

Ph 21 Homework 2

Emily Springer

January 31, 2020

1 Assignment 1.

1.1

Question 1: Prove for yourself that the definition of the Fourier series is consistent: show that inserting Eq. (3) in Eq. (2) yields an identity.

I began with the definition of $h(x)$ which is Eq. (2) and the definition of \tilde{h} which is Eq. (3).

$$h(x) = \lim_{N \rightarrow \infty} \sum_{k=-N}^N \tilde{h}_k e^{-2\pi i f_k x}$$

$$\tilde{h}_k = \frac{1}{L} \int_0^L h(x) e^{2\pi i f_k x} dx$$

After inserting Eq. (3) into Eq. (2) and switch the order of integration/summation, I got

$$h(\tilde{h}(x)) = \frac{1}{L} \lim_{N \rightarrow \infty} \int_0^L h(x') dx' \sum_{k=-N}^N e^{-2\pi i f_k (x' - x)}$$

This can be simplified to

$$\begin{aligned} &= \frac{1}{L} \lim_{N \rightarrow \infty} \int_0^L h(x') dx' \frac{e^{\frac{1}{L} 2\pi i N (x' - x)} - e^{-\frac{1}{L} 2\pi i N (x' - x)}}{\frac{1}{L} 2\pi i (x' - x)} \\ &= \frac{2}{L} \lim_{N \rightarrow \infty} \int_0^L h(x') dx' \frac{\sin(N)(x' - x)}{\frac{1}{L} 2\pi (x - x')} \\ &= \frac{2}{L} h(x) \lim_{N \rightarrow \infty} \int_0^L dx' \frac{\sin(N)(x' - x)}{\frac{1}{L} 2\pi (x - x')} \\ h(\tilde{h}(x)) &= \frac{1}{\pi} \int_0^L dy \frac{\sin(y)}{y} = \frac{1}{\pi} \pi = 1 \end{aligned}$$

□

1.2

Question 2: Show that a suitable linear combination of $e^{-\frac{2\pi i x}{L}}$ and $e^{\frac{2\pi i x}{L}}$ can represent any function of the form $A \sin(\frac{2\pi x}{L} + \phi)$, for any amplitude A and phase ϕ .

I recalled the sin expansion

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

Substituting $\frac{2\pi x}{L}$ for x , I got

$$\sin\left(\frac{2\pi x}{L}\right) = \frac{e^{\frac{2\pi i x}{L}} - e^{-\frac{2\pi i x}{L}}}{2i}$$

$$2i \sin\left(\frac{2\pi x}{L}\right) = e^{\frac{2\pi i x}{L}} - e^{-\frac{2\pi i x}{L}}$$

$2i \sin\left(\frac{2\pi x}{L}\right)$ is in the form $A \sin\left(\frac{2\pi x}{L} + \phi\right)$ and $e^{\frac{2\pi i x}{L}} - e^{-\frac{2\pi i x}{L}}$ is a linear combination. \square

1.3

Question 3: Show that for real functions $h(x)$, the Fourier coefficients \tilde{h}_k must satisfy the relation $\tilde{h}_{-k} = \tilde{h}_k^*$

Note: $(e^{if(k)})^* = e^{-if(k)}$. With this in mind, I considered \tilde{h}_k .

$$\tilde{h}_k = \frac{1}{L} \int_0^L h(x) e^{2\pi i f_k x} dx$$

where $f_k = \frac{k}{L}$

I substituted in $-k$ for k to find \tilde{h}_{-k} .

$$\tilde{h}_{-k} = \frac{1}{L} \int_0^L h(x) e^{2\pi i \frac{-k}{L} x} dx = \frac{1}{L} \int_0^L h(x) e^{-2\pi i \frac{k}{L} x} dx$$

I took the $*$ operation on \tilde{h}_k , where the imaginary part becomes negative, to get

$$\tilde{h}_k^* = \frac{1}{L} \int_0^L h(x) e^{-2\pi i \frac{k}{L} x} dx$$

Thus, $\tilde{h}_{-k} = \tilde{h}_k^*$. \square

1.4

Question 4: Convince yourself of the convolution theorem.

Starting with $h^{(1)}(x)h^{(2)}(x)$, we have

$$h^{(1)}(x)h^{(2)}(x) = \int_{-\infty}^{\infty} \tilde{h}_{k'}^{(2)} e^{-2\pi i \frac{x}{L} k'} dk' \int_{-\infty}^{\infty} \tilde{h}_q^{(1)} e^{-2\pi i \frac{x}{L} q} dq$$

We can combine the integrals to get

$$h^{(1)}(x)h^{(2)}(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{h}_q^{(1)} \tilde{h}_{k'}^{(2)} e^{-2\pi i \frac{x}{L} (q+k')} dq dk'$$

We can do a substitution where $q = k - k'$, so $dq = dk$. Then, we have

$$h^{(1)}(x)h^{(2)}(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{h}_{k-k'}^{(1)} \tilde{h}_{k'}^{(2)} e^{-2\pi i \frac{k}{L} x} dk dk'$$

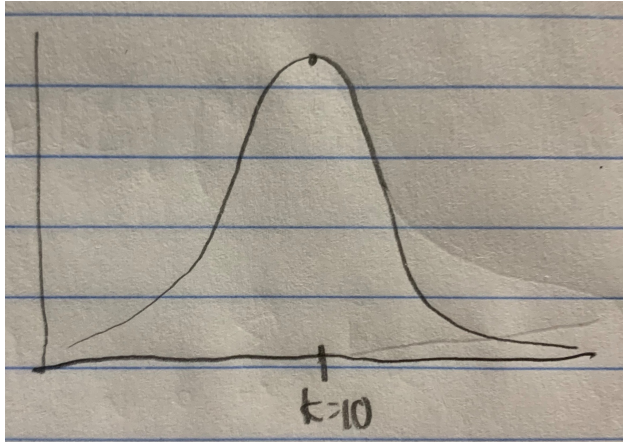
Reordering this, we have

$$h^{(1)}(x)h^{(2)}(x) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \tilde{h}_{k-k'}^{(1)} \tilde{h}_{k'}^{(2)} dk' \right] e^{-2\pi i \frac{k}{L} x} dk$$

Now this is the same as the convolution product $\tilde{h}_{k-k'}^{(1)} \tilde{h}_{k'}^{(2)}$ \square

Question 4.5: Sketch a graph of \tilde{H}_k for a smooth $\tilde{h}_{k'}^{(1)}$ around $k = 0$, and for a $\tilde{h}_{k'}^{(2)}$ consisting of a single unit pulse at $k = 10$.

With the convolution, $\tilde{h}_{k'}^{(2)}$ will shift the smooth graph to be centered around its center ($k = 10$). Therefore, the graph looks like this:



1.5

Question 5: Compute the Fourier Transform of $C + A \cos(ft + \phi)$ and $Ae^{-B(\frac{t-L}{2})^2}$.

To do this analytically, I substituted in the functions into the definition of $h(x)$ to get

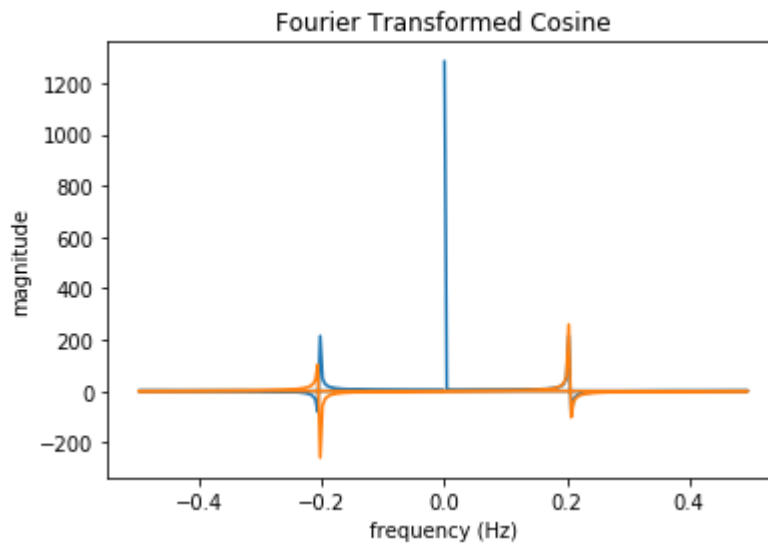
$$\sum_{-\infty}^{\infty} (C + A \cos(ft + \phi)) e^{-2\pi i f t}$$

$$\sum_{-\infty}^{\infty} (Ae^{-B(\frac{t-L}{2})^2}) e^{-2\pi i f t}$$

For the second part of the question, I wrote code using numpy to compute the Fourier Transform for the cosine and Gaussian functions. For the cosine function, I wrote

```
sp = np.fft.fft(C + A * np.cos(f * t + phi))
freq = np.fft.fftfreq(t.shape[-1])
plt.plot(freq, sp.real, freq, sp.imag)
plt.show()
```

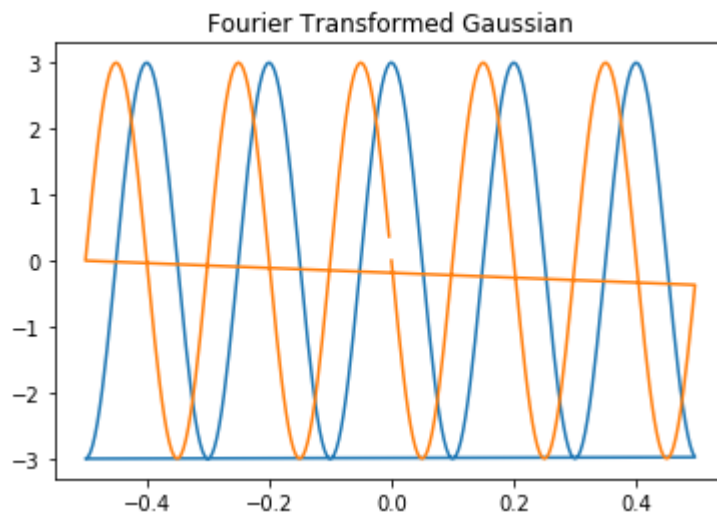
The resulting graph was



For the Gaussian curve, I wrote

```
sp2 = np.fft.fft(A * np.exp(-B * (t - f) ** 2))
plt.plot(freq, sp2.real, freq, sp2.imag)
plt.show()
```

The resulting graph was



2 Assignment 2.

2.1

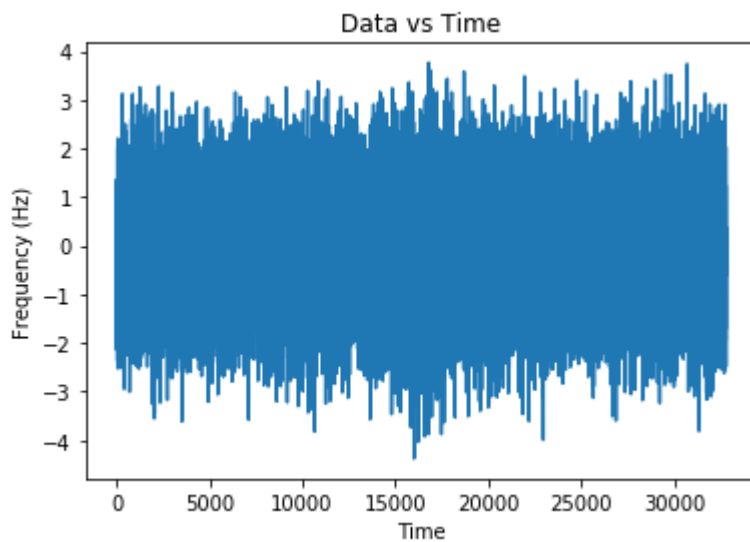
Question 1: Plot the data as a function of time. Using FFT to see signal and obtain frequency.

Using the text file `arecibo.txt`, I wrote this code to read from the file and keep the data in an array.

```
with open('arecibo1.txt') as f :  
    lines = f.readlines()  
    for line in lines :  
        values = [float(s) for s in line.split()]  
        x.append(i)  
        y.append(values[0])  
        i += 1;
```

I then plotted it against time.

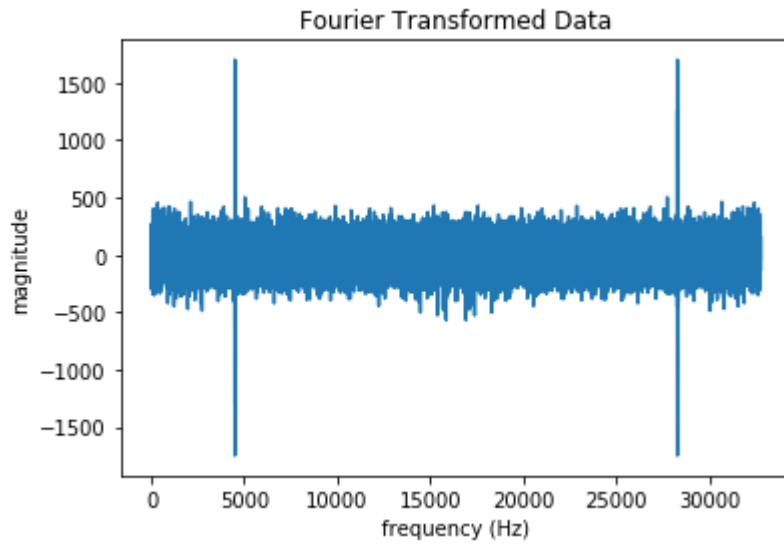
```
plt.plot(x, y)  
plt.show()
```



Then, I used numpy again to calculate the FFT using the line

```
spectrum = np.fft.fft(y)
```

Here was the resulting graph



I obtained the first frequency peak with

```
ff_ii = np.where(np.abs(spectrum) > 1000.0)[0][0]
```

I got a frequency of 4486.

2.2

Question 2: Use Gaussian envelope to find δt .

I added to my other code the lines

```
dt = 0.213
L = 1.
f = L/2
frequency = 3970.
t = np.linspace(0, L, 10000)
sp = np.fft.fft((np.sin(frequency * 2 * np.pi * t)) * np.exp(-1 * (t - f) ** 2 / dt ** 2))
freq = np.fft.fftfreq(t.shape[-1], L/30000.)
plt.plot(freq + 16380, abs(sp))
```

I had to play around with the numbers until the curve's peaks lined up with the original. The result was



That means δt is around 0.213.

3 Assignment 3.

3.1

Question 1: Find a Python implementation of the Numerical Recipes Lomb-Scargle algorithm.

After searching online, I found

```
from astropy.timeseries import LombScargle
```

3.2

Question 2: Use the python Lomb-Scargle routine to analyze the Gaussian distribution from Part I of the assignment, and the first arecibo data set, from Part II of the assignment.

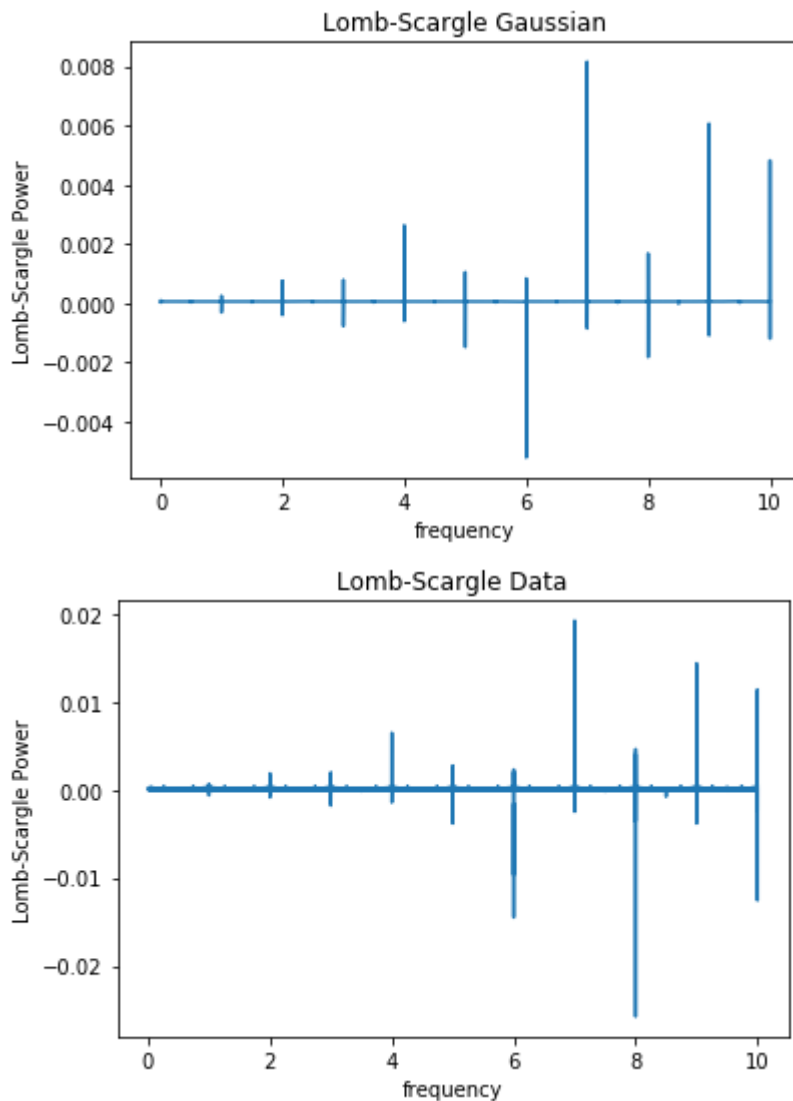
I set the Gaussian curve with

```
t = np.arange(32768)
gaus = 3 * np.exp(-100 * (t - 5) ** 2)
```

I retrived the data from the text in the same way as previously. To apply the Lomb-Scargle algorithm, I did

```
frequency, power = LombScargle(t, gaus).autopower(minimum_frequency = 0, maximum_frequency = 10)
frequency2, power2 = LombScargle(t, y).autopower(minimum_frequency = 0, maximum_frequency = 10)
```

The resulting graphs were



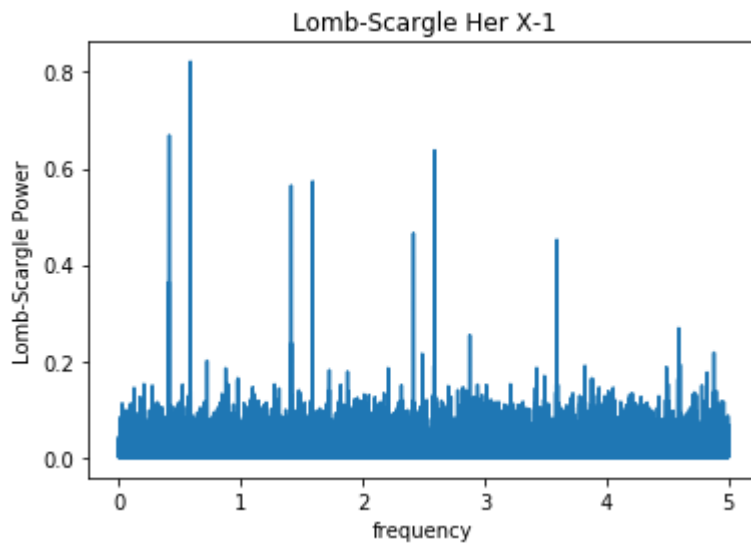
3.3

Question 3: Use the module developed for the Ph21.1 assignment to grab data from the Catalina Real Time Survey, in particular for the source Her X-1, a binary system containing a neutron star. Use Lomb-Scargle to find significant frequencies in the data.

Copying my code from previously, I had the x and y data in variables called xData and yData, so all I had to do was

```
frequency, power = LombScargle(xData, yData).autopower(minimum_frequency = 0, maximum_frequency = 5)
```

The resulting graph was



We can see a peak around 1.7 for the orbital period in days. We can also see many other significant frequencies. These could be due to other nearby celestial bodies' orbitals.