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Extension of gamma, beta and hypergeometric functions

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ABSTRACT

The main object of this paper is to present generalizations of gamma, beta and hypergeometric functions. Some recurrence relations, transformation formulas, operation formulas and integral representations are obtained for these new generalizations.

1. Introduction

In recent years, some extensions of the well known special functions have been considered by several authors [1–7]. In 1994, Chaudhry and Zubair [1] have introduced the following extension of gamma function

$$\Gamma_p(x) := \int_0^\infty t^{x-1} \exp\left(-t - pt^{-1}\right) \mathrm{d}t, \tag{1}$$

$\mathrm{Re}(p) > 0.$

In 1997, Chaudhry et al. [2] presented the following extension of Euler's beta function

$$B_p(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} \exp\left[-\frac{p}{t(1-t)}\right] \mathrm{d}t, \tag{2}$$

$(\mathrm{Re}(p) > 0, \mathrm{Re}(x) > 0, \mathrm{Re}(y) > 0)$

and they proved that this extension has connections with the Macdonald, error and Whittakers function. It is clearly seem that $\Gamma_0(x) = \Gamma(x)$ and $B_0(x, y) = B(x, y)$. Afterwards, Chaudhry et al. [8] used $B_p(x, y)$ to extend the hypergeometric functions (and confluent hypergeometric functions) as follows:

$$F_p(a, b; c; z) = \sum_{n=0}^{\infty} \frac{B_p(b+n, c-b)}{B(b, c-b)} (a)_n \frac{z^n}{n!}$$

$p \geq 0; \mathrm{Re}(c) > \mathrm{Re}(b) > 0,$

$$\phi_p(b; c; z) = \sum_{n=0}^{\infty} \frac{B_p(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}$$

$p \geq 0; \mathrm{Re}(c) > \mathrm{Re}(b) > 0,$

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where $(\lambda)_\nu$ denotes the Pochhammer symbol defined by

$$(\lambda)_0 \equiv 1 \quad \text{and} \quad (\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}$$

and gave the Euler type integral representation

$$F_p\,(a,b;c;z) = \frac{1}{B\,(b,c-b)} \int_0^1 t^{b-1}\,(1-t)^{c-b-1}\,(1-zt)^{-a} \exp\left[-\frac{p}{t(1-t)}\right] \mathrm{d}t$$
$$p>0;\, p=0 \quad \text{and} \quad |\arg(1-z)| < \pi < p;\, \mathrm{Re}(c) > \mathrm{Re}(b) > 0.$$

They called these functions extended Gauss hypergeometric function (EGHF) and extended confluent hypergeometric function (ECHF), respectively. They have discussed the differentiation properties and Mellin transforms of $F_p\,(a,b;c;z)$ and obtained transformation formulas, recurrence relations, summation and asymptotic formulas for this function. Note that $F_0\,(a,b;c;z) = {}_2F_1\,(a,b;c;z)$.

In this paper, we consider the following generalizations of gamma and Euler's beta functions

$$\Gamma_p^{(\alpha,\beta)}(x) := \int_0^\infty t^{x-1}\,{}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right)\mathrm{d}t \tag{3}$$
$$\mathrm{Re}(\alpha)>0,\, \mathrm{Re}(\beta)>0,\, \mathrm{Re}(p)>0,\, \mathrm{Re}(x)>0,$$

$$B_p^{(\alpha,\beta)}\,(x,y) := \int_0^1 t^{x-1}\,(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t, \tag{4}$$
$$(\mathrm{Re}(p)>0,\, \mathrm{Re}(x)>0,\, \mathrm{Re}(y)>0,\, \mathrm{Re}(\alpha)>0,\, \mathrm{Re}(\beta)>0)$$

respectively. It is obvious by (1), (3) and (2), (4) that, $\Gamma_p^{(\alpha,\alpha)}(x) = \Gamma_p\,(x)$, $\Gamma_0^{(\alpha,\alpha)}(x) = \Gamma(x)$, $B_p^{(\alpha,\alpha)}\,(x,y) = B_p\,(x,y)$ and $B_0^{(\alpha,\beta)}\,(x,y) = B\,(x,y)$.

In Section 2, different integral representations and some properties of new generalized Euler's beta function are obtained. Additionally, relations of new generalized gamma and beta functions are discussed. In Section 3, we generalize the hypergeometric function and confluent hypergeometric function, using $B_p^{(\alpha,\beta)}\,(x,y)$ obtain the integral representations of this new generalized Gauss hypergeometric functions. Furthermore we discussed the differentiation properties, Mellin transforms, transformation formulas, recurrence relations, summation formulas for these new hypergeometric functions.

2. Some properties of gamma and beta functions

It is important and useful to obtain different integral representations of the new generalized beta function, for later use. Also it is useful to discuss the relationships between classical gamma functions and new generalizations. For $p=0$, we have the following integral representation for $\Gamma_p^{(\alpha,\beta)}(x)$.

Theorem 2.1. *For the new generalized gamma function, we have*

$$\Gamma_p^{(\alpha,\beta)}(s) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 \Gamma_{p\mu^2}(s)\mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1}\mathrm{d}\mu.$$

Proof. Using the integral representation of confluent hypergeometric function, we have

$$\Gamma_p^{(\alpha,\beta)}(s) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^\infty \int_0^1 u^{s-1}e^{-ut-\frac{pt}{u}}t^{\alpha-1}(1-t)^{\beta-\alpha-1}\mathrm{d}t\mathrm{d}u.$$

Now using a one-to-one transformation (except possibly at the boundaries and maps the region onto itself) $\nu = ut$, $\mu = t$ in the above equality and considering that the Jacobian of the transformation is $J = \frac{1}{\mu}$, we get

$$\Gamma_p^{(\alpha,\beta)}(s) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^\infty \int_0^1 v^{s-1}e^{-v-\frac{p\mu^2}{v}}\mathrm{d}v\mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1}\mathrm{d}\mu.$$

From the uniform convergence of the integrals, the order of integration can be interchanged to yield that

$$\begin{aligned}\Gamma_p^{(\alpha,\beta)}(s) &= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 \left[\int_0^\infty v^{s-1}e^{-v-\frac{p\mu^2}{v}}\mathrm{d}v\right]\mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1}\mathrm{d}\mu \\ &= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 \Gamma_{p\mu^2}(s)\mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1}\mathrm{d}\mu.\end{aligned}$$

Whence the result. □

The case $p=0$ in the above Theorem gives (see [9, p. 192])

$$\Gamma^{(\alpha,\beta)}(s)=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 \Gamma(s)\mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1}\mathrm{d}\mu=\frac{\Gamma(\beta)\Gamma(\alpha-s)\Gamma(s)}{\Gamma(\alpha)\Gamma(\beta-s)}. \tag{5}$$

The next theorem gives the Mellin transform representation of the function $B_p^{(\alpha,\beta)}(x,y)$ in terms of the ordinary beta function and $\Gamma^{(\alpha,\beta)}(s)$.

Theorem 2.2. *Mellin transform representation of the new generalized beta function is given by*

$$\int_0^\infty p^{s-1}B_p^{(\alpha,\beta)}(x,y)\,\mathrm{d}p=B(s+x,y+s)\Gamma^{(\alpha,\beta)}(s), \tag{6}$$
$\mathrm{Re}(s)>0,\ \mathrm{Re}(x+s)>0,\ \mathrm{Re}(y+s)>0,$
$\mathrm{Re}(p)>0,\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0.$

Proof. Multiplying (4) by p^{s-1} and integrating with respect to p from $p=0$ to $p=\infty$, we get

$$\int_0^\infty p^{s-1}B_p^{(\alpha,\beta)}(x,y)\,\mathrm{d}p=\int_0^\infty p^{s-1}\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t\mathrm{d}p. \tag{7}$$

From the uniform convergence of the integral, the order of integration in (7) can be interchanged. Therefore, we have

$$\int_0^\infty p^{s-1}B_p^{(\alpha,\beta)}(x,y)\,\mathrm{d}p=\int_0^1 t^{x-1}(1-t)^{y-1}\int_0^\infty p^{s-1}{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}p\mathrm{d}t. \tag{8}$$

Now using the one-to-one transformation (except possibly at the boundaries and maps the region onto itself) $\nu=\frac{p}{t(1-t)}$, $\mu=t$ in (8), we get,

$$\int_0^\infty p^{s-1}B_p^{(\alpha,\beta)}(x,y)\,\mathrm{d}p=\int_0^1 \mu^{(s+x)-1}(1-\mu)^{(y+s)-1}\,\mathrm{d}\mu\int_0^\infty \nu^{s-1}{}_1F_1(\alpha;\beta;-\nu)\mathrm{d}\nu.$$

Therefore, using (5), we have

$$\int_0^\infty p^{s-1}B_p^{(\alpha,\beta)}(x,y)\,\mathrm{d}p=B(s+x,y+s)\Gamma^{(\alpha,\beta)}(s). \quad \square$$

Corollary 2.3. *By the Mellin inversion formula, we have the following complex integral representation for* $B_p^{(\alpha,\beta)}(x,y)$*:*

$$B_p^{(\alpha,\beta)}(x,y)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}B(s+x,y+s)\Gamma^{(\alpha,\beta)}(s)\,p^{-s}\mathrm{d}s.$$

Remark 2.1. Putting $s=1$ and considering that $\Gamma^{(\alpha,\beta)}(1)=\frac{\Gamma(\beta)\Gamma(\alpha-1)}{\Gamma(\alpha)\Gamma(\beta-1)}$ in (6), we get

$$\int_0^\infty B_p^{(\alpha,\beta)}(x,y)\,\mathrm{d}p=B(x+1,y+1)\frac{\Gamma(\beta)\Gamma(\alpha-1)}{\Gamma(\alpha)\Gamma(\beta-1)}.$$

Letting $B_p^{(\alpha,\alpha)}(x,y)=B_p(x,y)$, it reduces to Chaudhry's [2] interesting relation

$$\int_0^\infty B_p(x,y)\,\mathrm{d}p=B(x+1,y+1),$$
$\mathrm{Re}(x)>-1,\ \mathrm{Re}(y)>-1,$

between the classical and the extended beta functions.

Theorem 2.4. *For the new generalized beta function, we have the following integral representations:*

$$B_p^{(\alpha,\beta)}(x,y)=2\int_0^{\pi/2}\cos^{2x-1}\theta\sin^{2y-1}\theta\,{}_1F_1\left(\alpha;\beta;-p\sec^2\theta\csc^2\theta\right)\mathrm{d}\theta,$$
$$B_p^{(\alpha,\beta)}(x,y)=\int_0^\infty\frac{u^{x-1}}{(1+u)^{x+y}}\,{}_1F_1\left(\alpha;\beta;-2p-p\left(u+\frac{1}{u}\right)\right)\mathrm{d}u.$$

Proof. Letting $t=\cos^2\theta$ in (4), we get

$$\begin{aligned} B_p^{(\alpha,\beta)}(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t \\ &= 2\int_0^{\pi/2}\cos^{2x-1}\theta\sin^{2y-1}\theta\,{}_1F_1(\alpha;\beta;-p\sec^2\theta\csc^2\theta)\mathrm{d}\theta. \end{aligned}$$

On the other hand, letting $t=\frac{u}{1+u}$ in (4), we get

$$\begin{aligned} B_p^{(\alpha,\beta)}(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t \\ &= \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\,{}_1F_1\left(\alpha;\beta;-2p-p\left(u+\frac{1}{u}\right)\right)\mathrm{d}u. \quad \square \end{aligned}$$

Theorem 2.5. *For the new generalized beta function, we have the following functional relation:*

$$B_p^{(\alpha,\beta)}(x,y+1)+B_p^{(\alpha,\beta)}(x+1,y)=B_p^{(\alpha,\beta)}(x,y).$$

Proof. Direct calculations yield

$$\begin{aligned} B_p^{(\alpha,\beta)}(x,y+1)+B_p^{(\alpha,\beta)}(x+1,y) &= \int_0^1 t^{x}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t \\ &\quad + \int_0^1 t^{x-1}(1-t)^{y}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t \\ &= \int_0^1 \left[t^{x}(1-t)^{y-1}+t^{x-1}(1-t)^{y}\right]{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t \\ &= \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t = B_p^{(\alpha,\beta)}(x,y). \end{aligned}$$

Whence the result. $\square$

Theorem 2.6. *For the product of two new generalized gamma function, we have the following integral representation:*

$$\begin{aligned} \Gamma_p^{(\alpha,\beta)}(x)\Gamma_p^{(\alpha,\beta)}(y) &= 4\int_0^{\frac{\pi}{2}}\int_0^\infty r^{2(x+y)-1}\cos^{2x-1}\theta\sin^{2y-1}\theta \\ &\quad\times {}_1F_1\left(\alpha;\beta;-r^2\cos^2\theta-\frac{p}{r^2\cos^2\theta}\right){}_1F_1\left(\alpha;\beta;-r^2\sin^2\theta-\frac{p}{r^2\sin^2\theta}\right)\mathrm{d}r\mathrm{d}\theta. \end{aligned} \tag{9}$$

Proof. Substituting $t=\eta^2$ in (3), we get

$$\Gamma_p^{(\alpha,\beta)}(x)=2\int_0^\infty \eta^{2x-1}{}_1F_1\left(\alpha;\beta;-\eta^2-\frac{p}{\eta^2}\right)\mathrm{d}\eta.$$

Therefore

$$\Gamma_p^{(\alpha,\beta)}(x)\Gamma_p^{(\alpha,\beta)}(y)=4\int_0^\infty\int_0^\infty \eta^{2x-1}\xi^{2y-1}{}_1F_1\left(\alpha;\beta;-\eta^2-\frac{p}{\eta^2}\right){}_1F_1\left(\alpha;\beta;-\xi^2-\frac{p}{\xi^2}\right)\mathrm{d}\eta\mathrm{d}\xi.$$

Letting $\eta=r\cos\theta$ and $\xi=r\sin\theta$ in the above equality,

$$\begin{aligned} \Gamma_p^{(\alpha,\beta)}(x)\Gamma_p^{(\alpha,\beta)}(y) &= 4\int_0^{\frac{\pi}{2}}\int_0^\infty r^{2(x+y)-1}\cos^{2x-1}\theta\sin^{2y-1}\theta \\ &\quad\times {}_1F_1\left(\alpha;\beta;-r^2\cos^2\theta-\frac{p}{r^2\cos^2\theta}\right){}_1F_1\left(\alpha;\beta;-r^2\sin^2\theta-\frac{p}{r^2\sin^2\theta}\right)\mathrm{d}r\mathrm{d}\theta. \quad \square \end{aligned}$$

Remark 2.2. Putting $p=0$ and $\alpha=\beta$ in (9), we get the classical relation between the gamma and beta functions:

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Theorem 2.7. *For the new generalized beta function, we have the following summation relation:*

$$B_p^{(\alpha,\beta)}(x, 1-y) = \sum_{n=0}^{\infty} \frac{(y)_n}{n!} B_p^{(\alpha,\beta)}(x+n, 1),$$
$$\mathrm{Re}(p) > 0.$$

Proof. From the definition of the new generalized beta function, we get

$$B_p^{(\alpha,\beta)}(x, 1-y) = \int_0^1 t^{x-1} (1-t)^{-y} {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) \mathrm{d}t.$$

Using the following binomial series expansion

$$(1-t)^{-y} = \sum_{n=0}^{\infty} (y)_n \frac{t^n}{n!},$$
$$|t| < 1,$$

we obtain

$$B_p^{(\alpha,\beta)}(x, 1-y) = \int_0^1 \sum_{n=0}^{\infty} \frac{(y)_n}{n!} t^{x+n-1} {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) \mathrm{d}t.$$

Therefore, interchanging the order of integration and summation and then using (4), we obtain

$$\begin{aligned} B_p^{(\alpha,\beta)}(x, 1-y) &= \sum_{n=0}^{\infty} \frac{(y)_n}{n!} \int_0^1 t^{x+n-1} {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) \mathrm{d}t, \\ &= \sum_{n=0}^{\infty} \frac{(y)_n}{n!} B_p^{(\alpha,\beta)}(x+n, 1). \quad \square \end{aligned}$$

Now we obtain differential recurrence relations for generalized gamma and generalized beta functions defined by (3) and (4), respectively.

Theorem 2.8. *For the generalized gamma function, we have the following recurrence relation:*

$$\begin{aligned} &\frac{\mathrm{d}^2\left(\Gamma_p^{(\alpha,\beta)}(x+5)\right)}{\mathrm{d}p^2} + p\frac{\mathrm{d}^2\left(\Gamma_p^{(\alpha,\beta)}(x+3)\right)}{\mathrm{d}p^2} - \beta\frac{\mathrm{d}\left(\Gamma_p^{(\alpha,\beta)}(x+2)\right)}{\mathrm{d}p} \\ &\quad - \frac{\mathrm{d}\left(\Gamma_p^{(\alpha,\beta)}(x+3)\right)}{\mathrm{d}p} - p\frac{\mathrm{d}\left(\Gamma_p^{(\alpha,\beta)}(x+1)\right)}{\mathrm{d}p} + \alpha\Gamma_p^{(\alpha,\beta)}(x) = 0. \end{aligned}$$

Proof. Taking derivatives under the integral symbol by using the Leibnitz rule, we get

$$\begin{aligned} &\frac{\mathrm{d}^2\left(\Gamma_p^{(\alpha,\beta)}(x+5)\right)}{\mathrm{d}p^2} + p\frac{\mathrm{d}^2\left(\Gamma_p^{(\alpha,\beta)}(x+3)\right)}{\mathrm{d}p^2} - \beta\frac{\mathrm{d}\left(\Gamma_p^{(\alpha,\beta)}(x+2)\right)}{\mathrm{d}p} \\ &\quad - \frac{\mathrm{d}\left(\Gamma_p^{(\alpha,\beta)}(x+3)\right)}{\mathrm{d}p} - p\frac{\mathrm{d}\left(\Gamma_p^{(\alpha,\beta)}(x+1)\right)}{\mathrm{d}p} + \alpha\Gamma_p^{(\alpha,\beta)}(x) \\ &= \int_0^{\infty} t^{x-1}\left[\left(t^3+pt\right)\frac{\mathrm{d}^2z}{\mathrm{d}p^2} + \left(t^2+\beta t+p\right)\frac{\mathrm{d}z}{\mathrm{d}p} + \alpha z\right]\mathrm{d}t, \end{aligned}$$

where $z = {}_1F_1\left(\alpha; \beta; -t-\frac{p}{t}\right)$. On the other hand, since $z = {}_1F_1\left(\alpha; \beta; -t-\frac{p}{t}\right)$ is a solution of the equation

$$\left(t^3+pt\right)\frac{\mathrm{d}^2z}{\mathrm{d}p^2} + \left(t^2+\beta t+p\right)\frac{\mathrm{d}z}{\mathrm{d}p} + \alpha z = 0,$$

we get the result. $\square$

Theorem 2.9. *For the generalized beta function, we have the following recurrence relation:*

$$p\frac{\mathrm{d}^2\left(B_p^{(\alpha,\beta)}(x+3,y+3)\right)}{\mathrm{d}p^2}+\beta\frac{\mathrm{d}\left(B_p^{(\alpha,\beta)}(x+2,y+2)\right)}{\mathrm{d}p}+p\frac{\mathrm{d}\left(B_p^{(\alpha,\beta)}(x+1,y+1)\right)}{\mathrm{d}p}+\alpha B_p^{(\alpha,\beta)}(x,y)=0.$$

Proof. Let $\mathcal{S}$ denote the left-hand side of the above assertion. Taking derivatives under the integral symbol in (4) by using the Leibnitz rule, we get

$$\mathcal{S}=\int_0^1 t^{x-1}(1-t)^{y-1}\left[pt(1-t)\frac{\mathrm{d}^2z}{\mathrm{d}p^2}+(\beta t(1-t)+p)\frac{\mathrm{d}z}{\mathrm{d}p}+\alpha z\right]\mathrm{d}t,$$

where $z={}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)$. Since $z={}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)$ is a solution of the equation

$$pt(1-t)\frac{\mathrm{d}^2z}{\mathrm{d}p^2}+(\beta t(1-t)+p)\frac{\mathrm{d}z}{\mathrm{d}p}+\alpha z=0,$$

we get the result. □

3. Generalized Gauss hypergeometric and confluent hypergeometric functions

In this section we use the new generalization (4) of beta functions to generalize the hypergeometric and confluent hypergeometric functions defined by

$$F_p^{(\alpha,\beta)}(a,b;c;z):=\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}$$

and

$${}_1F_1^{(\alpha,\beta;p)}(b;c;z):=\sum_{n=0}^{\infty}\frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!},$$

respectively.

We call the $F_p^{(\alpha,\beta)}(a,b;c;z)$ by the generalized Gauss hypergeometric function (GGHF) and ${}_1F_1^{(\alpha,\beta;p)}(b;c;z)$ by the generalized confluent hypergeometric function (GCHF).

Observe that [8],

$$F_p^{(\alpha,\alpha)}(a,b;c;z)=F_p(a,b;c;z),$$
$$F_0^{(\alpha,\beta)}(a,b;c;z)={}_2F_1(a,b;c;z),$$

and

$${}_1F_1^{(\alpha,\alpha;p)}(b;c;z)={}_1F_1^{(p)}(b;c;z)=\phi_p(b;c;z),$$
$${}_1F_1^{(\alpha,\beta;0)}(b;c;z)={}_1F_1(b;c;z).$$

3.1. Integral representations of the GGHF and GCHF

The GGHF can be provided with an integral representation by using the definition of the new generalized beta function (4). We get

Theorem 3.1. *For the GGHF, we have the following integral representations:*

$$F_p^{(\alpha,\beta)}(a,b;c;z):=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)(1-zt)^{-a}\,\mathrm{d}t, \tag{10}$$

$\mathrm{Re}(p)>0$; $p=0$ *and* $|arg\,(1-z)|<\pi$; $\mathrm{Re}(c)>\mathrm{Re}(b)>0$;

$$F_p^{(\alpha,\beta)}(a,b;c;z):=\frac{1}{B(b,c-b)}\int_0^{\infty}u^{b-1}(1+u)^{a-c}[1+u(1-z)]^{-a}{}_1F_1\left(\alpha;\beta;-2p-p\left(u+\frac{1}{u}\right)\right)\mathrm{d}u,$$

$$F_p^{(\alpha,\beta)}(a,b;c;z):=\frac{2}{B(b,c-b)}\int_0^{\frac{\pi}{2}}\sin^{2b-1}\nu\cos^{2c-2b-1}\nu\left(1-z\sin^2\nu\right)^{-a}{}_1F_1\left(\alpha;\beta;\frac{-p}{\sin^2\nu\cos^2\nu}\right)\mathrm{d}\nu.$$

Proof. Direct calculations yield

$$\begin{aligned} F_p^{(\alpha,\beta)}(a,b;c;z) &:= \sum_{n=0}^{\infty}(a)_n \frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} \\ &= \frac{1}{B(b,c-b)}\sum_{n=0}^{\infty}(a)_n \int_0^1 t^{b+n-1}(1-t)^{c-b-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\frac{z^n}{n!}\mathrm{d}t \\ &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\sum_{n=0}^{\infty}(a)_n\frac{(zt)^n}{n!}\mathrm{d}t \\ &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)(1-zt)^{-a}\,\mathrm{d}t. \end{aligned}$$

Setting $u=\frac{t}{1-t}$ in (10), we get

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^{\infty} u^{b-1}(1+u)^{a-c}[1+u(1-z)]^{-a}\,{}_1F_1\left(\alpha;\beta;-2p-p\left(u+\frac{1}{u}\right)\right)\mathrm{d}u.$$

On the other hand, substituting $t=\sin^2\nu$ in (10), we have

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \frac{2}{B(b,c-b)}\int_0^{\frac{\pi}{2}} \sin^{2b-1}\nu\cos^{2c-2b-1}\nu\left(1-z\sin^2\nu\right)^{-a}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{\sin^2\nu\cos^2\nu}\right)\mathrm{d}\nu. \quad \square$$

A similar procedure yields an integral representation of the GCHF by using the definition of the new generalized beta function.

Theorem 3.2. *For the GCHF, we have the following integral representations:*

$$ {}_1F_1^{(\alpha,\beta;p)}(b;c;z) := \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}e^{zt}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t, \tag{11}$$

$$ {}_1F_1^{(\alpha,\beta;p)}(b;c;z) := \int_0^1 \frac{(1-u)^{b-1}u^{c-b-1}}{B(b,c-b)}e^{z(1-u)}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{u(1-u)}\right)\mathrm{d}u,$$

$p\ge 0$; *and* $\mathrm{Re}(c)>\mathrm{Re}(b)>0$.

Remark 3.1. Putting $p=0$ in (10) and (11), we get the integral representations of the classical GHF and CHF.

3.2. Differentiation formulas for the new GGHF's and new GCHF's

In this section, by using the formulas $B(b,c-b)=\frac{c}{b}B(b+1,c-b)$ and $(a)_{n+1}=a\,(a+1)_n$, we obtain new formulas including derivatives of GGHF and GCHF with respect to the variable z.

Theorem 3.3. *For GGHF, we have the following differentiation formula:*

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{F_p^{(\alpha,\beta)}(a,b;c;z)\right\} = \frac{(b)_n(a)_n}{(c)_n}F_p^{(\alpha,\beta)}(a+n,b+n;c+n;z).$$

Proof. Taking the derivative of $F_p^{(\alpha,\beta)}(a,b;c;z)$ with respect to z, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}\left\{F_p^{(\alpha,\beta)}(a,b;c;z)\right\} &= \frac{\mathrm{d}}{\mathrm{d}z}\left\{\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}\right\} \\ &= \sum_{n=1}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^{n-1}}{(n-1)!}. \end{aligned}$$

Replacing $n\to n+1$, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}\left\{F_p^{(\alpha,\beta)}(a,b;c;z)\right\} &= \frac{ba}{c}\sum_{n=0}^{\infty}(a+1)_n\frac{B_p^{(\alpha,\beta)}(b+n+1,c-b)}{B(b+1,c-b)}\frac{z^n}{n!} \\ &= \frac{ba}{c}F_p^{(\alpha,\beta)}(a+1,b+1;c+1;z). \end{aligned}$$

Recursive application of this procedure gives us the general form:

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{F_p^{(\alpha,\beta)}(a,b;c;z)\right\} = \frac{(b)_n\,(a)_n}{(c)_n}F_p^{(\alpha,\beta)}(a+n,b+n;c+n;z)\,. \quad \square$$

Theorem 3.4. *For GCHF, we have the following differentiation formula:*

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\,{}_1F_1^{(\alpha,\beta;p)}(b;c;z) = \frac{(b)_n}{(c)_n}\,{}_1F_1^{(\alpha,\beta;p)}(b+n;c+n;z)\,.$$

3.3. Mellin transform representation of the GGHF's and GCHF's

In this section, we obtain the Mellin transform representations of the GGHF and GCHF.

Theorem 3.5. *For the GGHF, we have the following Mellin transform representation:*

$$\mathfrak{M}\left\{F_p^{(\alpha,\beta)}(a,b;c;z):s\right\} := \frac{\Gamma^{(\alpha,\beta)}(s)B(b+s,c+s-b)}{B(b,c-b)}\,{}_2F_1(a,b+s;c+2s;z)\,.$$

Proof. To obtain the Mellin transform, we multiply both sides of (10) by p^{s-1} and integrate with respect to p over the interval $[0,\infty)$. Thus we get

$$\begin{aligned}\mathfrak{M}\left\{F_p^{(\alpha,\beta)}(a,b;c;z):s\right\} &:= \int_0^\infty p^{s-1}F_p^{(\alpha,\beta)}(a,b;c;z)\,\mathrm{d}p\\ &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\left[\int_0^\infty p^{s-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}p\right]\mathrm{d}t.\end{aligned} \tag{12}$$

Since substituting $u=\frac{p}{t(1-t)}$ in (12),

$$\begin{aligned}\int_0^\infty p^{s-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}p &= \int_0^\infty u^{s-1}t^s(1-t)^s\,{}_1F_1(\alpha;\beta;-u)\,\mathrm{d}u\\ &= t^s(1-t)^s\int_0^\infty {}_1F_1(\alpha;\beta;-u)\,\mathrm{d}u\\ &= t^s(1-t)^s\Gamma^{(\alpha,\beta)}(s).\end{aligned}$$

Thus we get

$$\begin{aligned}\mathfrak{M}\left\{F_p^{(\alpha,\beta)}(a,b;c;z):s\right\} &= \frac{1}{B(b,c-b)}\int_0^1 t^{b+s-1}(1-t)^{c+s-b-1}(1-zt)^{-a}\Gamma^{(\alpha,\beta)}(s)\mathrm{d}t\\ &= \frac{\Gamma^{(\alpha,\beta)}(s)B(b+s,c+s-b)}{B(b,c-b)}\frac{1}{B(b+s,c+s-b)}\int_0^1 t^{b+s-1}(1-t)^{c+2s-(b+s)-1}(1-zt)^{-a}\mathrm{d}t\\ &= \frac{\Gamma^{(\alpha,\beta)}(s)B(b+s,c+s-b)}{B(b,c-b)}\,{}_2F_1(a,b+s;c+2s;z)\,. \quad \square\end{aligned}$$

Corollary 3.6. *By the Mellin inversion formula, we have the following complex integral representation for $F_p^{(\alpha,\beta)}$:*

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma^{(\alpha,\beta)}(s)B(b+s,c+s-b)}{B(b,c-b)}\,{}_2F_1(a,b+s;c+2s;z)\;p^{-s}\mathrm{d}s.$$

Theorem 3.7. *For the new GCHF, we have the following Mellin transform representation:*

$$\mathfrak{M}\left\{{}_1F_1^{(\alpha,\beta;p)}(b;c;z):s\right\} := \frac{\Gamma^{(\alpha,\beta)}(s)B(b+s,c+s-b)}{B(b,c-b)}\,{}_1F_1(b+s;c+2s;z)\,.$$

Corollary 3.8. *By the Mellin inversion formula, we have the following complex integral representation for ${}_1F_1^{(\alpha,\beta;p)}$:*

$${}_1F_1^{(\alpha,\beta;p)}(b;c;z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma^{(\alpha,\beta)}(s)B(b+s,c+s-b)}{B(b,c-b)}\,{}_1F_1(b+s;c+2s;z)\;p^{-s}\mathrm{d}s.$$

3.4. Transformation formulas

Theorem 3.9. *For the new GGHF, we have the following transformation formula:*

$$F_p^{(\alpha,\beta)}(a,b;c;z) = (1-z)^{-a} F_p^{(\alpha,\beta)}\left(a, c-b; b; \frac{z}{z-1}\right),$$
$$|\arg(1-z)| < \pi.$$

Proof. By writing

$$[1-z(1-t)]^{-a} = (1-z)^{-a}\left(1+\frac{z}{1-z}t\right)^{-a}$$

and replacing $t \to 1-t$ in (10), we obtain

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \frac{(1-z)^{-a}}{B(b,c-b)}\int_0^1 (1-t)^{b-1} t^{c-b-1}\left(1-\frac{z}{z-1}t\right)^{-a} {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t$$
$$\mathrm{Re}(p) > 0;\ p = 0 \text{ and } |z| < \pi;\ \mathrm{Re}(c) > \mathrm{Re}(b) > 0.$$

Hence,

$$F_p^{(\alpha,\beta)}(a,b;c;z) = (1-z)^{-a} F_p^{(\alpha,\beta)}\left(a, c-b; b; \frac{z}{z-1}\right). \quad \square$$

Remark 3.2. Note that, replacing z by $1-\frac{1}{z}$ in Theorem 3.9, one easily obtains the following transformation formula

$$F_p^{(\alpha,\beta)}\left(a,b;c;1-\frac{1}{z}\right) = z^{\alpha} F_p^{(\alpha,\beta)}(a,c-b;b;1-z)$$
$$|\arg(z)| < \pi.$$

Furthermore, replacing z by $\frac{z}{1+z}$ in Theorem 3.9, we get the following transformation formula

$$F_p^{(\alpha,\beta)}\left(a,b;c;\frac{z}{1+z}\right) = (1+z)^{a} F_p^{(\alpha,\beta)}(a,c-b;b;z)$$
$$|\arg(1+z)| < \pi.$$

Theorem 3.10. *For the new GCHF, we have the following transformation formula:*

$${}_1F_1^{(\alpha,\beta;p)}(b;c;z) = \exp(z)\,{}_1F_1^{(\alpha,\beta;p)}(c-b;c;-z).$$

Remark 3.3. Setting $z = 1$ in (10), we have the following relation between new defined hypergeometric and beta functions:

$$\begin{aligned} F_p^{(\alpha,\beta)}(a,b;c;1) &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-a-b-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t \\ &= \frac{B_p^{(\alpha,\beta)}(b,c-a-b)}{B(b,c-b)}. \end{aligned}$$

3.5. Differential recurrence relations for GGHF's and GCHF's

In this subsection we obtain some differential recurrence relations for GGHF's and GCHF's. We start with the following theorem:

Theorem 3.11. *For GGHF's we have the following recurrence relation:*

$$\begin{aligned} &pB(b+3,c-b+3)\frac{d^2F_p^{(\alpha,\beta)}(a,b+3;c+6;z)}{\mathrm{d}p^2} - \beta B(b+2,c-b+2)\frac{\mathrm{d}F_p^{(\alpha,\beta)}(a,b+2;c+4;z)}{\mathrm{d}p} \\ &\quad - pB(b+1,c-b+1)\frac{\mathrm{d}F_p^{(\alpha,\beta)}(a,b+1;c+2;z)}{\mathrm{d}p} + \alpha F_p^{(\alpha,\beta)}(a,b;c;z) = 0. \end{aligned}$$

Proof. Let $\mathcal{S}$ denote the left-hand side of the above assertion. Taking derivatives under the integral symbol in (10) by using the Leibnitz rule, we get

$$\mathcal{S} = \int_0^1 t^{x-1}(1-t)^{y-1}\left[pt(1-t)\frac{\mathrm{d}^2 z}{\mathrm{d}p^2} + (\beta t(1-t)+p)\frac{\mathrm{d}z}{\mathrm{d}p} + \alpha z\right]\mathrm{d}t,$$

where $z = {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right)$. Since $z = {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right)$ is a solution of the equation

$$pt(1-t)\frac{\mathrm{d}^2 z}{\mathrm{d}p^2} + (\beta t(1-t)+p)\frac{\mathrm{d}z}{\mathrm{d}p} + \alpha z = 0,$$

we get the result. $\square$

In a similar manner, we have the following for GCHF's:

Theorem 3.12. *For GCHF's we have the following recurrence relation:*

$$pB(b+3, c-b+3)\frac{\mathrm{d}_1^2 F_1^{(\alpha,\beta;p)}(b+3; c+6; z)}{\mathrm{d}p^2} - \beta B(b+2, c-b+2)\frac{\mathrm{d}_1 F_1^{(\alpha,\beta;p)}(b+2; c+4; z)}{\mathrm{d}p}$$
$$- pB(b+1, c-b+1)\frac{\mathrm{d}_1 F_1^{(\alpha,\beta;p)}(b+1; c+2; z)}{\mathrm{d}p} + \alpha\, {}_1F_1^{(\alpha,\beta;p)}(b; c; z) = 0.$$

4. Concluding remarks

By using the CHF, we have defined generalizations of gamma and beta functions. In their special cases, these generalizations include the extension of gamma and beta functions which were proposed in [1,2], respectively. Using the generalization of the beta function, we have generalized GHF and CHF, which include extended hypergeometric and confluent hypergeometric functions considered in [8]. We have investigated some properties of these generalized functions, most of which are analogous with the original functions.

Most of the special functions of mathematical physics and engineering, such as Jacobi and Laguerre polynomials can be expressed in terms of GHF or CHF. Therefore, the corresponding extensions of several other familiar special functions are expected to be useful and need to be investigated.

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