Ph 21 Homework 2

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1 Assignment 1.

1.1

Question 1: Prove for yourself that the definition of the Fourier series is consistent: show that inserting Eq. (3) in Eq. (2) yields an identity.

I began with the definition of $h(x)$ which is Eq. (2) and the definition of $\tilde{h}$ which is Eq. (3).

$$h(x) = \lim_{N\to\infty} \sum_{k=-N}^{N} \tilde{h}_k e^{-2\pi i f_k x}$$

$$\tilde{h}_k = \frac{1}{L}\int_0^L h(x) e^{2\pi i f_k x} dx$$

After inserting Eq. (3) into Eq. (2) and switch the order of integration/summation, I got

$$h(\tilde{h}(x)) = \frac{1}{L}\lim_{N\to\infty}\int_0^L h(x')dx' \sum_{k=-N}^{N} e^{-2\pi i f_k (x'-x)}$$

This can be simplified to

$$\begin{aligned}
&= \frac{1}{L}\lim_{N\to\infty}\int_0^L h(x')dx' \frac{e^{\frac{1}{L}2\pi i N(x'-x)} - e^{-\frac{1}{L}2\pi i N(x'-x)}}{\frac{1}{L}2\pi i(x'-x)} \\
&= \frac{2}{L}\lim_{N\to\infty}\int_0^L h(x')dx' \frac{\sin(N)(x'-x)}{\frac{1}{L}2\pi(x-x')} \\
&= \frac{2}{L}h(x)\lim_{N\to\infty}\int_0^L dx' \frac{\sin(N)(x'-x)}{\frac{1}{L}2\pi(x-x')} \\
h(\tilde{h}(x)) &= \frac{1}{\pi}\int_0^L dy\frac{\sin(y)}{y} = \frac{1}{\pi}\pi = 1
\end{aligned}$$

□

1.2

Question 2: Show that a suitable linear combination of $e^{-\frac{2\pi i x}{L}}$ and $e^{\frac{2\pi i x}{L}}$ can represent any function of the form $A\sin(\frac{2\pi x}{L} + \phi)$, for any amplitude A and phase ϕ.

I recalled the sin expansion

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

Substituting $\frac{2\pi x}{L}$ for x, I got

$$\sin\left(\frac{2\pi x}{L}\right) = \frac{e^{\frac{2\pi i x}{L}} - e^{-\frac{2\pi i x}{L}}}{2i}$$

$$2i\sin\left(\frac{2\pi x}{L}\right) = e^{\frac{2\pi i x}{L}} - e^{-\frac{2\pi i x}{L}}$$

$2i\sin\left(\frac{2\pi x}{L}\right)$ is in the form $A\sin(\frac{2\pi x}{L} + \phi)$ and $e^{\frac{2\pi i x}{L}} - e^{-\frac{2\pi i x}{L}}$ is a linear combination. □

1.3

Question 3: Show that for real functions $h(x)$, the Fourier coefficients $\tilde{h}_k$ must satisfy the relation $\tilde{h}_{-k} = \tilde{h}_{k^*}$

Note: $(e^{if(k)})^* = e^{-if(k)}$. With this in mind, I considered $\tilde(h)_k$.

$$\tilde{h}_k = \frac{1}{L}\int_0^L h(x)e^{2\pi i f_k x}dx$$

where $f_k = \frac{k}{L}$

I substituted in $-k$ for k to find $\tilde{h}_{-k}$.

$$\tilde{h}_k = \frac{1}{L}\int_0^L h(x)e^{2\pi i \frac{-k}{L} x}dx = \frac{1}{L}\int_0^L h(x)e^{-2\pi i \frac{k}{L} x}dx$$

I took the $*$ operation on $\tilde{h}_k$, where the imaginary part becomes negative, to get

$$\tilde{h}_{k^*} = \frac{1}{L}\int_0^L h(x)e^{-2\pi i \frac{k}{L} x}dx$$

Thus, $\tilde{h}_{-k} = \tilde{h}_{k^*}$. □

1.4

Question 4: Convince yourself of the convolution theorem.

Starting with $h^{(1)}(x)h^{(2)}(x)$, we have

$$h^{(1)}(x)h^{(2)}(x) = \int_{-\infty}^{\infty} \tilde{h}^{(2)}_{k'} e^{-2\pi i \frac{x}{L} k'} dk' \int_{-\infty}^{\infty} \tilde{h}^{(1)}_{q} e^{-2\pi i \frac{x}{L} q} dq$$

We can combine the integrals to get

$$h^{(1)}(x)h^{(2)}(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{h}^{(1)}_{q}\tilde{h}^{(2)}_{k'} e^{-2\pi i \frac{x}{L}(q+k')} dq dk'$$

We can do a substitution where $q = k - k'$, so $dq = dk$. Then, we have

$$h^{(1)}(x)h^{(2)}(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{h}^{(1)}_{k-k'}\tilde{h}^{(2)}_{k'} e^{-2\pi i \frac{k}{L} x} dk dk'$$

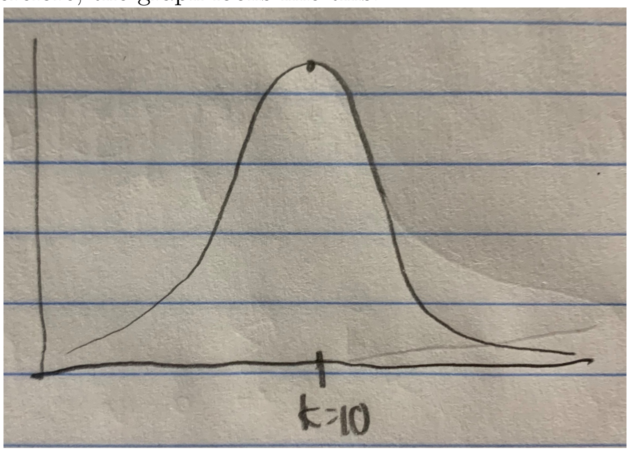
Reordering this, we have

$$h^{(1)}(x)h^{(2)}(x) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} \tilde{h}^{(1)}_{k-k'}\tilde{h}^{(2)}_{k'} dk'\right] e^{-2\pi i \frac{k}{L} x} dk$$

Now this is the same as the convolution product $\tilde{h}^{(1)}_{k-k'}\tilde{h}^{(2)}_{k'}$ □

Question 4.5: Sketch a graph of $\tilde{H}_k$ for a smooth $\tilde{h}^{(1)}_{k'}$ around $k = 0$, and for a $\tilde{h}^{(2)}_{k'}$ consisting of a single unit pulse at $k = 10$.

With the convolution, $\tilde{h}^{(2)}_{k'}$ will shift the smooth graph to be centered around its center ($k = 10$). Therefore, the graph looks like this:



1.5

Question 5: Compute the Fourier Transform of $C + A\cos(ft + \phi)$ and $Ae^{-B\left(\frac{t-L}{2}\right)^2}$.

To do this analytically, I substituted in the functions into the definition of $h(x)$ to get

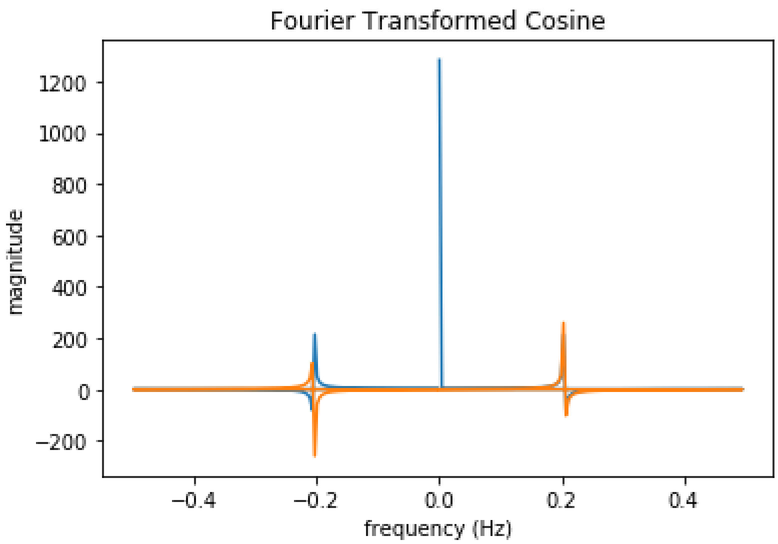
$$\sum_{-\infty}^{\infty}(C + A\cos(ft + \phi))e^{-2\pi ift}$$

$$\sum_{-\infty}^{\infty}(Ae^{-B\left(\frac{t-L}{2}\right)^2}e^{-2\pi ift}$$

For the second part of the question, I wrote code using numpy to compute the Fourier Transform for the cosine and Gaussian functions. For the cosine function, I wrote

$$sp = np.fft.fft(C + A * np.cos(f * t + phi))$$
$$freq = np.fft.fftfreq(t.shape[-1])$$
$$plt.plot(freq, sp.real, freq, sp.imag)$$
$$plt.show()$$

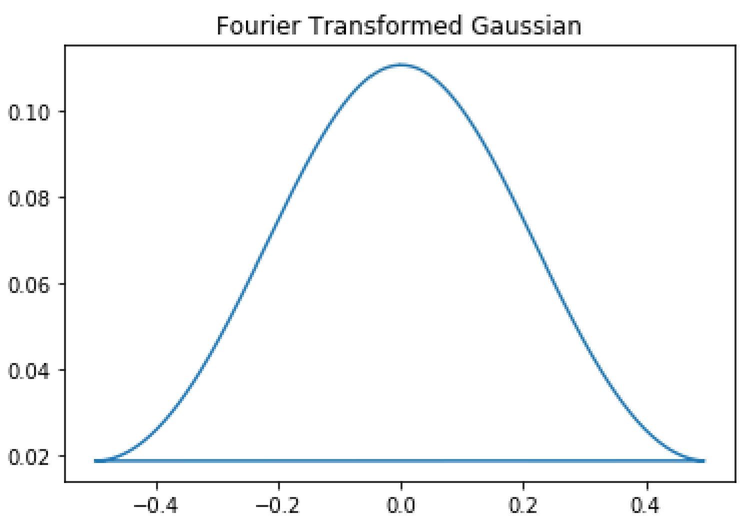
The resulting graph was



For the Gaussian curve, I wrote

$$y = A * np.exp(-B * (t - f) * *2)$$
$$y_f ft = (np.abs(np.fft.fft(y)))/np.sqrt(len(y))$$
$$plt.plot(freq, y_f ft)$$
$$plt.show()$$

The resulting graph was



2 Assignment 2.

2.1

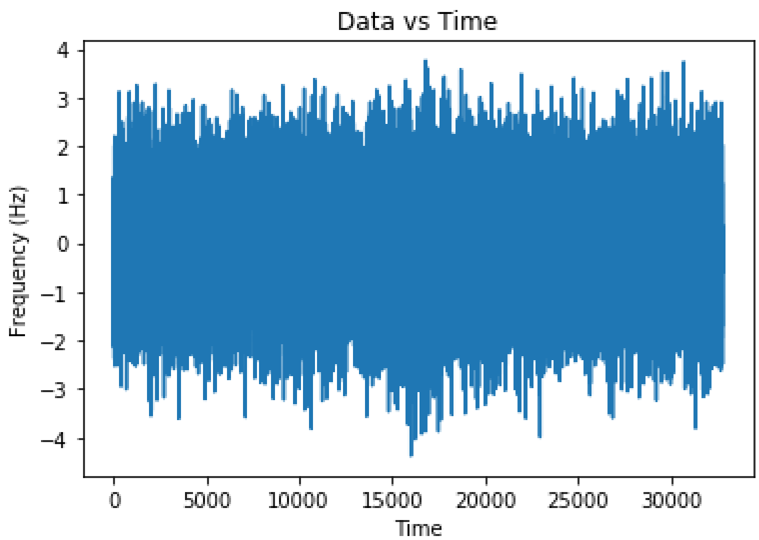
Question 1: Plot the data as a function of time. Using FFT to see signal and obtain frequency.

Using the text file arecibo.txt, I wrote this code to read from the file and keep the data in an array.

$$with\ open('arecibo1.txt')\ as\ f:$$
$$lines = f.readlines()$$
$$for\ line\ in\ lines:$$
$$values = [float(s)\ for\ s\ in\ line.split()]$$
$$x.append(i)$$
$$y.append(values[0])$$
$$i+=1;$$

I then plotted it against time.

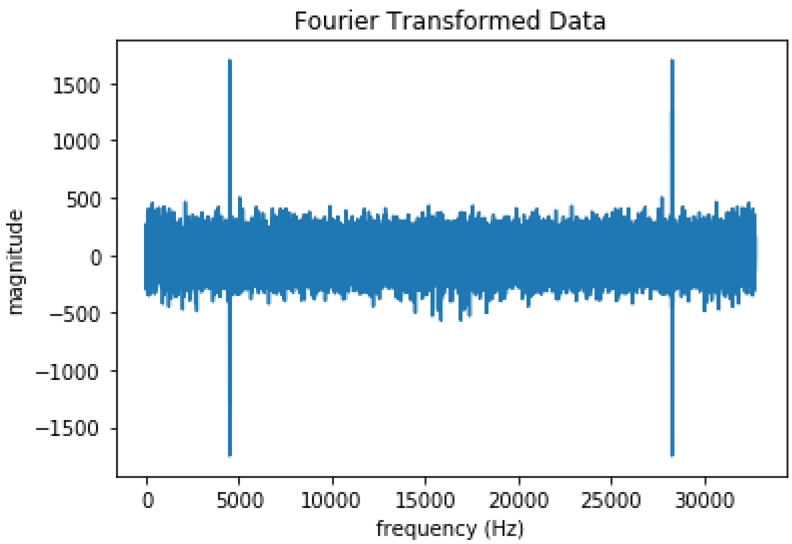
$$plt.plot(x, y)$$
$$plt.show()$$



Then, I used numpy again to calculate the FFT using the line

$$spectrum = np.fft.fft(y)$$

Here was the resulting graph



I obtained the first frequency peak with

$$ff\_ii = np.where(np.abs(spectrum) > 1000.0)[0][0]$$

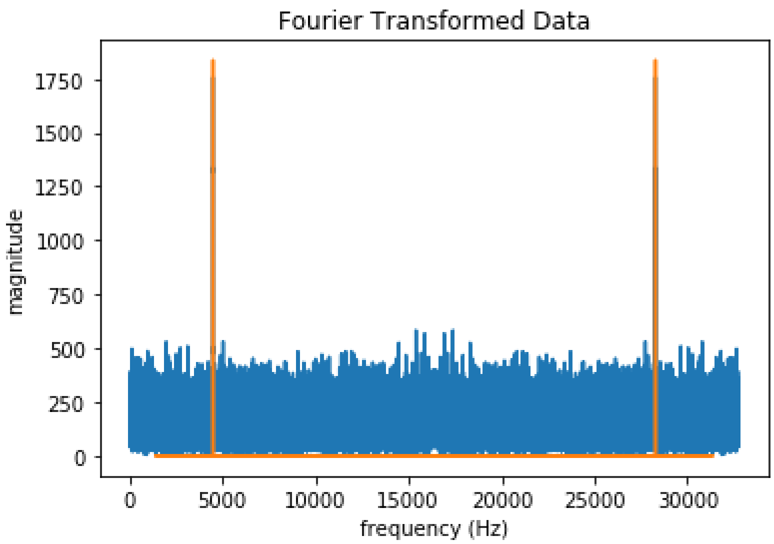
I got a frequency of 4486.

2.2

Question 2: Use Gaussian envelope to find δt.
I added to my other code the lines

$dt = 0.16$
$L = 1.$
$f = L/2$
$frequency = 3964.5$
$t = np.linspace(0, L, 10000)$
$sp = np.fft.fft(1.3 * (np.sin(frequency * 2 * np.pi * t)) * np.exp(-1 * (t - f) * *2/dt * *2))$
$freq = np.fft.fftfreq(t.shape[-1], L/30000.)$
$plt.plot(freq + 16380, abs(sp))$

I had to play around with the numbers until the curve's peaks lined up with the original. The result was



That means δt is around 0.213.

3 Assignment 3.

3.1

Question 1: Find a Python implementation of the Numerical Recipes Lomb-Scargle algorithm.

After searching online, I found

$$from\ astropy.timeseries\ import\ LombScargle$$

3.2

Question 2: Use the python Lomb-Scargle routine to analyze the Gaussian distribution from Part I of the assignment, and the first arecibo data set, from Part II of the assignment.

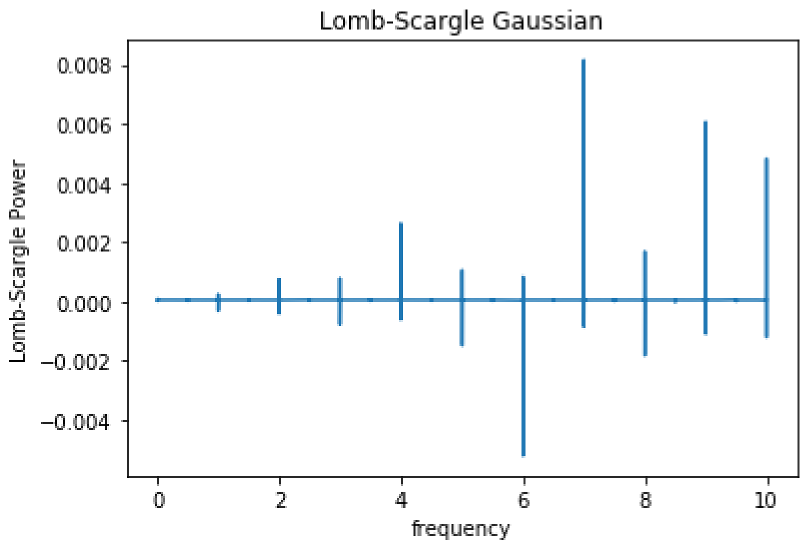
I set the Gaussian curve with

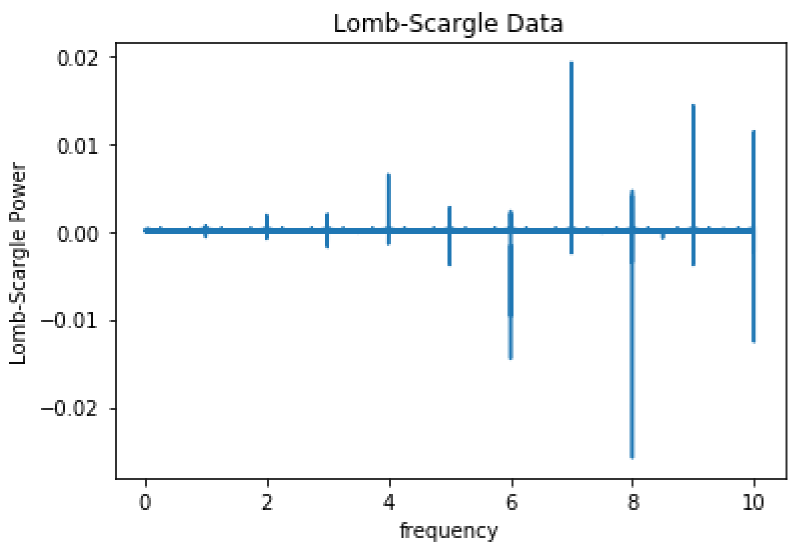
$$t = np.arange(32768)$$
$$gaus = 3 * np.exp(-100 * (t - 5) * *2)$$

I retrived the data from the text in the same way as previously. To apply the Lomb-Scargle algorithm, I did

$$frequency, power = LombScargle(t, gaus).autopower(minimum_f requency = 0, maximum_f requency = 10)$$
$$frequency2, power2 = LombScargle(t, y).autopower(minimum_f requency = 0, maximum_f requency = 10)$$

The resulting graphs were





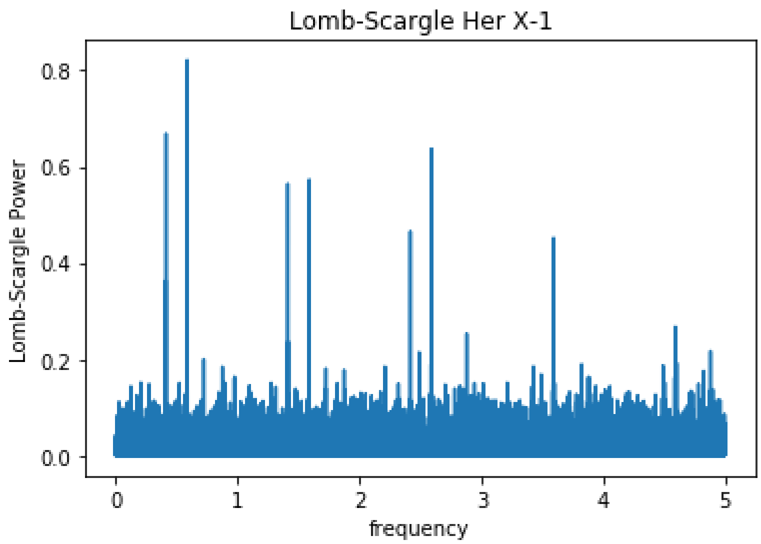
3.3

Question 3: Use the module developed for the Ph21.1 assignment to grab data from the Catalina Real Time Survey, in particular for the source Her X-1, a binary system containing a neutron star. Use Lomb-Scargle to find significant frequencies in the data.

Copying my code from previously, I had the x and y data in variables called xData and yData, so all I had to do was

$$frequency, power = LombScargle(xData, yData).autopower(minimum_frequency = 0, maximum_frequency = 5)$$

The resulting graph was



We can see a peak around 1.7 for the orbital period in days. We can also see many other significant frequencies. These could be due to other nearby celestial bodies' orbitals.