IIE3102 Production Planning

Inventory Control Subject to Uncertain Demand

Lecture Note to Accompany Chapter 05 of the Nahmias, S., & Olsen, T. L. (2015). *Production and operations analysis*. Waveland Press.

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Chapter Overview

Purpose:

To understand how one deals with uncertainty (randomness) in the demand when computing replenishment policies for a single inventory item.

Key Points:

1. The Nature of Randomness
2. Newsvendor Model
3. Lot Size Reorder Point Systems
4. Service Levels in (Q, R) Systems
5. Multiproduct Systems

The Nature of Randomness

Example 5.1

On consecutive Sundays, Mac, the owner of a local newsstand, purchases a number of copies of The Computer Journal, a popular weekly magazine. He pays 25 cents for each copy and sells each for 75 cents. Copies he has not sold during the week can be returned to his supplier for 10 cents each. The supplier is able to salvage the paper for printing future issues. Mac has kept careful records of the demand each week for the Journal. The observed demands during each of the last 52 weeks were

15	19	9	12	9	22	4	7	8	11
14	11	6	11	9	18	10	0	14	12
8	9	5	4	4	17	18	14	15	8
6	7	12	15	15	19	9	10	9	16
8	11	11	18	15	17	19	14	14	17
13	12								

Example 5.1 (continued)

There is no discernible pattern to these data, so it is difficult to predict the demand for the Journal in any given week.

- However, the frequency histogram can be used to estimate the empirical probability distribution.
- For example, the probability that demand is 10 is estimated to be 2/52=0.0385, and the probability that the demand is 15 is 5/52=0.0962.

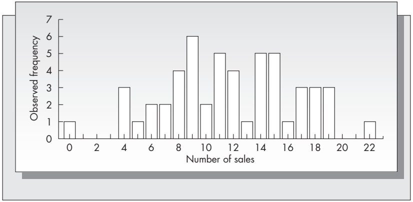


Figure 1: Frequency histogram for a 52-week history of sales of The Computer Journal at Mac's

Example 5.1 (continued)

Although empirical probabilities can be used in subsequent analysis, they are inconvenient for a number of reasons.

- They require maintaining a record of the demand history for every item.
- Distribution must be expressed (in this case) as 23 different probabilities.
- It is more difficult to compute optimal inventory policies with empirical distributions.

For these reasons, we generally approximate the demand history using a continuous distribution.

- The form of the distribution chosen depends upon the history of past demand and its ease of use. By far the most popular distribution for inventory applications is the normal.

Example 5.1 (continued)

The relative frequency histogram

- Obtained by dividing the y-axis entries of the frequency histogram by 52.
- Normal density function superimposed on the relative frequency histogram.

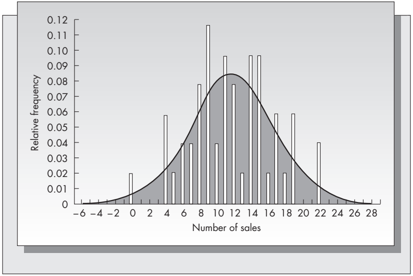


Figure 2: Frequency histogram and normal approximation

Optimization Criterion

When demand is random, the cost incurred is itself random.

- It is no longer obvious what the optimization criterion should be.
- Virtually all stochastic optimization techniques applied to inventory control assume that the goal is to minimize expected costs.
- The law of large numbers from probability theory says that the arithmetic average of many observations of a random variable will converge to the expected value of that random variable.
- In the context of the inventory problem, if we follow a control rule that minimizes expected costs, then the arithmetic average of the actual costs incurred over many periods will also be a minimum.
- In certain circumstances, the expected value may not be the best optimization criterion.

Newsvendor Model

Newsvendor Model

A single product is to be ordered at the beginning of a period and can be used only to satisfy demand during that period.

- Assume that the demand D is a continuous nonnegative random variable with pdf $f(x)$ and cdf $F(x)$.
  - Probability density function: $f(x) = Pr[X = x]$
  - Cumulative density function: $F(x) = Pr[X \leq x] = \int_0^x f(x)dx$
- Relevant costs
  - Overage cost c_o: cost per unit of positive inventory remaining at the end of the period.
  - Underage cost c_u: cost per unit of unsatisfied demand.
- Decision variable Q is the number of units to be purchased at the beginning of the period.
- The goal of the analysis is to determine Q to minimize the expected costs incurred at the end of the period.

Newsvendor Model: Development of the Cost Function

Define $G(Q, D)$ as the total overage and underage cost given Q and D.

- $\max\{Q - D, 0\}$ represents the number of remaining units and $\max\{D - Q, 0\}$ represents the unsatisfied demand. Then,

$$G(Q, D) = c_o \max(0, Q - D) + c_u \max(0, D - Q).$$

- Expected cost function, $G(Q)$, can be defined as

$$\begin{aligned} G(Q) &= E(G(Q, D)) \\ &= c_o \int_0^\infty \max(0, Q - x) f(x) dx + c_u \int_0^\infty \max(0, x - Q) f(x) dx \\ &= c_o \int_0^Q (Q - x) f(x) dx + c_u \int_Q^\infty (x - Q) f(x) dx. \end{aligned}$$

Determine the value of Q that minimizes the expected cost $G(Q)$.

- Investigate the cost function $G(Q)$.

$$\frac{dG(Q)}{dQ} = c_o \int_0^Q 1 f(x)dx + c_u \int_Q^\infty (-1)f(x)dx = c_o F(Q) - c_u(1 - F(Q))$$

$$\frac{d^2G(Q)}{dQ^2} = (c_o + c_u)f(Q) \geq 0 \quad \text{for all } Q \geq 0.$$

- Because the second derivative is nonnegative, the cost function $G(Q)$ is convex.

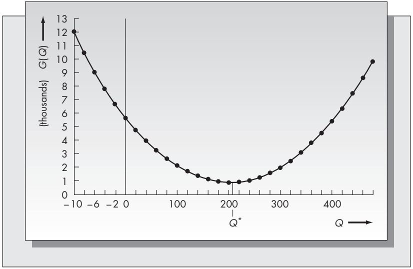


Figure 3: Expected cost function for newsvendor model

Newsvendor Model: Determine the Optimal Policy

Determine the value of Q that minimizes the expected cost $G(Q)$.

- The optimal solution, Q^*, occurs where the first derivative of $G(Q)$ equals to zero. That is,
$$G'(Q^*) = (c_o + c_u)F(Q^*) - c_u = 0.$$
- Rearranging this gives
$$F(Q^*) = \frac{c_u}{c_0 + c_u},$$
which is referred as the critical ratio.

Observations

- If c_o increases, then Q^* would decrease.
- If c_u increases, then Q^* would increase.

Example 5.1 (continued)

From past experience, the weekly demand for the Journal is approximately normally distributed with mean $\mu = 11.73$ and standard deviation $\sigma = 4.74$.

- The critical ratio $c_u/(c_o + c_u) = 0.77$.
- The optimal Q^* is the 77th percentile of the demand distribution as

$$Q^* = \sigma z + \mu = (4.74)(0.74) + 11.73 = 15.24 \approx 15.$$

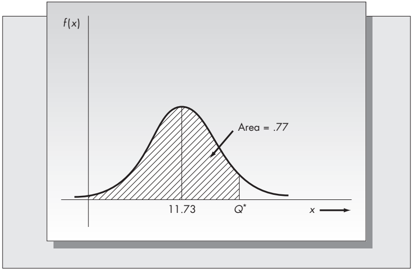
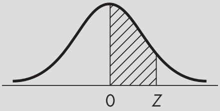


Figure 4: Determination of the optimal order quantity for the newsvendor example

Example 5.1 (continued)

Using the following table, we can obtain a standardized value of $z = 0.74$

TABLE A–1 **Areas under the Normal Curve**



Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.00	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
.10	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
.20	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
.30	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
.40	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
.50	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
.60	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
.70	.2580	.2611	.2642	.2673	.2703	.2734	.2764	.2793	.2823	.2852
.80	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
.90	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389

Figure 5: Appendix - Area under the Normal curve

Optimal Policy for Discrete Demand

- The optimal solution procedure is to locate the critical ratio between two values of F(Q) and choose the Q corresponding to the higher value.

Example 5.2

Solve the problem faced by Mac's newsstand using the empirical distribution.

- Because the critical ratio 0.77 is between $F(Q = 14)$ and $F(Q = 15)$, $Q^* = 15$.

Q	$f(Q)$	$F(Q)$	Q	$f(Q)$	$F(Q)$
0	1/52	1/52 (.0192)	12	4/52	30/52 (.5769)
1	0	1/52 (.0192)	13	1/52	31/52 (.5962)
2	0	1/52 (.0192)	14	5/52	36/52 (.6923)
3	0	1/52 (.0192)	15	5/52	41/52 (.7885)
4	3/52	4/52 (.0769)	16	1/52	42/52 (.8077)
5	1/52	5/52 (.0962)	17	3/52	45/52 (.8654)
6	2/52	7/52 (.1346)	18	3/52	48/52 (.9231)
7	2/52	9/52 (.1731)	19	3/52	51/52 (.9808)
8	4/52	13/52 (.2500)	20	0	51/52 (.9808)
9	6/52	19/52 (.3654)	21	0	51/52 (.9808)
10	2/52	21/52 (.4038)	22	1/52	52/52 (1.0000)
11	5/52	26/52 (.5000)			

Lot Size Reorder Point Systems

Lot Size Reorder Point Systems

Generalize EOQ to allow for random demand

- Recall, optimal policy driven by EOQ model is: when the level of on-hand inventory hits R (reorder point), place an order for Q units.
  - Only independent decision variable was Q, and the value of R was determined from Q, λ, and τ.
- We assume that the operating policy is of the (Q, R) form while treating Q and R as independent decision variables.

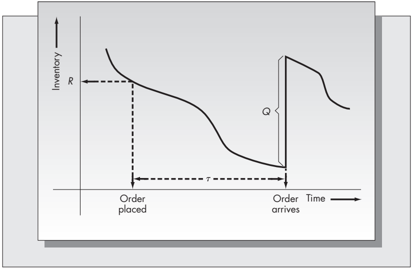


Figure 6: Changes in inventory over time for continuous-review (Q, R) system

Assumptions:

- The system is continuous review. That is, demands are recorded as they occur, and the level of on-hand inventory is known at all times.
- Demand is random and stationary. That means that although we cannot predict the value of demand, the expected value of demand over any time interval of fixed length is constant. Assume that the expected demand rate is λ units per year.
- There is a fixed positive lead time τ for placing an order.
- The following costs are assumed:
  - Setup cost at $K per order.
  - Holding cost at $h per unit held per year.
  - Proportional order cost of $c per item.
  - Stock-out cost of $p per unit of unsatisfied demand. This is also called the shortage cost or the penalty cost.

Derivation of the Expected Cost Function

- Holding Cost
  - Safety stock: $s = R - \lambda\tau$
  - Expected inventory: $s + Q/2 = R - \lambda\tau + Q/2$
  - Expected holding cost: $h(R - \lambda\tau + Q/2)$
- Setup Cost
  - Distance between successive arrivals of orders: $T = Q/\lambda$
  - Average setup cost incurred per unit time: $K/T = K\lambda/Q$

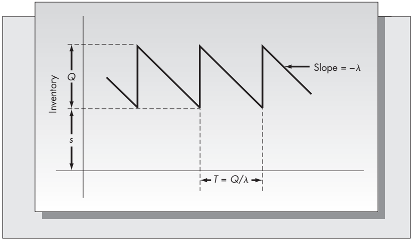


Figure 7: Expected inventory level for (Q, R) inventory model

Derivation of the Expected Cost Function (continued)

- Shortage Cost
    - Let x denote a demand during the lead time τ with pdf $f(x)$ and cdf $F(x)$.
    - If $x < R$, shortage $= 0$. Otherwise (if $x \geq R$), shortage $\geq x - R$.
    - Expected number of shortages that occur in one cycle, $n(R)$, can be defined

$$n(R) = \int_R^\infty (x - R) f(x) dx$$

    - Expected number of shortage cost incurred per unit time:

$$p \frac{n(R)}{T} = p \frac{\lambda}{Q} n(R)$$

- Expected average annual cost of holding, setup, and shortages, $G(Q, R)$, can be defined as follows:

$$G(Q, R) = h(\frac{Q}{2} + R - \lambda\tau) + K\frac{\lambda}{Q} + p\lambda\frac{n(R)}{Q}.$$

Objective if to choose Q and R to minimize $G(Q,R)$.

- Necessary condition for optimality is that $\partial G/\partial Q = \partial G/\partial R = 0$.

$$\frac{\partial G}{\partial Q} = \frac{h}{2} - \frac{K\lambda}{Q^2} - \frac{P\lambda n(R)}{Q^2} = 0, \tag{1}$$

$$\frac{\partial G}{\partial R} = h + \frac{p\lambda}{Q}\frac{\partial n(R)}{\partial R} = 0. \tag{2}$$

- Note that

$$\frac{\partial n(R)}{\partial R} = n'(R) = -(1 - F(R)). \tag{3}$$

- From (1),

$$Q^* = \sqrt{\frac{2\lambda[K + pn(R)]}{h}}. \tag{4}$$

- From (2),

$$1 - F(R) = \frac{Qh}{p\lambda}. \tag{5}$$

Iterate between (4) and (5) until two successive values of Q and R are the same.

- Iteration 0,

$$Q_0 = \sqrt{\frac{2K\lambda}{h}} = \text{EOQ}$$
$$1 - F(R_0) = \frac{Q_0 h}{p\lambda}$$

- Iteration k,

$$Q_k = \sqrt{\frac{2\lambda[K + pn(R_{k-1})}{h}}$$
$$1 - F(R_k) = \frac{Q_k h}{p\lambda}$$

- If $|Q_k - Q_{k-1}| < 1$ and $|R_k - R_{k-1}| < 1$, stop. Then, $Q^* = Q_k$ and $R^* = R_k$.

When the demand is normally distributed, $n(R)$ is computed by using the standardized loss function.

- The standardized loss function, $L(z)$, is defined as,

$$L(z) = \int_z^\infty (t - z)\phi(t)d(t),$$

  where $\phi(t)$ is the standard normal density.

- If lead time demand is normal with mean μ and standard deviation σ, i.e., $X(\mu, \sigma^2)$, then it can be shown that

$$n(R) = \sigma L\left(\frac{R - \mu}{\sigma}\right) = \sigma L(z).$$

- Hence,

$$L(z^*) = \frac{n(R)}{\sigma}$$
$$R = \mu + \sigma z^*.$$

Example 5.4

Harvey's Specialty Shop is a popular spot that specializes in international gourmet foods. One of the items that Harvey sells is a popular mustard that he purchases from an English company. The mustard costs Harvey \$10 a jar and requires a six-month lead time for replenishment of stock. Harvey uses a 20 percent annual interest rate to compute holding costs and estimates that if a customer requests the mustard when he is out of stock, the loss-of-goodwill cost is \$25 a jar. Bookkeeping expenses for placing an order amount to about \$50. During the six-month replenishment lead time, Harvey estimates that he sells an average of 100 jars, but there is substantial variation from one six-month period to the next. He estimates that the standard deviation of demand during each six-month period is 25. Assume that demand is described by a normal distribution. How should Harvey control the replenishment of the mustard?

Example 5.4

- Iteration 0,

$$Q_0 = \text{EOQ} = \sqrt{\frac{2K\lambda}{h}} = \sqrt{\frac{(2)(50)(200)}{(0.2)(100)}} = 100$$

$$1 - F(R_0) = \frac{Q_0 h}{p\lambda} = \frac{(100)(2)}{(25)(200)} = 0.04$$

$$z = 1.75 \text{ (from Table A-4)}$$

$$R_0 = \sigma z + \mu = (25)(1.75) + 100 = 144$$

$$L(z) = 0.0162 \text{ (from Table A-4)}$$

$$n(R_0) = \sigma L(z) = (25)(0.0162) = 0.405.$$

Example 5.4

- Iteration 1,

$$Q_1 = \sqrt{\frac{2\lambda[K + pn(R_0)]}{h}} = \sqrt{\frac{(2)(200)[50 + (25)(0.405)]}{2}} = 110$$

$$1 - F(R_1) = \frac{Q_1 h}{p\lambda} = \frac{(110)(2)}{(25)(200)} = 0.044$$

$$z = 1.70 \text{ (from Table A-4)}$$

$$R_1 = \sigma z + \mu = (25)(1.70) + 100 = 143$$

$$L(z) = 0.0183 \text{ (from Table A-4)}$$

$$n(R_1) = \sigma L(z) = (25)(0.0183) = 0.0.4575.$$

- Iteration 2,

$$Q_2 = \sqrt{\frac{2\lambda[K + pn(R_1)]}{h}} = \sqrt{\frac{(2)(200)[50 + (25)(0.4575)]}{2}} = 110.85 \approx 111$$

$$R_2 = 143$$

- Because both Q_2 and R_2 are within one unit of Q_1 and R_1, we may terminate computations. Optimal policy can be determined as $(Q, R) = (111, 143)$.

Example 5.4

Table A-4

Standardized Variate	Probabilities		Partial Expectations	
z	$F(z)$	$1 - F(z)$	$L(z)$	$L(-z)$
1.60	.9460	.0540	.0232	1.6232
1.61	.9463	.0537	.0227	1.6327
1.62	.9474	.0526	.0222	1.6422
1.63	.9484	.0516	.0216	1.6516
1.64	.9495	.0505	.0211	1.6611
1.65	.9505	.0495	.0206	1.6706
1.66	.9515	.0485	.0201	1.6801
1.67	.9525	.0475	.0197	1.6897
1.68	.9535	.0465	.0192	1.6992
1.69	.9545	.0455	.0187	1.7087
1.70	.9554	.0446	.0183	1.7183
1.71	.9564	.0436	.0178	1.7278
1.72	.9573	.0427	.0174	1.7374
1.73	.9582	.0418	.0170	1.7470
1.74	.9591	.0409	.0166	1.7566
1.75	.9599	.0401	.0162	1.7662

Figure 8: Appendix - Normal Probability Distribution and Partial Expectations

Example 5.4 (continued)

For the same example, determine the following:

1. Safety stock.
2. The average annual holding, setup, and penalty costs associated with the inventory control of the mustard.
3. The average time between placement of orders.
4. The proportion of order cycles in which no stock-outs occur.
5. The proportion of demands that are not met.

Service Levels in (Q, R) Systems

Service Levels in (Q, R) Systems

Difficult to estimate an exact value for the stock-out cost p

- A common substitute for a stock-out cost is a service level. In other words, it is common to set inventory levels to meet a specified service objective instead.
- Type 1 Service: Choose R so that the probability of not stocking out in the lead time is equal to a specified value α, which denotes *cycle service level*.
  - Determine R to satisfy the equation $F(R) = \alpha$.
  - Set $Q =$EOQ.
- Type 2 Service: Choose both Q and R so that the proportion of demands satisfied from stock equals a specified value β, which denotes fill rate.
  - $\beta = 1 - \frac{n(R)}{Q}$
  - EOQ is a good estimation of the lot size, Q, then $n(R) = \text{EOQ}(1 - \beta)$.

Example 5.5

Consider again Harvey's Specialty Shop, described in Example 5.4. Harvey feels uncomfortable with the assumption that the stock-out cost is $25 and decides to use a service level criterion instead. Suppose that he chooses to use a 98 percent service objective.

Type 1 service

- If we assume an α of .98, then we find R to solve $F(R) = 0.98$. We obtain $z = 2.05$. Setting $R = \sigma z + \mu$ gives $R = 151$.

Type 2 service

- Here $\beta = 0.98$. We are required to solve the equation $n(R) = \text{EOQ}(1 - \beta)$, which is equivalent to

$$L(z) = \text{EOQ}(1 - \beta)/\sigma.$$

- Substituting EOQ$= 100$ and $\beta = 0.98$, we obtain $L(z) = (100)(0.02)/25 = 0.08$ and $z = 1.02$. Setting $R = \sigma z + \mu$ gives $R = 126$.

Multiproduct Systems

ABC Analysis

- In multiproduct inventory systems, not all products are equally profitable. For this reason, it is important to differentiate profitable from unprofitable items.
- Pareto effect applied to inventory systems
  - Group A: Top 20 percent of the items account for about 80 percent of the annual dollar volume of sales.
  - Group B: The next 30 percent of the items for the next 15 percent of sales.
  - Group C: The remaining 50 percent for the last 5 percent of dollar volume.

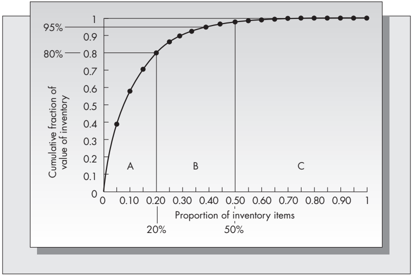


Figure 9: Pareto curve: Distribution of inventory by value

Example 5.10

A sample of 20 different stock items from Harvey's Specialty Shop is selected at random. These items vary in price from $0.25 to $24.99 and in average yearly demand from 12 to 786.

Part Number	Price	Yearly Demand	Dollar Volume
5497J	2.25	260	585.00
3K62	2.85	43	122.55
88450	1.50	21	31.50
P001	0.77	388	298.76
2M993	4.45	612	2,723.40
4040	6.10	220	1,342.00
W76	3.10	110	341.00
JJ335	1.32	786	1,037.52
R077	12.80	14	179.20
70779	24.99	334	8,346.66
4J65E	7.75	24	186.00
334Y	0.68	77	52.36
8ST4	0.25	56	14.00
16113	3.89	89	346.21
45000	7.70	675	5,197.50
7878	6.22	66	410.52
6193L	0.85	148	125.80
TTR77	0.77	690	531.30
39SS5	1.23	52	63.96
93939	4.05	12	48.60

Example 5.10 (continued)

Notice that only 4 of the 20 stock items account for over 80 percent of the annual dollar volume generated by the entire group.

Part Number	Price	Yearly Demand	Dollar Volume	Cumulative Dollar Value	
70779	24.99	334	8,346.66	8,346.66	A items:
45000	7.70	675	5,197.50	13,544.16	20% of items
2M993	4.45	612	2,723.40	16,267.56	account for 80.1%
4040	6.10	220	1,342.00	17,609.56	of total value.
JJ335	1.32	786	1,037.52	18,647.08	
5497J	2.25	260	585.00	19,232.08	B items:
TTR77	0.77	690	531.30	19,763.38	30% of items
7878	6.22	66	410.52	20,173.90	account for 14.8%
16113	3.89	89	346.21	20,520.11	of total value.
W76	3.10	110	341.00	20,861.11	
P001	0.77	388	298.76	21,159.87	
4J65E	7.75	24	186.00	21,345.87	
R077	12.80	14	179.20	21,525.07	
6193L	0.85	148	125.80	21,650.87	C items:
3K62	2.85	43	122.55	21,773.42	50% of items
39SS5	1.23	52	63.96	21,837.38	account for 5.1%
334Y	0.68	77	52.36	21,889.74	of total value.
93939	4.05	12	48.60	21,938.34	
88450	1.50	21	31.50	21,969.84	
8ST4	0.25	56	14.00	21,983.84	

Exchange Curves

We can treat the ratio K/I as a policy variable.

- If this ratio is large, lot sizes will be larger and the average investment in inventory will be greater.
- If this ratio is small, the number of annual replenishments will increase.

Derive a typical exchange curve

- Consider a system consisting of n products with varying demand rates $\lambda_1, \ldots, \lambda_n$ and item values $c_1, \ldots, c_n$. If EOQ values are used to replenish stock for each item, then

$$Q_i = \sqrt{\frac{2K\lambda_i}{Ic_i}} \quad \text{for } 1 \leq i \leq n.$$

- For item i
  - The cycle time is Q_i/λ_i, so that λ_i/Q_i is the number of replenishments in one year.
  - The total number of replenishments for the entire system is $\sum \lambda_i/Q_i$.
  - The average on-hand inventory of item i is $Q_i/2$, and the value of this inventory in dollars is $c_iQ_i/2$. Hence, the total value of the inventory is $\sum c_iQ_i/2$.

Derive a typical exchange curve (continued)

- Each choice of the ratio K/I will result in a different value of the number of replenishments per year and the dollar value of inventory.
- As K/I is varied, one traces out a curve such as the one pictured in the following figure.
- An exchange curve such as this one allows management to easily see the trade-off between the dollar investment in inventory and the frequency of stock replenishment.

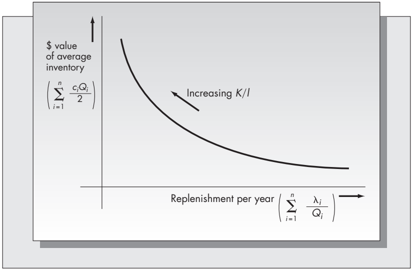


Figure 10: Exchange curve of replenishment frequency and inventory value

Exchange curves also can be used to compare safety stock and service level strategies.

- As an example, consider a system in which a fill rate constraint is used (i.e., Type 2 service) for all items. Furthermore, suppose that the lead time demand distribution for all items is normal, and each item gets equal service.
- The dollar value of the safety stock is $\sum c_i(R_i - \mu_i)$, and the annual value of back-ordered demand is $\sum c_i \lambda_i n(R_i)/Q_i$.
- A fixed value of the fill rate β will result in a set of values of the control variables $(Q_1, R_1), \ldots, (Q_n, R_n)$.

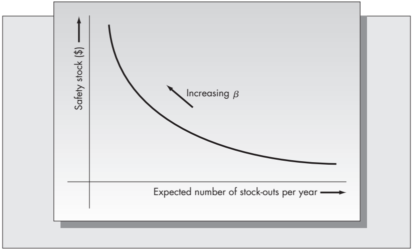


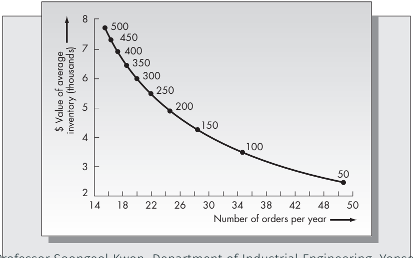
Figure 11: Exchange curve of the investment in safety stock and β

Exchange Curves

Example 5.11

Consider the 20 stock items listed in Tables 5–1 and 5–2. Suppose that Harvey, the owner of Harvey's Specialty Shop, is reconsidering his choices of the setup cost of \$50 and interest charge of 20 percent. Harvey uses the EOQ formula to compute lot sizes for the 20 items for a range of values of K/I from 50 to 500.

- Harvey is currently operating at $K/I = 50/0.2 = 250$, which results in approximately 22 orders per year and an average inventory cost of \$5,447 annually.
- By reducing K/I to 100, the inventory cost for these 20 items is reduced to \$3,445 and the order frequency is increased to 34 orders a year.



Questions?