Differential Equations

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Introduction

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Aim: Background on $\mathbb{R}$; Basic Existence Question of ODE's

1.1   Romeo and Juliet

$$\begin{cases} R' = aR + bJ \\ J' = cR + dJ \end{cases}$$

These equations model the rate of change of Romeo's and Juliet's feelings. We call this a **linear system of two coupled differential equations of first order in two unknowns**.

- What makes it linear is that the functions and variables appear in a linear fashion.

- What makes it coupled is that both equations have both R and J in them.

- An **uncoupled system** would look like:

$$\begin{cases} R' = aR \\ J' = bJ \end{cases}$$

- First-order refers to the fact that all the derivatives are the first derivatives.

"Identically cautious lovers":

$$\begin{aligned} R' &= aR + bJ \quad a < 0, b > 0 \\ J' &= bR + aJ \quad |a| > |b| \end{aligned}$$

We may have initial conditions, $R(0)$ and $J(0)$, and plot them on a **phase plane** with R against J. In this case, no matter where the starting point is, the trajectory will go towards a **stable node**.

In the case of $|a| < |b|$, points will move asymptotically towards $R = -J$ and $R = J$. In the case of $|a| = |b|$, points will cycle around the origin infinitely.
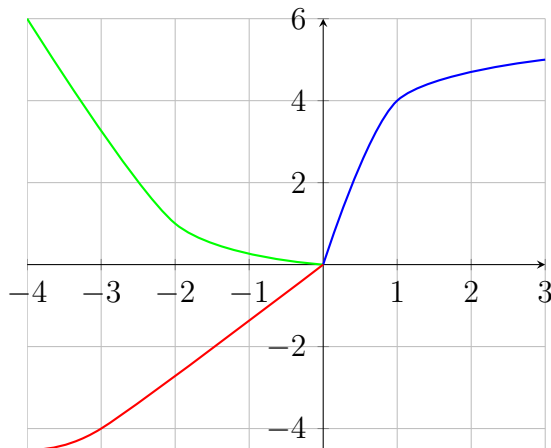
1.2   Supremum and Infimum of a Set $\mathcal{A} \subseteq \mathbb{R}$

- If $\mathcal{A} \in (-\infty, b]$ for some $b \in \mathbb{R}$, we say $\mathcal{A}$ is bounded above, and that b is an **upper bound** for A.

Theorem 1.1 (Supremum Theorem). *If $\mathcal{A} \in \mathbb{R}$, $\mathcal{A} \neq \emptyset$, and $A \subseteq (-\infty, b]$ for some $b \in \mathbb{R}$, then there exists $a \in \mathbb{R}$ such that $\mathcal{A} \subseteq (-\infty, a]$ but if $x < a$, then $\mathcal{A} \not\subseteq (-\infty, x]$.*
*We write $a = \sup \mathcal{A}$, call it the **supremum** of $\mathcal{A}$.*

Why is this necessary? Consider the set $\mathcal{A} = \{-\frac{1}{n} | n \in \mathbb{N}\}$. It does not have a maximum persay, but it has a supremum $\sup \mathcal{A} = 0$.

Consider this example: What is $\sup(-\mathbb{N})$? It is -1, which also happens to be the maximum of the set.

Theorem 1.2. *If max A exists as a real number, then $\sup A = \max A$.*

But to answer all these questions, we need to figure out: what exactly are the real numbers?

1.3   What is $\mathbb{R}$?

Let $x = (s, N, d_1, d_2, d_3, \ldots, d_k, \ldots)$, where:

- $s \in \{+1, -1\}$

- $N \in \mathbb{Z}$

- $d_k \in \mathbb{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

- $\neg(\exists k : d_{k+1} = d_{k+2} = \cdots = 0)$

In this case, "2.49" is shorthand for $(+1, 2, 4, 8, 9, 9, 9, \ldots)$