CH1

Monday, December 20, 2021 1:34 PM

Maxwell's equations:

- $\nabla\times\vec{E} = -\mu\frac{\partial H}{\partial t} = -\frac{\partial B}{\partial t}$
- $\nabla\cdot\vec{D} = \rho_v$
- $\nabla\times\vec{H} = \sigma\vec{E} + \epsilon\frac{\partial\vec{E}}{\partial t} + \vec{J}_{ext}$
- $\nabla\cdot\vec{B} = 0$

Pointing theorem:

→ From vector analsys: $\nabla\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{H})$

$\therefore \nabla\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot\left(-\mu\frac{\partial\vec{H}}{\partial t}\right) - \vec{E}\cdot\left(\sigma\vec{E} + \epsilon\frac{\partial\vec{E}}{\partial t} + \vec{J}_{ext}\right)$

$$= \underbrace{\frac{-1}{2}\mu\frac{\partial}{\partial t}(|\vec{H}|)^2}_{W/m^3} - \underbrace{\frac{1}{2}\epsilon\frac{\partial}{\partial t}(|\vec{E}|)^2}_{W/m^3} - \sigma|\vec{E}|^2 - \vec{E}\cdot\vec{J}_{ext}$$

$\frac{\partial}{\partial t}(\vec{H}\cdot\vec{H}) = \vec{H}\frac{\partial\vec{H}}{\partial t} + \vec{H}\frac{\partial\vec{H}}{\partial t}$

$\frac{\partial}{\partial t}(\vec{H})^2 = 2\vec{H}\frac{\partial\vec{H}}{\partial t}$

$\therefore \nabla\cdot(\vec{E}\times\vec{H})$ represent power density ⟶ $\therefore \iiint \nabla\cdot(\vec{E}\times\vec{H})\,dv$ represents power

$\because \oiint (\vec{E}\times\vec{H})\,ds = \iiint \nabla\cdot(\vec{E}\times\vec{H})\,dv$ "divergence theorem"

$\therefore \vec{E}\times\vec{H}$ represents power/m^2 ⟶ Pointing vector: $\vec{P} = \vec{E}\times\vec{H}$ "wave propogation direction"

Time Harmonic Fields:

→ $e^{j\omega t}$

→ $\vec{E}(r,t) = \underbrace{\vec{E}(r)}_{\text{space}}\ \underbrace{e^{j\omega t}}_{\text{time}}$ → $\vec{H}(r,t) = \vec{H}(r)\,e^{j\omega t}$

- maxwell's equations becomes:

- $\nabla\times\vec{E} = -j\omega\mu\vec{H}$
- $\nabla\cdot\vec{D} = \rho_v$
- $\nabla\times\vec{H} = \sigma\vec{E} + j\omega\epsilon\vec{E} + \vec{J}_{ext}$
- $\nabla\cdot\vec{B} = 0$

Wave equation:

$\boxed{\text{1. E wave equation}}$

→ take the curl for 1st eq.

$\nabla\times\nabla\times\vec{E} = -\mu\frac{\partial}{\partial t}(\nabla\times\vec{H})$ · we know $(\nabla\times\vec{H})$ from 2nd eq.

$\because \nabla\cdot\vec{D} = \rho_v$, $\because \vec{D} = \epsilon\vec{E}$

→ take the curl for 1^{st} eq.

$$\nabla\times\nabla\times\vec{E} = -\mu\frac{\partial}{\partial t}(\nabla\times\vec{H})$$, we know $(\nabla\times\vec{H})$ from 2^{nd} eq.

$\because \nabla\cdot\vec{D} = \rho_v$, $\because \vec{D} = \epsilon\vec{E}$

$\therefore \nabla\cdot(\epsilon\vec{E}) = \rho_v$

$\therefore \nabla\cdot\vec{E} = \frac{\rho_v}{\epsilon}$

$$\therefore \nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E} = -\mu\frac{\partial}{\partial t}\left[\sigma\vec{E} + \epsilon\frac{\partial\vec{E}}{\partial t} + J_{ext}\right]$$

$$\therefore \nabla\left(\frac{\rho_v}{\epsilon}\right) - \nabla^2\vec{E} = -\mu\sigma\frac{\partial\vec{E}}{\partial t} - \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} - \mu\frac{\partial\vec{J}}{\partial t}$$

$$\therefore \nabla^2\vec{E} - \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} - \mu\sigma\frac{\partial\vec{E}}{\partial t} = \nabla\left(\frac{\rho_v}{\epsilon}\right) + \mu\frac{\partial\vec{J}_{ext}}{\partial t}$$ → general electric field wave eq.

2. $\vec{H}$ wave equation:

→ take curl of 2^{nd} eq.

$$\nabla\times\nabla\times\vec{H} = \nabla\times\left[\sigma\vec{E} + \epsilon\frac{\partial\vec{E}}{\partial t} + \vec{J}_{ex}\right]$$

$$\nabla(\nabla\cdot\vec{H}) - \nabla^2(\vec{H}) = \sigma\,\nabla\times\vec{E} + \epsilon\frac{\partial}{\partial t}(\nabla\times\vec{E}) + \nabla\times\vec{J}_{ex}$$

$$0 - \nabla^2\vec{H} = \sigma\left(-\mu\frac{\partial\vec{H}}{\partial t}\right) + \epsilon\frac{\partial}{\partial t}\left(-\mu\frac{\partial\vec{H}}{\partial t}\right) + \nabla\times\vec{J}$$

$$-\nabla^2\vec{H} = -\sigma\mu\frac{\partial\vec{H}}{\partial t} - \epsilon\mu\frac{\partial^2\vec{H}}{\partial t^2} + \nabla\times\vec{J}_{ex}$$

$$\therefore \nabla^2\vec{H} - \mu\epsilon\frac{\partial^2\vec{H}}{\partial t^2} - \mu\sigma\frac{\partial\vec{H}}{\partial t} = -\nabla\times\vec{J}_{ex}$$ → general mag. wave eq.

☆ Solving the wave equation:

→ For a lossles dielectric source free medium "Free space"

$\therefore \sigma = 0$, $\rho_v = 0$, $J_{ext} = 0$

→ wave eqs. become:

- $\nabla^2\vec{E} - \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} = 0 \xrightarrow{\text{time harmonic}} \nabla^2\vec{E} + \underbrace{\omega^2\mu\epsilon}_{k^2}\vec{E} = 0 \longrightarrow \nabla^2\vec{E} + k^2\vec{E} = 0$
- $\nabla^2\vec{H} - \mu\epsilon\frac{\partial^2\vec{H}}{\partial t^2} = 0 \xrightarrow{\text{- - - - - -}} \nabla^2\vec{H} + \underbrace{\omega^2\mu\epsilon}_{k^2}\vec{H} = 0 \longrightarrow \nabla^2\vec{H} + k^2\vec{H} = 0$

Helmholt's eqs. form

wave number

$\because V = \frac{1}{\sqrt{\mu\epsilon}}$, $\because K = \frac{\omega}{V}$

$\therefore \mu\epsilon = \frac{1}{V^2} = \frac{1}{\omega^2/k^2} = \frac{k^2}{\omega^2}$

$\therefore k^2 = \omega^2\mu\epsilon$