UNIVERSITY OF TORONTO FACULTY OF ARTS AND SCIENCE

Test 2, Mar 15, 2024

MAT224H1 S

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Duration: 100 mins

This test has 12 pages.

Total: 50 marks

NO AIDS ALLOWED

No marks will be given for a completely wrong solution.

1. (3 marks) Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a set of orthogonal vectors in \mathbf{R}^3 . If $(-1,2,3)=5\mathbf{v}_1+8\mathbf{v}_2-2\mathbf{v}_3$ and $||\mathbf{v}_1||=2$. Find the inner product $\langle (-1,2,3),\mathbf{v}_1\rangle$.

2. (3 marks) Let $T: P_3(\mathbf{R}) \to P_2(\mathbf{R})$ be defined by T(p(x)) = p'(x) + 2. Show that T is not a linear mapping.

- 3. $(7 \text{ marks}) \text{ Let } P_{(1,3)}: \mathbf{R}^2 \to \mathbf{R}^2 \text{ be the orthogonal projection to span}\{(1,3)\}.$
 - (a) (2 marks) Show that there is a basis β for ${\bf R}^2$ such that the coordinate

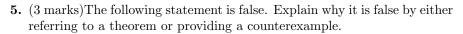
$$\text{matrix } [P_{(1,3)}]_{\beta}^{\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

- (b) (2 marks) Find $Ker(P_{(1,3)})$ and $Im(P_{(1,3)})$
- (c) (3 marks) Find eigenvalues and eigenvectors of $P_{(1,3)}$.

4. (7 marks) Let $S:U\to V$ and $T:V\to W$ be linear transforma-

tions. Let
$$\alpha$$
, β and γ be bases for U , V and W , respectively. Suppose $[TS]_{\alpha}^{\gamma} = \begin{bmatrix} 4 & 3 \\ 5 & 2 \end{bmatrix}$ and $[S]_{\alpha}^{\beta} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$.

- (a) (2 marks) Suppose $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathbf{x} = 2\mathbf{v}_1 \mathbf{v}_2$. Find $[S(\mathbf{x})]_{\beta}$.
- (b) (3 marks) Is S invertible? Justify your answer. If it is invertible, find $[S^{-1}]^{\alpha}_{\beta}$.
- (c) (2 marks) Find $[T]^{\gamma}_{\beta}$



If an $n \times n$ matrix A is invertible, then A is diagonalizable.

6. (4 marks) Prove that if $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ are orthogonal, nonzero vectors, $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent.

- 7. (7 marks) Consider a subspace $W = \operatorname{Span}(\{1, \sin x, \cos x\})$ of $C^{\infty}(\mathbf{R})$ with a basis $\{1, \sin x, \cos x\}$. Let $T: W \longrightarrow W$ be a linear transformation defined by $T(1) = 1 + 2\sin x + 3\cos x$, $T(\sin x) = 2\sin x + 3\cos x$, and $T(\cos x) = 2\cos x$.
 - (a) (5 marks) Find the eigenvalues and eigenvectors of T
 - (b) (2 marks) Determine whether the given linear mapping is diagonalizable. If it is diagonalizable, find a basis of $\mathrm{Span}(\{1,\sin x,\cos x\})$ consisting of eigenvectors.

8. (8 marks)

Let
$$V = \text{span}\{(1, 1, -1, 1), (1, 0, 0, -1)\}.$$

- (a) (3 marks) Find the orthogonal complement, V^{\perp} .
- (b) (4 marks) Find an orthogonal basis for $V^{\perp}.$
- (c) (1 mark) Find an orthonormal basis for V^{\perp} .

9. (8 marks)

Let
$$\mathbf{W} = \text{span}\{(-1,0,1),(1,-1,2)\}$$

- (a) (2 marks) Are $\{(-1,0,1),(1,-1,2)\}$ orthogonal?
- (b) (6 marks) Find the orthogonal projection $P_{\mathbf{W}}(0,1,1)$.