MAT 417 - HW9

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1 Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} + sinh(x), & 0 < x < 1, \quad t > 0 \\ u(x,0) = u_t(x,0) = 0, & 0 < x < 1 \\ u(0,t) = 3, & t > 0 \\ u(1,t) = 6, & t > 0 \end{cases}$$

Note: I realize the following steps are more obfuscated than necessary, but since I see that my work in a previous attempt is still valid, I chose to continue with the same general method.
Let $u(x,t) = v(x,t) + w(x,t)$ so that we obtain the IBVPs

$$\begin{cases} v_{tt} = c^2 v_{xx} + sinh(x), & 0 < x < 1, \quad t > 0 \\ v(x,0) = v_t(x,0) = 0, & 0 < x < 1 \\ v(0,t) = v(1,t) = 0 \end{cases} \quad \text{and} \quad \begin{cases} w_{tt} = c^2 w_{xx}, & 0 < x < 1, \quad t > 0 \\ w(x,0) = w_t(x,0) = 0, & 0 < x < 1, \\ w(0,t) = 3 \\ w(1,t) = 6 \end{cases}$$

Using my previous work on the Laplace transform for the first IBVP,

$$c^2 \mathcal{L}(v_{xx}) = \mathcal{L}(v_{tt}) - \mathcal{L}(sinh(x)) \quad \Rightarrow \quad c^2 \frac{d^2}{dx^2} V(x,s) = s^2 V(x,s) - \cancel{sv(x,0) - v_t(x,0)} - \frac{sinh(x)}{s}$$

$$\Rightarrow c^2 \frac{d^2}{dx^2} V(x,s) = s^2 V(x,s) - \frac{sinh(x)}{s}$$

Breaking this down further into a sum of the general solution of the homogeneous problem, $V_h(x,s)$ and a particular solution of the nonhomogeneous problem, $V_p(x,s)$ so that

$$\begin{cases} V_h(x,s) = C_1 e^{sx} + C_2 e^{-sx} \\ V_p(x,s) = Acos(\pi x) + Bsin(\pi x) \end{cases}$$

To find $V_p(x,s)$, note that

$$c^2 \frac{d^2}{dx^2} V_p(x,s) - s^2 V_p(x,s) = (-c^2\pi^2 - s^2)[Acos(\pi x) + Bsin\pi x] = -\frac{sinh(x)}{s}$$

Thus,

$$V_p(x,s) = Acos(\pi x) + Bsin(\pi x) = \frac{sinh(x)}{s(c^2\pi^2 + s^2)}$$

Applying the BCs to $V_h(x,s)$ to find C_1 and C_2, we get

$$V_h(0,s) = C_1 + C_2 = 0 \quad \text{and} \quad V_h(1,s) = C_1 e^s + C_2 e^{-s} = 0,$$

implying that $C_1 = C_2 = 0$ and thus $V_h(x,s) = 0$. So we then know that

$$V(x,s) = \frac{sinh(x)}{s(c^2\pi^2 + s^2)}$$

where applying the inverse Laplace transform gives

$$v(x,t) = \mathcal{L}^{-1}(V(x,s)) = \mathcal{L}^{-1}(\frac{sinh(x)}{s(c^2\pi^2+s^2)}) = \frac{sinh(x)[1-cos(\pi ct)]}{(c\pi)^2}$$

Now, to solve the IBVP of $w(x,t)$, let $w(x,t) = S(x,t) + 3 + 3x$ Thus, we have the new IBVP,

$$\begin{cases} S_{tt} = c^2 S_{xx} \\ S(x,0) = -3-3x \\ S_t(x,0) = 0 \\ S(0,t) = S(1,t) = 0 \end{cases}$$

Using the general solution for the wave equation with a length of 1,

$$S(x,t) = \sum_{n=1}^{\infty} sin(n\pi x)[A_n cos(n\pi ct) + B_n sin(n\pi ct)],$$

we can use our ICs to find A_n and B_n.

$$S_t(x,0) = \sum_{n=1}^{\infty} B_n(n\pi x) sin(n\pi x) = 0,$$

So $B_n = 0$, $\forall n$. The first IC then gives

$$S(x,0) = \sum_{n=1}^{\infty} A_n sin(n\pi x) = -3-3x$$

The Fourier sine series representation of A_n is then

$$A_n = 6\int_0^1 (-1-x)\, sin\,(n\pi x)\, dx \quad \Rightarrow \quad \frac{6\,(2\,(-1)^n - 1)}{\pi n}$$

Thus we have

$$S(x,t) = \sum_{n=1}^{\infty} \frac{(12\,(-1)^n - 6)}{\pi n}[sin(n\pi x)cos(n\pi ct)] \qquad \text{so}$$

$$w(x,t) = \sum_{n=1}^{\infty} \frac{(12\,(-1)^n - 6)}{\pi n}[sin(n\pi x)cos(n\pi ct)] + 3 + 3x$$

Finally, putting our first decomposition back together, we have our solution.

$$u(x,t) = \sum_{n=1}^{\infty} \frac{(12\,(-1)^n - 6)}{\pi n}[sin(n\pi x)cos(n\pi ct)] + 3 + 3x + \frac{sinh(x)[1-cos(\pi ct)]}{(c\pi)^2}$$

2 Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} + x^2, \quad 0 < x < 1, t > 0 \\ u(x,0) = x, \quad 0 < x < 1 \\ u_t(x,0) = 0, \quad 0 < x < 1 \\ u(0,t) = 0, \quad t > 0 \\ u(1,t) = 1, \quad t > 0 \quad \text{assuming } u(\pi,t) \text{ was a typo} \end{cases}$$

First, we use the superposition principle to let $u(x,t) = v(x,t) + W(x)$ and express the original IBVP as

$$\begin{cases} v_{tt} = c^2(v_{xx} + W''(x)) + x^2 \\ u(x,0) = v(x,0) + W(x) = x \\ u_t(x,0) = v_t(x,0) = 0 \\ u(0,t) = v(0,t) + W(0) = 0 \\ u(1,t) = v(1,t) + W(1) = 1 \end{cases}$$

To make $v(x,t)$ the homogeneous solution of $u(x,t)$, let $W(x)$ satisfy

$$c^2W''(x) + x^2 = 0 \quad \Rightarrow \quad W''(x) = -\frac{x^2}{c^2}$$

For this to be true, $W(x)$ must also satisfy $W(0) = 0$ and $W(1) = 1$. Then, integrating the above equation w.r.t. gives

$$\int W''(x)dx = \int -\frac{x^2}{c^2}dx \quad \Rightarrow \quad \int W'(x)dx = \int (-\frac{x^3}{3c^2} + C_1)dx \quad \Rightarrow \quad W(x)dx = -\frac{x^4}{12c^2} + C_1x + C_2$$

Using our initial BCs described above, we have the particular equation for $W(x)$,

$$W(x) = -\frac{x^4}{12c^2} + (1 + \frac{1}{12c^2})x$$

Now $v(x,t)$ can be expressed as the homogeneous problem,

$$\begin{cases} v_{tt} = c^2v_{xx} \\ v(x,0) = x - W(x) = -\frac{x^4}{12c^2} + \frac{1}{12c^2}x \\ v_t(x,0) = 0 \\ v(0,t) = 0 \\ v(1,t) = 0 \end{cases}$$

Using the general solution of the wave equation with a length of 1,

$$v(x,t) = \sum_{n=1}^{\infty} sin(n\pi x)[A_n cos(n\pi ct) + B_n sin(n\pi ct)]$$

Using the ICs to find A_n and B_n, we find

$$v(x,t) = \sum_{n=1}^{\infty} B_n(n\pi c) sin(n\pi x) = 0$$

Thus $B_n = 0 \quad \forall\, n \in N$. Then using the first IC,

$$v(x,0) = \sum_{n=1}^{\infty} A_n\, sin(n\pi x) = -\frac{x^4}{12c^2} + \frac{1}{12c^2}x$$

By the Fourier sine series, we then know that

$$A_n = (-\frac{x^4}{6c^2} + \frac{1}{6c^2}x) \int_0^1 (x^4 - x)\, sin(n\pi x)\, dx = \frac{2(-1)^{n+1}[2 - (n\pi)^2] + 4}{c^2\pi^5n^5}$$

Thus we have that

$$v(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}[2 - (n\pi)^2] + 4}{c^2\pi^5n^5}\, sin(n\pi x)\, cos(n\pi ct)$$

Putting this together with our solution to $W(x)$, we have the solution

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}[2 - (n\pi)^2] + 4}{c^2\pi^5n^5}\, sin(n\pi x)\, cos(n\pi ct) - \frac{x^4}{12c^2} + (1 + \frac{1}{12c^2})x$$

3 Reduce the following equation to canonical form: $6u_{xx} - u_{xy} + u = y^2$

From the general form of a second-order linear PDE, we have coefficients A = 6, B = -1, C = 0, D = 0, E = 0, F = 1, G = y^2
Since the discriminant is greater than 0, the PDE is hyperbolic.

$$B^2 - 4AC = 1 \quad \text{so} \quad B^2 - 4AC > 0$$

From the general canonical form of a general second-order linear PDE, we find ξ and η such that

$$\begin{cases} \frac{\xi_x}{\xi_y} = \frac{-B+\sqrt{B^2-4AC}}{2A} = \frac{1}{6} \\ \frac{\eta_x}{\eta_y} = \frac{-B-\sqrt{B^2-4AC}}{2A} = 0 \end{cases}$$

From chain rule, we know that

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}, \quad \frac{dy}{dx} = -\frac{\eta_x}{\eta_y}$$

So separation of variables of these ODEs gives

$$\begin{cases} y + \frac{1}{6}x = C_1 \\ y = C_2 \end{cases} \quad \text{so} \quad \begin{cases} \xi(x,y) = y + \frac{1}{6}x \\ \eta(x,y) = y \end{cases}$$

Thus we may find the new coefficients of the canonical form under this change of variables.

$$\overline{A} = 0, \quad \overline{B} = -\frac{1}{6}, \quad \overline{C} = 0, \quad \overline{D} = 0, \quad \overline{E} = 0, \quad \overline{F} = 1, \quad \overline{G} = y^2$$

Now we can write the PDE in its canonical form in terms of ξ and η, with $u(x,y) = v(\xi(x,y), \eta(x,y))$

$$-\frac{1}{6}v_{\xi\eta} + v = \eta^2$$

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Find the new characteristics coordinates for $u_{xx} + 4u_{xy} = 0$. Solve the transformed equation in the new coordinate system and the transform back to the original problem.

From the general form of a second-order linear PDE, we have coefficients A = 1, B = 4, C = 0, D = 0, E = 0, F = 0, and G = 0. We also know that the PDE is hyperbolic since $B^2 - 4AC = 16 > 0$, so there are two real roots of the auxillary equation such that the chain rule process in Question 3 gives these roots.

$$\lambda^2 + 4\lambda = 0 \quad \Rightarrow \quad \lambda = 0, -4$$

Then separation of variables gives

$$\begin{cases} y = C_1 \\ y - 4x = C_2 \end{cases} \quad \text{with } \xi \text{ and } \eta \text{ equal to } C_1 \text{ and } C_2 \text{ respectively, so} \quad \begin{cases} \xi = y \\ \eta = y - 4x \end{cases}$$

Finding our new coefficients of the canonical form, we have

$$\overline{A} = 0, \quad \overline{B} = -16, \quad \overline{C} = 0, \quad \overline{D} = 0, \quad \overline{E} = 0, \quad \overline{F} = 0, \quad \overline{G} = 0$$

So we have the canonical form of the PDE, $-16v_{\xi\eta} = 0 \quad \Rightarrow \quad v_{\xi\eta} = 0$
where $u(x,y) = v(\xi(x,y), \eta(x,y))$
First, integrating w.r.t. ξ gives

$$\int v_{\xi\eta} d\xi = \int 0 \quad \Rightarrow \quad v_\eta = \phi(\eta)$$

for some function ϕ. Then integrating again w.r.t. η gives

$$\int v_\eta d\eta = \int \phi(\eta) d\eta \quad \Rightarrow \quad v(\xi, \eta) = f(\eta) + g(\xi) \quad \text{for arbitrary functions f and g}$$

Since we let $u(x, y) = v(\xi, \eta)$, we then know that the canonical form in terms of the original variables is

$$u(x, y) = f(y - 4x) + g(y)$$