Exploration 7.4.2: $f(x) = e^x$ and $P_2(x)$ at $x_0 = 0, x_1 = 0.5, x_2 = 1$ for $x \in [0,1]$

$$|f(x) - P_2(x)| = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{j=0}^{n} (x - x_j) \text{ ; Here, } \xi(x) \in (0,1)$$

$f(x) = e^x$ is positive and increasing for $x \in [0,1]$

$$|f(x) - P_2(x)| = \frac{e^{\xi(x)}}{3!} x(x-0.5)(x-1) = \frac{e^{\xi(x)}}{6} [\overbrace{x^3 - 1.5x^2 + 0.5x}^{w}]$$

$w' = 3x^2 - 3x + 0.5$

$0 = 6x^2 - 6x + 1$

To bound $|x(x-0.5)(x-1)|$ on $[0,1]$,

c.p. $\frac{3 \pm \sqrt{3}}{6} \approx 0.7887, 0.2113$

$$\frac{3+\sqrt{3}}{6}\left(\frac{3+\sqrt{3}}{6} - 0.5\right)\left(\frac{3+\sqrt{3}}{6} - 1\right) \approx -0.0481 \quad \text{so}$$

$$|x^3 - 1.5x^2 + 0.5x| < 0.0481 \quad \text{on } [0,1]$$

$$\text{so } |e^x - P_2(x)| < \frac{e^{(1)}}{6}(0.0481)$$

Test: $P_2(x) = 1 + 1.2974(x) + 0.8417(x)(x-0.5) = 0.8417x^2 + 0.87655x + 1$

$|e^1 - P_2(1)| \approx 2.71828 - 2.71825 = 0.00003$

Ex. 7.4.3:

(7.17): $T_k(x) = \cos(k\cos^{-1}(x))$, $|x| \le 1$, $k = \{0, 1, 2, \dots\}$

(7.18): $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

(7.19): $\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$

Proposition (7.20): $T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$, $k \ge 1$

Proof: By (7.17), we can conclude $T_{k+1}(x) = \cos((k+1)\cos^{-1}(x)) \Rightarrow$ $T_{k+1}(x) = \cos(k\cos^{-1}(x) + \cos^{-1}(x))$. By (7.18), we then know that $T_{k+1}(x) = \cos(k\cos^{-1}(x))\cos(\cos^{-1}(x)) - \sin(k\cos^{-1}(x))\sin(\cos^{-1}(x))$ $\Rightarrow x\cos(k\cos^{-1}(x)) - (\sqrt{1-x^2})\sin(k\cos^{-1}(x))$.

$T_{k-1}(x)$ is then $\cos((k-1)\cos^{-1}(x)) = \cos(k\cos^{-1}(x) - \cos^{-1}(x))$, which by (7.19) is equal to $\cos(k\cos^{-1}(x))(x) + \sin(k\cos^{-1}(x))\sqrt{1-x^2}$. So the LHS in (7.20) is $2x\cos(k\cos^{-1}(x)) - x\cos(k\cos^{-1}(x)) + \sqrt{1-x^2}\sin(k\cos^{-1}(x))$, which is equivalent to $T_{k+1}(x) = x\cos(k\cos^{-1}(x)) - (\sqrt{1-x^2})\sin(k\cos^{-1}(x))$ $\square$

Ex. 7.4.4: (7.20): $T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$

Proposition: The leading coefficient of $T_k(x)$ is 2^{k-1} for $k \geq 1$

Proof: Let $k=1$ for $T_k(x)$. Then by (7.20), $T_2(x) = 2xT_1(x) - T_0(x)$
$\Rightarrow \cos(2\cos^{-1}(x)) = 2x\cos(\cos^{-1}(x)) - \cos(0) \Rightarrow T_2(x) = 2x^2 - 1$.
Rearranging in terms of $T_1(x)$ gives $T_1(x) = \frac{1}{2x}[(2x^2-1) + T_{k-1}(x)]$
$\Rightarrow T_1(x) = x - \frac{1}{2x} + \frac{1}{2x} = x$. So for $T_1(x)$, the leading coefficient is 2^0, or 2^{k-1} for $k=1$.

TST 2^{k-1} is the leading coefficient of $T_k(x)$, we NTS T_{k+1} has leading coefficient 2^k. Assuming the proposition to be true, we may express $T_k(x)$ as $2^{k-1}x^k + q_{k-1}(x)$ for polynomial q_{k-1} of degree $k-1$. Thus T_{k+1} may be expressed as $2^k x^{k+1} + q_k(x)$ for some polynomial $q_k(x)$ of degree $\leq k$.

Rewriting it as such gives $T_{k+1}(x) = 2x[2^{k-1}x^k + q_{k-1}(x)] - T_{k-1}(x)$
$\Rightarrow T_{k+1}(x) = 2^k x^{k+1} + 2xq_{k-1}(x) - T_{k-1}(x)$. Since the highest possible degree of both $q_{k-1}(x)$ and $T_{k-1}(x)$ is $k-1$, $2^k x^{k+1}$ is the leading term of $T_{k+1}(x)$ and 2^k is the leading coefficient. □

Ex 7.4.5: $T_k(x) = \cos(k\cos^{-1}(x)) = 0 \Rightarrow$ $\cancel{\cos^{-1}(x) = \frac{\pi}{2k}}$ $\cancel{x = \cos(\frac{\pi}{2k})}$. Since the range of $\cos(\theta)$ for $0 < \theta < \pi$ is $(-1,1)$ the roots $x \in (-1,1)$ as well. Since cosine has a period of π and is 0 at every odd multiple of $\pi/2$, we know that $\cos(\frac{\pi}{2}(2n+1)) = 0$, $\forall n \in \mathbb{Z}$. Thus we may write $T_k(x)$ as
$\cos(k\cos^{-1}(x)) = \cos(\frac{\pi}{2}(2n+1)) \Rightarrow k\cos^{-1}(x) = \frac{\pi}{2}(2n+1)$
$\Rightarrow \cos^{-1}(x) = \frac{\pi(2n+1)}{2k} \Rightarrow x = \cos\left(\frac{\pi(2n+1)}{2k}\right)$. Since for $n=k$,
$x = \cos\left(\pi + \frac{\pi}{2k}\right) = \cos(\pi)\cos\left(\frac{\pi}{2k}\right) - \sin(\pi)\sin\left(\frac{\pi}{2k}\right)$
$\Rightarrow x = -\cos\left(\frac{\pi}{2k}\right) - \sin\left(\frac{\pi}{2k}\right)$, where for $k \geq 1$, $x \notin (-1,1)$, so $n < k$ for $x = \cos\left(\frac{\pi(2n+1)}{2k}\right)$. Since the roots may be expressed as $x_k = \cos(\theta_k)$ for equally-spaced angles θ_k, and since cosine is injective on $[0,\pi]$, for $\theta_i, \theta_j \in [0,\pi]$, where $i,j \in \mathbb{N}$ and $i,j \leq k$, $\theta_i \neq \theta_j$. Thus $\cos(\theta_i) \neq \cos(\theta_j)$, so $x_i \neq x_j$ and the roots of $T_k(x)$ are distinct. Finally, since $\frac{(2n+1)}{2k} < 1$ for $n < k$, $\frac{\pi(2n+1)}{2k} < \pi$ and the roots of $T_k(x)$ are real, distinct roots on $[-1,1]$ □