Ex. 4.5.1

Proof:

Let x_0 be a point $\in \mathbb{R}$ and h be the length of $[x_{i-1}, x_i]$ for $i = 0, 1, \ldots, 2n$.
By Taylor expansion, we know for a given function f that

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f^{(4)}(x_0) + \frac{h^5 f^{(5)}(x_0)}{120} + \ldots + \frac{h^{2n-1}}{(2n-1)!}f^{(2n-1)}(x_0) + \frac{h^{2n}}{(2n)!}f^{(2n)}(\xi_1) \quad \text{for } \xi_1 \in (x_0, x_0+h) \text{ and}$$

$$f(x_0-h) = f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) - \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f^{(4)}(x_0) - \frac{h^5 f^{(5)}(x_0)}{120} + \ldots - \frac{h^{2n-1}}{(2n-1)!}f^{(2n-1)}(x_0) + \frac{h^{2n}}{(2n)!}f^{(2n)}(\xi_2) \quad \text{for } \xi_2 \in (x_0-h, x_0).$$

Finding $f(x_0+h) - f(x_0-h)$ gives $2\left[hf'(x_0) + \frac{h^3}{6}f'''(x_0) + \frac{h^5}{120}f^{(5)}(x_0) + \ldots + \frac{h^{2n-1}}{(2n-1)!}f^{(2n-1)}(x_0) + \frac{h^{2n}}{(2n)!}f^{(2n)}(\xi_1) - \frac{h^{2n}}{(2n)!}f^{(2n)}(\xi_2)\right]$.

Solving for $f'(x_0)$, we see $f'(x_0) = \frac{f(x_0+h)-f(x_0-h)}{2h} - \left[\frac{h^2}{6}f'''(x_0) + \frac{h^4}{120}f^{(5)}(x_0) + \ldots + \frac{h^{2n-2}}{(2n-1)!}f^{(2n-1)}(\xi)\right]$ for $\xi \in (x_0-h, x_0+h)$

Defining the function $D_1(h)$ to be $\frac{f(x_0+h)-f(x_0-h)}{2h}$, the centered difference about x_0 and abbreviating the error terms, we have

$f'(x_0) = D_1(h) - \frac{h^2}{6}f'''(x_0) - \frac{h^4}{120}f^{(5)} + \text{error}$. Now, finding $f'(x_0)$ for $h/2$,
$f'(x_0) = D_1(h/2) - \frac{1}{6}\left(\frac{h}{2}\right)^2 f'''(x_0) - \frac{1}{120}\left(\frac{h}{4}\right)^2 f^{(5)}(x_0) + \text{error}$, we have a system of equations, which by substitution yields

$$3f'(x_0) = (4D_1(h/2) - D_1(h)) + \frac{h^4}{160}f^{(5)}(x_0) \Rightarrow$$

$$f'(x_0) = \frac{4D_1(h/2) - D_1(h)}{3} + \left(\frac{1}{3}\right)\frac{h^4}{160}f^{(5)}(x) \Rightarrow$$

$$f'(x_0) = D_1(h/2) + \left(\frac{1}{3}\right)\left(D_1(h/2) - D_1(h)\right) + \frac{h^4}{6400}f^{(5)}(x_0).$$

So, using two expansions of order $O(h^2)$ gave a centered difference approximation of order $O(h^4)$. Now let $\{K_i\}_{i=1}^{2n} = \left\{\frac{1}{(i+2)!}f^{(i+2)}(x_0)\right\}$, so we have

$$f'(x_0) = D_1(h) - K_1h^2 - K_2h^4 - K_3h^6 - \ldots - \frac{h^{2n-2}}{(2n-1)!}f^{(2n-1)}(\xi) \text{ and}$$

$$f'(x_0) = D_1(h) - K_1\left(\frac{h}{2}\right)^2 - K_2\left(\frac{h}{2}\right)^4 - K_3\left(\frac{h}{2}\right)^6 - \ldots - \left(\frac{h}{2}\right)^{2n-2}\frac{1}{(2n-1)!}f^{(2n-1)}(\xi)$$

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9.5.1 (cont) Similar to the last step, we use this new system of equations,

$$\begin{cases} f'(x_0) = D_1(h) - K_1(h^2) - K_2(h^4) - \ldots - h^{2n-2}\frac{1}{(2n-1)!}f^{(2n-1)}(x_0) - h^{2n}\frac{1}{(2n+1)!}f^{(2n+1)}(\xi) \\ f'(x_0) = D_1(h/2) - (h/2)^2K_1 - (h/2)^4K_2 - \ldots - (h/2)^{2n-2}\frac{1}{(2n-1)!}f^{(2n-1)}(x_0) - \left(\frac{h}{2}\right)^{2n}\frac{1}{(2n+1)!}f^{(2n+1)}(\xi) \end{cases}$$

Multiplying the equation of step size h by (-1) and the equation of step size $h/2$ by $(2^{2(n-1)})$, we find

$$(2^{2(n-1)}-1)f'(x_0) = (2^{2(n-1)})D_1(h/2) - D_1(h) - (2^{2(n-1)}-1)[K_1h^2 + K_2h^4 + \ldots]$$
$$- (2^{2(n-1)})\left(\frac{1}{2}\right)^{2(n-1)}(h)^{2n-2}\frac{1}{(2n-1)!}f^{(2n-1)}(x_0)$$
$$+ h^{2n-2}\left(\frac{1}{(2n-1)!}\right)f^{(2n-1)}(x_0)$$
$$- (2^{2(n-1)})\left(\frac{1}{(2n+1)!}\right)(h^{2n})\left(\frac{1}{2}\right)^{2n}f^{(2n+1)}(x_0) + (h/2)^{2n}\left(\frac{1}{(2n+1)!}\right)f^{(2n+1)}(x_0)$$
$$+ \ldots$$

Thus
$$f'(x_0) = \left(\frac{1}{2^{2n-2}-1}\right)\Big[(2^{2n-2})D_1(h/2) - D_1(h) - (2^{2n-2}-1)[K_1h^2 + K_2h^4 + \ldots]$$
$$+ \left(\frac{h}{2}\right)^{2n}\left(\frac{1}{(2n+1)!}\right)f^{(2n+1)}(x_0) - \left(\frac{2^{2n-2}}{2^{2n}}\right)\left(\frac{1}{(2n+1)!}\right)(h^{2n})f^{(2n+1)}(x_0)$$

$$\Rightarrow f'(x_0) = \frac{2^{2n-2}D_1(h/2) - D_1(h)}{2^{2n-2}-1} - K_1h^2 - K_2h^4 - \ldots - O(h^{2n})$$

$$= \frac{2^{2n-2}D_1(h/2) + D_1(h/2) - D_1(h/2) - D_1(h)}{2^{2n-2}-1} - \{K_ih^{2i}\}_{i=1} - O(h^{2n})$$

$$f'(x_0) = D_1(h/2) + \frac{D_1(h/2) - D_1(h)}{4^{(n-1)}-1} - \{K_ih^{2i}\}_{i=1}^{n-2} - O(h^{2n}),$$

Thus, we see that this centered difference approximation has only even powers of h for any $n \in \mathbb{N}$ and the next most accurate approximation ~~where [illegible]~~ can be found with the above equation. $\square$

9.7.1 $\mu_k = \int_{-1}^{1} x^k dx$ for $k=0,1,2,3$

$\mu_0 = \int_{-1}^{1} dx = 2$, $\mu_1 = \int_{-1}^{1} x\,dx = 0$, $\mu_2 = \int_{-1}^{1} x^2 dx = 2/3$, $\mu_3 = \int_{-1}^{1} x^3 dx = 0$

Since we know that $\mu_k = \sum_{i=1}^{2} w_i x_i^k$, we know that

$\mu_0 = w_1 + w_2$,
$\mu_1 = x_1 w_1 + x_2 w_2$,
$\mu_2 = x_1^2 w_1 + x_2^2 w_2$, and
$\mu_3 = x_1^3 w_1 + x_2^3 w_2$.

Thus we have the system of equations

$$\begin{cases} w_1 + w_2 = 2 \\ x_1 w_1 + x_2 w_2 = 0 \\ x_1^2 w_1 + x_2^2 w_2 = 2/3 \\ x_1^3 w_1 + x_2^3 w_2 = 0 \end{cases} \Rightarrow \begin{cases} w_2 = 2 - w_1 \\ x_1 w_1 = -x_2 w_2 \\ x_1^2 w_1 + x_2^2 w_2 = 2/3 \\ x_1^3 w_1 = -x_2^3 w_2 \end{cases} \Rightarrow$$

$$\begin{cases} w_2 = 2 - w_1 \\ x_1 w_1 = x_2 (w_1 - 2) \\ x_1^2 w_1 + 2x_2^2 - x_2^2 w_1 = \frac{2}{3} \\ x_1^3 w_1 = x_2^3 (w_1 - 2) \end{cases} \Rightarrow \begin{cases} w_2 = 2 - w_1 \\ w_1 - 2 = \frac{x_1 w_1}{x_2} \\ x_1^2 w_1 + 2x_2^2 - w_1 x_2^2 = \frac{2}{3} \\ x_1^3 w_1 = x_2^3 \left(\frac{x_1 w_1}{x_2}\right) \end{cases}$$

$$\Rightarrow \begin{cases} w_2 = 2 - w_1 \\ w_1 - 2 = \frac{x_1 w_1}{x_2} \\ x_2^2 w_1 + 2x_2^2 - w_1 x_2^2 = \frac{2}{3} \\ x_1^2 = x_2^2 \end{cases} \Rightarrow \begin{cases} w_2 = 2 - w_1 \\ w_1 - 2 = \frac{x_1 w_1}{x_2} \\ x_2^2 = \frac{1}{3} \\ x_1^2 = x_2^2 \end{cases} \Rightarrow \begin{cases} w_2 = 2 - w_1 \\ w_1 - 2 = \frac{x_1 w_1}{x_2} \\ x_2 = \pm\sqrt{1/3} \\ x_1 = \pm\sqrt{1/3} \end{cases}.$$

Letting $x_1 = -\frac{1}{\sqrt{3}}$ and $x_2 = \frac{1}{\sqrt{3}}$, we see that

$$x_1 w_1 + x_2 w_2 = 0 \Rightarrow -\frac{1}{\sqrt{3}} w_1 + \frac{1}{\sqrt{3}} w_2 = 0.$$

Since we can't have negative weights, we know that $w_1 = w_2 = 1$.

9.7.5: Proposition: If f is $2n+2$ time continuously differentiable on $[a,b]$, then for $x_1,\dots,x_n$ and $w_1,\dots,w_n$, the nodes and weights of a Gauss quadrature rule on $[a,b]$, then we have that

$$I[f] = \int_a^b f(x)dx = \sum_{i=1}^{n} f(x_i)w_i + \frac{f^{(2n)}(\xi)}{(2n)!}\int_a^b \prod_{i=1}^{n}(x-x_i)^2 dx \text{ for } \xi \in (a,b)$$

Proof: Constructing the Hermite interpolating polynomial for $f(x)$ using the nodes $x_1,\dots,x_n$ to satisfy the $2n$ conditions $H_{2n-1}(x_i) = f(x_i)$ and $H'_{2n-1}(x_i) = f'(x_i)$ for $i=1,\dots,n$, we find $H_{2n-1}(x) = \sum_{i=1}^{n} f(x_i)H_{2n-1,i}(x) + \sum_{i=1}^{n} f'(x_i)K_{2n-1,i}(x)$. Integrating both sides with respect to $x\in[a,b]$, we then find that

$$\int_a^b H_{2n-1}(x)\,dx = \sum_{i=1}^{n} f(x_i)\int_a^b H_{2n-1,i}(x)dx + \sum_{i=1}^{n} f'(x_i)\int_a^b K_{2n-1,i}(x)dx.$$

We now define the function $\pi_n(x) = \prod_{i=1}^{n}(x-x_i)$, related to $H_{2n-1}(x)$ by $H_{2n-1,i}(x) = [\mathcal{L}_{n-1,i}(x)]^2[1-2\mathcal{L}'_{n-1,i}(x_i)(x-x_i)]$ and to $K_{2n-1}(x)$ by $K_{2n-1,i}(x) = (\mathcal{L}_{n-1,i}(x)^2)(x-x_i)$. for $i=1,\dots,n$. Thus,

$$\int_a^b H_{2n-1}(x)dx = \sum_{i=1}^{n} f(x_i)\int_a^b\left[(\mathcal{L}_{n-1,i}(x))^2 - 2\mathcal{L}'_{n-1,i}(x_i)(x-x_i)(\mathcal{L}_{n-1,i}(x))^2\right]dx + \sum_{i=1}^{n} f'(x_i)\int_a^b (\mathcal{L}_{n-1,i}(x))^2(x-x_i)dx \implies$$

$$\int_a^b H_{2n-1}(x)dx = \sum_{i=1}^{n} f(x_i)\left[\int_a^b(\mathcal{L}_{n-1,i}(x))^2 - \frac{2\mathcal{L}_{n-1,i}(x_i)}{\pi_n'(x_i)}\int_a^b \mathcal{L}_{n-1,i}(x)\pi_n(x)dx\right] + \sum_{i=1}^{n} f'(x_i)\left[\frac{1}{\pi_n(x_i)}\int_a^b \mathcal{L}_{n-1,i}(x)\,\pi_n(x)dx.\right.$$

Since $\pi_n(x)$ is of degree n and orthogonal to all lesser degree polynomials, such as $\mathcal{L}_{n-1,i}(x)$, we see that $\int_a^b K_{2n-1,i}(x)dx = 0$ and $\int_a^b H_{2n-1,i}(x)dx = \int_a^b(\mathcal{L}_{n-1,i}(x))^2dx$. Thus,

$$\int_a^b H_{2n-1}(x)dx = \sum_{i=1}^{n} f(x_i)\int_a^b(\mathcal{L}_{n-1,i}(x))^2dx.$$

Because $w_i = \int_a^b(\mathcal{L}_{n-1,i}(x))^2dx = \int_a^b(\mathcal{L}_{n-1,i}(x))dx$, we have that

$$\int_a^b H_{2n-1}(x)dx = \sum_{i=1}^{n} f(x_i)w_i.$$

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Using the error in the Hermite interpolating polynomial, we see

$$E[H_{2n-1}(x)] = \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_i) w_i \implies$$

$\int_a^b [f(x) - H_{2n-1}(x)]\,dx$. From Theorem 7.5.1, then

$$E[H_{2n-1}(x)] = \int_a^b \left[\frac{f^{(2n)}(\xi(x))}{(2n)!}(\pi_n(x))^2\right] dx.$$

Since $f^{(2n)}(\xi(x))$ is integrable on $[a,b]$ and $\pi_n(x)^2$ is continuous and does not change signs on $[a,b]$, we find that for some $\xi \in (a,b)$,

$$E[H_{2n-1}(x)] = \frac{f^{(2n)}(\xi)}{(2n)!} \int_a^b \prod_{i=1}^{n} (x - x_i)^2\,dx. \qquad \square$$