UNIVERSITY OF TORONTO
FACULTY OF ARTS AND SCIENCE

Test2, Nov 8, 2024

MAT224H1 F

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Duration: 100 mins

This test has 12 pages.

Total: 50 marks

-NO AIDS ALLOWED

-No marks will be given for a completely wrong solution.

-Unless specified, the vector addition and scalar multiplication of a vector space are all standard ones of the vector space.

1. (2 marks) Let $W = \{p(x) \in C(\mathbf{R}) | \, p(x) = a_0 + a_1 x + a_2 x^2, a_i \in \mathbf{R}, a_2 \neq 0\}$. Show that W is not a subspace of $C(\mathbf{R})$.

2. (4 marks) Let $W = \{ \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbf{R}) | \, a = d\}$.

(a) (3 marks) Find a basis for W.

(b) (1 marks) Show that W is isomorphic to $\mathbf{R}^2$ by constructing an isomorphism.

3. (8 marks) Let $T : V \longrightarrow V$ be a linear transformation and $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$ be an orthonormal basis for V. Suppose $[T]_\alpha^\alpha = \begin{bmatrix} 1 & -2 \\ 4 & -1 \end{bmatrix}$.

(a) (3 marks) Compute the inner product $\langle T(\mathbf{v}_1), \mathbf{v}_2 \rangle$

(b) (2 marks) Show that T invertible and find T^{-1}.

(c) (3 marks) Is T diagonalizble? Explain your answer.

4. (6 marks) Consider a subspace $W = \mathrm{Span}(\{1, e^x, e^{-x}\})$ of $C^\infty(\mathbf{R})$ with a basis $\{1, e^x, e^{-x}\}$. Let $T : W \longrightarrow W$ be a linear transformation defined by $T(1) = 2$, $T(e^x) = 1 + 3e^x$, and $T(e^{-x}) = 2 + 2e^{-x}$. Find all eigenvalues and their associated eigenspaces of T.

5. (7 marks) Let $\mathbf{W} = \text{span}\{(1,2,1)\} \subset \mathbf{R}^3$

(a) (3 marks) Find the orthogonal complement, $W^{\perp}$.

(b) (2 marks) Let $\mathbf{x} \neq 0 \in W$ and $\{\mathbf{y}, \mathbf{z}\}$ be an orthogonal basis for $W^{\perp}$. Show that $\alpha = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a basis of $\mathbf{R}^3$.

(c) (2 marks) Find $[P_{W^{\perp}}]_{\alpha}^{\alpha}$.

6. (6 marks) The following statements are false. Explain why they are false by providing a counterexample.

(a) (3 marks)
Let W_1 and W_2 be subspaces of a vector space V. The vectors in $W_1 + W_2$ are expressed uniquely by the addition of two vectors in W_1 and W_2.

(b) (3 marks) If a square matrix A is diagonalizable, A is invertible.

7. (3 marks) Let $T : V \longrightarrow V$ be a linear transformation. Suppose λ and μ are distinct eigenvalues of T. Show that any eigenvectors $\mathbf{x}$ and $\mathbf{y}$ associated with the eigenvalues λ and μ respectively, are linearly independent.

8. (6 marks)

Let $\mathbf{W} = \text{span}\{(-1, 0, 1, 0), (1, -1, 2, 1)\}$.

(a) (3 marks) Find an orthonormal basis of W using Gram-Schmidt Process.

(b) (3 marks) Find the orthogonal projection $P_{\mathbf{W}}(1, 1, 1, 1)$.

9. (8 marks) Let $\alpha = \{x, 1+x\}$ and $\beta = \{2x+1, x-3\}$ be bases for $P_1(\mathbf{R})$.

(a) (4 marks) Suppose for $p(x) \in P_1(\mathbf{R})$, $[p(x)]_\alpha = (2, 1)$. Find $[p(x)]_\beta$.

(b) (4 marks) Let $T : P_1(\mathbf{R}) \longrightarrow P_1(\mathbf{R})$ be defined by $T(x) = x+2$ and $T(1+x) = x-2$. Suppose $S : P_1(\mathbf{R}) \longrightarrow P_1(\mathbf{R})$ is a linear mapping such that $[ST]_\alpha^\alpha = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Find $[S]_\beta^\alpha$.

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