Sec 1.3 Linear Combinations

Def Let $S \subseteq V$, where V is a vector space over $\mathbb{R}$.

$a_1 x_1 + \cdots + a_n x_n$, where $a_i \in \mathbb{R}$ and $x_i \in S$, $1 \le i \le n$

is called a linear combination

Def Let V be a vector space over $\mathbb{R}$ and $S = \{x_1, \ldots x_n\} \subseteq V$.

$$\mathrm{Span}(S) = \{ a_1 x_1 + \cdots + a_n x_n \mid a_i \in \mathbb{R},\ x_i \in S,\ 1 \le i \le n \}$$

Note that $0x_1 + 0x_2 + \cdots + 0x_n = 0 \in \mathrm{Span}(S)$

Thm Span(S) is a subspace of V.

HW

Hint: $\forall\ a_1 x_1 + \cdots + a_n x_n,\ b_1 x_1 + \cdots + b_n x_n \in \mathrm{span}(S)$

$$(a_1 x_1 + \cdots + a_n x_n) + (b_1 x_1 + \cdots + b_n x_n)$$
$$= (a_1 + b_1) x_1 + (a_2 + b_2) x_2 + \cdots + (a_n + b_n) x_n$$

Ex 1 $W = \{ t(1,0,1) + s(-2,1,0) \mid t, s \in \mathbb{R} \}$

$= \mathrm{span}(\{(1,0,1), (-2,1,0)\})$ is a subspace of $\mathbb{R}^3$.

MAT223 example

$$t(1,0,1) + s(-2,1,0) = (t - 2s, s, t)$$

Let $x_1 = t - 2s$, $x_2 = s$, and $x_3 = t$

Then $x_1 = x_3 - 2x_2 \Rightarrow x_1 + 2x_2 - x_3 = 0$

$$W = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + 2x_2 - x_3 = 0 \}$$

a plane in $\mathbb{R}^3$

Ex 2 $S = \{1, x, x^2, \ldots x^n\} \subseteq P_n(\mathbb{R})$

$$\mathrm{Span}(S) = \{ \underbrace{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}_{\text{polynomial of a degree at most } n} \mid a_i \in \mathbb{R},\ 1 \le i \le n \}$$

$= P_n(\mathbb{R})$

Theorem 1 Let W_1 and W_2 be subspaces of a vector space V. Then $W_1+W_2=\{x+y \mid x\in W_1, y\in W_2\}$ is a subspace of V.

pf 1. $0\in W_1\cap W_2$, $0=0+0\in W_1+W_2$, not empty

2. $\forall\, x_1+y_1,\ x_2+y_2 \in W_1+W_2$, where $x_1, x_2\in W_1$ and $y_1, y_2\in W_2$

$(x_1+y_1)+(x_2+y_2)$

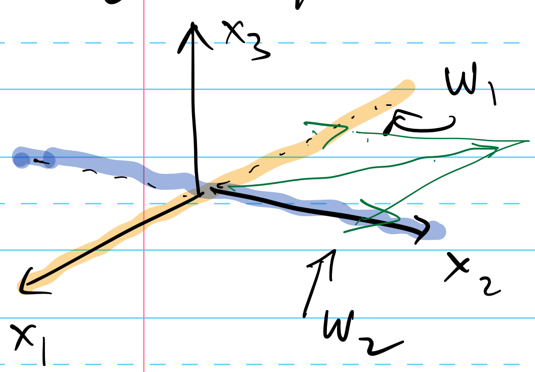
$=(x_1+x_2)+(y_1+y_2)\in W_1+W_2$

because $x_1+x_2\in W_1$ and $y_1+y_2\in W_2$

HW: prove it is closed under scalar multiplication

Remark If $W_1\cap W_2=\{0\}$, denote W_1+W_2 by $W_1\oplus W_2$ and we call $W_1\oplus W_2$ the direct sum of W_1 and W_2

Ex 3. $W_1=\{(x_1,0,0)\mid x_1\in\mathbb{R}\}$ $W_2=\{(0,x_2,0)\mid x_2\in\mathbb{R}\}$



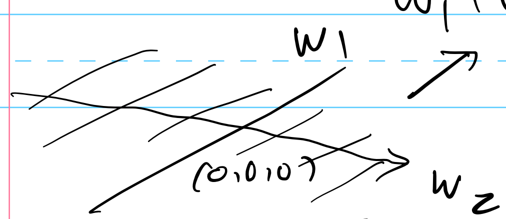
$W_1+W_2=W_1\oplus W_2$

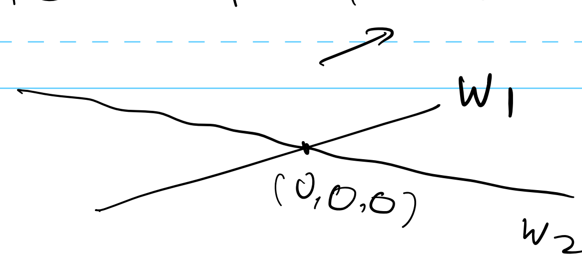
because $W_1\cap W_2=\{0\}$

and $W_1\oplus W_2=\{(x_1,x_2,0)\mid x_1,x_2\in\mathbb{R}\}$

the x_1x_2-plane $\cong\mathbb{R}^2$

$W_1+W_2=W_1\oplus W_2\neq W_1\cup W_2$

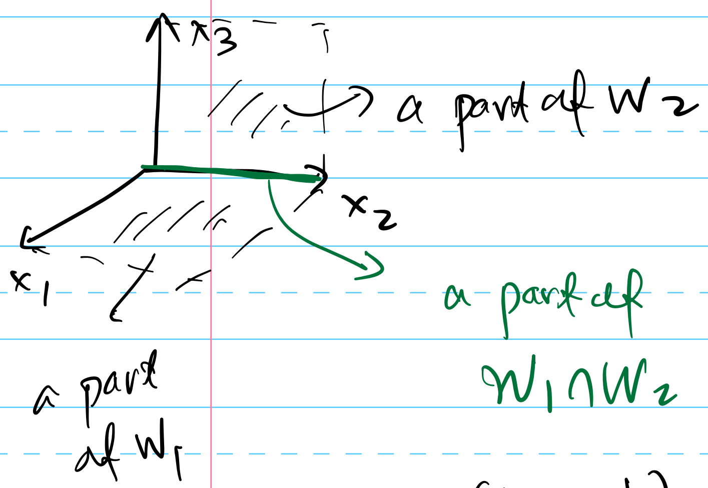




Ex 4. Elements of $W_1 + W_2$ are not expressed uniquely

$W_1 = \{(x_1, x_2, 0) \mid x_1, x_2 \in \mathbb{R}\}$ x_1x_2-plane

$W_2 = \{(0, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\}$ x_2x_3-plane



$W_1 \cap W_2 = \{(0, x_2, 0) \mid x_2 \in \mathbb{R}\}$

line $\neq \{0\}$

$W_1 + W_2 = \mathbb{R}^3$

$(1,1,1) \in \mathbb{R}^3$

$(1,1,1) = \underbrace{(1,1,0)}_{\text{in } W_1} + \underbrace{(0,0,1)}_{\text{in } W_2}$

$= \underbrace{(1,0,0)}_{\text{in } W_1} + \underbrace{(0,1,1)}_{\text{in } W_2}$

Sec 1.4 Linear Independence and Dependence

Sec 1.5 Solving Systems of Linear Equations

Def Let V be a vector space. $S = \{x_1, \dots x_n\} \subseteq V$

1. S is called linearly independent $\Longleftrightarrow$

$\forall\ a_1 x_1 + \cdots + a_n x_n \in \mathrm{span}(S)$,

if $a_1 x_1 + \cdots + a_n x_n = 0$, $a_1 = a_2 = \cdots = a_n = 0$

the additive identity   real number

2. S is called linearly dependent if S is not linearly independent.

$\Longleftrightarrow \exists\ (a_1, \dots, a_n) \neq (0, \cdots 0)$

such that $a_1 x_1 + \cdots + a_n x_n = 0$

Remark If $V = \mathbb{R}^n$,

matrix multiplication

$a_1 x_1 + \cdots + a_n x_n = 0$

$$\Leftrightarrow \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} x_1 & \cdots & x_n & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right] \sim \text{ a REF}$$

elementary row operations

S is linearly independent
$\Leftrightarrow$ the corresponding homogeneous system has only zero solution.

Ex 1 $S = \{(2,1,-1), (1,4,2), (-4,10,8)\} \subset \mathbb{R}^3$

Is S linearly independent?

Sol) Let $a_1(2,1,-1) + a_2(1,4,2) + a_3(-4,10,8) = (0,0,0)$

where $a_1, a_2, a_3 \in \mathbb{R}$

$$\left[\begin{array}{ccc|c} 2 & 1 & -4 & 0 \\ 1 & 4 & 10 & 0 \\ -1 & 2 & 8 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

optional

basic variables

free variable

Since there is a free variable,

there are infinitely many solutions

$\Rightarrow S$ is not linearly independent

or

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 - 2x_3 &= 0 \\ x_2 + 3x_3 &= 0 \end{aligned}$$

Let $x_3 = t$. Then $x_2 = -3t$

$x_1 = 2x_3 = 2t$

$$(x_1, x_2, x_3) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ -3t \\ t \end{bmatrix} = \underbrace{t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}}_{=\ \mathrm{Null}\left(\begin{bmatrix} 2 & 1 & -1 \\ 1 & 4 & 10 \\ -1 & 2 & 8 \end{bmatrix}\right)}$$

vector matrix

the dimension of the nullspace $= 1$

# of free variable
= # of parameters.

Ex2 Show that $S = \{\sin x, \cos x\} \subseteq F(\mathbb{R})$
is linearly independent

Notation: $F(\mathbb{R})$ is a set of real valued function from $\mathbb{R}$
$F(\mathbb{R})$ is a real vector space $\quad (f+g)(x) = f(x) + g(x)$
$cf(x) = cf(x)$

Pf

$$\underbrace{a_1 \sin x + a_2 \cos x}_{\text{a linear combination of } \sin x \text{ and } \cos x} = 0$$

zero function

$\boxed{\text{means } g(x) = 0}$

the additive identity

$x = \frac{\pi}{2}$, $\quad a_1 \sin(\frac{\pi}{2}) + a_2 \cos(\frac{\pi}{2}) = 0$
$a_1 + a_2 \cdot 0 = 0 \Rightarrow a_1 = 0$

$x = 0$, $\quad a_1 \sin(0) + a_2 \cos(0) = 0$
$a_1(0) + a_2(1) = 0 \Rightarrow a_2 = 0$

Another proof. Suppose $S = \{\sin x, \cos x\}$ is linearly dependent.

$\Longleftrightarrow \exists (a_1, a_2) \neq (0, 0)$ such that

$$a_1 \sin x + a_2 \cos x = 0$$

Say $a_1 \neq 0$, $\sin x = -\frac{a_2}{a_1} \cos x$ for all x

HW

$\downarrow$

Theorem Let $\{x_1, \ldots x_n\}$ be a linearly independent subset of a vector space V

Suppose $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b_1 x_1 + \cdots + b_n x_n$

Then $a_i = b_i$, $1 \leq i \leq n$.

Pf

$$a_1 x_1 + \cdots + a_n x_n = b_1 x_1 + \cdots + b_n x_n$$

$\Rightarrow (a_1 - b_1) x_1 + \cdots + (a_n - b_n) x_n = 0$

$\Rightarrow a_i - b_i = 0$ for all i because $\{x_1, \ldots x_n\}$ is linearly independent

$\Rightarrow a_i = b_i$ $1 \leq i \leq n$.

Ex3 (a) Show that $\{1, x, x^2\}$ is a linearly independent subset of $P_2(\mathbb{R})$

(b) Suppose $a + bx + cx^2 = 2x^2 + 3$.
Find a, b, and c.

Sol Let $a_1(1) + a_2 x + a_3 x^2 = 0$ ← 0 function polynomial

(a) when $x = 0$, $a_1 = 0$

when $x = 1$, $a_2 + a_3 = 0$
$x = -1$, $-a_2 + a_3 = 0$
$\Rightarrow a_2 = a_3 = 0$

(b) $a = 3$, $b = 0$, $c = 2$

Ex4 $S = \{ x^2+2x+1,\ x^2+4x+3,\ x^2+6x+5 \} \subseteq P_2(\mathbb{R})$

Is S linearly independent?

Sol $a_1(x^2+2x+1) + a_2(x^2+4x+3) + a_3(x^2+6x+5) = 0$

$\Rightarrow (a_1+a_2+a_3)x^2 + (2a_1+4a_2+6a_3)x + (a_1+3a_2+5a_3) = 0$

0 polynomial

Since $\{1, x, x^2\}$ is linearly independent,

$$\left.\begin{aligned} a_1+a_2+a_3 &= 0 \\ 2a_1+4a_2+6a_3 &= 0 \\ a_1+3a_2+5a_3 &= 0 \end{aligned}\right\} \Rightarrow$$

solve the system of the linear eqns for $a_1, a_2,$ and a_3

$$\Updownarrow \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 4 & 6 & 0 \\ 1 & 3 & 5 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

basic variables

free variable

There are infinitely many (a_1, a_2, a_3) solutions $\Rightarrow$ S is not linearly independent

Remark $\{ x^2+2x+1,\ x^2+4x+3 \}$ is linearly independent

Ex5 Show that $S = \left\{ \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ -4 & 1 \end{bmatrix} \right\} \subset M_{2\times 2}(\mathbb{R})$ is linearly independent

Sol $a_1 \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 3 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\mathbb{O}$

$$\Rightarrow \begin{bmatrix} a_1 & 2a_1 \\ -a_1 & a_1 \end{bmatrix} + \begin{bmatrix} a_2 & 3a_2 \\ -4a_2 & a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left.\begin{aligned} a_1+a_2&=0 \\ 2a_1+3a_2&=0 \\ -a_1-4a_2&=0 \\ a_1+a_2&=0 \end{aligned}\right\} \Rightarrow a_1=a_2=0$$