6-3:

NOTE: I believe the sample code provided in lecture has errors on its final lines

```
W{1,2} = (1/15)*(16*W{2,1}(1:2:end)-W{1,1});
```

I believe (if this was the intention for WT) that the LHS is meant to be the final 6th order approximation while the W{} on the RHS were meant to WT

I have since worked out a suitable Python implementation that is both atomized into more functions and allows for testing for what I believe are 6^{th} order approximations on arbitrarily fine meshes. For instance, setting num_meshes = 3 finds the 6^{th} order solution on the mid mesh. Setting num_meshes = 4 finds the 6^{th} order solution on the fine mesh. Setting num_meshes = 5 finds the 6^{th} order solution on the super-fine mesh. The accuracy seems to be going up in all tests.

Below is for tests with num_meshes = {3,4,5,6,7}

Output:

6th order approximation of u on the mid mesh:

```
[0.        0.25643446 0.509017   0.7539905  0.98778525 1.20710678
 1.409017   1.59100652 1.75105652 1.88768833 2.        2.08768833
 2.15105652 2.19100652 2.209017   2.20710678 2.18778525 2.1539905
 2.109017   2.05643446 2.       ]
```

error (inf norm) of approximation :

7.860259554348659e-09

6th order approximation of u on the fine mesh:

```
[0.        0.1284591  0.25643447 0.38344536 0.50901699 0.63268343
 0.7539905  0.87249856 0.98778525 1.09944805 1.20710678 1.31040597
 1.40901699 1.50264016 1.59100652 1.67387953 1.75105652 1.82236992
 1.88768834 1.94691733 2.        2.04691733 2.08768834 2.12236992
 2.15105652 2.17387953 2.19100652 2.20264016 2.20901699 2.21040597
 2.20710678 2.19944805 2.18778525 2.17249856 2.1539905  2.13268343
 2.10901699 2.08344536 2.05643447 2.0284591  2.       ]
```

error (inf norm) of approximation :

1.2385381609192336e-10

6th order approximation of u on the super_fine mesh:

```
[0.        0.06425982 0.1284591  0.1925374  0.25643447 0.32009032
 0.38344536 0.44644045 0.50901699 0.57111706 0.63268343 0.69365974
 0.7539905  0.81362124 0.87249856 0.93057023 0.98778525 1.04409395
 1.09944805 1.15380075 1.20710678 1.25932251 1.31040597 1.36031693
 1.40901699 1.45646961 1.50264016 1.54749601 1.59100652 1.63314317
 1.67387953 1.71319134 1.75105652 1.78745524 1.82236992 1.85578528
 1.88768834 1.91806846 1.94691733 1.97422904 2.        2.02422904
```

2.04691733 2.06806846 2.08768834 2.10578528 2.12236992 2.13745524
2.15105652 2.16319134 2.17387953 2.18314317 2.19100652 2.19749601
2.20264016 2.20646961 2.20901699 2.21031693 2.21040597 2.20932251
2.20710678 2.20380075 2.19944805 2.19409395 2.18778525 2.18057023
2.17249856 2.16362124 2.1539905 2.14365974 2.13268343 2.12111706
2.10901699 2.09644045 2.08344536 2.07009032 2.05643447 2.0425374
2.0284591 2.01425982 2.]

error (inf norm) of approximation :
1.964206575166827e-12

6th order approximation of u on the super_super_fine mesh:
[0. 0.03213369 0.06425982 0.0963708 0.1284591 0.16051714
0.1925374 0.22451234 0.25643447 0.28829628 0.32009032 0.35180915
0.38344536 0.41499158 0.44644045 0.47778468 0.50901699 0.54013018
0.57111706 0.6019705 0.63268343 0.66324883 0.69365974 0.72390924
0.7539905 0.78389674 0.81362124 0.84315737 0.87249856 0.90163832
0.93057023 0.95928796 0.98778525 1.01605594 1.04409395 1.07189328
1.09944805 1.12675244 1.15380075 1.18058736 1.20710678 1.2333536
1.25932251 1.28500833 1.31040597 1.33551045 1.36031693 1.38482065
1.40901699 1.43290144 1.45646961 1.47971723 1.50264016 1.52523439
1.54749601 1.56942126 1.59100652 1.61224828 1.63314317 1.65368796
1.67387953 1.69371493 1.71319134 1.73230605 1.75105652 1.76944034
1.78745524 1.80509909 1.82236992 1.83926588 1.85578528 1.87192657
1.88768834 1.90306934 1.91806846 1.93268473 1.94691733 1.96076561
1.97422904 1.98730724 2. 2.01230724 2.02422904 2.03576561
2.04691733 2.05768473 2.06806846 2.07806934 2.08768834 2.09692657
2.10578528 2.11426588 2.12236992 2.13009909 2.13745524 2.14444034
2.15105652 2.15730605 2.16319134 2.16871493 2.17387953 2.17868796
2.18314317 2.18724828 2.19100652 2.19442126 2.19749601 2.20023439
2.20264016 2.20471723 2.20646961 2.20790144 2.20901699 2.20982065
2.21031693 2.21051045 2.21040597 2.21000833 2.20932251 2.2083536
2.20710678 2.20558736 2.20380075 2.20175244 2.19944805 2.19689328
2.19409395 2.19105594 2.18778525 2.18428796 2.18057023 2.17663832
2.17249856 2.16815737 2.16362124 2.15889674 2.1539905 2.14890924
2.14365974 2.13824883 2.13268343 2.1269705 2.12111706 2.11513018
2.10901699 2.10278468 2.09644045 2.08999158 2.08344536 2.07680915
2.07009032 2.06329628 2.05643447 2.04951234 2.0425374 2.03551714
2.0284591 2.0213708 2.01425982 2.00713369 2.]

error (inf norm) of approximation :
1.8607337892717624e-13

6th order approximation of u on the super_super_super_fine mesh:
[0. 0.01606732 0.03213369 0.04819817 0.06425982 0.08031767
0.0963708 0.11241826 0.1284591 0.14449237 0.16051714 0.17653246
0.1925374 0.208531 0.22451234 0.24048047 0.25643447 0.27237338
0.28829628 0.30420224 0.32009032 0.3359596 0.35180915 0.36763805

0.38344536 0.39923018 0.41499158 0.43072864 0.44644045 0.4621261
0.47778468 0.49341528 0.50901699 0.52458893 0.54013018 0.55563985
0.57111706 0.5865609 0.6019705 0.61734497 0.63268343 0.64798501
0.66324883 0.67847403 0.69365974 0.70880509 0.72390924 0.73897133
0.7539905 0.76896592 0.78389674 0.79878212 0.81362124 0.82841327
0.84315737 0.85785274 0.87249856 0.88709403 0.90163832 0.91613066
0.93057023 0.94495626 0.95928796 0.97356455 0.98778525 1.0019493
1.01605594 1.03010441 1.04409395 1.05802382 1.07189328 1.0857016
1.09944805 1.1131319 1.12675244 1.14030895 1.15380075 1.16722711
1.18058736 1.19388081 1.20710678 1.2202646 1.2333536 1.24637312
1.25932251 1.27220113 1.28500833 1.29774348 1.31040597 1.32299516
1.33551045 1.34795124 1.36031693 1.37260693 1.38482065 1.39695753
1.40901699 1.42099848 1.43290144 1.44472533 1.45646961 1.46813375
1.47971723 1.49121954 1.50264016 1.51397861 1.52523439 1.53640701
1.54749601 1.55850091 1.56942126 1.58025662 1.59100652 1.60167056
1.61224828 1.62273929 1.63314317 1.64345953 1.65368796 1.66382808
1.67387953 1.68384194 1.69371493 1.70349818 1.71319134 1.72279407
1.73230605 1.74172697 1.75105652 1.7602944 1.76944034 1.77849404
1.78745524 1.79632367 1.80509909 1.81378125 1.82236992 1.83086487
1.83926588 1.84757275 1.85578528 1.86390328 1.87192657 1.87985498
1.88768834 1.89542651 1.90306934 1.9106167 1.91806846 1.9254245
1.93268473 1.93984903 1.94691733 1.95388955 1.96076561 1.96754546
1.97422904 1.98081631 1.98730724 1.99370181 2. 2.00620181
2.01230724 2.01831631 2.02422904 2.03004546 2.03576561 2.04138955
2.04691733 2.05234903 2.05768473 2.0629245 2.06806846 2.0731167
2.07806934 2.08292651 2.08768834 2.09235498 2.09692657 2.10140328
2.10578528 2.11007275 2.11426588 2.11836487 2.12236992 2.12628125
2.13009909 2.13382367 2.13745524 2.14099404 2.14444034 2.1477944
2.15105652 2.15422697 2.15730605 2.16029407 2.16319134 2.16599818
2.16871493 2.17134194 2.17387953 2.17632808 2.17868796 2.18095953
2.18314317 2.18523929 2.18724828 2.18917056 2.19100652 2.19275662
2.19442126 2.19600091 2.19749601 2.19890701 2.20023439 2.20147861
2.20264016 2.20371954 2.20471723 2.20563375 2.20646961 2.20722533
2.20790144 2.20849848 2.20901699 2.20945753 2.20982065 2.21010693
2.21031693 2.21045124 2.21051045 2.21049516 2.21040597 2.21024348
2.21000833 2.20970113 2.20932251 2.20887312 2.2083536 2.2077646
2.20710678 2.20638081 2.20558736 2.20472711 2.20380075 2.20280895
2.20175244 2.2006319 2.19944805 2.1982016 2.19689328 2.19552382
2.19409395 2.19260441 2.19105594 2.1894493 2.18778525 2.18606455
2.18428796 2.18245626 2.18057023 2.17863066 2.17663832 2.17459403
2.17249856 2.17035274 2.16815737 2.16591327 2.16362124 2.16128212
2.15889674 2.15646592 2.1539905 2.15147133 2.14890924 2.14630509
2.14365974 2.14097403 2.13824883 2.13548501 2.13268343 2.12984497
2.1269705 2.1240609 2.12111706 2.11813985 2.11513018 2.11208893
2.10901699 2.10591528 2.10278468 2.0996261 2.09644045 2.09322864
2.08999158 2.08673018 2.08344536 2.08013805 2.07680915 2.0734596
2.07009032 2.06670224 2.06329628 2.05987338 2.05643447 2.05298047
2.04951234 2.046031 2.0425374 2.03903246 2.03551714 2.03199237

2.0284591 2.02491826 2.0213708 2.01781767 2.01425982 2.01069817
2.00713369 2.00356732 2.   ]

error (inf norm) of approximation :
5.933031843596837e-13

Code:

```
import numpy as np
from scipy.sparse import diags

# 6.3 in HW 6

def get_A_b(interval: tuple, n: int, f, u) -> tuple:
    A = np.zeros((n+1,n+1))
    b = np.zeros(n+1)
    h = (interval[1]-interval[0])/n
    for i in range(1,n):
        A[i,i-1], A[i, i+1] = -1, -1
        A[i,i] = 2
        b[i] = (h**2)*f(interval[0]+i*h)
    A[0,0], A[n,n] = 1, 1
    b[0], b[n] = u(interval[0]), u(interval[1])
    return A, b

def get_W_expansion(W, f, h, n):
    W_hat = np.zeros(n+1)
    W_hat[0:(n+1):2] = W
    for k in range(1,n,2): # might be a problem with indices if this was supposed to be to n+1
        W_hat[k] = 0.5*(W_hat[k-1]+W_hat[k+1]) + ((h**2)/24)*(f((k-1)*h) + 10*f(k*h) + f((k+1)*h))
    return W_hat


# get sixth order solution up to arbitrarily fine meshes
def get_sixth_approx(W_hat, mesh_idx, n, u_exact):
    solution = ((1/15)*(16*W_hat[mesh_idx-1][:(n[mesh_idx]+1):2] - W_hat[mesh_idx-2]))
    error = np.linalg.norm(u_exact-solution, ord=np.inf)
    return solution, error

def get_mesh_names(num):
    name_list = ['coarse', 'mid', 'fine']
    if num > 3:
        for i in range(num-3):
            name_list.append(f'super_%s' % name_list[-1])
    return name_list
```

```
def Richardson_Extrapolation(u, f, interval, num_meshes):
    # partitioning into subintervals
    # number of subintervals for mesh i, with 10 initial points on BVP interval
    n = np.array([int(10*(2**i)) for i in range(num_meshes)])
    h = 1/n
    # generate spacial grid points for
    X = [np.linspace(interval[0], interval[1], n[i]+1) for i in range(num_meshes)]
    # preallocate u and approximations on the meshes
    U = [np.zeros(n[i]+1) for i in range(num_meshes)]
    V = [np.zeros(n[i]+1) for i in range(num_meshes)]
    W = [np.zeros(n[i]+1) for i in range(num_meshes)]
    W_hat = [np.zeros(n[i]+1) for i in range(1,num_meshes)]

    for i in range(num_meshes):
        U[i] = u(X[i]) # exact solutions to test error later
        A, b = get_A_b(interval, n[i], f, u) # get tridiagonal matrix and b = (k^2)*f_i where k =
[h, h/2, h/4]
        V[i] = np.linalg.solve(A,b) # solve for v_h, v_(h/2), and v_(h/4)
        # Perform Richardson Extrapolation
        if (i > 0) and (i < num_meshes):
            W[i-1] = (1/3)*(4*V[i][:len(V[i])+1:2]-V[i-1]) # 1 on algorithm 6.39
            W_hat[i-1] = get_W_expansion(W[i-1], f, h[i], n[i])
    return get_sixth_approx(W_hat, num_meshes-1, n, U[num_meshes-2])

# code below more heavily modified from lecture notes
if __name__ == '__main__':
    # BVP in ex. 6.3.9
    u = lambda x: np.sin(np.pi*x) + 2*x
    f = lambda x: (np.pi**2)*np.sin(np.pi*x)
    interval = [0,1]
    BVs = [0,2]

    ###########################################################################
    # mesh generation section

    for num_meshes in range(3,8):
        meshes = get_mesh_names(num_meshes)
        solution, error = Richardson_Extrapolation(u, f, interval, num_meshes)
        # solution, error = get_sixth_approx(W_hat, num_meshes-1, n, U[num_meshes-2])
        print(f'6th order approximation of u on the {meshes[num_meshes-2]} mesh: \n{solution}\n')
        print(f'error (inf norm) of approximation : \n{error}')
```