Multivariate Calculus

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Introduction

These are the notes from the course I took at Stuyvesant High School, Fall 2015. Note: these are not the exact notes that were taken; these notes skip some properties that should be obvious if you took precalculus.

1 3 Dimensional Space

A **plane** can be represented by the equation $Ax + By + Cz + D = 0$.

- When one of A, B, C is nonzero, the plane is parallel to a **coordinate plane** (the xy-plane, xz-plane, or yz-plane).
- When two of them are nonzero, the plane is paralle to a coordinate axis.
- When all of them are nonzero, the plane intersects all three axes.

A **cylinder** is a surface that consists of all lines parallel to a given line and passing through a given curve.

For example, a **right circular cylinder** can be represented by the equation $x^2 + (z - 3)^2 = 4$. This cylinder has an axis of symmetry parallel to the y-axis, and has radius 2.

To graph an arbitrary surface/cylinder, we look at the **traces** that the surface makes with the coordinates planes and any planes parallel to the coordinate axes. The traces are the intersection between the two. For example, if we wish to look at the traces that the cylinder makes with the xy-plane, we let $z = 0$ (the equation for the xy-plane) and see what equation that gives us on the plane. Likewise, to get the trace for any plane parallel to that xy-plane, we let $z = k$, and we find the intersection of the surface and the plane keeping k constant.

2 Vectors

2.1 Properties of Vectors

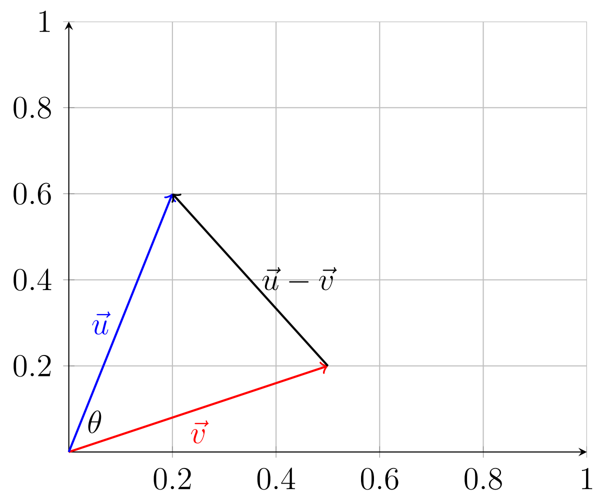
- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (Commutative)
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (Associative)
- $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$ (Identity Element under addition)
- $\vec{u} + (-\vec{u}) = \vec{0}$ (Additive Inverse)
- $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ (Distributive, vector)
- $(c + d)\vec{u} = c\vec{u} + d\vec{u}$ (Distributive, scalar)
- $c(d\vec{u}) = (cd)\vec{u}$ (Mixed Associative)
- $1\vec{u} = \vec{u}$ (Identity Element)

2.2 The Dot/Scalar Product

If $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$, then

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

Let θ denote the angle between $\vec{u}$ and $\vec{v}$. Using the Law of Cosines,



$$\begin{aligned} ||\vec{u} - \vec{v}||^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) &= ||\vec{u}||^2 + ||\vec{v}||^2 - 2||\vec{u}||||\vec{v}|| \cos\theta \\ ||\vec{u}||^2 - 2(\vec{u} \cdot \vec{v}) + ||\vec{v}||^2 &= ||\vec{u}||^2 + ||\vec{v}||^2 - 2||\vec{u}||||\vec{v}|| \cos\theta \\ &\to \boxed{\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}|| \cos\theta} \end{aligned}$$

This is the geometric definition of the dot product, which allows us to find the angle between two vectors easily.

2.3 The Cauchy-Schwarz Inequality, Triangle Inequality

Theorem 2.1 (Cauchy-Schwarz Inequality)**.** *If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then $|\vec{u} \cdot \vec{v}| \leq ||\vec{u}||||\vec{v}||$.*

Proof. Rearranging the geometric definition of the dot product, we get $\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||||\vec{v}||}$. And since $|\cos\theta| \leq 1$, we get that $\left|\frac{\vec{u} \cdot \vec{v}}{||\vec{u}||||\vec{v}||}\right| \leq 1 \to |\vec{u} \cdot \vec{v}| \leq ||\vec{u}||||\vec{v}||$. ■

Theorem 2.2 (Triangle Inequality)**.** *If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then $||\vec{u}+\vec{v}|| \leq ||\vec{u}||+||\vec{v}||$.*

Proof.

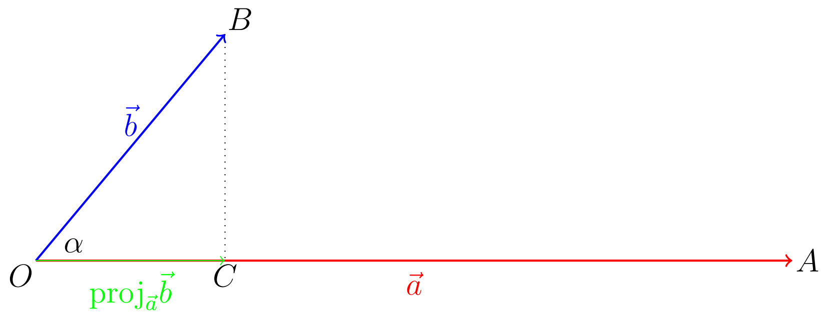
$$\begin{aligned} ||\vec{u} + \vec{v}||^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) \\ &= \vec{u} \cdot \vec{u} + 2(\vec{u} \cdot \vec{v}) + \vec{v} \cdot \vec{v} \\ &\leq ||\vec{u}||^2 + 2|\vec{u} \cdot \vec{v}| + ||\vec{v}||^2 \end{aligned}$$

By Cauchy-Schwarz:

$$\begin{aligned}||\vec{u}+\vec{v}||^2 &\leq ||\vec{u}||^2 + 2||\vec{u}||||\vec{v}|| + ||\vec{v}||^2 \\ &= (||\vec{u}|| + ||\vec{v}||)^2 \\ ||\vec{u}+\vec{v}|| &\leq ||\vec{u}|| + ||\vec{v}||\end{aligned}$$

$\blacksquare$

2.4 Projections



$\overrightarrow{OC}$ is the vector projection of $\overrightarrow{OB}$ ($\vec{b}$) onto $\overrightarrow{OA}$ ($\vec{a}$). This is denoted as $\text{proj}_{\vec{a}}\vec{b}$. The magnitude of this projection, $||\overrightarrow{OC}||$, is denoted as $\text{comp}_{\vec{a}}\vec{b}$.

To describe $\overrightarrow{OC}$, we need its direction and magnitude.

- Direction is given by the unit vector in the direction if $\vec{a}$, which is $\frac{\vec{a}}{||\vec{a}||}$.
- Magnitude is given by:

$$\begin{aligned}||\overrightarrow{OC}|| &= ||\vec{b}|| \cos\alpha \\ &= ||\vec{b}|| \frac{\vec{a}\cdot\vec{b}}{||\vec{a}||||\vec{b}||} = \frac{\vec{a}\cdot\vec{b}}{||\vec{a}||}\end{aligned}$$

Therefore,

$$\overrightarrow{OC} = \frac{\vec{a}\cdot\vec{b}}{||\vec{a}||}\left(\frac{\vec{a}}{||\vec{a}||}\right) = \frac{\vec{a}\cdot\vec{b}}{||\vec{a}||^2}$$

$$\boxed{\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{||\vec{a}||^2}\vec{a}}$$

We can also easily see that, from this, $\boxed{\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{||\vec{a}||}}$.

2.5 Direction Cosines

In 3-space, the direction cosines give the cosines of the angles formed between a vector and the three axes.

If α, β, γ are the angles that the vector $\vec{A} = \langle A_x, A_y, A_z \rangle$ makes with the x, y, and z axes respectively, then:

$$\begin{aligned}\vec{A} &= \langle A_x, A_y, A_z \rangle \\ &= \langle ||\vec{A}|| \cos\alpha, ||\vec{A}|| \cos\beta, ||\vec{A}|| \cos\gamma \\ &= ||\vec{A}|| \langle \cos\alpha, \cos\beta, \cos\gamma \rangle\end{aligned}$$

2.6 The Cross/Vector Product

The cross product returns a vector that is orthogonal (perpendicular) to the two vectors in question.

The proof is omitted, but if $\vec{a}$ and $\vec{b}$ are vectors, then

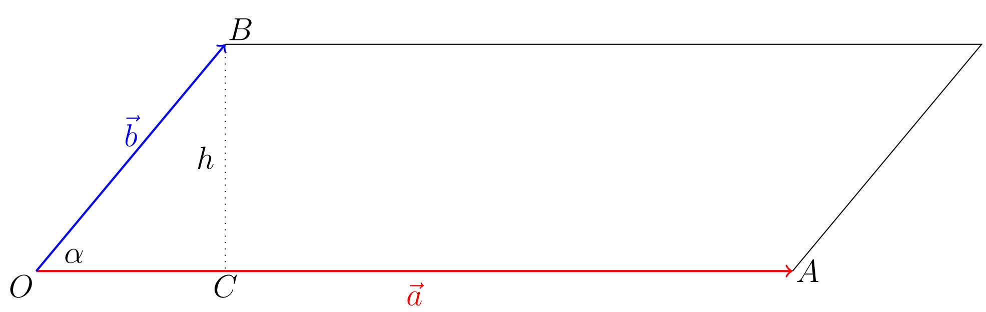
$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The magnitude of this vector is given by:

$$||\vec{a} \times \vec{b}|| = ||\vec{a}|| ||\vec{b}|| \sin\theta$$

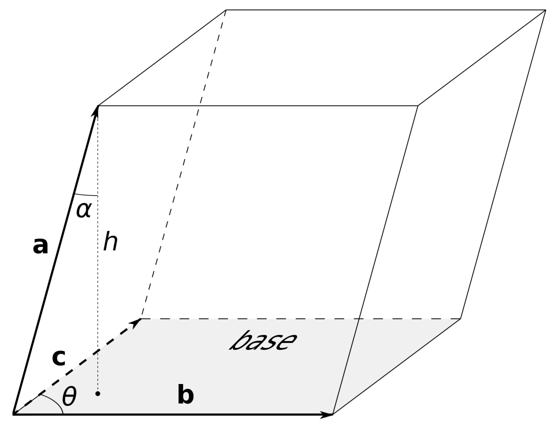
The direction of this vector is orthogonal to the vector, but which direction exactly? We have two different options. The convention is to use the right-hand rule: point your fingers towards $\vec{a}$ and sweep them towards your palm towards $\vec{b}$. Your thumb points in the direction of the product.

2.7 Volume of a Parallelogram



$$\begin{aligned}\text{Area} &= bh \\ b &= ||\vec{a}||, h = ||\vec{b}|| \sin\alpha \\ \text{Area} &= ||\vec{a}|| ||\vec{b}|| \sin\alpha \\ &\boxed{\text{Area} = ||\vec{a} \times \vec{b}||}\end{aligned}$$

2.8 Volume of a Parallelepiped



$$\text{Volume} = Bh$$

$$\boxed{\text{Volume} = (\vec{b} \times \vec{c}) \cdot \vec{a}}$$

3 Lines and Planes in Space

3.1 Lines

To find an equation for a line, we need a specific point on the plane, as well as a vector that points in the direction of the line. Representing the specific point as a vector, Let $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$ be the initial vector and $\vec{v} = \langle a, b, c \rangle$ be the vector that points in the direction of the line.

Then, we can hit all such points on the line and represent any arbitrary point as $\vec{r_0} + t\vec{v}$, where t is a parameter. As t varies, we will hit all the points on the line. This gives rise to three forms of the equation of a line:

- Vector form:
$$\vec{r} = \vec{r_0} + t\vec{v}$$
- Parametric form:
$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$
- Symmetric form[1]:
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

[1]This can be derived by isolating t in the above equations and setting them equal to each other.

3.2 Planes

Let $P_0(x_0, y_0, z_0)$ be a given point on a plane, and $P(x, y, z)$ an arbitrary point on the plane. Let $\vec{n}$ be the normal[2] to the plane at P_0. Now since $\vec{n} \perp \overrightarrow{P_0P}$, we have that

$$\begin{aligned}\vec{n} \cdot \overrightarrow{P_0P} &= 0\\ \langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle &= 0\\ \boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}\end{aligned}$$

This is the **point-normal form** of the equation of the line. We could also move the constant terms to one side to obtain $ax + by + cz = d$, the **standard form** of the equation.

4 Vector-Valued Functions

4.1 Definitions

A vector valued function $\vec{r}(t)$ is a function that takes in a scalar and returns a vector. We denote the function as such:

$$\begin{aligned}\vec{r}(t) &= f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}\\ &= \langle f(t), g(t), h(t)\rangle\end{aligned}$$

where $f(t), g(t), h(t)$ are scalar-valued functions. We can assess the domain of $\vec{r}(t)$ by looking at the domain of the individual scalar valued functions. We can also define a limit[3] as follows:

$$\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right\rangle$$

4.2 Continuity

$\vec{r}(t)$ is continuous at $t = a$ IFF:

1. $\lim_{t\to a} \vec{r}(t)$ exists
2. $\vec{r}(a)$ exists
3. $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$

4.3 Derivatives

We can also define the derivative of a vector-valued function as $\vec{r}\,'(t)$. The proof for finding the derivative of a vector-valued function is analagous to that of a scalar-valued function, just repeated multiple times, so it will be omitted here.

$$\vec{r}\,'(t) = f'(t)\hat{\mathbf{i}} + g'(t)\hat{\mathbf{j}} + h'(t)\hat{\mathbf{k}}$$

[2] Orthogonal/perpendicular.

[3] This limit is both additive and multiplicative with dot/cross products.

4.4 Unit Tangent/Normal Vectors

We define a **smooth** curve/function as one that is continuous everywhere and where $\vec{r}'(t) \neq 0$.

Here, we need two special vectors associated with tangent vectors. We can define the **unit tangent vector** as follows:

$$\vec{T}(t) = \frac{\vec{r}'(t)}{||\vec{r}'(t)||}$$

Now, suppose that $\vec{r}(t)$ is twice differentiable. Since $\vec{T}(t)$ is a unit vector,

$$\begin{aligned} \vec{T}(t) \cdot \vec{T}(t) &= 1 \\ \frac{d}{dt}(\vec{T}(t) \cdot \vec{T}(t)) &= \frac{d}{dt}(1) \\ 2[\vec{T}'(t) \cdot \vec{T}(t)] &= 0 \\ \vec{T}'(t) \cdot \vec{T}(t) &= 0 \end{aligned}$$

which means $\vec{T}'(t)$ is orthogonal to $\vec{T}(t)$. We can define a unit vector in the direction of $\vec{T}'(t)$, known as the **unit normal vector**:

$$\vec{N}(t) = \frac{\vec{T}'(t)}{||\vec{T}'(t)||}$$

4.5 Arc Length

The arc length along a curve $\vec{r}(t)$ from starting point $t = a$ is:

$$s(t) = \int_a^t ||\vec{r}'(t)||\ dt$$

We can parameterize a curve with arc length, meaning the input will be the length traveled along a curve and the function will return the position.

To do so, we first find the arc length function in terms of t. Then, we get t in terms of the arc length, and substitute this expression into the curve's function.

We prove two results of this:

Theorem 4.1. *This technique does yield an arc length parameterization.*

Proof. Let $\vec{r}(t)$ be a parameterization of the curve.

$$s(t) = \int_a^t ||\vec{r}'(u)||\ du \rightarrow s'(t) = ||\vec{r}'(t)|| > 0$$

which means $s(t)$ willl always have an inverse. Call this inverse $\varphi(s) = t$.

Let $\vec{r_1}(s) = \vec{r}(\varphi(s))$; this is our new parameterization. We wish to show that $||\vec{r_1}'(s)|| = 1$.[4]

[4] If this is the case, then the formula for arc length yields the proper arc length.

$$\begin{aligned}||\vec{r_1}'(s)|| &= ||\frac{d}{ds}[\vec{r}(\phi(s))]|| \\ &= ||\varphi'(s)\vec{r}'(\varphi(s))|| \\ &= ||\varphi'(s)||||\vec{r}'(\varphi(s))||\end{aligned}$$

Do note that $\vec{r}'(\phi(s))$ is $\vec{r}'(t)$ EVALUATED at $t = \varphi(s)$. Then,

$$\begin{aligned}\varphi'(s) = \frac{dt}{ds} &= \frac{1}{\frac{ds}{dt}} \\ &= \frac{1}{s'(t)} \\ &= \frac{1}{s'(\varphi(s))} \\ &= \frac{1}{||\vec{r}'(\varphi(s))||}\end{aligned}$$

Substituting:

$$||\vec{r_1}'(s)|| = ||\varphi'(s)||||\vec{r}'(\varphi(s))|| = \frac{1}{||\vec{r}'(\varphi(s))||}||\vec{r}'(\varphi(s))|| = 1$$

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Theorem 4.2. *The arc length is invariant under different parameterizations.*

Proof. Let $\vec{r}(t)$ and $\vec{R}(u)$ be two different parameterizations of the same curve. We will run the arc length along $a \leq t \leq b$ and $c \leq u \leq d$, so we relate the two paramters t and u with the equation $t = \phi(u)$. Note that $\vec{r}(\phi(u) = R(u)$.

We need to show that $\int_a^b ||\vec{r}'(t)||\ dt = \int_c^d ||\vec{R}'(u)||\ du$.

Noting that $dt = \phi'(u)\ du$, we do u-substitution:

$$\begin{aligned}\int_a^b ||\vec{r}'(t)||\ dt &= \int_c^d ||\vec{r}'(\phi(u))||\phi'(u)\ du \\ &= \int_c^d ||\phi'(u)\vec{r}'(\phi(u))||\ du \\ &= \int_c^d ||\frac{d}{du}\vec{r}(\phi(u))||\ du \\ &= \int_c^d ||\vec{R}'(u)||\ du\end{aligned}$$

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