q-analog chromatic polynomials

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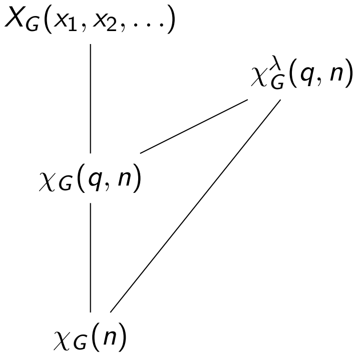
The big picture

Stanley's chromatic symmetric function $X_G(x_1, x_2, \ldots)$:

- Distinguishes some (all?) non-isomorphic trees

Chromatic polynomial $\chi_G(n)$:

- Polytopes perspective
- Deletion-contraction
- Does not distinguish trees

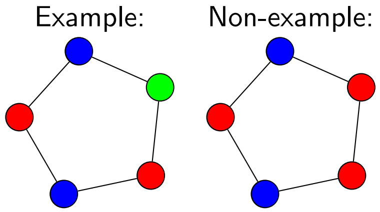


Proper colorings

A **proper n-coloring** of a graph $G = (V, E)$ is a function $c : V \to [n]$ such that

$$c(v) \neq c(w) \text{ if } \{v, w\} \in E.$$

The **chromatic number** $\chi(G)$ of G is the smallest positive integer such that G has a proper $\chi(G)$-coloring.



$\chi(C_5) = 3$

The chromatic polynomial

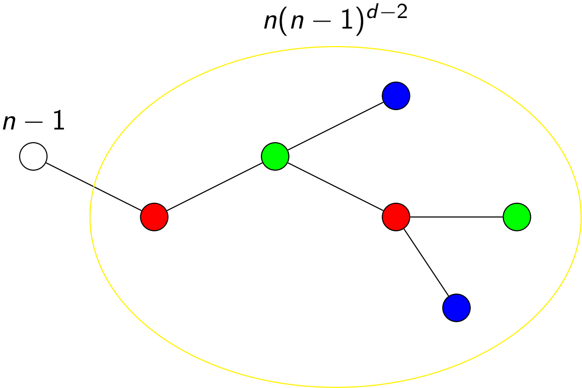
The number of proper n-colorings of a graph G agrees with a polynomial of degree $|V|$, called the **chromatic polynomial** $\chi_G(n)$ of G.

$$\chi_G(n) = \sum_{k=\chi(G)}^{|V|} \alpha_k \cdot n(n-1)\cdots(n-k+1),$$

where α_k is the number of partitions of V into k independent sets.

The chromatic polynomial does not distinguish trees

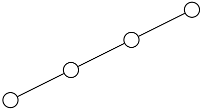
If T is a tree on d vertices, then $\chi_T(n) = n(n-1)^{d-1}$.



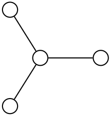
The chromatic symmetric function

Stanley's symmetric function generalization:

$$X_G(x_1, x_2, \ldots) = \sum_{\substack{\text{proper colorings} \\ c: V \to \mathbb{Z}^+}} x_1^{|c^{-1}(1)|} x_2^{|c^{-1}(2)|} x_3^{|c^{-1}(3)|} \cdots$$



$$X_{P_4}(x_1, x_2, 0, 0, \ldots) = 2x_1^2x_2^2$$



$$X_{S_4}(x_1, x_2, 0, 0, \ldots) = x_1^3x_2 + x_1x_2^3$$

Conjecture. (Stanley) For trees S and T, $X_S = X_T \iff S \cong T$.

The chromatic symmetric function in different bases

(Augmented) monomial basis

$$X_G(x_1, x_2, \ldots) = \sum_{\lambda \vdash |V|} \alpha_\lambda \widetilde{m}_\lambda,$$

where $\alpha_\lambda =$ number of partitions of type λ of V into independent sets and $\widetilde{m}_\lambda = r_1! r_2! \cdots m_\lambda$ ($r_i =$ number of parts of λ equal to i)

Power sum basis

$$X_G(x_1, x_2, \ldots) = \sum_{S \subseteq E} (-1)^{|S|} p_{\lambda(S)},$$

where $\lambda(S) =$ vector of sizes of connected components of (V, S)

Elementary basis

$$X_G(x_1, x_2, \ldots) = \sum_{\lambda \vdash |V|} c_\lambda e_\lambda,$$

is such that $\sum_{\substack{\lambda \text{ with} \\ j \text{ parts}}} c_\lambda =$ number of acyclic orientations of G with j sinks

Specializations of $X_G(x_1, x_2, \ldots)$

$$X_G(x_1, x_2, \ldots)$$

$$X_G(q, q^2, \ldots, q^n, 0, 0, \ldots)$$

$$X_G(\underbrace{1, \ldots, 1}_{n \text{ times}}, 0, 0, \ldots) = \chi_G(n)$$

Conjecture. (Loehr-Warrington) The **principal specialization** already distinguishes non-isomorphic trees!

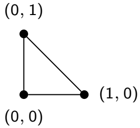
Lattice polytopes

A polytope is the convex hull of finitely many points in $\mathbb{R}^d$, equivalently a bounded intersection of finitely many halfspaces.

For P a lattice polytope (i.e. with vertices in $\mathbb{Z}^d$), we consider

$$\mathrm{ehr}_P(n) = \left| nP \cap \mathbb{Z}^d \right|.$$

Example:

$\Delta =$ 

$$\begin{aligned}\mathrm{ehr}_\Delta(n) &= |\{(x,y) \in \mathbb{Z}^2 \,:\, x, y \geq 0, x + y \leq n\}| \\ &= \binom{n+2}{2} = \frac{1}{2}n^2 + \frac{3}{2}n + 1\end{aligned}$$

Ehrhart theory

For any d-dimensional lattice polytope $P \subseteq \mathbb{R}^d$, $\mathrm{ehr}_P(n)$ is a polynomial of degree d, called the **Ehrhart polynomial**.

The **Ehrhart series** of P is its generating function

$$\mathrm{Ehr}_P(z) = \sum_{n \geq 0} \mathrm{ehr}_P(n) z^n.$$

Observe

$$\mathrm{Ehr}_P(z) = \sum_{x \in \mathrm{cone}(P) \cap \mathbb{Z}^{d+1}} z^{x_{d+1}},$$

where $\mathrm{cone}(P) = \{(tx, t) : x \in P, t \geq 0\}$.

Ehrhart theory of unimodular simplices

If Δ is a d-dimensional *unimodular* simplex with k missing facets (for some $0 \leq k \leq d+1$),

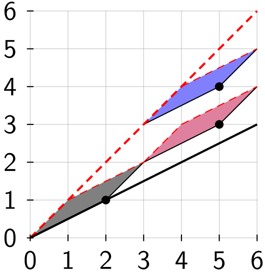
$$\mathrm{Ehr}_{\Delta}(z) = \frac{z^k}{(1-z)^{d+1}}.$$

Proof. The unique point in the "fundamental parallelepiped" of cone(Δ) is

$$\sum \binom{v_i}{i},$$

where the sum ranges over the k vertices of Δ that are opposite the missing facets.

cone((1, 2]) :



Ehrhart theory of order polytopes

The **order polytope** of a poset $\Pi = ([d], \preceq)$ is

$$\mathcal{O}(\Pi) = \{(x_1, \ldots, x_d) \in [0,1]^d \,:\, x_i \leq x_j \text{ if } i \preceq j\},$$

which has a disjoint *unimodular triangulation*

$$\mathcal{O}(\Pi) = \bigcup_{\sigma \in \mathcal{L}(\Pi)} \left\{0 \leq x_{\sigma_1} \leq \ldots \leq x_{\sigma_d} \leq 1,\ x_{\sigma_i} < x_{\sigma_{i+1}} \text{ if } i \in \mathrm{Des}(\sigma)\right\}.$$

Therefore,

$$\mathrm{Ehr}_{\mathcal{O}(\Pi)}(z) = \frac{\sum_{\sigma \in \mathcal{L}(\Pi)} z^{\mathrm{des}(\sigma)}}{(1-z)^{d+1}}.$$

Ehrhart theory of order polytopes, continued

$$\Longrightarrow \mathrm{ehr}_{\mathcal{O}(\Pi)}(n) = \sum_{\sigma \in \mathcal{L}(\Pi)} \binom{n + d - \mathrm{des}(\sigma)}{d}$$

$$\Longrightarrow \mathrm{ehr}_{\mathcal{O}(\Pi)^\circ}(n) = \sum_{\sigma \in \mathcal{L}(\Pi)} \binom{n + \mathrm{des}(\sigma) - 1}{d}$$

q-analog Ehrhart theory

Theorem. (Chapoton) If $P \subseteq \mathbb{R}^d$ is a d-dimensional lattice polytope and $\lambda : \mathbb{Z}^d \to \mathbb{Z}$ is a linear form that is nonnegative on the vertices of P,

$$\mathrm{ehr}_P^{\lambda}(q,n) = \sum_{x \in nP \cap \mathbb{Z}^d} q^{\lambda(x)}$$

agrees with a polynomial $\widetilde{\mathrm{ehr}}_P^{\lambda}(q,x) \in \mathbb{Q}(q)[x]$, evaluated at

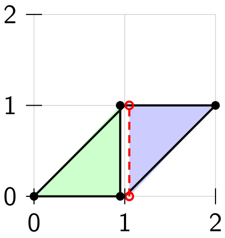
$$x = [n]_q := 1 + q + q^2 + \cdots + q^{n-1}.$$

If $\lambda((x_1,\ldots,x_d)) = x_1 + \cdots + x_d$, we omit it.

We are ignoring a condition called "genericity" that is needed, but we will not have to worry about it for the polytopes we are working with!

An example!

$$P = \mathrm{conv}\{(0,0),(1,0),(1,1),(2,1)\}$$



$$\mathrm{Ehr}_P(q,z) = \frac{1}{(1-z)(1-qz)(1-q^2z)} + \frac{q^3z}{(1-qz)(1-q^2z)(1-q^3z)}$$
$$= \frac{1-q^3z^2}{(1-z)(1-qz)(1-q^2z)(1-q^3z)}$$

$$\widetilde{\mathrm{ehr}}_P(q,x) = \frac{q^4-q^3}{q+1}x^3 + \frac{3q^3-q^2}{q+1}x^2 + \frac{3q^2+q}{q+1}x + 1$$

Properties of $\mathrm{Ehr}_P^\lambda(q,z)$ and $\widetilde{\mathrm{ehr}}_P^\lambda(q,x)$

(i) $\mathrm{Ehr}_P^\lambda(1,z) = \mathrm{Ehr}_P(z)$ and $\widetilde{\mathrm{ehr}}_P^\lambda(1,x) = \mathrm{ehr}_P(x)$

(ii) The denominator of $\mathrm{Ehr}_P^\lambda(q,z)$ divides $\displaystyle\prod_{\substack{\text{vertices} \\ v \text{ of } P}} (1 - q^{\lambda(v)} z)$.

(iii) $\deg(\widetilde{\mathrm{ehr}}_P^\lambda(q,x)) = \max_v \lambda(v)$

(iv) The poles of the coefficients of $\widetilde{\mathrm{ehr}}_P^\lambda(q,x)$ are roots of unity of order at most $\max_v \lambda(v)$.

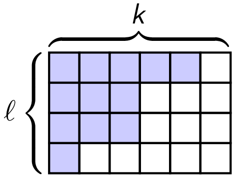
q-binomial coefficients

The q-binomial coefficient:

$$\begin{bmatrix} k+\ell \\ \ell \end{bmatrix}_q := \frac{[k+\ell]_q!}{[k]_q![\ell]_q!} = \frac{[k]_q[k-1]_q \cdots [k+1]_q}{[\ell]_q[\ell-1]_q \cdots [1]_q}$$

A combinatorial interpretation:

$$\begin{bmatrix} k+\ell \\ \ell \end{bmatrix}_q = \sum_{\mu \in \mathcal{R}(k,\ell)} q^{|\mu|}$$



Negative q-binomial theorem:

$$\frac{1}{(1-z)(1-qz)(1-q^2z)\cdots(1-q^dz)} = \sum_{n\geq 0} \begin{bmatrix} n+d \\ d \end{bmatrix}_q z^n$$

q-analog Ehrhart theory of order polytopes

The q-analog Ehrhart series of the order polytope $\mathcal{O}(\Pi)$, for $\Pi = ([d], E)$ a poset, is

$$\mathrm{Ehr}_{\mathcal{O}(\Pi)}(q, z) = \frac{\sum_{\sigma \in \mathcal{L}(\Pi)} q^{\mathrm{comaj}(\sigma)} z^{\mathrm{des}(\sigma)}}{(1-z)(1-qz)\cdots(1-q^d z)}.$$

Therefore,

$$\mathrm{ehr}_{\mathcal{O}(\Pi)}(q, n) = \sum_{\sigma \in \mathcal{L}(\Pi)} q^{\mathrm{comaj}(\sigma)} \begin{bmatrix} n + d - \mathrm{des}(\sigma) \\ d \end{bmatrix}_q.$$

Observe $[n+k]_q = q^k [n]_q + [k]_q$ and $[n-k]_q = \frac{[n]_q - [k]_q}{q^k}$, so $\widetilde{\mathrm{ehr}}_{\mathcal{O}(\Pi)}(q, x)$ has degree d and $[d]_q! \cdot \widetilde{\mathrm{ehr}}_{\mathcal{O}(\Pi)}(q, x) \in \mathbb{Z}(q)[x]$.

Proper colorings as lattice points

A coloring $c : [d] \to [n]$ of $G = ([d], E)$ can be thought of as a point

$$(c(1), \ldots, c(d)) \in \mathbb{Z}^d.$$

The proper n-colorings of G are points in

$$\left((0, n+1)^d \cap \mathbb{Z}^d\right) \setminus \left(\bigcup \mathcal{H}_G\right),$$

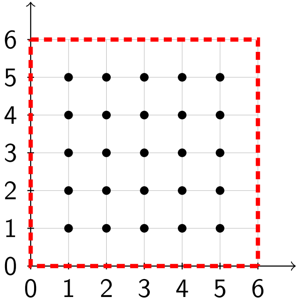
where $\mathcal{H}_G$ is the **graphical hyperplane arrangement**

$$\mathcal{H}_G = \{x_i = x_j \,:\, \{i, j\} \in E\}\,.$$

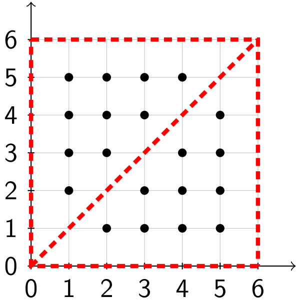
Proper colorings as lattice points, continued

Consider the path on two vertices, $P_2 =$ ○———○

5-colorings of P_2:



Proper 5-colorings of P_2:

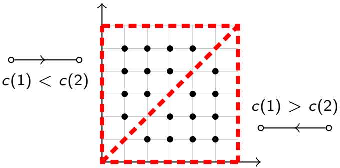


Proper colorings as lattice points, continued

$\left((0, n+1)^d \cap \mathbb{Z}^d\right) \setminus \left(\bigcup \mathcal{H}_G\right)$ has a region for each *acyclic orientation* ρ of G, given by

$$(0, n+1)^d \cap \left(\bigcap_{(i,j)\in\rho} \{x_i < x_j\} \right).$$

The region corresponding to ρ contains the proper colorings of G that "obey" ρ, i.e. for which $c(i) < c(j)$ if $(i, j) \in \rho$.



The chromatic polynomial is a sum of Ehrhart polynomials

Each region is the $(n+1)$st dilate of the open order polytope of the poset induced by ρ, which we call Π_ρ, therefore

$$\chi_G(n) = \sum_{\rho \in \mathcal{A}(G)} \mathrm{ehr}_{\mathcal{O}(\Pi_\rho)^\circ}(n+1)$$
$$= \sum_{\rho \in \mathcal{A}(G)} \sum_{\sigma \in \mathcal{L}(\Pi_\rho)} \binom{n + \mathrm{des}(\sigma)}{d}.$$

The linear extensions are of a *natural labeling* of the poset, not the vertex labels.

A q-analog of this: $\chi_G(q, n)$

$$X_G(q, q^2, \ldots, q^n, 0, \ldots) = \sum_{\substack{\text{proper} \\ c:[d]\to[n]}} q^{|c^{-1}(1)|+2|c^{-1}(2)|+\cdots+n|c^{-1}(n)|}$$

counts q raised to the *sum of the colors of each vertex* for each proper coloring, which is

$$\chi_G(q, n) := \sum_{\rho\in\mathcal{A}(G)} \mathrm{ehr}_{\mathcal{O}(\Pi_\rho)^\circ}(q, n+1).$$

Therefore,

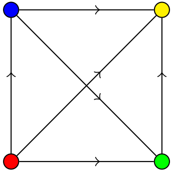
$$X_G(q, q^2, \ldots, q^n, 0, \ldots) = \sum_{\rho\in\mathcal{A}(G)} \sum_{\sigma\in\mathcal{L}(\Pi_\rho)} q^{\binom{d+1}{2}-\mathrm{comaj}(\sigma)} \begin{bmatrix} n+\mathrm{des}(\sigma) \\ d \end{bmatrix}_q$$

A (sort of boring) example

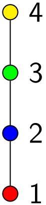
The acyclic orientations of the complete graph K_d are the total orderings of the vertices, which each have the chain as their induced poset.

$$\chi_{K_d}(q,n) = d! \cdot q^{\binom{d+1}{2}} \begin{bmatrix} n \\ d \end{bmatrix}_q$$

$\rho =$

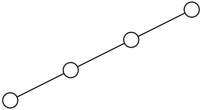


$\Pi_\rho =$

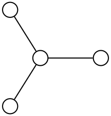


$\mathcal{L}(\Pi_\rho) = \{1234\}$

Some examples of $\chi_T(q,n)$ in the "h^*-basis"



$$8q^{10}\begin{bmatrix} n \\ 4 \end{bmatrix}_q + (4q^9+6q^8+4q^7)\begin{bmatrix} n+1 \\ 4 \end{bmatrix}_q + 2q^6\begin{bmatrix} n+2 \\ 4 \end{bmatrix}_q$$



$$8q^{10}\begin{bmatrix} n \\ 4 \end{bmatrix}_q + (5q^9+4q^8+5q^7)\begin{bmatrix} n+1 \\ 4 \end{bmatrix}_q + (q^7+q^5)\begin{bmatrix} n+2 \\ 4 \end{bmatrix}_q$$

The q-analog chromatic polynomial

There exists a polynomial $\widetilde{\chi}_G(q,x) \in \mathbb{Q}(q)[x]$, which we call the q-**analog chromatic polynomial**, such that

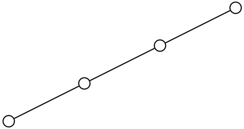
$$\widetilde{\chi}_G(q,[n]_q) = \chi_G(q,n) \quad \left(= X_G(q,q^2,,q^n,0,\dots)\right).$$

Theorem.

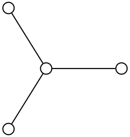
$$\widetilde{\chi}_G(q,x) = q^d \sum_{\text{flats } S\subseteq E} \mu(\varnothing,S) \prod_{\lambda_i\in\lambda(S)} \frac{1-(1+(q-1)x)^{\lambda_i}}{1-q^{\lambda_i}}$$

Question. Does the leading coefficient already distinguish trees?

Some examples of $[d]_q! \cdot \widetilde{\chi}_T(q,x)$



$$(2q^8+4q^7+6q^6+4q^5+8q^4)x^4+$$
$$(-6q^8-10q^7-18q^6-18q^5-20q^4)x^3+$$
$$(4q^8+10q^7+20q^6+22q^5+16q^4)x^2+$$
$$(-4q^7-8q^6-8q^5-4q^4)x$$



$$(q^9+6q^7+4q^6+5q^5+8q^4)x^4+$$
$$(-q^9-3q^8-14q^7-14q^6-21q^5-19q^4)x^3+$$
$$(3q^8+12q^7+18q^6+24q^5+15q^4)x^2+$$
$$(-4q^7-8q^6-8q^5-4q^4)x$$

Further work

Some classical properties of the chromatic polynomial:

(i) degree d, monic, no constant term

(ii) integer coefficients, alternating in sign

(iii) second coefficient is the number of edges

(iv) linear coefficient is the number of acyclic orientations with a unique sink at some fixed vertex

Question. Can we refine these for $\widetilde{\chi}$?

A reciprocity result

Theorem.

$$(-q)^d \cdot \widetilde{\chi}_G(1/q, -q[n]_q) = \sum_{(\rho,c)} q^{\sum c(i)},$$

where the sum ranges over all pairs of acyclic orientations ρ and *weakly* compatible colorings c (i.e. $c(i) \leq c(j)$ if $(i,j) \in \rho$).

Famous Case: $(-1)^d \cdot \chi_G(-1) = |\mathcal{A}(G)|$

The q, λ-analog chromatic polynomial

Chapoton's weighted Ehrhart theory applies to general linear forms λ, so we can also define:

$$\begin{aligned}\chi_G^\lambda(q,n) :&= \sum_{\substack{\text{proper} \\ c:[d]\to[n]}} q^{\lambda_1 c(1)+\cdots+\lambda_d c(d)} \\ &= \sum_{\rho\in\mathcal{A}(G)} \mathrm{ehr}^\lambda_{\mathcal{O}(\Pi_\rho)^\circ}(q, n+1).\end{aligned}$$

The bad news: For general λ, χ_G^λ is not a necessarily an instance of the chromatic symmetric function.

Why care about χ_G^λ (and $\widetilde{\chi}_G^\lambda$)?

Deletion-Contraction Lemma. Let $G = ([d], E)$ be a graph with $e = \{1,2\} \in E$. Then

$$\chi_G^{(\lambda_1,\ldots,\lambda_d)}(q,n) = \chi_{G\setminus e}^{(\lambda_1,\ldots,\lambda_d)}(q,n) - \chi_{G/e}^{(\lambda_1+\lambda_2,\ldots,\lambda_n)}(q,n).$$

$\cancel{X_G(x_1, x_2, \ldots)}$

$\chi_G^\lambda(q,n)$

$\cancel{\chi_G(q,n)}$

$\chi_G(n)$

Conjecture. If S and T are non-isomorphic trees, then there exists λ for which

$$\chi_S^\lambda(q,n) \neq \chi_T^\lambda(q,n).$$

References

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Thank you!! :)