Time Reversal

Random heterogeneous media

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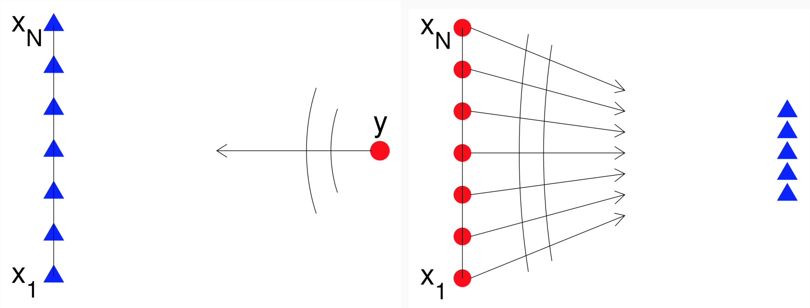


(a) 1st Step **(b)** 2nd Step

Figure 1: Time reversal experiment.

Paraxial Wave Equation

Paraxial Wave Equation

Due to the paraxial hypothesis the solution of a wave equation can be approximate as :

$$A(r) = \phi(r)e^{ikz}$$

where ϕ approximately solves :

$$\nabla^2\phi + 2ik\frac{\partial\phi}{\partial z} = 0$$

Schrodinger equation:

$$\partial_z\phi = \frac{i}{2k}\partial_x^2\phi \tag{1}$$

The paraxial approximation is valid if:

$$\left|\frac{\partial^2\phi}{\partial_{z^2}}\right| \ll \left|k\frac{\partial\phi}{\partial_z}\right| \tag{2}$$

Homogeneous medium

Propagation of time-harmonic waves : analytical solution

Solve the Schrodinger equation :

$$\partial_z \phi = \frac{i}{2k} \partial_x^2 \phi$$

With the initial condition at z = 0 :

$$\phi(z = 0, x) = \phi_0(x) = exp(-\frac{x^2}{r_0^2}) \tag{3}$$

By Fourier transform on x, the equation becomes :

$$\partial_z \hat{\phi} = -\frac{i\omega^2}{2k} \hat{\phi}$$

The solution is the following form, where C(w) only depends on w.

$$\hat{\phi}(z, w) = C(w) exp(\frac{-iw^2}{2k} z)$$

Propagation of time-harmonic waves : analytical solution and simulation

C(w) can be decided by the Equation 3, the solution becomes

$$\phi(z,x) \quad = \quad \mathfrak{F}^{-1}[\hat{\phi}_0 exp(\frac{-iw^2}{2k}z)](z,x) \tag{4}$$

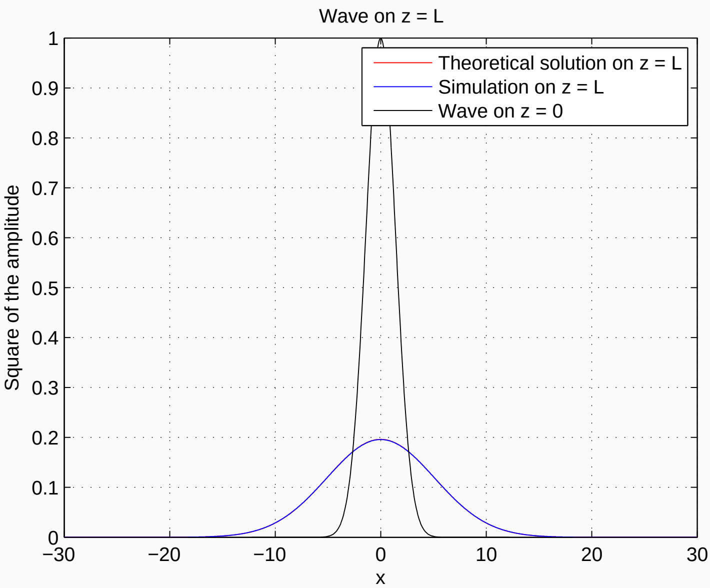
The equation 4 can be simulated easily, for the analytical form we simply apply the Fourier transform of a Gaussian function $f(x) = exp(-ax^2)$ which is given by :

$$\mathfrak{F}(exp(-ax^2))(w) \quad = \quad \sqrt{\frac{\pi}{a}} exp(-\frac{w^2}{4a})$$

The final solution on z = L is :

$$\phi_t(x) = \frac{r_0}{r_t} exp(-\frac{x^2}{r_t^2}), \qquad \text{with } r_t = r_0\sqrt{1 + \frac{4L^2}{k^2 r_0^4}}$$

Propagation of time-harmonic waves : result



Time reversal for time-harmonic

Denotes a time-reversal mirror χ_M at z = L, the time-reversal problem consists of solving the same equation but with different initial condition on z = L.

$$\partial_z \phi_{tr} = \frac{i}{2k} \partial_x^2 \phi_{tr}$$

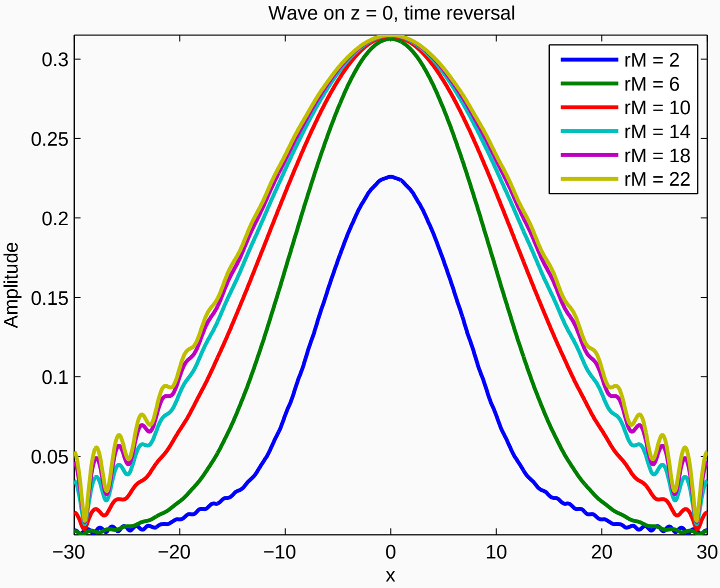
With

$$\phi_{tr}(z = L, x) = \overline{\phi}(z = L, x)\chi_M(x) = \overline{\phi_t}\chi_M$$

By exactly the same method, we can simulate the solution by Fourier transform.

$$\phi_{tr}(z, x) = \mathfrak{F}^{-1}[\mathfrak{F}[\overline{\phi_t}\chi_M] exp(\frac{iw^2(L-z)}{2k})] \tag{5}$$

Result for $\chi_M(x) = [1 - (\frac{x}{2r_M})^2]^2 \; 1_{[-2r_M, 2r_M]}(x)$



Analytical solution for $\chi_M(x) = exp(-\frac{x^2}{r_M^2})$

The mirror function is the form Gaussian, and the solution can thus be analytical resolved :

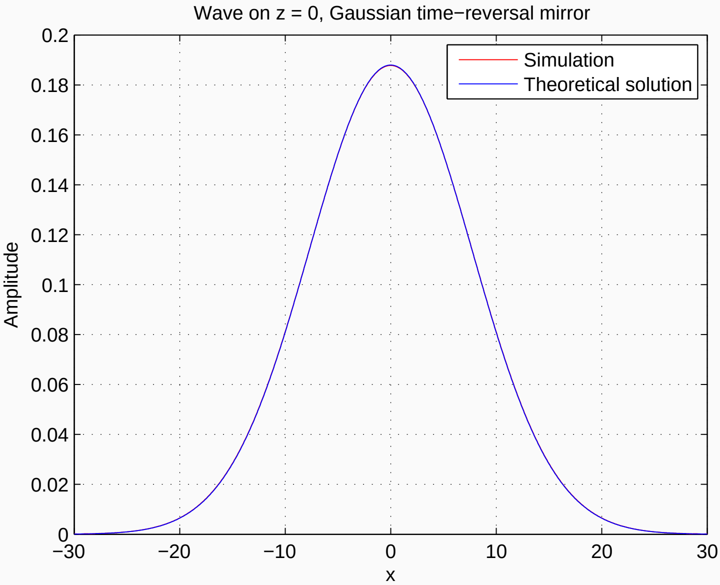
$$\phi_{tr}(z=0,x) = \frac{1}{a_{tr}} exp(-\frac{x^2}{r_{tr}^2})$$

With

$$a_{tr} = \sqrt{1 - \frac{4iL}{kr_0^2} - \frac{2iL}{kr_M^2} - \frac{4L^2}{k^2 r_0^2 r_M^2}}$$

$$r_{tr} = (\frac{1}{r_M^2} + \frac{1}{r_0^2 - \frac{2iL}{k}})^{-1} - \frac{2iL}{k}$$

Result for $\chi_M(x) = exp(-\frac{x^2}{r_M^2})$



Random medium

Random model

Schrodinger equation:

$$\begin{aligned} \partial_z \phi &= \frac{i}{2k}\partial_x^2 \phi + \frac{i}{2k}\mu(z,x)\phi \\ \mu(z,x) &= \mu_n(x) \text{ if } z \in [nz_c, (n+1)z_c] \end{aligned} \tag{6}$$

with $\mu_0(x), \mu_1(x), \ldots, \mu_{[L/z_c]}(x)$ independent realizations of a Gaussian Process with mean zero and covariance function

$$\mathbb{E}[\mu_n(x)\mu_n(x')] = \sigma^2 \exp(-(x-x')^2/x_c^2)$$

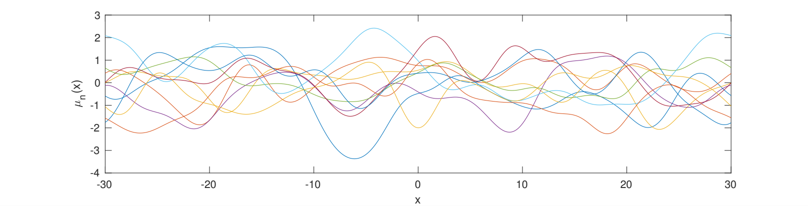


Figure 5: Gaussian processes samples.

Split-Step Fourier Method

Laplacian part:

$$\partial_z \phi(z,x) = \frac{ik}{2}\partial_x^2 \phi(z,x) \quad (7)$$

$$\Rightarrow \quad \phi(z+h,x) = \mathcal{F}^{-1}(\exp\{-h\frac{i}{2k}(2\pi i f)^2\}\mathcal{F}(\phi(z,x))) \quad (8)$$

Random medium part:

$$\partial_z \phi(z,x) = \frac{ik}{2}\mu_n(x)\phi(z,x) \quad (9)$$

$$\Rightarrow \quad \phi(z+h,x) = \exp\{h\frac{ik}{2}\mu_n(x)\}\phi(z,x) \quad (10)$$

Full step:

$$\phi(z+h,x) = \exp\{h\frac{ik}{2}\mu_n(x)\}\mathcal{F}^{-1}(\exp\{-h\frac{i}{2k}(2\pi i f)^2\}\mathcal{F}(\phi(z,x))) \quad (11)$$

Propagation of time-harmonic waves

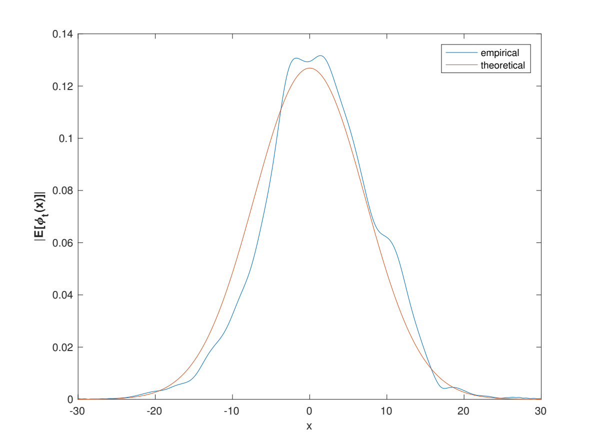
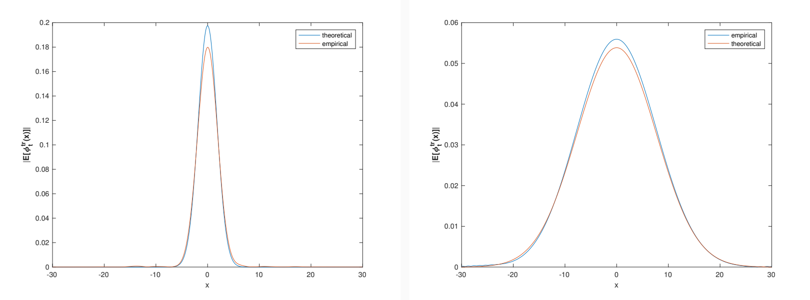


Figure 6: Mean transmitted wave profile $|\phi_t(x, z = L)|$ in random medium.

$$\mathbb{E}[\phi_t(x)] = \frac{r_0}{r_t} \exp(-\frac{x^2}{r_t^2}) \exp(-\frac{\gamma_0 \omega^2 L}{8})$$

Time reversal for time-harmonic waves



(a) Random, $r_M = 2$ **(b)** Homogeneous, $r_M = 2$

Figure 7: Mean refocused wave profiles ($z = 0$)

$$\mathbb{E}[\phi_r^{tr}(x)] = \frac{1}{a_{tr}} \exp(-\frac{x^2}{r_{tr}^2}) \exp(-\frac{x^2}{r_a^2}) \text{ (Figure (a))} \tag{12}$$

$$\mathbb{E}[\phi_r^{tr}(x)] = \frac{1}{a_{tr}} \exp(-\frac{x^2}{r_{tr}^2}) \exp(-\frac{\gamma_0 \omega^2 L}{8}) \text{ (Figure (b))} \tag{13}$$

Propagation of time-dependent waves

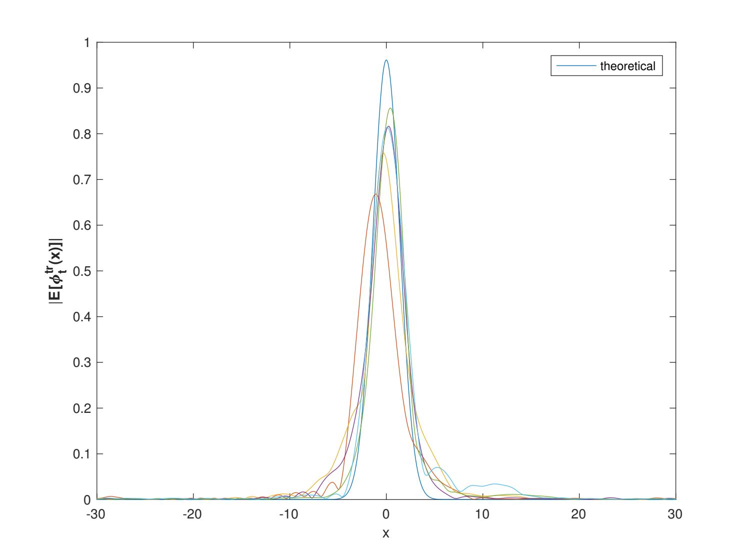


Figure 8: Refocused wave profile for a time-dependent initial condition.

Conclusion

Thank you for your attention !

Questions?