1.5 1.5

#1 (a) $(-1,0,2)+2(1,1,0)-3(1,3,1) = (-2,-7,-1)$. $\ker(T) = \mathrm{span}\{(-2,-7,-1)\}$

(b) $\dim(\mathrm{Im}(T)) = \dim(\mathbb{R}^3) - \dim(\ker(T)) = 3-1=2$. ← 1.5

Since $\dim(W)=4 > \dim(\mathrm{Im}(T))$, T is not surjective ← 1.5

#2 pf Suppose $\dim(V) \neq \dim(W)$.

1.5 If $\dim(V) > \dim(W)$.
Then T is not injective.
Therefore, T is not an isomorphism

1.5 If $\dim(V) < \dim(W)$
Then T is not surjective.
Therefore, T is not an isomorphism

or

1.5 pf T is an isomorphism
$\Rightarrow$ T is injective and surjective
Since T is injective, $\dim(V) \leq \dim(W)$
Since T is surjective, $\dim(V) \geq \dim(W)$
1.5 Therefore, $\dim(V) = \dim(W)$.

#3 (a) $f(x) = x$, $k=2$

a correct counter-example 1.5

explaining why it is not linear 1.5

for (a) and (b)

$T(2f(x)) = x^2(2f(x)+x) = 2x^2 f(x) + x^3 = 2x^3 + x^3 = 3x^3$

$2T(f(x)) = 2(x^2 f(x) + x^3) = 2x^2 f(x) + 2x^3 = 4x^3$

When $x=1$, $T(2f(1)) = 3$ and $2T(f(1)) = 4$

Therefore, $T(2f(x)) \neq 2T(f(x))$ for all $x \in \mathbb{R}$. This implies that T is not linear

(b) Let $V = \mathbb{R}^2$. $\underbrace{S_1 = \{(1,0),(0,1)\}}_{\text{linearly independent}}$ and $\underbrace{S_2 = \{(1,0),(-2,1)\}}_{\text{linearly independent}}$

because they are not parallel.

$S_1 \cup S_2 = \{(1,0),(0,1),(-2,1)\}$ is linearly dependent

$(-2,1) = -2(1,0) + 1(0,1)$

#4 $\forall f,g \in S = \{f \in \mathrm{Span}\{e^x, e^{2x}, e^{3x}\} \mid f(0) = f'(0) = 0\}$, $\forall k \in \mathbb{R}$

(a) Since f and $g \in \mathrm{Span}\{e^x, e^{2x}, e^{3x}\}$ and $\mathrm{Span}\{e^x, e^{2x}, e^{3x}\}$ is a subspace,

$f+g \in \mathrm{Span}\{e^x, e^{2x}, e^{3x}\}$ and $kf \in \mathrm{Span}\{e^x, e^{2x}, e^{3x}\}$. — 1

Also $(kf)(0) = kf(0) = 0$ and $(kf)'(0) = kf'(0) = 0$ $\Rightarrow$ $kf \in S$ — 1

$(f+g)(0) = f(0)+g(0) = 0+0=0$ and $(f+g)'(0) = f'(0)+g'(0) = 0+0=0$ $\Rightarrow$ $f+g \in S$ — 1

(b) $\forall f \in S$. Then $f = a_1 e^x + a_2 e^{2x} + a_3 e^{3x}$ for some $a_1, a_2, a_3 \in \mathbb{R}$.

and $\begin{cases} a_1 + a_2 + a_3 = 0 & \leftarrow f(0) = 0 \\ a_1 + 2a_2 + 3a_3 = 0 & \leftarrow f'(0) = 0 \end{cases}$ 2

$$\left[\begin{array}{ccc|c}1&1&1&0\\1&2&3&0\end{array}\right] \Rightarrow \left[\begin{array}{ccc|c}1&1&1&0\\0&1&2&0\end{array}\right] \Rightarrow \begin{array}{l}a_1+a_2+a_3=0\\a_2+2a_3=0\end{array} \Rightarrow \begin{array}{l}\text{Let } a_3=t.\\ \text{Then } a_2=-2t\end{array}$$

$$a_1 = -(a_2+a_3) = t$$

$$(a_1, a_2, a_3) = (t, -2t, t) = t(1,-2,1) \leftarrow 1$$

$$S = \text{span}\{(e^x - 2e^{2x} + e^{3x})\} \leftarrow 1$$

#5 $T(w_1) = 1w_1 + 5w_2$ $\quad T(w_2) = -2w_1 + 3w_2$

(a) $\Rightarrow [T]_\alpha^\beta = \begin{bmatrix}1 & -2\\5 & 3\end{bmatrix}$ 1 $\quad \det\left([T]_\alpha^\beta\right) = 3-(-10) = 13 \neq 0 \Rightarrow [T]_\alpha^\beta$ is invertible

1 $\Rightarrow T$ is invertible

(b) $[T^{-1}]_\beta^\alpha = \left([T]_\alpha^\beta\right)^{-1}$ 1 $= \frac{1}{13}\begin{bmatrix}3 & 2\\-5 & 1\end{bmatrix} = \begin{bmatrix}\frac{3}{13} & \frac{2}{13}\\-\frac{5}{13} & \frac{1}{13}\end{bmatrix} \leftarrow$ 1 computing $[T^{-1}]_\beta^\alpha$

$$\left.\begin{array}{l}\Rightarrow T^{-1}(w_1) = \frac{3}{13}w_1 - \frac{5}{13}w_2\\ T^{-1}(w_2) = \frac{2}{13}w_1 - \frac{1}{13}w_2\end{array}\right\} 1$$

1

(c) $[T(w)]_\beta = [T]_\alpha^\beta [w]_\alpha \Rightarrow [w]_\alpha = \left([T]_\alpha^\beta\right)^{-1}[T(w)]_\beta = \begin{bmatrix}\frac{3}{13} & \frac{2}{13}\\-\frac{5}{13} & \frac{1}{13}\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}$

$$= \left(-\frac{3}{13}+\frac{2}{13}, \frac{5}{13}+\frac{1}{13}\right) = \left(-\frac{1}{13}, \frac{6}{13}\right) \leftarrow 1$$

computation

#6 There are many correct answers:

For example, $x = (1, 2)$, $y = (2, 3)$, $\alpha = 2$

1.5 for a correct example

$$\left.\begin{array}{l}2\cdot(x+'y) = 2(2,6) = (4, 18)\\ 2\cdot x +' 2\cdot y = (2,4)+'(4,6) = (8, 24)\end{array}\right\} \text{different.}$$

1.5 for explanation

Or The additive identity of $x = (x_1, x_2) = (1, 1) \in \mathbb{R}^2$: $(x_1,x_2)+'(1,1) = (x_1,x_2)$

But the additive inverse of $x = (0,0)$ doesn't exist because for any $x = (x_1, x_2)$, $x +' (0,0) = (0,0) \neq (1,1)$

Or . . .

#7 Let $a_1 \sin x + a_2 \sin(2x) = 0$ for all $x \in \mathbb{R}$

When $x = \frac{\pi}{2}$, $0 = a_1\sin(\frac{\pi}{2}) + a_2\sin(\pi) = a_1 \Rightarrow a_1 = 0$

When $x = \frac{\pi}{4}$, $0 = a_1\sin(\frac{\pi}{4}) + a_2\sin(\frac{\pi}{2}) = \frac{a_1}{\sqrt{2}} + a_2 \Rightarrow a_2 = 0$, since $a_1 = 0$

1.5 1.5

Therefore, there is no $(a_1, a_2) \neq (0,0)$ such that $a_1\sin x + a_2\sin(2x) = 0$ for all $x \in \mathbb{R}$.

Or Suppose $\{\sin x, \sin(2x)\}$ is linearly dependent.

1 Then $\sin(2x) = k\sin(x)$ for $k \neq 0$ for all $x \in \mathbb{R}$.

1 But when $x = \pi/2$, $\sin(2\cdot\frac{\pi}{2}) = 0$ and $\sin(\pi/2) = 1$

$0 = k \cdot 1 \leftarrow$ contradiction.

1 Therefore, $\{\sin x, \sin(2x)\}$ is linearly independent.

#8 (a) $[(3,4)]_\beta = [I]_\alpha^\beta [(3,4)]_\alpha = \begin{bmatrix}1&1\\1&-1\end{bmatrix}[(3,4)]_\alpha$ 1

Let $[(3,4)]_\alpha = (a,b)$. Then $(3,4) = a(2,1) + b(3,1)$

So $\begin{bmatrix}2&3\\1&1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix} \Rightarrow \begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}2&3\\1&1\end{bmatrix}^{-1}\begin{bmatrix}3\\4\end{bmatrix} = -\begin{bmatrix}1&-3\\-1&2\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}9\\-5\end{bmatrix}$

1 1

Therefore, $[(3,4)]_\beta = \begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}9\\-5\end{bmatrix} = \begin{bmatrix}4\\14\end{bmatrix}$

0.5 computation 0.5

(b) Since $[I]_\alpha^\beta = \begin{bmatrix}[(2,1)]_\beta & [(3,1)]_\beta\end{bmatrix} = \begin{bmatrix}1&1\\1&-1\end{bmatrix}$,

1.5

$(2,1) = w_1 + w_2$
$(3,1) = w_1 - w_2$
$\Rightarrow$
$2w_1 = (5,2) \Rightarrow w_1 = (\frac{5}{2}, 1)$
$2w_2 = (-1,0) \Rightarrow w_2 = (-\frac{1}{2}, 0)$

1.5

$\Downarrow$

or

1.5

$\begin{cases}[w_1\ w_2]\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}\\ [w_1\ w_2]\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}\end{cases} \Rightarrow [w_1\ w_2]\begin{bmatrix}1&1\\1&-1\end{bmatrix} = \begin{bmatrix}2&3\\1&1\end{bmatrix} \Rightarrow [w_1\ w_2] = \begin{bmatrix}2&3\\1&1\end{bmatrix}\begin{bmatrix}1&1\\1&-1\end{bmatrix}^{-1}$

$\Rightarrow [w_1\ w_2] = -\frac{1}{2}\begin{bmatrix}2&3\\1&1\end{bmatrix}\begin{bmatrix}-1&-1\\-1&1\end{bmatrix} = \begin{bmatrix}5/2&-1/2\\1&0\end{bmatrix}$

1 1 computing

#9 (a) $\det(T) = \det([T]_\alpha^\alpha) = -1 - 6 = -7$

(b) $[T]_{\alpha'}^{\alpha'} = ([I]_{\alpha'}^{\alpha})^{-1}[T]_\alpha^\alpha [I]_{\alpha'}^{\alpha}$. Since $[I]_{\alpha'}^{\alpha} = \begin{bmatrix}[w_1]_\alpha & [w_2]_\alpha\end{bmatrix} = \begin{bmatrix}2&1\\1&4\end{bmatrix}$,

$([I]_{\alpha'}^{\alpha})^{-1} = \frac{1}{7}\begin{bmatrix}4&-1\\-1&2\end{bmatrix}$

$[T]_{\alpha'}^{\alpha'} = \frac{1}{7}\begin{bmatrix}4&-1\\-1&2\end{bmatrix}\begin{bmatrix}1&3\\2&-1\end{bmatrix}\begin{bmatrix}2&1\\1&4\end{bmatrix} = \frac{1}{7}\begin{bmatrix}17&54\\1&-17\end{bmatrix} = \begin{bmatrix}17/7&54/7\\1/7&-17/7\end{bmatrix}$

(c) $T(2w_1 - 4w_2) = 2T(w_1) - 4T(w_2)$ — 1

$= 2(17/7\, w_1 + 1/7\, w_2) - 4(54/7\, w_1 - 17/7\, w_2)$

0.5 0.5

$= \frac{182}{7} w_1 + \frac{70}{7} w_2$