Chapter 6

Let V be a complex vector space.

Let $T: V \to V$ be a linear mapping

Then $\det([T]_\alpha^\alpha - \lambda I) = 0$ has $\dim(V)$ roots (including repeated roots)

The main point of Chapter 6

We can find a basis α such that

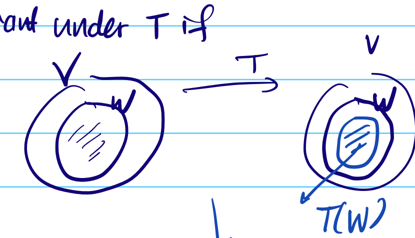
$[T]_\alpha^\alpha$ is an upper triangular

That is, $[T]_\alpha^\alpha = \begin{bmatrix} \ddots & * \\ 0 & \ddots \end{bmatrix}$ eigenvalues on the main diagonal

Sec 6.1 Triangular Form

Def Let $T: V \to V$ be a linear mapping

A subspace $W \subseteq V$ is said to be invariant under T if $T(W) \subseteq W$



$\boxed{T|_W : W \to W}$

Ex1 ⓐ $\{0\}$, V "trivial" invariant subspaces

HW

$T(0) = 0 \Rightarrow T(\{0\}) \subseteq \{0\}$

ⓑ $\ker(T)$ and $\operatorname{Im}(T)$ are invariant

HW

$\forall x \in \ker(T)$ $\quad T(x) = 0 \in \ker(T) \quad \Rightarrow T(\ker(T)) \subseteq \ker(T)$

Ex2 Let λ be an eigenvalue of a linear mapping $T: V \to V$

Then E_λ is invariant

We have proved it before:

$\forall x \in E_\lambda$

$T(T(x)) = T(\underbrace{\lambda x}_{x \in E_\lambda}) = \lambda T(x) \Rightarrow T(x) \in E_\lambda$

↑ by linearity $\quad \Rightarrow T(E_\lambda) \subseteq E_\lambda$

☆ Ex3 Suppose $T: \mathbb{C}^3 \to \mathbb{C}^3$ linear, λ is an eigenvalue of T and $\dim(E_\lambda) = 2$. Find $[T|_{E_\lambda}]_\alpha^\alpha$ where α is a basis for E_λ

Sol Since E_λ is invariant, $T|_{E_\lambda}: E_\lambda \to E_\lambda$ linear

Say $\alpha = \{v_1, v_2\}$. Then $[T|_{E_\lambda}]_\alpha^\alpha = \left[[T|_{E_\lambda}(v_1)]_\alpha \ [T|_{E_\lambda}(v_2)]_\alpha \right]$

$= \left[[\lambda v_1]_\alpha \ [\lambda v_2]_\alpha \right] = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

Remark

If T is diagonalizable, $\mathbb{C}^3 = E_\lambda \oplus E_\mu$ $\quad E_\mu = E_\lambda^\perp$ if T is self-adjoint

$\alpha = \alpha_1 \cup \alpha_2$

$[T|_{E_\lambda}]_{\alpha_1}^{\alpha_1}$

$$[T]_\alpha^\alpha = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

$[T|_{E_\mu}]_{\alpha_2}^{\alpha_2}$

HW1 $\quad T: V \to V$ linear mapping

$W = \text{span}\{\underbrace{x_1, \dots, x_k}_{\text{basis for } W}\} \subseteq V$ invariant $\iff T(x_i) \in W, \ 1 \le i \le k$

Ex 4 $\quad A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} : \mathbb{C}^3 \to \mathbb{C}^3$

columns $= v_1, v_2, v_3$. Notice that $\{v_1, v_2\}$ linearly independent

$v_3 = v_1 - \frac{1}{2} v_2$

$\text{span}\{v_1, v_2\} = \text{col}(A) = \text{Im}(A)$ invariant

$\text{span}\{v_1\} \subset \text{span}\{v_1, v_2\}$

$\downarrow$ not invariant because $A\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}3\\1\\2\end{bmatrix} \notin \text{span}(\{(1,1,0)\})$

Def $\quad T$ is triangularizable $\iff \exists$ a basis β for V such that $[T]_\beta^\beta$ is upper triangular

Theorem $\quad T: V \to V$ linear

$[T]_\beta^\beta$ is upper triangular where $\beta = \{x_1, \dots x_n\}$ is a basis for V

$\iff \{0\} \subset W_1 \subset W_2 \subset \dots \subset W_n = V$

where $W_i = \text{span}(\{x_1, \dots x_i\})$ $\quad 1 \le i \le n$ are invariant under T

$W_1 = \text{span}(\{x_1\})$

$W_2 = \text{span}(\{x_1, x_2\})$

Idea $\quad$ Say $\beta = \{x_1, x_2, x_3\}$ is a basis for $\mathbb{C}^3 = V$

$$[T]_\beta^\beta = \begin{bmatrix} \lambda_1 & a & b \\ 0 & \lambda_2 & c \\ 0 & 0 & \lambda_3 \end{bmatrix} \iff$$

① $T(x_1) = \lambda_1 x_1 \Rightarrow x_1 \in E_{\lambda_1}$, $T(x_1) \in \text{span}(\{x_1\}) \to \text{span}(\{x_1\})$ is invariant

② $T(x_2) = a x_1 + \lambda_2 x_2 \Rightarrow T(x_2) \in \text{span}(\{x_1, x_2\}) \Rightarrow$ with ①, $\text{span}(\{x_1, x_2\})$ is invariant

③ $T(x_3) = b x_1 + c x_2 + \lambda_3 x_3 \Rightarrow T(x_3) \in \text{span}(\{x_1, x_2, x_3\}) = \mathbb{C}^3 \Rightarrow$ with ① and ②, $\text{span}(\{x_1, x_2, x_3\})$ is invariant

Theorem $\quad$ Let V be a complex vector space and $T: V \to V$ be a linear mapping. Then T is triangularizable

Ex 5 $\quad$ Let $T: \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

example of

$\det(A - \lambda I) = (\lambda - 2)^2 \overset{\text{set}}{=} 0 \qquad \lambda = 2$

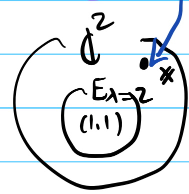
the theorem

$E_{\lambda=2} = \text{span}(\{(1,1)\})$. Since $\dim(E_{\lambda=2})=1$, T is not diagonalizable

We are looking for a basis β for $\mathbb{C}^2$ such that $\underset{A}{[T]_\beta^\beta} = \begin{bmatrix} 2 & a \\ 0 & 2 \end{bmatrix}$

must be an eigenvector of T

$\beta = \{(1,1), x\}$



$\dim(\mathbb{C}^2)=2$

$x \in \mathbb{C}^2 - E_{\lambda=2}$

$T(x) = A(x) = a(1,1) + 2x$

$\Rightarrow (A-2I)(x) = a(1,1)$

$\Rightarrow \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} x = a(1,1)$

Solve the system of linear eqns for $x = (x_1, x_2)$

say $a=1$,

$$\left[\begin{array}{cc|c} -1 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow -x_1 + x_2 = 1$$

Any (x_1, x_2) satisfying the eqn is correct

Choose $x_1 = 0$, $x_2 = 1$

Then $\beta = \{(1,1), (0,1)\}$ so that $[T]_\beta^\beta = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

Say $a=4$

$$\left[\begin{array}{cc|c} -1 & 1 & 4 \\ -1 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

$-x_1 + x_2 = 4$

If $x_1 = 0$, $x_2 = 4$

$\beta = \{(1,1), (0,4)\}$ $\quad [T]_\beta^\beta = \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}$

Summary

$T : \mathbb{C}^2 \to \mathbb{C}^2$ linear defined by A

diagonalizable

not diagonalizable $\Rightarrow$ one eigenvalue λ with $\dim(E_\lambda)=1$

there exists $\beta = \{x_1, x_2\}$ such that $[T]_\beta^\beta = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

eigenvector of T

$T(x_1) = \lambda x_1$

Case 1 two different eigenvalues λ and μ

in E_μ

$\exists\ \alpha = \{v_1, v_2\}$ such that

$[T]_\alpha^\alpha = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$

in E_λ

Case 2 one eigenvalue λ with $\dim(E_\lambda) = 2$

$\exists\ \alpha = \{w_1, w_2\} \subset E_\lambda$ such that

$$[T]_\alpha^\alpha = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

conclusion: $\underbrace{\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}}_{\text{from diagonalizable matrices}}, \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ are JCF of 2×2 matrices

$\boxed{\text{HW2}}$ Go back to Ex5 and show that $(A-2I)^2 = \mathbb{0}$.