

Differential Equations

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Contents

1	2/3/16: Background on \mathbb{R}; Basic Existence Question of ODE's	5
1.1	Romeo and Juliet	5
1.2	Supremum and Infimum of a Set $\mathcal{A} \subseteq \mathbb{R}$	6
1.3	What is \mathbb{R} ?	6
2	2/4/16: Background in \mathbb{R}; Fundamental Existence/Uniqueness Question	7
2.1	Supremums and Infimums in Integrals	7
2.2	Real Numbers, Again	8
3	2/5/16: Fundamental Existence of Uniqueness Theorem	8
3.1	Terminology	8
3.2	A Treatise on PDE's	9
4	2/9/16: Basic Existence and Uniqueness Theorem	9
4.1	Second-Order	10
5	2/10/16: The Flow Theorem	10
5.1	Application: n^{th} order initial value problem (IVP)	10
6	2/11/16: Proof of the Flow Theorem	11
7	2/12/16: Separable and First-Order Linear Equations	12
7.1	Multiplicatively Separable Functions	12
7.2	Separable ODE	13
7.3	Example	13
8	2/22/16: Separable Equations, First-Order Linear Equations; Uniqueness for C^1 IVP's	14
9	2/23/16: Uniqueness for C^1 IVP's	14
9.1	Uniqueness	15
10	2/24/16: Uniqueness & Existence for C^1 IVP's	15
10.1	Autonomous Equations and the Time Shift Property	15
10.2	Unique Solutions	16
11	2/25/16: Uniqueness/Existence for C^1 IVP's	16
12	2/26/16: Existence	18
12.1	Transforming to an Integral Equation	18
12.2	Picard's Method	18
13	2/29/16: Picard's Existence Proof, Continued...	19
13.1	Proving Well-Defined-ness	19

14 3/1/16: Finish Picard's Existence Proof	20
15 3/3/16	21
16 3/4/16: Existence of Solutions, Continued	21
16.1 Metric Spaces	21
16.2 Cauchy Sequences in Metric Spaces	22
17 3/7/16: Cauchy Sequences and Convergence in \mathbb{R} and in $C^0([a, b])$	22
18 3/8/16: Completeness of \mathbb{R}, $C^0([a, b])$	24
18.1 Putting it Together	25
19 3/9/16: Completeness of $C^0([a, b])$; Existence in the Flow Problem	25
20 3/10/16	26
21 3/11/16: Fact that the Picard limit function solves the IVP; Extension to Vector IVP's; Linear IVP's	27
22 3/17/16: Domain of Solutions for Linear Equations; Solving First-Order Linear Equations (Regular Case)	27
22.1 First-Order C^1 Vector IVP's	27
23 3/18/16	28
24 3/21/16: First-Order Linear Equation in One Unknown—Regular Case	28
25 3/22/16: First-Order Linear Equation—in One Unknown (Regular Case)	29
25.1 Form of Linear Equation	29
25.2 Euler's method of integrating factors	29
26 3/23/16: First-Order Linear Equations—Singular Case	30
26.1 Singular Case	30
26.2 Finding a Nonzero Solution	30
27 3/24/16	31
28 3/28/16	32
29 3/29/16	32
29.1 Rewriting Our Differential Equation	32
30 3/30/16: Theorems on Power Series, etc.	32
31 4/4/16: Series Solutions	34
32 4/6/16	35

33 4/13/16: Linear Equations with Constant Coefficients	35
33.1 Summary	35
33.2 Solving	36
34 4/14/16	36
34.1 Homogeneous and Inhomogeneous	36
34.2 Finding a Solution	36
35 4/15/16	37
36 4/18/16: Determination of the General Solution	38
37 4/20/16	39
37.1 Inhomogeneous Equation	39
37.1.1 Form of the Solution Set	39
37.1.2 Finding ψ_p	39
37.1.3 Variation of Parameters	40
38 4/22/16: n-th Order LDE's with Constant Coefficients—Homogeneous Case	41
39 5/3/16: Constant-Coefficient Linear Equations, Order n—Homogeneous Case	42

Introduction

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1 2/3/16: Background on \mathbb{R} ; Basic Existence Question of ODE's

1.1 Romeo and Juliet

$$\begin{cases} R' = aR + bJ \\ J' = cR + dJ \end{cases}$$

These equations model the rate of change of Romeo's and Juliet's feelings. We call this a **linear system of two coupled differential equations of first order in two unknowns**.

- What makes it linear is that the functions and variables appear in a linear fashion.
- What makes it coupled is that both equations have both R and J in them.
- An **uncoupled system** would look like:

$$\begin{cases} R' = aR \\ J' = bJ \end{cases}$$

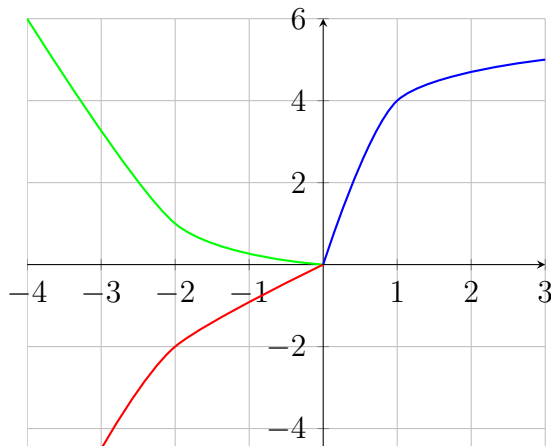
- First-order refers to the fact that all the derivatives are the first derivatives.

"Identically cautious lovers":

$$\begin{aligned} R' &= aR + bJ & a < 0, b > 0 \\ J' &= bR + aJ & |a| > |b| \end{aligned}$$

We may have initial conditions, $R(0)$ and $J(0)$, and plot them on a **phase plane** with R against J . In this case, no matter where the starting point is, the trajectory will go towards a **stable node**.

In the case of $|a| < |b|$, points will move asymptotically towards $R = -J$ and $R = J$. In the case of $|a| = |b|$, points will cycle around the origin infinitely.



1.2 Supremum and Infimum of a Set $\mathcal{A} \subseteq \mathbb{R}$

- If $\mathcal{A} \subseteq (-\infty, b]$ for some $b \in \mathbb{R}$, we say \mathcal{A} is bounded above, and that b is an **upper bound** for \mathcal{A} .

Theorem 1.1 (Supremum Theorem). *If $\mathcal{A} \subseteq \mathbb{R}$, $\mathcal{A} \neq \emptyset$, and $\mathcal{A} \subseteq (-\infty, b]$ for some $b \in \mathbb{R}$, then there exists $a \in \mathbb{R}$ such that $\mathcal{A} \subseteq (-\infty, a]$ but if $x < a$, then $\mathcal{A} \not\subseteq (-\infty, x]$. We write $a = \sup \mathcal{A}$, call it the **supremum** of \mathcal{A} .*

Why is this necessary? Consider the set $\mathcal{A} = \{-\frac{1}{n} | n \in \mathbb{N}\}$. It does not have a maximum per say, but it has a supremum $\sup \mathcal{A} = 0$.

Consider this example: What is $\sup(-\mathbb{N})$? It is -1, which also happens to be the maximum of the set. e

Theorem 1.2. *If $\max \mathcal{A}$ exists as a real number, then $\sup \mathcal{A} = \max \mathcal{A}$.*

But to answer all these questions, we need to figure out: what exactly are the real numbers?

1.3 What is \mathbb{R} ?

Let $x = (s, N, d_1, d_2, d_3, \dots, d_k, \dots)$, where:

- $s \in \{+1, -1\}$
- $N \in \mathbb{Z}$
- $d_k \in \mathbb{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- $\neg(\exists k : d_{k+1} = d_{k+2} = \dots = 0)$, this is to prevent multiple sequences from being the same number

In this case, “2.49” is shorthand for $(+1, 2, 4, 8, 9, 9, 9, \dots)$

2 2/4/16: Background in \mathbb{R} ; Fundamental Existence/Uniqueness Question

2.1 Supremums and Infimums in Integrals

Theorem 2.1 (Supremum/Infimum Theorem).

1. If \mathcal{A} is a non-empty set of \mathbb{R} , and is bounded above (i.e. $\mathcal{A} \subseteq (-\infty, b]$ for some $b \in \mathbb{R}$), then there is a least upper bound for \mathcal{A} , namely $a \in \mathbb{R}$ such that

$$(a) \mathcal{A} \subseteq (-\infty, a]$$

$$(b) \text{ if } x < a, \text{ then } \mathcal{A} \not\subseteq (-\infty, x]$$

This a is called the **supremum** of \mathcal{A} , written $\sup A$.

2. $\inf A$. This is the greatest lower bound for \mathcal{A} , or the **infimum**, provided $\mathcal{A} \neq \emptyset$ and \mathcal{A} has a lower bound at all.

Recall that the Riemann integral is taking the limit of a partition over an interval $[a, b]$. But when we take the limit, we make the mesh of the partition, $\|\mathcal{P}\|$, approach zero, where

$$\mathcal{P} = \max_{1 \leq i \leq n} \Delta x_i$$

To fix this, we can define:

$$\int_a^b f(x) dx = \sup \left\{ \sum_{i=1}^n [\inf\{f(x) \mid x_{i-1} \leq x \leq x_i\} \Delta x_i] \mid a = x_0 < x_1 < \cdots < x_n = b \right\}$$

This is a “down-and-up” procedure. The sum of the rectangle areas is a down approximation since we use the minimum possible height to find the area. Then, we take the supremum of that, since for any lower approximation there will always be a higher approximation. Turns out there will never be a maximum; that’s why we take the supremum. This is a **lower Riemann sum**.

We can also define the same thing for an **upper Riemann sum**:

$$\int_a^b f(x) dx = \inf \left\{ \sum_{i=1}^n [\sup\{f(x) \mid x_{i-1} \leq x \leq x_i\} \Delta x_i] \mid a = x_0 < x_1 < \cdots < x_n = b \right\}$$

Therefore, the following inequality is true:

$$\int_a^b f \leq \int_a^b f$$

If these two are equal, then we say that f is **Riemann integrable**.

Here’s an example of a function that is NOT Riemann integrable:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 1 & \text{if } x \in [0, 1] \setminus \mathbb{Q} \end{cases}$$

Note that $\int_0^1 f = 0$ and $\int_0^1 f = 1$, so this is not Riemann integrable.

2.2 Real Numbers, Again

We have shorthand for our previous definition of the real numbers.

$$\mathbb{R} = \{0\} \cup \{(s, N, d_1, d_2, \dots, d_k, \dots \mid s \in \{-1, +1\}, N \in \mathbb{Z}^+, d_k \in \mathbb{D}, \text{no 0-tail}\}$$

and the positive reals:

$$\mathbb{R}^+ = \{(s, N, d_1, d_2, \dots) \mid s = +1\}$$

Let us write $x = \underline{N.d_1d_2d_3\dots}$ and $y = \underline{M.e_1e_2e_3\dots}$.

We also define negation as:

$$-(s, N, d_1, d_2, \dots) := (-s, N, d_1, d_2, \dots)$$

Then we can define the “less than” operation as follows:

- If $x, y \in \mathbb{R}^+$, then $x < y$ if either $N < M$ or $N = M$ and $d_1 < e_1$ or $N = M$, $d_1 = e_1$ and $d_2 < e_2$, or...
- $0 < x$ if $x \in \mathbb{R}^+$
- $x < 0$ if $x \in \mathbb{R}^+$
- $x < y$ if $x \in \mathbb{R}^-, y \in \mathbb{R}^+$.
- $x, y \in \mathbb{R}^-$, and $x < y$ if $-y < -x$

3 2/5/16: Fundamental Existence of Uniqueness Theorem

3.1 Terminology

A **differential equation** is a relation between one or more unknown functions and at least some (but finitely many) of their derivatives, plus the independent variables.

Examples:

$$\begin{aligned} y' + 2xy - x^2 &= 3 \\ y''' + 2x^2y'' - 3x^3y' + xy - x^5 + 1 &= 0 \\ (y')^{y''} - e^{y'''} + x &= 0 \end{aligned}$$

Or,

$$\vec{y}' = \mathbf{A}(x)\vec{y}$$

where

$$\vec{y} = \vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}$$

$$\mathbf{A}(x) = \begin{bmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{bmatrix}$$

3.2 A Treatise on PDE's

There are two different types of differential equations: ODE's (ordinary, where all unknown functions depend on a single, same independent variable) and PDE's (partial, anything else).

$$\begin{aligned} \text{Wave equation: } \frac{\partial^2 u}{\partial x^2} &= c^2 \frac{\partial^2 u}{\partial t^2} \\ u &= g(x - t) + h(x + t) \end{aligned}$$

4 2/9/16: Basic Existence and Uniqueness Theorem

Theorem 4.1 (Flow Theorem). *Let $\vec{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), \dots, F_n(\vec{x}))$ be a vector field defined on some closed and bounded region $\mathcal{D} \subseteq \mathbb{R}^n$. Also assume \vec{F} is C^1 ; namely, $\frac{\partial F_i}{\partial x_j}$ is continuous everywhere interior to \mathcal{D} , for any i and j .*

Let \vec{p} be a specific point interior to \mathcal{D} . Then \exists a function $\vec{\sigma}(t)$ from some "time" interval $(-\varepsilon, \varepsilon)$ with $\varepsilon > 0$ into \mathcal{D} , such that $\vec{\sigma}(0) = \vec{p}$ and $\vec{\sigma}'(t) = \vec{F}(\vec{\sigma}(t))$ for any $t \in (-\varepsilon, \varepsilon)$.

This theorem basically says that in a vector field, we can use the vector field to get the velocity of a curve. We call $\vec{\sigma}(t)$ a **flow** for \vec{F} , starting at \vec{p} . This flow is, in fact, unique, in the sense that any two flows for the same \vec{F} starting at the same point must agree whenever they are both defined.

This is meaningful in that we can treat it as a differential equation:

$$\begin{cases} \sigma'_1 &= F_1(\sigma_1, \sigma_2, \dots, \sigma_n) \\ \sigma'_2 &= F_2(\sigma_1, \sigma_2, \dots, \sigma_n) \\ \vdots &\vdots \\ \sigma'_n &= F_n(\sigma_1, \sigma_2, \dots, \sigma_n) \end{cases}$$

$$\begin{cases} \sigma_1(0) &= p_1 \\ \sigma_2(0) &= p_2 \\ \vdots &\vdots \\ \sigma_n(0) &= p_n \end{cases}$$

4.1 Second-Order

$$mx'' = -kx, \quad x(0) = x_0, \quad x'(0) = v_0$$

$$x = x(t), \quad v = v(t) = x'(t), \quad a = a(t) = x''(t)$$

where $k > 0$ is the spring constant. We can rewrite this as:

$$\begin{cases} x' &= v = F_1(x, v) \\ v' &= -\frac{k}{m}x = F_2(x, v) \end{cases} \quad \text{and} \quad \begin{cases} x(0) &= x_0 \\ v(0) &= v_0 \end{cases}$$

The Flow Theorem will tell us there is a unique solution, for some time interval.

5 2/10/16: The Flow Theorem

5.1 Application: n^{th} order initial value problem (IVP)

$$\begin{cases} x &= x(t) \\ x^{(n)} &= F(t, x, x', x'', \dots, x^{(n-1)}) \\ x(t_0) &= x_{00} \\ x'(t_0) &= x_{10} \\ x''(t_0) &= x_{20} \\ \vdots & \\ x^{(n-1)}(t_0) &= x_{(n-1)0} \end{cases}$$

$f(t)x^{(n)} = F(t, x, x', x'', \dots, x^{(n-1)})$ is an n^{th} order ODE in standard form.

A **singularity** (or singular point) of this equation is a value t_0 where $f(t_0) = 0$. At this point, the equation ceases to be of n^{th} order. If $f(t)$ is of constant sign in the time interval on which we'd like to solve the equation, we just divide through by $f(t)$ to get our desired form (which is the regular case, as opposed to the singular case).

Here, the Flow Theorem says that there is a unique solution $x = x(t)$ defined in some time interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ where $\varepsilon > 0$.

To apply this:

$$\begin{cases} x_0(t) &= t \\ x_1 &= x_1(t) = x(t) \\ x_2 &= x_2(t) = x'(t) \\ x_3 &= x_3(t) = x''(t) \\ \vdots & \\ x_n &= x_n(t) = x^{(n-1)}(t) \end{cases}$$

becomes

$$\begin{cases} x'_0 &= 1 = F_0(x_0, x_1, x_2, \dots, x_n) \\ x'_1 &= x_2 = F_1(x_0, x_1, x_2, \dots, x_n) \\ x'_2 &= x_3 = F_2(x_0, x_1, x_2, \dots, x_n) \\ x'_3 &= x_4 = F_3(x_0, x_1, x_2, \dots, x_n) \\ \vdots &\vdots \\ x'_{n-1} &= x_n = F_{n-1}(x_0, x_1, x_2, \dots, x_n) \\ x'_n &= F(t, x_1, x_2, \dots, x_n) = F_n(\dots) \end{cases} \quad \text{and} \quad \begin{cases} x_0(t_0) &= t_0 \\ x_1(t_0) &= x_{00} \\ x_2(t_0) &= x_{10} \\ \vdots &\vdots \\ x_n(t_0) &= x_{(n-1)0} \end{cases}$$

This shows that we can recast an n^{th} order IVP into an $n + 1$ order system.

However, for the Flow Theorem to apply, \vec{F} needs to be C^1 . Therefore, our hypothesis in the IVP is that F is C^1 , meaning that $\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x'}, \dots$ are continuous.

6 2/11/16: Proof of the Flow Theorem

We will prove the Flow Theorem for two dimensions only; the proof can be extended to greater than two dimensions.

Proof. Let $\vec{F}(x, y) = (A(x, y), B(x, y))$ be a vector field. By hypothesis, A and B are defined on a closed, bounded region \mathcal{D} , and they are C^1 on \mathcal{D} . Then we need to solve the following equation:

$$\begin{aligned} \vec{x}' &= \vec{F}(\vec{x}) \\ \vec{x}(0) &= \vec{p} = \langle p, q \rangle \end{aligned}$$

We need to see how fast $A(x, y)$ is changing.

1.

$$\begin{aligned} |A(x_1, y_1) - A(x_2, y_2)| &= |A(x_1, y_1) - A(x_1, y_2) + A(x_1, y_2) - A(x_2, y_2)| \\ (\text{Triangle Inequality}) &\leq |A(x_1, y_1) - A(x_1, y_2)| + |A(x_1, y_2) - A(x_2, y_2)| \\ (\text{MVT}) &\leq \left| \frac{\partial A}{\partial y}(x_1, y^*)(y_1 - y_2) \right| + \left| \frac{\partial A}{\partial x}(x^*, y_2)(x_1 - x_2) \right| \end{aligned}$$

Take K to be some upper bound for all the partial derivatives of A and B on \mathcal{D} .

$$\left| \frac{\partial A}{\partial y}(x_1, y^*)(y_1 - y_2) \right| + \left| \frac{\partial A}{\partial x}(x^*, y_2)(x_1 - x_2) \right| \leq K(|x_1 - x_2| + |y_1 - y_2|)$$

Similarly:

$$|B(x_1, y_2) - B(x_2, y_2)| \leq K(|x_1 - x_2| + |y_1 - y_2|)$$

This is called the **Lipschitz Condition**.

2. Also, note that A and B are continuous in \mathcal{D} and so by the Extreme Value Theorem, we can find an upper bound M for $|A|$ and $|B|$ on \mathcal{D} , i.e.

$$M = \max \left(\max_{(x,y) \in \mathcal{D}} |A(x, y)|, \max_{(x,y) \in \mathcal{D}} |B(x, y)| \right)$$

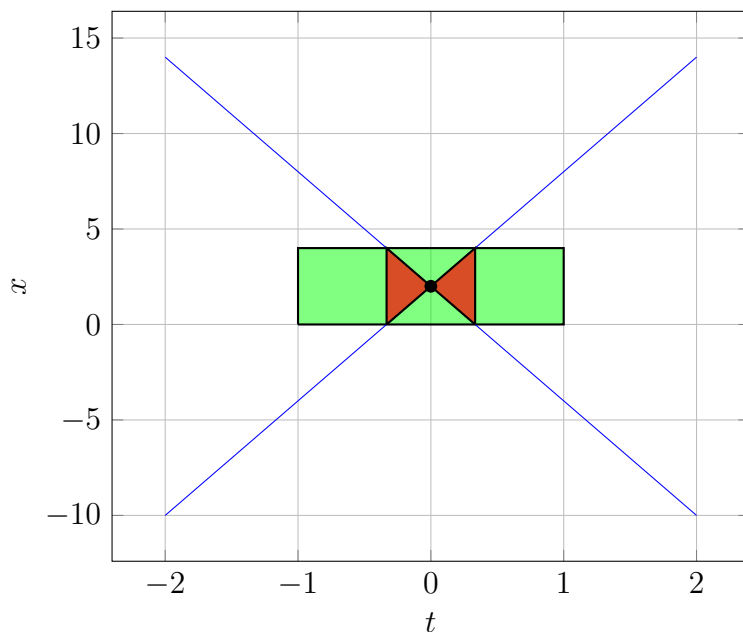
3. The point (p, q) is assumed to be interior to \mathcal{D} (not on the boundary).

We can therefore encase the point $(0, p)$ in a rectangle in the tx -plane defined by $R : [-r, r] \times [p - s, p + s] \subseteq \text{proj}_x \mathcal{D}$, $r, s > 0$. Draw two lines with slopes M and $-M$ through the point. We will consider the “bowtie” region formed by the intersections of the lines with the rectangle, joining them oppositely, and the lines themselves. Call the x -intersections $-h$ and h .

Define $h := \min(r, \frac{s}{M}) > 0$. This is to formally define the bowtie region and consider the two possible pictures depending on the size of M .

Now, we want to construct the solution of the differential equation within the bowtie region.

Whoopsies, there was a screwup here, proof to be fixed in the future.



■

7 2/12/16: Separable and First-Order Linear Equations

7.1 Multiplicatively Separable Functions

$$F(t, x) = f(t)g(x)$$

A non-example of a separable function is $F(t, x) = t^2 + x^2$. An example is $F(t, x) = t^2 x^3$.

For our purposes, we will work with first-order ODE's with scalar functions.

7.2 Separable ODE

$$\boxed{x' = f(t)g(x)}$$

There are other ways we can write this equation:

- **General Form:** $G(t, x, x') = 0$
- **Standard Form:** $\phi(t)x' = F(t, x)$
 - **Regular Case:** $x' = F(t, x)$, F is the “slope function”
 - **Singular Case:** This is when we solve in an interval $(t_0 - \delta, t_0 + \delta)$ where $\delta > 0$ and $\phi(t_0) = 0$.

To solve this type of equation:

$$\begin{aligned} x'(t) &\equiv f(t)g(x(t)) \\ \frac{x'(t)}{g(x(t))} &= f(t) \\ \int_a^t \frac{x'(\tau)}{g(x(\tau))} d\tau &= \int_a^t f(\tau) d\tau \end{aligned}$$

Letting $u = x(\tau)$ and $du = x'(\tau) d\tau$:

$$\underbrace{\int_{x(a)}^{x(t)} \frac{du}{g(u)}}_{G(x(t))} = \underbrace{\int_a^t f(\tau) d\tau}_{F(t)}$$

$$\boxed{G(x(t)) = F(t)}$$

7.3 Example

$$\begin{aligned} x' &= t^2 x^3 \\ \frac{x'}{x^3} &= t^2 \\ \int \frac{dx}{x^3} &= \int t^2 dt + C \\ \frac{x^{-2}}{-2} &= \frac{t^3}{3} + C \\ x^{-2} &= C - \frac{2}{3}t^3 \\ x &= \pm \frac{1}{\sqrt{C - \frac{2}{3}t^3}} \end{aligned}$$

8 2/22/16: Separable Equations, First-Order Linear Equations; Uniqueness for C^1 IVP's

Recall our form for the separable equation:

$$x' = f(t)g(x)$$

Assume f and g are continuous on their respective domains, f on $I = (t_0 - a, t_0 + a)$, $a > 0$, g on $J = (x_0 - b, x_0 + b)$, $b > 0$. Let $\mathcal{R} = I \times J$. If we also have $x(t_0) = x_0$, then we have an IVP (initial value problem) on our hands.

But the problem is, $\frac{1}{g(x)}$ isn't necessarily continuous.

Separately, solve the algebraic equation $g(x) = 0$ in the interval J . Assume for simplicity that the roots of g are isolated and C^∞ ("smooth"). Then, we can partition J into open subintervals J_1, J_2, \dots, J_n , i.e.

$$J = \{a, b, c, d\} \cup J_1 \cup J_2 \cup \dots \cup J_n$$

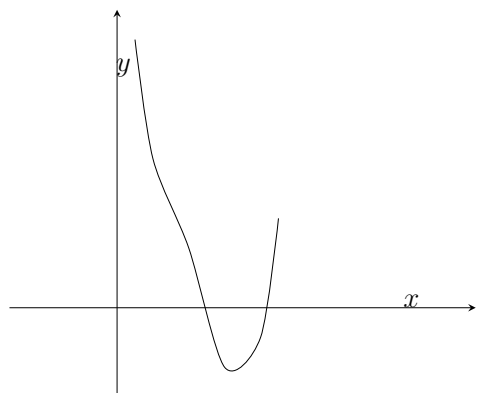
9 2/23/16: Uniqueness for C^1 IVP's

$$x' = f(t)g(x) \quad x = x(t)$$

$$x' \equiv f(t)g(x(t)) \text{ for all } t \in I$$

Assume: f, g have continuous derivatives on their respective domains. Then, all solutions are given as follows: Suppose $g(x)$ has domain J . If a and b are consecutive isolated roots of g , we can solve on (a, b) as we did yesterday:

$$\underbrace{\int_c^{x(t)} \frac{1}{g(u)} du}_{G_c(x(t))} = \underbrace{\int_{t_0}^t f(\tau) d\tau}_{F(t)} \quad \text{where } c \in (a, b) \text{ is arbitrary}$$



If a is a root of g (isolated or not) then claim: $x(t) \equiv a$ for $t \in \mathbb{R}$ is a solution of the differential equation.

9.1 Uniqueness

Are these all the solutions, however?

A first-order IVP in standard form (the regular case):

$$x' = \underbrace{F(t, x)}_{\text{slope function}}, \quad x(t_0) = x_0$$

Assumption: F is a C^1 function ($\frac{\partial F}{\partial t}$ and $\frac{\partial F}{\partial x}$ are both continuous) on a rectangle centered at the initial point (t_0, x_0) . Then, we have the following theorem:

Theorem 9.1. *If $\phi(t)$ and $\psi(t)$ are solutions of the IVP, defined on respective domains $I_\delta = (t_0 - \delta, t_0 + \delta)$ and $I_\varepsilon = (t_0 - \varepsilon, t_0 + \varepsilon)$ where $\delta > 0$ and $\varepsilon > 0$, then*

$$\phi(t) \equiv \psi(t)$$

for all $t \in I_\eta = (t_0 - \eta, t_0 + \eta)$ where $\eta > 0$.

Basic outline for the proof:

$$\text{IVP} \quad x'(t) \equiv F(t, x(t)), \quad x(t_0) = x_0$$

$$x(t) - x(t_0) = \int_{t_0}^t F(\tau, x(\tau)) \, d\tau$$

10 2/24/16: Uniqueness & Existence for C^1 IVP's

10.1 Autonomous Equations and the Time Shift Property

$$\begin{cases} x' = \sqrt{|x|} & (\text{autonomous} - \text{the independent variable makes no explicit appearance}) \\ x(0) = 0 \end{cases}$$

One important property of an autonomous differential equation is that it is time-independent, i.e. if $x = \phi(t)$ is a solution, then so is $x = \psi(t) := \phi(t + c)$. Without an initial condition, we have an infinite number of solutions.

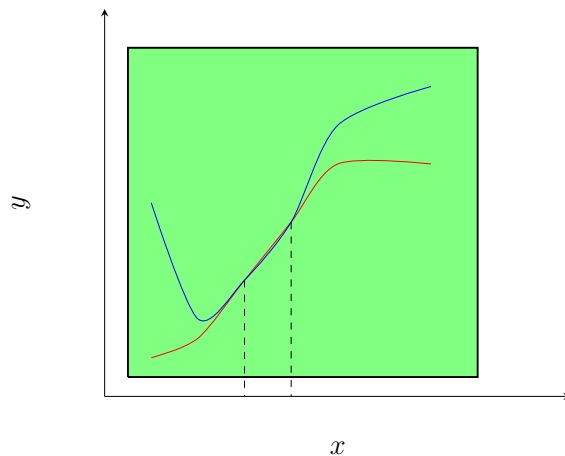
Let us try separation of variables:

$$\begin{aligned} \int_0^{x(t)} \frac{dx}{\sqrt{|x|}} &= \int_0^t d\tau = t \\ \lim_{\varepsilon \rightarrow 0^+} \int_\varepsilon^{x(t)} u^{-\frac{1}{2}} du &= \lim_{\varepsilon \rightarrow 0^+} \left[2u^{\frac{1}{2}} \right]_\varepsilon^{x(t)} \\ &= 2\sqrt{x(t)} - \lim_{\varepsilon \rightarrow 0^+} \sqrt{\varepsilon} \\ &= 2\sqrt{x(t)} \\ x(t) &= \frac{t^2}{4} > 0 \quad (\text{assuming } t \geq 0) \end{aligned}$$

We can similarly derive, for $t \leq 0$, that $x(t) = -\frac{t^2}{4}$. We can then construct our function:

$$x(t) = \begin{cases} \frac{t^2}{4}, & t \geq 0 \\ -\frac{t^2}{4}, & t < 0 \end{cases}$$

10.2 Unique Solutions



Proof. We begin by showing that our IVP is actually an integral equation.

$$\begin{cases} x'(t) \equiv F(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

$$\begin{aligned} x'(\tau) &\equiv F(\tau, x(\tau)) \\ \int_{t_0}^t x'(\tau) d\tau &= \int_{t_0}^t F(\tau, x(\tau)) d\tau \end{aligned}$$

$$x(t) = x_0 + \int_{t_0}^t F(\tau, x(\tau)) d\tau, \quad x(t) \text{ is continuous}$$

We have just proved one direction of equivalence. To prove the other direction, note that $x(t)$ is differentiable, since all of its parts are continuous and differentiable. ■

11 2/25/16: Uniqueness/Existence for C^1 IVP's

We're assuming: for $\delta > 0, \varepsilon > 0$:

$$\phi : I_\delta := (t_0 - \delta, t_0 + \delta) \rightarrow \mathbb{R} \text{ satisfies } \phi'(t) \equiv F(t, \phi(t)), \phi(t_0) = x_0$$

$$\psi : I_\varepsilon := (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow \mathbb{R} \text{ satisfies } \psi'(t) \equiv F(t, \psi(t)), \psi(t_0) = x_0$$

We want to show that for some $\eta > 0$, $\phi(t) \equiv \psi(t)$ on $I_\eta := (t_0 - \eta, t_0 + \eta)$.

First, we introduce the following concept:

Definition 11.1. If f is a bounded real-valued function on a set \mathcal{S} , then its **sup-norm** is defined as:

$$\|f\|_{\mathcal{S}} := \sup_{x \in \mathcal{S}} |f(x)|$$

If \mathcal{S} is a closed, bounded subset of \mathbb{R}^n , and f is continuous, then $\|f\|_{\mathcal{S}} = \max_{x \in \mathcal{S}} |f(x)|$, in which case is called the **max-norm**.

Note that:

- $\|f\|_{\mathcal{S}} \geq 0$
- $\|f\|_{\mathcal{S}} = 0$ iff $f(x) \equiv 0$ for all $x \in \mathcal{S}$
- $\|\alpha f\|_{\mathcal{S}} = |\alpha| \|f\|_{\mathcal{S}}$
- $\|f + g\|_{\mathcal{S}} \leq \|f\|_{\mathcal{S}} + \|g\|_{\mathcal{S}}$ where f and g are defined and bounded on \mathcal{S} .

We claim that $\|\phi - \psi\|_{I_{\eta}} \leq c \|\phi - \psi\|_{I_{\eta}}$, where $0 < c < 1$. This would mean that $\|\phi - \psi\|_{I_{\eta}} = 0$, then $\phi(t) - \psi(t) \equiv 0$ on I_{η} and $\phi(t) = \psi(t)$ on I_{η} .

Proof. Note that

$$\phi(t) \equiv x_0 + \int_{t_0}^t F(\tau, \phi(\tau)) \, d\tau$$

for all $t \in I_{\delta}$ and

$$\psi(t) \equiv x_0 + \int_{t_0}^t F(\tau, \psi(\tau)) \, d\tau$$

for all $t \in I_{\varepsilon}$. Both of these equations are true for all $t \in I_{\min(\delta, \varepsilon)}$.

Restrict $t \in I_{\eta}$ where $0 \leq \eta \leq \min(\delta, \varepsilon)$. Subtracting these two equations:

$$\begin{aligned}
 |\phi(t) - \psi(t)| &= \left| \int_{t_0}^t [F(\tau, \phi(\tau)) - F(\tau, \psi(\tau))] \, d\tau \right| \\
 &\leq \left| \int_{t_0}^t |F(\tau, \phi(\tau)) - F(\tau, \psi(\tau))| \, d\tau \right| \\
 \text{(MVT)} \quad &\leq \left| \int_{t_0}^t \underbrace{\left| \frac{\partial F}{\partial x}(x, \theta(\tau)) \right|}_{\leq M} |\phi(t) - \psi(t)| \, d\tau \right| \\
 &\leq M \left| \int_{t_0}^t \underbrace{|\phi(\tau) - \psi(\tau)|}_{\leq \|\phi - \psi\|_{I_{\eta}}} \, d\tau \right| \\
 &\leq M \|\phi - \psi\| (t - t_0) \leq M\eta \|\phi - \psi\|_{I_{\eta}}
 \end{aligned}$$

Now we simply pick η such that $M\eta = c < 1$, and we are done. ■

12 2/26/16: Existence

12.1 Transforming to an Integral Equation

Yesterday we proved the uniqueness of the solution of an IVP. Now we must prove the existence.

$$\begin{cases} x' = F(t, x), & x = x(t) \text{ is the unknown function} \\ x(t_0) = x_0 \end{cases}$$

C^1 IVP $\Leftrightarrow F(t, x)$ is C^1 in some rectangle \mathcal{R} centered at (x_0, y_0) .

Let us integrate our equation:

$$\begin{aligned} \int_{t_0}^t x'(\tau) d\tau &= \int_{t_0}^t F(\tau, x(\tau)) d\tau \\ x(t) - x(t_0) &= x(t) - x_0 = \int_{t_0}^t F(\tau, x(\tau)) d\tau \end{aligned}$$

So now our problem/equation becomes:

- $x(t) \equiv x_0 + \int_{t_0}^t F(\tau, x(\tau)) d\tau$
- $x(t)$ is a continuous function of t

Why do we need the continuity condition? If $x(t)$ is a solution, then it is differentiable, which implies it is continuous.

Now we prove the opposite direction. To prove that $x(t)$ is differentiable, note that $f(t) = x_0$ is differentiable, and the integral is also differentiable (since its derivative is $F(t, x(t))$, which is continuous). Therefore, by algebra, the two statements are equivalent.

12.2 Picard's Method

Define

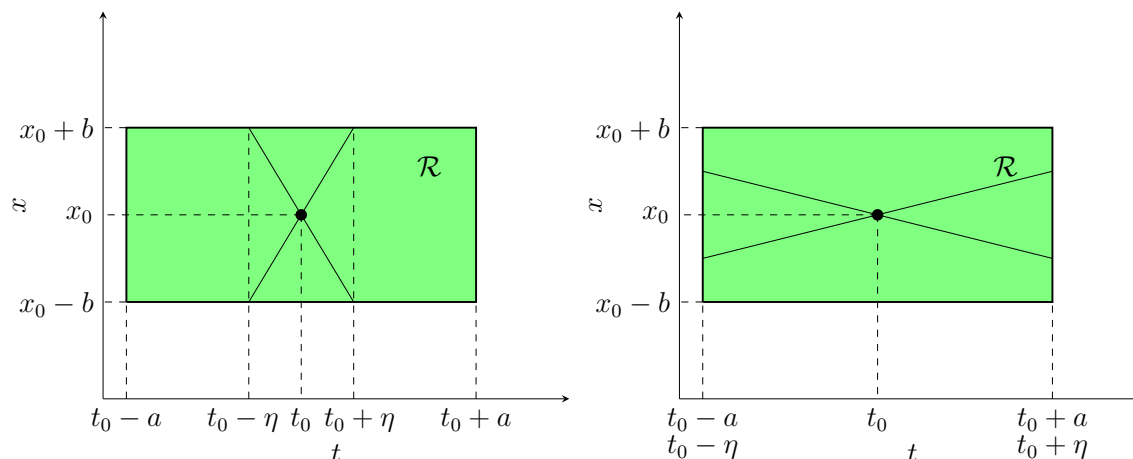
$$x_{n+1}(t) := x_0 + \int_{t_0}^t F(\tau, x_n(\tau)) d\tau$$

and

$$x_0(t) \equiv x_0 \quad \text{for all } t$$

In this section, we prove that for each $t \in I_\eta$, $\lim_{n \rightarrow \infty} x_n(t)$ exists, let's call it $x(t)$, and moreover:

- $x(t)$ is a continuous function of t on I_η
- $x(t) \equiv x_0 + \int_{t_0}^t F(\tau, x(\tau)) d\tau$



Define $M \geq \max_{(t,x) \in \mathcal{R}} |F(t,x)|$ (existence follows from EVT). Let $\eta := \min(a, \frac{b}{M}) > 0$. Assume $M > 0$; if $M = 0$, then the IVP is $x' \equiv 0$, $x(t_0) = x_0$ has a solution: $x(t) \equiv x_0$ for all $t \in (t_0 - a, t_0 + a)$.

13 2/29/16: Picard's Existence Proof, Continued...

Recapping: our base function is

$$x_0(t) \equiv x_0 \quad \text{for all } t \in I_\eta, \eta = \min\left(a, \frac{b}{M}\right) > 0$$

We need to choose the size of the rectangle for each function. In the example

$$\begin{cases} x' = t^2 + x^2 \\ x(0) = 1 \end{cases}$$

We need to find some $\eta > 0$ such that a solution is guaranteed to exist in $(-\eta, \eta)$. Let $\mathcal{R} = [-T, T] \times [1-r, 1+r]$ be our rectangle. Then take $M = T^2 + (1+r)^2$, then $|t^2 + x^2| \leq M$ when $(t, x) \in \mathcal{R}$.

13.1 Proving Well-Defined-ness

Theorem 13.1. *Each x_n is well-defined and continuous and satisfies $|x_n(t) - x_0| \leq b$ for all $t \in I_\eta$.*

Proof. The base case is trivial. Now assume true for $x_n(t)$; we now prove for $x_{n+1}(t)$. By assumption, the integrand $F(\tau, x_n(\tau))$ is well-defined and continuous (and therefore Riemann integrable on I_η) for all $t \in I_\eta$, which means the integral is well-defined. Therefore, $x_{n+1}(t)$ is well-defined for all $t \in I_\eta$.

Also, $x_{n+1}(t)$ is continuous on I_η by similar reasoning.

Now, we investigate $|x_{n+1}(t) - x_0|$. First, we claim that $(\tau, x_n(\tau)) \in \mathcal{R}$ for any τ between t_0 and t . But

$$|\tau - t_0| \leq |t - t_0| \leq \eta \leq a$$

$$|x_n(\tau) - x_0| \leq b$$

$$\begin{aligned} |x_{n+1}(t) - x_0| &= \left| \int_{t_0}^t F(\tau, x_n(\tau)) \, d\tau \right| \\ &\leq \left| \int_{t_0}^t \underbrace{|F(\tau, x_n(\tau))|}_{\leq M} \, d\tau \right| \\ &\leq M|t - t_0| \leq M\eta \leq b \end{aligned}$$

■

14 3/1/16: Finish Picard's Existence Proof

Theorem 14.1.

$$|x_{n+1}(t) - x_n(t)| \leq \frac{MK^n}{(n+1)!} |t - t_0|^{n+1}$$

for any $n \geq 0$ and any $t \in I_\eta$, where $K \geq \max_{(t,x) \in \mathcal{R}} \left| \frac{\partial F}{\partial x}(t, x) \right|$ (using the assumed C^1 -ness of F on \mathcal{R}).

Proof. We prove by induction. When $n = 0$:

$$|x_1(t) - x_0(t)| = |x_1(t) - x_0| = \left| \int_{t_0}^t F(\tau, x_0) \, d\tau \right| \leq M|t - t_0| = \frac{MK^0}{(0+1)!} |t - t_0|^{0+1}$$

Now assume the hypothesis, we want to prove that

$$|x_{n+2}(t) - x_{n+1}(t)| \leq \frac{MK^{n+1}}{(n+2)!} |t - t_0|^{n+2}$$

So:

$$\begin{aligned} |x_{n+2}(t) - x_{n+1}(t)| &= \left| \int_{t_0}^t F(\tau, x_{n+1}) - F(\tau, x_n) \, d\tau \right| \\ (\text{MVT, for some } y_n \in [x_n, x_{n+1}]) &= \left| \int_{t_0}^t \frac{\partial F}{\partial x}(\tau, y_n)(x_{n+1} - x_n) \, d\tau \right| \\ &\leq \left| \int_{t_0}^t \left| \frac{\partial F}{\partial x}(\tau, y_n) \right| |x_{n+1} - x_n| \, d\tau \right| \\ &\leq K \left| \int_{t_0}^t |x_{n+1} - x_n| \, d\tau \right| \\ &\leq K \left| \int_{t_0}^t \frac{MK^n}{(n+1)!} |t - t_0|^{n+1} \, d\tau \right| \\ &= \frac{MK^{n+1}}{(n+2)!} |t - t_0|^{n+2} \end{aligned}$$

■

15 3/3/16

For any $t \in I_\eta$,

$$|x_{n+p}(t) - x_n(t)| = |x_{n+p}(t) - x_{n+p-1}(t) + x_{n+p-1}(t) - x_{n+p-2}(t) + x_{n+p-2}(t) - \cdots - x_n(t)|$$

is bounded. By the Triangle Inequality,

$$\begin{aligned} |x_{n+p}(t) - x_n(t)| &\leq \sum_{j=n}^{n+p-1} |x_{j+1}(t) - x_j(t)| \\ &\leq \sum_{j=n}^{n+p-1} \frac{MK^j}{(j+1)!} |t - t_0|^{j+1} \\ &= \left(\frac{M}{K}\right) \sum_{j=n}^{n+p-1} \frac{(K|t - t_0|)^{j+1}}{(j+1)!} \\ &\leq \left(\frac{M}{K}\right) \sum_{j=n}^{\infty} \frac{(K|t - t_0|)^{j+1}}{(j+1)!} \\ &\leq \left(\frac{M}{K}\right) \sum_{j=n}^{\infty} \frac{(K\eta)^{j+1}}{(j+1)!} \\ &= \left(\frac{M}{K}\right) \left(e^{K\eta} - \sum_{j=0}^{n-1} \frac{(K\eta)^{j+1}}{(j+1)!} \right) \end{aligned}$$

Thus, we have an upper bound for any two terms in our sequence.

Now if we take $\|x_{n+p} - x_n\|_{I_\eta} = \sup_{t \in I_\eta} |x_{n+p}(t) - x_n(t)| \leq L$, then send $n \rightarrow \infty$. Then,

$$\lim_{n \rightarrow \infty} \|x_{n+p(n)} - x_n\|_{I_\eta} \leq 0$$

But this is also nonnegative, so it must be the case that the limit is zero, and thus this sequence is Cauchy.

16 3/4/16: Existence of Solutions, Continued

The last thing we proved was that (x_n) is a Cauchy sequence in the space of continuous functions on I_η , denoted $C^0(I_\eta)$, under the sup-norm, $\|f\|_{I_\eta} = \sup_{t \in I_\eta} |f(t)|$. This means:

$$\lim_{n \rightarrow \infty} \|x_{n+p(n)} - x_n\|_{I_\eta} = 0 \text{ for any } \mathbb{N}\text{-valued function } p(n).$$

16.1 Metric Spaces

Definition 16.1. A *metric space*, denoted (\mathcal{X}, d) , $\mathcal{X} \neq \emptyset$, d is a “distance” function, must satisfy the following:

1. $d : (\mathcal{X} \times \mathcal{X}) \rightarrow [0, \infty)$
2. $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) + d(y, z) \geq d(x, z)$

If we have a metric d on a vector space \mathcal{V} , we can also require $d(x, y) + d(y, z) = d(x, z)$ iff x - y - z (y is between x and z) or $x = y$ or $y = z$. To define betweenness: \vec{p} - \vec{q} - \vec{r} iff $\vec{q} = (1 - t)\vec{p} + t\vec{r}$.

Here, we define a metric on a vector space. Given a vector space \mathcal{V} , a **norm** on \mathcal{V} is a real-valued function $\|\cdot\|$ such that:

1. $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0 \Leftrightarrow \vec{v} = \vec{0}$ (Positive definiteness)
2. $\|c\vec{v}\| = |c|\|\vec{v}\|$ (Absolute homogeneity)
3. $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ (Triangle inequality)

This way, we can define

$$d(\vec{v}, \vec{w}) := \|\vec{v} - \vec{w}\|$$

16.2 Cauchy Sequences in Metric Spaces

In a metric space (\mathcal{X}, d) , a sequence of elements (x_n) converges to $x \in \mathcal{X}$ if $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.

A sequence (x_n) in (\mathcal{X}, d) is called Cauchy if $d(x_n, x_m) \rightarrow 0$ as $\min(n, m) \rightarrow \infty$.
Not all Cauchy sequences converge. As an example, take

$$\mathcal{X} = \mathbb{Q}, \quad d(q, \tilde{q}) = |q - \tilde{q}|$$

Take the sequence $(3, 3.1, 3.14, 3.141, \dots)$. This sequence is Cauchy since choosing two far-out values will differ very little. However, it converges to π , which is not in the metric space. Therefore, we call this an **incomplete metric space**.

17 3/7/16: Cauchy Sequences and Convergence in \mathbb{R} and in $C^0([a, b])$

A sequence (t_n) in \mathbb{R} is Cauchy if $|t_n - t_m| \rightarrow 0$ as $\min(n, m) \rightarrow \infty$. More precisely, $\forall \varepsilon > 0, \exists N : n, m \geq N \Rightarrow |t_n - t_m| < \varepsilon$.

We'd like to prove that every Cauchy sequence in \mathbb{R} converges to some $t \in \mathbb{R}$.

Theorem 17.1. $(\mathbb{R}, |\cdot - \cdot|)$ is a **complete metric space**, i.e. every Cauchy sequence of real numbers converges to a real limit.

We make use of several lemmas. First, a definition:

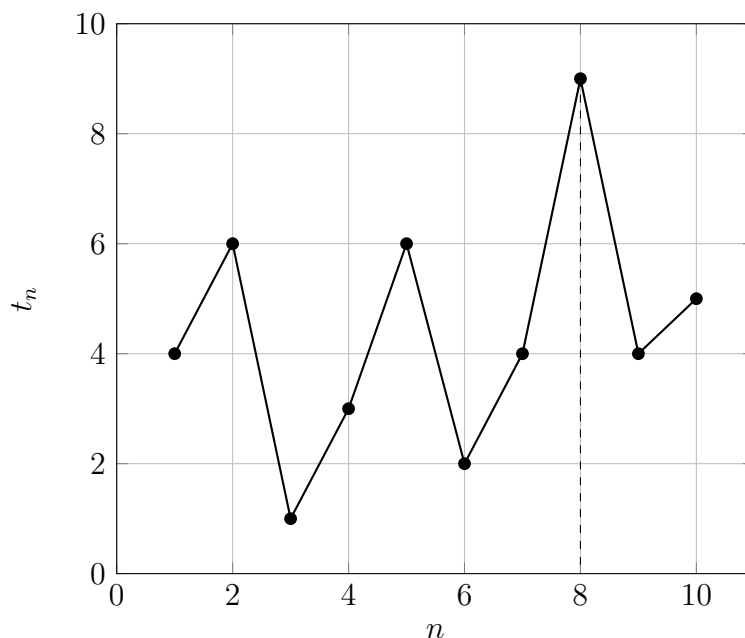
Definition 17.1. A **subsequence** of (t_n) is any sequence of the form (t_{k_n}) where (k_n) is a strictly increasing sequence of natural numbers: $1 \leq k_1 < k_2 < \dots < k_n < \dots$

Definition 17.2. A sequence (u_n) is called monotone if either $u_1 \leq u_2 \leq u_3 \leq \dots \leq u_n \leq \dots$ or $u_1 \geq u_2 \geq u_3 \geq \dots \geq u_n \geq \dots$.

Lemma 17.1. A subsequence of a subsequence of (t_n) is itself a subsequence of (t_n) .

Lemma 17.2. If (k_n) is a strictly increasing sequence in \mathbb{N} , then $k_n \rightarrow \infty$ as $n \rightarrow \infty$. In fact, $k_n \geq n$.

Lemma 17.3 (Rising Sun Lemma). Every sequence in \mathbb{R} has a monotone subsequence.



Proof. We call N a vista if $t_N > t_{N+k}$ for all $k \geq 1$. We consider two cases:

1. the set of vistas is infinite; call them $N_1 < N_2 < N_3 < \dots$. Then,

$$t_{N_1} > t_{N_2} > t_{N_3} > \dots$$

and we can take (t_{N_n}) as our subsequence—this is strictly decreasing, so certainly monotone down.

2. the set of vistas is finite (including possibly empty). Then, let N be one more than the greatest among the vistas. N is not a vista, so $\exists k_2 > k_1 = N : t_{k_2} \geq t_{k_1}$. k_2 is also not a vista, so $\exists k_3 > k_2 : t_{k_3} \geq t_{k_2}$. Then taking (t_{k_n}) , we have our monotone subsequence.

■

Lemma 17.4. *Every Cauchy sequence in \mathbb{R} (true in any metric space) is bounded:*

$$\exists M : |t_n| \leq M$$

for all $n \geq 1$.

Proof. By definition, we can choose some N such that

$$n, m \geq N \Rightarrow |t_n - t_m| < 1$$

Take $m = N$, then $|t_n - t_N| < 1$, and

$$|t_N| - 1 \leq |t_n| \leq |t_N| + 1 \quad \forall n \geq N$$

$$|t_n| \leq \max(|t_1|, |t_2|, |t_3|, \dots, |t_{N-1}|)$$

■

18 3/8/16: Completeness of \mathbb{R} , $C^0([a, b])$

Previously we proved that every cauchy sequence is bounded. Now, we can prove that:

Lemma 18.1. *Every cauchy sequence has a convergent subsequence.*

Proof. A monotone subsequence of a cauchy sequence is bounded. We need to show that a bounded monotone sequence must converge to a finite limit.

Assume

$$a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n \leq a_{n+1} \leq \dots \leq b$$

We claim that

$$l := \sup\{a_n | n \geq 1\} = \lim_{n \rightarrow \infty} a_n$$

■

Lemma 18.2. *If a subsequence of a cauchy sequence converges to some limit $t \in \mathbb{R}$, then the original sequence also converges to t .*

Proof. Since (t_n) has a subsequence (t_{k_n}) s.t. $t_{k_n} \rightarrow t$ as $n \rightarrow \infty$.

$$\begin{aligned} 0 \leq |t_n - t| &= |(t_n - t_{k_n}) + (t_{k_n} - t)| \\ &\leq |t_n - t_{k_n}| + |t_{k_n} - t| \\ &\leq 0 \end{aligned}$$

By the squeeze theorem, the proof is complete. ■

18.1 Putting it Together

Take the metric space

$$\mathcal{X} = C^0([a, b]) = \{f : [a, b] \rightarrow \mathbb{R} \mid f \text{ is continuous on } [a, b]\}$$

$$d(f, g) = \|f - g\|_I = \sup_{a \leq t \leq b} |f(t) - g(t)|$$

We claim that if (f_n) is a sequence in $C^0(I)$ that is cauchy $[\|f_n - f_m\|_I \rightarrow 0 \text{ as } \min n, m \rightarrow \infty]$, then (f_n) converges to some function $f \in C^0(I)$ $[\|f_n - f\|_I \rightarrow 0 \text{ as } n \rightarrow \infty]$.

We must propose some limit function. For each $t \in I$, define

$$f(t) := \lim_{n \rightarrow \infty} f_n(t)$$

We can see that $(f_n(t))$ converges in \mathbb{R} :

$$0 \leq |f_n(t) - f_m(t)| \leq \underbrace{\|f_n - f_m\|}_0 = \sup_{a \leq \tau \leq b} |f_n(\tau) - f_m(\tau)|$$

So we have our limit function, $f(t)$. However, we don't know yet if $f(t)$ is continuous; we have point-wise convergence, but not yet uniform convergence.

19 3/9/16: Completeness of $C^0([a, b])$; Existence in the Flow Problem

Let (f_n) be a cauchy sequence in $C^0(I)$ where $I := [a, b]$, so $\|f_n - f_m\| \rightarrow 0$ as $\min(n, m) \rightarrow \infty$. Define, for each $t \in I$ separately, $\sigma_t := (f_1(t), f_2(t), \dots, f_n(t), \dots)$, a sequence in \mathbb{R} .

We don't know yet, that

- The functions actually converge to $f(t)$ ($\|f_n - f\|_I \rightarrow 0$ as $n \rightarrow \infty$), known as uniform convergence.
- $f(t)$ is continuous

As an example, consider:

$$\begin{aligned} f_n(t) &:= t^n \\ n &= 1, 2, 3, \dots \\ t &\in I = [0, 1] \end{aligned}$$

$$f(t) = \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ 1 & \text{if } t = 1 \end{cases}$$

Then, $\forall t \in [0, 1]$, $f_n(t) \rightarrow f(t)$ as $n \rightarrow \infty$. We have point-wise convergence, but we do not have uniform convergence.

To prove continuity, we look at two points close together. Assuming continuity:

$$\begin{aligned}
 |f(t) - f(\tilde{t})| &= |f(t) - f_n(t) + f_n(t) - f_n(\tilde{t}) + f_n(\tilde{t}) - f(\tilde{t})| \\
 &\leq |f(t) - f_n(t)| + |f_n(t) - f_n(\tilde{t})| + |f_n(\tilde{t}) - f(\tilde{t})| \\
 &\leq \|f - f_n\|_I + |f_n(t) - f_n(\tilde{t})| + \|f_n - f\|_I \\
 &= 2\|f - f_n\|_I + |f_n(t) - f_n(\tilde{t})|
 \end{aligned}$$

We can take $\|f - f_n\|_I$ to be less than some fixed number $\frac{\varepsilon}{4}$, and we get that

$$|f(t) - f(\tilde{t})| < \varepsilon$$

For the other condition:

$$\|f_n - f_m\| \rightarrow 0$$

This is equivalent to saying

$$\forall \varepsilon > 0, \exists N(\varepsilon) : \forall n, m \geq N(\varepsilon), \|f_n - f_m\| < \varepsilon$$

Taking limits on both sides:

$$\lim_{m \rightarrow \infty} \|f_n - f_m\| \leq \underbrace{\lim_{m \rightarrow \infty} \varepsilon}_{\varepsilon}$$

$$\|f_n - f\|_I \leq \varepsilon \text{ for all } n \geq N(\varepsilon)$$

But by choosing ε small enough, then $\|f_n - f\|_I \rightarrow 0$.

20 3/10/16

Revisiting the Picard sequence, we now know that all such $x_n \in C^0(I_\eta)$. We also know that the Picard sequence is a Cauchy sequence in $C^0(I_\eta)$. We conclude that \exists a limit function $x(t) = \lim_{n \rightarrow \infty} x_n(t)$ for all $t \in I_\eta$.

Now we show that $x(t)$ solves our equation. Then,

$$\begin{aligned}
 x_{n+1}(t) &= x_0 + \int_{t_0}^t F(\tau, x_n(\tau)) \, d\tau \\
 \lim_{n \rightarrow \infty} x_{n+1}(t) &= \lim_{n \rightarrow \infty} x_0 + \int_{t_0}^t F(\tau, x_n(\tau)) \, d\tau \\
 x(t) &= x_0 + \int_{t_0}^t F(\tau, x(\tau)) \, d\tau
 \end{aligned}$$

It remains to show that

$$\lim_{n \rightarrow \infty} \int_{t_0}^t F(\tau, x_n(\tau)) \, d\tau = \int_{t_0}^t F(\tau, x(\tau)) \, d\tau$$

21 3/11/16: Fact that the Picard limit function solves the IVP; Extension to Vector IVP's; Linear IVP's

$\forall t \in I_\eta$, we must prove:

$$\lim_{n \rightarrow \infty} \int_{t_0}^t F(\tau, x_n(\tau)) d\tau = \int_{t_0}^t F(\tau, x(\tau)) d\tau$$

Examine the absolute difference between the integrals:

$$\begin{aligned} 0 \leq \left| \int_{t_0}^t F(\tau, x(\tau)) d\tau - \int_{t_0}^t F(\tau, x_n(\tau)) d\tau \right| &= \left| \int_{t_0}^t (F(\tau, x(\tau)) - F(\tau, x_n(\tau))) d\tau \right| \\ &\leq \left| \int_{t_0}^t |F(\tau, x(\tau)) - F(\tau, x_n(\tau))| d\tau \right| \\ &\leq K \left| \int_{t_0}^t |x(\tau) - x_n(\tau)| d\tau \right| \\ &\leq K \|x_n - x\|_{I_\eta} \left| \int_{t_0}^t d\tau \right| \\ &\leq K\eta \|x_n - x\|_{I_\eta} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Thus, the integrals are equivalent, and thus we have finally proved Picard's method.

22 3/17/16: Domain of Solutions for Linear Equations; Solving First-Order Linear Equations (Regular Case)

22.1 First-Order C^1 Vector IVP's

$$\begin{cases} \vec{x}(t) = (x_1(t), x_2(t), \dots, x_n(t)) \\ \vec{x}'(t) = \vec{F}(t, \vec{x}(t)) = \vec{F}(t, x_1(t), \dots, x_n(t)) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

To show that this has a unique solution, we can extend Picard's method:

$$\vec{x}_0(t) \equiv \vec{x}_0 \quad \text{for all } t \in I_\eta = [t_0 - \eta, t_0 + \eta], \eta = \min \left(a, \frac{b}{M} \right)$$

Our function now takes place in a box

$$\mathcal{B} = [t_0 - a, t_0 + a] \times [x_{01} - b, x_{01} + b] \times [x_{02} - b, x_{02} + b] \times \dots \times [x_{0n} - b, x_{0n} + b]$$

Here, we assume that

$$\frac{\partial \vec{F}}{\partial t}, \frac{\partial \vec{F}}{\partial x_1}, \dots, \frac{\partial \vec{F}}{\partial x_n} \quad \text{are continuous on } \mathcal{B}$$

We take $M \geq \|\vec{F}(t, \vec{x})\|$ for all $(t, \vec{x}) \in \mathcal{B} \in \mathbb{R}^{n+1}$.

We also take $K \geq \max\{\|\frac{\partial \vec{F}}{\partial x_1}(t, \vec{x})\|, \dots, \|\frac{\partial \vec{F}}{\partial x_n}(t, \vec{x})\|\}$

Now we wish to define the recursive step used in this method.

$$\vec{x}_{j+1}(t) := \vec{x}_0 + \int_{t_0}^t \vec{F}(\tau, \vec{x}_j(\tau)) d\tau \quad \text{for all } t \in I_\eta$$

The proof is similar.

23 3/18/16

$$\begin{cases} x'_1 = a_{11}(t)x_1 + a_{12}(t)x_2 + \dots + a_{1n}(t)x_n + b_1(t) \\ x'_2 = a_{21}(t)x_1 + a_{22}(t)x_2 + \dots + a_{2n}(t)x_n + b_2(t) \\ \vdots \\ x'_n = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \dots + a_{nn}(t)x_n + b_n(t) \end{cases}$$

Theorem 23.1. *If $a_{ij}(t)$ and $b_j(t)$ are continuous in a time interval I for all i, j , then $\vec{x}(t) = (x_1(t), x_2(t), \dots, x_n(t))$ exists for all $t \in I$.*

24 3/21/16: First-Order Linear Equation in One Unknown—Regular Case

$$\begin{cases} x'_1 = a_{11}(t)x_1 + a_{12}(t)x_2 + \dots + a_{1n}(t)x_n + b_1(t) \\ x'_2 = a_{21}(t)x_1 + a_{22}(t)x_2 + \dots + a_{2n}(t)x_n + b_2(t) \\ \vdots \\ x'_n = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \dots + a_{nn}(t)x_n + b_n(t) \end{cases}$$

$$\vec{x}'(t) = \vec{F}(t, \vec{x}(t)) = \mathbf{A}(t)\vec{x}(t) + \vec{b}(t)$$

Note that Picard's proof of existence & uniqueness of solutions for vector IVP's still works even if $\vec{F}(t, \vec{x})$ isn't quite C^1 on some box about (t_0, \vec{x}_0) , but satisfies a Lipschitz condition with respect to \vec{x} :

$$\|\vec{F}(t, \vec{x}) - \vec{F}(t, \vec{y})\| \leq K\|\vec{x} - \vec{y}\| \quad \text{for all } (t, \vec{x}), (t, \vec{y}) \in \mathcal{B}$$

where $\mathcal{B} = [t_0 - a, t_0 + a] \times [x_{01} - b, x_{01} + b] \times [x_{02} - b, x_{02} + b] \times \dots \times [x_{0n} - b, x_{0n} + b]$.

It suffices, if we wish to apply Picard's Theorem, to check that $\frac{\partial \vec{F}}{\partial x_1}, \frac{\partial \vec{F}}{\partial x_2}, \dots, \frac{\partial \vec{F}}{\partial x_n}$ are continuous in \mathcal{B} . (The partial w.r.t. t doesn't even need to be continuous.)

25 3/22/16: First-Order Linear Equation—in One Unknown (Regular Case)

25.1 Form of Linear Equation

The form of a linear equation is:

$$x' = F(t, x) = g(t)x + h(t) \quad \text{where } g, h \text{ are continuous on some interval } I = [t_0 - a, t_0 + a]$$

Here, the slope function is linear in x , but not necessarily in t .

25.2 Euler's method of integrating factors

$$\begin{aligned} x' - g(t)x &= h(t) && \text{(Let } u = u(t) \text{ be a function of } t \text{ TBD)} \\ \underbrace{ux' - (ug)x}_{u'} &= h \\ ux' + u'x &= u'x + h \\ (ux)' &= uh \end{aligned}$$

The problem then becomes finding such a u such that

$$u' = -gu$$

Solving this differential equation, we get that

$$\begin{aligned} u' &= -gu \\ (\ln u)' &= \frac{u'}{u} = -g(t) \\ \ln u &= - \int_{t_0}^t g(\tau) \, d\tau \end{aligned}$$

$$u = \exp \left\{ - \int_{t_0}^t g(\tau) \, d\tau \right\} = e^{- \int_{t_0}^t g(\tau) \, d\tau}$$

Going back to our equation:

$$\begin{aligned} (ux)' &= uh = h(t) \exp \left\{ - \int_{t_0}^t g(\tau) \, d\tau \right\} \\ u(t)x(t) &= \int_{t_0}^t h(\theta) \exp \left\{ - \int_{t_0}^{\theta} g(\tau) \, d\tau \right\} \, d\theta + C \\ x(t) &= \exp \left\{ \int_{t_0}^t g(\tau) \, d\tau \right\} \left[C + \int_{t_0}^t h(\theta) \exp \left\{ - \int_{t_0}^{\theta} g(\tau) \, d\tau \right\} \, d\theta \right] \end{aligned}$$

Example. Solve $x' = (2t)x - (1 + t)$, given $t_0 = 0$.

We first find u :

$$u(t) = \exp - \int_0^t 2\tau \, d\tau = e^{-t^2}$$

Then:

$$\begin{aligned}
 x' - (2t)x &= -(1+t) \\
 ux' - (2t)ux &= -u(1+t) \\
 (ux)' &= -(1+t)e^{-t^2} \\
 ux &= -\int_0^t (1+\theta)e^{-\theta^2} d\theta + C \\
 x(t) &= Ce^{t^2} - e^{t^2} \int_0^t (1+\theta)e^{-\theta^2} d\theta
 \end{aligned}$$

26 3/23/16: First-Order Linear Equations—Singular Case

26.1 Singular Case

$$f(t)x' + g(t)x = h(t)$$

We are interested in solving this equation near t_0 , such that $f(t_0) = 0$. f, g, h are defined and “nice” in some interval I . We cannot simply write the linear equation in the regular case, since $f(t)$ might be zero at some point.

We must find the solutions $x(t)$ in some punctured interval:

$$(t_0 - \varepsilon, t_0) \cup (t_0, t_0 + \varepsilon)$$

Take $h(t) \equiv 0$, and assume that $g(t)$ is analytic with radius of convergence $R > 0$

$$g(t) = \sum_{n=0}^{\infty} b_n(t-t_0)^n, \text{ convergent if } |t-t_0| < R$$

Also assume that

$$f(t) = (t-t_0) \underbrace{\sum_{n=0}^{\infty} a_n(t-t_0)^n}_{a(t)}, \text{ convergent if } |t-t_0| < R$$

The unusual form for $f(t)$ is due to the fact that the constant term of the power series must be 0.

26.2 Finding a Nonzero Solution

We can search for a solution in the form

$$x(t) = \sum_{n=0}^{\infty} c_n(t-t_0)^n$$

where $(c_n)_{n=1}^{\infty}$ is to be determined.

To make it easier for ourselves, we can define functions that will shift the t axis:

$$\begin{aligned}y(t) &:= x(t + t_0) \\ \tilde{f}(t) &:= f(t + t_0) \\ \tilde{g}(t) &:= g(t + t_0)\end{aligned}$$

So our equation becomes:

$$\tilde{f}(t)y' + \tilde{g}(t)y = 0$$

Therefore, WLOG we can assume that $t_0 = 0$. So now we have

$$\begin{aligned}x(t) &= \sum_{n=0}^{\infty} c_n t^n \\ f(t) &= t \sum_{n=0}^{\infty} a_n t^n \\ g(t) &= \sum_{n=0}^{\infty} b_n t^n\end{aligned}$$

We can calculate $x'(t)$ to be

$$x'(t) = \sum_{n=0}^{\infty} n c_n t^{n-1} = \sum_{n=1}^{\infty} n c_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} t^n$$

27 3/24/16

Let us assume that $g(0) \neq 0$.

$$\begin{aligned}ta(t)x' + g(t)x &= 0 \\ tx' &= \frac{g(t)}{a(t)}x = 0 \\ tx' + b(t)x &= 0\end{aligned}$$

where $b(t)$ is analytic at $t = 0$, with a radius of convergence of $R_1 > 0$. We need to show that $b(t)$ is analytic.

$$\begin{aligned}\frac{g(t)}{a(t)} &= \frac{b_0 + b_1 t + b_2 t^2 + \dots}{a_0 + a_1 t + a_2 t^2 + \dots} \\ &= \frac{1}{a_0} \frac{b_0 + b_1 t + b_2 t^2 + \dots}{1 + \left(\frac{a_1}{a_0}\right)t + \left(\frac{a_2}{a_0}\right)t^2 + \dots} \\ &= \frac{1}{a_0} g(t) \frac{1}{1 + J} \\ &= \frac{g(t)}{a_0} (1 - J + J^2 - J^3 + \dots)\end{aligned}$$

Plugging back in J , we get a power series, showing that $b(t)$ is analytic.

28 3/28/16

$$1 - \left(\sum_{k \geq 1} \frac{a_k}{a_0} t^k \right) + \left(\sum_{k \geq 1} \frac{a_k}{a_0} t^k \right)^2 - \left(\sum_{k \geq 1} \frac{a_k}{a_0} t^k \right)^3 + \dots$$

29 3/29/16

Lemma 29.1. Let $\sum_{n=0}^{\infty} a_n t^n$ have radius of convergence $R_1 > 0$, and let $\sum_{n=0}^{\infty} b_n t^n$ have radius of convergence $R_2 > 0$.

Then

$$\left(\sum_{n \geq 0} a_n t^n \right) \left(\sum_{n \geq 0} b_n t^n \right) = \sum_{n \geq 0} c_n t^n$$

is convergent for $|t| < R := \min(R_1, R_2)$.

Here, $c_n = \sum_{j=0}^n a_{n-j} b_j = a_n b_0 + a_{n-1} b_1 + a_{n-2} b_2 + \dots + a_0 b_n$.

Let us look at

$$\begin{aligned} \left(\sum_{n \geq 0} a_n t^n \right)^k &= \left(\sum_{n_1 \geq 0} a_{n_1} t^{n_1} \right) \left(\sum_{n_2 \geq 0} a_{n_2} t^{n_2} \right) \dots \left(\sum_{n_k \geq 0} a_{n_k} t^{n_k} \right) \\ &= \sum_{n \geq 0} \left(\sum_{n_1 + \dots + n_k = n} a_{n_1} a_{n_2} \dots a_{n_k} \right) t^n = \sum_{n \geq 0} c_{n,k} t^n \end{aligned}$$

The radius of convergence for this new series is the same as that of the original series.

29.1 Rewriting Our Differential Equation

$$tx' + \left(\sum_{n \geq 0} \beta_n t^n \right) x = 0$$

Fact: this equation has solutions of the form

$$x(t) = |t|^r \sum_{n \geq 0} c_n t^n$$

30 3/30/16: Theorems on Power Series, etc.

Theorem 30.1. Suppose $\sum_{n=0}^{\infty} a_n t^n$ has radius of convergence $R > 0$. Then $\sum_{n=0}^{\infty} |a_n t^n|$ converges, provided $t \in (-R, R)$, i.e. any power series with a positive radius of convergence is actually absolutely convergent in the interior of its interval of convergence.

Proof. Choose some value r such that $0 \leq |t| < r < R$. We claim that $\sum_{n=0}^{\infty} a_n r^n$ converges, because $r \in (-R, R)$. By the Divergence Test, $a_n r^n \rightarrow 0$ as $n \rightarrow \infty$. Any vanishing sequence is necessarily bounded, i.e.

$$\begin{aligned} \exists M \geq 0 : |a_n r^n| &\leq M \\ |a_n| &\leq M r^{-n} \\ \sum_{n=0}^{\infty} |a_n t^n| &= \sum_{n=0}^{\infty} |a_n| |t|^n \leq M \sum_{n=0}^{\infty} r^{-n} |t|^n \\ &= M \sum_{n=0}^{\infty} \left(\frac{|t|}{r} \right)^n \\ &< \infty \end{aligned}$$

The last sum is a geometric series with common ratio $\frac{|t|}{r} \in [0, 1)$. By the Comparison test, the smaller series converges, and we are done. ■

Theorem 30.2. Suppose $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are mutual rearrangements, i.e. \exists a one-to-one function $p : \mathbb{N} \rightarrow \mathbb{N}$ such that $\forall n, b_n = a_{p(n)}, a_m = b_{p^{-1}(m)}$. Then, if $\sum_{n=1}^{\infty} a_n$ is absolutely

convergent, then so is $\sum_{n=1}^{\infty} b_n$, and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$.

Proof. Define

$$\mathcal{S} = \{a_{n_1} + a_{n_2} + \cdots + a_{n_k} \mid n_1, n_2, \dots, n_k \in \mathbb{N}\}$$

Note that $\mathcal{S} \neq \emptyset$.

We claim that \mathcal{S} is bounded. Choose any $x \in \mathcal{S}$. We'll show: $|x| \leq A := \sum_{n=1}^{\infty} |a_n| < \infty$.

Say $x = a_{n_1} + a_{n_2} + \cdots + a_{n_k}$. By the Triangle Inequality:

$$\begin{aligned} x \leq |x| &\leq |a_{n_1}| + |a_{n_2}| + \cdots + |a_{n_k}| \\ &\leq |a_1| + |a_2| + \cdots + |a_N| \quad \text{where } N = \max(n_1, n_2, \dots, n_k) \\ &\leq |a_1| + |a_2| + \cdots = A \end{aligned}$$

Let $a := \sup \mathcal{S} \in \mathbb{R}$; in fact $a \leq A$. Let $A_n := a_1 + a_2 + \cdots + a_n$; note that $A_n \in \mathcal{S}$.

$$A_n \leq a \Rightarrow \lim_{n \rightarrow \infty} A_n \leq a$$

Define $B_n := b_1 + b_2 + \cdots + b_n$, note that $B_n \in \mathcal{S}$. So $B_n \leq a$.

Define $\tilde{B}_n := |b_1| + |b_2| + \cdots + |b_n| \leq A$. Therefore the b_n series is absolutely convergent. ■

31 4/4/16: Series Solutions

Recall, we want to solve $tx' + b(t)x = 0$, where $b(t) = \sum_{n \geq 0} \beta_n t^n$ is analytic, with radius $R > 0$

(so converges when $|t| < R$).

Test case: $b(t) = \beta_0$ (a constant). WLOG, can assume $\beta_0 \neq 0$.

$$tx' + \beta_0 x = 0$$

Consider when $t > 0$. We claim that there is a solution of the form $x(t) = t^r$; plugging this in:

$$rt^r + \beta_0 t^r \equiv 0$$

$$(r + \beta_0)t^r \equiv 0$$

so our solution is $x(t) = t^{-\beta_0}$.

Now consider when $t < 0$. Then $x(t) = (-t)^r$ (so then $tx' = r(-t)^r$):

$$r(-t)^r + \beta_0(-t)^r \equiv 0$$

$$(r + \beta_0)(-t)^r \equiv 0$$

so our solution here is $(-t)^{-\beta_0}$.

Combining these two solutions, we get our solution:

$$x(t) = |t|^{-\beta_0} \quad \text{for } t \neq 0$$

Our general solution turns out to be:

$$x(t) = c|t|^{-\beta_0} \quad \text{for } 0 < |t| < R$$

Now, what happens if $b(t)$ does not have such a nice form?

Ansatz:

$$x(t) = |t|^r \sum_{n \geq 0} c_n t^n$$

Let us restrict our solution to $t > 0$ to drop the absolute value sign.

$$x(t) = t^r \sum_{n \geq 0} c_n t^n$$

$$x'(t) = rt^{r-1} \sum_{n \geq 0} c_n t^n + t^r \sum_{n \geq 0} n c_n t^{n-1}$$

$$tx'(t) = rt^r \sum_{n \geq 0} c_n t^n + t^r \sum_{n \geq 0} n c_n t^n = t^r \sum_{n \geq 0} (r + n) c_n t^n$$

$$b(t)x(t) = t^r \sum_{n \geq 0} \beta_n t^n \sum_{n \geq 0} c_n t^n = t^r \sum_{n \geq 0} \left(\sum_{j=0}^n \beta_{n-j} c_j \right) t^n$$

Putting this in our differential equation:

$$t^r \sum_{n \geq 0} \left[(r+n)c_n + \sum_{j=0}^n \beta_{n-j} c_j \right] t^n \equiv 0$$

$$\sum_{n \geq 0} \left[(r+n)c_n + \sum_{j=0}^n \beta_{n-j} c_j \right] t^n \equiv 0$$

This is a power series, and since it is identically zero, its coefficients must all be zero as well.

32 4/6/16

For $t > 0$, we want a solution of the form $x(t) = t^r \sum_{n \geq 0} c_n t^n$. We found that we must have

$$\forall n \geq 0 : \quad (n+r)c_n + \sum_{j=0}^n \beta_{n-j} c_j = 0$$

Consider when $n > 0$:

$$(n - \beta_0)c_n + \beta c_n + \sum_{j=0}^{n-1} \beta_{n-j} c_j = 0$$

$$c_n = -\frac{1}{n} \sum_{j=0}^{n-1} \beta_{n-j} c_j = -\left(\frac{c_0 \beta_{n-1} + c_1 \beta_{n-2} + \cdots + c_{n-1} \beta_0}{n} \right)$$

Theorem 32.1. *If we define $(c_n)_{n=1}^\infty$ recursively by*

$$c_0 := 1, \quad c_n := -\frac{1}{n} \sum_{j=0}^{n-1} \beta_{n-j} c_j$$

then:

1. $\sum_{n=0}^{\infty} c_n t^n$ converges for all $t \in (-R, R)$
2. the function $x(t) := |t|^{-\beta_0} \sum_{n \geq 0} c_n t^n$ satisfies $tx' + b(t)x = 0$ for all $t \in (-R, R) \setminus \{0\}$

33 4/13/16: Linear Equations with Constant Coefficients

33.1 Summary

1. Separable Equations: $x' = f(t)g(x)$

2. **First-Order Linear Equation, regular case:** $x' = f(t)x + g(t)$
3. **First-Order Linear Equation, singular case:** $tx' = f(t)x + g(t)$
4. **Second-Order Linear Equation, regular case:** $x'' = f(t)x' + g(t)x + h(t)$
5. **Second-Order Linear Equation, singular case:** $t^2x'' = tf(t)x' + g(t)x + h(t)$

33.2 Solving

Let $L[x]$ be an **operator** such that

$$L[x] = ax'' + bx' + cx$$

where $x = x(t)$. The equation in question we want to solve is:

$$L[x] = h(t)$$

34 4/14/16

$$L[x] = ax'' + bx' + c$$

This is a linear differential operator of order 2 with constant coefficients. Note that a linear operator satisfies

$$F(s\psi + t\phi) = sF(\psi) + tF(\phi)$$

The domain of the operator L is

$$C^2(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f'' \text{ exists and is continuous}\}$$

Codomain of L : $C^0(\mathbb{R})$

34.1 Homogeneous and Inhomogeneous

The **homogeneous** equation looks like

$$L[x] = 0$$

34.2 Finding a Solution

$$ax'' + bx' + cx = 0$$

TRY $x(t) = e^{rt}$!!!

$$L[e^{rt}] = \underbrace{(ar^2 + br + c)}_{P_L(r) \text{ (characteristic polynomial)}} e^{rt}$$

$$P_L(r) = 0$$

$$r \in \{r^-, r^+\}$$

What happens when r is complex:

$$\begin{aligned} e^{(\alpha+i\beta)t} &= e^{\alpha t} e^{i\beta t} \\ &= e^{\alpha t} [\cos(\beta t) + i \sin(\beta t)] \end{aligned}$$

35 4/15/16

Definition 35.1. Two functions ψ and ϕ on a common domain $I \subseteq \mathbb{R}$ (an interval) are said to be independent if $\neg \exists c \in \mathbb{R} : \psi = c\phi$, i.e. $\frac{\psi}{\phi}$ is nonconstant on I .

We claim that if ψ and ϕ are two independent solutions, then the set of all possible solutions of $L[x] = 0$ is

$$\{r\psi + s\phi \mid r, s \in \mathbb{R}\}$$

The question becomes: can the quadratic yield suitable solutions?

- **Case 1:** $\Delta := b^2 - 4ac > 0$

Then \exists two real roots, $r_1 \neq r_2$:

$$r_1 = \frac{-b - \sqrt{\Delta}}{2a}, \quad r_2 = \frac{-b + \sqrt{\Delta}}{2a}$$

Then we have $\psi(t) = e^{r_1 t}$ and $\phi(t) = e^{r_2 t}$, both real-valued functions. We can easily see that they are linearly independent.

- **Case 2:** $\Delta < 0$

$$r_1 = \frac{-b - i\sqrt{|\Delta|}}{2a}, \quad r_2 = \frac{-b + i\sqrt{|\Delta|}}{2a}$$

Our solutions here are $\Psi(t) = e^{r_1 t}$ and $\Phi(t) = e^{r_2 t}$.

Write $\alpha := -\frac{b}{2a}, \beta := \frac{\sqrt{|\Delta|}}{2a}$. Expanding our functions:

$$\begin{aligned} \Psi(t) &= (e^{\alpha t} \cos \beta t) - i(e^{\alpha t} \sin \beta t) \\ \Phi(t) &= (e^{\alpha t} \cos \beta t) + i(e^{\alpha t} \sin \beta t) \end{aligned}$$

- **Case 3:** $\Delta = 0$, \exists a repeated root, $r_1 = -\frac{b}{2a}$.

$$\begin{aligned} L[e^{rt}] &\equiv p_L(r)e^{rt} \\ \frac{\partial}{\partial r} L[e^{rt}] &\equiv \frac{\partial}{\partial r} [p_L(r)e^{rt}] \\ L\left[\frac{\partial}{\partial r}(e^{rt})\right] &\equiv p_L(r)te^{rt} + p'_L(r)e^{rt} \\ L[te^{rt}] &\equiv [tp_L(r) + p'_L(r)]e^{rt} \\ &\equiv 0 \end{aligned}$$

Therefore, our two real-valued solutions are $\psi(t) = e^{rt}$ and $\phi(t) = te^{rt}$.

36 4/18/16: Determination of the General Solution

Recall:

$$L[x] = ax'' + bx' + cx$$

where $L : C^2(\mathbb{R}) \rightarrow C^0(\mathbb{R})$

Theorem 36.1. *If ψ and ϕ are linearly independent real-valued solutions of $L[x] = 0$, then every solution $X(t)$ of $L[x] = 0$ can be written as*

$$X(t) \equiv r\psi(t) + s\phi(t)$$

where $r, s \in \mathbb{R}$.

Proof. Proving that $X(t)$ is a solution is trivial:

$$L[r\psi + s\phi] = rL[\psi] + sL[\phi] \equiv 0$$

To prove the other direction, we set up an IVP:

$$\begin{aligned} ax'' + bx' + c &= 0 \\ x(0) &= X(0) \\ x'(0) &= X'(0) \end{aligned}$$

By the Picard Theorem, there is exactly one solution, namely $x(t) = X(t)$.

Our initial data gives us:

$$\begin{cases} r\psi(0) + s\phi(0) &= X(0) \\ r\psi'(0) + s\phi'(0) &= X'(0) \end{cases}$$

This is simply a system of 2 equations in r and s , which will have a solution if the determinant

$$\begin{vmatrix} \psi(0) & \phi(0) \\ \psi'(0) & \phi'(0) \end{vmatrix}$$

is nonzero at $t = 0$. It suffices to show that this is the case.

Define the **Wronskian** of ψ and ϕ to be

$$W(t) = W(\psi, \phi)(t) = \begin{vmatrix} \psi(t) & \phi(t) \\ \psi'(t) & \phi'(t) \end{vmatrix} = \psi(t)\phi'(t) - \psi'(t)\phi(t)$$

We first show that $W(t)$ is not identically zero (always zero).

Note that

$$\begin{aligned} W &= \psi\phi' - \phi'\psi \\ W' &= \psi\phi'' - \psi''\phi \\ aW' &= \psi(a\phi'') - (a\psi'')\phi \\ bW &= \psi(b\phi') - (b\psi')\phi \\ aW' + bW &= \phi(-c\phi) - (-c\psi)\phi = 0 \end{aligned}$$

Now we have an differential equation in W : $aw' + bw = 0$. The solution of this equation is $w = ce^{-\frac{bt}{a}}$. However, $c \neq 0$ because otherwise, $W(t)$ would be identically zero, and we just showed that it can't happen. Therefore, $W(t)$ is always nonzero, and then $W(0) \neq 0$. ■

37 4/20/16

37.1 Inhomogeneous Equation

We have found the solution for x in $L[x] = 0$, now we seek to find solution for x in $L[x] = f(t)$. Note that $f(t)$ doesn't have to be too nice, it just has to be nice enough, as long as it is Riemann Integrable. For simplicity's sake, we can restrict f to a piecewise continuous, and the domain of f is on some interval $I \subseteq \mathbb{R}$, and we're trying to find a solution $x(t)$ on I .

37.1.1 Form of the Solution Set

The solution set of $L[x] = f$ is:

$$\{\psi_p + y \mid L[y] = 0\}$$

where $L[\psi_p] = f$. We have to show two things for the bi-conditional:

1. all functions of the form $\psi_p + y$ is a solution to $L[x] = f$
2. all solution of $L[x]$ is of the form $\psi_p + y$

Let us first show that $\psi_p + y$ satisfies $L[x] = f$ for any y satisfying $L[y] = 0$. This is easy to prove, we just use the linearity of L :

$$L[\psi_p + y] = L[\psi_p] + L[y] = f + 0 = f$$

Now we have to go the other direction. We have to show that any function x such that $L[x] = f$ can be expressed as $\psi_p + y$ for some y with $L[y] = 0$.

Let us define $y := x - \psi_p$. We must show that $L[y] = 0$. Once again let us use the linearity of L :

$$L[y] = L[x - \psi_p] = L[x] - L[\psi_p] = f - f = 0$$

The result of this theorem is that we now only have to find one particular solution to $L[x] = f$, and once we have that we can generate the solution set to the non-homogeneous equation:

$$\{\psi_p + r\psi + s\phi \mid r, s \in \mathbb{R}\}$$

where ψ_p is a particular solution of $L[x] = f$ and ψ, ϕ are linearly independent solutions of the reduced equation.

37.1.2 Finding ψ_p

There are two methods of finding the solution:

1. Variation of Parameters – works for any Riemann Integrable function f .

2. Methods of Undetermined Coefficients – No integration is required, but only works for $f(t)$ in the form of:

$$f(t) = \sum_{k=1}^N P_k(t) e^{\sigma_k t} \text{trig}_k(\beta_k t)$$

Where $\text{trig}_k \in \{\sin, \cos\}$

37.1.3 Variation of Parameters

Ansatz: Try for a solution of the form $x = u\psi + v\phi$ where u, v are unknown functions of t (to be determined) and ψ, ϕ are the two linearly independent solutions of $L[x] = 0$, which we know how to find.

Given the form, let us calculate x' and x'' :

$$\begin{aligned} x &= u\psi + v\phi \\ x' &= u\psi' + u'\psi + v\phi' + v'\phi \\ x'' &= u\psi'' + 2u'\psi' + u''\psi + v\phi'' + 2v'\phi' + v''\psi \\ &= (u\psi'' + v\phi'') + (u''\psi + v''\phi) + 2(u'\psi' + v'\phi') \end{aligned}$$

Let us set $L[x] = f = ax'' + bx' + cx$, which looks like:

$$f = a(u\psi'' + v\phi'') + a(u''\psi + v''\phi) + 2a(u'\psi' + v'\phi') + bu\psi' + bu'\psi + bv\phi' + bv'\phi + cu\psi + cv\phi$$

Note that with some factoring, things begin to die:

$$\begin{aligned} f &= uL[\psi] + vL[\phi] + a(u''\psi + v''\phi) + 2a(u'\psi' + v'\phi') + b(u'\psi + v'\phi) \\ &= a(u''\psi + v''\phi) + 2a(u'\psi' + v'\phi') + b(u'\psi + v'\phi) \end{aligned}$$

Let us assume for the sake of argument that $u'\psi + v'\phi \equiv 0$. If we differentiate this, we get:

$$(u''\psi + v''\phi) + (u'\psi' + v'\phi') \equiv 0$$

note that both of these shows up in our definition for f , if we plug in this relationship we get:

$$f = a(u'\psi' + v'\phi')$$

Now we just have to find u', v' satisfying:

$$\begin{cases} u'\psi + v'\phi \equiv 0 \\ \psi'u' + \phi'v' = f/a \end{cases}$$

this is a system of two linear equations, we know that there is a unique solution because $W(t) \neq 0$. To solve this, let us multiply the top equation by ϕ' and the bottom equation by ψ :

$$\begin{aligned} \psi\phi'u' + \phi\phi'v' &= 0 \\ \phi\psi'u' + \phi\phi'v' &= \frac{1}{a}\phi f \end{aligned}$$

38 4/22/16: n -th Order LDE's with Constant Coefficients—Homogeneous Case

$$L[x] = a_0x^{(n)} + a_1x^{(n-1)} + a_2x^{(n-2)} + \cdots + a_{n-1}x' + a_nx$$

$$a_0, a_1, a_2, \dots, a_n \in \mathbb{R}$$

Homogeneous LDE based on: $L[x] = 0$.

Theorem 38.1. *The solution set*

$$L^{-1}[0] := \{x \in C^n(\mathbb{R}, \mathbb{R}) \mid L[x] = 0\}$$

is a subspace of $C^n(\mathbb{R}, \mathbb{R})$, in that it is closed under addition and scaling. That is, if $x_1, x_2 \in L^{-1}[0]$, then $x_1 + x_2 \in L^{-1}[0]$ and $cx_1 \in L^{-1}[0]$ for $\forall c \in \mathbb{R}$.

Our task now is to find n linearly independent real-valued solutions $\psi_1, \psi_2, \dots, \psi_n$ of $L[x] = 0$.

$$L[e^{rt}] = (a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n)e^{rt} = p_L(r)e^{rt}$$

Let us see what happens if we have polymultiplicative roots. Let

$$p_L(r) = a_0(r - r_1)^{m_1} \cdots (r - r_k)^{m_k} ((r - \alpha_1)^2 + \beta_1^2)^{n_1} \cdots ((r - \alpha_j)^2 + \beta_j^2)^{n_j}$$

Claim: For each real root r_l or p_L , we have m_l distinct solutions, as follows:

$$(e^{r_l t}, t e^{r_l t}, t^2 e^{r_l t}, \dots, t^{m_l-1} e^{r_l t})$$

To show that this is the case, we first introduce the **Leibniz Rule**:

$$(fg)^{(n)} = \sum_{k=0}^n \binom{n}{k} f^{(n-k)} g^{(k)}$$

which can be easily proved by induction.

$$\begin{aligned} L[e^{rt}] &= p_L(r)e^{rt} \\ L[e^{r_l t}] &= 0 \\ \frac{\partial^\mu}{\partial r^\mu} L[e^{rt}] &= \frac{\partial^\mu}{\partial r^\mu} [p_L(r)e^{rt}] \\ L \left[\frac{\partial^\mu}{\partial r^\mu} \right] &= \sum_{\nu=0}^{\mu} x \binom{\mu}{\nu} p_L^{(\mu-\nu)}(r) \{e^{rt}\}^{(\nu)} \end{aligned}$$

39 5/3/16: Constant-Coefficient Linear Equations, Order n —Homogeneous Case

$$L[x] = ax'' + bx' + cx = f(t), \quad a, b, c \in \mathbb{R}, a \neq 0$$

The general solution of the inhomogeneous equation $L[x] = f$ is

$$L^{-1}[f] = \{\psi_p + r\psi + s\phi \mid r, s \in \mathbb{R}\}$$

where ψ_p is a particular solution, i.e., $L[\psi_p] = f$, and ψ, ϕ are linearly independent solutions of $L[x] = 0$.

Now we consider the case of order n :

$$L[x] = a_0x^{(n)} + a_1x^{(n-1)} + \cdots + a_{n-1}x' + a_nx$$

$$(a_0, a_1, \dots, a_{n-1}, a_n) \in \mathbb{R}; a_0 \neq 0$$

To find our general solution:

$$L^{-1}[0] = \{x \in PC^n(\mathbb{R}, \mathbb{R}) \mid L[x] = 0\}$$

where $PC^n(\mathbb{R}, \mathbb{R}) = \{\text{piecewise } n\text{-differentiable functions } \mathbb{R} \rightarrow \mathbb{R}\}$.

It turns out that $\exists n$ linearly independent solutions $\psi_1, \psi_2, \dots, \psi_n$ (real-valued), and

$$L^{-1}[0] = \text{span}_{\mathbb{R}}(\psi_1, \psi_2, \dots, \psi_n) = \{c_1\psi_1 + c_2\psi_2 + \cdots + c_n\psi_n \mid c_1, c_2, \dots, c_n \in \mathbb{R}\}$$

These solutions are the fundamental solutions.

$$\begin{aligned} L[e^{rt}] &= p_L(r)e^{rt} \\ &= (a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n)e^{rt} \end{aligned}$$

For any real root ρ of p_L (if any exist), and if $\text{mult}(\rho; p_L) = k \geq 1$, ρ will contribute the following k fundamental solutions: $t^{j-1}e^{\rho t}$, where $1 \leq j \leq k$. To show that these are solutions:

$$\begin{aligned} L[e^{rt}] &\equiv p_L(r)e^{rt} \\ \frac{\partial^j}{\partial r^j} L[e^{rt}] &\equiv \frac{\partial^j}{\partial r^j} \{p_L(r)e^{rt}\} \\ L\left[\frac{\partial^j}{\partial r^j} e^{rt}\right] &\equiv \sum_{\mu=0}^j \binom{j}{\mu} p_L^{(j-\mu)}(r) t^\mu e^{rt} \\ L[t^j e^{\rho t}] &\equiv 0 \end{aligned}$$