Sec 4.3 Geometry in $\mathbb{R}^n$

$\forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\quad \mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1 \dots y_n)$

The dot product (= the standard inner product) of $\mathbf{x}$ and $\mathbf{y}$,

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n \in \mathbb{R}$$

For example, $\mathbf{x} = (2, 1, -1)$, $\mathbf{y} = (1, 0, 5)$

$$\langle \mathbf{x}, \mathbf{y} \rangle = (2)(1) + (1)(0) + (-1)(5) = 2 - 5 = -3 \in \mathbb{R}$$

The dot product has the following properties:

$\forall\, \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, $k \in \mathbb{R}$,

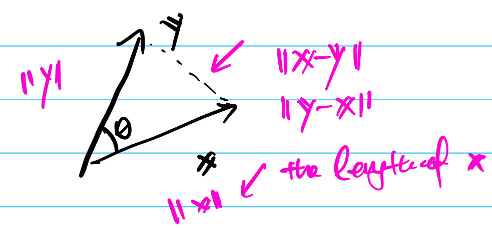
1. $\langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|^2 = x_1^2 + \cdots + x_n^2$, $\quad \mathbf{x} = (x_1, \dots x_n)$
   $(\langle \mathbf{x}, \mathbf{x} \rangle \geq 0)$
   $\uparrow$ when $\mathbf{x} = \mathbf{0}$
2. $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$
3. $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$
4. $k\langle \mathbf{x}, \mathbf{y} \rangle = \langle k\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, k\mathbf{y} \rangle$
   $2\langle (2,1), (3,-1) \rangle = \langle 2(2,1), (3,-1) \rangle$

$$\|\mathbf{x}+\mathbf{y}\|^2 = \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle$$
$$= \|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2$$

Geometric Meaning of the Dot Product

In $\mathbb{R}^2$,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\|\|\mathbf{y}\|\cos\theta \quad \text{where } \theta = \sphericalangle(\mathbf{x}, \mathbf{y}) \text{ (angle)} \quad 0 \leq \theta \leq \pi$$



by the cosine law

How do we define θ (the angle) between $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$

$$\cos\theta = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|\|\mathbf{y}\|}$$

$$\theta = \cos^{-1}\left(\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|\|\mathbf{y}\|}\right) = \cos^{-1}\left(\left\langle \frac{\mathbf{x}}{\|\mathbf{x}\|}, \frac{\mathbf{y}}{\|\mathbf{y}\|} \right\rangle\right)$$

unit vectors

For example,

Q: Find the angle between $\mathbf{x} = (1, 1, 1, 1)$ and $\mathbf{y} = (0, 1, 0, 1)$

A: $\theta = \cos^{-1}\left(\frac{2}{2\sqrt{2}}\right) = \underbrace{\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)}_{\text{correct answer too}}$ $\theta = \frac{\pi}{4}$

Remark: $\cos\theta = 0 \Leftrightarrow \theta = \pi/2 \ (0 \le \theta \le \pi) \Leftrightarrow \langle x, y \rangle = 0$

Def Two non-zero vectors $x, y \in \mathbb{R}^n$ are said to be <u>orthogonal</u> if $\langle x, y \rangle = 0$.

Ex1 Let $x = (2, 3, -1)$ and $y = (-1, a, 1)$.
Suppose x and y are orthogonal. Find a.
<u>Sol</u> $(2)(-1) + 3(a) + (-1)(1) = 0$ Solve it for a
Then $a = 1$

Ex2. Let $a = (1, 1)$. Find all vectors which are orthogonal to a.
<u>Ans</u> $\{x \in \mathbb{R}^2 \mid \langle x, a \rangle = 0\}$

<u>**Theorem**</u> If non-zero vectors x and $y \in \mathbb{R}^n$ are orthogonal, then $\{x, y\}$ is linearly independent.

pf $ax + by = 0$, where $a, b \in \mathbb{R}$.

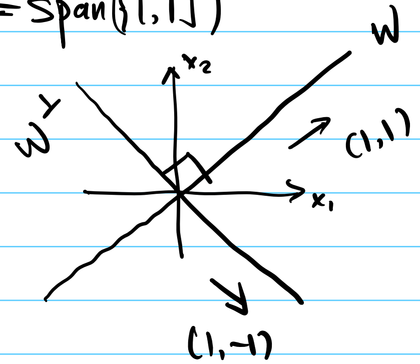
$0 = \langle ax + by, x \rangle = a\langle x, x \rangle + b\langle y, x \rangle$
0 vector
$= a\langle x, x \rangle$ because $\langle y, x \rangle = 0$
Since $x \neq 0$, $\langle x, x \rangle \neq 0$, so $a = 0$

HW: Show that $b = 0$
try $\langle ax + by, y \rangle$.

sec 4.4 Orthogonal Projections and the Gram-Schmidt Process

Def Let $W \subset \mathbb{R}^n$ be a subspace
$W^\perp = \{x \in \mathbb{R}^n \mid \langle x, w \rangle = 0 \text{ for all } w \in W\}$
we call $W^\perp$ the orthogonal complement of W or W perp

Ex1 $W = \text{span}(\{1, 1\})$



Let $x = (x_1, x_2)$
$0 = \langle x, a \rangle = x_1 + x_2$
for example, we can choose $(1, -1)$ as a basis for $W^\perp$
$W^\perp = \text{span}(\{(1, -1)\})$

Notice that $W^{\perp} \cap W = \{0\}$

$W^{\perp}$ is a subspace of $\mathbb{R}^2$

$W \oplus W^{\perp} = \mathbb{R}^2$

Remark of the definition of $W^{\perp}$

Let W be a subspace of $\mathbb{R}^n$.

$W = \text{span}(\underbrace{\{a_1, a_2, \dots a_k\}}_{\text{linearly independent}})$, so, a basis for W

Then $W^{\perp} = \{x \in \mathbb{R}^n \mid \langle x, a_i \rangle \overset{\shortparallel}{0} \quad 1 \le i \le k\} \overset{\text{let}}{=} W^*$

Pf Since $\{a_1, \dots a_k\} \subset W$, $W^{\perp} \subseteq W^*$ (HW: write it in detail)

Claim $W^* \subseteq W^{\perp}$

$\forall x \in W^*$, $\forall w \in W$ $w = \sum_{i=1}^{k} t_i a_i$

$\langle x, w \rangle = \langle x, \sum_{i=1}^{k} t_i a_i \rangle = \sum_{i=1}^{k} t_i \underbrace{\langle x_i, a_i \rangle}_{\shortparallel \atop 0} = 0$

$x \in W^{\perp}$, Therefore, $W^* \subseteq W^{\perp}$