Sec 2.5 Composition of Linear Transform.

Let U, V and W be real vector spaces

Theorem Let $S: U \to V$ and $T: V \to W$ be linear transformation. Then TS is a linear transformation from U to W

$(TS)(x)$
$= T(S(x))$

$$\begin{tikzcd} U \arrow[r, "S"] \arrow[rr, bend right, "TS"'] & V \arrow[r, "T"] & W \end{tikzcd}$$

Pf $\forall u_1, u_2 \in U$ $(TS)(u_1+u_2) = T(S(u_1+u_2)) \overset{S \text{ is linear}}{=} T(S(u_1)+S(u_2))$
$\overset{T \text{ is linear}}{=} T(S(u_1)) + T(S(u_2)) = (TS)(u_1)+(TS)(u_2)$

<u>Hw</u> Show that $\forall u \in U, \forall k \in \mathbb{R}$, $(TS)(ku) = k(TS)(u)$

Ex 1 $\mathbb{R}^3 \xrightarrow{S} \mathbb{R}^4 \xrightarrow{T} \mathbb{R}^2$

Review of MAT223

$S(x_1, x_2, x_3) = (x_1+x_2,\ x_1-x_2,\ x_3,\ x_1+x_3)$
$T(x_1, x_2, x_3, x_4) = (x_1 - x_4,\ x_2 + x_3)$
Find (TS)

the first / the second

<u>Sol</u> $(TS)(x_1,x_2,x_3) = T(S(x_1,x_2,x_3)) = T(x_1+x_2,\ x_1-x_2,\ x_3,\ x_1+x_3)$
$= ((x_1+x_2)-(x_1+x_3),\ (x_1-x_2)+x_3)$
$= (x_2-x_3,\ x_1-x_2+x_3)$

Remark: $[TS] = [T][S] = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

$(TS)(x_1, x_2, x_3) = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (x_2 - x_3,\ x_1 - x_2 + x_3)$

Ex 2 $S: P_2(\mathbb{R}) \to P_1(\mathbb{R})$ defined by $S(p(x)) = p'(x)$
$T: P_1(\mathbb{R}) \to P_2(\mathbb{R})$ " by $T(p(x)) = x\,p(x)$
Find TS

<u>Sol</u> $\forall p(x) \in P_2(\mathbb{R})$, $(TS)(p(x)) = T(S(p(x))) = T(p'(x)) = x\,p'(x)$

Theorem Let α, β and γ be bases for U, V and W respectively

$$\begin{tikzcd} U \arrow[r, "S"] \arrow[rr, bend right] & V \arrow[r, "T"] & W \end{tikzcd}$$

S, T: linear transformation

β

$TS: U \to W$

α γ

$$[TS]_\alpha^\gamma = [T]_\beta^\gamma [S]_\alpha^\beta$$

(Pf) Say $\alpha = \{u_1, \dots, u_n\}$. $[TS]_\alpha^\gamma = \left[[(TS)(u_1)]_\gamma \cdots [(TS)(u_i)]_\gamma \cdots [(TS)(u_n)]_\gamma \right]$

$[T(x)]_\gamma = [T]^\gamma_\beta [x]_\beta$

$[S(y)]_\beta = [S]^\beta_\alpha [y]_\alpha$

$AB = A[b_1\ b_2 \cdots b_n] = [Ab_1\ Ab_2 \cdots Ab_n]$

$$[(TS)(v_i)]_\gamma = [T(S(v_i))]_\gamma = [T]^\gamma_\beta [S(v_i)]_\beta = \underbrace{[T]^\gamma_\beta [S]^\beta_\alpha}_{A} [v_i]_\alpha$$

$v_1 = (1)v_1 + 0v_2 + \cdots + 0v_n$

$$[TS]^\gamma_\alpha = \left[A[v_1]_\alpha \ \ A[v_2]_\alpha \ \cdots \ A[v_n]_\alpha \right] = A\left[[v_1]_\alpha \ [v_2]_\alpha \cdots [v_n]_\alpha \right] = A \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \end{bmatrix} = A$$

memorize it

$$\longrightarrow [TS]^\gamma_\alpha = A = [T]^\gamma_\beta [S]^\beta_\alpha$$

Ex 3 $\mathbb{R}^3 \xrightarrow{S} \mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$ S, T: linear

$\alpha = \{u_1, u_2, u_3\}$ $\beta = \{v_1, v_2\}$ $\gamma = \{(2,3), (-1,2)\}$

Suppose $[T]^\gamma_\beta = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}$ and $[S]^\beta_\alpha = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

Find $(TS)(u_1 + 2u_2 - u_3)$. $TS : \mathbb{R}^3 \to \mathbb{R}^2$ (α, γ)

Sol $[TS]^\gamma_\alpha = [T]^\gamma_\beta [S]^\beta_\alpha$

$$= \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$$

$$[TS]^\gamma_\alpha [u_1 + 2u_2 - u_3]_\alpha = [TS]^\gamma_\alpha \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$[TS]^\gamma_\alpha \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} = [(TS)(u_1 + 2u_2 - u_3)]_\gamma$$

$$(TS)(u_1 + 2u_2 - u_3) = 0 \cdot (2,3) + 3(-1, 2) = (-3, 6)$$

Read the proposition in the book

$$\begin{tikzcd} U \arrow[r, "S"] \arrow[rr, bend right, "TS"'] & V \arrow[r, "T"] & W \end{tikzcd}$$

$\ker S \subseteq \ker(TS)$

$\operatorname{Im}(TS) \subseteq \operatorname{Im}(T)$ HW

Claim $\ker(S) \subseteq \ker(TS)$

Pf $\forall x \in \ker(S)$,

Then $S(x) = 0_V$

$\ker(TS) = \{x \in U \mid TS(x) = 0_W\}$

$T(S(x)) = 0_W$

$$T(S(x)) = T(0_V) \underset{\text{linear}}{=} 0_W \Rightarrow x \in \ker(TS)$$