Chapter 3

Theorem Let $T: V \to V$ be a linear transformation where $\dim(V)=n$. Then for any bases α and α' for V

$$\det\left(\underbrace{[T]_\alpha^\alpha}_{n\times n}\right)=\det\left(\underbrace{[T]_{\alpha'}^{\alpha'}}_{n\times n}\right)$$

$I: V \to V$

$I(x) = x$

$I^{-1}=I$

Pf In sec 2.7, $[T]_{\alpha'}^{\alpha'} = \underbrace{[I]_\alpha^{\alpha'}}_{=\left([I]_{\alpha'}^{\alpha}\right)^{-1}} [T]_\alpha^\alpha [I]_{\alpha'}^\alpha$

Recall

Let A and B be $n\times n$ matrices

$\det(AB)$
$=\det(A)\det(B)$

$\downarrow$

If A is invertible, $AA^{-1}=I$

So $\det(AA^{-1})=\det(I)$
$=1$

$\det(A^{-1})=\frac{1}{\det(A)}$

$$\det([T]_{\alpha'}^{\alpha'}) = \det\left(\left([I]_{\alpha'}^{\alpha}\right)^{-1} [T]_\alpha^\alpha [I]_{\alpha'}^\alpha\right)$$

$$= \det\left(\left([I]_{\alpha'}^{\alpha}\right)^{-1}\right) \det([T]_\alpha^\alpha) \det([I]_{\alpha'}^\alpha)$$

$$= \cancel{\frac{1}{\det([I]_{\alpha'}^{\alpha})}} \det([T]_\alpha^\alpha) \cancel{\det([I]_{\alpha'}^\alpha)} = \det([T]_\alpha^\alpha)$$

Def Let $T: V \to V$ be a linear mapping with $\dim(V)=n$.

Define $\det(T) = \det([T]_\alpha^\alpha)$ where α is any basis for V

Ex 1 in the textbook

$V = \text{span}(\{\cos x, \sin x, 2\cos x + 5\sin x, -7\sin x\})$

Let $T: V \to V$ be defined by $T(f) = f'$

Compute $\det(T)$.

Sol Notice that $V = \text{span}(\underbrace{\{\cos x, \sin x\}}_{\text{linearly independent}})$

Choose $\alpha = \{\cos x, \sin x\}$ which is a basis for V.

$$\det(T) = \det([T]_\alpha^\alpha)$$

$$[T]_\alpha^\alpha = \left[\, [T(\cos x)]_\alpha \ \ [T(\sin x)]_\alpha \,\right]$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$T(\cos x) = -\sin x$
$= (0)\cos x + (-1)\sin x$

$T(\sin x) = \cos x$
$= (1)\cos x + 0\sin x$

$\det([T]_\alpha^\alpha) = \det\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = 0 - (-1) = 1$

Chapter 4

Sec 4.1 Eigenvalues and Eigenvectors

Let V be a vector space over $\mathbb{R}$ ← in chapter 4

Def Let $T: V \to V$ be a linear mapping

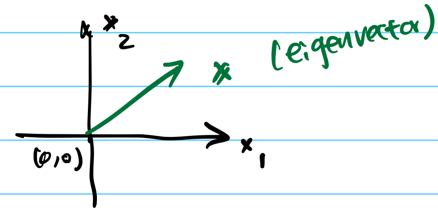
A vector $x \in V$ is called an eigenvector of T if $x \neq 0$ ↙ the additive identity of V

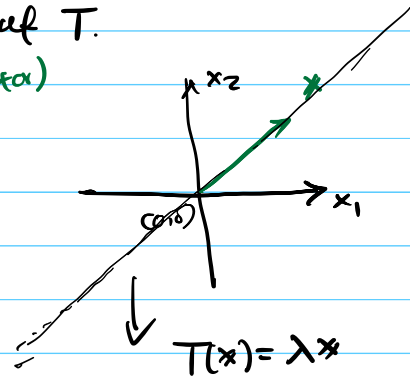
and $\exists \lambda \in \mathbb{R}$ such that $T(x) = \lambda x$

We call λ an eigenvalue of T.

In $\mathbb{R}^2$

$0 = (0,0)$





Remark ① $\forall k \in \mathbb{R}$ $k \neq 0$,

$T(\underset{\substack{\downarrow \\ \text{eigenvector}}}{kx}) = k\,T(x) = k\lambda x = \lambda(kx)$ ⇒ kx is an eigenvector of T associated with the same eigenvalue

② $T(\underset{\substack{\downarrow \\ \text{eigenvector}}}{T(x)}) = T(\lambda x) = \lambda T(x) = \lambda(\lambda x)$

$= \lambda^2 x$

$T^2 = TT$ ⇒ $(T^2)(x) = \lambda^2 x$

In general, $(T^k)(x) = \lambda^k x$

$T^k = \underbrace{TT\cdots T}_{k}$ ⇒ x is an eigenvector of T^k associated with λ^k eigenvalue.

Ex 1 $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1 + 2x_2,\ 2x_1 + x_2)$

$T(1,1) = (3,3) = 3(1,1)$

$(1,1)$ is an eigenvector of T associated with eigenvalue 3 $= \lambda$

example of Remark ② ⟶ $(1,1)$ is an eigenvector of T^2 associated with eigenvalue $3^2 = 9$

Ex 2 $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(p(x)) = x\,p'(x) + 2p(x)$

$T(x^2) = x(2x) + 2(x^2) = 2x^2 + 2x^2 = 4x^2$

x^2 is an eigenvector of T associated with the eigenvalue $\lambda = 4$

Ex 3 $T: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ defined by $T(f(x)) = f'(x)$

For example, $e^{\lambda x} = f(x)$.

$$f'(x) = \lambda e^{\lambda x} = \lambda f(x)$$

$$T(\underbrace{e^{\lambda x}}_{\text{an eigenvector}}) = \lambda e^{\lambda x} = \underbrace{\lambda}_{\text{eigenvalue}}(e^{\lambda x})$$

Since λ can be any real number, there are infinitely many eigenvalues of T

Recall

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping. How to find eigenvalues and eigenvectors?

$T(x) = [T][x]$

↳ the standard matrix

Let $[T] = A$

$x \neq 0$

Suppose x is an eigenvector associated with the eigenvalue λ.

$x = I(x)$

$T(x) = \lambda x \iff Ax = \lambda x$

$\lambda x = \lambda I(x)$

$\iff Ax - \lambda x = 0 = (0, \cdots 0)$

$\iff Ax - \lambda I(x) = 0$

$\iff \underbrace{(A - \lambda I)}_{\text{a new matrix}} x = 0$

not additive identity; not zero vector

$Ax = b$

Since x is a solution of the homogeneous system $(A - \lambda I)x = 0$,

$x \in \text{Null}(A - \lambda I) = \ker(A - \lambda I)$

$\text{Null}(A - \lambda I) - \{0\}$ = the set of eigenvectors of T (or $[T]$) associated with the eigenvalue λ.

Recall

B : $n \times n$ matrix

B is invertible

Then $Bx = 0$ has only zero solution.

because

$\underbrace{B^{-1}B}_{I} x = B^{-1} 0$

⇓

$x = 0$

Another fact: Since $A - \lambda I$ is not invertible,

(if $A - \lambda I$ is invertible, $(A - \lambda I)x = 0$ has only zero solution so x cannot be an eigenvector)

$$\det(A - \lambda I) = 0$$

By solving $\det(A - \lambda I) = 0$ we can find all eigenvalues

terminology

$0 = \det(\overset{n \times n}{A - \lambda I}) = p(\lambda) = a_n \lambda^n + \cdots + a_0$

We call $p(\lambda)$ characteristic polynomial.

We call $\text{Null}(A - \lambda I)$ the eigenspace of T associated with λ

We denote $\text{Null}(A - \lambda I)$ by E_λ

Ex4 (Revisit Ex1) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(x_1, x_2) = (x_1 + 2x_2,\ 2x_1 + x_2)$$

Find all eigenvalue(s) and eigenspace(s).

Sol $A = [T] = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

standard matrix

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$

$0 = \det(A - \lambda I) = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$

$\lambda = 3$ or $\lambda = -1$

Find $\underbrace{\mathrm{Null}(A - \lambda I)}_{E_\lambda}$ for $\lambda = 3$

$A - \lambda I = A - 3I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad x_1 - x_2 = 0$

free variable

Let $x_2 = t$. Then $x_1 = t$. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$E_\lambda = \mathrm{Span}(\{(1,1)\})$ $\quad \dim(E_\lambda) = 1$

Hw: Find the eigenspace $E_{\lambda=-1}$. Ans: $E_{\lambda=-1} = \mathrm{span}\{(1,-1)\}$

How to find eigenvalues and eigenvectors of T on non-Euclidean spaces?

$T: V \to V$ linear $\quad \dim(V) = n$

If $[x]_\alpha = (1,2,3)$ and $\alpha = \{v_1, v_2, v_3\}$, then $x = (1)v_1 + 2v_2 + 3v_3$

$T(x) = \lambda x \iff [T(x)]_\alpha = [\lambda x]_\alpha$ for any basis α for V

eigenvector eigenvalue

$[T(x)]_\alpha = [T]_\alpha^\alpha [x]_\alpha$, $\quad [\lambda x]_\alpha = \lambda [x]_\alpha$

$T(x) = \lambda x \iff \underbrace{[T]_\alpha^\alpha}_{n \times n \text{ matrix}} [x]_\alpha = \lambda [x]_\alpha$

$[x]_\alpha$: eigenvector of $[T]_\alpha^\alpha$ associated with λ

Ex5 (Revisit Ex2) $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(p) = xp' + 2p$, $p \in P_2(\mathbb{R})$

Find all eigenvalues and eigenspaces of T.

Sol $\alpha = \{1, x, x^2\}$ is the standard basis for $P_2(\mathbb{R})$.

$[T]_\alpha^\alpha = [\,[T(1)]_\alpha \ [T(x)]_\alpha \ [T(x^2)]_\alpha\,]$

$T(1) = x(1)' + 2(1) = 2 = 2\cdot(1) + 0(x) + (0)x^2$ $\quad [T(1)]_\alpha = (2, 0, 0)$

$T(x) = x(x)' + 2x = 3x = 0\cdot(1) + 3(x) + (0)x^2 \quad [T(x)]_\alpha = (0,3,0)$

$T(x^2) = x(x^2)' + 2(x^2) = 4x^2 = 0(1) + 0(x) + (4)x^2 \quad [T(x^2)]_\alpha = (0,0,4)$

$A \stackrel{\text{let}}{=} [T]_\alpha^\alpha = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad (A-\lambda I) = \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{bmatrix}$

$\det(A-\lambda I) = (2-\lambda)(3-\lambda)(4-\lambda) \stackrel{\text{let}}{=} 0 \qquad \lambda = 2, 3, 4$

$\lambda = 2, 3, 4 \implies$ eigenvalues of T

Find $\underbrace{Null(A-2I)}$: $A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_2 = 0$, $2x_3 = 0$

the eigenspace for $A = [T]_\alpha^\alpha$

free variable $\qquad x_2 = x_3 = 0$

$x_1 = t$

$Null(A-2I) = span(\underbrace{\{(1,0,0)\}})$ $\qquad (x_1, x_2, x_3) = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

eigenvector of $[T]_\alpha^\alpha$

$E_{\lambda=2}$ for T $= span(\{(1 + 0\cdot x + 0x^2)\}) = span(\{1\})$

<u>HW</u> Find $E_{\lambda=3}$ and $E_{\lambda=4}$

<u>Ans</u> $E_{\lambda=3} = span(\{(0)(1) + (1)x + (0)x^2\}) = span(\{x\})$

$E_{\lambda=4} = span(\{(0)(1) + (0)x + (1)x^2\}) = span(\{x^2\})$