#1

(a) $T(2,1) = a(2,1) + b\mathbb{W}$. (1) Since $T(2,1) = (10,5) = 5(2,1)$, $a=5$ and $b=0$ (1)

$T(\mathbb{W}) = 1(2,1) + 0\mathbb{W} = (2,1)$. Let $\mathbb{W} = (x_1, x_2)$ and solve $\begin{cases} 4x_1 + 2x_2 = 2 \\ 2x_1 + x_2 = 1 \end{cases}$ for x_1 and x_2.

(1) Then $2x_1 + x_2 = 1$. (1)

(b) Two matrices A and B are similar $\Leftrightarrow$ $\exists$ an invertible matrix P such that

$$A = P^{-1}BP$$

(1) for the definition

$$[T]_\alpha^\alpha = \left([I]_\alpha^S\right)^{-1}[T]_S^S[I]_\alpha^S$$ where the invertible matrix $[I]_\alpha^S$

Use it for the definition

$$= \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$ (1)

$\hookrightarrow$ any $\mathbb{W} = (x_1, x_2)$ satisfying $2x_1 + x_2 = 1$

#2

(a) $T(0_v) = T(0_v + 0_v) = T(0_v) + T(0_v)$ (1)

$\Rightarrow T(0_v) + (-T(0_v)) = T(0_v) + T(0_v) + (-T(0_v))$ (1)

It can be proved in different ways.

$\Rightarrow 0_w = T(0_v) + 0_w$ 0.5

$\Rightarrow 0_w = T(0_v)$ 0.5

(b) Let $S = \{T(v) \mid v \in V\}$.

(i) Since V is not empty, S is not empty. (or $T(0_v) = 0_w$) okay to skip it

(ii) $\forall k \in \mathbb{R}$, $\forall T(v) \in S$, $kT(v) \underset{\text{linear}}{=} T(kv)$. Since $kv \in V$, $T(kv) \in S$. 1.5

(iii) $\forall T(v_1)$ and $T(v_2) \in S$,

$T(v_1) + T(v_2) \underset{\text{linear}}{=} T(v_1 + v_2)$. Since $v_1 + v_2 \in V$, $T(v_1 + v_2) \in S$. 1.5

(c) $\underbrace{\text{Since } T(0_v) = 0_W}_{0.5}$, $\underbrace{\text{if } w \neq 0_w,\ 0_v \notin \{v \in V \mid T(v) = w\}}_{0.5}$. Therefore, $\{v \in V \mid T(v) = w\}$ is not a subspace

#3

(a) $[T(2v_1 - 3v_2)]_\beta = \underbrace{[2T(v_1) - 3T(v_2)]_\beta}_{(1)}$

$= 2[T(v_1)]_\beta - 3[T(v_2)]_\beta$ (1)

$= 2(2,-1) - 3(1,1)$ (1)

$= (1,-5)$

(b) $\det\left(\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\right) = 2-(-1) = 3 \neq 0$

So, T is invertible (1)

$[T^{-1}]_\beta^\alpha = \left([T]_\alpha^\beta\right)^{-1} = \left(\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\right)^{-1}$ (1)

$= \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 2/3 \end{bmatrix}$ (1)

(c) $T^{-1}(w_1 + w_2) = \underbrace{T^{-1}(w_1) + T^{-1}(w_2)}_{(1)} = \underbrace{\left(\tfrac{1}{3}v_1 + \tfrac{1}{3}v_2\right) + \left(-\tfrac{1}{3}v_1 + \tfrac{2}{3}v_2\right)}_{(1)} = v_2$

#4.

(a) Let $w_1 = 2v_1 + v_2$ and $w_2 = -v_1 + 3v_2$.

Suppose $aw_1 + bw_2 = 0$ for $a, b \in \mathbb{R}$. Then $a(2v_1+v_2) + b(-v_1+3v_2) = 0$

0.5 $\Rightarrow (2a-b)v_1 + (a+3b)v_2 = 0$ 1

1 Since $\{v_1, v_2\}$ is linearly independent, $2a-b=0$ and $a+3b=0$

0.5 Solve $\begin{cases} 2a-b=0 \\ a+3b=0 \end{cases}$ for a and b. Then $a=b=0$.

Therefore, $\{w_1, w_2\}$ is linealy independent. Since $\dim(V)=2$ $\{w_1, w_2\}$ is a basis for V.

0.5 0.5 ← explain why it is a basis

(b) $[T]_\beta^\beta = [TI]_\beta^\beta = [T]_\alpha^\beta [I]_\beta^\alpha$. $[I]_\beta^\alpha = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ 1

1.5

Therefore, $[T]_\beta^\beta = \underbrace{\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}}_{\text{given}} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 7 & 0 \end{bmatrix}$

computing 0.5

#5

(a) $\forall\, p(x) = a_0 + a_1x + a_2x^2,\ q(x) = b_0 + b_1x + b_2x^2 \in P_2(\mathbb{R}),\ \forall k \in \mathbb{R}$,

(i) $T(p(x)+q(x)) = T((a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2) = (a_0+b_0+a_1+b_1,\ a_1+b_1+a_2+b_2,\ a_2+b_2+a_0+b_0)$

$= (a_0+a_1, a_1+a_2, a_2+a_0) + (b_0+b_1, b_1+b_2, b_2+b_0) = T(p(x)) + T(q(x))$

(ii) $T(kp(x)) = T(ka_0 + (ka_1)x + (ka_2)x^2) = (ka_0+ka_1,\ ka_1+ka_2,\ ka_2+ka_0)$

$= k(a_0+a_1, a_1+a_2, a_2+a_0) = kT(p(x))$

(b) $\forall\, p(x) \in \ker(T)$. Then $T(p(x)) = (a_0+a_1, a_1+a_2, a_2+a_0) = (0,0,0)$

2.5

Solve $\begin{cases} a_0+a_1=0 \\ a_1+a_2=0 \\ a_2+a_0=0 \end{cases}$. $\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{matrix} a_0+a_1=0 \\ a_1=0 \\ a_2=0 \end{matrix}$

Showing T is injective or surjective

$\Rightarrow a_0=a_1=a_2=0 \Rightarrow p(x)=0 \Rightarrow T$ is injective

1.5 Since $\dim(\mathbb{R}^3) = \dim(P_2(\mathbb{R})) = 3$, T is surjective too.

#6

(a) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x) = 0$. ↙ linear transformation 1

$\alpha = \{(1,0),(0,1)\}$ is the standard basis for $\mathbb{R}^2$. But $T(1,0) = T(0,1) = 0$, and $\{0\}$ is not linearly independent

1 1

there are many correct answers

or $\operatorname{span}\{0\} \neq \mathbb{R}^2$

$\Rightarrow$ not a basis 1

(b) $V_1 = \text{span}(\underbrace{\{e_1, e_2\}}_{\text{linearly indep}}) \subset \mathbb{R}^3$ and $V_2 = \text{span}(\underbrace{\{e_2, e_3\}}_{\text{linearly indep}}) \subset \mathbb{R}^3$ ← correct example : 2

explain the example : 2

$\dim(V_1) + \dim(V_2) = 2+2=4$

$V_1+V_2 = \{ w_1 + w_2 \mid w_1 \in V_1 \text{ and } w_2 \in V_2 \}$

Since $e_1 = e_1 + 0$, $e_2 = e_2 + 0$, and $e_3 = 0 + e_3$ are in V_1+V_2, $V_1+V_2 = \mathbb{R}^3$.

Therefore, $\dim(V_1+V_2) = 3$, and $\dim(V_1+V_2) \neq \dim(V_1)+\dim(V_2)$

There are many correct answers

#7

(a) Claim $\{e^{2x}, \sin x, \cos x\}$ is linearly independent

Pf Let $ae^{2x} + \sin x + c\cos x = 0$ for all $x \in \mathbb{R}$. — ①

Then $2ae^{2x} + b\cos x - c\sin x = 0$ for all $x \in \mathbb{R}$ — ②

and $4ae^{2x} - b\sin x - c\cos x = 0$ for all $x \in \mathbb{R}$ — ③ 1.5

when $x=0$,

$$\begin{cases} a+c=0 & \text{— ①} \\ 2a+b=0 & \text{— ②} \\ 4a-c=0 & \text{— ③} \end{cases} \implies a=b=c=0 \rightarrow S \text{ is linearly independent}$$

1.5

Therefore, $\{e^{2x}, \sin x, \cos x\}$ is a basis

(b) $\forall f(x) = ae^{2x} + b\sin x + c\cos x \in S$. Then $\underbrace{f(0) = a+c \text{ and } f'(0) = 2a+b}_{1}$ (see (a))

$a+c=0$

0.5 $2a+b=0$ Let $c=t$. Then $b=2t$ and $a=-t$ 1 ← many different correct answers

$(a,b,c) = (-t, 2t, t) = t(-1, 2, 1)$

Therefore, $S = \text{span}(\{-e^{2x} + 2\sin x + \cos x\})$ with a basis $\underbrace{\{-e^{2x} + 2\sin x + \cos x\}}_{1}$

$\dim(S) = 1$

↑ 0.5