Q1

4marks (a) $T(v_1) = v_1 + 2v_2 \qquad T(v_2) = 3v_1 + 4v_2$ (2)

$$\langle T(v_1), T(v_2)\rangle = \langle v_1+2v_2, 3v_1+4v_2\rangle$$

(2)
$$= 3\underbrace{\langle v_1, v_1\rangle}_{1} + 4\underbrace{\langle v_1, v_2\rangle}_{0} + 6\underbrace{\langle v_2, v_1\rangle}_{0} + 8\underbrace{\langle v_2, v_2\rangle}_{1}$$

$$= 3+8 = 11.$$

3 marks (b) $[T^{-1}]_\alpha^\alpha = \frac{1}{4-6}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 4 & -3 \\ -2 & 1\end{bmatrix} = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2}\end{bmatrix}$ (1.5)

Since $T(v_1) = v_1 + 2v_2$ and T^{-1} exists, $T^{-1}(v_1+2v_2) = v_1$ (1.5)

or $T^{-1}(v_1+2v_2) = T^{-1}(v_1) + 2T^{-1}(v_2) = (-2v_1+v_2) + 2(\frac{3}{2}v_1 - \frac{1}{2}v_2)$ (0.5)

$= v_1$ (0.5) (0.5)

Q2

(a) $W^\perp = \{x \in \mathbb{R}^n \mid \langle x, w\rangle = 0 \text{ for all } w \in W\}$

Sol 1. Since $\langle 0, w\rangle = 0$ for all $w \in W$, $0 \in W^\perp \Rightarrow$ non-empty

(3marks) 2. $\forall x_1, x_2 \in W^\perp$, $\forall w \in W$, $\langle x_1 + x_2, w\rangle = \langle x_1, w\rangle + \langle x_2, w\rangle = 0+0=0$

(1.5) so, $x_1 + x_2 \in W^\perp$

(1.5) 3. $\forall x \in W^\perp$, $k \in \mathbb{R}$, $\forall w \in W$, $\langle kx, w\rangle = k\langle x, w\rangle = k\cdot 0 = 0$

so $kx \in W^\perp$.

(4marks) (b) $W = \text{span}\{(1,1,2,1), (-1,2,1,0)\}$. Find $W^\perp$.

Sol $W^\perp = \ker\left(\begin{bmatrix} 1 & 1 & 2 & 1 \\ -1 & 2 & 1 & 0\end{bmatrix}\right)$ (2)

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ -1 & 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 3 & 3 & 1\end{bmatrix} \qquad \begin{aligned} x_1 + x_2 + 2x_3 + x_4 &= 0 \\ 3x_2 + 3x_3 + x_4 &= 0 \end{aligned}$$

(2) Let $x_3 = t$ and $x_4 = s$. Then $x_2 = \frac{1}{3}(-3x_3 - x_4) = \frac{1}{3}(-3t - s)$

$$= -t - \frac{1}{3}s$$

$$x_1 = -x_2 - 2x_3 - x_4 = \cancel{+}t + \tfrac{1}{3}s - 2t - s = -t - \tfrac{2}{3}s$$

$$(x_1, x_2, x_3, x_4) = t(-1,-1,1,0) + s\left(-\tfrac{2}{3}, -\tfrac{1}{3}, 0, 1\right)$$

$$W^\perp = \operatorname{span}\left(\{(-1,-1,1,0), (-2,-1,0,3)\}\right)$$

Q3

(a) 6marks

$T(x\sin x) = (x\sin x)' = \sin x + x\cos x$, $[T(x\sin x)]_\alpha = (0,1,1,0)$

$T(x\cos x) = (x\cos x)' = \cos x - x\sin x$, $[T(x\cos x)]_\alpha = (-1,0,0,1)$

$T(\sin x) = (\sin x)' = \cos x$, $[T(\sin x)]_\alpha = (0,0,0,1)$

$T(\cos x) = (\cos x)' = -\sin x$, $[T(\cos x)]_\alpha = (0,0,-1,0)$ 2

2

$$[T]_\alpha^\alpha = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

rank 4

2

$\ker(T) = \{0\}$

$\operatorname{Im}(T) = \underbrace{W \text{ or } \operatorname{span}}_{\text{same}}(\{x\cos x + \sin x, -x\sin x + \cos x, \cos x, -\sin x\})$

(b) Since $\ker(T) = \{0\}$, T is injective.

1

Q4 Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $T(p(x)) = p(x) + p'(x)$

3marks (a) $\forall\, p_1(x), p_2(x) \in P_2(\mathbb{R})$, $T(p_1(x) + p_2(x)) = p_1(x) + p_2(x) + p_1'(x) + p_2'(x)$

$$= (p_1(x) + p_1'(x)) + (p_2(x) + p_2'(x))$$

1.5

$$= T(p_1(x)) + T(p_2(x))$$

$\forall\, p(x) \in P_2(\mathbb{R})$, $\forall k \in \mathbb{R}$, $T(kp(x)) = kp(x) + kp'(x)$

1.5

$$= k(p(x) + p'(x)) = kT(p(x))$$

So T is linear

4marks (b) Let $S = \{1, x, x^2\}$. $T(1) = 1$, $T(x) = x+1$, $T(x^2) = x^2 + 2x$

$$[T]_S^S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$ $\lambda = 1$ is the only eigenvalue of T

$$[T]_S^S - I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$ $x_1 = t$, $x_2 = 0$ and $x_3 = 0$ 2

$\Rightarrow \text{Null}([T]_S^S - I) = \text{span}(\{(1,0,0)\})$ 1

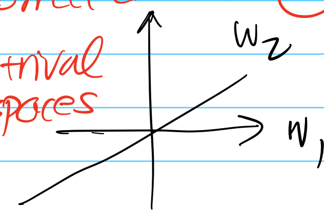
$\Rightarrow E_{\lambda=1} = \text{span}(\{1\})$ 1

0.5 0.5

(c) Since $\dim(E_{\lambda=1}) = 1$ and the algebraic multiplicity of $\lambda = 1$ is 3,
1 marks T is not diagonalizable.

a correct example 2

Q5 (a) $W_1 = \text{span}\{(1,0)\}$ and $W_2 = \text{span}\{(1,1)\}$. non-trivial subspaces



4 marks $\langle (1,0), (1,1) \rangle = 1 \neq 0 \Rightarrow W_1 \neq W_2^{\perp}$

explaining 1 But since $\dim(W_1) = \dim(W_2) = 1$ and $W_1 \cap W_2 = \{(0,0)\}$,
$W_1 \oplus W_2 = \mathbb{R}^2$

1

(b) $S_1 = \{(1,0,0), (0,1,0)\} \subset \text{span}(\{(1,0,0), (0,1,0)\})$

for a correct example 2 $S_2 = \{(0,2,0)\} \subset \text{span}(\{(0,1,0), (0,0,1)\})$

S_1 and S_2 are linearly independent because $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ has rank 2.

explaining 1 $S_1 \cup S_2 = \{(1,0,0), (0,1,0), (0,2,0)\}$ is linearly dependent

explaining 1 because $(0,2,0) = 0(1,0,0) + 2(0,1,0)$.

Remark : There are many different correct examples.

Q6

$W = \text{span}\{(2,1,2,0), (0,-1,2,1)\}$

not orthogonal

(a) Let $w_1 = (2,1,2,0)$ and $W = \text{span}(\{(2,1,2,0)\})$

(3 marks)

$w_2 = (0,-1,2,1) - \text{Proj}_W (0,-1,2,1) = (0,-1,2,1) - \frac{\langle (0,-1,2,1), (2,1,2,0) \rangle}{\|(2,1,2,0)\|^2} (2,1,2,0)$

0.5

1

$= (0,-1,2,1) - \frac{3}{9}(2,1,2,0) = (0,-1,2,1) - \frac{1}{3}(2,1,2,0)$

$= \frac{1}{3}((0,-3,6,3) - (2,1,2,0)) = \frac{1}{3}(-2,-4,4,3)$

1.5

for computing

$\{(2,1,2,0), (-2,-4,4,3)\}$ is an orthogonal basis for W.

(b) $P_W(1,0,0,1) = \frac{\langle (1,0,0,1), (2,1,2,0) \rangle}{\|(2,1,2,0)\|^2}(2,1,2,0) +$

(3 marks)

$\frac{\langle (1,0,0,1), (-2,-4,4,3) \rangle}{\|(-2,-4,4,3)\|^2}(-2,-4,4,3)$

1.5

$= \frac{2}{9}(2,1,2,0) + \frac{1}{45}(-2,-4,4,3)$

1.5

$= \frac{1}{45}(22,6,24,3)$

computing

Q7

(a) $T(x) = \lambda x$, $x \neq 0$. Then $T^2(x) = T(T(x)) = T(\lambda x) = \lambda T(x) = \lambda(\lambda x) = \lambda^2 x$

1 1 1

(3 marks)

$\Rightarrow \lambda^2$ is an eigenvalue of T^2.

(b) Let $t_1 x + t_2 T(x) = 0$ where $t_1, t_2 \in \mathbb{R}$.

(4 marks)

$T(t_1 x + t_2 T(x)) = T(0)$

1 1

$\Rightarrow t_1 T(x) + t_2 T^2(x) = 0 \Rightarrow t_1 T(x) + t_2 0 = 0 \Rightarrow t_1 T(x) = 0$

1

$\Rightarrow t_1 = 0$ because $T(x) \neq 0$.

Then $0x + t_2 T(x) = 0 \Rightarrow t_2 = 0$ too, because $T(x) \neq 0$. So $\{x, T(x)\}$ is linearly independent

1