Sec 5.3 Geometry in a complex vector space

Let V be a vector space over $\mathbb{C}$

Def A Hermitian inner product on V is a function $\langle \, , \, \rangle$ from $V \times V$ to $\mathbb{C}$ satisfying the following 3 conditions

① $\forall \, \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $\forall \, a, b \in \mathbb{C}$

$$\langle a\mathbf{u} + b\mathbf{v}, \mathbf{w} \rangle = a\langle \mathbf{u}, \mathbf{w} \rangle + b\langle \mathbf{v}, \mathbf{w} \rangle$$

② $\langle \mathbf{v}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{v} \rangle}$

③ $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, $\langle \mathbf{v}, \mathbf{v} \rangle = 0 \iff \mathbf{v} = \mathbf{0}$

Remark 1. $\langle a\mathbf{v}, \mathbf{w} \rangle = a\langle \mathbf{v}, \mathbf{w} \rangle$

$$\boxed{\langle \mathbf{v}, a\mathbf{w} \rangle} = \overline{\langle a\mathbf{w}, \mathbf{v} \rangle} \quad \text{by ②}$$

$$= \overline{a\langle \mathbf{w}, \mathbf{v} \rangle}$$

$$= \overline{a} \, \overline{\langle \mathbf{w}, \mathbf{v} \rangle} \quad \text{by} \quad \overline{z_1 z_2} = (\overline{z_1})(\overline{z_2})$$

$$\boxed{= \overline{a} \langle \mathbf{v}, \mathbf{w} \rangle}$$

For example

$$\langle \mathbf{v}, (2+i)\mathbf{w} \rangle = \overline{2+i} \langle \mathbf{v}, \mathbf{w} \rangle = (2-i) \langle \mathbf{v}, \mathbf{w} \rangle$$

Remark 2. $\underbrace{\langle \mathbf{v}, \mathbf{v} \rangle}_{\text{the same}} = \overline{\langle \mathbf{v}, \mathbf{v} \rangle}$ by ②

$\forall \, \mathbf{v} \in \mathbb{C}$ $\Rightarrow \langle \mathbf{v}, \mathbf{v} \rangle \in \mathbb{R}$

If $z = \overline{z}$, z is a real number

Standard Hermitian Inner Product on $\mathbb{C}^n$

$\langle \, , \, \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined by

Satisfies the 3 conditions $\longrightarrow$ $\langle \mathbf{x}, \mathbf{y} \rangle = x_1\overline{y_1} + x_2\overline{y_2} + \cdots + x_n\overline{y_n}$ where $\mathbf{x} = (x_1 \ldots x_n)$ and $\mathbf{y} = (y_1 \ldots y_n) \in \mathbb{C}^n$ complex numbers

Remark: If $x_i, y_i \in \mathbb{R}$, then $\langle \mathbf{x}, \mathbf{y} \rangle = x_1\overline{y_1} + \cdots + x_n\overline{y_n} = x_1 y_1 + \cdots + x_n y_n$

Ex1 $\langle \, , \, \rangle : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$

$$\langle (1+i, i), (-i, 2-i) \rangle = (1+i)\overline{(-i)} + (i)\overline{(2-i)}$$

$$= (1+i)(i) + (i)(2+i) = -2+3i \neq 0$$

Remark: $\{(1+i, i), (-i, 2-i)\}$ not orthogonal

Def $\| \cdot \|$ norm

$\forall \, \mathbf{v} \in \mathbb{C}^n$, $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \to$ a real number

Ex2 $\mathbf{v} = (1+i, i) \in \mathbb{C}^2$

$$\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle = \langle (1+i, i), (1+i, i) \rangle$$

$$= (1+i)\overline{(1+i)} + i\overline{i} = 3$$

$$\|\mathbf{v}\| = \sqrt{3}$$

Recall $\dim(\mathbb{C}^2)=2$

Ex 3 Construct an orthogonal basis for $\mathbb{C}^2$ using $\{(1+i, i), (-i, 2-i)\}$

Sol Let $w_1 = (1+i, i)$

$$w_2 = (-i, 2-i) - \frac{\langle(-i, 2-i), (1+i, i)\rangle}{\|(1+i, i)\|^2}(1+i, i)$$

$\|v\|^2 = \langle w, w\rangle$

$$= \frac{1}{3}(-1+2i, 3-i)$$

$\{w_1, w_2\}$ orthogonal

(*) Let $T: V \to V$ be a linear mapping where V is a complex vector space

Say $\alpha = \{w_1, \ldots w_n\}$ is an orthonormal basis for V.

Recall Sec 4.4

$\forall x \in V$ $\longrightarrow$

$x = \langle x, w_1\rangle w_1 + \cdots + \langle x, w_n\rangle w_n$

$$T(w_i) \in V$$

$$T(w_i) = \langle T(w_i), w_1\rangle w_1 + \langle T(w_i), w_2\rangle w_2 + \cdots + \langle T(w_i), w_n\rangle w_n$$

$$1 \le i \le n$$

$$[T(w_i)]_\alpha = (\langle T(w_i), w_1\rangle, \langle T(w_i), w_2\rangle, \ldots \langle T(w_i), w_n\rangle)$$

Say $\alpha = \{w_1, w_2\}$

$$[T]_\alpha^\alpha = \left[\, [T(w_1)]_\alpha \;\; [T(w_2)]_\alpha \,\right]$$

$$= \begin{bmatrix} \langle T(w_1), w_1\rangle & \langle T(w_2), w_1\rangle \\ \langle T(w_1), w_2\rangle & \langle T(w_2), w_2\rangle \end{bmatrix}$$

$x = (x_1, x_2)$

$T(x)$

$= A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Ex 4 Let $T: \mathbb{C}^2 \to \mathbb{C}^2$ be defined $A = \begin{bmatrix} i & 2i \\ 0 & 1-i \end{bmatrix}$

$\alpha = \left\{\left(\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right), \left(\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}$ orthonormal

Say $[T]_\alpha^\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find c.

Hermitian inner product

Sol: $$c = \langle T(w_1), w_2\rangle = \left\langle \underbrace{T\left(\tfrac{1}{\sqrt{2}}, \tfrac{i}{\sqrt{2}}\right)}, \left(\tfrac{i}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right)\right\rangle$$

$$T\left(\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right) = \begin{bmatrix} i & 2i \\ 0 & 1-i \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{i}{\sqrt{2}} + (2i)\left(\frac{i}{\sqrt{2}}\right) \\ 0 + (1-i)\left(\frac{i}{\sqrt{2}}\right) \end{bmatrix} = \begin{bmatrix} \frac{-2+i}{\sqrt{2}} \\ \frac{1+i}{\sqrt{2}} \end{bmatrix}$$

$$c = \left(\frac{-2+i}{\sqrt{2}}\right)\overline{\left(\frac{i}{\sqrt{2}}\right)} + \left(\frac{1+i}{\sqrt{2}}\right)\overline{\left(\frac{1}{\sqrt{2}}\right)} = \frac{2+3i}{2}$$

Def Let V be a finite dimensional Hermitian Inner Product Space.
Let α be an orthonormal basis for V.

Suppose $T: V \to V$ linear

The adjoint of T is a linear transformation denoted by T^* whose matrix w.r.t α is $[T^*]_\alpha^\alpha = \left(\overline{[T]_\alpha^\alpha}\right)^t$

Recall:
A: $n\times n$ matrix on $\mathbb{R}^n$
$\langle Ax, y\rangle = \langle x, A^t y\rangle$

A: $n\times n$ matrix on $\mathbb{C}^n$
$\langle Ax, y\rangle = \langle x, \overline{A}^t y\rangle$, $\overline{A}^t = (\overline{A})^t$

Def: $T: V \to V$ linear where V is a complex vector space. T is Hermitian or self-adjoint if $\langle T(x), y\rangle = \langle x, T(y)\rangle$ for any $x, y \in V$

If T is defined by a matrix A, then $A = A^*$

Exs $A = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$

(a) Find A^*
(b) Is A self-adjoint?

Sol

(a) $A^* = (\overline{A})^t$ $\quad \overline{A} = \begin{bmatrix} \overline{1} & \overline{1+i} \\ \overline{1-i} & \overline{2} \end{bmatrix} = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$

$\overline{A}^t = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$

(b) $A = A^*$ So A is Hermitian

A is self adjoint (Hermitian)

$\Leftrightarrow A = \begin{bmatrix} \text{real} & z \\ \overline{z} & \text{real} \end{bmatrix}$

Theorem The eigenvalues of a self-adjoint transformation T are "real"

Pf Suppose λ is eigenvalue of T and w is an "eigenvector" associated with λ. That is
$T(w) = \lambda w \quad (w \neq 0)$

$\langle T(w), w\rangle = \langle \lambda w, w\rangle = \lambda\langle w, w\rangle$

$\langle w, T(w)\rangle = \langle w, \lambda w\rangle = \overline{\lambda}\langle w, w\rangle$

Since T is self adjoint, $\langle T(w), w\rangle = \langle w, T(w)\rangle$,

$\lambda\langle w, w\rangle = \overline{\lambda}\langle w, w\rangle$

$\Rightarrow (\lambda - \overline{\lambda})\langle w, w\rangle = 0$

$\Rightarrow \lambda = \overline{\lambda}$

$\Rightarrow \lambda$ is a real number

HW Assume that $T: V \to V$ self-adjoint and $\lambda \neq \mu$ are eigenvalues of T. Then $E_\lambda \perp E_\mu$

Read the spectral theorem (the same as before)