Sec 6.2 Nilpotent mappings

Def Let V be a complex vector space and $\dim(V)=n$

Let $N: V \to V$ be linear

N is nilpotent if $\exists\, k \geq 1$ such that $N^k = 0$ ← 0-mapping

Remark If $l > k$, $N^l = N^{l-k+k} = (N^{l-k}) N^k = 0$

Ex 1 $A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$. $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ nilpotent

Theorem N is nilpotent $\Leftrightarrow$ N has only zero eigenvalue

we did it before

pf ($\Rightarrow$) Let λ be an eigenvalue of N

$\forall\, x \in E_\lambda - \{0\}$, $N(x) = \lambda x$

Since N is nilpotent, $\exists\, k \geq 1$ such that $N^k = 0$

$0 = N^k(x) = \lambda^k x \Rightarrow \lambda^k = 0 \Rightarrow \lambda = 0$

λ is an eigenvalue of N $\Downarrow$ N^k has an eigenvalue λ^k

($\Leftarrow$) See the textbook (need Caley-Hamilton theorem)

Remark: $E_\lambda = \ker(N - \lambda I)$

Since $\lambda = 0$, $E_\lambda = \ker(N)$

Ex 2

$\begin{bmatrix} 0 & 7 \\ 0 & 0 \end{bmatrix}$ nilpotent

$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 5 & 0 \end{bmatrix}$ nilpotent

by the theorem above

$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ has only zero eigenvalue

Theorem Suppose $N: V \to V$ is nilpotent, $\dim(V) = n$,

and k is the smallest integer satisfying $N^k = 0$

Then there exists $x \neq 0 \in V$ such that $N^{k-1}(x) \neq 0$

and $C(x) = \{N^{k-1}(x), N^{k-2}(x), \ldots x\}$ is linearly independent.

idea of proof

Say $k = 2$, $\exists\, \underset{x \neq 0}{x}$ such that $N(x) \neq 0$ and $N^2 = 0$

$C(x) = \{N(x), x\}$

Let $t_1 N(x) + t_2 x = 0$ (*)

$N(t_1 N(x) + t_2 x) = N(0) = 0$

$\Rightarrow$ $t_1 N^2(x) + t_2 N(x) = 0$

$\Rightarrow$ $0 + t_2 N(x) = 0 \Rightarrow t_2 = 0 \Rightarrow t_1 = 0$ from (*)

because $N(x) \neq 0$

terminology : $C(x) = \{N^{\ell}(x), N^{\ell-1}(x) \ldots x\}$ where $N^{\ell}(x) \neq 0$ but $N^{\ell+1}(x) = 0$ is called a cycle generated by x (a generator), and the length of the cycle = the number of the vectors in $C(x)$

Remark Notice that

① $N^{k-1}(x)$ is an eigenvector because $N(N^{k-1}(x)) = N^{k}(x) = 0$

② If $k = n = \dim(V)$, $C(x) = \underbrace{\{N^{n-1}(x), \ldots x\}}_{n \text{ linearly independent vectors}}$ is a basis for V

③ $k \leq n$ from ②

④ Suppose $k = n$ and let $C(x) = \alpha = \{N^{n-1}(x), \ldots x\}$

Then $[N]_{\alpha}^{\alpha} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & & \vdots \\ & & 0 & \ddots & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$

Say $n = 3 = k$. $C(x) = \{N^2(x), N(x), x\} \overset{\text{let}}{=} \alpha$

$$[N]_{\alpha}^{\alpha} = [[N(N^2(x))]_{\alpha} \; [N(N(x))]_{\alpha} \; [N(x)]_{\alpha}]$$
$$= [[N^3(x)]_{\alpha} \; [N^2(x)]_{\alpha} \; [N(x)]_{\alpha}]$$
$$= [[0]_{\alpha} \; [N^2(x)]_{\alpha} \; [N(x)]_{\alpha}]$$
$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$N^2(x) = 1N^2(x) + 0N(x) + 0x$

$N(x) = 0N^2(x) + 1N(x) + 0x$

HW Suppose $C(x_1) = \{N^{\ell_1}(x_1) \ldots, x_1\}$ ($\neq 0$)

$C(x_2) = \{\underbrace{N^{\ell_2}(x_2)}_{\neq 0} \ldots x_2\}$

If $N^{\ell_1}(x_1) \neq N^{\ell_2}(x_2)$, then $C(x_1) \cap C(x_2) = \phi$ empty set

Def let $N : V \to V$ be nilpotent
If a basis α for V is the union of a collection of non-overlapping cycles for N, we call α a canonical basis.
and $[N]_{\alpha}^{\alpha}$ a canonical form. (JCF)

Remark If $\dim(V) = n$ is the smallest integer satisfying $N^{\dim(V)} = 0$,
then $C(x) = \{N^{n-1}(x), \ldots x\}$ is a canonical basis consisting of only one cycle
and $[N]_{\alpha}^{\alpha}$ is a canonical form

Ex3 $N : \mathbb{C}^3 \to \mathbb{C}^3$ defined by $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Find the canonical form of N.

a canonical basis

Sol N is nilpotent because N has only zero eigenvalues

$$N^2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad N^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{3 = \dim(\mathbb{C}^3)}$

$\exists\, x \neq 0$ such that $N^2(x) \neq 0$

eigenvector of N

$C(x) = \{ N^2(x), N(x), x \}$ is a canonical basis

and $[N]_\alpha^\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is the canonical form.

How to find an explicit canonical basis?

sol

$E_{\lambda=0} = \ker(N)$. $\quad N = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ← a REF

free variable

$x_2 + x_3 = 0$ and $x_3 = 0 \Rightarrow x_2 = 0, \; x_3 = 0$

Let $x_1 = t$

$E_{\lambda=0} = \ker(N) = \mathrm{span}(\{(1,0,0)\})$ $\qquad \dim(E_{\lambda=0}) = 1$

Since $N^2(x)$ is an eigenvector of N, let $N^2(x) = (1,0,0)$ ← you can use any eigenvector

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to \text{solve} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\to x_3 = 1, \quad x_1 = t$ and $x_2 = s$ — parameters

I will choose, $(\underset{t}{0}, \underset{s}{0}, 1) = x$

$N(0,0,1) = (1,1,0)$ ← Check it

$C(x) = \{ (1,0,0), (1,1,0), (0,0,1) \}$ an explicit canonical basis

Ex 4 $\quad N : \mathbb{C}^5 \to \mathbb{C}^5$ nilpotent

an example of the definition of canonical bases

Suppose $\alpha_1 = \{ N^2(x_1), N(x_1), x_1 \}$ and $\alpha_2 = \{ N(x_2), x_2 \}$ are two cycles
$= C(x_1)$ $\qquad\qquad = C(x_2)$ $\qquad$ with $N^2(x_1) \neq N(x_2)$

$\alpha = \alpha_1 \cup \alpha_2$ is a canonical basis
$= \{ N^2(x_1), N(x_1), x_1, N(x_2), x_2 \}$

$[N]_\alpha^\alpha = \big[[N(N^2(x_1))]_\alpha \; [N(N(x_1))]_\alpha \; [N(x_1)]_\alpha \; [N(N(x_2))]_\alpha \; [N(x_2)]_\alpha \big]$
$= \big[[0]_\alpha \; [N^2(x_1)]_\alpha \; [N(x_1)]_\alpha \; [0]_\alpha \; [N(x_2)]_\alpha \big]$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{3} \times 3$ = the length of α_1

2×2, 2 = the length of α_2

Theory: If $N : V \to V$ is nilpotent, we can always find a canonical basis.?

Suppose $N^k = 0$. Then there exists $x \neq 0$ such that $N^{k-1}(x) \neq 0$

$V = \ker(N^k)$

and $\{N^{k-1}(x), N^{k-2}(x), \dots N^{k-i}(x), \dots x\}$ is linearly independent

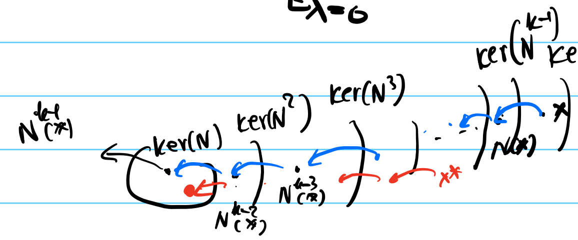
eigenvector $\leftarrow$ $N^{k-1}(x) \in \ker(N)$

Notice that ① $N^i(N^{k-i}(x)) = N^k(x) = 0 \Rightarrow N^{k-i}(x) \in \ker(N^i)$

For example, $N^2(N^{k-2}(x)) = N^k(x) = 0 \Rightarrow N^{k-2}(x) \in \ker(N^2)$

HW $\longrightarrow$ ② $\{0\} \subset \ker(N) \subset \ker(N^2) \subset \ker(N^3) \subset \cdots \subset \ker(N^k) = V$

$\ker(N) = E_{\lambda=0}$



Theory: the number of cycles of canonical basis $= \dim(\ker(N))$ $= E_{\lambda=0}$

the length of the cycles are depending on $\dim(\ker(N^i))$.

example of the theory

Ex5 $N: V \to V$ nilpotent $\dim(V) = 8$

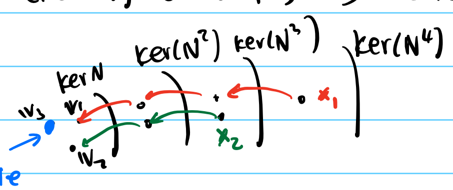
Suppose $\dim(\ker(N)) = 3$, $\dim(\ker(N^2)) = 5$

$\dim(\ker(N^3)) = 7$ and $\dim(\ker(N^4)) = 8$ $= \dim(V)$

Find a canonical basis and the canonical form.

Sol Since $\dim(\ker(N)) = \dim(E_{\lambda=0}) = 3$, canonical bases consist of "3 cycles" $\to$ 3 generators

$\ker(N) = \operatorname{span}(\underbrace{\{w_1, w_2, w_3\}}_{\text{eigenvectors}})$



w_1	$N^3(x_1)$	$N^2(x_1)$	$N(x_1)$	x_1
w_2	$N^2(x_2)$	$N(x_2)$	x_2	
	w_3			

A canonical basis $= \alpha_1 \cup \alpha_2 \cup \alpha_3$

where $\alpha_1 = \{N^3(x_1), N^2(x_1), N(x_1), x_1\}$, $[N]_\alpha^\alpha = \begin{bmatrix} 0&1&0&0&&&&\\ 0&0&1&0&&&&\\ 0&0&0&1&&&&\\ 0&0&0&0&&&&\\ &&&&0&1&0&\\ &&&&0&0&1&\\ &&&&0&0&0&\\ &&&&&&&0 \end{bmatrix}$

$N^3(x_1) = w_1$ $\qquad \alpha_2 = \{N^2(x_2), N(x_2), x_2\}$

$N^2(x_2) = w_2$ $\qquad \alpha_3 = \{w_3\}$

Ex6 Find a canonical basis and the canonical form of $A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{bmatrix} : \mathbb{C}^3 \to \mathbb{C}^3$

Sol Step1 A is nilpotent

$A^2 = \begin{bmatrix} 0&0&0\\0&0&0\\0&0&0 \end{bmatrix}$ 2 is the smallest integer satisfying $A^k = 0$ ← 0 matrix

Step2 Find $\dim(E_{\lambda=0}) = \dim(\ker A) = $ # of free variables of REF of A

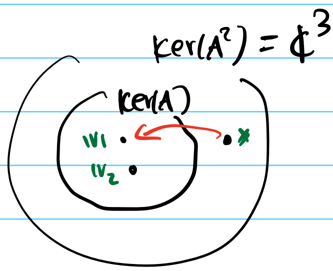
$$A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to 2 \text{ free variables}$$

$$\dim(\ker(A)) = 2 \to \text{two cycles}$$

w_1	x
w_2	

$w_1 = A(x)$

and $\{w_1, w_2\}$ are eigenvectors



$$\text{A canonical basis} = \{\underbrace{\overset{w_1}{\overset{\|}{N(x)}}, x}_{\text{length 2}}\} \cup \{\underbrace{w_2}_{\text{length 1}}\} = \alpha$$

$$[A]_\alpha^\alpha = \begin{bmatrix} \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} & \\ & \boxed{0} \end{bmatrix}$$