Differential Equations

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Introduction

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1.1 Romeo and Juliet

$$\begin{cases} R' = aR + bJ \\ J' = cR + dJ \end{cases}$$

These equations model the rate of change of Romeo's and Juliet's feelings. We call this a **linear system of two coupled differential equations of first order in two unknowns**.

- What makes it linear is that the functions and variables appear in a linear fashion.
- What makes it coupled is that both equations have both R and J in them.
- An **uncoupled system** would look like:

$$\begin{cases} R' = aR \\ J' = bJ \end{cases}$$

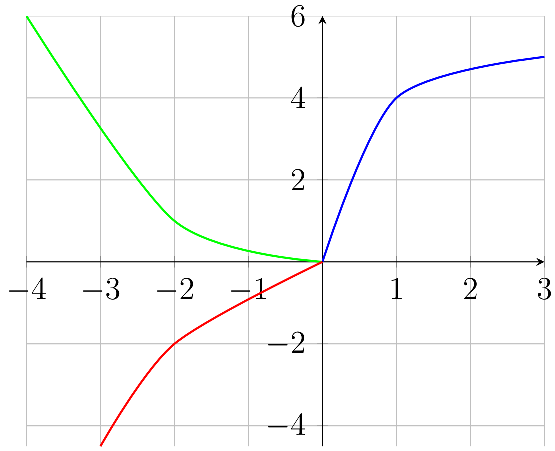
- First-order refers to the fact that all the derivatives are the first derivatives.

"Identically cautious lovers":

$$\begin{aligned} R' &= aR + bJ \quad a < 0, b > 0 \\ J' &= bR + aJ \quad |a| > |b| \end{aligned}$$

We may have initial conditions, $R(0)$ and $J(0)$, and plot them on a **phase plane** with R against J. In this case, no matter where the starting point is, the trajectory will go towards a **stable node**.

In the case of $|a| < |b|$, points will move asymptotically towards $R = -J$ and $R = J$. In the case of $|a| = |b|$, points will cycle around the origin infinitely.



1.2 Supremum and Infimum of a Set $\mathcal{A} \subseteq \mathbb{R}$

- If $\mathcal{A} \in (-\infty, b]$ for some $b \in \mathbb{R}$, we say $\mathcal{A}$ is bounded above, and that b is an **upper bound** for A.

Theorem 1.1 (Supremum Theorem)**.** *If $\mathcal{A} \in \mathbb{R}$, $A \neq \emptyset$, and $A \subseteq (-\infty, b]$ for some $b \in \mathbb{R}$, then there exists $a \in \mathbb{R}$ such that $\mathcal{A} \subseteq (-\infty, a]$ but if $x < a$, then $\mathcal{A} \not\subseteq (-\infty, x]$.*
We write $a = \sup \mathcal{A}$, call it the ***supremum*** *of $\mathcal{A}$.*

Why is this necessary? Consider the set $\mathcal{A} = \{-\frac{1}{n} | n \in \mathbb{N}\}$. It does not have a maximum persay, but it has a supremum $\sup \mathcal{A} = 0$.

Consider this example: What is $\sup(-\mathbb{N})$? It is -1, which also happens to be the maximum of the set. e

Theorem 1.2. *If max $\mathcal{A}$ exists as a real number, then $\sup \mathcal{A} = \max \mathcal{A}$.*

But to answer all these questions, we need to figure out: what exactly are the real numbers?

1.3 What is $\mathbb{R}$?

Let $x = (s, N, d_1, d_2, d_3, \ldots, d_k, \ldots)$, where:

- $s \in \{+1, -1\}$
- $N \in \mathbb{Z}$
- $d_k \in \mathbb{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- $\neg(\exists k : d_{k+1} = d_{k+2} = \cdots = 0)$, this is to prevent multiple sequences from being the same number

In this case, "2.49" is shorthand for $(+1, 2, 4, 8, 9, 9, 9, \ldots)$

2 2/4/16: Background in $\mathbb{R}$; Fundamental Existence/Uniqueness Question

2.1 Supremums and Infimums in Integrals

Theorem 2.1 (Supremum/Infimum Theorem)**.**

1. *If $\mathcal{A}$ is a non-empty set of $\mathbb{R}$, and is bounded above (i.e. $\mathcal{A} \subseteq (-\infty, b]$ for some $b \in \mathbb{R}$), then there is a least upper bound for $\mathcal{A}$, namely $a \in \mathbb{R}$ such that*
   (a) $\mathcal{A} \subseteq (-\infty, a]$
   (b) *if $x < a$, then $\mathcal{A} \not\subseteq (-\infty, x]$*

   *This a is called the called the **supremum** of $\mathcal{A}$, written* $\sup A$.
2. $\inf A$. *This is the greatest lower bound for $\mathcal{A}$, or the **infimum**, provided $\mathcal{A} \neq \emptyset$ and $\mathcal{A}$ has a lower bound at all.*

Recall that the Riemann integral is taking the limit of a partition over an interval $[a, b]$. But when we take the limit, we make the mesh of the partition, $\|\mathcal{P}\|$, approach zero, where

$$\mathcal{P} = \max_{1 \leq i \leq n} \Delta x_i$$

To fix this, we can define:

$$\underline{\int_a^b} f(x)\ dx = \sup \left\{ \sum_{i=1}^{n} [\inf\{f(x) \mid x_{i-1} \leq x \leq x_i\} \Delta x_i] \;\middle|\; a = x_0 < x_1 < \cdots < x_n = b \right\}$$

This is a "down-and-up" procedure. The sum of the rectangle areas is a down approximation since we use the minimum possible height to find the area. Then, we take the supremum of that, since for any lower approximation there will always be a higher approximation. Turns out there will never be a maximum; that's why we take the supremum. This is a **lower Riemann sum**.

We can also define the same thing for an **upper Riemann sum**:

$$\overline{\int_a^b} f(x)\ dx = \inf \left\{ \sum_{i=1}^{n} [\sup\{f(x) \mid x_{i-1} \leq x \leq x_i\} \Delta x_i] \;\middle|\; a = x_0 < x_1 < \cdots < x_n = b \right\}$$

Therefore, the following inequality is true:

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f$$

If these two are equal, then we say that f is **Riemann integrable**.

Here's an example of a function that is NOT Riemann integrable:

$$f(x) = \begin{cases} 0 \text{ if } x \in \mathbb{Q} \cap [0, 1] \\ 1 \text{ if } x \in [0, 1] \backslash \mathbb{Q} \end{cases}$$

Note that $\underline{\int_0^1} f = 0$ and $\overline{\int_0^1} f = 1$, so this is not Riemann integrable.

2.2 Real Numbers, Again

We have shorthand for our previous definition of the real numbers.

$$\mathbb{R} = \{0\} \cup \{(s, N, d_1, d_2, \dots, d_k, \dots \mid s \in \{-1, +1\}, N \in \mathbb{Z}^+, d_k \in \mathbb{D}, \text{no 0-tail}\}$$

and the positive reals:

$$\mathbb{R}^+ = \{(s, N, d_1, d_2, \dots) \mid s = +1\}$$

Let us write $x = \underline{N}.\underline{d_1 d_2 d_3 \dots}$ and $y = \underline{M}.\underline{e_1 e_2 e_3 \dots}$.

We also define negation as:

$$-(s, N, d_1, d_2, \dots) \coloneqq (-s, N, d_1, d_2, \dots)$$

Then we can define the "less than" operation as follows:

- If $x, y \in \mathbb{R}^+$, then $x < y$ if either $N < M$ or $N = M$ and $d_1 < e_1$ or $N = M$, $d_1 = e_1$ and $d_2 < e_2$, or...
- $0 < x$ if $x \in \mathbb{R}^+$
- $x < 0$ if $x \in \mathbb{R}^+$
- $x < y$ if $x \in \mathbb{R}^-, y \in \mathbb{R}^+$.
- $x, y \in \mathbb{R}^-$, and $x < y$ if $-y < -x$

3 2/5/16: Fundamental Existence of Uniqueness Theorem

3.1 Terminology

A **differential equation** is a relation between one or more unknown functions and at least some (but finitely many) of their derivatives, plus the independent variables.

Examples:

$$\begin{aligned} y' + 2xy - x^2 &= 3 \\ y''' + 2x^2 y'' - 3x^3 y' + xy - x^5 + 1 &= 0 \\ (y')^{y''} - e^{y'''} + x &= 0 \end{aligned}$$

Or,

$$\vec{y}\,' = \mathbf{A}(x)\vec{y}$$

where

$$\vec{y} = \vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}$$

$$\mathbf{A}(x) = \begin{bmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{bmatrix}$$

3.2 A Treatise on PDE's

There are two different types of differential equations: ODE's (ordinary, where all unknown functions depend on a single, same independent variable) and PDE's (partial, anything else).

$$\text{Wave equation: } \frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial t^2}$$
$$u = g(x-t) + h(x+t)$$

4 2/9/16: Basic Existence and Uniqueness Theorem

Theorem 4.1 (Flow Theorem)**.** *Let $\vec{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), \ldots, F_n(\vec{x}))$ be a vector field defined on some closed and bounded region $\mathcal{D} \subseteq \mathbb{R}^n$. Also assume $\vec{F}$ is C^1; namely, $\frac{\partial F_i}{\partial x_j}$ is continuous everywhere interior to D, for any i and j.*

Let $\vec{p}$ be a specific point interior to $\mathcal{D}$. Then $\exists$ a function $\vec{\sigma}(t)$ from some "time" interval $(-\varepsilon, \varepsilon)$ with $\varepsilon > 0$ into $\mathcal{D}$, such that $\vec{\sigma}(0) = \vec{p}$ and $\vec{\sigma}\,'(t) = \vec{F}(\vec{\sigma}(t))$ for any $t \in (-\varepsilon, \varepsilon)$.

This theorem basically says that in a vector field, we can use the vector field to get the velocity of a curve. We call $\vec{\sigma}(t)$ a **flow** for $\vec{F}$, starting at $\vec{p}$. This flow is, in fact, unique, in the sense that any two flows for the same $\vec{F}$ starting at the same point must agree whenever they are both defined.

This is meaningful in that we can treat it as a differential equation:

$$\begin{cases} \sigma_1' &= F_1(\sigma_1, \sigma_2, \ldots, \sigma_n) \\ \sigma_2' &= F_2(\sigma_1, \sigma_2, \ldots, \sigma_n) \\ \vdots & \vdots \\ \sigma_n' &= F_n(\sigma_1, \sigma_2, \ldots, \sigma_n) \end{cases}$$

$$\begin{cases} \sigma_1(0) &= p_1 \\ \sigma_2(0) &= p_2 \\ \vdots & \vdots \\ \sigma_n(0) &= p_n \end{cases}$$

4.1 Second-Order

$$mx'' = -kx, \quad x(0) = x_0, \quad x'(0) = v_0$$
$$x = x(t), \quad v = v(t) = x'(t), \quad a = a(t) = x''(t)$$

where $k > 0$ is the spring constant. We can rewrite this as:

$$\begin{cases} x' &= v = F_1(x, v) \\ v' &= -\frac{k}{m}x = F_2(x, v) \end{cases} \quad \text{and} \quad \begin{cases} x(0) &= x_0 \\ v(0) &= v_0 \end{cases}$$

The Flow Theorem will tell us there is a unique solution, for some time interval.

5 2/10/16: The Flow Theorem

5.1 Application: n^{th} order initial value problem (IVP)

$$\begin{cases} x &= x(t) \\ x^{(n)} &= F(t, x, x', x'', \dots, x^{(n-1)}) \\ x(t_0) &= x_{00} \\ x'(t_0) &= x_{10} \\ x''(t_0) &= x_{20} \\ \vdots & \\ x^{(n-1)}(t_0) &= x_{(n-1)0} \end{cases}$$

$f(t)x^{(n)} = F(t, x, x', x'', \dots, x^{(n-1)})$ is an n^{th} order ODE in standard form.

A **singularity** (or singular point) of this equation is a value t_0 where $f(t_0) = 0$. At this point, the equation ceases to be of n^{th} order. If $f(t)$ is of constant sign in the time interval on which we'd like to solve the equation, we just divide through by $f(t)$ to get our desired form (which is the regular case, as opposed to the singular case).

Here, the Flow Theorem says that there is a unique solution $x = x(t)$ defined in some time interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ where $\varepsilon > 0$.

To apply this:

$$\begin{cases} x_0(t) &= t \\ x_1 &= x_1(t) = x(t) \\ x_2 &= x_2(t) = x'(t) \\ x_3 &= x_3(t) = x''(t) \\ \vdots & \\ x_n &= x_n(t) = x^{(n-1)}(t) \end{cases}$$

becomes

$$\begin{cases} x_0' & = 1 = F_0(x_0, x_1, x_2, \dots, x_n) \\ x_1' & = x_2 = F_1(x_0, x_1, x_2, \dots, x_n) \\ x_2' & = x_3 = F_2(x_0, x_1, x_2, \dots, x_n) \\ x_3' & = x_4 = F_3(x_0, x_1, x_2, \dots, x_n) \\ \vdots & \vdots \\ x_{n-1}' & = x_n = F_{n-1}(x_0, x_1, x_2, \dots, x_n) \\ x_n' & = F(t, x_1, x_2, \dots, x_n) = F_n(\dots) \end{cases} \quad \text{and} \quad \begin{cases} x_0(t_0) & = t_0 \\ x_1(t_0) & = x_{00} \\ x_2(t_0) & = x_{10} \\ \vdots & \vdots \\ x_n(t_0) & = x_{(n-1)0} \end{cases}$$

This shows that we can recast an n^{th} order IVP into an $n+1$ order system.

However, for the Flow Theorem to apply, $\vec{F}$ needs to be C^1. Therefore, our hypothesis in the IVP is that F is C^1, meaning that $\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x'}, \dots$ are continuous.

6 2/11/16: Proof of the Flow Theorem

We will prove the Flow Theorem for two dimensions only; the proof can be extended to greater than two dimensions.

Proof. Let $\vec{F}(x, y) = (A(x, y), B(x, y))$ be a vector field. By hypothesis, A and B are defined on a closed, bounded region $\mathcal{D}$, and they are C^1 on $\mathcal{D}$. Then we need to solve the following equation:

$$\vec{x}' = \vec{F}(\vec{x})$$
$$\vec{x}(0) = \vec{p} = \langle p, q \rangle$$

We need to see how fast $A(x, y)$ is changing.

1.
$$\begin{aligned} |A(x_1, y_1) - A(x_2, y_2)| &= |A(x_1, y_1) - A(x_1, y_2) + A(x_1, y_2) - A(x_2, y_2)| \\ \text{(Triangle Inequality)} \qquad &\leq |A(x_1, y_1) - A(x_1, y_2)| + |A(x_1, y_2) - A(x_2, y_2)| \\ \text{(MVT)} \qquad &\leq \left|\frac{\partial A}{\partial y}(x_1, y^*)(y_1 - y_2)\right| + \left|\frac{\partial A}{\partial x}(x^*, y_2)(x_1 - x_2)\right| \end{aligned}$$

   Take K to be some upper bound for all the partial derivatives of A and B on $\mathcal{D}$.

$$\left|\frac{\partial A}{\partial y}(x_1, y^*)(y_1 - y_2)\right| + \left|\frac{\partial A}{\partial x}(x^*, y_2)(x_1 - x_2)\right| \leq K(|x_1 - x_2| + |y_1 - y_2|)$$

   Similarly:

$$|B(x_1, y_2) - B(x_2, y_2)| \leq K(|x_1 - x_2| + |y_1 - y_2|)$$

   This is called the **Lipschitz Condition**.

2. Also, note that A and B are continuous in $\mathcal{D}$ and so by the Extreme Value Theorem, we can find an upper bound M for $|A|$ and $|B|$ on $\mathcal{D}$, i.e.

$$M = \max\left(\max_{(x,y)\in\mathcal{D}} |A(x, y)|, \max_{(x,y)\in\mathcal{D}} |B(x, y)|\right)$$

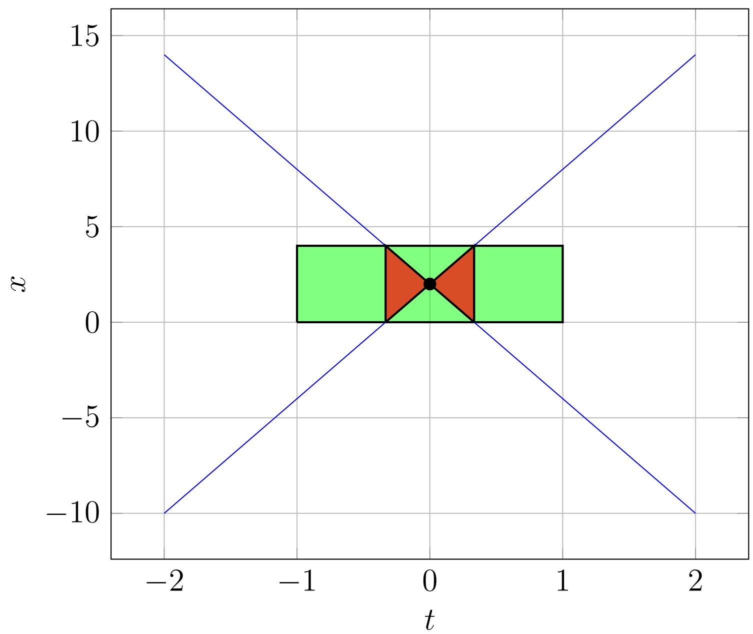
3. The point (p, q) is assumed to be interior to $\mathcal{D}$ (not on the boundary).

   We can therefore encase the point $(0, p)$ in a rectangle in the tx-plane defined by $R : [-r, r] \times [p - s, p + s] \subseteq \text{proj}_x \mathcal{D}, r, s > 0$. Draw two lines with slopes M and $-M$ through the point. We will consider the "bowtie" region formed by the intersections of the lines with the rectangle, joining them oppositely, and the lines themselves. Call the x-intersections $-h$ and h.

   Define $h \coloneqq \min\left(r, \frac{s}{M}\right) > 0$. This is to formally define the bowtie region and consider the two possible pictures depending on the size of M.

   Now, we want to construct the solution of the differential equation within the bowtie region.

   Whoopsies, there was a screwup here, proof to be fixed in the future.



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7 2/12/16: Separable and First-Order Linear Equations

7.1 Multiplicatively Separable Functions

$$F(t, x) = f(t)g(x)$$

A non-example of a separable function is $F(t, x) = t^2 + x^2$. An example is $F(t, x) = t^2x^3$.

For our purposes, we will work with first-order ODE's with scalar functions.

7.2 Separable ODE

$$\boxed{x' = f(t)g(x)}$$

There are other ways we can write this equation:

- **General Form**: $G(t, x, x') = 0$
- **Standard Form**: $\phi(t)x' = F(t, x)$
  - **Regular Case**: $x' = F(t, x)$, F is the "slope function"
  - **Singular Case**: This is when we solve in an interval $(t_0 - \delta, t_0 + \delta)$ where $\delta > 0$ and $\phi(t_0) = 0$.

To solve this type of equation:

$$\begin{aligned} x'(t) &\equiv f(t)g(x(t)) \\ \frac{x'(t)}{g(x(t))} &= f(t) \\ \int_a^t \frac{x'(\tau)}{g(x(\tau))}\ d\tau &= \int_a^t f(\tau)\ d\tau \end{aligned}$$

Letting $u = x(\tau)$ and $du = x'(\tau)\ d\tau$:

$$\underbrace{\int_{x(a)}^{x(t)} \frac{du}{g(u)}}_{G(x(t))} = \underbrace{\int_a^t f(\tau)\ d\tau}_{F(t)}$$

$$\boxed{G(x(t)) = F(t)}$$

7.3 Example

$$\begin{aligned} x' &= t^2 x^3 \\ \frac{x'}{x^3} &= t^2 \\ \int \frac{dx}{x^3} &= \int t^2\ dt + C \\ \frac{x^{-2}}{-2} &= \frac{t^3}{3} + C \\ x^{-2} &= C - \frac{2}{3}t^3 \\ x &= \pm\frac{1}{\sqrt{C - \frac{2}{3}t^3}} \end{aligned}$$

8 2/22/16: Separable Equations, First-Order Linear Equations; Uniqueness for C^1 IVP's

Recall our form for the separable equation:

$$x' = f(t)g(x)$$

Assume f and g are continuous on their respective domains, f on $I = (t_0 - a, t_0 + a), a > 0$, g on $J = (x_0 - b, x_0 + b), b > 0$. Let $\mathcal{R} = I \times J$. If we also have $x(t_0) = x_0$, then we have an IVP (initial value problem) on our hands.

But the problem is, $\frac{1}{g(x)}$ isn't necessarily continuous.

Separately, solve the algebraic equation $g(x) = 0$ in the interval J. Assume for simplicity that the roots of g are isolated and C^∞ ("smooth"). Then, we can partition J into open subintervals $J_1, J_2, \ldots, J_n$, i.e.

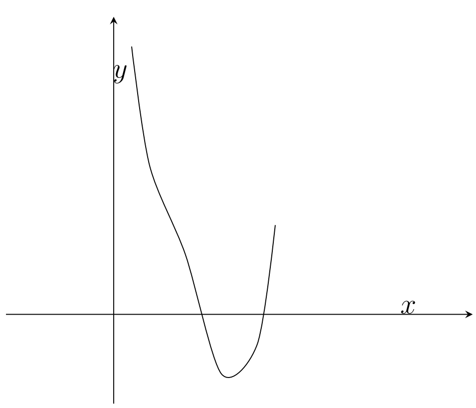
$$J = \{a, b, c, d\} \cup J_1 \cup J_2 \cup \cdots \cup J_n$$

9 2/23/16: Uniqueness for C^1 IVP's

$$x' = f(t)g(x) \qquad x = x(t)$$
$$x' \equiv f(t)g(x(t)) \text{ for all } t \in I$$

Assume: f, g have continuous derivatives on their respective domains. Then, all solutions are given as follows: Suppose $g(x)$ has domain J. If a and b are consecutive isolated roots of g, we can solve on (a, b) as we did yesterday:

$$\underbrace{\int_c^{x(t)} \frac{1}{g(u)}\, du}_{G_c(x(t))} = \underbrace{\int_{t_0}^t f(\tau)\, d\tau}_{F(t)} \qquad \text{where } c \in (a, b) \text{ is arbitrary}$$



If a is a root of g (isolated or not) then claim: $x(t) \equiv a$ for $t \in \mathbb{R}$ is a solution of the differential equation.

9.1 Uniqueness

Are these all the solutions, however?

A first-order IVP in standard form (the regular case):

$$x' = \underbrace{F(t,x)}_{\text{slope function}}, \qquad x(t_0) = x_0$$

<u>Assumption</u>: F is a C^1 function ($\frac{\partial F}{\partial t}$ and $\frac{\partial F}{\partial x}$ are both continuous) on a rectangle centered at the initial point (t_0, x_0). Then, we have the following theorem:

Theorem 9.1. *If $\phi(t)$ and $\psi(t)$ are solutions of the IVP, defined on respective domains $I_\delta = (t_0 - \delta, t_0 + \delta)$ and $I_\varepsilon = (t_0 - \varepsilon, t_0 + \varepsilon)$ where $\delta > 0$ and $\varepsilon > 0$, then*

$$\phi(t) \equiv \psi(t)$$

for all $t \in I_\eta = (t_0 - \eta, t_0 + \eta)$ where $\eta > 0$.

Basic outline for the proof:

$$\text{IVP} \qquad x'(t) \equiv F(t, x(t)), \qquad x(t_0) = x_0$$

$$x(t) - x(t_0) = \int_{t_0}^{t} F(\tau, x(\tau))\, d\tau$$

10 2/24/16: Uniqueness & Existence for C^1 IVP's

10.1 Autonomous Equations and the Time Shift Property

$$\begin{cases} x' = \sqrt{|x|} & \text{(autonomous – the independent variable makes no explicit appearance)} \\ x(0) = 0 \end{cases}$$

One important property of an autonomous differential equation is that it is time-indepedent, i.e. if $x = \phi(t)$ is a solution, then so is $x = \psi(t) := \phi(t + c)$. Without an initial condition, we have an infinite number of solutions.

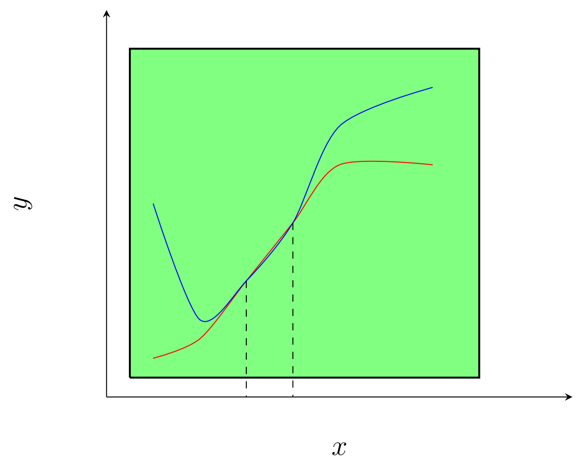
Let us try separation of variables:

$$\begin{aligned} \int_0^{x(t)} \frac{dx}{\sqrt{|x|}} &= \int_0^t d\tau = t \\ \lim_{\varepsilon \to 0^+} \int_\varepsilon^{x(t)} u^{-\frac{1}{2}}\, du &= \lim_{\varepsilon \to 0^+} \left[2u^{\frac{1}{2}}\right]_\varepsilon^{x(t)} \\ &= 2\sqrt{x(t)} - \lim_{\varepsilon \to 0^+} \sqrt{\varepsilon} \\ &= 2\sqrt{x(t)} \\ x(t) &= \frac{t^2}{4} > 0 \quad (\text{assuming } t \geq 0) \end{aligned}$$

We can similarly derive, for $t \leq 0$, that $x(t) = -\frac{t^2}{4}$. We can then construct our function:

$$x(t) = \begin{cases} \frac{t^2}{4}, & t \geq 0 \\ -\frac{t^2}{4}, & t < 0 \end{cases}$$

10.2 Unique Solutions



Proof. We begin by showing that our IVP is actually an integral equation.

$$\boxed{\begin{cases} x'(t) \equiv F(t, x(t)) \\ x(t_0) = x_0 \end{cases}}$$

$$\begin{aligned} x'(\tau) &\equiv F(\tau, x(\tau)) \\ \int_{t_0}^t x'(\tau)\, d\tau &= \int_{t_0}^t F(\tau, x(\tau))\, d\tau \\ &\boxed{x(t) = x_0 + \int_{t_0}^t F(\tau, x(\tau))\, d\tau, \quad x(t) \text{ is continuous}} \end{aligned}$$

We have just proved one direction of equivalence. To prove the other direction, note that $x(t)$ is differentiable, since all of its parts are continuous and differentiable. ■

11 2/25/16: Uniqueness/Existence for C^1 IVP's

We're assuming: for $\delta > 0$, $\varepsilon > 0$:

$$\phi : I_\delta := (t_0 - \delta, t_0 + \delta) \to \mathbb{R} \text{ satisfies } \phi'(t) \equiv F(t, \phi(t)), \phi(t_0) = x_0$$

$$\psi : I_\varepsilon := (t_0 - \varepsilon, t_0 + \varepsilon) \to \mathbb{R} \text{ satisfies } \psi'(t) \equiv F(t, \psi(t)), \psi(t_0) = x_0$$

We want to show that for some $\eta > 0$, $\phi(t) \equiv \psi(t)$ on $I_\eta := (t_0 - \eta, t_0 + \eta)$.

First, we introduce the following concept:

Definition 11.1. *If f is a bounded real-valued function on a set $\mathcal{S}$, then its **sup-norm** is defined as:*

$$\|f\|_{\mathcal{S}} \coloneqq \sup_{x \in \mathcal{S}} |f(x)|$$

*If $\mathcal{S}$ is a closed, bounded subset of $\mathbb{R}^n$, and f is continuous, then $\|f\|_{\mathcal{S}} = \max_{x \in \mathcal{S}} |f(x)|$, in which case is called the **max-norm**.*

Note that:

- $\|f\|_{\mathcal{S}} \geq 0$
- $\|f\|_{\mathcal{S}} = 0$ iff $f(x) \equiv 0$ for all $x \in \mathcal{S}$
- $\|\alpha f\|_{\mathcal{S}} = |\alpha| \|f\|_{\mathcal{S}}$
- $\|f + g\|_{\mathcal{S}} \leq \|f\|_{\mathcal{S}} + \|g\|_{\mathcal{S}}$ where f and g are defined and bounded on $\mathcal{S}$.

We claim that $\|\phi - \psi\|_{I_\eta} \leq c\|\phi - \psi\|_{I_\eta}$, where $0 < c < 1$. This would mean that $\|\phi - \psi\|_{I_\eta} = 0$, then $\phi(t) - \psi(t) \equiv 0$ on I_η and $\phi(t) = \psi(t)$ on I_η.

Proof. Note that

$$\phi(t) \equiv x_0 + \int_{t_0}^{t} F(\tau, \phi(\tau)) \, d\tau$$

for all $t \in I_\delta$ and

$$\psi(t) \equiv x_0 + \int_{t_0}^{t} F(\tau, \psi(\tau)) \, d\tau$$

for all $t \in I_\varepsilon$. Both of these equation are true for all $t \in I_{\min(\delta, \varepsilon)}$.

Restrict $t \in I_\eta$ where $0 \leq \eta \leq \min(\delta, \varepsilon)$. Subtracting these two equations:

$$\begin{aligned}
|\phi(t) - \psi(t)| &= \left| \int_{t_0}^{t} [F(\tau, \phi(\tau)) - F(\tau, \psi(\tau))] \, d\tau \right| \\
&\leq \left| \int_{t_0}^{t} |F(\tau, \phi(\tau)) - F(\tau, \psi(\tau))| \, d\tau \right| \\
\text{(MVT)} \qquad &\leq \left| \int_{t_0}^{t} \underbrace{\left| \frac{\partial F}{\partial x}(x, \theta(\tau)) \right|}_{\leq M} |\phi(t) - \psi(t)| \ d\tau \right| \\
&\leq M \left| \int_{t_0}^{t} \underbrace{|\phi(\tau) - \psi(\tau)|}_{\leq \|\phi - \psi\|_{I_\eta}} \ d\tau \right| \\
&\leq M \|\phi - \psi\| (t - t_0) \leq M\eta \|\phi - \psi\|_{I_\eta}
\end{aligned}$$

Now we simply pick η such that $M\eta = c < 1$, and we are done. ■

12 2/26/16: Existence

12.1 Transforming to an Integral Equation

Yesterday we proved the uniqueness of the solution of an IVP. Now we must prove the existence.

$$\begin{cases} x' = F(t,x), & x = x(t) \text{ is the unknown function} \\ x(t_0) = x_0 \end{cases}$$

C^1 IVP $\Leftrightarrow$ $F(t,x)$ is C^1 in some rectangle $\mathcal{R}$ centered at (x_0, y_0).

Let us integrate our equation:

$$\int_{t_0}^{t} x'(\tau)\ d\tau = \int_{t_0}^{t} F(\tau, x(\tau))\ d\tau$$
$$x(t) - x(t_0) = x(t) - x_0 = \int_{t_0}^{t} F(\tau, x(\tau))\ d\tau$$

So now our problem/equation becomes:

- $x(t) \equiv x_0 + \displaystyle\int_{t_0}^{t} F(\tau, x(\tau)) d\tau$
- $x(t)$ is a continuous function of t

Why do we need the continuity condition? If $x(t)$ is a solution, then it is differentiable, which implies it is continuous.

Now we prove the opposite direction. To prove that $x(t)$ is differentiable, note that $f(t) = x_0$ is differentiable, and the integral is also differentiable (since its derivative is $F(t, x(t))$, which is continuous. Therefore, by algebra, the two statements are equivalent.

12.2 Picard's Method

Define

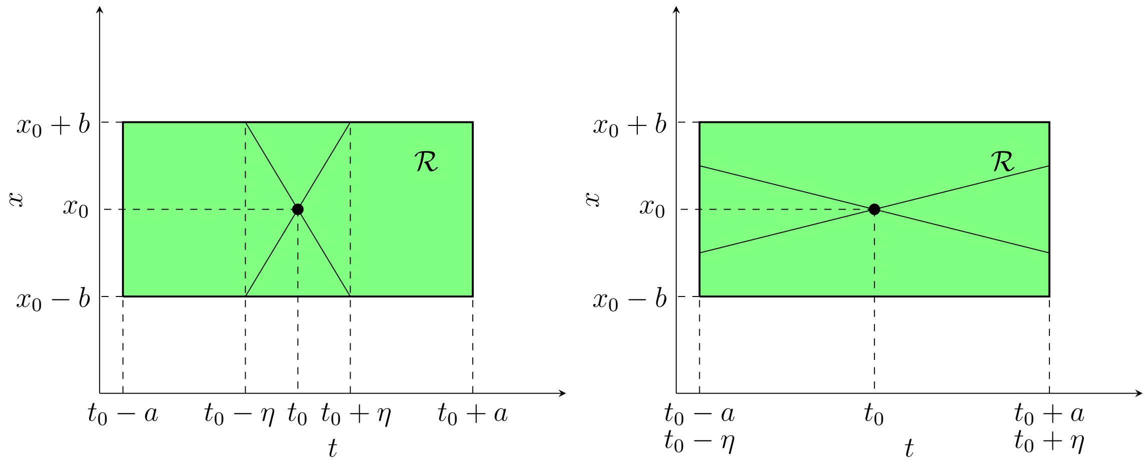
$$x_{n+1}(t) \coloneqq x_0 + \int_{t_0}^{t} F(\tau, x_n(\tau))\ d\tau$$

and

$$x_0(t) :\equiv x_0 \quad \text{for all } t$$

In this section, we prove that for each $t \in I_\eta$, $\lim\limits_{n\to\infty} x_n(t)$ exists, let's call it $x(t)$, and moreover:

- $x(t)$ is a continuous function of t on I_η
- $x(t) \equiv x_0 + \displaystyle\int_{t_0}^{t} F(\tau, x(\tau))\ d\tau$



Define $M \geq \max_{(t,x)\in\mathcal{R}} |F(t,x)|$ (existence follows from EVT). Let $\eta := \min(a, \frac{b}{M}) > 0$. Assume $M > 0$; if $M = 0$, then the IVP is $x' \equiv 0$, $x(t_0) = x_0$ has a solution: $x(t) \equiv x_0$ for all $t \in (t_0 - a, t_0 + a)$.

13 2/29/16: Picard's Existence Proof, Continued...

Recapping: our base function is

$$x_0(t) \equiv x_0 \qquad \text{for all } t \in I_\eta, \eta = \min\left(a, \frac{b}{M}\right) > 0$$

We need to choose the size of the rectangle for each function. In the example

$$\begin{cases} x' = t^2 + x^2 \\ x(0) = 1 \end{cases}$$

We need to find some $\eta > 0$ such that a solution is guaranteed to exist in $(-\eta, \eta)$. Let $\mathcal{R} = [-T, T] \times [1-r, 1+r]$ be our rectangle. Then take $M = T^2 + (1+r)^2$, then $|t^2 + x^2| \leq M$ when $(t, x) \in \mathcal{R}$.

13.1 Proving Well-Defined-ness

Theorem 13.1. *Each x_n is well-defined and continuous and satisfies $|x_n(t) - x_0| \leq b$ for all $t \in I_\eta$.*

Proof. The base case is trivial. Now assume true for $x_n(t)$; we now prove for $x_{n+1}(t)$. By assumption, the integrand $F(\tau, x_n(\tau))$ is well-defined and continuous (and therefore Riemann integrable on I_η) for all $t \in I_\eta$, which means the integral is well-defined. Therefore, $x_{n+1}(t)$ is well-defined for all $t \in I_\eta$.

Also, $x_{n+1}(t)$ is continuous on I_η by similar reasoning.

Now, we investigate $|x_{n+1}(t) - x_0$. First, we claim that $(\tau, x_n(\tau)) \in \mathcal{R}$ for any τ between t_0 and t. But

$$|\tau - t_0| \leq |t - t_0| \leq \eta \leq a$$

$$|x_n(\tau) - x_0| \leq b$$

$$\begin{aligned}
|x_{n+1}(t) - x_0| &= \left| \int_{t_0}^{t} F(\tau, x_n(\tau))\ d\tau \right| \\
&\leq \left| \int_{t_0}^{t} \underbrace{|F(\tau, x_n(\tau))|}_{\leq M}\ d\tau \right| \\
&\leq M|t - t_0| \leq M\eta \leq b
\end{aligned}$$

■

14 3/1/16: Finish Picard's Existence Proof

Theorem 14.1.

$$|x_{n+1}(t) - x_n(t)| \leq \frac{MK^n}{(n+1)!}|t - t_0|^{n+1}$$

for any $n \geq 0$ and any $t \in I_\eta$, where $K \geq \max_{(t,x)\in\mathcal{R}} \left|\frac{\partial F}{\partial x}(t, x)\right|$ (using the assumed C^1-ness of F on $\mathcal{R}$).

Proof. We prove by induction. When $n = 0$:

$$|x_1(t) - x_0(t)| = |x_1(t) - x_0| = \left| \int_{t_0}^{t} F(\tau, x_0)\ d\tau \right| \leq M|t - t_0| = \frac{MK^0}{(0+1)!}|t - t_0|^{0+1}$$

Now assume the hypothesis, we want to prove that

$$|x_{n+2}(t) - x_{n+1}(t)| \leq \frac{MK^{n+1}}{(n+2)!}|t - t_0|^{n+2}$$

So:

$$\begin{aligned}
|x_{n+2}(t) - x_{n+1}(t)| &= \left| \int_{t_0}^{t} F(\tau, x_{n+1}) - F(\tau, x_n)\ d\tau \right| \\
\text{(MVT, for some } y_n \in [x_n, x_{n+1}]) \qquad &= \left| \int_{t_0}^{t} \frac{\partial F}{\partial x}(\tau, y_n)(x_{n+1} - x_n)\ d\tau \right| \\
&\leq \left| \int_{t_0}^{t} \left|\frac{\partial F}{\partial x}(\tau, y_n)\right| |(x_{n+1} - x_n)|\ d\tau \right| \\
&\leq K \left| \int_{t_0}^{t} |x_{n+1} - x_n|\ d\tau \right| \\
&\leq K \left| \int_{t_0}^{t} \frac{MK^n}{(n+1)!}|t - t_0|^{n+1}\ d\tau \right| \\
&= \frac{MK^{n+1}}{(n+2)!}|t - t_0|^{n+2}
\end{aligned}$$

■

15 3/3/16

For any $t \in I_\eta$,

$$|x_{n+p}(t) - x_n(t)| = |x_{n+p}(t) - x_{n+p-1}(t) + x_{n+p-1}(t) - x_{n+p-2}(t) + x_{n+p-2}(t) - \cdots - x_n(t)$$

is bounded. By the Triangle Inequality,

$$\begin{aligned}
|x_{n+p}(t) - x_n(t)| &\leq \sum_{j=n}^{n+p-1} |x_{j+1}(t) - x_j(t)| \\
&\leq \sum_{j=n}^{n+p-1} \frac{MK^j}{(j+1)!}|t-t_0|^{j+1} \\
&= \left(\frac{M}{K}\right) \sum_{j=n}^{n+p-1} \frac{(K|t-t_0|)^{j+1}}{(j+1)!} \\
&\leq \left(\frac{M}{K}\right) \sum_{j=n}^{\infty} \frac{(K|t-t_0|)^{j+1}}{(j+1)!} \\
&\leq \left(\frac{M}{K}\right) \sum_{j=n}^{\infty} \frac{(K\eta)^{j+1}}{(j+1)!} \\
&= \left(\frac{M}{K}\right) \left(e^{k\eta} - \sum_{j=0}^{n-1} \frac{(k\eta)^{j+1}}{(j+1)!}\right)
\end{aligned}$$

Thus, we have an upper bound for any two terms in our sequence.

Now if we take $\|x_{n+p} - x_n\|_{I_\eta} = \sup_{t \in I_\eta} |x_{n+p}(t) - x_n(t)| \leq L$, then send $n \to \infty$. Then,

$$\lim_{n\to\infty} \|x_{n+p(n)} - x_n\|_{I_\eta} \leq 0$$

But this is also nonnegative, so it must be the case that the limit is zero, and thus this sequence is Cauchy.

16 3/4/16: Existence of Solutions, Continued

The last thing we proved was that (x_n) is a Cauchy sequence in the space of continuous functions on I_η, denoted $C^0(I_\eta)$, under the sup-norm, $\|f\|_{I_\eta} = \sup_{t\in I_\eta} |f(t)|$. This means: $\lim_{n\to\infty} \|x_{n+p(n)} - x_n\|_{I_\eta} = 0$ for any $\mathbb{N}$-valued function $p(n)$.

16.1 Metric Spaces

Definition 16.1. *A* ***metric space****, denoted $(\mathscr{X}, d)$, $\mathscr{X} \neq \emptyset$, d is a "distance" function, must satisfy the following:*

1. $d : (\mathscr{X} \times \mathscr{X}) \to [0, \infty)$

2. $d(x, y) \geq 0$ *and* $d(x, y) = 0 \Leftrightarrow x = y$

3. $d(x, y) = d(y, x)$

4. $d(x, y) + d(y, z) \geq d(x, z)$

If we have a metric d on a vector space $\mathcal{V}$, we can also require $d(x, y)+d(y, z) = d(x, z)$ iff x-y-z (y is between x and z) or $x = y$ or $y = z$. To define betweenness: $\vec{p}$-$\vec{q}$-$\vec{r}$ iff $\vec{q} = (1 - t)\vec{p} + t\vec{r}$.

Here, we define a metric on a vector space. Given a vector space $\mathcal{V}$, a **norm** on $\mathcal{V}$ is a real-valued function $\|\cdot\|$ such that:

1. $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0 \Leftrightarrow \vec{v} = \vec{0}$ (Positive definiteness)
2. $\|c\vec{v}\| = |c|\|\vec{v}\|$ (Absolute homogeneity)
3. $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ (Triangle inequality)

This way, we can define

$$d(\vec{v}, \vec{w}) \coloneqq \|\vec{v} - \vec{w}\|$$

16.2 Cauchy Sequences in Metric Spaces

In a metric space $(\mathscr{X}, d)$, a sequence of elements (x_n) converges to $x \in \mathscr{X}$ if $d(x_n, x) \to 0$ as $n \to \infty$.

A sequence (x_n) in $(\mathscr{X}, d)$ is called Cauchy if $d(x_n, x_m) \to 0$ as $\min(n, m) \to \infty$. Not all Cauchy sequences converge. As an example, take

$$\mathscr{X} = \mathbb{Q}, d(q, \tilde{q}) = |q - \tilde{q}|$$

Take the sequence $(3, 3.1, 3.14, 3.141, \cdots)$. This sequence is Cauchy since choosing two far-out values will differ very little. However, it converges to π, which is not in the metric space. Therefore, we call this an **incomplete metric space**.

Theorem 16.1. $(\mathbb{R}, |\cdot - \cdot|)$ *is a* ***complete metric space****, i.e. every Cauchy sequence of real numbers converges to a real limit.*