Sec 2.7 Change of Basis

V

V : vector space over $\mathbb{R}$
and $\dim(V) = n$.
Say α and α' are bases for V

$\forall x \in V$ $\quad [x]_\alpha$: the coordinates of x w.r.t α

$[x]_{\alpha'}$: " of x " α'

Change of basis : $[x]_\alpha \longrightarrow [x]_{\alpha'}$ $\quad$ How to change the coordinates?

Recall : $T : V \to W$ linear
$\qquad \nearrow_\alpha \quad \nwarrow_\beta$

$(*)$ $\quad [T(x)]_\beta = [T]_\alpha^\beta [x]_\alpha$

If $V = W$, $T = I_V$, $\therefore$ $\alpha = \alpha$, $\beta = \alpha'$

Then $(*)$ $\quad [x]_{\alpha'} = [I_V]_\alpha^{\alpha'} [x]_\alpha$

$$[x]_\alpha \xrightarrow{[I_V]_\alpha^{\alpha'}} [x]_{\alpha'}$$

Ex 1 Review

Let $\alpha = \{(1,1), (1,-1)\}$ and $\alpha' = \{(1,2), (-2,1)\}$ be bases for $\mathbb{R}^2$

Find $[I]_\alpha^{\alpha'}$

Sol $\quad [I]_\alpha^{\alpha'} = \left[\, [I(1,1)]_{\alpha'}, [I(1,-1)]_{\alpha'} \,\right] = \left[\, \underbrace{[(1,1)]_{\alpha'}}_{= \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}}, \underbrace{[(1,-1)]_{\alpha'}}_{= \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}} \,\right]$

Then $(1,1) = a_{11}(1,2) + a_{21}(-2,1) \rightarrow \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$(1,-1) = a_{12}(1,2) + a_{22}(-2,1) \rightarrow$

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Solve it for a_{11}, a_{21}, a_{12}, and a_{22}

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 1 \\ 2 & 1 & 1 & -1 \end{array}\right]$$

$$\int$$

HW

$$\left[\begin{array}{cc|c} 1 & 0 & \multirow{2}{*}{*} \\ 0 & 1 & \end{array}\right]$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{3}{5} \end{bmatrix}$$

Ex2 Suppose $\alpha = \{(1,1),(1,-1)\}$ and $\alpha' = \{w_1, w_2\}$ are bases for $\mathbb{R}^2$, and $[I]_{\alpha}^{\alpha'} = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$. Find w_1 and w_2

Sol ① $(1,1) = (-1)w_1 + 2w_2$

② $(1,-1) = 4w_1 + (1)w_2$

$\rightarrow \begin{bmatrix} w_1 & w_2 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\begin{bmatrix} w_1, & w_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}^{-1}$

①×4 : $(4,4) = -4w_1 + 8w_2$

② $+$ $(1,-1) = 4w_1 + w_2$

$(5,3) = 9w_2$

$w_2 = \frac{1}{9}(5,3) = \left(\frac{5}{9}, \frac{3}{9}\right)$

$\xrightarrow{\text{HW}}$ $w_1 = \left(\frac{1}{9}, -\frac{3}{9}\right)$

$V \xrightarrow{T} W$ T is linear

α and α' are bases for V

β and β' are bases for W

$[T]_{\alpha}^{\beta} \xrightarrow{?} [T]_{\alpha'}^{\beta'}$

$$\begin{tikzcd} V \text{ with } \alpha' \arrow[r, "I_V"] \arrow[d, "T"] & V \text{ with } \alpha \arrow[d, "T"] \\ W \text{ with } \beta' & W \text{ with } \beta \arrow[l, "I_W"] \end{tikzcd}$$

$[T]_{\alpha'}^{\beta'} \leftarrow T$ $T \rightarrow [T]_{\alpha}^{\beta}$

$\forall x \in V$

$T(x) = (I_W T I_V)(x)$

$[T]_{\alpha'}^{\beta'} = [I_W(T I_V)]_{\alpha'}^{\beta'} = [I_W]_{\beta}^{\beta'}[T I_V]_{\alpha'}^{\beta}$

$= [I_W]_{\beta}^{\beta'}[T]_{\alpha}^{\beta}[I_V]_{\alpha'}^{\alpha}$

(4) **Corollary**

$$V \xrightarrow{T} V = W$$

α and α' bases for V $\qquad$ α and α' are bases for V
$\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\overset{\shortparallel}{\beta}$ $\quad$ $\overset{\shortparallel}{\beta'}$

$$[T]_{\alpha}^{\alpha} \xrightarrow{?} [T]_{\alpha'}^{\alpha'}$$

$$\begin{tikzcd} V \arrow[r, "I_V"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ V & V \arrow[l, "I_V^{-1}"] \end{tikzcd}$$

$$[T]_{\alpha'}^{\alpha'} = [I_V]_{\alpha}^{\alpha'} [T]_{\alpha}^{\alpha} [I_V]_{\alpha'}^{\alpha}$$

$I_V = I_V^{-1}$

$$= [I_V^{-1}]_{\alpha}^{\alpha'} [T]_{\alpha}^{\alpha} [I_V]_{\alpha'}^{\alpha}$$

$$= \left([I_V]_{\alpha'}^{\alpha} \right)^{-1} [T]_{\alpha}^{\alpha} [I_V]_{\alpha'}^{\alpha}$$

Def Let A and B be $n \times n$ matrices.
A and B are said to be similar if $\exists$ an invertible $n \times n$ matrix Q
such that $B = Q^{-1} A Q$

Ex3 $[T]_{\alpha'}^{\alpha'}$ and $[T]_{\alpha}^{\alpha}$ are similar

Ex4 $P_1(\mathbb{R}) \xrightarrow{T} P_1(\mathbb{R})$ linear
α and $\alpha' = \{1+x, 1-x\}$ are bases for $P_1(\mathbb{R})$
Suppose $[T]_{\alpha}^{\alpha} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $[(1+x)]_{\alpha} = (1, 1)$ and $[(1-x)]_{\alpha} = (0, 1)$

put them together $[I]_{\alpha'}^{\alpha}$

Find $T(6-2x)$

Sol $[T(6-2x)]_{\alpha'} = [T]_{\alpha'}^{\alpha'} [6-2x]_{\alpha'}$

Let $6 - 2x = a(1+x) + b(1-x) \xrightarrow{HW} a = 2,\ b = 4$
$[6-2x]_{\alpha'} = (2, 4)$

$$[T]_{\alpha'}^{\alpha'} = \left([I]_{\alpha'}^{\alpha} \right)^{-1} [T]_{\alpha}^{\alpha} [I]_{\alpha'}^{\alpha} \quad I_{P_1(\mathbb{R})}$$

HW

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[T(6-2x)]_{\alpha'} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$T(6-2x) = 10(1+x) + 10(1-x) = 20$$