#1 $(-1,2,3) = 5v_1 + 8v_2 - 2v_3$ and $\|v_1\| = 2$.

$\langle (-1,2,3), v_1 \rangle = \langle 5v_1 + 8v_2 - 2v_3, v_1 \rangle = 5\langle v_1, v_1\rangle + 8\langle v_2, v_1\rangle - 2\langle v_3, v_1\rangle$

$= 5\langle v_1, v_1\rangle = 5\|v_1\|^2 = 5 \times 4 = 20$.

#2 For example, for $p(x) = x^3 \in P_3(\mathbb{R})$, $T(p(x)) = 3x^2+2 \in P_2(\mathbb{R})$

But $T(2x^3) = (2x^3)' + 2 = 6x^2 + 2$ and

$2T(x^3) = 2(3x^2+2) = 6x^2+4 \neq T(2x^3) \Rightarrow T$ is not linear

Remark: there are many different solutions

Marking Scheme: Choosing $p(x) \in P_3(\mathbb{R})$

Showing $\alpha T(p(x)) \neq T(\alpha p(x))$ for some α

or $T(p(x)+q(x)) \neq T(p(x)) + T(q(x))$

for some $p, q \in P_3(\mathbb{R})$

#3 (a) Choose a vector w such that w and $(1,3)$ are orthogonal.

For example, $w = (-3,1)$. Then $\langle (-3,1),(1,3)\rangle = -3+3=0$

Let $\beta = \{(1,3),(-3,1)\}$

$$[P_{(1,3)}]_\beta^\beta = \left[\, [P_{(1,3)}(1,3)]_\beta \quad [P_{(1,3)}(-3,1)]_\beta \,\right]$$

$$= \left[\, [(1,3)]_\beta \quad [0]_\beta \,\right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(b) $\ker(P_{(1,3)}) = \operatorname{span}\{(-3,1)\}$ $\quad \operatorname{Im}(P_{(1,3)}) = \operatorname{span}\{(1,3)\}$

(c) Since $[P_{(1,3)}]_\beta^\beta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, 1 and 0 are eigenvalues

and $P_{(1,3)}(1,3) = (1)(1,3)$ and $P_{(1,3)}(-3,1) = 0 = 0(-3,1)$

$E_{\lambda=1} = \operatorname{span}\{(1,3)\}$ or $\operatorname{Im}(P_{(1,3)})$

$E_{\lambda=0} = \operatorname{span}\{(-3,1)\}$ or $\ker(P_{(1,3)})$

Remark: Instead of $P_{(1,3)}(1,3) = (1)(1,3)$, you can find the eigenspace of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ first.

That is, $\ker\left(\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\right) = \operatorname{span}\{(1,0)\}$ the coordinates $(1)(1,3) + 0(-3,1)$

so $E_{\lambda=1} = \operatorname{span}\{(1,3)\}$

#4 (a) $[S(x)]_\beta = [S]_\alpha^\beta [x]_\alpha = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(b) $\det([S]_\alpha^\beta) = 1 - 2 = -1 \neq 0$ so $[S]_\alpha^\beta$ is invertible $\Rightarrow$ S is invertible.

$[S^{-1}]_\beta^\alpha = ([S]_\alpha^\beta)^{-1} = \frac{1}{1-2}\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$

(c) $[TS]_\alpha^\gamma = [T]_\beta^\gamma [S]_\alpha^\beta \Rightarrow [T]_\beta^\gamma = [TS]_\alpha^\gamma ([S]_\alpha^\beta)^{-1}$

$[T]_\beta^\gamma = \begin{bmatrix} 4 & 3 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

#5 $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $\det(A) = 1 \neq 0$ $\Rightarrow$ A is invertible

$\lambda = 1$ is only eigenvalue of A. So the algebraic multiplicity of $\lambda = 2$

But $A - I = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ $\Rightarrow$ $\dim(E_{\lambda=1}) = \dim(\ker(A-I)) = 1 \neq 2$

So A is not diagonalizable.

There are many different examples such as $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ $a \neq 0$, $b \neq 0$

#6 direct proof

Let $a_1 x + a_2 y = 0$. Since x and y are orthogonal, $a_1 = \langle 0, x \rangle = 0$ and $a_2 = \langle 0, y \rangle = 0$

So $\{x, y\}$ linearly independent

#6 Suppose $\{x, y\}$ is linearly dependent. Then $x = ky$ for some $k \neq 0$

Then $\langle x, y \rangle = \langle ky, y \rangle = k\langle y, y \rangle = k\|y\|^2 \neq 0$, since $k \neq 0$, $\|y\| \neq 0$.

Therefore, x and y are not orthogonal

#7 A basis of $\mathrm{span}(\{1, \sin x, \cos x\})$ is $\alpha = \{1, \sin x, \cos x\}$.

$[T(1)]_\alpha = (1, 2, 3)$, $[T(\sin x)]_\alpha = (0, 2, 3)$, and $[T(\cos x)]_\alpha = (0, 0, 2)$

the tutorial question.

So, $[T]_\alpha^\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 2 \end{bmatrix}$ and $\det([T]_\alpha^\alpha - \lambda I_3) = (1-\lambda)(2-\lambda)^2$.

The eigenvalues of $[T]_\alpha^\alpha$ are $\{1, 2\}$

When $\lambda = 1$, $[T]_\alpha^\alpha - I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 3 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{bmatrix}$

$\rightarrow$ $x_1 - \frac{1}{3}x_3 = 0$ and $x_2 + \frac{2}{3}x_3 = 0$

Let $x_3 = t$. Then $x_1 = \frac{1}{3}t$ and $x_2 = -\frac{2}{3}t$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}t \\ -\frac{2}{3}t \\ t \end{bmatrix} = \frac{1}{3}t\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ $E_{\lambda=1} = \mathrm{span}(\{1 - 2\sin x + 3\cos x\})$

coordinates of eigenvectors

When $\lambda = 2$, $[T]_\alpha^\alpha - 2I_3 = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\rightarrow \quad x_1 = 0, \quad x_2 = 0, \quad x_3 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad E_{\lambda=2} = \text{span}(\{\cos x\})$$

(a) $E_{\lambda=1} - \{0\}$ is the set of eigenvectors associated with eigenvalue $\lambda = 1$
$E_{\lambda=2} - \{0\}$ is the set of eigenvectors associated with eigenvalue $\lambda = 2$.

(b) $\dim(E_{\lambda=2}) = 1$ and the algebraic multiplicity of $\lambda=2$ is 2.

$\dim(E_{\lambda=2}) \neq 2 \quad \Rightarrow \quad T$ is not diagonalizable

#8 (a) $V^{\perp} = \ker\left(\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}\right)$ (or) Solve $x_1 + x_2 - x_3 + x_4 = 0$
$x_1 \qquad\qquad - x_4 = 0$

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 2 \end{bmatrix} \qquad \begin{matrix} x_1 - x_4 = 0 \\ x_2 - x_3 + 2x_4 = 0 \end{matrix}$$

Let $x_3 = t$ and $x_4 = s$.

Then $x_1 = x_4 = s$. $x_2 = x_3 - 2x_4 = t - 2s$

$(x_1, x_2, x_3, x_4) = (s, t-2s, t, s) = s(1,-2,0,1) + t(0,1,1,0)$

$V^{\perp} = \text{span}\{(1,-2,0,1), (0,1,1,0)\}$

(b) Let $w_1 = (1,-2,0,1)$.

$$w_2 = (0,1,1,0) - P_{(1,-2,0,1)}(0,1,1,0)$$

$$= (0,1,1,0) - \frac{\langle (0,1,1,0), (1,-2,0,1) \rangle}{\|(1,-2,0,1)\|^2}(1,-2,0,1)$$

$$= (0,1,1,0) + \frac{2}{6}(1,-2,0,1) = (0,1,1,0) + \frac{1}{3}(1,-2,0,1)$$

$$= \frac{1}{3}\left((0,3,3,0) + (1,-2,0,1)\right) = \frac{1}{3}(1,1,3,1)$$

$>$ $\{(1,-2,0,1), (1,1,3,1)\}$ orthogonal basis for $V^{\perp}$

(c) $\|(1,-2,0,1)\| = \sqrt{1+4+1} = \sqrt{6}$

$\|(1,1,3,1)\| = \sqrt{1+1+9+1} = \sqrt{12}$

$\left\{ \frac{1}{\sqrt{6}}(1,-2,0,1), \frac{1}{\sqrt{12}}(1,1,3,1) \right\}$ orthonormal basis for $V^{\perp}$.

#9 (a) $\langle (-1,0,1), (1,-1,2) \rangle = (-1)(1) + (0)(-1) + (1)(2) = -1+2 = 1 \neq 0$

So they are not orthogonal

(b) Method1 : find a orthogonal basis for W.

$w_1 = (-1,0,1)$ $\quad w_2 = (1,-1,2) - P_{(-1,0,1)}(1,-1,2)$

$$= (1,-1,2) - \frac{\langle (1,-1,2), (-1,0,1) \rangle}{\|(-1,0,1)\|^2}(-1,0,1)$$

$$= (1,-1,2) - \frac{1}{2}(-1,0,1) = \left(\frac{3}{2}, -1, \frac{3}{2}\right)$$

$\{(-1,0,1), (3,-2,3)\}$ orthogonal basis for W.

$$P_W(0,1,1) = \frac{\langle (0,1,1), (-1,0,1) \rangle}{\|(-1,0,1)\|^2}(-1,0,1) + \frac{\langle (0,1,1), (3,-2,3) \rangle}{\|(3,-2,3)\|^2}(3,-2,3)$$

$$= \frac{1}{2}(-1,0,1) + \frac{1}{22}(3,-2,3) = \frac{1}{22}\left((-11,0,11) + (3,-2,3)\right)$$

$$= \frac{1}{22}(-8,-2,14) = \left(-\frac{4}{11}, -\frac{1}{11}, \frac{7}{11}\right)$$

Method2 : $W^\perp = \mathrm{Ker}\left(\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}\right)$

$\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \end{bmatrix}$ $\qquad x_1 - x_3 = 0$, $\; x_2 - 3x_3 = 0$

Let $x_3 = t$. Then $x_2 = 3t$. $x_1 = t$ $\qquad W^\perp = \mathrm{span}\{(1,3,1)\}$

Let $(0,1,1) = x_1(-1,0,1) + x_2(1,-1,2) + x_3(1,3,1)$ $\quad (*)$

Then $P_W(0,1,1) = x_1(-1,0,1) + x_2(1,-1,2)$

Solve $(*)$ for (x_1, x_2, x_3).

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & -1 & 3 & 1 \\ 1 & 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5/11 \\ 0 & 1 & 0 & 1/11 \\ 0 & 0 & 1 & 4/11 \end{array}\right]$$

Therefore $P_W(0,1,1) = \frac{5}{11}(-1,0,1) + \frac{1}{11}(1,-1,2)$

$$= \left(\frac{-4}{11}, \frac{-1}{11}, \frac{7}{11}\right)$$