Differential Equations

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Introduction

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Aim: Background on $\mathbb{R}$; Basic Existence Question of ODE's

1.1 Romeo and Juliet

$$\begin{cases} R' = aR + bJ \\ J' = cR + dJ \end{cases}$$

These equations model the rate of change of Romeo's and Juliet's feelings. We call this a **linear system of two coupled differential equations of first order in two unknowns**.

- What makes it linear is that the functions and variables appear in a linear fashion.
- What makes it coupled is that both equations have both R and J in them.
- An **uncoupled system** would look like:

$$\begin{cases} R' = aR \\ J' = bJ \end{cases}$$

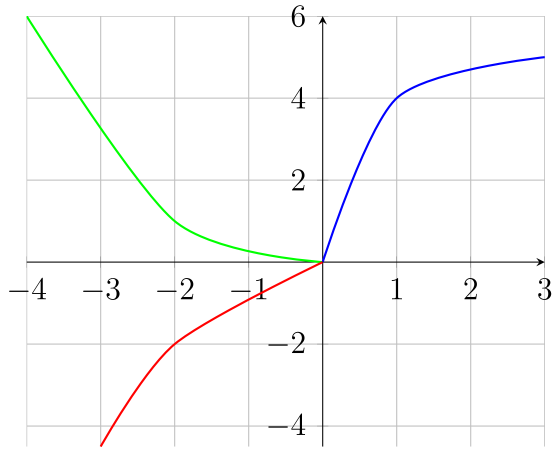
- First-order refers to the fact that all the derivatives are the first derivatives.

"Identically cautious lovers":

$$\begin{aligned} R' &= aR + bJ \quad a < 0, b > 0 \\ J' &= bR + aJ \quad |a| > |b| \end{aligned}$$

We may have initial conditions, $R(0)$ and $J(0)$, and plot them on a **phase plane** with R against J. In this case, no matter where the starting point is, the trajectory will go towards a **stable node**.

In the case of $|a| < |b|$, points will move asymptotically towards $R = -J$ and $R = J$. In the case of $|a| = |b|$, points will cycle around the origin infinitely.



1.2 Supremum and Infimum of a Set $\mathcal{A} \subseteq \mathbb{R}$

- If $\mathcal{A} \in (-\infty, b]$ for some $b \in \mathbb{R}$, we say $\mathcal{A}$ is bounded above, and that b is an **upper bound** for A.

Theorem 1.1 (Supremum Theorem)**.** *If $\mathcal{A} \in \mathbb{R}$, $A \neq \emptyset$, and $A \subseteq (-\infty, b]$ for some $b \in \mathbb{R}$, then there exists $a \in \mathbb{R}$ such that $\mathcal{A} \subseteq (-\infty, a]$ but if $x < a$, then $\mathcal{A} \not\subseteq (-\infty, x]$.*
*We write $a = \sup \mathcal{A}$, call it the **supremum** of $\mathcal{A}$.*

Why is this necessary? Consider the set $\mathcal{A} = \{-\frac{1}{n} | n \in \mathbb{N}\}$. It does not have a maximum persay, but it has a supremum $\sup \mathcal{A} = 0$.

Consider this example: What is $\sup(-\mathbb{N})$? It is -1, which also happens to be the maximum of the set.

Theorem 1.2. *If max A exists as a real number, then $\sup A = \max A$.*

But to answer all these questions, we need to figure out: what exactly are the real numbers?

1.3 What is $\mathbb{R}$?

Let $x = (s, N, d_1, d_2, d_3, \ldots, d_k, \ldots)$, where:

- $s \in \{+1, -1\}$
- $N \in \mathbb{Z}$
- $d_k \in \mathbb{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- $\neg(\exists k : d_{k+1} = d_{k+2} = \cdots = 0)$, this is to prevent multiple sequences from being the same number

In this case, "2.49" is shorthand for $(+1, 2, 4, 8, 9, 9, 9, \ldots)$

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Aim: Background in $\mathbb{R}$; Fundamental Existence/Uniqueness Question

2.1 Supremums and Infimums in Integrals

Theorem 2.1 (Supremum/Infimum Theorem)**.**

1. *If $\mathcal{A}$ is a non-empty set of $\mathbb{R}$, and is bounded above (i.e. $\subseteq(-\infty, b]$ for some $b \in \mathbb{R}$), then there is a least upper bound for $\mathcal{A}$, namely $a \in \mathbb{R}$ such that*

   (a) $\mathcal{A} \subseteq (-\infty, a]$

   (b) if $x < a$, then $\mathcal{A} \nsubseteq (-\infty, x]$

   This a is called the called the ***supremum*** *of $\mathcal{A}$, written* $\sup A$.

2. $\inf A$. *This is the greatest lower bound for $\mathcal{A}$, or the* ***infimum****, provided $\mathcal{A} \neq \emptyset$ and $\mathcal{A}$ has a lower bound at all.*

Recall that the Riemann integral is taking the limit of a partition over an interval $[a, b]$. But when we take the limit, we make the mesh of the partition, $||\mathcal{P}||$, approach zero, where

$$\mathcal{P} = \max_{1 \leq i \leq n} \Delta x_i$$

To fix this, we can define:

$$\underline{\int_a^b} f(x)\ dx = \sup \left\{ \sum_{i=1}^{n} [\inf\{f(x) \mid x_{i-1} \leq x \leq x_i\} \Delta x_i] \ \middle|\ a = x_0 < x_1 < \cdots < x_n = b \right\}$$

This is a “down-and-up” procedure. The sum of the rectangle areas is a down approximation since we use the minimum possible height to find the area. Then, we take the supremum of that, since for any lower approximation there will always be a higher approximation. Turns out there will never be a maximum; that's why we take the supremum. This is a **lower Riemann sum**.

We can also define the same thing for an **upper Riemann sum**:

$$\overline{\int_a^b} f(x)\ dx = \inf \left\{ \sum_{i=1}^{n} [\sup\{f(x) \mid x_{i-1} \leq x \leq x_i\} \Delta x_i] \ \middle|\ a = x_0 < x_1 < \cdots < x_n = b \right\}$$

Therefore, the following inequality is true:

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f$$

If these two are equal, then we say that f is **Riemann integrable**.

Here's an example of a function that is NOT Riemann integrable:

$$f(x) = \begin{cases} 0 \text{ if } x \in \mathbb{Q} \cap [0,1] \\ 1 \text{ if } x \in [0,1] \backslash \mathbb{Q} \end{cases}$$

Note that $\underline{\int_0^1} f = 0$ and $\overline{\int_0^1} f = 1$, so this is not Riemann integrable.

2.2 Real Numbers, Again

We have shorthand for our previous definition of the real numbers.

$$\mathbb{R} = \{0\} \cup \{(s, N, d_1, d_2, \dots, d_k, \dots \mid s \in \{-1, +1\}, N \in \mathbb{Z}^+, d_k \in \mathbb{D}, \text{no 0-tail}\}$$

and the positive reals:

$$\mathbb{R}^+ = \{(s, N, d_1, d_2, \dots) \mid s = +1\}$$

Let us write $x = \underline{N}.\underline{d_1 d_2 d_3 \dots}$ and $y = \underline{M}.\underline{e_1 e_2 e_3 \dots}$.

We also define negation as:

$$-(s, N, d_1, d_2, \dots) \coloneqq (-s, N, d_1, d_2, \dots)$$

Then we can define the "less than" operation as follows:

- If $x, y \in \mathbb{R}^+$, then $x < y$ if either $N < M$ or $N = M$ and $d_1 < e_1$ or $N = M$, $d_1 = e_1$ and $d_2 < e_2$, or...
- $0 < x$ if $x \in \mathbb{R}^+$
- $x < 0$ if $x \in \mathbb{R}^+$
- $x < y$ if $x \in \mathbb{R}^-, y \in \mathbb{R}^+$.
- $x, y \in \mathbb{R}^-$, and $x < y$ if $-y < -x$