Proof of the Second Partials Test

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1 Introduction

Recall the statement of the Second Partials Test.

Theorem 1.1 (Second Partials Test)**.** *Let $z = f(x, y)$ be a function of two variables with continuous first and second partial derivatives in some disk containing (a, b), where $f_x(a, b) = f_y(a, b) = 0$. Let the **discriminant** of f be $D = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$. Then:*

- *If $D > 0$ and $f_{xx}(a, b) < 0$, f has a relative maximum at (a, b).*
- *If $D > 0$ and $f_{xx}(a, b) > 0$, f has a relative minimum at (a, b).*
- *If $D < 0$, f has a **saddle point** at (a, b).*
- *If $D = 0$, the test is inconclusive.*

The proof of this theorem will rely on **quadratic forms**.

2 Quadratic Forms

A quadratic form is a function in 2 variables of the type

$$Q(h, k) = ah^2 + 2bhx + ck^2, \qquad a, b, c \in \mathbb{R}$$

In our proof, we want to analyze the sign of the quadratic form. depending on the values that h, k take on.

There is an easy way to find the sign of a quadratic form; by using the discriminant.

Lemma 2.1. *Let $Q(h, k) = Ah^2 + 2Bhk + Ck^2$ be a quadratic form with discriminant $D = AC - B^2$. Then:*

1. *If $D > 0$ and $A > 0$, then $Q(h, k) > 0$ for $(h, k) \neq (0, 0)$*
2. *If $D > 0$ and $A < 0$, then $Q(h, k) < 0$ for $(h, k) \neq (0, 0)$*
3. *If $D < 0$ then $Q(h, k)$ takes on both positive and negative values.*

Proof. Assume $A \neq 0$. Then

$$\begin{aligned}
Q(h, k) &= Ah^2 + 2Bhk + Ck^2 \\
&= A\left(h^2 + 2\frac{B}{A}hk + \frac{C}{A}k^2\right) \\
&= A\left(h^2 + 2\frac{B}{A}hk + \frac{B^2}{A^2}k^2 + \frac{C}{A}k^2 - \frac{B^2}{A^2}k^2\right) \\
&= A\left(h + \frac{B}{A}k\right)^2 + \left(C - \frac{B^2}{A}\right)k^2 \\
&= A\left(h + \frac{B}{A}k\right)^2 + \frac{D}{A}k^2
\end{aligned}$$

- If $D > 0$ and $A > 0$, then $\frac{D}{A} > 0$ and both terms above are positive; thus $Q(h,k) \geq 0$. If $Q(h,k) = 0$ then $k = 0$ and $Ah^2 = 0 \to h = 0$. Thus $Q(h,k) > 0$ when $(h,k) \neq (0,0)$.
- If $D > 0$ and $A < 0$, then $\frac{D}{A} < 0$ and both terms above are negative; thus (like above) $Q(h,k) < 0$ when $(h,k) \neq (0,0)$.
- If $D < 0$:
  - If $A \neq 0$, the terms above will have opposite signs, so $Q(h,k)$ takes on positive and negative values.
  - If $A = 0$, then $Q(h,k) = 2Bhk + Ck^2$; same as above.

■

3 Taylor Polynomials

Next, we recall something we learned from BC Calculus.

A **Taylor Polynomial** centered at a is a polynomial of the form

$$T(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Remember that this polynomial gave us an approximation of a function at $x = a$.

Theorem 3.1 (Taylor's Theorem)**.** *If f and its first n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on $[a,b]$ and $f^{(n)}$ is differentiable on (a,b), then there exists a number $c \in (a,b)$ such that*

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

4 The Proof

Proof. By hypothesis, f has continuous first and second partial derivatives in some region $\mathcal{R}$ about (a,b). Let the point $(a+h, b+k)$ be close enough to (a,b) such that $(a+h, b+k) \in \mathcal{R}$ and the segment joining them also is in $\mathcal{R}$.

Note that $\overline{PS}$ can be expressed parametrically as:

$$\begin{cases} x = a + ht \\ y = b + kt \end{cases} \qquad \text{where } 0 \leq t \leq 1$$

Let us investigate $f(x,y)$ over this line segment. Let $F(t) = f(a+ht, b+kt)$. Then,

$$\begin{aligned} F'(t) &= f_x(a+ht, b+kt)h + f_y(a+ht, b+kt)k \\ F''(t) &= (f_{xx}(a+ht, b+kt)h + f_{xy}(a+ht, b+kt)k)h \\ &\quad + (f_{yx}(a+ht, b+kt)h + f_{yy}(a+ht, b+kt)k)k \end{aligned}$$

so F and F' are continuous on $[0,1]$ and F' is differentiable on $(0,1)$. Then, by Taylor's Theorem, $\exists c \in (0,1)$ such that

$$\begin{aligned} F(1) &= F(0) + F'(0)(1-0) + \frac{F''(c)}{2!}(1-0)^2 \\ &= F(0) + F'(0) + \frac{F''(0)}{2} \\ F(1) - F(0) - F'(0) &= \frac{F''(c)}{2} \end{aligned}$$

Now, express F and its derivatives in terms of f:

$$\begin{aligned} F(1) &= f(a+h, b+k) \\ F(0) &= f(a,b) \\ F'(0) &= f_x(a,b)h + f_y(a,b)k \\ F''(c) &= h^2 f_{xx}(a+hc, b+kc) + 2hk f_{xy}(a+hc, b+kc) + k^2 f_{yy}(a+hc, b+kc) \end{aligned}$$

Substituting:

$$f(a+h, b+k) - f(a,b) - f_x(a,b)h - f_y(a,b)k = \frac{F''(c)}{2}$$

From the hypothesis of the theorem, $f_x(a,b) = f_y(a,b) = 0$, so

$$f(a+h,b+k) - f(a,b) = \frac{1}{2}(h^2 f_{xx}(a+hc,b+kc) + 2hk f_{xy}(a+hc,b+kc) + k^2 f_{yy}(a+hc,b+kc))$$

Notice that the sign of $f(a+h, b+k) - f(a,b)$ will enable us to determine if (a,b) harbors a maxima or minima.

Define

$$Q(t) = h^2 f_{xx}(a+ht, b+kt) + 2hk f_{xy}(a+ht, b+kt) + k^2 f_{yy}(a+ht, b+kt)$$

By this definition, $Q(c) = f(a+h, b+k) - f(a,b)$; we want to analyze the sign of Q(c). Also define $S(t) = 2Q(t)$; S and Q will have the same sign.

Since the second partials are continuous through $\mathcal{R}$, for sufficiently small h, k, their values at $(a+ch, b+ck)$ are nearly the same as the partials at (a,b). Therefore, the sign of $Q(c)$ will be nearly the same as $Q(0) = Q(h,k)$.

$$S(0) = 2Q(0) = 2Q(h,k) = h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)$$

Notice that this is a quadratic form, with

$$\begin{aligned} A &= f_{xx}(a,b) \\ B &= f_{xy}(a,b) \\ C &= f_{yy}(a,b) \\ D &= AC - B^2 = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2 \end{aligned}$$

And the proof follows from Lemma 2.1. ■