

Differential Equations

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Introduction

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1 2/3/16: Background on \mathbb{R} ; Basic Existence Question of ODE's

1.1 Romeo and Juliet

$$\begin{cases} R' = aR + bJ \\ J' = cR + dJ \end{cases}$$

These equations model the rate of change of Romeo's and Juliet's feelings. We call this a **linear system of two coupled differential equations of first order in two unknowns**.

- What makes it linear is that the functions and variables appear in a linear fashion.
- What makes it coupled is that both equations have both R and J in them.
- An **uncoupled system** would look like:

$$\begin{cases} R' = aR \\ J' = bJ \end{cases}$$

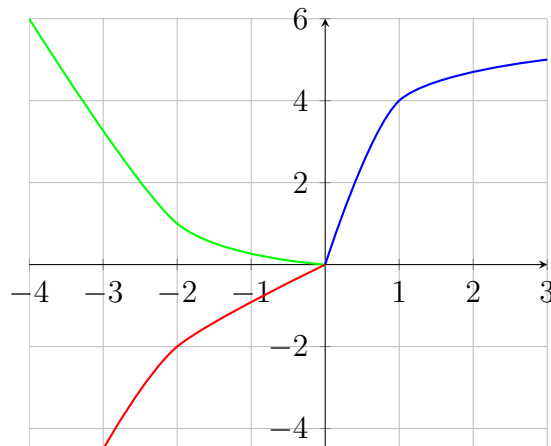
- First-order refers to the fact that all the derivatives are the first derivatives.

"Identically cautious lovers":

$$\begin{aligned} R' &= aR + bJ & a < 0, b > 0 \\ J' &= bR + aJ & |a| > |b| \end{aligned}$$

We may have initial conditions, $R(0)$ and $J(0)$, and plot them on a **phase plane** with R against J . In this case, no matter where the starting point is, the trajectory will go towards a **stable node**.

In the case of $|a| < |b|$, points will move asymptotically towards $R = -J$ and $R = J$. In the case of $|a| = |b|$, points will cycle around the origin infinitely.



1.2 Supremum and Infimum of a Set $\mathcal{A} \subseteq \mathbb{R}$

- If $\mathcal{A} \subseteq (-\infty, b]$ for some $b \in \mathbb{R}$, we say \mathcal{A} is bounded above, and that b is an **upper bound** for \mathcal{A} .

Theorem 1.1 (Supremum Theorem). *If $\mathcal{A} \subseteq \mathbb{R}$, $\mathcal{A} \neq \emptyset$, and $\mathcal{A} \subseteq (-\infty, b]$ for some $b \in \mathbb{R}$, then there exists $a \in \mathbb{R}$ such that $\mathcal{A} \subseteq (-\infty, a]$ but if $x < a$, then $\mathcal{A} \not\subseteq (-\infty, x]$. We write $a = \sup \mathcal{A}$, call it the **supremum** of \mathcal{A} .*

Why is this necessary? Consider the set $\mathcal{A} = \{-\frac{1}{n} | n \in \mathbb{N}\}$. It does not have a maximum per say, but it has a supremum $\sup \mathcal{A} = 0$.

Consider this example: What is $\sup(-\mathbb{N})$? It is -1, which also happens to be the maximum of the set. e

Theorem 1.2. *If $\max \mathcal{A}$ exists as a real number, then $\sup \mathcal{A} = \max \mathcal{A}$.*

But to answer all these questions, we need to figure out: what exactly are the real numbers?

1.3 What is \mathbb{R} ?

Let $x = (s, N, d_1, d_2, d_3, \dots, d_k, \dots)$, where:

- $s \in \{+1, -1\}$
- $N \in \mathbb{Z}$
- $d_k \in \mathbb{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- $\neg(\exists k : d_{k+1} = d_{k+2} = \dots = 0)$, this is to prevent multiple sequences from being the same number

In this case, “2.49” is shorthand for $(+1, 2, 4, 8, 9, 9, 9, \dots)$

2 2/4/16: Background in \mathbb{R} ; Fundamental Existence/Uniqueness Question

2.1 Supremums and Infimums in Integrals

Theorem 2.1 (Supremum/Infimum Theorem).

1. If \mathcal{A} is a non-empty set of \mathbb{R} , and is bounded above (i.e. $\mathcal{A} \subseteq (-\infty, b]$ for some $b \in \mathbb{R}$), then there is a least upper bound for \mathcal{A} , namely $a \in \mathbb{R}$ such that

$$(a) \mathcal{A} \subseteq (-\infty, a]$$

$$(b) \text{ if } x < a, \text{ then } \mathcal{A} \not\subseteq (-\infty, x]$$

This a is called the **supremum** of \mathcal{A} , written $\sup A$.

2. $\inf A$. This is the greatest lower bound for \mathcal{A} , or the **infimum**, provided $\mathcal{A} \neq \emptyset$ and \mathcal{A} has a lower bound at all.

Recall that the Riemann integral is taking the limit of a partition over an interval $[a, b]$. But when we take the limit, we make the mesh of the partition, $\|\mathcal{P}\|$, approach zero, where

$$\mathcal{P} = \max_{1 \leq i \leq n} \Delta x_i$$

To fix this, we can define:

$$\int_a^b f(x) dx = \sup \left\{ \sum_{i=1}^n [\inf\{f(x) \mid x_{i-1} \leq x \leq x_i\} \Delta x_i] \mid a = x_0 < x_1 < \cdots < x_n = b \right\}$$

This is a “down-and-up” procedure. The sum of the rectangle areas is a down approximation since we use the minimum possible height to find the area. Then, we take the supremum of that, since for any lower approximation there will always be a higher approximation. Turns out there will never be a maximum; that’s why we take the supremum. This is a **lower Riemann sum**.

We can also define the same thing for an **upper Riemann sum**:

$$\int_a^b f(x) dx = \inf \left\{ \sum_{i=1}^n [\sup\{f(x) \mid x_{i-1} \leq x \leq x_i\} \Delta x_i] \mid a = x_0 < x_1 < \cdots < x_n = b \right\}$$

Therefore, the following inequality is true:

$$\int_a^b f \leq \int_a^b f$$

If these two are equal, then we say that f is **Riemann integrable**.

Here’s an example of a function that is NOT Riemann integrable:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 1 & \text{if } x \in [0, 1] \setminus \mathbb{Q} \end{cases}$$

Note that $\int_0^1 f = 0$ and $\int_0^1 f = 1$, so this is not Riemann integrable.

2.2 Real Numbers, Again

We have shorthand for our previous definition of the real numbers.

$$\mathbb{R} = \{0\} \cup \{(s, N, d_1, d_2, \dots, d_k, \dots \mid s \in \{-1, +1\}, N \in \mathbb{Z}^+, d_k \in \mathbb{D}, \text{no 0-tail}\}$$

and the positive reals:

$$\mathbb{R}^+ = \{(s, N, d_1, d_2, \dots) \mid s = +1\}$$

Let us write $x = \underline{N.d_1d_2d_3\dots}$ and $y = \underline{M.e_1e_2e_3\dots}$.

We also define negation as:

$$-(s, N, d_1, d_2, \dots) := (-s, N, d_1, d_2, \dots)$$

Then we can define the “less than” operation as follows:

- If $x, y \in \mathbb{R}^+$, then $x < y$ if either $N < M$ or $N = M$ and $d_1 < e_1$ or $N = M$, $d_1 = e_1$ and $d_2 < e_2$, or...
- $0 < x$ if $x \in \mathbb{R}^+$
- $x < 0$ if $x \in \mathbb{R}^+$
- $x < y$ if $x \in \mathbb{R}^-, y \in \mathbb{R}^+$.
- $x, y \in \mathbb{R}^-$, and $x < y$ if $-y < -x$

3 2/5/16: Fundamental Existence of Uniqueness Theorem

3.1 Terminology

A **differential equation** is a relation between one or more unknown functions and at least some (but finitely many) of their derivatives, plus the independent variables.

Examples:

$$\begin{aligned} y' + 2xy - x^2 &= 3 \\ y''' + 2x^2y'' - 3x^3y' + xy - x^5 + 1 &= 0 \\ (y')^{y''} - e^{y'''} + x &= 0 \end{aligned}$$

Or,

$$\vec{y}' = \mathbf{A}(x)\vec{y}$$

where

$$\vec{y} = \vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}$$

$$\mathbf{A}(x) = \begin{bmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{bmatrix}$$

3.2 A Treatise on PDE's

There are two different types of differential equations: ODE's (ordinary, where all unknown functions depend on a single, same independent variable) and PDE's (partial, anything else).

$$\begin{aligned} \text{Wave equation: } \frac{\partial^2 u}{\partial x^2} &= c^2 \frac{\partial^2 u}{\partial t^2} \\ u &= g(x-t) + h(x+t) \end{aligned}$$

4 2/9/16: Basic Existence and Uniqueness Theorem

Theorem 4.1 (Flow Theorem). *Let $\vec{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), \dots, F_n(\vec{x}))$ be a vector field defined on some closed and bounded region $\mathcal{D} \subseteq \mathbb{R}^n$. Also assume \vec{F} is C^1 ; namely, $\frac{\partial F_i}{\partial x_j}$ is continuous everywhere interior to \mathcal{D} , for any i and j .*

Let \vec{p} be a specific point interior to \mathcal{D} . Then \exists a function $\vec{\sigma}(t)$ from some "time" interval $(-\varepsilon, \varepsilon)$ with $\varepsilon > 0$ into \mathcal{D} , such that $\vec{\sigma}(0) = \vec{p}$ and $\vec{\sigma}'(t) = \vec{F}(\vec{\sigma}(t))$ for any $t \in (-\varepsilon, \varepsilon)$.

This theorem basically says that in a vector field, we can use the vector field to get the velocity of a curve. We call $\vec{\sigma}(t)$ a **flow** for \vec{F} , starting at \vec{p} . This flow is, in fact, unique, in the sense that any two flows for the same \vec{F} starting at the same point must agree whenever they are both defined.

This is meaningful in that we can treat it as a differential equation:

$$\begin{cases} \sigma'_1 &= F_1(\sigma_1, \sigma_2, \dots, \sigma_n) \\ \sigma'_2 &= F_2(\sigma_1, \sigma_2, \dots, \sigma_n) \\ \vdots &\vdots \\ \sigma'_n &= F_n(\sigma_1, \sigma_2, \dots, \sigma_n) \end{cases}$$

$$\begin{cases} \sigma_1(0) &= p_1 \\ \sigma_2(0) &= p_2 \\ \vdots &\vdots \\ \sigma_n(0) &= p_n \end{cases}$$

4.1 Second-Order

$$mx'' = -kx, \quad x(0) = x_0, \quad x'(0) = v_0$$

$$x = x(t), \quad v = v(t) = x'(t), \quad a = a(t) = x''(t)$$

where $k > 0$ is the spring constant. We can rewrite this as:

$$\begin{cases} x' &= v = F_1(x, v) \\ v' &= -\frac{k}{m}x = F_2(x, v) \end{cases} \quad \text{and} \quad \begin{cases} x(0) &= x_0 \\ v(0) &= v_0 \end{cases}$$

The Flow Theorem will tell us there is a unique solution, for some time interval.

5 2/10/16: The Flow Theorem

5.1 Application: n^{th} order initial value problem (IVP)

$$\begin{cases} x &= x(t) \\ x^{(n)} &= F(t, x, x', x'', \dots, x^{(n-1)}) \\ x(t_0) &= x_{00} \\ x'(t_0) &= x_{10} \\ x''(t_0) &= x_{20} \\ \vdots & \\ x^{(n-1)}(t_0) &= x_{(n-1)0} \end{cases}$$

$f(t)x^{(n)} = F(t, x, x', x'', \dots, x^{(n-1)})$ is an n^{th} order ODE in standard form.

A **singularity** (or singular point) of this equation is a value t_0 where $f(t_0) = 0$. At this point, the equation ceases to be of n^{th} order. If $f(t)$ is of constant sign in the time interval on which we'd like to solve the equation, we just divide through by $f(t)$ to get our desired form (which is the regular case, as opposed to the singular case).

Here, the Flow Theorem says that there is a unique solution $x = x(t)$ defined in some time interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ where $\varepsilon > 0$.

To apply this:

$$\begin{cases} x_0(t) &= t \\ x_1 &= x_1(t) = x(t) \\ x_2 &= x_2(t) = x'(t) \\ x_3 &= x_3(t) = x''(t) \\ \vdots & \\ x_n &= x_n(t) = x^{(n-1)}(t) \end{cases}$$

becomes

$$\left\{ \begin{array}{l} x'_0 = 1 = F_0(x_0, x_1, x_2, \dots, x_n) \\ x'_1 = x_2 = F_1(x_0, x_1, x_2, \dots, x_n) \\ x'_2 = x_3 = F_2(x_0, x_1, x_2, \dots, x_n) \\ x'_3 = x_4 = F_3(x_0, x_1, x_2, \dots, x_n) \\ \vdots \\ x'_{n-1} = x_n = F_{n-1}(x_0, x_1, x_2, \dots, x_n) \\ x'_n = F(t, x_1, x_2, \dots, x_n) = F_n(\dots) \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} x_0(t_0) = t_0 \\ x_1(t_0) = x_{00} \\ x_2(t_0) = x_{10} \\ \vdots \\ x_n(t_0) = x_{(n-1)0} \end{array} \right.$$

This shows that we can recast an n^{th} order IVP into an $n + 1$ order system.

However, for the Flow Theorem to apply, \vec{F} needs to be C^1 . Therefore, our hypothesis in the IVP is that F is C^1 , meaning that $\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x'}, \dots$ are continuous.

6 2/11/16: Proof of the Flow Theorem

We will prove the Flow Theorem for two dimensions only; the proof can be extended to greater than two dimensions.

Proof. Let $\vec{F}(x, y) = (A(x, y), B(x, y))$ be a vector field. By hypothesis, A and B are defined on a closed, bounded region \mathcal{D} , and they are C^1 on \mathcal{D} . Then we need to solve the following equation:

$$\begin{aligned} \vec{x}' &= \vec{F}(\vec{x}) \\ \vec{x}(0) &= \vec{p} = \langle p, q \rangle \end{aligned}$$

We need to see how fast $A(x, y)$ is changing.

1.

$$\begin{aligned} |A(x_1, y_1) - A(x_2, y_2)| &= |A(x_1, y_1) - A(x_1, y_2) + A(x_1, y_2) - A(x_2, y_2)| \\ (\text{Triangle Inequality}) \quad &\leq |A(x_1, y_1) - A(x_1, y_2)| + |A(x_1, y_2) - A(x_2, y_2)| \\ (\text{MVT}) \quad &\leq \left| \frac{\partial A}{\partial y}(x_1, y^*)(y_1 - y_2) \right| + \left| \frac{\partial A}{\partial x}(x^*, y_2)(x_1 - x_2) \right| \end{aligned}$$

Take K to be some upper bound for all the partial derivatives of A and B on \mathcal{D} .

$$\left| \frac{\partial A}{\partial y}(x_1, y^*)(y_1 - y_2) \right| + \left| \frac{\partial A}{\partial x}(x^*, y_2)(x_1 - x_2) \right| \leq K(|x_1 - x_2| + |y_1 - y_2|)$$

Similarly:

$$|B(x_1, y_2) - B(x_2, y_2)| \leq K(|x_1 - x_2| + |y_1 - y_2|)$$

This is called the **Lipschitz Condition**.

2. Also, note that A and B are continuous in \mathcal{D} and so by the Extreme Value Theorem, we can find an upper bound M for $|A|$ and $|B|$ on \mathcal{D} , i.e.

$$M = \max \left(\max_{(x,y) \in \mathcal{D}} |A(x, y)|, \max_{(x,y) \in \mathcal{D}} |B(x, y)| \right)$$

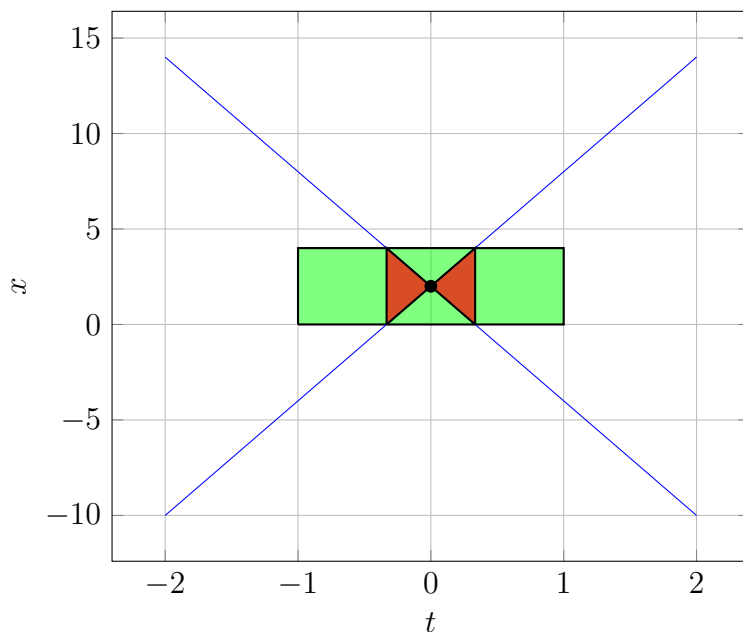
3. The point (p, q) is assumed to be interior to \mathcal{D} (not on the boundary).

We can therefore encase the point $(0, p)$ in a rectangle in the tx -plane defined by $R : [-r, r] \times [p - s, p + s] \subseteq \text{proj}_x \mathcal{D}$, $r, s > 0$. Draw two lines with slopes M and $-M$ through the point. We will consider the “bowtie” region formed by the intersections of the lines with the rectangle, joining them oppositely, and the lines themselves. Call the x -intersections $-h$ and h .

Define $h := \min(r, \frac{s}{M}) > 0$. This is to formally define the bowtie region and consider the two possible pictures depending on the size of M .

Now, we want to construct the solution of the differential equation within the bowtie region.

Whoopsies, there was a screwup here, proof to be fixed in the future.



■

7 2/12/16: Separable and First-Order Linear Equations

7.1 Multiplicatively Separable Functions

$$F(t, x) = f(t)g(x)$$

A non-example of a separable function is $F(t, x) = t^2 + x^2$. An example is $F(t, x) = t^2 x^3$.

For our purposes, we will work with first-order ODE's with scalar functions.

7.2 Separable ODE

$$\boxed{x' = f(t)g(x)}$$

There are other ways we can write this equation:

- **General Form:** $G(t, x, x') = 0$
- **Standard Form:** $\phi(t)x' = F(t, x)$
 - **Regular Case:** $x' = F(t, x)$, F is the “slope function”
 - **Singular Case:** This is when we solve in an interval $(t_0 - \delta, t_0 + \delta)$ where $\delta > 0$ and $\phi(t_0) = 0$.

To solve this type of equation:

$$\begin{aligned} x'(t) &\equiv f(t)g(x(t)) \\ \frac{x'(t)}{g(x(t))} &= f(t) \\ \int_a^t \frac{x'(\tau)}{g(x(\tau))} d\tau &= \int_a^t f(\tau) d\tau \end{aligned}$$

Letting $u = x(\tau)$ and $du = x'(\tau) d\tau$:

$$\underbrace{\int_{x(a)}^{x(t)} \frac{du}{g(u)}}_{G(x(t))} = \underbrace{\int_a^t f(\tau) d\tau}_{F(t)}$$

$$\boxed{G(x(t)) = F(t)}$$

7.3 Example

$$\begin{aligned} x' &= t^2 x^3 \\ \frac{x'}{x^3} &= t^2 \\ \int \frac{dx}{x^3} &= \int t^2 dt + C \\ \frac{x^{-2}}{-2} &= \frac{t^3}{3} + C \\ x^{-2} &= C - \frac{2}{3}t^3 \\ x &= \pm \frac{1}{\sqrt{C - \frac{2}{3}t^3}} \end{aligned}$$

8 2/22/16: Separable Equations, First-Order Linear Equations; Uniqueness for C^1 IVP's

Recall our form for the separable equation:

$$x' = f(t)g(x)$$

Assume f and g are continuous on their respective domains, f on $I = (t_0 - a, t_0 + a)$, $a > 0$, g on $J = (x_0 - b, x_0 + b)$, $b > 0$. Let $\mathcal{R} = I \times J$. If we also have $x(t_0) = x_0$, then we have an IVP (initial value problem) on our hands.

But the problem is, $\frac{1}{g(x)}$ isn't necessarily continuous.

Separately, solve the algebraic equation $g(x) = 0$ in the interval J . Assume for simplicity that the roots of g are isolated and C^∞ ("smooth"). Then, we can partition J into open subintervals J_1, J_2, \dots, J_n , i.e.

$$J = \{a, b, c, d\} \cup J_1 \cup J_2 \cup \dots \cup J_n$$

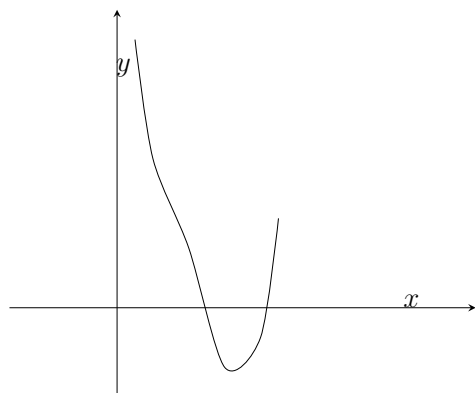
9 2/23/16: Uniqueness for C^1 IVP's

$$x' = f(t)g(x) \quad x = x(t)$$

$$x' \equiv f(t)g(x(t)) \text{ for all } t \in I$$

Assume: f, g have continuous derivatives on their respective domains. Then, all solutions are given as follows: Suppose $g(x)$ has domain J . If a and b are consecutive isolated roots of g , we can solve on (a, b) as we did yesterday:

$$\underbrace{\int_c^{x(t)} \frac{1}{g(u)} du}_{G_c(x(t))} = \underbrace{\int_{t_0}^t f(\tau) d\tau}_{F(t)} \quad \text{where } c \in (a, b) \text{ is arbitrary}$$



If a is a root of g (isolated or not) then claim: $x(t) \equiv a$ for $t \in \mathbb{R}$ is a solution of the differential equation.

9.1 Uniqueness

Are these all the solutions, however?

A first-order IVP in standard form (the regular case):

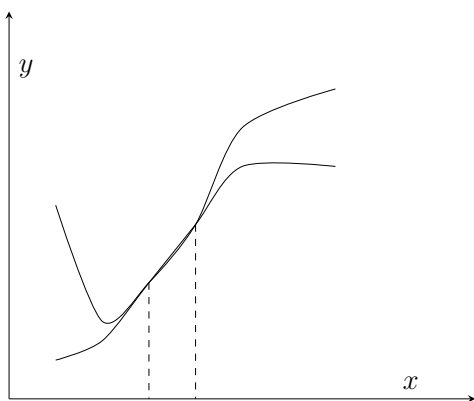
$$x' = \underbrace{F(t, x)}_{\text{slope function}}, \quad x(t_0) = x_0$$

Assumption: F is a C^1 function ($\frac{\partial F}{\partial t}$ and $\frac{\partial F}{\partial x}$ are both continuous) on a rectangle centered at the initial point (t_0, x_0) . Then, we have the following theorem:

Theorem 9.1. *If $\phi(t)$ and $\psi(t)$ are solutions of the IVP, defined on respective domains $I_\delta = (t_0 - \delta, t_0 + \delta)$ and $I_\epsilon = (t_0 - \epsilon, t_0 + \epsilon)$ where $\delta > 0$ and $\epsilon > 0$, then*

$$\phi(t) \equiv \psi(t)$$

for all $t \in I_\eta = (t_0 - \eta, t_0 + \eta)$ where $\eta > 0$.



Basic outline for the proof:

$$\text{IVP} \quad x'(t) \equiv F(t, x(t)), \quad x(t_0) = x_0$$

$$x(t) - x(t_0) = \int_{t_0}^t F(\tau, x(\tau)) \, d\tau$$