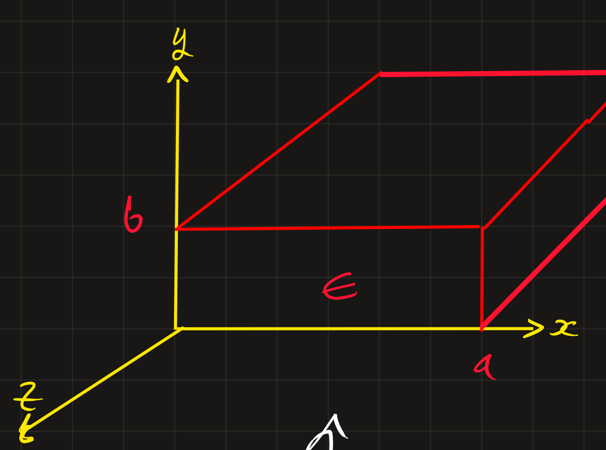
*Rectangular Waveguides:

$\nabla^2 \vec{E} + w^2 \mu \epsilon \vec{E} = 0 \longrightarrow (1)$

$\nabla^2 \vec{H} + w^2 \mu \epsilon \vec{H} = 0 \longrightarrow (2)$



→ We assume we have 6-components of $\vec{E}, \vec{H}$ ($E_x, E_y, E_z, H_x, H_y, H_z$)

→ $\nabla^2 \equiv \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}$

→ $H_z(x,y,z) = X(x)\, Y(y)\, Z(z) = X Y Z$

$\therefore$ (2): $YZ \frac{d^2X}{dx^2} + XZ \frac{d^2Y}{dy^2} + XY \frac{d^2Z}{dz^2} + w^2 \mu \epsilon XYZ = 0 \qquad \div XYZ$

$\therefore \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} + w^2 \mu \epsilon = 0$

→ since, the propagation direction in the z-direction

$\therefore \frac{1}{Z}\frac{d^2Z}{dz^2} = \gamma^2 \rightsquigarrow \therefore \frac{d^2Z}{dz^2} - \gamma^2 Z = 0$

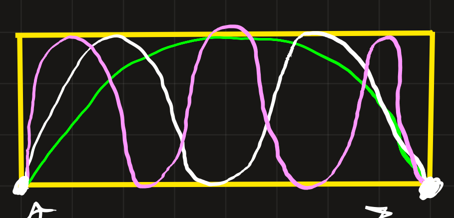
$\gamma = \alpha + j\beta$
$= j\beta$ "$\alpha = 0$ dielectric"

$\therefore Z = \underrightarrow{C_5 e^{-\gamma z}} + \underleftarrow{C_6 e^{\gamma z}}$

→ For a travelling wave inside the waveguide $\Rightarrow \boxed{Z = C_5 e^{-\gamma z}}$ no reflection

→ $\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{-A^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{-B^2} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{\gamma^2} + w^2 \mu \epsilon = 0$

- so far we know only one part "γ^2", we also do know something about the propagation component in x, y direction.
- we design the waveguide so that wave at the edges is minimum (0) and only sin & cos are the two waves which can be zero at the edges



- to get sin & cos in the differential solution the constants needs to be zero.
- this also true to get the equation $= 0$

$\therefore \frac{1}{X}\frac{d^2X}{dx^2} = -A^2 \rightsquigarrow \therefore \frac{d^2X}{dx^2} + A^2 X = 0 \rightsquigarrow \therefore \boxed{X = C_1 \sin(Ax) + C_2 \cos(Ax)}$

$\therefore \frac{1}{Y}\frac{d^2Y}{dy^2} = -B^2 \rightsquigarrow \therefore \frac{d^2Y}{dy^2} + B^2 Y = 0 \rightsquigarrow \therefore \boxed{Y = C_3 \sin(By) + C_4 \cos(By)}$

$\therefore \boxed{H_z(x,y,z) = [C_1 \sin(Ax) + C_2 \cos(Ax)]\,[C_3 \sin(By) + C_4 \cos(By)]\,[C_5 e^{-\gamma z}]}$

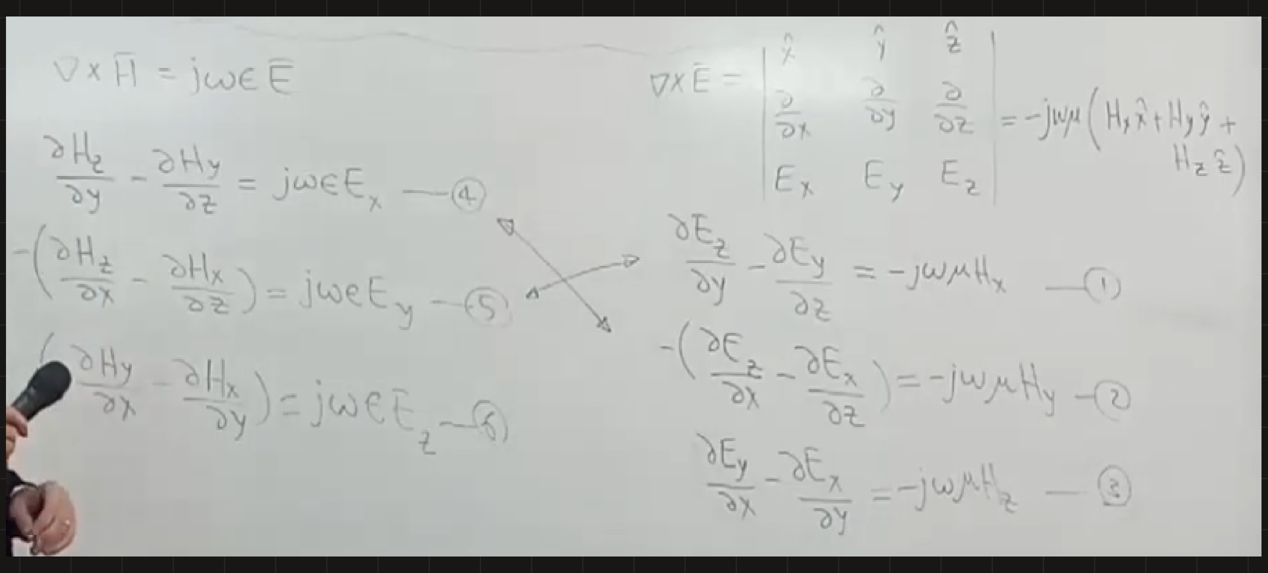
$\therefore -A^2 - B^2 + \gamma^2 + w^2 \mu \epsilon = 0$

$\therefore \boxed{K_c^2 = A^2 + B^2 = \gamma^2 + w^2 \mu \epsilon}$

* transverse components: E_x, E_y, H_x, H_y , Longitude components: E_z, H_z
* We have 6-unknown constants

→ From Maxwell's equations

- $\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} = -jw\mu \vec{H}$
- $\nabla \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t} = jw\epsilon \vec{E}$



From the 6-equations we can obtain the following Matrix:

$$\begin{bmatrix} E_x \\ H_y \\ E_y \\ H_x \end{bmatrix} = \frac{1}{K_c^2} \begin{bmatrix} -\gamma & -jw\mu & 0 & 0 \\ -jw\epsilon & -\gamma & 0 & 0 \\ 0 & 0 & -\gamma & jw\mu \\ 0 & 0 & jw\epsilon & -\gamma \end{bmatrix} \begin{bmatrix} \frac{\partial E_z}{\partial x} \\ \frac{\partial H_z}{\partial y} \\ \frac{\partial E_z}{\partial y} \\ \frac{\partial H_z}{\partial x} \end{bmatrix}$$

*Modes of waves inside the waveguide:

↳ 1) Transverse EM wave (TEM)

→ $E_z = 0$, $H_z = 0$
→ TEM can't propagate through the waveguide, because $E_x = E_y = H_x = H_y = 0$ from the Matrix.

↳ 2) TE Mode

→ $E_z = 0$, $H_z \neq 0$ ⇝ can propagate

↳ 3) TM Mode

→ $E_z \neq 0$, $H_z = 0$ ⇝ can propagate

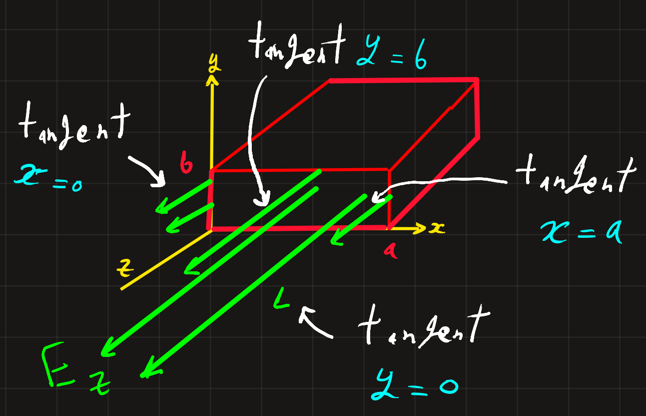
*TM Mode:

→ $H_z = 0$, $E_z = [C_1 \sin(Ax) + C_2 \cos(Ax)][C_3 \sin(By) + C_4 \cos(By)][C_5 e^{-j\beta}]$

→ $\gamma = j\beta$, $\alpha = 0$

* Boundry Conditions:

- $E_z|_{x=0} = 0$
- $E_z|_{x=a} = 0$
- $E_z|_{y=0} = 0$
- $E_z|_{y=b} = 0$



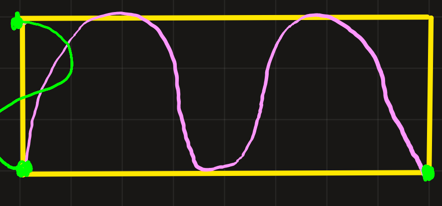
(1) $E_z|_{x=0} = 0$

$\therefore [C_1 \times 0 + C_2 \times 1][\cdots] = 0 \rightsquigarrow \therefore C_2 = 0 \rightsquigarrow \therefore E_z = [C_1 \sin(Ax)][C_3 \sin(By) + C_4 \cos(By)] C_5 e^{-\gamma z}$

(2) $E_z|_{y=0} = 0$

$\therefore [\cdots][C_3 \times 0 + C_4 \times 1][\cdots] = 0 \rightsquigarrow \therefore C_4 = 0 \rightsquigarrow \therefore E_z = C_1 \sin(Ax) \cdot C_3 \sin(By) \cdot C_5 e^{-\gamma z}$

. it's good to note that the fact the X, Y turned out to be sin does make sence because it's zero at the edges



$\therefore E_z = \underbrace{C_1 C_3 C_5}_{E_0} \sin(Ax) \sin(By) e^{-\gamma z}$

(3) $E_z|_{x=a} = 0$

$\therefore E_0 \sin(Aa) \sin(By) e^{-\gamma z} = 0$

$\therefore \sin(Aa) = 0 \rightsquigarrow \therefore Aa = m\pi$
$m = 1, 2, 3, \ldots$
reject $m = 0$

$\therefore \boxed{A = \frac{m\pi}{a}}$

(4) $E_z|_{y=b} = 0$

$\therefore E_0 \sin(\frac{m\pi}{a} x) \sin(Bb) e^{-\gamma z} = 0$

$\therefore \sin(Bb) = 0 \rightsquigarrow \therefore Bb = n\pi$
$n = 1, 2, 3, \ldots$
reject $n = 0$

$\therefore \boxed{B = \frac{n\pi}{b}}$

$\therefore$ For TM_{mn}

$\therefore \boxed{E_z = E_0 \sin(\frac{m\pi}{a} x) \sin(\frac{n\pi}{b} y) e^{-j\beta z}}$

$\therefore \boxed{K_c^2} = A^2 + B^2 = \gamma^2 + w^2 \mu \epsilon = \boxed{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2}$