MAT 417 - Lesson 22 - Extra Credit

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Transform the vibrating string problem (22.5) into dimensionless form (22.6) by means of the transformations

$$\xi = \frac{x}{L}, \qquad \tau = \frac{\alpha t}{L} \tag{1}$$

Equation 22.5:

$$\begin{array}{ll} \textbf{PDE} & u_{tt} = \alpha^2 u_{xx}, \quad 0 < x < L, \quad 0 < t < \infty \\ \textbf{BCs} & \begin{cases} u(0,t) = 0, \quad 0 < t < \infty \\ u(L,t) = 0 \end{cases} \\ \textbf{ICs} & \begin{cases} u(x,0) = sin(\frac{\pi x}{L}) + 0.5sin(\frac{3\pi x}{L}) \\ u_t(x,0) = 0, \quad 0 \le x \le L \end{cases} \end{array}$$

First, transforming the boundaries, we see that $0 < L\xi < L \Rightarrow 0 < \xi < 1$ in the PDE and $0 \le \xi \le 1$ in the IC.
Also, the time interval becomes $0 < \frac{L\tau}{\alpha} < \infty \quad \Rightarrow \quad 0 < \tau < \infty$ in the PDE.
The BCs become

$$u(0,t) = 0 \Rightarrow u(0, \frac{L\tau}{\alpha}) = 0 \Rightarrow u(0,\tau) = 0 \text{ since both are equal to 0.}$$

and

$$u(L,t) = 0 \Rightarrow u(1,\tau) = 0 \text{ since again, the BC is equal to 0.}$$

Similarly, the first IC may be transformed to

$$u(x,0) = sin(\frac{\pi x}{L}) + 0.5sin(\frac{3\pi x}{L}) \Rightarrow u(\xi,0) = sin(\pi\xi) + 0.5sin(3\pi\xi)$$

since we may substitute the $\frac{x}{L}$ part for ξ in the parameter list of each term

Since ξ is a function of x and τ is a function of t, and we want to transform $u(x,t)$ into dimensionless form $u(\xi(x),\tau(t))$, we know that the derivatives of the new problem can be found with the chain rule.

$$u_t = \frac{d\tau}{dt}u_\tau \Rightarrow u_t = (\frac{\alpha}{L})u_\tau \Rightarrow u_\tau = \frac{L}{\alpha}u_t$$

$$u_{tt} = \frac{\alpha}{L}\frac{\partial}{\partial t}(\frac{\partial u}{\partial \tau}) = \frac{\alpha}{L}(\frac{\partial^2 u}{\partial \tau^2}(\frac{d\tau}{dt})) = \frac{\alpha^2}{L^2}u_{\tau\tau} \quad \text{by substitution}$$

Then the derivative w.r.t x can be found in the same manner.

$$u_x = \frac{d\xi}{dx}u_\xi \Rightarrow u_x = (\frac{1}{L})u_\xi \Rightarrow u_\xi = \frac{u_x}{L}$$

$$u_{xx} = \frac{1}{L}\frac{\partial}{\partial x}(\frac{\partial u}{\partial \xi}) = \frac{1}{L}(\frac{\partial^2 u}{\partial \xi^2}(\frac{d\xi}{dx})) = \frac{1}{L^2}u_{\xi\xi}$$

Lastly, the second IC can now be found with u_t,

$$u_t(x,0) = 0 \Rightarrow u_t(x,0) = \frac{\alpha}{L}u_\tau(\xi,0) = 0 \Rightarrow u_\tau(\xi,0) = 0$$

Now substitutions produce the dimensionless form, (Equation 22.6)

PDE $\quad \mathrm{u}_{\tau\tau} = u_{\xi\xi}, \quad 0 < \xi < 1, \quad 0 < \tau < \infty$

BCs $\quad \begin{cases} u(0,\tau) = 0, \quad 0 < \tau < \infty \\ u(1,\tau) = 0 \end{cases}$

ICs $\quad \begin{cases} u(\xi,0) = sin(\pi\xi) + 0.5sin(3\pi\xi) \\ u_\tau(\xi,0) = 0, \quad 0 \le \xi \le 1 \end{cases}$

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Find the dimensionless formulation for the problem

PDE $\quad u_t = \alpha^2 u_{xx}, \quad 0 < x < L$

BCs $\quad \begin{cases} u(0,t) = T_1, & 0 < t < \infty \\ u(L,t) = 0 \end{cases}$

ICs $\quad u(x,0) = T_2, \quad 0 \le x \le L$

since u(x,t) is a linear, finite wave equation, using Lagrange interpolation on the boundary conditions to produce a change of variables.

$$w(x,t) = \frac{u(x,t) - T_1}{0 - T_1} \Rightarrow \frac{u(x,t) - T_1}{-T_1} \tag{2}$$

Since the given length is L, we can use $\xi = \frac{x}{L}$ for the dimensionless unit of length. Similarly, $\tau = \frac{\alpha^2 t}{L^2}$ for the dimensionless unit of time. Now the system is

Now by substitution, the BCs and IC become
$w(0,t) = 0 \Rightarrow w(0,\tau) = 0$
$w(L,t) = 1 \Rightarrow w(1,t) = 1$
$w(\xi, 0) = \frac{T_2 - T_1}{-T_1}$

$$w_x = \frac{u_x}{-T_1} \Rightarrow w_\xi = \frac{u_\xi}{-LT_1}$$
$$w_{xx} = \frac{u_{xx}}{-T_1} \Rightarrow w_{\xi\xi} = \frac{u_{\xi\xi}}{-L^2 T_1}$$
$$w_t = \frac{u_x}{-T_1} \Rightarrow w_\tau = \frac{L^2 u_\tau}{-\alpha^2 T_1}$$
$$w_{tt} = \frac{u_{tt}}{-T_1} \Rightarrow w_{\tau\tau} = \frac{L^2 u_{\tau\tau}}{-\alpha^2 T_1}$$

Letting the interval be defined by the change of variables, we have $0 < \xi < 1$ in the PDE and $0 \leq \xi \leq 1$ in the IC.
Also, the time interval becomes $0 < \tau < \infty$ in the PDE. So we know the problem becomes

PDE $\quad \mathrm{u}_\tau = u_{\xi\xi}, \quad 0 < \xi < 1, \quad 0 < \tau < \infty$

BCs $\quad \begin{cases} w(0,t) = 0 \Rightarrow w(0,\tau) = 0 \quad 0 < \tau < \infty \\ w(L,t) = 1 \Rightarrow w(1,t) = 1 \end{cases}$

ICs $\quad \mathrm{w}(\xi, 0) = \frac{T_2 - T_1}{-T_1}, 0 \leq \xi \leq 1$