USM - MAT 417 - Homework 7

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Consider the following wave equation:
$u_{tt} = 4u_{xx}$, $-\infty < x < \infty$, $t > 0$
$u(x,0)=\begin{cases} 1 - x^2, \ -1 \leq x \leq 1 \\ 0, otherwise \end{cases}$ and $u_t(x,0)=\begin{cases} 4, \ 1 \leq x \leq 2, \\ 0, otherwise \end{cases}$

(a) Find $u(0.5, 1)$, $u(-1, 2)$.

We first need to introduce a change of variable with c = 2,

$$\begin{cases} \xi = x + ct \\ \eta = x - ct \end{cases} \tag{1}$$

Then we use D'Alembert's solution,

$$u(\xi(x,t), \eta(x,t)) = \frac{1}{2}[f(\eta) + f(\xi)] + \frac{1}{2c}\int_{\eta}^{\xi} g(s)ds \tag{2}$$

where f is the IC $u(x,0)$ and g is the IC $u_t(x,0)$. and c=2 from our PDE.

Then if we're looking for $u(0.5, 1)$, we may use the first BC, $u(x,0) = f(x)$ as follows:
$f(\xi) = 0$ since $\xi(0.5, 1) = 2.5$ and is outside of the interval $-1 \leq x \leq 1$.
$f(\eta) = 0$ since $\eta(0.5, 1) = -1.5$ and is outside of the interval $-1 \leq x \leq 1$.

Thus our solution in (2) simplifies to

$$u(0.5,1) = \frac{1}{4}\int_{-1.5}^{2.5} g(s)ds$$

Then we need to use the second BC, $u_t(x,0) = g(x)$ to find our final solution. Since $g(s)$ is only nonzero on the subinterval $1 \le s \le 2$, we then find

$$u(0.5,1) = \frac{1}{4}\int_{-1.5}^{1} 0ds + \frac{1}{4}\int_{1}^{2} 4ds + \frac{1}{4}\int_{2}^{2.5} 0ds \Rightarrow u(0.5,1) = 1 + C$$

where C is an arbitrary constant

Repeating this process for $u(-1,2)$, we find that
$\xi(-1,2) = -1 + 2(2) = 3$ and $\eta(-1,2) = -1 - 2(2) = -5$
So (2) again reduces to just

$$u(-1,2) = \frac{1}{4}\int_{-5}^{3} g(s)ds = \frac{1}{4}\int_{1}^{2} 4ds + D = 1 + D$$

where D is an arbitrary constant

(b) find $\lim_{t\to\infty} \mathbf{u(5,t)}$

Now, using the same approach for $\lim_{t\to\infty} u(5,t)$, we note that
$\lim_{t\to\infty} \xi(5,t) = \lim_{t\to\infty} 5 + 2t$
$\lim_{t\to\infty} \eta(5,t) = \lim_{t\to\infty} 5 - 2t$
Since these values are both outside of the interval $-1 \le \xi, \eta \le 1$, we know $f(\xi) = 0$ and $f(\eta) = 0$.

Thus our solution reduces down to

$$\lim_{t\to\infty} u(5,t) = \lim_{t\to\infty} \frac{1}{4}\int_{5-2t}^{5+2t} g(s)ds$$

Again, since our bounds include the subinterval $1 \le x \le 2$ and is 0 elsewhere, we simplify this to

$$\lim_{t\to\infty} u(5,t) = \frac{1}{4}\int_{1}^{2} 4ds + C = 1 + C$$

where C is an arbitrary constant

2

Consider the following wave equation.
$u_{tt} = u_{xx}$, $-\infty < x < \infty$, $t > 0$

$$u(x,0) = \begin{cases} 0, x \in (-\infty, 1) \cup (1, \infty), \\ x+1, x \in [-1, 0], \\ 1-x, x \in [0,1] \end{cases} \quad \text{and } u_t(x,0) = \begin{cases} 0, x \in (-\infty, -1) \cup (1, \infty) \\ 1, x \in [-1, 1] \end{cases}$$

Find $u(1, 0.5)$ and $u(-1, 0.5)$.

First, we introduce the same change of variables as in 1.1 with t's coefficient equal to 1: $\begin{cases} \xi(x,t) = x+t \\ \eta(x,t) = x-t \end{cases}$

Then $\xi(1, 0.5) = 1.5$ and $\eta(1, 0.5) = 0.5$, so we know that $f(\xi(1, 0.5)) = 0$ and $f(\eta(1, 0.5)) = 1 - 0.5 = 0.5$.
Now, since g(x) is only nonzero on the interval $[-1, 1]$, which intersects the interval $(0.5, 1.5)$, using the intersection $(0.5, 1)$ we arrive at the following through D'Alembert's solution (in 1.2):

$$u(1, 0.5) = \frac{1}{4} + \frac{1}{2}\int_{0.5}^{1} ds + C = \frac{1}{2} + C$$

where C is an arbitrary constant.

Repeating the process for $u(-1, 0.5)$, we find $\xi(-1, 0.5) = -0.5$ and $\eta(-1, 0.5) = -1.5$. We then know that $f(\xi(-1, 0.5)) = -0.5 + 1 = 0.5$ and $f(\eta(1, 0.5)) = 0$.
Again, since g(x) is only nonzero on the interval $[-1, 1]$ and our bounds in the D'Alembert's solution will be $(-1.5, -0.5)$, we instead use their intersection, $(-1, -0.5)$ and our solution becomes:

$$u(-1, 0.5) = \frac{1}{4} + \frac{1}{2}\int_{-1}^{-0.5} ds + D = \frac{1}{2} + D$$

where D is an arbitrary constant.

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Solve the following wave equation

$$\begin{cases} u_{tt} = u_{xx}, & -\infty < x < \infty, t > 0 \\ u(x,0) = 0, & -\infty < x < \infty \\ u_t(x,0) = xe^{-x^2}, & -\infty < x < \infty \end{cases}$$

Using the change of variables defined in (1.1), we use $f(x) = u(x,0) = 0$ and $g(x) = u_t(x,0) = xe^{-x^2}$ to then find $f(\xi) = 0$ and $f(\eta) = 0$. Thus the remaining term in D'Alembert's solution is the integral

$$u(x,t) = \frac{1}{2c} \int_{\eta}^{\xi} g(s) ds \tag{3}$$

We know that $c = 1$ in this case and $g(s) = se^{-s^2}$ Thus,

$$u(x,t) = \frac{1}{2} \int_{x-t}^{x+t} se^{-s^2} = -\frac{1}{2}(e^{-x^2} e^{-t^2})(\frac{e^{2xt} - e^{-2xt}}{2}).$$

Letting $v = 2xt$, we use the fact that $sinh(v) = \frac{e^v - e^{-v}}{2}$ and get

$$\boxed{u(x,t) = \frac{-sinh(2xt)}{2e^{x^2+t^2}}}$$

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Evaluate $u(4,1)$ and $u(1,4)$ for the following problem.

$$\begin{cases} u_{tt} = u_{xx}, & 0 < x < \infty, t > 0 \\ u(x,0) = x^2, & 0 < x < \infty \\ u_t(x,0) = 6x, & 0 < x < \infty \\ u(0,t) = t^2, & t > 0. \end{cases}$$

To homogenize our initial condition, we may define a change of variables

$$u(x,t) = v(x,t) + w(x,t) \Rightarrow v(x,t) = u(x,t) - w(x,t) \tag{4}$$

Since we need w(x,t) to satisfy the PDE and the initial condition so that both are homogenized, let $w(x,t) = x^2 + t^2$. Thus the equivalent IBVP is

$$\begin{cases} v_{tt} = v_{xx}, \quad 0 < x < \infty, t > 0 \\ v(x,0) = 0, \quad 0 < x < \infty \\ v_t(x,0) = 6x \quad 0 < x < \infty \\ v(0,t) = 0, \quad t > 0. \end{cases} \tag{5}$$

Then, given that g(x) only exists where $x > 0$, D'Alembert's equation for v(x,t) using the change of variables in (1.1) reduces to

$$\begin{cases} v(x,t) = \frac{1}{2}\int_{t-x}^{x+t} g(s)ds, \quad 0 < x < t \\ v(x,t) = \frac{1}{2}\int_{x-t}^{x+t} g(s)ds, \quad x \geq t \end{cases} \Rightarrow \begin{cases} v(x,t) = 6tx, \quad 0 < x < t \\ v(x,t) = 6tx, \quad x \geq t \end{cases}$$

So finally, we know that $u(x,t) = x^2 + 6xt + t^2$
So $u(4,1) = 41$ **and** $u(1,4) = 41$

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Solve

$$\begin{cases} u_{tt} = u_{xx}, \quad 0 < x < \infty, t > 0 \\ u(x,0) = |sin(x)|, \quad 0 < x < \infty \\ u_t(x,0) = 0, \quad 0 < x < \infty \\ u(0,t) = 0, \quad t > 0. \end{cases}$$

Since $g(x) = 0$, we can focus only on the $f(x \pm t)$ terms. Using the modified D'Alembert solution for semi-infinite strings, we find

$$\begin{cases} u(x,t) = \frac{1}{2}(f(x+t) - f(t-x)) + \frac{1}{2}\int_{t-x}^{x+t} 0ds, \quad 0 < x < t \\ u(x,t) = \frac{1}{2}(f(x+t) + f(x-t) + \frac{1}{2}\int_{x-t}^{x+t} 0ds, \quad x \geq t \end{cases}$$

Thus our solution is

$$\begin{cases} u(x,t) = \frac{1}{2}(|sin(x+t)| - |sin(t-x)|) + C, \quad 0 < x < t \\ u(x,t) = \frac{1}{2}(|sin(x+t)| + |sin(x-t)|) + D, \quad x \geq t \end{cases}$$

where C and D are arbitrary constants.