Midterm

Marc Eskew

4/29/2022

Problem 1

1.1

TRUE This is true due to the property of inversion that says if X has a CDF $F(.)$ and $U \sim U(0,1)$, then $X = F^{-1}(U)$ and U and $1-U$ will have the same distribution.

1.2

FALSE For any RV X and Y with those conditions, this is saying that $P(X \leq z) \leq P(Y \leq z)$ and therefore $E(X) \geq E(Y)$. As the RV with the greater CDF has a higher probability of reaching a value z earlier, with an increasing function $h(\cdot)$, $E(h(X)) \geq E(h(Y))$.

1.3

TRUE If $f(\cdot)$ is symmetric then the CDF function must satisfy $F(X) + F(-X) = 1$. When applying the Gaussian copula, the resulting distribution will be inversely correlated with $\rho = -1$.

Problem 2

2.1

(R, Z) can be sampled from a method using acceptance/rejection and inversion. This will allow us to uniformly sample within the region A.

Step 1 Note that $z = \frac{1}{\sqrt{r}} - 1$ is a density as $\int_0^1 \frac{1}{\sqrt{r}} - 1 = 1$.

Step 2 This CDF for Z cannot be easily inverted, so we will use acceptance/rejection to sample from $f(r)$. We set $g(r) = \frac{1}{2\sqrt{r}}$ and find C.

$$\begin{aligned}
f(r) &\leq Cg(r) \\
\frac{1}{\sqrt{r}} - 1 &\leq \frac{C}{2\sqrt{r}} \\
2 - 2\sqrt{r} &\leq C \\
2 &\leq C
\end{aligned}$$

With $G(r) = \sqrt{r}$ then $R = U^2$ with $U \sim U(0,1)$.

Step 3 Sample R and perform acceptance/rejection for Z sampling V $U(0,1)$ iid from R and test to satisfy $f(r) \geq VCg(r)$. Accept sample (R, Z) if satisfies, reject otherwise.

2.2

We can sample X and Y by considering the RVs points within a circle of radius 1. We can sample these by taking random samples of length and radius from the origin.

$$\begin{aligned} l &= \sqrt{U} && \text{where } U \sim U(0,1) \\ \phi &= 2\pi V && \text{where } V \sim U(0,1) \end{aligned}$$

$$\begin{aligned} X &= lcos(\phi) \\ Y &= lsin(\phi) \end{aligned}$$

With X and Y satisfying the first condition; this is analogous to the first part with sampling R. We can use the same acceptance/rejection procedure outlined in part 1 to sample Z be setting $X^2 + Y^2 = R$.

2.3

We can use acceptance/rejection method to sample from the surface. This involves sampling X, Y, Z, normalizing and using the a/r method. Here we're setting $z = \frac{1}{\sqrt{\sqrt{x^2+y^2}}} - 1$

Step 1 Sample X, Y, Z using the method outlined previously.

Step 2 Set $\tilde{\theta} = (X, Y, Z)/\sqrt{(}X^2 + Y^2 + Z^2)$

Step 3 Sample $U \sim U(0,1)$

Step 4 If $U \leq \frac{f(\theta)}{Cg(\theta)}$ the set $\theta = \tilde{\theta}$. Otherwise, reject and resample.

Problem 3

3.1

The random variables (X, Y) are not independent. There is no way to put the density if $f(x, y) = f_1(x)f_2(y)$ where f_1 and f_2 are densities. This is due to the indicator function, $I(|x| + |y| \leq 1)$ in the numerator. Therefore, the two random variables are dependent.

3.2

$|x|$ and $|y|$ are both $[0,1]$ due to $I(|x| + |y| \leq 1)$. Therefore:

$$\begin{aligned} |x|I(|x| + |y|) \leq |x| \\ h(x) &= \frac{\alpha}{\sqrt{|x|}} I(-1 \leq x \leq 1) \\ h(y) &= \frac{\alpha}{\sqrt{|y|}} I(-1 \leq y \leq 1) \end{aligned}$$

Where:

$$\begin{aligned} \int_{-1}^{1} \alpha \frac{1}{\sqrt{|x|}} &= 1 \\ \alpha &= \frac{1}{4} \end{aligned}$$

We can then set our function $g(x, y)$ using independent functions $h(x)$ and $h(y)$:

$$
\begin{aligned}
h(x) &= \frac{1}{4\sqrt{|x|}} \\
h(y) &= \frac{1}{4\sqrt{|y|}} \\
g(x,y) &= h(x)h(y) \\
&= \frac{1}{16\sqrt{|x|}\sqrt{|y|}}
\end{aligned}
$$

We can find that there exists a C such that $f(x.y) \leq Cg(x,y)$ by solving for C:

$$
\begin{aligned}
\frac{I(|x|+|y| \leq 1)}{K\sqrt{|x|}\sqrt{|y|}} &\leq \frac{C}{16\sqrt{|x|}\sqrt{|y|}} \\
\frac{16I(|x|+|y| \leq 1)}{K} &\leq C \\
\frac{16}{K} &\leq C
\end{aligned}
$$

3.3

With $h(x)$ we can set a symmetric density function $h(x) = \frac{1}{4\sqrt{x}}I(x \geq 0) + \frac{1}{4\sqrt{-x}}I(x < 0)$ and determine the CDF $H(x)$. Using inversion, to sample from $h(x)$

Step 1 Sample from $U1, U2$ i.i.d. from $U(0,1)$

Step 2 Let $W_1 = \frac{1}{16U_1^2}$

Step 3 If $U_2 \leq .5$ then $W = W_1$ otherwise $W = -W_1$

Utilize the same procedure to sample $h(y)$.

3.4

The normalizing constant, K, cancels out when we check whether to accept the sample or not by using the equation:

$$
\frac{f(x,y)}{Cg(x,y)} = \frac{\frac{I(|x|+|y| \leq 1)}{K\sqrt{|x|}\sqrt{|y|}}}{\frac{16}{K}\frac{1}{16\sqrt{|x|}\sqrt{|y|}}}
$$

Therefore, K does not need to be explicitly known.

To use acceptance/rejection on $f(x,y)$ with the proposal density of $g(x,y)$ and C the following procedure will be used:

Step 1 Sample X, Y iid from $g(x,y)$.

Step 2 Sample $U \sim U(0,1)$ independently from X, Y.

Step 3 If $U \leq I(X+Y \leq 1)$ accept, otherwise reject and go back to step 1.