MAT 417 - Extra Credit 2

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Problem 3

Use the results of problem 2 to find the general solution of the system.

$$\begin{cases} \frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} = 0 \\ \frac{\partial u_2}{\partial t} + 4\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} = 0 \end{cases}$$

HINT: First rewrite the system in matrix form $u_t + Au_x = \vec{0}$

In matrix form, the system is

$$\begin{bmatrix} \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial t} \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial u_1}{\partial x} \\ \frac{\partial u_2}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

from part 2:

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

The eigenvalues λ of A satisfy $det(A - \lambda\, I_2) = 0$ where I_2 is the 2-dimensional identity matrix.

$$det \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} = 0 \quad \Rightarrow \quad (1-\lambda)^2 - 4 = 0 \quad \Rightarrow \quad \lambda = -1, 3$$

Thus there are 2 cases for the eigenvectors $\vec{v}$ of this matrix satisfying $(\lambda\, I_2 - A) \cdot \vec{v} = \vec{0}$.

$$(3I_2 - A) \cdot \vec{v} = \vec{0} \quad \Rightarrow \quad \begin{bmatrix} 2 & -1 & 0 \\ -4 & 2 & 0 \end{bmatrix} \quad \Rightarrow \quad 2v_1 - v_2 = 0 \quad \Rightarrow \quad \vec{v} = C \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$(-1I_2 - A) \cdot \vec{v} = \vec{0} \quad \Rightarrow \quad \begin{bmatrix} -2 & -1 & 0 \\ -4 & -2 & 0 \end{bmatrix} \quad \Rightarrow \quad 2v_1 + v_2 = 0 \quad \Rightarrow \quad \vec{v} = D \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

For some scalars C and D.

Let $\vec{u} = B\vec{w}$ where B is the matrix of eigenvectors of A. Substituting into the form $u_t + Au_x = 0$,

$$B\vec{w_t} + AB\vec{wx} = \vec{0}$$

Then left multiplying by B^{-1} gives

$$\vec{w_t} + B^{-1}AB\vec{wx} = \vec{0}$$

It follows from the eigendecomposition of A, $A = B(I_2\, \cdot\, \lambda_n)B^{-1}$ that

$$B^{-1}AB = I_2\, \cdot\, \lambda_n = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & -0.25 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

Thus we have

$$\vec{w_t} + \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \vec{wx} = \vec{0}$$

In the form of our system of PDEs,

$$\begin{cases} \frac{\partial w_1}{\partial t} + 3\frac{\partial w_1}{\partial x} = 0 \\ \frac{\partial w_2}{\partial t} - \frac{\partial w_2}{\partial x} = 0 \end{cases} \qquad \Rightarrow \quad \begin{cases} \frac{\partial w_1}{\partial t} = -3\frac{\partial w_1}{\partial x} \\ \frac{\partial w_2}{\partial t} = \frac{\partial w_2}{\partial x} \end{cases}$$

Thus we have general solutions

$$\begin{cases} w_1(x,t) = f(x-3t) \\ w_2(x,t) = g(x+t) \end{cases}$$

where f and g are arbitrary univariable functions. Thus we may find $\vec{u}$ from $\vec{u} = B\vec{w}$.

$$\vec{u} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} f(x-3t) \\ g(x+t) \end{bmatrix}$$

Finally, we know our general solution is of the form

$$\begin{cases} u_1 = f(x-3t) + g(x+t) \\ u_2 = 2f(x-3t) - 2g(x+t) \end{cases}$$