Q1 (a) T is not injective because $\ker(T) \neq \{0\}$

(b) $\dim(\ker(T)) + \dim(\mathrm{Im}(T)) = \dim(\mathbb{R}^4) = 4$

Since $\dim(\ker(T)) = 1$, $\dim(\mathrm{Im}(T)) = 3$.

(c) If T is surjective, $\dim(\mathrm{Im}(T)) = \dim(W)$. Therefore, $\dim(W) = 3$.

Q2

$$T(0_V) = T(0_V + 0_V) = T(0_V) + T(0_V)$$

$$\Rightarrow T(0_V) + (-T(0_V)) = T(0_V) + T(0_V) + (-T(0_V))$$

$$\Rightarrow 0_W = T(0_V) + 0_W \Rightarrow 0_W = T(0_V)$$

Q3 (a) i) $\forall p(x), q(x) \in P_2(\mathbb{R})$, $T(p+q) = \begin{bmatrix} (p+q)(0) & (p+q)'(0) \\ (p+q)'(0) & (p+q)''(0) \end{bmatrix} = \begin{bmatrix} p(0)+q(0) & p'(0)+q'(0) \\ p'(0)+q'(0) & p''(0)+q''(0) \end{bmatrix}$

$$= \begin{bmatrix} p(0) & p'(0) \\ p'(0) & p''(0) \end{bmatrix} + \begin{bmatrix} q(0) & q'(0) \\ q'(0) & q''(0) \end{bmatrix} = T(p) + T(q)$$

ii) $\forall p(x) \in P_2(\mathbb{R})$, $\forall c \in \mathbb{R}$,

$$T(cp) = \begin{bmatrix} (cp)(0) & (cp)'(0) \\ (cp)'(0) & (cp)''(0) \end{bmatrix} = \begin{bmatrix} cp(0) & cp'(0) \\ cp'(0) & cp''(0) \end{bmatrix} = c\begin{bmatrix} p(0) & p'(0) \\ p'(0) & p''(0) \end{bmatrix} = cT(p)$$

↑ it is okay to check both together

(b) $\forall p(x) = a_0 + a_1x + a_2x^2 \in \ker(T)$. Then $T(p(x)) = \begin{bmatrix} p(0) & p'(0) \\ p'(0) & p''(0) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow p(0) = 0$, $p'(0) = 0$, and $p''(0) = 0$ (*)

Since $p'(x) = a_1 + 2a_2x$ and $p''(x) = 2a_2$, by (*), $a_1 = 0$, $a_2 = 0$, and $a_3 = 0$

So $p(x) = 0$ polynomial. Therefore, $\ker(T) = \{0\}$ → zero polynomial

(b) or Suppose $T(p(x)) = T(q(x))$

⇓

$p(0) = q(0)$

$p'(0) = q'(0)$

$p''(0) = q''(0)$

⇓

$p = q$

(c) $\dim(P_2(\mathbb{R})) = 3 < \dim(M_{2\times2}(\mathbb{R})) = 4$, so T cannot be a surjective mapp

or (c) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $b \neq c$ cannot be in the $\mathrm{Im}(T)$ because $b \neq c$

Therefore, $\mathrm{Im}(T) \neq M_{2\times2}(\mathbb{R}) \Rightarrow$ not surjective

Q4 (a) $[T(v_1)]_\beta = (3, -1) \Rightarrow T(v_1) = 3(1,1) - (-2,3) = (3+2, 3-3) = (5, 0)$

$[T(v_2)]_\beta = (1, 2) \Rightarrow T(v_2) = (1,1) + 2(-2,3) = (1-4, 1+6) = (-3, 7)$

(b) $T(-v_1 + 2v_2) = -T(v_1) + 2T(v_2) = -(5,0) + 2(-3,7) = (-11, 14)$

(c) $[T^{-1}]_{\beta}^{\alpha} = ([T]_{\alpha}^{\beta})^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ \frac{1}{7} & \frac{3}{7} \end{bmatrix}$

or

(or) $\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{7} & -\frac{1}{7} \\ 0 & 1 & \frac{1}{7} & \frac{3}{7} \end{array}\right]$

Q5 (a) $(1,1) \in (\mathbb{R}^+)^2$ because $1 > 0$.

$\forall (x_1, x_2) \in (\mathbb{R}^+)^2, \quad (x_1, x_2) +' (1, 1) = ((x_1)(1), (x_2)(1)) = (x_1, x_2)$

(b) $(\frac{1}{2}, \frac{1}{3}) \in (\mathbb{R}^+)^2$ and $(2, 3) +' (\frac{1}{2}, \frac{1}{3}) = (2 \cdot \frac{1}{2}, 3 \cdot \frac{1}{3}) = (1, 1)$

(c) (i) $\forall x = (x_1, x_2), \forall y = (y_1, y_2) \in \mathbb{R}^2, \; T(x+y) = T(x_1+y_1, x_2+y_2) = (e^{x_1+y_1}, e^{x_2+y_2})$

$= (e^{x_1} \cdot e^{y_1}, e^{x_2} \cdot e^{y_2}) = T(x_1, x_2) +' T(y_1, y_2)$

(ii) $\forall x = (x_1, x_2) \in \mathbb{R}^2, \forall c \in \mathbb{R}, \quad T(cx) = T(cx_1, cx_2) = (e^{cx_1}, e^{cx_2})$

$= c \cdot (e^{x_1}, e^{x_2}) = c\,T(x_1, x_2)$

Q6 $S_1 = \{(x_1, 0) \mid x_1 \in \mathbb{R}\}$ and $S_2 = \{(0, x_2) \mid x_2 \in \mathbb{R}\}$ are subspaces of $\mathbb{R}^2$.

But $S_1 \cup S_2 = \{(x_1, x_2) \mid x_1 = 0 \text{ or } x_2 = 0\}$ is not a subspace

There are many counterexamples

because $(1, 0)$ and $(0, 1) \in S_1 \cup S_2$ but $(1, 0) + (0, 1) = (1, 1) \notin S_1 \cup S_2$

Q7

(i) $f(x) = x^2 \in S_1$ because $f(-x) = (-x)^2 = x^2 = f(x)$. $S \neq \emptyset$

(ii) $\forall f(x), g(x) \in S_1, \quad (f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$, so $f+g \in S$

(iii) $\forall f(x) \in S, \forall c \in S_1, \; (cf)(-x) = c(f(-x)) = c(f(x)) = (cf)(x)$, so $cf \in S_1$

Q8

(a) $\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ← free varial

$x_1 + x_2 + x_3 = 0$

$x_3 = 0$

Let $x_2 = t$. Then $x_1 = -t$

$(x_1, x_2, x_3) = (-t, t, 0) = t(-1, 1, 0)$ $\quad \ker([T]_{\alpha}^{\beta}) = \operatorname{span}\{(-1, 1, 0)\}$

(b) $\ker(T) = \operatorname{span}\{-e^x + e^{2x}\}$

(c) $\operatorname{Im}([T]_{\alpha}^{\beta}) = \operatorname{span}\{(1,-1,1), (1,1,-1)\}$. So $\operatorname{Im}(T) = \operatorname{span}\{w_1, w_2\}$

where $w_1 = (1,0,0) - (1,1,0) + (1,1,2) = (1,0,2)$

$w_2 = (1,0,0) + (1,1,0) - (1,1,2) = (1,0,-2)$

(d) $\{-e^x + e^{2x}, w_1, w_2\}$ where $[T(w_1)]_{\beta} = (1,-1,1)$ and $[T(w_2)]_{\beta} = (1,1,-1)$

So $\{-e^x + e^{2x}, \underset{w_1}{\underset{\shortparallel}{e^x}}, \underset{w_2}{\underset{\shortparallel}{e^{3x}}}\}$ is a basis of V containing $\{-e^x + e^{2x}\}$

There are many correct different answers

Q9. (a) Let $c_1(2w_1+w_2)+c_2(w_1-w_2)=0$

$(2c_1+c_2)w_1+(c_1-c_2)w_2=0$

Since $\{w_1, w_2\}$ is linearly independent, $\left.\begin{aligned}2c_1+c_2&=0\\ c_1-c_2&=0\end{aligned}\right\}$ Solve the system for c_1 and c_2

Then $3c_1=0$. So $c_1=0$ and $c_2=0$ by adding the equations.

$\Rightarrow \beta=\{2w_1+w_2,\ w_1-w_2\}$ is linearly independent

Since $\dim(\operatorname{span}(\beta))=2=\dim(V)$ and $\operatorname{span}(\beta)\subseteq V$,

$\operatorname{span}(\beta)=V$. Therefore, β is a basis of V.

or suppose β is linearly dependent $\downarrow$ contradiction

or Show $\operatorname{span}(\beta)=V$ directly.

(b) $[2w_1+w_2]_\alpha=(2,1)$ $\quad [w_1-w_2]_\alpha=(1,-1)$

So consider $[w]_\alpha=[I]_\beta^\alpha[w]_\beta$ where $I: V\to V$, identity

$$[I]_\beta^\alpha=\big[\,[2w_1+w_2]_\alpha\ \ [w_1-w_2]_\alpha\,\big]=\begin{bmatrix}2&1\\1&-1\end{bmatrix}$$

$$\text{So } [w]_\beta=\begin{bmatrix}2&1\\1&-1\end{bmatrix}^{-1}[w]_\alpha=-\frac{1}{3}\begin{bmatrix}-1&-1\\-1&2\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix}=-\frac{1}{3}\begin{bmatrix}-7\\5\end{bmatrix}=\left(\frac{7}{3},-\frac{5}{3}\right)$$

or

$[w]_\beta=[I]_\alpha^\beta[w]_\alpha$ where $I: V\to V$, identity

$[I]_\alpha^\beta=\big[\,[w_1]_\beta\ [w_2]_\beta\,\big]$ $\quad$ $[I]_\alpha^\beta=\left([I]_\beta^\alpha\right)^{-1}$ $\quad$ direct method

Let $[w_1]_\beta=(c_1, c_2)$

Then $w_1=c_1(2w_1+w_2)+c_2(w_1-w_2)$. $\quad (2c_1+c_2-1)w_1+(c_1-c_2)w_2=0$

$\Rightarrow \left.\begin{aligned}2c_1+c_2&=1\\ c_1-c_2&=0\end{aligned}\right\}$ Solve it for c_1 and c_2

Then $3c_1=1$. $\Rightarrow c_1=\frac{1}{3}$, So $c_2=\frac{1}{3}$

Let $[w_2]_\beta=(d_1,d_2)$. Then $w_2=d_1(2w_1+w_2)+d_2(w_1-w_2)$

$\left.\begin{aligned}2d_1+d_2&=0\\ d_1-d_2&=1\end{aligned}\right\}\Rightarrow 3d_1=1 \quad d_1=\frac{1}{3} \quad \text{so } d_2=-\frac{2}{3}$

$$[w]_\beta=\begin{bmatrix}\frac{1}{3}&\frac{1}{3}\\ \frac{1}{3}&-\frac{2}{3}\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}\frac{7}{3}\\ -\frac{5}{3}\end{bmatrix}$$