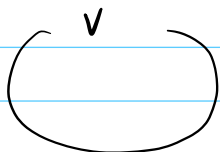


Sec 2.7 Change of Basis



V : vector space over \mathbb{R}
and $\dim(V) = n$.

Sup α and α' are bases for V

$\forall x \in V$ $[x]_{\alpha}$: the coordinates of x w.r.t α

$[x]_{\alpha'}$: the coordinates of x w.r.t α'

Change of basis: $[x]_{\alpha} \rightarrow [x]_{\alpha'}$ How to change the coordinates?

Recall: $T: V \rightarrow W$ linear
 \downarrow_{α} \downarrow_{β}

$$(*) [T(x)]_{\beta} = [T]_{\beta}^{\alpha} [x]_{\alpha}$$

If $V = W$, $T = I_V$, $\alpha = \alpha$, $\beta = \alpha'$

$$\text{Then } (*) [x]_{\alpha'} = [I_V]_{\alpha'}^{\alpha} [x]_{\alpha}$$

$$[x]_{\alpha} \xrightarrow{[I_V]_{\alpha'}^{\alpha}} [x]_{\alpha'}$$

Ex 1
Review

Let $\alpha = \{(1,1), (1,-1)\}$ and $\alpha' = \{(1,2), (-2,1)\}$ be bases for \mathbb{R}^2

Find $[I]_{\alpha'}^{\alpha}$

Sol

$$[I]_{\alpha'}^{\alpha} = \left[[I(1,1)]_{\alpha'}, [I(1,-1)]_{\alpha'} \right] = \left[\underbrace{[(1,1)]_{\alpha'}}_{\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}}, \underbrace{[(1,-1)]_{\alpha'}}_{\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}} \right]$$

$$\text{Then } (1,1) = a_{11}(1,2) + a_{21}(-2,1) \rightarrow \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(1,-1) = a_{12}(1,2) + a_{22}(-2,1) \rightarrow$$

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{Solve it for } a_{11}, a_{21}, a_{12}, \text{ and } a_{22}$$

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 1 \\ 2 & 1 & 1 & -1 \end{array} \right]$$

\int

$$\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \end{array} \right]$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{3}{5} \end{bmatrix}$$

HW

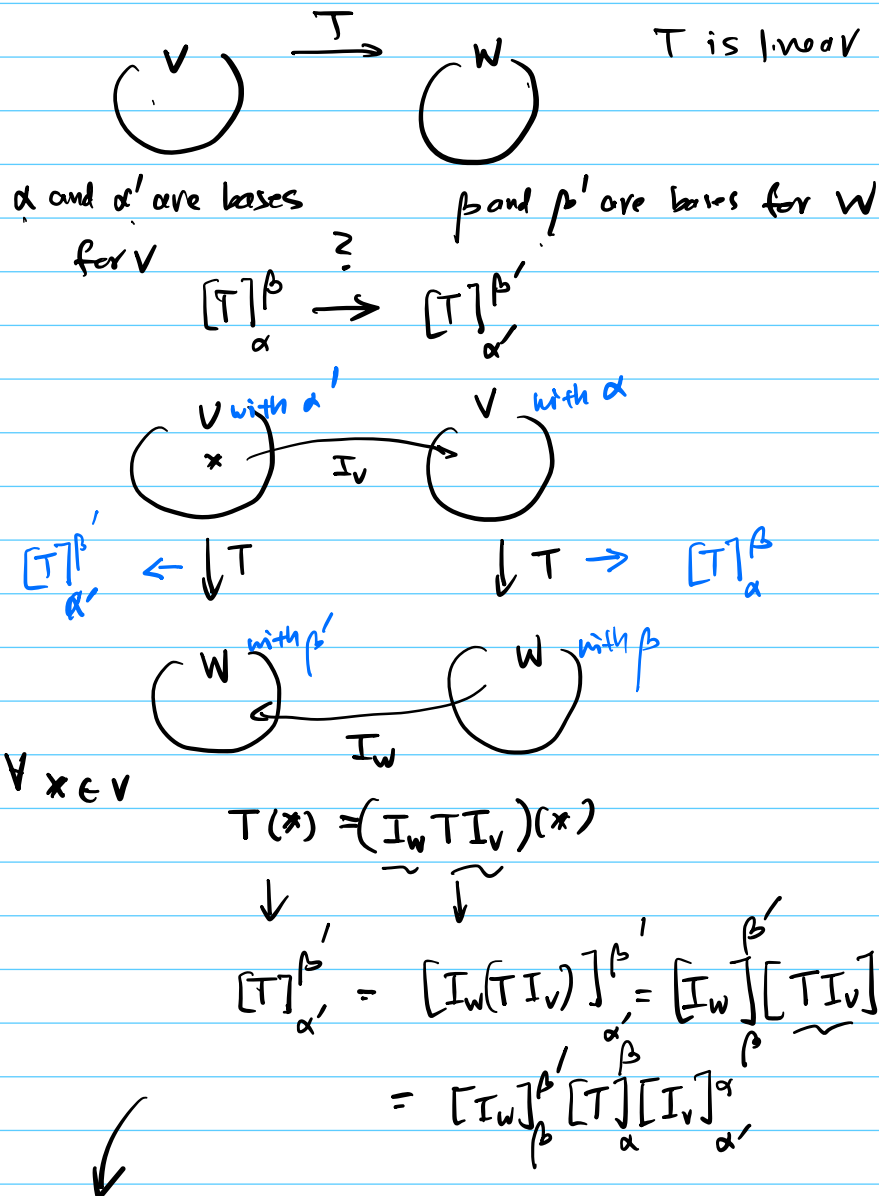
Ex2 Suppose $\alpha = \{(1,1), (1,-1)\}$ and $\alpha' = \{w_1, w_2\}$ are bases for \mathbb{R}^2 ,
and $[I]_{\alpha}^{\alpha'} = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$. Find w_1 and w_2

\downarrow \downarrow
 $\textcircled{1}$ $\textcircled{2}$

Sol

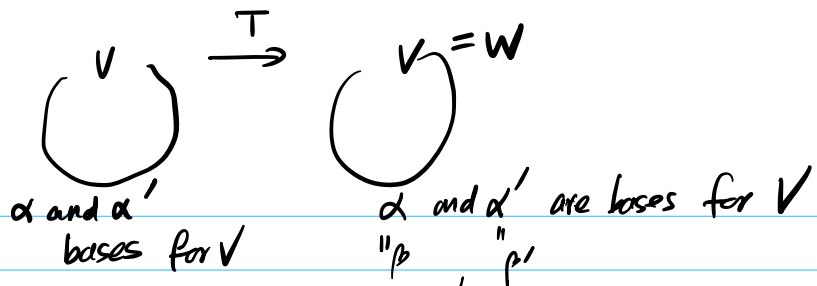
$$\begin{aligned} \textcircled{1} \quad (1,1) &= (-1)w_1 + 2w_2 & \rightarrow [w_1 \ w_2] \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ \textcircled{2} \quad (1,-1) &= 4w_1 + (1)w_2 \\ &\downarrow \\ \textcircled{1} \times 4 : (4,4) &= -4w_1 + 8w_2 \\ \textcircled{2} \quad + (1,-1) &= 4w_1 + w_2 \\ &\downarrow \\ (5,3) &= 9w_2 \\ w_2 &= \frac{1}{9}(5,3) = \left(\frac{5}{9}, \frac{3}{9}\right) \end{aligned}$$

$$\xrightarrow{HW} w_1 = \left(\frac{1}{9}, -\frac{3}{9}\right)$$



★

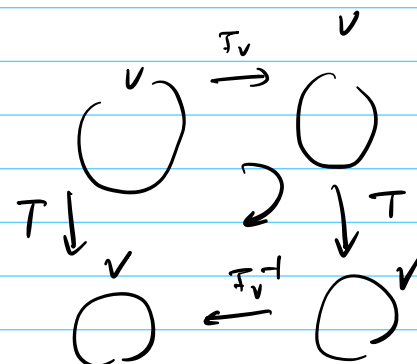
Corollary



$$[T]_{\alpha}^{\alpha} \xrightarrow{?} [T]_{\alpha'}^{\alpha'}$$

$$I_V = I_V^{-1}$$

$$\begin{aligned} [T]_{\alpha'}^{\alpha'} &= [I_V]_{\alpha}^{\alpha'} [T]_{\alpha}^{\alpha} [I_V]_{\alpha'}^{\alpha} \\ &= [I_V]_{\alpha}^{\alpha'} [T]_{\alpha}^{\alpha} [I_V]_{\alpha'}^{\alpha} \\ &= ([I_V]_{\alpha'}^{\alpha})^{-1} [T]_{\alpha}^{\alpha} [I_V]_{\alpha'}^{\alpha} \end{aligned}$$



Def Let A and B be $n \times n$ matrices

A and B are said to be similar if \exists an invertible $n \times n$ matrix Q such that $B = Q^{-1} A Q$

Ex3 $[T]_{\alpha}^{\alpha}$ and $[T]_{\alpha'}^{\alpha'}$ are similar

Ex4 $P_1(\mathbb{R}) \xrightarrow{T} P_1(\mathbb{R})$ linear

α and $\alpha' = \{1+x, 1-x\}$ are bases for $P_1(\mathbb{R})$

Suppose $[T]_{\alpha}^{\alpha} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $[(1+x)]_{\alpha} = (1, 1)$ and $[(1-x)]_{\alpha} = (0, 1)$

Find $T(6-2x)$

sol

$$[T(6-2x)]_{\alpha'} = [T]_{\alpha'}^{\alpha'} [6-2x]_{\alpha'}$$

put them together $[I]_{\alpha'}^{\alpha}$

$$\text{Let } 6-2x = a(1+x) + b(1-x) \xrightarrow{HW} a=2, b=4$$

$$[6-2x]_{\alpha'} = (2, 4)$$

$$[T]_{\alpha'}^{\alpha'} = ([I]_{\alpha'}^{\alpha})^{-1} [T]_{\alpha}^{\alpha} [I]_{\alpha}^{\alpha'}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[T(6-2x)]_{\alpha'} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$T(6-2x) = 10(1+x) + 10(1-x) = 20$$