N_1, y \in N_1]}{x \in N_1[x \in N_1, y \in N_1]} \\
\text{E-S } \frac{}{x \in N_1[x \in N_1, y \in N_1]}
\end{array}
\quad
\begin{array}{c}
\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type}[ ]} \\
\text{F-c } \frac{N_1 \text{ type}[ ]}{x \in N_1 \text{ cont}} (x \in N_1) \notin [ ] \\
\text{F-S } \frac{N_1 \text{ type}[x \in N_1]}{x \in N_1, y \in N_1 \text{ cont}} (y \in N_1) \\
\text{F-c } \frac{x \in N_1, y \in N_1 \text{ cont}}{x \in N_1, y \in N_1 \text{ cont}} \notin (x \in N_1) \\
\text{F-S } \frac{N_1 \text{ type}[x \in N_1, y \in N_1]}{x \in N_1, y \in N_1, z \in N_1 \text{ cont}} (z \in N_1) \notin \\
\text{F-c } \frac{x \in N_1, y \in N_1, z \in N_1 \text{ cont}}{N_1 \text{ type}[x \in N_1, y \in N_1, z \in N_1]} (x \in N_1, y \in N_1) \\
\text{F-S } \frac{N_1 \text{ type}[x \in N_1, y \in N_1, z \in N_1]}{N_1 \text{ type}[x \in N_1, y \in N_1, z \in N_1]} \\
\text{E-S } \frac{}{N_1 \text{ type}[x \in N_1, y \in N_1, z \in N_1]}
\end{array}
\quad
\begin{array}{c}
\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type } [ ]} \\
\text{F-c } \frac{[] \text{ cont}}{x \in N_1 \text{ cont}} (x \in N_1) \notin [ ] \\
\text{F-S } \frac{N_1 \text{ type}[ ]}{x \in N_1, y \in N_1 \text{ cont}} (y \in N_1) \\
\text{F-c } \frac{x \in N_1, y \in N_1 \text{ cont}}{x \in N_1, y \in N_1 \text{ cont}} \notin x \in N_1 \\
\text{var } \frac{x \in N_1, y \in N_1 \text{ cont}}{y \in N_1[x \in N_1, y \in N_1]} \notin x \in N_1 \\
\text{E-S } \frac{}{y \in N_1[x \in N_1, y \in N_1]}
\end{array}
\end{array}$$

Applicando la  $\beta$   $N_1$ -red allora  $\text{El}_{N_1}(x, y) \rightarrow_1 y$   
 $\text{El}_{N_1}(x, y)$  non è uguale definizionalmente.

5

$$\begin{array}{c}
\text{E-S } \frac{\text{5}_A \quad \text{5}_B \quad \text{5}_C}{\text{El}_{N_1}(\text{El}_{N_1}(*, y), \text{El}_{N_1}(x, *)) \in N_1[y \in N_1, x \in N_1]}
\end{array}$$

5<sub>A</sub>

$$\begin{array}{c}
\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type}[ ]} \\
\text{F-c } \frac{[] \text{ cont}}{y \in N_1 \text{ cont}} (y \in N_1) \notin [ ] \\
\text{F-S } \frac{N_1 \text{ type}[y \in N_1]}{y \in N_1, x \in N_1 \text{ cont}} (x \in N_1) \notin y \in N_1 \\
\text{F-c } \frac{N_1 \text{ type}[y \in N_1]}{y \in N_1, x \in N_1 \text{ cont}} (x \in N_1) \notin y \in N_1 \\
\text{ind-te } \frac{\text{3} \quad \text{F-c } \frac{N_1 \text{ type}[y \in N_1]}{y \in N_1, x \in N_1 \text{ cont}} (x \in N_1) \notin y \in N_1}{\text{El}_{N_1}(*, y) \in N_1[y \in N_1, x \in N_1]}
\end{array}$$

5<sub>B</sub>

$$\begin{array}{c}
\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type}[ ]} \\
\text{F-c } \frac{[] \text{ cont}}{y \in N_1 \text{ cont}} (y \in N_1) \notin [ ] \\
\text{F-S } \frac{N_1 \text{ type}[y \in N_1]}{y \in N_1, x \in N_1 \text{ cont}} (x \in N_1) \notin y \in N_1 \\
\text{F-c } \frac{N_1 \text{ type}[y \in N_1]}{y \in N_1, x \in N_1 \text{ cont}} (x \in N_1) \notin y \in N_1 \\
\text{F-S } \frac{N_1 \text{ type}[y \in N_1, x \in N_1]}{y \in N_1, x \in N_1, z \in N_1 \text{ cont}} (z \in N_1) \notin (y \in N_1, x \in N_1) \\
\text{F-c } \frac{N_1 \text{ type}[y \in N_1, x \in N_1]}{y \in N_1, x \in N_1, z \in N_1 \text{ cont}} (z \in N_1) \notin (y \in N_1, x \in N_1) \\
\text{F-S } \frac{N_1 \text{ type}[y \in N_1, x \in N_1, z \in N_1]}{N_1 \text{ type}[y \in N_1, x \in N_1, z \in N_1]}
\end{array}$$

5<sub>C</sub>

$$\text{ind-te} \frac{\text{ex-te} \frac{\text{2} \frac{\text{El}_{N_1}(x, *) \in N_1[x \in N_1]}{\text{El}_{N_1}(x, *) \in N_1[x \in N_1, y \in N_1]}}{\text{El}_{N_1}(x, *) \in N_1[x \in N_1, y \in N_1]}}{\text{El}_{N_1}(x, *) \in N_1[y \in N_1, x \in N_1]}$$

$$\frac{\text{F-S} \frac{[] \text{ cont}}{N_1 \text{ type } []} \quad \text{F-c} \frac{x \in N_1 \text{ cont}}{x \in N_1} \quad (x \in N_1) \notin [] \quad \text{F-S} \frac{[] \text{ cont}}{N_1 \text{ type } []} \quad \text{F-c} \frac{y \in N_1 \text{ count}}{y \in N_1} \quad (y \in N_1) \notin []}{\text{F-S} \frac{N_1 \text{ type}[x \in N_1]}{x \in N_1, y \in N_1 \text{ cont}} \quad \text{F-c} \frac{x \in N_1, y \in N_1 \text{ cont}}{x \in N_1, y \in N_1 \text{ cont}} \quad (y \in N_1) \notin x \in N_1 \quad \text{F-S} \frac{N_1 \text{ type}[y \in N_1]}{y \in N_1, x \in N_1 \text{ cont}} \quad \text{F-c} \frac{y \in N_1, x \in N_1 \text{ cont}}{y \in N_1, x \in N_1 \text{ cont}} \quad (x \in N_1) \notin y \in N_1}$$

Per le premesse  $El_{N_1}(*, y) \in N_1[y \in N_1]$  e  $El_{N_1}(x, *) \in N_1[x \in N_1]$  ho già dimostrato sopra (in 3 e 2) la loro tipabilità per il tipo singoletto. Applicando la  $red_I$  e  $\beta$   $N_1$ -red allora  $El_{N_1}(El_{N_1}(*, y), El_{N_1}(x, *)) \rightarrow_1 El_{N_1}(y, El_{N_1}(x, *))$ .

Più nel dettaglio la riduzione è la seguente

$$\text{red}_I \frac{\beta \text{ } N_1\text{-red} \frac{}{El_{N_1}(*, y) \rightarrow_1 y}}{El_{N_1}(El_{N_1}(*, y), El_{N_1}(x, *)) \rightarrow_1 El_{N_1}(y, El_{N_1}(x, *))}$$

$El_{N_1}(El_{N_1}(*, y), El_{N_1}(x, *))$  è uguale definizionalmente.

**2) Definire  $w + 2 \in \text{Nat}[w \in \text{Nat}]$ , ove 2 è l'abbreviazione del termine ottenuto applicando  $2 \equiv \text{succ}(\text{succ}(0))$ .**

**Soluzione**

La ricorsione la faccio su  $w$ , usando lo schema di ricorsione primitiva, vale che  $w + 2 \equiv El_{Nat}(w, 2, (x,y).\text{succ}(y))$ .

$$\text{E-Nat}_{dip} \frac{\text{F-Nat} \frac{[] \text{ cont}}{\text{Nat type } []} \quad \text{F-c} \frac{w \in \text{Nat cont}}{w \in \text{Nat cont}} \quad (w \in \text{Nat}) \notin [] \quad \text{I}_1\text{-Nat} \frac{[] \text{ cont}}{0 \in \text{Nat}[]} \quad \text{I}_2\text{-Nat} \frac{1 \in \text{Nat}[]}{} \quad \text{I}_2\text{-Nat}_{prog} \frac{\text{F-Nat} \frac{[] \text{ cont}}{\text{Nat type } []} \quad \text{F-c} \frac{x \in \text{Nat cont}}{x \in \text{Nat cont}} \quad (x \in \text{Nat}) \notin []}{\text{succ}(y) \in \text{Nat}[x \in \text{Nat}, y \in \text{Nat}]}}{El_{Nat}(w, 2, (x,y).\text{succ}(y)) \in \text{Nat}[w \in \text{Nat}]}$$

*Dimostrazione di correttezza di  $El_{Nat}(w, 2, (x,y).\text{succ}(y)) \in \text{Nat}[w \in \text{Nat}]$*

- $El_{Nat}(0, 2, (x,y).\text{succ}(y)) \rightarrow_1 2$  per  $\beta_{1Nat}\text{-red}$
- $El_{Nat}(\text{succ}(m), 2, (x,y).\text{succ}(y)) \rightarrow_1 \text{succ}(El_{Nat}(m, 2, (x,y).\text{succ}(y)))$  per  $\beta_{2Nat}\text{-red} \Rightarrow$  per  $m = 0 \equiv \text{succ}(El_{Nat}(0, 2, (x,y).\text{succ}(y))) \rightarrow_1 \text{succ}(2) \in \text{Nat} = 3$  (dal punto precedente).

**4) Si dimostri che esiste un *proof-term* pf del tipo**

$$\text{pf} \in \text{Id}(N_1, x, *)[x \in N_1]$$

**Soluzione**

$$\begin{aligned} El_{N_1}(z, c) &\equiv \text{pf} \equiv El_{N_1}(x, x.\text{id}(*)) \\ M(z) &\equiv \text{Id}(N_1, x, *) \\ c \in M(*)[\Gamma] &\equiv \text{id}(*)) \in \text{Id}(N_1, *, *)[] \end{aligned}$$

$$\begin{array}{c}
\begin{array}{c} \text{F-S} \frac{[] \text{ cont}}{N_1 \text{ type}[]} \\ \text{F-c} \frac{x \in N_1 \text{ cont}}{N_1 \text{ type}[x \in N_1]} \end{array} (x \in N_1) \notin [] \\
\text{F-Id} \frac{}{N_1 \text{ type}[x \in N_1]} \\
\hline
\text{E-S}_{dip} \frac{\text{Id}(N_1, x, *) \text{ type}[x \in N_1]}{\text{El}_{N_1}(x, x. \text{id}(*)) \in \text{Id}(N_1, x, *)[x \in N_1]}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c} \text{F-S} \frac{[] \text{ cont}}{N_1 \text{ type}[]} \\ \text{F-c} \frac{x \in N_1 \text{ cont}}{x \in N_1[x \in N_1]} \end{array} (x \in N_1) \notin [] \\
\text{var} \frac{}{x \in N_1[x \in N_1]} \\
\hline
\text{El}_{N_1}(x, x. \text{id}(*)) \in \text{Id}(N_1, x, *)[x \in N_1]
\end{array}
\quad
\begin{array}{c}
\begin{array}{c} \text{F-S} \frac{[] \text{ cont}}{N_1 \text{ type}[]} \\ \text{F-c} \frac{x \in N_1 \text{ cont}}{x \in N_1[x \in N_1]} \end{array} (x \in N_1) \notin [] \\
\text{I-S} \frac{}{* \in N_1[x \in N_1]} \\
\hline
\text{El}_{N_1}(x, x. \text{id}(*)) \in \text{Id}(N_1, x, *)[x \in N_1]
\end{array}
\quad
\begin{array}{c}
\begin{array}{c} \text{I-S} \frac{[] \text{ cont}}{* \in N_1[]} \\ \text{I-Id} \frac{}{\text{id}(*)) \in \text{Id}(N_1, *, *)[ ]} \end{array} \\
\hline
\text{El}_{N_1}(x, x. \text{id}(*)) \in \text{Id}(N_1, x, *)[x \in N_1]
\end{array}$$