Sec 6.3 Jordan Canonical Form (JCF) 3×3 matrices

$T: \mathbb{C}^3 \to \mathbb{C}^3$ linear

If T is diagonalizable, there exists a basis $\alpha = \{v_1, v_2, v_3\}$ consisting of eigenvectors such that $[T]_\alpha^\alpha = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ $\underbrace{\{\lambda_1, \lambda_2, \lambda_3\}}_{\text{they don't have be different eigenvalues}}$

$\alpha = \alpha_1 \cup \alpha_2 \cup \alpha_3$ where $\alpha_i = \{v_i\}$, $1 \le i \le 3$

is a canonical basis for $\mathbb{C}^3$ and $[T]_\alpha^\alpha$ is the canonical form

What if T is not diagonalizable?

→ Case 1 T has eigenvalues λ with algebraic multiplicity 2, μ, and $\dim(E_\lambda) = 1$

after case 1

case 2 T has eigenvalue λ with algebraic multiplicity 3.

I_n = $n \times n$ identity matrix

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$T = \underbrace{T - \lambda I_3}_{\text{nilpotent}} + \lambda I_3$

claim: $T - \lambda I_3$ is nilpotent

pf $\det(T - tI_3) = a(t-\lambda)^3$

$\det((T - \lambda I_3) - tI_3) = \det(T - (\lambda + t)I_3) = a(\cancel{\lambda} + t - \cancel{\lambda})^3 = at^3 \overset{\text{let}}{=} 0$

$\Rightarrow$ 0 is only eigenvalue with algebraic multiplicity 3.

$\Rightarrow$ $T - \lambda I_3$ is nilpotent

Since $T - \lambda I_3$ is nilpotent, there exists a canonical basis α for $\mathbb{C}^3$ such that $[T - \lambda I_3]_\alpha^\alpha$ is the canonical form

Therefore, $[T]_\alpha^\alpha = [T - \lambda I_3 + \lambda I_3]_\alpha^\alpha = [T - \lambda I_3]_\alpha^\alpha + [\lambda I_3]_\alpha^\alpha$

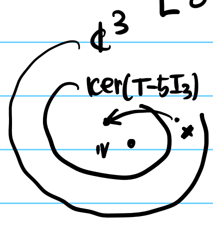
$= [T - \lambda I_3]_\alpha^\alpha + \lambda I_3$ JCF

Ex 1 $T: \mathbb{C}^3 \to \mathbb{C}^3$ defined by $\begin{bmatrix} 5 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Find a canonical basis and JCF.

Sol $\lambda = 5$ an eigenvalue with algebraic multiplicity 3.

$T = \underbrace{T - 5I_3}_{\text{nilpotent}} + 5I_3$

$T - 5I_3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ✓ a REF $\Rightarrow \dim(\ker(T - 5I_3)) = 2 \to$ 2 cycles



eigenvectors

A canonical basis for $\mathbb{C}^3 = \{(T-5I_3)(x), x\} \cup \{v\} \overset{\text{let}}{=} \alpha$

the canonical form of $T - 5I_3$ is

$[T - 5I_3]_\alpha^\alpha = \begin{bmatrix} 0 & 1 & \\ 0 & 0 & \\ & & 0 \end{bmatrix}$

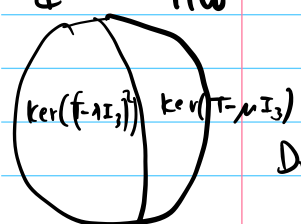
$[T]_\alpha^\alpha = [T - 5I_3]_\alpha^\alpha + 5I_3$

$$= \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix} + \begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix} = \begin{bmatrix}5&1&0\\0&5&0\\0&0&5\end{bmatrix}$$

Case 1 T has eigenvalues λ, λ and μ with $\dim(E_\lambda)=1$

$$\mathbb{C}^3 \neq \underset{\ker((T-\lambda I_3))}{\underset{\|}{E_\lambda}} \oplus E_\mu = \ker(T-\mu I_3)$$

$\mathbb{C}^3$



HW

$\ker(T-\lambda I_3) \subset \ker((T-\lambda I_3)^2)$

$\ker((T-\lambda I_3)^2) \cap \ker(T-\mu I_3) = \{0\}$

$\Rightarrow \dim(\ker(T-\lambda I_3)^2) = 2$ and $\mathbb{C}^3 = \underbrace{\ker((T-\lambda I_3)^2)}_{K_\lambda} \oplus \underbrace{\ker(T-\mu I_3)}_{K_\mu}$

Def : $K_\lambda = \ker((T-\lambda I)^m)$ where m is the algebraic multiplicity of λ, is called λ-generalized eigenspace of T.

Facts 1. K_λ is invariant, so $T|_{K_\lambda} : K_\lambda \to K_\lambda$ is linear (the same for $T|_{K_\mu}$)

2. $T|_{K_\lambda} = T|_{K_\lambda} - \lambda I_3 + \lambda I_3$ (the same for $T|_{K_\mu}$)

by case 2 $\longrightarrow$

Since $\overset{\text{nilpotent mapping}}{T|_{K_\lambda} - \lambda I_2}$ is nilpotent, there exists a canonical basis α_1 for K_λ

Such that $[T|_{K_\lambda}]_{\alpha_1}^{\alpha_1} = [T|_{K_\lambda} - \lambda I_3]_{\alpha_1}^{\alpha_1} + \lambda I_2$

$[\lambda I_3]_{\alpha_1}^{\alpha_1} = \lambda[I_3]_{\alpha_1}^{\alpha_1} = \lambda I_2$

Similarly, since $T|_{K_\mu} - \mu I_3$ is nilpotent, there exists a canonical basis α_2 for K_μ

Such that $[T|_{K_\mu}]_{\alpha_2}^{\alpha_2} = [T|_{K_\mu} - \mu I_3]_{\alpha_2}^{\alpha_2} + \mu I_1$

Let $\alpha = \alpha_1 \vee \alpha_2$. Then α is a canonical basis for $\mathbb{C}^3$

and $[T]_\alpha^\alpha = \begin{bmatrix} [T|_{K_\lambda}]_{\alpha_1}^{\alpha_1} & \\ & [T|_{K_\mu}]_{\alpha_2}^{\alpha_2} \end{bmatrix}$ JCF

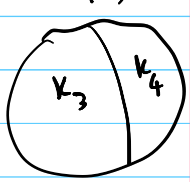
Jordan blocks

Ex2 $T: \mathbb{C}^3 \to \mathbb{C}^3$ defined by $\begin{bmatrix}3&1&1\\0&3&0\\0&0&4\end{bmatrix}$. Find a canonical basis and JCF

Sol $\lambda = 3, 3, 4$

$$T - 3I = \begin{bmatrix}0&1&1\\0&0&0\\0&0&1\end{bmatrix} \sim \underset{\swarrow \text{ a REF}}{\begin{bmatrix}0&1&1\\0&0&1\\0&0&0\end{bmatrix}} \Rightarrow \dim(E_{\lambda=3}) = \dim(\ker(T-3I)) = 1$$

$\mathbb{C}^3$



$T|_{K_3} : K_3 \to K_3$ linear, $\dim(K_3) = 2$

$T|_{K_3} = \underbrace{T|_{K_3} - 3I_3}_{\text{nilpotent}} + 3I_3$

There exists a canonical basis $\alpha_1 = \{(T|_{K_3} - 3I_2)(x), \overset{\text{eigenvector}}{x}\}$ for K_3

Such that $[T|_{K_3}]_{\alpha_1}^{\alpha_1} = \begin{bmatrix}0&1\\0&0\end{bmatrix} + 3\begin{bmatrix}1&0\\0&1\end{bmatrix} = \begin{bmatrix}3&1\\0&3\end{bmatrix}$

Likewise $T|_{K_4} : K_4 \to K_4$ linear, $\dim(K_4) = 1$

$$T|_{K_4} = \underbrace{T|_{K_4} - 4I_3}_{\text{nilpotent}} + 4I_3$$

eigenvector

There exists a canonical basis $\alpha_2 = \{v\}$ for K_4

Such that $[T|_{K_4}]_{\alpha_2}^{\alpha_2} = [0] + 4 = [4]$

$\alpha = \alpha_1 \cup \alpha_2$ is a canonical basis for $\mathbb{C}^3$

and $[T]_\alpha^\alpha = \begin{bmatrix} [T|_{K_3}]_{\alpha_1}^{\alpha_1} & \\ & [T|_{K_4}]_{\alpha_2}^{\alpha_2} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ JCF