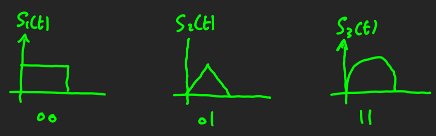
* Probabity of error :

- Generally for any vector $P_e = 1 - \sum_{i=1}^{M} P_i \int_{D_i} P(\vec{r}|\vec{s_i})\, d\vec{r}$, we never use it , Complex.

- P_e doesn't change under rotation or transiation ?
  - ↳ Same decision region area.
  - ↳ Same distance between Constellation Points.
  - ↳ Noise is spherically symetric.

* Bit vs symbol Prob. error :

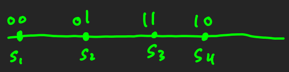


- P_e is the average symbol error , it doesn't give info about bit error.
  - ↳ if we sent S_1 and recived S_2, symbol error = 1 & bit error = 1
  - ↳ if we sent S_1 and recived S_3 , symbol error = 1 & bit error = 2

- Symbol error makes sence when send symbols info, like english letters.

- In general there's no relation between symbol error & bit error, except two cases :

Gray Coded Consellation



- $P_b \simeq \frac{P_e}{\log_2(M)}$ ← symbol error , # bits = $K = \log_2(M)$

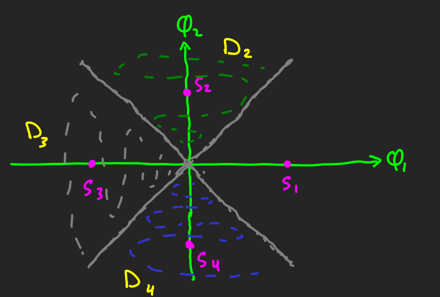
- assumes error happens between S_i and S_{i+1}

M'ary $M = 2^K$

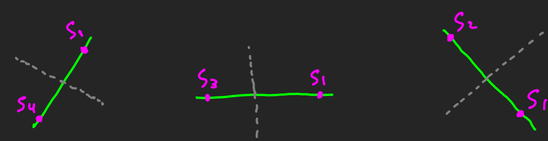
* $P_b = \frac{(M/2)}{M-1} P_e$ ← symbol error

* Bounds on P_e :

- As we saw , P_e equation is hard to solve. We also don't care about the exact value when desiging.
  We can take the upper bound "worst case".
  $P_e = P(\vec{r}$ Lies in the shaded region)
  $P_e = P[\vec{r} \in (D_2 \cup D_3 \cup D_4)]$



- We can approximate to 3 separate binary systems.
- We know the P_e in case of binary system $Q(\frac{d}{\sqrt{2N_o}})$
- We can add the 3 P_e of the 3 separate binary systems and that will be an upper bound.



$\therefore P_e \leq \sum_{\substack{K=1 \\ i \neq K}}^{M} P_{iK} \implies P_e \leq P_{12} + P_{13} + P_{14}$

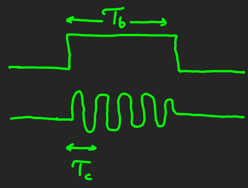
$\therefore P_e \leq \sum_{\substack{K=1 \\ i \neq K}}^{M} Q\left(\frac{d_{iK}}{\sqrt{2N_o}}\right)$. this approx. gives relativily high worst case
. higher than the actual system.

- We can approximate even more by taking only the smallest $d_{iK_{min}}$ and assuming all other d_{iK} is $d_{iK_{min}}$

$\therefore P_e \leq (M-1)\, Q\left(\frac{d_{min}}{\sqrt{2N_o}}\right)$

Important Concepts in digital Modulation:

Choice of Carrier Freq



$T_b = mT_c$

$$\frac{1}{T_b} = \frac{1}{m} \cdot \frac{1}{T_c} \rightsquigarrow \therefore R_b = \frac{f_c}{m}$$

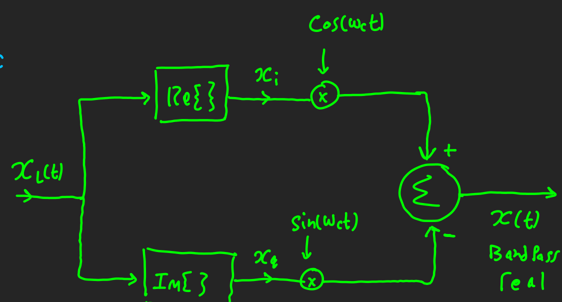
$$\boxed{\therefore f_c = m R_b}$$ Carrier freq has to be multible of the bit rate.

In Phase & Quadrature Components:

- any real, narrowband, bandpass signal $x(t)$ can be represented with these two Components.

$$x(t) = \underbrace{x_i(t)}_{\substack{\text{in-Phase} \\ \text{Component} \\ \text{real}}} + j\, \underbrace{x_q(t)}_{\substack{\text{Quadrature} \\ \text{component} \\ \text{real}}}$$

$$, \quad x(t) = \mathrm{Re}\{ \underbrace{x_L(t)}_{\text{Complex}}\, \underbrace{e^{j\omega_c t}}_{\text{seperate effect of high freq}} \}$$



$$x(t) = x_i(t) \cos(\omega_c t) - x_q(t) \sin(\omega_c t)$$

PSD & BW:

- For Linearly modulated signals "ASK, PSK, QAM", each signal is composed of an amplitude I_n and a shape $g(t)$ $\quad S(t) = I_n\, g(t)$, they can be expressed as:

$$X_L(t) = \sum_{n=-\infty}^{\infty} I(n)\, g(t - nT)$$

- in this expression PSD is given by $$S_{x_L}(f) = \frac{\overbrace{|G(f)|^2}^{F\{g(t)\}^2 \Rightarrow \text{ESD}}}{\underbrace{T}_{\text{symbol duration}}} \cdot \underbrace{F\{ \overbrace{R_I(k)}^{\text{Autocorrelation of amplitudes}} \}}_{\text{DFT of ACF}}$$

- We will work with the special case that I_n are i.i.d

$$\hookrightarrow R_I(k) = E\{ I_n I_{n+k} \} = \begin{cases} E[I_n^2] & k=0 \\ \mu_I^2 & \text{o.w} \end{cases}$$

- PSD of Modulated Signal

$$\hookrightarrow S_x(f) = \frac{1}{4}\left[S_{x_L}(f-f_c) + S_{x_L}(f+f_c) \right]$$

- example:

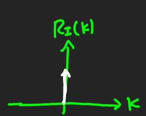
$I_n = \pm 1$, $g(t) = \mathrm{rect}\left(\frac{t}{T_b}\right)$ Polar NRZ, find B.W?

- $g(t) \xrightarrow{F} G(f)$
- $\mathrm{rect}\left(\frac{t}{T_b}\right) \xrightarrow{F} T_b \mathrm{sinc}(T_b f)$
- $R_I(k) = \begin{cases} E\{I_n^2\} & k=0 \\ \mu_I & \text{o.w} \end{cases}$

$\hookrightarrow E\{I_n^2\} = E\{1, 1, 1, \dots, N\} = \frac{1 * N}{N} = 1$

$\hookrightarrow \mu_I = \frac{1-1+1-1+\dots}{N} = \frac{0}{N} = 0$

$$\therefore R_I(k) = \left\{ \begin{matrix} 1 & k=0 \\ \mu_I & \text{o.T} \end{matrix} \right\} = \delta(k)$$



$$\because S_{x_L}(f) = \frac{G(f)^2}{T} \cdot F\{R_I(k)\}$$

$$\therefore S_{x_L}(f) = \frac{T_b^2 \mathrm{Sinc}^2(T_b f)}{T_b} \cdot F\{\delta(k)\}$$

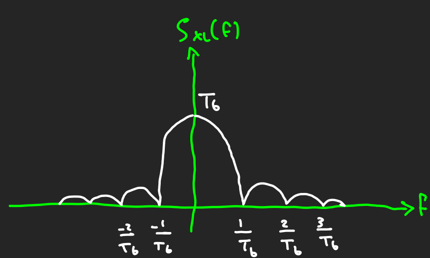
$$\therefore S_{x_L}(f) = T_b\, \mathrm{Sinc}^2(T_b f)$$

$\therefore$ BW null to null

$\hookrightarrow BW = \frac{1}{T_b} = R_B$

$\therefore$ BW 3dB

$\hookrightarrow BW = \frac{R_B}{2}$



* M'ary modulation:

bit stream → [bit-to-symbol conversion] → symbols → [Modulation] → M wave forms

take k bits and convert them to a symbol

- $R_s = \frac{R_b}{K} = \frac{R_b}{\log_2(M)}$ symbols/sec , $E_{b_{avg}} = \frac{E_{savg}}{K}$

- example: 4 PSK

$I_b = \underline{10}\,\underline{11}\,\underline{01}\,\underline{00}\,\underline{11}\,\underline{10}$

$M = 4 = 2^K \rightarrow K = 2$

bits	symbol	θ
00	0	0
01	1	$\pi/2$
11	3	π
10	2	$3\pi/2$

- why M'ary?
  - Less BW for the same rate $BW_{M'ary} = \frac{BW_b}{K}$
  - more information for same BW $R_s = K R_b$

* Signal Space:

- orthogonal basis functions $\varphi_1(t) = \sqrt{\frac{2}{T}} \sin(\omega_c t + \theta_c)$, $\varphi_2(t) = \sqrt{2/T} \cos(\omega_c t + \theta_c)$ $0 < t < T$

- $E_m = \| \vec{S}_m \|^2$, $\rho_{ij} = \cos(\theta_{ij})$

- $P_e = Q\left(\frac{d}{\sqrt{2N_0}}\right)$ (binary) $\leq (M-1) Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$

- Two Performance metrics:
  - Symbol efficiency $\eta_s = \frac{R_b}{BW}$ bits/Hz , it's a messure of how well we used the BW avalible Required $\eta_s = \frac{R_b}{BW} \uparrow\uparrow$ biggest bit rate possible for smallest BW possible
  - Power efficiency $\eta_p = \min \frac{E_b}{N_0}$ @ $P_e = 10^{-5}$ $\downarrow\downarrow$ Lowest energy possible for worst Noise.