assignment_2

October 9, 2022

0.1 Assignment 2 - 02601 Numerical algorithms

0.1.1 s214643

0.1.2 14/10-2022

0.2 1) Multiple choice

A) Answer: 4. There cannot be two points with the same x values.

B) Answer: 2. 0.50000

[ ]:
```
from functionality.interpolation import cardinal_polynomials
import matplotlib.pyplot as plt
import numpy as np
t = np.linspace(10,20,11, endpoint=True)
nodes = [15,18,22]
cd = cardinal_polynomials(nodes, 1, t)
cd[6]
```

[ ]:
```
0.5
```

C) Answer: 3. 23

[ ]:
```
import math
for n in range(1,30):
    h = 2/n
    M = 5**(n+1)*math.exp(5*2)
    error = 1 / (4*(n+1)) * h**(n+1) * M
    if error < 1e-6:
        print(n)
        break
```

```
23
```

D) Answer: 3, Simpson < Composite trapezoid < Trapezoid

[ ]:
```
def simpson(a,b, y):
    h = (b-a)/2
    return (h/3)*(y[0]+4*y[1]+y[2])

def trapezoid(a,b, y):
```

```
    h = b - a
    return (h/2)*(y[0]+y[-1])

def comp_trap(a,b, y):
    n = len(y)-1
    h = (b-a)/n
    edge_vals = 0.5*(y[0]+y[-1])
    return h*(edge_vals+sum(y[1:-1]))

print(f"Simpson: {round(simpson(1,3,[2,2,3]),2)}")
print(f"Trapezoid: {round(trapezoid(1,3,[2,2,3]),2)}")
print(f"Composite trapezoid: {round(comp_trap(1,3,[2,2,3]),2)}")
```

```
Simpson: 4.33
Trapezoid: 5.0
Composite trapezoid: 4.5
```

0.3 2) Interpolation

A):

$$\begin{aligned}
l_0(x) &= \left(\frac{x-2}{1-2}\right)\left(\frac{x-3}{1-3}\right) \\
l_1(x) &= \left(\frac{x-1}{2-1}\right)\left(\frac{x-3}{2-3}\right) \\
l_2(x) &= \left(\frac{x-1}{3-1}\right)\left(\frac{x-2}{3-2}\right)
\end{aligned}$$

$$P_2(x) = ln(1)l_0(x) + ln(2)l_1(x) + ln(3)l_2(x) = ln(2)l_1(x) + ln(3)l_2(x)$$

B):

$$\frac{1}{4(n+1)} M h^{n+1}$$

$$n = 2, \quad h = \frac{3-1}{n} = 1, \quad M = \max_{x\in[1,3]} \frac{2}{x^3} = 2$$

$$error = \frac{1}{4(2+1)} \cdot 2 \cdot 1^{2+1} = 6$$

0.4 3) Composite Simpsons rule

A) Implementing the function

[ ]:
```
def MySimpson(f, a,b,n):
    if n % 2 == 1:
        print(f"ERROR: n={n} is not even")
        return float("NaN")
```

```python
    h = (b-a) / n
    nodes = np.linspace(a,b,n+1, endpoint=True)
    comp_simpson = 0
    for a,b in zip(nodes[::2], nodes[1::2]):
        comp_simpson += (f(a) + 4*f(a+h) + f(b))

    return (h/3)*comp_simpson
```

[ ]:
```python
def MySimpson(f, a,b,n):
    if n % 2 == 1:
        print(f"ERROR: n={n} is not even")
        return float("NaN")
    h = (b-a) / n
    nodes = np.linspace(a,b,n+1, endpoint=True)
    comp_simpson = (h/3)*(f(a)+f(b))

    comp_simpson += ((4 * h)/3) * sum([f(a+(2*i-1)*h) for i in range(1, int(n/
↪2)+1)])
    comp_simpson += ((2 * h)/3) * sum([f(a+2*i*h) for i in range(1, int((n-2)/
↪2)+1)])

    return comp_simpson
```

B) Testing with ploynomial of degree 3

[ ]:
```python
import pandas as pd

## TEST FUNCTIONS

f = lambda x: 4*x**3 + 3*x**2 + 2*x + 1
int_f = lambda x: x**4 + x**3 + x**2 + x


## ERROR TABLE
a, b = 0, 1
actual = int_f(b)-int_f(a)
n_list, numerical, error = list(), list(), list()
for n in (2,3,20):
    estimat = MySimpson(f, a,b, n)
    n_list.append(n)
    numerical.append(estimat)
    error.append(abs(actual - estimat))

pd.DataFrame({"n": n_list, "Numerical estimation": numerical, "Error": error})
```

```
ERROR: n=3 is not even
```

```
[ ]:     n  Numerical estimation  Error
     0   2                   4.0    0.0
     1   3                   NaN    NaN
     2  20                   4.0    0.0
```

As n increases in the polynomial test, we know that a polynomial of degree 3 can be approximated without error. This checks out.

C) Testing with sine function

```python
import pandas as pd

## TEST FUNCTIONS

f = lambda x: math.sin(x)
int_f = lambda x: - math.cos(x)


## ERROR TABLE
a, b = 0, math.pi
actual = int_f(b)-int_f(a)
n_list, numerical, error = list(), list(), list()
for n in range(2,21,2):
    estimat = MySimpson(f, a,b, n)
    n_list.append(n)
    numerical.append(estimat)
    error.append(abs(actual- estimat))

pd.DataFrame({"n": n_list, "Numerical estimation": numerical, "Error": error})
```

```
[ ]:     n  Numerical estimation     Error
     0   2              2.094395  0.094395
     1   4              2.004560  0.004560
     2   6              2.000863  0.000863
     3   8              2.000269  0.000269
     4  10              2.000110  0.000110
     5  12              2.000053  0.000053
     6  14              2.000028  0.000028
     7  16              2.000017  0.000017
     8  18              2.000010  0.000010
     9  20              2.000007  0.000007
```

Here we test for a couple of more values of n, now testing the integral of a sine function. We see the error reducing rapidly as n increases. It seems this function is easier to estimate.

D) Upper bound for absolute error

Using the error theorem (8) from the book:

$$-\frac{1}{180}(b-a)h^4 f^{(4)}(\xi)$$

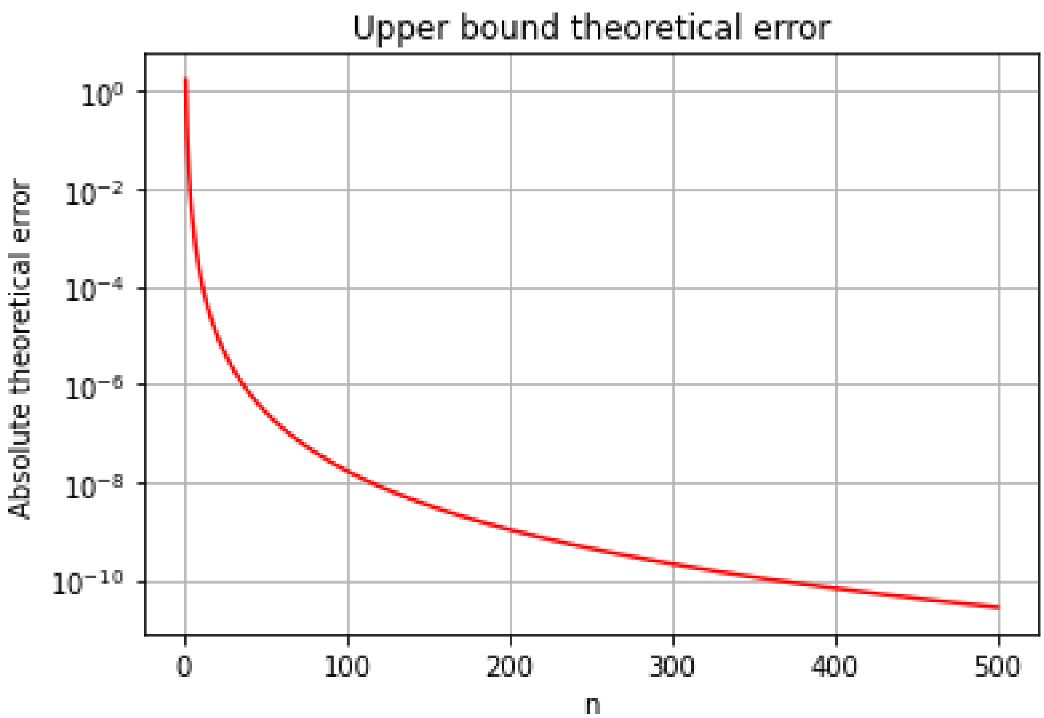
And inserting our values:

$$a = 0, \quad b = \pi, \quad h = \frac{\pi}{n}, \quad M = 1$$

We can find the theoretical upper bound for the error as:

$$-\frac{1}{180}\frac{\pi^5}{n^4}$$

[ ]:
```
x = np.linspace(1, 500, 500, endpoint=True)
plt.plot(x, [-(-1/180 * (math.pi**5)/(n**4)) for n in x], 'r')
plt.xlabel("n")
plt.ylabel("Absolute theoretical error")
plt.grid()
plt.yscale('log')
plt.title("Upper bound theoretical error")
plt.show()
```

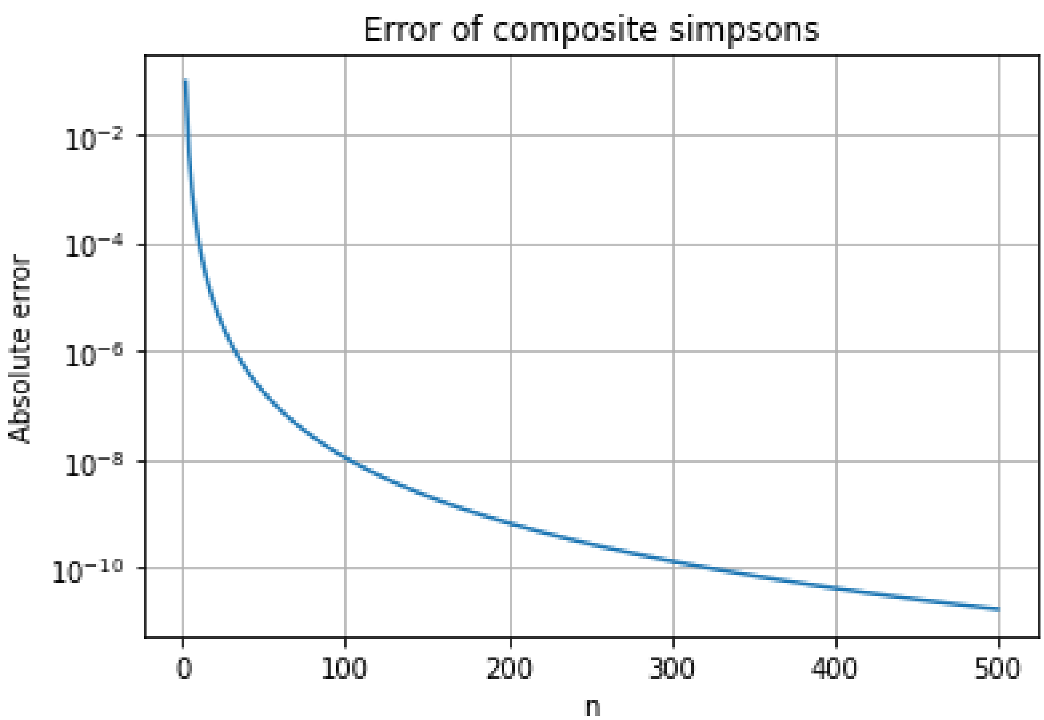


E) Testing the n for a maximum error of 1e-5, and checking with composite trapezoid function

[ ]:
```
## Test of minimal error (below 1e-5)
f = lambda x: math.sin(x)
int_f = lambda x: - math.cos(x)

a, b = 0, math.pi
actual = int_f(b)-int_f(a)
```

```
for n in range(2,500,2):
    estimat = MySimpson(f, a,b, n)
    error = abs(estimat-actual)
    if error < 1e-5:
        print(f"Lowest n with error below 1e-5 is: n = {n}")
        break

x = np.linspace(2,500, 499, endpoint=True)
plt.plot(x[::2], [abs(MySimpson(f, 0,math.pi, int(n))-2) for n in x[::2]])
plt.yscale('log')
plt.grid()
plt.xlabel("n")
plt.ylabel("Absolute error")
plt.title("Error of composite simpsons")
plt.show()
```

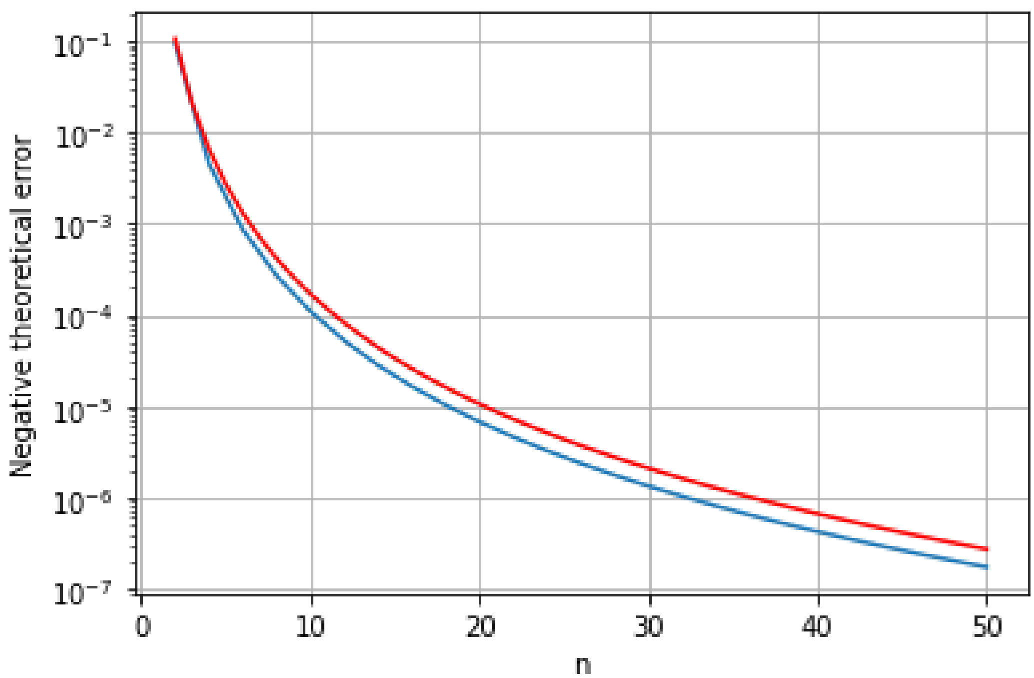
```
Lowest n with error below 1e-5 is: n = 20
```



The lowest n with an error less than 1e-5 is 20. This is fairly good.

Next: showing that the theoretical upper bound error is indeed an upper bound for the error

[ ]:
```
x = np.linspace(2, 50, 49, endpoint=True)
plt.plot(x[::2], [(abs(MySimpson(f, 0,math.pi, int(n))-2)) for n in x[::2]])
plt.plot(x, [((1/180 * (math.pi**5)/(n**4))) for n in x], 'r')
plt.xlabel("n")
plt.ylabel("Negative theoretical error")
plt.grid()
plt.yscale('log')
plt.show()
```



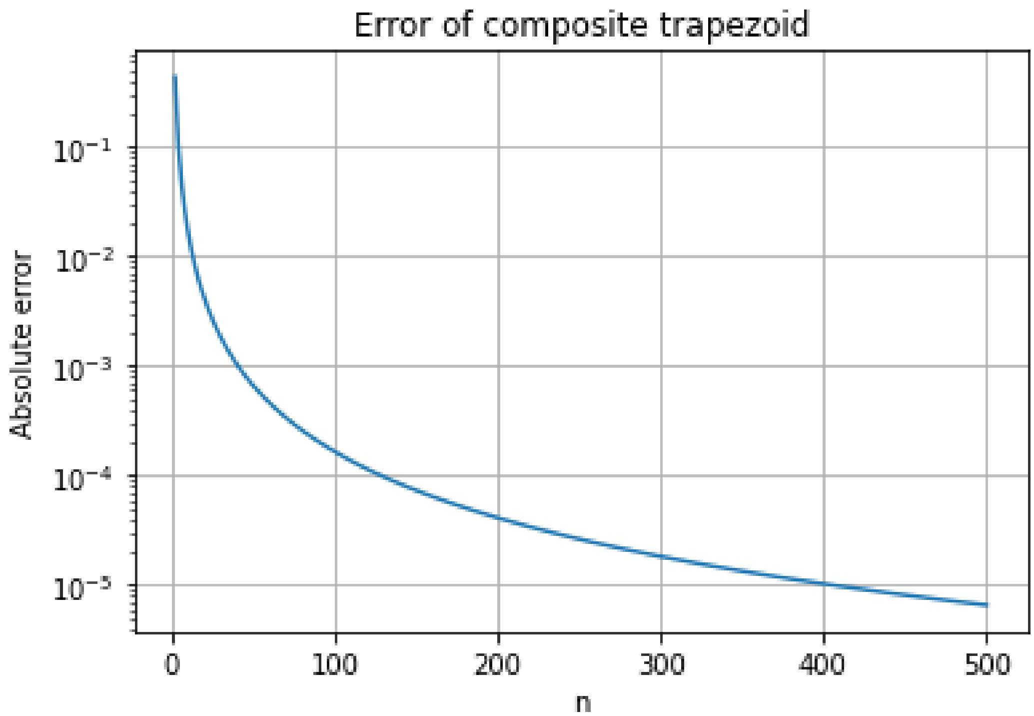
Except for a few outliers in the begining, it is clearly shown that the upper bound (in red) has a higher error for any number of intervals (n) than the calculated error (in blue).

Finding error of composite trapezoid function

[ ]:
```
## same check as above, but with composite trapezoid

def comp_trap(f, a,b, n):
    x = np.linspace(a,b,n+1,endpoint=True)
    h = (b-a) / n
    edge_vals = 0.5 * (f(x[0]) + f(x[-1]))
    return h * (edge_vals + sum([f(x_i) for x_i in x[1:-1]]))

f = lambda x: math.sin(x)
int_f = lambda x: - math.cos(x)
```

```
a, b = 0, math.pi
actual = int_f(b)-int_f(a)

for n in range(2,500,2):
    estimat = comp_trap(f, a,b, n)
    error = abs(estimat-actual)
    if error < 1e-5:
        print(f"Lowest n with error below 1e-5 is: n = {n}")
        break

x = np.linspace(2,500, 499, endpoint=True)
plt.plot(x[::2], [abs(comp_trap(f, 0,math.pi, int(n))-2) for n in x[::2]])
plt.yscale('log')
plt.grid()
plt.xlabel("n")
plt.ylabel("Absolute error")
plt.title("Error of composite trapezoid")
plt.show()
```

```
Lowest n with error below 1e-5 is: n = 406
```



The lowest n with an error less than 1e-5 is 406. It seems that the comp. trapezoid error falls with $\frac{1}{2^2}$, where the comp. simpsons falls with $\frac{1}{4^2}$. The Composite simpsons model is therefor the fastest of the two.