10.2.3.

Proposition: For f continuously differentiable on $[a,b]$ where $f(a)f(b)<0$, the sequence of midpoints of subintervals in $[a,b]$, $\{x^{(k)}\}_{n=1}^{\infty}$ converges to a number x^* in (a,b) such that $f(x^*)=0$, and each iterate $x^{(k)}$ satisfies $|x^{(k)}-x^*| \le \frac{b-a}{2^k}$,

Proof: Let the sequence $\{a_k\}_{k=0}^{\infty}$ be the left endpoints of the interval for each iteration k of the bisection method. Let $\{b_k\}_{k=0}^{\infty}$ be the right endpoints for each iteration.

We know $a_0 = a \le a_1 \le a_2 \le \ldots \le b_0 = b$ and $b_0 \ge b_1 \ge b_2 \ge \ldots \ge a_0 = a$.

Since $\{a_k\}$ is monotonically increasing and bounded above, and $\{b_k\}$ is monotonically decreasing and bounded below, both sequences converge by Monotone Convergence Theorem.

We know that $b_k - a_k = \frac{b_{k-1}-a_{k-1}}{2} = \ldots = \frac{b-a}{2^k}$ and $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = x^*$. Also $\lim_{k\to\infty} f(a_k)f(b_k) = (f(x^*))^2 \le 0$, implying $f(x^*)=0$.

If a_k and b_k converges to x^*, then $|x^{(k)}-x^*| \le \frac{b-a}{2^k}$ by Convergence of a Sequence $\square$