Sec 5.2 - Sec 5.3

Sec 5.2 Vector Spaces over a Field $\nearrow \mathbb{C}$

Let F be a field

$F^n = \underbrace{F \times F \times \cdots \times F}_{n\text{-tuple}}$ is a vector space over F by the operations

Vector addition: $\forall\, x, y \in F^n$, $x = (x_1, \ldots x_n)$, $y = (y_1, \ldots y_n)$

$$x + y = (x_1 + y_1, \ \ldots\ x_n + y_n)$$

Scalar multiplication: $\forall\, x \in F^n$, $c \in F$

$$cx = (cx_1, \ldots\ cx_n)$$

Ex1 $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ is a vector space over $\mathbb{C}$

Say $n = 2$.

say $x_1 = (1+i, i)$, $x_2 = (i, -1)$

$x_1 + x_2 = (1+i+i,\ i+(-1)) = (1+2i,\ -1+i)$

$\underset{\text{scalar}}{i}\, x_1 = i(1+i,\ i) = (i(1+i), (i)(i)) = (-1+i,\ -1)$

Basis for $\mathbb{C}^2$.

$\forall\, x \in C^2$, $x = (z_1, z_2)$ where $z_1, z_2 \in \mathbb{C}$

$= z_1(1, 0) + z_2(0, 1)$

$\mathbb{C}\ \ \mathbb{C}\ \ \mathbb{C}\ \ \mathbb{C}\ \ \mathbb{C}\ \ \mathbb{C}$

$(1, 0) = (1+0i, 0+0i)$

$\mathbb{C}^2 = \operatorname{span}\{(1,0), (0,1)\}$ $\dim(\mathbb{C}^2) = 2$.

Determinant

Ex2

ⓐ $\det\left(\begin{bmatrix} -i & 2+3i \\ 1 & 1-i \end{bmatrix}\right) = (-i)(1-i) - (1)(2+3i) = -3-4i$

ⓑ Is $\underbrace{\begin{bmatrix} -i & 2+3i \\ 1 & 1-i \end{bmatrix}}_{= A}$ invertible? <u>Ans</u> Yes $\det(A) \neq 0$

If it is invertible, find A^{-1}.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$

$A^{-1} = \dfrac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A^{-1} = \frac{1}{-3-4i} \begin{bmatrix} 1-i & -2-3i \\ -1 & -i \end{bmatrix}$$

$$\frac{1}{-3-4i} = \frac{(1)(-3+4i)}{(-3-4i)(-3+4i)} = \frac{-3+4i}{(-3)^2+(-4)^2}$$

$$= \begin{bmatrix} \left(\frac{-3+4i}{25}\right)(1-i) & \left(\frac{-3+4i}{25}\right)(-2-3i) \\ \left(\frac{-3+4i}{25}\right)(-1) & \left(\frac{-3+4i}{25}\right)(-i) \end{bmatrix}$$

Eigenvalues and Eigenvectors

Ex3 Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} : \mathbb{C}^2 \to \mathbb{C}^2$

Sol Let $\det(A - \lambda I) = 0 \Rightarrow \det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

eigenvalues of A

For $\lambda = i$,

$A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \sim \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix} \rightarrow i x_1 + x_2 = 0$

$\boxed{\frac{t}{i}} = \frac{i}{ii} = \frac{i}{-1} = \boxed{-i}$

Let $x_2 = t$, $i x_1 = -t \Rightarrow x_1 = -\frac{t}{i} = it$

$E_{\lambda=i} = \text{span}(\{(i, 1)\})$

HW For $\lambda = -i$ $E_{\lambda=-i} = \text{span}(\{(-i, 1)\})$

Notice that A is diagonalizable

$A \approx \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ with $\alpha = \{(i, 1), (-i, 1)\}$

similar

Sec 5.3 Geometry in a complex vector space

Let V be a vector space over $\mathbb{C}$

Def A Hermitian inner product on V is a function $\langle \, , \, \rangle$ from $V \times V$ to $\mathbb{C}$ satisfying the following 3 conditions

1. $\forall\, u, v, w \in V$, $\forall\, a, b \in \mathbb{C}$

   $\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$

2. $\langle v, w \rangle = \overline{\langle w, v \rangle}$

3. $\langle v, v \rangle \geq 0$, $\langle v, v \rangle = 0 \Leftrightarrow v = 0$

Remark 1. $\langle av, w \rangle = a\langle v, w \rangle$

$\boxed{\langle v, aw \rangle} = \overline{\langle aw, v \rangle}$ by (2)

$= \overline{a\langle w, v \rangle}$

$\overline{z_1 z_2} = (\overline{z_1})(\overline{z_2})$

$= \bar{a}\, \overline{\langle w, v \rangle}$ by

$\boxed{= \bar{a} \langle v, w \rangle}$

For example

$\langle v, (2+i)w \rangle = \overline{2+i} \langle v, w \rangle = (2-i)\langle v, w \rangle$

Remark 2. $\langle \mathbb{v}, \mathbb{v} \rangle = \overline{\langle \mathbb{v}, \mathbb{v} \rangle}$ by 2

two same $\mathbb{v} \in \mathbb{C}$ $\Rightarrow \langle \mathbb{v}, \mathbb{v} \rangle \in \mathbb{R}$

If $z = \bar{z}$,

z is a real number