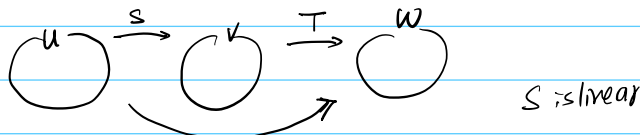


## Sec 2.5 Composition of Linear Transform.

Let  $U, V$  and  $W$  be real vector spaces

**Theorem** Let  $S: U \rightarrow V$  and  $T: V \rightarrow W$  be linear transformations  
Then  $TS$  is a linear transformation from  $U$  to  $W$

$$(TS)(x) = T(S(x))$$



$$\begin{aligned} \text{pf } \forall u_1, u_2 \in U \quad (TS)(u_1 + u_2) &= T(S(u_1 + u_2)) \xrightarrow{S \text{ is linear}} T(S(u_1) + S(u_2)) \\ &\xrightarrow{T \text{ is linear}} T(S(u_1)) + T(S(u_2)) = (TS)(u_1) + (TS)(u_2) \end{aligned}$$

$$\text{How show that } \forall u \in U, \forall k \in \mathbb{R}, (TS)(ku) = k(TS)(u)$$

Ex1  $\mathbb{R}^3 \xrightarrow{S} \mathbb{R}^4 \xrightarrow{T} \mathbb{R}^2$   
 $S(x_1, x_2, x_3) = (x_1 + x_2, x_1 - x_2, x_3, x_1 + x_3)$   
 $T(x_1, x_2, x_3, x_4) = (x_1 - x_4, x_2 + x_3)$   
 Find  $(TS)$

Sol  $(TS)(x_1, x_2, x_3) = T(S(x_1, x_2, x_3)) = T(x_1 + x_2, x_1 - x_2, x_3, x_1 + x_3)$   
 $= ((x_1 + x_2) - (x_1 + x_3), (x_1 - x_2) + x_3)$   
 $= (x_2 - x_3, x_1 - x_2 + x_3)$

**Remark:**  $[TS] = [T][S] = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

$$(TS)(x_1, x_2, x_3) = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (x_2 - x_3, x_1 - x_2 + x_3)$$

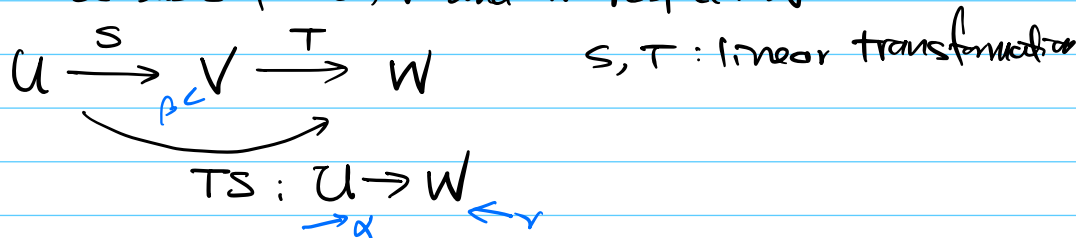
Ex2  $S: P_2(\mathbb{R}) \rightarrow P_1(\mathbb{R})$  defined by  $S(p(x)) = p'(x)$

$T: P_1(\mathbb{R}) \rightarrow P_2(\mathbb{R})$  " by  $T(p(x)) = x p(x)$

Find  $TS$

Sol  $\forall p(x) \in P_2(\mathbb{R}), (TS)(p(x)) = T(S(p(x))) = T(p'(x)) = x p'(x)$

**Theorem** Let  $\alpha, \beta$  and  $\gamma$  be bases for  $U, V$  and  $W$  respectively



$$[TS]_{\alpha}^{\gamma} = [T]_{\beta}^{\gamma} [S]_{\alpha}^{\beta}$$

(Pf)  $\text{Say } \alpha = \{u_1, \dots, u_n\}. [TS]_{\alpha}^{\gamma} = \begin{bmatrix} [(TS)(u_1)]_{\gamma} & \dots & [(TS)(u_n)]_{\gamma} \end{bmatrix}$

$$[TS](v_i)]_\gamma = [T(S(v_i))]_\gamma = [T]_\beta^\gamma [S(v_i)]_\beta$$

$$= [T]_\beta^\gamma [S]_\alpha^\beta [v_i]_\alpha$$

$$v_i = (1)v_1 + 0v_2 + \dots + 0v_n$$

$$[T(x)]_\gamma$$

$$= [T]_\beta^\gamma [x]_\beta$$

$$[S(y)]_\beta$$

$$= [S]_\alpha^\beta [y]_\alpha$$

$$[TS]_\alpha^\gamma = [A[v_1]_\alpha \ A[v_2]_\alpha \ \dots \ A[v_n]_\alpha] = A [v_1]_\alpha [v_2]_\alpha \dots [v_n]_\alpha$$

$$= A \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$$= A$$

memorize it

$$\longrightarrow [TS]_\alpha^\gamma = A = [T]_\beta^\gamma [S]_\alpha^\beta$$

AB

$$= A[b_1 \ b_2 \ \dots \ b_n]$$

$$= [Ab_1 \ Ab_2 \ \dots \ Ab_n]$$

Ex 3  $\mathbb{R}^3 \xrightarrow{S} \mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$   $S, T: \text{linear}$

$$\alpha = \{u_1, u_2, u_3\} \quad \beta = \{v_1, v_2\} \quad \gamma = \{(2, 3), (-1, 2)\}$$

Suppose  $[T]_\beta^\gamma = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}$  and  $[S]_\alpha^\beta = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

Find  $(TS)(u_1 + 2u_2 - u_3)$ .

$TS: \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$$\uparrow \alpha \quad \uparrow \gamma$$

sol  $[TS]_\alpha^\gamma = [T]_\beta^\gamma [S]_\alpha^\beta$

$$= \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$$

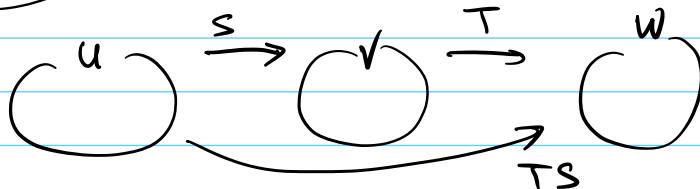
$$[TS]_\alpha^\gamma [u_1 + 2u_2 - u_3]_\alpha = [TS]_\alpha^\gamma \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$[TS]_\alpha^\gamma \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$(TS)(u_1 + 2u_2 - u_3) = 0 \cdot (2, 3) + 3 \cdot (-1, 2) \quad \parallel \quad [(TS)(u_1 + 2u_2 - u_3)]_\gamma$$

$$= (-3, 6)$$

Read the proposition in the book



$$\ker S \subseteq \ker(TS)$$

$$\text{Im}(TS) \subseteq \text{Im}(T) \quad (\text{HW})$$

claim  $\ker(S) \subseteq \ker(TS)$

pf  $\forall x \in \ker(S)$ ,

Then  $S(x) = 0_V$

$$\ker(TS) = \{x \in U \mid TS(x) = 0_W\}$$

$$\downarrow$$

$$T(S(x)) = 0_W$$

$$T(S(x)) \stackrel{\text{linear}}{=} T(0_W) = 0_W \Rightarrow x \in \ker(TS)$$