1. Verify $\|Ax'-b\|_2^2 = \|Ax-b\|_2^2 + \|Ae\|_2^2 \geq \|Ax-b\|_2^2$ for $x' = x+e$ where x is the minimizer of $\|Ax-b\|_2^2 = f(x)$.

$\|Ax'-b\|_2^2 = (Ax'-b)^T(Ax'-b)$. Since $(u+v)^T = u^T + v^T \ \forall u,v \in \mathbb{R}^n$ (1)

$(Ax'-b)^T(Ax'-b) = ((Ax')^T - b^T)(Ax'-b) = (Ax')^T Ax' - (Ax')^T b - b^T Ax' - b^T b$.

Since the last 3 terms are scalar, $(Ax'-b)^T(Ax'-b) = (x')^T A^T A x' - 2(x')^T A^T b + b^2$

$(x+e)^T A^T A(x+e) - 2(x+e)^T A^T b + b^2 = (x^T + e^T) A^T A(x+e) - 2(x^T + e^T) A^T b + b^2$

$\Rightarrow x^T A^T A x + x^T A^T A e + e^T A^T A x + e^T A^T A e - 2x^T A^T b - 2e^T A^T b + b^2$

$= (x^T A^T A x - 2x^T A^T b + b^2) + x^T A^T A e + e^T A^T A x + (e^T A^T A e) - 2e^T A^T b$.

By (1), $(Ax-b)^T(Ax-b) + x^T A^T A e + e^T A^T A x + (Ae)^T(Ae) - 2e^T A^T b$

$= \|Ax-b\|_2^2 + \|Ae\|_2^2 + x^T A^T A e + e^T (A^T A x) - 2e^T A^T b$

From the normal equation $A^T A x = A^T b$, (2)

$\Rightarrow \|Ax-b\|_2^2 + \|Ae\|_2^2 + x^T A^T A e + e^T A^T b - 2e^T A^T b$

$= \|Ax-b\|_2^2 + \|Ae\|_2^2 + x^T A^T A e - e^T A^T b$. To simplify further, note that $x^T A^T A e = ((A^T A)^T x)^T e = (A^T A x)^T e$ since $A^T A$ is symmetric.

By (2), $(A^T A x)^T e = (A^T b)^T e = (e^T A^T b)^T$. However, $e^T A^T b$ is a scalar, so $e^T A^T b = (e^T A^T b)^T$. Thus we know that $\|Ax'-b\|_2^2 = \|Ax-b\|_2^2 + \|Ae\|_2^2$.

Since an $\ell 2$-norm must be positive and since $Ae = 0$ iff $e = 0$ when A has full column rank, $\|Ae\|_2^2 \geq 0$

Thus $\|Ax'-b\|_2^2 = \|Ax-b\|_2^2 + \|Ae\|_2^2 \geq \|Ax-b\|_2^2$. $\square$

2. Let $x, y \in \mathbb{R}^n$ with $x \neq y$ but $\|x\|_2 = \|y\|_2$. Prove the uniqueness of the reflector P where $Px = y$. (pg. 99)

Proof: Consider Householder reflectors P_1 and P_2 such that $P_1 x = y$ and $P_2 x = y$ but $P_1 \neq P_2$.
Let $P_1 = I_n - \beta v v^T$ where $\beta = \frac{2}{\|v\|_2^2}$ and $P_2 = I_n - \alpha u u^T$ where $\alpha = \frac{2}{\|u\|_2^2}$.

Thus $P_1 x = y \Rightarrow (I_n - \beta v v^T) x = y$ and $P_2 x = y \Rightarrow (I_n - \alpha u u^T) x = y$,
so $P_1 x = P_2 x \Rightarrow (I_n - \beta v v^T) x = (I_n - \alpha u u^T) x$
$\Rightarrow I_n x - \beta v v^T x = I_n x - \alpha u u^T x \Rightarrow \beta v v^T x = \alpha u u^T x$.
By definition of α and β, $\frac{2 v v^T x}{\|v\|_2^2} = \frac{2 u u^T x}{\|u\|_2^2}$.
Therefore $\frac{2 v v^T}{\|v\|_2^2} = \frac{2 u u^T}{\|u\|_2^2} \Rightarrow \beta v v^T = \alpha u u^T$, so $I_n - \beta v v^T = I_n - \alpha u u^T$
$\Rightarrow P_1 = P_2$
So uniqueness of reflector P is proven by contradiction.
$\Longrightarrow\Longleftarrow$

$\square$