

# Differential Equations

Brandon Lin  
Stuyvesant High School  
Spring 2016  
Teacher: Mr. Stern  
February 11, 2016

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## Introduction

- Email: jstern@stuy.edu
- Office: Room 351, periods 5,7,9

## 1 2/3/16: Background on $\mathbb{R}$ ; Basic Existence Question of ODE's

### 1.1 Romeo and Juliet

$$\begin{cases} R' = aR + bJ \\ J' = cR + dJ \end{cases}$$

These equations model the rate of change of Romeo's and Juliet's feelings. We call this a **linear system of two coupled differential equations of first order in two unknowns**.

- What makes it linear is that the functions and variables appear in a linear fashion.
- What makes it coupled is that both equations have both  $R$  and  $J$  in them.
- An **uncoupled system** would look like:

$$\begin{cases} R' = aR \\ J' = bJ \end{cases}$$

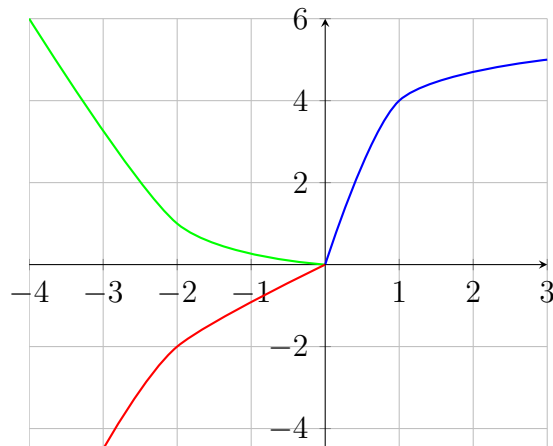
- First-order refers to the fact that all the derivatives are the first derivatives.

"Identically cautious lovers":

$$\begin{aligned} R' &= aR + bJ & a < 0, b > 0 \\ J' &= bR + aJ & |a| > |b| \end{aligned}$$

We may have initial conditions,  $R(0)$  and  $J(0)$ , and plot them on a **phase plane** with  $R$  against  $J$ . In this case, no matter where the starting point is, the trajectory will go towards a **stable node**.

In the case of  $|a| < |b|$ , points will move asymptotically towards  $R = -J$  and  $R = J$ . In the case of  $|a| = |b|$ , points will cycle around the origin infinitely.



## 1.2 Supremum and Infimum of a Set $\mathcal{A} \subseteq \mathbb{R}$

- If  $\mathcal{A} \subseteq (-\infty, b]$  for some  $b \in \mathbb{R}$ , we say  $\mathcal{A}$  is bounded above, and that  $b$  is an **upper bound** for  $\mathcal{A}$ .

**Theorem 1.1** (Supremum Theorem). *If  $\mathcal{A} \subseteq \mathbb{R}$ ,  $\mathcal{A} \neq \emptyset$ , and  $\mathcal{A} \subseteq (-\infty, b]$  for some  $b \in \mathbb{R}$ , then there exists  $a \in \mathbb{R}$  such that  $\mathcal{A} \subseteq (-\infty, a]$  but if  $x < a$ , then  $\mathcal{A} \not\subseteq (-\infty, x]$ . We write  $a = \sup \mathcal{A}$ , call it the **supremum** of  $\mathcal{A}$ .*

Why is this necessary? Consider the set  $\mathcal{A} = \{-\frac{1}{n} | n \in \mathbb{N}\}$ . It does not have a maximum per say, but it has a supremum  $\sup \mathcal{A} = 0$ .

Consider this example: What is  $\sup(-\mathbb{N})$ ? It is -1, which also happens to be the maximum of the set. e

**Theorem 1.2.** *If  $\max \mathcal{A}$  exists as a real number, then  $\sup \mathcal{A} = \max \mathcal{A}$ .*

But to answer all these questions, we need to figure out: what exactly are the real numbers?

## 1.3 What is $\mathbb{R}$ ?

Let  $x = (s, N, d_1, d_2, d_3, \dots, d_k, \dots)$ , where:

- $s \in \{+1, -1\}$
- $N \in \mathbb{Z}$
- $d_k \in \mathbb{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- $\neg(\exists k : d_{k+1} = d_{k+2} = \dots = 0)$ , this is to prevent multiple sequences from being the same number

In this case, “2.49” is shorthand for  $(+1, 2, 4, 8, 9, 9, 9, \dots)$

## 2 2/4/16: Background in $\mathbb{R}$ ; Fundamental Existence/Uniqueness Question

### 2.1 Supremums and Infimums in Integrals

**Theorem 2.1** (Supremum/Infimum Theorem).

1. If  $\mathcal{A}$  is a non-empty set of  $\mathbb{R}$ , and is bounded above (i.e.  $\mathcal{A} \subseteq (-\infty, b]$  for some  $b \in \mathbb{R}$ ), then there is a least upper bound for  $\mathcal{A}$ , namely  $a \in \mathbb{R}$  such that

$$(a) \mathcal{A} \subseteq (-\infty, a]$$

$$(b) \text{ if } x < a, \text{ then } \mathcal{A} \not\subseteq (-\infty, x]$$

This  $a$  is called the **supremum** of  $\mathcal{A}$ , written  $\sup A$ .

2.  $\inf A$ . This is the greatest lower bound for  $\mathcal{A}$ , or the **infimum**, provided  $\mathcal{A} \neq \emptyset$  and  $\mathcal{A}$  has a lower bound at all.

Recall that the Riemann integral is taking the limit of a partition over an interval  $[a, b]$ . But when we take the limit, we make the mesh of the partition,  $\|\mathcal{P}\|$ , approach zero, where

$$\mathcal{P} = \max_{1 \leq i \leq n} \Delta x_i$$

To fix this, we can define:

$$\int_a^b f(x) dx = \sup \left\{ \sum_{i=1}^n [\inf\{f(x) \mid x_{i-1} \leq x \leq x_i\} \Delta x_i] \mid a = x_0 < x_1 < \cdots < x_n = b \right\}$$

This is a “down-and-up” procedure. The sum of the rectangle areas is a down approximation since we use the minimum possible height to find the area. Then, we take the supremum of that, since for any lower approximation there will always be a higher approximation. Turns out there will never be a maximum; that’s why we take the supremum. This is a **lower Riemann sum**.

We can also define the same thing for an **upper Riemann sum**:

$$\int_a^b f(x) dx = \inf \left\{ \sum_{i=1}^n [\sup\{f(x) \mid x_{i-1} \leq x \leq x_i\} \Delta x_i] \mid a = x_0 < x_1 < \cdots < x_n = b \right\}$$

Therefore, the following inequality is true:

$$\int_a^b f \leq \int_a^b f$$

If these two are equal, then we say that  $f$  is **Riemann integrable**.

Here’s an example of a function that is NOT Riemann integrable:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 1 & \text{if } x \in [0, 1] \setminus \mathbb{Q} \end{cases}$$

Note that  $\int_0^1 f = 0$  and  $\int_0^1 f = 1$ , so this is not Riemann integrable.

## 2.2 Real Numbers, Again

We have shorthand for our previous definition of the real numbers.

$$\mathbb{R} = \{0\} \cup \{(s, N, d_1, d_2, \dots, d_k, \dots \mid s \in \{-1, +1\}, N \in \mathbb{Z}^+, d_k \in \mathbb{D}, \text{no 0-tail}\}$$

and the positive reals:

$$\mathbb{R}^+ = \{(s, N, d_1, d_2, \dots) \mid s = +1\}$$

Let us write  $x = \underline{N.d_1d_2d_3\dots}$  and  $y = \underline{M.e_1e_2e_3\dots}$ .

We also define negation as:

$$-(s, N, d_1, d_2, \dots) := (-s, N, d_1, d_2, \dots)$$

Then we can define the “less than” operation as follows:

- If  $x, y \in \mathbb{R}^+$ , then  $x < y$  if either  $N < M$  or  $N = M$  and  $d_1 < e_1$  or  $N = M$ ,  $d_1 = e_1$  and  $d_2 < e_2$ , or...
- $0 < x$  if  $x \in \mathbb{R}^+$
- $x < 0$  if  $x \in \mathbb{R}^+$
- $x < y$  if  $x \in \mathbb{R}^-, y \in \mathbb{R}^+$ .
- $x, y \in \mathbb{R}^-$ , and  $x < y$  if  $-y < -x$

## 3 2/5/16: Fundamental Existence of Uniqueness Theorem

### 3.1 Terminology

A **differential equation** is a relation between one or more unknown functions and at least some (but finitely many) of their derivatives, plus the independent variables.

Examples:

$$\begin{aligned} y' + 2xy - x^2 &= 3 \\ y''' + 2x^2y'' - 3x^3y' + xy - x^5 + 1 &= 0 \\ (y')^{y''} - e^{y'''} + x &= 0 \end{aligned}$$

Or,

$$\vec{y}' = \mathbf{A}(x)\vec{y}$$

where

$$\vec{y} = \vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}$$

$$\mathbf{A}(x) = \begin{bmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{bmatrix}$$

### 3.2 A Treatise on PDE's

There are two different types of differential equations: ODE's (ordinary, where all unknown functions depend on a single, same independent variable) and PDE's (partial, anything else).

$$\begin{aligned} \text{Wave equation: } \frac{\partial^2 u}{\partial x^2} &= c^2 \frac{\partial^2 u}{\partial t^2} \\ u &= g(x - t) + h(x + t) \end{aligned}$$

## 4 2/9/16: Basic Existence and Uniqueness Theorem

**Theorem 4.1** (Flow Theorem). *Let  $\vec{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), \dots, F_n(\vec{x}))$  be a vector field defined on some closed and bounded region  $\mathcal{D} \subseteq \mathbb{R}^n$ . Also assume  $\vec{F}$  is  $C^1$ ; namely,  $\frac{\partial F_i}{\partial x_j}$  is continuous everywhere interior to  $\mathcal{D}$ , for any  $i$  and  $j$ .*

*Let  $\vec{p}$  be a specific point interior to  $\mathcal{D}$ . Then  $\exists$  a function  $\vec{\sigma}(t)$  from some "time" interval  $(-\varepsilon, \varepsilon)$  with  $\varepsilon > 0$  into  $\mathcal{D}$ , such that  $\vec{\sigma}(0) = \vec{p}$  and  $\vec{\sigma}'(t) = \vec{F}(\vec{\sigma}(t))$  for any  $t \in (-\varepsilon, \varepsilon)$ .*

This theorem basically says that in a vector field, we can use the vector field to get the velocity of a curve. We call  $\vec{\sigma}(t)$  a **flow** for  $\vec{F}$ , starting at  $\vec{p}$ . This flow is, in fact, unique, in the sense that any two flows for the same  $\vec{F}$  starting at the same point must agree whenever they are both defined.

This is meaningful in that we can treat it as a differential equation:

$$\begin{cases} \sigma'_1 &= F_1(\sigma_1, \sigma_2, \dots, \sigma_n) \\ \sigma'_2 &= F_2(\sigma_1, \sigma_2, \dots, \sigma_n) \\ \vdots &\vdots \\ \sigma'_n &= F_n(\sigma_1, \sigma_2, \dots, \sigma_n) \end{cases}$$

$$\begin{cases} \sigma_1(0) &= p_1 \\ \sigma_2(0) &= p_2 \\ \vdots &\vdots \\ \sigma_n(0) &= p_n \end{cases}$$

## 4.1 Second-Order

$$mx'' = -kx, \quad x(0) = x_0, \quad x'(0) = v_0$$

$$x = x(t), \quad v = v(t) = x'(t), \quad a = a(t) = x''(t)$$

where  $k > 0$  is the spring constant. We can rewrite this as:

$$\begin{cases} x' &= v = F_1(x, v) \\ v' &= -\frac{k}{m}x = F_2(x, v) \end{cases} \quad \text{and} \quad \begin{cases} x(0) &= x_0 \\ v(0) &= v_0 \end{cases}$$

The Flow Theorem will tell us there is a unique solution, for some time interval.

## 5 2/10/16: The Flow Theorem

### 5.1 Application: $n^{\text{th}}$ order initial value problem (IVP)

$$\begin{cases} x &= x(t) \\ x^{(n)} &= F(t, x, x', x'', \dots, x^{(n-1)}) \\ x(t_0) &= x_{00} \\ x'(t_0) &= x_{10} \\ x''(t_0) &= x_{20} \\ \vdots & \\ x^{(n-1)}(t_0) &= x_{(n-1)0} \end{cases}$$

$f(t)x^{(n)} = F(t, x, x', x'', \dots, x^{(n-1)})$  is an  $n^{\text{th}}$  order ODE in standard form.

A **singularity** (or singular point) of this equation is a value  $t_0$  where  $f(t_0) = 0$ . At this point, the equation ceases to be of  $n^{\text{th}}$  order. If  $f(t)$  is of constant sign in the time interval on which we'd like to solve the equation, we just divide through by  $f(t)$  to get our desired form (which is the regular case, as opposed to the singular case).

Here, the Flow Theorem says that there is a unique solution  $x = x(t)$  defined in some time interval  $(t_0 - \varepsilon, t_0 + \varepsilon)$  where  $\varepsilon > 0$ .

To apply this:

$$\begin{cases} x_0(t) &= t \\ x_1 &= x_1(t) = x(t) \\ x_2 &= x_2(t) = x'(t) \\ x_3 &= x_3(t) = x''(t) \\ \vdots & \\ x_n &= x_n(t) = x^{(n-1)}(t) \end{cases}$$



becomes

$$\begin{cases} x'_0 &= 1 = F_0(x_0, x_1, x_2, \dots, x_n) \\ x'_1 &= x_2 = F_1(x_0, x_1, x_2, \dots, x_n) \\ x'_2 &= x_3 = F_2(x_0, x_1, x_2, \dots, x_n) \\ x'_3 &= x_4 = F_3(x_0, x_1, x_2, \dots, x_n) \\ \vdots &\vdots \\ x'_{n-1} &= x_n = F_{n-1}(x_0, x_1, x_2, \dots, x_n) \\ x'_n &= F(t, x_1, x_2, \dots, x_n) = F_n(\dots) \end{cases} \quad \text{and} \quad \begin{cases} x_0(t_0) &= t_0 \\ x_1(t_0) &= x_{00} \\ x_2(t_0) &= x_{10} \\ \vdots &\vdots \\ x_n(t_0) &= x_{(n-1)0} \end{cases}$$

This shows that we can recast an  $n^{\text{th}}$  order IVP into an  $n + 1$  order system.

However, for the Flow Theorem to apply,  $\vec{F}$  needs to be  $C^1$ . Therefore, our hypothesis in the IVP is that  $F$  is  $C^1$ , meaning that  $\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x'}, \dots$  are continuous.

## 6 2/11/16: Proof of the Flow Theorem

We will prove the Flow Theorem for two dimensions only; the proof can be extended to greater than two dimensions.

*Proof.* Let  $\vec{F}(x, y) = (A(x, y), B(x, y))$  be a vector field. By hypothesis,  $A$  and  $B$  are defined on a closed, bounded region  $\mathcal{D}$ , and they are  $C^1$  on  $\mathcal{D}$ . Then we need to solve the following equation:

$$\begin{aligned} \vec{x}' &= \vec{F}(\vec{x}) \\ \vec{x}(0) &= \vec{p} = \langle p, q \rangle \end{aligned}$$

We need to see how fast  $A(x, y)$  is changing.

1.

$$\begin{aligned} |A(x_1, y_1) - A(x_2, y_2)| &= |A(x_1, y_1) - A(x_1, y_2) + A(x_1, y_2) - A(x_2, y_2)| \\ (\text{Triangle Inequality}) &\leq |A(x_1, y_1) - A(x_1, y_2)| + |A(x_1, y_2) - A(x_2, y_2)| \\ (\text{MVT}) &\leq \left| \frac{\partial A}{\partial y}(x_1, y^*)(y_1 - y_2) \right| + \left| \frac{\partial A}{\partial x}(x^*, y_2)(x_1 - x_2) \right| \end{aligned}$$

Take  $K$  to be some upper bound for all the partial derivatives of  $A$  and  $B$  on  $\mathcal{D}$ .

$$\left| \frac{\partial A}{\partial y}(x_1, y^*)(y_1 - y_2) \right| + \left| \frac{\partial A}{\partial x}(x^*, y_2)(x_1 - x_2) \right| \leq K(|x_1 - x_2| + |y_1 - y_2|)$$

Similarly:

$$|B(x_1, y_2) - B(x_2, y_2)| \leq K(|x_1 - x_2| + |y_1 - y_2|)$$

This is called the **Lipschitz Condition**.

2. Also, note that  $A$  and  $B$  are continuous in  $\mathcal{D}$  and so by the Extreme Value Theorem, we can find an upper bound  $M$  for  $|A|$  and  $|B|$  on  $\mathcal{D}$ , i.e.

$$M = \max \left( \max_{(x,y) \in \mathcal{D}} |A(x, y)|, \max_{(x,y) \in \mathcal{D}} |B(x, y)| \right)$$

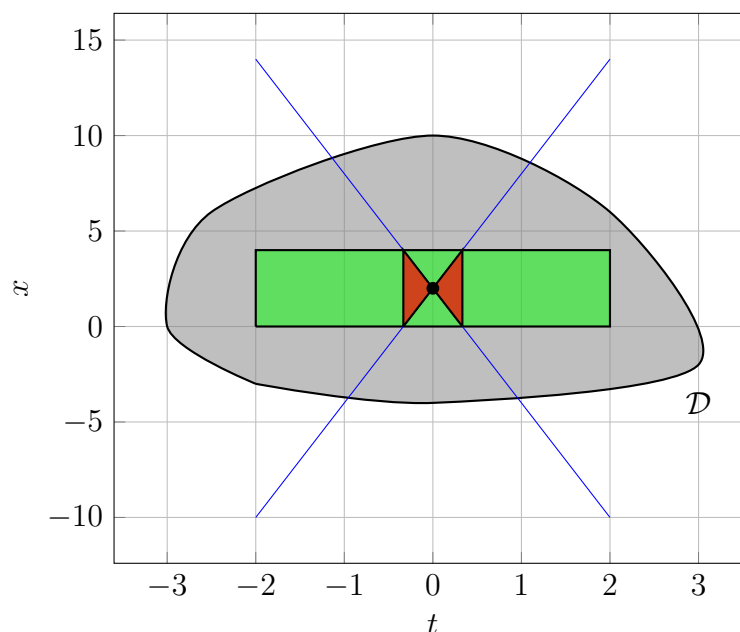
3. The point  $(p, q)$  is assumed to be interior to  $\mathcal{D}$  (not on the boundary).

We can therefore encase the point  $(0, p)$  in a rectangle in the  $tx$ -plane defined by  $R : [-r, r] \times [p - s, p + s] \subseteq \text{proj}_x \mathcal{D}, r, s > 0$ . Draw two lines with slopes  $M$  and  $-M$  through the point. We will consider the “bowtie” region formed by the intersections of the lines with the rectangle, joining them oppositely, and the lines themselves. Call the  $x$ -intersections  $-h$  and  $h$ .

Define  $h := \min\left(r, \frac{s}{M}\right) > 0$ . This is to formally define the bowtie region and consider the two possible pictures depending on the size of  $M$ .

Now, we want to construct the solution of the differential equation.

Whoopsies, there was a screwup here, proof to be fixed in the future.



■