4pts Question 1 Let $T: P_2(\mathbb{R}) \to \mathbb{R}^2$ be linear with bases $\alpha = \{P_1(x), P_2(x), P_3(x)\}$ and $\beta = \{(2,3), (-1,5)\}$ for $P_2(\mathbb{R})$ and $\mathbb{R}^2$ respectively. Suppose $[T]_\alpha^\beta = \begin{bmatrix} 1 & 4 & 3 \\ -1 & 1 & 5 \end{bmatrix}$.

(a) Find $T(P_3(x))$.

Sol $[T(P_3(x))]_\beta = (3,5) \Rightarrow T(P_3(x)) = 3(2,3) + 5(-1,5) = (1, 34)$ 1

(b) Suppose $p(x) = 3P_1(x) + P_2(x) - P_3(x)$. Find $T(p(x))$

Sol $[T(p(x))]_\beta = \underbrace{\begin{bmatrix} 1 & 4 & 3 \\ -1 & 1 & 5 \end{bmatrix}[p(x)]_\alpha}_{1} = \begin{bmatrix} 1 & 4 & 3 \\ -1 & 1 & 3 \end{bmatrix}\underbrace{\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}}_{1} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$ 0.5

$\Rightarrow T(p(x)) = 4(2,3) - 7(-1,5) = (15, -23)$ 0.5

4pts Question 2 Let $[\cdot]_\alpha : V \to \mathbb{R}^n$ be defined by $[x]_\alpha$, where $\alpha = \{v_1, \dots v_n\}$ is a basis for V. Show that $[\cdot]_\alpha$ is injective and surjective

pf $\forall x \in \ker([\cdot]_\alpha)$ Then $[x]_\alpha = (\underbrace{0, \dots, 0}_{n}) \Rightarrow x = 0v_1 + \dots + 0v_n = 0_V \Rightarrow [\cdot]_\alpha$ is injective 1 1

Since $\underbrace{\dim(V) = n = \dim(\mathbb{R}^n)}_{1}$ and $\underbrace{[\cdot]_\alpha \text{ is injective}}_{1}$, $[\cdot]_\alpha$ is surjective.

Or

$\forall (x_1, \dots x_n) \in \mathbb{R}^n$, $x_1v_1 + x_2v_2 + \dots + x_nv_n \in V$ 1 and $[x_1v_1 + x_2v_2 + \dots + x_nv_n]_\alpha = (x_1, x_2, \dots, x_n)$ 1

So $[\cdot]_\alpha$ is surjective

Since $\underbrace{\dim(V) = n = \dim(\mathbb{R}^n)}_{1}$ and $\underbrace{[\cdot]_\alpha \text{ is surjective}}_{1}$, $[\cdot]_\alpha$ is injective 1

3pts Sec 2.2 #3(c)

Let V be the vector space of $C(\mathbb{R})$ spanned by $\sin(x)$ and $\cos(x)$. Define $D: V \to V$ by $D(f(x)) = f'(x)$. What is the matrix of D with respect to the basis $\{\sin x, \cos x\}$.

1

Sol Let $\alpha = \{\sin x, \cos x\}$. Then $[D]_\alpha^\alpha = [[D(\sin x)]_\alpha \ [D(\cos x)]_\alpha] = [[\cos x]_\alpha \ [-\sin x]_\alpha]$

$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, since $\cos x = (0)\sin x + (1)\cos x$ and $-\sin x = (-1)\sin x + (0)\cos x$

2

4pts Sec 2.2 #6

Let V be a vector space of dimension n and let $\alpha = \{v_1, \dots, v_n\}$ be a basis for V

(a) Let $c \in \mathbb{R}$ be fixed. What is the matrix $[cI]_\alpha^\alpha$, where I is the identity transformation of V?

Sol $[cI]_\alpha^\alpha = [[cI(v_1)]_\alpha \dots [cI(v_n)]_\alpha] = [[cv_1]_\alpha \dots [cv_n]_\alpha] = \begin{bmatrix} c & 0 & 0 \\ 0 & c & \vdots \\ 0 & 0 & c \end{bmatrix} = cI$

1 1

(b) Let $T: V \to V$ be defined by $T(a_1 w_1 + \cdots + a_n w_n) = a_1 w_1 + \cdots + a_k w_k$ for $k < n$ fixed.

What is $[T]_\alpha^\alpha$?

sol $[T]_\alpha^\alpha = \left[[T(w_1)]_\alpha \cdots [T(w_n)]_\alpha \right]$ Since $w_i = (0)w_1 + \cdots + (1)w_i + \cdots + (0)w_n$,

$T(w_i) = T(0w_1 + \cdots + w_i + \cdots + 0w_n)$

$= \begin{cases} w_i & \text{if } i \le k \\ 0 & \text{if } i > k \end{cases}$ 1

Therefore, $[T]_\alpha^\alpha = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \vdots \\ \vdots & \\ 0 & 0 \end{bmatrix}}_{k} \underbrace{\begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}}_{n-k}$ 1

5pts

Sec 2.3 # 1(d)

Let $T: \mathbb{R}^4 \to P_2(\mathbb{R})$ defined by $T(a_1, \ldots a_4) = (a_1 + a_2) + (a_2 + a_3)x + (a_3 + a_4)x^2$

Find a basis for $\mathbb{R}^4$ so that the first $\dim(\ker T)$ vectors are a basis for $\ker(T)$.

Sol Let $[T]$ be the standard matrix for T. That is, $[T] = \left[[T(e_1)]_S \cdots [T(e_4)]_S \right]$ where $S = \{1, x, x^2\}$

1 $T(e_1) = T(1,0,0,0) = 1$, $T(e_2) = T(0,1,0,0) = 1 + x$

$T(e_3) = T(0,0,1,0) = x + x^2$, $T(e_4) = T(0,0,0,1) = x^2$

$\Rightarrow [T(e_1)]_S = (1,0,0)$, $[T(e_2)]_S = (1,1,0)$, $[T(e_3)]_S = (0,1,1)$, $[T(e_4)]_S = (0,0,1)$

1 $\Rightarrow [T] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ ← a ref $\rightarrow$ $\begin{aligned} x_1 + x_2 &= 0 \\ x_2 + x_3 &= 0 \\ x_3 + x_4 &= 0 \end{aligned}$ Let $x_4 = t$. Then $x_3 = -t$, $x_2 = t$, and $x_1 = -t$

basic variables free variable

So $\mathrm{Null}([T]) = \ker([T]) = \mathrm{span}(\{(-1, 1, -1, 1)\})$ 1

or $\Rightarrow \ker(T) = \mathrm{span}(\{-e_1 + e_2 - e_3 + e_4\}) = \mathrm{span}(\underbrace{\{(-1, 1, -1, 1)\}}_{\text{a basis}})$

Construct a basis $\alpha = \{(-1, 1, -1, 1), w_1, w_2, w_3\}$ for $\mathbb{R}^4$

2 Since the first three column vectors are linearly independent, choose $w_i \in \mathbb{R}^4$ such that $[T][w_i] =$ the i-th column vectors of $[T]$.

Then $\{[w_1] = (1,0,0,0), [w_2] = (0,1,0,0), [w_3] = (0,0,1,0)\}$ linearly independent

So $\{w_1 = (1,0,0,0), w_2 = (0,1,0,0), w_3 = (0,0,1,0)\}$ //

4pts

Sec 2.4 # 1(c)

Let $T: P_2(\mathbb{R}) \to \mathbb{R}^2$ be defined by $T(p(x)) = (p(0), p'(0))$. Is T injective, surjective, both or neither

sol Since $\dim(\underbrace{P_2(\mathbb{R})}_{\text{domain}}) = 3 > \dim(\underbrace{\mathbb{R}^2}_{\text{co-domain}}) = 2$, T is not injective 1

1 for explaining

$\forall (a,b) \in \mathbb{R}^2$, if T is surjective, $\exists\, p(x) = a_0 + a_1 x + a_2 x^2$ such that

$T(p(x)) = (p(0), p'(0)) = (a,b)$

That is, $p(0) = a_0 = a$ and
$p'(0) = a_1 = b$. from $p'(x) = a_1 + 2a_2 x$,

① explaining

Since $p(x) = a + bx \in P_2(\mathbb{R})$ and $T(p(x)) = (a, b)$, T is surjective ①

4 pts

sec2.4 #3

Let T be the linear transformation from V to W, $\dim(V)=4$ and $\dim(W)=3$, whose matrix

$$[T]_\alpha^\beta = \begin{bmatrix} 2 & -1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

with respect to the bases $\alpha = \{v_1, \ldots, v_4\}$ for V and $\beta = \{w_1, w_2, w_3\}$ for W.

a) or b) or c)

Determine $T^{-1}(\{w\})$ for the following:

a) $w = 0$ b) $w = 4w_1 + 2w_2 + w_3$ c) $w = w_3$

sol To find $T^{-1}(\{w\})$, first find $([T]_\alpha^\beta)^{-1}([w]_\beta)$. That is, solve the system of linear equations : $[T]_\alpha^\beta [x]_\alpha = [w]_\beta$

Since the coefficient matrix is the same for all a), b) and c),
solve the 3 systems of linear equations $[T]_\alpha^\beta [x]_\alpha = [0]_\beta$, $[T]_\alpha^\beta [x]_\alpha = [4w_1 + 2w_2 + w_3]_\beta$, and $[T]_\alpha^\beta [x]_\alpha = [w_3]_\beta$.

① for a ref

$$\left[\begin{array}{cccc|ccc} 2 & -1 & 1 & 1 & 0 & 4 & 0 \\ 0 & 1 & 2 & -1 & 0 & 2 & 0 \\ 1 & -1 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|ccc} 1 & -1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 & 0 & 2 & 0 \\ 2 & -1 & 1 & 1 & 0 & 4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|ccc} 1 & -1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 & 0 & 2 & 0 \\ 0 & 1 & 1 & -1 & 0 & 2 & -2 \end{array}\right] \sim \left[\begin{array}{cccc|ccc} 1 & -1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 \end{array}\right]$$

REF

①

a) $x_1 - x_2 + x_4 = 0$
$x_2 + 2x_3 - x_4 = 0$
$x_3 = 0$
$\Downarrow$
Let $x_4 = t$. Then $x_2 = t$, $x_1 = 0$
$\Downarrow$
① $([T]_\alpha^\beta)^{-1}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right) = \text{span}(\{(0,1,0,1)\})$
$\Downarrow$
① $T^{-1}(\{0\}) = \text{span}(\{v_2 + v_4\})$

b) $x_1 - x_2 + x_4 = 1$
$x_2 + 2x_3 - x_4 = 2$
$x_3 = 0$
$\Downarrow$
Let $x_4 = t$. Then $x_2 = 2+t$, $x_1 = 3$
$\Downarrow$
$([T]_\alpha^\beta)^{-1}\left(\begin{bmatrix}4\\2\\1\end{bmatrix}\right) = \left\{t\begin{bmatrix}0\\1\\0\\1\end{bmatrix} + \begin{bmatrix}3\\2\\0\\0\end{bmatrix} \,\middle|\, t \in \mathbb{R}\right\}$
$\Downarrow$
$T^{-1}(\{4w_1 + 2w_2 + w_3\})$
$= \{t(v_2 + v_4) + 3v_1 + 2v_2 \mid t \in \mathbb{R}\}$

c) $x_1 - x_2 + x_4 = 1$
$x_2 + 2x_3 - x_4 = 0$
$x_3 = 2$
$\Downarrow$
Let $x_4 = t$. Then $x_2 = t$, $x_1 = 1$
$\Downarrow$
$([T]_\alpha^\beta)^{-1}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \left\{t\begin{bmatrix}0\\1\\0\\1\end{bmatrix} + \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \,\middle|\, t \in \mathbb{R}\right\}$
$\Downarrow$
$T^{-1}(\{w_3\})$
$= \{t(v_2 + v_4) + v_1 \mid t \in \mathbb{R}\}$