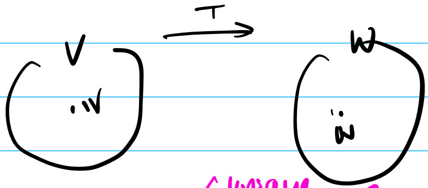
Sec 2.6 The inverse of a linear transformation



T is injective and surjective linear (bijective)

↙ unique ← from injectivity of T

$\forall w \in W$, $\exists! v \in V$ such that $T(v) = w$

↑ existence ← from surjectivity of T

Define $T^{-1}: W \to V$ by $T^{-1}(w) = v$ → We call T^{-1} the inverse transformation of T

Then T^{-1} is linear

pf $\forall w_1, w_2 \in W$ $\exists v_1, v_2 \in V$

such that $T(v_1) = w_1$ and $T(v_2) = w_2$

$$T^{-1}(w_1 + w_2) = T^{-1}(T(v_1) + T(v_2))$$

$$= \underbrace{T^{-1}(T}(v_1 + v_2)) \leftarrow T \text{ is linear}$$

$$= (T^{-1}T)(v_1 + v_2)$$

$(T^{-1}T) = I_V$ where I_V is the identity mapping from V to V

$$= I_V(v_1 + v_2)$$

$$= v_1 + v_2$$

$$= T^{-1}(w_1) + T^{-1}(w_2)$$

Hw: Show that $T^{-1}(kw) = kT^{-1}(w)$ for $\forall k \in \mathbb{R}, \forall w \in W$

Theorem Let $T: V \to W$ be a bijective linear mapping. Let α and β be bases for V and W respectively. Then

$$[T^{-1}]_\beta^\alpha = \left([T]_\alpha^\beta\right)^{-1}$$

$T^{-1}: W \to V$

$\nearrow \quad \nearrow$

$\beta \quad \alpha$

Pf

$$T^{-1}T = I_V$$

$$[T^{-1}T]_\alpha^\alpha = [I_V]_\alpha^\alpha$$

$$\Rightarrow [T^{-1}]_\beta^\alpha [T]_\alpha^\beta = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\Rightarrow [T^{-1}]_\beta^\alpha = \left([T]_\alpha^\beta\right)^{-1}$$

Say $\alpha = \{v_1, \ldots, v_n\}$

$$[I_V]_\alpha^\alpha = \left[[I_V(v_1)]_\alpha \cdots [I_V(v_n)]_\alpha\right]$$

$$= \left[[v_1]_\alpha \cdots [v_n]_\alpha\right]$$

$$= \begin{bmatrix} 1 & & 0 \\ 0 & & \vdots \\ \vdots & & \vdots \\ 0 & & 1 \end{bmatrix}$$

$AB = I$

$\Downarrow$

$A =$ the inverse matrix of B

$= B^{-1}$ ← notation for the inverse

Ex1 Let $T: V \to W$ defined by

$T(\mathbb{v}_1) = \mathbb{w}_1 + 2\mathbb{w}_2$ and $T(\mathbb{v}_2) = 2\mathbb{w}_1 - \mathbb{w}_2$

where $\alpha = \{\mathbb{v}_1, \mathbb{v}_2\}$ and $\beta = \{\mathbb{w}_1, \mathbb{w}_2\}$ are bases for V and W respectively.

Show that T is invertible (T^{-1} exists) and find T^{-1}.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\|$ A is invertible $\Leftrightarrow \det A \neq 0$

$A^{-1} = \dfrac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(or)

$\left[A \,\middle|\, \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right]$ $\wr$ $\left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \,\middle|\, \underset{\underset{A^{-1}}{\downarrow}}{*}\right]$

Sol

$[T]_\alpha^\beta = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \to$ invertible because $\det([T]_\alpha^\beta) = (1)(-1) - (2)(2) \neq 0$

$$\left([T]_\alpha^\beta\right)^{-1} = -\frac{1}{5}\begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix}$$

$\| \quad [T^{-1}]_\beta^\alpha$ $\qquad \downarrow \; [T^{-1}(\mathbb{w}_1)]_\alpha \quad \downarrow \; [T^{-1}(\mathbb{w}_2)]_\alpha$

$T^{-1}(\mathbb{w}_1) = \frac{1}{5}\mathbb{v}_1 + \frac{2}{5}\mathbb{v}_2 \qquad T^{-1}(\mathbb{w}_2) = \frac{2}{5}\mathbb{v}_1 - \frac{1}{5}\mathbb{v}_2 \leftarrow$ enough

Say $V = P_1(\mathbb{R})$ and $W = \mathbb{R}^2$

$\alpha = \{\underset{\mathbb{v}_1}{\overset{\|}{1}}, \underset{\mathbb{v}_2}{\overset{\|}{x}}\} \qquad = \{e_1, e_2\}$

Then $T^{-1}(\mathbb{w}_1) = T^{-1}(e_1) = \frac{1}{5}(1) + \frac{2}{5}(x) = \frac{1}{5} + \frac{2}{5}x$

$T^{-1}(\mathbb{w}_2) = T^{-1}(e_2) = \frac{2}{5}(1) - \frac{1}{5}(x) = \frac{2}{5} - \frac{1}{5}x$

Def If $T: V \to W$ is invertible, T is called an <u>isomorphism</u> and say V and W are isomorphic

Ex2 $[\cdot]_\alpha : V \to \mathbb{R}^n$ where $\dim(V) = n$

defined by $[x]_\alpha$ = the coordinates of x w.r.t α

quiz 2 ↗

$[\cdot]_\alpha$ is an isomorphism $\Rightarrow$ V and $\mathbb{R}^n$ are isomorphic

Theorem Let V and W be finite dimensional vector spaces

$\dim(V) = \dim(W) \Leftrightarrow$ There exists an isomorphism T from V to W

pf $(\Leftarrow)$ $\exists\, T: V \to W$ isomorphism

Since T is injective, $\dim(V) \leq \dim(W)$

Since T is surjective, $\dim(V) \geq \dim(W)$

$\dim(V)=\dim(W)$

$(\Rightarrow)$ assume that $\dim(V)=\dim(W)=n$

Say $\alpha=\{v_1\dots v_n\}$ and $\beta=\{w_1\dots w_n\}$ are bases for V and W respectively

Define $T: V\to W$ by $T(v_i)=w_i$ and linear

That is, $\forall v\in V$, $v=t_1v_1+\cdots+t_nv_n$

$T(v)=t_1T(v_1)+\cdots+t_nT(v_n)$

$=t_1w_1+\cdots+t_nw_n$

claim: T is injective.

$x\in\ker(T)$,

$0_W=T(x)=T(t_1v_1+\cdots+t_nv_n)$ $\downarrow$ by T

$=t_1w_1+\cdots+t_nw_n$

Since $\{w_1\dots w_n\}$ linearly independent, $t_1=\cdots=t_n=0$

Therefore, $x=0_V \Rightarrow \ker(T)=\{0_V\} \Rightarrow T$ is injective

Ex3 Are $P_3(\mathbb{R})$ and $M_{2\times2}(\mathbb{R})$ isomorphic?

Ans Since $\dim(P_3(\mathbb{R}))=\dim(M_{2\times2}(\mathbb{R}))=4$, they are isomorphic

Question: If $P_3(\mathbb{R})$ and $M_{2\times2}(\mathbb{R})$ are isomorphic, then construct an isomorphism.

Sol

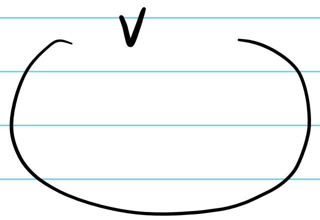
Standard basis for $P_3(\mathbb{R})=\{1, x, x^2, x^3\}$

" " " $M_{2\times2}(\mathbb{R})=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$

Define T by $T(1)=\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $T(x)=\begin{bmatrix}0&1\\0&0\end{bmatrix}$, $T(x^2)=\begin{bmatrix}0&0\\1&0\end{bmatrix}$

$T(x^3)=\begin{bmatrix}0&0\\0&1\end{bmatrix}$, and linear

Sec 2.7 Change of Basis



V : vector space over $\mathbb{R}$ and $\dim(V)=n$.

Say α and α' are bases for V

$\forall x\in V$ $[x]_\alpha$: the coordinates of x w.r.t α

$[x]_{\alpha'}$: " of x " α'

Change of basis : $[x]_\alpha \longrightarrow [x]_{\alpha'}$ How to change the coordinates?

Recall: $T: \underset{\alpha}{V} \to \underset{\beta}{W}$ linear

$\circledast$ $[T(x)]_\beta = [T]_\alpha^\beta [x]_\alpha$

If $V = W$, $T = I_V$, $\alpha = \alpha$, $\beta = \alpha'$

Then $\circledast$ $[x]_{\alpha'} = [I_V]_\alpha^{\alpha'} [x]_\alpha$

$$[x]_\alpha \xrightarrow{[I_V]_\alpha^{\alpha'}} [x]_{\alpha'}$$