Multivariate Calculus

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Introduction

These are the notes from the course I took at Stuyvesant High School, Fall 2015. Note: these are not the exact notes that were taken; these notes skip some properties that should be obvious if you took precalculus.

1   3 Dimensional Space

A **plane** can be represented by the equation $Ax + By + Cz + D = 0$.

- When one of A, B, C is nonzero, the plane is parallel to a **coordinate plane** (the xy-plane, xz-plane, or yz-plane).

- When two of them are nonzero, the plane is paralle to a coordinate axis.

- When all of them are nonzero, the plane intersects all three axes.

A **cylinder** is a surface that consists of all lines parallel to a given line and passing through a given curve.

For example, a **right circular cylinder** can be represented by the equation $x^2 + (z - 3)^2 = 4$. This cylinder has an axis of symmetry parallel to the y-axis, and has radius 2.

To graph an arbitrary surface/cylinder, we look at the **traces** that the surface makes with the coordinates planes and any planes parallel to the coordinate axes. The traces are the intersection between the two. For example, if we wish to look at the traces that the cylinder makes with the xy-plane, we let $z = 0$ (the equation for the xy-plane) and see what equation that gives us on the plane. Likewise, to get the trace for any plane parallel to that xy-plane, we let $z = k$, and we find the intersection of the surface and the plane keeping k constant.

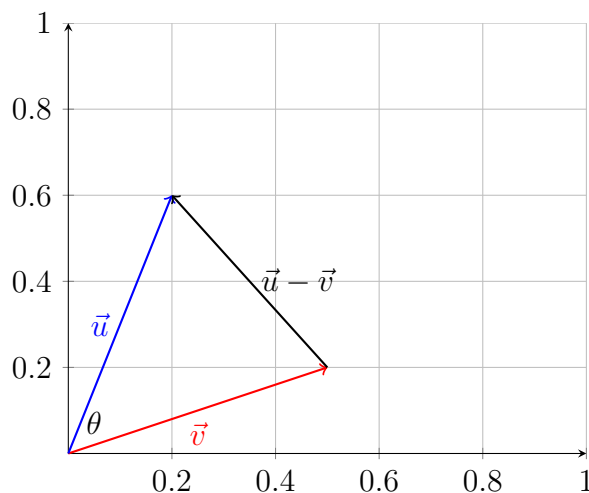# 2   Vectors

2.1   Properties of Vectors

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (Commutative)

- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (Associative)

- $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$ (Identity Element under addition)

- $\vec{u} + (-\vec{u}) = \vec{0}$ (Additive Inverse)

- $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ (Distributive, vector)

- $(c + d)\vec{u} = c\vec{u} + d\vec{u}$ (Distributive, scalar)

- $c(d\vec{u}) = (cd)\vec{u}$ (Mixed Associative)

- $1\vec{u} = \vec{u}$ (Identity Element)

2.2   The Dot/Scalar Product

If $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$, then

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

Let θ denote the angle between $\vec{u}$ and $\vec{v}$. Using the Law of Cosines,

$$||\vec{u} - \vec{v}||^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = ||\vec{u}||^2 + ||\vec{v}||^2 - 2||\vec{u}||||\vec{v}|| \cos\theta$$
$$||\vec{u}||^2 - 2(\vec{u} \cdot \vec{v}) + ||\vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2 - 2||\vec{u}||||\vec{v}|| \cos\theta$$
$$\rightarrow \boxed{\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}|| \cos\theta}$$

This is the geometric definition of the dot product, which allows us to find the angle between two vectors easily.

2.3   The Cauchy-Schwarz Inequality, Triangle Inequality

Theorem 2.1 (Cauchy-Schwarz Inequality)**.** *If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then $|\vec{u} \cdot \vec{v}| \leq ||\vec{u}||||\vec{v}||$.*

Proof. Rearranging the geometric definition of the dot product, we get $\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||||\vec{v}||}$. And since $|\cos\theta| \leq 1$, we get that $\left| \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||||\vec{v}||} \right| \leq 1 \rightarrow |\vec{u} \cdot \vec{v}| \leq ||\vec{u}||||\vec{v}||$. ■

Theorem 2.2 (Triangle Inequality)**.** *If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then $||\vec{u}+\vec{v}|| \leq ||\vec{u}||+||\vec{v}||$.*
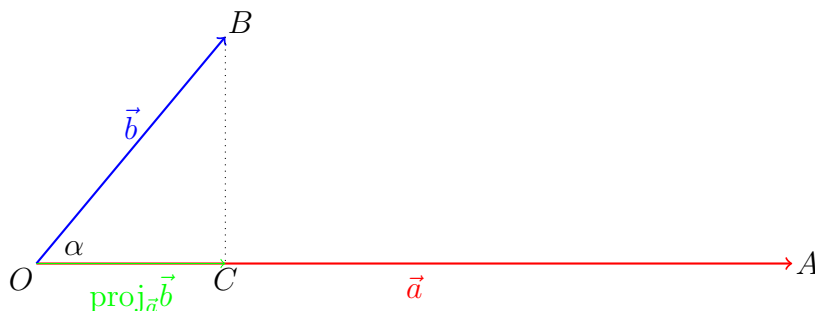
Proof.
$$||\vec{u} + \vec{v}||^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$$
$$= \vec{u} \cdot \vec{u} + 2(\vec{u} \cdot \vec{v}) + \vec{v} \cdot \vec{v}$$
$$\leq ||\vec{u}||^2 + 2|\vec{u} \cdot \vec{v}| + ||\vec{v}||^2$$

By Cauchy-Schwarz:
$$||\vec{u} + \vec{v}||^2 \leq ||\vec{u}||^2 + 2||\vec{u}||||\vec{v}|| + ||\vec{v}||^2$$
$$= (||\vec{u}|| + ||\vec{v}||)^2$$
$$||\vec{u} + \vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||$$

∎

2.4   Projections

$\overrightarrow{OC}$ is the vector projection of $\overrightarrow{OB}$ ($\vec{b}$) onto $\overrightarrow{OA}$ ($\vec{a}$). This is denoted as $\text{proj}_{\vec{a}}\vec{b}$. The magnitude of this projection, $||\overrightarrow{OC}||$, is denoted as $\text{comp}_{\vec{a}}\vec{b}$.

To describe $\overrightarrow{OC}$, we need its direction and magnitude.

- Direction is given by the unit vector in the direction if $\vec{a}$, which is $\frac{\vec{a}}{||\vec{a}||}$.

- Magnitude is given by:
$$||\overrightarrow{OC}|| = ||\vec{b}|| \cos \alpha$$
$$= ||\vec{b}|| \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||||\vec{b}||} = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||}$$

Therefore,
$$\overrightarrow{OC} = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||}\left(\frac{\vec{a}}{||\vec{a}||}\right) = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||^2}$$

$$\boxed{\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||^2}\vec{a}}$$

We can also easily see that, from this, $\boxed{\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||}}$.

2.5   Direction Cosines

In 3-space, the direction cosines give the cosines of the angles formed between a vector and the three axes.

If α, β, γ are the angles that the vector $\vec{A} = \langle A_x, A_y, A_z \rangle$ makes with the x, y, and z axes respectively, then:

$$\begin{aligned}
\vec{A} &= \langle A_x, A_y, A_z \rangle \\
&= \langle ||\vec{A}|| \cos \alpha, ||\vec{A}|| \cos \beta, ||\vec{A}|| \cos \gamma \rangle \\
&= ||\vec{A}|| \langle \cos \alpha, \cos \beta, \cos \gamma \rangle
\end{aligned}$$

2.6   The Cross/Vector Product

The cross product returns a vector that is orthogonal (perpendicular) to the two vectors in question.

The proof is omitted, but if $\vec{a}$ and $\vec{b}$ are vectors, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The magnitude of this vector is given by:

$$||\vec{a} \times \vec{b}|| = ||\vec{a}|| ||\vec{b}|| \sin \theta$$

The direction of this vector is orthogonal to the vector, but which direction exactly? We have two different options. The convention is to use the right-hand rule: point your fingers towards $\vec{a}$ and sweep them towards your palm towards $\vec{b}$. Your thumb points in the direction of the product.
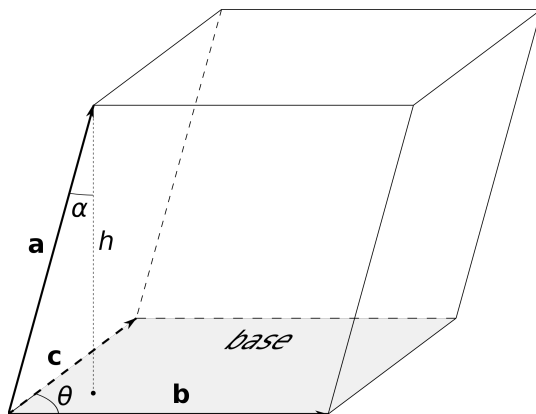
2.7   Volume of a Parallelogram
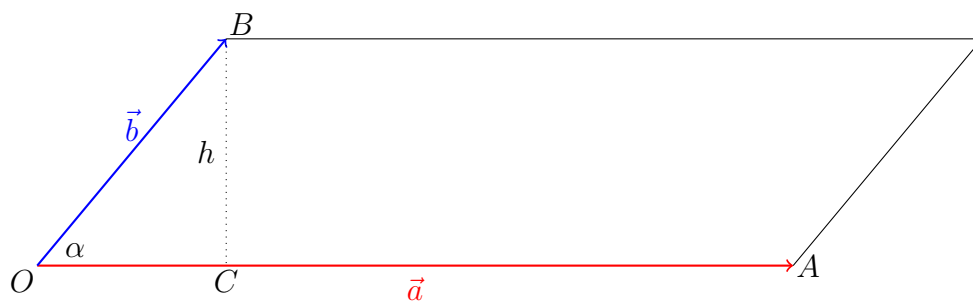
$$\text{Area} = bh$$
$$b = ||\vec{a}||, h = ||\vec{b}|| \sin \alpha$$
$$\text{Area} = ||\vec{a}|| ||\vec{b}|| \sin \alpha$$
$$\boxed{Area = ||\vec{a} \times \vec{b}||}$$

2.8   Volume of a Parallelepiped

$$\text{Volume} = Bh$$
$$= (\vec{b} \times \vec{c}) \cdot \vec{a}$$