Sec 4.6

3pts #4 (c) Suppose P and Q are orthogonal matrices

$(PQ)^t \overset{0.5}{=} Q^t P^t \overset{0.5}{=} Q^{-1}P^{-1} \overset{0.5}{=} (PQ)^{-1} \Rightarrow PQ$ is orthogonal

P and Q are orthogonal

(d) Suppose Q is orthogonal.

$(Q^{-1})^t \overset{0.5}{=} (Q^t)^t \overset{0.5}{=} Q \overset{0.5}{=} (Q^{-1})^{-1} \Rightarrow Q^{-1}$ is orthogonal

Q is orthogonal

#5 (a) $\forall\, x, y \in \mathbb{R}^n$, $\langle Qx, Qy \rangle \overset{1}{=} \langle x, Q^t(Qy) \rangle$ for any matrix Q.

3pts

($\Rightarrow$) Suppose Q is orthogonal. $\langle Qx, Qy \rangle = \langle x, (Q^tQ)y \rangle = \langle x, y \rangle$ (1)

$Q^tQ = I$ because Q is orthogonal

($\Leftarrow$) Suppose $\langle Qx, Qy \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{R}^n$.

$\langle x, (Q^tQ)y \rangle = \langle Qx, Qy \rangle = \langle x, y \rangle \Rightarrow Q^tQ = I$ (1)

by assumption

$\Rightarrow Q$ is orthogonal.

3pts → (b) Suppose Q is orthogonal. $\|Qx\|^2 \overset{0.5}{=} \langle Qx, Qx \rangle \overset{1}{=} \langle x, x \rangle \overset{0.5}{=} \|x\|^2$

From (a)

$\Rightarrow$ Since $\|Qx\| \geq 0$ and $\|x\| \geq 0$, $\|Qx\| = \|x\|$ (1)

3pts → (c) Let A be an $n \times n$ matrix. Suppose $\|Ax\| = \|x\|$ for all $x \in \mathbb{R}^n$.

$\langle Ax, Ay \rangle = \frac{1}{4}(\|Ax + Ay\|^2 - \|Ax - Ay\|^2)$ by polarization identity (1)

$= \frac{1}{4}(\|A(x+y)\|^2 - \|A(x-y)\|^2)$

$= \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$ by the assumption (1)

$= \langle x, y \rangle$ by polarization identity. (1)

$\Rightarrow A$ is orthogonal from #5(a)

#8 Suppose A is symmetric. That is, $A = A^t$

1pts → (i) A is orthogonal $\Rightarrow$ (ii) $A^2 = I$

proof A is orthogonal $\Rightarrow AA^t = I \Rightarrow AA = I \Rightarrow A^2 = I$ because $A = A^t$.

3pts → (ii) $A^2 = I \Rightarrow$ (iii) All eigenvalues of A are 1 or -1

proof Let λ be an eigenvalue of A. Then

$Ax = \lambda x$ for $\forall x \in E_\lambda$.

(1) $A(Ax) = A(\lambda x) \Rightarrow A^2(x) = \lambda(Ax) \Rightarrow x = \lambda^2 x$ (1) $A^2 = I$ and $Ax = \lambda x$

$\Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$ (1)

7pts → (iii) All eigenvalues of A are 1 or -1 $\Rightarrow$ A is orthogonal

proof Let λ be an eigenvalue of A.

Then $A\mathbf{x} = \lambda \mathbf{x}$ for $\forall \mathbf{x} \in E_\lambda$

$A^tA\mathbf{x} = A^t(\lambda\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A(\mathbf{x}) = \lambda^2\mathbf{x}$

0.5 — 1 — A is symmetric — 0.5

1 Since $\lambda = \pm 1$, $A^tA(\mathbf{x}) = \mathbf{x}$.

1 $\forall y \in \mathbb{R}^n$ $y = t_1\mathbf{x}_1 + \cdots + t_n\mathbf{x}_n$ where $\{\mathbf{x}_1 \ldots \mathbf{x}_n\}$ is a basis consisting of eigenvectors of A.

Therefore, $A^tA(y) = \sum_{i=1}^{n} t_i (A^tA)(\mathbf{x}_i) = \sum_{i=1}^{n} t_i\mathbf{x}_i = y$ 1 1

1 $\Rightarrow A^tA = I \Rightarrow A$ is orthogonal

Sec 5.1 #6 (5pts)

1pts → (a) Let $z^2+2z+2=0$. $z = -1 \pm \sqrt{1-2} = -1 \pm i$

1pts → (b) Let $z^3-z^2+2z-2=0$. $0 = z^3-z^2+2z-2 = z^2(z-1)+2(z-1) = (z^2+2)(z-1)$

$\Rightarrow z=1,\ z = \pm\sqrt{2}\, i$

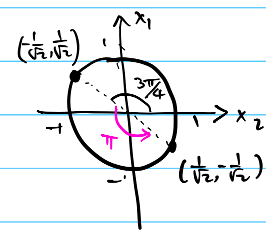
3pts → (c) Let $z^2+i=0$. Solve $z^2=-i$.

Since $-i = \cos\left(\frac{3\pi}{2}+2\pi k\right) + i\sin\left(\frac{3\pi}{2}+2\pi k\right)$, —1

$z = \cos\left(\frac{3\pi}{4}+\pi k\right) + i\sin\left(\frac{3\pi}{4}+\pi k\right)$, $k=0,1$

when $k=0$, $z = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ —1

when $k=1$, $z = \cos\left(\frac{3\pi}{4}+\pi\right) + i\sin\left(\frac{3\pi}{4}+\pi\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ —1



(5pts) #10 (a) $A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$ $\det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{bmatrix} = (1-\lambda)^2 + (i)^2 \overset{\text{let}}{=} 0$

$\lambda^2 - 2\lambda + 1 - 1 = 0 \Rightarrow \lambda^2 - 2\lambda = 0$

(1pt) $\Rightarrow \lambda(\lambda-2)=0$

$\Rightarrow \lambda=0$ or $\lambda=2$ ← eigenvalues

For $\lambda=0$,

2pts $A-\lambda I = A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \overset{R_1(i)+R_2}{\sim} \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \Rightarrow 1x_1 + i x_2 = 0$

Let $x_2 = t$.

Then $x_1 = -it$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -it \\ t \end{pmatrix} = t\begin{pmatrix} -i \\ 1 \end{pmatrix}$, So $E_{\lambda=0} = \text{span}\{(-i,1)\}$

For $\lambda=2$

2pts $A-2I = \begin{bmatrix} -1 & i \\ -i & -1 \end{bmatrix} \overset{R_1(-i)+R_2}{\sim} \begin{bmatrix} -1 & i \\ 0 & 0 \end{bmatrix} \Rightarrow -x_1 + i x_2 = 0$

Let $x_2 = t$. Then $x_1 = it$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} it \\ t \end{pmatrix} = t\begin{pmatrix} i \\ 1 \end{pmatrix}$, So $E_{\lambda=2} = \text{span}\{(i,1)\}$

Sec 5.2 # 12

(5pts) (a) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \overset{\text{let}}{=} A.$

$\det(A-\lambda I)=0$

$\Rightarrow (\cos\theta-\lambda)^2+\sin^2\theta=0$

$\Rightarrow \lambda^2-2\cos\theta\,\lambda+\cos^2\theta+\sin^2\theta=0$

$\Rightarrow \lambda^2-2\cos\theta\,\lambda+1=0$

(1) $\Rightarrow \lambda=\cos\theta\pm\sqrt{\cos^2\theta-1}=\cos\theta\pm\sqrt{-\sin^2\theta}$

$=\cos\theta\pm\sin\theta\, i$

(2) For $\lambda=\cos\theta+\sin\theta\, i$, $A-(\cos\theta+\sin\theta\, i)I$

$=\begin{bmatrix} (-\sin\theta)i & -\sin\theta \\ \sin\theta & (-\sin\theta)i \end{bmatrix}$

$\sim \begin{bmatrix} (\sin\theta)i & \sin\theta \\ 0 & 0 \end{bmatrix}$ $\quad (\sin\theta)i\,x_1+\sin\theta\,x_2=0$

$(\sin\theta)i\,x_1=-\sin\theta\,x_2$

Let $x_2=t$, then $x_1=-\frac{1}{i}x_2$

$=i\,x_2$ assuming $\sin\theta\neq 0$

$E_{\lambda=\cos\theta+\sin\theta i}=\text{span}\{(i,1)\}$

(2) For $\lambda=\cos\theta-\sin\theta\, i$, $A-(\cos\theta-\sin\theta\, i)I$

$=\begin{bmatrix} \sin\theta\, i & -\sin\theta \\ \sin\theta & \sin\theta\, i \end{bmatrix}$

$\sim \begin{bmatrix} \sin\theta\, i & -\sin\theta \\ 0 & 0 \end{bmatrix}$ $\quad \sin\theta\, i\,x_1-\sin\theta\,x_2=0$

Let $x_2=t$, then $x_1=\frac{1}{i}x_2=-it$ (assuming $\sin\theta\neq 0$)

$E_{\lambda=\cos\theta-\sin\theta i}=\text{span}((-i,1))$

(5pts)

(b) $\begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} \overset{\text{let}}{=} A$ $\quad a\neq 0$

$\det(A-\lambda I)=0 \Rightarrow (-\lambda)^2+a^2=0$

$\Rightarrow \lambda^2=-a^2 \Rightarrow \lambda=\pm a i$ (1)

For $\lambda=ai$, $A-(ai)I=\begin{bmatrix} -ai & -a \\ a & -ai \end{bmatrix}$

$\sim \begin{bmatrix} ai & a \\ 0 & 0 \end{bmatrix}$ $\quad ai\,x_1+a\,x_2=0$

Let $x_2=t$. Then $x_1=-\frac{1}{i}t$

$=it$ (2)

$E_{\lambda=ai}=\text{span}\{(i,1)\}$

For $\lambda=-ai$, $A+(ai)I=\begin{bmatrix} ai & -a \\ a & ai \end{bmatrix}$

$\sim \begin{bmatrix} ai & -a \\ 0 & 0 \end{bmatrix}$ $\quad ai\,x_1-ax_2=0$

Let $x_2=t$. Then $x_1=\frac{1}{i}t$

$=-it$ (2)

$E_{\lambda=-ai}=\text{span}\{(-i,1)\}$

Sec 5.3 #3 (a) (5pts)

$v_1=(i,-1,i)$ and $v_2=(1,1,0)$

Let $w_1=v_1$. (0.5)

$w_2=v_2-P_{\text{span}\{w_1\}}(v_2)=v_2-\frac{\langle v_2,w_1\rangle}{\|w_1\|^2}w_1$ (1.5)

$\langle v_2,w_1\rangle=\langle v_2,v_1\rangle=(1)(-i)+(1)(-1)+(0)(-i)=-1-i$ (1)

$\|w_1\|^2=\langle v_1,v_1\rangle=i(-i)+(-1)(-1)+(i)(-i)=1+1+1=3$ (1)

$w_2=(1,1,0)-\frac{(-1-i)}{3}(i,-1,i)=(1,1,0)+\frac{1+i}{3}(i,-1,i)$ (1)

$=\left(\frac{2}{3}+\frac{1}{3}i,\ \frac{2}{3}-\frac{1}{3}i,\ -\frac{1}{3}+\frac{1}{3}i\right)$

$\left\{(i,-1,i),\ \left(\frac{2}{3}+\frac{1}{3}i,\ \frac{2}{3}-\frac{1}{3}i,\ -\frac{1}{3}+\frac{1}{3}i\right)\right\}$ orthogonal