UNIVERSITY OF TORONTO
FACULTY OF ARTS AND SCIENCE

Test 2, Mar 15, 2024

MAT224H1 S

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Duration: 100 mins

This test has 12 pages.

Total: 50 marks

NO AIDS ALLOWED

No marks will be given for a completely wrong solution.

1. (3 marks) Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a set of orthogonal vectors in $\mathbf{R}^3$. If $(-1, 2, 3) = 5\mathbf{v}_1 + 8\mathbf{v}_2 - 2\mathbf{v}_3$ and $||\mathbf{v}_1|| = 2$. Find the inner product $\langle(-1, 2, 3), \mathbf{v}_1\rangle$.

2. (3 marks) Let $T : P_3(\mathbf{R}) \to P_2(\mathbf{R})$ be defined by $T(p(x)) = p'(x) + 2$. Show that T is not a linear mapping.

3. (7 marks) Let $P_{(1,3)} : \mathbf{R}^2 \to \mathbf{R}^2$ be the orthogonal projection to span$\{(1,3)\}$.

(a) (2 marks) Show that there is a basis β for $\mathbf{R}^2$ such that the coordinate matrix $[P_{(1,3)}]_\beta^\beta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(b) (2 marks) Find $\mathrm{Ker}(P_{(1,3)})$ and $\mathrm{Im}(P_{(1,3)})$

(c) (3 marks) Find eigenvalues and eigenvectors of $P_{(1,3)}$.

4. (7 marks) Let $S : U \to V$ and $T : V \to W$ be linear transformations. Let α, β and γ be bases for U, V and W, respectively. Suppose $[TS]_\alpha^\gamma = \begin{bmatrix} 4 & 3 \\ 5 & 2 \end{bmatrix}$ and $[S]_\alpha^\beta = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$.

(a) (2 marks) Suppose $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathbf{x} = 2\mathbf{v}_1 - \mathbf{v}_2$. Find $[S(\mathbf{x})]_\beta$.

(b) (3 marks) Is S invertible? Justify your answer. If it is invertible, find $[S^{-1}]_\beta^\alpha$.

(c) (2 marks) Find $[T]_\beta^\gamma$

5. (3 marks)The following statement is false. Explain why it is false by either referring to a theorem or providing a counterexample.

If an $n \times n$ matrix A is invertible, then A is diagonalizable.

6. (4 marks) Prove that if $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ are orthogonal, nonzero vectors, $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent.

7. (7 marks) Consider a subspace $W = \text{Span}(\{1, \sin x, \cos x\})$ of $C^\infty(\mathbf{R})$ with a basis $\{1, \sin x, \cos x\}$. Let $T : W \longrightarrow W$ be a linear transformation defined by $T(1) = 1 + 2\sin x + 3\cos x$, $T(\sin x) = 2\sin x + 3\cos x$, and $T(\cos x) = 2\cos x$.

(a) (5 marks) Find the eigenvalues and eigenvectors of T

(b) (2 marks) Determine whether the given linear mapping is diagonalizable. If it is diagonalizable, find a basis of $\text{Span}(\{1, \sin x, \cos x\})$ consisting of eigenvectors.

8. (8 marks)

Let $V = \text{span}\{(1,1,-1,1),(1,0,0,-1)\}$.

(a) (3 marks) Find the orthogonal complement, $V^{\perp}$.

(b) (4 marks) Find an orthogonal basis for $V^{\perp}$.

(c) (1 mark) Find an orthonormal basis for $V^{\perp}$.

9. (8 marks)

Let $\mathbf{W} = \text{span}\{(-1,0,1),(1,-1,2)\}$

(a) (2 marks) Are $\{(-1,0,1),(1,-1,2)\}$ orthogonal?

(b) (6 marks) Find the orthogonal projection $P_{\mathbf{W}}(0,1,1)$.

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