UNIVERSITY OF TORONTO
FACULTY OF ARTS AND SCIENCE

Test2, Mar 14, 2025

MAT224H1 S

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Duration: 100 mins

This test has 12 pages.

Total: 50 marks

NO AIDS ALLOWED

No marks will be given for a completely wrong solution.

1. (7 marks) Let $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ be a linear mapping.

Suppose $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal basis for $\mathbf{R}^2$ and $[T]_\alpha^\alpha = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

(a) (4 marks) Compute the inner product $\langle T(\mathbf{v}_1), T(\mathbf{v}_2) \rangle$

(b) (3 marks) Find $T^{-1}(\mathbf{v}_1 + 2\mathbf{v}_2)$.

2. (a) (3 marks) Let W be a subspace of $\mathbf{R}^n$. Show that $W^\perp$ is a subspace of $\mathbf{R}^n$.

(b) (4 marks) Suppose $W = \mathrm{Span}(\{(1, 1, 2, 1), (-1, 2, 1, 0)\})$. Find $W^\perp$.

3. (7 marks) Let C^∞ be the vector space equipped with the standard vector addition and scalar multiplication and $W = \text{Span}(\{x\sin x, x\cos x, \sin x, \cos x\}) \subset C^\infty$, where $\alpha = \{x\sin x, x\cos x, \sin x, \cos x\}$ is linearly independent. Let $T : W \longrightarrow W$ be defined by $T(f) = f'$, where $f \in W$.

(a) (6 marks) Find $\text{Ker}(T)$ and $\text{Im}(T)$.

(b) (1 mark) Is T injective?

4. (8 marks) Let $T : P_2(\mathbf{R}) \to P_2(\mathbf{R})$ be defined by $T(p(x)) = p(x) + p'(x)$.

(a) (3 marks) Show that T is a linear transformation.

(b) (4 marks) Find all eigenspace(s) of T

(c) (1 mark) Is T diagonalizable?

5. (8 marks) The following statements are all false. Explain why they are false by providing a counterexample.

Let W_1 and W_2 be non-trivial subspaces of $\mathbf{R}^n$

(a) (4 marks) If $W_1 \bigoplus W_2 = \mathbf{R}^n$, $W_1 = W_2^{\perp}$.

(b) (4 marks) Suppose S_1 and S_2 are linearly independent subsets of W_1 and W_2 respectively. Then $S_1 \cup S_2$ is linearly independent.

6. (6 marks) Let $\mathbf{W} = \text{span}\{(2,1,2,0),(0,-1,2,1)\}$.

(a) (3 marks) Find an orthogonal basis of W using Gram-Schmidt Process.

(b) (3 marks) Find the orthogonal projection $P_{\mathbf{W}}(1,0,0,1)$.

7. (7 marks) Let V be a finite dimensional vector space and $T : V \longrightarrow V$ a linear mapping.

(a) (3 marks) Suppose λ is an eigenvalue of T. Show that λ^2 is an eigenvalue of T^2.

(b) (4 marks) Suppose there exists a non-zero vector $\mathbf{x} \in V$ such that $T(\mathbf{x}) \neq \mathbf{0}$ but $T^2(\mathbf{x}) = \mathbf{0}$. Show that $\{\mathbf{x}, T(\mathbf{x})\}$ is linearly independent.

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