(3pts) Sec1.2 #2(b). Show that $V=\{f:\mathbb{R}\to\mathbb{R} \mid f(x)=f(-x) \text{ for all } x\in\mathbb{R}\}$ is a subspace of $F(\mathbb{R})$.

Sol (1) (i) $f(x)=0\in V$ because $f(x)=f(-x)=0$, not empty

(1) (ii) $\forall f,g\in V$, $(f+g)(x)=f(x)+g(x)=f(-x)+g(-x)=(f+g)(-x)$, so $f+g\in V$

(1) (iii) $\forall f\in V, \forall c\in\mathbb{R}$, $(cf)(x)=cf(x)=cf(-x)=(cf)(-x)$, so $cf\in V$

(3pts) Sec1.3 #3 Let $S=\{1, 1+x, 1+x+x^2\}$. Show that $\mathrm{span}(S)=P_2(\mathbb{R})$

Sol Since $\mathrm{span}(S)=\{a+b(1+x)+c(1+x+x^2) \mid a,b,c\in\mathbb{R}\}$, $\mathrm{span}(S)\subseteq P_2(\mathbb{R})$. (1)

$\forall p(x)=\alpha x^2+\beta x+\gamma\in P_2(\mathbb{R})$. Solve $a+b(1+x)+c(1+x+x^2)=\alpha x^2+\beta x+\gamma$ for a,b,c

$$\Rightarrow \begin{cases} (a+b+c)=\gamma & (1)\\ b+c=\beta & (2)\\ c=\alpha & (3)\end{cases} \Rightarrow c=\alpha,\ b=\beta-\alpha,\ a=\gamma-\beta$$

So $p(x)=(\gamma-\beta)+(\beta-\alpha)(1+x)+\alpha(1+x+x^2)\in\mathrm{span}(S)$

Therefore, $\mathrm{span}(S)=P_2(\mathbb{R})$

(2)

Sec1.4 #8 Let W_1 and W_2 be subspaces of a vector space satisfying $W_1\cap W_2=\{0\}$

(5pts) Show that if $S_1\subset W_1$ and $S_2\subset W_2$ are linearly independent, then $S_1\cup S_2$ is linearly independent.

Sol

Say $S_1\cup S_2=\{x_1,\dots,x_n, y_1,\dots,y_m\}$ $x_i\in S_1$, $y_j\in S_2$, $1\le i\le n$, $1\le j\le m$.

(1) Let $a_1x_1+a_2x_2+\cdots+a_nx_n+b_1y_1+\cdots+b_my_m=0$ where $a_i, b_j\in\mathbb{R}$

$a_1x_1+a_2x_2+\cdots+a_nx_n=-b_1y_1-b_2y_2-\cdots-b_my_m$ $1\le i\le n$, $1\le j\le m$.

(1) → Since W_1 and W_2 are subspaces, $a_1x_1+\cdots+a_nx_n\in W_1$ and $-b_1y_1-b_2y_2-\cdots-b_my_m\in W_2$

(1) → This implies that $a_1x_1+\cdots+a_nx_n=-b_1y_1-b_2y_2-\cdots-b_my_m\in W_1\cap W_2$

(1) → Since $W_1\cap W_2=\{0\}$, $a_1x_1+\cdots+a_nx_n=0$ and $-b_1y_1-\cdots-b_my_m=0$

(1) $\Rightarrow a_1=\cdots=a_n=0$ and $b_1=\cdots=b_m=0$ because S_1 and S_2 are linearly independent sets

$\Rightarrow S_1\cup S_2$ is linearly independent

(5pts)

Sec1.6 #2(d) Let $W=\{p\in P_3(\mathbb{R}) \mid p(2)=p(-1)=0\}$. Find a basis for W. What is the dimension of W?

Sol

$\forall p \in P_3(\mathbb{R})$. Then $p = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ for some a_0, a_1, a_2, and $a_3 \in \mathbb{R}$, and

1 $\begin{cases} p(2) = a_0 + 2a_1 + 4a_2 + 8a_3 = 0 \\ p(-1) = a_0 - a_1 + a_2 - a_3 = 0 \end{cases}$ } solve the homogenous system of linear equations for a_0, a_1, a_2, and a_3.

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & -1 & 1 & -1 \end{bmatrix} \xrightarrow[R_2 \to -R_1 + R_2]{} \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & -3 & -3 & -9 \end{bmatrix} \xrightarrow[R_2 \to -\frac{1}{3}R_2]{} \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 1 & 3 \end{bmatrix} \text{ REF}$$

free variables

2

Solving for a_0, a_1, a_2 and a_3

① $a_0 + 2a_1 + 4a_2 + 8a_3 = 0$
② $a_1 + a_2 + 3a_3 = 0$

$\Rightarrow$ Let $a_2 = t$ and $a_3 = s$.

From ②, $a_1 = -a_2 - 3a_3 = -t - 3s$

From ①, $a_0 = -2a_1 - 4a_2 - 8a_3 = -2(-t-3s) - 4t - 8s = -2t - 2s$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -2t-2s \\ -t-3s \\ t \\ s \end{bmatrix} = t \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

2 $\begin{cases} \text{Null}(A) = \text{span}\left(\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}\right) \\ \{p \in P_3(\mathbb{R}) \mid p(2) = p(-1) = 0\} = \text{span}(\underbrace{\{-2 - x + x^2, -2 - 3x + x^3\}}_{\text{a basis}}) \end{cases}$ and $\dim(W) = 2$

TUT1 Q3

(3pts) Let V be a real vector space and W_1 and W_2 be subspaces of V. Show that every vector of $W_1 \oplus W_2$ is expressed uniquely.

Sol Let $x_1 + y_1 = x_2 + y_2$, where $x_1, x_2 \in W_1$ and $y_1, y_2 \in W_2$. 1

Then $x_1 - x_2 = y_2 - y_1$ — (1)

Since W_1 and W_2 are subspaces of V, $x_1 - x_2 \in W_1$ and $y_2 - y_1 \in W_2$. 1

Also, by (1) $y_2 - y_1 \in W_1$ and $x_1 - x_2 \in W_2$, which means $x_1 - x_2, \; y_2 - y_1 \in W_1 \cap W_2$. 1

Since $W_1 \cap W_2 = \{0\}$, $x_1 - x_2 = 0$ and $y_2 - y_1 = 0$ 1

Therefore, $x_1 = x_2$ and $y_1 = y_2$. 1

(3 pts)

TUT2 Q3 Let V be real vector space and $\alpha = \{x_1, \dots x_n\}$ be a basis for V. Show that the function $[\cdot]_\alpha : V \to \mathbb{R}^n$ defined by $[x]_\alpha = (a_1, \dots a_n)$ where $x = a_1 x_1 + \cdots + a_n x_n$, is a linear transformation.

Sol

(i) $\forall x = a_1 x_1 + \cdots + a_n x_n, \; y = b_1 x_1 + \cdots + b_n x_n \in V$,

1.5

$$[x+y]_\alpha = [(a_1+b_1)x_1 + \cdots + (a_n+b_n)x_n]_\alpha = (a_1+b_1, \cdots, a_n+b_n)$$

$$= (a_1, \dots, a_n) + (b_1, \dots, b_n) = [x]_\alpha + [y]_\alpha$$

(ii) $\forall\, x = a_1x_1 + \cdots + a_nx_n \in V,\ \forall\, t \in \mathbb{R}$

1.5

$$[tx]_\alpha = [(ta_1)x_1 + \cdots + (ta_n)x_n] = (ta_1, \ldots, ta_n)$$
$$= t(a_1, \ldots, a_n) = t[x]_\alpha$$

(5pts)

TUT2 Q2 Let $S = \{f \in \text{span}(\{e^x, e^{2x}, e^{3x}\}) \mid f(0) = f'(0) = 0\}$. Find a basis for S. What is the dimension of S?

Sol $\forall f \in \{f \in \text{span}\{e^x, e^{2x}, e^{3x}\} \mid f(0) = f'(0) = 0\} \overset{\text{let}}{=} S$

Then $f(x) = ae^x + be^{2x} + ce^{3x}$ for some $a, b, c \in \mathbb{R}$, and

① $f(0) = a + b + c = 0$

② $f'(x) = ae^x + 2be^{2x} + 3ce^{3x} = 0 \Rightarrow f'(0) = a + 2b + 3c = 0$

1

Solve $\begin{cases} a+b+c=0 \\ a+2b+3c=0 \end{cases}$ for a, b, and c. $\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ RREF

$\Rightarrow \begin{matrix} a - c = 0 \\ b + 2c = 0 \end{matrix} \Rightarrow \begin{matrix} a = c \\ b = -2c \end{matrix}$

2 solve for a, b, c

2

So $f(x) = ce^x - 2ce^{2x} + ce^{3x} = c(e^x - 2e^{2x} + e^{3x})$ for any $c \in \mathbb{R}$

This implies that $S \subseteq \text{span}(\{e^x - 2e^{2x} + e^{3x}\})$. Since $e^x - 2e^{2x} + e^{3x} \in S$,

$$\{f \in \text{span}\{e^x, e^{2x}, e^{3x}\} \mid f(0) = f'(0) = 0\} = \text{span}\{e^x - 2e^{2x} + e^{3x}\}$$

A basis of $S = \{e^x - 2e^{2x} + e^{3x}\}$ and $\dim(S) = 1$

2pts

Sec 2.1 #5(a) Let $V = C^\infty(\mathbb{R})$, and let $D: V \to V$ be the mapping $D(f) = f'$. Show that D is linear mapping.

Sol $\forall f, g \in V,\ D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$ 1

$\forall f \in V, c \in \mathbb{R},\ D(cf) = (cf)' = cf' = cD(f)$ 1

1 pts

A question from Q1 set

The set $\alpha = \{5x+7, 2x-1\}$ is a basis for $P_1(\mathbb{R})$. Suppose $[p(x)]_\alpha = (-1, 5)$. Find $p(x)$

Sol $p(x) = -(5x+7) + 5(2x-1) = 5x - 12$ 1

Sec 1.1 #7(c) Let $F(\mathbb{R})$ be the set of functions from $\mathbb{R}$ to $\mathbb{R}$.

2pts

Is $F(\mathbb{R})$ a vector space with the operations: ① $f +' g = f \circ g$ and ② $c f = cf$?

① for a correct example

① for computing

There are infinitely many correct answers.

Sol No. $f +' g \neq g +' f$, For example, $f(x) = x^2$, $g(x) = x + 1$.

$(f +' g)(x) = (f \circ g)(x) = f(g(x)) = (x+1)^2 \quad (g +' f)(x) = g(f(x)) = x^2 + 1$

3pts

Sec 2.1 #11 Let $V=\mathbb{R}^2$ and $W=P_3(\mathbb{R})$. If a linear transformation T satisfies $T((1,1)) = x+x^2$ and $T((3,0)) = x-x^3$, what is $T((2,3))$?

Sol let $(2,3) = a(1,1)+b(3,0)$ — (1)

Then $a+3b=2 \Rightarrow 3b = 2-a = 2-3 = -1$

$a=3 \qquad b=-\frac{1}{3}$ (0.5)

$T((2,3)) = T\left(3(1,1)-\frac{1}{3}(3,0)\right) = 3T((1,1))-\frac{1}{3}T((3,0))$ (1)

$= 3(x+x^2)-\frac{1}{3}(x-x^3) = 3x+3x^2-\frac{1}{3}x+\frac{1}{3}x^3$ (0.5)

$= \frac{1}{3}x^3+3x^2+\frac{8}{3}x$