Examen session 1- Juin2021

Solutions de l'exercice (extrait du chap. 4 du livre de Cormen)

Note: In parts (a), (b), and (d) below, we are applying case 3 of the master theorem, which requires the regularity condition that $af(n/b) \leq cf(n)$ for some constant $c < 1$. In each of these parts, $f(n)$ has the form n^k. The regularity condition is satisfied because $af(n/b) = an^k/b^k = (a/b^k)n^k = (a/b^k)f(n)$, and in each of the cases below, a/b^k is a constant strictly less than 1.

a. $T(n) = 2T(n/2) + n^3 = \Theta(n^3)$. This is a divide-and-conquer recurrence with $a = 2$, $b = 2$, $f(n) = n^3$, and $n^{\log_b a} = n^{\log_2 2} = n$. Since $n^3 = \Omega(n^{\log_2 2+2})$ and $a/b^k = 2/2^3 = 1/4 < 1$, case 3 of the master theorem applies, and $T(n) = \Theta(n^3)$.

b. $T(n) = T(9n/10) + n = \Theta(n)$. This is a divide-and-conquer recurrence with $a = 1$, $b = 10/9$, $f(n) = n$, and $n^{\log_b a} = n^{\log_{10/9} 1} = n^0 = 1$. Since $n = \Omega(n^{\log_{10/9} 1+1})$ and $a/b^k = 1/(10/9)^1 = 9/10 < 1$, case 3 of the master theorem applies, and $T(n) = \Theta(n)$.

c. $T(n) = 16T(n/4) + n^2 = \Theta(n^2 \lg n)$. This is another divide-and-conquer recurrence with $a = 16$, $b = 4$, $f(n) = n^2$, and $n^{\log_b a} = n^{\log_4 16} = n^2$. Since $n^2 = \Theta(n^{\log_4 16})$, case 2 of the master theorem applies, and $T(n) = \Theta(n^2 \lg n)$.

d. $T(n) = 7T(n/3) + n^2 = \Theta(n^2)$. This is a divide-and-conquer recurrence with $a = 7$, $b = 3$, $f(n) = n^2$, and $n^{\log_b a} = n^{\log_3 7}$. Since $1 < \log_3 7 < 2$, we have that $n^2 = \Omega(n^{\log_3 7+\epsilon})$ for some constant $\epsilon > 0$. We also have $a/b^k = 7/3^2 = 7/9 < 1$, so that case 3 of the master theorem applies, and $T(n) = \Theta(n^2)$.

e. $T(n) = 7T(n/2) + n^2 = O(n^{\lg 7})$. This is a divide-and-conquer recurrence with $a = 7$, $b = 2$, $f(n) = n^2$, and $n^{\log_b a} = n^{\log_2 7}$. Since $2 < \lg 7 < 3$, we have that $n^2 = O(n^{\log_2 7-\epsilon})$ for some constant $\epsilon > 0$. Thus, case 1 of the master theorem applies, and $T(n) = \Theta(n^{\lg 7})$.

f. $T(n) = 2T(n/4) + \sqrt{n} = \Theta(\sqrt{n} \lg n)$. This is another divide-and-conquer recurrence with $a = 2$, $b = 4$, $f(n) = \sqrt{n}$, and $n^{\log_b a} = n^{\log_4 2} = \sqrt{n}$. Since $\sqrt{n} = \Theta(n^{\log_4 2})$, case 2 of the master theorem applies, and $T(n) = \Theta(\sqrt{n} \lg n)$.

h. $T(n) = T(\sqrt{n}) + 1$

The easy way to do this is with a change of variables, as on page 66 of the text. Let $m = \lg n$ and $S(m) = T(2^m)$. $T(2^m) = T(2^{m/2}) + 1$, so $S(m) = S(m/2) + 1$. Using the master theorem, $n^{\log_b a} = n^{\log_2 1} = n^0 = 1$ and $f(n) = 1$. Since $1 = \Theta(1)$, case 2 applies and $S(m) = \Theta(\lg m)$. Therefore, $T(n) = \Theta(\lg \lg n)$.