# Multivariate Calculus

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# Contents

# Introduction

These are the notes from the course I took at Stuyvesant High School, Fall 2015. Note: these are not the exact notes that were taken; these notes skip some properties that should be obvious if you took precalculus.

# 1 3 Dimensional Space

A **plane** can be represented by the equation $Ax + By + Cz + D = 0$.

- When one of $A, B, C$ is nonzero, the plane is parallel to a **coordinate plane** (the $xy$-plane, $xz$-plane, or $yz$-plane).
- When two of them are nonzero, the plane is paralle to a coordinate axis.
- When all of them are nonzero, the plane intersects all three axes.

A **cylinder** is a surface that consists of all lines parallel to a given line and passing through a given curve.

For example, a **right circular cylinder** can be represented by the equation $x^2 + (z - 3)^2 = 4$. This cylinder has an axis of symmetry parallel to the $y$-axis, and has radius 2.

To graph an arbitrary surface/cylinder, we look at the **traces** that the surface makes with the coordinates planes and any planes parallel to the coordinate axes. The traces are the intersection between the two. For example, if we wish to look at the traces that the cylinder makes with the $xy$-plane, we let $z = 0$ (the equation for the $xy$-plane) and see what equation that gives us on the plane. Likewise, to get the trace for any plane parallel to that $xy$-plane, we let $z = k$, and we find the intersection of the surface and the plane keeping $k$ constant.

# 2 Vectors

## 2.1 Properties of Vectors

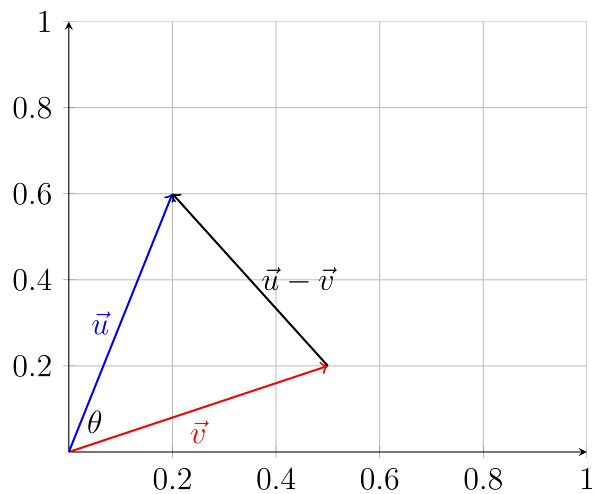
- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (Commutative)
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (Associative)
- $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$ (Identity Element under addition)
- $\vec{u} + (-\vec{u}) = \vec{0}$ (Additive Inverse)
- $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ (Distributive, vector)
- $(c + d)\vec{u} = c\vec{u} + d\vec{u}$ (Distributive, scalar)
- $c(d\vec{u}) = (cd)\vec{u}$ (Mixed Associative)
- $1\vec{u} = \vec{u}$ (Identity Element)

## 2.2 The Dot/Scalar Product

If $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$, then

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

Let $\theta$ denote the angle between $\vec{u}$ and $\vec{v}$. Using the Law of Cosines,




$$\begin{aligned} ||\vec{u} - \vec{v}||^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) &= ||\vec{u}||^2 + ||\vec{v}||^2 - 2||\vec{u}||||\vec{v}|| \cos\theta \\ ||\vec{u}||^2 - 2(\vec{u} \cdot \vec{v}) + ||\vec{v}||^2 &= ||\vec{u}||^2 + ||\vec{v}||^2 - 2||\vec{u}||||\vec{v}|| \cos\theta \\ &\rightarrow \boxed{\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}|| \cos\theta} \end{aligned}$$

This is the geometric definition of the dot product, which allows us to find the angle between two vectors easily.

## 2.3 The Cauchy-Schwarz Inequality, Triangle Inequality

**Theorem 2.1** (Cauchy-Schwarz Inequality). *If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then $|\vec{u} \cdot \vec{v}| \leq ||\vec{u}||||\vec{v}||$.*

*Proof.* Rearranging the geometric definition of the dot product, we get $\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||||\vec{v}||}$. And since $|\cos\theta| \leq 1$, we get that $\left| \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||||\vec{v}||} \right| \leq 1 \rightarrow |\vec{u} \cdot \vec{v}| \leq ||\vec{u}||||\vec{v}||$. ■

**Theorem 2.2** (Triangle Inequality). *If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then $||\vec{u} + \vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||$.*

*Proof.*

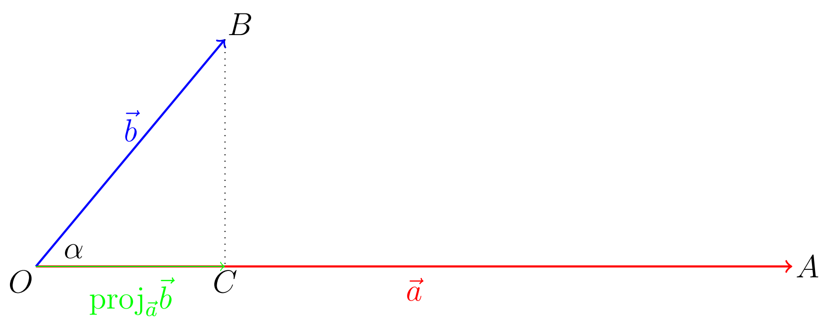
$$\begin{aligned} ||\vec{u} + \vec{v}||^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) \\ &= \vec{u} \cdot \vec{u} + 2(\vec{u} \cdot \vec{v}) + \vec{v} \cdot \vec{v} \\ &\leq ||\vec{u}||^2 + 2|\vec{u} \cdot \vec{v}| + ||\vec{v}||^2 \end{aligned}$$

By Cauchy-Schwarz:

$$\begin{aligned} ||\vec{u}+\vec{v}||^2 &\leq ||\vec{u}||^2 + 2||\vec{u}||||\vec{v}|| + ||\vec{v}||^2 \\ &= (||\vec{u}|| + ||\vec{v}||)^2 \\ ||\vec{u}+\vec{v}|| &\leq ||\vec{u}|| + ||\vec{v}|| \end{aligned}$$

■

## 2.4 Projections




$\overrightarrow{OC}$ is the vector projection of $\overrightarrow{OB}$ ($\vec{b}$) onto $\overrightarrow{OA}$ ($\vec{a}$). This is denoted as $\text{proj}_{\vec{a}}\vec{b}$. The magnitude of this projection, $||\overrightarrow{OC}||$, is denoted as $\text{comp}_{\vec{a}}\vec{b}$.

To describe $\overrightarrow{OC}$, we need its direction and magnitude.

- Direction is given by the unit vector in the direction if $\vec{a}$, which is $\frac{\vec{a}}{||\vec{a}||}$.
- Magnitude is given by:

$$\begin{aligned} ||\overrightarrow{OC}|| &= ||\vec{b}|| \cos\alpha \\ &= ||\vec{b}|| \frac{\vec{a}\cdot\vec{b}}{||\vec{a}||||\vec{b}||} = \frac{\vec{a}\cdot\vec{b}}{||\vec{a}||} \end{aligned}$$

  Therefore,

$$\overrightarrow{OC} = \frac{\vec{a}\cdot\vec{b}}{||\vec{a}||}\left(\frac{\vec{a}}{||\vec{a}||}\right) = \frac{\vec{a}\cdot\vec{b}}{||\vec{a}||^2}$$

$$\boxed{\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{||\vec{a}||^2}\vec{a}}$$

  We can also easily see that, from this, $\boxed{\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{||\vec{a}||}}$.

## 2.5 Direction Cosines

In 3-space, the direction cosines give the cosines of the angles formed between a vector and the three axes.

If $\alpha, \beta, \gamma$ are the angles that the vector $\vec{A} = \langle A_x, A_y, A_z \rangle$ makes with the $x$, $y$, and $z$ axes respectively, then:

$$\begin{aligned}\vec{A} &= \langle A_x, A_y, A_z \rangle \\ &= \langle ||\vec{A}|| \cos\alpha, ||\vec{A}|| \cos\beta, ||\vec{A}|| \cos\gamma \\ &= ||\vec{A}|| \langle \cos\alpha, \cos\beta, \cos\gamma \rangle\end{aligned}$$

## 2.6 The Cross/Vector Product

The cross product returns a vector that is orthogonal (perpendicular) to the two vectors in question.

The proof is omitted, but if $\vec{a}$ and $\vec{b}$ are vectors, then

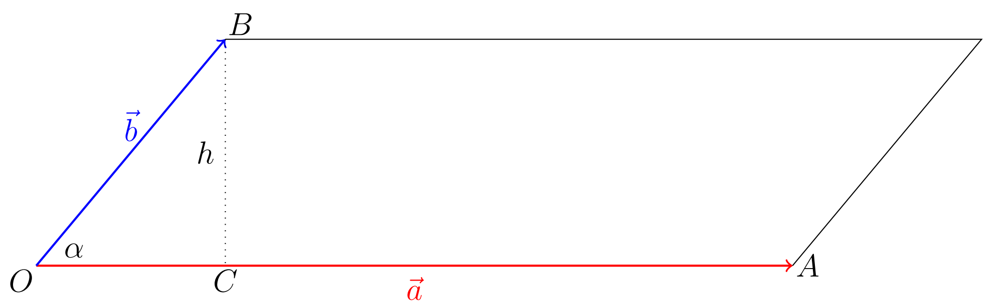
$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The magnitude of this vector is given by:

$$||\vec{a} \times \vec{b}|| = ||\vec{a}|| ||\vec{b}|| \sin\theta$$

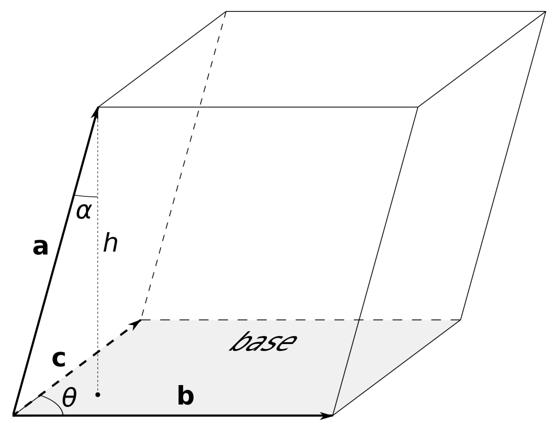
The direction of this vector is orthogonal to the vector, but which direction exactly? We have two different options. The convention is to use the right-hand rule: point your fingers towards $\vec{a}$ and sweep them towards your palm towards $\vec{b}$. Your thumb points in the direction of the product.

## 2.7 Volume of a Parallelogram




$$\begin{aligned}\text{Area} &= bh \\ b &= ||\vec{a}||, h = ||\vec{b}|| \sin\alpha \\ \text{Area} &= ||\vec{a}|| ||\vec{b}|| \sin\alpha \\ &\boxed{\text{Area} = ||\vec{a} \times \vec{b}||}\end{aligned}$$

### 2.8 Volume of a Parallelepiped




$$\text{Volume} = Bh$$

$$\boxed{\text{Volume} = (\vec{b} \times \vec{c}) \cdot \vec{a}}$$

# 3 Lines and Planes in Space

## 3.1 Lines

To find an equation for a line, we need a specific point on the plane, as well as a vector that points in the direction of the line. Representing the specific point as a vector, Let $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$ be the initial vector and $\vec{v} = \langle a, b, c \rangle$ be the vector that points in the direction of the line.

Then, we can hit all such points on the line and represent any arbitrary point as $\vec{r_0} + t\vec{v}$, where $t$ is a parameter. As $t$ varies, we will hit all the points on the line. This gives rise to three forms of the equation of a line:

- Vector form:
$$\vec{r} = \vec{r_0} + t\vec{v}$$
- Parametric form:
$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$
- Symmetric form[1]:
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

[1] This can be derived by isolating $t$ in the above equations and setting them equal to each other.

### 3.2 Planes

Let $P_0(x_0, y_0, z_0)$ be a given point on a plane, and $P(x, y, z)$ an arbitrary point on the plane. Let $\vec{n}$ be the normal[2] to the plane at $P_0$. Now since $\vec{n} \perp \overrightarrow{P_0P}$, we have that

$$\begin{aligned} \vec{n} \cdot \overrightarrow{P_0P} &= 0 \\ \langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle &= 0 \\ \boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0} \end{aligned}$$

This is the **point-normal form** of the equation of the line. We could also move the constant terms to one side to obtain $ax + by + cz = d$, the **standard form** of the equation.

# 4 Vector-Valued Functions

### 4.1 Definitions

A vector valued function $\vec{r}(t)$ is a function that takes in a scalar and returns a vector. We denote the function as such:

$$\begin{aligned} \vec{r}(t) &= f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}} \\ &= \langle f(t), g(t), h(t)\rangle \end{aligned}$$

where $f(t), g(t), h(t)$ are scalar-valued functions. We can assess the domain of $\vec{r}(t)$ by looking at the domain of the individual scalar valued functions. We can also define a limit[3] as follows:

$$\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

### 4.2 Continuity

$\vec{r}(t)$ is continuous at $t = a$ IFF:

1. $\lim_{t\to a} \vec{r}(t)$ exists
2. $\vec{r}(a)$ exists
3. $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$

### 4.3 Derivatives

We can also define the derivative of a vector-valued function as $\vec{r}\,'(t)$. The proof for finding the derivative of a vector-valued function is analagous to that of a scalar-valued function, just repeated multiple times, so it will be omitted here.

$$\vec{r}\,'(t) = f'(t)\hat{\mathbf{i}} + g'(t)\hat{\mathbf{j}} + h'(t)\hat{\mathbf{k}}$$

[2]Orthogonal/perpendicular.

[3]This limit is both additive and multiplicative with dot/cross products.

## 4.4 Unit Tangent/Normal Vectors

We define a **smooth** curve/function as one that is continuous everywhere and where $\vec{r}'(t) \neq 0$.

Here, we need two special vectors associated with tangent vectors. We can define the **unit tangent vector** as follows:

$$\vec{T}(t) = \frac{\vec{r}'(t)}{||\vec{r}'(t)||}$$

Now, suppose that $\vec{r}(t)$ is twice differentiable. Since $\vec{T}(t)$ is a unit vector,

$$\begin{aligned}
\vec{T}(t) \cdot \vec{T}(t) &= 1 \\
\frac{d}{dt}(\vec{T}(t) \cdot \vec{T}(t)) &= \frac{d}{dt}(1) \\
2[\vec{T}'(t) \cdot \vec{T}(t)] &= 0 \\
\vec{T}'(t) \cdot \vec{T}(t) &= 0
\end{aligned}$$

which means $\vec{T}'(t)$ is orthogonal to $\vec{T}(t)$. We can define a unit vector in the direction of $\vec{T}'(t)$, known as the **unit normal vector**:

$$\vec{N}(t) = \frac{\vec{T}'(t)}{||\vec{T}'(t)||}$$

## 4.5 Arc Length

The arc length along a curve $\vec{r}(t)$ from starting point $t = a$ is:

$$s(t) = \int_a^t ||\vec{r}'(t)||\ dt$$

We can parameterize a curve with arc length, meaning the input will be the length traveled along a curve and the function will return the position.

To do so, we first find the arc length function in terms of $t$. Then, we get $t$ in terms of the arc length, and substitute this expression into the curve's function.

We prove two results of this:

**Theorem 4.1.** *This technique does yield an arc length parameterization.*

*Proof.* Let $\vec{r}(t)$ be a parameterization of the curve.

$$s(t) = \int_a^t ||\vec{r}'(u)||\ du \to s'(t) = ||\vec{r}'(t)|| > 0$$

which means $s(t)$ willl always have an inverse. Call this inverse $\varphi(s) = t$.

Let $\vec{r_1}(s) = \vec{r}(\varphi(s))$; this is our new parameterization. We wish to show that $||\vec{r_1}'(s)|| = 1$.[4]

[4] If this is the case, then the formula for arc length yields the proper arc length.

$$\begin{aligned}
||\vec{r_1}\,'(s)|| &= ||\frac{d}{ds}[\vec{r}(\phi(s))]|| \\
&= ||\varphi'(s)\vec{r}\,'(\varphi(s))|| \\
&= ||\varphi'(s)||||\vec{r}\,'(\varphi(s))||
\end{aligned}$$

Do note that $\vec{r}\,'(\phi(s))$ is $\vec{r}\,'(t)$ EVALUATED at $t = \varphi(s)$. Then,

$$\begin{aligned}
\varphi'(s) = \frac{dt}{ds} &= \frac{1}{\frac{ds}{dt}} \\
&= \frac{1}{s'(t)} \\
&= \frac{1}{s'(\varphi(s))} \\
&= \frac{1}{||\vec{r}\,'(\varphi(s))||}
\end{aligned}$$

Substituting:

$$||\vec{r_1}\,'(s)|| = ||\varphi'(s)||||\vec{r}\,'(\varphi(s))|| = \frac{1}{||\vec{r}\,'(\varphi(s))||}||\vec{r}\,'(\varphi(s))|| = 1$$

■

**Theorem 4.2.** *The arc length is invariant under different parameterizations.*

*Proof.* Let $\vec{r}(t)$ and $\vec{R}(u)$ be two different parameterizations of the same curve. We will run the arc length along $a \leq t \leq b$ and $c \leq u \leq d$, so we relate the two paramters $t$ and $u$ with the equation $t = \phi(u)$. Note that $\vec{r}(\phi(u) = R(u)$.

We need to show that $\int_a^b ||\vec{r}\,'(t)||\ dt = \int_c^d ||\vec{R}\,'(u)||\ du$.

Noting that $dt = \phi'(u)\ du$, we do u-substitution:

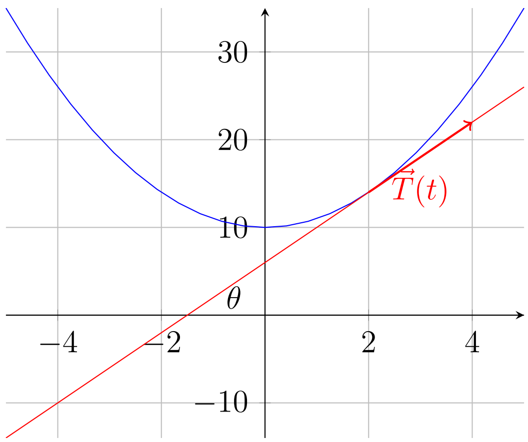
$$\begin{aligned}
\int_a^b ||\vec{r}\,'(t)||\ dt &= \int_c^d ||\vec{r}\,'(\phi(u))||\phi'(u)\ du \\
&= \int_c^d ||\phi'(u)\vec{r}\,'(\phi(u))||\ du \\
&= \int_c^d ||\frac{d}{du}\vec{r}(\phi(u))||\ du \\
&= \int_c^d ||\vec{R}\,'(u)||\ du
\end{aligned}$$

■

## 4.6 Curvature

**Curvature**, represented by the Greek letter $\kappa$ (kappa), is how fast the direction of a certain curve is changing. We will look at two definitions of curvature. We prove the result for plane curves, but the results can be extended to space curves.

1. $\kappa = \left|\frac{d\theta}{ds}\right|$
   Suppose $\vec{r}(t) = \langle x(t), y(t) \rangle$. Note that




$$\tan\theta = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}$$

And $\theta = \tan^{-1}\left(\frac{y'(t)}{x'(t)}\right)$. Since $\frac{d\theta}{ds} = \frac{d\theta}{dt}\frac{dt}{ds}$ by the Chain Rule, then,

$$\frac{d\theta}{dt} = \frac{1}{1+[\frac{y'(t)}{x'(t)}]^2} \cdot \frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right)$$

$$\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{\sqrt{[x'(t)]^2 + [y'(t)]^2}}$$

$$\frac{d\theta}{ds} = \frac{1}{1+[\frac{y'(t)}{x'(t)}]^2} \cdot \frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right) \cdot \frac{1}{\sqrt{[x'(t)]^2 + [y'(t)]^2}}$$

$$= \frac{x'y'' - y'x''}{[(x')^2 + (y')^2]^{\frac{3}{2}}}$$

$$\boxed{\kappa = \frac{||\vec{r}'(t) \times \vec{r}''(t)||}{||\vec{r}'(t)||^3}}$$

2. $\kappa = ||\frac{d\vec{T}}{ds}||$

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt}\frac{dt}{ds} = \frac{\vec{T}'(t)}{||\vec{r}'(t)||}$$

$$\left|\left|\frac{d\vec{T}}{ds}\right|\right| = \boxed{\kappa = \frac{||\vec{T}'(t)||}{||\vec{r}'(t)||}}$$

Now we show these two expressions are equivalent, namely the following theorem:

**Theorem 4.3.** *The curvature of a curve* $\vec{r}(t)$ *is given by* $\kappa = \left|\left|\frac{d\vec{T}}{ds}\right|\right| = \dfrac{||\vec{r}'(t) \times \vec{r}''(t)||}{||\vec{r}'(t)||^3}$

*Proof.* By definition, $\vec{T}(t) = \frac{\vec{r}'(t)}{||\vec{r}'(t)||}$ and $||\vec{r}'(t)|| = s'(t)$, so we can write

$$\vec{r}'(t) = s'(t)\vec{T}'(t)$$

Differentiating, we get

$$\vec{r}''(t) = s''(t)\vec{T}(t) + s'(t)\vec{T}'(t)$$

Then,

$$\begin{aligned}
\vec{r}'(t) \times \vec{r}''(t) &= s'(t)\vec{T}(t) \times [s''(t)\vec{T}(t) + s'(t)\vec{T}'(t)] \\
&= [s'(t)\vec{T}(t) \times s''(t)\vec{T}(t)] + [s'(t)\vec{T}(t) \times s'(t)\vec{T}'(t)] \\
&= \underbrace{s'(t)s''(t)[\vec{T}(t) \times \vec{T}(t)]}_{\vec{0}} + [s'(t)]^2[\vec{T}(t) \times \vec{T}'(t)] \\
&= [s'(t)]^2[\vec{T}(t) \times \vec{T}'(t)]
\end{aligned}$$

$$\begin{aligned}
||\vec{T}(t) \times \vec{T}'(t)|| &= ||\vec{T}(t)||||\vec{T}'(t)|| \sin\frac{\pi}{2} \\
&= ||\vec{T}'(t)||
\end{aligned}$$

$$\begin{aligned}
||\vec{r}(t) \times \vec{r}'(t)|| &= [s'(t)]^2||\vec{T}'(t)|| \\
&= ||\vec{r}'(t)||^2||\vec{T}'(t)|| \\
&= ||\vec{r}'(t)||^2(||\vec{r}'(t)||\kappa) \\
\kappa &= \frac{||\vec{r}'(t) \times \vec{r}''(t)||}{||\vec{r}'(t)||^3}
\end{aligned}$$

■

By considering the curve $\vec{r}(t) = \langle a\cos t, a\sin t, 0\rangle$, which is the path of a circle, you can prove that the curvature of a circle with radius $a$ is $\kappa = \frac{1}{a}$.

It is also possible to prove (the details are not shown here) that for a function $y = f(x)$, the curvature is given by:

$$\kappa(x) = \frac{|f''(x)|}{[1 + [f'(x)]^2]^{\frac{3}{2}}}$$

## 4.7 The Osculating Circle

An **osculating circle** at a point $P$ on a curve $C$:

- Has the same curvature as $C$ at $P$
- Same tangent vector/tangent line

- Same principal normal vector
- Has radius $\frac{1}{\kappa}$

## 4.8 Binormal Vector

The **binormal vector**, defined as $\vec{B} = \vec{T} \times \vec{N}$, enables us to create a moving coordinate system of a particle moving along a curve. We can define the coordinate planes of this reference frame:

- The **osculating plane**, consisting of $\vec{T}$ and $\vec{N}$
- The **normal plane**, consisting of $\vec{N}$ and $\vec{B}$
- The **rectifying plane**, consisting of $\vec{T}$ and $\vec{B}$

## 4.9 Motion in Space

Using basic knowledge of physics, the velocity vector of a particle following the path $\vec{r}(t)$ is given by $\vec{v}(t) = \vec{r}'(t)$, and the acceleration is given by $\vec{a}(t) = \vec{r}''(t)$. We can also get the speed as the magnitude of the velocity, which is $v(t) = ||\vec{r}'(t)||$.

However, it is useful to know what the tangential and normal components of acceleration are rather than solely the $x$ and $y$ components. Here, we derive the formulas for those.

Recall that

$$\vec{T}(t) = \frac{\vec{r}'(t)}{||\vec{r}'(t)||} = \frac{v(t)}{||\vec{v}(t)||} \rightarrow \vec{v}(t) = v(t)\vec{T}(t)$$

Differentiating:

$$\vec{a}(t) = v'(t)\vec{T}(t) + v(t)\vec{T}'(t)$$

Recall that $\vec{N}(t) = \frac{\vec{T}'(t)}{||\vec{T}'(t)||} \rightarrow \vec{T}'(t) = ||\vec{T}'(t)||\vec{N}(t)$, and that $\kappa = \frac{||\vec{T}'(t)||}{||\vec{r}'(t)||} \rightarrow ||\vec{T}'(t)|| = \kappa v(t)$

Substituting:

$$\vec{a}(t) = v'(t)\vec{T}(t) + \kappa[v(t)]^2\vec{N}(t)$$

The tangential component is given by $a_T = v'(t)$, and the normal component is given by $a_N = \kappa[v(t)]^2$.

There are other formulas for these:

$$a_T = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{||\vec{r}'(t)||}$$
$$a_N = \frac{||\vec{r}'(t) \times \vec{r}''(t)||}{||\vec{r}'(t)||}$$

I don't have the proofs of these on me now; they're on a handout somewhere that I'm trying to find.

# 5 Multivariate Functions

A function of $n$-variables $x_1, x_2, \ldots, x_n$ is a rule that maps/assigns to each $n$-tuple in a set $D \subseteq \mathbb{R}^n$ a unique value $z = f(x_1, x_2, \ldots, x_n) \in \mathbb{R}$. In other words, $f : \mathbb{R}^n \to \mathbb{R}$.

We introduce some terminology to discuss sets in Euclidean space:

- $(x_0, y_0)$ is an **interior point** of a region $\mathcal{R}$ if it is the center of a disk contained entirely within $\mathcal{R}$.
- $(x_0, y_0)$ is a **boundary point** of a region $\mathcal{R}$ if all such disks centered at $(x_0, y_0)$ contain both exterior and interior points.
- An **open region** consists solely of interior points.
- A **closed region** consists of both interior and boundary points.
- A **bounded region** can be contained within a disk of finite radius.
- An **unbounded region** cannot be contained within a disk of finite radius.

## 5.1 Function Notation

There are 2 forms:

- Explicit form: $z = f(x_1, x_2, \ldots, x_n)$
- Implicit form: $F(x_1, x_2, \ldots, x_n, z) = 0$

The graph of a multivariate function of 2 variables $z = f(x, y)$ is called a **surface** and consists of all points $(x, y, z)$ for which $z = f(x, y)$ and $(x, y)$ is in the domain of $f$. We can extend this to functions of 3-variables; however, we can't physically draw the graph as it would involve 4-dimensional space...

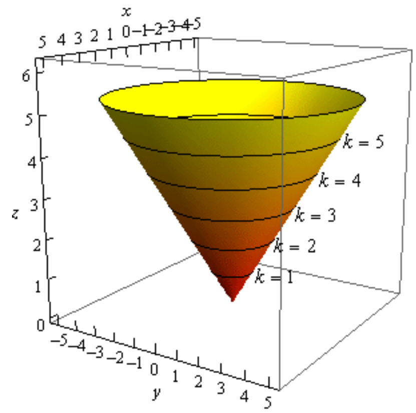
## 5.2 Level Curves

A **level curve** is the set of all points $(x, y)$ in the domain of $f$ such that $f(x, y) = k$. If we envision the surface on a graph, the level curves would give all the points at the same elevation from the $xy$-plane. By combining multiple level curves, we can create a contour map of a function's graph.

Note that two level curves will never intersect; if they intersected, say at a point $(x_0, y_0)$, then $f(x_0, y_0)$ would take on two different values, a contradiction.

## 5.3 Limits of Multivariate Functions

Previously when we defined limits for single-variable functions, $x$ would approach a certain value $a$ along the $x$-axis either from the positive direction or the negative direction, so we could check both to make sure that they exist and are equal. However, in a multivariate function, there are infinitely many ways to approach a point $(x_0, y_0)$, via different paths.




Therefore, we cannot take limits as easily as before; there is no special L'Hopital's Rule for this.

For example, consider:

$$\lim_{(x,y)\to(0,0)} \frac{-3xy}{x^2+y^2}$$

If we try and plug in $(0,0)$ directly, we get $\frac{0}{0}$, which is not possible. So instead, we choose a path that we can take the limit on.

We can first let $y=0$; then the limit becomes

$$\lim_{(x,0)\to(0,0)} \frac{-3x(0)}{x^2+(0)^2} = \lim_{(x,0)\to(0,0)} 0 = 0$$

However, if we let $y=x$:

$$\lim_{(x,x)\to(0,0)} \frac{-3x^2}{2x^2} = \lim_{(x,x)\to(0,0)} -\frac{3}{2} = \frac{3}{2}$$

But we have obtained two different values for the limit. Therefore, this limit does not exist.

One other way you can compute limits in two dimensions is to use polar coordinates. Note that as $(x,y)\to(0,0)$, then $r\to 0$, so we can convert our initial problem into a single variable limit. In our example:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{-3xy}{x^2+y^2} &= \lim_{r\to 0} \frac{-3(r\cos\theta)(r\sin\theta)}{(r\cos\theta)^2+(r\sin\theta)^2} \\
&= \lim_{r\to 0} \frac{-3r^2\cos\theta\sin\theta}{r^2(\cos^2\theta+\sin^2\theta)} \\
&= \lim_{r\to 0} -\frac{3}{2}\sin 2\theta
\end{aligned}$$

But this limit does not exist, since we know nothing about the value of $\theta$ or where it's approaching.

## 5.4 Partial Derivatives

Now, we investigate rates of changes in multivariate functions. In a function of two variables $z = f(x, y)$, it is often useful to see how the function behaves in the $x$ and the $y$ directions. This can be accomplished by holding one of the variables constant, and allowing the other one to vary.

If we hold $y$ constant, say at $y = y_0$, then $z = f(x, y_0)$ is now a function of one variable, and we can analyze the rate of change with respect to $x$ by taking $\frac{d}{dx}$. In multivariate functions, these are known as **partial derivatives**, denoted and defined as:

$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$
$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Note that the partial derivative (in this case, w.r.t. $x$) actually give us the slope of the tangent line at $(a, b)$ to the curve of intersection of the surface $z = f(x, y)$ with the plane $y = b$.

We can denote higher-order partial derivatives (commonly known as "partials") as:

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}(x, y) \text{ (second partial)} \quad \text{and} \quad \frac{\partial^2 f}{\partial y\, \partial x} = f_{xy}(x, y) \text{ (partial w.r.t. } x\text{, then } y\text{)}$$

## 5.5 Equality of Mixed Partials Theorem/Clairaut's Theorem

It turns out that if the partials are continuous within $f$, the following powerful theorem is true:

**Theorem 5.1** (Equality of Mixed Partials Theorem/Clairaut's Theorem)**.** *If $f(x, y)$ and its partial derivatives $f_x$, $f_y$, $f_{xy}$, $f_{yx}$ are defined in a region containing $(a, b)$ and all are continuous in said region, then $f_{xy}(a, b) = f_{yx}(a, b)$.*

*Proof.* Pick $h, k > 0$ such that $(a + h, b)$, $(a, b + k)$, and $(a + h, b + k)$ are all within the same said region. Our goal is to show that

$$f_{xy}(a, b) = f_{yx}(a, b) = \lim_{hk \to 0} \frac{f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b)}{hk}$$

We will prove this for $f_{xy}(a, b)$; the proof for $f_{yx}(a, b)$ is analogous and will be left as an exercise.

Let $g(x) = f(x, b + k) - f(x, b)$. Notice that $g$ is continuous and differentiable on $(a, a + k)$, and that

$$g(a + h) - g(a) = f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b)$$

Applying the Mean Value Theorem on $g$, we get that $\exists c \in (a, a + k)$ such that

$$g'(c) = \frac{g(a + h) - g(a)}{h} \to g(a + h) - g(a) = hg'(c)$$

But $g'(x) = f_x(x, b+k) - f_x(x, b)$ from how we defined $g$, so

$$g(a+h) - g(a) = h[f_x(c, b+k) - f_x(c, b)]$$

Now, we apply the Mean Value Theorem to $f_x$ (w.r.t $y$) on $(b, b+k)$: $\exists e \in (b, b+k)$ such that

$$f_{xy}(c, e) = \frac{f_x(c, b+k) - f_x(c, b)}{k} \to f_x(c, b+k) - f_x(c, b) = k f_{xy}(c, e)$$

Substituting:

$$\begin{aligned}
hk f_{xy}(c, e) &= g(a+h) - g(a) \\
f_{xy}(c, e) &= \frac{g(a+h) - g(a)}{hk} \\
&= \frac{f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)}{hk}
\end{aligned}$$

And

$$\begin{aligned}
\lim_{hk \to 0} f_{xy}(c, e) &= \lim_{(c,e) \to (a,b)} f_{xy}(c, e) \\
&= f_{xy}(a, b) \\
f_{xy}(a, b) &= \lim_{hk \to 0} \frac{f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)}{hk}
\end{aligned}$$

■

## 5.6 Differentiability

For functions of one variable, continuity doesn't imply differentiability, but differentiability implies continuity.

In two or more variables:

1. $$\begin{aligned} \text{Existence of } f_x, f_y &\not\Rightarrow f \text{ is continuous} \\ &\not\Rightarrow f \text{ is differentiable} \end{aligned}$$

2. $$\begin{aligned} \text{Continuity of } f_x, f_y &\Rightarrow f \text{ is continuous} \\ &\Rightarrow f \text{ is differentiable} \end{aligned}$$

3. $f$ is differentiable $\not\Rightarrow$ $f$ is continuous

In one variable, the necessary condition for a function $f$ to be differentiable at $x_0$ if we can write $\Delta f$ as:

$$\Delta f = f(x + \Delta x) - f(x) = f'(x_0)\Delta x + \varepsilon \Delta x$$

where as $\Delta x \to 0$, $\varepsilon \to 0$.

In two variables:

- Suppose $f(x, y)$ is defined at each point in a disk centered at $(x_0, y_0)$ and that contains $(x + \Delta x, y + \Delta y)$.

- Then we say $f(x,y)$ is differentiable at $(x_0, y_0)$ if $\Delta f$ can be written as:

$$\Delta f = f(x+\Delta x, y+\Delta y) - f(x,y) = f_x(x,y)\Delta x + f_y(x,y)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$.

To illustrate this tedious definition, we introduce an example: show that $z = f(x,y) = x^2 y - 1$ is differentiable.

Note that $f_x = 2xy$ and $f_y = x^2$. We compute:

$$\begin{aligned}
\Delta f &= f(x+\Delta x, y+\Delta y) - f(x,y) \\
&= (x+\Delta x)^2(y+\Delta y) - 1 - (x^2 y - 1) \\
&= x^2 y + 2xy\Delta x + y(\Delta x)^2 + x^2 \Delta y + 2x\Delta x \Delta y + \Delta y(\Delta x)^2 - x^2 y \\
&= (2xy)\Delta x + (x^2)\Delta y + (y\Delta x)\Delta x + (2x\Delta x + (\Delta x)^2)\Delta y
\end{aligned}$$

By taking $\varepsilon_1 = y\Delta x$, $\varepsilon_2 = 2x\Delta x + (\Delta x)^2$, we have satisfied our condition for differentiability.

## 5.7 Tangent Planes

We wish to investigate tangent planes to a surface $z = f(x,y)$.

Let the plane be tangent at $P_0(x_0, y_0, z_0)$, and $\langle A, B, C \rangle$ be the normal to the plane. Let $P(x,y,z)$ be an arbitrary point in the plane. Then $\overrightarrow{P_0 P} = \langle x - x_0, y - y_0, z - z_0 \rangle$.

Note that $\langle A, B, C \rangle \cdot \overrightarrow{P_0 P} = 0$, and therefore, $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ (or $-\frac{A}{C}(x - x_0) - \frac{B}{C}(y - y_0) - (z - z_0) = 0$).

Letting $x = x_0$, we get the tangent line to the surface on the plane $x = x_0$, which has slope $f_y(x_0, y_0)$, so $\frac{-B}{C} = f_y(x_0, y_0)$. By an analagous argument, $\frac{-A}{C} = f_x(x_0, y_0)$.

Our equation for the tangent plane then becomes:

$$\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

## 5.8 Approximations

Just like we can use tangent lines to approximate values of a single-variable function, we can approximate values of a multivariate function by using a tangent plane.

For some $(x_0, y_0)$ very close to a point $(a, b)$:

$$\boxed{f(x_0, y_0) \approx f(a,b) + f_x(a,b)(x_0 - a) + f_y(a,b)(y_0 - b)}$$

Notice how the expression gives you the value if the point were on the tangent plane to $(a, b)$.

We can also estimate the change in a function given the change in $x$ and $y$. Looking at the equation for a tangent plane, we can set the differential changes $dz = z - f(a,b)$, $dx = x - a$ and $dy = y - a$ to arrive at an expression for the function value change from $(a,b) \to (x_0, y_0)$:

$$\boxed{\Delta z \approx dz = f_x(a,b)dx + f_y(a,b)dy}$$

## 5.9 Chain Rule

We state the general case of the Chain Rule here:

**Theorem 5.2** (Multivariate Chain Rule)**.** *Given a differentiable function $z = f(x_1, x_2, \dots, x_n)$ where $x_i = x_i(t_1, t_2, \dots, t_m)$ for $i \in [1, n]$, the partial of $z$ with respect to $t_j$ with $j \in [1, m]$ is given by:*

$$\frac{\partial z}{\partial t_j} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_j} = \sum_{i=1}^{n}\left(\frac{\partial z}{\partial x_i}\frac{\partial x_i}{\partial t_j}\right)$$

*Proof.* We can extend our definition of differentiability to $n$ variables; i.e. if $z = f(x_1, x_2, \dots, x_n)$ then $f$ is differentiable if we can write $\Delta z$ as:

$$\Delta z = \sum_{i=1}^{n} \frac{\partial z}{\partial x_i}\Delta x_i + \sum_{i=1}^{n} \varepsilon_i \Delta x_i$$

where $\varepsilon_i \to 0$ as $\Delta x_i \to 0$. Then:

$$\begin{aligned}
\frac{\Delta z}{\Delta t_j} &= \sum_{i=1}^{n} \frac{\partial z}{\partial x_i}\frac{\Delta x_i}{\Delta t_j} + \sum_{i=1}^{n} \varepsilon_i \frac{\Delta x_i}{\Delta t_j} \\
\lim_{\Delta t_j \to 0} \frac{\Delta z}{\Delta t_j} &= \sum_{i=1}^{n}\left(\frac{\partial z}{\partial x_i} \lim_{\Delta t_j \to 0}\frac{\Delta x_i}{\Delta t_j}\right) + \sum_{i=1}^{n}\left(\lim_{\Delta t_j \to 0}\varepsilon_i \lim_{\Delta t_j \to 0}\frac{\Delta x_i}{\Delta t_j}\right) \\
\frac{\partial z}{\partial t_j} &= \sum_{i=1}^{n}\left(\frac{\partial z}{\partial x_i}\frac{\partial x_i}{\partial t_j}\right) + \sum_{i=1}^{n}\left(0 \cdot \frac{\partial x_i}{\partial t_j}\right) \\
&\boxed{\frac{\partial z}{\partial t_j} = \sum_{i=1}^{n}\left(\frac{\partial z}{\partial x_i}\frac{\partial x_i}{\partial t_j}\right) = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_j}}
\end{aligned}$$

∎

## 5.10 Implicit Function Theorem

Note that we can define a function $z = f(x, y)$ implicitly as $F(x, y, f(x, y)) = 0$. This is useful whenever we cannot solve for the value of $f$ directly.

We formulate a theorem that gives the value of a function's partial derivative if it is defined implicitly.

**Theorem 5.3** (Implicit Function Theorem)**.** *If $z = f(x, y)$ is a function that is given implicitly by the function $F(x, y, z) = F(x, y, f(x, y)) = 0$, then*

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \textit{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

*Proof.*

$$
\begin{aligned}
F(x,y,z) &= 0 \\
\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} &= 0 \\
\frac{\partial F}{\partial x}\cdot 1 + \frac{\partial F}{\partial y}\cdot 0 + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} &= 0 \\
\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} &= 0 \\
\frac{\partial z}{\partial x} &= -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z}
\end{aligned}
$$

Analogously,

$$
\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z}
$$

■

## 5.11 Directional Derivatives and the Gradient

Up until now, we have only explored rates of changes in the directions of $x$ and $y$. In this section, we explore rates of change in any arbitrary direction.

We specify a certain direction with a unit vector $\vec{u} = \langle a, b \rangle$ in the $xy$-plane. Let $P(x_0, y_0, z_0)$ be a specific point on the surface $z = f(x,y)$ and let $P(x_0, y_0, 0)$ be its projection onto the $xy$-plane. Choose an arbitrary point $Q(x,y,z)$ such that its $xy$ projection $Q'(x,y,0)$ forms the vector $\overrightarrow{P'Q'} = h\vec{u} = \langle x - x_0, y - y_0, 0 \rangle = \langle ha, hb \rangle$.

Our average rate of change from $P$ to $Q$ is given by:

$$
\text{Average rate of change} = \frac{z - z_0}{h} = \frac{f(x,y) - f(x_0,y_0)}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}
$$

To find our instantaneous rate of change, we take a limit:

$$
\text{Instantaneous rate of change} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}
$$

This instantaneous rate of change gives our directional derivative in the direction of $\vec{u}$, denoted $D_{\vec{u}} f(x_0, y_0)$.

If we look at $\frac{df}{dh}$:

$$
\begin{aligned}
\frac{df}{dh} &= \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} \\
&= \frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b \\
&= \langle f_x, f_y \rangle \cdot \langle a, b \rangle
\end{aligned}
$$

The vector $\langle f_x, f_y \rangle$ is called the **gradient** of $f$, denoted by the symbol $\vec{\nabla}$, and has many useful properties we will explore. Generalizing for a function $f(x_1, x_2, \ldots, x_n)$:

$$\vec{\nabla} f(x, y) = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle$$

Therefore,

$$\boxed{D_{\vec{u}} f(x, y) = \vec{\nabla} f(x, y) \cdot \vec{u}}$$

Let us look at two properties of the gradient.

1. Where does $f(x, y)$ increase most rapidly? It turns out the direction in which $f(x, y)$ increases most rapidly in given by the gradient vector. To see this, we write

$$D_{\vec{u}} f(x, y) = \vec{\nabla} f(x, y) \cdot \vec{u} = ||\vec{\nabla} f(x, y)||||\vec{u}|| \cos\theta = ||\vec{\nabla} f(x, y)|| \cos\theta$$

   If $\vec{\nabla} f(x, y) \neq \vec{0}$, then the rate of change is maximal when $\cos\theta = 1$, or when $\vec{u}$ and $\vec{\nabla} f(x, y)$ are in the same direction. Thus, the gradient gives the direction of maximal increase.
   On the other hand, the rate of change is minimal when $\cos\theta = -1$, or when $\vec{u}$ and $\vec{\nabla} f(x, y)$ are in opposite directions, so $-\vec{\nabla} f(x, y)$ gives the minimal increase.

2. Let us investigate the relation between level curves of a function and the gradient. Given a level curve $f(x, y) = k$, the function value is constant. Therefore, at any point on the curve the directional derivative is 0.

   Also note that the unit tangent vector $\vec{T}$ is given by $\vec{u}$. Since $D_{\vec{u}} f(x, y) = \vec{\nabla} f(x, y) \cdot \vec{u} = 0$, we get that $\vec{\nabla} f(x, y)$ and $\vec{u}$ are orthogonal, and therefore the gradient is normal to the level curve. Formally, $\vec{\nabla} f(x_0, y_0)$ is normal to the level curve through $(x_0, y_0)$.

   Note that for functions of three variables, the gradient is orthogonal to the level surface rather than the level curve.

### 5.12 Maxima/Minima

## 6 Iterated Integrals

### 6.1 Double Integrals

### 6.2 Reversing the Order of Integration

### 6.3 Triple Integrals

### 6.4 Change of Variables

## 7 Vector Calculus

### 7.1 Line Integrals

### 7.2 Conservative Vector Fields

### 7.3 Green's Theorem

### 7.4 Surface Integrals

### 7.5 Divergence Theorem

### 7.6 Stokes' Theorem