Sec 4.1

#3(g) $V = \mathrm{span}\{e^x, xe^x, e^{2x}, xe^{2x}\}$ $T(f) = f''$

sol So $\alpha = \{e^x, xe^x, e^{2x}, xe^{2x}\}$ is a basis for V.

8 pts

$$\left.\begin{aligned} T(e^x) &= (e^x)'' = e^x \\ T(xe^x) &= (xe^x)'' = (e^x + xe^x)' = e^x + e^x + xe^x = 2e^x + xe^x \\ T(e^{2x}) &= (e^{2x})'' = (2e^{2x})' = 4e^{2x} \\ T(xe^{2x}) &= (xe^{2x})'' = (e^{2x} + 2xe^{2x})' = 2e^{2x} + 2e^{2x} + 4xe^{2x} = 4e^{2x} + 4xe^{2x} \end{aligned}\right\} 2$$

$$[T]_\alpha^\alpha = \left[[T(e^x)]_\alpha\ [T(xe^x)]_\alpha\ [T(e^{2x})]_\alpha\ [T(xe^{2x})]_\alpha\right] = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad 1$$

1 $\rightarrow$ $\det([T]_\alpha^\alpha - \lambda I) = (1-\lambda)^2(4-\lambda)^2 \stackrel{\text{Let}}{=} 0$ Then $\lambda = 1, 4$

When $\lambda = 1$, $[T]_\alpha^\alpha - I = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

↑ free variable

Let $x_1 = t$
$3x_4 = 0 \Rightarrow x_4 = 0$
$3x_3 + 4x_4 = 0 \Rightarrow x_3 = 0$
$2x_2 = 0 \Rightarrow x_2 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad E_{\lambda=1} = \mathrm{span}\{e^x\}$$

↑ a basis

2

When $\lambda = 4$, $[T]_\alpha^\alpha - 4I = \begin{bmatrix} -3 & 2 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

↑ free variable

Let $x_3 = t$
$4x_4 = 0 \Rightarrow x_4 = 0$
$-3x_2 = 0 \Rightarrow x_2 = 0$
$-3x_1 + 2x_2 = 0 \Rightarrow x_1 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad E_{\lambda=4} = \mathrm{span}\{e^{2x}\}$$

↑ a basis

2

#13 (a) $Ax = \lambda x$ $x \neq 0$.

Sol $A^2x = A(Ax) = A(\lambda x) = \lambda Ax = \lambda^2 x$

Suppose $A^k x = \lambda^k x$.

Then $A^{k+1}x = A(A^k x) = A(\lambda^k x) = \lambda^k Ax = \lambda^k(\lambda x) = \lambda^{k+1}x$.

λ^k is an eigenvalue of A^k, $k \geq 1$

(b) Let $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k$.

Suppose λ is an eigenvalue of A. Then $Ax = \lambda x$, $x \neq 0$.

$p(A) = a_0 I_n + a_1 A + a_2 A^2 + \cdots + a_k A^k$

$$\begin{aligned} p(A)x &= a_0 I_n x + a_1 Ax + a_2 A^2 x + \cdots + a_k A^k x \\ &= a_0 x + a_1(\lambda x) + a_2(\lambda^2 x) + \cdots + a_k \lambda^k x \quad \text{by (a)} \\ &= (a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_k\lambda^k)x \\ &= p(\lambda)x \end{aligned}$$

$\Rightarrow p(\lambda)$ is an eigenvalue of $p(A)$

Sec 4.2 #1(g)

6pts

Sol $V = \text{span}\{\sin x, \cos x, e^x, e^{-x}, 1\}$. Let $\alpha = \{\sin x, \cos x, e^x, e^{-x}, 1\}$, which is a basis for V

$T(\sin x) = (\sin x)'' - 4(\sin x)' + 3(\sin x) = -\sin x - 4\cos x + 3\sin x = 2\sin x - 4\cos x$

2

$\Rightarrow [T(\sin x)]_\alpha = (2, -4, 0, 0, 0)$

$T(\cos x) = (\cos x)'' - 4(\cos x)' + 3\cos x = -\cos x + 4\sin x + 3\cos x = 4\sin x + 2\cos x$

$\Rightarrow [T(\cos x)]_\alpha = (4, 2, 0, 0, 0)$

$T(e^x) = (e^x)'' - 4(e^x)' + 3(e^x) = e^x - 4e^x + 3e^x = 0$

$\Rightarrow [T(e^x)]_\alpha = (0, 0, 0, 0, 0)$

$T(e^{-x}) = (e^{-x})'' - 4(e^{-x})' + 3(e^{-x}) = e^{-x} + 4e^{-x} + 3e^{-x} = 8e^{-x}$

$\Rightarrow [T(e^{-x})]_\alpha = (0, 0, 0, 8, 0)$

$T(1) = (1)'' - 4(1)' + 3(1) = 3$

$\Rightarrow [T(1)]_\alpha = (0, 0, 0, 0, 3)$

3

$$[T]_\alpha^\alpha = \begin{bmatrix} 2 & 4 & 0 & 0 & 0 \\ -4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \quad \det([T]_\alpha^\alpha - \lambda I) = \begin{vmatrix} 2-\lambda & 4 & 0 & 0 & 0 \\ -4 & 2-\lambda & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 8-\lambda & 0 \\ 0 & 0 & 0 & 0 & 3-\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} 2-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & 8-\lambda & 0 \\ 0 & 0 & 0 & 3-\lambda \end{vmatrix} + 4 \begin{vmatrix} 4 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & 8-\lambda & 0 \\ 0 & 0 & 0 & 3-\lambda \end{vmatrix}$$

upper triangular

upper triangular

$$= (2-\lambda)(2-\lambda)(-\lambda)(8-\lambda)(3-\lambda) + (4)\,4(-\lambda)(8-\lambda)(3-\lambda)$$

$$= (-\lambda)(8-\lambda)(3-\lambda)\left[\underbrace{(2-\lambda)^2 + 16}_{\text{positive}}\right]$$

1

Let $\det([T]_\alpha^\alpha - \lambda I) = 0$. Then $\lambda = 0, 8, 3$ with algebraic multiplicity 1 for each.

Since the addition of the algebraic multiplicities is 3, which is less than $\dim(V) = 5$, T is not diagonalizable

Sec 4.2 #2 (4 pts)

Sol Let $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$. $A - \lambda I = \begin{bmatrix} a-\lambda & b \\ 0 & a-\lambda \end{bmatrix}$ (2)

$A : \mathbb{R}^2 \to \mathbb{R}^2$

$\det(A - \lambda I) = (a-\lambda)^2 = 0$ $\lambda = a$ and the algebraic multiplicity of $\lambda = a$ is $2 = \dim(\mathbb{R}^2)$

$A - aI = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ 1) If $b=0$, then x_1 and x_2 are free variables, so $\dim(E_{\lambda=a}) = 2$ which is the same as the algebraic multiplicity of $\lambda = a$. Therefore, A is diagonalizable (1)

2) If $b \neq 0$, then $x_2 = 0$ and x_1 is a free variable. So $\dim(E_{\lambda=a}) = 1$ < the algebraic multiplicity of $\lambda = a$ $= 2$. Therefore, A is not diagonalizable (1)

(2 pts)

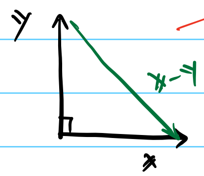
Sec 4.3 #4

Suppose x and y are orthogonal.

$\|x-y\|^2 = \langle x-y, x-y \rangle = \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle$ (0.5)

$= \|x\|^2 - 2\langle x,y\rangle + \|y\|^2 = \|x\|^2 + \|y\|^2$, Since $\langle x,y\rangle = 0$.

(0.5) (0.5) (0.5)



Pythagorean theorem

Extra Question (2 pts)

Sol Suppose $\{x_1, \ldots, x_k\}$ is a set of non-zero orthogononal vectors

Let $t_1 x_1 + \cdots + t_k x_k = 0$, $t_i \in \mathbb{R}$, $1 \le i \le k$.

For any i,

$0 = \langle t_1 x_1 + \cdots + t_k x_k, x_i \rangle = \sum_{j=1}^{k} t_j \langle x_j, x_i \rangle$ (0.5)

(0.5) $t_1 x_1 + \cdots + t_k x_k = 0$

$= t_i \langle x_i, x_i \rangle$ because $\langle x_j, x_i \rangle = 0$ if $j \neq i$ (0.5)

Since $x_i \neq 0$, $\langle x_i, x_i \rangle > 0$,

so $t_i = 0$. (0.5)

8pts

tutorial 7

problem 1 (Sol) A basis of $\text{span}(\{1, \sin x, \cos x\})$ is $\alpha = \{1, \sin x, \cos x\}$.

$[T(1)]_\alpha = (1,2,3)$, $[T(\sin x)]_\alpha = (0,2,3)$, and $[T(\cos x)]_\alpha = (0,0,2)$

So, $[T]_\alpha^\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 2 \end{bmatrix}$ and $\det\left([T]_\alpha^\alpha - \lambda I_3\right) = (1-\lambda)(2-\lambda)^2$. (1) (1)

The eigenvalues of $[T]_\alpha^\alpha$ are $\{1, 2\}$.

When $\lambda = 1$, $[T]_\alpha^\alpha - I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$

(1) $\to x_1 - \frac{1}{3}x_3 = 0$ and $x_2 + \frac{2}{3}x_3 = 0$

Let $x_3 = t$. Then $x_1 = \frac{1}{3}t$ and $x_2 = -\frac{2}{3}t$ (1)

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}t \\ -\frac{2}{3}t \\ t \end{bmatrix} = \frac{1}{3}t \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ $\quad E_{\lambda=1} = \text{span}(\{1 - 2\sin x + 3\cos x\})$

↑ coordinates of eigenvectors

When $\lambda = 2$, $[T]_\alpha^\alpha - 2I_3 = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(1) $\to x_1 = 0,\ x_2 = 0,\ x_3 = t$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad E_{\lambda=2} = \text{span}(\{\cos x\})$ (1)

(a) $E_{\lambda=1} - \{0\}$ is the set of eigenvectors associated with eigenvalue $\lambda = 1$

(1) $E_{\lambda=2} - \{0\}$ is the set of eigenvectors associated with eigenvalue $\lambda = 2$.

(1) (b) Since $\dim(E_{\lambda=2}) = 1$, T is not diagonalizable.

Problem 2 (8pts)

tutorial 7

(a) $V^\perp = \ker\left(\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}\right)$ (1.5)

3pts

$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + x_2 - x_3 + x_4 &= 0 \\ x_2 - x_3 + 2x_4 &= 0 \end{aligned}$

Let $x_3 = t$ and $x_4 = s$. Then $x_2 = x_3 - 2x_4 = t - 2s$.

$x_1 = -x_2 + x_3 - x_4 = -(t - 2s) + t - s = s$

$(x_1, x_2, x_3, x_4) = (s, t-2s, t, s) = s(1, -2, 0, 1) + t(0, 1, 1, 0)$ (1.5)

$V^\perp = \text{span}\{\underbrace{(1,-2,0,1), (0,1,1,0)}_{\text{not orthogonal}}\}$

3 pts

(b) Let $a_1 = (1,-2,0,1)$. $a_2 = (0,1,1,0) - \text{proj}_{\{a_1\}}(0,1,1,0)$ $\rightarrow$ 1

$$\text{proj}_{\{a_1\}}(0,1,1,0) = \frac{\langle (0,1,1,0),(1,-2,0,1)\rangle}{\|(1,-2,0,1)\|^2}(1,-2,0,1)$$

$$= \frac{-2}{6}(1,-2,0,1) = -\frac{1}{3}(1,-2,0,1)$$

So, $a_2 = (0,1,1,0) + \frac{1}{3}(1,-2,0,1) = \frac{1}{3}(1,1,3,1)$

2

$\{(1,-2,0,1),(1,1,3,1)\}$ is an orthogonal basis of $V^\perp$.

(c) $\|(1,-2,0,1)\| = \sqrt{1+4+1} = \sqrt{6}$, $\|(1,1,3,1)\| = \sqrt{1+1+9+1} = \sqrt{12}$

$\left\{\frac{1}{\sqrt{6}}(1,-2,0,1), \frac{1}{\sqrt{12}}(1,1,3,1)\right\}$ is an orthonormal basis of $V^\perp$.

2 pts