MAT 417 - HW8

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1

Solve the following IBVP.

$$\begin{cases} u_{tt} = u_{xx}, \quad 0 < x < \pi, \quad t > 0 \\ u(x,0) = 3sin(x), \quad 0 < x < \pi \\ u_t(x,0) = 0, \quad 0 < x < \pi \\ u(0,t) = u(\pi,t) = 0, \quad t > 0 \end{cases}$$

For a finite vibrating string, we use the D'Alembert solution with transformations $x+t$ and $x-t$, with respect to the reflexive property of the line $x+t$ about $x=0$. So we define

$$f(\bar{x}) = \begin{cases} f(x), \quad 0 < x < \pi, \\ -f(-x), \quad -\pi < x < 0 \end{cases} \quad \text{and} \quad g(\bar{x}) = \begin{cases} g(x), \quad 0 < x < \pi, \\ -g(-x), \quad -\pi < x < 0 \end{cases}$$

with $u(x,0) = f(x) = 3sin(x)$ and $u_t(x,0) = g(x) = 0$. Since g(x) = 0, the solution simplifies to

$$u(x,t) = \frac{\bar{f}(x+t) + \bar{f}(x-t)}{2}$$

We know that for $x-t < 0$, $x < t$ and for $x+t < 0$, $x < -t \Rightarrow t < x$. Converting back to our original notation, we then have

$$u(x,t) = \frac{3sin(x+t) + 3sin(x-t)}{2} = 3sin(x)cos(t), \quad 0 < t < x$$

and

$$u(x,t) = \frac{3sin(x+t) - 3sin(x-t)}{2} = 3sin(t)cos(x), \quad 0 < x < t$$

by sum and difference identities of sine and by the D'Alembert solution for finite strings.

2

What is the solution of the vibrating-string problem (20.1) if the ICs are changed to the following? What does the graph of the solution look like for various values of time?

$$\begin{cases} u_{tt} = \alpha^2 u_{xx}, \quad 0 < x < L, \quad 0 < t < \infty \\ u(0,t) = u(L,t) = 0, \quad 0 < t < \infty \\ u(x,0) = 0, \quad 0 \le x \le L \\ u_t(x,0) = sin(\frac{3\pi x}{L}), \quad 0 \le x \le L \end{cases}$$

To solve by separation of variables, we express our solution in terms of the shapes of the wave $X_n(x)$,the vibration times $T_n(t)$, and the coefficients C_n that satisfy the ICs:

$$u(x,t) = \sum_{n=1}^{\infty} C_n X_n(x) T_n(t)$$

Since we may first consider just solutions to the PDE, we can ignore the C_n and substitute into the PDE to find the ODEs

$$\begin{cases} T''(t) - \alpha^2 \lambda^2 T(t) = 0 \\ X''(x) - \lambda^2 X(x) = 0 \end{cases}$$

This gives solutions for $X(x)$ as

$$X_n(x) = sin(\frac{n\pi x}{L}), \quad n \in N$$

and solutions for $T(t)$ as

$$T_n(t) = A_n cos(\frac{n\pi\alpha t}{L}) + B_n sin(\frac{n\pi\alpha t}{L}), \quad n \in N$$

So the solution has the general form

$$u(x,t) = \sum_{n=1}^{\infty} sin(\frac{n\pi x}{L})[A_n cos(\frac{n\pi\alpha t}{L}) + B_n sin(\frac{n\pi\alpha t}{L})] \tag{1}$$

Using our initial conditions,

$$u(x,0) = \sum_{n=1}^{\infty} A_n sin(\frac{n\pi x}{L}) = 0$$

meaning that A_n must necessarily be 0 for all $n \in N$ and

$$u_t(x,0) = \sum_{n=1}^{\infty} sin(\frac{n\pi x}{L})[B_n \frac{n\pi\alpha}{L}] = sin(\frac{3\pi x}{L})$$

By comparing the right and left sides like with A_n, we know that B_3 is logically the only nonzero term and has a value of $\frac{L}{3\pi\alpha}$ Thus we finally find a solution of the form

$$u(x,t) = \frac{L}{3\pi\alpha} sin(\frac{3\pi x}{L}) sin(\frac{3\pi\alpha t}{L})$$

From using the graphing tools on geogebra.org/3d and rewriting the expressions of the angles, I would say the graph has a period of $\frac{2\pi}{3}$ since it returns to the state of the ICs at those coefficients of t for various times.

3

Show that for $\lambda \geq 0$ in Figure 20.2. the solutions $u(x,t) = X(x)T(t)$ are either unbounded or 0.

First consider the simpler case, where $\lambda = 0$. In that case, we have the ODEs

$$\begin{cases} T(t) = At + B \\ X(x) = Cx + D \end{cases}$$

So in the case that either A is nonzero and T(t) is unbounded as $t \to \infty$ or C is nonzero and X(x) is unbounded as $x \to \infty$ or if they are both nonzero and both terms are unbounded, then $u(x,t) = X(x)T(t)$ is unbounded. In the case that both A and C are 0, Then u(x,t) is a constant, which is impossible under the given problem unless $u(x,t) = 0$, since the higher order derivatives of u(x,t) are all equivalent, violating the ICs.

In the case that $\lambda > 0$,

$$\begin{cases} T(t) = Ae^{(\alpha\beta)t} + Be^{-(\alpha\beta)t} \\ X(x) = Ce^{\beta x} + De^{-\beta x} \end{cases}$$

In T(t), if $A = 0$, then the other term increases without bound as $t \to \infty$ and vice versa. If both A and B are equal to 0, then $X(x)T(t) = 0$. Similarly, if C or D are equal to 0, the other term increases without bound as $x \to \infty$. If both C and D are 0, then $X(x)T(t) = 0$.

4

What is the solution of the following vibrating-string problem?

$$\begin{cases} u_{tt} = \alpha^2 u_{xx}, \quad 0 < x < L, \quad t > 0 \\ u(x,0) = sin(\frac{3\pi x}{L}), \quad 0 < x < L \\ u_t(x,0) = (\frac{3\pi\alpha}{L}) sin(\frac{3\pi x}{L}), \quad 0 < x < L \\ u(0,t) = u(L,t) = 0, \quad t > 0 \end{cases}$$

We can again use the general form of the solution from (1) where

$$u(x,t) = \sum_{n=1}^{\infty} sin(\frac{n\pi x}{L})[A_n cos(\frac{n\pi\alpha t}{L}) + B_n sin(\frac{n\pi\alpha t}{L})]$$

Again, using the ICs, we find that

$$u(x,0) = \sum_{n=1}^{\infty} A_n sin(\frac{n\pi x}{L}) = sin(\frac{3\pi x}{L}) \text{ and}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n(\frac{n\pi\alpha}{L})sin(\frac{n\pi x}{L}) = (\frac{3\pi\alpha}{L})sin(\frac{3\pi x}{L})$$

By comparison, we see that the only nonzero A_n term is necessarily $A_3 = 1$. Likewise, we know by comparison that the infinite sum is satisfied by the only nonzero being $B_3 = 1$. Thus,

$$u(x,t) = sin(\frac{3\pi x}{L})[cos(\frac{3\pi\alpha t}{L}) + sin(\frac{3\pi\alpha t}{L})]$$

5

A stretched string of unit length lies along the x-axis with ends fixed at (0,0) and (1,0). If the string is initially displaced into the curve $Asin^3(\pi x)$, where A is a small constant, and then let go from rest. Show that the subsequent displacements are given by

$$u(x,t) = \frac{3A}{4}sin(\pi x)cos(c\pi t) - \frac{A}{4}sin(3\pi x)cos(3c\pi t)$$

Since the string has ends fixed at (0,0) and (1,0), we have homogeneous BCs at $x = 0$ and $x = 1$. Since it's initially displaced (at $t = 0$) onto the given curve, we know that $u(x,0) = Asin^3(\pi x)$. Since it was initially at rest, then $u_t(x,0) = 0$ Using the general solution in (1) with these ICs of a string length 1 gives

$$\begin{cases} u(x,0) = \sum_{n=1}^{\infty} A_n sin(n\pi x) = Asin^3(\pi x) \\ u_t(x,0) = \sum_{n=1}^{\infty} B_n(nc\pi)sin(n\pi x) = 0 \end{cases}$$

Now if we rewrite the first IC using the formula

$$sin^3(x) = \frac{3sin(x) - sin(3x)}{4} \tag{2}$$

We can again use direct comparison to find A_n and B_n, we find that $B_n = 0$ $\forall n \in N$ and A_n's only nonzero terms are the terms which satisfy

$$\frac{3A}{4}sin(\pi x) - \frac{A}{4}sin(3\pi x).$$

These terms are $A_1 = \frac{3A}{4}$ and $A_3 = \frac{-A}{4}$. Thus substituting in the general solution from (1), we find that for $n = 1$ and $n = 3$, we have

$$u(x,t) = (\frac{3A}{4})sin(\pi x)cos(\pi ct) - (\frac{A}{4})sin(3\pi x)cos(3\pi ct)$$

6

What is the solution to the simply-supported beam problem (at both ends) with ICs

$$\begin{cases} u(x,0) = sin(\pi x), & 0 \le x \le 1 \\ u_t(x,0) = sin(\pi x) \end{cases}$$

The beam problem is given by

$$\begin{cases} u_{tt} = -u_{xxxx}, & 0 < x < 1, 0 < t < \infty \\ u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, & 0 < t < \infty \\ u(x,0) = u_t(x,0) = sin(\pi x) \end{cases}$$

Using the general solution,

$$u(x,t) = \sum_{n=1}^{\infty} (sin(n\pi x))[A_n sin(n^2\pi^2 t) + B_n cos(n^2\pi^2 t)] \tag{3}$$

with the first IC,

$$u(x,0) = \sum_{n=1}^{\infty} B_n sin(n\pi x) = sin(\pi x),$$

direct comparison shows that the only nonzero term of B_n is $B_1 = 1$. Using this and the time derivative of (3) gives

$$u_t(x,0) = \sum_{n=1}^{\infty} A_n(n^2\pi^2) sin(n\pi x) = sin(\pi x)$$

Direct comparison of each side of this IC shows that the only nonzero term of A_n is $A_1 = \frac{1}{\pi^2}$. Then substitution of these coefficients into the general solution in (3) produces the particular solution

$$u(x,t) = sin(\pi x)[\frac{1}{\pi^2} sin(\pi^2 t) + cos(\pi^2 t)]$$