UNIVERSITY OF TORONTO
FACULTY OF ARTS AND SCIENCE

Test1, Oct 18, 2024

MAT224H1 S

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Duration: 100 mins

This test has 12 pages.

Total: 50 marks

-NO AIDS ALLOWED

-No marks will be given for a completely wrong solution.

-Unless specified, the vector addition and scalar multiplication of a vector space are all standard ones of the vector space.

1. (6 marks)Let $T : \mathbf{R}^3 \to W$ be a linear transformation with bases $\alpha = \{(-1, 0, 2), (1, 1, 0), (1, 3, 1)\}$ and $\beta = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ for $\mathbf{R}^3$ and W respectively. Suppose $\mathrm{Ker}([T]_\alpha^\beta) = \mathrm{span}\{(1, 2, -3)\}$.

(a) (3 marks) Find $\mathrm{Ker}(T)$.

(b) (3 marks) Find the dimension of $\mathrm{Im}(T)$. Is T surjective?

2. (3 marks) Suppose $T : V \to W$ is an isomorphism, where V and W are finite dimensional vector spaces. Show that $\dim(V) = \dim(W)$.

3. (6 marks) The following statements are false. Explain why they are false by providing a counterexample.

(a) (3 marks) The mapping $T : C^\infty(\mathbf{R}) \longrightarrow C^\infty(\mathbf{R})$ defined by $T(f(x)) = x^2(f(x) + x)$ is linear.

(b) (3 marks) Let V be a vector space. If S_1 and S_2 are linearly independent subsets of V, then $S_1 \cup S_2$ is also linearly independent.

4. (7 marks) Let $S = \{f \in \mathrm{Span}\{e^x, e^{2x}, e^{3x}\} |\, f(0) = f'(0) = 0\}$.

(a) (3 marks) Show that S is a subspace of $C^\infty(\mathbf{R})$.

(b) (4 marks) Find a basis of S.

5. (7 marks) Let $T : V \longrightarrow W$ be a linear transformation defined by $T(\mathbf{v}_1) = \mathbf{w}_1 + 5\mathbf{w}_2$ and $T(\mathbf{v}_2) = -2\mathbf{w}_1 + 3\mathbf{w}_2$, where $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\beta = \{\mathbf{w}_1, \mathbf{w}_2\}$ be bases of the vector spaces V and W respectively.

(a) (2 marks) Show that T invertible.

(b) (3 marks) Find $T^{-1}(\mathbf{w}_1)$ and $T^{-1}(\mathbf{w}_2)$.

(c) (2 marks) If $[T(\mathbf{v})]_\beta = (-1, 1)$, Compute $[\mathbf{v}]_\alpha$.

6. (3 marks) Suppose $\mathbf{R}^2$ has the following vector addition and scalar multiplication: for any $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathbf{R}^2$, and any $c \in \mathbf{R}$,

$\mathbf{x} +' \mathbf{y} = (x_1y_1, x_2y_2)$ and $c \cdot \mathbf{x} = (cx_1, cx_2)$.

Show that $\mathbf{R}^2$ is not a vector space with the vector addition and scalar multiplication above.

7. (3 marks) Let $S = \{\sin x, \sin(2x)\}$ be a subset of the vector space $C^\infty(\mathbf{R})$ with the standard vector addition and scalar multiplication. Show that S is linearly independent.

8. (7 marks) Let $\alpha = \{(2,1),(3,1)\}$ and $\beta = \{\mathbf{w}_1, \mathbf{w}_2\}$ be bases for $\mathbf{R}^2$. Suppose $[I]_\alpha^\beta = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, where I is the identity mapping from $\mathbf{R}^2$ to $\mathbf{R}^2$.

(a) (4 marks) Find $[(3,4)]_\beta$.

(b) (3 marks) Find $\mathbf{w}_1$ and $\mathbf{w}_2$.

9. (8 marks) Let T be a linear transformation from a vector space V to V, $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis for V, and

Let $[T]_\alpha^\alpha = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$.

Suppose $\alpha' = \{\mathbf{w}_1, \mathbf{w}_2\}$ is a new basis for V, $[\mathbf{w}_1]_\alpha = (2, 1)$ and $[\mathbf{w}_2]_\alpha = (1, 4)$.

(a) (2 marks) Compute $\det(T)$.

(b) (4 marks) Find $[T]_{\alpha'}^{\alpha'}$.

(c) (2 marks) Find $T(2\mathbf{w}_1 - 4\mathbf{w}_2)$.

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