

# Differential Equations

Brandon Lin  
Stuyvesant High School  
Spring 2016  
Teacher: Mr. Stern  
February 4, 2016

# Introduction

## 1 2/3/16

**Aim:** Background on  $\mathbb{R}$ ; Basic Existence Question of ODE's

### 1.1 Romeo and Juliet

$$\begin{cases} R' = aR + bJ \\ J' = cR + dJ \end{cases}$$

These equations model the rate of change of Romeo's and Juliet's feelings. We call this a **linear system of two coupled differential equations of first order in two unknowns**.

- What makes it linear is that the functions and variables appear in a linear fashion.
- What makes it coupled is that both equations have both  $R$  and  $J$  in them.
- An **uncoupled system** would look like:

$$\begin{cases} R' = aR \\ J' = bJ \end{cases}$$

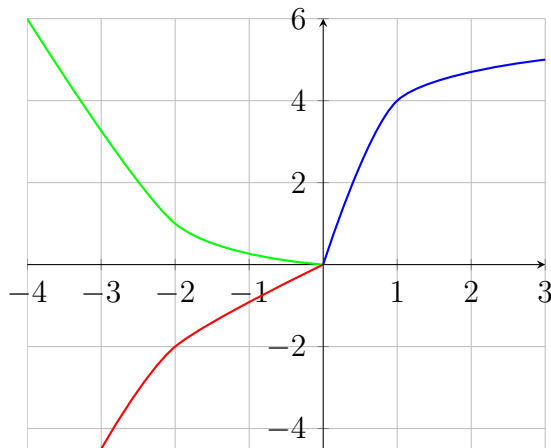
- First-order refers to the fact that all the derivatives are the first derivatives.

“Identically cautious lovers”:

$$\begin{aligned} R' &= aR + bJ & a < 0, b > 0 \\ J' &= bR + aJ & |a| > |b| \end{aligned}$$

We may have initial conditions,  $R(0)$  and  $J(0)$ , and plot them on a **phase plane** with  $R$  against  $J$ . In this case, no matter where the starting point is, the trajectory will go towards a **stable node**.

In the case of  $|a| < |b|$ , points will move asymptotically towards  $R = -J$  and  $R = J$ . In the case of  $|a| = |b|$ , points will cycle around the origin infinitely.



## 1.2 Supremum and Infimum of a Set $\mathcal{A} \subseteq \mathbb{R}$

- If  $\mathcal{A} \subseteq (-\infty, b]$  for some  $b \in \mathbb{R}$ , we say  $\mathcal{A}$  is bounded above, and that  $b$  is an **upper bound** for  $\mathcal{A}$ .

**Theorem 1.1** (Supremum Theorem). *If  $\mathcal{A} \subseteq \mathbb{R}$ ,  $\mathcal{A} \neq \emptyset$ , and  $\mathcal{A} \subseteq (-\infty, b]$  for some  $b \in \mathbb{R}$ , then there exists  $a \in \mathbb{R}$  such that  $\mathcal{A} \subseteq (-\infty, a]$  but if  $x < a$ , then  $\mathcal{A} \not\subseteq (-\infty, x]$ . We write  $a = \sup \mathcal{A}$ , call it the **supremum** of  $\mathcal{A}$ .*

Why is this necessary? Consider the set  $\mathcal{A} = \{-\frac{1}{n} | n \in \mathbb{N}\}$ . It does not have a maximum per say, but it has a supremum  $\sup \mathcal{A} = 0$ .

Consider this example: What is  $\sup(-\mathbb{N})$ ? It is -1, which also happens to be the maximum of the set.

**Theorem 1.2.** *If  $\max A$  exists as a real number, then  $\sup A = \max A$ .*

But to answer all these questions, we need to figure out: what exactly are the real numbers?

## 1.3 What is $\mathbb{R}$ ?

Let  $x = (s, N, d_1, d_2, d_3, \dots, d_k, \dots)$ , where:

- $s \in \{+1, -1\}$
- $N \in \mathbb{Z}$
- $d_k \in \mathbb{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- $\neg(\exists k : d_{k+1} = d_{k+2} = \dots = 0)$ , this is to prevent multiple sequences from being the same number

In this case, “2.49” is shorthand for  $(+1, 2, 4, 8, 9, 9, 9, \dots)$

## 2 2/4/16

**Aim:** Background in  $\mathbb{R}$ ; Fundamental Existence/Uniqueness Question

Recall:

**Theorem 2.1** (Supremum/Infimum Theorem).

1. If  $\mathcal{A}$  is a non-empty set of  $\mathbb{R}$ , and is bounded above (i.e.  $\subseteq (-\infty, b]$  for some  $b \in \mathbb{R}$ ), then there is a least upper bound for  $\mathcal{A}$ , namely  $a \in \mathbb{R}$  such that

$$(a) \mathcal{A} \subseteq (-\infty, a]$$

$$(b) \text{ if } x < a, \text{ then } \mathcal{A} \not\subseteq (-\infty, x]$$

This  $a$  is called the **supremum** of  $\mathcal{A}$ , written  $\sup A$ .

2.  $\inf A$ . This is the greatest lower bound for  $\mathcal{A}$ , or the **infimum**, provided  $\mathcal{A} \neq \emptyset$  and  $\mathcal{A}$  has a lower bound at all.

Recall that the Riemann integral is taking the limit of a partition over an interval  $[a, b]$ . But when we take the limit, we make the mesh of the partition,  $\|\mathcal{P}\|$ , approach zero, where

$$\mathcal{P} = \max_{1 \leq i \leq n} \Delta x_i$$

To fix this, we can define:

$$\int_a^b f(x) dx = \sup \left\{ \sum_{i=1}^n [\inf\{f(x) \mid x_{i-1} \leq x \leq x_i\} \Delta x_i] \mid a = x_0 < x_1 < \cdots < x_n = b \right\}$$

This is a “down-and-up” procedure. The sum of the rectangle areas is a down approximation since we use the minimum possible height to find the area. Then, we take the supremum of that, since for any lower approximation there will always be a higher approximation. Turns out there will never be a maximum; that’s why we take the supremum. This is a **lower Riemann sum**

We can also define the same thing for an **upper Riemann sum**:

$$\int_a^b f(x) dx = \inf \left\{ \sum_{i=1}^n [\sup\{f(x) \mid x_{i-1} \leq x \leq x_i\} \Delta x_i] \mid a = x_0 < x_1 < \cdots < x_n = b \right\}$$

Therefore, the following inequality is true:

$$\int_a^b f \leq \int_a^b \bar{f}$$

If these two are equal, then we say that  $f$  is **Riemann integrable**.

Here’s an example of a function that is NOT Riemann integrable:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 1 & \text{if } x \in [0, 1] \setminus \mathbb{Q} \end{cases}$$

Note that  $\int_0^1 f = 0$  and  $\int_0^1 \bar{f} = 1$ , so this is not Riemann integrable.

## 2.1 Real Numbers, Again

We have shorthand for our previous definition of the real numbers.

$$\mathbb{R} = \{0\} \cup \{(s, N, d_1, d_2, \dots, d_k, \dots \mid s \in \{-1, +1\}, N \in \mathbb{Z}^+, d_k \in \mathbb{D}, \text{no 0-tail}\}$$

and the positive reals:

$$\mathbb{R}^+ = \{(s, N, d_1, d_2, \dots) \mid s = +1\}$$

Let us write  $x = \underline{N}.d_1d_2d_3\dots$  and  $y = \underline{M}.e_1e_2e_3\dots$ .

We also define negation as:

$$-(s, N, d_1, d_2, \dots) = (-s, N, d_1, d_2, \dots)$$

Then we can define the “less than” operation as follows:

- If  $x, y \in \mathbb{R}^+$ , then  $x < y$  if either  $N < M$  or  $N = M$  and  $d_1 < e_1$  or  $N = M$ ,  $d_1 = e_1$  and  $d_2 < e_2$ , or...
- $0 < x$  if  $x \in \mathbb{R}^+$
- $x < 0$  if  $x \in \mathbb{R}^+$
- $x < y$  if  $x \in \mathbb{R}^-, y \in \mathbb{R}^+$ .
- $x, y \in \mathbb{R}^-$ , and  $x < y$  if  $-y < -x$