Differential Equations

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Spring 2016
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February 10, 2016

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Introduction

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1.1 Romeo and Juliet

$$\begin{cases} R' = aR + bJ \\ J' = cR + dJ \end{cases}$$

These equations model the rate of change of Romeo's and Juliet's feelings. We call this a **linear system of two coupled differential equations of first order in two unknowns**.

- What makes it linear is that the functions and variables appear in a linear fashion.
- What makes it coupled is that both equations have both R and J in them.
- An **uncoupled system** would look like:

$$\begin{cases} R' = aR \\ J' = bJ \end{cases}$$

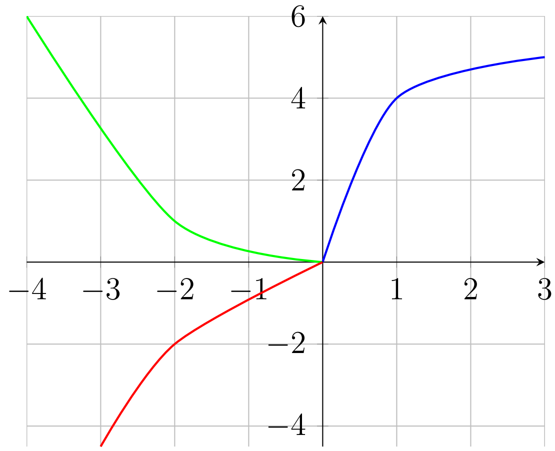
- First-order refers to the fact that all the derivatives are the first derivatives.

"Identically cautious lovers":

$$\begin{aligned} R' &= aR + bJ \quad & a < 0, b > 0 \\ J' &= bR + aJ \quad & |a| > |b| \end{aligned}$$

We may have initial conditions, $R(0)$ and $J(0)$, and plot them on a **phase plane** with R against J. In this case, no matter where the starting point is, the trajectory will go towards a **stable node**.

In the case of $|a| < |b|$, points will move asymptotically towards $R = -J$ and $R = J$. In the case of $|a| = |b|$, points will cycle around the origin infinitely.



1.2 Supremum and Infimum of a Set $\mathcal{A} \subseteq \mathbb{R}$

- If $\mathcal{A} \in (-\infty, b]$ for some $b \in \mathbb{R}$, we say $\mathcal{A}$ is bounded above, and that b is an **upper bound** for A.

Theorem 1.1 (Supremum Theorem)**.** *If $\mathcal{A} \in \mathbb{R}$, $A \neq \emptyset$, and $A \subseteq (-\infty, b]$ for some $b \in \mathbb{R}$, then there exists $a \in \mathbb{R}$ such that $\mathcal{A} \subseteq (-\infty, a]$ but if $x < a$, then $\mathcal{A} \not\subseteq (-\infty, x]$.*
*We write $a = \sup \mathcal{A}$, call it the **supremum** of $\mathcal{A}$.*

Why is this necessary? Consider the set $\mathcal{A} = \{-\frac{1}{n} | n \in \mathbb{N}\}$. It does not have a maximum persay, but it has a supremum $\sup \mathcal{A} = 0$.

Consider this example: What is $\sup(-\mathbb{N})$? It is -1, which also happens to be the maximum of the set. e

Theorem 1.2. *If max $\mathcal{A}$ exists as a real number, then $\sup \mathcal{A} = \max \mathcal{A}$.*

But to answer all these questions, we need to figure out: what exactly are the real numbers?

1.3 What is $\mathbb{R}$?

Let $x = (s, N, d_1, d_2, d_3, \dots, d_k, \dots)$, where:

- $s \in \{+1, -1\}$
- $N \in \mathbb{Z}$
- $d_k \in \mathbb{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- $\neg(\exists k : d_{k+1} = d_{k+2} = \cdots = 0)$, this is to prevent multiple sequences from being the same number

In this case, "2.49" is shorthand for $(+1, 2, 4, 8, 9, 9, 9, \dots)$

2 2/4/16: Background in $\mathbb{R}$; Fundamental Existence/Uniqueness Question

2.1 Supremums and Infimums in Integrals

Theorem 2.1 (Supremum/Infimum Theorem)**.**

1. *If $\mathcal{A}$ is a non-empty set of $\mathbb{R}$, and is bounded above (i.e. $\mathcal{A} \subseteq (-\infty, b]$ for some $b \in \mathbb{R}$), then there is a least upper bound for $\mathcal{A}$, namely $a \in \mathbb{R}$ such that*
   (a) $\mathcal{A} \subseteq (-\infty, a]$
   (b) *if $x < a$, then $\mathcal{A} \not\subseteq (-\infty, x]$*

   This a is called the called the ***supremum*** *of $\mathcal{A}$, written* $\sup A$.
2. $\inf A$. *This is the greatest lower bound for $\mathcal{A}$, or the* ***infimum***, *provided $\mathcal{A} \neq \emptyset$ and $\mathcal{A}$ has a lower bound at all.*

Recall that the Riemann integral is taking the limit of a partition over an interval $[a, b]$. But when we take the limit, we make the mesh of the partition, $||\mathcal{P}||$, approach zero, where

$$\mathcal{P} = \max_{1 \le i \le n} \Delta x_i$$

To fix this, we can define:

$$\underline{\int_a^b} f(x)\ dx = \sup \left\{ \sum_{i=1}^{n} [\inf\{f(x) \mid x_{i-1} \le x \le x_i\} \Delta x_i] \;\middle|\; a = x_0 < x_1 < \cdots < x_n = b \right\}$$

This is a "down-and-up" procedure. The sum of the rectangle areas is a down approximation since we use the minimum possible height to find the area. Then, we take the supremum of that, since for any lower approximation there will always be a higher approximation. Turns out there will never be a maximum; that's why we take the supremum. This is a **lower Riemann sum**.

We can also define the same thing for an **upper Riemann sum**:

$$\overline{\int_a^b} f(x)\ dx = \inf \left\{ \sum_{i=1}^{n} [\sup\{f(x) \mid x_{i-1} \le x \le x_i\} \Delta x_i] \;\middle|\; a = x_0 < x_1 < \cdots < x_n = b \right\}$$

Therefore, the following inequality is true:

$$\underline{\int_a^b} f \le \overline{\int_a^b} f$$

If these two are equal, then we say that f is **Riemann integrable**.

Here's an example of a function that is NOT Riemann integrable:

$$f(x) = \begin{cases} 0 \text{ if } x \in \mathbb{Q} \cap [0, 1] \\ 1 \text{ if } x \in [0, 1] \backslash \mathbb{Q} \end{cases}$$

Note that $\underline{\int_0^1} f = 0$ and $\overline{\int_0^1} f = 1$, so this is not Riemann integrable.

2.2 Real Numbers, Again

We have shorthand for our previous definition of the real numbers.

$$\mathbb{R} = \{0\} \cup \{(s, N, d_1, d_2, \dots, d_k, \dots \mid s \in \{-1, +1\}, N \in \mathbb{Z}^+, d_k \in \mathbb{D}, \text{no 0-tail}\}$$

and the positive reals:

$$\mathbb{R}^+ = \{(s, N, d_1, d_2, \dots) \mid s = +1\}$$

Let us write $x = \underline{N}.\underline{d_1 d_2 d_3 \dots}$ and $y = \underline{M}.\underline{e_1 e_2 e_3 \dots}$.

We also define negation as:

$$-(s, N, d_1, d_2, \dots) := (-s, N, d_1, d_2, \dots)$$

Then we can define the "less than" operation as follows:

- If $x, y \in \mathbb{R}^+$, then $x < y$ if either $N < M$ or $N = M$ and $d_1 < e_1$ or $N = M$, $d_1 = e_1$ and $d_2 < e_2$, or...
- $0 < x$ if $x \in \mathbb{R}^+$
- $x < 0$ if $x \in \mathbb{R}^+$
- $x < y$ if $x \in \mathbb{R}^-, y \in \mathbb{R}^+$.
- $x, y \in \mathbb{R}^-$, and $x < y$ if $-y < -x$

3 2/5/16: Fundamental Existence of Uniqueness Theorem

3.1 Terminology

A **differential equation** is a relation between one or more unknown functions and at least some (but finitely many) of their derivatives, plus the independent variables.

Examples:

$$\begin{aligned} y' + 2xy - x^2 &= 3 \\ y''' + 2x^2 y'' - 3x^3 y' + xy - x^5 + 1 &= 0 \\ (y')^{y''} - e^{y'''} + x &= 0 \end{aligned}$$

Or,

$$\vec{y}\,' = \mathbf{A}(x)\vec{y}$$

where

$$\vec{y} = \vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}$$

$$\mathbf{A}(x) = \begin{bmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{bmatrix}$$

3.2 A Treatise on PDE's

There are two different types of differential equations: ODE's (ordinary, where all unknown functions depend on a single, same independent variable) and PDE's (partial, anything else).

$$\text{Wave equation: } \frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial t^2}$$
$$u = g(x-t) + h(x+t)$$

4 2/9/16: Basic Existence and Uniqueness Theorem

Theorem 4.1 (Flow Theorem)**.** *Let $\vec{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), \ldots, F_n(\vec{x}))$ be a vector field defined on some closed and bounded region $\mathcal{D} \subseteq \mathbb{R}^n$. Also assume $\vec{F}$ is C^1; namely, $\frac{\partial F_i}{\partial x_j}$ is continuous everywhere interior to D, for any i and j.*

Let $\vec{p}$ be a specific point interior to $\mathcal{D}$. Then $\exists$ a function $\vec{\sigma}(t)$ from some "time" interval $(-\varepsilon, \varepsilon)$ with $\varepsilon > 0$ into $\mathcal{D}$, such that $\vec{\sigma}(0) = \vec{p}$ and $\vec{\sigma}\,'(t) = \vec{F}(\vec{\sigma}(t))$ for any $t \in (-\varepsilon, \varepsilon)$.

This theorem basically says that in a vector field, we can use the vector field to get the velocity of a curve. We call $\vec{\sigma}(t)$ a **flow** for $\vec{F}$, starting at $\vec{p}$. This flow is, in fact, unique, in the sense that any two flows for the same $\vec{F}$ starting at the same point must agree whenever they are both defined.

This is meaningful in that we can treat it as a differential equation:

$$\begin{cases} \sigma_1' &= F_1(\sigma_1, \sigma_2, \ldots, \sigma_n) \\ \sigma_2' &= F_2(\sigma_1, \sigma_2, \ldots, \sigma_n) \\ \vdots & \vdots \\ \sigma_n' &= F_n(\sigma_1, \sigma_2, \ldots, \sigma_n) \end{cases}$$

$$\begin{cases} \sigma_1(0) &= p_1 \\ \sigma_2(0) &= p_2 \\ \vdots & \vdots \\ \sigma_n(0) &= p_n \end{cases}$$

4.1 Second-Order

$$mx'' = -kx, \quad x(0) = x_0, \quad x'(0) = v_0$$
$$x = x(t), \quad v = v(t) = x'(t), \quad a = a(t) = x''(t)$$

where $k > 0$ is the spring constant. We can rewrite this as:

$$\begin{cases} x' &= v = F_1(x, v) \\ v' &= -\frac{k}{m}x = F_2(x, v) \end{cases} \quad \text{and} \quad \begin{cases} x(0) &= x_0 \\ v(0) &= v_0 \end{cases}$$

The Flow Theorem will tell us there is a unique solution, for some time interval.

5 2/10/16: The Flow Theorem

5.1 Application: n^{th} order initial value problem (IVP)

$$\begin{cases} x &= x(t) \\ x^{(n)} &= F(t, x, x', x'', \ldots, x^{(n-1)}) \\ x(t_0) &= x_{00} \\ x'(t_0) &= x_{10} \\ x''(t_0) &= x_{20} \\ \vdots & \\ x^{(n-1)}(t_0) &= x_{(n-1)0} \end{cases}$$

$f(t)x^{(n)} = F(t, x, x', x'', \ldots, x^{(n-1)})$ is an n^{th} order ODE in standard form.

A **singularity** (or singular point) of this equation is a value t_0 where $f(t_0) = 0$. At this point, the equation ceases to be of n^{th} order. If $f(t)$ is of constant sign in the time interval on which we'd like to solve the equation, we just divide through by $f(t)$ to get our desired form (which is the regular case, as opposed to the singular case).

Here, the Flow Theorem says that there is a unique solution $x = x(t)$ defined in some time interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ where $\varepsilon > 0$.

To apply this:

$$\begin{cases} x_0(t) &= t \\ x_1 &= x_1(t) = x(t) \\ x_2 &= x_2(t) = x'(t) \\ x_3 &= x_3(t) = x''(t) \\ \vdots & \\ x_n &= x_n(t) = x^{(n-1)}(t) \end{cases}$$

becomes

$$\begin{cases} x_0' &= 1 = F_0(x_0, x_1, x_2, \dots, x_n) \\ x_1' &= x_2 = F_1(x_0, x_1, x_2, \dots, x_n) \\ x_2' &= x_3 = F_2(x_0, x_1, x_2, \dots, x_n) \\ x_3' &= x_4 = F_3(x_0, x_1, x_2, \dots, x_n) \\ \vdots & \vdots \\ x_{n-1}' &= x_n = F_{n-1}(x_0, x_1, x_2, \dots, x_n) \\ x_n' &= F(t, x_1, x_2, \dots, x_n) = F_n(\dots) \end{cases} \quad \text{and} \quad \begin{cases} x_0(t_0) &= t_0 \\ x_1(t_0) &= x_{00} \\ x_2(t_0) &= x_{10} \\ \vdots & \vdots \\ x_n(t_0) &= x_{(n-1)0} \end{cases}$$

This shows that we can recast an n^{th} order IVP into an $n+1$ order system.

However, for the Flow Theorem to apply, $\vec{F}$ needs to be C^1. Therefore, our hypothesis in the IVP is that F is C^1, meaning that $\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x'}, \dots$ are continuous.

We will prove the Flow Theorem for two dimensions only; the proof can be extended to greater than two dimensions.

Proof. ■