**CSE 211: Discrete Mathematics** **(Due: 24/12/19)**

# Homework #4

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**Course Policy**: Read all the instructions below carefully before you start working on the assignment, and before you make a submission.

- It is not a group homework. Do not share your answers to anyone in any circumstance. Any cheating means at least -100 for both sides.
- Do not take any information from Internet.
- No late homework will be accepted.
- For any questions about the homework, send an email to gizemsungu@gtu.edu.tr
- Submit your homework into Assignments/Homework4 directory of the CoCalc project CSE211-2019-2020.

**Problem 1: Nonhomogeneous Linear Recurrence Relations** (15+15=30 points)

Consider the nonhomogeneous linear recurrence relation $a_n = 3a_{n-1} + 2^n$ .

**(a)** Show that whether $a_n = -2^{n+1}$ is a solution of the given recurrence relation or not. Show your work step by step.

***(Solution)***

$a_{n-1} = -2^{n-1+1} = -2^n$

*Put the given equation into the main equation:*

$$\begin{aligned} 3a_{n-1} + 2^n &= 3(-2^n) + 2^n \\ &= -3(2^n) + 2^n \\ &= -2(2^n) \\ &= -2^{n+1} \\ &= a_n \end{aligned}$$

***So $a_n = -2^{n+1}$ is a solution of the given recurrence relation.***

**(b)** Find the solution with $a_0 = 1$.

***(Solution)***

$$a_n = \{{a_n}^{(h)}\} + \{{a_n}^{(p)}\}$$

①***Characteristic equation*** *(degree 1)*
$r = 3$

②***Solution for homogeneous part*** $\{{a_n}^{(h)}\}$
$a_n = \alpha r^n$
$a_n^{(h)} = \alpha 3^n$

③***Solution for particular part*** $\{{a_n}^{(p)}\}$
*Let* $F(n) = 2^n$
$a_n = c(2^n)$

$c(2^n) = 3(c2^{n-1}) + 2^n$
$2^n c - 2^n - 3(2^n)2^{-1}c = 0$
$2^n(c - 1 - 3(2^{-1})c) = 0$
$2^n(-\frac{c}{2} - 1) = 0$
$2^n$ *cannot be zero so* $(-\frac{c}{2} - 1) = 0$
$c = -2$

*So* $\{{a_n}^{(p)}\} = -2(2^n)$
$= -2^{n+1}$ *(same with part a)*

④***So :***
$a_n = \alpha 3^n - 2^{n+1}$
$a_0 = 1$
$a_0 = \alpha 3^0 - 2^{0+1}$
$1 = \alpha - 2$
$\alpha = 3$

④***So the solution of the nonhomogeneous linear recurrence is :***
$a_n = 3(3^n) - 2^{n+1}$
$a_n = 3^{n+1} - 2^{n+1}$

**Problem 2: Linear Recurrence Relations** (35 points)

Find all solutions of the recurrence relation $a_n = 7a_{n-1}$ - $16a_{n-2}$ + $12a_{n-3}$ + n$4^n$ with $a_0$ = -2, $a_1$ = 0, and $a_2 = 5$.

***(Solution)***

$$a_n = \{a_n{}^{(h)}\} + \{a_n{}^{(p)}\}$$

① ***Characteristic equation*** *(degree 3)*

$r^3 = 7r^2 - 16r + 12$
$0 = r^3 - 2r^2 - 5r^2 + 10r + 6r - 12$
$0 = r^2(r-2) - 5r(r-2) + 6(r-2)$
$0 = (r-2)(r^2 - 5r + 6)$
$0 = (r-2)(r^2 - 2r - 3r + 6)$
$0 = (r-2)(r-2)(r-3)$

$r_1 = 2$ *(double root)*
$r_2 = 3$

② ***Solution for homogeneous part*** $\{a_n{}^{(h)}\}$

$7a_{n-1} - 16a_{n-2} + 12a_{n-3}$ *is homogeneous.*

$$\{a_n{}^{(h)}\} = (\alpha_{1,0} + \alpha_{1,1}n)r_1^n + \alpha_{2,0}(r_2^n)$$
$$= (\alpha_{1,0} + \alpha_{1,1}n)2^n + \alpha_{2,0}3^n$$

③ ***Check F(n) (s = r OR s ≠ r)***

*In nonhomogeneous linear recurrence relation;*

$F(n) = (b_t n^t + b_{t-1}n^{t-1} + ... + b_1 n + b_0)s^n$

*Suppose* $b_0, b_1, ...,$ *s are real numbers.*

*Let* $F(n) = n4^n$
$s = 4$ , $deg(p(n)) = 1$

*In our equation , s is 4. And it is not a root of the characteristic equation. Then we should write:*

④ ***Solution for particular part*** $\{a_n{}^{(p)}\}$

$$\{a_n{}^{(p)}\} = (p_t n^t + p_{t-1}n^{t-1} + ... + p_1 n + p_0)s^n$$
$$\{a_n{}^{(p)}\} = (p_1 n + p_0)4^n$$

⑤ ***Put particular part on implicit equation :***

$$\begin{aligned} p_1.n.4^n + p_0.4^n = {} & 7.p_1.(n-1).4^{n-1} + 7.p_0.4^{n-1} \\ & -16.p_1.(n-2).4^{n-2} - 16.p_0.4^{n-2} \\ & +12.p_1.(n-3).4^{n-3} + 12.p_0.4^{n-3} \\ & +n.4^n \end{aligned}$$

*Write all equations* $(4^{n-3})$ *form :*

$$\begin{aligned} 4^3.p_1.n.4^{n-3} + 4^3.p_0.4^{n-3} = {} & 4^2.7.p_1.(n-1).4^{n-3} + 4^2.7.p_0.4^{n-3} \\ & -4.16.p_1.(n-2).4^{n-3} - 4.16.p_0.4^{n-3} \\ & +12.p_1.(n-3).4^{n-3} + 12.p_0.4^{n-3} \\ & +4^3.n.4^n \end{aligned}$$

*Simplify equations :*

$4^{n-3}[64.p_1.n + 64.p_0 - 60.p_1.n + 20.p_1 - 60.p_0 - 64.n] = 0$
$4^{n-3}[n(4.p_1 - 64) + (4.p_0 + 20.p_1)] = 0$

$4^{n-3}$ *cannot be zero. So :*

| | |
|---|---|
| $n(4.p_1 - 64) = 0$ | $(4.p_0 + 20.p_1) = 0$ |
| $4.p_1 - 64 = 0$ | $p_0 = -80$ |
| $p_1 = 16$ | |

$\{a_n{}^{(p)}\} = 16.n.4^n - 80.4^n$

⑥ ***Write :*** $a_n = \{a_n{}^{(h)}\} + \{a_n{}^{(p)}\}$

$$a_n = \alpha_{1,0}.2^n + \alpha_{1,1}.n.2^n + \alpha_{2,0}.3^n + 16.n.4^n - 80.4^n$$

*Apply* $a_0 = -2, a_1 = 0$ *and* $a_2 = 5$ *:*

$$-2 = \alpha_{1,0}.2^0 + \alpha_{1,1}.0.2^0 + \alpha_{2,0}.3^0 + 16.0.4^0 - 80.4^0$$
$$0 = \alpha_{1,0}.2^1 + \alpha_{1,1}.1.2^1 + \alpha_{2,0}.3^1 + 16.1.4^1 - 80.4^1$$
$$5 = \alpha_{1,0}.2^2 + \alpha_{1,1}.2.2^2 + \alpha_{2,0}.3^2 + 16.2.4^2 - 80.4^2$$

*Continue to simplify :*

$$-2 = \alpha_{1,0} + \alpha_{2,0} - 80$$
$$0 = 2.\alpha_{1,0} + 2.\alpha_{1,1} + 3.\alpha_{2,0} - 256$$
$$5 = 4.\alpha_{1,0} + 8.\alpha_{1,1} + 9.\alpha_{2,0} - 768$$

*Multiply first equation by -4 and add with third equation :*

$$78 = \alpha_{1,0} + \alpha_{2,0}$$
$$-251 = -4.\alpha_{1,0} - 3.\alpha_{2,0}$$

*Multiply first equation by 4 and add with second equation :*

$$61 = \alpha_{2,0}$$
$$17 = \alpha_{1,0}$$
$$\alpha_{1,1} = \frac{-2.17-3.61+256}{2}$$

*So :*

$$a_n = \{a_n{}^{(h)}\} + \{a_n{}^{(p)}\}$$
$$= 17.2^n + \frac{39}{2}.n.2^n + 61.3^n + 16.n.4^n - 80.4^n$$

$a_n = 17.2^n + 39.n.2^{n-1} + 61.3^n + 4^n(16.n - 80)$ *is the result.*

**Problem 3: Linear Homogeneous Recurrence Relations** (20+15 = 35 points)

Consider the linear homogeneous recurrence relation $a_n = 2a_{n-1}$ - $2a_{n-2}$.

**(a)** Find the characteristic roots of the recurrence relation.

***(Solution)***

***Characteristic equation****(degree 3)*

$$r^2 = 2.r - 2$$
$$r^2 - 2.r + 2 = 0$$

$$\Delta = b^2 - 4.a.c$$
$$= 4 - 4.1.2$$
$$= -4$$
$$\Delta < 0$$

*So we have complex roots :*

$$r_{1,2} = \frac{-b \mp \sqrt{\Delta}}{2a}$$
$$r_{1,2} = \frac{2 \mp \sqrt{-4}}{2.1}$$
$$r_{1,2} = \frac{2 \mp 2.\sqrt{-1}}{2}$$

$r_{1,2} = 1 \pm i$ ***are roots of the recurrence relation.***

**(b)** Find the solution of the recurrence relation with $a_0 = 1$ and $a_1 = 2$.

***(Solution)***

① ***In part (a) we write char. eq.***

② ***Create*** $a_n$

$$a_n = \alpha_1.(r_1)^n + \alpha_2.(r_2)^n$$
$$a_n = \alpha_1.(1+i)^n + \alpha_2.(1-i)^n$$

③ ***Use initial condition to find unknowns :***

$$a_0 = \alpha_1.(1+i)^0 + \alpha_2.(1-i)^0$$
$$a_0 = \alpha_1 + \alpha_2 = 1$$

$$a_1 = \alpha_1.(1+i)^1 + \alpha_2.(1-i)^1$$
$$a_1 = \alpha_1.(1+i) + \alpha_2.(1-i) = 2$$

*Multiply* $a_0$ *with -(1+i) :*

$$\alpha_1.(-1-i) + \alpha_2.(-1-i) = (-1-i)$$
$$\alpha_1.(1+i) + \alpha_2.(1-i) = 2$$

*Then add them together :*

$$\alpha_2(-2i) = 1 - i$$

$$\alpha_2 = \tfrac{1-i}{-2i} = \tfrac{(1-i).i}{-2i.i} = \tfrac{i+1}{2}$$
$$\alpha_1 + \alpha_2 = 1$$
$$\alpha_1 + \tfrac{i+1}{2} = 1$$
$$\alpha_1 = \tfrac{1-i}{2}$$

④ ***Write explicit form of*** $a_n$

$$a_n = \alpha_1.(r_1)^n + \alpha_2.(r_2)^n$$

$a_n = (\tfrac{1-i}{2}).(1+i)^n + (\tfrac{i+1}{2}).(1-i)^n$ *is the result.*