Sec 4.4 #1 (a) 5 marks

Sol : $W = \text{span}\{(0,1,1,-1),\ (3,1,4,2)\}$

$\forall\ \vec{x} \in W^{\perp}$, $\vec{x} = (x_1, x_2, x_3, x_4)$,

$\langle \vec{x}, (0,1,1,-1) \rangle = 0$ and $\langle \vec{x}, (3,1,4,2) \rangle = 0$ $\Rightarrow$ $x_2 + x_3 - x_4 = 0$, $3x_1 + x_2 + 4x_3 + 2x_4 = 0$ okay to skip the step

So, $\vec{x} \in \ker(A)$ where $A = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 3 & 1 & 4 & 2 \end{bmatrix}$

$A \sim \begin{bmatrix} 3 & 1 & 4 & 2 \\ 0 & 1 & \underbrace{1 \quad -1}_{\text{free variables}} \end{bmatrix}$ (2)

Let $x_3 = t$ and $x_4 = s$.

Then $x_2 = x_4 - x_3 = s - t$

$x_1 = \frac{1}{3}(-x_2 - 4x_3 - 2x_4) = \frac{1}{3}(t - s - 4t - 2s) = \frac{1}{3}(-3t - 3s)$
$= -t - s$

Therefore, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t-s \\ s-t \\ t \\ s \end{bmatrix} = t\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ (2)

$W^{\perp} = \ker(A) = \text{span}\left(\underbrace{\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right\}}_{\text{basis}}\right)$ (1)

Sec 4.4 #2 Suppose $W_1 \subset W_2$.

3 marks

Pf: Let $x \in W_2^{\perp}$. Then $\langle x, y \rangle = 0$ for all $y \in W_2$ (1)

Since $W_1 \subset W_2$, $\langle x, y \rangle = 0$ for all $y \in W_1$ (1)

This implies that $x \in W_1^{\perp}$. Therefore, $W_2^{\perp} \subset W_1^{\perp}$

(1)

Sec 4.4 #3

Sol (a) claim: $(W_1 + W_2)^{\perp} \subseteq W_1^{\perp} \cap W_2^{\perp}$

Pf : $\forall\ x \in (W_1 + W_2)^{\perp}$. Then $\langle x, y \rangle = 0$ for all $y \in W_1 + W_2$

Since $W_1 \subseteq W_1 + W_2$ and $W_2 \subseteq W_1 + W_2$, $\langle x, w_1 \rangle = 0$ and $\langle x, w_2 \rangle = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$. $\Rightarrow x \in W_1^{\perp}$ and $x \in W_2^{\perp} \Rightarrow x \in W_1^{\perp} \cap W_2^{\perp}$

Pf or

Since $W_1 \subseteq W_1 + W_2$ and $W_2 \subseteq W_1 + W_2$, by #2, $(W_1 + W_2)^{\perp} \subseteq W_1^{\perp}$ and $(W_1 + W_2)^{\perp} \subseteq W_2^{\perp}$ $\Rightarrow (W_1 + W_2)^{\perp} \subseteq W_1^{\perp} \cap W_2^{\perp}$

claim: $W_1^{\perp} \cap W_2^{\perp} \subseteq (W_1 + W_2)^{\perp}$

pf : $\forall\ x \in W_1^{\perp} \cap W_2^{\perp}$. $\forall\ y \in W_1 + W_2$, $y = w_1 + w_2$ for some $w_1 \in W_1$ and for some $w_2 \in W_2$. $\langle x, y \rangle = \langle x, w_1 + w_2 \rangle = \langle x, w_1 \rangle + \langle x, w_2 \rangle = 0 + 0$ because $x \in W_1^{\perp}$ and $x \in W_2^{\perp}$

Therefore, $x \in (W_1 + W_2)^{\perp}$.

(c) claim: $W_1 \subseteq ((W_1)^{\perp})^{\perp}$.

(pf) Let $x \in W_1$. Then $\langle x, y \rangle = 0$ for $\forall\ y \in W_1^{\perp}$ $\Rightarrow$

$x \in (W_1^{\perp})^{\perp} = \{x \in \mathbb{R}^n \mid \langle x, y \rangle = 0, \text{ for } \forall\ y \in (W_1)^{\perp}\}$

claim: $(W_1^{\perp})^{\perp} \subseteq W_1$

or

(pf) Let $x \in (W_1^{\perp})^{\perp}$. Then $\langle x, y \rangle = 0$ for all $y \in W_1^{\perp}$.

Since $W_1 \subseteq (W_1^{\perp})^{\perp}$ and $\mathbb{R}^n = W_1 \oplus W_1^{\perp} = W_1^{\perp} \oplus (W_1^{\perp})^{\perp}$,

Since for some $x_{W_1} \in W_1$ and $x_{W_1^{\perp}} \in W_1^{\perp}$, $x = x_{W_1} + x_{W_1^{\perp}}$,

$W_1 = (W_1^{\perp})^{\perp}$

$0 = \langle x, y\rangle = \underbrace{\langle x_{W_1}, y\rangle}_{=0} + \langle x_{W_1^\perp}, y\rangle = \langle x_{W_1^\perp}, y\rangle$, This implies that $x_{W_1^\perp} = 0$

so $x = x_{W_1} \in W_1$

(b) $(W_1^\perp + W_2^\perp)^\perp \overset{\text{by (a)}}{=} (W_1^\perp)^\perp \cap (W_2^\perp)^\perp \overset{\text{by (c)}}{=} W_1 \cap W_2$

Therefore, $W_1^\perp + W_2^\perp = (W_1 \cap W_2)^\perp$ by (c)

Remark: We can prove (b) separately too.

Sec 4.4 #4 (b) 5 marks

Let $\alpha = \{ \frac{1}{\sqrt{2}}(1,0,1,0), \frac{1}{\sqrt{2}}(0,1,0,1)\}$ and $W = \text{span}\{\alpha\}$

1 $[P_W] = [P_W(e_1) \ P_W(e_2) \ P_W(e_3) \ P_W(e_4)]$ where $\{e_1, e_2, e_3, e_4\}$ is the standard basis for $\mathbb{R}^4$.

1 $P_W(e_1) = \frac{1}{\sqrt{2}}\langle (1,0,0,0), \frac{1}{\sqrt{2}}(1,0,1,0)\rangle (1,0,1,0)$

$+ \frac{1}{\sqrt{2}}\langle (1,0,0,0), \frac{1}{\sqrt{2}}(0,1,0,1)\rangle (0,1,0,1)$

$= \frac{1}{2}(1,0,1,0) + 0 = (\frac{1}{2}, 0, \frac{1}{2}, 0)$

1 $P_W(e_2) = \frac{1}{\sqrt{2}}\langle (0,1,0,0), \frac{1}{\sqrt{2}}(1,0,1,0)\rangle (1,0,1,0) + \frac{1}{\sqrt{2}}\langle (0,1,0,0), \frac{1}{\sqrt{2}}(0,1,0,1)\rangle (0,1,0,1)$

$= 0 + \frac{1}{2}(0,1,0,1) = (0, \frac{1}{2}, 0, \frac{1}{2})$

1 $P_W(e_3) = \frac{1}{\sqrt{2}}\langle (0,0,1,0), \frac{1}{\sqrt{2}}(1,0,1,0)\rangle (1,0,1,0) + \frac{1}{\sqrt{2}}\langle (0,0,1,0), \frac{1}{\sqrt{2}}(0,1,0,1)\rangle (0,1,0,1)$

$= (\frac{1}{2}, 0, \frac{1}{2}, 0) + 0 = (\frac{1}{2}, 0, \frac{1}{2}, 0)$

1 $P_W(e_4) = \frac{1}{\sqrt{2}}\langle (0,0,0,1), \frac{1}{\sqrt{2}}(1,0,1,0)\rangle (1,0,1,0) + \frac{1}{\sqrt{2}}\langle (0,0,0,1), \frac{1}{\sqrt{2}}(0,1,0,1)\rangle (0,1,0,1)$

$= 0 + \frac{1}{2}(0,1,0,1) = (0, \frac{1}{2}, 0, \frac{1}{2})$

$$[P_W] = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Sec 4.5 #4 3 marks

pf Let λ and μ be distinct eigenvalues of A.

Since A is symmetric, $E_\lambda \perp E_\mu$. — 2

1 For any $x \in E_\lambda$ and $y \in E_\mu$, $\langle x, y\rangle = 0 \Rightarrow x \in E_\mu^\perp = \{x \in \mathbb{R}^n \mid \langle x, y\rangle \text{ for all } y \in E_\mu\}$

$\Rightarrow E_\lambda \subseteq E_\mu^\perp$

3 marks Sec 4.5 #8 (b) $W = \{f \in C[0,1] \mid f'' \text{ exists}, f'' \in C[0,1] \text{ and } f(0) = f(1) = 0\}$

Sol $\forall f, g \in W$, $\langle T(f), g\rangle = \langle f'', g\rangle = \int_0^1 f''(t) g(t)\, dt$

$= f'(t)g(t)\Big|_0^1 - \int_0^1 f'(t) g'(t)\, dt$ } 1.5

$= (f'(1)\overset{0}{g(1)} - f'(0)\overset{0}{g(0)}) - [f(t)g'(t)\Big|_0^1 - \int_0^1 f(t) g''(t)\, dt]$

$$= -\left[\underset{0}{f(1)}g'(1) - \underset{0}{f(0)}g'(0) \right] + \int_0^1 f(t)g''(t)dt$$ 1.5

$$= \int_0^1 f(t)g''(t)dt = \langle f(t), g''(t) \rangle = \langle f, T(g) \rangle$$

Therefore, T is symmetric

Sec 4.5 #10 (a)

Sol
$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

$$[x_1\ x_2] \underbrace{\begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{bmatrix}}_{\text{Symmetric}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$$

5 marks Sec 4.6 #2.

Pf Suppose A is symmetric and λ is an eigenvalue of A.

Then $\mathbb{R}^n = E_\lambda \oplus \sum_{\lambda \neq \mu} E_\mu$ because A is diagonalizable. 1

Since A is symmetric, $E_\lambda \perp E_\mu$ when $\lambda \neq \mu$. This implies that $E_\lambda \perp \sum_{\lambda \neq \mu} E_\mu$ because for any $x \in E_\lambda$ $\langle x, x_{\mu_1} + \cdots + x_{\mu_k} \rangle = \langle x, x_{\mu_1} \rangle + \cdots + \langle x, x_{\mu_k} \rangle = 0$ where $x_{\mu_i} \in E_{\mu_i}$, $\mu_i \neq \lambda$, $i \le i \le k$. $\Rightarrow \sum_{\lambda \neq \mu} E_\mu \subseteq E_\lambda^\perp$ 2 1

1 $\to$ Therefore, since $\mathbb{R}^n = E_\lambda \oplus E_\lambda^\perp$, $\sum_{\lambda \neq \mu} E_\mu = E_\lambda^\perp$

8 marks Sec 4.6 #1 (a)

Sol $\det(A - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{bmatrix} \right) = (2-\lambda)^2 - 9 \overset{\text{let}}{=} 0$

$\lambda^2 - 4\lambda + 4 - 9 = 0 \Rightarrow \lambda^2 - 4\lambda - 5 = 0 \Rightarrow (\lambda - 5)(\lambda + 1) = 0 \Rightarrow \lambda = 5, -1$ 1

1 For $\lambda = -1$, $A - \lambda I = A + I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

unit vector

so $E_{\lambda=-1} = \text{span}(\{(-1, 1)\}) = \text{span}(\{(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\})$

1 For $\lambda = 5$, $A - \lambda I = A - 5I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$

unit vector

so $E_{\lambda=5} = \text{span}(\{(1, 1)\}) = \text{span}(\{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\})$

Let $\alpha = \{ (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \}$. Then α is an orthonormal basis for $\mathbb{R}^2$.

1 Since $[P_{E_{\lambda=-1}}] = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and $[P_{E_{\lambda=5}}] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $\leftarrow$ 2

verification of the spectral theorem

$$[-P_{E_{\lambda=-1}} + 5P_{E_{\lambda=5}}] = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{5}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{5}{2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} = [A]$$ $\leftarrow$ 2

$$\Rightarrow -P_{E_{\lambda=-1}} + 5P_{E_{\lambda=5}} = A$$

or

$$[A]_\alpha^\alpha = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}, \quad [P_{E_{\lambda=-1}}]_\alpha^\alpha = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad [P_{E_{\lambda=5}}]_\alpha^\alpha = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad — ②$$

$$\text{So } [A]_\alpha^\alpha = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 5\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad — ②$$

$$= -[P_{E_{\lambda=-1}}]_\alpha^\alpha + 5[P_{E_{\lambda=5}}]_\alpha^\alpha$$

$$\Rightarrow A = -P_{E_{\lambda=-1}} + 5P_{E_{\lambda=5}}$$

orthogonal

or

$$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1)P_{E_{\lambda=-1}}(1,-1) + 5P_{E_{\lambda=5}}(1,-1)$$

$$= (-1)(1,-1) + 5\cdot(0,0) = (-1,1)$$

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = (-1)P_{E_{\lambda=-1}}(1,1) + 5P_{E_{\lambda=5}}(1,1)$$

$$= (-1)(0,0) + 5(1,1) = (5,5)$$

$$\text{So } A = (-1)P_{E_{\lambda=-1}} + 5P_{E_{\lambda=5}}$$

or

many different solutions