UNIVERSITY OF TORONTO
FACULTY OF ARTS AND SCIENCE

Test1, Feb 26, 2025

MAT224H1 S

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Duration: 100 mins

This test has 12 pages.

Total: 50 marks

NO AIDS ALLOWED

No marks will be given for a completely wrong solution.

1. (6 marks)

Let $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ be defined by $T(x_1, x_2) = (4x_1 + 2x_2, 2x_1 + x_2)$.

Suppose $\alpha = \{(2, 1), \mathbf{w}\}$ is a basis for $\mathbf{R}^2$ and $[T]_\alpha^\alpha = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}$.

(a) (4 marks) Find a, b, and all possible $\mathbf{w}$.

(b) (2 marks) Let s be the standard basis for $\mathbf{R}^2$. Write the definition of similar matrices, and find an invertible matrix P such that $[T]_\alpha^\alpha$ and $[T]_s^s$ are similar.

2. (7 marks) Suppose T is a linear transformation from a vector space V to a vector space W. Let $\mathbf{0_v}$ and $\mathbf{0_w}$ be the additive identities of V and W respectively.

(a) (3 marks) Show that $T(\mathbf{0_v}) = \mathbf{0_w}$.

(b) (3 marks) Show that $\{T(\mathbf{v})|\, \mathbf{v} \in V\}$ is a subspace of W.

(c) (1 mark) Show that $\{\mathbf{v} \in V|\, T(\mathbf{v}) = \mathbf{w}\}$ is not a subspace of V if $\mathbf{w}$ is not $\mathbf{0_w}$.

3. (8 marks) Let $T : \mathbf{V} \longrightarrow \mathbf{W}$ be a linear transformation with bases $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\beta = \{\mathbf{w}_1, \mathbf{w}_2\}$ of domain and codomain respectively.

Suppose the matrix of T with respect to α and β is $= \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$.

(a) (3 marks) Find the coordinates of $T(2\mathbf{v}_1 - 3\mathbf{v}_2)$ with respect to β.

(b) (3 marks) Show that T is invertible and find the matrix of T^{-1} with respect to β and α.

(c) (2 marks) Find $T^{-1}(\mathbf{w}_1 + \mathbf{w}_2)$.

4. (7 marks) Suppose T be a linear mapping from a vector space V to V and $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for V.

(a) (4 marks) Show that $\beta = \{2\mathbf{v}_1 + \mathbf{v}_2, -\mathbf{v}_1 + 3\mathbf{v}_2\}$ is also a basis for V.

(b) (3 marks) Suppose the matrix $[T]_\alpha^\beta$ is $= \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$. Find $[T]_\beta^\beta$.

5. (7 marks) Let $T : P_2(\mathbf{R}) \to \mathbf{R}^3$ be defined by $T(p(x)) = (a_0 + a_1, a_1 + a_2, a_2 + a_0)$, where $p(x) = a_0 + a_1 x + a_2 x^2$.

(a) (3 marks) Show that T is a linear transformation.

(b) (4 marks) Is T injective, surjective, both or neither?

6. (8 marks) The following statements are all false. Explain why they are false by providing a counterexample.

(a) (4 marks) Let $T : \mathbf{V} \longrightarrow \mathbf{W}$ be a linear transformation with the basis $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$ for V. Then $\{T(\mathbf{v}_1), T(\mathbf{v}_2)\}$ is a basis for W.

(b) (4 marks) Suppose V_1 and V_2 are different subspaces of a finite dimensional vector space V. Then $\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2)$.

7. (7 marks) Let C^∞ be the vector space equipped with the standard vector addition and scalar multiplication and $\mathrm{Span}(\{e^{2x}, \sin x, \cos x\}) \subset C^\infty$.

(a) (3 marks) Find a basis for $\mathrm{Span}(\{e^{2x}, \sin x, \cos x\})$. Explain your answer.

(b) (4 marks) Let $S = \{f \in \mathrm{Span}(\{e^{2x}, \sin x, \cos x\}) |\, f(0) = f'(0) = 0\}$. Find a basis and the dimension of S. Explain your answer.

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