Jacob Kutch

10/10/19

MAT 460 - HW4

pg. 1

7.1.3 $P_n(x_i) = Q_n(x_i) = y_i$ for $i = 0, 1, \ldots, n$, where $P_n(x)$ and $Q_n(x)$ are of degree n.

Proposition: $P_n(x) \equiv Q_n(x)\ \forall x$.

Proof: Assume $P_n(x) \not\equiv Q(x)\ \forall x$. So $\exists x_i$ such that $P(x_i) \neq Q(x_i)$.

Let $R_n(x) = P_n(x) - Q_n(x)$ so that $R_n(x)$ is of degree n.

If $P_n(x) \neq Q(x)$, then $R_n(x) \neq 0$.

So $R_n(x)$ can be written as $(x - x_0)(x - x_1)\cdots(x - x_n) = x^{n+1} + \ldots$ for x_i where $i = 0, 1, \ldots, n$.

This implies $R_n(x)$ is of degree $n+1$, unless $R_n(x) = 0$, since $(x - x_0)(x - x_1)\cdots(x - x_n) = 0 \Rightarrow (x - x_0)(x - x_1)\cdots(x - x_{n-1}) = 0 \Rightarrow x^n + \ldots = 0$

This contradicts the previous assumption that $R_n(x) \neq 0$. So

$P_n(x) - Q_n(x) = 0 \Rightarrow P_n(x_i) = Q_n(x_i)\ \forall x_i$. $\square$

7.1.4 Suppose $P_1(x) = a_0(x - x_1) + a_1(x - x_0)$.

$$\begin{cases} p_1(x_0) = y_0 \\ p_1(x_1) = y_1 \end{cases} \Rightarrow \begin{cases} p_1(x_0) = a_0(x_0 - x_1) + a_1(x_0 - x_0) = y_0 \\ p_1(x_1) = a_0(x_1 - x_1) + a_1(x_1 - x_0) = y_1 \end{cases}$$

$$\begin{bmatrix} (x_0 - x_1) & 0 \\ 0 & (x_1 - x_0) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$ For $P_1(x) = \sum_{i=0}^{1} c_i Q_i(x)$

Goal: $$\begin{bmatrix} ? & 0 & 0 \\ 0 & ? & 0 \\ 0 & 0 & ? \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} \Rightarrow \begin{bmatrix} (x_0-x_1)(x_0-x_2) & (x_1-x_1)(x_1-x_2) & (x_2-x_1)(x_2-x_2) \\ (x_0-x_0)(x_0-x_2) & (x_1-x_0)(x_1-x_2) & (x_2-x_0)(x_2-x_2) \\ (x_0-x_0)(x_0-x_1) & (x_1-x_0)(x_1-x_1) & (x_2-x_0)(x_2-x_1) \end{bmatrix}$$

$$\mathcal{L}_{n,j}(x) = \prod_{i=0, i\neq j}^{n} \frac{x - x_i}{x_j - x_i}$$

$$\mathcal{L}_{1,j}(x) = \prod_{i=0, i\neq j}^{1} \frac{x - x_i}{x_j - x_i}$$

$$\begin{bmatrix} (x-x_1)(x-x_2) & 0 & 0 \\ 0 & (x-x_0)(x-x_2) & 0 \\ 0 & 0 & (x-x_0)(x-x_1) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

$\mathcal{L}_{1,0}(x) = DNE$

$\mathcal{L}_{1,1}(x) = \dfrac{x - x_0}{x_1 - x_0}$

$\mathcal{L}_{1,0}(x) = \dfrac{x - x_1}{x_0 - x_1}$

for (i,j)

$(0,1)$

$(1,0)$

Permutations of $n=2$

$(0,1), (0,2),$

$(1,0), (1,2)$

$(2,0), (2,1)$

$\mathcal{L}_{2,0}(x) = \left(\dfrac{x - x_1}{x_0 - x_1}\right)\left(\dfrac{x - x_2}{x_0 - x_2}\right)$

$\mathcal{L}_{2,1}(x) = \left(\dfrac{x - x_0}{x_1 - x_0}\right)\left(\dfrac{x - x_2}{x_1 - x_2}\right)$

$\mathcal{L}_{2,2}(x) = \left(\dfrac{x - x_0}{x_2 - x_0}\right)\left(\dfrac{x - x_1}{x_2 - x_1}\right)$

7.21 Prop: $P_n(x) = \sum_{j=0}^{n} y_j \mathcal{L}_{n,j}(x)$ is the unique n^{th} degree polynomial satifying $P_n(x_i) = y_i$ for $i = 0, 1, \ldots, n$ with $x_0, x_1, \ldots, x_n$ distinct in $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$

Proof: $\mathcal{L}_{n,j}(x) = \begin{cases} 1 \text{ if } i \neq j \\ 0 \text{ if } i = j \end{cases} \Rightarrow \mathcal{L}_{n,j}(x) = \prod_{i=0, i\neq j}^{n} \frac{x - x_i}{x_j - x_i}$.

We NTS that $\sum_{j=0}^{n} y_j \mathcal{L}_{n,j}(x) = y_j \quad \forall j \in [0, n]$.

Assume $n=3$ and $j = 0, 1, 2$ and $i = 0, 1, 2$ for a base case of

$$\mathcal{L}_{3,0}(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}, \quad \mathcal{L}_{3,1}(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)},$$

$$\text{and } \mathcal{L}_{3,2}(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}.$$

For general n and $i, j = 0, 1, 2, \ldots, n$ then,

$$\mathcal{L}_{n,n}(x) = \frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})} \quad \text{and}$$

$$\cancel{\mathcal{L}_{n+1,n+1}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(x_{n+1}-x_0)(x_{n+1}-x_1)\cdots(x_{n+1}-x_n)}}.$$

So for $P_n(x) = \sum_{j=0}^{n} [y_j \mathcal{L}_{n,j}(x)]$, we know $\sum_{j=0}^{n} \mathcal{L}_{n,j}(x) = 1$

$$\text{so} \quad P_n(x) = \left(\sum_{j=0}^{n} y_j\right)\left(\sum_{j=0}^{n} \mathcal{L}_{n,j}(x)\right) = \sum_{j=0}^{n} y_j = y.$$

Thus $P_n(x) = y \Rightarrow P_n(x_i) = y_i \quad \forall i \in [0, n]$ $\square$