Part I: Theory and Calculation

1. The joint PMF of two discrete random variables X and Y is given by

$$p_{XY}(x,\ y) = \begin{cases} kxy & x = 1,\ 2,\ 3;\ \ y = 1,\ 2,\ 3 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Determine the value of the constant k.
   (b) Find the marginal PMFs of X and Y.
   (c) Find $P[1 \le X \le 2,\ Y \le 2]$.

2. The joint CDF of two continuous random variables X and Y is given by

$$F_{XY}(x,\ y) = \begin{cases} 1 - e^{-ax} - e^{-by} + e^{-(ax+by)} & x \ge 0,\ \ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the marginal PDFs of X and Y
   (b) Carefully show why or why not X and Y are independent.

3. The joint PDF of two random variables X and Y is given by

$$f_{XY}(x,\ y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \qquad 0 < x < 1,\ \ 0 < y < 2$$

   (a) What is the CDF, $F_X(x)$, of X?
   (b) Find $P[X > Y]$
   (c) Find $P[Y > \frac{1}{2} \mid X < \frac{1}{2}]$

4. Let the random variables X and Y have the joint PDF $f_{XY}(x,\ y) = 2e^{-(x+2y)},\ \ x \ge 0,\ \ y \ge 0.$ Find the conditional expectation of
   (a) X given Y
   (b) Y given X

5. Two discrete random variables X and Y have the joint PMF given by

$$p_{XY}(x,\ y) = \begin{cases} 0 & x = -1,\ \ y = 0 \\ \frac{1}{3} & x = -1,\ \ y = 1 \\ \frac{1}{3} & x = 0,\ \ y = 0 \\ 0 & x = 0,\ \ y = 1 \\ 0 & x = 1,\ \ y = 0 \\ \frac{1}{3} & x = 1,\ \ y = 1 \end{cases}$$

   (a) Are X and Y independent?
   (b) What is the covariance of X and Y?

6. Two events A and B are such that $P[A] = \frac{1}{4}$, $P[B \mid A] = \frac{1}{2}$ and $P[A \mid B] = \frac{1}{4}$. Let the random variable X be defined such that $X = 1$ if event A occurs and $X = 0$ if event A does not occur. Similarly, let the random variable Y be defined such that $Y = 1$ if event B occurs and $Y = 0$ if event B does not occur.
   (a) Find $E[X]$ and the variance of X.
   (b) Find $E[Y]$ and the variance of Y.
   (c) Find ρ_{XY} and determine whether or not X and Y are uncorrelated.

7. The students in one college have the following rating system for their professors: excellent, good, fair, and bad. In a recent poll of the students, it was found that they believe that 20% of the professors are excellent, 50% are good, 20% are fair, and 10% are bad. Assume that 12 professors are randomly selected from the college.
   (a) What is the probability that 6 are excellent, 4 are good, 1 is fair, and 1 is bad?
   (b) What is the probability that 6 are excellent, 4 are good, and 2 are fair?
   (c) What is the probability that 6 are excellent and 6 are good?
   (d) What is the probability that 4 are excellent and 3 are good?
   (e) What is the probability that 4 are bad?
   (f) What is the probability that none is bad?

Remark: for simplicity, we treat the process of selecting 12 professors as performing 12 times of `selecting one professor from the set of all professors' and assume that these selections of `one single professors' are independent and identically distributed (if there are sufficiently many professors then this is a reasonable assumption).

8. Suppose X is a random variable and $Y = aX - b$, where a and b are constants. Find the PDF, expected value and variance of Y.

9. If $Y = aX^2$, where $a > 0$ is a constant and the mean and other moments of X are known, determine the following in terms of the moments of X:
   (a) the mean of Y
   (b) the variance of Y.

10. Two independent random variables X and Y have variances $\sigma_X^2 = 9$ and $\sigma_Y^2 = 25$, respectively. If we define two new random variables U and V as follows:

$$U = 2X + 3Y$$
$$V = 4X - 2Y$$

   (a) Find the variances of U and V
   (b) Find the correlation coefficient of U and V
   (c) Find the joint PDF of U and V in terms of $f_{XY}(x, y)$.

Part II: Simulation

Guidelines and rules for this part (please read carefully):

- For each problem, you are expected to first do necessary mathematical analysis (key words: 'calculate') of the problem, then use simulation to verify your analysis results.
  - For mathematical part, please show process of the solution in the report. Answers with only results will not earn full grade.
  - For the simulation part, in your report, please clearly state how you set up the simulations, your ideas and the necessary intermediate results and figures/tables you got from simulations. You will use MATLAB to run the computer simulations. The codes should run without problems on the TA's computer.
- For each problem, please turn in: (i) source codes, and (ii) a lab report to the Gradescope.
- Penalty for late submission: 10% penalty if late within one day; Not acceptable (0 grade) if late more than one day.
- You may discuss coding/debugging issues with your classmates. You cannot, however, share code. It is a violation of ethical policies if you request code from someone else, or if you give code to someone else.
- Main reference for lab session: *Introduction to Probability by Charles M. Grinstead and J. Laurie Snell*, free open book at: http://www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/amsbook.mac.pdf

1. Please use Example 2.3 in the Grinstead's open book to simulate the Buffon's Needle experiments. Please calculate, simulate and store the estimated number of π using 100, 1000, 10000, 100000 trials, and plot the estimated numbers versus trial numbers to visualize the convergence (better use logarithmic coordinate for the number of trials).

2. Please use simulation to estimate the area under the graph of y = 1/(x + 1) in the unit square (x in [0,1], y in [0,1]) in the same way as in Fig. 2.3 of the Grinstead book. Calculate the true value of this area and use your simulation results to estimate the value of log2. How accurate is your estimation?

3. If X is a geometric random variable with $p = 0.25$, what is the probability that $X \geq 4$? Calculate the result and then verify your result by performing a computer simulation.