UNIVERSITY OF TORONTO
FACULTY OF ARTS AND SCIENCE

Test1, Feb 16, 2024

MAT224H1 S

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Duration: 100 mins

This test has 12 pages.

Total: 50 marks

NO AIDS ALLOWED

No marks will be given for a completely wrong solution.

1. (6 marks)Let $T : R^4 \to W$ be a linear transformation. Suppose W is an n-dimensional vector space and $\mathrm{Ker}(T) = \mathrm{span}\{(1,1,1,1)\}$.

(a) (2 marks) Is T injective?

(b) (2 marks) Find the dimension of Im(T).

(c) (2 marks) If T is surjective, what is the dimension of W?

2. (4 marks) Let V and W be vector spaces over $\mathbf{R}$. Show that $T(\mathbf{0}_v) = \mathbf{0}_w$, where T is any linear mapping from V to W, and $\mathbf{0}_v$ and $\mathbf{0}_w$ are the additive identities of V and W respectively.

3. (6 marks) Let $T : P_2(\mathbf{R}) \to M_{2\times 2}(\mathbf{R})$ be defined by $T(p(x)) = \begin{bmatrix} p(0) & p'(0) \\ p'(0) & p''(0) \end{bmatrix}$.

(a) (2 marks) Show that T is a linear transformation.

(b) (2 marks) Is T injective? Explain your answer.

(c) (2 marks) Is T surjective? Explain your answer.

4. (7 marks) Let $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ be a linear transformation with bases $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\beta = \{(1,1), (-2,3)\}$ of domain and codomain respectively.

Suppose the coordinate matrix $[T]_\alpha^\beta = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$.

(a) (2 marks) Find $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$.

(b) (2 marks) Find $T(-\mathbf{v}_1 + 2\mathbf{v}_2)$.

(c) (3 marks) If T is invertible, find $[T^{-1}]_\beta^\alpha$.

5. (7 marks) Suppose $\mathbf{R}^2$ has the standard vector structure and $\left(\mathbf{R}^{+}\right)^2$ has the following vector structure: for any $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \left(\mathbf{R}^{+}\right)^2$, and any $c \in \mathbf{R}$,

$\mathbf{x} +' \mathbf{y} = (x_1 y_1, x_2 y_2)$ and $c \cdot \mathbf{x} = ({x_1}^c, {x_2}^c)$.

(a) (2 marks) Show that $(1, 1)$ is the additive identity of $\left(\mathbf{R}^{+}\right)^2$.

(b) (1 marks) Find the additive inverse of $(2, 3) \in \left(\mathbf{R}^{+}\right)^2$.

(c) (4 marks) Let $T : \mathbf{R}^2 \to \left(\mathbf{R}^{+}\right)^2$ be defined by $T(x_1, x_2) = (e^{x_1}, e^{x_2})$. Show that T is a linear mapping.

6. (3 marks) The following statement is false. Explain why it is false by providing a counterexample.

Let V be a vector space. If S_1 and S_2 are subspaces of V, then $S_1 \cup S_2$ is also a subspace of V.

7. (3 marks) Let $F(\mathbf{R})$ be the set of functions from $\mathbf{R}$ to $\mathbf{R}$ with standard vector addition and scalar multiplication and $S_1 = \{f \in F(\mathbf{R}) |\, f(x) = f(-x)\}$. Show that S_1 is a subspace of $F(\mathbf{R})$.

8. (6 marks) Let $T : V = \text{span}\{e^x, e^{2x}, e^{3x}\} \to \mathbf{R}^3$ be a linear transformation and $\alpha = \{e^x, e^{2x}, e^{3x}\}$ and $\beta = \{(1,0,0),(1,1,0),(1,1,2)\}$ be bases of V and $\mathbf{R}^3$ respectively.

Suppose $[T]_\alpha^\beta = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

(a) (1 marks) Find $\text{Ker}([T]_\alpha^\beta)$.

(b) (1 marks) Find $\text{Ker}(T)$.

(c) (2 marks) Find a basis of $\text{Im}(T)$.

(d) (2 marks) Find a basis of V including the basis of $\text{Ker}(T)$ in (b).

9. (8 marks) Let V be a vector space over $\mathbf{R}$. Suppose $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of V.

(a) (4 marks) Show that $\beta = \{2\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\}$ is a basis of V.

(b) (4 marks) Suppose $\mathbf{v} \in V$ and $[\mathbf{v}]_\alpha = (3, 4)$. Find $[\mathbf{v}]_\beta$.

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