assignment_1

September 20, 2022

0.1 Numerical algorithms 1, assignment 1

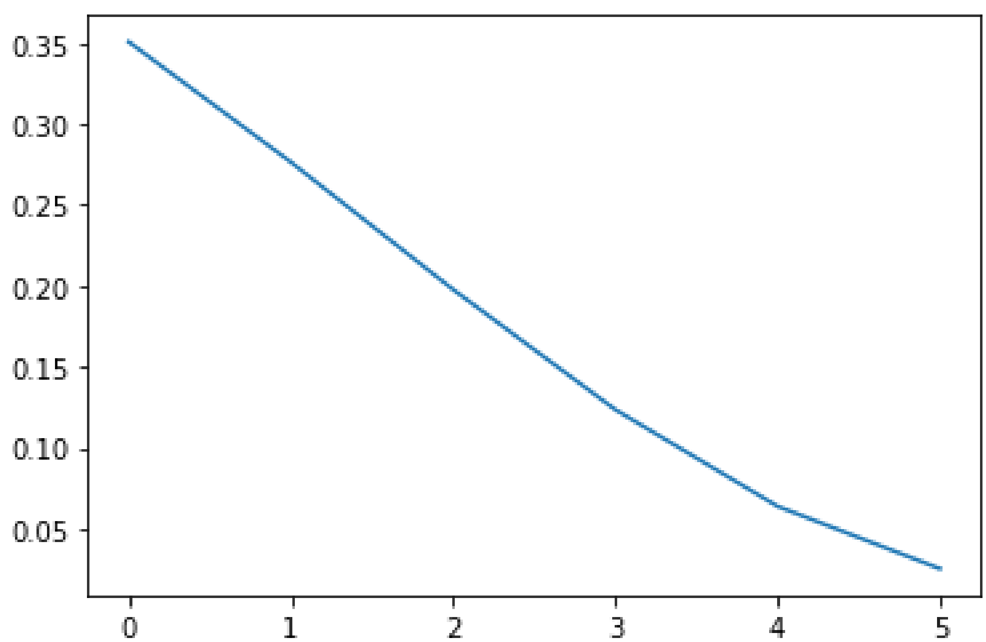
0.1.1 1 Multiple choice

```
A - errors, correct answer: 1
B - Data-fitting, correct answer: 2
C - Bisection method, correct answer: 2
D - Convergence, correct answer: 3
```

[ ]:
```python
import numpy as np
import matplotlib.pyplot as plt
```

[ ]:
```python
# For task D in multiple choice

x = np.arange(6)
err = np.array([1.23e-1, 7.64e-2, 3.91e-2, 1.53e-2, 4.10e-3, 6.53e-4])
plt.plot(x,np.sqrt(err))
plt.show()
```



0.1.2 2 Data fitting

2.1

[ ]:
```
# Initialize arrays with data
x = np.array([0.0, 0.5, 1.0, 1.5, 2.0])
y = np.array([2.24, 0.80, 0.38, 0.09, 0.06])

# Linearize
x1 = np.exp(-x)
x2 = np.exp(-2*x)

# Find the coefficient matrix
A = np.array([x1,x2]).T

print("System matrix")
print(A,"\n")
print("Right hand side")
print(np.dot(A.T,y))
```

```
System matrix
[[1.         1.        ]
 [0.60653066 0.36787944]
 [0.36787944 0.13533528]
 [0.22313016 0.04978707]
 [0.13533528 0.01831564]]

Right hand side
[2.89322055 2.59131074]
```

2.2

[ ]:
```
a,b = np.linalg.solve(A.T@A,np.dot(A.T,y))

def f(a,b,x):
    return a*np.exp(-x) + b*np.exp(-2*x)

print("Coefficients: a and b")
print(f"a: {a:#.4}, b: {b:#.4}\n")
```

```
Coefficients: a and b
a: 0.07511, b: 2.157
```

2.3

[ ]:
```
print("Absolute error")
print([float(f"{abs(y[i]-f(a,b,x_point)):#.2}") for i,x_point in enumerate(x)])
```

```
Absolute error
[0.0077, 0.039, 0.06, 0.034, 0.01]
```

0.1.3 3 Convergence of Newton's method for double root

Question 1

```
[ ]: def newton(starting_point, nmax, f, f_div):
         x = starting_point
         xs = [x]
         for _ in range(nmax):
             fx = f(x)
             fp = f_div(x)
             x = x - (fx / fp)
             xs.append(x)
         return xs

     f = lambda x: (x-2)**2 * (x-8)
     f_div = lambda x: (x-8)*(2*x-4) + (x-2)**2

     x_hat = newton(4.6, 12, f, f_div)
     error = [abs(2-x_point) for x_point in x_hat]
     quad_error = [abs(error[i]/(error[i-1]**2)) for i in range(1, len(error))]
```

```
[ ]: import pandas as pd
     quad_error.insert(0, "-")
     pd.DataFrame({"x_n": x_hat, "|e_n|": error, "|en/e(n-1)**2|": quad_error})
```

[ ]:

	x_n	|e_n|	|en/e(n-1)**2|
0	4.600000	2.600000	-
1	2.495238	0.495238	0.0732601
2	2.235956	0.235956	0.962061
3	2.115513	0.115513	2.07476
4	2.057184	0.057184	4.28562
5	2.028454	0.028454	8.70146
6	2.014193	0.014193	17.5304
7	2.007088	0.007088	35.1871
8	2.003542	0.003542	70.4999
9	2.001770	0.001770	141.125
10	2.000885	0.000885	282.375
11	2.000443	0.000443	564.876
12	2.000221	0.000221	1129.88

Question 2 First we define equation (4) from the assignment.

$$e_{n+1} = x_{n+1} - r = e_n - 2\frac{f(x_n)}{f'(x_n)}$$

Next we define the taylor series polynomiums.

$$f(x_n) = f(r) + f'(r)e_n + \frac{1}{2}f''(r)e_n^2 + \frac{1}{6} + f'''(\xi_n)e_n^3$$
$$f'(x_n) = f'(r)e_n + f''(r)e_n + \frac{1}{2} + f'''(\zeta)e_n^2$$

We'll reduce the above expressions using that $f(r)$ is a double root, and substitute them in to equation (4) from the assignment.

$$e_{n+1} = x_{n+1} - r = e_n - 2\frac{\frac{1}{2}f''(r)e_n^2 + \frac{1}{6} + f'''(\xi_n)e_n^3}{f''(r)e_n + \frac{1}{2} + f'''(\zeta)e_n^2} = e_n - \frac{f''(r)e_n^2 + \frac{1}{3} + f'''(\xi_n)e_n^3}{f''(r)e_n + \frac{1}{2} + f'''(\zeta)e_n^2}$$

Then we find a common denominator for e_n and the fraction.

$$e_{n+1} = \left(\frac{f''(r) + \frac{1}{2}f'''(\zeta_n)e_n - f''(r) + \frac{1}{3}f'''(\xi_n)e_n}{f''(r)e_n + \frac{1}{2}f'''(\zeta)e_n^2}\right)e_n^2 = \left(\frac{\frac{1}{2}f'''(\zeta_n)e_n - \frac{1}{3}f'''(\xi_n)e_n}{f''(r)e_n + \frac{1}{2}f'''(\zeta)e_n^2}\right)e_n^2$$

Which can be further reduced to:

$$e_{n+1} = \left(\frac{\frac{1}{2}f'''(\zeta_n) - \frac{1}{3}f'''(\xi_n)}{f''(r) + \frac{1}{2}f'''(\zeta)e_n}\right)e_n^2 = \left(\frac{3f'''(\zeta_n) - 2f'''(\xi_n)}{6f''(r) + 3f'''(\zeta)e_n}\right)e_n^2$$

Question 3 Now we impliment expression (3) from the assignment in to the newton function.

[ ]:
```
def newton(starting_point, nmax, f, f_div, m=2):
    x = starting_point
    xs = [x]
    for _ in range(nmax):
        fx = f(x)
        fp = f_div(x)
        x = x - m*(fx / fp)
        xs.append(x)
    return xs

f = lambda x: (x-2)**2 * (x-8)
f_div = lambda x: (x-8)*(2*x-4) + (x-2)**2

x_hat = newton(4.6, 5, f, f_div)
error = [abs(2-x_point) for x_point in x_hat]
quad_error = [abs(error[i]/(error[i-1]**2)) for i in range(1, len(error))]
```

[ ]:
```
quad_error.insert(0, "-")
pd.DataFrame({"x_n": x_hat, "|e_n|": error, "|en/e(n-1)**2|": quad_error})
```

[ ]:

	x_n	|e_n|	|en/e(n-1)**2|
0	4.600000	2.600000e+00	-
1	0.390476	1.609524e+00	0.238095

```
2  1.846061  1.539386e-01      0.0594228
3  1.998098  1.901576e-03      0.0802451
4  2.000000  3.011896e-07      0.0832937
5  2.000000  7.549517e-15      0.0832222
```

As you can see on the above table the convergence ratio converges on a constant $c = 0.083$. Hence the function must converge quadratically.

Question 4 We start by double and triple differentiating the function (2) from the assignment.

[ ]:
```
ddf = lambda x: 6*(x-4)
dddf = lambda c=0: 6
```

As we can see, the tripple derivative is a constant $c = 6$. This makes it possible for us to completely determine our convergence ratio. Furthermore we know, that the error converges on zero, as n goes towards infinity.

[ ]:
```
con_rat = lambda x: ((3*dddf()-2*dddf())/(6*ddf(x)))
con_rat(2)
```

[ ]:
```
-0.08333333333333333
```

The convergence ratio converges to $\frac{1}{12}$, which is relatively close to the value found in question 3.