Chapter 5

Sec 5.1 Complex Numbers

Def (Field) A field is a set F with two operations defined on ordered pairs of elements of F

addition: $\forall x, y \in F$, $x+y \in F$

multiplication: $\forall x, y \in F$ $xy \in F$

Satisfying the following conditions: $\forall x, y, z \in F$

1. $x+y = y+x$
2. $(x+y)+z = x+(y+z)$
3. $\exists\, 0 \in F$ such that $x+0 = x$ — called the additive identity of F
4. For $x \in F$, there exists $-x \in F$ such that $x+(-x) = 0$ — additive inverse
5. $xy = yx$
6. $(xy)z = x(yz)$
7. $(x+y)z = xz+yz$
8. $\exists\, 1 \in F$, $1 \cdot x = x$ "1" is called the multiplicative identity of F
9. If $x \neq 0$, there exists $x^{-1} \in F$ such that $x \cdot x^{-1} = 1$ — multiplicative inverse of x

Ex1 $\mathbb{R}$: 0 is the additive identity (zero)

1 is " multiplicative identity

Ex2 $\mathbb{C}$ is the set of complex numbers

Definition: $\mathbb{C}$ is the set of ordered pairs of real numbers (a, b) with the operations of addition and multiplication by

addition: $(a, b) + (c, d) = (a+c, b+d)$

multiplication: $(a, b) \cdot (c, d) = (ac-bd, ad+bc)$ (a, b)

These operations satisfy the 9 axioms above.
So, $\mathbb{C}$ is a field.

Symbol: i i is a solution of $x^2 = -1$
That is, $i^2 = -1$.

Notation: $(a, b) = a+bi = z$

a = real part of z $a = \mathrm{Re}(z)$
b = imaginary part of z. $b = \mathrm{Im}(z)$

$$(a, b) \cdot (c, d) = (a+bi)(c+di)$$
$$= ac + adi + bci + bdi^2 \quad \text{" } -1$$
$$= ac - bd + (ad+bc)i$$

the additive identity
$= (0, 0) = 0 + 0i = 0$

multiplicative identity $= (1,0)$

$(1,0)\cdot(a,b) = (a,b)$

real Imaginary

$= (ac-bd,\ ad+bc)$

Ex3 ⓐ $(1+i)+(3-7i) = (1+3)+(1-7)i = 4-4i$

ⓑ $(1+i)\cdot(3-7i) = (1)(3)+(1)(-7)i + 3i - 7(i)^2$ $\quad i^2=-1$

$= (3+7) + (-7+3)i$

$= 10-4i$

Remark: $a \in \mathbb{R}$, $(a,0) = a+0i = a$

$a(b+ci) = ab+aci$

Multiplicative inverse of z when $z \neq 0 = 0+0i$

Say $z = a+bi \neq 0$. $\quad z^{-1}\cdot z = 1$

conjugate of z

$$z^{-1} = \frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2} \leftarrow \quad i^2=-1$$

$$= \left(\frac{a}{a^2+b^2}\right) - \frac{b}{(a^2+b^2)}\, i$$

Notation: $\bar{z}$ is the conjugate of z

If $z = a+bi$, $\bar{z} = a-bi$, $\quad z\bar{z} = a^2+b^2$

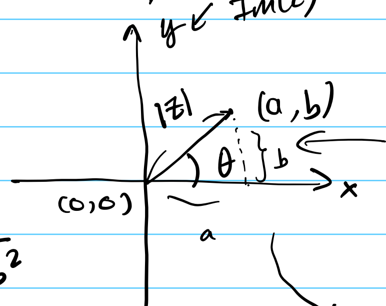
Ex4 $$\frac{1+i}{3-7i} = \frac{(1+i)(3+7i)}{(3-7i)(3+7i)} = \frac{-4+10i}{3^2+7^2} = -\frac{4}{58} + \frac{10}{58} i$$

Solve $(3-7i)z = 1+i$ for z. (sol)

Polar Form of Complex Numbers

$z = a+bi = (a,b)$

$\uparrow$ Re(z) $\quad \uparrow$ Im(z)



counter clockwise from the positive x-axis.

$a = |z|\cos\theta$

$b = |z|\sin\theta$

Def The absolute value of $|z| = \sqrt{a^2+b^2}$

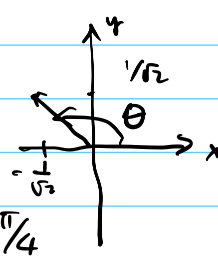
ⓐ For example, $z = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$

$$|z| = \sqrt{\left(-\tfrac{1}{\sqrt{2}}\right)^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2} = \sqrt{\tfrac{1}{2}+\tfrac{1}{2}} = 1$$

ⓑ The angle θ is called the argument of z and denoted by Arg(z)

For example, $z = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$

$\qquad\qquad\quad\ \ {}^{\shortparallel}x \quad\ {}^{\shortparallel}y$



$$\tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}}\right) = \tan^{-1}(-1)$$

$= -\pi/4$ $\qquad$ $\mathrm{Arg}(\theta) = \pi - \pi/4 = 3\pi/4$

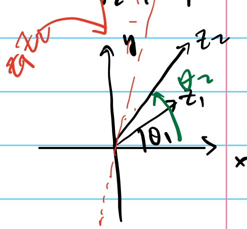
$$z = (a, b) = a + bi = |z|\cos\theta + (|z|\sin\theta)i$$
$$= |z|(\cos\theta + i\sin\theta)$$

Multiplication of polar form

$r_1 = |z_1|$

$r_2 = |z_2|$

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$



$$z_1 z_2 = r_2 r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2\left(\boxed{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2} + \boxed{(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)}\, i\right)$$
$$= r_1 r_2\left(\boxed{\cos(\theta_1+\theta_2)} + \boxed{\sin(\theta_1+\theta_2)}\, i\right)$$

Ex5 $z_1 = 2\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$ $z_2 = \frac{1}{3}\left(\cos\frac{\pi}{2} + i\sin\left(\frac{\pi}{2}\right)\right)$

(a)
$$z_1 z_2 = (2)\left(\tfrac{1}{3}\right)\left(\cos\left(\tfrac{\pi}{4}+\tfrac{\pi}{2}\right) + i\sin\left(\tfrac{\pi}{4}+\tfrac{\pi}{2}\right) i\right)$$

(b) $z = r(\cos\theta + i\sin\theta)$

$$z^{10} = \underbrace{r\times\cdots\times r}_{10}\left(\cos(\underbrace{\theta+\cdots+\theta}_{10\theta}) + i\sin(\underbrace{\theta+\cdots+\theta}_{10\theta})\right)$$

" r^{10}

In general $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\cdot\frac{\bar{z}_2}{\bar{z}_2} \overset{HW}{=} \left(\frac{r_1}{r_2}\right)\left(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right)$$

Remark
$$\boxed{\frac{1}{z}} = \frac{1(\cos(0) + i\sin(0))}{r(\cos\theta + i\sin\theta)} = \frac{1}{r}\left(\cos(0-\theta) + i\sin(0-\theta)\right)$$
$$= \frac{1}{r}\left(\cos(-\theta) + i\sin(-\theta)\right)$$
$$= \boxed{\frac{1}{r}(\cos\theta - i\sin\theta)}$$

Ex6
$$\frac{z_1}{z_2} = \frac{2\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)}{\frac{1}{3}\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)} = \frac{2}{\frac{1}{3}}\left(\cos\left(\tfrac{\pi}{4}-\tfrac{\pi}{2}\right) + i\sin\left(\tfrac{\pi}{4}-\tfrac{\pi}{2}\right)\right)$$

" $-\frac{\pi}{4}$

$$= 6\left(\cos\tfrac{\pi}{4} - i\sin\tfrac{\pi}{4}\right)$$ ← polar form

$$= 6\left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\right) = \frac{6}{\sqrt{2}} - \frac{6}{\sqrt{2}}i$$

↑ the standard form

Ex 7 Compute $\left(\frac{1}{2} + \frac{1}{2}i\right)^5$

Sol Let $z=\frac{1}{2}+\frac{1}{2}i$. $|z|=\sqrt{\frac{1}{4}+\frac{1}{4}}=\sqrt{\frac{1}{2}}=\frac{1}{\sqrt{2}}$

$$\underset{\uparrow \text{ not in general}}{\mathrm{Arg}(\theta)=}\tan^{-1}\left(\frac{1/2}{1/2}\right)=\tan^{-1}(1)=\pi/4$$

$$\left(\frac{1}{2}+\frac{1}{2}i\right)=\frac{1}{\sqrt{2}}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)$$

$$\left(\frac{1}{2}+\frac{1}{2}i\right)^5=\left(\frac{1}{\sqrt{2}}\right)^5\left(\cos\left(\frac{\pi}{4}\times5\right)+i\sin\left(\frac{\pi}{4}\times5\right)\right)$$

$$=\frac{1}{4\sqrt{2}}\left(-\cos\left(\frac{\pi}{4}\right)-i\sin\left(\frac{\pi}{4}\right)\right) \leftarrow \text{polar form}$$

$$=\frac{1}{4\sqrt{2}}\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=-\frac{1}{8}-\frac{1}{8}i$$

Theorem $P(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_0$, $a_i\in\mathbb{C}$, $a_n\neq0$ degree n polynomial has n roots counted with multiplicity in $\mathbb{C}$.

Ex 8 (a) Solve $z^2+1=0$ Sol $z=\pm i$ $\leftarrow$ $z^2=-1\Rightarrow z=\pm\sqrt{-1}$ $=i$

(b) Solve $z^2+3=0$ Sol $z=\pm\sqrt{3}i$

(c) Solve $z^3-1=0$ Sol $z^3-1=(z-1)(z^2+z+1)=0$

$\Rightarrow z=1$ or $z^2+z+1=0$

$$z=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{-3}}{2}=\frac{-1\pm\sqrt{3}i}{2}$$

$az^2+bz+c=0$

$\Downarrow$

$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$z=1,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i$

the conjugate of $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

because the coefficients of $z^3-1=0$ are real numbers

Ex9 Solve $z^3-8i=0$ (complex number)

Ans: $\sqrt{3}+i,\ -\sqrt{3}+i,\ -2i$

Solve it later

In general, how to solve $z^n=w\in\mathbb{C}$?

Called the n-th root of w

Step1 $W=r(\cos\theta+i\sin\theta)$

$=r(\cos(\theta+2\pi k)+i\sin(\theta+2\pi k))$ $k=0,1,2,3\ldots$

Step2 $z^n=w\Rightarrow z=w^{\frac{1}{n}}$

$$w^{\frac{1}{n}}=r^{\frac{1}{n}}\left(\cos\left(\frac{\theta+2\pi k}{n}\right)+i\sin\left(\frac{\theta+2\pi k}{n}\right)\right) \leftarrow \text{solutions of } z^n=w$$

$$=r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n}+\left(\frac{2\pi}{n}\right)k\right)+i\sin\left(\frac{\theta}{n}+\left(\frac{2\pi}{n}\right)k\right)\right)$$

Step3 When $k=0$, $z = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right) \overset{\text{let}}{=} z_0$

n solutions

$k=1$, $z = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n}+\frac{2\pi}{n}\right) + i\sin\left(\frac{\theta}{n}+\frac{2\pi}{n}\right)\right) = z_1$

$k=2$, $z = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n}+\left(\frac{2\pi}{n}\right)\cdot 2\right) + i\sin\left(\frac{\theta}{n}+\frac{2\pi}{n}\cdot 2\right)\right) \overset{\text{let}}{=} z_2$

$\vdots$

$k=n-1$, $z = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n}+\left(\frac{2\pi}{n}\right)(n-1)\right) + i\sin\left(\frac{\theta}{n}+\left(\frac{2\pi}{n}\right)(n-1)\right)\right) \overset{\text{let}}{=} z_{n-1}$

$k=n$, $z = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n}+\frac{2\pi}{\cancel{n}}\cdot\cancel{n}\right) + i\sin\left(\frac{\theta}{n}+\frac{2\pi}{\cancel{n}}\cancel{n}\right)\right) = z_0$

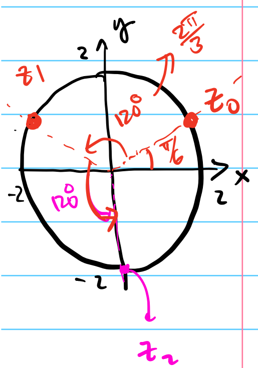
Go back to ex9: Solve $z^3 = 8i$

Step1 : $8i = 8\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 8\left(\cos\left(\frac{\pi}{2}+2\pi k\right) + i\sin\left(\frac{\pi}{2}+2\pi k\right)\right)$

Step2 : $z = (8i)^{1/3}$ $\qquad k=0,1,2.$

$= 8^{1/3}\left(\cos\left(\frac{\pi}{2}\cdot\frac{1}{3}+\frac{2\pi}{3}k\right) + i\sin\left(\frac{\pi}{2}\cdot\frac{1}{3}+\frac{2\pi}{3}k\right)\right)$

$= 2\left(\cos\left(\frac{\pi}{6}+\frac{2\pi}{3}k\right) + i\sin\left(\frac{\pi}{6}+\frac{2\pi}{3}k\right)\right)$



Step3 : When $k=0$, $z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\cdot\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) = \sqrt{3}+i$ $\quad // \; z_0$

When $k=1$,

$z = 2\left(\cos\left(\frac{\pi}{6}+\frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)\right)$

$= 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right) = 2\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) = -\sqrt{3}+i$

$= z_1$

When $k=2$, $z = 2\left(\cos\left(\frac{\pi}{6}+\frac{4\pi}{3}\right) + i\sin\left(\frac{\pi}{6}+\frac{4\pi}{3}\right)\right)$

$= 2\left(\cos\left(\frac{3}{2}\pi\right) + i\sin\left(\frac{3}{2}\pi\right)\right) = -2i = z_2$

HW (suggested exercise)

Let z_1 and z_2 be complex numbers

① $\overline{\overline{z_1}} = z_1$ ② $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

③ $\overline{z_1+z_2} = \overline{z_1}+\overline{z_2}$ ④ $z_1 = \overline{z_1}$ if z_1 is a real number

HW: $p(z) = a_n z^n + \cdots + a_0$, $a_i \in \mathbb{R}$ $\quad a_n \neq 0$.

If z is a solution of $p(z)=0$, then $\overline{z}$ is also a solution of $p(z)$

Hint: Use the properties above

Start: Since z is a solution of $p(z)=0$

$a_n z^n + \cdots + a_0 = 0$

$\overline{a_n z^n + \cdots + a_0} = \overline{0} = 0$

HW :

$a_n(\bar{z})^n + a_{n-1}(\bar{z})^{n-1} + \cdots + a_0 = 0 \Rightarrow \bar{z}$ is a solution of $p(z)$