Sec 1.6 Bases and Dimension

Def Let V be a vector space and $S \subseteq V$

S is called a basis for V $\Leftrightarrow$ $V = \text{span}(S)$ and S is linearly independent

Ex 1 $\mathbb{R}^n = V$. A basis of $\mathbb{R}^n = \{e_1, \ldots e_n\}$ where $e_i = (0, \ldots 1, \cdots 0)$

$\uparrow$ ith place

For example, $V = \mathbb{R}^2$ $\{\underbrace{(1,0)}_{= e_1}, \underbrace{(0,1)}_{= e_2}\}$ is a basis for $\mathbb{R}^2$.

Remark : Basis is not unique

For example, $\mathbb{R}^2 = \text{span}\{e_1, e_2\} = \text{span}\{\underbrace{(1,1), (-1,1)}_{\text{H.W: linearly independent}}\}$

Ex 2 $P_n(\mathbb{R}) = \text{span}\{\underbrace{1, x, \ldots x^n}_{\text{linearly independent}}\}$

Therefore, $\{1, x, \ldots, x^n\} \subseteq P_n(\mathbb{R})$ is a basis for $P_n(\mathbb{R})$

Also, it is called the standard basis of $P_n(\mathbb{R})$.

when $n=2$, $P_2(\mathbb{R}) = \text{Span}\{1, x, x^2\}$

Ex 3 $M_{2\times 2}(\mathbb{R}) = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

because $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

for any $a, b, c, d \in \mathbb{R}$, so $M_{2\times 2}(\mathbb{R}) \subseteq \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

and since $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \subset M_{2\times 2}(\mathbb{R})$,

$\text{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \subseteq M_{2\times 2}(\mathbb{R})$

Therefore, $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $M_{2\times 2}(\mathbb{R})$

$A = B$

$\Leftrightarrow A \subseteq B$

and $B \subseteq A$

$A \subseteq B$

$\Leftrightarrow \forall x \in A$,

$x \in B$

Ex 4 $W = \text{span}\{\sin x, \cos x\}$. Find a basis for W

Sol Since $\{\sin x, \cos x\}$ is linearly independent (we have shown it before),

$\{\sin x, \cos x\}$ is a basis

Theorem Let $S=\{x_1, \dots, x_n\}$ be a subspace of a vector space V

S is a basis for $V \iff \forall x \in V$ is written uniquely as as a linear combination of vectors in S.

$(\Rightarrow)$ We have done it

$(\Leftarrow)$ Since $\forall x \in V$ is written as a linear combination of vectors in S. $V = \text{span}\{x \dots\}$

Let $a_1 x_1 + \cdots + a_n x_n = 0 = 0x_1 + 0x_2 + \cdots + 0x_n$

By the assumption (unique expression), $a_1 = 0, a_2 = 0 \dots a_n = 0$

$\Rightarrow \{x_1, \dots x_n\}$ is linearly independent

Therefore, S is a basis

Notation: Let $S=\{x_1 \dots x_n\}$ be a basis for V

$\forall x \in V, \quad x = a_1 x_1 + \dots + a_n x_n$

unique

$[x]_S = (a_1, a_2, \dots a_n)$ We call $[x]_S$ the "coordinates" of x with respect to S

For example, $\mathbb{R}^2 = \text{span}\{e_1, e_2\} = \text{span}\{(1,1), (1,-1)\}$

Let $S=\{e_1, e_2\}$ and $\alpha = \{(1,1), (1,-1)\}$

$\forall (a,b) \in \mathbb{R}^2$

$[(a,b)]_S = (a,b)$

$x = 2(1,1) + 3(1,-1) \in \mathbb{R}^2$

$[x]_S = (5,-1)) \leftarrow$ $2(1,1)+3(1,-1)$
$= (5,-1)$
$= 5(1,0) + (-1)(0,1)$

$[x]_\alpha = (2,3)$

Theorem Let $S=\{x_1 \dots x_n\} \subseteq V$

Suppose S is linearly independent. Let $x \notin S$

$S \cup \{x\}$ is linearly independent $\iff x \notin \text{span}(S)$

independent

Pf $(\Rightarrow)$ Suppose $S \cup \{x\}$ is linearly independent

But what if $x \in \text{span}(S)$?

Then $x = t_1 x_1 + \dots + t_n x_n$ for some $t_1, \dots t_n \in \mathbb{R}$

$\Rightarrow t_1 x_1 + \dots + t_n x_n - x = 0$

But $(t_1, \dots t_n, -1) \neq (0, \dots 0)$ contradiction to the assumption

($\Leftarrow$) $x \notin \mathrm{span}(S)$

Let $ax + a_1 x_1 + \cdots + a_n x_n = 0$

If $a \neq 0$, $x = -\frac{a_1}{a} x_1 - \cdots \frac{a_n}{a} x_n \in \mathrm{span}(S)$

So $a = 0$. That is, $0 = a_1 x_1 + \cdots + a_n x_n$

Since $\{x_1 \ldots x_n\}$ is linearly independent, $a_1 = \cdots = a_n = 0$

Therefore, $\{x, x_1 \ldots x_n\}$ is linearly independent.

Ex5 $S = \{x^2 + 2x + 1, x^2 + 4x + 3\}$ linearly independent

Find $p(x) \in P_2(\mathbb{R})$ such that $S \cup \{p(x)\}$ is linearly independent

Sol Find a $p(x) \in P_2(\mathbb{R})$ such that $p(x) \notin \mathrm{span}(S)$ $\Leftrightarrow$

Find $p(x) = px^2 + qx + r \in P_2(\mathbb{R})$ such that

$px^2 + qx + r$ is not a linear combination of S.

$\Leftrightarrow$ $a_1(x^2 + 2x + 1) + a_2(x^2 + 4x + 3) = px^2 + qx + r$

Let

$$\begin{aligned} a_1 + a_2 &= p \\ 2a_1 + 4a_2 &= q \\ a_1 + 3a_2 &= r \end{aligned}$$ doesn't have a solution

REF

$$\Leftrightarrow \underbrace{\underbrace{\left[\begin{array}{cc|c} 1 & 1 & p \\ 2 & 4 & q \\ 1 & 3 & r \end{array}\right]}_{\text{coefficient matrix}}}_{\text{the augmented matrix}} \sim \underbrace{\left[\begin{array}{cc|c} 1 & 1 & p \\ 0 & 2 & q-2p \\ 0 & 0 & r-q+p \end{array}\right]}$$

the rank of the coefficient matrix is 2

In order for the system not to have a solution,

$r - q + p \neq 0$

For example, $p = 0$, $q = 0$, $r = 1$ $\Rightarrow$ $p(x) = 1$

$\{x^2 + 2x + 1, x^2 + 4x + 3, 1\}$ linearly independent

Theorem Let V be a vector space over $\mathbb{R}$

Let $B_1 = \{x_1, x_2, \ldots x_n\}$ and $B_2 = \{y_1, \ldots y_m\}$ be bases for V. Then $n=m$.

Def The number of elements of any bases of a vector space V is called the dimension of V, and we denote it by $\dim(V)$

Ex 6 ⓐ $\dim(P_2(\mathbb{R}))=3$ because $P_2(\mathbb{R})=\operatorname{span}\{\underbrace{1, x, x^2}_{\text{linearly indep}}\}$

ⓑ $\dim(M_{2\times 3}(\mathbb{R}))=6$ because $M_{2\times3}(\mathbb{R})=\operatorname{span}\left\{\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}, \dots \begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}\right\}$

a set of 2x3 matrices

Theorem Let W be a subspace of a finite dimensional vector space V

Then ① $\dim(W)\leq\dim(V)$
② $\dim(W)=\dim(V) \iff W=V$

Ex 7 Is $S=\{1, x^2+2x+1, x^2+4x+3\}$ a basis of $P_2(\mathbb{R})$?

Sol show that S is linearly independent (Look at the previous ex)
Since $S\subset P_2(\mathbb{R})$ and $\dim(\operatorname{span}(S))=3=\dim(P_2(\mathbb{R}))$,
$\operatorname{span}(S)=P_2(\mathbb{R})$ by the theorem above.
Therefore, S is a basis for $P_2(\mathbb{R})$.