#1 (2 marks) 0 is the additive identity of $C(\mathbb{R})$ but $0 \notin W$ because all the vectors (polynomials) in W are none-zero.

(there are many correct answers)

#2 (a) (3 marks)

$\forall \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \in W$, $\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & a \end{bmatrix}$

$\begin{bmatrix} a & 0 \\ c & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. So,

$W = \text{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$

Let $t_1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t_2\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} t_1 & 0 \\ t_2 & t_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow t_1 = t_2 = 0$

$\Rightarrow \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$ is linearly independent

A basis for $W = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$

(b) (1 mark) Define $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = (1, 0)$ and $T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = (0, 1)$ linear

Then T is an isomorphism

#3 (a) (3 marks) $T(\mathbf{v}_1) = \mathbf{v}_1 + 4\mathbf{v}_2$

$\langle T(\mathbf{v}_1), \mathbf{v}_2\rangle = \langle \mathbf{v}_1 + 4\mathbf{v}_2, \mathbf{v}_2\rangle = \langle \mathbf{v}_1, \mathbf{v}_2\rangle + 4\langle \mathbf{v}_2, \mathbf{v}_2\rangle$

$= 0 + 4 = 4$

(b) (2 marks) $\det(T) = \det([T]_\alpha^\alpha) = -1-(-8) = 7 \neq 0$

$\Rightarrow T$ is invertible

$[T^{-1}]_\alpha^\alpha = ([T]_\alpha^\alpha)^{-1} = \frac{1}{7}\begin{bmatrix} -1 & 2 \\ -4 & 1 \end{bmatrix}$

$T^{-1}(x_1, x_2) = \left(-\frac{1}{7}x_1 + \frac{2}{7}x_2, -\frac{4}{7}x_1 + \frac{1}{7}x_2\right)$

(c) (3 marks) $\det([T]_\alpha^\alpha - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & -2 \\ 4 & -1-\lambda \end{bmatrix}\right)$

$= (1-\lambda)(-1-\lambda) + 8 = 0$

$\Rightarrow \lambda^2 + 7 = 0 \Rightarrow$ no real eigenvalue

So T is not diagonalizable.

#4 (6 marks) Let $\alpha = \{1, e^x, e^{-x}\}$

$[T]_\alpha^\alpha = [[T(1)]_\alpha \; [T(e^x)]_\alpha \; [T(e^{-x})]_\alpha]$

$= [[2]_\alpha \; [1+3e^x]_\alpha \; [2+2e^{-x}]_\alpha] = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

The eigenvalues of $[T]_\alpha^\alpha$ are 2 and 3.

Therefore, 2 and 3 are eigenvalues of T

For $\lambda = 2$. $[T]_\alpha^\alpha - 2I = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_2 + 2x_3 = 0$, $x_3 = 0$ $\Downarrow$ $x_2 = x_3 = 0$, $x_1 = t$

The eigenspace of $[T]_\alpha^\alpha$ for $\lambda = 2 = \text{span}\{(1,0,0)\}$

$E_{\lambda=2} = \text{span}\{1\}$

For $\lambda = 3$, $[T]_\alpha^\alpha - 3I = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{cases} x_1 - x_2 - 2x_3 = 0 \\ x_3 = 0 \end{cases} \Rightarrow x_1 = x_2$

Let $x_2 = t$. Then $x_1 = t$,

The eigenspace of $[T]_\alpha^\alpha$ for $\lambda = 3$, $\text{span}\{(1,1,0)\}$

$E_{\lambda=3} = \text{span}\{1 + e^x\}$

#5 (a) $W^\perp = \ker([1\ 2\ 1])$
(3marks) Let $x_2 = t$ and $x_3 = s$

Then $x_1 = -2x_2 - x_3 = -2t - s$

$(x_1, x_2, x_3) = t(-2, 1, 0) + s(-1, 0, 1)$

$W^\perp = \text{span}\{(-2,1,0), (-1,0,1)\}$

(b) Since $W + W^\perp$, $\{x, y, z\}$ is orthogonal.
(2marks) $\Rightarrow \{x, y, z\}$ is linear independent

Since $\dim(\mathbb{R}^3) = 3$, $\alpha = \{x, y, z\}$ is a basis

(2marks) (c) $[P_{W^\perp}]_\alpha^\alpha = [[P_{W^\perp}(x)]\ [P_{W^\perp}(y)]\ [P_{W^\perp}(z)]]_\alpha = [[0]_\alpha\ [y]_\alpha\ [z]_\alpha] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

#6 (a) Let $W_1 = \{(x_1, x_2, x_3) \mid x_1 = 0\}$, $W_2 = \{(x_1, x_2, x_3) \mid x_2 = 0\}$, and $V = \mathbb{R}^3$.
(3marks) Then W_1 and W_2 are subspaces of $\mathbb{R}^3$ and $W_1 + W_2 = \mathbb{R}^3$

$(1, 2, 3) \in \mathbb{R}^3 = W_1 + W_2$, $(1,2,3) = (0, 2, 3) + (1, 0, 0)$ There are many different
$= (0, 2, 1) + (1, 0, 2)$ linear combinations

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonalizable because there are two different
3marks eigenvalues $\lambda = 1$ and 0

But A is not invertible because $\det(A) = 0$

or existance of zero eigenvalue

#7 Suppose $x = ky$, $k \neq 0$ linearly dependent.
(3marks) $T(x) = \lambda x$ because x is an eigenvector associated with λ

$T(x) = T(ky) = kT(y) = k\mu y$ because y " " " μ.

$T(x) = \lambda x = k\mu y \Rightarrow \lambda(ky) = k\mu y \Rightarrow k(\lambda - \mu)y = 0$

Since $k \neq 0$ and $y \neq 0$; $\lambda = \mu$ $\leftarrow$ contradiction.

#8 $v_1 = (-1, 0, 1, 0)$, $v_2 = (1,-1,2,1) - \text{proj}_{\text{span}\{v_1\}}(1,-1,2,1) = (1,-1,2,1) - \dfrac{\langle (1,-1,2,1), (-1,0,1,0)\rangle}{\|(-1,0,1,0)\|^2}(-1,0,1,0)$
(a)
(3marks) $= (1,-1,2,1) - \frac{1}{2}(-1,0,1,0) = \frac{1}{2}(2,-2,4,2) - \frac{1}{2}(-1,0,1,0)$

$= \frac{1}{2}(3, -2, 3, 2)$

$\|v_1\| = \sqrt{1+1} = \sqrt{2}$, $\|v_2\| = \frac{1}{2}\sqrt{9+4+9+1} = \frac{1}{2}\sqrt{26}$

An orthonormal basis: $\left\{ \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{3}{\sqrt{26}}, -\frac{2}{\sqrt{26}}, \frac{3}{\sqrt{26}}, \frac{2}{\sqrt{26}}\right) \right\}$

(b) $P_W(1,1,1,1) = \langle (1,1,1,1), (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0)\rangle (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0) +$
(3marks) $\langle (1,1,1,1), (\frac{3}{\sqrt{26}}, \frac{-2}{\sqrt{26}}, \frac{3}{\sqrt{26}}, \frac{2}{\sqrt{26}})\rangle (\frac{3}{\sqrt{26}}, \frac{-2}{\sqrt{26}}, \frac{3}{\sqrt{26}}, \frac{2}{\sqrt{26}})$

$= 0 + \frac{6}{\sqrt{26}}\left(\frac{3}{\sqrt{26}}, -\frac{2}{\sqrt{26}}, \frac{3}{\sqrt{26}}, \frac{2}{\sqrt{26}}\right) = \left(\frac{18}{26}, \frac{-12}{26}, \frac{18}{26}, \frac{12}{26}\right)$

$= \left(\frac{9}{13}, \frac{-6}{13}, \frac{9}{13}, \frac{6}{13}\right)$

#9

(a) (4 marks)

$[p(x)]_\beta = [I]_\alpha^\beta [p(x)]_\alpha$ where $[I]_\alpha^\beta = \left[\ [x]_\beta \quad [1+x]_\beta\ \right] \overset{\text{let}}{=} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

Then $x = a(2x+1) + b(x-3)$ and $1+x = c(2x+1) + d(x-3)$

Slove $\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & -3 & 0 \end{array}\right]$ for (a,b) and $\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & -3 & 1 \end{array}\right]$ for (c,d)

$$\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \frac{-1}{7}\begin{bmatrix} -3 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} -3 & -4 \\ -1 & 1 \end{bmatrix}$$

$$[p(x)]_\beta = [I]_\alpha^\beta [p(x)]_\alpha = -\frac{1}{7}\begin{bmatrix} -3 & -4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10/7 \\ 1/7 \end{bmatrix}$$

(b) (4 marks)

$[ST]_\alpha^\alpha = [S]_\beta^\alpha [T]_\alpha^\beta$.

$[T]_\alpha^\beta = \left[\ [T(x)]_\beta \quad [T(1+x)]_\beta\ \right] = \left[\ [x+2]_\beta \quad [x-2]_\beta\ \right] \overset{\text{let}}{=} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

Like (a), solve $\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$ for $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

Then $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} -3 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} -5 & -1 \\ 3 & -5 \end{bmatrix}$

$$[S]_\beta^\alpha = [ST]_\alpha^\alpha \left([T]_\alpha^\beta\right)^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 3 & 5 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 & 4 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1/2 & -3/2 \end{bmatrix}$$