2.1 : Plane Wave Solutions

Assume wave Propagation in x-axis, thus: $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$

$\because \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$

$$\therefore \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & 0 & 0 \\ E_x & E_y & E_z \end{vmatrix} = -\mu \frac{\partial}{\partial t}\left(H_x \hat{x} + H_y \hat{y} + H_z \hat{z} \right)$$

$\therefore 0 \hat{x} = -\mu \frac{\partial}{\partial t} H_x \hat{x}$

$\therefore \frac{\partial H_x}{\partial t} = 0 \rightarrow H_x = \text{Const}$ ✗

$\rightarrow H_x = 0$ ✓

no mag. field comp. along the direction of Prop.

• lossless medium

$\because \nabla \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t}$ "Source free + lossless" ($J = 0$, $\sigma = 0$)

$\therefore \frac{\partial E_x}{\partial t} = 0 \rightarrow E_x = \text{Const.}$ ✗

$\rightarrow E_x = 0$ ✓

• lossy medium:

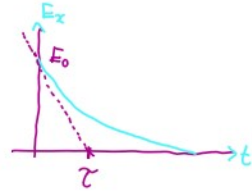
$\nabla \times \vec{H} = | \quad | = \sigma \langle E_x, E_y, E_z \rangle + \epsilon \frac{\partial}{\partial t} \langle E_x, E_y, E_z \rangle$

$\therefore \sigma E_x + \epsilon \frac{\partial E_x}{\partial t} = 0$

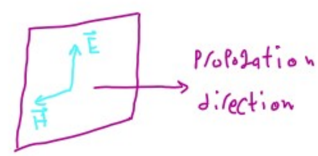
$\therefore \frac{\partial E_x}{\partial t} = \frac{-\sigma}{\epsilon} E_x$

$\therefore E_x = E_0 e^{-t/\tau}$, $\tau = \frac{\epsilon}{\sigma} \rightsquigarrow 0$

$\therefore E_x \rightsquigarrow 0$



→ there's no Component of $\vec{E}, \vec{H}$ along the direction of Propagation, $\vec{E}, \vec{H}$ are found in the Plane normal to Prop. direction. $\vec{P} = \vec{E} \times \vec{H}$



→ therefor, $\vec{E}$ wave equation become:

$\nabla_x^2 \vec{E} + \omega^2 \mu \epsilon \vec{E} = 0$, $\vec{E} = E_y \hat{y} + E_z \hat{z}$

$\therefore \frac{\partial^2 E_y}{\partial x^2} - \gamma E_y = 0$, $\gamma = -\omega^2 \mu \epsilon$

$$\therefore E_y(x) = E_{oy^+} e^{-\gamma x} + E_{oy^-} e^{\gamma x}$$

Incident Forword ; Reflecte backword

Similar for z Component:

$$\therefore E_z(x) = E_{oz^+} e^{-\gamma x} + E_{oz^-} e^{-\gamma x}$$

→ $\vec{H}$ wave eq:

• $\nabla_x^2 \vec{H} - \gamma \vec{H} = 0$, $\vec{H} = H_y \hat{y} + H_z \hat{z}$

→ y comp:

$\therefore \frac{\partial^2 H_y}{\partial x^2} - \gamma H_y = 0$

$$\therefore H_y(x) = H_{oy^+} e^{-\gamma x} + H_{oy^-} e^{\gamma x}$$

→ z comp:

$\therefore \frac{\partial^2 H_z}{\partial x^2} - \gamma H_z = 0$

$$\therefore H_z(x) = H_{oz^+} e^{-\gamma x} + H_{oz^-} e^{\gamma x}$$

→ We send the electric signal in the Tx, So we know E_{oy}, E_{oz} Constants. Mag field is generated due to the electric field

→ We send the electric signal in the Tx, so we know E_{oy}, E_{oz} constants. Mag field is generated due to the electric field, and we don't know the H_{oy}, H_{oz} const., so we need to find a relation between H, E const. to know H_{oy}, H_{oz}.

* Relation between E, H Constants

we will get the relation from maxwell eq.

$\nabla \times \vec{E} = -J\omega\mu\vec{H}$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & 0 & 0 \\ E_x & E_y & E_z \end{vmatrix} = -J\omega\mu \langle 0, H_y, H_z \rangle$$

y-dir: $-\left(\frac{\partial E_z}{\partial x}\right) = -J\omega\mu H_y$

z-dir: $\frac{\partial E_y}{\partial x} = -J\omega\mu H_z$

$\therefore \frac{\partial E_y}{\partial x} = -J\omega\mu H_z$, $\because E_y = E_{oy+}e^{-\gamma x} + E_{oy^-}e^{\gamma x}$, $\because H_z = H_{oz^+}e^{-\gamma x} + H_{oz^-}e^{\gamma x}$

→ For incident wave

$\therefore \cancel{-}\gamma E_{oy^+} e^{-\gamma x} = \cancel{-}J\omega\mu (H_{oz^+} e^{-\gamma x})$

$\therefore H_{oz^+} = \frac{E_{oy^+}}{J\omega\mu/\gamma} = \frac{E_{oy^+}}{\eta}$

$$\boxed{\therefore H_{oz^+} = \frac{E_{oy^+}}{\eta}}$$

→ For reflected wave

$$\boxed{\therefore H_{oz^-} = -\frac{E_{oy^-}}{\eta}}$$

η: wave Impedance, $\eta = \frac{J\omega\mu}{\gamma}$

$$\therefore E_y(x) = E_{oy^+} e^{-\gamma x} + E_{oy^-} e^{\gamma x} \qquad \therefore H_y(x) = -\frac{E_{oz}^+}{\eta} e^{-\gamma x} + \frac{E_{oz}^-}{\eta} e^{\gamma x}$$

$$\therefore E_z(x) = E_{oz^+} e^{-\gamma x} + E_{oz^-} e^{\gamma x} \qquad \therefore H_z(x) = \frac{E_{oy}^+}{\eta} e^{-\gamma x} - \frac{E_{oy}^-}{\eta} e^{\gamma x}$$

→ it's important to notice the signs of each forward & backward component in the above two eqs, The direction of propagation is ⊥ E & H "in this case E_z, H_y", meaning the propagation direction is in $\hat{x}$, now if we get the cross product of $E_z(x)$ & $H_y(x)$ of the forward part, it will be $(E_{oz^+} e^{-\gamma x})\hat{z} \otimes (\frac{-E_{oz}}{\eta} e^{-\gamma x})\hat{y}$ which is $\hat{z} \otimes -\hat{y}$, which is $\hat{x}$ "The propagation direction", for the backword component; $\hat{z} \otimes \hat{y} = -\hat{x}$ which is the direction of the reflected wave.

* Prove that $\vec{E}$ & $\vec{H}$ are orthogonal

$\because \vec{E}_F = \langle 0, E_{oy}^+ e^{-\gamma x}, E_{oz^+} e^{-\gamma x} \rangle$

$\because \vec{H}_F = \langle 0, \frac{-E_{oz}^+}{\eta} e^{-\gamma x}, \frac{E_{oz}^+}{\eta} e^{\gamma x} \rangle$

$\vec{E}_F \cdot \vec{H}_F = E_{oy}^+ e^{-\gamma x} \cdot \frac{-E_{oz}^+}{\eta} e^{-\gamma x} + E_{oz}^+ e^{-\gamma x} \cdot \frac{E_{oz}^+}{\eta} e^{\gamma x}$

$\vec{E}_F \cdot \vec{H}_F = 0$

$\therefore \boxed{\vec{E}_F \perp \vec{H}_F}$

$\because \vec{E}_r = \langle 0, E_{oy}^- e^{\gamma x}, E_{oz^-} e^{\gamma x} \rangle$

$\therefore \vec{H}_r = \langle 0, -\frac{E_{oz}^-}{\eta} e^{\gamma x}, -\frac{E_{oy}^-}{\eta} e^{\gamma x} \rangle$

$\therefore \vec{E}_r \cdot \vec{H}_r = 0$

$\therefore \boxed{E_r \perp H_r}$

$\therefore \vec{E} \cdot \vec{H} = 0$

$\boxed{\therefore E \perp H}$

Complex Propagation Constant

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→ For a lossy medium, The Maxwell equation:

$$\nabla \times \vec{H} = \sigma \vec{E} + j\omega\epsilon\vec{E} = j\omega(\epsilon + \frac{\sigma}{j\omega})\vec{E} = j\omega\,\epsilon_{eff}\,\vec{E}$$

∴ $\epsilon_{eff} = \epsilon - j\frac{\sigma}{\omega}$, therefore, for a dielectric medium $\gamma = \sqrt{-\omega^2\mu\epsilon}$ but for a lossy medium $\gamma = \sqrt{-\omega^2\mu\underbrace{\epsilon_{eff}}_{\text{Complex}}}$

→ γ is now a Complex number

$$\gamma = \sqrt{-\omega^2\mu(\epsilon - j\frac{\sigma}{\omega})} = \underbrace{\alpha}_{\text{attenuation Constant}} + j\underbrace{\beta}_{\text{Phase shift}}$$

∴ $\eta = \sqrt{\frac{\mu}{\epsilon_{eff}}}$ → η is complex too due to ϵ_{eff}

∴ The Final form of the wave eqn

$$\therefore E_y(x,t) = E_{oy}\,e^{-\alpha x}\,e^{j(\omega t - \beta x)} + E_{o\bar{y}}\,e^{\alpha x}\,e^{j(\omega t + \beta x)}$$

$$\therefore E_z(x,t) = E_{oz}^{+}\,e^{-\alpha x}\,e^{j(\omega t - \beta x)} + E_{\bar{oz}}\,e^{\alpha x}\,e^{j(\omega t + \beta x)}$$

$$\therefore H_y(x,t) = \frac{-E_{oz}^{+}}{\eta}\,e^{-\alpha x}\,e^{j(\omega t - \beta x)} + \frac{E_{o\bar{z}}}{\eta}\,e^{\alpha x}\,e^{j(\omega t + \beta x)}$$

$$\therefore H_z(x,t) = \frac{E_{oy}^{+}}{\eta}\,e^{-\alpha x}\,e^{j(\omega t - \beta x)} + \frac{-E_{o\bar{y}}}{\eta}\,e^{\alpha x}\,e^{j(\omega t + \beta x)}$$

→ note that α has an undiserble effect on the wave, as the wave prologate the $x\uparrow\uparrow$, $e^{-\alpha x}\downarrow\downarrow$ meaning the magntiude ↓↓ "attenuation", this happens in the lossy medium.

→ before in the lossless dielectric medium γ ~~whe~~ was pure imaginary, therefor $\alpha = 0$ thus there were no attenuation, only the change was in the phase.

Velocities:

→ Phase velocity $V_{ph} = \frac{\omega}{\beta}$ is the velocity of the wave front "the plane"

→ Group velocity $V_g = \frac{\partial\omega}{\partial\beta}$ - - - - - - - - - - - - - - - - - - energy

$V_{ph} \geq V_g$

→ The optimum value of $\alpha = 0$ or constant that is known to account for at the T_x & R_x, for β = linear with freq. ?

2.2. Propagation in Dielectric Medium

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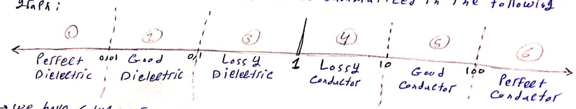
→ Conduction current density $J_c = \sigma\vec{E}$, displacment current density $J_D = \frac{\partial D}{\partial t} = j\omega\epsilon\vec{E}$

→ The ration between the magntiudes $\left|\frac{J_c}{J_D}\right| = \frac{\sigma}{\omega\epsilon}$

if $\frac{\sigma}{\omega\epsilon} \ll 1$, The medium is considered as dielectric, while if $\frac{\sigma}{\omega\epsilon} \gg 1$ The medium is considered conducting.

→ The classification of mediums can be summarized in the following graph:



→ we have 6 types of mediums, we need to know how to get α, β for each.

→ we have 3 types of dielectric mediums

1. Lossy dielectric medium $0,1 < \frac{\sigma}{\omega\epsilon} < 1$

The propagation constant γ is given as $\gamma = \sqrt{-\omega^2\mu\epsilon_{eff}} = \sqrt{-\omega^2\mu(\epsilon - j\frac{\sigma}{\omega})}$

$$\therefore \gamma = j\omega\sqrt{\mu\epsilon}\sqrt{1 - j\frac{\sigma}{\omega\epsilon}}$$

$$\because \sqrt{1 - j\frac{\sigma}{\omega\epsilon}} \simeq 1 - \frac{1}{2}j\frac{\sigma}{\omega\epsilon} + \frac{1}{8}\left(\frac{\sigma}{\omega\epsilon}\right)^2 + \frac{j}{16}\left(\frac{\sigma}{\omega\epsilon}\right)^3$$

$\sqrt{1-x} = 1 - \frac{x}{2} - \frac{1}{8}x^2 - \frac{1}{16}x^3 \dots$ (F $x < 1$)

$$\therefore \gamma = j\omega\sqrt{\mu\epsilon}\,[\quad] = \alpha + j\beta$$

" Second degree in ω "

$$\therefore \alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}\left[1 - \frac{\sigma^2}{8\omega^2\epsilon^2}\right], \quad \beta = \omega\sqrt{\mu\epsilon}\left[1 + \frac{\sigma^2}{8\omega^2\epsilon^2}\right]$$

→ neither α is constant nor β is linear with ω :")

2. Good dielectric medium $0,01 < \frac{\sigma}{\omega\epsilon} < 0,1$

→ in this case $\sqrt{1-x} \simeq 1 - \frac{x}{2}$ as $x \ll 1$

thus, $\gamma = j\omega\sqrt{\mu\epsilon}\sqrt{1 - j\frac{\sigma}{\omega\epsilon}} \simeq j\omega\sqrt{\mu\epsilon}\left(1 - \frac{j}{2}\left(\frac{\sigma}{\omega\epsilon}\right)\right) = \alpha + j\beta$

$\therefore \alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$ ⇝ Constant "solved by amplification"

$\therefore \beta = \omega\sqrt{\mu\epsilon}$ ⇝ Linear with freq.

3. Perfect dielectric $\frac{\sigma}{\omega\epsilon} < 0,01$

→ $\sqrt{1-x} \simeq 1$ as $x \lll 1$

$\gamma = j\omega\sqrt{\mu\epsilon}\sqrt{1 - j\frac{\sigma}{\omega\epsilon}} \simeq j\omega\sqrt{\mu\epsilon} = \alpha + j\beta$

$\therefore \left.\begin{array}{l}\alpha = 0 \\ \beta = \omega\sqrt{\mu\epsilon}\end{array}\right\}$ optimum values

Skin Depth (δ)

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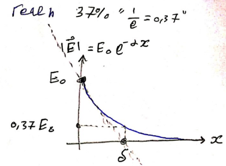
δ is the depth at which the magntiude of $\vec{E}$ reach 37% "$\frac{1}{e} = 0,37$" its initial value.

$$|E(x)| = E_o\,e^{-\alpha x}$$



at $x = \delta$:

$$E_o\,e^{-\alpha\delta} = 0{,}37\,E_o = E_o\,e^{-1}$$

$$\therefore \alpha\delta = 1 \rightsquigarrow \boxed{\delta = \frac{1}{\alpha}}$$

→ ∴ we conclude: we knew that that the attenuation is α for a medium, ∴ we know that the maxmium distance between the T_x & R_x is $\delta = \frac{1}{\alpha}$, any further The R_x will consider it as noise.