The theory of nuclear operators

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Translator's note.

This text is one of a series of translations of various papers into English. What follows is a translation (last updated July 14, 2020) of the French paper:*

SCHWARTZ, L. La théorie des opérateurs nucléaires. *Séminaire Schwartz*, Volume **1** (1953-1954), Talk no. 12, 7 p. `http://www.numdam.org/item/SLS_1953-1954__1__A13_0/`

Contents

1 The trace

p. 1

Let E be a *Banach* space, whose dual we will call E'. We know, by definition, that there exists a bijective and isometric correspondence between the space $\mathcal{B}(E, E')$ of continuous bilinear forms on $E \times E'$ and the dual $E\widehat{\otimes}_\pi E'$. To the canonical bilinear form $(x, x') \mapsto \langle x, x'\rangle$ thus corresponds a continuous linear form on $E\widehat{\otimes}_\pi E'$ that we call "the trace", and that we denote by Tr. If $u = \sum_v x_v \otimes y'_v$ then, by definition, $\mathrm{Tr}(u) = \sum_v \langle x_v, y'_v\rangle$. The trace form is of norm 1. Furthermore, every $u \in E\widehat{\otimes}_\pi E'$ can be written in the form $u = \sum_{n\geqslant 0} x_n \otimes y'_n$

*`https://github.com/thosgood/translations`

with $\sum_{n\geqslant 0}\|x_n\|\|y'_n\|$ finite, and so the series $\sum_{n\geqslant 0}\langle x_n, y'_n\rangle$ converges absolutely, and, since the trace is continuous, we have that

$$\mathrm{Tr}(u) = \sum_{n\geqslant 0}\langle x_n, y'_n\rangle.$$

To justify the name "trace", recall that we can identify $E \otimes E'$ with the space of endomorphisms of finite rank of E, and that, if E is of finite dimensions, then the trace form agrees with the usual trace of operators.

There exists a canonical continuous map $E'\widehat{\otimes}_\pi E \to \mathcal{L}(E;E)$. If we do not know whether or not it is bijective, we can only speak of the trace of an element of $E'\widehat{\otimes}_\pi E$, and not the trace of the image of the operator in $\mathcal{L}(E;E)$.

Recall as well that there exists an isomorphism S (for symmetry) between $E \otimes E'$ and $E' \otimes E$, defined by

$$S\colon \sum_v x_v \otimes y'_v \mapsto \sum_v y'_v \otimes x_v$$

for $x_v \in E$ and $y'_v \in E'$.

If we identify $E \otimes E'$ with the space of maps of finite rank from E to E, and $E' \otimes E \subset E' \otimes (E')'$ with a space of transformations of E', then the map *p. 2* S corresponds to the transposition of operators. Thanks to S, the trace is also defined on $E'\widehat{\otimes}_\pi E$. We can thus understand the duality between $E\widehat{\otimes}_\pi F$ and $\mathcal{B}(E,F)$ by means of the trace: let $A \in \mathcal{B}(E,F) \subset \mathcal{L}(E;F')$. If 1 is the identity in F, then $A \otimes 1$ sends $E\widehat{\otimes}_\pi F$ to $F'\widehat{\otimes}_\pi F$. So if $u \in E\widehat{\otimes}_\pi F$, then we can take the trace of $(A \otimes 1)(u) \in \mathcal{L}(F;F')$, and we have

$$\langle u, A\rangle = \mathrm{Tr}((A \otimes 1)(u)). \tag{1}$$

Indeed, both sides of the equation (for fixed A) are continuous linear forms in u, and are equal for $u = x \otimes y$.

2 The map $E'\widehat{\otimes}_\pi F \to \mathcal{L}_b(E_\tau;F)$ for E and F locally convex

The subscript $_b$ denotes the uniform convergence topology on bounded subsets of a space of linear maps.

Let E and F be arbitrary locally convex separated spaces. Elements of $E' \otimes F$ correspond to continuous linear maps of finite rank from E to F. So $E' \otimes F \subset \mathcal{L}(E_\tau;F)$, since the latter is the space of weakly continuous maps (see Exposé 8, §1).

Proposition 1. *The topology induced on $E' \otimes F$ by $\mathcal{L}_b(E_\tau;F)$ is identical to the topology of $E'_b\otimes_\varepsilon F$.*

Proof. The topology of $E'_b\otimes_\varepsilon F$ is, by definition, the topology induced on $E' \otimes F$ by $\mathcal{L}_\varepsilon((E'')_\tau;F)$. But an equicontinuous subset of E'' is the polar of a neighbourhood of 0 in E', which is itself the polar of a bounded subset of E, and thus (by the bipolar theorem) is the weakly closed convex balanced hull of a bounded subset of E. But, in a $\mathfrak{S}$-topology, we can replace the sets of $\mathfrak{S}$ by their closed convex balanced hull. Thus $\mathcal{L}_\varepsilon((E'')_\tau;F)$ and $\mathcal{L}_b(E_\tau;F)$ induce the same topology on $E'\otimes F$. $\square$

Corollary 1. *If E and F are complete, then there exists a continuous map φ from $E'\otimes_\pi F$ to $\mathcal{L}_b(E_\tau;F)$ that extends the identity on $E'\otimes F$.*

Proof. Indeed, the π-topology being finer than the ε-topology, there exists a canonical map $E'\widehat{\otimes}_\pi F\to E'\widehat{\otimes}_\varepsilon F$ which we can compose with the map $E'\widehat{\otimes}_\varepsilon F\to\mathcal{L}_b(E_\tau;F)$. $\square$

3 Definition of nuclear maps — the case of Banach spaces

From now on, the only tensor product that we will consider is the π-product; thus $E\widehat{\otimes}F$ means $E\widehat{\otimes}_\pi F$. p. 3

Definition 1. If E and F are Banach spaces, then we write $L^1(E;F)$ to denote the subspace $\varphi(E'\widehat{\otimes}F)$ of $\mathcal{L}(E;F)$. The elements of $L^1(E;F)$ are called *nuclear* (or *Fredholm*) operators. Note that $L^1(E;F)$ is a *quotient space* of $E\widehat{\otimes}F$. The quotient norm of the π-norm will be called the *trace norm*, or the *nuclear norm*, denoted by $\|\cdot\|_1$ or $\|\cdot\|_{\mathrm{Tr}}$.

We do not know a case where φ is not bijective, but we do not know how to prove this in general.

Since $E'\otimes F$ is dense in $E'\widehat{\otimes}F$, and φ is continuous, every nuclear operator is the *"uniform" limit* (in $\mathcal{L}_b$) *of operators of finite rank*, and is thus, in particular, *compact* (since the image in F of a ball in E is relatively compact).

Remark. If $E=F$ is a Hilbert space, then the *hermitian* nuclear operators are exactly the completely continuous operators u such that the sequence (λ_n) of eigenvalues is summable and such that

$$\|u\|_1=\sum_n|\lambda_n|.$$

In the general Banach case, every nuclear operator u admits a decomposition

$$u=\sum\lambda_i x_i'\otimes y_i$$

where $x_i'\in E'$ are such that $\|x_i'\|_1\leqslant 1$ and $y_i\in F$ are such that $\|y_i\|\leqslant 1$, and such that $\sum|\lambda_i|<\infty$; the lower bound of $\sum|\lambda_i|$ for any such decomposition is exactly $\|u\|_1$.

Proposition 2. *Let $u\colon E\to F$ be a nuclear operator, and let $A\colon H\to E$ and $B\colon F\to G$ be continuous maps. Then $B\circ u\circ A$ is a nuclear operator, and $\|B\circ u\circ A\|_1\leqslant\|A\|\|u\|_1\|B\|$.*

Proof. We have the commutative diagram

$$\begin{tikzcd} E'\widehat{\otimes}F \arrow[r, "t_{A\otimes B}"] \arrow[d, "\varphi"'] & H'\widehat{\otimes}G \arrow[d, "\varphi"] \\ \mathcal{L}(E;F) \arrow[r, "u\mapsto B\circ u\circ A"'] & \mathcal{L}(H;G) \end{tikzcd}$$

(since the two maps from $E'\widehat{\otimes}F$ to $\mathcal{L}(H;G)$ that define this diagram are continuous, and agree for $u_0 \in E'\widehat{\otimes}F$ of the form $x'\otimes y$). So, if u is nuclear, with u_0 an element of $E\widehat{\otimes}F$ such that $\varphi(u_0) = u$, then $({}^tA\otimes B)u_0 \in H'\widehat{\otimes}G$, and $B\circ u\circ A = \varphi(({}^tA\otimes B)(u_0))$, and so $B\circ u\circ A$ is nuclear. Taking into account the fact that $\|{}^tA\otimes B\| = \|A\|\|B\|$, we have that | *p. 4*

$$\|B\circ u\circ A\|_1 \leqslant \inf_{\varphi(u_0)=u} ({}^tA\otimes B)(u_0) \leqslant \inf_{\varphi(u_0)=u} \|A\|\|B\|\|u_0\| = \|A\|\|B\|\|u\|_1.$$

□

4 Definition of nuclear operators — the general case

Definition 2. We say that a linear map $u\colon E\to F$, where E and F are locally convex separated spaces, is *nuclear* if there exist Banach spaces E_1 and F_1, a nuclear operator $\beta\colon E_1\to F_1$, and continuous operators $\alpha\colon E\to E_1$ and $\gamma\colon F_1\to F$ such that $u=\gamma\circ\beta\circ\alpha$, i.e. such that

$$\begin{tikzcd} E \arrow[r, "\alpha"] \arrow[rrr, bend right, "u"'] & E_1 \arrow[r, "\beta"] & F_1 \arrow[r, "\gamma"] & F \end{tikzcd}$$

commutes.

Remark. It suffices for F_1 to be Banach and E_1 to be normed, since we can extend β to $\widehat{E_1}$.

To simplify, we call any convex balanced set a *disc*. By replacing E_1 with $\alpha(E)$, and F_1 with $F_1/\gamma^{-1}(0)$, we can assume that α is an epijection and γ is an injection; but we know (exposé 7) that E_1 will be isomorphic to E_{U_1}, and F_1 to F_{B_1}, for U_1 some open disc of E, and B_1 some **complétante**[2] subset of F. Since the dual of E_{U_1} is $\widehat{E'_{A'_1}}$, where $A'_1 = U_1^0$, we know that β comes from an element β_0 of $E'_{A'_1}\widehat{\otimes}F_{B_1}$, and that $u=\gamma\circ\beta\circ\alpha$ comes from an element $u_0 = ({}^t\alpha\otimes\gamma)(\beta_0)$ of $E'\widehat{\otimes}F$; but ${}^t\alpha\otimes\gamma$ is exactly the canonical map from $E'_{A'_1}\widehat{\otimes}F_{B_1}$ in $E'\widehat{\otimes}F$. Even stronger: u comes from some $u_0\in E'_{A'_1}\widehat{\otimes}F_{B_1}$, but we know (exposé 5) that u_0 belongs to a set of the form $\Gamma(A'\otimes B)$, where A and B are compact subsets of $E'_{A'_1}$ and F_{B_1} (respectively). Whence:

[2] *[Translator]. I was unable to find a translation for this term, but I **think** it refers to the following property: an absolutely convex subset S of a topological vector space is said to be **complétante** if S_A is a Banach space, where S_A is the subset absorbed by S.*

Proposition 3. *An operator $u\colon E \to F$ is nuclear if and only if it is defined by an element of some $E'_{A'} \otimes F_B$, where A' and B are compact convex balanced (and thus* ***complétante****) subsets. We can thus suppose, in Definition 2, that α and β are compact maps.*

Note also that, since $E'_{A'}$ is the dual of $E_{A'_0}$, we have two "canonical" continuous maps:

$$E'_{A'}\widehat{\otimes}F_B \to \mathcal{L}_b(E_{A'_0};F_B) \to \mathcal{L}_b(E;F).$$

Since any element of $E'_{A'}\widehat{\otimes}F_B$ can be written in the form

$$u = \sum \lambda_i x'_i \otimes y_i$$

we have such an equality in $\mathcal{L}_b(E;F)$. *p. 5*

Conversely, if $\sum|\lambda_i| < +\infty$, if (x'_i) is an equicontinuous sequence, and if (y_i) is contained inside a **complétante** subset of F, then $\sum \lambda_i x'_i \otimes y_i$ converges in $\mathcal{L}_b(E;F)$, and defines a nuclear operator.

Proposition 4. *For an operator u to be nuclear, it is necessary and sufficient for it to be of the form $u = \sum \lambda_i x'_i \otimes y_i$, where $\sum|\lambda_i| < +\infty$, (x'_i) is an equicontinuous sequence, and (y_i) a sequence contained inside some* ***complétante*** *subset.*

Proposition 5. *If u is nuclear, then $B \circ u \circ A$ is nuclear (Proposition 2).*

Corollary 1. *If $u\colon E \to F$ is nuclear, then it remains nuclear when we strengthen the topology of E and weaken the topology of F; if E_1 is a subspace of E, and F a subspace of F_1, then the restriction $u\colon E_1 \to F_1$ is nuclear.*

However, if u is a nuclear map from E to F, and if $u(E)$ is contained in a subspace F_2 of F, then $u\colon E \to F_2$ is not necessarily nuclear. Similarly, if u is zero on a subspace E_2 of E, then $u\colon E/E_2 \to F$ is not necessarily nuclear.

5 Transpose of a nuclear map

Proposition 6. *Let E and F be Banach spaces, and $u \in L^1(E;F)$. Then ${}^t u \in L^1(F';E')$, and $\|{}^t u\|_1 \leqslant \|u\|_1$. Conversely, if F is reflexive, and ${}^t u$ is nuclear, then u is nuclear, and $\|{}^t u\|_1 = \|u\|)1$.*

Proof. Let $u \in L^1(E;F)$ with $u = \varphi(u_0)$, where $u_0 \in E'\widehat{\otimes}F$. Let i be the injection from $F\widehat{\otimes}E'$ to $F''\widehat{\otimes}E'$. Then ${}^t u\colon F' \to E'$ is given by ${}^t u = \varphi(i(S(u_0)))$, and so ${}^t u$ is nuclear. Since S is an isometry, and $\|i\| \leqslant 1$ (in fact, we can even show that i is an isometry), we have that

$$\|{}^t u\|_1 \leqslant \inf_{\varphi(u_0)=u} \|i(S(u_0))\| \leqslant \inf_{\varphi(u_0)=u} \|u_0\| = \|u\|_1.$$

Finally, if F is *reflexive*, and ${}^t u$ is nuclear, then ${}^{tt}u\colon E'' \to F'' = F$ is nuclear, and so $u\colon E \to F$ is nuclear. In all known cases, this property still holds true even without the reflexivity hypothesis on F. □

Corollary 1. *Let E and F be locally convex separated spaces; if $u\colon E \to F$ is nuclear, then ${}^t u\colon F'_c \to E'_b$ is nuclear, and, a fortiori, ${}^t u\colon F'_b \to E'_b$ or ${}^t u\colon F'_c \to E'_c$.*

Proof. Indeed, ${}^t u = {}^t\alpha{}^t\beta{}^t\gamma$ (see Definition 2), with ${}^t\beta$ nuclear, ${}^t\alpha$ continuous, and ${}^t\gamma$ continuous from F'_c to F'_1 if γ is compact, which we have the right to assume. □

6 Lifting properties

Proposition 7. *Let E, F, and G be locally convex separated spaces, with $E \subset F$; let* p. 6
$u\colon E \to G$ be a nuclear map. Then there exists a nuclear map $v\colon F \to G$ extending u. Furthermore, in the Banach case, we can assume that $\|v\|_1 \leqslant \|u\|_1 + \varepsilon$.

Proof. We restrict ourselves to proving the Banach case.

Consider the diagram that we have already seen (Proposition 2):

$$\begin{tikzcd} F'\widehat{\otimes}G \arrow[r, "{}^t i\otimes 1"] \arrow[d, "\varphi"'] & E'\widehat{\otimes}G \arrow[d, "\varphi"] \\ L^1(F;G) \arrow[r, "u\mapsto i\circ u"'] & L^1(E;G) \end{tikzcd}$$

where i is the injection of E into F. Then the path ↴ is a metric epimorphism, and thus so too is the path ↳. □

Proposition 8. *Let E, F, and G be locally convex separated spaces, with $F \subset E$, and F closed; suppose that every compact disc of E/F is the image of a* ***complétante*** *subset of E. Then every nuclear map $u\colon G \to E/F$ comes from the image (under taking the quotient) of a nuclear map $v\colon G \to E$. Furthermore, in the Banach case, we can assume that $\|v\|_1 \leqslant \|u\|1 + \varepsilon$.*

Proof. Let $H = E/F$. Suppose that u comes from some element u_0 of $G'_{A'}\widehat{\otimes}H_B$, where B is a *compact* **complétante** subset of H (Proposition 3). Let B_1 be a **complétante** subset of E that projects onto B. We have an epimorphism $E_{B_1} \to H_B$, and it suffices to show that u_0 can be obtained from an element of $G'_{A'}\widehat{\otimes}E_{B_1}$ by projection, i.e. that we can reduce to the Banach case. But in this case, we have the following diagram:

$$\begin{tikzcd} G'\widehat{\otimes}E \arrow[r, "1\otimes P"] \arrow[d, "\varphi"'] & G'\widehat{\otimes}H \arrow[d, "\varphi"] \\ L^1(G;E) \arrow[r, "u\mapsto P\circ u"'] & L^1(G;H) \end{tikzcd}$$

and, again, ↴ is a epimorphism, and thus so too is ↳. □

Remark. 1. The conditions of Proposition 8 are satisfied if E is a Fréchet space (or if E is a dual of a Fréchet space) and F is weakly closed. p. 7

2. Returning to Proposition 7: if we use Proposition 4, then we can write u in the form $u = \sum \lambda_i y'_i \otimes z_i$, and, if we simultaneously extend (by Hahn-Banach) the y'_i to equicontinuous forms $\overline{y'_i}$ on F, then we can set $v = \sum \lambda_i \overline{y'_i} \otimes z_i$, and v extends u, which gives another proof of the proposition (and similarly for Proposition 8)