Efficient and Fair Allocation under Uncertainty: A School Assignment Approach

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Abstract

This paper presents a novel algorithm for the allocation of indivisible goods, specifically addressing the school assignment problem when preferences are uncertain. My approach goes beyond traditional methods that prioritize either fairness or efficiency, instead achieving both by recognizing that students often have only a general sense of their preferences. I introduce a fairness measure, interim envy-freeness, that ensures each student's expected outcome is as favorable as any alternative and characterize the conditions that make this achievable, offering new insights into feasible fair allocations. Utilizing data from Mexico City's centralized school admissions, where students' choices reveal varying levels of certainty, I show that my algorithm outperforms Deferred Acceptance in fairness and delivers welfare outcomes comparable to utility-maximizing methods. The findings highlight how ignoring preference uncertainty can lead to less favorable student outcomes, underscoring the importance of addressing it in school assignment processes.

Keywords: Fairness, Market Design, Indivisible Object Allocation, Education Economics

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1 Introduction

Imagine two students, Ariel and Ben, applying to two schools, School A and School B. They have similar grades and test scores, making them equally strong candidates, and each has reasons for preferring one school over the other. Ariel is drawn to School A's academics but would gladly attend School B if offered a scholarship. Ben, on the other hand, is nearly indifferent but slightly favors School A, believing it has better sports facilities. With only one seat available at each school, how should we decide on their assignments? Ideally, we would want to fill both seats, admitting one student to each.

The usual approach would be to ask Ariel and Ben to submit their ranked preferences, which both put School A first and School B second. Given they have similar academic achievements and exam scores, the only way to achieve efficiency in this case is to randomly select who will attend School A. Suppose Ariel is assigned to School A, and Ben to School B. Ariel is satisfied with getting her top choice, but Ben feels the allocation is unfair. Even if his preference for School A was not strong, it was still his top choice, and he is equally qualified. The allocation is efficient, as no seats are left empty, but is it fair? Random selection gives equal opportunity, but if Ben had the chance to switch places with Ariel, he would take it, suggesting that equal opportunity alone is not enough to prevent envy.

This example illustrates a common challenge in distributing limited resources among individuals with competing needs. When these resources, like a school seat, are non-shareable, the decision becomes even more complex, as only one person can benefit, leaving others without. In cases like Ariel and Ben's, where resources are limited and indivisible, we face a choice between efficiency and fairness. Most commonly used mechanisms worldwide differ in how they tend to favor one objective over the other. In this paper, I explore cases where both goals can be achieved by considering the practical realities agents face when making their choices. While the results and solution approach proposed here apply broadly to any allocation of indivisible resources, I have chosen to focus on this issue within the context of school assignments.

Consider the problem broadly: we aim to allocate school seats to students. Traditional methods assume students are certain about their preferences, asking them to submit a ranked list of schools from most to least preferred. Is this realistic? Do students have enough information to decide what is best for them, or could education authorities help guide them in making these decisions? A survey by Gallup (2017) found that 51% of Americans with postsecondary education would redo their choice of degree, institution, or major if given the chance. Similarly, an experimental study by Chen and He (2021) suggests that providing students with more information can improve welfare. Thus, before making allocation decisions, I factor in students' uncertainty over their preferences.

Another issue with preference rankings is that they reveal little about preference intensity. In the example, both Ariel and Ben submitted the same list, but without knowing how much each values one option over another, we miss the reasoning and strength behind their choices. To address this, I use cardinal utilities to capture the intensity of preferences, providing a clearer view of how much each student values one choice over another.

Recent legal cases, such as *Parents Involved in Community Schools v. Seattle School District No. 1* (2007), which examined the fairness of using racial status as a tiebreaker in school assignments, and *Students for Fair Admissions v. Harvard* (2014) and *Students for Fair Admissions v. University of North Carolina*, which scrutinized race-conscious admissions practices, highlight the need for fair assignment mechanisms. These cases emphasize the importance of respecting individual qualifications and choices while avoiding criteria that could unintentionally disadvantage specific groups. Together with efficiency, I aim to develop a fair mechanism that balances these considerations, ensuring equitable access for all students without compromising overall system effectiveness.

In this paper, I ask: *How should a principal fairly allocate indivisible objects to agents to maximize welfare in an environment where agents are uncertain about their own preferences?*

To address this question, in Section 2, I formally establish the setup for the allocation problem, define the structure of agents' preferences, and outline a mechanism that assigns objects to agents based on an announced allocation rule. I incorporate uncertainty in preferences by allowing for multiple possible utility values that agents might receive across different scenarios. I then introduce a new fairness definition, interim envy-freeness, tailored to this uncertain setting. An allocation mechanism is considered interim envy-free if the announced rule allows agents to form expectations about their utilities such that each agent's expected utility from their assigned allocation is at least as high as the expected utility they would receive from another agent's allocation, essentially making sure that if we were to switch two people's assignments, they would still feel they were treated fairly.

In Section 3, I illustrate interim envy-freeness with an example and show how the results of the proposed approach differ from benchmarks such as the utility-maximizing allocation, Deferred Acceptance, and Random Serial Dictatorship. This example demonstrates that interim envy-freeness captures a unique level of fairness, as even Random Serial Dictatorship does not yield an interim envy-free solution. I conclude the section by detailing my Fairness-Welfare Optimization algorithm, discussing its complexity, and proposing a method to manage this complexity.

Section 4 examines the necessary conditions for the existence of an interim envy-free solution. In Proposition 1, I establish that when agents' preferences lack variation, achieving interim envy-freeness is impossible. In Theorem 1, I characterize preference structures that ensure the existence of an interim envy-free solution.

Furthermore, in Section 5, I utilize a student-level administrative dataset from the centralized high school admission system in Mexico City to compare my approach with other traditional methods. I begin by providing background information on the dataset and the centralized admission system in Mexico City, demonstrating the inefficiencies of the current system. To account for uncertainty in students' preferences, I construct a simulation environment based on reported choices and underlying characteristics. I conclude the section by interpreting the results in terms of welfare and fairness violations, showing that my algorithm outperforms Deferred Acceptance in fairness and achieves welfare outcomes comparable to utility-maximizing methods.

Related Work

My work mainly contributes to two strands of literature. Firstly, I contribute to the fairness literature in computer science and economics by introducing a new fairness definition that is less stringent than ex-post envy-freeness yet stronger than ex-ante envy-freeness. This extensive body of work originates with the seminal contribution of Foley (1967). My work further advances this literature by developing a novel algorithm based on this new definition and establishing bounds for fairness. Aziz et al. (2024) address similar challenges in balancing fairness standards with practical applicability, proposing a framework that combines exact ex-ante fairness with approximate ex-post fairness in randomized allocations. Aziz (2020) has extended some of these findings in randomized settings. In contrast, my approach introduces a fairness standard with moderate strictness that is specifically designed for deterministic settings. The term interim envy-freeness has been previously used in the literature, beginning with Palfrey and Srivastava (1987) and later explored by Veszteg (2004) and de Clippel (2008), though these works apply the term in different contexts and with definitions distinct from mine. A more recent study by Caragiannis et al. (2021) introduces a fairness concept also termed interim envy-freeness, which is closer to my definition but differs in setup and is weaker than the notion I propose. In their definition, an allocation is interim envy-free if an agent's utility from their own allocation is at least as good as the utility they would obtain from the allocation of a randomly selected agent. If an allocation is interim envy-free in my framework, it should also be interim envy-free by their definition; however, an allocation that is interim envy-free by their definition may not satisfy mine. The main distinction lies in the source of uncertainty: in their setup, uncertainty arises from the agent's lack of knowledge about what they could have received, while in my setup, it stems from variations in agents' preferences.

This paper also contributes to the matching literature, which often assumes that agents have perfect knowledge of their preferences (Gale and Shapley (1962); Roth and Sotomayor (1990); Abdulkadiroğlu and Sönmez (2003)). Recent theoretical studies by Bade (2015), Harless and Manjunath (2015), Artemov (2021), and Chen and He (2021) explore how matching mechanisms influence agents' information gathering. In particular, Grenet et al. (2022) investigate how mechanisms can provide students with information about admission probabilities, helping them better understand their preferences. Unlike these studies, my paper directly incorporates preference uncertainty, proposing a mechanism that optimizes both fairness and efficiency within school assignments under uncertain preferences. The work most closely related to mine is by Dasgupta (2024), which examines how a planner can optimally share information to maximize welfare when agents initially lack private information. While their focus is on identifying welfare-maximizing allocations, my research develops an interim envy-free mechanism that also aims to maximize welfare by prioritizing fairness through interim envy-freeness and achieving efficiency through non-wastefulness.

Finally, my analysis of the Mexican centralized admissions system for high schools in Mexico City contributes to the discussion on the system's inefficiencies. Bobba et al. (2023) study an intervention within this system that offers students feedback on their academic performance before they submit their school preferences. Their findings suggest that, although students' preferences do not change on average, such feedback can improve the alignment between students' skills and school placements. Similarly, Bobba and Frisancho (2022) demonstrate that individualized feedback in this context enhances students' self-assessments, with significant variations in how students update their beliefs based on ability, gender, and socioeconomic status.

The remainder of the paper is structured as follows: Section 2 discusses the theoretical model. Section 3 presents the solution approach and details the algorithm. Section 4 characterizes the preference structures that ensure the existence of an interim envy-free solution. Section 5 describes the centralized school assignment system in Mexico City and the dataset used for the study. It also simulates the assignment process, incorporates uncertainty in students' preferences, and analyzes the results. Section 6 concludes the paper.

2 Model

We consider a setting with a finite set of agents $\mathcal{I} = \{1, \ldots, n\}$ and a finite set of indivisible objects $\mathcal{N} = \{1, \ldots, n\}$, where the number of agents equals the number of objects. Each agent $i \in \mathcal{I}$ has a cardinal utility u_{ik} for object $k \in \mathcal{N}$, with $u_{ik} \in \mathbb{R}_+$. These utilities form a utility vector $u_i = (u_{i1}, u_{i2}, \ldots, u_{in})$, representing agent i's preferences over the set of objects. Each agent's utility vector u_i lies within a compact set $\mathcal{U}_i \subset \mathbb{R}^n_+$.

The joint utility space, $\mathcal{U} = \times_{i \in \mathcal{I}} \mathcal{U}_i$, represents all possible combinations of individual utility vectors for each agent, with a typical element denoted by $u = (u_1, \ldots, u_n)$. Each utility profile, $u^{(\ell)}$, reflects the preferences of all agents combined, where ℓ indexes the possible profiles.

Agents are uncertain about the exact realization of the utility profile and only know the joint distribution of their cardinal utilities. This uncertainty is captured by a common prior distribution μ over the joint utility space $\mathcal{U}$, where $\mu(u)$ is the prior probability that profile u is the true utility profile.

2.1 Allocation Mechanism

An allocation $m \in \mathcal{M}$ is represented by a binary matrix of size $n \times n$, where $m_{ik} = 1$ if object k is assigned to agent i, and $m_{ik} = 0$ otherwise. The set $\mathcal{M}$ contains all possible allocations that assign each object to exactly one agent and each agent receives exactly one object.

The principal uses a mechanism $\nu : \mathcal{U} \to \Delta(\mathcal{M})$, which assigns a probability distribution over the set of allocations $\mathcal{M}$ for each utility profile $u \in \mathcal{U}$. Specifically, $\nu(m \mid u)$ represents the probability of selecting an allocation $m \in \mathcal{M}$ given the utility profile u. The mechanism ν is designed to optimize efficiency by maximizing welfare, subject to fairness constraints, while accounting for the uncertainty about agents' true preferences.

2.2 Fairness Condition: Interim Envy-Freeness

We introduce the concept of *Interim Envy-Freeness*, a fairness condition defined with respect to agents' updated beliefs, which depend on the allocation mechanism. When the allocation mechanism is announced, agents form expectations about the utility they will receive based on probabilistic beliefs over possible utility profiles. Interim Envy-Freeness ensures that, given these beliefs, no agent envies another's allocation in expectation. Unlike ex-ante and ex-post fairness, this concept addresses fairness in the interim stage, guaranteeing that each realized allocation is perceived as fair before agents' preferences are fully revealed.

After the principal announces the mechanism and an allocation $m = a$ is realized, agents update their beliefs about which utility profile u might have occurred using Bayes' Rule. The posterior probability that a utility profile u occurred, given that allocation $m = a$ was chosen, is given by:

$$P(u \mid m = a) = \frac{\nu(a \mid u) \cdot \mu(u)}{\sum_{u' \in \mathcal{U}} \nu(a \mid u') \cdot \mu(u')}$$

Using these updated beliefs, we can express the expected utility for each agent. The notation $\mathbb{E}_{u\sim P(\cdot|m=a)}$ represents the expectation over the possible utility profiles u, weighted by their posterior probabilities $P(u \mid m = a)$. In this context, it captures the expected utility that an agent i receives, considering the uncertainty about the exact utility profile that led to the observed allocation.

The interim envy-freeness condition requires that, for any allocation $m = a$, the expected utility that an agent i receives from their assigned allocation must be at least as large as the expected utility they would receive if they were assigned another agent j's allocation. An allocation outcome that satisfies this condition is termed an Interim Envy-Free Allocation.

$$\mathbb{E}_{u\sim P(\cdot|m=a)}[a_i \cdot u_i] \geq \mathbb{E}_{u\sim P(\cdot|m=a)}[a_j \cdot u_i], \quad \forall i, j.$$

This expression can be rewritten with the posterior probabilities as:

$$\sum_{u\in\mathcal{U}} P(u \mid m = a) \cdot (a_i \cdot u_i) \geq \sum_{u\in\mathcal{U}} P(u \mid m = a) \cdot (a_j \cdot u_i), \quad \forall i, j \tag{1}$$

or equivalently:

$$\sum_{u\in\mathcal{U}} \frac{\nu(a \mid u) \cdot \mu(u)}{\sum_{u'\in\mathcal{U}} \nu(a \mid u') \cdot \mu(u')} \cdot (a_i \cdot u_i) \geq \sum_{u\in\mathcal{U}} \frac{\nu(a \mid u) \cdot \mu(u)}{\sum_{u'\in\mathcal{U}} \nu(a \mid u') \cdot \mu(u')} \cdot (a_j \cdot u_i). \tag{2}$$

Definition: *Interim Envy-Freeness*

An allocation mechanism satisfies *Interim Envy-Freeness* if, for any realized allocation $m = a$, the following condition holds:

$$\mathbb{E}_{u\sim P(\cdot|m=a)}[a_i \cdot u_i] \geq \mathbb{E}_{u\sim P(\cdot|m=a)}[a_j \cdot u_i], \quad \forall i, j.$$

2.3 The Principal's Problem

The principal aims to maximize the expected total welfare of agents by selecting an allocation mechanism $\nu : \mathcal{U} \to \Delta(\mathcal{M})$, subject to interim envy-freeness and allocation feasibility constraints. The objective is to maximize the expected total welfare across all possible utility profiles, weighted by their probability:

$$\max_{\nu} \sum_{u \in \mathcal{U}} \mu(u) \sum_{m \in \mathcal{M}} \nu(m \mid u) \sum_{i \in \mathcal{I}} m_i \cdot u_i$$

subject to:

1. **Allocation Feasibility:** Each agent receives exactly one object, and each object is assigned to exactly one agent:

$$\sum_{k \in \mathcal{N}} m_{ik} = 1, \quad \forall i \in \mathcal{I},$$
$$\sum_{i \in \mathcal{I}} m_{ik} = 1, \quad \forall k \in \mathcal{N}.$$

2. **Interim Envy-Freeness:** The allocation mechanism should satisfy the following condition: For any realized allocation $m = a$,

$$\sum_{u \in \mathcal{U}} P(u \mid m = a) \cdot (a_i \cdot u_i) \geq \sum_{u \in \mathcal{U}} P(u \mid m = a) \cdot (a_j \cdot u_i), \quad \forall i, j.$$

3 Solution Approach

This section introduces the concept of interim envy-freeness and illustrates how the interim envy-free mechanism differs from other existing mechanisms. It concludes by presenting the proposed algorithm, Fairness-Welfare Optimization, which finds the best welfare outcome within the set of interim envy-free solutions.

3.1 Illustrative Example

We provide an example to clarify the concept of Interim Envy-Freeness, demonstrating how the proposed mechanism balances fairness and welfare maximization under uncertainty. To highlight its unique contributions, we compare it with the Deferred Acceptance (DA) mechanism, known for fairness through stability; the Random Serial Dictatorship (RSD), valued for its simplicity and fairness via random priority; and a utility-maximizing allocation method.

Each is adapted to our setup to highlight the distinguishing features of the proposed interim envy-free allocation rule.

Consider a scenario with two agents, each have two possible utility vectors, equally likely to occur. Agent 1's utility vectors are $u_1 = (11, 4)$ and $u_1' = (2, 4)$, while Agent 2's possible vectors are $u_2 = (7, 8)$ and $u_2' = (6, 4)$.

The joint utility space $\mathcal{U}$ captures all possible combinations of the agents' utility vectors, accounting for the uncertainty in each agent's preferences. Since both agents have two equally likely utility vectors, there are four possible utility profiles in $\mathcal{U}$, each with a probability of 0.25. We denote each profile as $u^{(\ell)}$ for $\ell \in \{1, 2, 3, 4\}$:

$$\mathcal{U} = \left\{ \begin{array}{l} u^{(1)} = ((11,4),(7,8)), \\ u^{(2)} = ((11,4),(6,4)), \\ u^{(3)} = \;\; ((2,4),(7,8)), \\ u^{(4)} = \;\; ((2,4),(6,4)) \end{array} \right\}.$$

Each profile lists Agent 1's utility vector first and Agent 2's second, with the uniform probability distribution reflecting the independent and equally likely realizations of both agents' preferences.

Utility-maximizing Allocation

If the principal's objective were simply to maximize total welfare, the objects would be allocated in a way that maximizes the sum of utilities in each utility profile. The resulting allocations are underlined below, and the mechanism yields an expected utility of 13.75:

$$\mathcal{U} = \left\{ \begin{array}{l} u^{(1)} = ((\underline{11},4),(7,\underline{8})), \\ u^{(2)} = ((\underline{11},4),(6,\underline{4})), \\ u^{(3)} = \;\; ((2,\underline{4}),(\underline{7},8)), \\ u^{(4)} = \;\; ((2,\underline{4}),(\underline{6},4)) \end{array} \right\}.$$

In $u^{(1)}$ and $u^{(2)}$, Agent 1 receives Object 1 and Agent 2 receives Object 2, whereas in $u^{(3)}$ and $u^{(4)}$, Agent 1 receives Object 2 and Agent 2 receives Object 1. The expected welfare is calculated by looking at the utility each agent gains from the object they are assigned in each scenario and averaging those utilities based on how likely each utility profile is. In this example, each utility profile has an equal probability of occurring 0.25. By summing up the utilities assigned in each scenario and averaging them, the total expected utility comes out to 13.75.

While this mechanism maximizes overall welfare, it is unfair to Agent 2. When the allocation rule assigns the first object to Agent 1 and the second object to Agent 2, both agents believe they are in either $u^{(1)}$ or $u^{(2)}$, each with a probability of 0.5. Agent 1 knows his utility is 11 regardless of the utility profile, and he is satisfied as he received his most preferred object. However, Agent 2, without knowing her exact utility realization, is aware that she did not receive the allocation that maximizes her expected utility.

Deferred Acceptance Mechanism

In the Deferred Acceptance (DA) mechanism, agents submit a ranked list of preferences and apply to their top choice. Each choice tentatively accepts applicants based on priority, rejecting lower-ranked applicants, and this process continues until all agents are matched. Since the DA mechanism is not designed to incorporate uncertainty in preferences and requires agents to rank objects directly, we apply DA separately for each utility profile to establish an allocation rule.

First, assuming that all objects prefer Agent 1 over Agent 2, we apply DA to each profile to determine the allocation outcome. In each profile, DA assigns Agent 1 his highest-ranked object, leaving Agent 2 with the remaining one. This allocation results in an expected welfare of 13.75, aligning with the utility-maximizing allocation. However, similar to the utility-maximizing outcome, this allocation causes Agent 2 to feel envy toward Agent 1, as her expected utility would increase by 0.5 if she could switch allocations with him.

Alternatively, if all objects prefer Agent 2 over Agent 1, DA assigns Agent 2 her most preferred object in each profile, leaving Agent 1 with the other. The specific allocations in each profile are as follows:

$$\mathcal{U} = \left\{ \begin{array}{l} u^{(1)} = ((\underline{11}, 4), (7, \underline{8})), \\ u^{(2)} = ((11, \underline{4}), (\underline{6}, 4)), \\ u^{(3)} = \ \ ((\underline{2}, 4), (7, \underline{8})), \\ u^{(4)} = \ \ ((2, \underline{4}), (\underline{6}, 4)) \end{array} \right\}.$$

In this case, when the mechanism assigns Object 1 to Agent 2 and Object 2 to Agent 1, both agents believe the utility profile is either $u^{(2)}$ or $u^{(4)}$, each with a probability of 0.5. Agent 2 is satisfied with this allocation since she receives her most preferred object. However, Agent 1 realizes that if he were assigned Object 1 instead, his expected utility would increase to 6.5 rather than remain at 4, which results in envy.

Random Serial Dictatorship

Under the Random Serial Dictatorship (RSD) mechanism, agents are randomly assigned a priority order, allowing each agent to choose their most preferred available object in sequence. To incorporate uncertainty in preferences, we apply RSD separately to each utility profile.

In $u^{(1)}$, the allocation assigns Object 1 to Agent 1 and Object 2 to Agent 2, staying the same under both priority orders. Similarly, in $u^{(4)}$, Object 1 is given to Agent 2 and Object 2 to Agent 1 in both priority orders. In $u^{(2)}$ and $u^{(3)}$, however, the allocation depends on the priority order: the agent who goes first receives Object 1 in $u^{(2)}$ and Object 2 in $u^{(3)}$. Thus, each agent has a 0.5 probability of receiving Object 1 in these profiles. This allocation mechanisms results in an expected utility of 13.

If the allocation mechanism is announced this way, and the realized allocation is that Agent 1 receives Object 2 and Agent 2 receives Object 1, then Agent 1 will feel envy toward Agent 2. This is because both agents would update their beliefs, assigning a probability of 0.5 to $u^{(4)}$ and 0.25 each to $u^{(2)}$ and $u^{(3)}$. Based on these beliefs, Agent 1's expected utility is 4, but he would have an expected utility of 4.25 if he were assigned Agent 2's allocation, resulting in an envy of 0.25.

Interim Envy-Free Mechanism

If the principal announces the following allocation mechanism, the objects will be distributed as follows: Object 1 will be given to Agent 1 and Object 2 to Agent 2 in utility profiles $u^{(1)}$, $u^{(2)}$, and $u^{(3)}$. In utility profile $u^{(4)}$, Object 2 is given to Agent 1, and Object 1 to Agent 2. The specific allocations within each utility profile are highlighted below:

$$\mathcal{U} = \left\{ \begin{array}{rl} u^{(1)} = & ((\underline{11}, 4), (7, \underline{8})), \\ u^{(2)} = & ((\underline{11}, 4), (6, \underline{4})), \\ u^{(3)} = & ((\underline{2}, 4), (7, \underline{8})), \\ u^{(4)} = & ((2, \underline{4}), (\underline{6}, 4)) \end{array} \right\}.$$

This allocation rule results in an expected utility of 13.5, which is slightly lower than both the utility-maximizing allocation and the outcome produced by the deferred acceptance algorithm. However, it successfully addresses fairness by eliminating envy between agents, which is present in other mechanisms.

When Object 1 is assigned to Agent 1 and Object 2 to Agent 2, both agents assume they are in one of the utility profiles $u^{(1)}$, $u^{(2)}$, or $u^{(3)}$, each with a probability of one-third. Agent 1 does not envy Agent 2 because her expected utility in this arrangement is greater than what he would gain from Agent 2's allocation ($8 > 4$). Similarly, Agent 2 does not experience envy, as her expected utility from these objects is the same in each profile. In the case where Object 2 is assigned to Agent 1 and Object 1 to Agent 2, both agents know they are in the last utility profile, $u^{(4)}$, where each agent receives their most preferred object.

This example illustrates the trade-offs between welfare and fairness across different allocation mechanisms. Both the utility-maximizing and Deferred Acceptance (DA) mechanisms achieve a high expected welfare of 13.75 but leave one agent envious. In the DA mechanism, fairness varies with agent priority: when Agent 1 has priority, Agent 2 experiences envy with a potential utility gain of 0.5, and conversely, when Agent 2 has priority, Agent 1 is envious, as his expected utility would increase by 1.5 if he could switch. The Random Serial Dictatorship (RSD) mechanism results in the lowest expected welfare but allows each agent a fair chance to select objects based on priority, though it still leads to envy in some cases. For example, when Agent 1 is left with an expected utility of 4, he could achieve 4.25 if he were assigned Agent 2's allocation. In contrast, the Interim Envy-Free mechanism achieves a slightly lower expected welfare of 13.5 but eliminates envy entirely, ensuring that neither agent would benefit from switching allocations.

3.2 Algorithm Details

The algorithm, Fairness-Welfare Optimization, is designed to find the interim envy-free allocation rule by solving an optimization model. It computes an allocation mechanism that maximizes expected total welfare, subject to interim envy-freeness and allocation feasibility constraints. The interim envy-freeness condition requires comparing agents' expected utilities across multiple utility profiles to ensure no agent prefers another agent's allocation in expectation, while allocation feasibility ensures that each agent receives exactly one object and each object is assigned to only one agent.

The algorithm takes as inputs utility profiles, which represent all possible combinations of agents' preferences over the objects, and their associated probabilities, which capture the likelihood of each profile. These probabilities may be derived from sources such as historical data, surveys, or expert judgments, though the precise method of collecting them is beyond the scope of this paper. The algorithm assumes this data is available and uses it to compute optimal allocations. Based on the inputs, the algorithm generates all possible deterministic allocations, representing every potential way objects can be assigned to agents. For each allocation and utility profile, it defines a decision variable that represents the probability of selecting that specific allocation under the given utility profile, forming the foundation of the optimization problem.

To ensure fairness, the algorithm imposes constraints guaranteeing that no agent prefers another agent's allocation over their own in expectation. This condition is applied across all utility profiles where the allocation remains the same. Feasibility is enforced by requiring that each allocation satisfies the one-to-one correspondence between agents and objects. Additionally, the algorithm ensures the probabilities of all possible allocations under each utility profile sum to one, forming a valid probability distribution. The objective is to maximize the total expected welfare, calculated as the weighted sum of agents' utilities over all allocations and utility profiles, with weights determined by the probabilities of the profiles and the decision variables. Solving this optimization problem yields the optimal deterministic allocation probabilities and the maximum achievable welfare.

Algorithm 1 Fairness - Welfare Optimization

Input: Utility profiles $\mathcal{U}$, Probability distribution μ
Output: Optimal deterministic allocation probabilities and total expected welfare

1. **Generate Deterministic Allocations:** Identify all potential ways objects can be assigned to agents, ensuring that each object is assigned to exactly one agent, and each agent receives exactly one object.
2. **Define Decision Variable:** Let ν_{ua} represent the probability of selecting allocation a under utility profile u.
3. **Set Fairness Constraints:** For each allocation a and pair of agents i, j, where $i \neq j$:

   Ensure that, in expectation across utility profiles, agent i's utility from their assigned object is at least as high as the utility they would derive from agent j's assigned object.

4. **Set Feasibility Constraints:** Ensure each allocation satisfies:
   - Each object is assigned to exactly one agent.
   - Each agent receives exactly one object.
5. **Set Probability Constraints:** For each utility profile u, ensure that the probabilities ν_{ua} across all allocations sum to 1, forming a valid probability distribution.
6. **Define Objective Function:** Maximize the total expected welfare, calculated as the weighted sum of agents' utilities across all profiles and allocations, weighted by $\mu(u)$ and ν_{ua}.
7. **Optimize and Return Results:** Solve the optimization problem and return the optimal allocation probabilities along with the total expected welfare achieved.

As the number of agents, objects, and utility profiles increases, the complexity of the problem grows significantly due to the expanding set of possible allocations and constraints. To address these computational challenges, the approach could be adapted to reduce complexity by narrowing the search space. For instance, starting with utility-maximizing allocations and making targeted adjustments to eliminate envy could avoid the need to evaluate all possible allocations. While this adaptation is not implemented in the current algorithm, it offers a promising direction for scaling the approach to larger systems.

Interim envy-free allocations are not guaranteed to exist. Although finding an interim envy-free allocation is generally more feasible than achieving ex-post envy-freeness, there are specific conditions under which interim envy-freeness remains unattainable. The following section provides a detailed examination of these conditions.

4 Existence of Interim Envy-Free Allocations

Achieving interim envy-freeness can present a substantial challenge, as it requires eliminating envy after agents observe the realized allocation and update their beliefs accordingly. This challenge becomes unavoidable when agents consistently prefer the same object across all utility profiles[1]. Without preference variation, no deterministic allocation can prevent envy, making interim envy-freeness impossible to achieve.

Proposition 1 (Impossibility Result). *For n agents and n objects, where each agent strictly prefers the same object across all utility profiles, no mechanism can satisfy interim envy-freeness.*

Proof. Assume that each agent's most preferred object is Object k in every utility profile. For any given allocation $m = a$, only one agent can be assigned Object k. Suppose that, under allocation a, Agent i is not assigned their most preferred Object k, but assigned some other Object k'. Since $u_{ik} > u_{ik'}$ for all utility profiles, the expected utility for Agent i from receiving Object k' is strictly less than their expected utility from Object k:

$$\mathbb{E}_{u \sim P(\cdot|m=a)}[u_i \cdot a_i] < u_{ik}.$$

This result implies that Agent i will envy the agent who is assigned Object k, which contradicts the interim envy-freeness condition. Therefore, no allocation can satisfy interim envy-freeness.

[1] An agent i strictly prefers Object k in every profile if, for all ℓ and $k' \neq k$, $u_{ik}^{(\ell)} > u_{ik'}^{(\ell)}$.

This impossibility result also highlights why distributing objects uniformly at random among agents, regardless of the utility profiles, does not achieve interim envy-freeness. When agents only know the distribution over objects, they anticipate an equal chance of receiving their most preferred object, which preserves fairness in expectation. However, once they observe the realized allocation, they update their beliefs about the utility profiles, specifically recognizing scenarios where they do not receive their most preferred object. This knowledge may lead to envy, as agents compare their allocations to those of others who may have received a more desirable object.

To better understand the structure and limitations of interim envy-free allocations, we explore a simple case with two agents and two objects, labeled Object x and Object y. Here, each agent has two equally likely utility vectors, yielding four possible utility profiles with probabilities of 0.25 each. This setup provides insights into how the impossibility of interim envy-free allocations emerges in a basic setting, offering a foundation for more complex cases.

Each agent's preferences are represented by two utility vectors: (x_i, y_i) and (x'_i, y'_i). In (x_i, y_i), Agent i assigns utility x_i to Object x and y_i to Object y. In (x'_i, y'_i), these utilities are x'_i and y'_i, respectively. We explore variations in preference intensities across profiles, examining how these differences impact interim envy-freeness.

Case 1: Aligned Preferences.

If both agents prefer the same object across all profiles, the impossibility result implies that interim envy-freeness cannot be achieved.

Case 2: Shared Expected Preference, Profile-Specific Differences.

If both agents prefer the same object in expectation but have varying preferences across profiles, the situation becomes more nuanced. Assume both agents strictly prefer Object x over Object y in expectation, meaning $x_i + x'_i > y_i + y'_i$ for both agents.

Scenario 1: No Interim Envy-Free Solution.

If one agent always prefers Object x over Object y in both utility vectors, then it is not possible to find interim envy-free allocation. This agent knows that she always prefers Object x to Object y, even if she is uncertain about the exact intensity of her preference. To prevent envy, an interim envy-free allocation would need to assign Object x to this agent. However, since the other agent also prefers Object x in expectation, this assignment leads to envy from him.

Scenario 2: Conditions for a Non-Empty Interim Envy-Free Set.

If agents prefer different objects across their vectors, with preferences such as $x > y$ and $y' > x'$ (or $x < y$ and $y' < x'$), the interim envy-free set is non-empty if:

$$
\begin{aligned}
2|x_i - y_i| &\geq |y'_i - x'_i|, \\
2|y'_j - x'_j| &\geq |x_j - y_j|.
\end{aligned}
$$

Both agents share the same ex-ante preference, though their utility vectors rank different objects as their top choice. The inequality condition distinguishes these agents based on the intensity of their preferences for each object. Specifically, this condition indicates that interim envy-freeness can be achieved if one agent views their least preferred object as a close substitute, reducing the likelihood of envy.

Case 3: Agents Prefer Different Objects in Expectation.

If each agent has a distinct most-preferred object in expectation, assigning each agent their ex-ante top object across profiles achieves interim envy-freeness.

Observations from this two-agent exercise suggest that interim envy-free allocations can exist when agents have the same ex-ante ranking of objects but show varying intensities in their preferences. In this context, we identified an intensity condition that supports interim envy-freeness. To expand on this insight, we now investigate the conditions under which interim envy-free allocations might hold when agents share identical ordinal preferences. For simplicity, we assume agents are symmetric, meaning they have identical utility vectors. The expected utility for Agent i from Object k, given the distribution μ, is denoted by $\mathbb{E}_\mu[u_{ik}] \equiv \tilde{u}_i(k)$.

Theorem 1 (Existence of Interim Envy-Free Mechanism under Weak Preference Disparity)**.**

Assume agents are symmetric[2] *and each Agent i has strictly ordered expected utilities such that*

$$\tilde{u}_i(1) > \tilde{u}_i(2) > \cdots > \tilde{u}_i(n),$$

There exists a bound $\epsilon_n > 0$ such that if

$$\frac{\tilde{u}_i(1)}{\tilde{u}_i(n)} \leq 1 + \epsilon_n,$$

then an interim envy-free allocation exists.

Proof. We prove the statement by induction. For $n = 2$, we can use the bounds from the discussion in Case 2 by manipulating the inequality for $i = j$. Therefore, we have $\frac{x + x'}{y + y'} \leq 1 + \epsilon_2$ where $\epsilon_2 = \frac{y' - x'}{y + y'}$. For the induction hypothesis, assume the statement is true for n, that is there exists ϵ_n such that if $\frac{\tilde{u}_i(1)}{\tilde{u}_i(n)} \leq 1 + \epsilon_n$ then an interim envy-free allocation exists. We need show it holds for $n + 1$.

Take $\epsilon_{n+1} = \epsilon_n$, and assume

$$\frac{\tilde{u}_i(1)}{\tilde{u}_i(n+1)} \leq 1 + \epsilon_{n+1}$$

then we want to show an interim envy-free allocation exists. To construct the allocation, consider a market with agents $\{2, \ldots, n+1\}$ and the same set of objects. By the assumption $\tilde{u}_i(1) > \tilde{u}_i(2)$, we have

$$\frac{\tilde{u}_i(2)}{\tilde{u}_i(n+1)} < \frac{\tilde{u}_i(1)}{\tilde{u}_i(n+1)} \leq 1 + \epsilon_{n+1}$$

Therefore, by the induction hypothesis, there exists an interim envy-free allocation rule, denoted by ν^*. Now, consider the following allocation rule, which assigns objects according to ν^* for agents $\{2, \ldots, n+1\}$ based on their utility profile u_{-1}, while assigning first object to first agent.

$$\nu(m \mid u) = (1, \nu^*(u_{-1})).$$

We claim that ν is interim envy-free. To see this, first note that Agent 1 always receives

[2] We say agents are symmetric if for any two different agents, $i \neq j$, and Object k, $u_{ik} = u_{jk}$. That is, all agents share common uncertain utility.

her favorite object and, moreover, in all realized allocations, assigns equal probability to each of her utility vectors. Since her updated beliefs remain the same as ex ante, she does not envy any other agent. For all other agents, the information about the utility profiles is identical under ν and ν^*. Therefore, since ν^* is interim envy-free, it follows that no agent in $\{2, \ldots, n+1\}$ wants to deviate from the assignment given by ν. This completes the proof.

The theorem establishes that an interim envy-free allocation is possible when agents' preferences are moderate, making the options closer to substitutes. Specifically, it states that if the ratio of an agent's expected utility for their most preferred to least preferred option is within a bound of $1 + \epsilon_n$ (where ϵ_n is a small threshold), an interim envy-free outcome can be achieved. This bound limits how much more an agent values their top choice over others, ensuring that no option stands out as overwhelmingly superior. As a result, agents are more willing to accept different allocations without significant loss in expected utility, viewing them as nearly interchangeable.

5 Practical Applications on Mexico City

In this section, I describe the centralized school assignment system in Mexico City and the dataset used for this study. I provide descriptive statistics that highlight inefficiencies in the system and demonstrate that students' preference lists do not align with their chances of being assigned to each school. I then explain how uncertainty in students' preferences is incorporated and simulate the assignment process with six representative students, varying in GPA and gender. I conclude the section with the analysis of the simulation results.

5.1 Centralized School Assignment in Mexico City

In Mexico City, students have access to various types of schools for their secondary education. Public high schools managed by a local commission (COMIPEMS, by its Spanish acronym) represent a centralized system that oversees admissions for a specific group of 16 upper secondary institutions, with placements based on student ranked preferences and entrance exam scores. There are also options without the centralized assignment process: open-admission public schools that allow students to enroll based on available capacity and private schools which set their own admission criteria.

Students apply to the centralized system during the application period, which typically opens in early March of their final year in middle school (ninth grade). Prior to registration, students receive a booklet that outlines the application timeline, detailed instructions, and a list of available schools along with their key characteristics. In addition to the registration form, students complete a socio-demographic survey and submit a ranked list of up to 20 schools. The entrance examination is administered in early June, and priority in the assignment system is determined based on total scores from this standardized test. The matching algorithm sequentially assigns priority-ranked applicants to their most preferred available school.

In the system, each student is assigned to one school based on their preferences and exam scores. When a tie occurs for the last spot at a school, the local commission decides whether to accept all tied students or none. If students are not placed, they can apply to schools with open seats afterward or look for schools outside of the system. Those unhappy with their assignment can also request admission to a different school. The system encourages students to carefully consider their preferences, as participating in the second round usually leads to placements outside their original choices, which may affect their satisfaction.

5.2 Dataset Overview

I use a student-level administrative dataset from the centralized high school admission system in Mexico City that includes data from 302,709 students who participated in the 2020 admission cycle. It contains student characteristics (e.g., gender, GPA, residence, exam scores), student preferences (e.g., the rank-ordered lists), and the final school assignment outcomes, including multiple rounds of assignment. It has a nearly equal distribution of male and female students, with 50.17% of the students identified as female and 49.83% as male. Approximately 94.57% of the students in the dataset took the exam, and the remaining 5.43% did not participate.

The dataset mostly includes students from Mexico City and its metropolitan zone. The

Metropolitan Area of Mexico City excluding CDMX accounts for 45.51% of the students (137,767), containing municipalities from the State of Mexico and Hidalgo that are closely connected to the city. Additionally, 44.23% of the students (133,874) come from the 16 boroughs of Mexico City (CDMX). A smaller portion, 10.26% (31,068), is from regions outside of Mexico City and its metropolitan zone. The highest number of observations in the dataset comes from the municipality of Iztapalapa, with 30,143 students residing there. This significant representation indicates that Iztapalapa is a major contributor to the overall student population applying to schools in Mexico City.

The dataset reveals that 84.31% of students received assignments in the first round, 10.26% were assigned in the second round, while 5.43% were not assigned to any school by the assignment mechanism. A quarter of students listed 6 or fewer schools, while the median is 10, indicating that half of the students listed more than 10 schools. A quarter of students were assigned to their first ranked school.

5.3 District Selection and Subgroup Characteristics

Iztapalapa was chosen as the study focus because the largest proportion of observed students reside in this district and it provides a good representation of Mexico City's youth and working-class demographics. With over 1.8 million residents with diverse socioeconomic conditions, the district offers a range of public schools with varying quality and admission criteria. Its proximity to Coyoacán, a municipality known for high-quality schools, offers students the chance to pursue prestigious education options. Public schools remain the preferred choice for Iztapalapa's residents, providing accessible and affordable education that meets the community's diverse needs. For these reasons, insights gained from studying Iztapalapa can inform broader policy discussions aimed at improving access to quality education across Mexico City.

Within the Iztapalapa observations, 89.6% of students were assigned in the first round, while 6.9% were not assigned by the mechanism. I focus specifically on students who took the exam in the morning and received a school assignment in the first round, resulting in a subgroup of 14,354 observations. The gender distribution in this subgroup is nearly balanced, with approximately 50.71% male and 49.29% female students. The average length of the submitted rank-ordered list is 11.66 schools, ranging from 1 to 20 with a standard deviation of 4.46. The average student is placed in a school ranked 5.36th on their list. Nearly 20% of students in the subgroup were assigned to their first-choice school.

5.3.1 Most Commonly Assigned Schools

Given that students are, on average, assigned to their 5.36th ranked school, this section describes the six schools that receive the most students from the Iztapalapa municipality. Admissions to these schools are primarily determined by entrance exam scores, with each program's cutoff score set by the lowest exam score among students admitted during the assignment process. Schools with higher cutoff scores require students to achieve higher exam results to gain admission, making them more competitive.

School Name	**Number of Students Assigned**
CB 6	1497
CB 7	824
CB 3	775
CB 4	632
CCH Oriente, UNAM	545
CECyT 7, IPN	525

Table 1: Assignment Counts for the Six Most Assigned Schools

The six most commonly assigned schools include four campuses from the Colegio de Bachilleres (CB) network: Plantel 6 'Vicente Guerrero' (CB 6) and Plantel 7 'Iztapalapa' (CB 7), both located in Iztapalapa, with CB 6 being slightly more competitive. Other CB campuses, such as Plantel 3 'Iztacalco' (CB 3) and Plantel 4 'Culhuacán' (CB 4), are also in high demand, with CB 4 typically requiring higher scores for admission.

In addition to these, the Colegio de Ciencias y Humanidades, Plantel Oriente (CCH Oriente, UNAM) is part of the prestigious UNAM system, recognized for its rigorous academic standards and high entry requirements. Finally, CECyT NÚM. 7 'Cuauhtémoc' (CECyT 7, IPN), a technical school under the IPN system, attracts students with its specialized programs in fields such as engineering. These two institutions further require applicants to have a minimum GPA of 7 out of 10. Table 1 shows the number of students from Iztapalapa assigned to each school.

5.3.2 Exam Scores and GPA

Exam scores of the students range from 17 to 126, with a possible minimum of 0 and a maximum of 128. Figure 1 shows that the scores are spread across various levels, indicating heterogeneity in student performance. There is a concentration around the 60-70 range, where most students scored, while the symmetric shape suggests a wide spread, with fewer students at the higher and lower extremes.

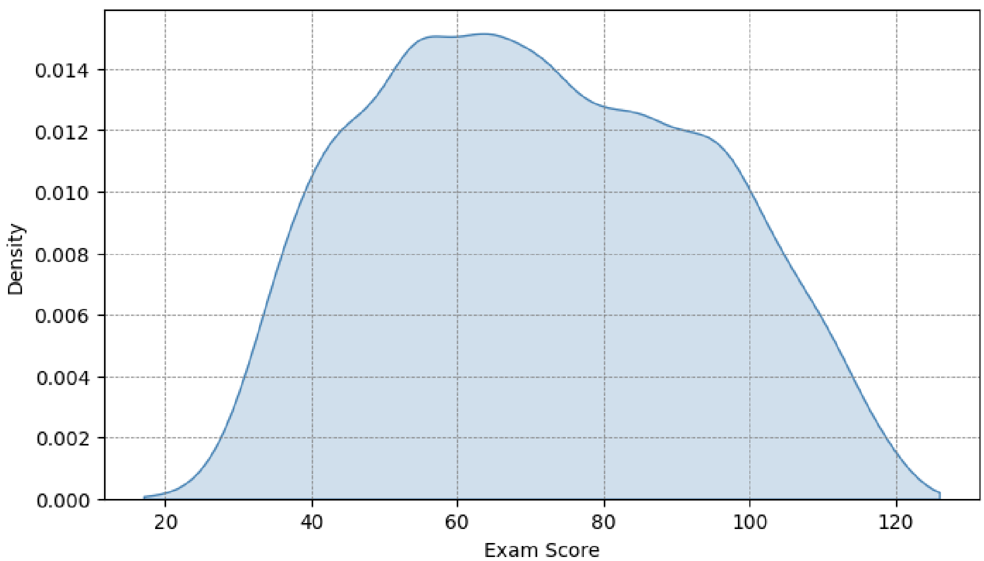

Figure 1: Distribution of Exam Scores

Figure 2 displays the distribution of exam scores conditional on students receiving their k-th ranked option as their final assignment. The first column shows the score distribution for students assigned to their top-choice school, the second column for those assigned to their second choice, and so forth. Additionally, the score distributions are color-coded by gender: red for female students and blue for male students, providing a visual comparison of performance across assigned preferences. Wider sections in the plots indicate higher concentrations of scores, and the distributions are relatively symmetrical, suggesting similar performance patterns across genders without significant advantage. Overall, maximum scores decrease as assignment rank increases, indicating that higher-scoring students tend to secure more favorable placements. However, both high- and low-scoring students often receive one of their top four choices, and in some cases, lower-scoring students are placed in more desirable schools than their higher-scoring peers, indicating a potential misalignment in the allocation mechanism.

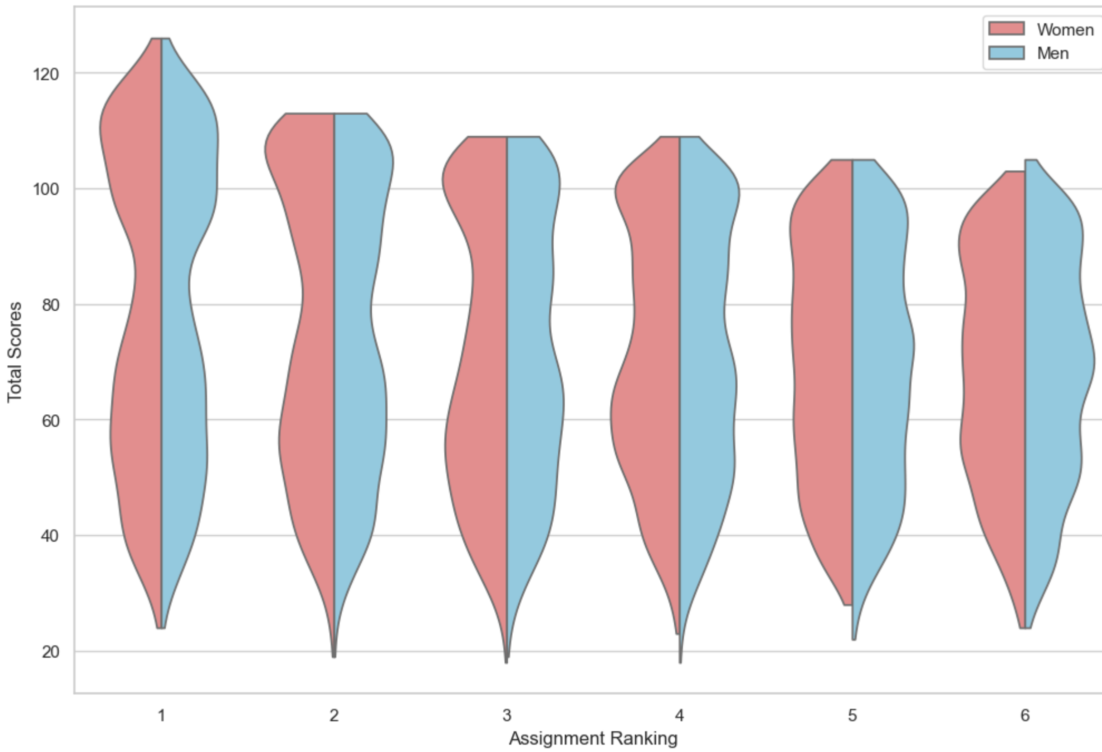

Figure 2: Distribution of Exam Scores for Top 6 Choices.

When students submit their preferences, they only know their GPA. GPA ranges from 6.0 to 10.0 as 6.0 is the minimum required to graduate. I categorized the students based on their GPA using the 25th (Q1) and 75th (Q3) percentiles. Students above the 75th percentile in GPA are labeled as high, those below the 25th percentile as low, and those in between are classified as medium. Figure 3 shows that total exam score and GPA are positively correlated, indicating that students with higher GPAs tend to achieve higher scores on the exam. However, GPA alone does not fully account for exam performance, as there is considerable variation in scores within each GPA category. Outliers are particularly common in both the low and high GPA groups. The high variability observed at the 6.0 GPA threshold may reflect that some students focus primarily on meeting the minimum passing requirements while concentrating their efforts on exam preparation, given the importance of exam scores in assignment decisions.

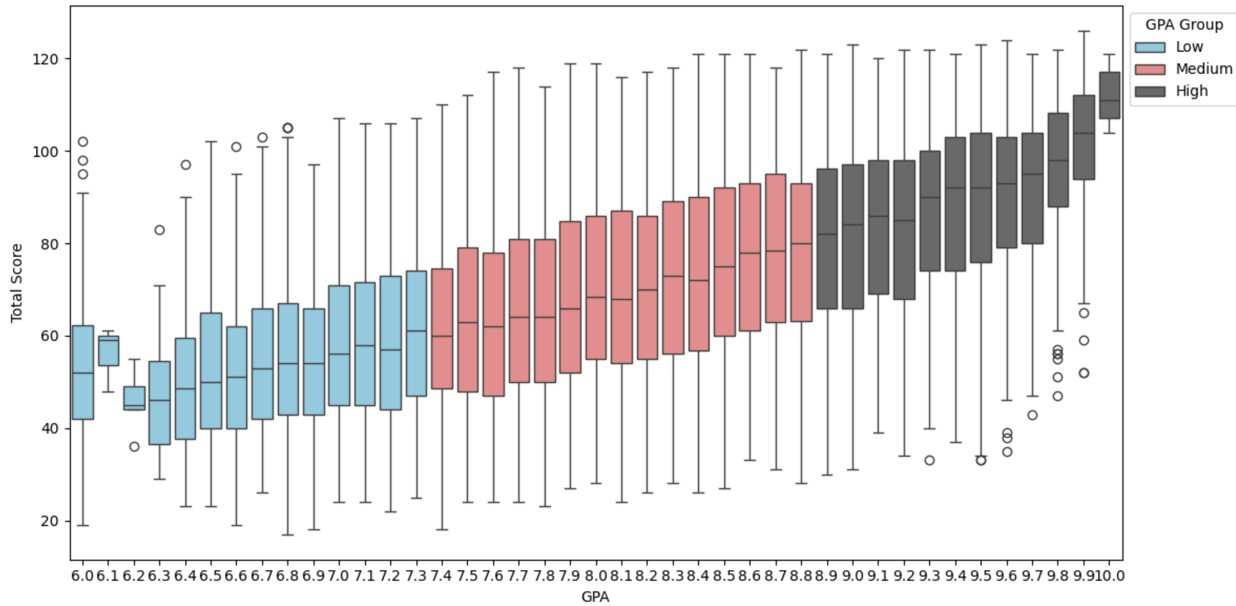

Figure 3: Total Scores by GPA Type

To further analyze the relationship between GPA and exam scores, I regressed exam score percentage on GPA (converted to a percentage) and a gender variable, where male is assigned a value of 1. Results indicate that both GPA and gender significantly affect exam scores, with male students scoring on average 6.11 percentage points higher than female students, holding GPA constant. The regression findings are summarized below in Table 2.

Variable	Coefficient	Std. Error	t-Statistic	P-value
Constant	30.0797***	0.370	81.243	0.000
GPA	0.4246***	0.006	76.127	0.000
Gender	6.1054***	0.254	23.996	0.000
Model Summary:				
R-squared		0.288		
Adjusted R-squared		0.288		
F-statistic		2904		
Prob (F-statistic)		0.000		
Number of Observations		14353		

Table 2: Regression Analysis of Exam Scores on GPA and Gender

5.4 Evidence of Inefficiencies in the Centralized System

Using data from Iztapalapa, this section highlights inefficiencies within the allocation system by examining the relationship between student exam scores and their assigned schools. Although the system is designed to align assignments with student preferences and performance, observed patterns reveal possible misalignments. These findings encourage a closer examination of how assignments vary across score levels and raise questions about potential feelings of envy and associated welfare losses.

We categorize students based on their exam scores: those above the 75th percentile (high threshold) are labeled as high, those below the 25th percentile (low threshold) as low, and those in between as medium. As shown in Table 3, the lists submitted by each group are similar in terms of their length.

Metric	Statistic	Low Score	Medium Score	High Score
Length of Submitted List	Mean	11.49	11.91	11.32
	Std Dev	4.44	4.42	4.53
	Min	1	1	1
	25%	9	9	8
	50%	11	11	11
	75%	14	15	14
	Max	20	20	20
Assignment Ranking	Mean	5.89	6.07	3.41
	Std Dev	4.26	3.94	2.69
	Min	1	1	1
	25%	2	3	1
	50%	5	6	3
	75%	8	9	5
	Max	20	20	18
Assignment Cutoff Scores	Mean	29.98	54.95	93.08
	Std Dev	9.96	14.62	14.63
	Min	11	11	17
	25%	21	50	91
	50%	30	53	96
	75%	37	65	101
	Max	52	88	114

Table 3: Summary Statistics for Low, Medium, and High Score Students.

Interestingly, the median student with a high score receives their 3rd choice, while a median student in the low-score group gets their 5th choice, and in the medium-score group, their 6th choice. If preferences were similar, we would expect low-score students to receive lower-ranked options than medium-score students, but this is not observed. Further evidence of this inconsistency is that the first quantile of the medium group is assigned to schools with a cutoff score of 50, despite having exam scores above 53. These patterns suggest that either medium-score students prefer less selective schools, or that low-score students are favored by the mechanism.

To examine this further, Figure 4 presents the conditional probability of a student securing *at least* their k-th choice based on their exam score, where $k \in \{1, 2, \dots, 6\}$. The rationale is that higher-scoring students should consistently obtain more preferred options, and for the system to prevent envy, this probability should increase with score. For each k-th choice, we create an indicator for whether a student achieved at least that rank and group students by total score. For each unique score, we calculate the average proportion of students who obtained at least their k-th choice, providing a measure of how score influences placement likelihood across preference rankings.

The students with nearly perfect exam scores are consistently assigned to their top-ranked school, and this pattern holds when considering the probability of securing at least their preferred k-th choice. However, Figure 4 also reveals an unexpected result: students with lower scores (around 25) are often more likely to receive at least their k-th preferred school compared to students with moderate scores (around 80). This inconsistency is concerning, as it implies that higher scores do not necessarily translate to better outcomes.

5.5 Preferences from Submitted Lists

While students' submitted preference lists may not entirely capture their true choices, examining differences across these lists can still provide valuable insights. Such analysis allows us to explore how students approach school selection and to identify potential factors that shape their preferences.

A closer look at the submitted lists for each GPA group (Figure 5) reveals that high- and medium-GPA students tend to rank competitive schools high on their lists, with outliers

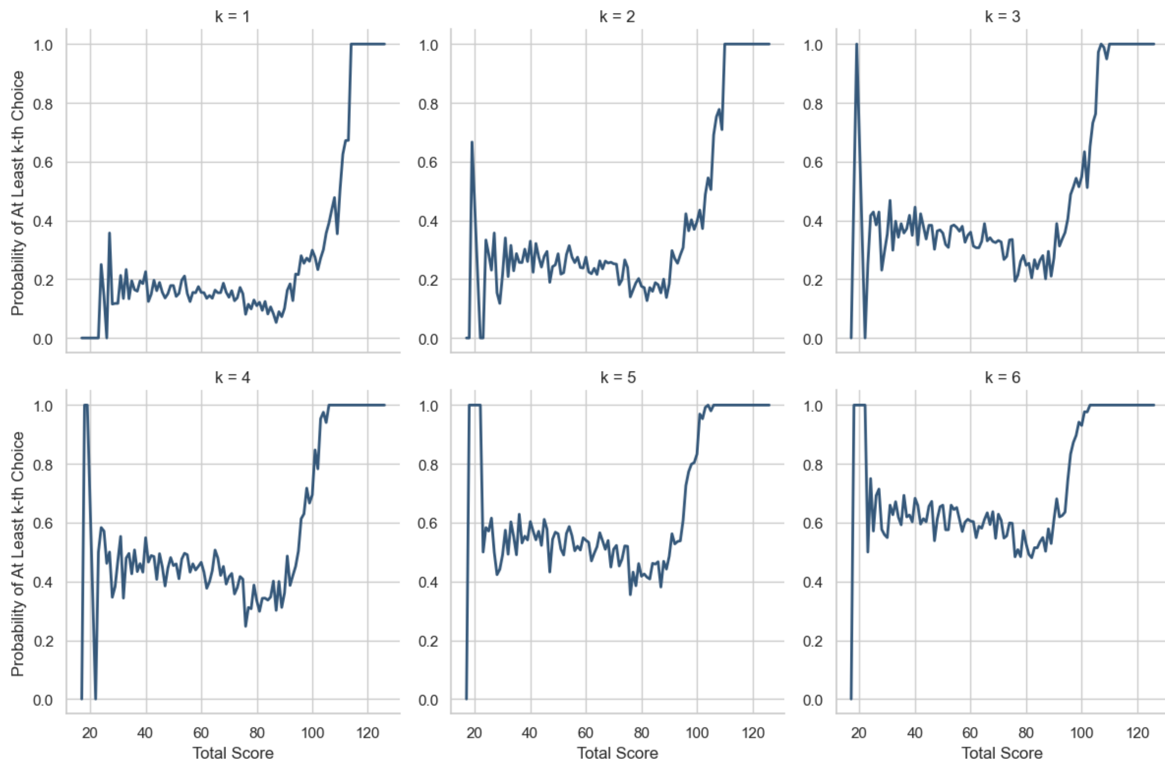

Figure 4: Conditional Probability of Getting at least the k-th preferred school, condition on exam score.

appearing in groups who also include very safe choices with low cutoffs. The median cutoff of schools ranked by the medium- and high-GPA groups decreases only gradually, reflecting a consistent focus on competitive options. In contrast, low-GPA students display no outliers and primarily list very safe schools. The box for the low-GPA group shows minimal overlap with the medium-GPA group in their top choices, while medium-GPA students frequently compete with high-GPA students for more competitive schools. Notably, each group includes at least one student who ranks a highly competitive school in their list.

The lack of outliers in the low-GPA group may be due to the GPA requirements of certain schools, which could restrict low-GPA students to applying for only safer choices. However, Figure 5 suggests there may be differences between these groups beyond just GPA restrictions. To investigate further, we exclude students with a GPA below 7 and examine the gap between their exam scores and the cutoff of the schools they were assigned to by the mechanism. The regression in Table 4 shows that GPA becomes insignificant. The intercept for the medium type is larger than the one for low type, indicating that the distance

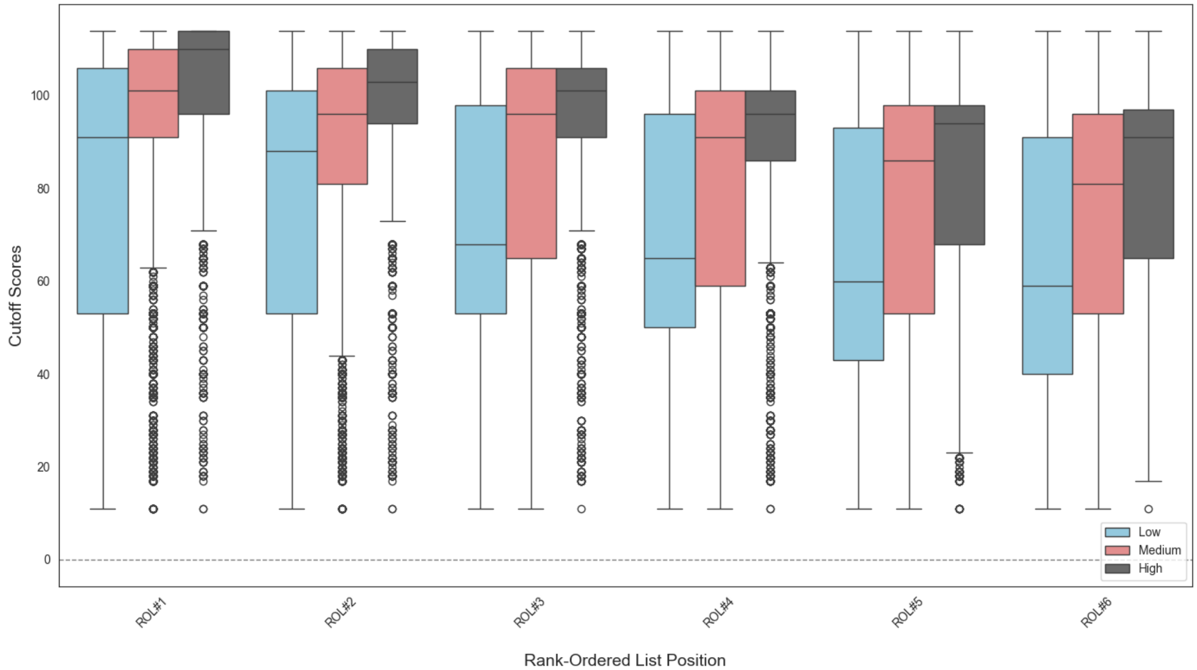

Figure 5: Distribution of Cutoff Scores Across Rankings in the Rank-Ordered List (ROL) for Low, Medium, and High GPA Groups

between their score and the assigned choice cutoff is larger. This can be either because they do not know their score so they do not know their potential or they prefer safer schools. Nevertheless, the regression results, along with Figure 5, imply that the differences between the lists of schools across groups cannot be fully explained by the GPA requirements of some schools. In particular, the medium and low groups seem to rank schools according to personal preferences rather than purely based on their likelihood of admission.

Variable	Coefficient	Std. Error	t-Statistic	P-value
Constant	10.5288***	1.086	9.693	0.000
Medium Score Type	3.2662***	0.264	12.351	0.000
High Score Type	-4.2596***	0.323	-13.196	0.000
GPA	0.0521	0.137	0.381	0.703
Model Summary:				
R-squared		0.073		
Adjusted R-squared		0.073		
F-statistic		336.6		
Prob (F-statistic)		0		
Number of Observations		12838		

Table 4: Regression Analysis of Score Difference on Score Type and GPA

5.6 Numerical Simulations

Building on the previous analysis, I group students by GPA and gender to create representative categories. I use GPA instead of exam scores since students do not know their scores at the time of preference submission, and Figure 5 shows that preferences differ across groups. The goal of creating these categories is to construct an environment that accounts for the uncertainty in preferences that rank-ordered lists cannot capture.

I use the six schools most frequently assigned to students in Iztapalapa and analyze their frequency in students' Rank Ordered Lists (ROLs) across different demographic groups (e.g., female high GPA, male low GPA). By calculating the proportion of ROL positions for each school, I convert these counts into probabilities that reflect the likelihood of each school being selected at each ROL position. This transformation allows the simulation to better capture student preferences in a structured environment. The probability distributions are shown in Table 5.

I use probability distributions to randomly generate two preference rankings, representing ordinal utilities. For each agent, I then create two cardinal utility vectors using the values 10, 8, 6, 4, 2, and 1 to represent preference strength. These utilities provide a quantitative basis for preferences, simulating variability in choice behavior. Next, I created utility profiles for each group by taking the Cartesian product of their respective utility vectors, thereby capturing all possible combinations of preferences. By assigning a uniform probability to each profile, the model assumes each utility configuration is equally likely, allowing for a balanced and comprehensive exploration of potential preference patterns within the allocation system.

Fairness-Welfare Optimization

The algorithm described in Section 3.2 used to find the allocation rule that improves welfare and eliminates envy in the each possible scenario. While the algorithm can achieve fairness without imposing a priority order among students, I choose to assume that high GPA students are prioritized. This decision reflects societal norms that often prioritize academic merit, making it more realistic to consider an allocation fair when students with higher academic achievement receive precedence. Thus, I modified the algorithm to prevent envy from

Table 5: School Preferences by Gender and GPA Type

School	Rank Ordered List					
	#1	#2	#3	#4	#5	#6
Women with High GPA						
CCH Oriente, UNAM	0.66	0.65	0.56	0.46	0.44	0.31
CECyT 7, IPN	0.26	0.21	0.22	0.22	0.19	0.17
CB 3	0.02	0.04	0.05	0.08	0.10	0.14
CB 4	0.01	0.01	0.02	0.03	0.04	0.07
CB 6	0.03	0.05	0.07	0.12	0.14	0.15
CB 7	0.01	0.05	0.08	0.08	0.10	0.16
Men with High GPA						
CCH Oriente, UNAM	0.36	0.52	0.48	0.43	0.37	0.31
CB 3	0.02	0.05	0.05	0.12	0.09	0.12
CB 6	0.03	0.05	0.09	0.10	0.11	0.15
CECyT 7, IPN	0.58	0.36	0.32	0.24	0.23	0.22
CB 4	0.01	0.01	0.02	0.03	0.06	0.08
CB 7	0.01	0.01	0.05	0.09	0.13	0.12
Women with Medium GPA						
CCH Oriente, UNAM	0.59	0.47	0.39	0.32	0.31	0.24
CB 7	0.03	0.09	0.11	0.17	0.15	0.18
CB 4	0.03	0.04	0.05	0.06	0.07	0.08
CB 3	0.04	0.08	0.12	0.12	0.15	0.17
CB 6	0.09	0.13	0.14	0.18	0.19	0.19
CECyT 7, IPN	0.21	0.20	0.19	0.16	0.14	0.14
Men with Medium GPA						
CCH Oriente, UNAM	0.40	0.44	0.40	0.33	0.32	0.25
CECyT 7, IPN	0.49	0.33	0.24	0.21	0.19	0.19
CB 3	0.03	0.04	0.09	0.11	0.13	0.13
CB 6	0.05	0.08	0.15	0.16	0.16	0.19
CB 7	0.02	0.08	0.08	0.14	0.14	0.17
CB 4	0.02	0.03	0.04	0.05	0.07	0.07
Women with Low GPA						
CCH Oriente, UNAM	0.43	0.32	0.29	0.24	0.19	0.14
CB 6	0.18	0.18	0.22	0.20	0.19	0.26
CECyT 7, IPN	0.18	0.16	0.11	0.13	0.10	0.12
CB 4	0.05	0.06	0.05	0.06	0.10	0.09
CB 7	0.06	0.17	0.17	0.22	0.24	0.22
CB 3	0.09	0.11	0.17	0.15	0.19	0.17
Men with Low GPA						
CCH Oriente, UNAM	0.31	0.32	0.24	0.20	0.22	0.18
CECyT 7, IPN	0.33	0.19	0.17	0.15	0.14	0.16
CB 6	0.14	0.17	0.21	0.21	0.22	0.20
CB 7	0.08	0.14	0.15	0.20	0.18	0.21
CB 3	0.10	0.11	0.16	0.15	0.15	0.18
CB 4	0.05	0.06	0.07	0.08	0.08	0.08

high GPA students toward those with lower GPAs and among students within the same GPA category.

Deferred Acceptance (DA)

In this setup, I run DA for each utility profile, and the allocation mechanism assigns the allocation found for that specific utility profile. We assume students with high GPA have the highest priority, followed by students with medium GPA, and finally, students with low GPA. In each utility profile, we have two students from each GPA category, and there is only one seat available at each school. Due to this, we assume that DA prioritizes male students over female students within the same GPA group to establish a priority ranking for school assignments.

Maximum Nash Welfare (MNW)

The Maximum Nash Welfare (MNW) mechanism aims to maximize the Nash product of utilities, ensuring that the allocation maximizes total welfare while balancing the interests of all agents. This mechanism seeks to identify an allocation that achieves the highest level of joint satisfaction. The MNW allocation is derived from the outcomes of the MNW algorithm applied to each utility profile.

5.7 Analysis of Simulation Results

This section summarizes the findings from comparing the algorithms in terms of expected welfare and fairness violations, where a fairness violation occurs if an agent's expected utility from another agent's allocation exceeds that from her own allocation. These results are based on 100 simulated environments.

Welfare Comparisons

Expected welfare is assessed by evaluating the average utility that agents derive from the allocation of objects under different mechanisms. Each mechanism produces different outcomes based on agents' preferences and the associated probabilities of these preferences. The expected welfare is calculated by taking into account the probabilities of each utility profile and the resulting utilities from the allocations.

The Welfare Statistics table, Table 6, compares the performance of four allocation mechanisms. The table highlights that while Utility Maximizing and MNW algorithms produce the highest average and median welfare outcomes, they prioritize efficiency at the cost of fairness. These algorithms also demonstrate the most consistent welfare results, with higher minimum values and lower variability. The Fairness-Welfare Optimization (Fairness Optimization), while slightly lower in welfare outcomes, achieves a balance by maintaining fairness without significantly sacrificing overall welfare. In contrast, DA achieves the lowest welfare outcomes, as its primary goal is to ensure stability rather than maximizing efficiency.

Table 6: Welfare Statistics

Algorithm	Avg	Median	Min	Max	Std Dev	95% CI
Fairness Optimization	45.23	45.00	35.50	52.16	2.99	(40.26, 51.23)
DA	34.68	34.94	25.00	42.88	3.79	(28.49, 42.75)
Utility Maximizing	48.96	49.07	44.34	53.06	1.92	(45.26, 52.85)
MNW	48.88	48.92	42.45	53.06	2.05	(45.19, 52.85)

Fairness Comparisons

The Fairness Comparisons in Table 7 highlights how different allocation mechanisms perform in terms of fairness by measuring fairness violations. A fairness violation is calculated as the difference between agent i's expected utility from agent j's allocation and her own allocation, in cases where agent i prefers agent j's allocation. This metric ensures that violations are quantified based on agents' preferences, capturing how much better off one agent perceives they would be with another agent's allocation. This measure reveals not only whether a mechanism meets fairness criteria but also the extent of unfairness in situations where it fails.

The Fairness Optimization mechanism stands out with zero fairness violations, confirming that it effectively ensures fairness in every allocation scenario. On the other hand, the Utility Maximizing mechanism, while focusing on maximizing welfare, shows the highest fairness violations, indicating it often overlooks individual fairness to achieve higher overall efficiency.

Similarly, the DA mechanism has substantial fairness violations, with an average fairness violation of 3.74 units, suggesting agents frequently feel they could benefit significantly from another allocation. MNW shows fewer fairness violations than DA but still fails to fully address fairness concerns, indicating it provides a better balance between welfare and fairness than DA, but not to the level of Fairness Optimization.

Table 7: Fairness Violation Statistics

Algorithm	Avg	Median	Min	Max	Std Dev	95% CI
Fairness Optimization	0.00	0.00	0.00	0.00	0.00	(0.00, 0.00)
DA	3.74	2.77	0.38	10.97	2.64	(2.98, 4.50)
Utility Maximizing	4.16	3.38	0.81	14.92	2.70	(3.39, 4.94)
MNW	3.66	3.33	0.72	11.95	2.22	(3.02, 4.30)

To interpret these simulation results, consider fairness violations as a form of welfare loss. In this regard, if fairness is equally important to a central planner, the expected benefits of the utility-maximizing mechanism are outweighed by its fairness violations. This comparison highlights how the fairness-optimized mechanism achieves similar welfare gains without generating any welfare loss.

6 Conclusion

This paper addresses the conflicting objectives of fairness and welfare in the allocation of indivisible objects, with a focus on school assignments. A key distinction of this framework compared to previous ones is its departure from assuming that students enter the assignment process with clear, well-defined preferences. Instead, it acknowledges that students may lack fixed or fully formed preferences and incorporates a model of preference uncertainty to better reflect real-world decision-making complexities. This adaptation allows the model to generate allocations more closely aligned with students' likely preferences, even when those preferences are not fully determined at the outset. Consequently, the proposed approach seeks to achieve fairer outcomes that respect the inherent unpredictability in student choices while maintaining a high standard of welfare.

To conclude, we revisit the example of Ariel and Ben, who share the same ordinal preferences for School A and School B but differ in the intensity and reasons behind their choices. The analysis in this paper reveals that collecting detailed information about students' preferences can significantly improve allocation outcomes. For example, the results suggest that if education authorities knew that a scholarship played a significant role in Ariel's decision, they could inform her of its availability, allowing her to reconsider her top choice and alleviating any potential envy from Ben. Similarly, the findings indicate that if Ben's satisfaction depends on the quality of sports facilities, providing him with relevant information could ensure he is satisfied with his final outcome. By obtaining specific information on individual preferences, authorities can better align assignments with individual needs, enhancing both fairness and satisfaction. Overall, this study recommends that education authorities collect comprehensive information on students' preferences and communicate relevant insights back to students to support a more personalized decision-making process. Ideally, authorities could understand students' priorities so well that they can help guide choices that align with each student's genuine needs.

In countries like Turkey, for instance, school reputation often drives student preferences, leading many to rank the same top schools. By providing students with targeted information, such as details about attainable schools that align with their broader needs, including academic quality and specialized programs, authorities can encourage students to make more informed and satisfying choices. This approach allows students to consider schools where they are more likely to thrive, reducing the competition around top-ranked preferences by highlighting factors beyond reputation.

A practical policy implication could be a two-step application process. In the first step, students submit their initial preferences, which are analyzed by authorities to identify trends and assignment probabilities. In the second step, authorities provide individualized feedback, suggesting realistic choices aligned with each student's preferences and likely outcomes. This iterative feedback empowers students to refine their final choices, resulting in placements that better meet both individual and system-wide needs.

The main objective of this paper is not to propose a new algorithm to replace assignment systems in Mexico City or elsewhere. Instead, it aims to uncover the inefficiencies within current systems, bringing attention to underlying causes and aspects we often overlook or assume by default. To truly improve outcomes, we need to change our approach, recognizing that effective school assignments depend on a deep understanding of students' varied needs and priorities, and on turning this understanding into meaningful actions. By questioning assumptions and rethinking these challenges, this paper contributes to the growing literature and lays the groundwork for future efforts to develop assignment systems that genuinely empower students, respect their goals, and strengthen the foundation for equitable education. An important next step involves finding practical ways to capture the specifics of students' preferences and incorporate these insights into allocation processes. Such advancements would bring us closer to a system that truly aligns with individual needs and promotes fairness.

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