

Esercizi finali

1) Sia T_1 la teoria dei tipi definita del tipo singoletto con le regole strutturali, definite all'interno di "*Esame di Teoria dei Tipi*", incluse quelle di sostituzione e indebolimento. Allora stabilire se i seguenti termini sono tipabili come termini del tipo singoletto, secondo T_1 e quali sono uguali definizionalmente.

- $\text{El}_{N_1}(*, *)$
- $\text{El}_{N_1}(x, *)$
- $\text{El}_{N_1}(*, y)$
- $\text{El}_{N_1}(x, y)$
- $\text{El}_{N_1}(\text{El}_{N_1}(*, y), \text{El}_{N_1}(x, *))$

Soluzione

1

$$\frac{\text{I-S } \frac{[] \text{ cont}}{* \in N_1[]} \quad \text{F-S } \frac{\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type}[]} (z \in N_1) \notin [] \quad \text{F-c } \frac{z \in N_1 \text{ cont}}{N_1 \text{ type}[z \in N_1]}}{\text{E-S } \frac{}{\text{El}_{N_1}(*, *) \in N_1[]}}$$

Applicando la β N_1 -red allora $\text{El}_{N_1}(*, *) \rightarrow_1 *$
 $\text{El}_{N_1}(*, *)$ è uguale definizionalmente.

2

$$\frac{\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type}[]} \quad \text{F-c } \frac{x \in N_1 \text{ cont}}{x \in N_1[x \in N_1]} \quad \text{F-S } \frac{\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type}[]} (x \in N_1) \notin [] \quad \text{F-c } \frac{x \in N_1, z \in N_1 \text{ cont}}{N_1 \text{ type}[x \in N_1]} \quad \text{F-S } \frac{x \in N_1, z \in N_1 \text{ cont}}{N_1 \text{ type}[x \in N_1, z \in N_1]} \quad \text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type}[]} (x \in N_1) \notin [] \quad \text{F-c } \frac{x \in N_1 \text{ cont}}{x \in N_1[x \in N_1]} \quad \text{I-S } \frac{* \in N_1[x \in N_1]}{* \in N_1[x \in N_1]}}{\text{E-S } \frac{}{\text{El}_{N_1}(x, *) \in N_1[x \in N_1]}}$$

Applicando la β N_1 -red allora $\text{El}_{N_1}(x, *) \not\rightarrow_1$
 $\text{El}_{N_1}(x, *)$ non è uguale definizionalmente.

3

$$\frac{\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type}[]} \quad \text{F-c } \frac{y \in N_1 \text{ cont}}{y \in N_1[y \in N_1]} \quad \text{F-S } \frac{\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type}[]} (y \in N_1) \notin [] \quad \text{F-c } \frac{y \in N_1, z \in N_1 \text{ cont}}{N_1 \text{ type}[y \in N_1]} \quad \text{F-S } \frac{y \in N_1, z \in N_1 \text{ cont}}{N_1 \text{ type}[y \in N_1, z \in N_1]} \quad \text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type}[]} (y \in N_1) \notin [] \quad \text{F-c } \frac{y \in N_1 \text{ cont}}{y \in N_1[y \in N_1]} \quad \text{I-S } \frac{* \in N_1[y \in N_1]}{* \in N_1[y \in N_1]} \quad \text{var } \frac{y \in N_1[y \in N_1]}{y \in N_1[y \in N_1]}}{\text{E-S } \frac{}{\text{El}_{N_1}(*, y) \in N_1[y \in N_1]}}$$

2

Applicando la β N_1 -red allora $\text{El}_{N_1}(*, y) \rightarrow_1 y$
 $\text{El}_{N_1}(*, y)$ è uguale definizionalmente.

4

$$\begin{array}{c}
\begin{array}{c}
\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type } []} \\
\text{F-c } \frac{N_1 \text{ type } []}{x \in N_1 \text{ cont}} (x \in N_1) \notin [] \\
\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type } [x \in N_1]} (y \in N_1) \\
\text{F-c } \frac{N_1 \text{ type } [x \in N_1]}{x \in N_1, y \in N_1 \text{ cont}} \notin x \in N_1 \\
\text{var } \frac{x \in N_1, y \in N_1 \text{ cont}}{x \in N_1[x \in N_1, y \in N_1]} \\
\text{E-S }
\end{array}
\quad
\begin{array}{c}
\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type } []} \\
\text{F-c } \frac{N_1 \text{ type } []}{x \in N_1 \text{ cont}} (x \in N_1) \notin [] \\
\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type } [x \in N_1]} (y \in N_1) \\
\text{F-c } \frac{N_1 \text{ type } [x \in N_1]}{x \in N_1, y \in N_1 \text{ cont}} \notin (x \in N_1) \\
\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type } [x \in N_1, y \in N_1]} (z \in N_1) \notin \\
\text{F-c } \frac{N_1 \text{ type } [x \in N_1, y \in N_1]}{x \in N_1, y \in N_1, z \in N_1 \text{ cont}} (x \in N_1, y \in N_1) \\
\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type } [x \in N_1, y \in N_1, z \in N_1]} \\
\text{E-S }
\end{array}
\quad
\begin{array}{c}
\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type } []} \\
\text{F-c } \frac{N_1 \text{ type } []}{x \in N_1 \text{ cont}} (x \in N_1) \notin [] \\
\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type } [x \in N_1]} (y \in N_1) \\
\text{F-c } \frac{N_1 \text{ type } [x \in N_1]}{x \in N_1, y \in N_1 \text{ cont}} \notin x \in N_1 \\
\text{var } \frac{x \in N_1, y \in N_1 \text{ cont}}{y \in N_1[x \in N_1, y \in N_1]} \\
\text{E-S }
\end{array}
\end{array}
\frac{}{\text{El}_{N_1}(x, y) \in N_1[x \in N_1, y \in N_1]}$$

Applicando la β N_1 -red allora $\text{El}_{N_1}(x, y) \rightarrow_1$
 $\text{El}_{N_1}(x, y)$ non è uguale definizionalmente.

5

$$\begin{array}{c}
\mathbf{5}_A \quad \mathbf{5}_B \quad \mathbf{5}_C \\
\text{E-S } \frac{\frac{\text{El}_{N_1}(*, y) \in N_1[y \in N_1, x \in N_1]}{\text{El}_{N_1}(*, y) \in N_1[y \in N_1, x \in N_1]} \quad \frac{N_1 \text{ type}[y \in N_1, x \in N_1, z \in N_1]}{\text{El}_{N_1}(x, *) \in N_1[y \in N_1, x \in N_1]} \quad \frac{\text{El}_{N_1}(x, *) \in N_1[y \in N_1, x \in N_1]}{\text{El}_{N_1}(\text{El}_{N_1}(*, y), \text{El}_{N_1}(x, *)) \in N_1[y \in N_1, x \in N_1]}}{}
\end{array}$$

$\mathbf{5}_A$

$$\begin{array}{c}
\mathbf{3} \\
\text{ind-te } \frac{\frac{\text{El}_{N_1}(*, y) \in N_1[y \in N_1]}{\text{El}_{N_1}(*, y) \in N_1[y \in N_1]} \quad \frac{\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type } []} \quad \text{F-c } \frac{N_1 \text{ type } []}{y \in N_1 \text{ cont}} (y \in N_1) \notin [] \quad \text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type } [y \in N_1]} (x \in N_1) \notin y \in N_1}{\text{El}_{N_1}(*, y) \in N_1[y \in N_1, x \in N_1]}
\end{array}$$

$\mathbf{5}_B$

$$\begin{array}{c}
\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type } []} \\
\text{F-c } \frac{N_1 \text{ type } []}{y \in N_1 \text{ cont}} (y \in N_1) \notin [] \\
\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type } [y \in N_1]} (x \in N_1) \notin y \in N_1 \\
\text{F-c } \frac{N_1 \text{ type } [y \in N_1]}{y \in N_1, x \in N_1 \text{ cont}} (x \in N_1) \notin y \in N_1 \\
\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type } [y \in N_1, x \in N_1]} (z \in N_1) \notin (y \in N_1, x \in N_1) \\
\text{F-c } \frac{N_1 \text{ type } [y \in N_1, x \in N_1]}{y \in N_1, x \in N_1, z \in N_1 \text{ cont}} (z \in N_1) \notin (y \in N_1, x \in N_1) \\
\text{F-S } \frac{[] \text{ cont}}{N_1 \text{ type } [y \in N_1, x \in N_1, z \in N_1]}
\end{array}$$

5_C

$$\text{ind-te} \frac{\text{ex-te} \frac{\text{F-S} \frac{[] \text{ cont}}{N_1 \text{ type } []} \text{F-c} \frac{x \in N_1 \text{ cont}}{N_1 \text{ type}[x \in N_1]} (x \in N_1) \notin [] \quad \text{F-S} \frac{[] \text{ cont}}{N_1 \text{ type } []} \text{F-c} \frac{y \in N_1 \text{ count}}{N_1 \text{ type}[y \in N_1]} (y \in N_1) \notin []}{\text{El}_{N_1}(x, *) \in N_1[x \in N_1] \quad \text{El}_{N_1}(x, *) \in N_1[y \in N_1, x \in N_1]} \quad \text{F-S} \frac{[] \text{ cont}}{N_1 \text{ type } []} \text{F-c} \frac{y \in N_1 \text{ count}}{N_1 \text{ type}[y \in N_1]} (x \in N_1) \notin y \in N_1}{\text{El}_{N_1}(x, *) \in N_1[x \in N_1, y \in N_1]} \quad \text{F-S} \frac{[] \text{ cont}}{N_1 \text{ type } []} \text{F-c} \frac{y \in N_1 \text{ count}}{N_1 \text{ type}[y \in N_1]} (x \in N_1) \notin y \in N_1$$

Per le premesse $El_{N_1}(*, y) \in N_1[y \in N_1]$ e $El_{N_1}(x, *) \in N_1[x \in N_1]$ ho già dimostrato sopra (in 3 e 2) la loro tipabilità per il tipo singoletto. Applicando la red_I e β N_1 -red allora $El_{N_1}(El_{N_1}(*, y), El_{N_1}(x, *)) \rightarrow_1 El_{N_1}(y, El_{N_1}(x, *))$.

Più nel dettaglio la riduzione è la seguente

$$\text{red}_I \frac{\beta \text{ } N_1\text{-red} \frac{}{El_{N_1}(*, y) \rightarrow_1 y}}{El_{N_1}(El_{N_1}(*, y), El_{N_1}(x, *)) \rightarrow_1 El_{N_1}(y, El_{N_1}(x, *))}$$

$El_{N_1}(El_{N_1}(*, y), El_{N_1}(x, *))$ è uguale definizionalmente.

2) Definire $w + 2 \in \text{Nat}[w \in \text{Nat}]$, ove 2 è l'abbreviazione del termine ottenuto applicando $2 \equiv \text{succ}(\text{succ}(0))$.

Soluzione

La ricorsione la faccio su w , usando lo schema di ricorsione primitiva, vale che $w + 2 \equiv El_{Nat}(w, 2, (x,y).\text{succ}(y))$.

$$\text{E-Nat}_{dip} \frac{\text{F-Nat} \frac{[] \text{ cont}}{\text{Nat type } []} \text{F-c} \frac{w \in \text{Nat cont}}{\text{Nat type}[w \in \text{Nat}]} (w \in \text{Nat}) \notin [] \quad \text{I}_1\text{-Nat} \frac{[] \text{ cont}}{0 \in \text{Nat}[]} \text{I}_2\text{-Nat} \frac{1 \in \text{Nat}[]}{} \quad \text{I}_2\text{-Nat}_{prog} \frac{\text{F-Nat} \frac{[] \text{ cont}}{\text{Nat type } []} \text{F-c} \frac{x \in \text{Nat cont}}{\text{succ}(y) \in \text{Nat}[x \in \text{Nat}, y \in \text{Nat}]} (x \in \text{Nat}) \notin []}{\text{El}_{Nat}(w, 2, (x,y).\text{succ}(y)) \in \text{Nat}[w \in \text{Nat}]}$$

Dimostrazione di correttezza di $El_{Nat}(w, 2, (x,y).\text{succ}(y)) \in \text{Nat}[w \in \text{Nat}]$

- $El_{Nat}(0, 2, (x,y).\text{succ}(y)) \rightarrow_1 2$ per $\beta_{1Nat\text{-red}}$
- $El_{Nat}(\text{succ}(m), 2, (x,y).\text{succ}(y)) \rightarrow_1 \text{succ}(El_{Nat}(m, 2, (x,y).\text{succ}(y)))$ per $\beta_{2Nat\text{-red}} \Rightarrow$ per $m = 0 \equiv \text{succ}(El_{Nat}(0, 2, (x,y).\text{succ}(y))) \rightarrow_1 \text{succ}(2) \in \text{Nat} = 3$ (dal punto precedente).

4) Si dimostri che esiste un *proof-term* pf del tipo

$$\text{pf} \in \text{Id}(N_1, x, *) [x \in N_1]$$

Soluzione

$$\begin{aligned} El_{N_1}(z, c) &\equiv \text{pf} \equiv El_{N_1}(x, x.\text{id}(*)) \\ M(z) &\equiv \text{Id}(N_1, x, *) \\ c \in M(*)[\Gamma] &\equiv \text{id}(*). \in \text{Id}(N_1, *, *) [] \end{aligned}$$

$$\begin{array}{c}
\text{F-Id} \frac{\overline{N_1 \text{ type}[x \in N_1]} \quad \overline{x \in N_1[x \in N_1]} \quad \overline{* \in N_1[x \in N_1]}}{\text{E-}S_{dip} \frac{\text{Id}(N_1, x, *) \text{ type}[x \in N_1]}{\text{El}_{N_1}(x, x.\text{id}(*)) \in \text{Id}(N_1, x, *)[x \in N_1]}} \quad \text{I-Id} \frac{\text{I-S} \frac{[] \text{ cont}}{* \in N_1[]}}{\text{id}(*) \in \text{Id}(N_1, *, *)[]}}
\end{array}$$