Problem 1 from Tutorial 4

8pts

(a) $[S]_\alpha^\beta = \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 1 & -3 \end{bmatrix}$ and $[T]_\beta^\gamma = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & -2 \end{bmatrix}$ — 1

$$[T]_\beta^\gamma [S]_\alpha^\beta = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -2 & 9 \end{bmatrix}$$ — 1

$(TS)(w_1) = 4w_1 - 2w_2 \qquad (TS)(w_2) = 9w_2$ — 1

(b) $[TS]_\alpha^\gamma [2w_1 + 3w_2]_\alpha = \begin{bmatrix} 4 & 0 \\ -2 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 23 \end{bmatrix}$. Therefore, $(TS)(2w_1 + 3w_2)$ 1

$= 8w_1 + 23 w_2$.

(c) $\det \begin{pmatrix} 4 & 0 \\ -2 & 9 \end{pmatrix} = 36 \neq 0$ invertible

Method 1
$$\left[\begin{array}{cc|cc} 4 & 0 & 1 & 0 \\ -2 & 9 & 0 & 1 \end{array}\right] \xrightarrow[R_2:\ \frac{1}{2}R_1 + R_2]{} \left[\begin{array}{cc|cc} 4 & 0 & 1 & 0 \\ 0 & 9 & \frac{1}{2} & 1 \end{array}\right] \xrightarrow[\frac{1}{9}R_2]{\frac{1}{4}R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{18} & \frac{1}{9} \end{array}\right]$$ explaining

the inverse of $[TS]_\alpha^\gamma$

or

Therefore, $[(TS)^{-1}]_\gamma^\alpha = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{18} & \frac{1}{9} \end{bmatrix}$ ← 1

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Method 2 $$[(TS)^{-1}]_\gamma^\alpha = \frac{1}{\det([TS]_\alpha^\gamma)} \begin{bmatrix} 9 & 0 \\ 2 & 4 \end{bmatrix} = \frac{1}{36} \begin{bmatrix} 9 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{18} & \frac{1}{9} \end{bmatrix}$$

$(TS)^{-1}(w_1) = \frac{1}{4} w_1 + \frac{1}{18} w_2$ and $(TS)^{-1}(w_2) = \frac{1}{9} w_2$ ← 1

8pts

Sec 2.5 #8 (Sol)

$\alpha = \{1, x, x^2, x^3\} \quad \beta = \{1, x, x^2, x^3, x^4\}$

(a) $D(1) = 0$, $[D(1)]_\alpha = (0,0,0,0)$

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$D(x) = 1$, $[D(x)]_\alpha = (1,0,0,0)$

$D(x^2) = 2x$. $[D(x^2)]_\alpha = (0,2,0,0)$

$D(x^3) = 3x^2$ $[D(x^3)]_\alpha = (0,0,3,0)$

$D(x^4) = 4x^3$, $[D(x^4)]_\alpha = (0,0,0,4)$

So, $[D]_\beta^\alpha = [\,[D(1)]_\alpha \ [D(x)]_\alpha \ [D(x^2)]_\alpha \ [D(x^3)]_\alpha \ [D(x^4)]_\alpha\,]$
$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

2

$\mathrm{Int}(1) = \int_0^x dt = x$, $[\mathrm{Int}(1)]_\beta = (0,1,0,0,0)$

$\mathrm{Int}(x) = \int_0^x t\,dt = \frac{1}{2}x^2$, $[\mathrm{Int}(x)]_\beta = (0,0,\frac{1}{2},0,0)$

$\mathrm{Int}(x^2) = \int_0^x t^2\,dt = \frac{1}{3}x^3$, $[\mathrm{Int}(x^2)]_\beta = (0,0,0,\frac{1}{3},0)$

$\mathrm{Int}(x^3) = \int_0^x t^3\,dt = \frac{1}{4}x^4$, $[\mathrm{Int}(x^3)]_\beta = (0,0,0,0,\frac{1}{4})$

So, $[\mathrm{Int}]_\alpha^\beta = [\,[\mathrm{Int}(1)]_\beta \ [\mathrm{Int}(x)]_\beta \ [\mathrm{Int}(x^2)]_\beta \ [\mathrm{Int}(x^3)]_\beta\,]$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

(b) $$[D\,\mathrm{Int}]_\alpha^\alpha = [D]_\beta^\alpha [\mathrm{Int}]_\alpha^\beta = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ 2

$$[\mathrm{Int}\,D]_\beta^\beta = [\mathrm{Int}]_\alpha^\beta [D]_\beta^\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ 2

$(D\,\mathrm{Int})(p(x)) = D\left(\int_0^x p(t)\,dt\right) = p(x)$ $\Rightarrow$ Fundamental theorem of calculus I

Sec 2.5 #12 (Sol)

$$\begin{tikzcd} U \arrow[r, "S"] & V \arrow[r, "T"] & W \end{tikzcd}$$

$TS : U \to W$

3pts (a) Claim: TS is injective if T and S are injective   (or) Show if $(TS)(u_1) = (TS)(u_2)$, $u_1 = u_2$

Pf $\forall\, u \in \mathrm{Ker}(TS)$. Then $(TS)(u) = T(S(u)) = 0_W$ (1)

(1) Since T is injective, $S(u) = 0_V$.

(1) Since S is injective $u = 0_U$. Therefore, TS is injective.

3pts (b) Claim: TS is surjective if T and S are surjective.

(or) Show $\mathrm{Im}(TS) = W$

Pf $\forall\, w \in W$. Since T is surjective, there exists $w_0 \in V$ such that (1) $T(w_0) = w$

Since S is surjective, there exists $u_0 \in U$ such that $S(u_0) = w_0$ (1)

Therefore, $(TS)(u_0) = T(S(u_0)) = T(w_0) = w$ (1)

$\Rightarrow$ TS is surjective

8pts Sec 2.6 #1(c) $T: P_2(\mathbb{R}) \to \mathbb{R}^3$ defined by $T(p(x)) = (p(0), p(1), p(2))$

Sol $\alpha = \{1, x, x^2\}$ is a standard basis of $P_2(\mathbb{R})$ and $\beta = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is a standard basis of $\mathbb{R}^3$

$T(1) = (1,1,1)$ $T(x) = (0,1,2)$, $T(x^2) = (0,1,4)$

(3) $[T]_\alpha^\beta = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$. Try to find $\left([T]_\alpha^\beta\right)^{-1}$.   (or) $\det\left([T]_\alpha^\beta\right) = 2 \neq 0$ so $\left([T]_\alpha^\beta\right)^{-1}$ exists. $\to$ T is invertible

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right] \Rightarrow T \text{ is invertible and } [T^{-1}]_\beta^\alpha = \left[[T]_\alpha^\beta\right]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 2 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$ (2)

$\underbrace{\qquad\qquad}_{\text{the inverse of } [T]_\alpha^\beta}$

$\forall\, a \in \mathbb{R}^3$, $a = (a_1, a_2, a_3)$

$$[T^{-1}(a)]_\alpha = [T^{-1}]_\beta^\alpha [a]_\beta = [T^{-1}]_\beta^\alpha \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 2 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ -\frac{3}{2}a_1 + 2a_2 - \frac{1}{2}a_3 \\ \frac{1}{2}a_1 - a_2 + \frac{1}{2}a_3 \end{bmatrix}.$$ (1)

Therefore $T^{-1}(\vec{a}) = a_1 + \left(-\frac{3}{2}a_1 + 2a_2 - \frac{1}{2}a_3\right)x + \left(\frac{1}{2}a_1 - a_2 + \frac{1}{2}a_3\right)x^2$ (1)

Sec 2.6 #8 (sol)

3pts ($\Rightarrow$) Suppose $\{w_1, \dots w_k\}$ is linearly independent

Let $a_1 T(w_1) + \dots + a_k T(w_k) = 0_W$ where $a_i \in \mathbb{R}$, $1 \le i \le k$.

1 $\rightarrow T(a_1 w_1 + \dots + a_k w_k) = 0_W$

by linearity

Since T is injective, $a_1 w_1 + \dots + a_k w_k = 0_V$ 1

Since $\{w_1, \dots w_k\}$ is linearly independent, $a_1 = \dots = a_k = 0$ 1

Therefore, $\{T(w_1), \dots, T(w_k)\}$ is linearly independent.

3pts ($\Leftarrow$) Suppose $\{T(w_1), \dots, T(w_k)\}$ is linearly independent

Let $a_1 w_1 + \dots + a_k w_k = 0_V$. where $a_i \in \mathbb{R}$, $1 \le i \le k$

Then $T(a_1 w_1 + \dots + a_k w_k) = T(0_V) = 0_W$ because T is linear. 1

1 $\rightarrow a_1 T(w_1) + \dots + a_k T(w_k) = 0_W$

by linearity

Since $\{T(w_1), \dots, T(w_k)\}$ is linearly independent,

1 $a_1 = \dots = a_k = 0$

Therefore, $\{w_1, \dots, w_k\}$ is linearly independent

Sec 2.7 #1(a)

5pts Sol Consider the associated matrix with the rotation : $\underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{1} \overset{\text{let}}{=} A$

Then $A = [T]_\alpha^\alpha$ where $\alpha = \{e_1, e_2\}$

Let $\beta = \{(2,1), (1,-2)\}$. Then the matrix of T with respect to β

$$[T]_\beta^\beta = \left([I]_\beta^\alpha\right)^{-1} [T]_\alpha^\alpha [I]_\beta^\alpha \quad \text{— } 1$$

$$[I]_\beta^\alpha = \left[[(2,1)]_\alpha \ [(1,-2)]_\alpha \right] = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \quad 1$$

$$\left([I]_\beta^\alpha\right)^{-1} = \frac{1}{-4-1} \begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix} \quad 1$$

$$\text{Therefore, } [T]_\beta^\beta = \underbrace{\begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{1}$$

Sec 2.7 #2

8pts

Sol. $T(1,1,1)=(2,2,2)$, $T(1,1,0)=(3,3,0)$, $T(1,0,0)=(-1,0,0)$

$\alpha=\{(1,1,1),(1,1,0),(1,0,0)\}$

$\beta=\{(1,0,0),(0,1,0),(0,0,1)\}$

2

$[T]_\beta^\beta = [T\underset{\text{identity}}{I}]_\beta^\beta = [T]_\alpha^\beta [I]_\beta^\alpha$ where $[T]_\alpha^\beta = \left[[T(1,1,1)]_\beta \ [T(1,1,0)]_\beta \ [T(1,0,0)]_\beta\right]$

$$= \begin{bmatrix} 2 & 3 & -1 \\ 2 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

2

and $[I]_\beta^\alpha = \left[[(1,0,0)]_\alpha \ [(0,1,0)]_\alpha \ [(0,0,1)]_\alpha\right] \overset{\text{let}}{=} \begin{bmatrix} a_{11} & a_{12} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then $(1,0,0)=a_{11}(1,1,1)+a_{21}(1,1,0)+a_{31}(1,0,0)$

$(0,1,0)=a_{12}(1,1,1)+a_{22}(1,1,0)+a_{32}(1,0,0)$

$(0,0,1)=a_{31}(1,1,1)+a_{32}(1,1,0)+a_{33}(1,0,0)$

$\Downarrow$

2 $\longrightarrow$ $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} [I]_\beta^\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow [I]_\beta^\alpha = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

the inverse

Ans $\begin{bmatrix} 2 & 3 & -1 \\ 2 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 4 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

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