# MA 8463 - Homework 2

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2. Performing a sample matrix multiplication $LL^T$ for $L \in \mathbb{R}^{4\times 4}$ is done below to work out a more general formula for $L \in \mathbb{R}^{n \times n}$

$$A = LL^T =$$

$$\begin{bmatrix} l_{11}^2 & l_{21}l_{11} & l_{31}l_{11} & l_{11}l_{41} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{31}l_{21} + l_{31}l_{22} & l_{21}l_{41} + l_{22}l_{42} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 & l_{31}l_{41} + l_{32}l_{42} + l_{33}l_{43} \\ l_{11}l_{41} & l_{21}l_{41} + l_{22}l_{42} & l_{31}l_{41} + l_{32}l_{42} + l_{33}l_{43} & l_{41}^2 + l_{42}^2 + l_{43}^2 + l_{44}^2 \end{bmatrix}$$

$a_{11} = l_{11}^2$

$a_{22} = l_{21}^2 + l_{22}^2$

$a_{33} = l_{31}^2 + l_{32}^2 + l_{33}^2$

$a_{44} = l_{41}^2 + l_{42}^2 + l_{43}^2 + l_{44}^2$

$a_{12} = a_{21} = l_{21}l_{11}$

$a_{13} = a_{31} = l_{31}l_{11}$

$a_{23} = a_{32} = l_{31}l_{21} + l_{32}l_{22}$

$\dots$

For elements of A outside the main diagonal, the formula for $i > j$ is $a_{ij} = \sum_{k=1}^{j} l_{ik}l_{jk} \Rightarrow a_{ij} = l_{ij}l_{jj} + \sum_{k=1}^{j-1} l_{ik}l_{jk}$. For elements on the diagonal, where $i=j$, $a_{ii} = \sum_{k=1}^{i} l_{ik}^2$. For a single formula, note that when $i=j$,

$$a_{ij} = a_{ii} = l_{ii}l_{ii} + \sum_{k=1}^{i-1} l_{ik}l_{ik} \Rightarrow a_{ii} = l_{ii}^2 + \sum_{k=1}^{i-1} l_{ik}^2.$$

For elements of L, note $l_{ii}^2 = a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2 \Rightarrow l_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2}$. For $i > j$,

$$l_{ij} = \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik}l_{jk}\right) / l_{jj}.$$

Since only lower triangular elements need to be found, we consider only cases $i > j$ and $i = j$, leading to the algorithm on the right.

**Cholesky Factorization**

for $j = 1:n$

$\quad l_{jj} = \left(a_{jj} - \sum_{k=1}^{j-1} l_{jk}^2\right)^{\frac{1}{2}}$ for $j \in [2, n]$

$\quad$ for $i = (j+1):n$

$\quad\quad l_{ij} = \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik}l_{jk}\right) / l_{jj}$

3. Let $L = [l_{ij}]$ and $M = [m_{ij}]$ be lower-triangular matrices.

a. Prove that $LM$ is lower triangular.

Let $L, M \in \mathbb{R}^{n \times n}$ (judging by details in part b's prompt)

Note that for lower triangular matrices $L$ and $M$, $l_{ij} = m_{ij} = 0$ for $i<j$.

Let $A = LM$ so that $a_{ij} = \sum_{k=1}^{n} l_{ik} m_{kj}$.

$$a_{ij} = \sum_{k=1}^{i} l_{ik} m_{kj} + \sum_{k=i+1}^{n} l_{ik} m_{kj}$$

Since $l_{ik} m_{kj} = 0$ if $i<k$ or $k<j$, consider for $i<j$,

$a_{ij} = \sum_{k=1}^{i} l_{ik} m_{kj} = 0$ since $m_{kj} = 0$ for $k = 1, 2, \ldots, i$ and

$a_{ij} = \sum_{k=i+1}^{n} l_{ik} m_{kj} = 0$ since $l_{ik} = 0$ for $i<k$, which is necessarily true for $k = i+1, \ldots, n$.

Thus, while $i<j$, $a_{ij} = 0$ and $A$ must be lower triangular. $\square$

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b. Prove that the entries of the main diagonal of $LM$ are $l_{11}m_{11}, l_{22}m_{22}, \ldots, l_{nn}m_{nn}$

Let $A = LM$ so that $a_{ij} = \sum_{k=1}^{n} l_{ik} m_{kj}$. Concerned only with the main diagonal of $A$ where $i=j$, we examine $a_{ii} = \sum_{k=1}^{n} l_{ik} m_{ki}$. Since $l_{ik} = 0$ for $i<k$ and $m_{ki} = 0$ for $i>k$, the only nonzero elements occur when $i=k$. Thus $a_{ii} = \sum_{k=1}^{n} l_{ik} m_{ki} = l_{ii} m_{ii}$ $\forall i$.

$\square$

4. $Ly_i = e_i$ for $i \in [1,n] \subseteq \mathbb{N}$

NTS $A^{-1}$ can be computed in $2n^3 + O(n^2)$ flops

$L^{-1} = [y_1, \dots, y_n] \Rightarrow LL^{-1} = [e_1, \dots, e_n] \Rightarrow LL^{-1} = L[y_1, \dots, y_n] = [Ly_1, \dots, Ly_n]$

For $Ax = b$, solve $x = A^{-1}b$

Let $M = A^{-1} \Rightarrow AM = I$

$Am_i = e_i$ for $i = 1, 2, \dots, n$

$PAm_i = Pe_i \Rightarrow LUm_i = Pe_i$

Let $y_i = Um_i$, so that $Ly_i = Pe_i$

To compute columns of $A^{-1}$, $m_i$, $Ly_i = Pe_i$ must first be computed. Since $L$ is lower triangular, $L^{-1} = [y_1, y_2, \dots, y_n]$ is also lower triangular. Only the lower triangular portion of $L$ needs to be utilized using for solving. Furthermore, in using forward substitution, since there will only be a single nonzero element in $Pe_i$, all elements $y_{ij}$ of $y_i$ can be assumed to be $0$ until the first nonzero element in $Pe_i$ is encountered, in which case $l_{ij} \cdot y_{ij} = 1$. All rows of $L$ prior to encountering this $j^{th}$ element may be skipped in the forward substitution algorithm $y_i = Pe_i - \sum_{k=1}^{i-1} l_{ik} y_{ik}$, becoming $y_i = Pe_i - \sum_{k=j}^{i-1} l_{ik} y_{ik}$, resulting in a total savings of $\frac{2}{3}n^3$ flops over $n$ iterations of $i$. Thus considering the original $\frac{8}{3}n^3 + O(n^2)$ flops needed to find $A^{-1}$ using this method, $A^{-1}$ can now be found in $2n^3 + O(n^2)$ flops.

5. Use LU decomposition with partial pivoting to solve $Ax=b$ for

$$A=\begin{bmatrix} 1 & -2 & -1 & 3 \\ 1 & -2 & 0 & 1 \\ -3 & -2 & 1 & 7 \\ 0 & -2 & 8 & 5 \end{bmatrix} \text{ and } b=\begin{bmatrix} -12 \\ -5 \\ -14 \\ -7 \end{bmatrix}$$

$Ax=b$

$PAx=Pb$, $PA=LU$

$LUx=Pb$

Let $y=Ux$ so we may solve new system

$Ly=Pb$

$Ux=y$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} -3 & -2 & 1 & 7 \\ 1 & -2 & 0 & 1 \\ 1 & -2 & -1 & 3 \\ 0 & -2 & 8 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} -3 & -2 & 1 & 7 \\ 0 & -\frac{8}{3} & \frac{1}{3} & \frac{10}{3} \\ 0 & -\frac{8}{3} & -\frac{2}{3} & \frac{16}{3} \\ 0 & -2 & 8 & 5 \end{bmatrix} \rightarrow$$

$R_2 \leftarrow \frac{1}{3}R_1 + R_2$

$R_3 \leftarrow \frac{1}{3}R_1 + R_3$

$R_3 \leftarrow -R_2 + R_3$

$R_4 \leftarrow -\frac{3}{4}R_2 + R_4$

$$\begin{bmatrix} -3 & -2 & 1 & 7 \\ 0 & -\frac{8}{3} & \frac{1}{3} & \frac{10}{3} \\ 0 & 0 & -1 & 2 \\ 0 & 0 & \frac{17}{2} & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -3 & -2 & 1 & 7 \\ 0 & -\frac{8}{3} & \frac{1}{3} & \frac{10}{3} \\ 0 & 0 & \frac{17}{2} & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} -3 & -2 & 1 & 7 \\ 0 & -\frac{8}{3} & \frac{1}{3} & \frac{10}{3} \\ 0 & 0 & \frac{17}{2} & 1 \\ 0 & 0 & 0 & \frac{36}{17} \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$R_4 \leftarrow \frac{2}{17}R_3 + R_4$

$$P_1=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$P_2 = P_1$

$$P_3=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$P=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad L=\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 1 & 0 \\ 0 & \frac{3}{4} & -\frac{2}{17} & 1 \end{bmatrix}, \quad U=\begin{bmatrix} -3 & -2 & 1 & 7 \\ 0 & -\frac{8}{3} & \frac{1}{3} & \frac{10}{3} \\ 0 & 0 & \frac{17}{2} & 1 \\ 0 & 0 & 0 & \frac{36}{17} \end{bmatrix}$$

$Ly=Pb$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -14 \\ -\frac{1}{3} & 1 & 0 & 0 & -5 \\ -\frac{1}{3} & 1 & 1 & 0 & -7 \\ 0 & \frac{3}{4} & -\frac{2}{17} & 1 & -12 \end{array}\right] \Rightarrow y = \begin{bmatrix} -14 \\ -\frac{29}{3} \\ -2 \\ -\frac{339}{68} \end{bmatrix}$$

$Ux=y$

$$\left[\begin{array}{cccc|c} -3 & -2 & 1 & 7 & -14 \\ 0 & -\frac{8}{3} & \frac{1}{3} & \frac{10}{3} & -\frac{29}{3} \\ 0 & 0 & \frac{17}{2} & 1 & -2 \\ 0 & 0 & 0 & \frac{36}{17} & -\frac{339}{68} \end{array}\right] \Rightarrow x = \begin{bmatrix} -61/48 \\ 11/16 \\ 1/24 \\ -\frac{113}{48} \end{bmatrix}$$

6. Let $A$ be a nonsingular symmetric matrix with factorization $A = LDM^T$ where $L$ and $M$ are unit lower triangular matrices and $D$ is a diagonal matrix. Show that $L = M$.

$$A = LDM^T$$

$$\Rightarrow A(M^T)^{-1} = LD \qquad (1)$$

$$\Rightarrow M^{-1}A(M^T)^{-1} = M^{-1}LD \qquad (2)$$

Observing the transpose of $M^{-1}A(M^T)^{-1}$, we see that

$$\left(M^{-1}A(M^T)^{-1}\right)^T = \left(M^{-1}\left(A(M^{-1})^T\right)\right)^T = \left(A(M^{-1})^T\right)^T (M^{-1})^T = M^{-1}A^T(M^{-1})^T$$

Since $A$ is symmetric so that $A = A^T$, $M^{-1}A^T(M^{-1})^T = M^{-1}A(M^T)^{-1}$.

Thus $\left(M^{-1}A(M^T)^{-1}\right)^T = M^{-1}A(M^T)^{-1}$, so $M^{-1}A(M^T)^{-1}$ is symmetric.

This implies that $M^{-1}LD$ is also symmetric from equation (2).

Since the inverse of a lower triangular matrix is lower triangular, $M^{-1}$ is lower triangular. Given that the product of two lower triangular matrices is lower triangular, $LD$ is lower triangular. By the same reasoning, $M^{-1}(LD)$ is also lower triangular. Since $M^{-1}LD$ is both symmetric and lower triangular, $M^{-1}LD$ must be diagonal.

Since $M^{-1}$ and $L$ are both unit lower triangular, $M^{-1}L$ is unit lower triangular with $(M^{-1}L)_{ij} = 1$ for $i=j$, So $(M^{-1}LD)_{ij} = D_{ij}$ for $i=j$.

Because $M^{-1}LD$ is diagonal with the same main diagonal as $D$, $M^{-1}LD = D$. This implies that $M^{-1}L = I$, which will only be the case when $L = M$. $\square$