MAT 461- HW2

3.2.4 (a) Proposition: The product of two upper/lower triangular matrices is upper/lower triangular.

Proof: Let A and B be nxn upper triangular matrices. Let the elements of A be denoted a_{ij} and elements of B be denoted b_{ij} for the i'th row and j'th column.

For upper triangular matrices, every element below the main diagonal when $i>j$ is equal to 0.

From the algorithm for matrix multiplication, the product $(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$, the dot products of A's i'th row vector and B's j'th column vector. Thus $a_{ik} b_{kj} = 0$ if either $i>k$ or $k>j$.

Let some $(AB)_{ij} = \sum_{k=i+1}^{n} a_{ik} b_{kj}$ where $i>j$. since $k>i$, we see that $k>j$, so $b_{kj}=0$ for $k=i+1,\ldots,n$. Since $(AB)_{ij}$ is then equal to 0 when $i>j$, AB is thus an upper triangular matrix. □

Since A^T and B^T are lower triangular, the exact same algorithm will apply since it would then refer to the j'th row and i^{th} column.

b. Proposition: The inverse of an invertible upper/lower triangular matrix is upper/lower triangular.

Proof: Let A be an upper triangular of size nxn and let $B=A^{-1}$ be an nxn matrix that is not upper triangular. Then $\exists\, b_{ij} \neq 0$ for $i>j$. Let $b_{ik} \neq$ for the smallest $k \in \mathbb{N}$ s.t. $k<i$. For $C=BA$, $c_{ik} = \sum_{j=1}^{n} b_{ij} a_{jk} = b_{i1} a_{1k} + b_{i2} a_{2k} + \ldots + b_{ik} a_{kk} + \ldots + b_{in} a_{nk}$. Since b_{ik} is the first nonzero element in that row, all $b_{ij} a_{jk}$ in the sum where $j<k$ vanish.

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cont. So $c_{ik} = b_{ik}a_{kk} + \dots + b_{in}a_{nk}$. However, since A is upper triangular and a_{jk}'s row indices are greater than its column indices, all $b_{ik}a_{jk} = 0$ for $j > k$.
Thus $c_{ik} = b_{ik}a_{kk} \neq 0$.

Since $C = A^{-1}A = I$, C should have nonzero elements only on the diagonal, which can only be true if $i = k$. Since $k < i$, there is a contradiction.

Hence, $B = A^{-1}$ is upper triangular. $\square$
Again, the same logic applies since A^T is lower triangular and $(A^T)^{-1} = (A^{-1})^T$.

3.2.5. For a matrix to be nonsingular, its determinant needs to be nonzero. For some $n \times n$ unit lower triangular matrix, we see that its determinant will always be 1.

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{21} & 1 & 0 & \cdots & 0 \\ a_{31} & a_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 1 \end{bmatrix}; \quad \det(A) = 1 \begin{vmatrix} 1 & 0 & \cdots & 0 \\ a_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n2} & a_{n3} & \cdots & 1 \end{vmatrix}$$

$$\Rightarrow 1 \begin{vmatrix} 1 & 0 & \cdots & 0 \\ a_{43} & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n3} & a_{n4} & \cdots & 1 \end{vmatrix} \Rightarrow \dots \Rightarrow 1 \begin{vmatrix} 1 & 0 \\ a_{n,n-1} & 1 \end{vmatrix} = 1$$

Thus a unit lower triangular matrix will always be nonsingular. The same applies to unit upper triangular matrices since the same process may be repeated with columns of A.

3.2.10:

$$A = \begin{bmatrix} 5 & 0 & 2 & -5 \\ -1 & -5 & 3 & 2 \\ 2 & -2 & 1 & -3 \\ 3 & 3 & -1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 2 & -5 \\ 0 & -5 & \frac{17}{5} & 1 \\ 0 & -2 & \frac{1}{5} & -1 \\ 0 & 3 & -\frac{11}{5} & 3 \end{bmatrix}$$

$m_{21} = \frac{1}{5}$

$m_{31} = -\frac{2}{5}$

$m_{41} = -\frac{3}{5}$

$$\begin{bmatrix} 5 & 0 & 2 & -5 \\ 0 & -5 & \frac{17}{5} & 1 \\ 0 & 0 & -\frac{29}{25} & -\frac{7}{5} \\ 0 & 0 & -\frac{4}{25} & \frac{18}{5} \end{bmatrix}$$

$m_{32} = -\frac{2}{5}$

$m_{42} = \frac{3}{5}$

$\frac{1}{5} - \frac{34}{25} = -\frac{29}{25}$

$-\frac{51}{25} - \frac{55}{25}$

$$\begin{bmatrix} 5 & 0 & 2 & -5 \\ 0 & -5 & \frac{17}{5} & 1 \\ 0 & 0 & -\frac{29}{25} & -\frac{7}{5} \\ 0 & 0 & 0 & \frac{110}{29} \end{bmatrix}$$

$m_{43} = -\frac{4}{29}$

$-\frac{28}{145} + \frac{18}{5}$

$\frac{28}{145} + \frac{18}{5}$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{5} & 1 & 0 & 0 \\ \frac{2}{5} & \frac{2}{5} & 1 & 0 \\ \frac{3}{5} & -\frac{3}{5} & \frac{4}{29} & 1 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{5} & 1 & 0 & 0 \\ \frac{2}{5} & \frac{2}{5} & 1 & 0 \\ \frac{3}{5} & -\frac{3}{5} & \frac{4}{29} & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 2 & -5 \\ 0 & -5 & \frac{17}{5} & 1 \\ 0 & 0 & -\frac{29}{25} & -\frac{7}{5} \\ 0 & 0 & 0 & \frac{110}{29} \end{bmatrix}$$

3.2.14. Gaussian elimination w/ partial pivoting of A.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix} \Rightarrow \overset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 & 2 \\ 0 & -1 & -1 & 1 & 2 \\ 0 & -1 & -1 & -1 & 2 \end{bmatrix}}$$

$$\Rightarrow \overset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & -1 & 1 & 4 \\ 0 & 0 & -1 & -1 & 4 \end{bmatrix}}$$

$$\Rightarrow \overset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & 0 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & -1 & 8 \end{bmatrix}}$$

$$\Rightarrow \overset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 16 \end{bmatrix}}$$

The entries of U's column 5 are growing by powers of 2 in each iteration of the algorithm because of the multiplier used. Partial pivoting was not need here since $|1| = |-1|$

3.2.15. $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 & 2 \\ 0 & -1 & -1 & 1 & 2 \\ 0 & -1 & -1 & -1 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & -1 \\ 0 & 2 & -1 & 1 & -1 \\ 0 & 2 & -1 & -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & 1 & -2 \\ 0 & 0 & -1 & -1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & -2 & 1 & -1 \\ 0 & 0 & -2 & -1 & -1 \end{bmatrix}$$

swapped

swapped

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

swapped

The entries of the upper triangular matrix's last column no longer grow like in the case of partial pivoting. Since pivoting was actually applied this time, we have $A = P^{-1}LU$ rather than $A = LU$. Applying the permutation matrix would show that we can get to the original through matrix multiplication.