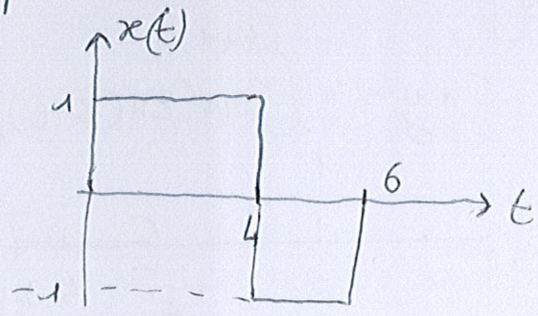
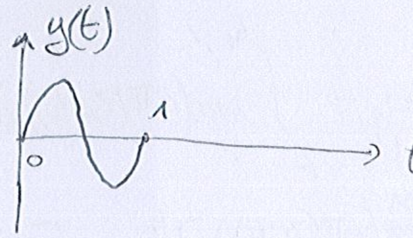
TD2 Analyse Spectrale

exo 1 détection des discontinuités par intercorrélation

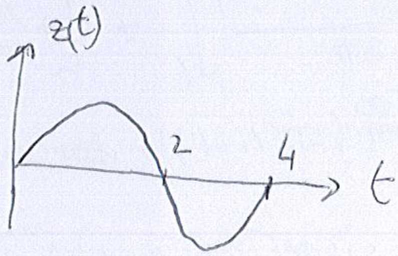
- $x(t) = \begin{cases} 0 & t > 6 \text{ et } t < 0 \\ 1 & \text{si } t \in (0;4] \\ -1 & \text{si } t \in [4;6] \end{cases}$



- $y(t) = rect_1(t-0,5)\sin(2\pi F t)$
  avec $F = 1$

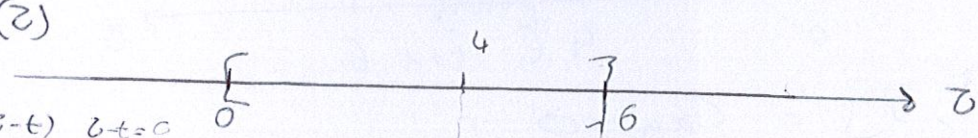


- $z(t) = y(t/4)$



- calcul de $C_{xy}(t) = \int x(\tau)\, \hat{y}(\tau - t)\, d\tau$

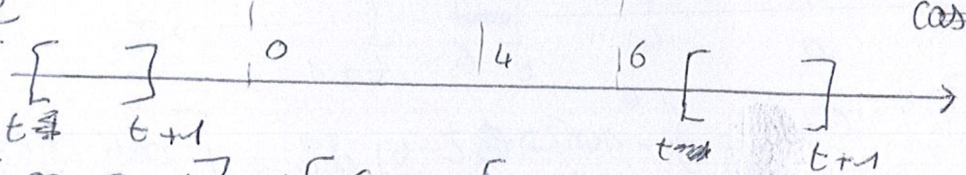
support de $x(\tau)$



$(\tau - t)$ $\tau - t = 0$

support de $g(\tau - t)$

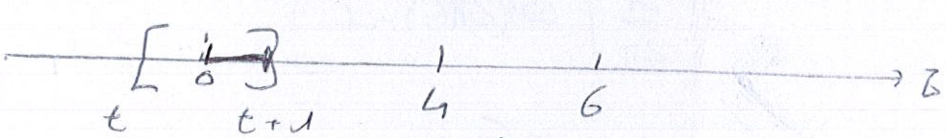
cas n° 1 et 2



~~cas no 1 et 2~~

$t \in]-\infty ; -1] \cup [6 ; +\infty[$

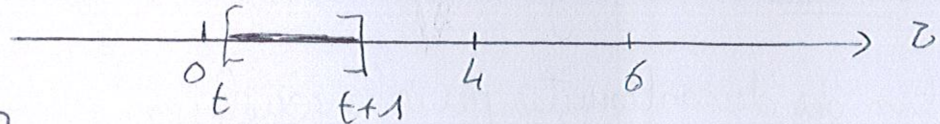
cas n° 3 :



$t \in [-1 ; 0]$: $C_{xy} = \int_0^{t+1} \sin(2\pi(\tau - t))\, d\tau$

$C_{xy} = \left[-\frac{\cos(2\pi(\tau - t))}{2\pi} \right]_0^{t+1} = \frac{1}{2\pi}\left[-1 + \cos(2\pi t) \right]$

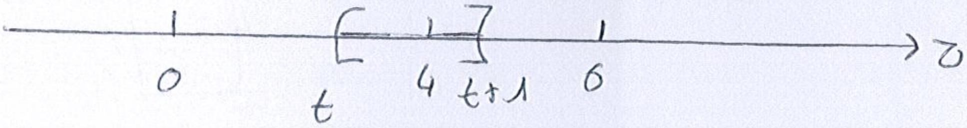
cas n°4 :



$t \in [0\,;3]$

$$C_{xy} = \int_t^{t+1} \sin(2\pi(\tau - t))\,d\tau = \left[-\frac{\cos 2\pi(\tau - t)}{2\pi}\right]_t^{t+1} = 0$$

cas n°5 :



$t \in [3\,;4]$

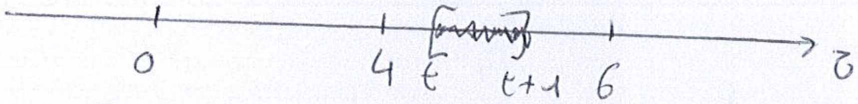
$$C_{xy} = \int_t^{4} \sin(2\pi(\tau - t))\,d\tau + \int_4^{t+1} -\sin(2\pi(\tau - t))\,d\tau$$

$$C_{xy} = \left[-\frac{\cancel{\sin}\cos(2\pi(\tau - t))}{2\pi}\right]_t^{4} + \left[+\frac{\cancel{\sin}\cos(2\pi(\tau - t))}{2\pi}\right]_4^{t+1}$$

$$= \frac{1}{2\pi}\left[-\cancel{\sin}\cos(2\pi(4-t)) + \cancel{\sin}\cos(0) + \cos(2\pi) - \cos(2\pi(4-t))\right]$$

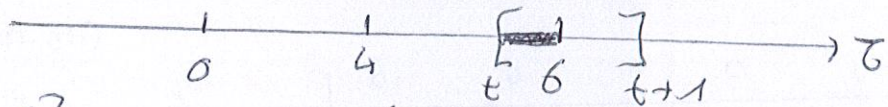
$$= \frac{1}{2\pi}\left[2 - 2\cos(2\pi(4-t))\right] = \frac{1}{\pi}\left[1 - \cos(2\pi t)\right] \quad [\text{formule trigo}]$$

cas n°6 :



$t \in [4\,;5] = C_{xy} = 0$

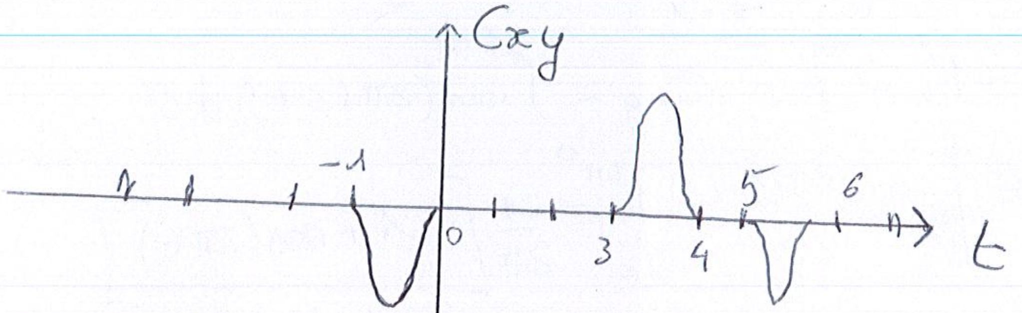
cas n°7



$t \in [5\,;6]$: $C_{xy} = \int_t^{6} -\sin(2\pi(\tau - t))\,d\tau = \left[\frac{\cos(2\pi(\tau - t))}{2\pi}\right]_t^{6}$

$$= \frac{1}{2\pi}\left[\cos(2\pi t) - 1\right]$$

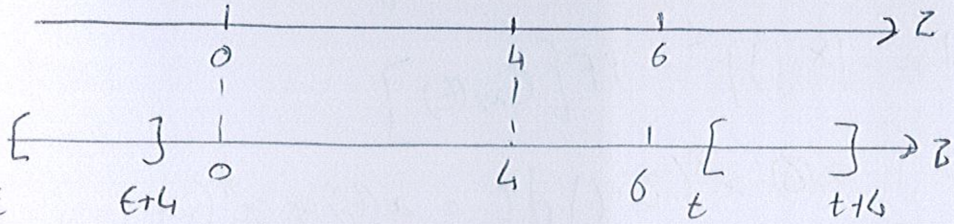
bilan :



détection des discontinuités : si le signal varie rapidement + de précision.

⇒ Radar : impulsion.

Cxz :



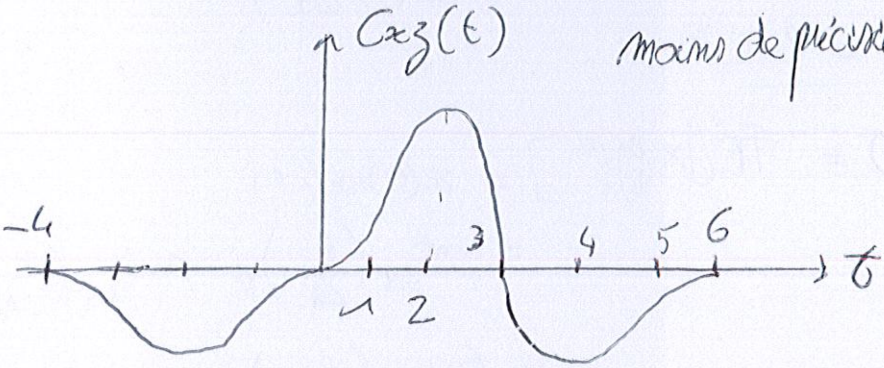
$t \leq -4$ et $t \geq 6$: $Cxz(t) = 0$

$t \in [-4;0]$: $Cxz(t) = \int_0^{t+4} \sin\left(\frac{\pi}{2}(z-t)\right) dz = \frac{2}{\pi}\left(\cos(\pi/2\, t) - 1\right)$

$t \in [0;2]$: $Cxy(t) = \frac{4}{\pi}\left(1 - \cos(\pi/2\, t)\right)$

$t \in [2;4]$: $Cxy(t) = \frac{2}{\pi}\left(1 - 3\cos(\pi/2\, t)\right)$

$t \in [4,6]$: $Cxz = \frac{2}{\pi}\left(\cos(\pi/2\, t) + 1\right)$



moins de précision.

exo 2 propriétés énergétiques des signaux :

1- signaux d'énergie finie :

11- théorème de Parseval : $\int_{\mathbb{R}} |x(t)|^2 dt = \int_{\mathbb{R}} |X(f)|^2 df$

$$\int_{\mathbb{R}} x(t)\, y^*(t)\, dt = \int_{\mathbb{R}} x(t)\, y^*(t)\, e^{2\pi j f t} dt \Big|_{f=0} = TF\left[x(t)\, y^*(t)\right]_{f=0}$$

$$= TF[x(t)] * TF[y^*(t)] \Big|_{f=0} = X(f) * Y^*(-f) \Big|_{f=0}$$

$$= \int_{\mathbb{R}} X(u)\, Y^*(-(f-u))\, du \Big|_{f=0} = \int_{\mathbb{R}} X(u)\, Y^*(u)\, du$$

si $X = Y \Rightarrow$ nous avons $\int_{\mathbb{R}} |x(t)|^2 dt = \int_{\mathbb{R}} |X(f)|^2 df$

1-2- $|X(f)|^2$: densité spectrale d'énergie,

Rq : $dse = |X(f)|^2 = TF[C_{xx}(t)]$

$$C_{xx}(t) = \int_{\mathbb{R}} x(\tau)\, x^*(\tau - t)\, d\tau = x(t) * x^*(-t)$$

$$TF[C_{xx}(t)] = TF[x(t) * x^*(-t)] = TF[x(t)] . TF[x^*(-t)]$$
$$= X(f) . X^*(f) = |X(f)|^2 = dse$$

<u>en $t=0$</u> : $C_{xx}(t) = \int_{\mathbb{R}} x(\tau)\, x^*(\tau)\, d\tau = E_x$: énergie du signal.

1-3 - énergie total de $x(t) = rect_{2a}(t/2a)$

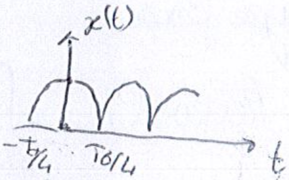
$$\underline{E_x = 2a}$$

$$C_{xx}(t) = TF[|X(f)|^2] = x(t) * x^*(-t)$$
$$= rect_{2A}\left(\frac{t}{2A}\right) * rect_{2A}\left(\frac{t}{2A}\right)$$
$$= tri_{4A}(t/4A)$$

$$C_{xx}(t=0) = tri_{4A}\left(\frac{0}{4A}\right) = 2A$$

2- signaux de puissance finie

2-1- soit $x(t) = |\cos(2\pi f_0 t)|$



$\begin{cases} \cos^2 + \sin^2 = 1 \\ \cos^2 - \sin^2 = \cos 2\theta \end{cases}$

def : signal de période T_0 : $P_x = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt$

$$P_x = 2f_0 \int_{-T_0/4}^{T_0/4} \cos^2(2\pi f_0 t)\, dt = 2f_0 \int_{-T_0/4}^{T_0/4} \cos^2(2\pi f_0 t)\, dt$$

or $\cos^2\theta = (1 + \cos 2\theta)\frac{1}{2}$ => $f_0 \int_{-T_0/4}^{T_0/4} (1 + \cos 2\theta)\, d\theta$

$$P_x = f_0 \int_{-T_0/4}^{T_0/4} (1 + \cos(4\pi f_0 t))\, dt = f_0 \left[t + \frac{\sin(4\pi f_0 t)}{4\pi f_0} \right]_{-T_0/4}^{T_0/4}$$

$$P_x = f_0\left[\frac{\sin(\pi f_0 T_0)}{4\pi f_0} - \frac{\sin(-\pi f_0 T_0)}{4\pi f_0}\right] + f_0\left[T_0/4 + T_0/4\right]$$

$$= f_0\left[0\right] + f_0\left[\frac{T_0}{2}\right]$$

$$P_x = 1/2$$

développement en série de Fourier :

$$c_n = \frac{1}{\Delta}\int_\Delta x(t)\, e^{-2\pi j \frac{n t}{\Delta}}\, dt$$

$$c_n = \frac{1}{T_0/2}\int_{-T_0/4}^{T_0/4} |\cos(2\pi f_0 t)|\, e^{-2\pi j n t\left(\frac{1}{T_0/2}\right)} dt$$

$$= \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} \frac{e^{j2\pi f_0 t} + e^{-2\pi j f_0 t}}{2}\, \overbrace{e^{-2\pi j n \frac{2}{T_0} t}}^{-4\pi j n f_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} e^{2\pi j f_0 t(1-2n)} + e^{-2\pi j f_0 t(1+2n)}\, dt$$

$$= \frac{1}{T_0}\left[\frac{e^{2\pi j f_0 \frac{T_0}{4}(1-2n)} - e^{-2\pi j f_0 \frac{T_0}{4}(1-2n)}}{2\pi j f_0(1-2n)} - \frac{e^{-2\pi j f_0 \frac{T_0}{4}(1+2n)} - e^{2\pi j f_0 \frac{T_0}{4}(1+2n)}}{2\pi j f_0(1+2n)}\right]$$

$$= \frac{1}{T_0}\left[\frac{e^{\pi j \frac{1}{2}(1-2n)} - e^{-\frac{\pi}{2} j(1-2n)}}{2\pi j f_0(1-2n)} - \frac{e^{-\pi/2\, j(1+2n)} - e^{\pi/2\, j(1+2n)}}{2\pi j f_0(1+2n)}\right]$$

$$e^{\pi/2\, j(1-2n)} = e^{\pi/2\, j}\, e^{-j\pi n} = j(-1)^n = e^{j\pi/2(1+2n)}$$

$$e^{-\pi/2\, j(1+2n)} = -j(-1)^n$$

$$= \frac{j(-1)^n + j(-1)^n}{2\pi j(1-2n)} + \frac{j(-1)^n + j(-1)^n}{2\pi j(1+2n)} = \frac{(-1)^n}{\pi(1-2n)} + \frac{(-1)^n}{\pi(1+2n)}$$

$$\boxed{c_n = \frac{2}{\pi}\,\frac{(-1)^n}{1-4n^2}}$$

on a donc :

$$s(t) = \sum \frac{2}{\pi} \frac{(-1)^n}{1-4n^2} e^{4\pi j n f_0 t}$$

TF

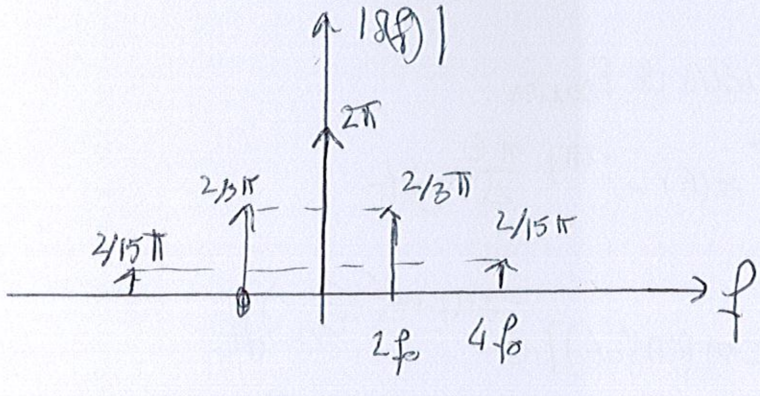
$$S(f) = \sum \left(\frac{2}{\pi} \frac{(-1)^n}{1-4n^2}\right) \delta(f - 2nf_0)$$

$\cos(2\pi f_0 t)\ \text{rect}\ T_0/2\ \ \sum \delta(t - nT_0/2)$

$2F_0 \sum \delta(f - n2F_0) \times [\frac{1}{2} \delta(f-f_0) + \delta(f+f_0)$

$* \frac{T_0}{2} \text{sinc}(\pi f T_0/2)$

$= \sum \delta(f - 2nF_0) \times \frac{1}{2} \left(\text{sinc}\left(\pi \frac{f - f_0}{2f_0}\right) + \text{sinc}\left(\pi \frac{f+f_0}{2f_0}\right) \right)$



$$\sum_{n=-3}^{3} |S_n|^2 = 1/2 = P_s \Rightarrow$$ on a la puissance du signal dans les 3 en harmonique

2.2 - fonction d'autocorrélation $s(t)$ en fonction de c_n.

$$C_s(t) = s(t) * s^*(-t) = \frac{1}{T_0/2} \int_{(T_0/2)} s(\tau)\, s^*(\tau - t)\, d\tau$$

$$= \frac{2}{T_0} \int_{-T_0/4}^{T_0/4} \sum_{n\in\mathbb{Z}} c_n e^{4\pi j n f_0 \tau} \left(\sum_{m \in \mathbb{Z}} c_m e^{4\pi j m f_0 (\tau - t)} \right)^* d\tau$$

$$= \frac{2}{T_0} \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} c_n c_m^* \int_{-T_0/4}^{T_0/4} e^{4\pi j f_0 \tau n}\, e^{-4\pi j m f_0 (\tau - t)}\, d\tau$$

$$= \underbrace{\frac{2}{T_0} \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} c_n c_m^* e^{4\pi j f_0 m t}}_{} \int_{-T_0/4}^{T_0/4} e^{4\pi j f_0 (n-m)\tau}\, d\tau$$

$$= A \left[\frac{e^{4\pi j f_0 (n-m)\tau}}{4\pi j f_0 (n-m)} \right]_{-T_0/4}^{T_0/4} = A \times \frac{1}{4\pi j f_0 (n-m)} \left[e^{\pi j (n-m)} - e^{-\pi j (n-m)} \right]$$

$$= A \times \frac{\sin \pi (n-m)}{2\pi f_0 (n-m)\, 2j} = A \frac{1}{2f_0} \underbrace{\text{sinc}(\pi(n-m))}_{\substack{= 1 \text{ si } n = m \\ = 0 \text{ sinon}}}$$

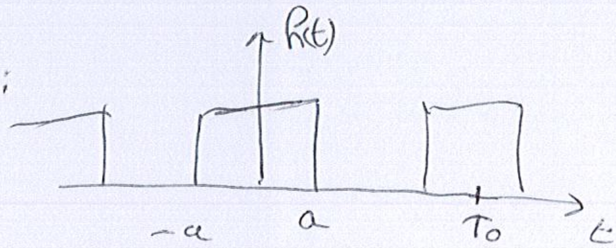
$$C_s(t) = \sum_{n \in \mathbb{Z}} \frac{2}{T_0} |C_n|^2 e^{4\pi j f_0 n t} \times \frac{1}{2f_0}$$

$$C_s(t) = \sum_{n \in \mathbb{Z}} |C_n|^2 e^{4\pi j f_0 n t}$$

pour $t=0 \Rightarrow C_s(t) = \sum_{n \in \mathbb{Z}} |C_n|^2 = P_s$

La fonction d'autocorrélation d'un signal périodique de période $\frac{1}{T} = F$ est $C_s(t) = \sum_{n \in \mathbb{Z}} |C_n|^2 e^{2\pi j n F t}$

2.3 - signal périodique T_0 :



$$h(t) = rect_{2a}\left(\frac{t}{2a}\right) * \sum_{n \in \mathbb{Z}} \delta(t - nT_0)$$

$$H(f) = 2a \operatorname{sinc}(\pi f 2a) \times F_0 \sum \delta(f - nF_0)$$

$$P_x = \frac{1}{T_0} \int_{-a}^{a} 1\, dt = \frac{2a}{T_0} \Rightarrow 1/2 \text{ si } a = T_0/2$$