

Differential Equations

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Introduction

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1 2/3/16: Background on \mathbb{R} ; Basic Existence Question of ODE's

1.1 Romeo and Juliet

$$\begin{cases} R' = aR + bJ \\ J' = cR + dJ \end{cases}$$

These equations model the rate of change of Romeo's and Juliet's feelings. We call this a **linear system of two coupled differential equations of first order in two unknowns**.

- What makes it linear is that the functions and variables appear in a linear fashion.
- What makes it coupled is that both equations have both R and J in them.
- An **uncoupled system** would look like:

$$\begin{cases} R' = aR \\ J' = bJ \end{cases}$$

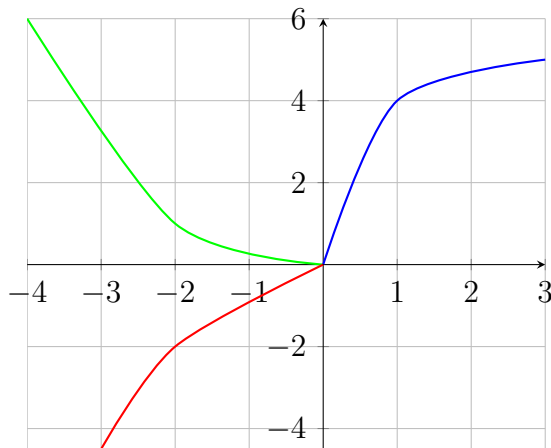
- First-order refers to the fact that all the derivatives are the first derivatives.

"Identically cautious lovers":

$$\begin{aligned} R' &= aR + bJ & a < 0, b > 0 \\ J' &= bR + aJ & |a| > |b| \end{aligned}$$

We may have initial conditions, $R(0)$ and $J(0)$, and plot them on a **phase plane** with R against J . In this case, no matter where the starting point is, the trajectory will go towards a **stable node**.

In the case of $|a| < |b|$, points will move asymptotically towards $R = -J$ and $R = J$. In the case of $|a| = |b|$, points will cycle around the origin infinitely.



1.2 Supremum and Infimum of a Set $\mathcal{A} \subseteq \mathbb{R}$

- If $\mathcal{A} \subseteq (-\infty, b]$ for some $b \in \mathbb{R}$, we say \mathcal{A} is bounded above, and that b is an **upper bound** for \mathcal{A} .

Theorem 1.1 (Supremum Theorem). *If $\mathcal{A} \subseteq \mathbb{R}$, $\mathcal{A} \neq \emptyset$, and $\mathcal{A} \subseteq (-\infty, b]$ for some $b \in \mathbb{R}$, then there exists $a \in \mathbb{R}$ such that $\mathcal{A} \subseteq (-\infty, a]$ but if $x < a$, then $\mathcal{A} \not\subseteq (-\infty, x]$. We write $a = \sup \mathcal{A}$, call it the **supremum** of \mathcal{A} .*

Why is this necessary? Consider the set $\mathcal{A} = \{-\frac{1}{n} | n \in \mathbb{N}\}$. It does not have a maximum per say, but it has a supremum $\sup \mathcal{A} = 0$.

Consider this example: What is $\sup(-\mathbb{N})$? It is -1, which also happens to be the maximum of the set. e

Theorem 1.2. *If $\max \mathcal{A}$ exists as a real number, then $\sup \mathcal{A} = \max \mathcal{A}$.*

But to answer all these questions, we need to figure out: what exactly are the real numbers?

1.3 What is \mathbb{R} ?

Let $x = (s, N, d_1, d_2, d_3, \dots, d_k, \dots)$, where:

- $s \in \{+1, -1\}$
- $N \in \mathbb{Z}$
- $d_k \in \mathbb{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- $\neg(\exists k : d_{k+1} = d_{k+2} = \dots = 0)$, this is to prevent multiple sequences from being the same number

In this case, “2.49” is shorthand for $(+1, 2, 4, 8, 9, 9, 9, \dots)$

2 2/4/16: Background in \mathbb{R} ; Fundamental Existence/Uniqueness Question

2.1 Supremums and Infimums in Integrals

Theorem 2.1 (Supremum/Infimum Theorem).

1. If \mathcal{A} is a non-empty set of \mathbb{R} , and is bounded above (i.e. $\mathcal{A} \subseteq (-\infty, b]$ for some $b \in \mathbb{R}$), then there is a least upper bound for \mathcal{A} , namely $a \in \mathbb{R}$ such that

$$(a) \mathcal{A} \subseteq (-\infty, a]$$

$$(b) \text{ if } x < a, \text{ then } \mathcal{A} \not\subseteq (-\infty, x]$$

This a is called the **supremum** of \mathcal{A} , written $\sup A$.

2. $\inf A$. This is the greatest lower bound for \mathcal{A} , or the **infimum**, provided $\mathcal{A} \neq \emptyset$ and \mathcal{A} has a lower bound at all.

Recall that the Riemann integral is taking the limit of a partition over an interval $[a, b]$. But when we take the limit, we make the mesh of the partition, $\|\mathcal{P}\|$, approach zero, where

$$\mathcal{P} = \max_{1 \leq i \leq n} \Delta x_i$$

To fix this, we can define:

$$\int_a^b f(x) dx = \sup \left\{ \sum_{i=1}^n [\inf\{f(x) \mid x_{i-1} \leq x \leq x_i\} \Delta x_i] \mid a = x_0 < x_1 < \cdots < x_n = b \right\}$$

This is a “down-and-up” procedure. The sum of the rectangle areas is a down approximation since we use the minimum possible height to find the area. Then, we take the supremum of that, since for any lower approximation there will always be a higher approximation. Turns out there will never be a maximum; that’s why we take the supremum. This is a **lower Riemann sum**.

We can also define the same thing for an **upper Riemann sum**:

$$\int_a^b f(x) dx = \inf \left\{ \sum_{i=1}^n [\sup\{f(x) \mid x_{i-1} \leq x \leq x_i\} \Delta x_i] \mid a = x_0 < x_1 < \cdots < x_n = b \right\}$$

Therefore, the following inequality is true:

$$\int_a^b f \leq \int_a^b f$$

If these two are equal, then we say that f is **Riemann integrable**.

Here’s an example of a function that is NOT Riemann integrable:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 1 & \text{if } x \in [0, 1] \setminus \mathbb{Q} \end{cases}$$

Note that $\int_0^1 f = 0$ and $\bar{\int}_0^1 f = 1$, so this is not Riemann integrable.

2.2 Real Numbers, Again

We have shorthand for our previous definition of the real numbers.

$$\mathbb{R} = \{0\} \cup \{(s, N, d_1, d_2, \dots, d_k, \dots \mid s \in \{-1, +1\}, N \in \mathbb{Z}^+, d_k \in \mathbb{D}, \text{no 0-tail}\}$$

and the positive reals:

$$\mathbb{R}^+ = \{(s, N, d_1, d_2, \dots) \mid s = +1\}$$

Let us write $x = \underline{N.d_1d_2d_3\dots}$ and $y = \underline{M.e_1e_2e_3\dots}$.

We also define negation as:

$$-(s, N, d_1, d_2, \dots) := (-s, N, d_1, d_2, \dots)$$

Then we can define the “less than” operation as follows:

- If $x, y \in \mathbb{R}^+$, then $x < y$ if either $N < M$ or $N = M$ and $d_1 < e_1$ or $N = M$, $d_1 = e_1$ and $d_2 < e_2$, or...
- $0 < x$ if $x \in \mathbb{R}^+$
- $x < 0$ if $x \in \mathbb{R}^+$
- $x < y$ if $x \in \mathbb{R}^-, y \in \mathbb{R}^+$.
- $x, y \in \mathbb{R}^-$, and $x < y$ if $-y < -x$

3 2/5/16: Fundamental Existence of Uniqueness Theorem

3.1 Terminology

A **differential equation** is a relation between one or more unknown functions and at least some (but finitely many) of their derivatives, plus the independent variables.

Examples:

$$\begin{aligned} y' + 2xy - x^2 &= 3 \\ y''' + 2x^2y'' - 3x^3y' + xy - x^5 + 1 &= 0 \\ (y')^{y''} - e^{y'''} + x &= 0 \end{aligned}$$

Or,

$$\vec{y}' = \mathbf{A}(x)\vec{y}$$

where

$$\vec{y} = \vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}$$

$$\mathbf{A}(x) = \begin{bmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{bmatrix}$$

3.2 A Treatise on PDE's

There are two different types of differential equations: ODE's (ordinary, where all unknown functions depend on a single, same independent variable) and PDE's (partial, anything else).

$$\begin{aligned} \text{Wave equation: } \frac{\partial^2 u}{\partial x^2} &= c^2 \frac{\partial^2 u}{\partial t^2} \\ u &= g(x-t) + h(x+t) \end{aligned}$$

4 2/9/16: Basic Existence and Uniqueness Theorem

Theorem 4.1 (Flow Theorem). *Let $\vec{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), \dots, F_n(\vec{x}))$ be a vector field defined on some closed and bounded region $\mathcal{D} \subseteq \mathbb{R}^n$. Also assume \vec{F} is C^1 ; namely, $\frac{\partial F_i}{\partial x_j}$ is continuous everywhere interior to \mathcal{D} , for any i and j .*

Let \vec{p} be a specific point interior to \mathcal{D} . Then \exists a function $\vec{\sigma}(t)$ from some "time" interval $(-\varepsilon, \varepsilon)$ with $\varepsilon > 0$ into \mathcal{D} , such that $\vec{\sigma}(0) = \vec{p}$ and $\vec{\sigma}'(t) = \vec{F}(\vec{\sigma}(t))$ for any $t \in (-\varepsilon, \varepsilon)$.

This theorem basically says that in a vector field, we can use the vector field to get the velocity of a curve. We call $\vec{\sigma}(t)$ a **flow** for \vec{F} , starting at \vec{p} . This flow is, in fact, unique, in the sense that any two flows for the same \vec{F} starting at the same point must agree whenever they are both defined.

This is meaningful in that we can treat it as a differential equation:

$$\begin{cases} \sigma'_1 &= F_1(\sigma_1, \sigma_2, \dots, \sigma_n) \\ \sigma'_2 &= F_2(\sigma_1, \sigma_2, \dots, \sigma_n) \\ \vdots &\vdots \\ \sigma'_n &= F_n(\sigma_1, \sigma_2, \dots, \sigma_n) \end{cases}$$

$$\begin{cases} \sigma_1(0) &= p_1 \\ \sigma_2(0) &= p_2 \\ \vdots &\vdots \\ \sigma_n(0) &= p_n \end{cases}$$

4.1 Second-Order

$$mx'' = -kx, \quad x(0) = x_0, \quad x'(0) = v_0$$

$$x = x(t), \quad v = v(t) = x'(t), \quad a = a(t) = x''(t)$$

where $k > 0$ is the spring constant. We can rewrite this as:

$$\begin{cases} x' &= v = F_1(x, v) \\ v' &= -\frac{k}{m}x = F_2(x, v) \end{cases} \quad \text{and} \quad \begin{cases} x(0) &= x_0 \\ v(0) &= v_0 \end{cases}$$

The Flow Theorem will tell us there is a unique solution, for some time interval.

5 2/10/16: The Flow Theorem

5.1 Application: n^{th} order initial value problem (IVP)

$$\begin{cases} x &= x(t) \\ x^{(n)} &= F(t, x, x', x'', \dots, x^{(n-1)}) \\ x(t_0) &= x_{00} \\ x'(t_0) &= x_{10} \\ x''(t_0) &= x_{20} \\ \vdots & \\ x^{(n-1)}(t_0) &= x_{(n-1)0} \end{cases}$$

$f(t)x^{(n)} = F(t, x, x', x'', \dots, x^{(n-1)})$ is an n^{th} order ODE in standard form.

A **singularity** (or singular point) of this equation is a value t_0 where $f(t_0) = 0$. At this point, the equation ceases to be of n^{th} order. If $f(t)$ is of constant sign in the time interval on which we'd like to solve the equation, we just divide through by $f(t)$ to get our desired form (which is the regular case, as opposed to the singular case).

Here, the Flow Theorem says that there is a unique solution $x = x(t)$ defined in some time interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ where $\varepsilon > 0$.

To apply this:

$$\begin{cases} x_0(t) &= t \\ x_1 &= x_1(t) = x(t) \\ x_2 &= x_2(t) = x'(t) \\ x_3 &= x_3(t) = x''(t) \\ \vdots & \\ x_n &= x_n(t) = x^{(n-1)}(t) \end{cases}$$

becomes

$$\left\{ \begin{array}{l} x'_0 = 1 = F_0(x_0, x_1, x_2, \dots, x_n) \\ x'_1 = x_2 = F_1(x_0, x_1, x_2, \dots, x_n) \\ x'_2 = x_3 = F_2(x_0, x_1, x_2, \dots, x_n) \\ x'_3 = x_4 = F_3(x_0, x_1, x_2, \dots, x_n) \\ \vdots \\ x'_{n-1} = x_n = F_{n-1}(x_0, x_1, x_2, \dots, x_n) \\ x'_n = F(t, x_1, x_2, \dots, x_n) = F_n(\dots) \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} x_0(t_0) = t_0 \\ x_1(t_0) = x_{00} \\ x_2(t_0) = x_{10} \\ \vdots \\ x_n(t_0) = x_{(n-1)0} \end{array} \right.$$

This shows that we can recast an n^{th} order IVP into an $n + 1$ order system.

However, for the Flow Theorem to apply, \vec{F} needs to be C^1 . Therefore, our hypothesis in the IVP is that F is C^1 , meaning that $\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x'}, \dots$ are continuous.

6 2/11/16: Proof of the Flow Theorem

We will prove the Flow Theorem for two dimensions only; the proof can be extended to greater than two dimensions.

Proof. Let $\vec{F}(x, y) = (A(x, y), B(x, y))$ be a vector field. By hypothesis, A and B are defined on a closed, bounded region \mathcal{D} , and they are C^1 on \mathcal{D} . Then we need to solve the following equation:

$$\begin{aligned} \vec{x}' &= \vec{F}(\vec{x}) \\ \vec{x}(0) &= \vec{p} = \langle p, q \rangle \end{aligned}$$

We need to see how fast $A(x, y)$ is changing.

1.

$$\begin{aligned} |A(x_1, y_1) - A(x_2, y_2)| &= |A(x_1, y_1) - A(x_1, y_2) + A(x_1, y_2) - A(x_2, y_2)| \\ (\text{Triangle Inequality}) \quad &\leq |A(x_1, y_1) - A(x_1, y_2)| + |A(x_1, y_2) - A(x_2, y_2)| \\ (\text{MVT}) \quad &\leq \left| \frac{\partial A}{\partial y}(x_1, y^*)(y_1 - y_2) \right| + \left| \frac{\partial A}{\partial x}(x^*, y_2)(x_1 - x_2) \right| \end{aligned}$$

Take K to be some upper bound for all the partial derivatives of A and B on \mathcal{D} .

$$\left| \frac{\partial A}{\partial y}(x_1, y^*)(y_1 - y_2) \right| + \left| \frac{\partial A}{\partial x}(x^*, y_2)(x_1 - x_2) \right| \leq K(|x_1 - x_2| + |y_1 - y_2|)$$

Similarly:

$$|B(x_1, y_2) - B(x_2, y_2)| \leq K(|x_1 - x_2| + |y_1 - y_2|)$$

This is called the **Lipschitz Condition**.

2. Also, note that A and B are continuous in \mathcal{D} and so by the Extreme Value Theorem, we can find an upper bound M for $|A|$ and $|B|$ on \mathcal{D} , i.e.

$$M = \max \left(\max_{(x,y) \in \mathcal{D}} |A(x, y)|, \max_{(x,y) \in \mathcal{D}} |B(x, y)| \right)$$

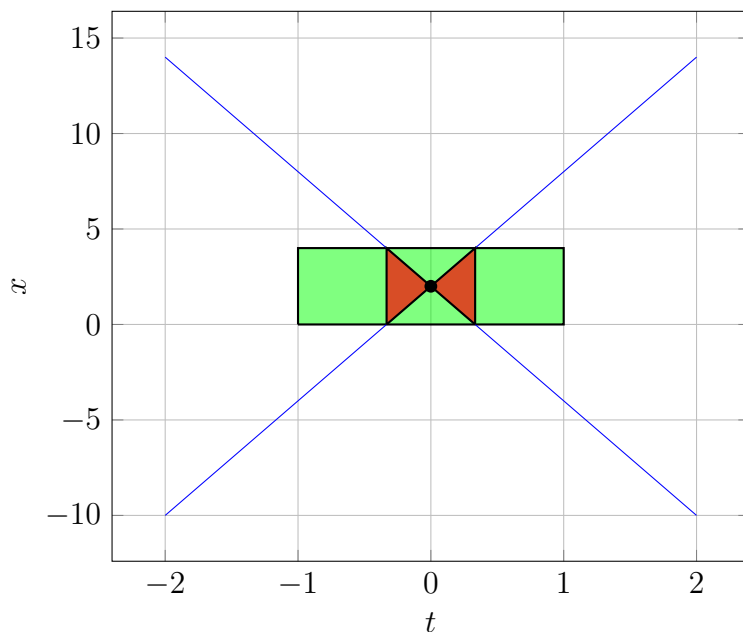
3. The point (p, q) is assumed to be interior to \mathcal{D} (not on the boundary).

We can therefore encase the point $(0, p)$ in a rectangle in the tx -plane defined by $R : [-r, r] \times [p - s, p + s] \subseteq \text{proj}_x \mathcal{D}, r, s > 0$. Draw two lines with slopes M and $-M$ through the point. We will consider the “bowtie” region formed by the intersections of the lines with the rectangle, joining them oppositely, and the lines themselves. Call the x -intersections $-h$ and h .

Define $h := \min(r, \frac{s}{M}) > 0$. This is to formally define the bowtie region and consider the two possible pictures depending on the size of M .

Now, we want to construct the solution of the differential equation within the bowtie region.

Whoopsies, there was a screwup here, proof to be fixed in the future.



■

7 2/12/16: Separable and First-Order Linear Equations

7.1 Multiplicatively Separable Functions

$$F(t, x) = f(t)g(x)$$

A non-example of a separable function is $F(t, x) = t^2 + x^2$. An example is $F(t, x) = t^2 x^3$.

For our purposes, we will work with first-order ODE's with scalar functions.

7.2 Separable ODE

$$\boxed{x' = f(t)g(x)}$$

There are other ways we can write this equation:

- **General Form:** $G(t, x, x') = 0$
- **Standard Form:** $\phi(t)x' = F(t, x)$
 - **Regular Case:** $x' = F(t, x)$, F is the “slope function”
 - **Singular Case:** This is when we solve in an interval $(t_0 - \delta, t_0 + \delta)$ where $\delta > 0$ and $\phi(t_0) = 0$.

To solve this type of equation:

$$\begin{aligned} x'(t) &\equiv f(t)g(x(t)) \\ \frac{x'(t)}{g(x(t))} &= f(t) \\ \int_a^t \frac{x'(\tau)}{g(x(\tau))} d\tau &= \int_a^t f(\tau) d\tau \end{aligned}$$

Letting $u = x(\tau)$ and $du = x'(\tau) d\tau$:

$$\underbrace{\int_{x(a)}^{x(t)} \frac{du}{g(u)}}_{G(x(t))} = \underbrace{\int_a^t f(\tau) d\tau}_{F(t)}$$

$$\boxed{G(x(t)) = F(t)}$$

7.3 Example

$$\begin{aligned} x' &= t^2 x^3 \\ \frac{x'}{x^3} &= t^2 \\ \int \frac{dx}{x^3} &= \int t^2 dt + C \\ \frac{x^{-2}}{-2} &= \frac{t^3}{3} + C \\ x^{-2} &= C - \frac{2}{3}t^3 \\ x &= \pm \frac{1}{\sqrt{C - \frac{2}{3}t^3}} \end{aligned}$$

8 2/22/16: Separable Equations, First-Order Linear Equations; Uniqueness for C^1 IVP's

Recall our form for the separable equation:

$$x' = f(t)g(x)$$

Assume f and g are continuous on their respective domains, f on $I = (t_0 - a, t_0 + a)$, $a > 0$, g on $J = (x_0 - b, x_0 + b)$, $b > 0$. Let $\mathcal{R} = I \times J$. If we also have $x(t_0) = x_0$, then we have an IVP (initial value problem) on our hands.

But the problem is, $\frac{1}{g(x)}$ isn't necessarily continuous.

Separately, solve the algebraic equation $g(x) = 0$ in the interval J . Assume for simplicity that the roots of g are isolated and C^∞ ("smooth"). Then, we can partition J into open subintervals J_1, J_2, \dots, J_n , i.e.

$$J = \{a, b, c, d\} \cup J_1 \cup J_2 \cup \dots \cup J_n$$

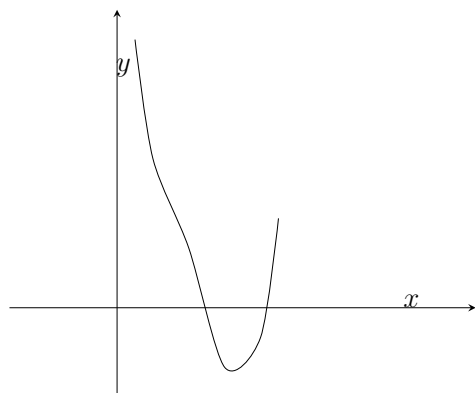
9 2/23/16: Uniqueness for C^1 IVP's

$$x' = f(t)g(x) \quad x = x(t)$$

$$x' \equiv f(t)g(x(t)) \text{ for all } t \in I$$

Assume: f, g have continuous derivatives on their respective domains. Then, all solutions are given as follows: Suppose $g(x)$ has domain J . If a and b are consecutive isolated roots of g , we can solve on (a, b) as we did yesterday:

$$\underbrace{\int_c^{x(t)} \frac{1}{g(u)} du}_{G_c(x(t))} = \underbrace{\int_{t_0}^t f(\tau) d\tau}_{F(t)} \quad \text{where } c \in (a, b) \text{ is arbitrary}$$



If a is a root of g (isolated or not) then claim: $x(t) \equiv a$ for $t \in \mathbb{R}$ is a solution of the differential equation.

9.1 Uniqueness

Are these all the solutions, however?

A first-order IVP in standard form (the regular case):

$$x' = \underbrace{F(t, x)}_{\text{slope function}}, \quad x(t_0) = x_0$$

Assumption: F is a C^1 function ($\frac{\partial F}{\partial t}$ and $\frac{\partial F}{\partial x}$ are both continuous) on a rectangle centered at the initial point (t_0, x_0) . Then, we have the following theorem:

Theorem 9.1. *If $\phi(t)$ and $\psi(t)$ are solutions of the IVP, defined on respective domains $I_\delta = (t_0 - \delta, t_0 + \delta)$ and $I_\varepsilon = (t_0 - \varepsilon, t_0 + \varepsilon)$ where $\delta > 0$ and $\varepsilon > 0$, then*

$$\phi(t) \equiv \psi(t)$$

for all $t \in I_\eta = (t_0 - \eta, t_0 + \eta)$ where $\eta > 0$.

Basic outline for the proof:

$$\text{IVP} \quad x'(t) \equiv F(t, x(t)), \quad x(t_0) = x_0$$

$$x(t) - x(t_0) = \int_{t_0}^t F(\tau, x(\tau)) \, d\tau$$

10 2/24/16: Uniqueness & Existence for C^1 IVP's

10.1 Autonomous Equations and the Time Shift Property

$$\begin{cases} x' = \sqrt{|x|} & (\text{autonomous} - \text{the independent variable makes no explicit appearance}) \\ x(0) = 0 \end{cases}$$

One important property of an autonomous differential equation is that it is time-independent, i.e. if $x = \phi(t)$ is a solution, then so is $x = \psi(t) := \phi(t + c)$. Without an initial condition, we have an infinite number of solutions.

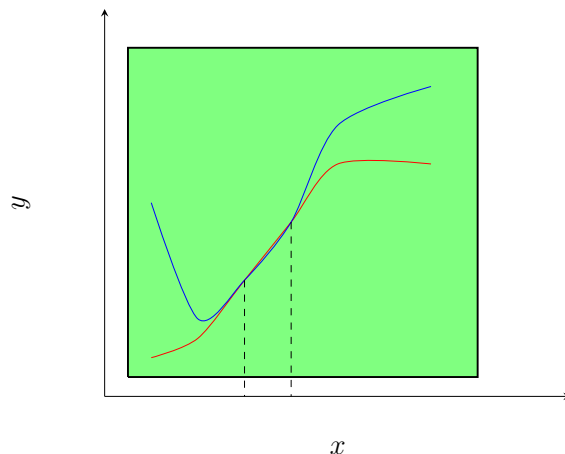
Let us try separation of variables:

$$\begin{aligned} \int_0^{x(t)} \frac{dx}{\sqrt{|x|}} &= \int_0^t d\tau = t \\ \lim_{\varepsilon \rightarrow 0^+} \int_\varepsilon^{x(t)} u^{-\frac{1}{2}} du &= \lim_{\varepsilon \rightarrow 0^+} \left[2u^{\frac{1}{2}} \right]_\varepsilon^{x(t)} \\ &= 2\sqrt{x(t)} - \lim_{\varepsilon \rightarrow 0^+} \sqrt{\varepsilon} \\ &= 2\sqrt{x(t)} \\ x(t) &= \frac{t^2}{4} > 0 \quad (\text{assuming } t \geq 0) \end{aligned}$$

We can similarly derive, for $t \leq 0$, that $x(t) = -\frac{t^2}{4}$. We can then construct our function:

$$x(t) = \begin{cases} \frac{t^2}{4}, & t \geq 0 \\ -\frac{t^2}{4}, & t < 0 \end{cases}$$

10.2 Unique Solutions



Proof. We begin by showing that our IVP is actually an integral equation.

$$\begin{cases} x'(t) \equiv F(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

$$\begin{aligned} x'(\tau) &\equiv F(\tau, x(\tau)) \\ \int_{t_0}^t x'(\tau) d\tau &= \int_{t_0}^t F(\tau, x(\tau)) d\tau \end{aligned}$$

$$x(t) = x_0 + \int_{t_0}^t F(\tau, x(\tau)) d\tau, \quad x(t) \text{ is continuous}$$

We have just proved one direction of equivalence. To prove the other direction, note that $x(t)$ is differentiable, since all of its parts are continuous and differentiable. ■

11 2/25/16: Uniqueness/Existence for C^1 IVP's

We're assuming: for $\delta > 0, \varepsilon > 0$:

$$\phi : I_\delta := (t_0 - \delta, t_0 + \delta) \rightarrow \mathbb{R} \text{ satisfies } \phi'(t) \equiv F(t, \phi(t)), \phi(t_0) = x_0$$

$$\psi : I_\varepsilon := (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow \mathbb{R} \text{ satisfies } \psi'(t) \equiv F(t, \psi(t)), \psi(t_0) = x_0$$

We want to show that for some $\eta > 0$, $\phi(t) \equiv \psi(t)$ on $I_\eta := (t_0 - \eta, t_0 + \eta)$.

First, we introduce the following concept:

Definition 11.1. If f is a bounded real-valued function on a set \mathcal{S} , then its **sup-norm** is defined as:

$$\|f\|_{\mathcal{S}} := \sup_{x \in \mathcal{S}} |f(x)|$$

If \mathcal{S} is a closed, bounded subset of \mathbb{R}^n , and f is continuous, then $\|f\|_{\mathcal{S}} = \max_{x \in \mathcal{S}} |f(x)|$, in which case is called the **max-norm**.

Note that:

- $\|f\|_{\mathcal{S}} \geq 0$
- $\|f\|_{\mathcal{S}} = 0$ iff $f(x) \equiv 0$ for all $x \in \mathcal{S}$
- $\|\alpha f\|_{\mathcal{S}} = |\alpha| \|f\|_{\mathcal{S}}$
- $\|f + g\|_{\mathcal{S}} \leq \|f\|_{\mathcal{S}} + \|g\|_{\mathcal{S}}$ where f and g are defined and bounded on \mathcal{S} .

We claim that $\|\phi - \psi\|_{I_{\eta}} \leq c \|\phi - \psi\|_{I_{\eta}}$, where $0 < c < 1$. This would mean that $\|\phi - \psi\|_{I_{\eta}} = 0$, then $\phi(t) - \psi(t) \equiv 0$ on I_{η} and $\phi(t) = \psi(t)$ on I_{η} .

Proof. Note that

$$\phi(t) \equiv x_0 + \int_{t_0}^t F(\tau, \phi(\tau)) \, d\tau$$

for all $t \in I_{\delta}$ and

$$\psi(t) \equiv x_0 + \int_{t_0}^t F(\tau, \psi(\tau)) \, d\tau$$

for all $t \in I_{\varepsilon}$. Both of these equations are true for all $t \in I_{\min(\delta, \varepsilon)}$.

Restrict $t \in I_{\eta}$ where $0 \leq \eta \leq \min(\delta, \varepsilon)$. Subtracting these two equations:

$$\begin{aligned}
 |\phi(t) - \psi(t)| &= \left| \int_{t_0}^t [F(\tau, \phi(\tau)) - F(\tau, \psi(\tau))] \, d\tau \right| \\
 &\leq \left| \int_{t_0}^t |F(\tau, \phi(\tau)) - F(\tau, \psi(\tau))| \, d\tau \right| \\
 \text{(MVT)} \quad &\leq \left| \int_{t_0}^t \underbrace{\left| \frac{\partial F}{\partial x}(x, \theta(\tau)) \right|}_{\leq M} |\phi(t) - \psi(t)| \, d\tau \right| \\
 &\leq M \left| \int_{t_0}^t \underbrace{|\phi(\tau) - \psi(\tau)|}_{\leq \|\phi - \psi\|_{I_{\eta}}} \, d\tau \right| \\
 &\leq M \|\phi - \psi\| (t - t_0) \leq M\eta \|\phi - \psi\|_{I_{\eta}}
 \end{aligned}$$

Now we simply pick η such that $M\eta = c < 1$, and we are done. ■

12 2/26/16: Existence

12.1 Transforming to an Integral Equation

Yesterday we proved the uniqueness of the solution of an IVP. Now we must prove the existence.

$$\begin{cases} x' = F(t, x), & x = x(t) \text{ is the unknown function} \\ x(t_0) = x_0 \end{cases}$$

C^1 IVP $\Leftrightarrow F(t, x)$ is C^1 in some rectangle \mathcal{R} centered at (x_0, y_0) .

Let us integrate our equation:

$$\begin{aligned} \int_{t_0}^t x'(\tau) d\tau &= \int_{t_0}^t F(\tau, x(\tau)) d\tau \\ x(t) - x(t_0) &= x(t) - x_0 = \int_{t_0}^t F(\tau, x(\tau)) d\tau \end{aligned}$$

So now our problem/equation becomes:

- $x(t) \equiv x_0 + \int_{t_0}^t F(\tau, x(\tau)) d\tau$
- $x(t)$ is a continuous function of t

Why do we need the continuity condition? If $x(t)$ is a solution, then it is differentiable, which implies it is continuous.

Now we prove the opposite direction. To prove that $x(t)$ is differentiable, note that $f(t) = x_0$ is differentiable, and the integral is also differentiable (since its derivative is $F(t, x(t))$, which is continuous). Therefore, by algebra, the two statements are equivalent.

12.2 Picard's Method

Define

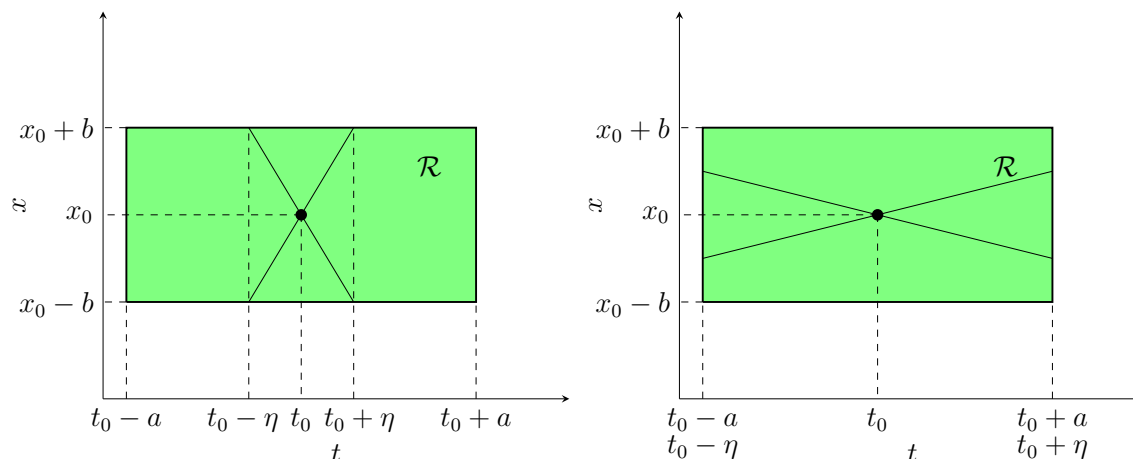
$$x_{n+1}(t) := x_0 + \int_{t_0}^t F(\tau, x_n(\tau)) d\tau$$

and

$$x_0(t) \equiv x_0 \quad \text{for all } t$$

In this section, we prove that for each $t \in I_\eta$, $\lim_{n \rightarrow \infty} x_n(t)$ exists, let's call it $x(t)$, and moreover:

- $x(t)$ is a continuous function of t on I_η
- $x(t) \equiv x_0 + \int_{t_0}^t F(\tau, x(\tau)) d\tau$



Define $M \geq \max_{(t,x) \in \mathcal{R}} |F(t,x)|$ (existence follows from EVT). Let $\eta := \min(a, \frac{b}{M}) > 0$. Assume $M > 0$; if $M = 0$, then the IVP is $x' \equiv 0$, $x(t_0) = x_0$ has a solution: $x(t) \equiv x_0$ for all $t \in (t_0 - a, t_0 + a)$.

13 2/29/16: Picard's Existence Proof, Continued...

Recapping: our base function is

$$x_0(t) \equiv x_0 \quad \text{for all } t \in I_\eta, \eta = \min\left(a, \frac{b}{M}\right) > 0$$

We need to choose the size of the rectangle for each function. In the example

$$\begin{cases} x' = t^2 + x^2 \\ x(0) = 1 \end{cases}$$

We need to find some $\eta > 0$ such that a solution is guaranteed to exist in $(-\eta, \eta)$. Let $\mathcal{R} = [-T, T] \times [1-r, 1+r]$ be our rectangle. Then take $M = T^2 + (1+r)^2$, then $|t^2 + x^2| \leq M$ when $(t, x) \in \mathcal{R}$.

13.1 Proving Well-Defined-ness

Theorem 13.1. *Each x_n is well-defined and continuous and satisfies $|x_n(t) - x_0| \leq b$ for all $t \in I_\eta$.*

Proof. The base case is trivial. Now assume true for $x_n(t)$; we now prove for $x_{n+1}(t)$. By assumption, the integrand $F(\tau, x_n(\tau))$ is well-defined and continuous (and therefore Riemann integrable on I_η) for all $t \in I_\eta$, which means the integral is well-defined. Therefore, $x_{n+1}(t)$ is well-defined for all $t \in I_\eta$.

Also, $x_{n+1}(t)$ is continuous on I_η by similar reasoning.

Now, we investigate $|x_{n+1}(t) - x_0|$. First, we claim that $(\tau, x_n(\tau)) \in \mathcal{R}$ for any τ between t_0 and t . But

$$|\tau - t_0| \leq |t - t_0| \leq \eta \leq a$$

$$|x_n(\tau) - x_0| \leq b$$

$$\begin{aligned} |x_{n+1}(t) - x_0| &= \left| \int_{t_0}^t F(\tau, x_n(\tau)) \, d\tau \right| \\ &\leq \left| \int_{t_0}^t \underbrace{|F(\tau, x_n(\tau))|}_{\leq M} \, d\tau \right| \\ &\leq M|t - t_0| \leq M\eta \leq b \end{aligned}$$

■

14 3/1/16: Finish Picard's Existence Proof

Theorem 14.1.

$$|x_{n+1}(t) - x_n(t)| \leq \frac{MK^n}{(n+1)!} |t - t_0|^{n+1}$$

for any $n \geq 0$ and any $t \in I_\eta$, where $K \geq \max_{(t,x) \in \mathcal{R}} \left| \frac{\partial F}{\partial x}(t, x) \right|$ (using the assumed C^1 -ness of F on \mathcal{R}).

Proof. We prove by induction. When $n = 0$:

$$|x_1(t) - x_0(t)| = |x_1(t) - x_0| = \left| \int_{t_0}^t F(\tau, x_0) \, d\tau \right| \leq M|t - t_0| = \frac{MK^0}{(0+1)!} |t - t_0|^{0+1}$$

Now assume the hypothesis, we want to prove that

$$|x_{n+2}(t) - x_{n+1}(t)| \leq \frac{MK^{n+1}}{(n+2)!} |t - t_0|^{n+2}$$

So:

$$\begin{aligned} |x_{n+2}(t) - x_{n+1}(t)| &= \left| \int_{t_0}^t F(\tau, x_{n+1}) - F(\tau, x_n) \, d\tau \right| \\ (\text{MVT, for some } y_n \in [x_n, x_{n+1}]) &= \left| \int_{t_0}^t \frac{\partial F}{\partial x}(\tau, y_n)(x_{n+1} - x_n) \, d\tau \right| \\ &\leq \left| \int_{t_0}^t \left| \frac{\partial F}{\partial x}(\tau, y_n) \right| |x_{n+1} - x_n| \, d\tau \right| \\ &\leq K \left| \int_{t_0}^t |x_{n+1} - x_n| \, d\tau \right| \\ &\leq K \left| \int_{t_0}^t \frac{MK^n}{(n+1)!} |t - t_0|^{n+1} \, d\tau \right| \\ &= \frac{MK^{n+1}}{(n+2)!} |t - t_0|^{n+2} \end{aligned}$$

■

15 3/3/16

For any $t \in I_\eta$,

$$|x_{n+p}(t) - x_n(t)| = |x_{n+p}(t) - x_{n+p-1}(t) + x_{n+p-1}(t) - x_{n+p-2}(t) + x_{n+p-2}(t) - \cdots - x_n(t)|$$

is bounded. By the Triangle Inequality,

$$\begin{aligned} |x_{n+p}(t) - x_n(t)| &\leq \sum_{j=n}^{n+p-1} |x_{j+1}(t) - x_j(t)| \\ &\leq \sum_{j=n}^{n+p-1} \frac{MK^j}{(j+1)!} |t - t_0|^{j+1} \\ &= \left(\frac{M}{K}\right) \sum_{j=n}^{n+p-1} \frac{(K|t - t_0|)^{j+1}}{(j+1)!} \\ &\leq \left(\frac{M}{K}\right) \sum_{j=n}^{\infty} \frac{(K|t - t_0|)^{j+1}}{(j+1)!} \\ &\leq \left(\frac{M}{K}\right) \sum_{j=n}^{\infty} \frac{(K\eta)^{j+1}}{(j+1)!} \\ &= \left(\frac{M}{K}\right) \left(e^{K\eta} - \sum_{j=0}^{n-1} \frac{(K\eta)^{j+1}}{(j+1)!} \right) \end{aligned}$$

Thus, we have an upper bound for any two terms in our sequence.

Now if we take $\|x_{n+p} - x_n\|_{I_\eta} = \sup_{t \in I_\eta} |x_{n+p}(t) - x_n(t)| \leq L$, then send $n \rightarrow \infty$. Then,

$$\lim_{n \rightarrow \infty} \|x_{n+p(n)} - x_n\|_{I_\eta} \leq 0$$

But this is also nonnegative, so it must be the case that the limit is zero, and thus this sequence is Cauchy.

16 3/4/16: Existence of Solutions, Continued

The last thing we proved was that (x_n) is a Cauchy sequence in the space of continuous functions on I_η , denoted $C^0(I_\eta)$, under the sup-norm, $\|f\|_{I_\eta} = \sup_{t \in I_\eta} |f(t)|$. This means:

$$\lim_{n \rightarrow \infty} \|x_{n+p(n)} - x_n\|_{I_\eta} = 0 \text{ for any } \mathbb{N}\text{-valued function } p(n).$$

16.1 Metric Spaces

Definition 16.1. A *metric space*, denoted (\mathcal{X}, d) , $\mathcal{X} \neq \emptyset$, d is a “distance” function, must satisfy the following:

1. $d : (\mathcal{X} \times \mathcal{X}) \rightarrow [0, \infty)$
2. $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) + d(y, z) \geq d(x, z)$

If we have a metric d on a vector space \mathcal{V} , we can also require $d(x, y) + d(y, z) = d(x, z)$ iff x - y - z (y is between x and z) or $x = y$ or $y = z$. To define betweenness: \vec{p} - \vec{q} - \vec{r} iff $\vec{q} = (1 - t)\vec{p} + t\vec{r}$.

Here, we define a metric on a vector space. Given a vector space \mathcal{V} , a **norm** on \mathcal{V} is a real-valued function $\|\cdot\|$ such that:

1. $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0 \Leftrightarrow \vec{v} = \vec{0}$ (Positive definiteness)
2. $\|c\vec{v}\| = |c|\|\vec{v}\|$ (Absolute homogeneity)
3. $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ (Triangle inequality)

This way, we can define

$$d(\vec{v}, \vec{w}) := \|\vec{v} - \vec{w}\|$$

16.2 Cauchy Sequences in Metric Spaces

In a metric space (\mathcal{X}, d) , a sequence of elements (x_n) converges to $x \in \mathcal{X}$ if $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.

A sequence (x_n) in (\mathcal{X}, d) is called Cauchy if $d(x_n, x_m) \rightarrow 0$ as $\min(n, m) \rightarrow \infty$.
Not all Cauchy sequences converge. As an example, take

$$\mathcal{X} = \mathbb{Q}, \quad d(q, \tilde{q}) = |q - \tilde{q}|$$

Take the sequence $(3, 3.1, 3.14, 3.141, \dots)$. This sequence is Cauchy since choosing two far-out values will differ very little. However, it converges to π , which is not in the metric space. Therefore, we call this an **incomplete metric space**.

17 3/7/16: Cauchy Sequences and Convergence in \mathbb{R} and in $C^0([a, b])$

A sequence (t_n) in \mathbb{R} is Cauchy if $|t_n - t_m| \rightarrow 0$ as $\min(n, m) \rightarrow \infty$. More precisely, $\forall \varepsilon > 0, \exists N : n, m \geq N \Rightarrow |t_n - t_m| < \varepsilon$.

We'd like to prove that every Cauchy sequence in \mathbb{R} converges to some $t \in \mathbb{R}$.

Theorem 17.1. $(\mathbb{R}, |\cdot - \cdot|)$ is a **complete metric space**, i.e. every Cauchy sequence of real numbers converges to a real limit.

We make use of several lemmas. First, a definition:

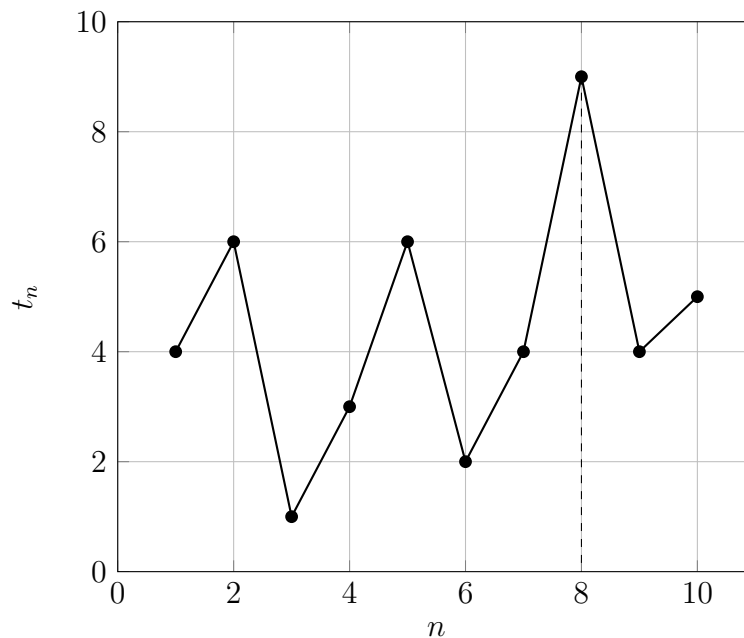
Definition 17.1. A **subsequence** of (t_n) is any sequence of the form (t_{k_n}) where (k_n) is a strictly increasing sequence of natural numbers: $1 \leq k_1 < k_2 < \cdots < k_n < \cdots$

Definition 17.2. A sequence (u_n) is called monotone if either $u_1 \leq u_2 \leq u_3 \leq \cdots \leq u_n \leq \cdots$ or $u_1 \geq u_2 \geq u_3 \geq \cdots \geq u_n \geq \cdots$.

Lemma 17.1. A subsequence of a subsequence of (t_n) is itself a subsequence of (t_n) .

Lemma 17.2. If (k_n) is a strictly increasing sequence in \mathbb{N} , then $k_n \rightarrow \infty$ as $n \rightarrow \infty$. In fact, $k_n \geq n$.

Lemma 17.3 (Rising Sun Lemma). Every sequence in \mathbb{R} has a monotone subsequence.



Proof. We call N a vista if $t_N > t_{N+k}$ for all $k \geq 1$. We consider two cases:

1. the set of vistas is infinite; call them $N_1 < N_2 < N_3 < \cdots$. Then,

$$t_{N_1} > t_{N_2} > t_{N_3} > \cdots$$

and we can take (t_{N_n}) as our subsequence—this is strictly decreasing, so certainly monotone down.

2. the set of vistas is finite (including possibly empty). Then, let N be one more than the greatest among the vistas. N is not a vista, so $\exists k_2 > k_1 = N : t_{k_2} \geq t_{k_1}$. k_2 is also not a vista, so $\exists k_3 > k_2 : t_{k_3} \geq t_{k_2}$. Then taking (t_{k_n}) , we have our monotone subsequence.

■

Lemma 17.4. *Every Cauchy sequence in \mathbb{R} (true in any metric space) is bounded:*

$$\exists M : |t_n| \leq M$$

for all $n \geq 1$.

Proof. By definition, we can choose some N such that

$$n, m \geq N \Rightarrow |t_n - t_m| < 1$$

Take $m = N$, then $|t_n - t_N| < 1$, and

$$|t_N| - 1 \leq |t_n| \leq |t_N| + 1 \quad \forall n \geq N$$

$$|t_n| \leq \max(|t_1|, |t_2|, |t_3|, \dots, |t_{N-1}|)$$

■

18 3/8/16: Completeness of \mathbb{R} , $C^0([a, b])$

Previously we proved that every cauchy sequence is bounded. Now, we can prove that:

Lemma 18.1. *Every cauchy sequence has a convergent subsequence.*

Proof. A monotone subsequence of a cauchy sequence is bounded. We need to show that a bounded monotone sequence must converge to a finite limit.

Assume

$$a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n \leq a_{n+1} \leq \dots \leq b$$

We claim that

$$l := \sup\{a_n | n \geq 1\} = \lim_{n \rightarrow \infty} a_n$$

■

Lemma 18.2. *If a subsequence of a cauchy sequence converges to some limit $t \in \mathbb{R}$, then the original sequence also converges to t .*

Proof. Since (t_n) has a subsequence (t_{k_n}) s.t. $t_{k_n} \rightarrow t$ as $n \rightarrow \infty$.

$$\begin{aligned} 0 \leq |t_n - t| &= |(t_n - t_{k_n}) + (t_{k_n} - t)| \\ &\leq |t_n - t_{k_n}| + |t_{k_n} - t| \\ &\leq 0 \end{aligned}$$

By the squeeze theorem, the proof is complete. ■

18.1 Putting it Together

Take the metric space

$$\mathcal{X} = C^0([a, b]) = \{f : [a, b] \rightarrow \mathbb{R} \mid f \text{ is continuous on } [a, b]\}$$

$$d(f, g) = \|f - g\|_I = \sup_{a \leq t \leq b} |f(t) - g(t)|$$

We claim that if (f_n) is a sequence in $C^0(I)$ that is cauchy $[\|f_n - f_m\|_I \rightarrow 0 \text{ as } \min n, m \rightarrow \infty]$, then (f_n) converges to some function $f \in C^0(I)$ $[\|f_n - f\|_I \rightarrow 0 \text{ as } n \rightarrow \infty]$.

We must propose some limit function. For each $t \in I$, define

$$f(t) := \lim_{n \rightarrow \infty} f_n(t)$$

We can see that $(f_n(t))$ converges in \mathbb{R} :

$$0 \leq |f_n(t) - f_m(t)| \leq \underbrace{\|f_n - f_m\|}_0 = \sup_{a \leq \tau \leq b} |f_n(\tau) - f_m(\tau)|$$

So we have our limit function, $f(t)$. However, we don't know yet if $f(t)$ is continuous; we have point-wise convergence, but not yet uniform convergence.

19 3/9/16: Completeness of $C^0([a, b])$; Existence in the Flow Problem

Let (f_n) be a cauchy sequence in $C^0(I)$ where $I := [a, b]$, so $\|f_n - f_m\| \rightarrow 0$ as $\min(n, m) \rightarrow \infty$. Define, for each $t \in I$ separately, $\sigma_t := (f_1(t), f_2(t), \dots, f_n(t), \dots)$, a sequence in \mathbb{R} .

We don't know yet, that

- The functions actually converge to $f(t)$ ($\|f_n - f\|_I \rightarrow 0$ as $n \rightarrow \infty$), known as uniform convergence.
- $f(t)$ is continuous

As an example, consider:

$$\begin{aligned} f_n(t) &:= t^n \\ n &= 1, 2, 3, \dots \\ t &\in I = [0, 1] \end{aligned}$$

$$f(t) = \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ 1 & \text{if } t = 1 \end{cases}$$

Then, $\forall t \in [0, 1]$, $f_n(t) \rightarrow f(t)$ as $n \rightarrow \infty$. We have point-wise convergence, but we do not have uniform convergence.

To prove continuity, we look at two points close together. Assuming continuity:

$$\begin{aligned}
 |f(t) - f(\tilde{t})| &= |f(t) - f_n(t) + f_n(t) - f_n(\tilde{t}) + f_n(\tilde{t}) - f(\tilde{t})| \\
 &\leq |f(t) - f_n(t)| + |f_n(t) - f_n(\tilde{t})| + |f_n(\tilde{t}) - f(\tilde{t})| \\
 &\leq \|f - f_n\|_I + |f_n(t) - f_n(\tilde{t})| + \|f_n - f\|_I \\
 &= 2\|f - f_n\|_I + |f_n(t) - f_n(\tilde{t})|
 \end{aligned}$$

We can take $\|f - f_n\|_I$ to be less than some fixed number $\frac{\varepsilon}{4}$, and we get that

$$|f(t) - f(\tilde{t})| < \varepsilon$$

For the other condition:

$$\|f_n - f_m\| \rightarrow 0$$

This is equivalent to saying

$$\forall \varepsilon > 0, \exists N(\varepsilon) : \forall n, m \geq N(\varepsilon), \|f_n - f_m\| < \varepsilon$$

Taking limits on both sides:

$$\lim_{m \rightarrow \infty} \|f_n - f_m\| \leq \underbrace{\lim_{m \rightarrow \infty} \varepsilon}_{\varepsilon}$$

$$\|f_n - f\|_I \leq \varepsilon \text{ for all } n \geq N(\varepsilon)$$

But by choosing ε small enough, then $\|f_n - f\|_I \rightarrow 0$